Marginal Deformations of $\mathcal{N}=4$ SYM and of its Supersymmetric Orbifold Descendants

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Contents

Abstract

There was a great burst of research and interest in $\mathcal{N} = 4$ Super Yang-Mills in recent years. Most of the interest is due to the relation of this theory to the **AdS/CFT** correspondence (see [3], [1]). $\mathcal{N} = 4$ SU(N) SYM appears in this context as an effective description of **N** coincident D3-branes. It was argued ([18], [25], [26]) that after orbifolding the configuration above one can obtain effective descriptions in terms of conformal field theories with less supersymmetries ($\mathcal{N} = 2, \mathcal{N} = 1, \mathcal{N} = 0$). There is a great interest in looking on marginal deformations of such theories. In the **AdS/CFT** [1] correspondence such deformations on the field theory side correspond to moduli of the string theory. For instance, as we will see (see [18]), in the $\mathcal{N} = 2$ Z_k orbifold theory there are **k** marginal deformations which don't break the $\mathcal{N} = 2$ SUSY. These deformations correspond to the string coupling and the ALE blow-up modes on the string side.

In this work we investigate marginal deformations, which keep at least $\mathcal{N} = 1$ SUSY unbroken, of $\mathcal{N} = 4$ SYM and its orbifold $\mathcal{N} = 2, 1$ descendants. There are well known *exactly* marginal deformations of this type for the $\mathcal{N} = 4$ SYM ([14]), in our work we argue that these are the only *exactly* marginal deformations of this theory. It was also argued ([21], [22]) that the planar diagram contribution in the orbifold theory is the same as in the $\mathcal{N} = 4$ theory, leading to a conclusion that in the large **N** limit many correlation functions in these theories coincide up to some gauge coupling rescaling. One could expect that the orbifold theories possess similar *exactly* marginal deformations. We will see that this is actually the case, and we get additional *exactly* marginal operators from the twisted sectors. In some cases we find that the dimension of the space of *exactly* marginal deformations

at low orders in perturbation theory is smaller than the general analysis implies. Another interesting observation which is made is about gauge theories with gauge group SU(N=3). Here we find a very large number of *exactly* marginal deformations. These deformations can not be directly related to the string theory because the correspondence is well understood only in the large $\mathbf{N}$ limit.

Chapter 1

General Introduction

1.1 AdS/CFT

The aim of this work is investigating marginal deformations of a specific class of super-symmetric field theories. These theories appear in the framework of the **AdS/CFT** cor-respondence. Here we will review the general idea of this correspondence and make a link to the specific research presented in the thesis.

It was already since t'Hooft's [2] work anticipated that string theories are linked to gauge theories. The general idea of t'Hooft was based on two simple facts. The first fact is that pure gauge theories consist of fields in the adjoint representation of the gauge group which appears in the product of fundamental and anti-fundamental representations (for SU(N)), and thus every adjoint index can be described by two indices - one fundamental and one anti-fundamental. In the Feynman diagrams we can describe each index by a line, so the propagators in pure gauge theories can be represented by two lines. The other simple fact is that in the double line notation we can attach for any Feynman diagram to every index loop a surface. Thus, we can view a diagram as a decomposition of some closed surface (for vacuum or gauge invariant diagrams). The main result here is that if we take the large **N** limit with fixed $g_{YM}^2 N$ then the **N** dependence of any diagram will now be determined by the topology of the surface which it decomposes: the power of **N** we get is $(2 - 2\mathbf{g})$, where **g** is the genus of the surface. For a sphere **g**=0, for a torus **g**=1 etc, **g** simply counts the handles of the surface (In other words: when we say that a diagram has

a topology of surface $\mathcal{G}$, we mean that it can be drawn on surface $\mathcal{G}$ in double line notation without any line crossings).

When the number of colors, $\mathbf{N}$, is taken to infinity[1], one can expand the path integral in a power series in $\frac{1}{\mathbf{N}}$, such that the leading contribution is of the planar diagrams. So we get a power series with powers being linear functions of the genera of oriented closed surfaces, exactly like in oriented closed string theory (If we add matter fields in (anti)fundamental representation we get open surfaces, leading to an open string theory-like expansion, and if we look at gauge group SO(N) for example we get an unoriented string theories-like expansion, because here the adjoint is a product of two fundamentals). We stress that we don't see from here any well defined string theory appearing, but only that the perturbation series is very similar to the perturbation series of string theory, with the string coupling constant being $\frac{1}{\mathbf{N}}$.

It was Maldacena's work [1] that for the first time translated t'Hooft's idea of similarity between large N gauge theories and string theories to a definite, although still conjectured, relation between a subclass of conformal field theories and a class of well defined string theories. Maldacena's conjecture was based on the following observation. In superstring theories appear solitonic, non-perturbative, objects called Dq-branes. These objects have at least two descriptions:

- In string-perturbative language they are defined as manifolds (extended in q directions) on which an open string can end.

- In the supergravity language, which is supposed to describe the low energy limit of string theories, they are defined as extended (in q directions) black hole solutions.

We now look at a system of N coincident D3-branes. In the string-perturbation theory language, in the low energy limit, the physics of the system is described by $\mathcal{N} = 4$ SYM with U(N) gauge group[2] on the brane and by supergravity in the bulk, with these two systems decoupled. It is well known that in order for field theory perturbation theory to work $\mathbf{g_{YM}^2 N}$ should be much smaller than one. In the supergravity language we will have some black hole solution, which in the near horizon limit is described by $\mathbf{AdS_5 \times S^5}$

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[1]$\mathbf{N}$ is taken to infinity while keeping $\mathbf{g_{YM}^2 N}$ fixed.

[2] The U(1) part is free so we will discuss essentially only the SU(N) part.

geometry. Here again we can describe our physics by two decoupled systems: supergravity in the bulk and the type IIB string theory on $\mathbf{AdS_5} \times \mathbf{S^5}$. The supergravity solution is valid only if the radius of curvature is much larger than the string scale, which leads us to demand large $\mathbf{N}$, since the radius in string units of $\mathbf{AdS}$ and the radius of the sphere are both proportional to $(\mathbf{g_{string}N})^{\frac{1}{4}}$. Thus, we see that the same object is described on one hand by field theory and supergravity and on the other hand by string theory and supergravity. This led Maldacena to conjecture that :

$\mathcal{N} = 4$ d=4 SU($\mathbf{N}$) SYM is equivalent to type IIB string theory on $\mathbf{AdS_5} \times \mathbf{S^5}$ in the large $\mathbf{N}$ limit.

There is also a stronger conjecture that these theories describe the same physics for every value of $\mathbf{N}$.

There are many different indirect checks of this conjecture. One such check is the striking property of S-duality. S-duality relates two theories, one with small and the other with large coupling. Both $\mathcal{N} = 4$ SYM and type IIB string theory are believed to be self dual under the S-duality.

The $\mathbf{AdS/CFT}$ correspondence relates expectation values in string theory to coupling constants in the field theory. For instance we get from the correspondence that $\mathbf{g_{YM}^2} \propto \mathbf{g_{string}}$, and from string theory we know that $\mathbf{g_{string}}$ is related to the vacuum expectation value of the dilaton field. Thus we conclude that changing the gauge coupling on the field theory side, which is done by adding some marginal operator, is equivalent to changing the expectation value of the dilaton field on the string theory side. The marginal operators of the field theory are related to some moduli of the string theory[3]. In general, scalar supergravity fields ϕ which live in $\mathbf{AdS}$ couple to operators $\mathcal{O}$ which live on the boundary of $\mathbf{AdS}$ via $\int_{\mathbb{R}^{3,1}} \phi_0 \mathcal{O}$, where ϕ_0 is a restriction of ϕ to the boundary (up to some power of the radial coordinate). The dimension of $\mathcal{O}$, Δ, is related to the mass m^2 of the scalar field by:

$$m^2 = \Delta(\Delta - d). \tag{1.1}$$

Here $\mathbf{d}$ is the dimensionality of the space-time which the field theory lives in. We

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[3]We need operators to be marginal in order not to spoil the conformal properties of the field theory.

see that the massless, massive and tachyonic fields on the supergravity side correspond to marginal, irrelevant and relevant operators, respectively, on the field theory side.

The classification of operators to marginal, relevant and irrelevant in this way is meaningful before we deform our theory with them. After we deform our theory with these oprators the conformal dimensions of operators can receive corrections (via the anomalous dimensions). The marginality of an operator, as defined in the previous paragraph, can not assure that it will remain marginal after deforming the theory: the operators can be exactly marginal, marginally relevant or marginally irrelevant.

On the field theory side adding an irrelevant operator strongly affects the UV limit of the theory. Thus, because usually we define field theories in the UV and then flow to the IR, it does not make sense to discuss theories with irrelevant deformations. On the other hand, relevant deformations affect weakly the UV limit but break the conformal invariance. Finally, the exactly marginal operators keep the conformal properties of the theory.

On the string theory side giving a VEV to a massive field will change significantly the behavior on the boundary of $\mathbf{AdS}$, which is equivalent to demanding a new UV description on the field theory side. The tachyonic fields will go to zero on the boundary, thus this deformation will affect only the interior and asymptotically we will still have an $\mathbf{AdS}$ background. Giving a VEV to a massless field (if it corresponds to an *exactly* marginal operator) will always leave us with an $\mathbf{AdS}$ factor.

Thus we see that by finding *exactly* marginal operators on the field theory side we can learn about the moduli of string theory.

Adding additional operators to the theory will in general change the supergravity background. The $\mathbf{AdS_5}$ space has as its symmetry group SO(2,4), which is exactly the conformal group in four dimensions (the boundary of $\mathbf{AdS_5}$ is four dimensional). Thus, if we demand conformality, this factor will remain even after deforming the original theory. The second factor (the five-sphere) is related to the SU(4) global symmetry of the SYM in some sense, and thus can be and will be deformed after deforming the original theory, if we break some supersymmetry[4]. Thus, we can say that if we deform the original $\mathcal{N} = 4$,

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[4]The breaking of supersymmetry here is inevitable, because essentially there is only one renormalizable, consistent $\mathcal{N} = 4$ theory in d=4 which is the SYM theory. Thus, by adding additional operators, other than the change of the gauge coupling, we always break the $\mathcal{N} = 4$ SUSY.

d=4 SYM by some marginal operators, then on the string theory side we have to deform the supergravity solution: $\mathbf{AdS_5} \times \mathbf{S^5} \to \mathbf{AdS_5} \times \mathbf{M}$, where $\mathbf{M}$ is some five dimensional compact manifold (For marginal deformations of $\mathcal{N} = 4$ SYM from supergravity side see for example [4]), and sometimes we will also have to turn on some fields.

It is really striking that two so different mathematical tools like string theory and field theory may describe the same object.

To summarize, learning the marginal deformations of the field theory side can contribute to a better understanding of string theories in general and of the $\mathbf{AdS/CFT}$ correspondence in particular.

Another motivation comes from pure field theoretic considerations. New $\mathbf{CFT}$s can be used as UV fixed points, leading in the IR to various field theories, including perhaps phenomenologically interesting theories.

1.2 Marginal operators

In this work we discuss the marginal deformations of field theories coming from a system of N coincident D3-branes. The whole discussion is done from the field theory perspective. The theories we discuss have $\mathcal{N} = 4, 2, 1$ supersymmetry.

An operator is *exactly* marginal if upon adding it to the original conformal theory all the β-functions still vanish. Generally if we have **p** couplings in the theory we also have **p** β-functions. The conditions for the theory to be conformal are:

$$
\begin{aligned}
0 &= \beta_{g_1}(g_i, h_j) \\
&\vdots \\
0 &= \beta_{g_n}(g_i, h_j) \\
0 &= \beta_{h_1}(g_i, h_j) \\
&\vdots \\
0 &= \beta_{h_k}(g_i, h_j)
\end{aligned}
\tag{1.2}
$$

(Here h_is are the couplings and g_is are the gauge couplings of the system.) We have **n+k** equations in **n+k** variables. Thus, in general, we expect to have isolated, if at all we will have any, solutions of this system of equations.

However, in supersymmetric field theories we have several simplifications. The first one is that from nonrenormalization of the superpotential in supersymmetric theories we get a relation between the anomalous dimensions of the fields and the coupling associated with the superpotential term(see [9]). For a superpotential $W = \frac{1}{6}Y^{ijk}\Phi_i\Phi_j\Phi_k$ we get:

$$
\beta_Y^{ijk} = Y^{p(ij}\gamma_p^{k)} = Y^{ijp}\gamma_p^k + (k \leftrightarrow i) + (k \leftrightarrow j).
\tag{1.3}
$$

Here γ_p^k is the anomalous dimension related to the $\langle \Phi^{\dagger k}\Phi_p \rangle$ Z factor. The second simplification is the relation between the gauge coupling and the anomalous dimensions - the NSVZ β-function ([11], [12], [13]),

$$
\beta_g = \frac{g^3}{16\pi^2}\left[\frac{Q - 2r^{-1}Tr\left[\gamma C(R)\right]}{1 - 2C(G)g^2(16\pi^2)^{-1}}\right].
\tag{1.4}
$$

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The symbols appearing here will be defined later. We conclude that in a supersymmetric field theory, in order to find *exactly* marginal deformations we have to solve a set of linear equations in the anomalous dimensions. These equations can be linearly dependent, giving a manifold of solutions [14]. The equations (1.2) become:

$$
\begin{aligned}
0 &= \beta_{g_1}(\gamma_l, g_i, h_j) \\
&\vdots \\
0 &= \beta_{g_n}(\gamma_l, g_i, h_j) \\
0 &= \beta_{h_1}(\gamma_l, g_i, h_j) \\
&\vdots \\
0 &= \beta_{h_k}(\gamma_l, g_i, h_j)
\end{aligned}
\tag{1.5}
$$

Here usually we will get that the righthand sides of these equations depend only on γs which will greatly simplify our job.

In order to find *exactly* marginal directions we have to solve a set of linear equations, to find the possible values for the anomalous dimensions (γs) such that all βs vanish. Then, by loop calculations we calculate the dependence of the γs on the couplings and other parameters of the theory, and finally we impose the conditions from the first step on the γs and see if they can be satisfied. This will be the strategy in our search for *exactly* marginal deformations throughout this work.

When solving the set of linear equations (1.5) we can get possible solutions which will be ruled out from the loop calculations[5]. We will see examples of this below.

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[5] We can count on the loop calculations only in the weak coupling regime. Thus, we can not rule out these solutions from appearing in the strong coupling regime.

Chapter 2

$\mathcal{N} = 4$ theory

First we review some basic properties of $\mathcal{N} = 4$ SYM with gauge group SU(N). In $\mathcal{N} = 0$ language the theory contains six scalar fields, four Weyl fermions and a real vector field. All fields are in the adjoint representation of $SU(N)$. In $\mathcal{N} = 1$ language the six scalars can be coupled to form three complex scalars which together with three Weyl fermions form three chiral superfields Φ_i, while the vector and the remaining Weyl spinor can be joined to form a vector superfield V. The Lagrangian in $\mathcal{N} = 1$ language is then:

$$
\begin{aligned}
L &= \int d^4\theta \sum_{i=1}^{3} Tr(e^{-gV}\bar{\Phi}_i e^{gV}\Phi^i) + \frac{1}{64g^2}\int d^2\theta Tr(W^\alpha W_\alpha) \\
&+ \left(\frac{ig\sqrt{2}}{3!}\int d^2\theta \epsilon_{ijk}Tr(\Phi^i[\Phi^j, \Phi^k]) + h.c.\right).
\end{aligned}
\tag{2.1}
$$

Here traces are taken in the fundamental representation of SU(N).

This is a pure Yang Mills theory with sixteen supercharges, in particular for $U(1)$ gauge group this theory becomes free. The β-function of the gauge coupling vanishes identically (at one loop it's a trivial consequence of having three chiral superfields in the adjoint representation), thus it is a conformal theory. It is believed to be exactly self S-dual. This symmetry of the theory exchanges the strong coupling regime with a weak coupling regime, and the perturbative, electric, degrees of freedom with non-perturbative, magnetic degrees of freedom.

In string theory we get $\mathcal{N} = 4$ SYM with SU(N) gauge group by putting N D3-branes in type IIB string theory together. In this picture we have six "vibrational" modes of the

branes (which are related to the six transverse directions to the brane) which become six scalars, which in turn when joined in pairs comprise the $\mathcal{N} = 1$ scalar part of three complex chiral supermultiplets. The possibility of the fundamental string to end on one of the N branes gives an SU(N) gauge group and puts the scalars (as well as the other fields) in the adjoint representation. To all these integer spin fields we have fermionic counterparts, and all in all we get an SU(N) gauge group in 4d with three chiral and one vector multiplets. In type IIB superstrings we have 32 supercharges, D-branes are BPS states and thus they break half of the supersymmetry. Finally we have 16 supercharges in d=4 which give us $\mathcal{N} = 4$ supersymmetry. Three $\mathcal{N} = 1$ chiral multiplets and the vector multiplet in $\mathcal{N} = 4$ language give an $\mathcal{N} = 4$ vector multiplet. Thus to summarize we get on the D-branes $\mathcal{N} = 4$ pure Yang Mills with SU(N) gauge group.

Another, related, way to obtain $\mathcal{N} = 4$ SYM in d=4 is [8] to look at pure $\mathcal{N} = 1$ SYM in d=10 and then do the dimensional reduction procedure to d=4. In d=10 we had only the vector multiplet, six scalar components of which lose their vector nature after the reduction. They can be coupled in pairs to form complex scalars which will be the scalars of three chiral $\mathcal{N} = 1$, d=4 multiplets. Of course in this procedure we have a global $SO(6) \sim SU(4)$ symmetry, which becomes the $\mathcal{R}$ symmetry of $\mathcal{N} = 4$.

There is extensive literature on this field theory, in particular regarding its finiteness and the exact S-duality of this theory. There is also research concerning the relevant deformations of $\mathcal{N} = 4$ ([5], [6], [7] for example). Relevant deformations break the conformal invariance by introducing a scale to the theory. We will be interested only in marginal deformations throughout this work.

