Black String Perturbations in RS1 Model

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Abstract

We present a general formalism for black string perturbations in Randall-Sundrum 1 model (RS1). First, we derive the master equation for the electric part of the Weyl tensor $E_{\mu\nu}$. Solving the master equation using the gradient expansion method, we give the effective Teukolsky equation on the brane at low energy. It is useful to estimate gravitational waves emitted by perturbed rotating black strings. We also argue the effect of the Gregory-Laflamme instability on the brane using our formalism.

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I. INTRODUCTION

The existence of extra-dimensions is the most exciting prediction of the superstring theory. The evidence of extra dimensions should be sought in the early stage of the universe or in the features of gravitational waves from the black hole. We explore the latter possibility here.

In this paper, we concentrate on a two-brane model proposed by Randall and Sundrum as a simple model [1]. In this model, the black hole on the brane with the large gravitational radius compared to the curvature scale in the bulk can be considered as a section of a black string [2]. Hence, it is important to clarify how the gravitational waves are generated in the perturbed black string system and how the effects of the Gregory-Laflamme instability come into the observed signal of gravitational waves. For these purposes, we need a basic formalism for black string perturbations in RS1 model.

In general relativity, the perturbation around Kerr black hole is elegantly treated in the Newman-Penrose formalism [3]. Indeed, Teukolsky derived a separable master equation for the gravitational waves in the Kerr black hole background [4]. One purpose of this paper is to extend the Teukolsky formalism to the braneworld context and derive the effective Teukolsky equation [5].

When the horizon size of the black string gets close to the Compton wavelength of the lowest Kaluza-Klein mode, the Gregory-Laflamme instability commences. The other purpose of this paper is to explore this phenomena from the brane point of view.

The organization of this paper is as follows. In sec.II, we describe the black string in two-brane system and highlight the role of Weyl tensor $E_{\mu\nu}$ in deriving the effective Teukolsky equation. In sec.III, a master equation and junction conditions for $E_{\mu\nu}$ are obtained. In sec.IV, we solve the master equation at low energy using the gradient expansion method and give the explicit solution for $E_{\mu\nu}$. Then, the effective Teukolsky equation is obtained. In sec.V, we analyze the effect of the Gregory-Laflamme instability on the braneworld. The final section is devoted to the conclusion.

II.   BLACK STRING IN RS1 AND TEUKOLSKY EQUATION

Let us recall some basics of the black hole perturbations. Based on the Newmann-Penrose (NP) null-tetrad formalism, in which the tetrad components of the curvature tensor are the fundamental variables, a master equation for the curvature perturbation was developed by Teukolsky for a Kerr black hole with source. The master equation is called the Teukolsky equation, and it is a wave equation for a null-tetrad component of the Weyl tensor $\Psi_0 = -C_{pqrs}\ell^p m^q \ell^r m^s$ or $\Psi_4 = -C_{pqrs}n^p \bar{m}^q n^r \bar{m}^s$, where C_{pqrs} is the Weyl tensor and $\ell, n, m, \bar{m}$ are null basis in the NP formalism. All information about the gravitational radiation flux at infinity and at the event horizon can be extracted from Ψ_0 and Ψ_4. The Teukolsky equation is constructed by combining the Bianchi identity with the Einstein equations. The Riemann tensor in the Bianchi identity is written in terms of the Weyl tensor and the Ricci tensor. The Ricci tensor is replaced by the matter fields using the Einstein equations. In this way, the Bianchi identity becomes no longer identity and one can get a master equation in which the curvature tensor is the fundamental variable [6].

It is of interest to extend the Teukolsky equation to the braneworld. In the RS1 model, the two 3-branes are embedded in AdS_5 with the curvature radius ℓ and the brane tensions given by $\sigma_\oplus = 6/(\kappa^2 \ell)$ and $\sigma_\ominus = -6/(\kappa^2 \ell)$. Our system is described by the action

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-\overset{(5)}{g}} \left(\mathcal{R} + \frac{12}{\ell^2} \right) - \sum_{i=\oplus,\ominus} \sigma_i \int d^4x \sqrt{-g^{i\text{-brane}}} + \sum_{i=\oplus,\ominus} \int d^4x \sqrt{-g^{i\text{-brane}}} \mathcal{L}^i_{\text{matter}} \quad (1)$$

where $\overset{(5)}{g}_{\mu\nu}$, $\mathcal{R}$, $g^{i\text{-brane}}_{\mu\nu}$, and κ^2 are the 5-dimensional metric, the 5-dimensional scalar curvature, the induced metric on the i-brane, and the 5-dimensional gravitational constant, respectively. It is easy to show the metric

$$ds^2 = dy^2 + e^{-2y/\ell} g_{\mu\nu}(x) dx^\mu dx^\nu \quad (2)$$

is the exact vacuum solution of the two-brane system provided that the $g_{\mu\nu}$ satisfies the Ricci flat condition

$$R_{\mu\nu}(g) = 0 \ . \quad (3)$$

Thus, a black hole on the brane is expected to be a section of black string. In fact, Kerr metric leads to the exact solution for this model:

$$ds^2 = dy^2 + e^{-2\frac{y}{\ell}} \left[(1 - 2Mr/\Sigma)dt^2 + (4Mar\sin^2\theta/\Sigma)dtd\varphi - (\Sigma/\triangle)dr^2 - \Sigma d\theta^2 \right.$$
$$\left. - \sin^2\theta(r^2 + a^2 + 2Ma^2r\sin^2\theta/\Sigma)d\varphi^2) \right] \ , \quad (4)$$

3

where y is the coordinate of extra-dimension, M is the mass of the black hole, aM its angular momentum, $\Sigma = r^2 + a^2 \cos^2 \theta$ and $\triangle = r^2 - 2Mr + a^2$. When $a = 0$, the metric reduces to the Schwarzschild metric on the brane.

