PROOF OF A LORENTZ AND LEVI-CIVITA THESIS

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Abstract. A formal proof of the thesis by Lorentz and Levi-Civita that the left-hand side of Einstein field equations represents the real energy-momentum-stress tensor of the gravitational field.

Summary – **1**. Introduction. Aim of the paper. – **2**. Mathematical preliminaries. – **3**. Proof that the left-hand side of the Einstein field equations gives the true energy-momentum-stress tensor of the gravitational field. – **4**. A fundamental consequence. – *Appendix*: On the pseudo energy-tensor.

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1. – As it has been remarked [1], if I is the *action* integral of any field (of any tensorial nature) – say $\varphi(x)$, $x \equiv (x^0, x^1, x^2, x^3)$ – acting in a pseudo-Riemannian spacetime, and we perform the variation of I – say $\delta_g I$ – generated by the variation δg_{jk}, $(j, k = 0, 1, 2, 3)$, of the metric tensor $g_{jk}(x)$ (possibly interacting with $\varphi(x)$),

$$(1) \qquad \delta_g I = \int_D (\ldots)^{jk} \, \delta g_{jk} \sqrt{-g} \; \mathrm{d}^4 x \quad ,$$

– where D is a fixed spacetime domain – , the expression $(\ldots)^{jk}$ is a symmetrical tensor, which represents the energy-momentum-stress tensor of $\varphi(x)$. This statement has been *verified* for various fields [1]. And its *general* validity can be intuitively understood bearing in mind that I is an action integral, with the Lagrange density of $\varphi(x)$ as integrand.

We shall prove that the above statement holds also if $\varphi(x) \equiv g_{jk}(x)$, thus corroborating a famous (and debated!) thesis by Lorentz [2] and Levi-Civita [3] – see also Pauli [4] (and the references therein).

The essential merit of the following demonstration is its ***independence*** *of the Einstein field equations* (and of the Bianchi relations).

2. – Let $\sqrt{-g}\; S\; [g_{jk}(x), g_{jk,m}(x), g_{jk,mn}(x), \ldots]$ be a generic scalar density which is a function of the metric $g_{jk}(x)$ and of a finite number of its ordinary derivatives [5]. We do ***not*** assume that $\sqrt{-g}\; S$ is a Lagrange density, and therefore the integral

$$\mathcal{J} = \int_D S\,\sqrt{-g}\;\mathrm{d}^4x \tag{2}$$

is ***not*** an action integral. We have:

$$\delta_g\,\mathcal{J} = \int_D \frac{\delta(S\,\sqrt{-g})}{\delta g_{jk}}\,\delta g_{jk}\,\mathrm{d}^4x \quad ; \tag{3}$$

the variational derivative $\delta(S\,\sqrt{-g})/\delta g_{jk}$ is equal to

$$\frac{\partial(S\,\sqrt{-g})}{\partial g_{jk}} - \frac{\partial}{\partial x^m}\left[\frac{\partial(S\,\sqrt{-g})}{\partial g_{jk,m}}\right] + \frac{\partial^2}{\partial x^m x^n}\left[\frac{\partial(S\,\sqrt{-g})}{\partial g_{jk,mn}}\right] - \ldots \quad ; \tag{4}$$

putting $\delta(S\,\sqrt{-g})/\delta g_{jk} := P^{jk}\,\sqrt{-g}$, we can write:

$$\delta_g\,\mathcal{J} = \int_D P^{jk}\,\sqrt{-g}\;\delta g_{jk}\,\mathrm{d}^4x \quad . \tag{3'}$$

Let us now consider the *particular* δg_{jk} – say $\delta^* g_{jk}$ –, which is generated by an infinitesimal change of the co-ordinates x:

$$x'^j = x^j + \varepsilon^j(x) \quad ; \tag{5}$$

we assume that $\varepsilon^j(x)$ is zero on the bounding surface ∂D. The corresponding variation of $\mathcal{J}$ – say $\delta^*_g\mathcal{J}$ – will be equal to *zero*, because $\mathcal{J}$ is an invariant.

