Existence of solutions of the master equation in the smooth case

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Abstract

We give a different proof of a theorem of W. Gangbo and A. Swiech on the short time existence of solutions of the master equation.

Introduction

Mean Field Games are games with a continuum of players, each of which sees only the "mean field" generated by the other ones. They attracted the attention of a wider set of analysts after the lectures of P. L. Lions at the Collège de France, which are available in video streaming (see also the written presentation [11]). They can model a wide array of phenomena in physics and mathematical economics; we dwell a little on one aspect of the latter. Actually, the idea of considering a continuum of players came up naturally in mathematical economy, where it was used ([6], see also [14] for a more elementary presentation) to model the formation of prices in a market with perfect concurrence. Quoting from [6], "the essential idea of this notion is that the economy under consideration has a "very large" number of participants, and that the influence of each participant is "negligible"".

To be more precise, let us look at the situation of [15]: we have a probability measure μ_s on the d-dimensional torus $\mathbf{T}^d = \frac{\mathbf{R}^d}{\mathbf{Z}^d}$ which models the distribution of the players at time s; we fix an initial time t < 0, an initial distribution $\bar{\mu}$ and we suppose that μ_s evolves according to the continuity equation, forward in time,

$$\begin{cases} \partial_s \mu_s + \operatorname{div}(X\mu_s) = 0 & s > t \\ \mu_t = \bar{\mu} \end{cases}$$
 (1)

where the vector field X is a control which we are free to choose in the following.

Let us call $\mathcal{P}(\mathbf{T}^d)$ the space of the Borel probability measures on \mathbf{T}^d , and let us suppose that we are given two potentials $\mathcal{F}, \mathcal{U}_0: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$. We would like the whole society to minimize the value function

$$\mathcal{V}(t,\bar{\mu}) := \inf \left\{ \int_t^0 \mathrm{d}s \left[\int_{\mathbf{T}^d} \frac{1}{2} |X^2(s,x)|^2 \mathrm{d}\mu_s(x) - \mathcal{F}(\mu_s) \right] + \mathcal{U}_0(\mu_0) \right\}$$
 (2)

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where the inf is over all curves which satisfy (1) and all controls X. It turns out that under suitable hypotheses on \mathcal{F} and \mathcal{U}_0 the inf is a minimum: there is a vector field X minimizing in (2); by (1), we also have a minimal trajectory μ_s .

In (2), we minimize the cost for the whole society, but what about its members? One possible notion is that of Nash equilibrium: roughly, we are on a Nash equilibrium if no one can get a better deal by a unilateral change of strategy. It happens that, in our case, the optimum for the whole society is a Nash equilibrium. Actually, under suitable hypotheses on \mathcal{F} and \mathcal{U}_0 , we shall be able to define two functions $F(x,\mu)$ and $u_0(x,\mu)$ which, heuristically, are the "mean field" potentials felt by the particle placed at x, provided the other ones are distributed as μ . We shall see that the drift X in (1) optimal for the whole group is also best for the single particle; namely, $X(s,q) = -\partial_x v(s,q)$ where v solves the Hamilton-Jacobi equation with time reversed

$$\begin{cases}
-\partial_t v(s,q) + \frac{1}{2} |\partial_q v(s,q)|^2 + F(q,\mu_s) = 0 \quad s \le 0 \\
v(0,q) = u_0(q,\mu_0).
\end{cases}$$
(3)

Equivalently, the particle initially placed at q minimizes its cost:

$$\int_{t}^{0} \frac{1}{2} [|\dot{q}(s)|^{2} + F(q(s), \mu_{s})] ds + u_{0}(q(0), \mu_{0})$$

if it follows the vector field X.

Since the value function $V(t, \mu)$ of (2) is defined on the metric space $\mathcal{P}(\mathbf{T}^d)$, this approach calls for a study of the Hamilton-Jacobi equation in metric spaces; we refer the reader to [3], [16] and [20] for three definitions of viscosity solutions of H-J in metric spaces.

In this framework, the task is to solve the coupled equations (1) and (3); it turns out that, formally, these two equations are equivalent to the so-called master equation, i. e. formula (6) below. Heuristically, the solution of the master equation is a value function both for the single particle and the whole community. In [15] it is shown that, under suitable hypotheses on \mathcal{F} and \mathcal{U} , the master equation has a smooth solution for t negative and small and that the master equation is equivalent (this time rigorously) to (1) and (3).

In this paper, we want to give a different proof of the results of [15]. Instead of working in $\mathcal{P}(\mathbf{T}^d)$, we take up a suggestion of [11] (see also [18], [19]) and work in the space of L^2 parametrizations of particles: a parametrization for μ will be a function $\sigma \in L^2([0,1)^d, \mathbf{R}^d)$ whose law, when projected on \mathbf{T}^d , is μ . In other words, we are choosing $[0,1)^d$ as parameter space.

We shall see that this approach is equivalent to that of [15]; as in [15], the implicit function theorem is at the core of our proof, but we are going to use it in a way that is closer to the original approach of [10].

We set $M = L^2([0,1)^d, \mathbf{R}^d)$ and denote by AC([a,b],X) the set of the absolutely continuous functions from [a,b] to a space X; throughout the paper, we shall denote by ∇ , D and d the gradients of functions on \mathbf{T}^d , M and $\mathcal{P}(\mathbf{T}^d)$ respectively.

We want to prove the following.

Theorem 1. Let $\hat{\mathcal{F}}, \hat{\mathcal{U}}_0: M \to \mathbf{R}$ be respectively a potential and a final condition satisfying the hypotheses of section 2 below. Then, the following points hold.

1) There is T > 0 such that, if $t \in [-T, 0]$ and $\psi \in M$, the minimum

$$\hat{\mathcal{U}}(t,\psi) := \min \left\{ \int_{t}^{0} \left[\frac{1}{2} ||\dot{\sigma}_{s}||_{M}^{2} - \hat{\mathcal{F}}(\sigma_{s}) \right] \mathrm{d}s + \hat{\mathcal{U}}_{0}(\sigma_{0}) : \sigma \in AC([t,0], M), \quad \sigma_{t} = \psi \right\}$$
(4)

is attained on a unique curve $\sigma^{(t,\psi)} \in AC([t,0],M)$.

- 2) The maps: $(t, \psi) \to \sigma^{(t, \psi)}$ and $: (t, \psi) \to \hat{\mathcal{U}}(t, \psi)$ are of class C^2 ; moreover, they are $L^2_{\mathbf{Z}}$ and H-equivariant in the last variable for the groups $L^2_{\mathbf{Z}}$ and H defined in section 1 below.
- 3) There are two functions of class C^3

$$\hat{F}, \hat{u}_0: \mathbf{T}^d \times M \to \mathbf{R}$$

such that, if we set

$$u(t, x, \psi) = \min \left\{ \int_{t}^{0} \left[\frac{1}{2} |\dot{q}(s)|^{2} - \hat{F}(q(s), \sigma_{s}^{(t,\psi)}) \right] ds + \hat{u}_{0}(q(0), \sigma_{0}^{(t,\psi)}) :$$

$$q \in AC([t, 0], \mathbf{T}^{d}), \quad q(t) = x \right\}$$
(5)

then u is of class C^2 in $[-T,0] \times \mathbf{T}^d \times M$ and satisfies the master equation

$$-\partial_t u(t,q,\psi) + \frac{1}{2} |\nabla u(t,q,\psi)|^2 + F(q,\psi) + \langle \nabla u(t,\psi(\cdot),\psi), Du(t,q,\psi) \rangle_M = 0 \qquad \forall (t,x,\psi) \in [-T,0] \times \mathbf{T}^d \times M$$
(6)

where $\langle \cdot, \cdot \rangle_M$ denotes the inner product in M. To districate the inner product above, we note that $Du(t,q,\psi) \in M$ because it is the gradient with respect to the M variable; moreover, $: x \to \nabla u(t,\psi(x),\psi)$ belongs to M since it is the C^2 function $u(t,\cdot,\psi)$ composed with ψ . The function $u(t,\cdot,\psi)$ second variable and $L^2_{\mathbf{Z}}$ and H-equivariant in the last one.

- 4) Let the law of ψ be absolutely continuous with respect to the Lebesgue measure; then, for $s \in [-T, 0]$ the law of $\sigma_s^{(t,\psi)}$ is absolutely continuous too.
- 5) For \mathcal{L}^d a. e. $x \in [0,1)^d$ we have that, for all $s \in [-T,0]$,

$$\dot{\sigma}_s^{(t,x)}(x) = -\nabla u(s, \sigma_s^{(t,x)}(x), \sigma_s^{(t,x)}).$$

In other words, the orbit q(s) minimal in (5) coincides with $\sigma_s^{(t,\psi)}(x)$ if they start at the same point of \mathbf{T}^d ; equivalently, $: s \to \sigma_s^{(t,\psi)}(x)$ minimizes the one-particle problem (5) for \mathcal{L}^d a. e. $x \in [0,1)^d$.

Recently the master equation has been studied extensively, expecially from the stochastic viewpoint; we refer the reader to [7], [8], [9], [12] and [13].

The paper is organized as follows: section 1 contains the notation and a theorem of [11] about the relationship between differentiability on parametrizations and on measures; section 2 recalls the hypotheses used in [15] from section 6 onwards; in section 3 we recall the method of [10] for the minimum of (4), in section 4 we deal with the master equation (6).

Preliminaries and notation

We denote by $\pi: \mathbf{R}^d \to \mathbf{T}^d := \frac{\mathbf{R}^d}{\mathbf{Z}^d}$ the natural projection, and by $|\cdot|_{\mathbf{T}^d}$ the distance on \mathbf{T}^d given by

$$|x - y|_{\mathbf{T}^d} = \min\{|\tilde{x} - \tilde{y}| : \pi(\tilde{x}) = x, \quad \pi(\tilde{y}) = y\}.$$

We let $\mathcal{P}(\mathbf{T}^d)$ be the space of Borel probability measures on \mathbf{T}^d ; if $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{T}^d)$, we denote by $\Gamma(\mu_1, \mu_2)$ the set of all the Borel probability measures on $\mathbf{T}^d \times \mathbf{T}^d$ whose first and second marginals are, respectively, μ_1 and μ_2 . For $\lambda \geq 1$ we define the λ -Wasserstein distance on $\mathcal{P}(\mathbf{T}^d)$ by

$$W_{\lambda}(\mu_1, \mu_2)^{\lambda} = \min_{\gamma \in \Gamma(\mu_1, \mu_2)} \int_{\mathbf{T}^d \times \mathbf{T}^d} |x - y|_{\mathbf{T}^d}^{\lambda} d\gamma(x, y). \tag{1.1}$$

We refer the reader to [4] or [23] for the proof that the minimum is attained and that $(\mathcal{P}(\mathbf{T}^d), \mathcal{W}_{\lambda})$ is a compact metric space.

When $\lambda = 2$ (which is the only case we consider in this paper) we denote by $\Gamma_o(\mu_1, \mu_2)$ the set of the minimizers in (1.1).

We want to parametrize $\mu \in \mathcal{P}(\mathbf{T}^d)$ with a map $\sigma \in M := L^2([0,1)^d, \mathbf{R}^d)$. To do this, we begin to define $\mathcal{P}_2(\mathbf{R}^d)$ as the set of the Borel probability measures on \mathbf{R}^d with finite second moment. Following [19], we push forward $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ to $\tilde{\mu} := \pi_{\sharp} \mu \in \mathcal{P}(\mathbf{T}^d)$. By the definition of push-forward, this is tantamount to

$$\int_{\mathbf{T}^d} f(x) d\tilde{\mu}(x) = \int_{\mathbf{R}^d} f(x) d\mu(x) \qquad \forall f \in C(\mathbf{T}^d, \mathbf{R})$$

where we have identified f with its lift to a periodic function on \mathbf{R}^d .

If $\pi_{\sharp}\mu_{1} = \pi_{\sharp}\mu_{2} = \tilde{\mu}$, we say with [19] that μ_{1} and μ_{2} are two representatives of $\tilde{\mu}$. By lemma 1.2 of [19], it is possible to lift any couple of measures on \mathbf{T}^{d} to measures on \mathbf{R}^{d} in such a way to preserve the 2-Wasserstein distance. More precisely, if $\tilde{\mu}_{1}, \tilde{\mu}_{2} \in \mathcal{P}(\mathbf{T}^{d})$, then there are two representatives $\mu_{1}, \mu_{2} \in \mathcal{P}_{2}(\mathbf{R}^{d})$ such that μ_{1} is supported in $[0, 1]^{d}$, μ_{2} in $[-1, 2]^{d}$ and

$$W_2(\tilde{\mu}_1, \tilde{\mu}_2)^2 = W_2(\mu_1, \mu_2)^2 := \min_{\gamma \in \Gamma(\mu_1, \mu_2)} \int_{\mathbf{R}^d \times \mathbf{R}^d} |x - y|^2 d\gamma(x, y)$$
(1.2)

where we have denoted by W_2 the 2-Wasserstein distance on $\mathcal{P}_2(\mathbf{R}^d)$.

