Heegaard Splittings and Virtually Haken Dehn Filling

J. Masters[1], W. Menasco and X. Zhang[2]

Abstract. We use Heegaard splittings to give some examples of virtually Haken 3-manifolds.

A compact connected 3-manifold is said to be virtually Haken if it has a finite sheeted covering space which is Haken. The virtual Haken conjecture states that every compact, connected, orientable, irreducible 3-manifold with infinite fundamental group is virtually Haken. Since virtually Haken 3-manifolds and Haken 3-manifolds possess similar properties, such as geometric decompositions and, in the closed case, topological rigidity, the resolution of this conjecture would provide solutions to several fundamental problems about compact 3-manifolds with infinite fundamental groups.

Some recent results in attacking the conjecture can be found in [CL] [BZ] [M] [DT]. A summary of earlier results can be found in [K, Problem 3.2]. For connections between the virtual Haken conjecture, Heegaard splittings, and the Property τ conjecture, see [L].

Motivated by the work of Casson and Gordon ([CG]), we shall show that lifted Heegaard surfaces can often be compressed to become essential. Our techniques can be used to produce many families of non-Haken but virtually Haken 3-manifolds, a few of which are given here to illustrate the method. A more general result will be proved in a forthcoming paper.

The first named author wishes to thank Cameron Gordon for many useful conversations, and the University of Texas at Austin for its hospitality.

We proceed to give the examples. Let K_{2n+1} be the twist knot in S^3 as shown in Figure 1. Let M_n be the exterior of K_{2n+1}, with standard meridian-longitude framing on ∂M_n. Recall that a connected, compact, orientable 3-manifold whose boundary is a torus is called *small* if every closed, orientable, embedded, incompressible surface is parallel to the boundary, and called *large* otherwise.

Theorem 1 *The 3-fold cyclic cover of M_n is large for every $n > 0$. Every Dehn filling of M_n with slope $3p/q$, $(3p, q) = 1$, $|p| > 1$, yields a virtually Haken 3-manifold.*

Note that by [HT], M_n is hyperbolic, small, and has exactly three boundary slopes, for every $n > 0$. It follows (combining with [CGLS, Theorem 2.0.3]) that all but exactly three

[1]Supported by NSF Postdoctoral Fellowship

[2]Partially supported by NSF grant DMS 0204428

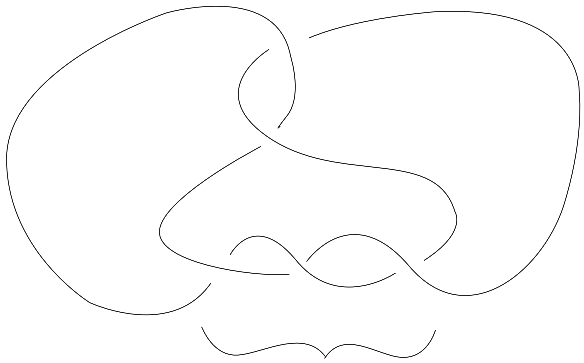


Figure 1: The twisted knot K_{2n+1}

Dehn fillings of M_n give irreducible non-Haken 3-manifolds. Also note that each K_{2n+1}, $n > 0$, is a non-fiberd knot with a genus one Seifert surface, and thus by [CL] it was known that every m-fold cyclic cover of M_n, $m \geq 4$, is large and every Dehn filling of M_n with slope p/q, $(p, q) = 1$, $|p| \geq 8$, is virtually Haken.

Proof. Let $\tilde{M}_n$ be the 3-fold cyclic cover of M_n with induced meridian-longitude framing on $\partial\tilde{M}_n$. We shall show that $\tilde{M}_n$ contains a connected, essential (i.e. orientable, incompressible, non-boundary-parallel) genus two closed surface which has an essential simple closed curve isotopic to a longitude curve of the cover. It follows from [CGLS, Theorem 2.4.3] that the surface remains incompressible in every Dehn filling of $\tilde{M}_n$ with slope p/q, $(p, q) = 1$, $|p| > 1$. As every Dehn filling of M_n with slope $3p/q$, $(3p, q) = 1$, $|p| > 1$, is free covered by Dehn filling of $\tilde{M}_n$ with slope p/q, $(p, q) = 1$, $|p| > 1$, the second conclusion of the theorem will follow.

To make the illustration simple, we first prove the theorem with all details in case $n = 1$, i.e. for the 5_2 knot $K = K_3$. The knot K is tunnel number one, and Figure 2 shows an unknotting tunnel. Also pictured in Figure 2 is a longitude λ of K. Let N be a regular neighborhood of K in S^3, $M = M_1 = \overline{S^3 - N}$, B a regular neighborhood of the unknotting tunnel in M, and $H = \overline{M - B}$. Then H is a handle body of genus two. Let D be a meridian disk of the 1-handle B whose boundary is shown in Figure 2. We deform the handle body $H' = N \cup B$ by an isotopy in S^3 so that its exterior H can be recognized as a standard handle body in S^3 and at the same time we trace the corresponding deformation of ∂D and λ under the isotopy. The process is shown through Figures 3-6.

A *meridian disk system* of a handlebody of genus g is a set of g properly embedded