2.1 Marginal deformations

In this section we investigate some *exactly* marginal deformations of $\mathcal{N} = 4$ SYM. There are essentially only three types of marginal deformations which one can add to the lagrangian above[1]. The obvious deformation is just changing the gauge coupling constant, the two other types are superpotentials of the form [2]:

$$\frac{i\delta\lambda\sqrt{2}}{3!}\epsilon_{ijk}Tr(\Phi^i[\Phi^j,\Phi^k])$$
$$\frac{h_{ijk}}{3!}Tr(\Phi^i\{\Phi^j,\Phi^k\}), \tag{2.2}$$

where h_{ijk} is totally symmetric. These operators are marginal (by power counting), obey gauge invariance and preserve $\mathcal{N} = 1$ supersymmetry. So by adding these operators we get $\mathcal{N} = 1$ SQCD. What has to be determined is under what conditions these marginal deformations are *exactly* marginal, i.e. the β-functions vanish to all orders in perturbation theory.

$\mathcal{N} = 1$ SQCD was analyzed for general superpotentials and general simple gauge group G([10] and references therein). We will briefly summarize the general results:

We write the superpotential as:

$$W = \frac{1}{6}Y^{ijk}\Phi_i\Phi_j\Phi_k. \tag{2.3}$$

We assume that the gauge group is simple and that there are no gauge singlets. The $\beta - function$ of Y can be written in terms of the anomalous dimensions:

$$\beta_Y^{ijk} = Y^{p(ij}\gamma_p^{k)} = Y^{ijp}\gamma_p^k + (k \leftrightarrow i) + (k \leftrightarrow j). \tag{2.4}$$

The one loop gauge β-function and the anomalous dimensions are given by:

---

[1]There are also relevant deformations, inserting mass terms for the fields, and they were discussed in [5], [6].

[2]We will assume everywhere, unless stated otherwise, that the couplings are real. The extension to complex couplings is trivial $\rightarrow$ the actual manifold of fixed points is a complex manifold of same dimensions.

$$16\pi^2\beta_g^{(1)} = g^3Q, \quad \text{and} \quad 16\pi^2\gamma_j^{(1)i} = P_j^i, \tag{2.5}$$

where we have defined:

$$Q = T(R) - 3C(G), \quad \text{and} \quad P_j^i = \frac{1}{2}Y^{ikl}Y_{jkl} - 2g^2C(R)^i{}_j, \tag{2.6}$$

and:

$$T(R)\delta_{AB} = Tr(R_AR_B), \quad C(G)\delta_{AB} = f_{ACD}f_{BCD} \quad \text{and} \quad C(R)_j^i = (R_AR_A)_j^i. \tag{2.7}$$

R_A is the representation of the chiral superfields. A,B,C,D are indices in the adjoint representation. For the gauge coupling we use the NSVZ $\beta - function$ [11], [12], [13]:

$$\beta_g = \frac{g^3}{16\pi^2}\left[\frac{Q - 2r^{-1}Tr\left[\gamma C(R)\right]}{1 - 2C(G)g^2(16\pi^2)^{-1}}\right], \tag{2.8}$$

(here $r = \delta_{AA}$) which at one loop gives (as in(2.5)):

$$16\pi^2\beta_g^{(1)} = g^3Q. \tag{2.9}$$

Now we have set the general stage and return to the specific marginally deformed $\mathcal{N} = 4$ theory. The superpotential can be rewritten in the form:

$$W = (-\lambda\sqrt{2}f_{abc}\epsilon_{ijk} + d_{abc}h_{ijk})\frac{1}{6}\Phi_i^a\Phi_j^b\Phi_k^c, \tag{2.10}$$

$$Y_{abc}^{ijk} = (-\lambda\sqrt{2}f_{abc}\epsilon_{ijk} + d_{abc}h_{ijk}), \tag{2.11}$$

where:

$$d_{abc} \equiv Tr\left[T_a\{T_b, T_c\}\right]. \tag{2.12}$$

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Here T_a are the generators of the Lie algebra of G in the fundamental representation of the group and $\lambda = g + \delta\lambda$. The groups for which d_{abc} is not vanishing are only $SU(N \geq 3)$ or E_8. Here we discuss only the SU(N) case.

Φ_i are in the adjoint of SU(N):

$$
R_A = \begin{pmatrix} T_A^{adj} & 0 & 0 \\ 0 & T_A^{adj} & 0 \\ 0 & 0 & T_A^{adj} \end{pmatrix} \tag{2.13}
$$

Here T_A^{adj} are the adjoint representation matrices. From here we can calculate the parameters appearing in the general setup above:

$$
\begin{aligned}
C(G)\delta_{AB} &= f_{ACD}f_{BCD} \equiv C_1\delta_{AB} \tag{2.14} \\
T(R)\delta_B^A &= Tr(R_A R_B) = 3Tr(T_A^a T_B^a) = 3C_1\delta_B^A(= -3f_{ACD}f_{BDC}) \\
C(R)_{Bj}^{Ai} &= (R_D R_D)_{Bj}^{Ai} = C_1\delta_B^A\delta_j^i \\
r &= N^2 - 1.
\end{aligned}
$$

C_1 depends on the normalization of the Lie algebra generators, for the moment we will keep it arbitrary which will not affect our results. The one loop gauge $\beta - function$ is proportional to Q:

$$
Q = T(R) - 3C(G) = 3C_1 - 3C_1 = 0. \tag{2.15}
$$

So the gauge $\beta - function$ vanishes at one-loop. The gauge $\beta - function$ vanishes at one loop in general gauge theories with three chiral superfields in the adjoint representation.

First we do the general Leigh-Strassler analysis [14]. We have here:

$$
\begin{aligned}
\beta_g &\propto Tr\gamma \\
\beta_{\delta\lambda} &\propto Tr\gamma \tag{2.16}
\end{aligned}
$$

So in general we have 10 h_{ijk}s, $\delta\lambda$ and the gauge coupling, total of 12 couplings, we have to demand that $Tr\gamma = 0$ and $\beta_{h_{ijk}} = 0$ giving a total of 11 conditions. So we expect

a one dimensional manifold of the fixed points which we have already in $\mathcal{N} = 4$ and it is parameterized by the gauge coupling, with $\delta\lambda = h_{ijk} = 0$. But we can do a more complicated thing. If we assume also that γ is proportional to identity matrix[3] we get $\beta_{h_{ijk}} \propto Tr\gamma$. So we will have 12 couplings, one condition $Tr\gamma = 0$ and 8 conditions for $\gamma_i^j \propto \delta_i^j$, giving a total of 12-8-1=3 free parameters. So we expect to have a three dimensional manifold of exactly marginal deformations. We will see below that we essentially get only these three marginal directions.

Now we continue with the perturbation theory analysis. For SU(N) : $d_{acd}d_{bcd} = 2\frac{N^2-4}{N}C_2^3\delta_{ab}$, where $C_2\delta_{ab} \equiv Tr(T_aT_b)$. So we can write:

$$P_{bj}^{ai} = (2C_1(\lambda^2 - g^2)\delta_{ij} + \frac{N^2 - 4}{N}C_2^3h_{ij}^{(2)})\delta_{ab}. \tag{2.17}$$

Here $h_{ij}^{(2)} \equiv h_{ilm}h_{jlm}^*$. And finally we get the one-loop anomalous dimensions and $\beta - functions$:

$$
\begin{align}
\gamma_{bj}^{(1)ai} &= \frac{1}{16\pi^2}(2C_1(\lambda^2 - g^2)\delta_{ij} + \frac{N^2 - 4}{N}C_2^3h_{ij}^{(2)})\delta_{ab} \tag{2.18} \\
\beta_{abc}^{(1)ijk} &= \frac{1}{16\pi^2}\left\{6C_1(\lambda^2 - g^2)Y_{abc}^{ijk} + \frac{N^2 - 4}{N}C_2^3\left(-\sqrt{2}\lambda f_{abc}\tilde{h}_{ijk} + d_{abc}h_{ijk}^{(3)}\right)\right\} \tag{2.19} \\
h_{ijk}^{(3)} &\equiv h_{plm}^*(h_{ijp}h_{klm} + h_{kjp}h_{ilm} + h_{ikp}h_{jlm}) \tag{2.20} \\
\tilde{h}_{ijk} &\equiv \epsilon_{ijp}h_{kp}^{(2)} + \epsilon_{pjk}h_{ip}^{(2)} + \epsilon_{ipk}h_{jp}^{(2)} \tag{2.21}
\end{align}
$$

$\tilde{h}_{ijk}$ is totally antisymmetric, thus because (i j k) run over (1 2 3), $\tilde{h}_{ijk}$ has only one independent component:

$$
\begin{align}
\tilde{h}_{123} &= \epsilon_{123}h_{33}^{(2)} + \epsilon_{123}h_{22}^{(2)} + \epsilon_{123}h_{11}^{(2)} = Tr(h^{(2)}) \tag{2.22} \\
\tilde{h}_{ijk} &= Tr(h^{(2)})\epsilon_{ijk}. \tag{2.23}
\end{align}
$$

Now we can look separately on the part of the $\beta - function$ proportional to f_{abc} and on the part proportional to d_{abc}:

---

[3] There are also other restrictions we can make on the γs and get the same dimensionality of the manifold of fixed points, but they all are related by the global SU(3) symmetry we have here.

$$\beta_\lambda^{(1)} = \frac{\lambda}{16\pi^2}\left\{6C_1(\lambda^2 - g^2) + \frac{N^2 - 4}{N}C_2^3 Tr(h^{(2)})\right\} \tag{2.24}$$

$$\beta_{ijk}^{(1)} = \frac{1}{16\pi^2}\left\{6C_1(\lambda^2 - g^2)h_{ijk} + \frac{N^2 - 4}{N}C_2^3 h_{ijk}^{(3)}\right\}. \tag{2.25}$$

When we constrain ourselves only to the case of $h_{123}, h_{111} = h_{222} = h_{333}, \lambda$ non zero (which is the only case where we will get *exactly* marginal deformations as we will see later), we get:

$$h_{ijk}^{(3)} = h_{ijk} Tr(h^{(2)}). \tag{2.26}$$

And in this case:

$$\frac{\beta_\lambda}{\lambda} = \frac{\beta_{ijk}}{h_{ijk}}. \tag{2.27}$$

So if we are looking for fixed points we have only one condition on four couplings, and thus we have a three dimensional manifold of fixed points in the coupling constants space [14].

2.2   RG flow analysis

Here we will analyze the $\beta - functions$ obtained in the previous section. The equations will simplify if we rescale the coupling constants:

$$g \to \frac{\sqrt{C_1}}{4\pi}g \quad \lambda \to \frac{\sqrt{C_1}}{4\pi}\lambda \quad \text{and} \quad h_{ijk} \to \frac{\sqrt{C_2^3}}{4\pi}\sqrt{\frac{N^2 - 4}{N}}h_{ijk}. \tag{2.28}$$

The $\beta - functions$ become:

$$\beta_g = -\frac{2g^3}{1 - 2g^2}Tr\gamma, \quad \beta_\lambda = \lambda Tr\gamma. \tag{2.29}$$

Here the trace is taken only over the SU(3) indices and not over gauge indices. From these $\beta - functions$ we can obtain a differential equation:

$$-\frac{1}{2g^3}dg + \frac{1}{g}dg = \frac{d\lambda}{\lambda}. \tag{2.30}$$

This can be easily solved to give:

$$\frac{\lambda}{\lambda_0} = \frac{ge^{\frac{1}{4g^2}}}{g_0 e^{\frac{1}{4g_0^2}}}. \tag{2.31}$$

This result means that the RG flow lines in the $\lambda - g$ plane are exactly known (to the extent that we can count on the NSVZ $\beta - function$). It is easy to convince oneself that there is no line with the couplings going to zero in the UV, except the trivial case when one of the couplings is constantly zero. This implies that there is no choice of coupling constants for which this theory is asymptotically free.

Another interesting question is the existence of fixed points. In order to have a fixed point we have to satisfy $Tr\gamma = 0$ which implies at one loop that:

$$Tr(h^{(2)}) = -6(\lambda^2 - g^2) \tag{2.32}$$

And we can substitute this into β_{ijk} to get another condition:

$$Tr(h^{(2)})h_{ijk} = h_{ijk}^{(3)}. \tag{2.33}$$

We will argue that these conditions can be satisfied (in the limit $g \to 0$) only if the anomalous dimensions matrix is proportional to identity matrix.

First we don't assume any special property of γ. By multiplying (2.33) on both sides by h_{ijk}^* we get:

$$3Tr((h^{(2)})^2) = (Tr(h^{(2)}))^2. \tag{2.34}$$

We denote:

$$h^{(2)} \equiv \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \tag{2.35}$$

$$\tag{2.36}$$

And then:

$$(Tr(h^{(2)}))^2 = (a + e + k)^2 \tag{2.37}$$

$$Tr((h^{(2)})^2) = (a^2 + e^2 + k^2) + 2(bd + cg + hf). \tag{2.38}$$

So (2.34) implies:

$$(a - e)^2 + (a - k)^2 + (k - e)^2 + 6(cg + bd + fh) = 0 \tag{2.39}$$

But remembering that $h^{(2)}$ is hermitian:

$$c = g^* \quad f = h^* \quad b = d^*, \tag{2.40}$$

we get that the only possibility for (2.39) to hold is if:

$$a = e = k, \quad h = f = b = g = c = d = 0, \tag{2.41}$$

which implies: $h^{(2)}_{ij} = \alpha^2 \delta_{ij}$ and γ which is proportional to identity matrix. So the theory at weak coupling only has fixed points when the anomalous dimensions matrix is proportional to identity matrix.

If the anomalous dimensions matrix is proportional to the identity then:

$$\gamma^i_j \equiv \rho \delta^i_j. \tag{2.42}$$

From here and from (2.4) we obtain:

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$$\beta_{ijk} = Tr\gamma \cdot h_{ijk}. \tag{2.43}$$

The one loop γ implies further that:

$$h_{ij}^{(2)} = \alpha^2 \delta_{ij}, \quad \alpha^2 \equiv \frac{1}{3} \sum_{i,j,k} |h_{ijk}|^2. \tag{2.44}$$

So the condition (2.33) is automatically satisfied and from (2.32) we get:

$$\alpha^2 = -2(\lambda^2 - g^2). \tag{2.45}$$

The fixed points we found are essentially IR stable fixed points, we have:

$$Tr\gamma = 3(2(\lambda^2 - g^2) + \alpha^2), \tag{2.46}$$

and the condition for a fixed point is $Tr\gamma = 0$. From the $\beta - functions$ we calculated we see that if we increase one of the couplings λ, h_{ijk}, $Tr\gamma$ becomes positive thus decreasing these couplings and increasing the gauge coupling in IR, till we get again zero. And the same if we decrease the couplings. Thus we can conclude that in the weak coupling limit all fixed points that exist imply diagonal γ and are IR stable. Consequently nothing is known of the UV behavior of the theory, and we can unambiguously define it only at the conformal fixed points. (A simple calculation shows also that the fixed points we find are IR stable even if we go out of the special γ regime.)

All the calculations in this section were done based on the one loop anomalous dimensions. An interesting question is how the results are altered by higher loop calculations. Again following [10] general formulae, we get for couplings such that $\gamma^{(1)} = 0$ [4] :

---

[4]At three loop level the calculation is renormalization scheme dependent (the above result is in $\overline{MS}$). As described in [10] one can redefine the coupling constant or equivalently change the renormalization scheme and get that γ vanishes also at three loop order exactly, including the non-planar graphs.

$$
\begin{aligned}
\gamma_j^{(2)i} &= -2(2(\lambda^2 - g^2) + \alpha^2)(2\lambda^2 + g^2 + \alpha^2)\delta_j^i \\
\gamma_j^{(3)i} &= 2\kappa g^2(2(\lambda^2 - g^2) + \alpha^2)(2(2\lambda^2 + \alpha^2) + 3g^2)\delta_j^i.
\end{aligned}
\tag{2.47}
$$

Here $\kappa = 6\zeta(3)$ and in $\gamma^{(3)}$ we have omitted terms proportional to $\gamma^{(1)}$ in the general formulae. We have omitted contributions from the non-planar diagrams (figure (2.1)) coming only from the superpotential (We do it under the assumption that $g, \lambda \gg \alpha$). These extra contributions are proportional to:

$$
\Delta_{ax}^{ij} \equiv \frac{\kappa}{4} Y_{abc}^{*ikl} Y_{bed}^{kmn} Y_{cfg}^{lrs} Y_{hef}^{*pmr} Y_{odg}^{*qns} Y_{xho}^{jpq}.
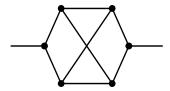\tag{2.48}
$$

Figure 2.1: Typical non-planar graph appearing in three loop

Δ is proportional to at least one power of α^2 (because in the $\mathcal{N} = 4$ theory Δ vanishes), thus the condition above is necessary for neglecting it. So we see that up to third order γ is proportional to $2(\lambda^2 - g^2) + \alpha^2$ which is the one loop $\gamma - function$, and the only deviation from it comes from the non-planar graph (which first appears at three loops). So the fixed points we find at one loop are essentially correct fixed points even up to three loops, assuming $\lambda, g \gg \alpha$. We will encounter the same assumption also in the S-duality context and it simply means that we are close to the $\mathcal{N} = 4$ theory.