Let \mathcal{L}^d denote the *d*-dimensional Lebesgue measure on $[0,1)^d$ and let $\mu \in \mathcal{P}_2(\mathbf{R}^d)$; it is standard ([4] or [23]) that there is a map $\psi \in M$ (actually, ψ is the gradient of a convex function) such that $\psi_{\sharp}\mathcal{L}^d = \mu$. The trivial converse is that, if $\psi \in M$, then $\psi_{\sharp}\mathcal{L}^d \in \mathcal{P}_2(\mathbf{R}^d)$. The map ψ is called the Brenier map, or the parametrization of μ .

For completeness' sake, we give a well-known extension of lemma 6.4 of [11].

Lemma 1.1. 1) Let $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbf{R}^d)$, let $\psi_1, \psi_2 \in M$ be two parametrizations of μ_1, μ_2 respectively and let $\gamma \in \Gamma(\mu_1, \mu_2)$. Then, there is a sequence of invertible, measure-preserving maps $h_n: [0,1)^d \to [0,1)^d$ such

that $(\psi_1 \circ h_n, \psi_2)_{\sharp} \mathcal{L}^d$ converges weak* to γ . Moreover, for all functions $f \in C(\mathbf{T}^d \times \mathbf{R}^d, \mathbf{R})$ such that $\frac{f(x,v)}{1+|v|^2}$ is bounded, we have that

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} f(x, x - y) d\gamma(x, y) = \lim_{n \to +\infty} \int_{[0, 1)^d} f(\psi_1 \circ h_n(x), \psi_2(x) - \psi_1 \circ h_n(x)) dx. \tag{1.3}$$

2) Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{P}(\mathbf{T}^d)$ and let $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbf{R}^d)$ be two representatives such that (1.2) holds. Let $\psi_1, \psi_2 \in M$ be as in point 1). Then,

$$W_2(\tilde{\mu}_1, \tilde{\mu}_2)^2 = W_2(\mu_1, \mu_2)^2 = \inf \int_{[0,1)^d} |\psi_1 \circ h(x) - \psi_2(x)|^2 dx$$
(1.4)

where the inf is over all invertible, measure-preserving maps $h:[0,1)^d \to [0,1)^d$.

Proof. As for (1.4), the first equality comes from (1.2). For the second one, we note that, since $(\psi_1 \circ h, \psi_2)_{\sharp} \mathcal{L}^d \in \Gamma(\mu_1, \mu_2)$, we have that

$$W_2(\mu_1, \mu_2)^2 \le \inf_h \int_{[0,1)^d} |\psi_2(x) - \psi_1 \circ h(x)|^2 dx.$$

The opposite inequality follows immediately from point 1), which we prove it in the steps below using a variation of the technique of [11].

Step 1. We begin to suppose that μ_1 and μ_2 are supported in a common cube, say $\tilde{Q}^l = [-l, l)^d$. We partition \tilde{Q}^l into smaller cubes

$$Q_k = \frac{2kl}{2n} + \frac{1}{2n}\tilde{Q}^l$$

with $k = (k_1, ..., k_d) \in \mathbf{Z}^d$ such that $-2^n + 1 \le k_i \le 2^n - 1$. Next, we relabel the Q_k to Q_i , with i in a finite set of \mathbf{N} .

In the step 3, 4 and 5 below we are going to find maps h_n such that

$$\mathcal{L}^{d}[(\psi_1 \circ h_n, \psi_2)^{-1}(Q_i \times Q_j)] = \gamma(Q_i \times Q_j) \quad \text{for all} \quad i, j.$$
(1.5)

Using the fact that the sides of Q_i have length $\frac{2l}{2^n}$ and that μ_1 and μ_2 are supported in \tilde{Q}_l , the formula above easily implies that $(\psi_1 \circ h_n, \psi_2)_{\sharp} \mathcal{L}^d$ converges to γ in the weak* topology. Formula (1.3) now follows because γ and $(\psi_1 \circ h_n, \psi_2)_{\sharp} \mathcal{L}^d$ are supported in $\tilde{Q}^l \times \tilde{Q}^l$, a compact set on which $:(x,y) \to f(x,y-x)$ is continuous.

Step 2. Before showing (1.5) for the case with bounded support, let us show how it implies (1.3) in the general case.

Let $h: [0,1)^d \to [0,1)^d$ be measure preserving. The equality below comes from the definition of pushforward; in the inequality, \tilde{Q}^l is the cube of step 1.

$$\left| \int_{[0,1)^d} f(\psi_1 \circ h(x), \psi_2(x) - \psi_1 \circ h(x)) dx - \int_{\mathbf{R}^d \times \mathbf{R}^d} f(x, y - x) d\gamma(x, y) \right| =$$

$$\left| \int_{\mathbf{R}^d \times \mathbf{R}^d} f(x, y - x) \mathrm{d}(\psi_1 \circ h, \psi_2)_{\sharp} \mathcal{L}^d(x, y) - \int_{\mathbf{R}^d \times \mathbf{R}^d} f(x, y - x) \mathrm{d}\gamma(x, y) \right| \le$$

$$\int_{(\tilde{Q}^l \times \tilde{Q}^l)^c} |f(x, y - x)| d(\psi_1 \circ h, \psi_2)_{\sharp} \mathcal{L}^d(x, y) + \tag{1.6}_a$$

$$\int_{(\tilde{Q}^l \times \tilde{Q}^l)^c} |f(x, y - x)| d\gamma(x, y) + \tag{1.6}_b$$

$$\left| \int_{(\tilde{Q}^l \times \tilde{Q}^l)} f(x, y) \mathrm{d}(\psi_1 \circ h, \psi_2)_{\sharp} \mathcal{L}^p(x, y) - \int_{(\tilde{Q}^l \times \tilde{Q}^l)} f(x, y - x) \mathrm{d}\gamma(x, y) \right|. \tag{1.6}_c$$

Let $\epsilon > 0$; from the formula above we see that (1.3) follows if we prove that we can find $l \in \mathbb{N}$ such that

$$(1.6)_a < \epsilon$$

for all measure-preserving h,

$$(1.6)_b < \epsilon$$

and that, once l is fixed in this way, we can find a measure-preserving h such that

$$(1.6)_c \leq \epsilon$$
.

The last formula comes immediately from step 1; $(1.6)_b < \epsilon$ follows because the measure $|f(x, y - x)|\gamma$ is finite and $\bigcap_l (\tilde{Q}_l \times \tilde{Q}_l)^c = \emptyset$.

As for $(1.6)_a \leq \epsilon$, it suffices to prove that $|f(x, y - x)|(\psi_1 \circ h, \psi_2)_{\sharp} \mathcal{L}^d$ is a tight set of measures as h varies in the measure-preserving maps of $[0,1)^d$. By our hypotheses on f, this follows if we show that $(1+|y-x|^2)(\psi_1 \circ h, \psi_2)_{\sharp} \mathcal{L}^d$ is tight. This is equivalent to say that $|\psi_1 \circ h - \psi_2|^2$ is uniformly integrable as h varies among the measure-preserving maps, which follows if we prove that $|\psi_1 \circ h|^2$ is uniformly integrable; we leave the easy proof of this to the reader.

Step 3. In this step, we define the pre-images of the cubes Q_i , which the map h_n of step 1 will permute in a Rubik cube fashion. We set

$$A_i = \psi_1^{-1}(Q_i) \subset [0,1)^d, \qquad B_i = \psi_2^{-1}(Q_i) \subset [0,1)^d.$$

The equalities on the left in the two formulas below follow since $\gamma \in \Gamma(\mu_1, \mu_2)$; those on the right come from the fact that $\mu_j = (\psi_j)_{\sharp} \mathcal{L}^d$ for j = 1, 2.

$$\gamma(Q_i \times [-l, l)^d) = \mu_1(Q_i) = \mathcal{L}^d(A_i), \qquad \gamma([-l, l)^d \times Q_i) = \mu_2(Q_i) = \mathcal{L}^d(B_i).$$
 (1.7)

In the next two steps, we shall settle the first row of cubes, say $\{A_i \times B_1\}_i$. The idea is to partition B_1 into sets $B_{i,1}$ and to find sets $A_{i,1} \subset A_i$ such that $\mathcal{L}^d(A_{i,1}) = \mathcal{L}^d(B_{i,1}) = \gamma(Q_i \times Q_1)$; then, we shall send $A_{i,1}$ into $B_{i,1}$ by a measure-preserving map. We shall see that this yields (1.5) for j = 1.

Step 4. We assert that we can find sets $A_{i,1} \subset A_i$ such that

$$\mathcal{L}^d(A_{i,1}) = \gamma(Q_i \times Q_1) \quad \text{and} \quad \sum_i \mathcal{L}^d(A_{i,1}) = \mathcal{L}^d(B_1). \tag{1.8}$$

Note that the sets $A_{i,1}$ are disjoint since the A_i are disjoint. Moreover, we can find sets $B_{i,1} \subset B_1$ such that

$$\begin{cases}
\mathcal{L}^{d}(B_{i,1}) = \mathcal{L}^{d}(A_{i,1}) \\
\text{the } B_{i,1} \text{ are disjoint} \\
\mathcal{L}^{d}(B_{1} \setminus \bigcup_{i} B_{i,1}) = 0 \\
B_{i,1} \supset A_{i,1} \cap B_{1} \\
B_{i,1} \cap A_{j,1} = \emptyset \quad \text{if} \quad j \neq i.
\end{cases}$$
(1.9)

We begin to show that the first equality of (1.8) implies the second one: the first equality below follows since the Q_i partition $[-l, l)^d$, the second one follows since γ has μ_2 as the second marginal, the third one since $(\psi_2)_{\sharp}\mathcal{L}^d = \mu_2$ and the fourth one from the definition of B_1 .

$$\sum_{i} \gamma(Q_i \times Q_1) = \gamma([-l, l)^d \times Q_1) = \mu_2(Q_1) = \mathcal{L}^d(\psi_2^{-1}(Q_1)) = \mathcal{L}^d(B_1).$$

Thus, we only have to find sets $A_{i,1} \subset A_i$ which satisfy the first formula of (1.8); since \mathcal{L}^d is non-atomic and, by (1.7),

$$\mathcal{L}^d(A_i) = \gamma(Q_i \times [-l, l)^d) \ge \gamma(Q_i \times Q_1)$$

this is standard.

Now, we find the sets $B_{i,1}$ which satisfy (1.9). First of all we note that, by (1.8),

$$\mathcal{L}^d(B_1 \setminus \bigcup_{i>2} A_{i,1}) \ge \mathcal{L}^d(A_{1,1}).$$

Since the $A_{i,1}$ are disjoint,we also have that $B_1 \cap A_{1,1}$ does not intersect $A_{i,1}$ for $i \geq 2$; moreover, $\mathcal{L}^d(B_1 \cap A_{1,1}) \leq \mathcal{L}^d(A_{1,1})$. Thus, we can find $B_{1,1} \subset B_1$ such that

- a) $B_{1,1} \supset A_{1,1} \cap B_1$,
- b) $\mathcal{L}^d(B_{1,1}) = \mathcal{L}^d(A_{1,1}),$
- c) $B_{1,1}$ is disjoint from $A_{i,1}$ for $i \geq 2$.

Point c) follows by the last formula: in $B_1 \setminus \bigcup_{i \geq 2} A_{i,1}$ there is enough space to accommodate a $B_{1,1}$ satisfying b).

We show the next step of the induction, namely how to find $B_{2,1}$. By (1.8) and the aforesaid,

$$\mathcal{L}^d\left(B_1\setminus\left(B_{1,1}\cup\bigcup_{i\neq 2}A_{i,1}\right)\right)\geq \mathcal{L}^d(A_{2,1}).$$

Using this, we can find $B_{2,1} \subset B_1$ such that

- $a') B_{2,1} \supset A_{2,1} \cap B_1,$
- $b') \mathcal{L}^d(B_{2,1}) = \mathcal{L}^d(A_{2,1}),$
- c') $B_{2,1}$ is disjoint from $B_{1,1}$ and from $A_{i,1}$ for $i \neq 2$.

Iterating, we get the sets $B_{i,1}$; the first, second, fourth and fifth formulas of (1.9) follow by construction, the third one by the first formula of (1.9), (1.8) and the fact that the $B_{i,1}$ are disjoint.

