Towards the solution of Schwinger-Dyson equations in Minkowski space

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1 Introduction

This PhD thesis is devoted to studies of quantum field models with strong coupling. The *Schwinger-Dyson equations* (SDEs) in momentum representation are solved in Minkowski space. The gauge invariant non-perturbative regularization scheme is constructed to treat correctly non-asymptotically free models and propagators of the strong-coupling QED in 3+1 dimension are calculated. As for QCD, our method in its present form is not able to deal with the Faddeev-Popov ghosts. However, the analyticity of the gluon propagator is exploited to continue some recent lattice data to the timelike momentum axis.

In this introduction our motivations, broad contexts and structure of the thesis are briefly summarized.

These days the experimentally accessible particle physics is being described in the framework of the (effective) quantum field theory (QFT) by the Standard model of the strong and electroweak interactions. In the perturbation theory (PT) the S-matrix elements are generated from n-point *Green's functions* (GFs) which are calculated and regularized/renormalized order by order according to the standard rules. Order by order PT can be naturally generated from expansion of the SDEs: an infinite tower of coupled equations connecting successively higher and higher points GFs.

Going beyond the PT requires development of elaborated approaches and sophisticated tools, especially in the strong coupling regime. Number of such approaches has been pursued: lattice theories, Feynman-Schwinger representation, non-perturbative treatments of SDEs, bag models . . . . These days neither of them provides such a clear and unambiguous understanding of the physics as we are accustomed to in the perturbative regime.

In this work we deal with the non-perturbative solutions of the SDEs. If one could solve the full infinite set of SDEs, the complete solution of the QFT will be available. In reality one has to truncate this set of equations: the main weakness of SDEs phenomenology is the necessity to employ some Ansatze for higher Green functions. The reliability of these approximations can be estimated by comparison with the known PT result in the regime of

a soft coupling constant and by comparison with the lattice data (and/or results of alternative approaches) where available. The recent results (see e.g. reviews [12, 13, 14, 15] and references therein) provide some encouragement: they suggest that SDEs are viable tool for obtaining (eventually) the truly non-perturbative answers for plethora of fundamental questions: *dynamical chiral symmetry breaking*, *confinement* of colored objects in QCD, the high temperature superconductivity in condensed matter (modeled [16] by QED_{2+1}). In the QCD sector there is also a decent agreement with some experimental data: for meson decay constants like F_π, evolution of quark masses to their phenomenologically known (constituent) values, bound states properties obtained from Bethe-Salpeter or Faddeev equations (the part of SDEs). The recent solutions of SDEs [15, 17] nicely agree with the lattice data [10, 18, 19] (the explicit comparison is made in [15, 17, 20]). Thus, encouraging connections between results of SDEs studies and fundamental theory (QCD) and/or various phenomenological approaches (chiral perturbation theory, constituent quark model, vector meson dominance ...) are emerging.

Most of these results were obtained by solution of the SDEs in Euclidean space. One of the main goals of this work is to develop an alternative approach allowing to solve them directly in Minkowski space. In this method (generalized) *spectral decompositions* of the GFs are employed, based on their analytical properties. The main merit of this approach is possibility to get solutions both for spacelike and timelike momenta. Techniques of solving SDEs directly in Minkowski space are much less developed than the corresponding Euclidean ones. Therefore, most of this thesis deals with developing and testing such solutions on some simple QFT models.

For large class of *non-asymptotically free* (NAF) theories running couplings – calculated from PT – diverge at some finite, but usually large, spacelike scale. This is why we pay special attention to behavior of our NAF solutions in the region of large momenta.

One should mention a possibility of getting solutions of the SDEs in Minkowski space with the help of analytical continuation of the Euclidean ones. In simple cases this can be successful: we have recently done this for the

gluon form factor and we present the results in this thesis in Section devoted to the QCD. We have found the non-positive absorptive part contribution to the gluon propagator in the Landau gauge, in agreement with some recent analyzes.

The thesis starts with an introduction of essential ideas of the integral representation in QFT, in particular of the Perturbation Theory Integral Representation (PTIR), and with discussion of the renormalization in the framework of non-perturbative SDEs. In Chapters 2 and 3 of the thesis some well-known general results and useful textbooks formulas are collected and basics of the formalism employed are discussed. The derivation of SDEs is reminded for the well-known example of QED. Chapter 3 deals with integral representation for various GFs. The spectral representation of one-particle GF is derived. Then the generalization to higher point Greens function is presented with some examples. One of these examples – the derivation of the dispersion relation for the sunset diagram – is original work of the author.