Since the Bianchi identity is independent of dimensions, what we need is the projected Einstein equations on the brane derived by Shiromizu, Maeda and Sasaki [7]. The first order perturbation of the projected Einstein equation is

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \delta E_{\mu\nu} \ , \tag{5}$$

where $8\pi G = \kappa^2/\ell$. If we replace the Ricci curvature in the Bianchi identity to the matter fields and $E_{\mu\nu}$ using Eq. (5), then the projected Teukolsky equation on the brane is written in the following form,

$$
\begin{aligned}
[(\triangle + 3\gamma & - \gamma^* + 4\mu + \mu^*)(D + 4\epsilon - \rho) - (\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi)(\delta - \tau + 4\beta) - 3\Psi_2]\delta\Psi_4 \\
= \frac{1}{2}&(\triangle + 3\gamma - \gamma^* + 4\mu + \mu^*)[(\delta^* - 2\tau^* + 2\alpha)(8\pi G T_{nm^*} - \delta E_{nm^*}) \\
& \qquad\qquad -(\triangle + 2\gamma - 2\gamma^* + \mu^*)(8\pi G T_{m^*m^*} - \delta E_{m^*m^*})] \\
+ \frac{1}{2}&(\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi)[(\triangle + 2\gamma + 2\mu^*)(8\pi G T_{nm^*} - \delta E_{nm^*}) \\
& \qquad\qquad -(\delta^* - \tau^* + 2\beta^* + 2\alpha)(8\pi G T_{nn} - \delta E_{nn})] \ . \tag{6}
\end{aligned}
$$

Here our notation follows that of [4]. We see the effects of a fifth dimension, $\delta E_{\mu\nu}$, is described as a source term in the projected Teukolsky equation. It should be stressed that the projected Teukolsky equation on the brane Eq. (6) is not a closed system yet. One must solve the gravitational field in the bulk to obtain $\delta E_{\mu\nu}$.

III. FORMALISM

In this section, we will derive the basic equations needed to analyze the gravitational waves emitted from the black string.

Firstly, we write down 5-dimensional Einstein equations and the junction conditions at the brane positions. Let n be a unit normal vector field to branes. Using the extrinsic

curvature $K_{\mu\nu} = -(1/2)\pounds_n g_{\mu\nu}$, 5-dimensional Einstein equations in the bulk become

$$-\frac{1}{2}\overset{(4)}{R} + \frac{1}{2}K^2 - \frac{1}{2}K^{\alpha\beta}K_{\alpha\beta} = \frac{6}{\ell^2} \ , \tag{7}$$

$$K_\mu{}^\lambda{}_{|\lambda} - K_{|\mu} = 0 \ , \tag{8}$$

$$\overset{(4)}{G}{}^\mu{}_\nu = -E^\mu{}_\nu + \frac{3}{\ell^2}\delta^\mu_\nu - K^{\mu\alpha}K_{\alpha\nu} + KK^\mu{}_\nu + \frac{1}{2}\delta^\mu_\nu\left(K^{\alpha\beta}K_{\alpha\beta} - K^2\right) \ , \tag{9}$$

where "the electric part" of the Weyl tensor

$$E_{\mu\nu} = \pounds_n K_{\mu\nu} + K_\mu{}^\alpha K_{\alpha\nu} + \frac{1}{\ell^2}g_{\mu\nu} \tag{10}$$

is defined. Here, (4) represents the 4-dimensional quantity and | is the covariant derivative with respect to the metric $g_{\mu\nu}(y, x^\mu)$. As the branes act as the singular sources, we also have the junction conditions

$$\left[K^\mu{}_\nu - \delta^\mu_\nu K\right]\Big|_{\oplus} = \frac{\kappa^2}{2}\left(-\overset{\oplus}{\sigma}\delta^\mu_\nu + \overset{\oplus}{T}{}^\mu{}_\nu\right) \ , \tag{11}$$

$$\left[K^\mu{}_\nu - \delta^\mu_\nu K\right]\Big|_{\ominus} = -\frac{\kappa^2}{2}\left(-\overset{\ominus}{\sigma}\delta^\mu_\nu + \overset{\ominus}{T}{}^\mu{}_\nu\right) \ . \tag{12}$$