Step 5. In this step, we define h_n on the first row of cubes: we want to find an invertible, bi-measurable map \hat{h}_1 which preserve Lebesgue measure and such that, for all i,

$$\begin{cases} \hat{h}_1(x) = x & \text{if } x \notin \bigcup_i (A_{i,1} \cup B_{i,1}) \\ (\psi_1 \circ \hat{h}_1, \psi_2)^{-1} (Q_i \times Q_1) = B_{i,1}. \end{cases}$$
 (1.10)

Before proving this, note that $\mathcal{L}^d(B_{i,1}) = \gamma(Q_i \times Q_1)$ by (1.8) and (1.9); this and (1.10) proves that (1.5) holds for the first row of cubes $\{Q_i \times Q_1\}_i$. The other rows will follow by induction, as we shall see in step 6.

We prove (1.10). First of all, there are invertible maps $\phi_i: B_{i,1} \to A_{i,1}$ which preserve Lebesgue measure and which are the identity on $A_{i,1} \cap B_{i,1}$. This is easy to do: we set $\phi_i(x) = x$ on $A_{i,1} \cap B_{i,1}$; then, we use theorem 15.5.16 of [22] to get an invertible, measure-preserving map ϕ_i from $B_{i,1} \setminus A_{i,1}$ to $A_{i,1} \setminus B_{i,1}$; recall that these sets have the same Lebesgue measure by the first one of (1.9).

Next, we glue together the maps ϕ_i in the following way:

$$\hat{h}_{1}(x) = \begin{cases} x & \text{if } x \notin \bigcup_{i} (A_{i,1} \cup B_{i,1}) \\ \phi_{i}(x) & \text{if } x \in B_{i,1} \\ \phi_{i}^{-1}(x) & \text{if } x \in A_{i,1}. \end{cases}$$

The definition is well-posed: since by (1.9) the $B_{i,1}$ are disjoint, and since we saw above that the $A_{i,1}$ are disjoint, the only possible conflict is when $x \in B_{i,1} \cap A_{j,1}$. But then by (1.9) j = i; now on $B_{i,1} \cap A_{i,1} \phi_i$ and ϕ_i^{-1} coincide, since both are the identity on this set.

To check (1.10), we begin to note that its first formula comes straight from the definition of \hat{h}_1 . As for the second one, if $x \in (\psi_1 \circ \hat{h}_1, \psi_2)^{-1}(Q_i \times Q_1)$, then $x \in \psi_2^{-1}(Q_1) = B_1$ and $\hat{h}_1(x) \in \psi_1^{-1}(Q_i) = A_i$. Now B_1 is partitioned by the $B_{j,1}$ and the only $B_{j,1}$ which \hat{h}_1 sends to A_i is $B_{i,1}$. Thus, $x \in B_{i,1}$, proving that $(\psi_1 \circ \hat{h}_1, \psi_2)^{-1}(Q_i \times Q_1) = B_{i,1}$.

Step 6. We saw above that (1.5) follows if we show (1.10) for all the other rows; we do this by iteration. By the last step, the pre-image of $\bigcup_i (Q_i \times Q_1)$ by $(\psi_1 \circ \hat{h}_1, \psi_2)$ is B_1 . We want to adjust the second row of cubes without touching B_1 . To do this, we restrict $(\psi_1 \circ \hat{h}_1, \psi_2)$ to B_1^c ; its image will fall in

$$\bigcup_{j\neq 1} (Q_i \times Q_j).$$

Now we apply the procedure of the first step to the second row, i. e. to $\{Q_i \times Q_2\}_i$ and to $(\psi_1 \circ \hat{h}_1, \psi_2)$. We get a map \hat{h}_2 from B_1^c to itself such that $(\psi_1 \circ \hat{h}_1 \circ \hat{h}_2, \psi_2)$ satisfies (1.5) for j = 2. Now we extend \hat{h}_2 to be the identity on B_1 , and we get that $(\psi_1 \circ \hat{h}_1 \circ \hat{h}_2, \psi_2)$ satisfies (1.5) for j = 1 too. To close, it suffices to call h_n the last step of the iteration, the one in which all the rows are settled.

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We can look at W_2 on $\mathcal{P}(\mathbf{T}^d)$ keeping track of the action of \mathbf{R}^d on \mathbf{T}^d . Let us define

$$\pi_{\mathbf{T}^d} \colon \mathbf{T}^d \times \mathbf{R}^d \to \mathbf{T}^d$$

as the projection on the first coordinate, and let us set

$$\alpha: \mathbf{T}^d \times \mathbf{R}^d \to \mathbf{T}^d, \qquad \alpha: (x, v) \to x + v.$$

Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{P}(\mathbf{T}^d)$; we say that $\gamma \in \mathcal{P}_2(\mathbf{T}^d \times \mathbf{R}^d)$ belongs to $\Psi(\tilde{\mu}_1, \tilde{\mu}_2)$ if $(\pi_{\mathbf{T}^d})\sharp \gamma = \tilde{\mu}_1$ and $\alpha_{\sharp} \gamma = \tilde{\mu}_2$; we leave to the reader the simple proof that

$$\mathcal{W}_2^2(\tilde{\mu}_1, \tilde{\mu}_2) = \min_{\gamma \in \Psi(\tilde{\mu}_1, \tilde{\mu}_2)} \int_{\mathbf{T}^d \times \mathbf{R}^d} |v|^2 d\gamma(x, v). \tag{1.11}$$

We denote by $\Psi_o(\tilde{\mu}_1, \tilde{\mu}_2)$ the set of minimals.

In the following, we shall denote by L^2_{μ} a space of L^2 functions for the measure μ ; we shall omit the μ when it is the Lebesgue measure.

Let now $G: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$ be a function; we say that G is differentiable at $\tilde{\mu} \in \mathcal{P}(\mathbf{T}^d)$ if there is a vector field $\xi \in L^2_{\tilde{\mu}}(\mathbf{T}^d, \mathbf{R}^d)$ such that

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \int_{\mathbf{T}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle d\gamma(x, v) \right| = o(\mathcal{W}_2(\tilde{\mu}, \tilde{\nu}))$$

for all $\tilde{\nu} \in \mathcal{P}(\mathbf{T}^d)$ and all $\gamma \in \Psi_o(\tilde{\mu}, \tilde{\nu})$; we have denoted by $\langle \cdot, \cdot \rangle$ the inner product in \mathbf{R}^d .

Following [15], we say that G is strongly differentiable at $\tilde{\mu}$ if there is k > 0 such that

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \int_{\mathbf{T}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle d\gamma(x, v) \right| \le k \int_{\mathbf{T}^d \times \mathbf{R}^d} |v|^2 d\gamma(x, v)$$

for all $\tilde{\nu} \in \mathcal{P}(\mathbf{T}^d)$ and all $\gamma \in \Psi(\tilde{\mu}, \tilde{\nu})$. Note that we don't restrict the transfer plan γ to be in $\Psi_o(\tilde{\mu}, \tilde{\nu})$; it is immediate that strong differentiability implies differentiability. Of course, there are parallel definitions of differentiability and strong differentiability in $\mathcal{P}_2(\mathbf{R}^d)$, which we forego to state.

If $G: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$, we can define

$$\bar{G}: \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}, \qquad \bar{G}(\mu) = G(\pi_{\dagger}\mu).$$
 (1.12)

Lemma 1.2. Let $G: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$ be strongly differentiable at $\tilde{\mu}$ and let $\bar{G}: \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ be defined as in (1.12). Then, \bar{G} is strongly differentiable at any $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ such that $\pi_{\sharp}\mu = \tilde{\mu}$.

Conversely, if $\bar{G}: \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ quotients to a map $G: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$ and is strongly differentiable at μ , then G is strongly differentiable at $\tilde{\mu} = \pi_{\sharp}\mu$.

Proof. We begin with the direct statement. Let $\tilde{\xi} \in L^2(\mathbf{T}^d, \tilde{\mu})$ be the derivative of G at $\tilde{\mu}$; we define $\xi \colon \mathbf{R}^d \to \mathbf{R}^d$ by $\xi(y) = \tilde{\xi}(\pi(y))$. We assert that $\xi \in L^2(\mathbf{R}^d, \mu)$; indeed, since $\pi_{\sharp}\mu = \tilde{\mu}$ we get the equality below, while the inequality comes from the fact that $\tilde{\xi} \in L^2_{\tilde{\mu}}$.

$$\int_{\mathbf{R}^d} |\xi(x)|^2 \mathrm{d}\mu(x) = \int_{\mathbf{T}^d} |\tilde{\xi}(x)|^2 \mathrm{d}\tilde{\mu}(x) < +\infty.$$

We prove that ξ is the derivative of \bar{G} at μ . Let $\nu \in \mathcal{P}_2(\mathbf{R}^d)$ project on $\tilde{\nu} \in \mathcal{P}(\mathbf{T}^d)$ and let $\gamma \in \Psi(\mu, \nu)$; if we define $\tilde{\gamma} = (\pi \times id)_{\sharp} \gamma$ we see easily that $\tilde{\gamma} \in \Psi(\tilde{\mu}, \tilde{\nu})$. We disintegrate γ as $\mu \otimes \gamma_x$ and $\tilde{\gamma}$ as $\tilde{\mu} \otimes \tilde{\gamma}_q$, where γ_x and $\tilde{\gamma}_q$ are measures on \mathbf{R}^d . An easy check shows that, if $f \in C(\mathbf{T}^d \times \mathbf{R}^d)$ with $\frac{f(x,v)}{1+|v|^2}$ bounded, then

$$\int_{\mathbf{R}^d} \mathrm{d}\mu(x) \int_{\mathbf{R}^d} f(x,y) \mathrm{d}\gamma_x(y) = \int_{\mathbf{T}^d} \mathrm{d}\tilde{\mu}(q) \int_{\mathbf{R}^d} f(q,y) \mathrm{d}\tilde{\gamma}_q(y).$$

The first equality below comes from (1.12) and the disintegration of γ ; the second one comes from the definition of ξ using the fact that $\tilde{\mu} = \pi_{\sharp} \mu$ and the formula above. The third equality comes from the disintegration of $\tilde{\gamma}$. The first inequality comes from the fact that G is strongly differentiable, while the last equality is obvious.

$$\left| \bar{G}(\nu) - \bar{G}(\mu) - \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle d\gamma(x, v) \right| =$$

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \langle \int_{\mathbf{R}^d} \xi(x) d\mu(x), \int_{\mathbf{R}^d} v d\gamma_x(v) \rangle \right| =$$

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \langle \int_{\mathbf{T}^d} \tilde{\xi}(q) d\tilde{\mu}(q), \int_{\mathbf{R}^d} v d\tilde{\gamma}_q(v) \rangle \right| =$$

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \int_{\mathbf{T}^d \times \mathbf{R}^d} \langle \tilde{\xi}(q), v \rangle d\tilde{\gamma}(q, v) \right| \leq$$

$$k \int_{\mathbf{T}^d \times \mathbf{R}^d} |v|^2 d\tilde{\gamma}(x, v) = k \int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 d\gamma(x, v).$$

Since this is the definition of strong differentiability in $\mathcal{P}_2(\mathbf{R}^d)$, we are done.

We prove the converse.

Step 1. Let $\tilde{\mu}, \tilde{\nu} \in \mathcal{P}(\mathbf{T}^d)$, let $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ be such that $\pi_{\sharp}\mu = \tilde{\mu}$ and let $\tilde{\gamma} \in \Psi(\tilde{\mu}, \tilde{\nu})$. Recall that we have defined a map $\alpha: (x, v) \to x + v$. We assert that we can find $\gamma \in \mathcal{P}_2(\mathbf{R}^d \times \mathbf{R}^d)$ and $\nu \in \mathcal{P}_2(\mathbf{R}^d)$ such that

- a) the first marginal of γ is μ ,
- b) $(\pi \times id)_{\sharp} \gamma = \tilde{\gamma}$ and
- c) $\alpha_{\sharp} \gamma = \nu$ and $\pi_{\sharp} \nu = \tilde{\nu}$; in particular, $\gamma \in \Psi(\mu, \nu)$.

To find γ , we disintegrate μ as $\mu = \beta_q \otimes \tilde{\mu}$, with β_q a probability measure on the fiber $\{q + \mathbf{Z}^d\}$; in other words, $\beta_q(z) \geq 0$ and

$$\sum_{z \in \mathbf{Z}^d} \beta_q(z) = 1.$$

Then, we can define γ by

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} f(x, v) d\gamma(x, v) = \int_{\mathbf{T}^d \times \mathbf{R}^d} \left[\sum_{z \in \mathbf{Z}^d} \beta_q(z) f(q + z, v) \right] d\tilde{\gamma}(q, v)$$

for all continuous functions $f: \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ such that $\frac{f(x,v)}{1+|v|^2}$ is bounded. Setting $\nu = \alpha_{\sharp} \gamma$ we easily check that γ and ν satisfy a), b) and c).

