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# Coherent and Squeezed States in Shape Invariant Potentials Obtained from the Master Function

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**Abstract**

A general algorithm has been given for the generation of Coherent and Squeezed states, in one-dimensional hamiltonians with shape invariant potential, obtained from the master function. The minimum uncertainty states of these potentials are expressed in terms of the well-known special functions.

**Keywords: Squeezed States, Coherent States, Shape Invariance, Special Functions.**

**PACs Index: 42.50.Dv, 02.30. +g or 02.90. +p.**

# 1 Introduction

Coherent states, known as the closest states to classical ones, play an important role in many different contexts of the theoretical and experimental physics, specially quantum optics[7, 24] and multiparticle dynamics[13]. Schrodinger first discovered the coherent states of the harmonic oscillator potential in 1926 [6] and much work has been done since then on their properties and applications[7, 17, 24]. The coherent states have also been found in systems with the Lie group symmety [12, 22, 23]. Recently, coherent states have been found in special Hamiltonians [3, 4, 5]. These coherent states are called minimum uncertainty coherent states(MUCS). In coherent states the standard deviation of $x$ and $p$ are equal and their product is minimum over these states. There are also quantum states where, though we have minimum uncertainty for the standard deviation of coordinate and momentum, they are not equal any more; these states are called squeezed states. These quantum states are as important as coherent ones and their generation play an important role in many different branch of physics and communication engineering[14, 26]. Nieto and his colleagues have developed an interesting algorithm for the generation of the coherent and squeezed states for special potential, where the product of the standard deviation generalized Harmonic phase variable $X_c$ and $P_c$ are minimum over these states. Here in this article following the algorithm of reference [3, 4, 5] we obtain these coherent and squeezed states for all the shape invariant potential obtained from the master function of reference [8]. The Hamiltonians of the reference[4, 5] are the special cases of the most general shape invariant Hamiltonian to be treated in this paper and the result thus obtained in this work, are in good agreement with those of thier reference[4, 5] in these few special cases.

This paper is organized as follows. In section II we explain very briefly the shape invariant potential obtained from the master functions. In section III first we show that the generalized Harmonic variable $X_c$ of reference [3] is linear function of the $x$ coordinate of orthogonal function and $P_c$ the generalized Harmonic momentum is proportional to master function. In section IV following the reference [3], we obtain the raising and lowering operator of these potentials. In section V we obtain the most general minimum uncertainty states for the general potential obtained from the master function. For particular choice of even master and weight function together with the symmetric interval we can obtain even or odd minimum uncertainty coherent (squeezed)states or well known Schrodinger-Cat state. It is also shown that the ground state of the Hamiltonian is one of the MUCS of the system. Eigenstates of Annihilation operators namlely the generalization of annihilation operators coherent states(AOCS) is derived in section VI. Here in this section it is shown that in general the minimum uncertainty states are different from AOCS ones. Section VII devoted to investigate the time evolution of the Minimum Uncertainty states. Here in this section it is shown that, the time evolution of the generlized quantum phase coordinates is almost similar to the time evolution of the phase coordinates of the qunantum oscillator except for the appearance of the constant phase $\omega_0$ and also the Hamiltonian depenence of the frequency $\omega_H$. The paper ended with a brief conclusion.

## 2 The Shape InvariantPotentials Obtained From the Master function

According to references [9] by introducing the master function $A(x)$ as a polynomial of at most second order one can define a non-negative weight function $W(x)$ in the interval $[a, b]$ such that the expression $\frac{1}{W(x)}\frac{d}{dx}(A(x)W(x))$ be a polynomial of at most first order and the function $A(x)W(x)$ to vanish at the ends of the interval. Now we can define second order differential operator $L = \frac{1}{W(x)}\frac{d}{dx}A(x)W(x)\frac{d}{dx}$ with the following properties:

1) $L$ is a self-adjoint linear operator.

2) $L$ transforms a given polynomial of order $m$ to another polynomial of order $m$ at most.

3) The expression $\frac{1}{W(x)}(\frac{d}{dx})^n(A^n(x)W(x))$ is a polynomial of order at most $n$, which is indeed Rodrigues formula for the classical orthogonal polynomials.

4) The polynomials

$$\phi_n(x) = \frac{a_n}{W(x)}(\frac{d}{dx})^n(A^n(x)W(x))$$

are orthogonal with respect to the weight function $W(x)$ in the interval $[a, b]$ as defined above, and one can find $a_n$ simply by comparing the coefficient of highest power of $\phi_n(x)$ with those of the traditionally defined special orthogonal polynomials.

5) The polynomials $\phi_n(x)$ are eigenfunctions of operator $L$, and therefore satisfy the following second order linear differential equation

$$\frac{1}{W(x)}\frac{d}{dx}(A(x)W(x)\frac{d}{dx}\phi_n(x)) = -\gamma_n\phi_n(x). \tag{2-1}$$

In order the differential $Eq.(2-1)$ have polynomial solution of degree $n$, $\gamma_n$ must be given by

$$\gamma_n = -n(\frac{(A(x)W(x))'}{W(x)})' - \frac{n(n-1)}{2}A''(x).$$

Thus, the general form of the differential equation is as follows

$$A(x)\phi_n''(x) + \frac{(A(x)W(x))'}{W(x)}\phi_n'(x) - [n(\frac{(A(x)W(x))'}{W(x)})' + \frac{n(n-1)}{2}A''(x)]\phi_n(x) = 0. \tag{2-2}$$

By differentiating the differential Eq.$(2-2)$ $m$ times and then multiplying it by $(-1)^m A^{\frac{m}{2}}(x)$ we get the following associated differential equation

$$A(x)\phi''_{n,m}(x) + \frac{(A(x)W(x))'}{W(x)}\phi'_{n,m}(x) + [-\frac{1}{2}(n^2+n-m^2)A''(x) + (m-n)(\frac{A(x)W'(x)}{W(x)})'$$

$$-\frac{m^2}{4}\frac{(A'(x))^2}{A(x)} - \frac{m}{2}\frac{A'(x)W'(x)}{W(x)}]\phi_{n,m}(x) = 0 \qquad \text{(2-3)}$$

where

$$\phi_{n,m}(x) = (-1)^m A^{\frac{m}{2}}(x)(\frac{d}{dx})^m \phi_n(x).$$

Now, changing the variable $\frac{dx}{d\xi} = \sqrt{A(x)}$ in associated differential equations of Eq.$(2-3)$ and defining the new function $\psi_n^m(\xi) = A^{\frac{1}{4}}(x)W^{\frac{1}{2}}(x)\phi_{n,m}(x)$ we obtain the Schrödinger equation [8]