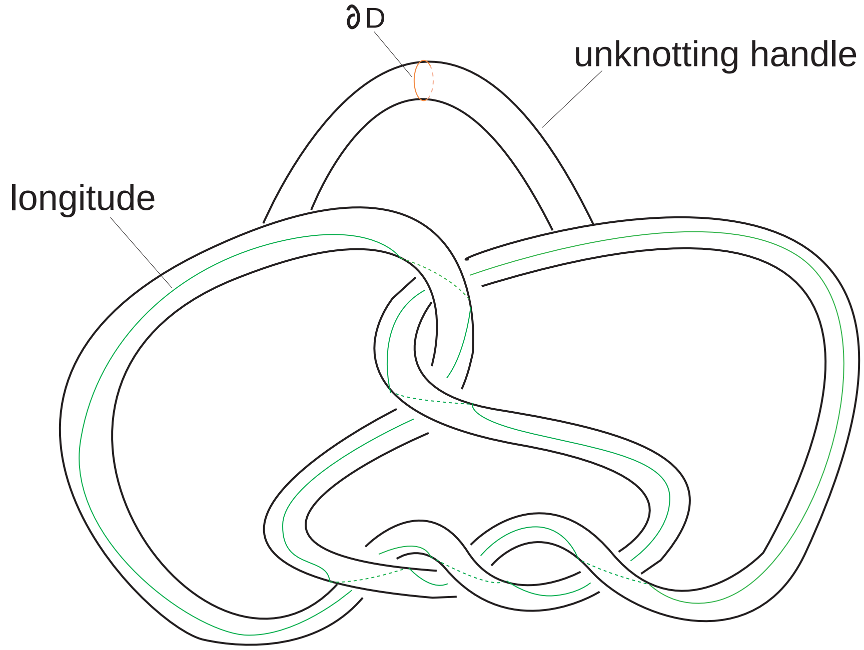


Figure 2: An unknotting tunnel, its co-core ∂D and a standard longitude of K

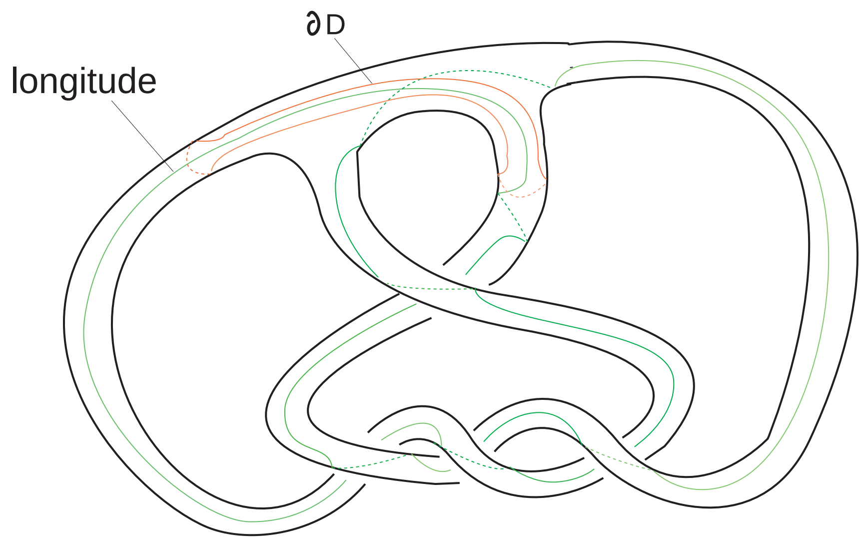


Figure 3: The deformation of H', ∂D and λ (part a)

mutually disjoint disks in the handle body such that cutting the handlebody along these disks results in a 3-ball. Let $\{X, Y\}$ be a meridian disk system of H whose boundary are shown in Figure 6. Following ∂D in the given orientation, we get a geometric presentation

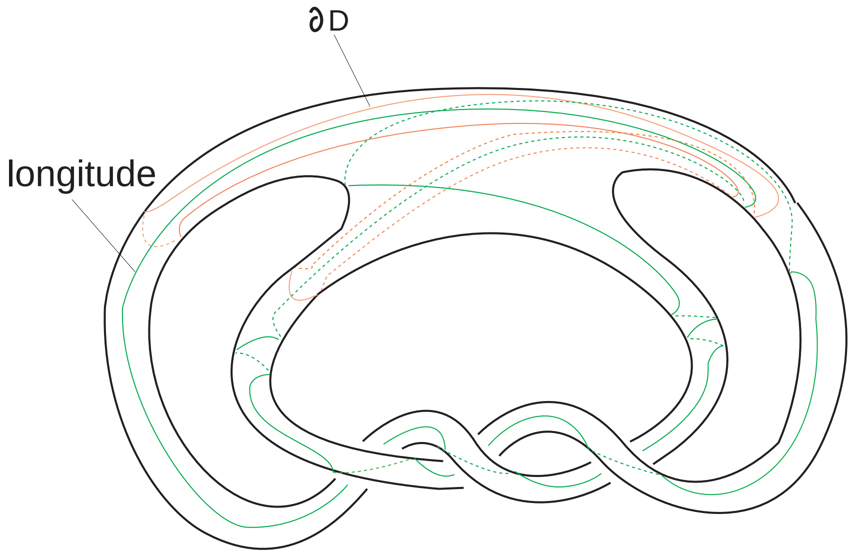


Figure 4: The deformation of H', ∂D and λ (part b)

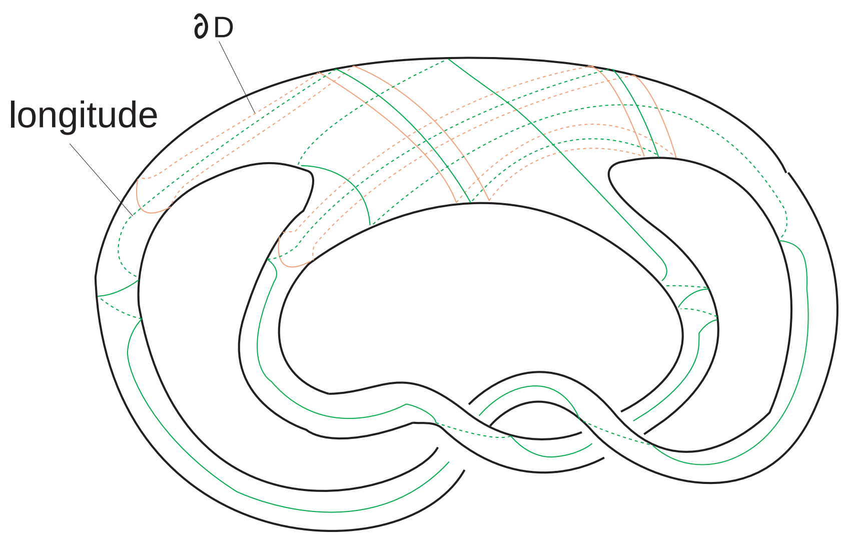


Figure 5: The deformation of H', ∂D and λ (part c)

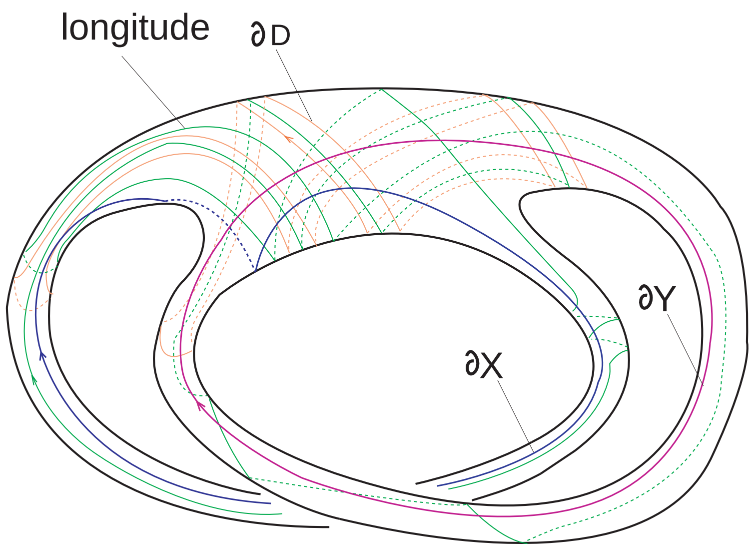


Figure 6: The deformation of H', ∂D and λ (part d)

of the fundamental group $\pi_1(M)$ of M:

$$\pi_1(M) =< x, y; x^{-1}y^{-1}x^{-1}yxyxy^{-1}x^{-1}y^{-1}xyxy >,$$

where x is chosen such that it has a representative curve which is a simple closed curve in ∂H which is disjoint from ∂Y and intersects ∂X exactly once and y is also chosen similarly. (We shall call such generators *dual to* the disk system.) Also we can read off the longitude in terms of these two generators:

$$\lambda = yxyx^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^{-1}x^{-1}yxyx^2.$$

Cutting H along X and Y, we get a 3-ball. Figure 7 shows the boundary 2-sphere of the 3-ball, which records X^+, X^-, Y^+, Y^- and ∂D. Figure 8 shows H in a standard position, and ∂D in ∂H.

The exterior of H in M is a compression body which we denote by C. Topologically, C is $\partial M \times [0,1]$ with a 1-handle attached on $\partial M \times \{1\}$. It has two boundary components: one is $\partial M = \partial M \times \{0\}$ and the other is the genus two surface ∂H. We have that $H \cup_{\partial H} C$ is a Heegaard splitting of M.

Let $\tilde{M} = \tilde{M_1}$ be the 3-fold cyclic cover of $M = M_1$. Note that each of x and y is a generator of $H_1(M;\mathbb{Z}) = \mathbb{Z}$. Let $\tilde{M}$ have the induced Heegaard splitting from that of M.