To obtain a master equation for $\delta E_{\mu\nu}$, we start with the 5-dimensional Bianchi identities. Using the Gauss equation and the Codacci equation, we obtain

$$\pounds_n B_{\mu\nu\lambda} + E_{\mu\nu|\lambda} - E_{\mu\lambda|\nu} + K^\alpha{}_\mu B_{\alpha\nu\lambda} + K^\alpha{}_\lambda B_{\nu\alpha\mu} - K^\alpha{}_\nu B_{\lambda\alpha\mu} = 0 \ , \tag{13}$$

$$B_{\mu[\nu\lambda|\rho]} + K^\sigma{}_{[\rho}\overset{(4)}{R}{}_{\nu\lambda]\sigma\mu} = 0 \ , \tag{14}$$

$$\pounds_n \overset{(4)}{R}{}_{\mu\nu\lambda\rho} + K^\alpha{}_\mu \overset{(4)}{R}{}_{\alpha\nu\lambda\rho} - K^\alpha{}_\nu \overset{(4)}{R}{}_{\alpha\mu\lambda\rho} + B_{\lambda\mu\nu|\rho} - B_{\rho\mu\nu|\lambda} = 0 \ , \tag{15}$$

$$\overset{(4)}{R}{}_{\mu\nu[\lambda\rho|\sigma]} = 0 \ , \tag{16}$$

where we have defined "the magnetic part" of the Weyl tensor

$$B_{\mu\nu\lambda} = K_{\mu\lambda|\nu} - K_{\mu\nu|\lambda} \ . \tag{17}$$

After combining (7) with (9) and putting the result into Eq. (15), we have

$$\pounds_n E_{\mu\nu} = B_{(\mu\nu)\rho}{}^{|\rho} - \Sigma^{\alpha\beta}\overset{(4)}{C}{}_{\mu\alpha\nu\beta} + \frac{1}{2}g_{\mu\nu}E_{\alpha\beta}\Sigma^{\alpha\beta} - 2(\Sigma^\alpha{}_\mu E_{\alpha\nu} + \Sigma^\alpha{}_\nu E_{\alpha\mu})$$
$$+ \frac{1}{2}KE_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Sigma_\alpha{}^\beta \Sigma_\beta{}^\gamma \Sigma_\gamma{}^\alpha - 2\Sigma_\mu{}^\alpha \Sigma_\alpha{}^\beta \Sigma_{\beta\nu} + \frac{7}{6}\Sigma_{\mu\nu}\Sigma^\alpha{}_\beta \Sigma^\beta{}_\alpha \ , \tag{18}$$

where we decomposed the extrinsic curvature into the traceless part and the trace part

$$K_{\mu\nu} = \Sigma_{\mu\nu} + \frac{1}{4}g_{\mu\nu}K \ . \tag{19}$$

Now we consider the perturbation of these equations. We generalize the background metric Eq. (4) to a Ricci flat string without source ($T_{\mu\nu} = 0$) whose metric is written as

$$ds^2 = dy^2 + e^{-2\frac{y}{\ell}}g_{\mu\nu}(x^\mu)dx^\mu dx^\nu , \tag{20}$$

where $g_{\mu\nu}(x^\mu)$ is supposed to be the Ricci flat metric. In the case of Kerr metric, the metric (20) represents the rotating black string [8]. The variables $K_{\mu\nu}$, $E_{\mu\nu}$ and $B_{\mu\nu}$ for this background are given by

$$K_{\mu\nu} = \frac{1}{\ell}g_{\mu\nu} , \qquad E_{\mu\nu} = B_{\mu\nu\lambda} = 0 . \tag{21}$$

Linearizing Eq. (18) around this background, we obtain

$$\delta E_{\mu\nu,y} = \delta B_{(\mu\nu)\rho}^{\ \ |\rho} - \overset{(4)}{C}_{\mu\alpha\nu\beta}\delta\Sigma^{\alpha\beta} + \frac{2}{\ell}\delta E_{\mu\nu} . \tag{22}$$

Similarly, Eq. (13) reduces to

$$\delta B_{(\mu\nu)\alpha}^{\ \ \ |\alpha} = -\delta E_{\mu\nu|\lambda} + \delta E_{\mu\lambda|\nu} . \tag{23}$$

Using the following relation

$$\left(\delta B_{(\mu\nu)\lambda,y}\right)^{|\lambda} = \left(\delta B_{(\mu\nu)\lambda}^{\ \ \ |\lambda}\right)_{,y} - \frac{2}{\ell}\delta B_{(\mu\nu)\lambda}^{\ \ \ |\lambda} , \tag{24}$$

we can eliminate $B_{\mu\nu\lambda}$ from Eqs. (22) and (23). Thus, the master equation for $\delta E_{\mu\nu}$ is found as

$$\left(\partial_y^2 - \frac{4}{\ell}\partial_y + \frac{4}{\ell^2}\right)\delta E_{\mu\nu} = -e^{2\frac{y}{\ell}}\hat{\mathcal{L}}_{\mu\nu}^{\ \ \alpha\beta}\delta E_{\alpha\beta}$$
$$\equiv -e^{2\frac{y}{\ell}}\hat{\mathcal{L}}\delta E_{\mu\nu} , \tag{25}$$

where $\hat{\mathcal{L}}_{\mu\nu}^{\ \ \alpha\beta} = \Box\delta_\mu^\alpha\delta_\nu^\beta + 2R_\mu^{\ \alpha}{}_\nu^{\ \beta}$ stands for the Lichnerowicz operator. Here, the covariant derivative and the Riemann tensor are constructed from $g_{\mu\nu}(x)$.

