New approach for solving master equation of open atomic system[*]

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We describe a new approach called Ket-Bra Entangled State (KBES) Method which enables one convert master equations into Schrödinger-like equation. In sharply contrast to the super-operator method, the KBES method is applicable for any master equation of finite-level system in theory, and the calculation can be completed by computer. With this method, we obtain the exact dynamic evolution of a radioactivity damped 2-level atom in time-dependent external field, and a 3-level atom coupled with bath; Moreover, the master equation of N-qubits Heisenberg chain each qubit coupled with a reservoir is also resolved in Sec.III; Besides, the paper briefly discuss the physical implications of the solution.

I. INTRODUCTION

The theory of open quantum systems which describes the dynamic evolution of system coupled with environment is a fundamental approach to understand dissipation and decoherence in quantum optics[1]. One of the most important instruments of open quantum systems is represented by master equation[2], which plays a significant role in fundamental aspects, like decoherence[3, 4], disentanglement[5, 6], quantum dynamics[7], non-equilibrium thermodynamics[8], and applied ones like atom emission[9], quantum transport[10], Bose-Einstein condensates, Brownian motion[8], laser field[11].

Recently one topic has mostly drawn attention is the derivation, solution, and systemic theory analysis of master equation which is a set complicated differential equations of reduced density operator. Generally, the master equations are solved by means of phase space representation of density operator which can transform master equation into c-number equation such as Fokker-Planck equation and Langevin equation[12]. There is also some other way like super-operator[13, 14] method, which is applicable only for that the Lindblad can be expressed in super-operator generators of Su(1,1), Su(2)[22, 23] and some other Lie algebra. We shall briefly review their method and compare it with our new method in Sec.II

In this contribution, a new procedure called Ket-Bra Entangled State(KBES) method have been developed[24], which allow one to transform conventional master equation into Schrödinger-like equation, then the time evolution of density operator can be deduced; We can obtain the solution of Schrödinger-like equation(see Sec.II). This method is applicable for any master equation of finite-level system in theory. With this KBES method, we obtain the exact solutions of three different physical models in Sec.III, which include a qubit coupled with reservoir in time-dependent external field, a qutrit with Bose-bath, and a N-qubit Heisenberg chains coupled with reservoir.

II. SUPER-OPERATOR METHOD VERSUS KBES METHOD

In this section, after briefly reviewing the general super-operator method, we introduce Fan's method that constructs bosonic thermal entangled representation to mutual transform general operator between real and fictitious mode[20, 21]; As a generalization and development of previous, we put forward KBES method which can convert operator master equation into Schrödinger-like equation.

To explain the super-operator method[13–15], consider an usual master equation

$$\dot{\rho} = \mathscr{L}\rho, \tag{1}$$

whose Lindblad operator is

$$\mathscr{L}\rho = \frac{\gamma}{2}(n+1)(2\sigma^-\rho\sigma^+ - \sigma^+\sigma^-\rho - \rho\sigma^+\sigma^-) \\ + \frac{\gamma}{2}n(2\sigma^+\rho\sigma^- - \sigma^-\sigma^+\rho - \rho\sigma^-\sigma^+). \tag{2}$$

If $\mathscr{L}$ is time independent, the form solution to Eq. (1) is

$$\rho(t) = e^{\mathscr{L}t}\rho(0). \tag{3}$$

$\rho(t)$ has the explicit expression if $\mathscr{L}$ consist of super-operator generators of Lie algebras, and there is a wide calss of master equation whose $\mathscr{L}$ can be expressed in terms of super-operator generators of Su(2) or Su(1,1) Lie algebra.

Previous literatures have defined the super-operators

$$L_+\rho = \sigma^+\rho\sigma^-, L_-\rho = \sigma^-\rho\sigma^+, \\ L_z\rho = \frac{1}{2}(\sigma^+\sigma^-\rho - \rho\sigma^-\sigma^+). \tag{4}$$

where $[L_+, L_-] = 2L_z, [L_z, L_\pm] = \pm L_\pm$ obeying Su(2) Lie algebra. The explicit expression of $\mathscr{L}$ is

$$\mathscr{L} = -\frac{1}{2}(2n+1)\gamma + n\gamma L_- + (n+1)\gamma L_+ - \gamma L_z. \tag{5}$$

Thus Eq. (3) can be represented as

$$\rho(t) = e^{-\frac{1}{2}(2n+1)\gamma t + n\gamma L_- + (n+1)\gamma t L_+ - \gamma t L_z}\rho(0), \\ = e^{x_+(t)L_+}e^{\ln x_z(t)L_z}e^{x_-(t)L_-}\rho(0), \tag{6}$$

where $x_\pm(t), x_z(t)$ is given in Ref. [22, 23]. Finally, the Kraus operator solution of Eq. (6) is obtained by transforming the super operator L_+, L_-, L_z into gereral operator

$$\rho(t) = \sum_i K_i \rho(0) K_i^\dagger, \tag{7}$$

K_i is the Kraus operator and satisfy $\sum_i K_i^\dagger K_i = I$, where K_i is given by

$$
\begin{aligned}
K_1 &= \frac{e^{-(2n+1)\gamma/4}}{\sqrt[4]{x_z}} x_+(t) \sigma^+, \\
K_2 &= \frac{e^{-(2n+1)\gamma/4}}{\sqrt[4]{x_z}} x_-(t) \sigma^-, \\
K_3 &= \frac{e^{-(2n+1)\gamma/4}}{\sqrt[4]{x_z}} x_+(t) x_-(t) \sigma^+ \sigma^-, \\
K_4 &= \frac{e^{-(2n+1)\gamma/4}}{\sqrt[4]{x_z}} \left[I + \sigma^+ \sigma^- \left(\sqrt{x_z} - 1 \right) \right].
\end{aligned} \tag{8}
$$