The one-loop solution can be easily argued to extend to all-loops. Our condition for zero βs is $\gamma_j^i = 0$. Now if we restrict ourselves only to the case of $h_{111} = h_{222} = h_{333} \equiv h \neq 0$ and $h_{123} \equiv h' \neq 0$ (which is consistent with the one loop solution), with all other h's vanishing, then from the symmetry of the interactions we see that γ_j^i is proportional to

the identity($\equiv \gamma \delta^i_j$). Thus let's parameterize our solutions by λ, h and h'. The vanishing of γ will give an equation of the form $g = g(\lambda, h, h')$. We can expand this relation as a power series in its arguments. Assume we have determined $g(\lambda, h, h')$ up to (n-1)th order[5]. Now we determine it to n'th order. We define $\tilde{\gamma}^{(n)} \equiv \sum_{i=2}^{n} \gamma^{(i)}$ where we insert $g(\lambda, h, h')$ such that this expression will be of n'th order. This expression consists only of already determined quantities, since the undetermined n'th order of $g(\lambda, h, h')$ appears only at one loop. Thus we get:

$$\tilde{\gamma}^{(n)} - \frac{1}{16\pi^2} 2C_1(g^2)^{(n)} = \gamma^{(n)}. \tag{2.49}$$

Thus from the demand of vanishing γ we get $(g^2)^{(n)} = (\frac{1}{16\pi^2} 2C_1)^{-1} \tilde{\gamma}^{(n)}$, and we can extend our solution to any order of perturbation theory (see [19] , and a slightly different approach [20]).

---

[5](n-1)th order in λ^2, h^2, h'^2.

2.3   S-duality

S-duality was first conjectured by Montonen and Olive [15] (see [16] for a review with references) as a symmetry of Yang-Mills theory interchanging magnetic and electric degrees of freedom. S-duality involves a conjecture that a theory with weak coupling constant is equivalent to another theory with a strong one. This conjecture can be meaningful only if the notion of a coupling constant for a theory is well defined. In general field theory coupling constants depend on the energy scale at which experiments are done, namely the couplings flow. The cases when the couplings are independent of a scale happen when the theory possesses exact conformal invariance, even at a quantum level. So in order to have a Montonen-Olive duality a theory is expected to be conformal. The only finite quantum theories known include some amount of supersymmetry. The $\mathcal{N} = 4$ SYM under discussion is believed to possess the Montonen-Olive duality, and moreover to be selfdual in this sense.

The MO conjecture can be schematically written as:

$$\langle ... \rangle = \int D[\phi](...)e^{-\frac{1}{g^2}S[\phi]} = \int D[\tilde{\phi}](...)e^{-g^2 S[\tilde{\phi}]}. \tag{2.50}$$

This duality is hard to check because it involves strong coupling. There are several checks of this duality, mainly through some BPS arguments - counting of degrees of freedom of some stable configurations in both cases. As we mentioned above the $\mathcal{N} = 4$ SYM theory is believed to be selfdual. We can write the lagrangian as:

$$L = \frac{1}{g^2}Tr(-\frac{1}{4}F_{\mu\nu}^2 + \mathcal{D}_\mu\phi^I\mathcal{D}^\mu\phi^I + [\phi^I, \phi^J]^2 + fermions) \tag{2.51}$$

Under the SL(2,$\mathbb{Z}$)[6] transformation $g^2 \to \frac{1}{g^2}$ and for the theory to remain invariant the operators appearing in the lagrangian have also to undergo some kind of transformation. It is convenient to define the lowest components of chiral primary operators of the $\mathcal{N} = 4$ superconformal algebra to be:

---

[6] In its complete form, the S-duality transformations involve also the θ term of the lagrangian (by taking it to $\theta + 2\pi$). If we define $\tau \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi}$, the S-duality acts as $\tau \to \frac{a\tau+b}{c\tau+d}$ (ad-bc=1 and $a, b, c, d \in \mathbb{Z}$). This is the SL(2,$\mathbb{Z}$) transformation.

$$\mathcal{O}^{(p)}_{I_1 I_2 \ldots I_p} \equiv N (g_{YM}^2 N)^{-p/2} Tr(\phi^{\{I_1} \phi^{I_2} \ldots \phi^{I_p\}}). \tag{2.52}$$

The other single trace (gauge invariant) chiral operators of the theory can be obtained from these by the supersymmetry generators. These operators are SL(2,$\mathbb{Z}$) invariant [17]. This is consistent with the U(N=1) case where the theory is free and thus nothing depends on the coupling, and the normalization we chose is just scaling the fields such that the coupling doesn't appear in the lagrangian. The action of the $\mathcal{N} = 4$ supersymmetry algebra may be schematically written as:

$$
\begin{aligned}
[Q_\alpha^A, \phi^I] \quad &\sim \lambda_{\alpha B} \\
\{Q_\alpha^A, \lambda_{\beta B}\} \quad &\sim (\sigma^{\mu\nu})_{\alpha\beta} F_{\mu\nu} + \epsilon_{\alpha\beta}[\phi^I, \phi^J] \\
\{Q_\alpha^A, \bar{\lambda}_{\dot{\beta}}^B\} \quad &\sim (\sigma^\mu)_{\alpha\dot{\beta}} \mathcal{D}_\mu \phi^I \\
[Q_\alpha^A, A_\mu] \quad &\sim (\sigma_\mu)_{\alpha\dot{\alpha}} \bar{\lambda}_{\dot{\beta}}^A \epsilon^{\dot{\alpha}\dot{\beta}}.
\end{aligned}
\tag{2.53}
$$

We wish to extend our discussion in the previous sections to the strong coupling limit using the S-duality of $\mathcal{N} = 4$. We wish to learn how the marginal deformations we found look like at strong coupling. As we mentioned we believe in S-duality in $\mathcal{N} = 4$ SYM, so this transition has to assume that we are somehow very close to the $\mathcal{N} = 4$ theory. We will assume that $\frac{\delta\lambda}{g} \ll 1$ and $\frac{h}{g} \ll 1$. Now we have to obtain the transformation of the operators we add to the lagrangian under the SL(2,$\mathbb{Z}$). Namely we want to find $\bar{\lambda}$ and $\bar{h}$ such that if we add operators $\lambda\mathcal{O}, h\mathcal{O}'$ in the weak coupling limit, we have to add $\bar{\lambda}\mathcal{O}, \bar{h}\bar{\mathcal{O}}'$ in the strong coupling limit. So we first calculate what scalar operators are we actually adding to the $\mathcal{N} = 4$ action.

By adding the superpotential:

$$Y_{abc}^{ijk} = (-\delta\lambda\sqrt{2} f_{abc}\epsilon_{ijk} + d_{abc} h_{ijk}) \tag{2.54}$$

we actually change the F term and thus the scalar potential. The F term now is equal to:

$$F_a^i = -\frac{1}{2}(-\lambda\sqrt{2}f_{abc}\epsilon_{ijk} + d_{abc}h_{ijk})\Phi_j^{*b}\Phi_k^{*c} \tag{2.55}$$

So after integrating out F we get the F-term contribution to the scalar potential:

$$-\frac{1}{4}Y_{abc}^{ijk}Y_{aed}^{ilm}\phi_j^{*b}\phi_k^{*c}\phi_l^e\phi_m^d \tag{2.56}$$

The D term contribution is like in $\mathcal{N} = 4$:

$$-\frac{1}{2}(f_{abc}\phi_a^{*i}\phi_b^i)^2. \tag{2.57}$$

Now if we input the exact expression for Y_{abc}^{ijk} in the F term contribution we get:

$$
\begin{aligned}
(\frac{1}{2}g^2 f_{abc}f_{aed}\epsilon_{ijk}\epsilon_{ilm} &+ \frac{1}{2}\delta\lambda^2 f_{abc}f_{aed}\epsilon_{ijk}\epsilon_{ilm}, \\
+ \quad \frac{1}{2}(\delta\lambda + \delta\lambda^*)g f_{abc}f_{aed}\epsilon_{ijk}\epsilon_{ilm} &+ \frac{1}{4}d_{abc}d_{aed}h_{ijk}^*h_{ilm}, \\
- \quad \left\{\frac{1}{\sqrt{2}\cdot 2}(g+\delta\lambda)f_{aed}d_{abc}\epsilon_{ilm}h_{ijk}^* + c.c\right\}&)\phi_j^{*b}\phi_k^{*c}\phi_l^e\phi_m^d.
\end{aligned}
\tag{2.58}
$$

The fermionic couplings can be obtained from here by supersymmetry.

The term proportional to g^2 with the D term gives the usual $\mathcal{N} = 4$ scalar potential $Tr[\phi^I, \phi^J]^2$. The remaining terms are the perturbation scalar potential. We would like to bring them to the form (2.52) or its supersymmetric descendants. By using the following equations:

$$
\begin{aligned}
\frac{1}{C_2}Tr[T_a, T_b][T_e, T_d] &= -f_{abc}f_{edc} \\
Tr\{T_a, T_b\}[T_e, T_d] &= \frac{1}{C_2}id_{abc}f_{edc} \\
Tr\{T_a, T_b\}\{T_e, T_d\} &= \frac{4}{N^2}\delta_{ab}\delta_{ed} + \frac{1}{C_2^3}d_{abc}d_{edc}
\end{aligned}
\tag{2.59}
$$

we get that: the $\delta\lambda g$ term is proportional to an operator of the form $Tr[\phi, \phi]^2$ which is part of an operator of the form $Q^4\mathcal{O}^{(2)}$. The $g \cdot h$ term is proportional to an operator of

the form $Tr[\phi, \phi]\{\phi, \phi\}$ which is of the form $Q^2\mathcal{O}^{(3)}$. All the other terms, which are not linear in the deformations, can be obtained by the supersymmetry considerations. From here we can get the transformations of our couplings. First however we have to get to the form (2.51) and this is done by scaling all the fields by factor of g.

$$
\begin{aligned}
\delta\lambda \cdot g \cdot Tr[\phi, \phi]^2 \quad &\rightarrow \quad \frac{\delta\lambda}{g^3} \cdot Tr[\tilde{\phi}, \tilde{\phi}]^2 \rightarrow \frac{\delta\lambda}{g^3} \cdot Q^4\tilde{\phi}^2 \rightarrow \frac{\delta\lambda}{g} \cdot Q^4\mathcal{O}^{(2)} \\
h \cdot g \cdot Tr[\phi, \phi]\{\phi, \phi\} \quad &\rightarrow \quad \frac{h}{g^3} \cdot Tr[\tilde{\phi}, \tilde{\phi}]\left\{\tilde{\phi}, \tilde{\phi}\right\} \rightarrow \frac{h}{g^3} \cdot Q^2\tilde{\phi}^3 \rightarrow h \cdot Q^2\mathcal{O}^{(3)} \quad (2.60)
\end{aligned}
$$

Now we finally can write down the transformations of the couplings $g^2 \rightarrow \frac{1}{g^2}$:

$$
\begin{aligned}
\frac{\delta\lambda}{g}Q^4\mathcal{O}^{(2)} \rightarrow \delta\tilde{\lambda}gQ^4\mathcal{O}^{(2)} \Rightarrow \qquad & \delta\lambda \rightarrow \frac{\delta\lambda}{g^2} = \delta\tilde{\lambda} \qquad (2.61) \\
hQ^2\mathcal{O}^{(3)} \rightarrow \tilde{h}Q^2\mathcal{O}^{(3)} \Rightarrow \qquad & h \rightarrow h = \tilde{h}
\end{aligned}
$$

Here we got the coupling constant transformations only from the terms linear in the deformations $\delta\lambda$ and h, the other, quadratic terms, are automatically transformed in the right way by supersymmetry transformations.

In (2.45) we got the condition for a fixed point, so now we can apply this equation to the strong coupling limit using the S-duality:

$$
\begin{aligned}
\alpha^2 \quad &= \quad -2(\lambda^2 - g^2) \rightarrow \alpha^2 \cong 4\delta\lambda g \qquad g \rightarrow 0 \\
\alpha^2 \quad &\cong \quad \frac{4\delta\lambda}{g^3} \qquad\qquad\qquad\qquad\qquad g \rightarrow \infty \qquad (2.62)
\end{aligned}
$$

The second line is the condition for having a conformal theory at large coupling. In order to use S-duality we have to assume that the strong coupling quantities satisfy $\frac{1}{g} \gg \delta\lambda, \alpha$ and of course to use the perturbation theory $\frac{1}{g} \ll 1$. From the conditions above we see that the result is consistent with these demands.

The only knowledge we have about the fixed points, as discussed above, is in the small coupling region and in the limit where $g \rightarrow \infty$ and $\delta\lambda g \ll 1$. We established that in the regions of our knowledge there are no UV fixed points, namely there is no line going in the UV to zero couplings, or no asymptotically free regime.

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Chapter 3

$\mathcal{N} = 2$ theory

It is possible to reduce the number of supersymmeties of the d=4 $\mathcal{N} = 4$ SYM, which we dealt with in the previous chapter, via the orbifolding procedure([25], [26], [18], see also discussion in [27] and for supergravity part [30], [31]). In this chapter we will concentrate on the case with $\mathcal{N} = 2$ and in the next one we will deal with $\mathcal{N} = 1$ theories.

We will look on N coincident D3-branes at the $\mathbb{Z}_k$ orbifold singularity of an A_{k-1} ALE space. If we consider the low energy theory on D-branes in the bulk and not on the singularity then essentially we are back to the non-orbifold case. The orbifold group acts on the $\mathbb{C}^3$ (we put the $\mathbb{R}^6$ coordinates in pairs, for instance $(Z_1, Z_2, Z_3) = (X_4 + iX_5, X_6 + iX_7, X_8 + iX_9)$ assuming that the branes lie in 0123 directions) perpendicular space of the D3-branes as:

$$Z^i \to \omega^{a_i} Z^i, \tag{3.1}$$

where $\omega \equiv e^{\frac{2\pi i}{k}}$ and $(a_1, a_2, a_3) \equiv (1, 0, -1)$[1].

---

[1] The vector $\vec{a}$ has to satisfy $\sum_i a_i = 0 \pmod{k}$ in order that the orbifold action will be part of SU(3) and not U(3), in other words to preserve supersymmetry. So if we choose all the components to be non zero we have $\mathbb{Z}_k \subset SU(3)$ thus leaving us with one supersymmetry. If we choose one of the components zero then we can have (n,0,-n)(mod k) case. This case is equivalent to (1,0,-1) case, and the important fact is that in this case $\mathbb{Z}_k \subset SU(2)$ and we have $\mathcal{N} = 2$ supersymmetry. So there is only one choice giving an $\mathcal{N} = 2$ theory here. If we put two components of $\vec{a}$ to zero we will have to take the remaining one to zero too, thus remaining in the $\mathcal{N} = 4$ case. If we consider orbifolds $\mathbb{R}^6/\mathbb{Z}_k$ rather than $\mathbb{C}^3/\mathbb{Z}_k$ we can have non-supersymmetric theories [28] [29].

To see[2] how the orbifold can act on the D-branes we put kN D3-branes on the covering ALE space and group them in N sets of k branes. We put each set of the D-branes in the regular representation of the $\mathbb{Z}_k \to \gamma_j^i(g) \equiv \delta_j^i \omega^i$ where g is the generator of $\mathbb{Z}_k$. As we see the regular representation is reducible and is the direct sum of k one dimensional irreducible representations ($\mathcal{R}_r = \bigoplus_{n=0}^{k-1} \mathcal{R}_n$), parameterized by the integer n, given by ω^n (ω defined above). We define the N set indices by I and the brane indices within each set are $i \in (0, 1, .., k-1)$.

From here we conclude that the projection on the gauge vector fields is (We write the gauge fields in double index notation - upper fundamental,lower anti-fundamental):

$$A_{J,j}^{I,i} = \omega^{i-j} A_{J,j}^{I,i} \tag{3.2}$$

And the projection on the bosonic components of the chiral multiplets (which are related to the transverse 6 directions) is:

$$Z_{J,j}^{lI,i} = \omega^{i-j+a_l} Z_{J,j}^{lI,i} \tag{3.3}$$

This projection comes from acting on the Chan-Paton indices as well as on the space-time index.

Thus we see that in our case the fields which survive the projection are: gauge fields $A_{J,j}^{I,i}$ where both indices lie in the same irreducible representation of $\mathbb{Z}_k$ (as defined above), giving a total of k copies of U(N). The bosonic part of the matter fields which survive are: $Z_{J,j}^{2I,i}$ survives if $i-j = 0$, exactly as for the gauge field $\to$ we get fields in the adjoint. $Z_{J,j}^{1I,i}$ survives when we have $i - j + 1 = 0$, thus these fields are in the fundamental of the **i**'th gauge group and the antifundamental of the **(i+1)**'th gauge group. $Z_{J,j}^{3I,i}$ survives when we have $i - j - 1 = 0$, thus these fields are in the fundamental of the **(i+1)**'th gauge group and the antifundamental of the **i**'th gauge group.

The same projection can be made for the fermions. Giving a total of three types of chiral super fields:

---

[2]There are other possible choices for the orbifold to act on the brane indices [27].

- For each U(N) group an adjoint field which is a singlet of the other groups → denote it by Φ_i.

- Fields which are in the fundamental of the i'th U(N) and in the anti-fundamental of the (i+1)'th U(N) → denote them by Q_i.

- Fields which are in the fundamental of the (i+1)'th U(N) and in the anti-fundamental of the i'th U(N) → denote them by $\tilde{Q}_i$.

So to summarize, we have an $\mathcal{N} = 2$ theory with gauge group[3] $U(N)^k$ and hypermultiplets in the representations:

$$(N, \bar{N}, 1, .., 1) \oplus (1, N, \bar{N}, 1, .., 1) \oplus \cdots \oplus (\bar{N}, 1, .., 1, N). \tag{3.4}$$

The matter content of this theory can be summarized in a so called "quiver" diagram, where vertices represent the gauge groups, and oriented lines represent chiral multiplets in the fundamental of the group to which they point and the antifundamental of the second group. Unoriented lines are lines pointing in both directions → lines in fundamental and anti-fundamental of the groups which they connect, i.e. they represent fields in the adjoint if they end on same group vertex . In figure 3.1 we have an example for the $\mathbb{Z}_3$ orbifold theory.

The superpotential of this theory can be easily read off from the $\mathcal{N} = 4$ superpotential by restricting it only to $\mathbb{Z}_k$ invariant fields. The easier way, however, is: we know that the theory has $\mathcal{N} = 2$ supersymmetry, and we know the gauge group and matter content → from here the theory is uniquely defined by supersymmetry considerations [4].