In order to deduce the junction conditions for $\delta E_{\mu\nu}$ from Eqs. (11) and (12), we use Eq. (22) and

$$\delta B_{\mu\nu\rho} = \delta K_{\mu\rho|\nu} - \delta K_{\mu\nu|\rho} . \tag{26}$$

As a result, the junction conditions on each branes become

$$e^{2\frac{y}{\ell}}\left[e^{-2\frac{y}{\ell}}\delta E_{\mu\nu}\right]_{,y}\Bigg|_{y=0} = -\frac{\kappa^2}{6}\overset{\oplus}{T}_{|\mu\nu} - \frac{\kappa^2}{2}\hat{\mathcal{L}}_{\mu\nu}^{\ \ \alpha\beta}\left(\overset{\oplus}{T}_{\alpha\beta} - \frac{1}{3}g_{\alpha\beta}\overset{\oplus}{T}\right) , \tag{27}$$

$$e^{2\frac{y}{\ell}}\left[e^{-2\frac{y}{\ell}}\delta E_{\mu\nu}\right]_{,y}\Bigg|_{y=d} = \frac{\kappa^2}{6}\overset{\ominus}{T}_{|\mu\nu} + \frac{\kappa^2}{2}\hat{\mathcal{L}}_{\mu\nu}^{\ \ \alpha\beta}\left(\overset{\ominus}{T}_{\alpha\beta} - \frac{1}{3}g_{\alpha\beta}\overset{\ominus}{T}\right) . \tag{28}$$

Let $\overset{\oplus}{\phi}(x)$ and $\overset{\ominus}{\phi}(x)$ be the scalar fields on each branes which satisfy

$$\Box \overset{\oplus}{\phi} = \frac{\kappa^2}{6} \overset{\oplus}{T} , \tag{29}$$

$$\Box \overset{\ominus}{\phi} = \frac{\kappa^2}{6} \overset{\ominus}{T} , \tag{30}$$

respectively [9]. In the Ricci flat space-time, the identity

$$(\Box \phi)_{|\mu\nu} = \hat{\mathcal{L}}_{\mu\nu}{}^{\alpha\beta} \left(\phi_{|\alpha\beta} \right) \tag{31}$$

holds. Thus, Eqs.(27) and (28) can be rewritten as

$$e^{2\frac{y}{\ell}} \left[e^{-2\frac{y}{\ell}} \delta E_{\mu\nu} \right]_{,y} \bigg|_{y=0} = -\hat{\mathcal{L}} \left(\overset{\oplus}{\phi}_{|\mu\nu} - g_{\mu\nu} \Box \overset{\oplus}{\phi} \right) - \frac{\kappa^2}{2} \hat{\mathcal{L}} \overset{\oplus}{T}_{\mu\nu} \equiv \hat{\mathcal{L}} \overset{\oplus}{S}_{\mu\nu} , \tag{32}$$

$$e^{2\frac{y}{\ell}} \left[e^{-2\frac{y}{\ell}} \delta E_{\mu\nu} \right]_{,y} \bigg|_{y=d} = \hat{\mathcal{L}} \left(\overset{\ominus}{\phi}_{|\alpha\beta} - g_{\mu\nu} \Box \overset{\ominus}{\phi} \right) + \frac{\kappa^2}{2} \hat{\mathcal{L}} \overset{\ominus}{T}_{\mu\nu} \equiv -\hat{\mathcal{L}} \overset{\ominus}{S}_{\mu\nu} . \tag{33}$$

The scalar fields $\overset{\oplus}{\phi}$ and $\overset{\ominus}{\phi}$ can be interpreted as the brane fluctuation modes.

IV.   EFFECTIVE TEUKOLSKY EQUATION AT LOW ENERGY

A.   Gradient Expansion Method

It is known that the Gregory-Laflamme instability occurs if the curvature length scale of the black hole L is less than the Compton wavelength of Kaluza-Klein (KK) modes $\sim \ell \exp(d/\ell)$ [10]. As we are interested in the stable rotating black string,

$$\epsilon = \left(\frac{\ell}{L} \right)^2 \ll 1 \tag{34}$$

is assumed. This means that the curvature on the brane can be neglected compared with the derivative with respect to y. Our iteration scheme consists in writing the Weyl tensor $E_{\mu\nu}$ in the order of ϵ [11]. Hence, we will seek the Weyl tensor as a perturbative series

$$\delta E_{\mu\nu}(y, x^\mu) = \delta \overset{(1)}{E}_{\mu\nu}(y, x^\mu) + \delta \overset{(2)}{E}_{\mu\nu}(y, x^\mu) + \delta \overset{(3)}{E}_{\mu\nu}(y, x^\mu) + \cdots . \tag{35}$$

1.   First order

At first order, we can neglect the Lichnerowicz operator term. Then Eq. (25) become

$$\left(\partial_y^2 - \frac{4}{\ell} \partial_y + \frac{4}{\ell^2} \right) \delta \overset{(1)}{E}_{\mu\nu} = 0 , \tag{36}$$

7

where the superscript (1) represents the order of the derivative expansion. This can be readily integrated into

$$\delta \overset{(1)}{E}_{\mu\nu} = e^{2\frac{y}{\ell}} \left(\overset{(1)}{C}_{\mu\nu} \frac{y}{\ell} + \overset{(1)}{\chi}_{\mu\nu} \right) , \tag{37}$$

where $\overset{(1)}{C}_{\mu\nu}$ and $\overset{(1)}{\chi}_{\mu\nu}$ are the constants of integration which depend only on x^μ and satisfy the transverse $\overset{(1)}{C}{}^\mu{}_{\nu|\mu} = \overset{(1)}{\chi}{}^\mu{}_{\nu|\mu} = 0$ and traceless $\overset{(1)}{C}{}^\mu{}_\mu = \overset{(1)}{\chi}{}^\mu{}_\mu = 0$ constraints. The junction conditions on each branes at this order are

$$e^{2\frac{y}{\ell}} \left[e^{-2\frac{y}{\ell}} \delta \overset{(1)}{E}_{\mu\nu} \right]_{,y} \bigg|_{y=0,d} = 0 . \tag{38}$$