$$-\frac{d^2}{d\xi^2}\psi_n^m(\xi) + V_m(x(\xi))\psi_n^m(\xi) = E(n,m)\psi_n^m(\xi), \qquad m = 0,1,2,\cdots,n, \qquad \text{(2-4)}$$

where the most general shape invariant potential is

$$V_m(x) = W^2(x) + \frac{d}{d\xi}W(x) = -\frac{1}{2}(\frac{A(x)W'(x)}{W(x)})' - \frac{2m-1}{4}A''(x) + \frac{1}{4A(x)}(\frac{A(x)W'(x)}{W(x)})^2 +$$

$$\frac{m}{2}\frac{A'(x)W'(x)}{W(x)} + \frac{4m^2-1}{16}\frac{A'^2(x)}{A(x)}, \qquad \text{(2-5)}$$

and prime stands for derivative with respect to $x$. The spectrum $E(n,m)$ is

$$E(n,m) = -(n-m+1)[(\frac{A(x)W'(x)}{W(x)})' + \frac{1}{2}(n+m)A''(x)]. \qquad \text{(2-6)}$$

# 3 Generalized Harmonic Phase space variables $X_c$ and $P_c$

Following the prescription of references [3, 4, 5] in a one dimensional hamiltonian:

$$H = \frac{1}{2}(\frac{d\xi}{dt})^2 + V_m(x(\xi)), \qquad \text{(3-1)}$$

with $V_m(x(\xi))$ given in Eq.$(2-5)$, the classical paths of constant energy around the minimum points of the potential form closed paths in the phase space $(\xi, P_\xi)$. Therefore, there is an injective canonical map from the phase space $(\xi, P_\xi)$ into the new phase space $(X_c, P_c)$ , such that the closed constant energy paths turn into elliptic constant energy paths. Hence, in the phase space $\xi - P_\xi$ the time dependence of these closed paths can be written as

$$X_c = A(E)\sin\omega_c(E)t, \qquad \text{(3-2)}$$

$$P_c = mA(E)\omega_c(E)\cos\omega_c(E)t.$$

From the constancy of the hamiltonian, Eq.$(3-1)$, along the paths we have

$$t + t_0 = \int \frac{d\xi}{\sqrt{\frac{2}{m}(E - V_m(x(\xi))}}.$$

By changing the variable $\frac{dx}{\sqrt{A(x)}} = d\xi$, we get

$$t + t_0 = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{A(x)(E - V_m(x))}}. \tag{3-3}$$

Inserting the expression $(2-5)$ for $V_m(x)$ and considering the fact that $A(x)$ is at most second order and $A(x)\frac{d\log W(x)}{dx}$ is at most first order, one can show that the expression $A(x)(E - V_m(x))$ is quadratic. Hence, integrating Eq.$(3-5)$ we obtain

$$x + \frac{\eta_2}{2\eta_1} = \sqrt{(\frac{\eta_2}{2\eta_1})^2 - \frac{\eta_3}{\eta_1}} \sin\sqrt{\frac{-2\eta_1}{m}}(t + t_0) \tag{3-4}$$

with

$$\eta_1 = \frac{1}{2}A''(E - \gamma + \frac{2m-1}{4}A'') + \frac{1-2m}{4}A''(\frac{AW'}{W})' - \frac{1}{4}(\frac{AW'}{W})' - \frac{4m^2-1}{16}(\frac{AW'}{W})'$$

$$\eta_2 = A'(0)(E - \gamma + \frac{2m-1}{4}A'') + \frac{1}{2}A'(0)(\frac{AW'}{W})' - \frac{1}{2}(\frac{AW'}{W})'(\frac{AW'}{W})(0) - \frac{m}{2}(A''(0) + A'(0))\frac{AW'}{W})(0)$$

$$\eta_3 = A(0)(E - \gamma + \frac{2m-1}{4}A'') + \frac{1}{2}A(0)(\frac{AW'}{W})' - \frac{1}{4}(\frac{AW'}{W}(0))^2 - \frac{m}{2}A'(0)(\frac{AW'}{W})(0)\frac{4m^2-1}{16}(A'(0))^2. \tag{3-5}$$

Comparing the relations $(3-2)$ and $(3-6)$ we have

$$X_c = x_0(x + \frac{\eta_2}{2\eta_1})$$

$$\omega_c = \sqrt{\frac{-2\eta_1}{m}}$$

$$A(E) = x_0\sqrt{(\frac{\eta_2}{2\eta_1})^2 - \frac{\eta_3}{\eta_1}}, \tag{3-6}$$

where $x_0$ and $t_0$ are arbitrary constants of integration. Also $\gamma$ is a constant which is added to the potential $V_m(x(\xi))$ for convenience. Similarly, for the momentum $P_c = m\frac{dX_c}{dt}$ we have

$$P_c = mx_0\frac{dx}{dt} = x_0 m\frac{d\xi}{dt}\frac{dx}{d\xi} = x_0 m P_\xi\sqrt{A(x)}. \tag{3-7}$$

The quantum operators corresponding to $X_c$ and $P_c$, denoted by $\widehat{X}$ and $\widehat{P}$, are defined as following according to [3, 4, 5]:

$$\widehat{X} = X_c(x) = x_0(x + \frac{\eta_2}{2\eta_1})$$

$$\widehat{P} = \frac{1}{2i}(X_c' p + p X_c').$$

Making a change variable from $\xi$ to $x$, we get

$$\widehat{P} = \frac{p_0}{2i}(A(x)\frac{d}{dx} + \frac{d}{dx}A(x))$$

where $x_0$ and $p_0$ are arbitrary constants.