Ulteriorly, prof. Fan[20, 21] constructs the thermal entangled state $|I\rangle = e^{a^\dagger \tilde{a}^\dagger}|0, \tilde{0}\rangle$ by introducing an extra fictitious mode, for the Bose operator

$$a|I\rangle = \tilde{a}^\dagger|I\rangle, \quad a^\dagger|I\rangle = \tilde{a}|I\rangle, \tag{9}$$

thus, for the super operator $L_\pm$ that consist of creation and annihilation operator

$$
\begin{aligned}
L_- \rho|I\rangle &\equiv a\rho a^\dagger|I\rangle = a\tilde{a}\rho|I\rangle \Rightarrow L_- = a\tilde{a}, \\
L_+ \rho|I\rangle &\equiv a^\dagger \rho a|I\rangle = a^\dagger \tilde{a}^\dagger \rho|I\rangle \Rightarrow L_+ = a^\dagger \tilde{a}^\dagger,
\end{aligned} \tag{10}
$$

The thermal entangled state $|I\rangle$ can transform general operator into fictitious mode, with the characteristic it be used to slove master equation of Bose system.

Next, we shall introduce our new method, consider a density operator ρ in Hilbert space $\mathscr{H}$

$$\rho = \sum_{m,n} \rho_{m,n}|m\rangle\langle n|, \tag{11}$$

where $|m\rangle$ constitutes any complete orthogonal basis in $\mathscr{H}$. By introducing an extra fictitious mode, we construct the Ket-Bran entangled states

$$|\eta\rangle = \sum_m |m, \tilde{m}\rangle \tag{12}$$

fot the density operator ρ

$$|\rho\rangle = \rho|\eta\rangle = \sum_{m,n} \rho_{m,n}|m, \tilde{n}\rangle. \tag{13}$$

With the defining of $|\eta\rangle$, for any operator $A_{mn} \equiv |m\rangle\langle n|$ in $\mathscr{H}$

$$A_{mn}|\eta\rangle = |m, \tilde{n}\rangle = |\tilde{n}\rangle\langle\tilde{m}||\eta\rangle = \tilde{A}_{mn}^\dagger|\eta\rangle. \tag{14}$$

Besides, Eq. (14) is valid for any $A \equiv \sum_{m,n} \mathfrak{a}_{mn} A_{mn}$ where $\mathfrak{a}_{mn}$ is real, namely $A|\eta\rangle = A^\dagger|\eta\rangle$. Eq. (14) show that the general operator can also be transformed between real mode and fictitious mode by KBES just as Fan's method.

Obviously the fictitious mode indeed represent the bra vector of system, so we called $|\eta\rangle$ Ket-Bra Entangled State (KBES). Density operator ρ can be translated into pure state

$|\rho\rangle$ and general operator can mutual transform between real and fictitious mode by KBES $|\eta\rangle$, which enable one to convert master equation into Schrödinger-like equation.

The case of Eq. (1) has been solved by KBES method in our previous work [24], here we consider the Lindblad equation, which is most general form of Markovian and time-homogeneous master equation describing non-unitary evolution of the density matrix ρ, that is

$$
\begin{aligned}
\frac{d\rho}{dt} = &-\frac{i}{\hbar}[H, \rho] + \sum_{n,m=1}^{N^2-1} h_{n,m}[L_n \rho L_m^\dagger \\
&- \frac{1}{2}\left(\rho L_m^\dagger L_n + L_m^\dagger L_n \rho\right)],
\end{aligned} \tag{15}
$$

where H is Hamiltonian part, L_m is arbitrary linear operators in system's Hilbert space, and $h_{n,m}$ is constant. With the KBES $|\eta_L\rangle = \sum_{n=1}^N |n, \tilde{n}\rangle$, Eq. (15) can be transformed into Schrödinger-like equation

$$
\begin{aligned}
\frac{d}{dt}|\rho\rangle &= -\frac{i}{\hbar}\left\{\left(H - \tilde{H}\right) + \mathscr{L}_{lin}\right\}|\rho\rangle, \\
&\equiv \mathscr{F}_L|\rho\rangle,
\end{aligned} \tag{16}
$$

where

$$\mathscr{L}_{lin} = \sum_{n,m=1}^{N^2-1} h_{n,m}\left[L_n \tilde{L}_m - \frac{1}{2}\left(\tilde{L}_n^\dagger \tilde{L}_m + L_m^\dagger L_n\right)\right]. \tag{17}$$

Schrödinger-like Eq. (16) can be solved by two different approaches. The first is the evolution operator method. assume $\mathscr{F}_L$ is independent of time t, then

$$|\rho(t)\rangle = e^{\mathscr{F}_L t}|\rho(0)\rangle. \tag{18}$$

where the explicit matrix represent of $\mathscr{F}_L$ is demanded, then as mentioned before, the calculation of $e^{\mathscr{F}_L t}$ need to diagonalize a N^2-order matrix $\mathscr{F}_L$, the calculation is complicate but can be done by computer. For few simple situations, we can also decompose $e^{F_L t}$ into several exponential form of operators with Lie algebra (see Eq. (6)) just like super operator method.