Imposing this junction condition Eq. (38) on the solution (37), we see $\overset{(1)}{C}_{\mu\nu} = 0$. Thus, we get the first order Weyl tensor

$$\delta \overset{(1)}{E}_{\mu\nu} = e^{2\frac{y}{\ell}} \overset{(1)}{\chi}_{\mu\nu}(x) , \tag{39}$$

where $\overset{(1)}{\chi}_{\mu\nu}(x)$ is arbitrary at this order. This should be determined from the next order analysis.

2. Second order

The next order solutions are obtained by taking into account the terms neglected at first order. At second order, Eq. (25) becomes

$$\left(\partial_y^2 - \frac{4}{\ell} \partial_y + \frac{4}{\ell^2} \right) \delta \overset{(2)}{E}_{\mu\nu} = -e^{2\frac{y}{\ell}} \hat{\mathcal{L}} \delta \overset{(1)}{E}_{\mu\nu} . \tag{40}$$

Substituting the first order $\overset{(1)}{E}_{\alpha\beta}$ into the right hand side of Eq. (40), we obtain

$$\delta \overset{(2)}{E}_{\mu\nu} = e^{2\frac{y}{\ell}} \left(\overset{(2)}{C}_{\mu\nu} \frac{y}{\ell} + \overset{(2)}{\chi}_{\mu\nu} \right) - \frac{\ell^2}{4} e^{4\frac{y}{\ell}} \hat{\mathcal{L}} \overset{(1)}{\chi}_{\mu\nu} , \tag{41}$$

where $\overset{(2)}{C}_{\mu\nu}$ and $\overset{(2)}{\chi}_{\mu\nu}$ are again the constants of integration at this order and satisfy the transverse and traceless constraint ($\overset{(2)}{\chi}{}^\mu{}_\mu = \overset{(2)}{\chi}{}^\mu{}_{\nu|\mu} = 0$, etc.). The junction conditions at this order give

$$\left[e^{-2\frac{y}{\ell}} \delta \overset{(2)}{E}_{\mu\nu} \right]_{,y} \bigg|_{y=0} = \hat{\mathcal{L}} \hat{S}{}^\oplus_{\mu\nu}{}^{(1)} , \tag{42}$$

$$\left[e^{-2\frac{y}{\ell}} \delta \overset{(2)}{E}_{\mu\nu} \right]_{,y} \bigg|_{y=d} = -\Omega^2 \hat{\mathcal{L}} \hat{S}{}^\ominus_{\mu\nu}{}^{(1)} . \tag{43}$$

8

Here $\Omega^2 = \exp[-2\frac{d}{\ell}]$ is a conformal factor that relates the metric on the $\oplus$-brane to that on the $\ominus$-brane [11]. Substituting Eq. (41) into the above junction conditions, we get

$$\frac{1}{\ell}\overset{(2)}{C}_{\mu\nu} - \frac{\ell}{2}\hat{\mathcal{L}}\overset{(1)}{\chi}_{\mu\nu} = \hat{\mathcal{L}}\overset{\oplus}{S}_{\mu\nu}^{(1)} , \tag{44}$$

$$\frac{1}{\ell}\overset{(2)}{C}_{\mu\nu} - \frac{\ell}{2}\frac{1}{\Omega^2}\hat{\mathcal{L}}\overset{(1)}{\chi}_{\mu\nu} = -\Omega^2\hat{\mathcal{L}}\overset{\ominus}{S}_{\mu\nu}^{(1)} . \tag{45}$$

Eliminating $\overset{(1)}{\chi}_{\alpha\beta}$ from these equations, we obtain one of the constants of integration

$$\overset{(2)}{C}_{\mu\nu} = -\frac{\ell}{1-\Omega^2}\left(\hat{\mathcal{L}}\overset{\oplus}{S}_{\mu\nu}^{(1)} + \Omega^4\hat{\mathcal{L}}\overset{\ominus}{S}_{\mu\nu}^{(1)}\right) . \tag{46}$$

Similarly, eliminating $\overset{(2)}{C}_{\mu\nu}$ from Eqs.(44) and (45), we obtain the equation

$$\hat{\mathcal{L}}\overset{(1)}{\chi}_{\mu\nu} = \frac{2}{\ell}\frac{\Omega^2}{1-\Omega^2}\left(\hat{\mathcal{L}}\overset{\oplus}{S}_{\mu\nu}^{(1)} + \Omega^2\hat{\mathcal{L}}\overset{\ominus}{S}_{\mu\nu}^{(1)}\right) , \tag{47}$$

which is easily integrated as

$$\overset{(1)}{\chi}_{\mu\nu} = \frac{2}{\ell}\frac{\Omega^2}{1-\Omega^2}\left(\overset{\oplus}{S}_{\mu\nu}^{(1)} + \Omega^2\overset{\ominus}{S}_{\mu\nu}^{(1)}\right) . \tag{48}$$

Comparing Eq.(46) with the analysis for the perturbations around the flat two-brane background, we see the above result corresponds to the zero mode contribution [11]. Hence, the KK corrections come from the second order corrections which are not yet determined.