## 4 Raising And Lowering Operators

In the algorithm of generation of the minimum uncertainty states of the hamiltonian $-\frac{d^2}{d\xi^2} + V_m(x(\xi))$, the raising and lowering operators of its discrete eigenstates $\psi_n^m$, play an important role. These operators are denoted by $\tilde{B}_{n,m}$ and $\tilde{A}_{n,m}$ respectively. To obtain them, following the refernce [9], first we factorize the differential equation $(2-3)$ in a shape invariant form as:

$$\begin{cases} B(n,m)A(n,m)\phi_{n,m}(x) = E(n,m)\phi_{n,m}(x) \\ A(n,m)B(n,m)\phi_{n-1,m}(x) = E(n,m)\phi_{n-1,m}(x). \end{cases} \tag{4-1}$$

From the equations $(4-1)$ it is straightforward to derive the following recursion relations

$$\begin{cases} B(n,m)\phi_{n-1,m}(x) = \mu_{m,n}\phi_{n,m}(x) \\ A(n,m)\phi_{n,m}(x) = \frac{E(n,m)}{\mu_{m,n}}\phi_{n-1,m}(x), \end{cases} \tag{4-2}$$

where $E(n,m)$, $B(n,m)$ and $A(n,m)$ are given by

$$E(n,m) =$$

$$\frac{[(\frac{A(x)W'(x)}{W(x)})'\frac{AW'}{W}(0) + n^2 A''(x)A'(0) + (2n-m)(\frac{A(x)W'(x)}{W(x)})'A'(0) + mA''(x)\frac{AW'}{W}(0)]^2}{4[(\frac{A(x)W'(x)}{W(x)})' + nA''(x)]^2}$$

$$-\frac{1}{4}(\frac{AW'}{W}(0))^2 - (n-m)(\frac{A(x)W'(x)}{W(x)})'A(0) - \frac{1}{4}m^2(A'(0))^2 - \frac{1}{2}mA'(0)(\frac{AW'}{W})(0)$$

$$-\frac{1}{2}(n^2-m^2)A''(x)A(0)$$

$$B(n,m)=-A(x)\frac{d}{dx}-((\frac{A(x)W'(x)}{W(x)})'+\frac{1}{2}nA''(x))x$$
$$-\frac{2(\frac{A(x)W'(x)}{W(x)})'\frac{AW'}{W}(0)+n^2A''(x)A'(0)+(2n-m)(\frac{A(x)W'(x)}{W(x)})'A'(0)+(m+n)A''(x)\frac{AW'}{W}(0)}{2[(\frac{A(x)W'(x)}{W(x)})'+nA''(x)]}$$

$$A(n,m)=A(x)\tfrac{d}{dx}-\tfrac{1}{2}nA''(x)x$$
$$-\frac{n^2A''(x)A'(0)+(2n-m)(\frac{A(x)W'(x)}{W(x)})'A'(0)+(m-n)A''(x)\frac{AW'}{W}(0)}{2[(\frac{A(x)W'(x)}{W(x)})'+nA''(x)]}. \tag{4-3}$$

In order to evaluate $\mu_{n,m}$, it is sufficient to divide both sides of Eq.$(4-2)$ by $(A(x))^{\frac{m}{2}}$ and compare the coefficients of the highest degree terms of boths sides, resulting in

$$\mu_{n,m}=[-\frac{1}{2}A''(x)(n-1)-((\frac{AW'}{W})'+\frac{1}{2}nA'')]. \tag{4-4}$$

From $\psi_n^m(\xi)=A^{\frac{1}{4}}W^{\frac{1}{2}}\phi_{n,m}$, it follows that $\widehat{A}_{n,m}$ and $\widehat{B}_{n,m}$, that is

$$\begin{cases}\tilde{A}_{n,m}=A^{\frac{1}{4}}W^{\frac{1}{2}}A_{n,m}A^{\frac{-1}{4}}W^{\frac{-1}{2}}\\ \tilde{B}_{n,m}=A^{\frac{1}{4}}W^{\frac{1}{2}}B_{n,m}A^{\frac{-1}{4}}W^{\frac{-1}{2}}\end{cases} \tag{4-5}$$

are the required raising and lowering operators of the wave functions, that is, we have

$$\begin{cases}\tilde{B}_{n,m}\psi_{n-1}^m(\xi)=\mu_{n,m}\psi_n^m(\xi)\\ \tilde{A}_{n,m}\psi_n^m(\xi)=\frac{E(n,m)}{\mu_{n,m}}\psi_{n-1}^m(\xi).\end{cases} \tag{4-6}$$

where $\tilde{A}_{n,m}$ and $\tilde{B}_{n,m}$ are

$$\tilde{A}=A(x)\frac{d}{dx}-(\frac{1}{2}nA''(x)+\frac{1}{4}A''(x)+\frac{1}{2}(\frac{A(x)W'(x)}{W(x)})')x$$
$$-\frac{1}{2}A'(0)-\frac{A(x)W'(x)}{W(x)}(0)-$$
$$\frac{n^2A''(x)A'(0)+(2n-m)A1A'(0)+(m-n)A''(x)C}{2(A1+nA''(x))}$$

$$\tilde{B}=-A(x)\frac{d}{dx}-(\frac{1}{2}nA''(x)+(\frac{AW'}{W(x)})'-(\frac{1}{2}\frac{A(x)W'(x)}{W(x)})'-\frac{1}{4}A''(x))x$$

$$+\frac{1}{2}A'(0)+\frac{A(x)W'(x)}{W(x)}(0)-$$

$$\frac{2A1C+n^2A''(x)A'(0)+(2n-m)A1A'(0)+(m+n)A''(x)C}{2(A1+nA''(x))} \tag{4-7}$$

with

$$A1=(\frac{AW'}{W(x)})'$$

$$C=\frac{AW'}{W}(0).$$

The raising and lowering operators of the one dimensional shape invariant potentials, obtained from the master function, and also all other necessary information in constructing their minimum uncertainty state are given in Table I. The Hermitian quantum operators associated with the generalized Harmonic quantum phase variables $\widehat{X}$ and $\widehat{P}$ can be expressed in terms of the operators $\tilde{A},\tilde{B}$ and their Hermitian conjugates as

$$\widehat{X}=x_0[\tilde{A}+\tilde{A}^\dagger+\tilde{B}+\tilde{B}^\dagger]$$

$$\widehat{P}=\frac{p_0}{2i}[\tilde{A}+\tilde{B}^\dagger-(\tilde{A}^\dagger+\tilde{B})], \tag{4-8}$$

where $\tilde{A}^\dagger$ and $\tilde{B}^\dagger$, the Hermitian conjugate of the raising and lowering operators are

$$\tilde{A}^\dagger=(\frac{f_1-f_3-A''(x)}{f_1+f_3})\tilde{A}+(\frac{2f_1-A''(x)}{f_1+f_3})\tilde{B}+$$

$$\frac{(f_2-A'(0))(f_1+f_3)-(f_1-A''(x))(f_2+f_4)+\frac{1}{2}(f_2-f_4)(f_1+f_3)-\frac{1}{2}(f_1-f_3)(f_2+f_4)}{f_1+f_3}$$