The second approach is the stationary state method. For some high-dimensional systems the calculation of $e^{\mathscr{F}_L t}$ may be difficult even with computer, thus we calculate the eigenstates and eigenvalues of $\mathscr{F}_L$

$$\mathscr{F}_L|\varphi_i\rangle = \lambda_i|\varphi_i\rangle. \tag{19}$$

Then the solution can be represented as follow:

$$|\rho(t)\rangle = \sum_i C_i e^{\lambda_i t}|\varphi_i\rangle, \tag{20}$$

where C_i is constant which can be determined by initially $\rho(0)$ and characters of density operator. **Obviously, all eigenvalues $\lambda_i \leq 0$ and the eigenstate $|\varphi\rangle$ whose eigenvalue $\lambda = 0$ corresponding the final state $\rho(\infty)$, For $t \to \infty$, only the eigenvector term whose eigenvalue is zero left, other term's coefficient $C_i e^{\lambda_i}$ shall disappear along with the growth of t,**

this characteristic can obtain the final density operator **directly.** Furthermore, most methods of Schrödinger equation previously can also be used to solve master equation with KBES.

Super-operator method is concise and the solution is applicable for any initial state, however restricted by Lie algebra, it show a narrow applicable range, and even slightly changes of master equation may lead to unsolvable effect for super operator method. Compared with the both method, our KBES method have three merits: **1. The procedure is applicable for any master equation of finite-level systems in theoretical; 2. The process of resolution is most concise and can be fully completed by computer; 3. The method can convert master equation into Schrödinger-like equation, which means most methods of Schrödinger equation can be used to solve master equation. These ascendances shall be proved in the process of solving follow models.**

III.   SEVERAL PHYSICAL MODELS

To further concretely explain our KBES method, we shall introduce three different physical models in this section, moreover each model can't be solved by general super-operator method. Especially the third model, to best of our knowledge is still unsolved for large N, whereas all these can be solved by KBES method.

A.   Damped 2-level Atom Model

Consider an atom in cavity full of external field that is radiatively damped by its interaction with the various modes of bath in thermal equilibrium at temperature T just as Fig.1. The
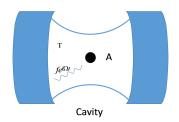

Fig. 1: Qubit with Cavity Damping in External Field

Hamiltonian is given in the rotating-wave and dipole approximations as follow:

$$H = \omega_0 \sigma_z + \sum_k \omega_k a_k^\dagger a_k + f\left(\sigma^+ e^{i\Omega t} + \sigma^- e^{-i\Omega t}\right)$$
$$+ \sigma^- \sum_k g_k^* a_k^\dagger + \sigma^+ \sum_k g_k a_k, \qquad (21)$$

where f and Ω represent the amplitude and frequency of external field respectively. In the interaction picture

$$H_I = e^{iH_0 t}(H - H_0) e^{-iH_0 t}$$
$$= H' + H_I', \qquad (22)$$

where $\Lambda = \omega_0 - \Omega$ and

$$H_0 = \Omega \sigma_z + \sum_k \omega_k a_k^\dagger a_k$$
$$H' \equiv \Lambda \sigma_z + f(\sigma^+ + \sigma_-)$$
$$H_I' = \sigma^- e^{i\Omega t} \sum_k g_k^* a_k^\dagger e^{i\omega_k t} + \sigma^+ e^{-i\Omega t} \sum_k g_k a_k e^{-i\omega_k t}. \qquad (23)$$

Through the derivation of master equation in Ref. [2], we can obtain the corresponding master equation of this model as follow

$$\frac{d\rho}{dt} = -i\left[H', \rho\right] + \frac{\gamma}{2} n \left(2\sigma^+ \rho \sigma^- - \sigma^- \sigma^+ \rho - \rho \sigma^- \sigma^+\right)$$
$$+ \frac{\gamma}{2}(n+1)\left(2\sigma^- \rho \sigma^+ - \sigma^+ \sigma^- \rho - \rho \sigma^+ \sigma^-\right), \qquad (24)$$

where H' is the entangled Hamiltonian of qubit and external field, γ is the spontaneous emission rate and $n = n(\Omega, T)$ is the photon number of frequency Ω in temperature T.

