# Master Integrals, Superintegrability and Quadratic Algebras

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#### Abstract

In this paper we use a generalization of Oevel's theorem about master symmetries to relate them with superintegrability and quadratic algebras.

# I Introduction

In this article a general framework is built in terms of master symmetries and recursion operator to provide superintegrability and quadratic algebras.

This is applied to the isotropic harmonic oscillator, the rational Calogero-Moser system and the "Goldfish" model.

## II Background

Let M be a differentiable  $(C^{\infty})$  manifold of finite dimension and  $\Lambda$  a bivector (a 2-times contravariant skew-symmetric tensor field) on M. Associated with  $\Lambda$  there is a natural

morphism  $\Lambda^{\sharp}$  from the cotangent bundle  $T^{*}M$  into the tangent bundle TM defined, for all  $\alpha, \beta \in T^{*}M$ , by

$$<\Lambda^{\sharp}(\alpha), \beta> = \Lambda(\alpha, \beta),$$
 (2.1)

where  $\langle .,. \rangle$  denotes the usual coupling between 1-forms and vector fields.

We also define a bilinear map from  $C^{\infty}(M) \times C^{\infty}(M)$  into  $C^{\infty}(M)$  by

$$\{f,g\} = \Lambda(df,dg), \quad f,g \in C^{\infty}(M). \tag{2.2}$$

Due to the properties of  $\Lambda$ , this bracket satisfies

 $\mathbf{PB1}\ \{f,g\} = -\{g,f\}\ \textit{skew-symmetry}$ 

PB2 
$$\{fh,g\} = f\{h,g\} + \{f,h\}g$$
 Leibniz rule

We say that  $(M, \Lambda)$  is a Poisson manifold, and  $\Lambda$  is a Poisson tensor, if, in addition, the bracket (2.2) satisfies the Jacobi identity

**PB3** 
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

which is equivalent to the vanishing of the Schouten-Nijenhuis bracket  $[\Lambda, \Lambda]$ .

We call a vector field X an infinitesimal Poisson automorphism if it satisfies

$$X\{f,g\} = \{X(f),g\} + \{f,X(g)\}, \quad f,g \in C^{\infty}(M).$$

Given a differentiable function H on M the Hamiltonian vector field associated with H is the vector field defined by

$$X_H(x) = \Lambda^{\sharp}(x)(dH(x)), \quad x \in M. \tag{2.3}$$

If  $\Lambda$  is a Poisson tensor then

$$[X_f, X_g] = [\Lambda^{\sharp}(df), \Lambda^{\sharp}(dg)] = \Lambda^{\sharp}(d\{f, g\}) = X_{\{f, g\}}$$
 (2.4)

that is,  $\Lambda^{\sharp}$  is a Lie algebra homomorphism between the Lie algebra of differentiable functions  $(C^{\infty}(M), \{.,.\})$  and the Lie algebra of vector fields (A(M), [.,.]).

An integral of motion of H, or of  $X_H$ , is a differentiable function F such that

$$\{H, F\} = X_H(F) = 0,$$
 (2.5)

so is constant along the orbits of the Hamiltonian system  $X_H$ 

$$\frac{d\phi}{dt} = X_H(\phi). \tag{2.6}$$

By a Nijenhuis operator R in a manifold M we mean a (1,1)-tensor satisfying, for all vector fields Z in M,

$$\mathcal{L}_{R(Z)}R = R\mathcal{L}_Z R. \tag{2.7}$$

The Nijenhuis operators transform closed 1-forms into closed 1-forms in the following sense

**Proposition 1** Let R be a Nijenhuis operator and  $\alpha$  a closed 1-form such that  $\alpha_1 = {}^tR\alpha$  is also closed then, for all  $i \in \mathbb{N}$ ,  $\alpha_i = {}^tR^i\alpha_1$  (where  ${}^tR^i$  means  $i^{th}$  iterates of  ${}^tR$ ) are closed.

By a conformal vector field of a tensor W we mean a vector field Z such that  $\mathcal{L}_Z W = cW$ , for some constant  $c \in \mathbb{R}$ .

With Oevel's Theorem [11], a Nijenhuis operators helps to define new symmetries if a conformal vector field is known.