$$\tilde{B}^\dagger=(\frac{2f_3+A''(x)}{f_1+f_3})\tilde{A}+(\frac{f_3-f_1+A''(x)}{f_1+f_3})\tilde{B}+$$

$$\frac{(f_4+A'(0))(f_1+f_3)-(f_3+A''(x))(f_2+f_4)+\frac{1}{2}(f_1-f_3)(f_2+f_4)-\frac{1}{2}(f_2-f_4)(f_1+f_3)}{f_1+f_3}$$

with $f_1,f_2,f_3$ and $f_4$:

$$f_1=-\frac{1}{2}A''(x)-\frac{1}{2}(\frac{A(x)W'(x)}{W(x)})'-\frac{1}{4}A''(x),$$

$$f_2=-\frac{1}{2}\frac{A(x)W'(x)}{W(x)}(0)-\frac{1}{4}A'(0)-\frac{n^2A''(x)A'(0)+(2n-m)(\frac{A(x)W'(x)}{W(x)})'A'(0)+(m-n)A''(x)C}{2((\frac{A(x)W'(x)}{W(x)})'+nA''(x))},$$

$$f_3=-\frac{1}{2}\frac{A(x)W'(x)}{W(x)}(0)-\frac{1}{2}A''(x)+\frac{1}{4}A''(x),$$

and

$$f_4 = \frac{1}{2}\frac{A(x)W'(x)}{W(x)}(0) + \frac{1}{4}A'(0) -$$

$$\frac{n^2 A''(x)A'(0) + (2n-m)(\frac{A(x)W'(x)}{W(x)})'A'(0) + (m+n)A''(x)C + 2(\frac{A(x)W'(x)}{W(x)})'C}{2((\frac{A(x)W'(x)}{W(x)})' + nA''(x))}. \tag{4-9}$$

Generally speaking, the number $n$ should not appear anywhere and it must be replaced by the Hamiltonian. This is done by expressing $n$ in terms of $E(n,m)$ and replacing $E(n,m)$ by the Hamiltonian. Since the set of eigenfunctions $\psi_n^m$ are complete and we can expand every function in our Hilbert space in terms of them, therefore it is sufficient to consider the effect of the operators on these base. Therefore, in order not to make things too complicated, we do not bother to replace $n$ in terms of Hamiltonian, as in reference[3], except for the operators $\widehat{X}$ and $\widehat{P}$ which are going to be functions of the Hamiltonian.

# 5 The Most General Minimum Uncertainty States

The coherent and squeezed states are generally the minimum uncertainty states of general harmonic phase variables $\widehat{X}$ and $\widehat{P}$, which can be obtained by solving the eigenfunction equation of the operators $\widehat{X}+i\frac{<G>}{2(\Delta p)^2}\widehat{P}$ [2, 3, 4, 5], that is, we solve

$$(\widehat{X} + i\frac{< G >}{2(\delta p)^2}\widehat{P})\psi_{MUCS} = C\psi_{MUCS} \tag{5-1}$$

where $G$ is proportional to the commutation of $\widehat{X}$ and $\widehat{P}$ as $[\widehat{X},\widehat{P}] = iG$, and where $C =< \widehat{X} > +i\frac{<G>}{2(\Delta p)^2} < \widehat{P} >$.

In order to solve the Eq.$(5-1)$, we expand $\psi_{MUCS}$ in terms of $\psi_n^m$, the eigenstate of Schrodinger equation $(2-4)$,

$$\psi_{MUCS} = \sum_{j=m}^{\infty} a_j \psi_j^m(x). \tag{5-2}$$

Inserting the above expansion in Eq.$(5-1)$ and using the independence of the eigenstates $\psi_j^m$, we get a recursion relation between the coefficients $a_j$. This is possible if we know how the adjoint operators $\tilde{A}^\dagger$ and $\tilde{B}^\dagger$ act over $\psi_j^m$ by writing them in terms of the raising and lowering operators $\tilde{A}$ and $\tilde{B}$ as

$$\tilde{A}^\dagger = (\frac{f_1 - f_3 - A''(x)}{f_1 + f_3})\tilde{A} + (\frac{2f_1 - A''(x)}{f_1 + f_3})\tilde{B} +$$

$$\frac{(f_2 - A'(0))(f_1 + f_3) - (f_1 - A''(x))(f_2 + f_4) + \frac{1}{2}(f_2 - f_4)(f_1 + f_3) - \frac{1}{2}(f_1 - f_3)(f_2 + f_4)}{f_1 + f_3}$$

| $A(x)$ and Name | $W(x)$ and Interval of $x$ | $x=x(t)$ | $\mu_{n,m}$ | $V_m(t)$ and $E(n,m)$ |
|---|---|---|---|---|
| $1$ | $e^{-\frac{1}{2}\alpha x^2}$ <br> $\alpha>0$ | $x=t-\frac{2b}{\alpha}$ | $\frac{n-m}{n}\sqrt{n}$ | $\frac{1}{4}\omega^2(t-\frac{2b}{\omega})^2+\frac{\omega}{2}$ |
| shifted oscillator | $-\infty<x<+\infty$ | | | $\alpha(n-m+1)$ |
| $x$ | $x^\alpha e^{-\beta x}$ <br> $\alpha>-1$ <br> $\beta>0$ | $x=\frac{1}{4}t^2$ | $-(n-m)\sqrt{\frac{n+\alpha}{n}}$ | $\frac{1}{4}\omega^2t^2+\frac{l(l+1)}{t^2}-(l-\frac{1}{2})\omega$ |
| three dimensional oscillator | $0<x<+\infty$ | | | $\beta(n-m+1)$ |
| $x^2$ | $x^\alpha e^{-\frac{\beta}{x}}$ <br> $\alpha<-2$ <br> $\beta>0$ | $x=e^t$ | $-\frac{(n+m)(n+\alpha)}{2n+\alpha}\times$ <br> $\sqrt{\frac{n+\alpha}{n}}$ | $A^2+B^2e^{-2t}-B(2A+1)e^{-t}$ |
| Morse | $0<x<+\infty$ | | | $-(\alpha+n+m)(n-m+1)$ |
| $1+x^2$ | $(1+x^2)^\alpha e^{\beta Arctanx}$ <br> $\alpha<-1$ <br> $-\infty<\beta<+\infty$ | $x=sinht$ | $\frac{(n-m)(n+2\alpha)}{2n+2\alpha}\times$ <br> $\sqrt{\frac{2m-1}{2}}$ | $A^2+(B^2-A^2-A)sech^2t+$ <br> $B(2A+1)tanhtsecht$ |
| Scarf II (hyperbolic) | $-\infty<x<+\infty$ | | | $-(2\alpha+n+m)(n-m+1)$ |
| $x(1-x)$ | $x^\alpha(1-x)^\beta$ <br> $\alpha,\beta>-1$ | $x=\frac{1+sint}{2}$ | $-\frac{n(n+\alpha+\beta)}{2n+\alpha+\beta}\times$ <br> $\sqrt{\frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta-1)}{n(n+\alpha+\beta)(2n+\alpha+\beta+1)}}$ | $-A^2+(A^2+B^2+A)sec^2t+$ <br> $-B(2A+1)tantsect$ |
| Scarf I (trigonometric) | $0<x<+1$ | | | $(\alpha+\beta+n+m)(n-m+1)$ |
| $x^2-1$ | $(x-1)^\alpha(x+1)^\beta$ <br> $\alpha,\beta>-1$ | $x=cosht$ | $\frac{2n(n+\alpha+\beta)}{2n+\alpha+\beta}\times$ <br> $\sqrt{\frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta-1)}{n(n+\alpha+\beta)(2n+\alpha+\beta+1)}}$ | $A^2+(A^2+B^2+A)cosech^2t$ <br> $-B(2A+1)cotanhtcosecht$ |
| generalized Pöschl-teller | $-1<x<+1$ | | | $-(\alpha+\beta+n+m)(n-m+1)$ |