Step 2. Let ξ be the derivative of \bar{G} at μ ; we assert that $\xi = \tilde{\xi} \circ \pi$, where $\tilde{\xi}$ is a vector field on \mathbf{T}^d . This is easy to see: for instance, taking a vector field η supported in a small ball $B(x_0, r)$ of \mathbf{R}^d , considering

 $\gamma_{\epsilon,z} = \mu \otimes (id + \epsilon \eta(\cdot + z))_{\sharp} \mathcal{L}^d$ for $z \in \mathbf{Z}^d$, setting $\nu_{\epsilon,z} = \alpha_{\sharp} \gamma_{\epsilon,z}$ and noting that $\bar{G}(\nu_{\epsilon,z})$, which quotients on $\mathcal{P}(\mathbf{T}^d)$, depends on z only through $\mu(B(z_0,r))$.

End of the proof. The two steps above yield the first equality below, while the inequality comes from the fact that \bar{G} is strongly differentiable at μ .

$$\left| G(\tilde{\nu}) - G(\tilde{\mu}) - \int_{\mathbf{T}^d \times \mathbf{R}^d} \langle \tilde{\xi}(q), v \rangle d\tilde{\gamma}(q, v) \right| =$$

$$\left| \bar{G}(\nu) - \bar{G}(\mu) - \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle d\gamma(x, v) \right| \le k \int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 d\gamma(x, v) = k \int_{\mathbf{T}^d \times \mathbf{R}^d} |v|^2 d\tilde{\gamma}(q, v).$$
\(\begin{align*} \begin{align*} \left\left\text{\text{\$\times \text{\$\text{\$\times \text{\$\text{\$\times \text{\$\text{\$\text{\$\times \text{\$\texit{\$\text{\$\text{\$\text{\$\text{\$\text{\$\exitit{\$\text{\$\text{\$\text{\$\text{\$\text{\$\t

We shall denote by H the group of all bi-measurable maps $h: [0,1)^d \to [0,1)^d$ which preserve Lebesgue measure; we also set $L^2_{\mathbf{Z}} := L^2([0,1)^d, \mathbf{Z}^d)$, which is a group under addition.

Given $G: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$, we can define a function

$$\hat{G}: M \to \mathbf{R}, \qquad \hat{G}(\psi) = G(\pi_{\sharp} \circ \psi_{\sharp} \mathcal{L}^d).$$
 (1.13)

Clearly, the map \hat{G} defined above is H and $L_{\mathbf{Z}}^2$ -equivariant, i. e.

$$\hat{G}(\psi \circ h + z) = \hat{G}(\psi) \qquad \forall (\psi, h, z) \in M \times H \times L^{2}_{\mathbf{Z}}. \tag{1.14}$$

Going in the opposite direction, if $\hat{G}: M \to \mathbf{R}$ is a continuous map such that (1.14) holds, we can define

$$\bar{G}: \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}, \qquad \bar{G}(\mu) = \hat{G}(\psi)$$
 (1.15)

where $\psi \in M$ is such that $\psi_{\sharp} \mathcal{L}^p = \mu$. We prove that \bar{G} is well-defined on $\mathcal{P}_2(\mathbf{R}^d)$: actually, we are going to see that \bar{G} quotients to a function G on $\mathcal{P}(\mathbf{T}^d)$. Indeed, if $\psi_1, \psi_2 \in M$ are such that $\pi_{\sharp}(\psi_i)_{\sharp} \mathcal{L}^p = \tilde{\mu} \in \mathcal{P}(\mathbf{T}^d)$ for i = 1, 2, then it is standard (lemma 6.4 of [11] or lemma 1.1 above) that there are $h_n \in H$ and $z_n \in L^2_{\mathbf{Z}}$ such that

$$||\psi_1 - \psi_2 \circ h_n - z_n||_M \to 0 \text{ as } n \to +\infty.$$

The equality below comes from (1.14), while the limit comes from the formula above and the continuity of \hat{G} .

$$\hat{G}(\psi_1) - \hat{G}(\psi_2) = \hat{G}(\psi_1) - \hat{G}(\psi_2 \circ h_n + z_n) \to 0.$$

This proves that \hat{G} is well defined; as for the differentiability of \hat{G} , we recall theorems 6.2 and 6.5 of [11].

Proposition 1.3. Let $\hat{G}: M \to \mathbf{R}$ be continuous and let it satisfy (1.14). Then, the following happens. 1) If \hat{G} is differentiable at ψ , then \hat{G} is differentiable at η for all $\eta \in M$ such that $\eta_{\sharp} \mathcal{L}^{d} = \psi_{\sharp} \mathcal{L}^{d}$. Moreover, the law of $D\hat{G}(\psi)$ does not depend on the choice of η .

2) Let us suppose that \hat{G} is of class C^1 and let $\mu \in \mathcal{P}_2(\mathbf{R}^d)$. Then, there is $\xi \in L^2_{\mu}(\mathbf{R}^d, \mathbf{R}^d)$ such that, for all ψ satisfying $\psi_{\sharp}\mathcal{L}^d = \mu$, we have

$$D\hat{G}(\psi)(x) = \xi \circ \psi(x)$$
 for \mathcal{L}^p a. e. x .

3) Let $\hat{G} \in C^2(M, \mathbf{R})$ with a bounded second derivative and let it satisfy (1.14); then, the function $\bar{G}: \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ defined by (1.15) is strongly differentiable. By lemma 1.2 this implies that its quotient G on $\mathcal{P}(\mathbf{T}^d)$ is strongly differentiable.

Proof. Point 1) is theorem 6.2 of [11], point 2 theorem 6.5. We prove the easy consequence 3).

We want to show that \bar{G} is strongly differentiable at any $\mu \in \mathcal{P}_2(\mathbf{R}^d)$. Thus, let $\nu \in \mathcal{P}_2(\mathbf{R}^d)$ and let $\psi, \eta \in M$ be such that $\psi_{\sharp} \mathcal{L}^p = \mu$, $\eta_{\sharp} \mathcal{L}^p = \nu$; let $\lambda \in \Psi(\mu, \nu)$ and let ξ be as in point 2) above. Let $\beta: (x, v) \to (x, x + v)$; since $\lambda \in \Psi(\mu, \nu)$ it is easy to check that $\gamma:=\beta_{\sharp}\lambda$ belongs to $\Gamma(\mu, \nu)$. By formula (1.3) of lemma 1.1 we can find $h_n \in H$ such that

$$\int_{[0,1)^d} |\psi(x) - \eta \circ h_n(x)|^2 dx \to \int_{\mathbf{R}^d \times \mathbf{R}^d} |q - q'|^2 d\gamma(q, q')$$

or equivalently, setting $\lambda_n := (\psi, \eta \circ h_n - \psi)_{\sharp} \mathcal{L}^d$,

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 d\lambda_n(x, v) \to \int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 d\lambda(x, v). \tag{1.16}$$

We assert that

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle d\lambda_n(x, v) \to \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle d\lambda(x, v). \tag{1.17}$$

Indeed, if ξ were continuous, this would follow from (1.3). In the general case, we can find a continuous vector field ξ' such that $||\xi - \xi'||_{L^2_{\mu}} < \epsilon$; the first inequalities in the two formulas below are Hölder while the second ones come from (1.16).

$$\left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi - \xi', v \rangle d\lambda_n(x, v) \right| \leq ||\xi - \xi'||_{L^2_{\mu}} \left[\int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 d\lambda_n(x, v) \right]^{\frac{1}{2}} \leq M||\xi - \xi'||_{L^2_{\mu}} \leq M\epsilon,$$

$$\left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi - \xi', v \rangle d\lambda(x, v) \right| \leq ||\xi - \xi'||_{L^2_{\mu}} \left[\int_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 d\lambda(x, v) \right]^{\frac{1}{2}} \leq M||\xi - \xi'||_{L^2_{\mu}} \leq M\epsilon.$$

These two formulas imply the second inequality below; the third one follows from (1.3) taking n large enough.

$$\left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle \mathrm{d}(\lambda_n - \lambda)(x, v) \right| \le$$

$$\left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi - \xi', v \rangle \mathrm{d}(\lambda_n - \lambda) \right| + \left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi', v \rangle \mathrm{d}(\lambda_n - \lambda) \right| \le$$

$$2\epsilon M + \left| \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi', v \rangle \mathrm{d}(\lambda_n - \lambda) \right| \le 2\epsilon M + \epsilon.$$

This proves (1.17). By (1.17), there is $\epsilon_n \to 0$ such that the first inequality below holds. The second one follows if we take k to be the sup of $\frac{1}{2}||D^2\hat{G}||$, which is finite by hypothesis. The last inequality follows from (1.16).

$$\left| \bar{G}(\nu) - \bar{G}(\mu) - \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(x), v \rangle d\lambda(x, v) \right| \le$$

$$\left| \hat{G}(\eta \circ h_n) - \hat{G}(\psi) - \int_{[0,1)^d} \langle \xi(\psi(x)), \eta \circ h_n(x) - \psi(x) \rangle dx \right| + \epsilon_n \le$$

$$k \int_{[0,1)^d} |\eta \circ h_n(x) - \psi(x)|^2 dx + \epsilon_n \le k \int_{\mathbf{T}^d \times \mathbf{R}^d} |v|^2 d\lambda(x,v) + 2\epsilon_n.$$

Letting $n \to +\infty$, we recover the definition of strong differentiability at μ .

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In the opposite direction, we have the following.

Lemma 1.4. Let $G: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$ be a function and let $\hat{G}: M \to \mathbf{R}$ be defined as in (1.13). Let us suppose that G is strongly differentiable at $\tilde{\mu} \in \mathcal{P}(\mathbf{T}^d)$, let $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ be a representative of $\tilde{\mu}$ and let $\psi \in M$ such that $\psi_{\sharp} \mathcal{L}^d = \mu$. Then, \hat{G} is differentiable at $\psi \circ h + z$ for all $(h, z) \in H \times L^2_{\mathbf{Z}}$, and

$$D\hat{G}(u \circ h + z) = D\hat{G}(u) \circ h. \tag{1.18}$$

Proof. We define $\bar{G}: \mathcal{P}_2(\mathbf{R}^d) \to \mathbf{R}$ as in (1.12); by lemma 1.2, \bar{G} is strongly differentiable at any representative μ of $\tilde{\mu}$.

Let ξ be the derivative of \bar{G} at μ and let $\psi \in M$ be such that $(\psi)_{\sharp} \mathcal{L}^p = \mu$. Let $\eta \in M$ and let us set $\nu = \eta_{\sharp} \mathcal{L}^p$. If we define $\lambda = (\psi, \eta - \psi)_{\sharp} \mathcal{L}^p$, we get the first equality below. Now $\lambda \in \Psi(\mu, \nu)$ and G is strongly differentiable at μ with differential ξ ; for some k > 0 this implies the inequality below, while the last equality comes from the definitions of \hat{G} and λ .

$$k \int_{[0,1)^d} |\psi(x) - \eta(x)|^2 dx = k \int_{\mathbf{T}^p \times \mathbf{R}^d} |v|^2 d\lambda(x,v) \ge$$

$$\left| \bar{G}(\nu) - \bar{G}(\mu) - \int_{\mathbf{R}^d \times \mathbf{R}^d} \langle \xi(q), v \rangle d\lambda(q,v) \right| =$$

$$\left| \hat{G}(\eta) - \hat{G}(\psi) - \int_{[0,1)^d} \langle \xi \circ \psi(x), \eta(x) - \psi(x) \rangle dx \right|.$$

The last formula implies that \hat{G} is differentiable at ψ .

As for point 2), this is a general property of equivariant functions: if T_h is a set of bounded linear operators from M to M having the group property

$$T_{h_1} \circ T_{h_2} = T_{h_1 h_2}$$

then it is standard that

$$D\hat{G}(T_h u) = [T_{h^{-1}}^T D\hat{G}(u)]$$

where A^T denotes the adjoint operator of A. Setting $T_h u := u \circ h$ and substituting, we get (1.18).

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Assumptions on the potential and the final condition

We recall the assumptions used in [15] from section 6 onward.

We begin to suppose that we are given $U^0, U^1, \phi \in C^3(\mathbf{T}^d)$ such that the lifts of ϕ and U^1 to \mathbf{R}^d are even.