**Theorem 2 (Oevel)** Let R be a recursion operator of  $X_0 \in A(M)$ , this means  $\mathcal{L}_X R = 0$ , and  $Z_0 \in A(M)$  a conformal vector field of  $X_0$  and R such that

$$\mathcal{L}_{Z_0}X_0 = \lambda X_0, \quad \mathcal{L}_{Z_0}R = \mu R, \quad \lambda, \mu \in \mathbb{R}.$$

If R is also a Nijenhuis operator then, defining the sequences  $X_n = R^n X_0$  and  $Z_n = R^n Z_0$ ,  $n \in \mathbb{N}$ , we have, for all  $n, m \in \mathbb{N}_0$ 

$$\mathcal{L}_{Z_n}R = \mu R^{n+1},$$

$$[Z_n, Z_m] = \mu(m-n)Z_{n+m}$$

and

$$[Z_n, X_m] = (\lambda + m\mu)X_{n+m}.$$

The  $Z_i$ 's are called master symmetries or symmetries of second order of the vector fields  $X_j$  because  $[[Z_i, X_j], X_j] = 0$  but  $[Z_i, X_j] \neq 0$  and they help us define new symmetries of the system. We call master integrals to functions G which may not be integrals of motion of the system but they induce an integral X(G).

Now let us consider two linearly independent Poisson structures  $\Lambda_0$  and  $\Lambda_1$  in M. We say that they are compatible if  $\Lambda_0 + \Lambda_1$  is again a Poisson tensor. The compatibility condition is equivalent to the vanishing of the Schouten-Nijenhuis bracket  $[\Lambda_0, \Lambda_1]$ .

A vector field  $X \in A(M)$  is said to be *bihamiltonian* if it is Hamiltonian with respect to two independent compatible Poisson tensors, that is, if there exist two functions  $H, F \in C^{\infty}(M)$  such that

$$X = \Lambda_0^{\sharp}(dH) = \Lambda_1^{\sharp}(dF). \tag{2.8}$$

There is an important example of bihamiltonian manifold: the Poisson-Nijenhuis manifold [9]. This manifold is special because one of the Poisson tensors is obtain from the other by means of a Nijenhuis operator. For instance, a bihamiltonian manifold  $(M, \Lambda_0, \Lambda_1)$  such that the Poisson tensor  $\Lambda_0$  is non-singular (symplectic manifold), is a Poisson-Nijenhuis manifold with Nijenhuis operator  $R = \Lambda_1^{\sharp} \Lambda_0^{\sharp -1}$ . If  $X_1 = \Lambda_0^{\sharp} (dH_1) = \Lambda_1^{\sharp} (dH_0)$  is a bihamiltonian system then we may define a sequence of symmetries of X,  $X_i = R^{i-1}X_1$  and, if the first cohomology group is trivial, a sequence of integrals of motion in involution  $(H_i)_{i \in \mathbb{N}}$  such that  $X_i = \Lambda_0^{\sharp} (dH_i)$ , ie  $dH_i = {}^t R^i (dH_0)$ ,  $i \in \mathbb{N}$ .

# III The superintegrability and the cubic algebra

Let X be a vector field on a manifold M of dimension n. It is called maximally superintegrable if it possesses n-1 functionally independent first integrals.

There are several examples of maximally superintegrable systems, some of them are shown in the last section.

The superintegrability may be a consequence of the existence of a sufficient number of master integrals, as the next proposition shows.

**Proposition 3** Let X be a vector field on a manifold M and  $G, F \in C^{\infty}(M)$  master integrals of X. Then the function

$$L = X(G)F - X(F)G \tag{3.9}$$

is an integral of the vector field X.

#### **Proof:**

Just notice that

$$X(L) = X(X(G))F + X(G)X(F) - X(X(F))G - X(F)X(G)$$
  
=  $X(X(G))F - X(X(F))G = 0$ ,

because that X(F) and X(G) are integrals of the system.

**Remark 4** If X is a vector field on a manifold of dimension 2n and if n functionally independent master integrals are known  $G_1, \ldots, G_n$  then we can define the integrals of motion  $F_i = X(G_i)$  and  $L_{i,j} = X(G_i)G_j - G_iX(G_j)$  which may provide the superintegrability of the system if 2n-1 of them are functionally independent.

**Theorem 5** Let X be a Poisson infinitesimal automorphism on a Poisson manifold  $(M, \{.,.\})$ . Suppose there exist master integrals of X,  $G_i$ , such that  $\{G_i, X(G_i), i \in J \subset \mathbb{N}\}$  is a basis of a Lie subalgebra of  $(C^{\infty}(M), \{.,.\})$ 

Then, for all  $i, j \in \mathbb{N}$  the functions  $X(G_i)$  and  $L_{G_i,G_j} = X(G_i)G_j - X(G_j)G_i$  generate a cubic algebra for the Poison bracket.