| $A(x)$ and Name | $\tilde{A}$ and $\tilde{B}$ | $\omega_c(E)$ | $G$ and $a_n/a_0$ |
|---|---|---|---|
| $1$ | $\frac{\alpha x}{2} + \frac{d}{dx}$ | $\omega_c = \alpha$ | $x_0 p_0$ |
| shifted oscillator | $\frac{\alpha x}{2} - \frac{d}{ax}$ | | $\frac{(k_0/\alpha)^n}{\sqrt{\Gamma(n+1)}}$ |
| $x$ | $\frac{\beta x}{2} + x\frac{d}{dx}$ $-\frac{1}{2}(2n+\alpha-m+\frac{1}{2})$ | $\omega_c = \beta$ | $\frac{x_0 p_0(\alpha+m-\frac{1}{2})}{\beta}\times$ $(1+\frac{2H}{(\alpha+m-1)\beta})$ |
| three dimensional oscillator | $\frac{\beta x}{2} - x\frac{d}{dx}$ $-\frac{1}{2}(2n+\alpha-m+\frac{3}{2})$ | | $\frac{(-k_0)^n}{\sqrt{\Gamma(n+1)\Gamma(n+\alpha+1)}}$ |
| $x^2$ | $-(n+\frac{\alpha+1}{2})x + x^2\frac{d}{dx}$ $-\frac{\beta}{2} - \frac{\beta(m-n)}{2n+\alpha}$ | $\omega_c =$ $2\sqrt{\frac{\alpha^2}{4} + \frac{4m^2-1}{8} - E}$ | $x_0 p_0 e^{2t}$ |
| Morse | $-(n+\frac{\alpha-1}{2})x - x^2\frac{d}{dx}$ $+\frac{\beta}{2} - \frac{\beta(m+n+\alpha)}{2n+\alpha}$ | | $(2k_0)^n\Gamma(n+\alpha/2+1)\sqrt{\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}}$ |
| $1+x^2$ | $-(n+\alpha+\frac{1}{2})x + (1+x^2)\frac{d}{dx}$ $-\frac{\beta}{2} - \frac{\beta}{2}\frac{m-n}{n+\alpha}$ | $s_1 = \alpha^2 + \frac{4m^2-1}{8}$ $s_2 = (2m-1)(\alpha-\frac{1}{2}) - E$ | $x_0 p_0 \cosh^2(t)$ |
| Scarf II (hyperbolic) | $-(n+\alpha-\frac{1}{2})x - (1+x^2)\frac{d}{dx}$ $+\frac{\beta}{2} - \frac{\beta}{2}\frac{m+n+2\alpha}{2(n+\alpha)}$ | $\omega_c = 2\sqrt{s_1+s_2}$ | $(2k_0)^n\frac{\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)\Gamma(n+m+2\alpha)}$ |
| $x(1-x)$ | $(n+\frac{\alpha+\beta+1}{2})x + x(1-x)\frac{d}{dx} - \frac{2\alpha+1}{4}$ $-\frac{n(n+\beta)}{2n+\alpha+\beta} - \frac{m(\alpha-\beta)}{2(2n+\alpha+\beta)}$ | $s_1 = \frac{(\alpha+\beta)^2}{4}$ $s_2 = (m-\frac{1}{2})(\alpha+\beta-1)$ $s_3 = -\frac{4m^2-1}{8} + E$ | $x_0 p_0 \cos^2(t)$ |
| Scarf I (trigonometric) | $(n+\frac{\alpha+\beta-1}{2})x - x(1-x)\frac{d}{dx} + \frac{2\alpha+1}{4}$ $-\frac{(n+\alpha)(n+\alpha+\beta)}{2n+\alpha+\beta} - \frac{m(\alpha-\beta)}{2(2n+\alpha+\beta)}$ | $\omega_c = 2\sqrt{s_1+s_2+s_3}$ | $(-2k_0)^n\Gamma(n+\frac{\alpha+\beta}{2})\sqrt{\frac{\Gamma(n+\frac{\alpha+\beta-1}{2})}{\Gamma(n+\frac{\alpha+\beta+1}{2})}}\times$ $\sqrt{\frac{1}{\Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}}$ |
| $x^2-1$ | $-(n+\frac{\alpha+\beta+1}{2})x + (x^2-1)\frac{d}{dx}$ $-\frac{\alpha-\beta}{2} - \frac{(\alpha-\beta)(m-n)}{2n+\alpha+\beta}$ | $s_1 = \frac{(\alpha+\beta)^2}{4}$ $s_2 = \frac{4m^2-1}{8} - E$ $s_3 = \frac{2m-1}{2}(\alpha+\beta-1)$ | $x_0 p_0 \sinh^2(t)$ |
| generalized | $-(n+\frac{\alpha+\beta-1}{2})x - (x^2-1)\frac{d}{dx}$ | $\omega_c = 2\sqrt{s_1+s_2+s_3}$ | $\Gamma(n+\frac{\alpha+\beta}{2})\sqrt{\frac{\Gamma(n+\frac{\alpha+\beta-1}{2})}{\Gamma(n+\frac{\alpha+\beta+1}{2})}}\times$ |