With the corresponding KBES $|\eta_Q\rangle = |0, \tilde{0}\rangle + |1, \tilde{1}\rangle$, we can deduce the Schrödinger-like equation of Eq. (24):

$$\frac{d}{dt}|\rho\rangle = \mathscr{F}|\rho\rangle, \qquad (25)$$

where $\alpha \equiv \gamma(n+1)/2, \beta \equiv \gamma n/2$ and

$$\mathscr{F} = i\left(\tilde{H}' - H'\right) + \alpha\left(2\sigma^- \tilde{\sigma}^- - \sigma^+ \sigma^- - \tilde{\sigma}^+ \tilde{\sigma}^-\right)$$
$$+ \beta\left(2\sigma^+ \tilde{\sigma}^+ - \sigma^- \sigma^+ - \tilde{\sigma}^- \tilde{\sigma}^+\right). \qquad (26)$$

The formal solution is $|\rho(t)\rangle = e^{\mathscr{F} t}|\rho(0)\rangle$, and the explicit formulation of $\mathscr{F}$ in Kronecker product space can be given:

$$\mathscr{F} = \begin{pmatrix} -2\alpha & if & -if & 2\beta \\ if & -\alpha - \beta - 2i\Lambda & 0 & -if \\ -if & 0 & -\alpha - \beta + 2i\Lambda & if \\ 2\alpha & -if & if & -2\beta \end{pmatrix}, \qquad (27)$$

where $\sigma^\pm = \sigma^\pm \otimes \tilde{I}, \tilde{\sigma}^\pm = I \otimes \tilde{\sigma}^\pm$. In fact, $e^{\mathscr{F} t}$ can be obtained through the diagonalization of Eq. (27), however the explicit expression of $\rho(t)$ is too long and complicated to be presented in here. As we have mentioned in Sec.II, **the final density $\rho(\infty)$ is represented by the eigenvector of $\mathscr{F}$ whose corresponding eigenvalue equal to zero.** The eigenvector whose eigenvalue equals to zero is

$$|\varphi\rangle = \begin{pmatrix} \frac{(f^2 + \beta(\alpha+\beta)) + 4\Lambda^2 \beta/(\alpha+\beta)}{2(f^2 + 2\Lambda^2) + (\alpha+\beta)^2} \\ \frac{-if(\alpha-\beta)(\alpha+\beta-2i\Lambda)}{2(f^2 + 2\Lambda^2) + (\alpha+\beta)^2} \\ \frac{if(\alpha-\beta)(\alpha+\beta-2i\Lambda)}{2(f^2 + 2\Lambda^2) + (\alpha+\beta)^2} \\ \frac{(f^2 + \alpha(\alpha+\beta)) + 4\Lambda^2 \alpha/(\alpha+\beta)}{2(f^2 + 2\Lambda^2) + (\alpha+\beta)^2} \end{pmatrix}. \qquad (28)$$

Thus $\rho(\infty)$ is given by

$$\rho(\infty) = \frac{1}{2n+1}\begin{pmatrix} n + \frac{f_\gamma^2}{M(n, f_\gamma, \Lambda_\gamma)} & \frac{-if_\gamma(n+1/2 - 2i\Lambda_\gamma)}{M(n, f_\gamma, \Lambda_\gamma)} \\ \frac{if_\gamma(n+1/2 + 2i\Lambda_\gamma)}{M(n, f_\gamma, \Lambda_\gamma)} & \frac{(n+1)M(n, f_\gamma, \Lambda_\gamma) - f_\gamma^2}{M(n, f_\gamma, \Lambda_\gamma)} \end{pmatrix}, \qquad (29)$$

where $f_\gamma \equiv 2f/\gamma$, $\Lambda_\gamma \equiv 2\Lambda/\gamma$ and

$$M\left(n, f_\gamma, \Lambda_\gamma\right) = (n + 1/2)^2 + 2\left(f_\gamma^2 + 2\Lambda_\gamma^2\right). \qquad (30)$$

Plot the figure of $\rho_{11}(t)$ verus γt and Λ_γ on different conditions with initial state $\rho(0) = |\varphi\rangle\langle\varphi|$, $|\varphi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ as follow:
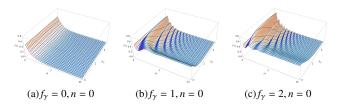

(a)$f_\gamma = 0, n = 0$ (b)$f_\gamma = 1, n = 0$ (c)$f_\gamma = 2, n = 0$

Fig. 2: ρ_{11} versus γt and Λ_γ.

Obviously, $\rho_{11}(\infty)$ that is severely influenced by amplitude of external field f and frequency Λ; For a fixed Λ, $\rho_{11}(\infty)$ show a positive correlation with f, namely $\rho_{11}(\infty)$ increases along with the augmentation of f; However for a fixed $f > 0$, there is a negative correlation between $\rho_{11}(\infty)$ and $|\Lambda|$ which lead to a local maximum of $\rho_{11}(\infty)$ at $\Lambda = 0$; While for $f = 0$, the change of Λ make no difference to $\rho_{11}(t)$. For any $f \neq 0$, we find larger $|\Lambda|$ shall lead to sharp and frequent fluctuations on the outiline of $\rho_{11}(t)$.

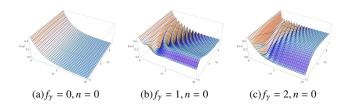While the figure of non-diagonal element $|\rho_{10}(t)|$ is: Eq.

(a)$f_\gamma = 0, n = 0$ (b)$f_\gamma = 1, n = 0$ (c)$f_\gamma = 2, n = 0$

Fig. 3: $|\rho_{10}|$ versus γt and Λ_γ.