#### **Proof:**

Once  $\{G_i, X(G_i), i \in J\}$  generates a Lie algebra, for each  $i, j, k \in J$  there exist constants  $a_{i,j}^k, b_{i,j}^k$  such that

$$\{G_i, G_j\} = \sum_{k \in J} (a_{i,j}^k G_k + b_{i,j}^k X(G_k)).$$
(3.10)

Applying X twice to the last equation we obtain

$$X({X(G_i), (G_j)} + {G_i, X(G_j)}) = X(\sum_{k \in J} a_{i,j}^k X(G_k)),$$

SO

$${X(G_i), X(G_j)} = 0.$$

Writing  $\{G_i, X(G_j)\} = \sum_{k \in J} (c_{i,j}^k X(G_k) + d_{i,j}^k G_k)$ , with  $c_{i,j}^k$ ,  $d_{i,j}^k$  constants, and noticing that

$$X(\{G_i, X(G_j)\}) = \{X(G_i), X(G_j)\} = 0,$$

we have  $\sum_{k \in J} d_{i,j}^k X(G_k) = 0$ , which yields  $d_{i,j}^k = 0$ . Thus

$$\{L_{G_i,G_j}, X(G_k)\} = X(G_i)\{G_j, X(G_k)\} - X(G_j)\{G_i, X(G_k)\}$$
$$= \sum_{l} [c_{j,k}^{l} X(G_i) X(G_l) - c_{i,k}^{l} X(G_j) X(G_l)]$$

and

$$\{L_{G_{i},G_{j}}, L_{G_{k},G_{h}}\} =$$

$$= X(G_{k})[L_{\{G_{i},G_{h}\},G_{j}} - L_{\{G_{j},G_{h}\},G_{i}}] + X(G_{h})[L_{\{G_{j},G_{k}\},G_{i}} - L_{\{G_{i},G_{k}\},G_{j}}]$$

$$+ L_{G_{j},G_{k}}\{G_{i}, X(G_{h})\} + L_{G_{i},G_{h}}\{G_{j}, X(G_{k})\}$$

$$- L_{G_{j},G_{h}}\{G_{i}, X(G_{k})\} - L_{G_{i},G_{k}}\{G_{j}, X(G_{h})\}$$

$$= \sum_{l}[X(G_{k})(a_{i,h}^{l}L_{G_{l},G_{j}} + b_{i,h}^{l}L_{X(G_{l}),G_{j}} - a_{j,h}^{l}L_{G_{l},G_{i}} - b_{j,h}^{l}L_{X(G_{l}),G_{i}})$$

$$+ X(G_{h})(a_{j,k}^{l}L_{G_{l},G_{i}} + b_{j,k}^{l}L_{X(G_{l}),G_{i}} - a_{i,k}^{l}L_{G_{l},G_{j}} - b_{i,k}^{l}L_{X(G_{l}),G_{j}})$$

$$+ c_{i,h}^{l}L_{G_{j},G_{k}}X(G_{l}) + c_{j,k}^{l}L_{G_{l},G_{k}}X(G_{l}) - c_{i,k}^{l}L_{G_{j},G_{k}}X(G_{l}) - c_{j,h}^{l}L_{G_{l},G_{k}}X(G_{l})]$$

$$= \sum_{l}[X(G_{k})(a_{i,h}^{l}L_{G_{l},G_{j}} - a_{j,h}^{l}L_{G_{l},G_{i}}) + X(G_{h})(a_{j,k}^{l}L_{G_{l},G_{i}} - a_{i,k}^{l}L_{G_{l},G_{j}})$$

$$+ c_{i,h}^{l}L_{G_{j},G_{k}}X(G_{l}) + c_{j,k}^{l}L_{G_{l},G_{k}}X(G_{l}) - c_{i,k}^{l}L_{G_{j},G_{k}}X(G_{l}) - c_{j,h}^{l}L_{G_{l},G_{k}}X(G_{l})$$

$$+ (b_{i,h}^{l}X(G_{j}) - b_{i,h}^{l}X(G_{i}))X(G_{l})X(G_{l}) + (b_{i,k}^{l}X(G_{i}) - b_{i,k}^{l}X(G_{j}))X(G_{l})X(G_{h})]$$

This last expression being cubic in the quantities  $X(G_i)$  and  $L_{G_i,G_j}$ , we refer to this result as that they generate a cubic algebra.

Corollary 6 If for each  $i, j \in \mathbb{N}$ ,  $\sum_{l} b_{i,j}^{l} X(G_{l}) = b_{i} X(G_{j}) - b_{j} X(G_{i})$  or all the constants  $b_{i,j}^{k}$  are zero then the integrals of motion generate a quadratic algebra.