We have:

$$g_{mn}(x) = \frac{\partial x'^j}{\partial x^m}\,\frac{\partial x'^k}{\partial x^n}\;g'_{jk}(x') \quad , \tag{6}$$

and we consider the $\delta^* g_{jk}$ for *fixed* values of the coordinates, *i.e.*:

$$\delta^* g_{jk} := g'_{jk}(x') - g_{jk}(x') = g'_{jk}(x') - g_{jk}(x) - g_{jk,s}(x)\,\varepsilon^s \quad . \tag{6'}$$

It follows immediately from eqs.(5), (6), (6′) that

$$\delta^* g_{mn} = -g_{mn,j}\,\varepsilon^j - g_{mj}\,\varepsilon^j_{,n} - g_{nj}\,\varepsilon^s_{,m} \quad ; \tag{7}$$

from eq.(3′) we get:

$$\begin{aligned}
\delta^*_g\mathcal{J} \;&=\; \int_D P^{mn}\,\sqrt{-g}\,\delta^* g_{mn}\,\mathrm{d}^4x = \\
&= \int_D P^{mn}\left(-g_{m};\varepsilon^j_{,n} - g_{nj}\,\varepsilon^j_{,m} - g_{mn,j};\varepsilon^j\right)\,\sqrt{-g}\,\mathrm{d}^4x = \\
&= \int_D \left[2\,(P^n_j\,\sqrt{-g})_{,n} - g_{mn,j}\,P^{mn}\,\sqrt{-g})\right]\,\varepsilon^j\,\mathrm{d}^4x = \\
&= 2\int_D P^m_{j:m}\,\varepsilon^j\,\sqrt{-g}\,\mathrm{d}^4x = 0 \quad ,
\end{aligned} \tag{8}$$

if the colon denotes a covariant derivative; in the last passage we use the following property of any symmetrical tensor S^{mn}:

$$S^{m}_{j:m}\sqrt{-g} = \left(S^{n}_{j}\sqrt{-g}\right)_{,n} - \frac{1}{2}\, g_{mn,j}\, S^{mn}\sqrt{-g} \quad . \tag{8'}$$

Accordingly:

$$P^{m}_{j:m} = 0 \quad ; \quad (j = 0, 1, 2, 3) \quad . \tag{9}$$

3. – The result (9) has a mere mathematical interest. It becomes physically significant when $\mathcal{J}$ is the action integral, say A, given by

$$A = \int_D R\sqrt{-g}\,\mathrm{d}^4x \quad , \tag{10}$$

where $R = R^{jk}g_{jk}$ is the Ricci scalar. We shall not use the fact that the g_{jk}'s are (*a priori*) independent variables, because we do not wish to deduce from the action A the Einstein field equations.

Standard procedures (see, *e.g.*, Hilbert's method in *Appendix*, $\boldsymbol{\beta}$)) tell us that

$$\delta_g A = \int_D \left(R^{jk} - \frac{1}{2}\, g^{jk}\, R\right)\sqrt{-g}\;\delta g_{jk}\,\mathrm{d}^4x \quad ; \tag{11}$$

the analogue of eq.(8) is:

$$\delta^*_g A = 2\int_D \left(R^{k}_{j} - \frac{1}{2}\,\delta^{j}_{k}\, R\right)_{:k} \varepsilon^j\sqrt{-g}\;\mathrm{d}^4x = 0 \quad , \tag{12}$$

from which:

$$\left(R^{jk} - \frac{1}{2}\, g^{jk}\, R\right)_{:k} = 0 \quad , \quad (j = 0, 1, 2, 3) \quad . \tag{13}$$

Thus, quite independently of the field equations, we see that the *symmetrical* tensor $R^{jk} - (1/2)g^{jk}R$ satisfies four *conservation equations*. Of course, eqs.(13) are identically satisfied by virtue of Bianchi relations, but the above method – which is essentially due to the conceptions of Emmy Noether [6] – evidences the conservative property of $R^{jk} - (1/2)g^{jk}R$, and attributes it the nature of an energy-momentum-stress tensor. Properly speaking, $[R^{jk} - (1/2)g^{jk}R]/\kappa$, if κ is the Newton-Einstein gravitational constant, represents the Einsteinian energy tensor, as it was emphasized by Lorentz [2] and Levi-Civita [3]. And the fact that this tensor is a function *only* of the potential g^{jk} implies that it is *the unique* energy-momentum-stress tensor of the gravitational field.