| $A(x)$ and Name | $\tilde{A}$ and $\tilde{B}$ | $\omega_c(E)$ | $G$ and $a_n/a_0$ |
|---|---|---|---|
| $4x^2-1$ | $-4(n+\frac{\alpha+\beta+1}{2})x+(4x^2-1)\frac{d}{dx}$ $-(\alpha-\beta)-\frac{2(\alpha-\beta)(m-n)}{2n+\alpha+\beta}$ | $s_1=\alpha+\beta)^2$ <br> $s_2=\frac{4m^2-1}{8}-E$ <br> $s_3=2(2m-1)(\alpha+\beta-1)$ <br> $\omega_c=4\sqrt{s_1+s_2+s_3}$ | $x_0p_0\sinh^2(2t)$ |
| Natanzon | $-4(n+\frac{\alpha+\beta-1}{2})x-(4x^2-1)\frac{d}{dx}$ $+(\alpha-\beta)-\frac{2(\alpha-\beta)(m+n+\alpha+\beta)}{2n+\alpha+\beta}$ | | $(k_0/4)^n\Gamma(n+\frac{\alpha+\beta}{2})\sqrt{\frac{\Gamma(n+\frac{\alpha+\beta-1}{2})}{\Gamma(n+\frac{\alpha+\beta+1}{2})}}\times$ $\sqrt{\frac{1}{\Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}}$ |

$$\tilde{B}^\dagger=(\frac{2f_3+A''(x)}{f_1+f_3})\tilde{A}+(\frac{f_3-f_1+A''(x)}{f_1+f_3})\tilde{B}+$$
$$\frac{(f_4+A'(0))(f_1+f_3)-(f_3+A''(x))(f_2+f_4)+\frac{1}{2}(f_1-f_3)(f_2+f_4)-\frac{1}{2}(f_2-f_4)(f_1+f_3)}{f_1+f_3}$$

Therefore, the most general quantum harmonic phase coordinate $\widehat{X}$ and momentum $\widehat{P}$, given in Eq.$(5-1)$, can be expressed in terms of the raising and lowering operators.

$$\widehat{X}=2x_0(\tilde{A}+\tilde{B})$$

$$\frac{2i}{p_0}\widehat{P}=\frac{4f_3+2A''(x)}{f_1+f_3}\tilde{A}+\frac{-4f_1+2A''(x)}{f_1+f_3}\tilde{B}+$$
$$\frac{2(f_1+f_3)(f_4-f_2)+2(f_2+f_4)(f_1-f_3)+2A'(0)(f_1+f_3)-2A''(x)(f_2+f_4)}{f_1+f_3}$$

Substituting the operators $\widehat{X}$ and $\widehat{P}$, written only in terms of the raising and lowering operators as above, in Eq.$(5-1)$ and using the expansion of $\psi_{MUCS}(x)$ in terms of $\psi_n^m(x)$, we obtain the following recursion relation for $a_{n+m}$:

$$a_{n+m+1}=\frac{1}{2x_0+\frac{p_0<G>}{4(\Delta p)^2}\frac{4f_3+2A''(x)}{f_1+f_3}}\frac{\mu_{n+1,m}}{E(n+1,m)}\times$$
$$[(C-\frac{p_0<G>}{4(\Delta p)^2}g)a_{n+m}-(2x_0-\frac{p_0<G>}{4(\Delta p)^2}\frac{4f_1-2A''(x)}{f_1+f_3})\mu_{n,m}a_{n+m-1}]. \qquad \text{(5-3)}$$

where
$$g=\frac{2(f_1+f_3)(f_4-f_2)+2(f_2+f_4)(f_1-f_3)+2A'(0)(f_1+f_3)-2A''(x)(f_2+f_4)}{f_1+f_3}.$$

In principle, by iterating the recursion relation (5-3) we can determine $a_{n+m}$ in terms of $a_0$ and $a_{m+1}$. In general, due to the appearance of $a_{n+m}$ and $a_{n+m-1}$ on the right hand side of this recursion relation, we are not able to obtain a closed form for the coefficient $a_{n+m}$ in terms of $a_m$ and $a_{m+1}$. Note that, by an appropriate choice of the parameters appearing in MUCS, as e.g. taking such as the averge of the quantum phase coordinates, the average of the commutator of the quatum phase coordinates or the squeezing percentage that is the ratio standard deviations, $(2x_0 - \frac{p_0<G>}{4(\Delta p)^2}\frac{4f_1-2A''(x)}{f_1+f_3})$ vanishes. We iterate

$$\frac{a_{n+m+1}}{a_{n+m}} = \frac{(C - \frac{p_0<G>}{4(\Delta p)^2}g)}{2x_0 + \frac{p_0<G>}{4(\Delta p)^2}\frac{4f_3+2A''(x)}{f_1+f_3}}\frac{\mu_{n+1,m}}{E(n+1,m)}$$

to obtain $a_{n+m}$ only in terms of $a_m$:

$$\frac{a_{n+m}}{a_m} = k_0^n \prod_{j=m}^{n+m} \frac{\mu_j}{E(j)} \tag{5-4}$$

with

$$k_0 = \frac{(C - \frac{p_0<G>}{4(\Delta p)^2}g)}{2x_0 + \frac{p_0<G>}{4(\Delta p)^2}\frac{4f_3+2A''(x)}{f_1+f_3}}.$$

Substituting the coefficient $a_{n+m}$, given in Eq.$(5-4)$, into the expansion $\psi_{MUCS}$, we get

$$\psi_{MUCS} = \sum_{n=m}^{\infty} a_m k_0^n \prod_{j=m}^{n+m} \frac{\mu_j}{E(j)} \psi_n^m. \tag{5-5}$$

In the rest of this section we investigate some important special cases:

a- For $A(x) = 1$ we get the harmonic oscillator with eigenfunction

$$\psi_n(x) = e^{-\frac{\alpha x^2}{4}} H_n(x)$$

and the coherent states yield

$$\psi_{MUCS}(x) = e^{tx - t^2/2}$$

, where $t = \frac{k_0}{\alpha}$.