**Remark 7** Notice that in the above proposition we could have just demanded that  $d\{G_i, G_j\}$  be a linear combination of the dG's and the dX(G)'s.

**Theorem 8 (Generalization of Oevel's theorem)** Let X be a vector field on a manifold M, R a Nijenhuis operator which is also a recursion operator of X and P a (1,1)-tensor satisfying

$$\mathcal{L}_X P = a(R) \tag{3.11}$$

and

$$\mathcal{L}_{PX}R = b(R), \tag{3.12}$$

with a(R), b(R) polynomials in R. Then, defining the sequences  $X_i = R^i X$ ,  $Y_i = R^i (PX)$ ,  $i \in \mathbb{N}_0$ , we have

$$[X_i, X_j] = 0, (3.13)$$

$$[X_i, Y_j] = a(R)(X_{i+j}) - ib(R)(X_{i+j-1})$$
(3.14)

$$[Y_i, Y_j] = (j-i)b(R)Y_{i+j-1}. (3.15)$$

#### **Proof:**

The proof is similar to the original Oevel's theorem proof, which can be seen at [11].

Suppose that M has trivial first cohomology group and is endowed with a non-degenerated Poisson structure  $\Lambda$  such that  $R\Lambda^{\sharp} = \Lambda^{\sharp t}R$  (this means that the tensor  $R\Lambda$  is a bivector). Furthermore suppose that there exist functions such that  $X = \Lambda^{\sharp}(dH_1) = R\Lambda^{\sharp}(dH_0)$  and  $Y = \Lambda^{\sharp}(dG_1) = R\Lambda^{\sharp}(dG_0)$ .

Then Proposition 1 ensures us that the 1-forms

$$\alpha_i = {}^t R^i(dH_1), \quad \beta_i = {}^t R^i(dG_1), \ i \in \mathbb{N}$$

are closed and we can consider them exact because of the triviality of the first cohomology group.

Write  $\alpha_i = dH_i$  and  $\beta_i = dG_i$ , for all  $i \in \mathbb{N}$ .

First notice that  $R\Lambda$  being a bivector yields

$$X_i(H_j) = \langle X_i, dH_j \rangle = \langle X_{i+j}, dH_1 \rangle$$
  
=  $R^{i+j} \Lambda^{\sharp} (dH_1, dH_1) = 0.$ 

Moreover (3.14) ensures that the G's are master integrals of the X's because

$$X_i(X_i(G_j)) = X_i(\{H_i, G_j\}) = -[X_i, Y_j](H_i)$$
  
=  $(ib(R)X_{i+j-1} - a(R)X_{i+j})(H_i) = 0,$ 

relation (3.15) implies

$$d\{G_i, G_j\} = (j - i)b({}^{t}R)dG_{i+j-1}$$

and

$$\begin{aligned} \{X_i(G_j), G_k\} &= -Y_k(X_i(G_j)) = [X_i, Y_k](G_j) - X_i(Y_k(G_j)) \\ &= ib(R)X_{i+k-1}(G_j) - a(R)X_{i+k}(G_j) - X_i(\{G_k, G_j\}) \\ &= ib(R)X_i(G_{j+k-1}) - a(R)X_i(G_{j+k}) - (j-k)b(R)X_i(G_{k+j-1}) \\ &= (i+k-j)b(R)X_i(G_{j+k-1}) - a(R)X_i(G_{j+k}). \end{aligned}$$

So  $\{X_i(G_j), G_k\}$  can be written as a linear combination of the  $X_i(G)$ 's.

Now we can apply Theorem 5, with the b's equal to zero, and guarantee that, for each  $i \in \mathbb{N}$  the integrals of  $X_i$ ,  $X_i(G_j)$  and  $L_{k,j}^i = X_i(G_k)G_j - X_i(G_j)G_k$ ,  $j, k \in \mathbb{N}_0$ , close quadratically under the Poisson bracket.

Futhermore notice that

$$[R\Lambda, \Lambda](\Lambda^{\sharp -1}Y) = \mathcal{L}_{RY}\Lambda + (\mathcal{L}_{Y}\Lambda) \circ {}^{t}R + (\mathcal{L}_{Y}R) \circ \Lambda$$

so, as Y and RY are Hamiltonian vector fields, we have

$$[R\Lambda, \Lambda](\Lambda^{\sharp -1}Y) = (\mathcal{L}_Y R) \circ \Lambda = b(R)\Lambda.$$

Thus, if  $R\Lambda$  is a Poisson tensor then it is compatible with  $\Lambda$ , because R is a Nijenhuis operator, and  $b(R)\Lambda = 0$ .