The $\mathcal{N} = 2$ theory vector multiplet consists of an $\mathcal{N} = 1$ vector multiplet V and an $\mathcal{N} = 1$ chiral multiplet Φ, both in the adjoint representation of the group. The hypermultiplet consists of chiral and antichiral fields $(\tilde{Q}^\dagger, Q)$, both transforming under the same representation. The general lagrangian is:

---

[3]The U(1) factors are expected to decouple in the IR so we will deal with an $SU(N)^k$ group.
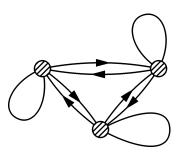
[4]Of course we have here no mass terms.

Figure 3.1: $\mathbb{C}^3/\mathbb{Z}_3$ (1,-1,0) quiver diagram

$$\sum_i \int d^4\theta(Q_i^\dagger e^{-2gV} Q_i + \tilde{Q}_i^\dagger e^{2gV} \tilde{Q}_i) + \int d^2\theta \sqrt{2} g \tilde{Q}_i \Phi Q_i + \cdots \qquad (3.5)$$

Where dots mean the $\mathcal{N} = 2$ pure SYM action. The $\mathcal{N} = 2$ theories have only one coupling[5] , the gauge coupling. In our case we denote for any one of the k SU(N)'s the chiral field of the vector multiplet by Φ^i and the vector field V^i. The hypermultiplets will be denoted by Q^i and $\tilde{Q}^i$ where each such field has two indices one in N and one in $\bar{N}$ of i'th and (i+1)'th gauge group. $\mathcal{N} = 2$ theories are renormalized only at one-loop. The general one-loop gauge $\beta - function$ is:

$$\beta(g) \propto -\frac{1}{16\pi^2} \left[3C_2(G) - \sum_A T_A(R) \right]. \qquad (3.6)$$

In $\mathcal{N} = 2$ SQCD we have two chiral multiplets from each hypermultiplet (the antichiral one can be treated as the complex conjugate of a chiral) and one chiral multiplet coming from the vector multiplet, so if the chirals are in the fundamental of SU(N) (like in our case) $T(R) = \frac{1}{2}$ and $C_2 = N$. For the adjoint representation $T(R) = N$, so from (3.6) we get for m hypermultiplets:

$$\beta(g) \propto -\frac{1}{16\pi^2} [3N - N - m] = -\frac{1}{16\pi^2} [2N - m]. \qquad (3.7)$$

---

[5]For each simple factor of the gauge group.

In our case we effectively have 2N hypermultiplets for every gauge group: $Q^i_{\alpha\tilde{\beta}}$ is N chirals of the i'th SU(N) (and also N antichirals of the (i+1)'th SU(N)), and $Q^{(i-1)}_{\alpha\tilde{\beta}}$ is N antichirals of the i'th SU(N). So m=2N and the β-functions vanish $\rightarrow$ the theory is finite for any value of the couplings.

3.1 Marginal deformations

Now we consider the marginal deformations of the theory which preserve some $\mathcal{N} = 1$ supersymmetry. First, obviously we can change the couplings of the $\mathcal{N} = 2$ superpotentials to be different from the gauge couplings and change the gauge couplings themselves. Another option is to add a superpotential for the Φ's: $h^i d_{abc} \Phi_i^a \Phi_i^b \Phi_i^c$ (where $d_{abc} \equiv Tr\,[T_a\,\{T_b, T_c\}]$, T_a in the fundamental of SU(N)). We can add also a superpotential for Q's: $s_{lmn}^{(i)abc} Q_a^{(i)l} Q_b^{(i)m} Q_c^{(i)n}$ (and similarly for $\tilde{Q}$). Here the **s** couplings are constrained from gauge invariance to satisfy:

$$(T^\alpha)_e^a s_{lmn}^{(i)ebc} + (T^\alpha)_e^b s_{lmn}^{(i)aec} + (T^\alpha)_e^c s_{lmn}^{(i)abe} = 0. \tag{3.8}$$

The same condition we get also for the lower indices. These constrains we can satisfy only for $SU(N = 3)$, because otherwise $N \otimes N \otimes N$ doesn't include gauge singlets. Obviously the **s** couplings have to be symmetric. The only possible choice for the **s** couplings is thus: $s_{abc}^{ijk} \propto \epsilon^{ijk} \epsilon_{abc}$.

The terms of the form $p_{amn}^{(i)lbc} \tilde{Q}_l^{(i)a} Q_b^{(i)m} Q_c^{(i)n}$ and $q_{abn}^{(i)lmc} \tilde{Q}_l^{(i)a} \tilde{Q}_m^{(i)b} Q_c^{(i)n}$ are ruled out by the constraints (3.8), since $N \otimes \bar{N} \otimes N$ doesn't include gauge singlets.

In k=3 case we also can have operators of the form $\frac{\kappa}{3!} Q_1 Q_2 Q_3$ (and similarly for $\tilde{Q}$).

So to summarize, we have the following marginal deformations of our finite theory:

$$
\begin{aligned}
W_1 &= \frac{1}{6}(\alpha^i \tilde{Q}_i \Phi^i Q_i + \delta^i Q_i \Phi^{i+1} \tilde{Q}_i) \\
W_2 &= \frac{1}{6} h^i d_{abc} \Phi_i^a \Phi_i^b \Phi_i^c \\
W_3 &= \frac{\rho_i}{3!} \epsilon_{lmn} \epsilon^{abc} Q_a^{(i)l} Q_b^{(i)m} Q_c^{(i)n} \\
W_4 &= \frac{\tilde{\rho}_i}{3!} \epsilon_{lmn} \epsilon^{abc} \tilde{Q}_a^{(i)l} \tilde{Q}_b^{(i)m} \tilde{Q}_c^{(i)n} \\
W_5 &= \frac{\kappa}{3!} Q_1 Q_2 Q_3 \\
W_6 &= \frac{\tilde{\kappa}}{3!} \tilde{Q}_1 \tilde{Q}_2 \tilde{Q}_3
\end{aligned}
\tag{3.9}
$$

Where the W_3, W_4 can be added only for $SU(N = 3)$ and W_5, W_6 only for k=3. So for general N and k the only marginal deformations are changing the couplings of the $\mathcal{N} = 2$

superpotential and adding a superpotential for the Φ_i. As we will see later, there are also additional interactions in the k=2 case.

Now we can engage in the search for fixed points. The one-loop gauge β function will still vanish, because it is not affected by the superpotential.

The Q's and $\tilde{Q}$'s appear symmetrically in the case of turning on only W_1 and W_2, thus we expect their γ-functions to be the same. Moreover obviously the fields appearing here don't mix in renormalization (for $k > 2$), thus all the anomalous dimensions are diagonal. Now we wish to obtain the β-functions via the anomalous dimensions.

For the gauge β-functions we have the NSVZ formula(2.8), in which for a gauge group (i) we have to take only the Φ_i and $Q_i, Q_{i-1}, \tilde{Q}_i, \tilde{Q}_{i-1}$ anomalous dimensions into account. For the superpotential couplings we have from the $\mathcal{N} = 1$ non-renormalization theorem:

$$\beta_Y^{ijk} = Y^{p(ij}\gamma_p^{k)} = Y^{ijp}\gamma_p^k + (k \leftrightarrow i) + (k \leftrightarrow j) \tag{3.10}$$

Where we arrange the superpotential to the form: $W = \frac{1}{6}Y^{ijk}S_iS_jS_k$. Here the indices run over the gauge indices as well as over the field indices. From this we get:

$$
\begin{aligned}
\beta_{g_i} &= -\frac{2g^3}{16\pi^2}\frac{N}{1 - \frac{2Ng^2}{16\pi^2}}(\frac{1}{2}(\gamma_{Q_i} + \gamma_{\tilde{Q}_i} + \gamma_{Q_{i-1}} + \gamma_{\tilde{Q}_{i-1}}) + \gamma_{\Phi_i}) \\
\beta_{h_i} &= 3h_i \cdot \gamma_{\Phi_i} \\
\beta_{\alpha_i} &= \alpha_i(\gamma_{Q_i} + \gamma_{\tilde{Q}_i} + \gamma_{\Phi_i}) \\
\beta_{\delta_i} &= \delta_i(\gamma_{Q_i} + \gamma_{\tilde{Q}_i} + \gamma_{\Phi_{i+1}})
\end{aligned}
\tag{3.11}
$$

From here we see that if we want the β-functions to vanish a sufficient condition is the vanishing of the γ-functions. We also notice that $\beta_{g_i} \propto \frac{\beta_{\alpha_i}}{\alpha_i} + \frac{\beta_{\delta_{i-1}}}{\delta_{i-1}}$. Thus essentially we only have to worry about the vanishing superpotential coupling β-functions, and the gauge β-function will vanish automatically. The gauge couplings contribute with negative sign to the anomalous dimensions and the other couplings with positive sign (at one-loop), so because of the signs of the β-functions the fixed points we get in this way are IR-stable fixed points.

3.1.1 General k case

Let's first analyze the general expressions(3.11) following [14]. The interactions we have here, as shown above, are symmetric in respect to Q_i and $\tilde{Q}_i$, thus $\gamma_{Q_i} = \gamma_{\tilde{Q}_i}$. First we analyze the case without h_i's. We see that in order that the β's will vanish all γ_{Φ_i}'s have to be equal $\rightarrow \forall i : \gamma_{\Phi_i} \equiv \gamma$. And also all the γ_{Q_i}'s have to be equal and equal to $-\frac{1}{2}\gamma$. So we have 2k conditions on the parameters, 3k couplings and one additional parameter γ. So totally we expect for a $3k + 1 - 2k = k + 1$ dimensional manifold of fixed points. This is one dimension more than the space of the $\mathcal{N} = 2$ theory.

If the h_i's don't vanish then we have to have $\gamma_{\Phi_i} = 0$ and so don't have the additional parameter. In this case we have 4k couplings and 2k conditions and so we expect a 2k dimensional manifold of fixed points.

Now we will see if we get the expected results from loop calculations. The one-loop anomalous dimensions are[6]:

$$\gamma_{\Phi_i} = \frac{1}{16\pi^2} \left\{ \frac{1}{8} \frac{N^2 - 4}{N} h_i^2 + \frac{N}{4} ((\delta_{i-1}^2 + \alpha_i^2) - 8g_i^2) \right\}. \tag{3.12}$$

$$\gamma_{Q_i} = \frac{1}{16\pi^2} \frac{N^2 - 1}{4N} \left\{ (\delta_i^2 + \alpha_i^2) - 4(g_i^2 + g_{i+1}^2) \right\}.$$

In particular we see that in the case that all h_i vanish and $\alpha_i = \delta_{i-1} = 2g_i$ the anomalous dimensions vanish, and this is exactly the $\mathcal{N} = 2$ case.

- **Vanishing h_is**

First we put all h_i's to zero. Let's define: $B_i \equiv \delta_{i-1}^2 - 4g_i^2$, $A_i \equiv \alpha_i^2 - 4g_i^2$ and $16\pi^2 \frac{4N}{N^2-1} \gamma \rightarrow \gamma$. Then, the requirement of vanishing β-functions becomes :

$$B_I + A_I = \frac{N^2 - 1}{N^2} \gamma, \tag{3.13}$$

$$B_{I+1} + A_I = -\frac{1}{2}\gamma.$$

---

[6]For one loop calculation we can use the results of [10], although here we have a non-simple gauge group. This is because at one loop every diagram contains at most one gauge group (out of k), thus we can simply sum the contributions from every gauge group.

From here, by subtracting the first line from the second and summing over i, we get that $\gamma = 0$. Thus in the one loop precision the γ parameter has to vanish. As we will see later this is not necessarily true for higher loop calculations. The case of vanishing γ is the case of vanishing of all anomalous dimensions. We see that in this case the condition for having zero β-functions is: $\forall i \{B_i \equiv X = -A_i\}$ where $\mathbf{X}$ is a parameter. From here we find a family of solutions parameterized by $\mathbf{X}$ and the gauge couplings:

$$
\begin{aligned}
\delta_{i-1}^2 &= X + 4g_i^2 \\
\alpha_i^2 &= 4g_i^2 - X.
\end{aligned}
\tag{3.14}
$$

We see that the parameter $\mathbf{X}$ is constrained to be: $-\min_i \{4g_i^2\} \le X \le \min_i \{4g_i^2\}$. The case X=0 is the case of $\mathcal{N} = 2$ SUSY.

Thus to summarize, we get a **k+1** dimensional solution. We expected a **k+1** dimensional manifold from the Leigh-Strassler analysis, but the **+1** was due to the γ parameter, here at one loop we find that $\gamma = 0$ but nevertheless we get a **k+1** dimensional space of solutions. The question is whether the vanishing of γ extends to higher loops and whether we can find the parameter $\mathbf{X}$ at higher loops.

We will prove now that the non-vanishing $\mathbf{X}$ solution doesn't disappear at higher loops. First we will represent a general solution as a function of the gauge couplings and the $\mathbf{X}$ parameter. The procedure we use here is similar to the coupling constant reduction procedure described in [19].

We define the most general solution for α_i and δ_i depending on $\mathbf{X}$, g_i, and consistent with the one loop analysis and with the $\mathcal{N} = 2$ case (which is known to be exactly conformal):

$$
\begin{aligned}
\delta_{i-1}^2 &= 4g_i^2 + X(1 + \sum_{m,j,l_s} a_{l_1...l_j}^{(i)m} X^m g_{l_1}^2 ... g_{l_j}^2) \\
\alpha_i^2 &= 4g_i^2 - X(1 + \sum_{m,j,l_s} b_{l_1...l_j}^{(i)m} X^m g_{l_1}^2 ... g_{l_j}^2)
\end{aligned}
\tag{3.15}
$$

Here $\mathbf{a}$, $\mathbf{b}$ are some constants and $m+j > 0$. Now assume we have computed the $\mathbf{a}$ and $\mathbf{b}$ parameters in these solutions up to (n-1)'th order in g^2, $\mathbf{X}$. We look at the n'th order. First we calculate the γ_{Q_i} and γ_{Φ_i}. We write them as:

$$\gamma_{Q_i}^{(n)} = \gamma_{Q_i}^{(n)(1-loop)} + \gamma_{Q_i}^{(n)(2..n-loops)} \tag{3.16}$$

$$\gamma_{\Phi_i}^{(n)} = \gamma_{\Phi_i}^{(n)(1-loop)} + \gamma_{\Phi_i}^{(n)(2..n-loops)}.$$

We define:

$$\tilde{B}_i^{(n)} \equiv \Delta\delta_{i-1}^2 = X \cdot \left(\sum_{m,j,l_s,m+j=(n-1)} a_{l_1...l_j}^{(i)m} X^m g_{l_1}^2...g_{l_j}^2 \right) \tag{3.17}$$

$$\tilde{A}_i^{(n)} \equiv \Delta\alpha_i^2 = -X \cdot \left(\sum_{m,j,l_s,m+j=(n-1)} b_{l_1...l_j}^{(i)m} X^m g_{l_1}^2...g_{l_j}^2 \right)$$

The $\gamma_{Q_i}^{(1-loop)}$ and $\gamma_{\Phi_i}^{(1-loop)}$ have a special structure, giving:

$$\tilde{B}_{i+1}^{(n)} + \tilde{A}_i^{(n)} = \gamma_{Q_i}^{(n)(1-loop)} \tag{3.18}$$

$$\tilde{B}_i^{(n)} + \tilde{A}_i^{(n)} = \frac{N^2-1}{N^2} \gamma_{\Phi_i}^{(n)(1-loop)}$$

Now we parameterize the remaining contributions to γs as:

$$\gamma_{Q_i}^{(n)(2..n-loops)} \equiv T_i^{(n)} + S_{i+1}^{(n)} - \frac{1}{2}\tilde{\gamma}^{(n)} \tag{3.19}$$

$$\frac{N^2-1}{N^2}\gamma_{\Phi_i}^{(n)(2..n-loops)} \equiv T_i^{(n)} + S_i^{(n)} + \frac{N^2-1}{N^2}\tilde{\gamma}^{(n)}$$

Where the different quantities are defined as :

$$-k\left(\frac{1}{2} + \frac{N^2-1}{N^2}\right)\tilde{\gamma}^{(n)} \equiv \sum_i \gamma_{Q_i}^{(n)(2..n-loops)} - \frac{N^2-1}{N^2}\gamma_{\Phi_i}^{(n)(2..n-loops)}$$

$$\Delta X^{(n)} = S_1^{(n)}(\equiv 0) \tag{3.20}$$

$$\gamma_{Q_i}^{(n)(2..n-loops)} - \frac{N^2-1}{N^2}\gamma_{\Phi_i}^{(n)(2..n-loops)} = S_{i+1}^{(n)} - S_i^{(n)} - \left(\frac{1}{2} + \frac{N^2-1}{N^2}\right)\tilde{\gamma}^{(n)}$$

$\Delta X^{(n)}$ is just a redefinition of X, so we can set it to zero without any loss of generality, and the $T_i^{(n)}$s are automatically determined from above. We see that the definitions above are well and uniquely defined.

The crucial point is that in order to calculate the one loop contribution to the n'th order we use the n'th order components of (3.15), and for two loops we use the (n-1)'th order of (3.15),..., for n'th order we use the first order of (3.15). Thus because we have already determined (3.15)up to n-1'th order, $\gamma_{\Phi_i}^{(n)(2..n-loops)}$, $\gamma_{Q_i}^{(n)(2..n-loops)}$ depend only on already determined quantities. The yet undetermined quantities appear only in one loop.

From the β-function analysis we know that $\gamma_{Q_i} = -\frac{1}{2}\gamma$ and $\gamma_{\Phi_i} = \gamma$. Thus the demand (3.16) is translated to:

$$
\begin{aligned}
\tilde{A}_i^{(n)} &= -T_i^{(n)} \\
\tilde{B}_i^{(n)} &= -S_i^{(n)} \\
\tilde{\gamma}^{(n)} &= \gamma^{(n)}
\end{aligned}
\tag{3.21}
$$

Here, in the first two lines, we are defining the yet undetermined **a**'s and **b**'s. And in the third line we compute the γ parameter. We see that it does not have to be zero in higher loops.

Again this procedure is well defined and unique, and can be extended to any order in perturbation theory.