Our potential is the function $\mathcal{F}: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$ defined by

$$\mathcal{F}(\mu) = \frac{1}{2} \int_{\mathbf{T}^d} (\phi * \mu)(z) d\mu(z) = \frac{1}{2} \int_{\mathbf{T}^d \times \mathbf{T}^d} \phi(z - z') d\mu(z) d\mu(z')$$

where the symbol * denotes, as usual, convolution. The final condition is the function $\mathcal{U}_0: \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}$ given by

$$\mathcal{U}_0(\mu) = \int_{\mathbf{T}^d} [U^0(z) + \frac{1}{2} (U^1 * \mu)(z)] d\mu(z) =$$
$$\int_{\mathbf{T}^d \times \mathbf{T}^d} [U^0(z) + \frac{1}{2} U^1(z - z')] d\mu(z) d\mu(z').$$

It is shown in [15] that \mathcal{F} and \mathcal{U} are strongly differentiable.

We recall from the introduction that we denote by d the differential of functions on $\mathcal{P}(\mathbf{T}^d)$, by D and ∇ that of functions on M and on \mathbf{R}^d respectively.

Always by [15], we have that

$$d\mathcal{F}(\mu) = \nabla F(q, \mu)$$
 and $d\mathcal{U}_0(\mu) = \nabla u_0(q, \mu)$

where

$$F(q,\mu) = (\phi * \mu)(q)$$
 and $u_0(q,\mu) = U^0(q) + (U^1 * \mu)(q)$.

By (1.13), \mathcal{F} and \mathcal{U} induce functions $\hat{\mathcal{F}}$ and $\hat{\mathcal{U}}_0$ on M; by the definition of push-forward we see that, if $\sigma \in M$,

$$\hat{\mathcal{F}}(\sigma) = \frac{1}{2} \int_{[0,1)^d \times [0,1)^d} \phi[\sigma(x) - \sigma(y)] dx dy, \qquad (2.1)_a$$

$$\hat{\mathcal{U}}_0(\sigma) = \int_{[0,1)^d \times [0,1)^d} \{ U^0(\sigma(x)) + \frac{1}{2} U^1[\sigma(x) - \sigma(y)] \} dx dy.$$
 (2.1)_b

Also the functions F and u_0 extend to parametrizations:

$$\hat{F}: \mathbf{R}^d \times M \to \mathbf{R}^d, \qquad \hat{F}(q, \sigma) = \int_{[0,1)^d} \phi[q - \sigma(x)] dx,$$
 (2.2)_a

$$\hat{u}_0: \mathbf{R}^d \times M \to \mathbf{R}^d, \qquad \hat{u}_0(q, \sigma) = U^0(q) + \int_{[0, 1)^d} U^1[q - \sigma(x)] dx.$$
 (2.2)_b

We forego the proof of the following lemma, which follows from our hypotheses on ϕ , U^0 , U^1 and standard facts about the Nemitsky operators (see for instance [2]).

Lemma 2.1. Let $\hat{\mathcal{F}}, \hat{\mathcal{U}}_0: M \to \mathbf{R}$ be defined as in (2.1), let \hat{F}, \hat{u}_0 be as in (2.2). Then, $\hat{\mathcal{F}}$ and $\hat{\mathcal{U}}_0$ are functions of class C^3 on M. Denoting by $\langle \cdot, \cdot \rangle_M$ the inner products in \mathbf{R}^d and in M respectively, we have that

$$D\hat{\mathcal{F}}(\sigma)\psi = \int_{[0,1)^d \times [0,1)^d} \langle \nabla \phi[\sigma(x) - \sigma(y)], \psi(x) \rangle \mathrm{d}x \mathrm{d}y = \langle \nabla \hat{F}(\sigma(\cdot), \sigma), \psi \rangle_M$$

and

$$D\hat{\mathcal{U}}_0(\sigma)\psi = \int_{[0,1)^d \times [0,1)^d} \langle \nabla U^0(\sigma(x)) + \nabla U^1[\sigma(x) - \sigma(y)], \psi(x) \rangle dxdy = \langle \nabla \hat{u}_0(\sigma(\cdot), \sigma), \psi \rangle_M.$$

In other words, $D\hat{\mathcal{F}}(\sigma)$ is represented by the function $\nabla \hat{F}(\sigma(\cdot), \sigma) \in M$, $D\hat{\mathcal{U}}_0(\sigma)$ by the funtion $\nabla \hat{u}_0(\sigma(\cdot), \sigma) \in M$. The functions \hat{F} and \hat{u}_0 are of class C^3 in both variables, with bounded first, second and third derivatives. Moreover, \hat{F} and \hat{u}_0 are \mathbf{Z}^d -equivariant in the first variable; they are also $L^2_{\mathbf{Z}}$ and H-equivariant in the second one.

§**3**

Minima on short time intervals

In lemmas 3.2-3.5 below, we recall the method of [10] for the minimals of the value function; in lemma 3.1, we prove that the value functions on measures and on parametrizations coincide.

Definitions. Let μ : $(t,0) \to \mathcal{P}(\mathbf{T}^d)$ be a curve of measures satisfying, in the weak sense (the precise definition is in the proof of lemma 3.1 below), the continuity equation

$$\partial_s \mu_s + \operatorname{div}(X\mu_s) = 0 \tag{3.1}$$

for a drift $X \in L^2((t,0) \times \mathbf{T}^d, \mathcal{L}^1 \otimes \mu_t)$. We define the augmented action of (μ_s, X) as

$$\mathcal{A}(t, \mu_s, X) = \int_t^0 \left[\frac{1}{2} ||X(s, \cdot)||_{L_{\mu_s}^2}^2 - \mathcal{F}(\mu_s) \right] ds + \mathcal{U}_0(\mu_0).$$

The value function on $\mathcal{P}(\mathbf{T}^d)$ is defined by

$$\mathcal{U}: (-\infty, 0] \times \mathcal{P}(\mathbf{T}^d) \to \mathbf{R}, \qquad \mathcal{U}(t, \bar{\mu}) = \inf \mathcal{A}(t, \mu_s, X)$$
 (3.2)

where the inf is over all paths (μ_s, X) which satisfy (3.1) and such that $\mu_t = \bar{\mu}$. We are not going to need this, but the inf is actually a minimum.

Augmented action and value function lift in a natural way to the space M. Given $t \leq 0$ and a curve $\sigma \in AC((t,0),M)$, we can define

$$\hat{\mathcal{A}}(t,\sigma) = \int_t^0 \left[\frac{1}{2}||\dot{\sigma}_s||_M^2 - \hat{\mathcal{F}}(\sigma_s)\right] \mathrm{d}s + \hat{\mathcal{U}}_0(\sigma_0).$$

For $t \leq 0$ and $\psi \in M$, we set

$$\hat{\mathcal{U}}(t,\psi) = \inf{\{\hat{\mathcal{A}}(t,\sigma) : \sigma \in AC((t,0), M) \text{ and } \sigma_t = \psi\}}.$$

Lemma 3.1. Let \mathcal{U} and $\hat{\mathcal{U}}$ be defined as above. Then, the following holds.

1) The function $\hat{\mathcal{U}}$ is continuous. Moreover, it is H and $L^2_{\mathbf{Z}}$ -equivariant, i. e.

$$\hat{\mathcal{U}}(t,\psi) = \hat{\mathcal{U}}(t,\psi \circ h + z) \qquad \forall (t,\psi,h,z) \in (-\infty,0] \times M \times H \times L_{\mathbf{Z}}^2.$$

2) Let $\tilde{\mu} \in \mathcal{P}(\mathbf{T}^d)$ and let $\psi \in M$ be such that $(\pi \circ \psi)_{\sharp} \mathcal{L}^d = \tilde{\mu}$. Then,

$$\mathcal{U}(t,\tilde{\mu}) = \hat{\mathcal{U}}(t,\psi).$$

Proof. Point 1) is easy to dispatch, since continuity is standard; we follow [18] for equivariance. If σ_s is an AC curve with $\sigma_t = \psi$, $h \in H$ and $z \in L^2_{\mathbf{Z}}$, then $\tilde{\sigma}_s = \sigma_s \circ h + z$ is AC and satisfies $\tilde{\sigma}_t = \psi \circ h + z$; moreover, since the Lagrangian and $\hat{\mathcal{U}}_0$ are $L^2_{\mathbf{Z}}$ and H-equivariant, we see immediately that

$$\mathcal{A}(t,\sigma) = \mathcal{A}(t,\tilde{\sigma}).$$

Clearly, this implies that $\hat{\mathcal{U}}(t, \psi \circ h + z) \leq \hat{\mathcal{U}}(t, \psi)$; the opposite inequality is similar.

As for point 2), we begin to prove that

$$\hat{\mathcal{U}}(t,\psi) \le \mathcal{U}(t,\tilde{\mu}). \tag{3.3}$$

We assert that this follows if we show that, for any curve (μ_s, X) satisfying (3.1) with $\mu_t = \tilde{\mu}$ we can find $\sigma \in AC([t, 0], M)$ such that

- $i) (\pi \circ \sigma_t)_{\sharp} \mathcal{L}^d = (\pi \circ \psi)_{\sharp} \mathcal{L}^d = \tilde{\mu},$
- $ii) \ \mathcal{A}(t, \mu_s, X) = \hat{\mathcal{A}}(t, \sigma).$

Indeed, we saw after formula (1.15) that i) together with point 1) of this lemma implies that $\hat{\mathcal{U}}(t, \sigma_0) = \hat{\mathcal{U}}(t, \psi)$; since ii) implies that $\hat{\mathcal{U}}(t, \sigma_0) \leq \mathcal{U}(t, \tilde{\mu})$, formula (3.3) follows.

Thus, let (μ_s, X) be a weak solution of (3.1) with $\mu_t = \tilde{\mu}$. By proposition 4.21 of [5] (or theorem 8.2.1 of [4]) there is a measure Ξ on $C([t, 0], \mathbf{T}^d)$ such that, denoting by $\eta_s: C([t, 0], \mathbf{T}^d) \to \mathbf{T}^d$ the evaluation map $\eta_s: \gamma \to \gamma_s$, we have

$$(\eta_s)_{\sharp} \Xi = \mu_s \quad \text{for all} \quad s \in [t, 0]. \tag{3.4}$$

Moreover, Ξ concentrates on absolutely continuous curves and

$$\int_{C([a,b],\mathbf{T}^d)} d\Xi(\gamma) \int_t^0 |\dot{\gamma}(s)|^2 ds = \int_t^0 ||X(s,x)||_{L^2_{\mu_s}}^2 ds.$$
 (3.5)

It is standard (see for instance theorem 15.5.16 of [22]) that there is a Borel map $B: [0,1)^d \to C([t,0], \mathbf{T}^d)$ such that $\Xi = B_{\sharp} \mathcal{L}^d$. We set

$$\sigma_s(x) = B(x)(s) = \eta_s \circ B(x).$$

Now point i) follows from (3.4), since $(\sigma_t)_{\sharp}\mathcal{L}^d = (\eta_t \circ B)_{\sharp}\mathcal{L}^d = (\eta_t)_{\sharp}\Xi = \mu_t$. We prove point ii).

The first equality below is the definition of \mathcal{A} , the second one is implied by (3.4) and (3.5) while the third one follows because $\Xi = B_{\sharp}\mathcal{L}^d$ and $(\eta_0)_{\sharp}\Xi = \mu_0 = (\sigma_0)_{\sharp}\mathcal{L}^d$. The last equality is the definition of $\hat{\mathcal{A}}$.

$$\mathcal{A}(t, \mu_s, X) = \int_t^0 \left[\frac{1}{2} ||X(s, \cdot)||_{L_{\mu_s}^2}^2 - \frac{1}{2} \int_{\mathbf{T}^d \times \mathbf{T}^d} \phi(q - q') d\mu_s(q) d\mu_s(q') \right] ds + \mathcal{U}_0(\mu_0) =$$

$$\int_{t}^{0} ds \left[\int_{C([a,b],\mathbf{T}^{d})} \frac{1}{2} |\dot{\gamma}(s)|^{2} d\Xi(\gamma) - \frac{1}{2} \int_{C([a,b],\mathbf{T}^{d}) \times C([a,b],\mathbf{T}^{d})} \phi(\gamma(s) - \gamma'(s)) d\Xi(\gamma) d\Xi(\gamma') \right] + \frac{1}{2} \left[\int_{C([a,b],\mathbf{T}^{d}) \times C([a,b],\mathbf{T}^{d})} \frac{1}{2} |\dot{\gamma}(s)|^{2} d\Xi(\gamma) - \frac{1}{2} \int_{C([a,b],\mathbf{T}^{d}) \times C([a,b],\mathbf{T}^{d})} \phi(\gamma(s) - \gamma'(s)) d\Xi(\gamma) d\Xi(\gamma') d$$

$$+\mathcal{U}_0((\eta_0)_{\sharp}\Xi) = \int_t^0 \left[\frac{1}{2}||\dot{\sigma}_s||_M^2 ds - \int_t^0 \hat{\mathcal{F}}(\sigma_s)ds\right] + \hat{\mathcal{U}}_0(\sigma_0) = \hat{\mathcal{A}}(t,\sigma).$$

To prove the inequality opposite to (3.3), we let $\sigma \in AC((t,0),M)$ with $\sigma_0 = \psi$ and we define

$$\mu_s = (\pi \circ \sigma_s)_{\dagger} \mathcal{L}^d \quad \text{for} \quad s \in (t, 0). \tag{3.6}$$

We want to show

- a) that μ satisfies (3.1) for a suitable drift X and
- b) that the augmented action of (μ_s, X) isn't larger than the augmented action of σ .