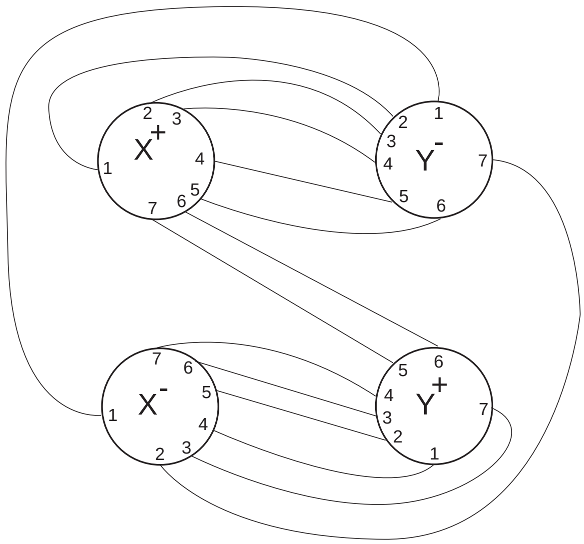


Figure 7: ∂D on the sphere $\partial(\overline{H - \{X \times I \cup Y \times I\}})$

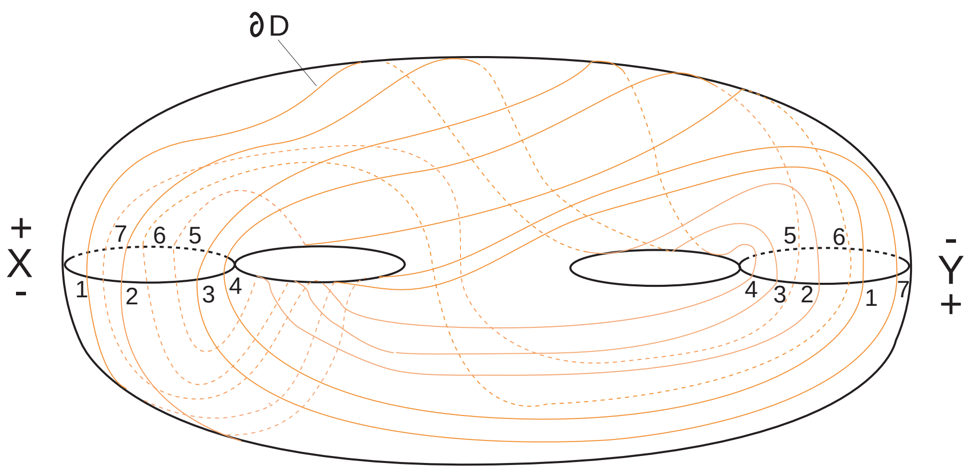


Figure 8: H and ∂D in standard position

We can easily give the Heegaard diagram of $\tilde{M}$, as shown in Figure 9. The genus four handle body $\tilde{H}$ in Figure 9 is the corresponding cover of H. The corresponding cover $\tilde{C}$ of C is a compression body obtained by attaching three 1-handles to $\partial\tilde{M} \times [0,1]$ on the side $\partial\tilde{M} \times \{1\}$. The disk X lifts to three disks X_1, X_2, X_3; and the disk Y lifts to three disks Y_1, Y_2, Y_3, as shown in Figure 9. Pick the meridian disk X_4 of $\tilde{H}$ as shown in Figure 9. Then $\{X_1,\ X_2,\ X_3,\ X_4\}$ forms a disk system of $\tilde{H}$. The disk D lifts to three disks $\{W_1, W_2,\ W_3\}$ whose boundary $\{\partial W_1,\ \partial W_2,\ \partial W_3\}$ is shown in Figure 9. Figure 9 also shows the longitude $\tilde{\lambda}$ of $\tilde{M}$, which is a lift of λ.

This Heegaard splitting of $\tilde{M}$ is weakly reducible: ∂X_4 is disjoint from ∂W_3. We now show that the closed, genus 2 surface S obtained by compressing the Heegaard surface $\partial\tilde{H}$ using the disks W_3 and X_4 is essential in $\tilde{M}$. It is enough to show that the surface S is incompressible in $\tilde{M}(2)$, which is the manifold obtained by Dehn filling $\tilde{M}$ with the slope 2. $\tilde{M}(2)$ has the induced Heegaard splitting $\tilde{H}\cup\tilde{C}(2)$. Note that $\tilde{M}(2)$ is the free 3-fold cyclic cover of $M(6)$, extending the cover $\tilde{M}\to M$, and that $\tilde{C}(2)$ is a handle body of genus four covering the handle body $C(6)$ of genus two, extending the cover $\tilde{C}\to C$. Let $\tilde{V}$ be the filling solid torus in $\tilde{M}(2)$ and let W_4 be a meridian disk of $\tilde{V}$. Then $\{W_1, W_2, W_3, W_4\}$ is a disk system of the handle body $\tilde{C}(2)$.

Cutting $\tilde{H}$ along X_4, we get a handle body $H_\#$ of genus three, and $\{X_1, X_2, X_3\}$ is a disk system of $H_\#$. Using the Whitehead algorithm [S], we see that $\partial H_\# - \partial W_3$ is incompressible in $H_\#$. In fact, from Figure 9, we can read off the Whitehead graph of ∂W_3 with respect to the disk system $\{X_1, X_2, X_3\}$ of $H_\#$, which is given as Figure 10. The graph is connected with no cut vertex, which means, by the Whitehead algorithm, that ∂W_3 must intersect every essential disk of $H_\#$. Now by the Handle Addition Lemma due to Przytycki [P] and Jaco [J], the manifold $H_\#\cup W_3\times I$, obtained by attaching the 2-handle $W_3\times I$ to $H_\#$, has incompressible boundary.

On the other hand, cutting the handle body $\tilde{C}(2)$ along the disk W_3, we get a handle body H_*, which is homeomorphic to $\tilde{V}$ with the two 1-handles $W_1\times I$ and $W_2\times I$ attached on $\partial\tilde{V}$. The genus of H_* is three, and $\{W_1, W_2, W_4\}$ gives a disk system. Let $\alpha\subset\partial M$ be an essential simple closed curve of slope 6. We can easily see that with respect to the generators x, y of $\pi_1(M)$,

$$\alpha = \lambda x^6 = yxyx^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^{-1}x^{-1}yxyx^8.$$

Let $\tilde{\alpha}\subset\partial\tilde{M}$ be a lift of α. Then $\tilde{\alpha}$ has slope 2 in $\partial\tilde{M}$ which can be considered as the boundary of the disk W_4. Figure 11 shows $\tilde{\alpha}=\partial W_4$, ∂W_1 and ∂W_2 in $\partial\tilde{H}$.

Again using the Whitehead algorithm, we see that $\partial H_* - \partial X_4$ is incompressible in $\tilde{H}_*$. In fact, from Figure 11, we can read off the Whitehead graph of ∂X_4 with respect to the disk system $\{W_1, W_2, W_4\}$, which is given as Figure 12. The graph is connected with no cut vertex, which means, by the Whitehead algorithm, that $\partial H_* - \partial X_4$ is incompressible in H_*. Again by the Handle Addition Lemma, the manifold $H_*\cup X_4\times I$ has incompressible boundary of genus two. Note that $\partial(H_*\cup X_4\times I) = \partial(\tilde{H}_\#\cup Y_3\times I) = S$ (up to a small isotopy), and thus S is incompressible in $\tilde{M}(2)$. But the surface S is contained $\tilde{M}$, and thus it is an essential surface in $\tilde{M}$.