4. – The fact that $[R^{jk} - (1/2)g^{jk}R]/\kappa$ is *the true* energy-momentum-stress tensor of the gravitational field has a very important consequence [3]: the mathematical undulatory solutions of the equations $R^{jk} - (1/2)g^{jk}R = 0 = R^{jk}$ are quite devoid of physical meaning, because they do not transport energy, momentum, stress. This was the *first* demonstration of the physical non-existence of the gravitational waves. Quite different demonstrations have been given in recent years, see *e.g.* [7], and references therein.

In his fundamental memoir [3], Levi-Civita proved also the nature of mere *mathematical fiction* (Eddington [8]) of the well-known pseudo energy-tensor of the metric field g_{jk}. –

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APPENDIX

α) The full illogicality of the notion of pseudo energy-tensor can be seen also in the following way. The usual definition of this pseudo tensor is:

$$\text{(A.1)} \qquad \sqrt{-g}\, t_m^{\ n} \stackrel{DEF}{=} \frac{\partial(L\sqrt{-g})}{\partial g_{jk,n}}\, g_{jk,m} - \delta_m^n\, L\, \sqrt{-g} \quad ;$$

the function L:

$$\text{(A.2)} \qquad L \equiv g^{mn}\, (\Gamma^s_{mn}\, \Gamma^r_{sr} - \Gamma^r_{ms}\, \Gamma^s_{nr})$$

yields the Lagrangean field equations:

$$\text{(A.3)} \qquad \frac{\partial(L\sqrt{-g})}{\partial g_{jk}} - \frac{\partial}{\partial x^n}\left[\frac{\partial(L\sqrt{-g})}{\partial g_{jk,n}}\right] = 0 \quad .$$

Now, the left-hand side of (A.3) is ***not*** equal to

$$\text{(A.4)} \qquad -\left(R^{jk} - \frac{1}{2}g^{jk}R\right)\sqrt{-g}$$

as it is commonly affirmed. Indeed:

i) A non tensor entity cannot be equal to a tensor density –

ii) The above affirmed equality has its origin in a "negligence": in the customary variational deduction of the Einstein field equations the variation of $\int_D R\sqrt{-g}\, \mathrm{d}^4x$ is "reduced" to the variation of $\int_D L\sqrt{-g}\, \mathrm{d}^4x$. But in his "reduction" two perfect differentials in the integrand have been omitted, because on the boundary ∂D the variations of the g_{jk} and of their first derivatives are zero (by assumption): this omission *destroys* the tensor-density character of the initial expressions. –

β) It is likely that the pseudo energy-tensor would not have been invented if the authors had followed Hilbert's procedure [9]. This Author started from the fact that (with our previous notations) the explicit evaluation of the variational derivative $\delta(R\sqrt{-g})/\delta g^{mn}$ gives the following Lagrangean expressions:

$$\frac{\partial(R\sqrt{-g})}{\partial g^{mn}} - \frac{\partial}{\partial x^k}\left[\frac{\partial(R\sqrt{-g})}{\partial g^{mn}_{,k}}\right] + \frac{\partial^2}{\partial x^k x^l}\left[\frac{\partial(R\sqrt{-g})}{\partial g^{mn}_{,kl}}\right] \quad ; \tag{A.5}$$

Hilbert wrote: "... specializiere man zunächst das Koordinatensystem so, daß für den betrachteten Weltpunkt die $g^{mn}_{,s}$ sämtlich verschwinden.". *I.e.*, he chose a *local* coordinate-system for which the *first* derivatives of g^{mn} are equal to zero. Thus, only the first term of (A.5) gives a non-zero contribution, and we have that (A.5) is equal to

$$\sqrt{-g}\left(R_{mn} - \frac{1}{2}\, g_{mn}\, R\right) \quad . \tag{A.6}$$

There is no room in this procedure for false (pseudo) tensor entities.

References

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[2] H.A. Lorentz, *Amst. Versl.*, **25** (1916) 468; (this memoir is written in Dutch – an English translation would be desirable).

[3] T. Levi-Civita, *Rend. Acc. Lincei*, **26** (1917) 381; an English translation in *arXiv:physics/9906004* (June 2nd, 1999). See also: *Idem*, *ibid.*, **11** (s.6^{a}) (1930) 3 and 113.

[4] See Pauli [1], sects. **23** and **61**.

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[6] E. Noether, *Gött Nachr.*, (1918) 235 ("Invariante Variationsprobleme").

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