(29) show that $|\rho_{10}(\infty)|^2$ take the maximum

$$|\rho_{10}(\infty)|^2_{Max} = \frac{f_\gamma^2}{4(n + 1/2)^2 - 1 + 8f_\gamma^2}, \qquad (31)$$

when $8\Lambda_\gamma^2 + 1 = 2(n + 1/2)^2 + 4f_\gamma^2$. For zero temperature $n = 0$, $|\rho_{10}(\infty)|$ has a maximum $1/2\sqrt{2}$ at $16\Lambda^2 + \gamma^2 = 8f^2$, that has been presented in Fig.3. Despite these, for $f = 0$, change of Λ make no difference and larger $|\Lambda|$ lead to more frequent fluctuations on the outiline.

All those can be easily interpreted, f represents the amplitude of external field, so larger f means strong external field which lead to the transition of qubit from ground to excited state, i.e. the increasing of $\rho_{11}(\infty)$ and decreasing of $\rho_{00}(\infty)$. When $\Lambda = 0$ means the qubit and external field have a same frequency $\omega_0 = \Omega$, there is resonance between qubit and external field, so $\rho_{11}(\infty)$ have a local maximum value in here; The lack of space forbids a further detail discussion of the off-diagonal elements $\rho_{10}(t)$ in here.

B. 3-level V-type Atom Model

Now we extend the research of qubit to the case of V-type three-level qutrit. Interesting example come from the three-level V-type atomic system where spontaneous emission may take place from two excited levels to the ground state but direct transition between excited levels is forbidden. However the indirect coupling between excited states can appear due to interaction with the ground state (quantum interference).
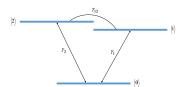

Fig. 4: V-type Damped Qutrit

Consider a three-level V-type atom as in Fig.4, in which the atom has two nondegenerate excited states $|1\rangle$, $|2\rangle$ with transition frequencies to ground state $|0\rangle$ given by ω_1, ω_2 and spontaneous emission rate γ_1, γ_2 respectively. The indirecting coupling between $|1\rangle$ and $|2\rangle$ can represent as cross damping constant

$$\gamma_{12} = \beta_I \sqrt{\gamma_1\gamma_2}, \qquad (32)$$

where β_I change from 0 to 1 represents the mutual orientation of transition dipole moments.

The dynamics of such system is given by the master equation[16, 17]

$$\frac{d\rho}{dt} = \frac{\gamma_1}{2}\left(2\sigma_{01}\rho\sigma_{10} - \{\sigma_{11}, \rho\}\right) + \frac{\gamma_2}{2}\left(2\sigma_{02}\rho\sigma_{20} - \{\sigma_{22}, \rho\}\right)$$
$$+ \frac{\gamma_{12}}{2}\left(2\sigma_{01}\rho\sigma_{20} + 2\sigma_{02}\rho\sigma_{10} - \{\sigma_{12} + \sigma_{21}, \rho\}\right), \quad (33)$$

where $\{A, B\} = AB + BA$ is the anticommutation and $\sigma_{mn} = |m\rangle\langle n|$ means the transition from $|n\rangle$ to $|m\rangle$. Ref. [16, 17] solve analogous equation with the conditions $\gamma_{12} = 0$ and $\gamma_1 = \gamma_2$. We shall solve this master equation without any assumption and approximation in present work. Construct the corresponding KBES

$$|\eta_T\rangle = |0\tilde{0}\rangle + |1\tilde{1}\rangle + |2\tilde{2}\rangle, \qquad (34)$$

Eq. (33) can be rewrite as Schrödinger-like equation

$$\frac{d}{dt}|\rho\rangle = \mathscr{F}_T|\rho\rangle, \qquad (35)$$

where

$$\mathscr{F}_T = \frac{\gamma_1}{2}\left(2\sigma_{01}\tilde{\sigma}_{01} - \sigma_{11} - \tilde{\sigma}_{11}\right) + \frac{\gamma_2}{2}\left(2\sigma_{02}\tilde{\sigma}_{02} - \sigma_{22} - \tilde{\sigma}_{22}\right)$$
$$+ \frac{\gamma_{12}}{2}\left(2\sigma_{01}\tilde{\sigma}_{02} + 2\sigma_{02}\tilde{\sigma}_{01} - \sigma_{12} - \tilde{\sigma}_{21} - \sigma_{21} - \tilde{\sigma}_{12}\right). \quad (36)$$

Then the formal solution is:

$$|\rho(t)\rangle = e^{\mathscr{F}_T}|\rho(0)\rangle. \qquad (37)$$

The explicit solution of Eq. (37) can be given by diagonalizing $\mathscr{F}_T$ to get the explicit expression of $e^{\mathscr{F}_T t}$, which can be finished by computer with Mathematica. However the explicit expression is too complicated to be given here, so we give the solution for $\gamma_1 = \gamma_2 = \gamma, \gamma_{12} = \beta_I \gamma$, the five independent matrix elements is:

$$\rho_{10}(t) = \frac{1}{2} e^{-\frac{1}{2}(1+\beta_I)\gamma t}\left[\rho_{10} + \rho_{20} + e^{\beta_I \gamma t}(\rho_{10} - \rho_{20})\right],$$

$$\rho_{20}(t) = \frac{1}{2} e^{-\frac{1}{2}(1+\beta_I)\gamma t}\left[\rho_{10} + \rho_{20} + e^{\beta_I \gamma t}(\rho_{20} - \rho_{10})\right],$$