But this implies that

$$b(R)X_i = b(R)\Lambda^{\sharp}(dH_i) = 0$$
 and  $b(R)Y_i = b(R)\Lambda^{\sharp}(dG_i) = 0$ ,

so the relations (3.13), (3.14) and (3.15) become

$$[X_i, X_j] = [Y_i, Y_j] = 0;$$
  $[X_i, Y_j] = a(R)X_{i+j}$ 

and

$$d\{G_i, G_j\} = 0;$$
  $d\{G_k, X_i(G_j)\} = a(R)X_i(G_{j+k}).$ 

Although in the last procedure we need two master integrals, Hamiltonians of the master symmetries, to generate a new sequence of integrals of motion, we may construct it only knowing one master integral of all the vector fields.

**Proposition 9** Under the conditions of proposition 8, suppose there exists a master integral G of all the vector fields  $X_i$ ,  $i \in \mathbb{N}_0$ , then the functions  $G_i = Y_i(G)$ ,  $i \in \mathbb{N}_0$  are also master integrals of the same vector fields and, for each  $k \in \mathbb{N}_0$ ,

$$L_{i,j}^k = X_k(G_i)G_j - X_k(G_j)G_i$$
, for all  $i, j \in \mathbb{N}_0$ ,

are integrals of  $X_k$ .

#### **Proof:**

Due to relation (3.14), we have, for all  $i, k \in \mathbb{N}_0$ 

$$X_k(G_i) = [X_k, Y_i](G) + Y_i(X_k(G))$$
  
=  $a(R)X_{i+k}(G) - kb(R)X_{i+k-1}(G) + Y_i(X_k(G)).$ 

But

$$X_k(Y_i(X_k(G))) = a(R)X_{i+k}(X_k(G)) - kb(R)X_{i+k-1}(X_k(G)) + Y_i(X_k(X_k(G))) = 0,$$

because  $X_k(G)$  is an integral of all the  $X_j$ , so  $Y_i(X_k(G))$  is an integral of  $X_k$  and  $X_k(G_i)$  is then an integral of  $X_k$ . Thus  $G_i$  is a master integral of all the  $X_k$ .

**Remark 10** It seems that in the previous procedure only one Hamiltonian  $G_0$  would be necessary, but note that the procedure applied to  $G_0$  yields  $G_i = Y_i(G_0) = R^i \Lambda(dG_0, dG_0) = 0$ , i.e. all master integrals are zero.

## IV Examples

### Example 11 (Isotropic Harmonic Oscillator and the Fernandes' Theorem)

The Isotropic Harmonic Oscillator is the Hamiltonian system in  $(\mathbb{R}^{2n}, (q_i, p_i))$ , defined by the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{n} \frac{1}{2} (p_i^2 + q_i^2). \tag{4.16}$$

It is well known that this system is completely integrable with constants of motion in involution

$$E_i = \frac{1}{2}(p_i^2 + q_i^2), \quad i = 1, \dots, n,$$
 (4.17)

and it is bi-Hamiltonian [10] with respect to the compatible Poisson tensors

$$\Lambda_0 = \sum_{i=1}^n \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} \tag{4.18}$$

and

$$\Lambda_1 = \sum_{i=1}^n E_i(\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}). \tag{4.19}$$

The Hamiltonian vector field can be expressed as

$$X_{\mathcal{H}} = \sum_{i=1}^{n} (p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i}) = \Lambda_0^{\sharp}(d\mathcal{H}) = \Lambda_1^{\sharp}(dH_0), \tag{4.20}$$

with  $H_0 = \ln(E_1) + \ldots + \ln(E_n)$ . Defining the recursion operator of the system

$$R = \Lambda_1^{\sharp} \Lambda_0^{\sharp - 1} = \sum_{i=1}^n E_i \left( \frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i \right), \tag{4.21}$$

a sequence of Hamiltonians and of Hamiltonian vector fields can be defined

$$X_i = N^{i-1} X_{\mathcal{H}} = \Lambda_0^{\sharp} (dH_i), \quad i = 1, \dots, n.$$
 (4.22)

Notice that the (1,1)-tensor

$$P = \sum_{i=1}^{n} \varphi_i(\frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i), \tag{4.23}$$

with  $\varphi_i = \arcsin(\frac{q_i}{\sqrt{q_i^2 + p_i^2}})$ , satisfies the conditions of Theorem 8 with a(R) = Id and b(R) = 0. So defining the sequence

$$Y_k = R^k P X = \sum_{i=1}^n \varphi_i E_i^k \left( p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i} \right), \tag{4.24}$$

we have

$$[Y_i, Y_j] = 0$$
$$[Y_i, X_j] = -X_{i+j}$$

and also

$$dZ_i(H_j) = 0$$
$$[Y_i, R^j \Lambda_0] = -R^{i+j} \Lambda_0$$

i.e., the vector fields  $Y_i$  are not Hamiltonian with respect to any of the Poisson structures.