Clearly, a) and b) imply the inequality opposite to (3.3), from which the thesis follows. We begin with a): the idea is that X(s,q) is the average of the velocities $\dot{\sigma}_s(x)$ of the curves which satisfy $\sigma_s(x) = q$.

The measure $\mathcal{L}^1 \otimes (\pi \circ \sigma_s, \dot{\sigma}_s)_{\sharp} \mathcal{L}^d$ on $[t, 0] \times \mathbf{T}^d \times \mathbf{R}^d$ has marginal $\mathcal{L}^1 \otimes (\pi \circ \sigma_s)_{\sharp} \mathcal{L}^d$ on $[t, 0] \times \mathbf{T}^d$; we disintegrate $\mathcal{L}^1 \otimes (\pi \circ \sigma_s, \dot{\sigma}_s)_{\sharp} \mathcal{L}^d = \mathcal{L}^1 \otimes (\pi \circ \sigma_s)_{\sharp} \mathcal{L}^d \otimes \nu_{s,q}$ where $\nu_{s,q}$ is a measure on \mathbf{R}^d , depending in a Borel way on $(s,q) \in [t,0] \times \mathbf{T}^d$. In other words, if $f \in C(\mathbf{T}^d \times \mathbf{R}^d)$ is such that $\frac{|f(x,v)|}{1+|v|^2}$ is bounded, then the first equality below holds for \mathcal{L}^1 a. e. $s \in [a,b]$; the second equality comes from (3.6).

$$\int_{[0,1)^d} f(\sigma_s(x), \dot{\sigma}_s(x)) dx = \int_{[0,1)^d} dx \int_{\mathbf{R}^d} f(\sigma_s(x), v) d\nu_{s,\sigma_s(x)}(v) = \int_{\mathbf{T}^d} d\mu_s(q) \int_{\mathbf{R}^d} f(q, v) d\nu_{s,q}(v).$$
(3.7)

We set

$$X(s,q) = \int_{\mathbf{R}^d} v \mathrm{d}\nu_{s,q}(v)$$

Let now $\phi \in C_c^{\infty}((t,0) \times \mathbf{T}^d)$; the first equality below comes from (3.6), the second one from the definition of X and the third one from (3.7). The last equality follows since ϕ has compact support in $(t,0) \times \mathbf{T}^d$.

$$\int_{t}^{0} ds \int_{\mathbf{T}^{d}} [\partial_{s}\phi(s,q) + \langle \nabla\phi(s,q), X(s,q) \rangle] d\mu_{s}(q) =$$

$$\int_{t}^{0} ds \int_{[0,1)^{d}} [\partial_{s}\phi(s,\sigma_{s}(x)) + \langle \nabla\phi(s,\sigma_{s}(x)), X(s,\sigma_{s}(x)) \rangle] dx =$$

$$\int_{t}^{0} ds \int_{[0,1)^{d}} \left[\partial_{s}\phi(s,\sigma_{s}(x)) + \langle \nabla\phi(s,\sigma_{s}(x)), \int_{\mathbf{R}^{d}} v d\nu_{s,\sigma_{s}(x)}(v) \rangle \right] dx =$$

$$\int_{t}^{0} ds \int_{[0,1)^{d}} [\partial_{s}\phi(s,\sigma_{s}(x)) + \langle \nabla\phi(s,\sigma_{s}(x)), \dot{\sigma}_{s}(x) \rangle] dx =$$

$$\int_{t}^{0} \left[\frac{d}{ds} \int_{[0,1)^{d}} \phi(s,\sigma_{s}(x)) dx \right] ds = 0.$$

This means that (μ_s, X) is a weak solution of (3.1), i. e. point a) holds.

As for b), it is the same calculation, up to the use of Jensen's inequality:

$$\int_{t}^{0} \left[\frac{1}{2} \int_{\mathbf{T}^{d}} |X(s,q)|^{2} d\mu_{s}(q) - \mathcal{F}(\mu_{s}) \right] ds + \mathcal{U}_{0}(\mu_{0}) \leq$$

$$\int_{t}^{0} \left[\frac{1}{2} \int_{\mathbf{R}^{d}} |v|^{2} d\nu_{s,q}(v) - \hat{\mathcal{F}}(\sigma_{s}) \right] ds + \hat{\mathcal{U}}(\sigma_{0}) = \int_{t}^{0} \left[\frac{1}{2} ||\dot{\sigma}_{s}||_{M}^{2} - \hat{\mathcal{F}}(\sigma_{s}) \right] ds + \hat{\mathcal{U}}(\sigma_{0}).$$

Secured by the last lemma, from now on we shall concentrate on $\hat{\mathcal{A}}$ and $\hat{\mathcal{U}}$.

Definition. By $H^1_M(t,0)$ we denote the space of the maps $\sigma \in AC((t,0),M)$ such that

$$||\sigma||_{H_M^1}^2 := ||\sigma_t||_M^2 + \int_t^0 ||\dot{\sigma}_s||_M^2 ds < +\infty.$$

It is standard ([1]) that this is a Hilbert space for the inner product

$$\langle \sigma, \eta \rangle_{H_M^1} := \langle \sigma_t, \eta_t \rangle_M + \int_t^0 \langle \dot{\sigma}_s, \dot{\eta}_s \rangle \mathrm{d}s.$$

We recall the Poincaré-Wirtinger inequality

$$\sup_{s \in (t,0)} ||\sigma_s||_M \le ||\sigma_t||_M + |t|^{\frac{1}{2}} \cdot ||\sigma||_{H_M^1}.$$

Lemma 3.2. For t < 0, let us consider the functional

$$I: H_M^1(t,0) \to \mathbf{R}, \qquad I: \sigma \to \hat{\mathcal{A}}(t,\sigma)$$

where the augmented action \hat{A} has been defined at the beginning of this section. Then, the following points hold.

1) The functional I is of class C^1 on $H_M^1(t,0)$. For \hat{F} and \hat{u}_0 defined as in (2.2), we have

$$I'(\sigma)(h) = \int_{t}^{0} [\langle \dot{\sigma}_{s}, \dot{h}_{s} \rangle_{M} - \langle \nabla \hat{F}(\sigma_{s}(\cdot), \sigma_{s}), h_{s} \rangle_{M}] ds + \langle \nabla \hat{u}(\sigma_{0}(\cdot), \sigma_{0})), h_{0} \rangle_{M} =$$

$$\int_{t}^{0} \langle \dot{\sigma}_{s}, \dot{h}_{s} \rangle_{M} ds - \int_{t}^{0} ds \int_{[0,1)^{d} \times [0,1)^{d}} \langle \nabla \phi(\sigma_{s}(x) - \sigma_{s}(y)), h_{s}(x) \rangle dx dy +$$

$$\int_{[0,1)^{d} \times [0,1)^{d}} \langle \nabla U^{0}(\sigma_{0}(x)) + \nabla U^{1}(\sigma_{0}(x) - \sigma_{0}(y)), h_{0}(x) \rangle dx dy. \tag{3.8}$$

To explain the notation, we recall that $\nabla \hat{F}(\cdot, \sigma_s)$ is a C^2 function from \mathbf{T}^d to \mathbf{R}^d and thus $\nabla \hat{F}(\sigma_s(\cdot), \sigma_s) \in M$. 2) Let $\sigma \in H^1_M(t, 0)$ be minimal in the definition of $\hat{\mathcal{U}}(t, \psi)$; then, σ solves

$$\begin{cases}
\ddot{\sigma}_s(x) = -(\nabla \phi * \mu_s)(\sigma_s(x)) = -\nabla \hat{F}(\sigma_s(x), \sigma_s) & \text{for } s \in (t, 0) \\
\sigma_t(x) = \psi(x) \\
\dot{\sigma}_0(x) = -\nabla U^0(\sigma_0(x)) - (\nabla U^1 * \mu_0)(\sigma_0(x)) = -\nabla \hat{u}_0(\sigma_0(x), \sigma_0)
\end{cases}$$
(3.9)

where we have set $\mu_s = (\sigma_s)_{\sharp} \mathcal{L}^p$. The equalities are in the space M, i. e. they hold for a. e. $x \in [0,1)^d$.

Proof. Since the potential $\hat{\mathcal{F}}$ and the final condition $\hat{\mathcal{U}}$ are defined by (2.1), the proof of (3.8) is classical (see for instance [2]) and we forego it.

We recall the proof of point 2), which again is classical. Since I is of class C^1 by point 1), if σ minimizes I under the constraint $\sigma_t = \psi$, then we must have that

$$I'(\sigma)(h) = 0$$
 for all $h \in H_M^1(t,0)$ with $h_t = 0$.

Integrating by parts in (3.8), this implies that

$$\int_{t}^{0} \left\langle -\ddot{\sigma}_{s} - \left(\nabla \hat{F}(\sigma_{s}(\cdot), \sigma_{s}), h_{s}\right)_{M} ds + \left\langle \dot{\sigma}_{0}, h_{0}\right\rangle_{M} + \left\langle \nabla \hat{u}_{0}(\sigma_{0}(\cdot), \sigma_{0}), h_{0}\right\rangle_{M} = 0$$

for all $h \in H_M^1(t,0)$ with $h_t = 0$. Clearly, this implies the first and third formulas of (3.9), while the second one comes from the boundary conditions on the minimal σ .

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Finding minima of I is a delicate proposition (see for instance [21]) because Tonelli's theorem does not apply to the infinite-dimensional space M. However, in our case the implicit function theorem comes to the rescue: in the next three lemmas we recall the approach of [10] in our situation. In the next lemma, we denote by $B_X(\psi, r)$ the ball in X of radius r and centered in ψ .

Lemma 3.3. There are T, r > 0 such that the following holds. Let $t \in [-T, 0]$, and let $\psi \in M$; we shall denote by ψ both the element of M and the function of $H^1_M(t, 0)$ constantly equal to ψ .

- 1) There is a unique function $\sigma^{(t,\psi)} \in C^1([-T,0],M)$ such that
- i) $\sigma_s^{(t,\psi)} \in B_M(\psi,r)$ for $s \in [-T,0]$, and
- ii) $\sigma^{(t,\psi)}$ satisfies (3.9).

By the Poincaré-Wirtinger inequality, this implies that (3.9) has a unique solution in $B_{H_M^1(-T,0)}(\psi,r')$ for some r'>0.

2) The map

$$\Phi: [-T,0] \times M \to H^1_M(-T,0), \qquad \Phi: (t,\psi) \to \sigma^{(t,\psi)}$$

is of class C^2 and equivariant, i. e. $\sigma^{(t,\psi\circ h+z)}=\sigma^{(t,\psi)}\circ h+z$ for all $h\in H$ and $z\in L^2_{\mathbf{Z}}$.

Proof. Let us consider the map

$$\Sigma: [-T, 0] \times M \to M, \qquad \Sigma: (s, \tilde{\psi}) \to \sigma_s$$

where σ_s solves the Cauchy problem

$$\begin{cases}
\ddot{\sigma}_s(x) = -\nabla \hat{F}(\sigma_s(x), \sigma_s) \\
\sigma_0 = \tilde{\psi} \\
\dot{\sigma}_0(x) = -\nabla \hat{u}_0(\sigma_0(x), \sigma_0) = -\nabla \hat{u}_0(\tilde{\psi}(x), \psi)
\end{cases}$$
(3.10)

for the functions \hat{F} and \hat{u} which have been defined in (2.2). Since these two functions are of class C^3 by lemma 2.1, their gradients are in C^2 and the map Σ is of class C^2 by the continuous dependence theorem.

Step 1. We assert that points 1) and 2) follow if we show that there is a C^2 function $\tilde{\psi}$: $[-T, 0] \times M \to M$ which is, for all $\psi \in M$, the unique solution in $B(\psi, r)$ of

$$\Sigma(t, \tilde{\psi}(t, \psi)) = \psi. \tag{3.11}$$

Indeed, if this holds we can set

$$\sigma_s^{(t,\psi)} = \Sigma(s, \tilde{\psi}(t, \psi)) \tag{3.12}$$

and (3.11) immediately implies that

$$\sigma_t^{(t,\psi)} = \psi$$

i. e. $\sigma^{(t,\psi)}$ satisfies the second equation of (3.9).