$$\rho_{21}(t) = \frac{1}{2} e^{-\gamma t}\left[\begin{array}{c} \rho_{21} - \rho_{12} + (\rho_{12} + \rho_{21})e^{\beta_I \gamma t} \\ -(\rho_{11} + \rho_{22})\sinh(\beta_I \gamma t) \end{array}\right],$$

$$\rho_{11}(t) = \frac{1}{2} e^{-\gamma t}\left[\begin{array}{c} \rho_{11} - \rho_{22} + (\rho_{11} + \rho_{22})\cosh(\beta_I \gamma t) \\ -(\rho_{12} + \rho_{21})\sinh(\beta_I \gamma t) \end{array}\right],$$

$$\rho_{22}(t) = \frac{1}{2} e^{-\gamma t}\left[\begin{array}{c} \rho_{22} - \rho_{11} + (\rho_{11} + \rho_{22})\cosh(\beta_I \gamma t) \\ -(\rho_{12} + \rho_{21})\sinh(\beta_I \gamma t) \end{array}\right]. \quad (38)$$

Remark $\rho_{mn} = \rho_{mn}(0)$ in the equation for simplify, other elements can be easily obtained with $\rho_{ij}(t) = \rho_{ji}^*(t)$ and $\rho_{00}(t) = 1 - \rho_{22}(t) - \rho_{11}(t)$.

C. N-qubit XXZ Heisenberg model

Here we consider a N-qubit anisotropic XYZ Heisenberg chain, the Hamiltonian is given by

$$H_S = \sum_{i=1}^{N}\left[J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z\right], \quad (39)$$

where σ_i^ε ($\varepsilon = x, y, z$) are the Pauli matrices of the i-th qubit. J_ε ($\varepsilon = x, y, z$) are the strengths of the spin interaction. For the interaction, when $J_x = J_y \neq J_z$, the model can be called XXZ chain. Eq. (39) is rewriten as

$$H_S = \sum_i \left[J\left(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+\right) + J_z \sigma_i^z \sigma_{i+1}^z\right], \quad (40)$$

in which $J = J_x + J_y$. The eigenvector of H_S can be exactly solved by Jordan-Wigner transformation[25]. Now, we only consider the "one-particle" eigenvector (i.e. only one qubit in spin up)

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \exp\left(\frac{i2\pi nk}{N}\right) \sigma_n^+ |0^{\otimes N}\rangle, \quad (41)$$

and the inverse transformation is

$$\sigma_n^+ |0^{\otimes N}\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \exp\left(\frac{-i2\pi nk}{N}\right) |k\rangle. \quad (42)$$

For the vector $|k\rangle$ and $|0^{\otimes N}\rangle$, the eigenequation is

$$\begin{cases} H_S |k\rangle = E_k |k\rangle, \\ H_S |0^{\otimes N}\rangle = E_0 |0^{\otimes N}\rangle, \end{cases} \quad (43)$$

the eigenvalues are

$$\begin{cases} E_0 = NJ_z, \\ E_k = J\cos\left(\frac{2\pi k}{N}\right) + (N-4)J_z. \end{cases} \quad (44)$$

When the XXZ chain couple with reservoir as Eq. (21), the evolution is no longer unitary and this dynamic process can be described by the master equation:

$$\frac{d\rho}{dt} = -i[H_S, \rho] + \gamma \sum_{i=1}^{N}\left(2\sigma_i^- \rho \sigma_i^+ - \sigma_i^+ \sigma_i^- \rho - \rho \sigma_i^+ \sigma_i^-\right) \quad (45)$$

Through the previous models Eq. (24,33) can be solved by c-number or other method, this model Eq. (45) is still unsolved. To solve the master equation, we construct the corresponding Ket-Bra Entangled State

$$|\eta_S\rangle = \sum_{i=1}^{N} \sum_{S_i=0,1} |S_1, S_2, \cdots, S_N, \tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_N\rangle \quad (46)$$

where $\tilde{S}_i = S_i = 0, 1$ represents the spin of i-th qubit. Similar with Eq. (12), one can found

$$\rho|\eta_S\rangle = \sum \rho_{\varphi,\psi}|\varphi, \tilde{\psi}\rangle \equiv |\rho\rangle,$$
$$\sigma_i^-|\eta_S\rangle = \tilde{\sigma}_i^+|\eta_S\rangle, \sigma_i^+|\eta_S\rangle = \tilde{\sigma}_i^-|\eta_S\rangle. \quad (47)$$

Thus Eq. (45) can be converted into Schrödinger-like equation:

$$\frac{d}{dt}|\rho\rangle = \mathscr{F}_S|\rho\rangle = (H_F + G)|\rho\rangle, \quad (48)$$

where

$$G = 2\gamma \sum_i \sigma_i^- \tilde{\sigma}_i^-,$$
$$H_F = i\left(\tilde{H}_S - H_S\right) - \gamma \sum_i \left(\sigma_i^+ \sigma_i^- + \tilde{\sigma}_i^+ \tilde{\sigma}_i^-\right). \quad (49)$$

The formal solution of Eq. (48) is

$$|\rho(t)\rangle = e^{\mathscr{F}_S t}|\rho(0)\rangle = e^{(H_F + G)t}|\rho(0)\rangle. \quad (50)$$