We see that we found a solution to our problem which obviously exists in any order of perturbation theory. So we have proven that there exists a **k+1** dimensional manifold of fixed points parameterized by the gauge couplings and the **X** parameter (or equivalently the γ parameter if it is non zero starting from some order in perturbation series), in all orders of perturbation theory.

- **Non-vanishing h_is**

Now we turn our attention to the non zero h_i case. In this case we are constrained to have $\gamma_{\Phi_i} = 0$ or equivalently $\gamma = 0$. At one-loop we get: (define $C_i \equiv \frac{1}{8}\frac{N^2-4}{N}h_i^2$)

$$
\begin{aligned}
B_i + A_i &= -C_i \\
B_{i+1} + A_i &= 0
\end{aligned}
\tag{3.22}
$$

From here:

$$B_{i+1} - B_i = C_i. \tag{3.23}$$

But from the cyclic nature of our couplings ($\alpha_{k+1} = \alpha_1$ etc.) we get $\sum_i \{B_{i+1} - B_i\} = 0$ $\to \sum_i C_i = 0$, but this is impossible unless all h_i vanish because C_i is positive definite. So we conclude that there are no fixed points with non vanishing h_i. So at one loop level we get no new marginal directions in this case.

We now proceed in search of all loop solutions like we did above. The general expressions for α_i and δ_i now depend also on h_is, and we proceed as before:

$$\gamma_{Q_i}^{(n)} = \gamma_{Q_i}^{(n)(1-loop)} + \gamma_{Q_i}^{(n)(2..n-loops)} \tag{3.24}$$
$$\gamma_{\Phi_i}^{(n)} = \gamma_{\Phi_i}^{(n)(1-loop)} + \gamma_{\Phi_i}^{(n)(2..n-loops)}$$

The $\gamma_{Q_i}^{(1-loop)}$ and $\gamma_{\Phi_i}^{(1-loop)}$ have a special structure, giving:

$$\tilde{B}_{i+1}^{(n)} + \tilde{A}_i^{(n)} = \gamma_{Q_i}^{(n)(1-loop)} \tag{3.25}$$
$$\tilde{B}_i^{(n)} + \tilde{A}_i^{(n)} + C_i^{(n)} = \frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(1-loop)}$$

Now we parameterize the remaining contributions to γs as:

$$\gamma_{Q_i}^{(n)(2..n-loops)} \equiv T_i^{(n)} + S_{i+1}^{(n)} \tag{3.26}$$
$$\frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(2..n-loops)} \equiv T_i^{(n)} + S_i^{(n)} - C_i^{(n)}$$

Where the different quantities are obtained from here as :

$$\sum_i C_i^{(n)} = \sum_i \gamma_{Q_i}^{(n)(2..n-loops)} - \frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(2..n-loops)}$$
$$\Delta X^{(n)} = S_1^{(n)}(\equiv 0) \tag{3.27}$$
$$\gamma_{Q_i}^{(n)(2..n-loops)} - \frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(2..n-loops)} = S_{i+1}^{(n)} - S_i^{(n)} + C_i^{(n)}$$

$\Delta X^{(n)}$ is just a redefinition of $\mathbf{X}$, so we can set it to zero without any loss of generality, and the $T_i^{(n)}$s are automatically determined from above. We see that the definitions above are well and uniquely defined, except for one caveat. The first equation above cannot always be satisfied: $C_i \equiv \sum_n C_i^{(n)} \geq 0$, implying that in the lowest order where $C_i^{(n)}$ is not zero it has to be positive, thus in that order $\sum_i \gamma_{Q_i}^{(n)(2..n-loops)} - \frac{N^2-1}{N^2} \gamma_{\Phi_i}^{(n)(2..n-loops)}$ has to be positive. At one loop and two loops this quantity is zero, and in Appendix A we make a calculation and find that at three loops this is also zero, thus implying that no h_is can be turned on at this order.

If this quantity has the right sign we proceed like in the previous case to obtain a solution, by demanding that $\gamma_{Q_i} = \gamma_{\Phi_i} = 0$:

$$
\begin{aligned}
\tilde{A}_i^{(n)} &= -T_i^{(n)} \\
\tilde{B}_i^{(n)} &= -S_i^{(n)}
\end{aligned}
\tag{3.28}
$$

Here we are defining the yet undetermined $\mathbf{a}$'s and $\mathbf{b}$'s.

Again this procedure is well defined and unique, and can be extended to any order in perturbation theory, considering the caveat above is satisfied.

In the calculation of the first non-vanishing contribution to the γ parameter we will take advantage of the fact that if the $\mathbf{X}$ parameter is zero then we are in the $\mathcal{N} = 2$ case and we know that $\gamma = 0$, and that following the analysis of ([21], [22]) we know that the theory we are concerned with (if all the gauge couplings are equal) is the same as the $\mathcal{N} = 4$ theory up to the non-planar diagrams. Thus, we expect contributions to γ only from these diagrams.

The two-loop γ-function vanishes here if the one-loop γ-function vanishes so it doesn't bring any new features. At three loops however we get several non-planar diagrams . As we see in Appendix A the contribution proportional to $\mathbf{X}$ will vanish because it always comes in a $X(g_i^4 - g_{i+1}^4)$ combination, and the graphs with three gauge interactions don't include $\mathbf{X}$ parameter, and thus are not interesting due to the $\mathcal{N} = 2$ case. The explicit calculations (see Appendix A) show that the contribution of the rest of the diagrams is zero at three loops, implying that the γ parameter is zero and thus there is no possibility of satisfying (in three loop precision):

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$$\sum_i C_i \;=\; \sum_i \gamma_{Q_i} - \frac{N^2 - 1}{N^2} \gamma_{\Phi_i}. \tag{3.29}$$

Thus there are two possibilities: either we have non zero γ at higher loops, or the γ parameter is strictly zero.

So to summarize, in the general **k** case there are always **k+1** *exactly* marginal directions which can be parameterized by the **k** gauge couplings and a parameter **X**. If at higher loops $\sum_i \gamma_{Q_i} - \frac{N^2-1}{N^2} \gamma_{\Phi_i}$ is turned on with the correct sign, then we obtain another **k-1** exactly marginal deformations as anticipated from the Leigh-Strassler analysis.

3.1.2   k=3 case

If we consider the special case of k=3 we can add additional marginal operators: $\frac{\kappa}{3!}Q_1Q_2Q_3$ and $\frac{\tilde{\kappa}}{3!}\tilde{Q}_1\tilde{Q}_2\tilde{Q}_3$. Again first we deal with vanishing h_i's. We get two additional β-functions:

$$\beta_\kappa = \kappa(\gamma_{Q_1} + \gamma_{Q_2} + \gamma_{Q_3})$$
$$\beta_{\tilde{\kappa}} = \tilde{\kappa}(\gamma_{\tilde{Q}_1} + \gamma_{\tilde{Q}_2} + \gamma_{\tilde{Q}_3}). \tag{3.30}$$

From here and (3.11) we get that in order that the β-functions will vanish we have to demand $\gamma_{\Phi_i} = 0$ for all i, and $\gamma_{Q_i} = -\gamma_{\tilde{Q}_i} \equiv \gamma_i$, $\sum_i \gamma_i = 0$. Thus we have 3k+2 couplings, k parameters γ_i and 3k+1 conditions, so the expected dimension of the manifold of fixed points is 3k+2+k-(3k+1)=k+1=4. We get the same prediction as for general k and thus we expect the solution to be exactly as there.

The extra interactions of the k=3 case will add a term of the form $\frac{\kappa^2 N}{2}(\equiv D)$ to the one-loop γ_{Q_i} and $\frac{\tilde{\kappa}^2 N}{2}(\equiv \tilde{D})$ to the one-loop $\gamma_{\tilde{Q}_i}$, and won't effect the one-loop γ_{Φ_i}. Thus now the general condition for vanishing of the β-functions becomes:(defining: $\gamma_i \to 16\pi^2 \frac{4}{N}\gamma_i$)

$$
\begin{aligned}
B_i + A_i &= 0 \\
B_{i+1} + A_i + D &= \gamma_i \\
B_{i+1} + A_i + \tilde{D} &= -\gamma_i.
\end{aligned}
\tag{3.31}
$$

By subtracting the first equation from the second and the third ones and summing over i we get two conditions: $k\tilde{D} = -\sum_i \gamma_i = 0$ and $kD = \sum_i \gamma_i = 0$. So we see that κ and $\tilde{\kappa}$ have to be zero at one loop precision because D and $\tilde{D}$ are positive definite. Subtracting the second equation from the third we get: $2\gamma_i = D - \tilde{D} = 0$. We see that all γ_i's have to vanish $\to$ for all i $\gamma_i = 0$. Again, as in the previous subsection, we can write a general solution:

$$B_{i+1} = B_i. \tag{3.32}$$

From here we see that also all A_i have to be equal. So we get that a solution is parameterized by the gauge couplings and by B_1. Thus the dimension of the manifold

of fixed points is k+1=4. We get the expected result. Which is exactly the same as the general **k** solution.

Now we turn on non vanishing h_i's. Again we will have to demand $\gamma_{\Phi_i} = 0$, as was the case also in the vanishing h_i's case, which implies also $\gamma_{Q_i} + \gamma_{\tilde{Q}_i} = 0$. From here we get k parameters corresponding to the value of each $\gamma_{Q_i} \equiv \gamma_i$, with $\sum_i \gamma_i = 0$. Thus we have 4k+2 couplings, k parameters and 3k+1 conditions giving a naive expectation for the dimension of the manifold of fixed points 4k+2+k-(3k+1)=2k+1=7. The one loop analysis gives: (C_i defined in previous subsection)

$$
\begin{aligned}
B_i + A_i &= -C_i \\
B_{i+1} + A_i + D &= \gamma_i \\
B_{i+1} + A_i + \tilde{D} &= -\gamma_i
\end{aligned}
\tag{3.33}
$$

By subtracting the second equation from the third we get: $\tilde{D} - D = -2\gamma_i$, thus as in the previous case all γ_i have to be equal (to some γ). But $\sum_i \gamma_i = 0$, so $\gamma = 0$. By subtracting the first equation from the second and summing over i we get: $kD = \sum_i C_i$. And the general solution is:

$$
B_{i+1} = B_i + C_i - D
\tag{3.34}
$$

So, if we choose $B_1 \equiv X$ the solution is:

$$
\begin{aligned}
B_i &= X + \sum_{n=1}^{i-1} C_n - (i-1)D \\
A_i &= -B_i - C_i = -(X + \sum_{n=1}^{i} C_n - (i-1)D).
\end{aligned}
\tag{3.35}
$$

Obviously we have a constraint on D: $\sum_i C_i = kD$. For the αs and δs we get:

$$
\begin{aligned}
\delta_{i-1}^2 &= X + 4g_i^2 + \sum_{n=1}^{i-1} C_n - (i-1)D \\
\alpha_i^2 &= 4g_i^2 - X - (\sum_{n=1}^{i} C_n - (i-1)D).
\end{aligned}
\tag{3.36}
$$

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It is parameterized by $\mathbf{X}$, gauge couplings, h_is and D subject to a constraint $\rightarrow$ giving a total of (1+3+3+1)-1=7 dimensional manifold of marginal deformations.

Here again we can extend the solution to higher loops using the procedure we described in general k case, again here we will have:

$$
\begin{aligned}
\tilde{B}_{i+1}^{(n)} + \tilde{A}_i^{(n)} + D^{(n)} &= \gamma_{Q_i}^{(n)(1-loop)} \\
\tilde{B}_{i+1}^{(n)} + \tilde{A}_i^{(n)} + \tilde{D}^{(n)} &= \gamma_{\tilde{Q}_i}^{(n)(1-loop)} \\
\tilde{B}_i^{(n)} + \tilde{A}_i^{(n)} + C_i^{(n)} &= \frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(1-loop)}
\end{aligned}
\tag{3.37}
$$

Now we parameterize the remaining contributions to γs as:

$$
\begin{aligned}
\gamma_{Q_i}^{(n)(2..n-loops)} &\equiv T_i^{(n)} + S_{i+1}^{(n)} - D^{(n)} + \tilde{\gamma}_i^{(n)} \\
\gamma_{\tilde{Q}_i}^{(n)(2..n-loops)} &\equiv T_i^{(n)} + S_{i+1}^{(n)} - \tilde{D}^{(n)} - \tilde{\gamma}_i^{(n)} \\
\frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(2..n-loops)} &\equiv T_i^{(n)} + S_i^{(n)} - C_i^{(n)}
\end{aligned}
\tag{3.38}
$$

Where the different quantities are defined as :

$$
\begin{aligned}
\sum_i C_i^{(n)} - kD^{(n)} &\equiv \sum_i \gamma_{Q_i}^{(n)(2..n-loops)} - \frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(2..n-loops)} \\
\sum_i C_i^{(n)} - k\tilde{D}^{(n)} &\equiv \sum_i \gamma_{\tilde{Q}_i}^{(n)(2..n-loops)} - \frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(2..n-loops)} \\
\Delta X^{(n)} &= S_1^{(n)} (\equiv 0) \\
S_{i+1}^{(n)} - S_i^{(n)} + C_i^{(n)} - D^{(n)} + \tilde{\gamma}_i^{(n)} &= \gamma_{Q_i}^{(n)(2..n-loops)} - \frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(2..n-loops)} \\
2\gamma_i^{(n)} &= \gamma_{Q_i}^{(n)(2..n-loops)} - \gamma_{\tilde{Q}_i}^{(n)(2..n-loops)} - (D^{(n)} - \tilde{D}^{(n)}).
\end{aligned}
\tag{3.39}
$$

We can consider solution to be defined by h_is, $\mathbf{X}$ and the gauge couplings, and then all other quantities can be expressed in terms of these 7 ones as we see above. The first two expressions can be seen as defining D and $\tilde{D}$ (we can set $C_i^{(n)} = 0$ for $n > 1$, setting nonzero value is just redefinition of h_is), the fifth equation is definition of $\tilde{\gamma}_i$, and the fourth is definition of S_i, again T_i can be defined from here.

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The solution as before is (demanding $\gamma_{Q_i} = -\gamma_{\tilde{Q}_i} = \gamma_i, \gamma_{\Phi_i} = 0$):

$$\tilde{A}_i^{(n)} = -T_i^{(n)} \tag{3.40}$$
$$\tilde{B}_i^{(n)} = -S_i^{(n)}$$
$$\tilde{\gamma}_i^{(n)} = \gamma_i^{(n)}$$

(We can see that $\sum_i \tilde{\gamma}_i^{(n)} = 0$ as required for γ_i).

Again, we have proven existence of an all orders solution. In the k=3 case the switching on of the h_i's doesn't produce any new conditions, so unlike the general k case here the case of $h_i = 0$ and the case $h_i \neq 0$ can be treated together, thus giving that in this case the space of fixed points is **7** dimensional.

Here the **k+1=4** directions are as in the general **k** case, and **1** deformation coming from κ and $\tilde{\kappa}$ is due to the h_{iii} interactions of the $\mathcal{N} = 4$ theory which survive orbifolding[7]. We get **2** additional *exactly* marginal deformations which we don't see in the general case and do not come from the $\mathcal{N} = 4$ theory $\rightarrow$ this is a prediction of our analysis.

---

[7]So we know that it is *exactly* marginal at least in the large **N** limit.

3.1.3   k=2 case

In this case the fields Q_1 and $\tilde{Q}_2$ are in the same gauge group representations, and also the fields Q_2 and $\tilde{Q}_1$ are. What this implies is that we have here a global $SU(2)^2$ symmetry rotating these fields and we can write the most general marginal deformations of this theory as:

$$W = \Phi_1 \begin{pmatrix} Q_1 & \tilde{Q}_2 \end{pmatrix} \begin{pmatrix} \alpha & s \\ p & \delta \end{pmatrix} \begin{pmatrix} \tilde{Q}_1 \\ Q_2 \end{pmatrix} + \Phi_2 \begin{pmatrix} Q_2 & \tilde{Q}_1 \end{pmatrix} \begin{pmatrix} \alpha' & s' \\ p' & \delta' \end{pmatrix} \begin{pmatrix} \tilde{Q}_2 \\ Q_1 \end{pmatrix} \tag{3.41}$$

Lets make the Leigh-Strassler analysis. Using the global $SU(2)^2$ symmetry we can diagonalize the anomalous dimensions (apriori we can have mixing terms here). Thus we will have here additional β-functions:

$$\beta_{s,s'} \propto \gamma_{Q_{1,2}} + \gamma_{Q_{2,1}} + \gamma_{\Phi_{1,2}} = 0$$
$$\beta_{p',p} \propto \gamma_{\tilde{Q}_{1,2}} + \gamma_{\tilde{Q}_{2,1}} + \gamma_{\Phi_{2,1}} = 0 \tag{3.42}$$

From here and as before we conclude that $\gamma_{\Phi_1} = \gamma_{\Phi_2} \equiv \gamma$, $\gamma_{Q_1} = \gamma_{\tilde{Q}_2} \equiv \tilde{\gamma}$ and $\gamma_{Q_2} = \gamma_{\tilde{Q}_1} \equiv -\gamma - \tilde{\gamma}$.

So we have here **10** couplings, **2** γs $\rightarrow$ **12** parameters. We have **8** anomalous dimensions $\rightarrow$ we have **8** constraints. Thus, we expect a **4** dimensional manifold of fixed points with non zero new interactions.

This case is a special one of the $\mathbb{Z}_k$ **(a,a,-2a)** orbifold theory (with $\mathbb{Z}_{k=2}$ and **a=1**). In the $\mathcal{N} = 1$ chapter we will deal with this case extensively and so we will postpone our discussion until then. In particular we will see that actually we get only a **3** dimensional manifold of fixed points because of the one-loop structure.