3. Third order

In order to obtain KK corrections, we need to fix $\overset{(2)}{\chi}_{\mu\nu}$. For that purpose, we must proceed to third order analysis. At third order, we have the solution

$$\delta\overset{(3)}{E}_{\mu\nu} = e^{2\frac{y}{\ell}}\left(\overset{(3)}{C}_{\mu\nu}\frac{y}{\ell} + \overset{(3)}{\chi}_{\mu\nu}\right) - \frac{\ell^2}{4}e^{4\frac{y}{\ell}}\hat{\mathcal{L}}\overset{(2)}{\chi}_{\mu\nu} - \frac{\ell}{4}(y-\ell)e^{4\frac{y}{\ell}}\hat{\mathcal{L}}\overset{(2)}{C}_{\mu\nu} + \frac{\ell^4}{64}e^{6\frac{y}{\ell}}\hat{\mathcal{L}}^2\overset{(1)}{\chi}_{\mu\nu} , \tag{49}$$

where $\overset{(3)}{C}_{\mu\nu}$ and $\overset{(3)}{\chi}_{\mu\nu}$ are the constants of integration at this order. Junction conditions yield

$$\frac{\ell^3}{16}\hat{\mathcal{L}}^2\overset{(1)}{\chi}_{\mu\nu} + \frac{\ell}{4}\hat{\mathcal{L}}\overset{(2)}{C}_{\mu\nu} - \frac{\ell}{2}\hat{\mathcal{L}}\overset{(2)}{\chi}_{\mu\nu} + \frac{1}{\ell}\overset{(3)}{C}_{\mu\nu} = \hat{\mathcal{L}}\overset{\oplus}{S}_{\mu\nu} , \tag{50}$$

$$-\frac{1}{2\Omega^2}(d-\frac{\ell}{2})\hat{\mathcal{L}}\overset{(2)}{C}_{\mu\nu} + \frac{\ell^3}{16\Omega^4}\hat{\mathcal{L}}^2\overset{(1)}{\chi}_{\mu\nu} - \frac{\ell}{2\Omega^2}\hat{\mathcal{L}}\overset{(2)}{\chi}_{\mu\nu} + \frac{1}{\ell}\overset{(3)}{C}_{\mu\nu} = -\Omega^2\hat{\mathcal{L}}\overset{\ominus}{S}_{\mu\nu} . \tag{51}$$

9

We get $\overset{(3)}{C}_{\mu\nu}$ from above equations as

$$\overset{(3)}{C}_{\mu\nu} = -\frac{\ell d}{2}\frac{1}{1-\Omega^2}\hat{\mathcal{L}}\overset{(2)}{C}_{\mu\nu} + \frac{\ell^4}{16}\frac{1}{\Omega^2}\hat{\mathcal{L}}^2\overset{(1)}{\chi}_{\mu\nu} + \frac{\ell}{1-\Omega^2}\left[\hat{\mathcal{L}}\overset{\oplus}{S}^{(2)}_{\mu\nu} + \Omega^4\hat{\mathcal{L}}\overset{\ominus}{S}^{(2)}_{\mu\nu}\right] \tag{52}$$

and

$$\hat{\mathcal{L}}\overset{(2)}{\chi}_{\mu\nu} = \left[\frac{1}{2} + \frac{d}{\ell}\frac{1}{\Omega^2-1}\right]\hat{\mathcal{L}}\overset{(2)}{C}_{\mu\nu} + \frac{\ell^2}{8}\left(1+\frac{1}{\Omega^2}\right)\hat{\mathcal{L}}^2\overset{(1)}{\chi}_{\mu\nu} + \frac{2}{\ell}\frac{\Omega^2}{1-\Omega^2}\left[\hat{\mathcal{L}}\overset{\oplus}{S}^{(2)}_{\mu\nu} + \Omega^2\hat{\mathcal{L}}\overset{\ominus}{S}^{(2)}_{\mu\nu}\right] \tag{53}$$

It is easy to obtain

$$\overset{(2)}{\chi}_{\mu\nu} = \left[\frac{1}{2} + \frac{d}{\ell}\frac{1}{\Omega^2-1}\right]\overset{(2)}{C}_{\mu\nu} + \frac{\ell^2}{8}\left(1+\frac{1}{\Omega^2}\right)\hat{\mathcal{L}}\overset{(1)}{\chi}_{\mu\nu} + \frac{2}{\ell}\frac{\Omega^2}{1-\Omega^2}\left[\overset{\oplus}{S}^{(2)}_{\mu\nu} + \Omega^2\overset{\ominus}{S}^{(2)}_{\mu\nu}\right] \ . \tag{54}$$

Thus, we have obtained KK corrections. In principle, we can continue this perturbative calculations to any order.