Introduce the two-mode vector of $|k\rangle$

$$|k, \tilde{k}'\rangle = |k\rangle\langle k'\||\eta_S\rangle = \frac{1}{N} \sum_{m,n=1}^{N} e^{\frac{i2\pi(nk-mk')}{N}} \sigma_n^+ \tilde{\sigma}_m^+ |0, \tilde{0}\rangle. \quad (51)$$

Easy to found

$$\begin{array}{ll} G|0, \tilde{0}\rangle = 0, & H_F|0, \tilde{0}\rangle = 0; \\ G|0, \tilde{k}\rangle = 0, & H_F|0, \tilde{k}\rangle = h_{0,k}|0, \tilde{k}\rangle; \\ G|k, \tilde{0}\rangle = 0, & H_F|k, \tilde{0}\rangle = h_{0,k}^*|k, \tilde{0}\rangle; \\ G|k, \tilde{k}'\rangle = g_{k,k'}|0, \tilde{0}\rangle, & H_F|k, \tilde{k}'\rangle = h_{k,k'}|k, \tilde{k}'\rangle; \end{array} \quad (52)$$

where

$$\begin{array}{l} g_{k,k'} = 2\gamma \delta_{k,k'}, \\ h_{0,k} = i(E_k - E_0) - \gamma, \\ h_{k,k'} = i(E_{k'} - E_k) - 2\gamma. \end{array} \quad (53)$$

With Eq. (52), for $|0, \tilde{k}\rangle$ and $|k, \tilde{0}\rangle$ we obtain

$$\begin{array}{l} |\rho_{0,k}(t)\rangle = e^{(H_F + G)t}|0, \tilde{k}\rangle = e^{h_{0,k}t}|0, \tilde{k}\rangle \\ |\rho_{k,0}(t)\rangle = e^{(H_F + G)t}|k, \tilde{0}\rangle = e^{h_{0,k}^*t}|k, \tilde{0}\rangle \end{array} \quad (54)$$

while for $|k, \tilde{k}'\rangle$

$$|\rho_{k,k'}(t)\rangle = e^{(H_F + G)t}|k, \tilde{k}'\rangle = \sum_{n=0}^{\infty} \frac{t^n}{n!}\left(H_F^n + GH_F^{n-1}\right)|k, \tilde{k}'\rangle,$$

$$= e^{h_{k,k'}t}|k, \tilde{k}'\rangle + \frac{g_{k,k'}}{h_{k,k'}}\left(e^{h_{k,k'}t} - 1\right)|0, \tilde{0}\rangle. \quad (55)$$

Consider the initial state is $\rho(0) = |k\rangle\langle k|$ i.e. $|\rho(0)\rangle = |k, \tilde{k}\rangle$ according to Eq. (53) and Eq. (55),

$$|\rho(t)\rangle = e^{-2\gamma t}|k, \tilde{k}\rangle + \left(1 - e^{-2\gamma t}\right)|0, \tilde{0}\rangle. \quad (56)$$

Moreover, fix the initial state $\rho(0) = |\varphi\rangle\langle\varphi|, |\varphi\rangle = a|0^{\otimes N}\rangle + b\sigma_1^+|0^{\otimes N}\rangle$, where a is real and $a^2 + |b|^2 = 1$. Accoding to Eq. (42) and Eq. (47), the solution can be expressed as

$$|\rho(0)\rangle = \left(|b|^2\sigma_1^+\tilde{\sigma}_1^+ + ab\sigma_1^+ + ab^*\tilde{\sigma}_1^+ + a^2\right)|0, \tilde{0}\rangle,$$

$$= \frac{|b|^2}{N}\sum_{k,k'=1}^{N} e^{\frac{i2\pi(k'-k)}{N}}|k, \tilde{k}'\rangle + a^2|0, \tilde{0}\rangle$$

$$+ \frac{ab}{\sqrt{N}}\sum_{k=1}^{N} e^{\frac{-i2\pi k}{N}}|k, \tilde{0}\rangle + \frac{ab^*}{\sqrt{N}}\sum_{k=1}^{N} e^{\frac{i2\pi k}{N}}|0, \tilde{k}\rangle \quad (57)$$

As what we have done in Eq. (54) and Eq. (55), the solution is given by

$$|\rho(t)\rangle = \left(1 - |b|^2 e^{-2\gamma t}\right)|0, \tilde{0}\rangle + \frac{ab}{N}\sum_{n,k=1}^{N} e^{\frac{i2\pi(n-1)k}{N} + h_{0,k}^* t}\sigma_n^+|0^{\otimes N}, \tilde{0}^{\otimes N}\rangle$$

$$+ \frac{|b|^2}{N^2}\sum_{m,n,k,k'=1}^{N} e^{\frac{i2\pi[(n-1)k-(m-1)k']}{N} + h_{k,k'}t}\sigma_n^+\tilde{\sigma}_m^+|0^{\otimes N}, \tilde{0}^{\otimes N}\rangle$$

$$+ \frac{ab^*}{N}\sum_{n,k=1}^{N} e^{\frac{-i2\pi(n-1)k}{N} + h_{0,k}t}\tilde{\sigma}_n^+|0^{\otimes N}, \tilde{0}^{\otimes N}\rangle. \quad (58)$$