But let us define the functions  $G = \sum_{i=1}^{\infty} E_k \varphi_k$  and notice that

$$X_i(G) = \sum_k E_k^{i-1} \left(p_k \frac{\partial}{\partial q_k} - q_k \frac{\partial}{\partial p_k}\right)(G) = iH_i.$$
 (4.25)

So the functions  $G_i = Y_i(G) = \sum_k \varphi_k E_k^{i+1}$  satisfy

$$X_i(G_j) = X_{i+j}(G) = (i+j)H_{i+j},$$

$$\begin{aligned}
\{X_i(G_j), G_k\} &= \{(i+j)H_{i+j}, G_k\} \\
&= (i+j)X_{i+j}(G_k) = (i+j)X_{i+j+k}(G) \\
&= (i+j)(i+j+k)H_{i+j+k}
\end{aligned}$$

and

$$\{G_i, G_j\} = \Lambda_0(dG_i, dG_j) = \Lambda_0(dZ_i(G), dZ_j(G))$$

$$= \Lambda_0(d(\sum_{k=1}^n E_k^{i+1} \varphi_k), d(\sum_{h=1}^n E_h^{j+1} \varphi_h)) = (i-j)Z_{i+j}(G) = (j-i)G_{i+j}.$$

Therefore, for each  $k \in \mathbb{N}_0$  the integrals of motions of  $X_k$ ,  $X_k(G_j)$ 's, and  $L_{i,j}^k = X_k(G_i)G_j - X_k(G_j)G_i = (i+k)H_{i+k}G_j - (j+k)H_{j+k}G_i$  close quadratically under the Poisson bracket defined by  $\Lambda_0$ .

Now let us consider a little more general configuration, in which the isotropic harmonic oscillator is a particular case.

Given a completely integrable Hamiltonian system  $(M^{2n}, \omega, H)$  in a symplectic manifold, Fernandes [5] establishes necessary and sufficient conditions for the existence of a second Poisson structure giving the complete integrability of the system, in a neighborhood of a fixed invariant torus. Without lost of generality let us consider

$$(M^{2n} = \mathbb{R}^n \times \mathbb{T}^n, (s_i, \theta_i)_{i=1}^n), \quad H = H(s_1, \dots, s_n) \text{ and } \omega = \sum_i ds_i \wedge d\theta_i.$$
 (4.26)

**Definition 12** Let  $(x^1, \ldots, x^{n+1})$  be affine coordinates in a (n+1)-dimensional affine space  $A^{n+1}$ . A hypersurface in  $A^{n+1}$  is called a hypersurface of translation if it admits a parametrization of the form

$$(y^1, \dots, y^n) \to x^j(y^1, \dots, y^n) = a_1^j(y_1) + \dots + a_n^j(y^n), \quad (j = 1, \dots, n+1).$$
 (4.27)

**Theorem 13** ([5]) The completely integrable Hamiltonian system (4.26) admits a second Poisson structure, giving its complete integrability if and only if the graph of the Hamiltonian function is a hypersurface of translation relative to the affine structure determined by the action variables.

We present the "only if" part of the proof because in what follows the new Poisson structure will be needed.

#### **Proof:**

Assume the  $(M^{2n}, \omega, H)$  is a completely integrable system and that the graph of H is a hypersurface of translation relative to the action variables  $(s^i)$ , so it has a parametrization of the form (4.27) with  $x^i = s^i$ , i = 1, ..., n and  $x^{n+1} = H$ . We can choose the parameters  $(y^i)$  so that the Hamiltonian takes the simple form

$$H(y^1, \dots, y^n) = y^1 + \dots + y^n.$$

If  $(\varphi^1, \dots, \varphi^n)$  are the coordinates conjugated to  $(y^1, \dots, y^n)$ , we define a second Poisson structure by the formula

$$\Lambda_1 = \sum_{i=1}^n y^i \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial \varphi^i}.$$

One checks easily that the two Poisson structures are compatible, and that the recursion operator is given by

$$R = \sum_{i=1}^{n} y^{i} \left( \frac{\partial}{\partial y^{i}} \otimes dy^{i} + \frac{\partial}{\partial \varphi^{i}} \otimes d\varphi^{i} \right).$$

It is now clear from the expression of the Hamiltonian function in the y-coordinates that  $\mathcal{L}_{X_H}R = 0$ , so the vector field  $X_H$  is bi-Hamiltonian.