Then, the technique based on the integral representation of GFs is used to solve SDEs for several quantum field models. Chapter 4 is devoted to the simple scalar model. The SDEs are briefly rederived and then the momentum space SDE is converted into the *Unitary Equation* (UE) for propagator spectral density. The UE is shown to be a real equation, in which the singularities accompanying usual Minkowski space calculations are avoided. The relativistic bound-state problem as an intrinsically non-perturbative phenomenon is treated in Chapter 5.

Large part of the thesis is devoted to solutions of analogous problems in more complicated theories. The systems of the SDEs for QED is considered in two approximation, one of which, the so-called ladder approximations, is widely used as a pedagogical introduction to the subject. The second approximation employs the Ball-Chiu vertex, consistent with the Ward-Takahashi identity. These studies are based on papers [1, 2]. Further insights and improvements of rather technical character can be found in recent author's paper [3], where some obstacles due to singular integral kernels are avoided.

The Yukawa theory, i.e., the quantum field theory describing the interaction of spinless boson with Dirac fermion is considered in rather simple

bare-vertex approximation in Chapter 7 of the thesis. A generalized spectral Ansatz is employed to obtain the spectral function from recent lattice data (and recent solution of SDEs for gluon propagator in Euclidean space) in Chapter 8.

In what follows we briefly summarize the main results of the thesis, including also some basic information on current status of particular problems.

2   Scalar toy-model

We solve the SDEs for scalar Φ^3 and $\Phi_i^2\Phi_j$ theories, the second model is referred to as the (generalized massive) Wick-Cutkosky model (WCM). The solution is performed with the help of the spectral technique directly in Minkowski space. The first aim of this exercise is to test the employed method of solution of integral equations (by iterations) and its actual implementation (numerical accuracy) by varying numerical procedure and by comparing to results obtained in Euclidean space. The second aim is to obtain the dressed propagators to be used in the bound-state studies. The SDEs have been solved in the bare vertex approximation and also with non-trivial improvement of the interaction vertex. We discuss *multiplicative renormalization* in its non-perturbative context. Our method provides the solutions only for coupling constants smaller than certain critical value. It is explained that the method fails when the field renormalization constant approaches zero.

The (metastable) scalar models often serve as an useful methodological tool. The main reason is, of course, their inherent simplicity: absence of spin degrees of freedom leads to lowest possible number of independent amplitudes and the Φ^3-like models involve also simplest possible vertices. The Φ^3 theory has already been employed as a suitable playground for studies of various phenomena [22, 23, 24, 25], including non-perturbative asymptotic freedom and non-perturbative renormalization.

Dealing with massive theories we will use the generic spectral decompo-

sition of the renormalized propagator of the stable and unconfined particle:

$$G(p^2) = \frac{r}{p^2 - m^2 + i\epsilon} + \int\limits_{\alpha_{th}}^{\infty} d\alpha \, \frac{\sigma(\alpha)}{p^2 - \alpha + i\varepsilon} \,, \qquad (1)$$

where - as will be confirmed by actual evaluation - the spectral function $\sigma(\alpha)$ is positive, regular and is spread smoothly from zero at the threshold $\alpha_{th} = 4m^2$.

Putting the spectral decomposition of the propagators and the expression for the vertex function into the SDE allows one to derive a real integral equation for the weight function $\sigma(\alpha)$. This equation can be solved numerically by iterations. Since all momentum integrations are performed analytically, there is no numerical uncertainty following from the renormalization, which is usually not the case in the Euclidean formalism. In the thesis the renormalization procedure is performed analytically with the help of direct subtraction in momentum space. The super-renormalizability makes our models particularly suitable for model studies. It implies the finiteness of the field renormalization constant Z, which therefore need not be considered at all. Nevertheless, for the Φ^3 theory the field renormalization is not fully omitted, but with the help of an appropriate choice of the finite constant Z we choose the renormalization scheme. Minkowski results obtained in the bare vertex approximation and with trivial field renormalization $Z = 1$ are compared with the appropriate Euclidean solutions.

In the Section devoted to Φ^3 theory it is shown how to properly renormalize the SDEs. The key point is the requirement of multiplicative renormalization, then by the construction the S-matrix puzzled from the GFs is invariant under the choice of renormalization schemes in which these GFs have been calculated.

The Lagrangian density for this model reads

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\phi_0(x)\partial^\mu\phi_0(x) - \frac{1}{2}m_0^2\phi_0^2(x) - g_0\phi_0^3(x) \,, \qquad (2)$$

where the subscript 0 indicates the unrenormalized quantities.