Moreover, the map $:(t,\psi,s)\to\sigma_s^{(t,\psi)}$ is of class C^2 because of (3.12) and the fact that Σ and $\tilde{\psi}$ are of class C^2 ; in particular, $\sigma^{(t,\psi)}\in H^1_M(-T,0)$. The map $\sigma^{(t,\psi)}$ solves the first equation of (3.9) because $:s\to\Sigma(s,\tilde{\psi}(t,\psi))$ solves it by the definition of Σ . Finally, $\sigma^{(t,\psi)}$ satisfies the third equation of (3.9) simply because it satisfies the third equation of (3.10). Uniqueness follows because, if (3.9) had two different solutions in $B_M(\psi,r)$, then also (3.11) would have two different solutions in $B_M(\psi,r)$, and we are supposing that this is not the case.

We prove the last assertion of the lemma, equivariance. Recall that \hat{F} and \hat{u}_0 are H and $L^2_{\mathbf{Z}}$ -equivariant; in particular, if $\sigma^{(t,\psi)}$ satisfies (3.9) and $(h,z) \in H \times L^2_{\mathbf{Z}}$, then also $\sigma^{(t,\psi)} \circ h + z$ satisfies (3.9) for the initial condition $\psi \circ h + z$. By the uniqueness of point 1), this implies that $\sigma^{(t,\psi)} \circ h + z$ for all $h \in H$ and $z \in L^2_{\mathbf{Z}}$.

Step 2. In this step and in the following ones, we check that we can apply the implicit function theorem to solve for ψ in (3.11).

First of all, we saw above that the map Σ is C^2 . By definition, $\Sigma(0,\psi) = \psi$ for all $\psi \in M$, which implies that

$$D\Sigma(0,\psi_0)=Id \qquad \forall \psi_0 \in M.$$

Thus, the implicit function theorem yields the existence of a C^2 function $\tilde{\psi}(t,\psi)$ defined in $[-T_0,0] \times B_M(\psi_0,r)$ which solves (3.1).

In step 3 below, we shall see that T_0 and r do not depend on ψ_0 ; in step 4, we shall use the monodromy theorem to glue the local solutions into a solution defined globally on $[-T_0, 0] \times M$.

Step 3. We prove that we can choose T_0 and r independent on ψ_0 .

If we look at the proof of the implicit function theorem, we see that $T_0, r > 0$ must be chosen in order that the Lipschitz constant of $: \psi \to \Sigma(t, \psi) - \psi$ is smaller than, say, $\frac{1}{2}$ in $[-T_0, 0] \times B(\psi_0, r)$; by the Lagrange theorem, this follows if $||D\Sigma(t, \psi) - Id|| \le \frac{1}{2}$ in $[-T_0, 0] \times B(\psi_0, r)$. This follows by a Taylor development, since we saw above that $D\Sigma(0, \psi) - Id = 0$ for all ψ and that $||\partial_t D\Sigma(t, \psi)||$ is bounded in $[-1, 0] \times M$.

Step 4. By the last step, in each neighbourhood $[-T_0, 0] \times B(\psi_0, r)$ we can define a function $\tilde{\psi}$ which satisfies (3.12); since M is simply connected, we can use the monodromy theorem (see for instance theorem 1.8 of chapter 3 of [2]) to define globally a function $\tilde{\psi}$: $[-T_0, 0] \times M \to M$ satisfying (3.11).

Definition. From now on, $\sigma_s^{(t,\psi)}$ will be defined as in the last lemma.

Since the map $:(t,\psi)\to\sigma^{(t,\psi)}$ is of class C^2 , the next lemma reduces to a classical computation ([10]) which we are only going to sketch; we continue in our practice of denoting by D the derivative in the M variable.

Lemma 3.4. We set

$$\hat{\mathcal{V}}(t,\psi) = \int_{t}^{0} \left[\frac{1}{2} ||\dot{\sigma}_{s}^{(t,\psi)}||_{M}^{2} - \hat{\mathcal{F}}(\sigma_{s}^{(t,\psi)}) \right] ds + \hat{\mathcal{U}}_{0}(\sigma_{0}^{(t,\psi)}). \tag{3.13}$$

Then, $\hat{\mathcal{V}} \in C^2([-T,0] \times M)$ and we have

$$\begin{cases} -\partial_t \hat{\mathcal{V}}(t, \psi) + \frac{1}{2} ||D\hat{\mathcal{V}}(t, \psi)||_M^2 + \hat{\mathcal{F}}(\psi) = 0 & \text{for } (t, \psi) \in [-T, 0] \times M \\ \hat{\mathcal{V}}(0, \psi) = \hat{\mathcal{U}}_0(\psi). \end{cases}$$
(3.14)

Moreover,

$$\dot{\sigma}_s^{(t,\psi)} = -D\hat{\mathcal{V}}(s, \sigma_s^{(t,\psi)}) \quad \text{for all} \quad s, t \in [-T, 0]. \tag{3.15}$$

Proof. First of all, $\hat{\mathcal{V}} \in C^2([-T,0] \times M)$ by point 2) of lemma 3.3. Next, we differentiate with respect to ψ both terms of (3.13); after using (3.8) and (3.9) we get that

$$\dot{\sigma}_t^{(t,\psi)} = -D\hat{\mathcal{V}}(t,\sigma_t^{(t,\psi)}) = -D\hat{\mathcal{V}}(t,\psi). \tag{3.16}$$

Now we differentiate in (3.13) with respect to t; after an integration by parts, we get that

$$\begin{split} \partial_t \hat{\mathcal{V}}(t,\psi) &= -\frac{1}{2} ||\dot{\sigma}_t^{(t,\psi)}||_M^2 + \hat{\mathcal{F}}(\sigma_t^{(t,\psi)}) + \\ &\int_t^0 \langle -\ddot{\sigma}_s^{(t,\psi)} - D\hat{\mathcal{F}}(\sigma_s^{(t,\psi)}), \partial_t \sigma_t^{(s,\psi)} \rangle_M \mathrm{d}s + \\ &\langle \dot{\sigma}_s^{(t,\psi)}, \partial_t \sigma_s^{(t,\psi)} \rangle_M |_{s=t}^{s=0} + \langle D\hat{\mathcal{U}}(\sigma_0^{(t,\psi)}), \partial_t \sigma_0^{(t,\psi)} \rangle_M. \end{split}$$

We note that the integral term is zero by the first equation of (3.9). Since $\sigma_t^{(t,\psi)} = \psi$ for all t, differentiating we get that

$$\partial_t \sigma_s^{(t,\psi)}|_{s=t} = -\dot{\sigma}_t^{(t,\psi)}.$$

Together with the last equation of (3.9), the last two equations imply that

$$\partial_t \hat{\mathcal{V}}(t,\psi) = \frac{1}{2} ||\dot{\sigma}_t^{(t,\psi)}||_M^2 + \hat{\mathcal{F}}(\sigma_t^{(t,\psi)}).$$

Bt (3.16), this implies (3.14).

Next, we assert that (3.15) follows from (3.16) if we show that, for all $t, s, \tau \in [-T, 0]$, we have that

$$\sigma_{\tau}^{(t,\psi)} = \sigma_{\tau}^{(s,\sigma_s^{(t,\psi)})}.\tag{3.17}$$

To show the assertion, we denote by the dot the derivative in the τ variable; now (3.17) implies the first equality below, (3.16) the second one.

$$\dot{\sigma}_{\tau}^{(t,\psi)}|_{\tau=s} = \dot{\sigma}_{\tau}^{(s,\sigma_{s}^{(t,\psi)})}|_{\tau=s} = -D\hat{\mathcal{V}}(s,\sigma_{s}^{(t,\psi)}).$$

To show (3.17), by the uniqueness of lemma 3.3 it suffices to show that $: \tau \to \sigma_{\tau}^{(t,\psi)}$ satisfies

$$\begin{cases} \ddot{\sigma}_{\tau}^{(t,\psi)}(x) = -\nabla \hat{F}(\sigma_{\tau}^{(t,\psi)}(x), \sigma_{\tau}^{(t,\psi)}) \\ \sigma_{s}^{(t,\psi)}(x) = \sigma_{s}^{(t,\psi)}(x) \\ \dot{\sigma}_{0}^{(t,\psi)}(x) = -\nabla \hat{u}_{0}(\sigma_{0}^{(t,\psi)}(x), \sigma_{0}^{(t,\psi)}) \end{cases}$$

which is obvious since $\sigma^{(t,\psi)}$ satisfies (3.9).

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Lemma 3.5. Let $t \in [-T, 0]$ and let $\psi \in M$. Then,

- 1) for all $s \in [-T, 0]$, $\sigma^{(t,\psi)}$ is the unique minimal in the definition of $\hat{\mathcal{U}}(s, \sigma_s^{(t,\psi)})$.
- 2) $\hat{\mathcal{U}}(t,\psi) = \hat{\mathcal{V}}(t,\psi)$ for $(t,\psi) \in [-T,0] \times M$.

Proof. Point 2) follows immediately from point 1) and the definitions of $\hat{\mathcal{U}}$ and $\hat{\mathcal{V}}$; we recall the classical proof of [10] for point 1). Let $\hat{\mathcal{V}}$ be as in the last lemma and let us consider the functional

$$J_s: H^1_M(t,0) \to \mathbf{R}.$$

$$J_s: \sigma \to \int_{0}^{0} \left[\frac{1}{2} ||\dot{\sigma}_{\tau}||_{M}^{2} - \mathcal{F}(\sigma_{\tau}) + \partial_{\tau} \hat{\mathcal{V}}(\tau, \sigma_{\tau}) + \langle D\hat{\mathcal{V}}(\tau, \sigma_{\tau}), \dot{\sigma}_{\tau} \rangle_{M}\right] d\tau.$$
 (3.18)

Since $\hat{\mathcal{V}}$ is of class C^2 by lemma 3.4, we get the first equality below, while the second one follows from the second formula of (3.14) and the definition of $\hat{\mathcal{A}}$ at the beginning of this section.

$$J_s(\sigma) = \int_s^0 \left[\frac{1}{2} ||\dot{\sigma}_\tau||_M^2 - \mathcal{F}(\sigma_\tau) \right] d\tau + \hat{\mathcal{V}}(0, \sigma_0) - \hat{\mathcal{V}}(s, \sigma_s) =$$

$$\hat{\mathcal{A}}(s, \sigma) - \hat{\mathcal{V}}(s, \sigma_s). \tag{3.19}$$

Thus, if we restrict to the curves $\sigma \in H^1_M(s,0)$ with $\sigma_s = \sigma_s^{(t,\psi)}$, minimizing J_s is the same as minimizing $\hat{\mathcal{A}}(s,\sigma)$: the thesis follows if we check that $\sigma^{(t,\psi)}$ is minimal for J_s . Actually, we are going to show that the integrand of J_s is constantly equal to its minimum along $(\tau, \sigma_{\tau}^{(t,\psi)}, \dot{\sigma}_{\tau}^{(t,\psi)})$.

Clearly, for all $(\tau, \eta) \in [-T, 0] \times M$ the minimum of the Lagrangian of J_s

$$B_{\tau,n}:M\to\mathbf{R}$$

$$B_{\tau,\eta} : \dot{\lambda} \to \frac{1}{2} ||\dot{\lambda}||_M^2 - \mathcal{F}(\eta) + \partial_\tau \hat{\mathcal{V}}(\tau,\eta) + \langle D_\eta \hat{\mathcal{V}}(\tau,\eta), \dot{\lambda} \rangle_M$$

is attained at $\dot{\lambda} = -D_{\eta}\hat{\mathcal{V}}(\tau, \eta)$; substituting this value into the expression for $B_{\tau,\eta}$ we get the inequality below, while the equality is the first formula of (3.14).

$$B_{\tau,\eta}(\dot{\lambda}) \ge -\frac{1}{2} ||D_{\eta} \hat{\mathcal{V}}(\tau,\eta)||_{M}^{2} - \mathcal{F}(\eta) + \partial_{\tau} \hat{\mathcal{V}}(\tau,\eta) = 0 \qquad \forall \dot{\lambda} \in M.$$
 (3.20)

On the other side, (3.15) implies the second equality below, (3.14) the third one.

$$B_{\tau, \dot{\sigma}_{\tau}^{(t,\psi)}}(\dot{\sigma}_{\tau}^{(t,\psi)}) = \frac{1}{2} ||\dot{\sigma}_{\tau}^{(t,\psi)}||_{M}^{2} - \mathcal{F}(\sigma_{\tau}^{(t,\psi)}) + \partial_{\tau} \hat{\mathcal{V}}(\tau, \sigma_{\tau}^{(t,\psi)}) + \langle D\hat{\mathcal{V}}(\tau, \sigma_{\tau}^{(t,\psi)}), \dot{\sigma}_{\tau}^{(t,\psi)} \rangle_{M} =$$

$$-\frac{1}{2} ||D\hat{\mathcal{V}}(\tau, \sigma_{\tau}^{(t,\psi)})||_{M}^{2} - \mathcal{F}(\sigma_{\tau}^{(t,\psi)}) + \partial_{\tau} \hat{\mathcal{V}}(\tau, \sigma_{\tau}^{(t,\psi)}) = 0.$$

The last two formulas imply that $: \tau \to \sigma_{\tau}^{(t,\psi)}$ minimizes J_s , as we wanted.