b- $A(x) = x, \beta = 1, m = 0, and \alpha = \lambda + \frac{1}{2}$ lead to

$$\psi_n(x) = x^{\frac{\alpha+1/2}{2}} e^{-x/2} n! L_n^{\alpha}(x)$$

and the coherent states yield

$$\psi_{MUCS}(x) = x^{\frac{\alpha+1/2}{2}} e^{-x/2} I_{\lambda+1/2}(2\sqrt{-k_0 x}),$$

in agreement with reference[4].

c- $A(x) = x(1-x), \alpha = \beta = 0, m = 1/2 - \lambda, and n = n + \lambda - 1/2$ lead to

$$\psi_n(x) = \sqrt{\cos(t)} P_{n+\lambda-1/2}^{1/2-\lambda}(\sin(t))$$

and the coherent states yield

$$\psi_{MUCS}(x) =$$

in agreement with reference[4].

d- $A(x) = 1 + x^2, \alpha = -1/2, and \beta = 0$, lead to

$$\psi_n(x) = P^n_{m-1}(\tanh(t))$$

and the coherent states yield

$$\psi_{MUCS}(x) =$$

in agreement with reference[5].

e- $A(x) = x^2, \beta = 1, \lambda = n - \alpha/2$ lead

$$\psi_n(x) = x^{-\frac{\alpha+1}{2}} e^{-x/2} L_n^{-\alpha-1}(x)$$

and the coherent states yield

$$\psi_{MUCS}(x) =$$

in agreement with reference[5]. If the coeffient of $a_n$ in the recursion relation (5-3) vanishes, that is for:

$$C = \frac{p_0 < G >}{4(\Delta p)^2} g$$

we encounter another interesting special case. This is again possible by an appropriate choice of the parameters of MUCS. In this case the MUCS can consist only of the odd or even associative orthogonal polynomials. In particular, if we choose even master function together with the corresponding even weight function, then the associative orthogonal function will be even for even $n + m$ and will be odd for odd $n + m$, provided that we choose a symmetric interval,that is $[-a, a]$. Therefore, MUCS is either even or odd known as even or odd coherent (squeezed) states in the special case of the harmonic Hamiltonian (Schrodinger-Cat states) [18, 19, 20, 21]. Finally we can choose the parameters of MUCS such that the coefficients of $a_{n+m}$ and $a_{n+m-1}$ on the right hand side of the recursion (5-3) become null. This means that in the expansion of MUCS, given in (5-2), only the ground state of the Hamiltonian will remain, that is, the ground state of the system belongs to its MUCS ensemble. At the end of this section it should be remined that, eventhough the closed form of MUCS is only available in seldom special case. But this is not a serious prblem, since one can iterate the recursion relation (5-3) numerically, to calculate the MUCS and related average quantities over these states which is under separate investigation.

# 6 Eigenstates of Annihilation Operators

It is well-known that the coherent states of harmonic oscillators are eigenstates of the annihilation operator $a$, that is, we have

$$a \mid \alpha >= \alpha \mid \alpha > .$$

Here we try to generalize this idea to the general shape invariant potentials $(V_m(x(\xi)))$. This is possible if we can find the eigenstate of annihilation operator $(A \mid \alpha >= \alpha \mid \alpha >)$ associated with these potentials. Here in this section, by multiplying the annihilation operator by an appropriate function of Hamiltonian and using the result of reference [15], we get a closed form for their eigenstates. Denoting the eigenstates of the annihilation operator $A$ by $\psi_{AOCS}(\beta_0, x)$, we have the following eigen-equation

$$F(g^{-1}(H))A\psi_{AOCS}(\beta_0, x) = \beta_0 \psi_{AOCS}(\beta_0, x),$$

where $F(g^{-1}(H))$ is an arbitrary function of Hamiltonian which is to be determined below. Expanding $\psi_{AOCS}(\beta_0, x)$ in terms of the associated orthogonal functions $\phi_{n,m}(x)$ and using the relation

$$F(g^{-1}(H))A\sum_{n=0}^{\infty} C_n \phi_{n,m} = \beta_0 \sum_{n=0}^{\infty} C_n \phi_{n,m},$$

we get the following recursion relation

$$C_{n+1}\frac{F(n+1)E(n+1,m)}{\mu_{n+1,m}} = \beta_0 C_n. \tag{6-1}$$

Substituting for $\mu_{n+1,m}$ we get

$$\frac{C_{n+1}}{C_n} = \beta_0 \frac{\frac{-1}{2}nA''(x) - B_0}{E(n+1,m)F(n+1)}\frac{a_n}{a_{n+1}}$$

with

$$B_0 = \frac{1}{2}(n+1)A''(x) + (\frac{A(x)W'(x)}{W(x)})'.$$

By defining $F(g^{-1}(H))$ as

$$F(n+1) = \frac{(n+1)(\frac{-1}{2}nA''(x) - B_0)}{E(n+1,m)}$$

we can solve the recursive relation $(6-1)$ if we choose $C_n = \frac{\lambda^n}{n!a_n}$ and $\lambda = \beta_0$. Substituting these in the expansion of $\psi_{AOCS}(\beta_0, x)$ we get

$$\psi_{AOCS}(\beta_0, x) = \sum_{n=0}^{\infty}\frac{\beta_0^n}{n!a_n}\phi_{n,m}.$$

Now, using the result of reference [15] we obtain

$$\psi_{AOCS}(\beta_0, x) = (-1)^m (A(x))^{\frac{m}{2}} (\frac{d}{dx})^m (\frac{W(z)}{W(x)} \frac{dz}{dx})$$

with $z = x + tA(z)$. From the result thus obtained it is clear that in general, the $\psi_{AOCS}$ states are different from the corresponding $\psi_{MUCS}$ states and they only coincide in the case of harmonioc oscillator. To see their difference more explicitly, using the result of reference[15], we give in the rest of this section some of the $\psi_{AOCS}$ states.

I-The choice of $A(x) = x, \beta = 1, m = 0, \alpha = \lambda + \frac{1}{2}$ leeds

$$\psi_n(x) = x^{\frac{\alpha+1/2}{2}} e^{-x/2} n! L_n^{\alpha}(x)$$

and the using coherent states

$$\psi_{MUCS}(x) = x^{\frac{\alpha+1/2}{2}} e^{-x/2} I_{\lambda+1/2}(2\sqrt{-k_0 x})$$

which is in agreement with reference[4].