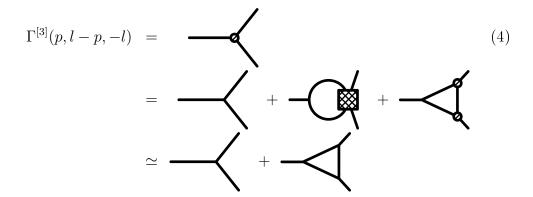$$\Gamma^{[3]}(p, l - p, -l) \quad = \tag{4}$$

Figure 1: Diagrammatical representation of the SDE for the vertex function. As in previous figure all internal propagators are dressed. Blobs represent the full vertex, the box stands for $\Gamma^{[4]}$.

The propagator SDE in the momentum space reads:

$$G_0^{-1}(p^2) \;=\; p^2 - m_0^2 - \Pi_0(p^2) \,,$$

$$\Pi_0(p^2) \;=\; \qquad =$$

$$i 3 g_0^2 \int \frac{d^d q}{(2\pi)^d} \Gamma_0^{[3]}(p, q - p, -q) G_0(q - p) G_0(q) \,, \tag{3}$$

where the arguments of $\Gamma_0^{[3]}(p_1, p_2, p_3)$ are the incoming momenta (and $p_1 + p_2 + p_3 = 0$).

In the thesis the system of SDEs is closed already at the level of the equation for the proper vertex. Replacing the full vertex by its tree approximation defines the *bare vertex approximation* (BVA), in our *dressed vertex approximation* (DVA) the next term of the skeleton expansion of the SDE for the vertex is also included. The equation for the propagator is solved in the BVA and DVA for two renormalization schemes. The on-mass-shell subtraction is employed to renormalize the self-energy, for an extension to the off-shell renormalization scheme see [3].

After the renormalization some simple algebra allows us to convert the momentum space SDE (3) into two equations for σ and the absorptive part of self-energy $Im\, \Pi = \pi\rho$. The latter is obtained from the dispersion relation for renormalized self-energy, which e.g. for the Minimal momentum subtraction (MMS) renormalization scheme reads:

$$\Pi_R(p^2) = \int\limits_{4m^2}^{\infty} d\alpha \, \frac{\rho(\alpha)(p^2 - m^2)}{(p^2 - \alpha + i\epsilon)(\alpha - m^2)} \quad , \tag{5}$$

where m is the "physical" (pole) mass of the scalar.

Having solved the system of equations for σ and ρ the GFs are calculated through their integral representation. This is the essence of the spectral approach to SDEs in Minkowski space.

For the MMS RS the comparison with more usual Euclidean solution was made. Up to rather small (numerical) deviation the results for propagators agree in the spacelike regime, where solutions are available for both approaches (Fig. 1). Also the comparison to the usual perturbation theory was made and the critical value of the coupling constant was determined.

2.1 The relativistic bound-states

In quantum field theory the two-body bound state is described by the bound state vertex function or, equivalently, by Bethe-Salpeter (BS) amplitude, both of them are solutions of the corresponding (see Fig. 3) covariant four-dimensional Bethe-Salpeter equations (BSE) [30]. In most studies the kernel of the BSE is approximated by a single boson exchange (ladder-approximation). Besides, all single particle propagators are very often replaced by the free ones. In the thesis we move beyond bare ladder approximation and include the fully dress single particle GFs.

We considered simple super-renormalizable model of three massive fields with cubic interaction $\phi_i^2 \phi_3 \,; i = 1, 2$ (the massive Wick-Cutkosky model). Having the propagators calculated within the bare vertex truncation of the SDEs we combine them with the dressed ladder BSE for the scalar s-wave bound state amplitudes, following the treatment of ref. [5] where the spectral
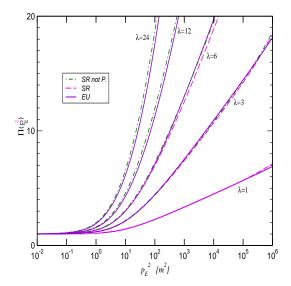
Figure 2: The function $\Pi(p_E^2)$ calculated in the spectral (dashed line) and Euclidean formalism (solid line). Comparison is made for various couplings $\lambda = \{1, 3, 6\}$, for the strongest case $\lambda = 12, 24$ the spectral solution has not been found. The dashed line shows the solution of the SDEs in which the principal value integrals were omitted.

technique was used to obtain the accurate results directly in Minkowski space. In this paper the analytic formula has been derived for the kernel of the resulting equation which significantly simplifies the numerical treatment. For the extensive, but certainly not exhaustive, up-to-date review of methods to treat the scalar models see also [5].

Analogously to the treatment of the SDEs reviewed above, the BSE written in momentum space is converted into a real integral equation for a real weight function. This then allows us to treat the ladder BSE in which all propagators (of constituents and of the exchanged particle) are fully dressed. Having solved equations for BSE spectral functions, one can easily determine the BS amplitudes in an arbitrary reference frame.