Obviously the longitude $\tilde{\lambda}$ in $\partial\tilde{M}$ is isotopic to an essential simple closed curve in the

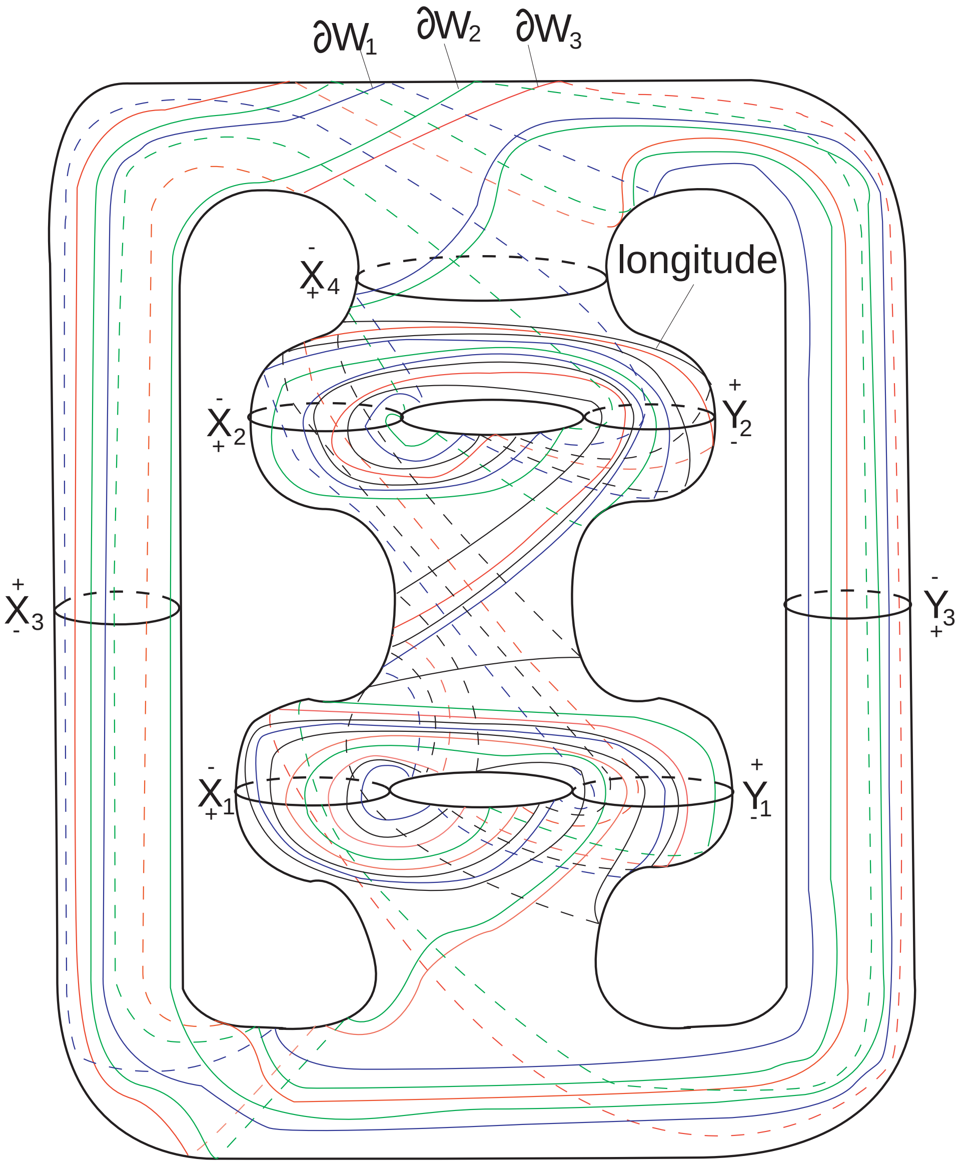


Figure 9: the Heegaard diagram of the 3-fold cyclic cover $\tilde{M}$ and the longitude $\tilde{\lambda}$

surface S, as shown in Figure 9. The proof of Theorem 1 is complete for $n = 1$.

The proof for general K_{2n+1}, $n > 0$, is similar. The knot K_{2n+1} is tunnel number one, with an unknotting tunnel shown in Figure 2 (replacing the bottom three crossings by $2n+1$ crossings). Let M_n be the exterior of K_{2n+1}, H' the handlebody which is a regular neighborhood of the knot and its unknotting tunnel, $H = \overline{M_n - H'}$, and D a meridian disk of the unknotting tunnel. There is a corresponding Heegaard splitting $M_n = H \cup_{\partial H} C$, where C is a compression body. We let λ be a standard longitude, and again we deform the

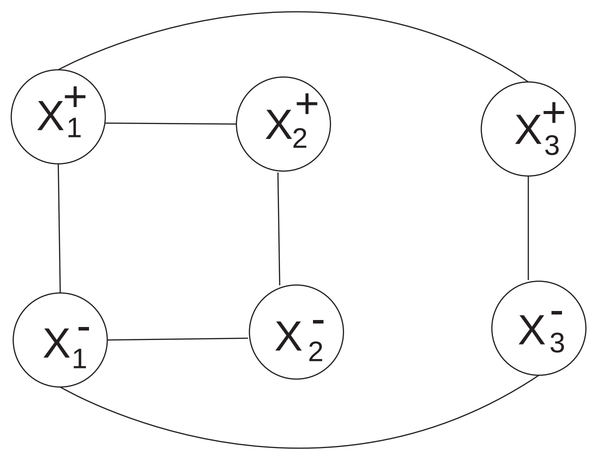


Figure 10: The Whitehead graph of ∂W_3 with respect to the disk system $\{X_1, X_2, X_3\}$ of the handle body $H_\#$

handlebody H' by an isotopy in S^3 so that its exterior H can be recognized as a standard handlebody in S^3, while tracing the corresponding deformations of ∂D and λ under the isotopy. We thus get two essential disks X and Y which form a disk system of H. From ∂D, we get a geometric presentation of the fundamental group $\pi_1(M_n)$ of M_n with respect to the disk system $\{X, Y\}$:

$$\pi_1(M) =< x, y; (x^{-1}y^{-1})^{2n-1}x^{-1}(yx)^{n+1}y^{-1}(x^{-1}y^{-1})^{2n-1}(xy)^{n+1} > .$$

Also we get

$$\lambda = y(xy)^n(x^{-1}y^{-1})^n x^{-1}y^{-2}(x^{-1}y^{-1})^n x^{-1}(yx)^{n+1}x.$$