Thus a completely Hamiltonian system whose Hamiltonian's graph is a hypersurface of translation relative to the affine structure determined by the action variables, is bi-Hamiltonian

$$X_H = \Lambda_0^{\sharp}(dH) = \Lambda_1^{\sharp}(dH_0)$$

with 
$$H_0 = \sum_{i=1}^n \ln(y^i)$$
.  
The  $(1, 1)$ -tensor

$$P = \sum_{i=1}^{n} (\varphi^{i} \frac{\partial}{\partial \varphi^{i}} \otimes d\varphi_{i} + \frac{\partial}{\partial dy_{i}} \otimes dy_{i})$$

satisfies

$$\mathcal{L}_{X_H}P = Id \text{ and } \mathcal{L}_{PX}R = 0$$

so we can apply Theorem 8 and conclude that the vector fields  $Y_k = R^k P X = \sum_{i=1}^n y_k \varphi^i \frac{\partial}{\partial \varphi^i}$  and the function  $G = \sum_{i=1}^n y^i \varphi^i$  allows us to define the sequence of functions  $G_i = Y_i(G)$  ( $i = \sum_{i=1}^n y^i \varphi^i$ ).

 $0, 1, \ldots$ ), such that

$$X_i(G) = iH_i,$$
  
 $X_i(G_j) = X_{i+j}(G) = (i+j)H_{i+j},$   
 $\{X_i(G_j), G_k\} = (i+j)X_i(G_{k+j})$ 

and

$${G_i, G_j}_0 = (i - j)G_{i+j}.$$

Thus, for each  $i \in \mathbb{N}$ , the integrals of motion of  $X_i$ ,  $L_{k,j}^i = X_i(G_k)G_j - X_i(G_j)G_k$  and  $X_i(G_j)$ , close quadratically under the Poisson bracket defined by  $\Lambda_0$ .

#### Example 14 (The Rational Calogero-Moser System)

The rational Calogero-Moser system is an integrable Hamiltonian system defined by

$$\mathcal{H} = \sum_{i=1}^{n} \left(\frac{p_i^2}{2} + \frac{g^2}{2} \sum_{j \neq i} (q_i - q_j)^{-2}\right). \tag{4.28}$$

It admits the pair of matrices (L, M),

$$L_{ij} = p_i \delta_i^j + g \sqrt{-1} (q_i - q_j)^{-1} (1 - \delta_i^j), \tag{4.29}$$

$$M_{ij} = g\sqrt{-1}\sum_{h \neq i} (q_i - q_h)^{-2}\delta_i^j - g\sqrt{-1}(q_i - q_j)^{-2}(1 - \delta_i^j)$$
(4.30)

as a Lax pair.

The Hamiltonian vector field with respect to the canonical Poisson structure in  $\mathbb{R}^{2n}$  is

$$X_H = \sum_{i} (p_i \frac{\partial}{\partial q_i} + \sum_{j \neq i} 2(q_i - q_j)^{-3} \frac{\partial}{\partial p_i}). \tag{4.31}$$

This system is completely integrable when we consider the sequence of integrals of motion  $F_i = Tr(L^i), i = 1, ... n$ .

Moreover, following [12], if we consider the functions  $G_i = Tr(QL^{i-1})$ , which provide the algebraic linearization of the system [2],[3], the Hamiltonian vector field becomes

$$X_1 = \sum_{i} F_i \frac{\partial}{\partial G_i} \tag{4.32}$$

and we may define the following compatible Poisson tensors [10]

$$\Lambda_0 = \sum_i \frac{\partial}{\partial F_i} \wedge \frac{\partial}{\partial G_i},\tag{4.33}$$

$$\Lambda_1 = \sum_i F_i \frac{\partial}{\partial F_i} \wedge \frac{\partial}{\partial G_i}.$$
 (4.34)

The system  $\dot{u} = X_1(u)$  is bi-Hamiltonian with respect to these Poisson structures and the bi-Hamiltonian sequence of integrals of motion is

$$\Lambda_0^{\sharp}(dh_i) = \Lambda_1^{\sharp}(dh_{i-1}), \quad j = 0, 1, \dots$$
(4.35)

where

$$h_{-1} = \ln(F_1 \dots F_n),$$

$$h_j = \frac{1}{2(j+1)} Tr(\Lambda_1^{\sharp} \Lambda_0^{\sharp -1})^{j+1} = \frac{1}{j+1} \sum_i (F_i)^{j+1}, \quad j = 0, 1, \dots$$
(4.36)