The BS amplitude for bound state (ϕ_1, ϕ_2) in momentum space is defined through the Fourier transform of

$$\langle 0|T\phi_1(x_1)\phi_2(x_2)|P\rangle = e^{-iP\cdot X} \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \Phi(p, P) \ . \tag{6}$$

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Figure 3: Diagrammatic representation of the BSE for the bound state vertex function.

The corresponding scalar BS vertex function $\Gamma = iG_{(1)}^{-1}G_{(2)}^{-1}\Phi$ satisfies

$$\Gamma(p, P) = i \int \frac{d^4k}{(2\pi)^4} V(p, k; P)G_{(1)}(k + P/2)G_{(2)}(-k + P/2)\Gamma(k, P), \quad (7)$$

where V is the kernel of the BSE and bracketed subscript label the particles. When the mass of exchanged particle is nonzero, the appropriate integral representation for the BS vertex function is two dimensional:

$$\Gamma(p, P) = \int\limits_{-1}^{1} dz \int\limits_{\alpha_{min}(z)}^{\infty} d\alpha \; \frac{\rho^{[n]}(\alpha, z)}{[\alpha - (p^2 + zp \cdot P + P^2/4) - i\epsilon]^n} \; . \quad (8)$$

The positive integer n is a dummy parameter constrained only by convergence of the BSE, $n = 2$ was found to be the most suitable choice. The BSE can be converted into the following real integral equation for the real spectral function:

$$\rho^{[2]}(\alpha', z') = \frac{g^2}{(4\pi)^2} \int\limits_{-1}^{1} dz \int\limits_{\alpha_{min}(z)}^{\infty} d\alpha \; V^{[2]}(\alpha', z'; \alpha, z) \, \rho^{[2]}(\alpha, z) \, . \quad (9)$$

The BSE interaction kernel is $V(p, k; P) = g^2 G_{(3)}(p-k)$ in the dressed ladder approximation, in which all propagators are fully dressed. For this case the kernel $V^{[2]}(\alpha', z'; \alpha, z)$ in spectral equation (9) has been derived in [5]. The explicit formula for $V^{[2]}$ involves the integrals over the spectral function $\bar{\sigma}_{(i)}$ from the SR of the propagator of the i-th field.

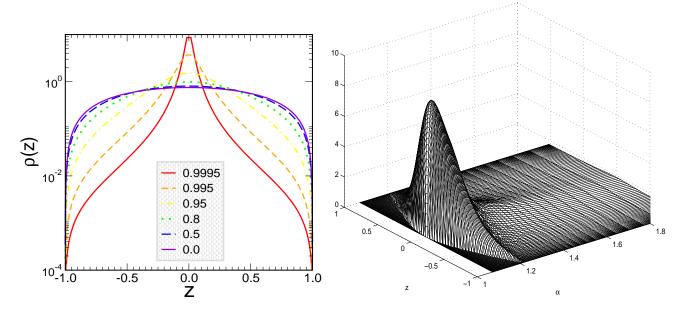In the ladder approximation the obtained energy spectrum agree with

Figure 4: Left figure displays the weight function $\rho(z)$ for the original WCM ($m_3 = 0$) for several values of $\eta = \sqrt{P^2}/2m$. The right figure displays rescaled weight function for the massive WCM for $m_3 = 0.1m$ and $\eta = 0.95$.

ones obtained by other techniques [32, 33, 34, 35]. In the dressed ladder approximation there is no reliable published result to compare with. However, the qualitative agreement with paper [34] was found: The critical value of coupling constant g_{crit} (defined by the value of g for which the renormalization constant becomes zero: $Z(g_{crit}) = 0$) determined from SDEs gives the domain of applicability of the BSE. The couplings below the critical one allow only solutions for relatively weakly bound states.

In the original Wick-Cutkosky model [31] the exchanged boson is massless and no radiative corrections are considered. This model is particularly interesting because it is the only example of the nontrivial BSE which is solvable exactly [31]. For this model the s-wave bound-state PTIR reduces to one variable spectral integral and the equation for the spectral function simplifies. The resulting weight functions $\rho(z)$ for various are displayed in Fig. 7 for several values of η.

The electromagnetic form factors parametrize the response of bound systems to external electromagnetic field. The calculation of these observables
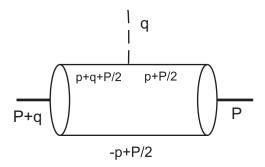
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Figure 5: Diagrammatic representation of the electromagnetic current bound state matrix element.

within the BS framework proceeds along the Mandelstam's formalism [36].