Let $\tilde{M}_n$ be the 3-fold cyclic cover of M_n and let $\tilde{M}_n = \tilde{H} \cup_{\partial \tilde{H}} \tilde{C}$ have the induced Heegaard splitting from that of M_n, where $\tilde{H}$ is a genus four handle body which is the corresponding 3-fold cyclic cover of H and $\tilde{C}$ a compression body which covers C. Again the disk X lifts to three disks X_1, X_2, X_3; and the disk Y lifts to three disks Y_1, Y_2, Y_3, as shown in Figure 9 (ignore the ∂W_i and $\tilde{\lambda}$ part), and we pick the meridian disk X_4 of $\tilde{H}$ as shown in Figure 9. Then $\{X_1,\ X_2,\ X_3,\ X_4\}$ forms a disk system of $\tilde{H}$. The disk D lifts to three disks $\{W_1, W_2,\ W_3\}$ which form a disk system of $\tilde{C}$. Again exactly one of the disks $\{W_1,\ W_2,\ W_3\}$, say W_3, is disjoint from X_4, which shows that the Heegaard splitting of $\tilde{M}_n$ is weekly reducible. Again one can show that the surface S obtained by compressing the Heegaard surface $\partial \tilde{H}$ using the disks W_3 and X_4 is an essential closed genus two surface in $\tilde{M}_n$. In fact, cutting $\tilde{H}$ along X_4, we get a handle body $H_\#$ of genus three and $\{X_1, X_2, X_3\}$ is a disk system of $H_\#$. The Whitehead graph of ∂W_3 with respect to the disk system $\{X_1, X_2, X_3\}$ of $H_\#$ is given as Figure 13. The graph is connected with no

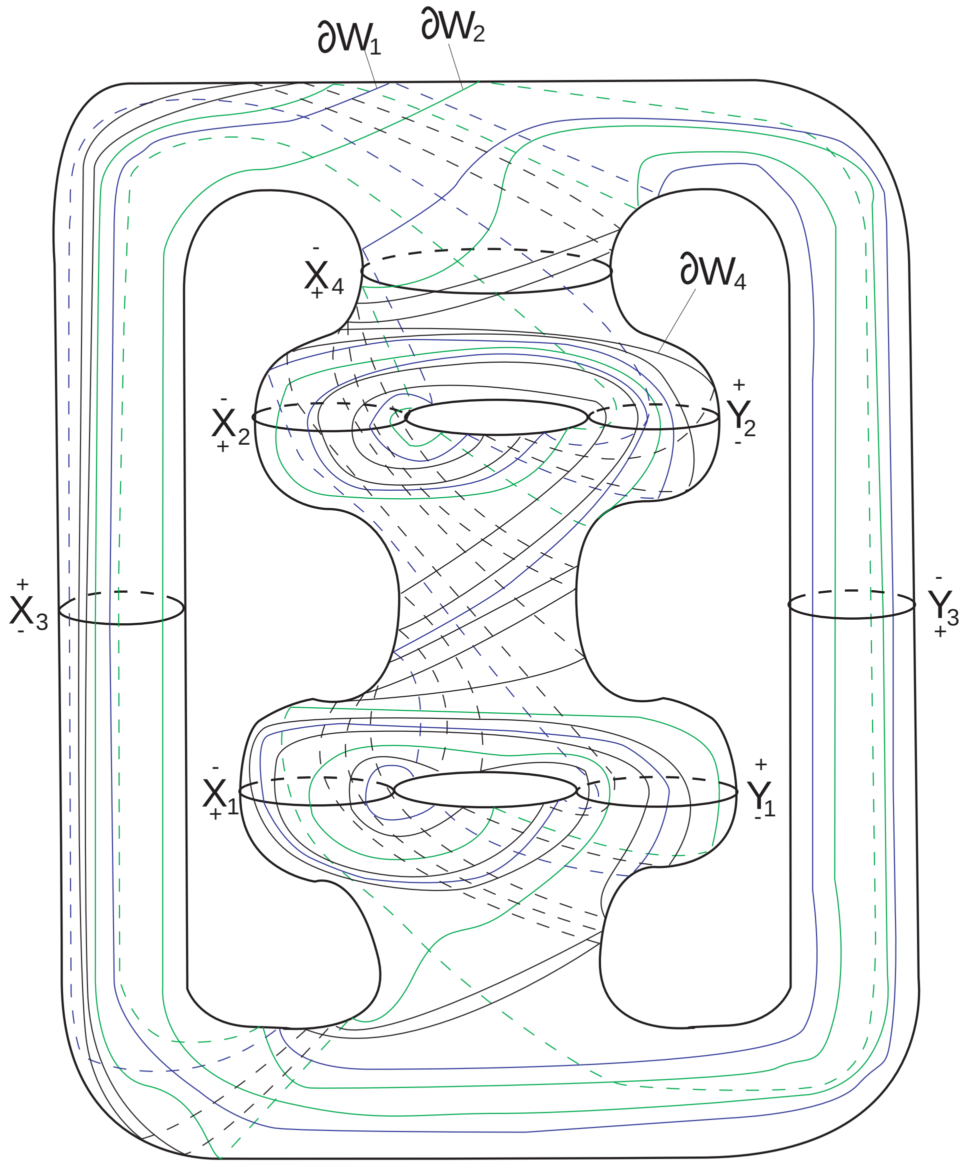


Figure 11: $\partial W_4 = \tilde{\alpha}$, ∂W_1 and ∂W_2 on the Heegaard surface $\partial \tilde{H}$

cut vertex, which means that $\partial H_\# - \partial W_3$ is incompressible. Thus by the handle addition lemma, the manifold $H_\# \cup W_3 \times I$, obtained by attaching the 2-handle $W_3 \times I$ to $H_\#$, has incompressible boundary.

On the other hand, letting $\tilde{C}(2)$ be the handle body obtained by Dehn filling $\tilde{C}$ with slope 2 and letting W_4 be a meridian disk of the filling solid torus, then $\{W_1, W_2, W_3, W_4\}$ forms a disk system of $\tilde{C}(2)$. Cutting $\tilde{C}(2)$ along the disk W_3, we get a handlebody H_* with disk system $\{W_1, W_2, W_4\}$. Let $\alpha \subset \partial M$ be an essential simple closed curve of slope

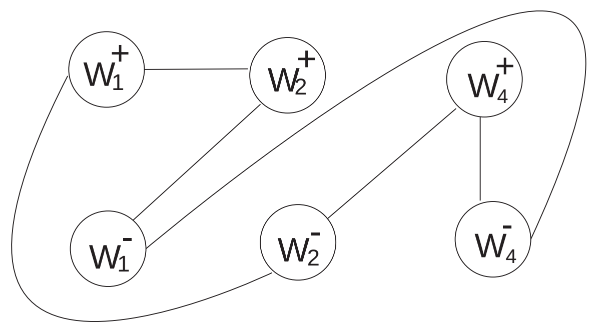


Figure 12: The Whitehead graph of ∂X_4 with respect to the disk system $\{W_1, W_2, W_4\}$ of the handle body H_*

6. Then with respect to the generators x, y of $\pi_1(M)$,

$$\alpha = \lambda x^6 = y(xy)^n(x^{-1}y^{-1})^n x^{-1}y^{-2}(x^{-1}y^{-1})^n x^{-1}(yx)^{n+1}x^6.$$

We may consider ∂W_4 as a lift of α. From the word α, we can draw ∂W_4 on $\partial\tilde{H}$. Consequently we can read off the Whitehead graph of ∂X_4 with respect to the disk system $\{W_1, W_2, W_4\}$ and see that the graph is the same as that shown in Figure 12, showing that $\partial H_* - \partial X_4$ is incompressible in H_*. Thus the manifold $H_* \cup X_4 \times I$ has incompressible boundary of genus two. We thus have justified the incompressibility of the surface S in $\tilde{M}_n(2)$ and thus in $\tilde{M}_n$.