II-The choice of $A(x) = x(1-x), \alpha = \beta = 0, m = 1/2 - \lambda, n = n + \lambda - 1/2$ leeds

$$\psi_n(x) = \sqrt{\cos(t)} P_{n+\lambda-1/2}^{1/2-\lambda}(\sin(t))$$

and the using coherent states

$$\psi_{MUCS}(x) =$$

which is in agreement with reference[4].

III-The choice of $A(x) = 1 + x^2, \alpha = -1/2, \beta = 0,$ leeds

$$\psi_n(x) = P_{m-1}^{n}(\tanh(t))$$

and the using coherent states

$$\psi_{MUCS}(x) =$$

which is in agreement with reference[5].

IV-The choice of $A(x) = x^2, \beta = 1, \lambda = n - \alpha/2$ leeds

$$\psi_n(x) = x^{-\frac{\alpha+1}{2}} e^{-x/2} L_n^{-\alpha-1}(x)$$

and the using coherent states

$$\psi_{MUCS}(x) =$$

which is in agreement with reference[5].

# 7 Time Evolution of the Minimum Uncertainty States

In this section we investigate the time evolution of the generalized Harmonic quantum phase variable $\widehat{X}$ and $\widehat{P}$. Since they do not have any explicit time dependence, thus their time dependence can be written as

$$\widehat{X}(t) = e^{\frac{iHt}{\hbar}} \widehat{X} e^{-\frac{iHt}{\hbar}}, \tag{7-1a}$$

$$\widehat{P}(t) = e^{\frac{iHt}{\hbar}} \widehat{P} e^{-\frac{iHt}{\hbar}}, \tag{7-1b}$$

which follows from the Heisenberg equations of motion

$$\dot{\widehat{X}} = \frac{1}{i\hbar}[\widehat{X}, H] = \frac{\widehat{P}}{m}$$

$$\dot{\widehat{P}} = \frac{1}{i\hbar}[\widehat{P}, H] = \widehat{X} B_1(H) + i\widehat{P} B_0$$

with $B_1(H)$ and $B_0$ defined as:

$$B_0 = -\frac{1}{2m}\hbar A''(x)$$

$$B_1 = A''(H - \gamma + \frac{2m-1}{4}A'') + \frac{1-2m}{2}A''(\frac{AW'}{W})' - \frac{1}{2}((\frac{AW'}{W})')^2 - \frac{4m^2-1}{8}A''.$$

Now, using the Baker-Hausdorf formula in Eq.$(7-1a)$, we can calulate $\widehat{X}(t)$ :

$$\widehat{X}(t) = \sum_{n=0}^{\infty} (\frac{it}{\hbar})^n \frac{1}{n!} X_n$$

with

$$X_n = \widehat{X} f_n(H) + \widehat{P} g_n(H) \tag{7-2}$$

By iterating the following recursion relations

$$f_{n+1}(H) = \frac{\hbar}{i} B_1(H) g_n(H) \tag{7-3}$$

$$g_{n+1}(H) = \frac{\hbar}{i}(\frac{1}{m} f_n(H) + (iB_0) g_n(H)) \tag{7-4}$$

with

we get the following expression for the function $g_n(H)$

$$g_n(H) = Ar_+^n + Br_-^n$$

where $r_-, r_+, A$ and $B$ are

$$r_+ = \frac{1}{2}\hbar B_0 + \frac{1}{2}\hbar\sqrt{B_0^2 - 4B_1(H)/m} = \hbar\omega_0 + \hbar\omega_H$$
$$r_- = \frac{1}{2}\hbar B_0 - \frac{1}{2}\hbar\sqrt{B_0^2 - 4B_1(H)/m} = \hbar\omega_0 - \hbar\omega_H$$
$$A = -B = \frac{-i\hbar}{2m\omega_H}$$

with $\omega_0$ and $\omega_H$ defined as

$$\omega_0 = B_0/2 \quad \textit{and} \quad \omega_H = \sqrt{B_0^2 - 4B_1(H)/m}/2,$$

respectively. Substituting the result thus obtained for $g_n(H)$ in (7-4a) we can determine $f_n(H)$. Now, inserting all these in (7-3), we obtain the closed form for $\widehat{X}(t)$ and $\widehat{P}(t)$ as follows:

$$\widehat{X}(t) = \widehat{X}e^{i\omega_0 t}[\cos(\omega_H t) - i\frac{\omega_0}{\omega_H}\sin(\omega_H t)]$$

$$+ \widehat{P}e^{i\omega_0 t}2\frac{\omega_0}{\omega_H}\sin(\omega_H t) \tag{7-5}$$

Starting from the formula (7-1b) and performing some calculation which is similar to the above calculation, we will obtain

$$\widehat{P}(t) = \widehat{P}e^{i\omega_0 t}[\cos(\omega_H t) + i\frac{\omega_0}{\omega_H}\sin(\omega_H t)]$$
$$+ \widehat{X}e^{i\omega_0 t}(2\frac{\omega_0}{\omega_H})\sin(\omega_H t) \tag{7-6}$$

Again the result thus obatained are in agreement with refernces[4, 5] for the special caes provided that we insert the corresponding eigen value of the operators of eigen frequency $\omega_H$, given in Table I. The time dependence of the generlized quantum phase coordinates given in (7-5) and (7-6) is almost similar to the time evolution of the phase coordinates of the qunantum oscillator except for the appearance of the $\omega_0$ and also the energy depenence of the $\omega_H$. We see that in case of the general shape invarint hamiltonian the frequency is a hamiltonian dependent operator and it is only constant in special case of oscillator.

# 8 CONCLUSION

In this paper a general algorithm has been given for the generation of the minimum uncertainty coherent and squeezed states in some one-dimensional hamiltonians with shape invariant potential, obtained from the master function. It looks like that the shape invariance symmetry of these hamiltonian might be the reason for the observation the MUCS. Since Solvability of these quantum systems are mainly due to the existance of this symmetry [8, 25]. But this not the only reason , Actually the main role belongs the existence of the lowering and raising operators or ladder ones [2, 3, 4, 5], which map different energy eigenstates of a given hamiltonian into each other. As quoted in the introduction, the coherent and squeezed states generated by harmonic oscillator have already play such an importantat role in different branches of physics. Definetly the MUCS have been generated in refrence [3] and here will soon play very important role in almost all branches of physics. Therefore, it deserve to find all other hamiltonian which can generate MUCS, which is under investigation.

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