The current conservation implies the parametrization of the current matrix element G^μ in terms of the single real form factor $G(Q^2)$

$$G^\mu(P_f, P_i) = G(Q^2)(P_i + P_f)^\mu, \tag{10}$$

where $Q^2 = -q^2$, so that Q^2 is positive for elastic kinematics.

The matrix element of the current in relativistic impulse approximation (RIA) is diagrammatically depicted in Fig. 5. The matrix element is given in terms of the BS vertex functions as

$$G^\mu(P + q, P) = i \int \frac{d^4p}{(2\pi)^4} \bar{\Gamma}(p + \frac{q}{2}, P + q)$$
$$\left[D(p_f; m_1^2) j_1^\mu(p_f, p_i) D(p_i; m_1^2) D(-p + P/2; m_2^2) \right] \Gamma(p, P), \tag{11}$$

where we denote $P = P_i$ and j_1^μ represents one-body current for particle ϕ_1, which for the bare particle reads $j_1^\mu(p_f, p_i) = p_f^\mu + p_i^\mu$, where p_i, p_f is initial and final momentum of charged particle inside the loop in Fig. 5.

We have rewritten the r.h.s. of Eq. (11) directly in terms of the spectral weights of the bound state vertex function. It allows the evaluation of the form factor by calculating the dispersion relation:

$$G(Q^2) = \int d\omega \frac{\rho_G(\omega)}{Q^2 + \omega - i\epsilon} \tag{12}$$

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without having to reconstruct the vertex functions $\Gamma(p, P)$ from their spectral representation.

3 Gauge theories

Large part of the thesis deals with gauge theories, with main stress on the strong-coupling Quantum Electrodynamics, treated in Chapter 6.

In the last Chapter of the thesis we employ the generalized spectral decomposition for the gluon propagator (formally identical to the Lehmann one, but with the spectral function which is not required to be positive) to $\sigma(\omega)$ from recent Euclidean lattice and SDE results and analytically continue them into the timelike region.

3.1 SDEs of one-flavor strong QED

The brief textbook derivation of the Schwinger-Dyson equations for QED (Fig. 6) is given already in the introductory Chapter 2 of the thesis. In Chapter 6 we investigate SDEs of the 3+1 dimensional QED, in a first attempt to move beyond the class of scalar models. We review the methods and results of papers [1, 2], in which the solutions for this theory in Euclidean and Minkowski space were compared for the first time. Since no non-trivial non-perturbative effects (i.e., effects not known from PT) were found in the photon propagator, our attention is mainly concentrated on the SDE for the fermion one. We deal mostly with the strong coupling regime, which is far from the "real-life QED", for which α_{QED} is small (in experimentally accessible energy region) and use of the PT is fully justified. Of course, all our solutions agree in weak coupling limit with the perturbation theory (PT).

The strong-coupling Abelian dynamics was considered as one of the candidates for explanation of the electroweak symmetry breaking [37]. Although we do not think this to be a realistic model for a mass generation of the SM fermions, it could be a candidate for the strong coupling sector of theories like Technicolors, e.g., Slowly Walking Technicolors [38, 39, 40, 41], for which the LA of a QED-like theory seems to provide a reasonable model of
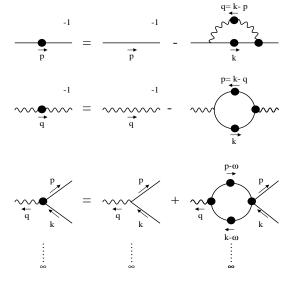
Figure 6: Diagrammatical representation of the Schwinger-Dyson equations in the quantum electrodynamics. The black blobs represent the full propagators and vertices. Wavy lines stand for the photon and full lines for the fermion propagator.

(Techni)lepton propagators. Moreover, the strong-coupling Abelian model is not only a suitable playground for studies of supercritical phenomena like dynamical mass generation, it is suitable also for investigations of the analytical structure of the fermion propagator [1, 43]. It was 30 years ago when Fukuda and Kugo [43] observed the disappearance of the real pole of the fermion propagator in the ladder QED and it was argued that this is a signal for confinement of the fermion. Again, the main motivation is the simplicity of the model, in alternative more complicated models the non-perturbative phenomena are even more difficult to study quantitatively.