3.1.4 SU(N=3) case

In this case one can add an operator $\frac{\rho_i}{3!}\epsilon_{lmn}\epsilon^{abc}Q_a^{(i)l}Q_b^{(i)m}Q_c^{(i)n}$ and $\frac{\tilde{\rho}_i}{3!}\epsilon_{lmn}\epsilon^{abc}\tilde{Q}_a^{(i)l}\tilde{Q}_b^{(i)m}\tilde{Q}_c^{(i)n}$. Again we begin by analyzing the vanishing h_i's case.

$$\beta_{\rho_i} = 3\rho_i \cdot \gamma_{Q_i}$$
$$\beta_{\tilde{\rho}_i} = 3\tilde{\rho}_i \cdot \gamma_{\tilde{Q}_i} \tag{3.43}$$

So from here and (3.11) we have for all i $\gamma_{Q_i} = 0$, $\gamma_{\tilde{Q}_i} = 0$, $\gamma_{\Phi_i} = \gamma$ and $\gamma_{Q_i} + \gamma_{\tilde{Q}_i} = -\gamma \rightarrow \gamma_{\Phi_i} = 0$. Thus we see that as in the k=3 case adding non zero h_is doesn't change the conditions for the γs (γ_{Φ_i} has to be zero anyway), so we can consider them together. We have here 6k couplings and 3k conditions, leading naively to a 3k dimensional manifold of fixed points. The interactions we add affect the one loop γ_{Φ_i} as in (3.22,) and we add a term $2\rho_i^2(\equiv K_i)$ for Q and $2\tilde{\rho}_i^2(\equiv \tilde{K}_i)$ for $\tilde{Q}$:

$$
\begin{aligned}
B_i + A_i &= -C_i \\
B_{i+1} + A_i &= -K_i \\
B_{i+1} + A_i &= -\tilde{K}_i
\end{aligned}
\tag{3.44}
$$

Again we get here that at one loop there are some conditions:

$$
\begin{aligned}
\sum_i C_i &= \sum_i K_i \\
K_i &= \tilde{K}_i
\end{aligned}
\tag{3.45}
$$

And the one loop solution is:

$$
\begin{aligned}
B_i &= X + \sum_{n=1}^{i-1}(C_n - K_n) \\
A_i &= -C_i - B_i = -X + \sum_{n=1}^{i-1} K_n - \sum_{n=1}^{i} C_n
\end{aligned}
\tag{3.46}
$$

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Implying for αs and δs:

$$
\begin{aligned}
\delta_{i-1}^2 &= X + 4g_i^2 + \sum_{n=1}^{i-1}(C_n - K_n) \\
\alpha_i^2 &= 4g_i^2 - X + \sum_{n=1}^{i-1} K_n - \sum_{n=1}^{i} C_n.
\end{aligned}
\tag{3.47}
$$

So the general solution is parameterized by k gauge couplings, the **X** parameter, k ρ_is and k h_is subject to the condition (3.45) above, giving a total of 3k parameters $\rightarrow$ we get 3k dimensional manifold of fixed points as expected.

In the k=3 case we can have two additional marginal operators (see section 3.1.2), they will add 2 parameters to our analysis and no extra conditions. So we will expect to find a 3k+2=11 dimensional manifold of fixed points.

We can here also extend[8] the solution to all orders in perturbation theory. We choose to parameterize our solution by k gauge couplings, the **X** parameter, k K_is and (k-1) C_is ($i \in (1,..,(k-1))$). Again we can write:

$$
\begin{aligned}
\tilde{B}_{i+1}^{(n)} + \tilde{A}_i^{(n)} + K_i^{(n)} &= \gamma_{Q_i}^{(n)(1-loop)} \\
\tilde{B}_{i+1}^{(n)} + \tilde{A}_i^{(n)} + \tilde{K}_i^{(n)} &= \gamma_{\tilde{Q}_i}^{(n)(1-loop)} \\
\tilde{B}_i^{(n)} + \tilde{A}_i^{(n)} + C_i^{(n)} &= \frac{N^2-1}{N^2}\gamma_{\Phi_i}^{(n)(1-loop)}.
\end{aligned}
\tag{3.48}
$$

Now we parameterize the remaining contributions to γs as:

$$
\begin{aligned}
\gamma_{Q_i}^{(n)(2..n-loops)} &\equiv T_i^{(n)} + S_{i+1}^{(n)} - K_i^{(n)} \\
\gamma_{\tilde{Q}_i}^{(n)(2..n-loops)} &\equiv T_i^{(n)} + S_{i+1}^{(n)} - \tilde{K}_i^{(n)} \\
\frac{N^2-1}{N^2}\gamma_{\Phi_i}^{(n)(2..n-loops)} &\equiv T_i^{(n)} + S_i^{(n)} - C_i^{(n)},
\end{aligned}
\tag{3.49}
$$

where the different quantities are defined as:

[8]Here essentially because the solution we get at one loop is exactly as predicted from the Leigh-Strassler analysis the existence of the solution to all orders is guaranteed. Nevertheless we write the all loop extension for completeness.

$$\sum_i (C_i^{(n)} - K_i^{(n)}) \equiv \sum_i \gamma_{Q_i}^{(n)(2..n-loops)} - \frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(2..n-loops)}$$

$$\sum_i (C_i^{(n)} - \tilde{K}_i^{(n)}) \equiv \sum_i \gamma_{\tilde{Q}_i}^{(n)(2..n-loops)} - \frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(2..n-loops)}$$

$$\Delta X^{(n)} = S_1^{(n)} (\equiv 0) \tag{3.50}$$

$$S_{i+1}^{(n)} - S_i^{(n)} + C_i^{(n)} - K_i^{(n)} = \gamma_{Q_i}^{(n)(2..n-loops)} - \frac{N^2 - 1}{N^2} \gamma_{\Phi_i}^{(n)(2..n-loops)}$$

$$\gamma_{Q_i}^{(n)(2..n-loops)} - \gamma_{\tilde{Q}_i}^{(n)(2..n-loops)} = (K_i^{(n)} - \tilde{K}_i^{(n)}).$$

The last line can be seen as the definition of $\tilde{K}_i$ and the first equation can be seen as the definition of C_k. And finally again we demand:

$$\tilde{A}_i^{(n)} = -T_i^{(n)} \tag{3.51}$$

$$\tilde{B}_i^{(n)} = -S_i^{(n)}.$$

$$\tag{3.52}$$

In the SU(3) k=3 case the calculation is very similar.

So to conclude: in general k with SU(3) group we get **3k** *exactly* marginal directions, and in k=3, SU(3) we get **11** *exactly* marginal directions.

3.2 RG flow analysis

As we saw in the $\mathcal{N} = 4$ case the dependence of the β- functions on the anomalous dimensions can teach us about the flow lines in the space of the coupling constants. The main point is that from (3.11) we get relations between the couplings which are obeyed during the RG flow. We rewrite (3.11) here:

$$
\begin{aligned}
\beta_{g_i} &= -\frac{2g^3}{16\pi^2} \frac{N}{1 - \frac{2Ng^2}{16\pi^2}} \left(\frac{1}{2} (\gamma_{Q_i} + \gamma_{\tilde{Q}_i} + \gamma_{Q_{i-1}} + \gamma_{\tilde{Q}_{i-1}}) + \gamma_{\Phi_i} \right) \\
\beta_{h_i} &= 3h_i \cdot \gamma_{\Phi_i} \\
\beta_{\alpha_i} &= \alpha_i (\gamma_{Q_i} + \gamma_{\tilde{Q}_i} + \gamma_{\Phi_i}) \\
\beta_{\delta_i} &= \delta_i (\gamma_{Q_i} + \gamma_{\tilde{Q}_i} + \gamma_{\Phi_{i+1}}).
\end{aligned}
\tag{3.53}
$$

From here we conclude that (We do same the rescaling of couplings as in the $\mathcal{N} = 4$ section):

$$
\begin{aligned}
-\frac{1 - 2g_i^2}{g_i^3} \frac{\partial g_i}{\partial \ln \mu} &= \frac{\partial \ln \delta_{i-1}}{\partial \ln \mu} + \frac{\partial \ln \alpha_i}{\partial \ln \mu} \\
\frac{\partial \ln \delta_i}{\partial \ln \mu} - \frac{\partial \ln \alpha_i}{\partial \ln \mu} &= \frac{1}{3} \left(\frac{\partial \ln h_{i+1}}{\partial \ln \mu} - \frac{\partial \ln h_i}{\partial \ln \mu} \right)
\end{aligned}
\tag{3.54}
$$

So we get :

$$
\begin{aligned}
\alpha_i \delta_{i-1} &\propto g_i^2 exp(\frac{1}{2g_i^2}), \\
\left(\frac{\delta_i}{\alpha_i} \right)^3 &\propto \frac{h_{i+1}}{h_i}.
\end{aligned}
\tag{3.55}
$$

Here the proportionality factors are determined by the initial conditions and are not changed by the RG flow. In the $\mathcal{N} = 4$ case these constraints on the flow were sufficient to determine the flow lines. Here however we see that we only have some relations which have to be obeyed during the flow.

The expressions above are of explicit non perturbative nature. As an example of the use of these relations (3.55) consider this: we can use the solutions above to find to what

value of $\mathbf{X}$ the couplings will flow from given initial conditions, in terms of the final gauge couplings. We have:$\alpha_i \delta_{i-1} = \mathcal{A}g_i^2 exp(\frac{1}{2g_i^2})$ (Where the initial conditions are encoded in $\mathcal{A}$). So using the one loop solution from general k section we obtain: $X^2 = 16g_i^4 - \mathcal{A}^2 g_i^4 exp(\frac{1}{g_i^2})$. Thus we see that if we know the final gauge couplings and the initial parameters we know which value of $\mathbf{X}$ we will flow to[9].

---

[9]Because the $\mathbf{X}$ parameter is obtained from perturbation theory the argument is only valid if we end up in the small coupling regime.

Chapter 4

$\mathcal{N} = 1$ theory

From the discussion in the previous chapter we see that to get $\mathcal{N} = 1$ we need to have an $\overrightarrow{a}$ vector satisfying: no one of the a_is is zero and $a_1 + a_2 + a_3 = 0 (mod\mathbf{k})$. We get gauge group $SU(N)^k$, and we have matter in the representations (chiral fields):

$$\oplus_{i=0}^{k-1} (1, 1...1, N_{(i)}, 1...1, \bar{N}_{(i+a_1)}, 1....1, 1)$$
$$\oplus_{i=0}^{k-1} (1, 1...1, N_{(i)}, 1...1, \bar{N}_{(i+a_2)}, 1....1, 1)$$
$$\oplus_{i=0}^{k-1} (1, 1...1, N_{(i)}, 1...1, \bar{N}_{(i+a_3)}, 1....1, 1). \tag{4.1}$$

We will denote the fields in our theory by Q_l^I where $I \in (0, ..., k-1)$, $l \in (1, 2, 3)$. The index $\mathbf{I}$ denotes the group SU(N) of which the field is in the fundamental representation, and $\mathbf{l}$ denotes the $\overrightarrow{a}$ component.

In this chapter we won't be interested in the potential coming from the orbifold theory but rather in the most general superpotential with this matter content. The only superpotential we can get here is of the type we got in the k=3 case in the previous chapter, because we don't have here fields in the adjoint representation. This implies that the only superpotential[1] possible here is of the form:

$$W = h_{lmn}^I Q_l^I Q_m^{I+a_l} Q_n^{I+a_l+a_m}, \tag{4.2}$$

---

[1]It is very easy to read out all the possible interactions (for $SU(N \neq 3)$) from the quiver diagrams: all oriented triangles in the diagram correspond to an interaction term (see the examples in the next sections).

where we have to assume that $a_l + a_m + a_n = 0(mod\mathbf{k})$. The definition of the couplings h^I_{lmn} in this way is redundant, we see that:

$$h^I_{lmn} = h^{I+a_l}_{mnl} = h^{I+a_l+a_m}_{nlm}. \tag{4.3}$$

Obviously for a general choice of k and the $\overrightarrow{a}$ vector, the only possibility for this to be true is by taking (l,m,n) to be some permutation of $(1,2,3)^2$, so we will start by constraining our research to this case.

From the string theory arguments we have to have here at least one exactly marginal direction at large $\mathbf{N}$ parameterized by the string coupling (g_{string}) of the dual string theory. We will see now that we have for any k another marginal deformation.

First look at the β-functions. The one loop gauge β-function is proportional to: $3C_2(G) - \sum_A T_A$. In our case $C_2(G) = N$ for SU(N) and T_A is $\frac{1}{2}$ for (anti)fundamental representations of SU(N). For each gauge group the matter content above implies that we effectively have 6N chiral multiplets[3], so the one loop result is proportional to $3N - \frac{1}{2}6N = 0$.

Further we see that $\langle Q^{\dagger I}_l Q^J_l \rangle \propto \delta_{IJ}$ from gauge symmetry considerations, so the only mixing allowed is between the lower indices $\rightarrow$ the γs can be written as γ^I_{lm}, and obviously γ^I_{lm} can be non vanishing only if $a_l = a_m$. We conclude from NSVZ that:

$$\beta_{g_I} \propto Tr\gamma^I + \gamma^{I-a_1}_{11} + \gamma^{I-a_2}_{22} + \gamma^{I-a_3}_{33}. \tag{4.4}$$

For superpotential couplings we have the usual expression coming from the general formula (2.4):

$$\beta_{h^I_{lmn}} \propto h^I_{pmn}\gamma^I_{pl} + h^I_{lpn}\gamma^{I+a_l}_{pm} + h^I_{lmp}\gamma^{I+a_l+a_m}_{pn}. \tag{4.5}$$

[2]We will see examples with larger possibilities, in particular an example of $\mathbb{C}^3/\mathbb{Z}_3$ where the possibilities are much larger.

[3]Work this out for the i'th gauge group: we have $Q^i_l, Q^{i-a_1}_1, Q^{i-a_2}_2, Q^{i-a_3}_3$ matter fields transforming non trivially under this group. The first one is $3 \times N$ fields, and the three others give N fields each (N being the index of the other gauge group in which they transform), giving a total of 6N.

Obviously if we demand $\gamma = 0$ for all γs the β-functions will vanish (we will see that actually we will have solutions also for non vanishing γs in some cases).

Let us constrain our theory further to get some general result: we will assume that all k gauge couplings are equal and that $h^I_{lmn} \equiv h$ for (lmn) an even permutation of (123), and $h^I_{lmn} \equiv h'$ for (lmn) an odd permutation of (123)[4]. It is easy to see that in this case we have two discrete symmetries: by taking $Q^I_l \to Q^I_{l+1}$ we remain with the same theory, and also by taking $Q^I_l \to Q^{I+1}_l$ we also remain with same theory. These symmetries obviously imply that all the γ function of the matter fields here are the same. And from the demand for vanishing β-functions we will have to demand $\gamma = 0$. γ is a function of three parameters and thus we expect a two dimensional manifold of fixed points.

Let us look at the one loop contributions to the γ-functions. We can schematically write them as[5]:

$$\gamma_{Q^I_l} = \mathcal{A}(h^2 + h'^2) - 2\mathcal{B}g^2 = 0. \tag{4.6}$$

Thus we see that we have the same demand for all the γ-functions, so we will have a solution:

$$h'^2(g, h) = 2\frac{\mathcal{B}}{\mathcal{A}}g^2 - h^2. \tag{4.7}$$

This can easily be seen to generalize to all orders of perturbation theory (see the next section for the details). Thus we see that in any $\mathcal{N} = 1$ orbifold theory we have at least two *exactly* marginal directions at weak coupling, parameterized by the gauge coupling **g** and the parameter **h**.

When the gauge group is SU(N=3) the manifold of fixed points is larger. In this case we can add a superpotential of the form: $(Q^I_l)^3$. Again from considerations of vanishing of the β-functions we can constrain ourselves to the case where all the couplings of the new superpotential are equal (equal to ρ).

---

[4]The superpotential coming from the orbifold is of this form [27]: $\sum_I Q^I_1 Q^{I+a_1}_2 Q^{I+a_1+a_2}_3 - Q^I_2 Q^{I+a_2}_1 Q^{I+a_1+a_2}_3$.

[5]We define $\mathcal{B} = \frac{1}{16\pi^2}\frac{N^2-1}{N}$, $\mathcal{A} = \frac{1}{16\pi^2}\frac{N}{2}$.

Thus again if we constrain ourselves as above (all gauge couplings equal and the potential including only **h** and **h'** couplings) we have the discrete symmetry and we can schematically write the anomalous dimensions at one loop as:

$$\gamma_{Q_l^I} = \mathcal{A}(h^2 + h'^2) + \mathcal{C}\rho^2 - 2\mathcal{B}g^2 = 0. \tag{4.8}$$

This leads to a solution:

$$h'^2(g, h, \rho) = 2\frac{\mathcal{B}}{\mathcal{A}}g^2 - h^2 - \frac{\mathcal{C}}{\mathcal{A}}\rho^2. \tag{4.9}$$

Thus we see that for $SU(N = 3)$ and general k we have at least a **3** dimensional manifold of fixed points. Again we have to stress that the actual manifold for every case can be much larger, depending on the specific choice of the $\overrightarrow{a}$ vector.

Now we wish to classify all the possibilities with a larger number of possible superpotentials. First we denote: $\overrightarrow{a} \equiv (a, b, -a - b)$, and without any loss of generality we assume that $\frac{k}{2} \geq a, b > 0$.

The possibilities to have other superpotentials than the one described above are:

- **(I)** b+2a=k

  From here we conclude that $\overrightarrow{a} \equiv (a, k - 2a, a - k) = (a, -2a, a)$.

- **(I')** 2a-(a+b)=0

  From here we conclude that $\overrightarrow{a} \equiv (a, a, -2a)$, so we see it's essentially the same case as above.

- **(II)** 3b=k

  From here we conclude: $\overrightarrow{a} \equiv (a, \frac{k}{3}, -a - \frac{k}{3})$.

- **(III)** 3(a+b)=k

  From here we conclude: $\overrightarrow{a} \equiv (a, \frac{k}{3} - a, -\frac{k}{3})$, we see that this case is essentially the same as the previous one (by taking $\overrightarrow{a} \to -\overrightarrow{a}$ and $a \to -a$).