Notice that  $X_1 = \Lambda_0^{\sharp}(dh_1) = \Lambda_1^{\sharp}(dh_0)$  and if we define the sequence of Hamiltonian vector fields

$$X_{i} = (\Lambda_{1}^{\sharp} \Lambda_{0}^{\sharp -1})^{i-1} X_{1} = \sum_{k} (F_{k})^{i} \frac{\partial}{\partial G_{k}} \quad i = 1, 2, \dots,$$

$$(4.37)$$

the following relation holds

$$X_i(\sum_k G_k F_k) = (i+1)h_i. (4.38)$$

Now consider the (1,1)-tensor

$$P = \sum_{i=1}^{n} \frac{\partial}{\partial F_i} \otimes dG_i$$

that satisfies

$$\mathcal{L}_{PX}R = R$$
 and  $\mathcal{L}_{X}P = Id$ 

and define the sequence of master symmetries

$$Y_i = (\Lambda_1^{\sharp} \Lambda_0^{\sharp - 1})^i P X_1. \tag{4.39}$$

Considering the functions  $g_i = Z_i(\sum_k (G_k F_k)) = \sum_k F_k^{j+1} G_k$  we have

$$X_i(g_j) = \sum_k F_k^{i+j+1},$$

$${X_i(g_j), g_k}_0 = (i+j+1)X_i(g_{j+k})$$

and

$${g_i, g_j}_0 = (i - j)g_{i+j}.$$

Now the Propositions 5 and 9 ensure that for each  $X_i$ ,  $i \in \mathbb{N}$ , the integrals  $X_i(g_j)$  and  $L_{k,j}^i = X_i(g_k)g_j - X_i(g_j)g_k$  close quadratically under  $\{.,.\}_0$ .

**Example 15 (The Goldfish System)** The Goldfish system was first introduced by Calogero [1] and is the Hamiltonian system in  $(\mathbb{R}^{2n}, (q_i, p_i))$  defined by  $X = \Lambda_0^{\sharp}(dH)$ , where  $\Lambda_0$  is the canonical Poisson tensor and

$$H = \sum_{i=1}^{n} \frac{g_i(q_i)}{\prod_{j \neq i} (q_i - q_j)} e^{ap_i}, \tag{4.40}$$

with  $g_i$  arbitrary smooth functions, each one depending only on the corresponding coordinate  $q_i$  and a an arbitrary constant.

Defining the Nijenhuis tensor

$$R = q_i \left(\frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i\right) \tag{4.41}$$

and the function

$$K = \sum_{i=1}^{n} \frac{\rho_i g_i(q_i)}{\prod_{j \neq i} (q_i - q_j)} e^{ap_i}, \quad \rho_i = \prod_{j \neq i} q_j$$
 (4.42)

this system becomes completely integrable and quasi-bihamiltonian [10]. The functions  $c_i$ , coefficients of the minimal polynomial of R  $(q^n + \sum_i c_i q^{n-i} = \prod_{i=1}^n (q - q_i))$ , together with the integrals of motion

$$F_k = \sum_{i=1}^n \frac{\partial c_k}{\partial q_i} \frac{g_i(q_i) e^{ap_i}}{\prod_{j \neq i} (q_i - q_j)}$$

$$\tag{4.43}$$

linearize algebraically the system because  $\dot{c}_k = aF_k$ .

In the coordinates  $(c_i, F_i)$  the system has a simple bi-Hamiltonian structure defined by the compatible Poisson tensors

$$Q_0 = \sum_{i=1}^n \frac{\partial}{\partial c_i} \wedge \frac{\partial}{\partial F_i}, \quad Q_1 = \sum_{i=1}^n F_i \frac{\partial}{\partial c_i} \wedge \frac{\partial}{\partial F_i}$$
 (4.44)

and the Hamiltonians

$$H_0 = F_1 + \dots + F_n, \quad H_1 = \frac{1}{2}(F_1^2 + \dots + F_n^2).$$
 (4.45)

Similarly to last example, the (1,1)-tensor

$$P = \sum_{i} \frac{\partial}{\partial F_i} \otimes dc_i$$

and the function  $G = \sum_{i=1}^{n} F_i c_i$  allow us to define the integrals of motion that close quadratically.

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