3+1 QED has been studied frequently, mostly in Euclidean formalism employing various approximations. The truncation of the SDEs should not violate the gauge invariance. At least two technically different ways were proposed to deal with SDEs of QED. Ball and Chiu [44] derived the formula for the proper vertex which– being written in the terms of propagator functions A and B– closes the SDEs system in a way respecting WTI. The second approach is known as the "Gauge Technique". In this framework the spectral Anzatz is made for the untruncated vertex function, i.e., for $G^\mu = S\Gamma^\mu S$. The

technical advantage of the Gauge Technique is that the resultant equation for the fermion propagator is linearized in a spectral function. On the other hand, this is one of essential weaknesses of this approach: it cannot be true in general, even for the case of electron propagator. It is clear that beyond the simplest approximations employed in the literature [45, 46, 47, 48, 49] such linearization does not take place.

The first part of Chapter 6 of the thesis is devoted to the solution for the electron propagator in the *ladder approximation* (LA), in the second part extension to the *unquenched* (with the photon polarization included) case is made. In the latter case the running coupling is considered *self-consistently*: we follow the paper [2], where the fermion mass and photon polarization function have been calculated by solving the coupled SDEs for electron and photon propagators in the Landau gauge.

In both approximations we are looking both for solutions with zero and non-zero bare electron mass in the Lagrangian. In the first case the chirally symmetric solution (with massless electrons) always exists. In the Euclidean formalism we obtain, in agreement with many previous studies, for sufficiently large α also non-trivial solution for the mass function M. On the other hand, no such solution was found in our spectral Minkowski approach. This is a strong indication that the fermion mass function has in this case a complicated analytical structure, which is not reflected by assumptions of our spectral Anzatz.

In explicit chiral symmetry breaking (EχSB) case, in which the non-zero electron mass exists from the very beginning, both approaches– Minkowski and Euclidean– offer approximately the same results (in the spacelike domain, of course) in the regime where both solutions were obtained. It is interesting that in the unquenched approximation the EχSB spectral solution (with explicit mass term) fails almost exactly for the same value of the coupling at which the SχSB solution occur.

In order to compare the Minkowski solution with some recent Euclidean results [50, 51, 52] the same approximations has to be made and the same schemes has to be picked up. By these we mean the truncation of the SDEs, the same choice of the gauge and of the renormalization scheme. The result-
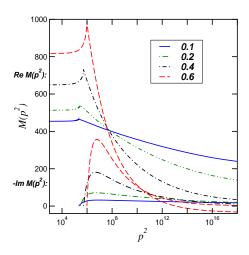
Figure 7: Figure presents the Minkowski solutions of the SDE for time-like momenta, i.e., in the range where solutions are not available in much more common Euclidean approach. The electron mass function M [from the propagator $S = F(p^2)/(\not{p} - M(p^2))$] calculated in the ladder approximation in the Landau gauge is shown. The renormalization is chosen such that $M(-10^8) = 400$ and the lines are labelled by the coupling constant α. The cusps correspond to the physical pole masses, while the absorptive parts of M correspond to the smooth lines. The results in the spacelike regime agree with the ones obtained in Euclidean calculations.
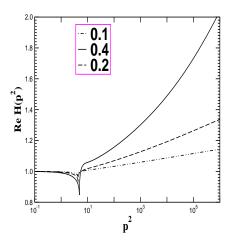

Figure 8: Figure shows timelike behavior of the charge renormalization function, i.e., the running coupling constant normalized to 1 at zero momentum, as it was obtained from the SDEs for the unquenched QED in the Landau gauge. The vertex function was approximated by the bare vertex γ_μ. The down-oriented peak corresponds to the threshold $4m^2$, the mass of the fermion was renormalized such that $M(0) = 1$. The lines are labelled by the renormalized couplings at zero momenta: $\alpha(0) = 0.1, 0.2, 0.4$.

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ing integral equations are solved in their full form, the linearization inherent to the Gauge technique is avoided. The unitary equations and dispersion relations for self-energies are derived both for ladder and unquenched approximations. For both of them also the Euclidean formalism is reviewed. The numerical results are briefly summarized below.

The solutions of the SDEs were obtained for several values of the coupling constant and for several renormalization choices. In the ladder approximation the expected damping of the mass function to its negative values is observed in the supercritical phase of QED [51, 43]. The Euclidean and Minkowski solutions for the fermion mass functions are compared in several figures. The confirmation of disappearance of the real pole (first indicated in [43]) in the supercritical ladder QED is one of the main physical results of this section and of paper [1]. We also argue that the fermion SR is absent in the supercritical QED. We have but a numerical evidence for such a statement, hence it should remit to further investigation.

When the vacuum polarization effects are taken into account self-consistently we confirm triviality of QED. It implies that some kind of regularization is needed beyond and irrespective of the renormalization technique employed. We explain how to introduce such cut-off in our Minkowski spectral treatment. In numerics, this cut-off is taken to be $\Lambda^2 = 10^7 M^2(0)$, where $M(0)$ is the infrared electron mass. The numerical solution fails when α exceeds the value $\alpha_R(0) = 0.41$. Note that at ultraviolet region the running coupling $\alpha_R(\Lambda^2)$ is approximately 2.6 times larger then $\alpha_R(0)$. We are not able to see Landau pole directly (it corresponds to $\alpha_\mp \to \pm\infty$). However, the observed large growth of the running coupling and of its derivative can be understood as the evidence for the Landau singularity somewhere above the cut-off.