We see that the three distinct cases above have intersections: $\overrightarrow{a} \equiv (\frac{k}{3}, \frac{k}{3}, \frac{k}{3})$ is in all of the cases above and $\overrightarrow{a} \equiv (\frac{k}{6}, \frac{k}{6}, \frac{2k}{3})$ is in the intersection of **(I)** and **(III)**. Thus to summarize we have these different cases to deal with:

- general $k \rightarrow \overrightarrow{a} \equiv (a, a, -2a)$,

- $k = 3k'$ $\overrightarrow{a} \equiv (a, \frac{k}{3} - a, -\frac{k}{3})$,

And two special cases: $\overrightarrow{a} \equiv (\frac{k}{3}, \frac{k}{3}, \frac{k}{3})$, $\overrightarrow{a} \equiv (\frac{k}{6}, \frac{k}{6}, \frac{2k}{3})$.

When **a,b** and **k** have a common divisor larger than one (which we denote by **J**), then essentially we are in the $\mathbb{Z}_{\frac{k}{J}}$, $\frac{1}{J}\overrightarrow{a}$ theory. In particular there is no meaning to discussing the $\overrightarrow{a} \equiv (\frac{k}{3}, \frac{k}{3}, \frac{k}{3})$, $\overrightarrow{a} \equiv (\frac{k}{6}, \frac{k}{6}, \frac{2k}{3})$ theories for general **k**, they all are equivalent to the ones with k=3,6 respectively. So we will assume that **a,b** and **k** have no non trivial common divisor.

Now we will deal with the specific cases. First with the most general one, then with the case where two of the $\overrightarrow{a}$ components equal, then with the k=3k' case and finally with the two special cases.

4.1 (a,b,-a-b)

In this section we consider a theory which does not have any additional interactions (except the ones which exist in any orbifold theory). As an example of such a theory we will keep in mind the $\mathbb{C}^3/\mathbb{Z}_7$ orbifold theory with $\overrightarrow{a} = (1, 2, 4)$. We see that the only way here that $a_l + a_m + a_n = 0(mod\mathbf{k})$ is if $(a_l, a_m, a_n) = (1, 2, 4)$ (as sets). This theory can be represented by the following quiver diagram:
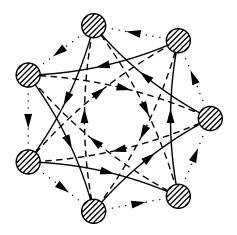

Figure 4.1: $\mathbb{C}^3/\mathbb{Z}_7$ (1,2,4) quiver diagram

Here the dotted lines represent the $a_l = 1$ sector, the plain lines represent $a_l = 2$ sector and the dashed lines represent the $a_l = 4$ sector.

We use here the notations from the first section of this chapter, denoting the fields Q_l^I. From the redundancy condition (4.3) we can write:

$$\begin{aligned}
h_{123}^I &= h_{231}^{I+a} = h_{312}^{I+a+b} \\
h_{132}^I &= h_{321}^{I+a} = h_{213}^{I-b}.
\end{aligned} \tag{4.10}$$

Thus we see that we have actually only $2 \times k$ independent couplings here, $h_{123}^I \equiv h_I$, $h_{132}^I \equiv h_I'$.

First we do the Leigh-Strassler analysis. From the general β-functions (4.4, 4.5) we obtain, using the fact that in our case there is no mixing between the fields:

$$\beta_{h_I} \;\propto\; \gamma_1^I + \gamma_2^{I+a} + \gamma_3^{I+a+b}$$

$$\beta_{h_I'} \;\propto\; \gamma_1^I + \gamma_3^{I+a} + \gamma_2^{I-b}$$

$$\beta_{g_I} \;\propto\; \gamma_1^I + \gamma_2^I + \gamma_3^I + \gamma_1^{I-a} + \gamma_2^{I-b} + \gamma_3^{I+a+b}. \tag{4.11}$$

Now let's define the largest common divisor of k and $\mathbf{a}$ by α, of k and $\mathbf{b}$ by β and of k with $(\mathbf{a+b})$ by γ. Let's define $\mathcal{S}_{a_i}^I$ to be the set of indices $(I, I+a_i, I+2a_i, ...)$.

From these definitions we calculate[6]:

$$\sum_{\mathcal{S}_a^I} \frac{\beta_{h_I}}{h_I} \;\propto\; \sum_{\mathcal{S}_a^I}(\gamma_1^I + \gamma_2^I) + \sum_{\mathcal{S}_a^{I+b}} \gamma_3^{I+b}$$

$$\sum_{\mathcal{S}_a^I} \frac{\beta_{h_I'}}{h_I'} \;\propto\; \sum_{\mathcal{S}_a^I}(\gamma_1^I + \gamma_3^I) + \sum_{\mathcal{S}_a^{I-b}} \gamma_2^{I-b}$$

$$\sum_{\mathcal{S}_a^I} \frac{\beta_{g_I}}{f(g_I)} \;\propto\; \sum_{\mathcal{S}_a^I}(2\gamma_1^I + \gamma_2^I + \gamma_3^I) + \sum_{\mathcal{S}_a^{I-b}} \gamma_2^{I-b} + \sum_{\mathcal{S}_a^{I+b}} \gamma_3^{I+b}. \tag{4.12}$$

From here we see that $\sum_{\mathcal{S}_a^I} \frac{\beta_{g_I}}{f(g_I)} \propto \sum_{\mathcal{S}_a^I} \frac{\beta_{h_I}}{h_I} + \sum_{\mathcal{S}_a^I} \frac{\beta_{h_I'}}{h_I'}$, so our system of linear equations is dependent. The number of such dependencies is obviously α (because there are $\frac{k}{\alpha}$ elements in $\mathcal{S}_a^I$). We can do the same procedure for $\mathbf{b}$ and $(\mathbf{a+b})$. We will get $\sum_{\mathcal{S}_b^I} \frac{\beta_{g_I}}{f(g_I)} \propto$ $\sum_{\mathcal{S}_b^I} \frac{\beta_{h_I}}{h_I} + \sum_{\mathcal{S}_b^I} \frac{\beta_{h_I'}}{h_I'}$, $\sum_{\mathcal{S}_{a+b}^I} \frac{\beta_{g_I}}{f(g_I)} \propto \sum_{\mathcal{S}_{a+b}^I} \frac{\beta_{h_I}}{h_I} + \sum_{\mathcal{S}_{a+b}^I} \frac{\beta_{h_I'}}{h_I'}$. These three relations are not completely independent. Obviously by summing over $\mathbf{I}$ the three relations we get same constraint $\rightarrow$ we get from here $(\alpha + \beta + \gamma - 2)$ relations.

The β_{h_I} and $\beta_{h_I'}$ are also not completely independent: $\sum_I \frac{\beta_{h_I}}{h_I} = \sum_I \frac{\beta_{h_I'}}{h_I'}$, which gives another relation.

Thus, we have $(\alpha + \beta + \gamma - 1)$ linear relations between the β-functions.

Finally, we count our degrees of freedom: we have $\mathbf{k}$ gauge couplings, $\mathbf{k}$ h_Is and h_I's $\rightarrow$ a total of 3k parameters. We have $3k - (\alpha + \beta + \gamma - 1)$ independent equations $\rightarrow$ we expect an $(\alpha + \beta + \gamma - 1)$ dimensional manifold of fixed points. As a byproduct of this the anomalous dimensions don't have to vanish.

---

[6]Here $f(g) = \frac{1}{16\pi^2} \frac{2g^3 C_1}{1 - \frac{2C_1 g^2}{16\pi^2}}$.

Let's now calculate the possible values for the anomalous dimensions (without any loss of generality we assume $\gamma \leq \alpha, \beta$) . From (4.11) we get:

$$
\begin{aligned}
-\gamma_3^I &= \gamma_2^{I-b} + \gamma_1^{I-a-b} \\
-\gamma_3^I &= \gamma_1^{I-a} + \gamma_2^{I-a-b} \\
-\gamma_3^I - \gamma_3^{I+a+b} &= \gamma_2^{I-b} + \gamma_1^{I-a} + \gamma_1^I + \gamma_2^I.
\end{aligned}
\tag{4.13}
$$

From here:

$$
\begin{aligned}
\gamma_2^{I-b} + \gamma_1^{I-a-b} &= \gamma_2^{I-b-a} + \gamma_1^{I-a} \\
\gamma_2^{I-b} + \gamma_1^I &= \gamma_2^{I-a-b} + \gamma_1^{I+b}.
\end{aligned}
\tag{4.14}
$$

And finally we obtain:

$$
\gamma_1^{I-a-b} - \gamma_1^{I-a} = \gamma_1^I - \gamma_1^{I+b}.
\tag{4.15}
$$

We see that if we define $\gamma_1^I - \gamma_1^{I+b} \equiv K_I$ then $K_I = K_{I-a-b}$, so we have γ independent K_Is. And we see that $\sum_{\mathcal{S}_b^I} K_I = 0$.

We can look on $\sum_{\mathcal{S}_b^I, \mathcal{S}_{-a-b}^I} K_I = \frac{k}{\gamma} \sum_{\mathcal{S}_b^I} K_I$. The left hand side is obviously invariant under $I \to I+a, I+b, I-a-b$, thus we conclude that if $\sum_{\mathcal{S}_b^I} K_I = 0$ for some $\mathbf{I}$ then it is true for any $\mathbf{I}$. And also by the same arguments if $\sum_{\mathcal{S}_b^I} K_I = 0$ then also $\sum_{\mathcal{S}_a^I} K_I = 0$. Thus essentially we have one constraint on γ K_Is.

Further we get that:

$$
\begin{aligned}
\gamma_1^I + K_I &= \gamma_1^{I+b} \\
\gamma_2^I + K_I &= \gamma_2^{I+a} \\
-\gamma_3^I &= \gamma_1^{I-a} + \gamma_2^{I-a-b}.
\end{aligned}
\tag{4.16}
$$

So finally we conclude that we have β independent γ_1^Is, α independent γ_2^Is and $\gamma - 1$ independent K_Is $\to$ we can have $(\alpha + \beta + \gamma - 1)$ independent γ-functions as expected.

Let's now write the one loop conditions for vanishing β-functions:

$$\begin{aligned}
\gamma_{Q_1^I} &= \mathcal{A}(h_{123}^{I2} + h_{132}^{I2}) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma_1^I \\
\gamma_{Q_2^I} &= \mathcal{A}(h_{231}^{I2} + h_{213}^{I2}) - \mathcal{B}(g_I^2 + g_{I+b}^2) = \gamma_2^I \\
\gamma_{Q_3^I} &= \mathcal{A}(h_{312}^{I2} + h_{321}^{I2}) - \mathcal{B}(g_I^2 + g_{I-a-b}^2) = \gamma_3^I.
\end{aligned} \tag{4.17}$$

And so we get:

$$\begin{aligned}
\gamma_{Q_1^I} &= \mathcal{A}(h_I^2 + h_I'^2) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma_1^I \\
\gamma_{Q_2^I} &= \mathcal{A}(h_{I-a}^2 + h_{I+b}'^2) - \mathcal{B}(g_I^2 + g_{I+b}^2) = \gamma_2^I \\
\gamma_{Q_3^I} &= \mathcal{A}(h_{I-a-b}^2 + h_{I-a}'^2) - \mathcal{B}(g_I^2 + g_{I-a-b}^2) = \gamma_3^I.
\end{aligned} \tag{4.18}$$

From here we define:

$$\begin{aligned}
A_{I+a} &\equiv \mathcal{A}h_I^2 - \mathcal{B}g_{I+a}^2 \qquad B_I \equiv \mathcal{A}h_I'^2 - \mathcal{B}g_I^2 \\
\frac{1}{\mathcal{B}}C_I &\equiv g_{I-a}^2 + g_{I-b}^2 - g_I^2 - g_{I-a-b}^2.
\end{aligned} \tag{4.19}$$

And further we obtain using (4.16):

$$\begin{aligned}
\gamma_{Q_1^I} &= A_{I+a} + B_I = \gamma_1^I \\
\gamma_{Q_2^I} &= A_I + B_{I+b} = \gamma_2^I \\
\gamma_{Q_3^{I+a+b}} &= A_{I+a} + B_{I+b} + C_{I+a+b} = -(\gamma_1^I + \gamma_2^I + K_I)
\end{aligned} \tag{4.20}$$

By subtracting the first equation from the third and summing over $\mathcal{S}_b^I$ we obtain that $\sum_{\mathcal{S}_b^I}(2\gamma_1^I + \gamma_2^I + K_I) = \sum_{\mathcal{S}_b^I}(2\gamma_1^I + \gamma_2^I) = 0$, and further by using (4.16): $-\frac{2k}{\beta}\gamma_1^I = \sum_{\mathcal{S}_b^I}\gamma_2^I + 2\sum_{j=0}^{\frac{k}{\beta}-1}((\frac{k}{\beta} - 1 - j) + 1)K_{I+jb}$.

By subtracting the second equation from the third and summing over $\mathcal{S}_a^I$ we obtain that $\sum_{\mathcal{S}_a^I}(\gamma_1^I + 2\gamma_2^I + K_I) = \sum_{\mathcal{S}_a^I}(\gamma_1^I + 2\gamma_2^I) = 0$, and further by using (4.16): $-\frac{2k}{\alpha}\gamma_2^I = \sum_{\mathcal{S}_a^I}\gamma_1^I + 2\sum_{j=0}^{\frac{k}{\alpha}-1}((\frac{k}{\alpha} - 1 - j) + 1)K_{I+ja}$.

Using the two relations above we get:

$$\frac{4k^2}{\alpha\beta}\gamma_1^I = \sum_{\mathcal{S}_a^I,\mathcal{S}_b^I}\gamma_1^I + 2\sum_{l,j}(\frac{k}{\alpha}-j)K_{I+ja+lb} - \frac{4k}{\alpha}\sum_{j=0}^{\frac{k}{\beta}-1}(\frac{k}{\beta}-j)K_{I+jb} =$$

$$= \sum_{\mathcal{S}_a^I,\mathcal{S}_b^I}\gamma_1^I + \frac{4k}{\alpha}\sum_{j=0}^{\frac{k}{\beta}-1}jK_{I+jb}. \tag{4.21}$$

In last line we used $\sum_{\mathcal{S}_{a,b}^I}K^I = 0$. Obviously from here $\gamma_1^I + K_I = \gamma_1^{I+b}$ as we demand.

Now by subtracting the first two equations from one another in (4.20) and summing over $\mathcal{S}_{a+b}^I$ we get $\sum_{\mathcal{S}_{a+b}^{I+b}}\gamma_1^{I+b} = \sum_{\mathcal{S}_{a+b}^I}\gamma_2^I$, and from here also $\sum_I\gamma_1^I = \sum_I\gamma_2^I \to \sum_I\gamma_1^I = \sum_I\gamma_2^I = 0$. From here and from (4.21) we see that $\gamma_1^I = \beta\sum_{j=0}^{\frac{k}{\beta}-1}jK_{I+jb}$ and $\gamma_2^I = \alpha\sum_{j=0}^{\frac{k}{\alpha}-1}jK_{I+ja}$. We see that $\sum_{\mathcal{S}_{a+b}^{I+b}}\gamma_1^{I+b} \propto \gamma_1^{I+b}$ and $\sum_{\mathcal{S}_{a+b}^I}\gamma_2^I \propto \gamma_2^I$, thus $\gamma_1^{I+b} = \gamma_2^I$.

We see also that $\gamma_1^I = \gamma_1^{I+a+b}$ and from (4.16) we get:

$$\gamma_1^{I+a+b} = \gamma_2^{I+a}$$
$$\downarrow$$
$$\gamma_1^{I+a+b} = \gamma_2^I + K_I = \gamma_1^{I+b} + K_I$$
$$\downarrow$$
$$\gamma_1^{I+a+b} = \gamma_1^I + 2K_I$$
$$\downarrow$$
$$K_I = 0. \tag{4.22}$$

And so all K^I have to vanish $\to$ all the $\gamma_{1,2,3}^I$ have to vanish. So at one loop order we can not turn on any non vanishing anomalous dimensions. This is similar to what we found in the $\mathcal{N}=2$ case.

From the first and second equations in (4.20) we see that $A_I = A_{I+a+b}$, thus we will parameterize our solution by γA_Is. The B_Is are obtained from the A_Is. Now from the third and the first equations we get $A_{I+a} - A_I = -C_{I+a+b}$, from here we get the C_Is. From the definition of C_Is we get $\sum_{\mathcal{S}_b^I}C_I, \sum_{\mathcal{S}_a^I}C_I = 0$, thus there are only $k+1-\alpha-\beta$ independent C_Is (the $+\mathbf{1}$ is because $\sum_I C_I$ follows from both constraints). From the C_Is

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we get constraints on **k** gauge couplings $\rightarrow$ we get $\alpha + \beta - 1$ independent gauge couplings. The h_Is and h'_Is can be obtained from the B_Is, the A_Is and the gauge couplings.

Thus to summarize, we have $\alpha + \beta + \gamma - 1$ parameters for our solution, as expected.

Let's now try to prove the all loop existence of the solutions above. As in previous sections we divide our n'th order contribution to the anomalous dimensions at one loop and the rest:

$$\gamma^{(n)} = \gamma^{(n)(1-loop)} + \gamma^{(n)(2..n-loops)} \tag{4.23}$$

Again we say that the yet undetermined parameters appear only in one loop. The one loop contribution will have the structure (4.20):

$$
\begin{aligned}
A_{I+a}^{(n)(1-loop)} + B_I^{(n)(1-loop)} &= \gamma_1^{I(n)(1-loop)} \\
A_I^{(n)(1-loop)} + B_{I+b}^{(n)(1-loop)} &= \gamma_2^{I(n)(1-loop)} \\
A_{I+a}^{(n)(1-loop)} + B_{I+b}^{(n)(1-loop)} + C_{I+a+b}^{(n)(1-loop)} &= \gamma_3^{I(n)(1-loop)}.
\end{aligned}
\tag{4.24}
$$

We will try to parameterize the $\gamma^{(n)(2..n-loops)}$ in the similar way:

$$
\begin{aligned}
S_{I+a} + T_I - \tilde{\gamma}_1^I &= \gamma_1^{I(n)(2..n-loop)} \\
S_I + T_{I+b} - \tilde{\gamma}_2^I &= \gamma_2^{I(n)(2..n-loop)} \\
S_{I+a} + T_{I+b} + P_{I+a+b} + (\tilde{\gamma}_1^I + \tilde{\gamma}_2^I + \tilde{K}_I) &= \gamma_3^{I(n)(2..n-loop)}.
\end{aligned}
\tag{4.25}
$$

Our aim is obviously to say that $A_I = -S_I, B_I = -T_I, C_I = -P_I$ and the anomalous dimensions at n'th order are $\tilde{\gamma}_j^I$. We just have to check the consistency of this.