B. Effective Teukolsky Equation

Schematically, Teukolsky equation (9) takes the following form

$$\hat{P}\delta\Psi_4 = \hat{Q}\left(8\pi G T_{nm^*} - \delta E_{nm^*}\right) + \cdots \ , \tag{55}$$

where $\hat{P}$ and $\hat{Q}$ are the operators in Eq.(9). What we needed is $\delta E_{\mu\nu}$ in the above equation. Now, we can write down $\delta E_{\mu\nu}$ on the brane up to the second order as

$$\delta E_{\mu\nu}\Big|_{y=0} = \frac{2}{\ell}\frac{\Omega^2}{1-\Omega^2}\left[\overset{\oplus}{S}_{\mu\nu} + \Omega^2\overset{\ominus}{S}_{\mu\nu}\right] + \left[\frac{1}{2}+\frac{d}{\ell}\frac{1}{\Omega^2-1}\right]\frac{\ell}{1-\Omega^2}\left[\hat{\mathcal{L}}\overset{\oplus}{S}_{\mu\nu} + \Omega^4\hat{\mathcal{L}}\overset{\ominus}{S}_{\mu\nu}\right]$$
$$+\frac{\ell}{4}\left[\hat{\mathcal{L}}\overset{\oplus}{S}_{\mu\nu} + \Omega^2\hat{\mathcal{L}}\overset{\ominus}{S}_{\mu\nu}\right] \ , \tag{56}$$

where

$$\overset{\oplus}{S}_{\mu\nu} = -\overset{\oplus}{\phi}_{|\mu\nu} + g_{\mu\nu}\Box\overset{\oplus}{\phi} - \frac{\kappa^2}{2}\overset{\oplus}{T}_{\mu\nu} \ , \tag{57}$$

$$\overset{\ominus}{S}_{\mu\nu} = -\overset{\ominus}{\phi}_{|\mu\nu} + g_{\mu\nu}\Box\overset{\ominus}{\phi} - \frac{\kappa^2}{2}\overset{\ominus}{T}_{\mu\nu} \ . \tag{58}$$

Substituting this $\delta E_{\mu\nu}$ into (6), we get the effective Teukolsky equation on the brane. From Eq. (56), we see KK corrections give extra sources to Teukolsky equation. To obtain quantitative results, we must resort to numerical calculations. It should be stressed that the effective Teukolsky equation is separable like as the conventional Teukolsky equation. Therefore, it is suitable for numerical treatment.

V. GREGORY-LAFLAMME INSTABILITY:VIEW FROM THE BRANE

It is possible to apply our master equation (25) to the transition phenomena from black string to black hole. Interestingly, our master equation coincides with the 5-dimensional Einstein equations in the harmonic gauge. Hence, it must exhibit the Gregory-Laflamme instability. It would be interesting to see this instability from the brane point of view.

The Gregory-Laflamme instability takes place in the regime $\epsilon \approx 1$ where the low energy gradient expansion method cannot be applied. Hence, we will examine qualitative features of this phenomena through a formal Green function representation of the solution. To solve equation for $\delta E_{\mu\nu}$, we introduce the Green function

$$\left[\left(\partial_y^2 - \frac{4}{\ell}\partial_y + \frac{4}{\ell^2}\right)\delta_\mu^\alpha\delta_\nu^\beta + e^{2\frac{y}{\ell}}\hat{\mathcal{L}}_{\mu\nu}{}^{\alpha\beta}\right]G_{\alpha\beta}{}^{\lambda\rho}(x,y;x',y')$$
$$= -\frac{e^{4\frac{y}{\ell}}}{\sqrt{-g}}\delta^4(x-x')\delta(y-y')\left(\delta_\mu^{(\alpha}\delta_\nu^{\rho)} - \frac{1}{4}g_{\mu\nu}g^{\lambda\rho}\right) \quad (59)$$

with the boundary conditions

$$\partial_y\left[e^{-2\frac{y}{\ell}}G_{\mu\nu}{}^{\alpha\beta}\right]\bigg|_{y=0,d} = 0 . \quad (60)$$

Then, the formal solution is given by

$$\delta E_{\mu\nu}(x,y) = \int d^4x'\sqrt{-g}\left[e^{-2\frac{y}{\ell}}G_{\mu\nu}{}^{\lambda\rho}\partial_y\left(e^{-2\frac{y}{\ell}}\delta E_{\lambda\rho}\right)\right]\bigg|_{y=0}^{y=d} . \quad (61)$$

Using junction conditions (32) and (33), we obtain

$$\delta E_{\mu\nu}(x',y') = \int d^4x\sqrt{-g}\left[G_{\mu\nu}{}^{\lambda\rho}(x,d;x',y')\Omega^4\hat{\mathcal{L}}\overset{\ominus}{S}_{\lambda\rho} + G_{\mu\nu}{}^{\lambda\rho}(x,0;x',y')\hat{\mathcal{L}}\overset{\oplus}{S}_{\lambda\rho}\right] , \quad (62)$$

which can be rewritten as

$$\delta E_{\mu\nu}(x',y') = \int d^4x\sqrt{-g}\left[\hat{\mathcal{L}}_{\lambda\rho}{}^{\alpha\beta}G_{\mu\nu}{}^{\lambda\rho}(x,d;x',y')\Omega^4\overset{\ominus}{S}_{\lambda\rho} + \hat{\mathcal{L}}_{\lambda\rho}{}^{\alpha\beta}G_{\mu\nu}{}^{\lambda\rho}(x,0;x',y')\overset{\oplus}{S}_{\lambda\rho}\right] \quad (63)$$

where we dropped the surface term. If we use the 4-dimensional Green function

$$\left[\hat{\mathcal{L}}_{\mu\nu}{}^{\alpha\beta} - \lambda_n^2\delta_\mu^\alpha\delta_\nu^\beta\right]\overset{(4)}{G}(x,x';\lambda_n^2)_{\alpha\beta}{}^{\lambda\rho} = -\frac{\delta(x-x')}{\sqrt{-g}}\delta_\mu^\lambda\delta_\nu^\rho , \quad (64)$$

the Green function can be expressed as

$$G_{\mu\nu}{}^{\alpha\beta}(x,y;x',y') = \sum_n \varphi_n(y)\varphi_n(y')\overset{(4)}{G}(x,x';\lambda_n^2)_{\mu\nu}{}^{\alpha\beta} . \quad (65)$$