4   QCD

The non-Abelian character of the QCD makes it difficult to convert the momentum SDEs into equations for spectral functions. We were not able to do this yet (the main obstacle is the ghost SR, mainly due to zero momen-

tum behavior). Therefore, we first briefly review the symmetry preserving *gauge invariant* solution obtained by Cornwall two decades ago [64]. To our knowledge this is the best published example, in which the behavior of the QCD Green function in the whole range of Minkowski formalism is addressed within the framework of the SDEs. Instead of solving the SDEs we use the generalized spectral representation to fit the spectral function to Euclidean solutions obtained in recent lattice simulations and in the SDEs formalism.

The Quantum Chromodynamics (QCD) is the only experimentally studied strongly interacting relativistic quantum field theory. This non-Abelian gauge theory with a gauge group $SU(3)$ has many interesting properties. The dynamical spontaneous breaking of chiral symmetry explains why the pions are light, identifying them with the pseudo-Goldstone bosons associated with the symmetry breaking of the group $SU(2)_L \times SU(2)_R$ to $SU(2)_V$ (in flavor space). Asymptotic freedom [53, 54] implies that the coupling constant of the strong interaction decreases in the ultraviolet region. For less than $33/2$ quark flavors the QCD at high energy becomes predictable by the PT. However, in the infrared region the PT does not work and non-perturbative techniques have to be applied.

One of the most straightforward non-perturbative approaches is a solution of the SDEs for QCD. The extensive studies were undertaken for a quark SDE, based on various model assumptions for a gluon propagator. These approximate solutions, often accompanied by a solution of the fermion-antifermion BSE for meson states, have become an efficient tool for studies of many non-perturbative problems, e.g., the chiral symmetry breaking, low energy electroweak hadron form factors, strong form factors of exclusive processes, etc (see reviews [12, 13, 14, 55] and also recent papers [15, 56, 57, 58, 59]).

However, to take gluons into account consistently is much more difficult than to solve the quark SDE alone. The SDE for the gluon propagator is more non-linear than the quark one. Moreover, the Faddeev-Popov ghosts have to be included [60] in a class of Lorentz gauges. In recent papers [17, 15, 61, 62, 63] studies of the coupled SDEs for gluon and ghost propagators in the Landau gauge in Euclidean space were performed in various approximations.
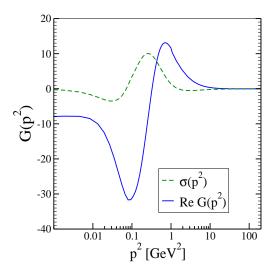
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Figure 9: The gluon propagator for timelike momenta.

In Chapter 8 we consider the implication of analyticity for the solution of SDEs for gluon propagator. As in previous Chapters we assume analyticity of the propagators. The generic SR for the renormalized gluon propagator in the Landau gauge reads

$$
\begin{aligned}
G_{AB}^{\alpha,\beta}(q) &= \delta_{AB} \left[-g^{\alpha\beta} + \frac{q^\alpha q^\beta}{q^2} \right] G(q^2)\,, \\
G(q^2) &= \int\limits_0^\infty d\omega\, \frac{\sigma(\omega; g(\xi), \xi)}{q^2 - \omega + i\epsilon}\,,
\end{aligned}
\tag{13}
$$

where only here full dependencies of the continuous function $\sigma(\omega; g(\xi), \xi)$ are indicated explicitly. The Ansatz (13) should be consider the generalized spectral representation, since we do not assume (and do not obtain) the spectral function $\sigma(\omega)$ positive for all values of ω.

To find $\sigma(\omega)$ we first represented some recent lattice [18] and SDEs results [15] (both defined in spacelike region) by an analytical formula. Our spectral decomposition is fitted to these "data" for spacelike momenta and then predicts the gluon propagator in the timelike region. The solutions are plotted

in Fig. 9. The spectral function has a smooth peak around $p^2 = (0.7\text{GeV})^2$ with the width $\approx 1\text{GeV}$. It becomes negative for asymptotically large p^2, as expected already from the PT. The gluon propagator should describe the confined particle, so the "unusual" shape (violation of the spectral function positivity) is in accord with our physical expectations.