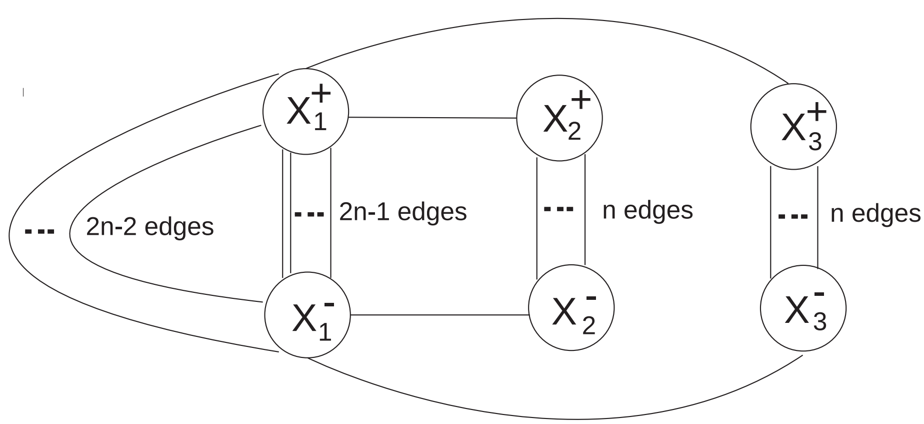


Figure 13: The Whitehead graph of ∂W_3 with respect to the disk system $\{X_1, X_2, X_3\}$ of the handle body $H_\#$

Finally the longitude $\tilde{\lambda}$ in $\partial\tilde{M}$ is isotopic to an essential simple closed curve in the surface S, which is obvious. The proof for the general case is complete. $\diamondsuit$

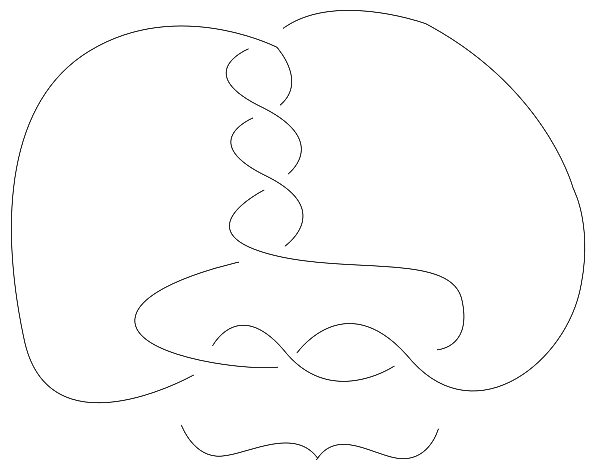


Figure 14: The knot J_{2n+1}

Let J_{2n+1}, $n > 0$, be the family of two bridge knots shown in Figure 14. Note that these knots are hyperbolic, small and non-fibered with genus two Seifert surfaces.

Theorem 2 *The* 5*-fold cyclic cover of the exterior of* J_{2n+1} *is large and every Dehn filling of the exterior of* J_{2n+1} *with slope* $5p/q$*,* $(5p, q) = 1$*,* $|p| > 1$*, yields a virtually Haken 3-manifold, for every* $n > 0$*.*

This theorem gives another family of non-Haken, virtually Haken 3-manifolds to which the results of [CL] do not apply. As the proof of Theorem 2 is very similar to that of Theorem 1, we omit the details and indicate only the steps. In fact the exterior of J_{2n+1} is tunnel number one and a genus two Heegard splitting of it can be explicitly given as in the case for the exterior of the twist knot K_{2n+1}. In the 5-fold cyclic cover of the exterior of J_{2n+1}, the lifted Heegaard surface is of genus 6 and can be compressed along two reducing disks, one on each side of the Heegaard surface, to a closed incompressible surface of genus 4. Also a lift of the longitude can be isotoped into the resulting incompressible surface.

We now go back to the twist knots K_{2n+1} and prove the following Theorem 3. Although the result of the theorem is covered by [CL], we have included it primarily because its proof illustrates two complications which arise in more general settings. First, we have to deal with multi 2-handle additions, which requires the multi 2-handle addition theorem of Lei [L]. Also, one of the Whitehead graphs contains a cut vertex, and must be simplified using Whitehead moves.

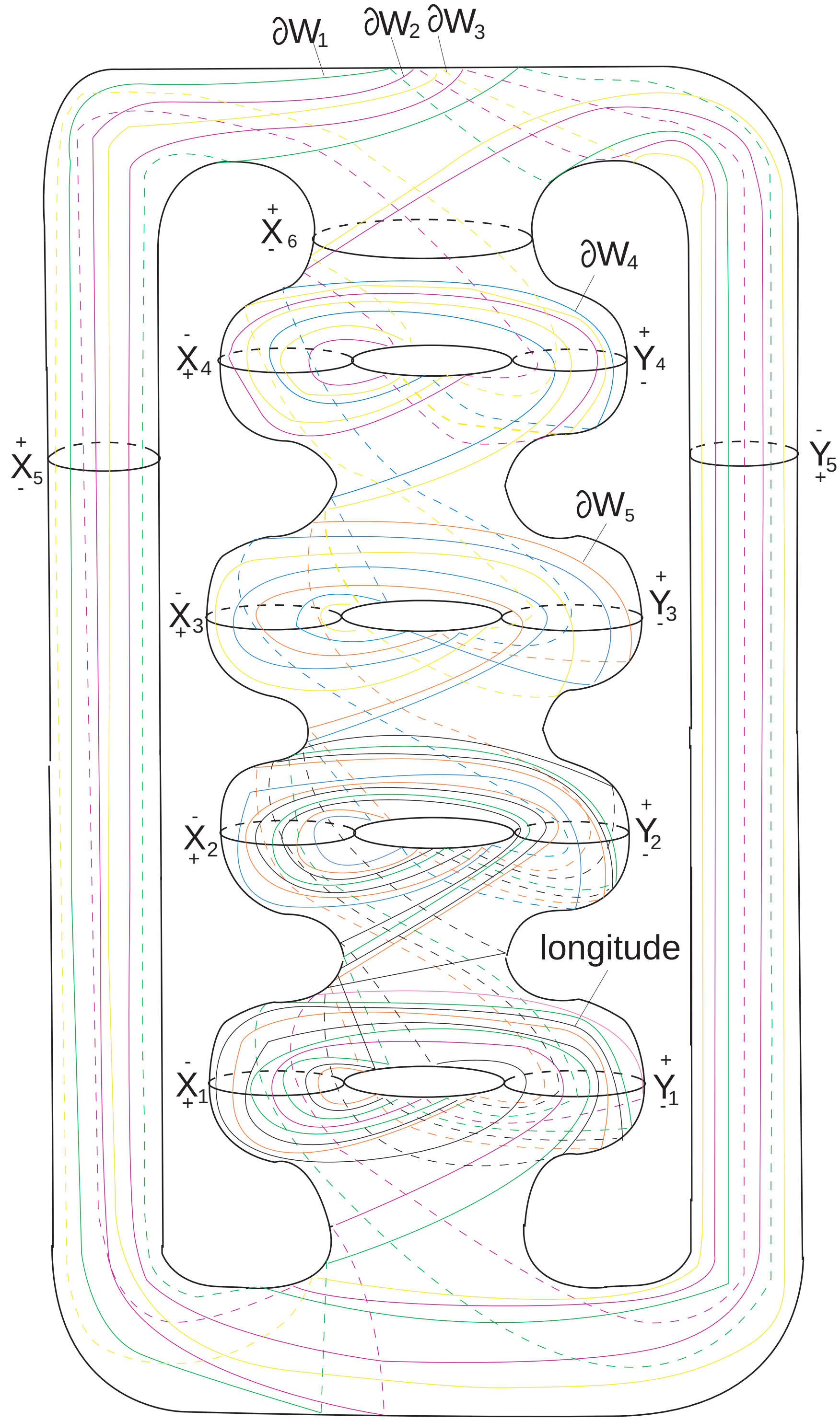


Figure 15: The Heegaard splitting of the 5-fold cover of M

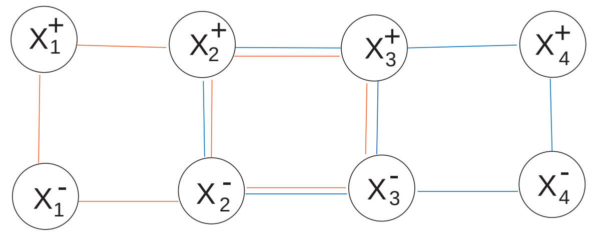


Figure 16: The Whitehead graph of $\{\partial W_4, \partial W_5\}$ with respect to the disk system $\{X_1, X_2, X_3, X_4\}$ of the handle body $H_\#$