We prove uniqueness: by the aforesaid, if σ_{τ} minimizes, then the integrand of J_s must be zero along σ_{τ} . By (3.20), this implies that $\dot{\sigma}_{\tau} = -D\mathcal{V}(\tau, \sigma_{\tau})$. By (3.15) this implies that σ_{τ} and $\sigma_{\tau}^{(t,\psi)}$ satisfy the same differential equation; we recall from lemma 3.4 that $-D\hat{\mathcal{V}}(t,\psi)$ is Lipschitz. Since $\sigma_s = \sigma_s^{(t,\psi)}$ by hypothesis, we get that $\sigma_{\tau} = \sigma_{\tau}^{(t,\psi)}$ for $\tau \in [-T,0]$ by the existence and uniqueness theorem.

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§**4**

The master equation

In this section, we are going to define the value function for the single particle; we shall see that it determines the movement of the whole pack and that it satisfies the master equation.

Definition. We define

$$v: [-T, 0] \times \mathbf{T}^{d} \times [-T, 0] \times M \to \mathbf{R},$$

$$v(s, q|t, \psi) = \min \left\{ \int_{s}^{0} \left[\frac{1}{2} |\dot{y}(\tau)|^{2} - \hat{F}(y(\tau), \sigma_{\tau}^{(t,\psi)}) \right] d\tau + \hat{u}_{0}(y(0), \sigma_{0}^{(t,\psi)}) \right\}$$
(4.1)

where the minimum (whose existence is guaranteed by Tonelli's theorem) is over all $y \in AC((s,0), \mathbf{T}^p)$ such that y(s) = q. In the notation for v we have inaugurated the practice of placing the "parameters", in this case (t, ψ) , after the vertical slash. In other words, we are interested in the equation solved by v in the first two variables. If we freeze (t, ψ) , then $v(s, q|t, \psi)$ is the value function of the particle q, given that the whole pack moves like $\sigma^{(t,\psi)}$. Thus, v solves, in its first two variables, the Hamilton-Jacobi equation.

Lemma 4.1. Up to reducing T, the following holds.

1) For $s,t \in [-T,0]$, the minimum in the definition of $v(s,q|t,\psi)$ is attained on a unique function

$$: \tau \to y(\tau|s,q,t,\psi).$$

Again, the parameters of the orbit (i. e. the initial conditions of the single particle and of the whole pack) are on the right of the vertical slash.

2) The map

$$: (\tau, s, q, t, \psi) \rightarrow y(\tau | s, q, t, \psi)$$

is of class C^2 .

3) The value function

$$: (s, q, t, \psi) \rightarrow v(s, q|t, \psi)$$

is of class C^2 with bounded first and second derivatives. It is \mathbf{Z}^d -equivariant in the second variable, H and $L^2_{\mathbf{Z}}$ -equivariant in the fourth one. For all $(t,\psi) \in [-T,0] \times M$ it satisfies the Hamilton-Jacobi equation with time reversed

$$\begin{cases}
-\partial_s v(s, q|t, \psi) + \frac{1}{2} |\nabla v(s, q|t, \psi)|^2 + \hat{F}(q, \sigma_s^{(t, \psi)}) = 0 \quad (s, q) \in [-T, 0] \times \mathbf{T}^d \\
v(0, q|t, \psi) = \hat{u}_0(q, \sigma_0^{(t, \psi)})
\end{cases}$$
(4.2)

in the classical sense. Recall that we denote the gradient in the \mathbf{T}^p variable by ∇ , in the M variable by D.

4) We have that, for \mathcal{L}^p a. e. $x \in [0,1)^d$ and all $t, s, \tau \in [-T,0]$,

$$\dot{y}(\tau|s,\sigma_s^{(t,\psi)}(x),t,\psi) = \dot{\sigma}_\tau^{(t,\psi)}(x) = -\nabla v(\tau,y(\tau|s,\sigma_s^{(t,\psi)}(x),t,\psi)|t,\psi) = -D\hat{\mathcal{V}}(\tau,\sigma_\tau^{(t,\psi)})(x).$$

5) Let us define the function S as the flow of $-\nabla v$, i. e. as

$$S(s, q, \tau | t, \psi) = y(\tau)$$

where y solves

$$\begin{cases} \dot{y}(\tau) = -\nabla v(\tau, y(\tau)|t, \psi) \\ y(s) = q. \end{cases}$$
(4.3)

Then, up to reducing T, there is $D_2 > 0$ independent of $(s, q, \tau, t, \psi) \in [-T, 0] \times \mathbf{T}^d \times [-T, 0]^2 \times M$ such that

$$\frac{1}{D_2} \le \det \frac{\partial S(s, q, \tau | t, \psi)}{\partial q} \le D_2.$$

Proof. We fix (t, ψ) as the initial condition of the whole pack; we consider the time dependent Lagrangian

$$\mathcal{L}(s, q, \dot{q}) = \frac{1}{2} |\dot{q}|^2 - \hat{F}(q, \sigma_s^{(t, \psi)})$$

and the final condition

$$: q \to \hat{u}_0(q, \sigma_0^{(t,\psi)}).$$

Note that, by lemma 2.1, \mathcal{L} is C^3 in (s, q, \dot{q}) ; it depends in a C^2 way on the parameters (t, ψ) by lemma 3.3. Analogously, \hat{u}_0 is C^3 in the variable q and C^2 in (t, ψ) . Now points 1), 2) and 3) follow by the argument of [10], which we have seen in lemmas 3.3, 3.4 and 3.5 above. As for point 4), formula (3.15) gives that, for all $\tau \in [-T, 0]$,

$$\dot{\sigma}_{\tau}^{(t,\psi)}(x) = -D\hat{\mathcal{V}}(\tau, \sigma_{\tau}^{(t,\psi)})(x) \quad \text{for } \mathcal{L}^p \text{ a. e.} \quad x \in [0,1)^d.$$

On the other side, with exactly the same proof we used for formula (3.15) we see that

$$\dot{y}(\tau|s,\sigma_s^{(t,\psi)}(x),t,\psi) = -\nabla v(\tau,y(\tau|s,\sigma_s^{(t,\psi)}(x),t,\psi)|t,\psi) \quad \text{for} \quad t,s,\tau \in [-T,0].$$

Thus, it suffices to show the first equality of point 4). Classical Hamilton-Jacobi theory (which we recalled above in lemmas 3.3 to 3.5) implies that the minimizer

$$: \tau \to y(\tau|s,q,t,\psi)$$

satisfies

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}y(\tau|s,q,t,\psi) = -\nabla \hat{F}(y(\tau|s,q,t,\psi),\sigma_{\tau}^{(t,\psi)}) \\ y(s|s,q,t,\psi) = q \\ \dot{y}(0|s,q,t,\psi) = -\nabla \hat{u}_0(y(0|s,q,t,\psi),\sigma_0^{(t,\psi)}). \end{cases}$$

If $q = \sigma_s^{(t,\psi)}(x)$ then, by (3.9), this is the same equation that is satisfied by $: \tau \to \sigma_\tau^{(t,\psi)}(x)$ for \mathcal{L}^d a. e. $x \in [0,1)^d$; by the uniqueness of lemma 3.3 this implies the first equality of point 4).

We prove point 5). Since $S(s,q,s|t,\psi)=q$ by definition, we see that $\partial_q S(s,q,s|t,\psi)=Id$; thus, point 5) follows if we show that the map $:\tau\to\partial_q S(s,q,\tau|t,\psi)$ is Lipschitz uniformly in (s,q,τ,t,ψ) ; in other words, we have to show that the norm of $\partial_{q\tau}^2 S(s,q,\tau|t,\psi)$ is bounded. This follows easily by (4.3), the differentiable dependence theorem and point 3) of this lemma, which implies

$$|\partial_{q,q}^2 v(s,q|t,\psi)| \le M$$
 $\forall (s,q,t,\psi) \in [-T,0] \times \mathbf{T}^d \times [-T,0] \times M.$

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We can apply to the value function $v(s,q|t,\psi)$ a change of coordinates: namely, instead of seeing it as a function of $\sigma_s^{(t,\psi)} = \psi$, we can see it as a function of $\sigma_s^{(t,\psi)}$. In other words, we can define a function u as

$$u(s, q|\sigma_s^{(t,\psi)}) := v(s, q|t, \psi).$$

Equivalently, by (3.17) we get that, for $\psi \in M$, $\psi = \sigma_t^{(s,\sigma_s^{(t,\psi)})}$; setting $\eta = \sigma_s^{(t,\psi)}$ and substituting in the formula above, we get that

$$u(s,q|\eta) = v(s,q|t,\sigma_t^{(s,\eta)}) \quad \text{for all} \quad t \in [-T,0], \quad \eta \in M$$

$$\tag{4.4}$$

which incidentally proves that the definition of u is well posed. The first equality below comes from (4.4), since $\sigma_s^{(s,\psi)} = \psi$; the second one is (4.1).

$$u(s, q|\psi) = v(s, q|s, \psi) =$$

$$\min \left\{ \int_{s}^{0} \left[\frac{1}{2} |\dot{y}(\tau)|^{2} - \hat{F}(y(\tau), \sigma_{\tau}^{(s,\psi)}) \right] d\tau + \hat{u}_{0}(y(0), \sigma_{0}^{(s,\psi)}) : y \in AC((s,0), \mathbf{T}^{p}), \quad y(s) = q \right\}.$$
 (4.5)

Lemma 4.2. Let

$$u: [-T, 0] \times \mathbf{T}^d \times M \to \mathbf{R}$$

be defined as in (4.4) or as in (4.5), which is the same. Then, u is of class C^2 in all its variables and satisfies the master equation

$$-\partial_t u(t,q|\psi) + \frac{1}{2} |\nabla u(t,q|\psi)|^2 + F(q,\psi) + \langle \nabla u(t,\psi(\cdot)|\psi), Du(t,q|\psi) \rangle_M = 0.$$

Proof. By (4.4), lemma 4.1 and the chain rule we get that u is of class C^2 in all its variables. Since $\sigma_t^{(t,\psi)} = \psi$ for all t, differentiating we get that

$$\frac{\partial}{\partial s} \sigma_t^{(s,\psi)}|_{s=t} = -\dot{\sigma}_t^{(t,\psi)}. \tag{4.6}$$

The first equality of (4.5) implies the equalities below.

$$Du(t, q|\psi) = Dv(t, q|t, \psi), \qquad \nabla u(t, q|\psi) = \nabla v(t, q|t, \psi).$$
 (4.7)

The first equality below is point 4) of lemma 4.1, the second one comes from (4.7).

$$\dot{\sigma}_t^{(t,\psi)}(x) = -\nabla v(t,\psi(x)|t,\psi) = -\nabla u(t,\psi(x)|\psi). \tag{4.8}$$

If we differentiate (4.4) in s, we get the first equality below; the second one comes from (4.2) and (4.6); the last one comes from (4.7) and (4.8).

$$\begin{split} \partial_s u(s,q|\psi)|_{s=t} &= \partial_s v(s,q|t,\sigma_t^{(s,\psi)})|_{s=t} + \langle Dv(s,q|t,\sigma_t^{(s,\psi)}),\frac{\partial}{\partial s}\sigma_t^{(s,\psi)}\rangle_M|_{s=t} = \\ & \frac{1}{2}|\nabla v(t,q|t,\psi)|^2 + \hat{F}(q,\psi) - \langle Dv(t,q|t,\psi),\dot{\sigma}_t^{(t,\psi)}\rangle_M = \\ & \frac{1}{2}|\nabla u(t,q|\psi)|^2 + \hat{F}(q,\psi) + \langle Du(t,q|\psi),\nabla u(t,\psi(\cdot)|\psi)\rangle_M. \end{split}$$

End of the proof of theorem 1. Point 1) follows from lemma 3.5; point 2) is point 2) of lemma 3.3; point 3) is lemma 4.2; point 4) follows from point 5) of lemma 4.1; point 5) is point 4) of lemma 4.1 and (4.7).

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Remark. By the results of section 1, $u(t, q|\psi)$ quotients to a function on measures which is strongly differentiable, with continuous derivative; it satisfies the master equation in the classical sense, i. e. taking derivatives at their face value.

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