5   Yukawa theory

In Chapter 7 of the thesis the SDEs of the massive Yukawa theory are studied. The Yukawa interaction appears in many models of Nature. Original motivation for developing the Yukawa theory (YT) was modeling of the nuclear forces. Several decades from its birth the Yukawa interaction has become the firm part of the Standard Model and of many models beyond it. In alternatives to the minimal SM, such as its SUSY extensions [65], the Little Higgs models [66], [67], the Zee model of the neutrino sector [68], [69], and in many others, the interaction of spinless particles with matter generates the masses of particles. This can happen at tree level due to the nonzero vacuum expectation values of the Higgs fields (like in the SM) or the masses can be generated dynamically through the radiative corrections. Of course, the combination of both effects is also possible.

As for the vectorial interaction from the previous Chapter, this study of the YT SDEs should be regarded as a first step in building intrinsically non-perturbative methodological tools within the spectral formalism in Minkowski space. Such tools should next help to reveal the quantitative face of dynamical phenomena, such as the triviality of this QFT, the quantum stability or the dynamical mass generation for the YT with a given parameter space. Some attempts to study such phenomena in the YT (or the SUSY YT) have already been made [9, 70].

The SDEs for the YM are derived from the following Langrangian:

$$
\begin{aligned}
\mathcal{L} &= i\bar{\Psi}_0 \slashed{\partial} \Psi_0 - m_0 \bar{\Psi}_0 \Psi_0 + \frac{1}{2}\partial_\mu \Phi_0\, \partial^\mu \Phi_0 - \frac{m_{0\phi}^2}{2}\Phi_0^2 \\
&- ig_0\, \bar{\Psi}_0 \gamma_5 \Psi_0 \Phi - h_0 \Phi_0^4 \,,
\end{aligned}
\tag{14}
$$

where we assume (mainly for historical reasons) that the field Φ_0 is pseudoscalar. If the Yukawa coupling g_0 is nonzero, the term $h_0\Phi_0^4$ has to be present in the Lagrangian due to the general requirement of renormalizability. For the sake of simplicity, the Yukawa vertex is modeled by its tree approximation and it is assumed that the renormalized constants satisfy $h << g$, therefore we also neglect the quartic meson self-interaction. For the renormalization the momentum subtraction scheme is used. The numerical results are given for the absorptive and dispersive parts of the self-energies. The SDEs results are in agreement with our expectation supported by the perturbation analysis: one gets rather small corrections to the fermion self-energy, whereas the boson self-energy exhibits a quadratic behavior for large momenta squared (quadratic divergences turn quadratic dependence after the renormalization).

6   Summary, conclusion and outlook

A connection between the Euclidean space and the physical Minkowski space is an important question, in field theory in general and in the SDE approach in particular. We have explicitly demonstrated (albeit on simple models and employing some simplifying approximations) that SDEs studies are feasible directly in Minkowski space. A reasonable agreement of the spectral SDEs solutions (at spacelike momenta) with the ones obtained by more conventional strategy in Euclidean space gives us strong belief in a relevance of developed methods.

We started an introduction of the spectral concept into the formalism of SDEs with specific attention to scalar models and QED. We performed studies of the strong-coupling QED in various approximations. One of them is unquenched QED, in which the running of the coupling constant has been correctly taken into account. In this case the QED triviality plays its crucial role for asymptotically large spacelike momenta. We explain how to deal with this trivial theory within the formalism of spectral representations and dispersion relations.

The strong coupling QED is often regarded as an ideal pedagogical tool

for SDE studies and their application to QCD. We also attempted to extend the discussion to QCD, for which the direct Minkowski space formulation and solution is not yet fully developed, since the theory is much more non-linear and complicated.

While many important steps have been made, it is obvious that much more needs to be done. Up to now all studies of profoundly non-perturbative phenomena, such as the chiral symmetry breaking or dynamical mass generation, were carried out in Euclidean space. It is not quite clear how to perform such symmetry breaking SDE studies directly in Minkowski space.

Another interesting perspective is to further explore the timelike infrared behavior of Green functions obtained by analytical continuation of lattice results. We offered a solution for gluon propagator in the Landau gauge, but information on the timelike structure of the quark propagator is currently unavailable.

We also solved the Bethe-Salpeter bound-state equation in (3+1) Minkowski space within the spectral framework in the dressed ladder approximation. It would be interesting to extend this technique to more complicated BS kernels: e.g., to include cross-box contributions, s and u channel interactions etc.

Further developed, the Minkowski space spectral technique should become rather efficient tool of hadronic physics. It was already demonstrated that obstacles due to the fermionic degrees of freedom can be overcome. The solution of the spinless bound state of two quarks interacting via dressed gluon exchange (a pion) is under auspicious consideration.

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