Theorem 3 *The 5-fold cyclic cover of the exterior M_n of K_{2n+1} is large for every $n > 0$. Every Dehn filling of M_n with slope $5p/q$, $(5p, q) = 1$, $|p| > 1$, yields a virtually Haken 3-manifold.*

Proof. Again we give details only for the $n = 1$ case. We continue to use the Heegaard splitting of $M = M_1 = H \cup C$ as given in the proof of Theorem 1. Let $\tilde{M}$ be the 5-fold cyclic cover of M with the induced Heegaard splitting from that of M. The Heegaard diagram of $\tilde{M}$ is shown in Figure 15. The genus six handle body of Figure 14 is $\tilde{H}$ which covers H. The disks X and Y of H lift to disks $X_1, ..., X_5$ and $Y_1, ..., Y_5$, as shown in Figure 15. Pick the meridian disk X_6 of $\tilde{H}$ as shown in Figure 15. Then $\{X_1, X_2, X_3, X_4, Y_5, X_6\}$ forms a disk system of $\tilde{H}$. The disk D lifts to five disks $\{W_1, W_2, W_3, W_4, W_5\}$ whose boundaries are shown in Figure 15. Figure 15 also shows a longitude $\tilde{\lambda}$ of $\tilde{M}$, which is a lift of the longitude λ of M.

This Heegaard splitting of $\tilde{M}$ is weakly reducible: $\{\partial Y_5, \partial X_6\}$ is disjoint from $\{\partial W_4, \partial W_5\}$. We now show that the surface S obtained by compressing the Heegaard surface $\partial\tilde{H}$ using these four disks is an essential closed genus two surface in $\tilde{M}$. It is enough to show that the surface S is incompressible in $\tilde{M}(2)$, which is the free 5-fold cyclic cover of $M(10) = H \cup C(10)$, and has the induced Heegaard splitting $\tilde{H} \cup \tilde{C}(2)$. Let $\tilde{V}$ be the filling solid torus in $\tilde{M}(2)$ and let W_6 be a meridian disk of $\tilde{V}$. Then $\{W_1, ..., W_5, W_6\}$ is a disk system of the handle body $\tilde{C}(2)$.

Cutting $\tilde{H}$ along Y_5, X_6, we get a handle body $H_\#$ of genus four and $\{X_1, X_2, X_3, X_4\}$ is a disk system of $H_\#$. The Whitehead graph of $\{\partial W_4, \partial W_5\}$ with respect to the disk system $\{X_1, ..., X_4\}$ of $H_\#$ is given in Figure 16. The graph is connected with no cut vertex, which means that the surface $\partial H_\# - \{\partial W_4, \partial W_5\}$ is incompressible in $H_\#$. Moreover as ∂W_4 is disjoint from the disk X_1, and ∂W_5 is disjoint from the disk X_4, each of the surfaces $\partial H_\# - \partial W_4$ and $\partial H_\# - \partial W_5$ is compressible in $H_\#$. Therefore all the conditions of the

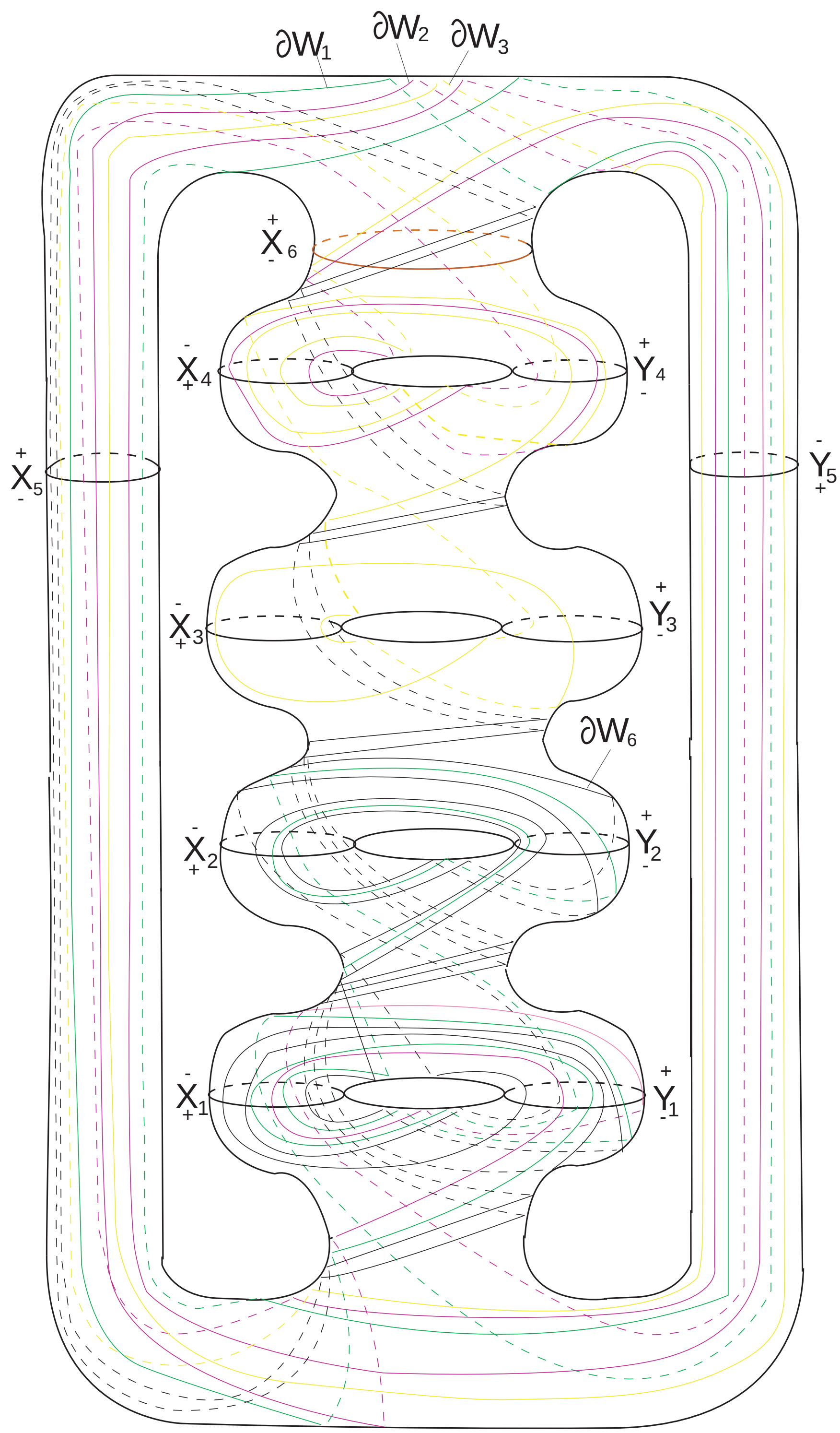


Figure 17: $\partial W_6 = \tilde{\alpha}$, ∂W_1, ∂W_2, ∂W_3 on the Heegaard surface $\partial \tilde{H}$