Trace to other qubit except j-qubit, the density of j-qubit is obtained

$$\rho_{01}^j(t) = \frac{ab^*}{N}\sum_{k=1}^{N} e^{\frac{-i2\pi(j-1)k}{N} + h_{0,k}t}$$
$$\rho_{11}^j(t) = \frac{|b|^2}{N^2}\sum_{k,k'=1}^{N} e^{\frac{i2\pi(j-1)(k-k')}{N} + h_{k,k'}t} \quad (59)$$

other elements can be given by $\rho_{00}^j(t) = 1 - \rho_{11}^j(t)$ and $\rho_{10}^j(t) = \rho_{01}^{j*}(t)$. To analysis the decoherence evolution of the Heisenberg chain, we plot $\rho_{11}^j(t)$ and $|\rho_{01}^j(t)|$ as functions of the dimensionless time γt

In Fig.5, the evolution of $\rho_{11}^j(t)$ and $|\rho_{01}^j(t)|$ is no longer monotone decreasing but fluctuate with different frequency, the frequency increases with the growth of J/γ. Fig.5 show that there are some cross points for $N = 3, 4$, while $N = 5, 6$ is not; Moreover, the cross points only occurs when $N \leq 4$, and disappear for $N \geq 5$.

For the case $N = 4$, the abscissa of cross points can be given by

$$\rho_{11}^1(t) = \rho_{11}^2(t) = \rho_{11}^3(t) = \rho_{11}^4(t). \quad (60)$$
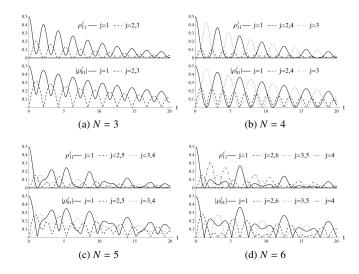

(a) $N = 3$ (b) $N = 4$ (c) $N = 5$ (d) $N = 6$

Fig. 5: $\rho_{11}^j(t)$ and $|\rho_{01}^j(t)|$ when $J = 2, \gamma = 1/220, a = |b| = 1/\sqrt{2}$.

The solution is given by

$$Jt_n^{\pm} = 2n\pi \pm \frac{\pi}{2} \quad (61)$$

Substituting the value of $t_n^{\pm}$ into Eq. (58), the density ρ at cross points is

$$\rho(t_n^{\pm}) = |\psi(t_n^{\pm})\rangle\langle\psi(t_n^{\pm})| + |b|^2\left(1 - e^{-2\gamma t_n^{\pm}}\right)|0^{\otimes 4}\rangle\langle0^{\otimes 4}|,$$

$$|\psi(t_n^{\pm})\rangle \equiv be^{-\gamma t_n}|\varphi^{\pm}\rangle + ae^{-4iJ_z t_n^{\pm}}|0^{\otimes 4}\rangle, \quad (62)$$

where $|\varphi^{\pm}\rangle$ is the W-states for 4 qubit

$$|\varphi^{\pm}\rangle = \frac{1}{2}\left(|1000\rangle \pm i|0100\rangle - |0010\rangle \pm i|0001\rangle\right). \quad (63)$$

That means $\rho(t_n^{\pm})$ consist of two parts, one is all spin down states $|0^{\otimes 4}\rangle$, the other is $|\psi(t_n^{\pm})\rangle$ a superposition state of W-states $|\varphi^{\pm}\rangle$ and $|0^{\otimes 4}\rangle$. $\rho(t_n^{\pm})$ reduced to a pure W-states $|\varphi^{\pm}\rangle\langle\varphi^{\pm}|$ for $b = 1, \gamma = 0$. Eq. (61) exhibit that the frequency of cross points $f_{cp} = J/\pi$ is independent of γ, while Eq. (61) show that γ only influence the exponential damping of both functions. All these imply that the occur of cross point and W-state should be attribute to the interact of Heisenberg model, γ only effects on dissipation process.

In order that a certain state occurs periodically in the system, the necessary and unsufficient condition is that the ratio of any two frequencies is a rational number. Eq. (44) show that the condition is satisfied for 2, 3, 4 and 6 qubits. For 6-qubits the $\rho_{11}^j(t)$ evolve decaying periodically with time, however no cross point exist namely 6-qubit W-states can't be created by this model. All the solution Eq. (58), W-states Eq. (63), and entanglement evolution still need further investigation. However, in present paper our primary purpose is to introduce our new Ket-Bra Entangled State method, so we shall research those problems in the near future.

IV. CONCLUSION

In this paper we present a new method (Sec. II) that map a master equation into a Schrödinger-like equation, so most procedure of Schrödinger equation can be used to solve the master equation. All master equation of finite dimension system can be resolved by this KBES method in theory. To solve master equation of N-level system, the calculation of N^2-order matrix's exponent is necessary (see Eq. (37)), whereas this tedious matrix operation can be finished by computer. For other special cases, the way of stationary Schördinger equation method may simplify the calculation effectively.

Through this method, we solve the model of a damped qubit in time-dependent external field and a qutrit coupled to reservoir, then we resolve N-qubit Heisenberg chain each coupled with reservoir at zero temperature, and preliminarily analyze the dissipation dynamics and decoherence dynamics find that W-states plays an important role in this process (see Sec.III.c). All these cases show that KBES method is a generalization and systematization method for solving master equation, which can greatly oversimplify the resolution of master equation.

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