Here, a mode function satisfies

$$e^{-2\frac{y}{\ell}} \left(\partial_y - \frac{2}{\ell} \right)^2 \varphi_n(y) = -\lambda_n^2 \varphi_n(y) \ . \tag{66}$$

where λ_n^2 is an eigenvalue. The solution is given by

$$\varphi_n(y) = N_n e^{2\frac{y}{\ell}} \left[J_0(\lambda_n \ell e^{\frac{y}{\ell}}) - \frac{J_1(\lambda_n \ell)}{N_1(\lambda_n \ell)} N_0(\lambda_n \ell e^{\frac{y}{\ell}}) \right] \ , \tag{67}$$

where N_n is a normalization constant determined by

$$\int dy e^{-2\frac{y}{\ell}} \varphi_n(y) \varphi_m(y) = \delta_{nm} \ . \tag{68}$$

The junction condition for the negative tension brane gives a KK-spectrum as

$$J_1(\lambda_n \ell e^{\frac{d}{\ell}}) - \frac{J_1(\lambda_n \ell)}{N_1(\lambda_n \ell)} N_1(\lambda_n \ell e^{\frac{d}{\ell}}) = 0 \ . \tag{69}$$

Substituting Eq. (65) into (71), we obtain

$$\delta E_{\mu\nu}(x', y') = \int d^4 x \sqrt{-g} \sum_n \left[\varphi_n(d)\varphi_n(y')\Omega^4 \hat{\mathcal{L}}_{\lambda\rho}{}^{\alpha\beta} \overset{(4)}{G}(x, x')_{\mu\nu}{}^{\lambda\rho} \overset{\ominus}{S}_{\alpha\beta} \right.$$

$$\left. + \varphi_n(0)\varphi_n(y') \hat{\mathcal{L}}_{\lambda\rho}{}^{\alpha\beta} \overset{(4)}{G}(x, x')_{\mu\nu}{}^{\lambda\rho} \overset{\oplus}{S}_{\alpha\beta} \right] \ . \tag{70}$$

Using Eq. (64), we can write $\delta E_{\mu\nu}$ as

$$\delta E_{\mu\nu}(x', y') = -\sum_n \varphi_n(d)\varphi_n(y')\Omega^4 \overset{\ominus}{S}_{\alpha\beta} - \sum_n \varphi_n(0)\varphi_n(y') \overset{\oplus}{S}_{\alpha\beta}$$

$$+ \int d^4 x \sqrt{-g} \sum_n \lambda_n^2 \overset{(4)}{G}(x, x')_{\mu\nu}{}^{\lambda\rho} \left[\Omega^4 \varphi_n(d)\varphi_n(y') \overset{\ominus}{S}_{\lambda\rho} + \varphi_n(0)\varphi_n(y') \overset{\oplus}{S}_{\lambda\rho} \right] \ . \tag{71}$$

Let us introduce a new field, which corresponds to a 4-dimensional massive graviton (KK mode),

$$\hat{\mathcal{L}}_{\mu\nu}{}^{\alpha\beta} \psi_{\alpha\beta}^n = \mu_n^2 \psi_{\mu\nu}^n \tag{72}$$

where μ_n^2 is an eigenvalue. Then, the 4-dimensional Green function is formally written as

$$\overset{(4)}{G}(x, x'; \lambda_m^2)_{\alpha\beta}{}^{\lambda\rho} = \sum_n \frac{\psi_{\alpha\beta}^n(x)\psi_n^{\lambda\rho}(x')}{\lambda_m^2 - \mu_n^2} \tag{73}$$

Note that it is known that this massive graviton causes the Gregory-Laflamme instability. Therefore, $\delta E_{\mu\nu}$ also becomes unstable. This implies that there is a possibility that there exists the scalar wave emition from the rotating black string due to the Gregory-Laflamme instability.

12

VI. CONCLUSION

We have formulated the black string perturbation theory around the Ricci flat two-brane system. In particular, the master equation for $\delta E_{\mu\nu}$ is derived. At low energy, the gradient expansion method is utilized to get a series solution. This gives the closed system of equations, i.e. the effective Teukolsky equations. This can be used for estimating the Kaluza-Klein corrections on the gravitational waves emitted from the perturbed rotating black string. Our effective Teukolsky equation is completely separable, hence the numerical scheme can be developed in a similar manner as was done in the case of 4-dimensional Teukolsky equation [12, 13]. We have also applied our formalism to the Gregory-Laflamme instability. We argued the possibility of scalar wave emition from the black string due to the Gregory-Laflamme instability.

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