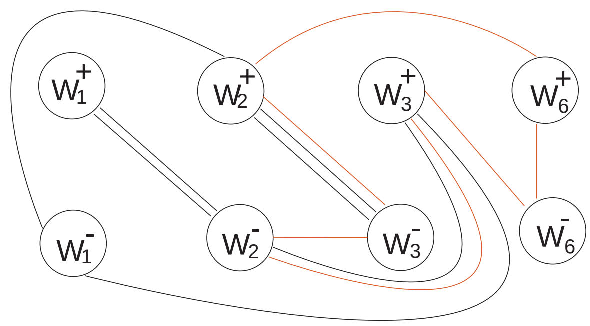


Figure 18: The Whitehead graph of $\{\partial Y_5,\ \partial X_6\}$ with respect to the disk system $\{W_1, W_2, W_3, W_6\}$ of the handle body H_*

multi-handle addition theorem of [L] are satisfied, and thus the manifold $H_\# \cup W_4 \times I \cup W_5 \times I$ has incompressible boundary.

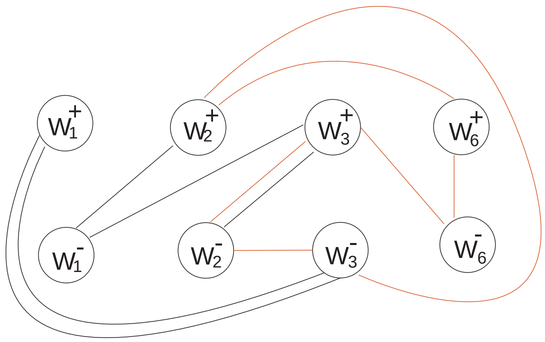
On the other hand, cutting the handle body $\tilde{C}(2)$ along the disks W_4 and W_5, we get a handle body H_*, with disk system $\{W_1, W_2, W_3, W_6\}$. Let $\alpha \subset \partial M$ be an essential simple closed curve of slope 10. Then

$$\alpha = \lambda x^{10} = yxyx^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^{-1}x^{-1}yxyx^{12}.$$

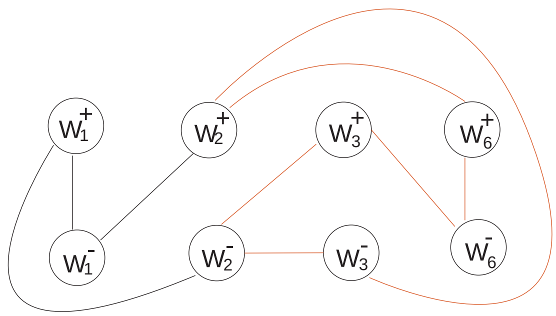
Let $\tilde{\alpha} \subset \partial \tilde{M}$ be a lift α. Then $\tilde{\alpha}$, which can be considered as the boundary of the disk W_6, has slope 2 in $\partial \tilde{M}$. Figure 17 shows $\tilde{\alpha} = \partial W_6$, ∂W_1,∂W_2, ∂W_3 in $\partial \tilde{H}$.

From Figure 17, we can read off the Whitehead graph of $\{\partial Y_5,\ \partial X_6\}$ with respect to the disk system $\{W_1, W_2, W_3, W_6\}$ of H_*, which is given as Figure 18. The graph is connected but has a cut vertex (the vertex W_2^-). Applying Whitehead moves to the graph twice with results shown in Figure 19, we end up with a graph (shown in Figure 19 (b)) which is connected with no cut vertex. This means that the surface $\partial H_* - \{\partial Y_5 \cup \partial X_6\}$ is incompressible in H_*. From Figure 16, we also see that ∂Y_5 is disjoint from ∂W_6 and ∂X_6 is disjoint from ∂W_1. Thus each of the surfaces $\partial H_* - \partial Y_5$ and $\partial H_* - \partial X_6$ is compressible in H_*. Again the multi-handle addition theorem of [L] implies that the manifold $H_* \cup X_6 \times I \cup Y_5 \times I$ has incompressible boundary. Therefore the genus two surface $S = \partial(H_* \cup X_6 \times I \cup Y_5 \times I) = \partial(H_\# \cup W_4 \times I \cup W_5 \times I)$ is incompressible in $\tilde{M}(2)$ and thus is essential in $\tilde{M}$.

Obviously $\tilde{\lambda}$ can be isotoped into S. The proof of Theorem 3 is complete in case $n = 1$. The proof for the general case is similar (cf the proof of Theorem 1 in general case). We leave the details to the reader to verify. $\Diamond$



(a)



(b)

Figure 19: (a) The resulting graph after the Whitehead move with respect to the cut vertex W_2^- of Figure 18. (b) The resulting graph after the Whitehead move with respect to the cut vertex W_3^- of part (a)

References

[BZ] S. Boyer and X. Zhang, Virtual Haken 3-manifolds and Dehn filling, Topology 39 (2000) 103-114.

[CG] A. Casson and C. Gordon, Reducing Heegaard splittings. Topology Appl. 27 (1987) 275–283.

[CGLS] M. Culler, C. Gordon, J. Luecke and P. Shalen, Dehn surgery on knots, Ann. of Math. 125 (1987) 237-300.

[CL] D. Cooper and D. Long, Virtually Haken Dehn-filling, J. Differential Geom. 52 (1999) 173–187.

[DT] N. Dunfield and W. Thurston, The virtusl Haken conjecture: experiments and examples, preprint.

[HT] A. Hatcher and W. Thurston, Incompressible surfaces in 2-bridge knot complements. Invent. Math. 79 (1985) 225–246.

[J] W. Jaco, Adding a 2-handle to a 3-manifold: an application to property R. Proc. Amer. Math. Soc. 92 (1984) 288–292.

[K] R. Kirby, Problems in Low-Dimensional Topology, Geometric topology. Edited by William H. Kazez. AMS/IP Studies in Advanced Mathematics, 2.2. (1997) 35-473.

[La] M. Lackenby, Heegaard splittings, the virtually Haken conjecture and Property tau, preprint.

[L] F. Lei, A proof of Przytycki's conjecture on n-relator 3-manifolds, Topology (1995) 473-476.

[M] J. Masters, Virtual homology of surgered torus bundles. Pacific J. Math. 195 (2000) 205–223.

[P] J. Przytycki, Incompressibility of surfaces after Dehn surgery. Michigan Math. J. 30 (1983) 289–308.

[S] J. Stallings, Whitehead graphs on handlebodies, preprint.

Mathematics Department, Rice University, Houston, TX 77005
mastersj@rice.edu

Mathematics Department, SUNY at Buffalo, Buffalo, NY 14260-2900
menasco@math.buffalo.edu

Mathematics Department, SUNY at Buffalo, Buffalo, NY 14260-2900
xinzhang@math.buffalo.edu