By subtracting the first equation from the third and summing over $\mathcal{S}_b^I$ we obtain that $\sum_{\mathcal{S}_b^I}(2\tilde{\gamma}_1^I + \tilde{\gamma}_2^I + \tilde{K}_I) = \sum_{\mathcal{S}_b^I}(\gamma_3^{I(n)(2..n-loop)} - \gamma_1^{I(n)(2..n-loop)})$, and further by using (4.16):

$$\frac{2k}{\beta}\tilde{\gamma}_1^I = -\sum_{\mathcal{S}_b^I}\gamma_2^I - 2\sum_{j=0}^{\frac{k}{\beta}-1}j\tilde{K}_{I+jb} + \sum_{\mathcal{S}_b^I}(\gamma_3^{I(n)(2..n-loop)} - \gamma_1^{I(n)(2..n-loop)}). \tag{4.26}$$

By subtracting the second equation from the third and summing over $\mathcal{S}_a^I$ we obtain that $\sum_{\mathcal{S}_a^I}(\tilde{\gamma}_1^I + 2\tilde{\gamma}_2^I + \tilde{K}_I) = \sum_{\mathcal{S}_a^I}(\gamma_3^{I(n)(2..n-loop)} - \gamma_2^{I(n)(2..n-loop)})$, and further by using (4.16):

$$\frac{2k}{\alpha}\tilde{\gamma}_2^I = -\sum_{\mathcal{S}_a^I}\gamma_1^I - 2\sum_{j=0}^{\frac{k}{\alpha}-1}j\tilde{K}_{I+ja} + \sum_{\mathcal{S}_a^I}(\gamma_3^{I(n)(2..n-loop)} - \gamma_2^{I(n)(2..n-loop)}). \qquad (4.27)$$

From these two observations we see that:

$$\begin{aligned}
\frac{4k^2}{\alpha\beta}\tilde{\gamma}_1^I &= \sum_{\mathcal{S}_a^I,\mathcal{S}_b^I}\tilde{\gamma}_1^I - \sum_{\mathcal{S}_a^I,\mathcal{S}_b^I}(\gamma_3^{I(n)(2..n-loop)} - \gamma_2^{I(n)(2..n-loop)}) + \frac{2k}{\alpha}\sum_{j=0}^{\frac{k}{\beta}-1}j\tilde{K}_{I+jb} + \\
&+ \frac{4k}{\alpha}\sum_{\mathcal{S}_b^I}(\gamma_3^{I(n)(2..n-loop)} - \gamma_1^{I(n)(2..n-loop)}) \equiv \\
&\equiv \sum_{\mathcal{S}_a^I,\mathcal{S}_b^I}\tilde{\gamma}_1^I + \frac{4k}{\alpha}\sum_{j=0}^{\frac{k}{\beta}-1}j\tilde{K}_{I+jb} + F(\gamma_l^{I(n)(2..n-loop)}). \qquad (4.28)
\end{aligned}$$

We see that again $\tilde{\gamma}_1^I + \tilde{K}_I = \tilde{\gamma}_1^{I+b}$.

From the first and second equations in (4.25) we see that we have to get:

$$\sum_{\mathcal{S}_{a+b}^I}(\tilde{\gamma}_2^I - \tilde{\gamma}_1^I) = \sum_{\mathcal{S}_{a+b}^I}(\gamma_1^{I(n)(2..n-loop)} - \gamma_2^{I(n)(2..n-loop)}). \qquad (4.29)$$

From these equations we can determine $\tilde{\gamma}_{1,2}^I$ and $\tilde{K}_I$.

After we determined the anomalous dimensions we can proceed to determine other quantities as we did in one loop:

$$\begin{aligned}
T_{I+b} - T_I &= -P_{I+a+b} + \gamma_3^{I(n)(2..n-loop)} - \gamma_1^{I(n)(2..n-loop)} + (2\tilde{\gamma}_1^I + \tilde{\gamma}_2^I) \\
S_{I+a} - S_I &= -P_{I+a+b} + \gamma_3^{I(n)(2..n-loop)} - \gamma_2^{I(n)(2..n-loop)} + (\tilde{\gamma}_1^I + 2\tilde{\gamma}_2^I). \qquad (4.30)
\end{aligned}$$

It seems that we have here $\alpha + \beta + \gamma - 1$ additional parameters like we did at one loop, but as a matter of fact these parameters are just redefinitions of our one loop parameters.

Thus to conclude, we can extend our one loop solutions to higher loops. The price we will have to pay is turning on non zero anomalous dimensions.

In our example of (1,2,4) $\mathbb{C}^3/\mathbb{Z}_7$ we have $\alpha = \beta = \gamma = 1$. So we expect to have a **2** dimensional manifold of solutions. It is easy to see that in this case (4.20) implies that all the gauge couplings are equal. And from here all the h'_Is are equal (to h') and also the h^Is are equal (to h).

The condition we get in this case at one loop is:

$$h'^2(g,h) = 2\frac{\mathcal{B}}{\mathcal{A}}g^2 - h^2. \tag{4.31}$$

So here with general SU(N) group we have only two marginal deformations, parameterized by **g** and **h**. For **k** prime we always have only a **2** dimensional manifold of fixed points.

So we see that there are actually cases "saturating" our lower bound for the number of marginal directions that we found in the beginning of the chapter.

4.1.1 $SU(N = 3)$

In the special case of $SU(N = 3)^k$ gauge group we can add additional interactions:

$$\frac{\rho_j^I}{3!}\epsilon_{lmn}\epsilon^{abc}(Q_j^I)_a^l(Q_j^I)_b^m(Q_j^I)_c^n. \tag{4.32}$$

Obviously these interactions don't contribute to any mixing. The extra β-functions are:

$$\beta_{\rho_j^I} \propto 3\gamma_{Q_j^I}. \tag{4.33}$$

Thus, for vanishing β-functions we have to demand vanishing anomalous dimensions. We have **3k** anomalous dimensions and **6k** couplings $\rightarrow$ we expect a **3k** dimensional manifold of fixed points.

The one loop expressions we found above (4.18) are modified:

$$\gamma_{Q_1^I} = \mathcal{A}(h_I^2 + h_I'^2) + \mathcal{C}\rho_1^{I2} - \mathcal{B}(g_I^2 + g_{I+a}^2) = 0$$
$$\gamma_{Q_2^I} = \mathcal{A}(h_{I-a}^2 + h_{I+b}'^2) + \mathcal{C}\rho_2^{I2} - \mathcal{B}(g_I^2 + g_{I+b}^2) = 0$$
$$\gamma_{Q_3^I} = \mathcal{A}(h_{I-a-b}^2 + h_{I-a}'^2) + \mathcal{C}\rho_3^{I2} - \mathcal{B}(g_I^2 + g_{I-a-b}^2) = 0 \tag{4.34}$$

By the same procedure as above we obtain ($\mathcal{C}\rho_j^{I2} \equiv D_j^I$):

$$\gamma_{Q_1^I} = A_{I+a} + B_I + D_1^I = 0$$
$$\gamma_{Q_2^I} = A_I + B_{I+b} + D_2^I = 0$$
$$\gamma_{Q_3^{I+a+b}} = A_{I+a} + B_{I+b} + C_{I+a+b} + D_3^{I+a+b} = 0. \tag{4.35}$$

By subtracting these equations one from another and summing over **i** we get that $\sum_{\mathcal{S}_{a+b}^{I+a}} D_1^{I+a} = \sum_{\mathcal{S}_{a+b}^I} D_2^I, \sum_{\mathcal{S}_b^I} D_1^I = \sum_{\mathcal{S}_b^{I+a}} D_3^{I+a}, \sum_{\mathcal{S}_a^I} D_2^I = \sum_{\mathcal{S}_a^{I+b}} D_3^{I+b}$. These relations put $\gamma + \alpha + \beta - 1$ constraints on the **3k** D's.

We parameterize our solution by the D's. From the first two equations above we get that:

$$A_{I+a+b} - A_I = D_2^I - D_1^{I+b} \tag{4.36}$$

Thus we get γ independent A_Is, and the B_Is are determined from the D's and the A_Is. From the second and the third equations we get:

$$A_{I+a} - A_I = D_2^I - D_3^{I+a+b} - C_{I+a+b}. \tag{4.37}$$

From here we can determine the gauge couplings, as before we see that there are $\alpha + \beta - 1$ independent gauge couplings.

So totally we will have γ A_Is, $\alpha + \beta - 1$ independent gauge couplings and $3k - (\gamma + \alpha + \beta - 1)$ independent γ's $\to$ we have here a **3k** dimensional manifold of fixed points, exactly as in the $\mathcal{N} = 2$ case. The extension to all orders is done exactly as in the previous cases.

4.2 (a,a,-2a)

Here we deal with the $\vec{a} \equiv (a, a, -2a)$ case. As an example we will keep in mind the $\mathbb{C}^3/\mathbb{Z}_4$ (1,1,2) case which can be represented by the following quiver diagram:
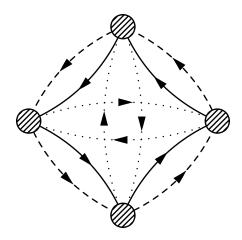

Figure 4.2: $\mathbb{C}^3/\mathbb{Z}_4$ (1,1,2) quiver diagram

Here the three types of lines represent the three sectors of the theory.

The possible interactions of the theory are the ones we saw for general case, i.e h^I_{123} etc, and also h^I_{113}, h^I_{223}. From the redundancy condition (4.3) we see that:

$$
\begin{aligned}
h^I_{123} &= h^{I+a}_{231} = h^{I+2a}_{312} \\
h^I_{132} &= h^{I+a}_{321} = h^{I-a}_{213} \\
h^I_{113} &= h^{I+a}_{131} = h^{I+2a}_{311} \\
h^I_{223} &= h^{I+a}_{232} = h^{I+2a}_{322}.
\end{aligned}
\tag{4.38}
$$

Thus we see that all the couplings can be defined in terms of: $h_I \equiv h^I_{123}, h'_I \equiv h^I_{132}, p_I \equiv h^I_{113}$ and $s_I \equiv h^I_{223}$.

In this theory we have a global $SU(2)^k$ symmetry which rotates Q^I_1 and Q^I_2. We had a very similar symmetry in the $\mathcal{N} = 4$ case (there it was an SU(3) symmetry). In this case in general we expect the fields to mix: $\langle Q^{\dagger I}_i Q^I_j \rangle \neq 0$ for $i \neq j$. However, by using the $SU(2)^k$ symmetry we can make the anomalous dimensions diagonal $\langle Q^{\dagger I}_i Q^I_j \rangle \to U^I_{li} U^{\dagger I}_{mj} \langle Q^{\dagger I}_i Q^I_j \rangle$.

Let's now make the Leigh-Strassler analysis assuming that the anomalous dimensions are diagonal. The β-functions have to satisfy:

$$
\begin{aligned}
\beta_{h_I} &\propto \gamma_1^I + \gamma_2^{I+a} + \gamma_3^{I+2a} \\
\beta_{h'_I} &\propto \gamma_1^I + \gamma_3^{I+a} + \gamma_2^{I-a} \\
\beta_{g_I} &\propto \gamma_1^I + \gamma_2^I + \gamma_3^I + \gamma_1^{I-a} + \gamma_2^{I-a} + \gamma_3^{I+2a} \\
\beta_{s_I} &\propto \gamma_2^I + \gamma_2^{I+a} + \gamma_3^{I+2a} \\
\beta_{p_I} &\propto \gamma_1^I + \gamma_1^{I+a} + \gamma_3^{I+2a}
\end{aligned}
\tag{4.39}
$$

We see that from the first and the last equations, in order to have vanishing β-functions we will have to have $\gamma_2^{I+a} = \gamma_1^{I+a} \equiv \gamma^I$.[7] Assuming this the first two equations are essentially the same.

From the third equation we get that:

$$
\gamma_1^I - \gamma_1^{I+a} = \gamma_1^{I-2a} - \gamma_1^{I-a}.
\tag{4.40}
$$

Thus we see that: $\gamma_1^I - \gamma_1^{I+a} \equiv K_I$ where $K_I = K_{I+2a}$. The largest common divisor of $\mathbf{a}$ with $\mathbf{k}$ has to be $\mathbf{1}$ because otherwise also $\mathbf{2a}$ will have a non trivial largest common divisor with $\mathbf{k} \rightarrow$ thus we see that if $\mathbf{k}$ is odd we will have only one K_I, and if $\mathbf{k}$ is even we will have two different K_Is (say A,B). Further, by summing the definition of K_I over $\mathbf{I}$ we get that:

- $\mathbf{k}$ odd $\rightarrow K_I = 0$

- $\mathbf{k}$ even $\rightarrow A = -B$

From here in $\mathbf{k}$ odd case we get that all the anomalous dimensions γ_1^I, γ_2^I are equal, and in $\mathbf{k}$ even case we get only two different anomalous dimensions (which we denote by δ_1, δ_2): $\gamma_{1,2}^{I+2a} = \gamma_{1,2}^I$. We also see that all the γ_3^Is are equal and equal to $-(\delta_1 + \delta_2)$.

[7]We see that the anomalous dimensions are proportional to the identity in the $(\mathbf{1\ 2})$ directions. No $SU(2)$ rotation will change this fact, so it will remain true on the fixed manifold for any $SU(2)^k$ rotation, and we can use the $SU(2)^k$ symmetry again in our discussion.

So to summarize, we can parameterize our solutions here by one or two parameters.

Let's now write the one loop conditions for vanishing β-functions. First we write the diagonal contributions to the anomalous dimensions:

$$
\begin{aligned}
\gamma_{Q_1^I} &= \mathcal{A}(h_{123}^{I2} + h_{132}^{I2}) + \mathcal{C}(h_{113}^{I2} + h_{131}^{I2}) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma^I \\
\gamma_{Q_2^I} &= \mathcal{A}(h_{231}^{I2} + h_{213}^{I2}) + \mathcal{C}(h_{223}^{I2} + h_{232}^{I2}) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma^I \\
\gamma_{Q_3^I} &= \mathcal{A}(h_{312}^{I2} + h_{321}^{I2}) + \mathcal{C}(h_{322}^{I2} + h_{311}^{I2}) - \mathcal{B}(g_I^2 + g_{I-2a}^2) = -(\delta_1 + \delta_2). \quad (4.41)
\end{aligned}
$$

We can rewrite this as:

$$
\begin{aligned}
\gamma_{Q_1^I} &= \mathcal{A}(h_I^2 + h_I'^2) + \mathcal{C}(p_I^2 + p_{I-a}^2) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma^I \\
\gamma_{Q_2^I} &= \mathcal{A}(h_{I-a}^2 + h_{I+a}'^2) + \mathcal{C}(s_I^2 + s_{I-a}^2) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma^I \\
\gamma_{Q_3^I} &= \mathcal{A}(h_{I-2a}^2 + h_{I-a}'^2) + \mathcal{C}(s_{I-2a}^2 + p_{I-2a}^2) - \mathcal{B}(g_I^2 + g_{I-2a}^2) = -(\delta_1 + \delta_2).
\end{aligned}
$$
$$(4.42)$$

From here:

$$
\begin{aligned}
h_I^2 &= -h_I'^2 - \frac{1}{\mathcal{A}}(\mathcal{C}(p_I^2 + p_{I-a}^2) - \mathcal{B}(g_I^2 + g_{I+a}^2)) + \gamma^I \\
h_I^2 &= -h_{I+2a}'^2 - \frac{1}{\mathcal{A}}(\mathcal{C}(s_{I+a}^2 + s_I^2) - \mathcal{B}(g_{I+2a}^2 + g_{I+a}^2)) + \gamma^{I+a} \\
h_I^2 &= -h_{I+a}'^2 - \frac{1}{\mathcal{A}}(\mathcal{C}(s_I^2 + p_I^2) - \mathcal{B}(g_{I+2a}^2 + g_I^2)) + (\delta_1 + \delta_2). \quad (4.43)
\end{aligned}
$$

From here we can write:

$$
\begin{aligned}
-\gamma^I + h_I'^2 + \frac{1}{\mathcal{A}}(\mathcal{C}(p_I^2 + p_{I-a}^2) - \mathcal{B}g_I^2) &= -\gamma^{I+a} + h_{I+2a}'^2 + \frac{1}{\mathcal{A}}(\mathcal{C}(s_{I+a}^2 + s_I^2) - \mathcal{B}g_{I+2a}^2) \\
-\gamma^{I+a} + h_{I+2a}'^2 + \frac{1}{\mathcal{A}}(\mathcal{C}s_{I+a}^2 - \mathcal{B}g_{I+a}^2) &= -(\delta_1 + \delta_2) + h_{I+a}'^2 + \frac{1}{\mathcal{A}}(\mathcal{C}p_I^2 - \mathcal{B}g_I^2) \\
-(\delta_1 + \delta_2) + h_{I+a}'^2 + \frac{1}{\mathcal{A}}(\mathcal{C}s_I^2 - \mathcal{B}g_{I+2a}^2) &= -\gamma^I + h_I'^2 + \frac{1}{\mathcal{A}}(\mathcal{C}p_{I-a}^2 - \mathcal{B}g_{I+a}^2).
\end{aligned}
$$
$$(4.44)$$

Obviously the last two equations imply the first one. Now from the last two equations we obtain:

$$2(\delta_1 + \delta_2 - \gamma^I) + \mathcal{B}(g_{I-a}^2 - g_I^2) = \mathcal{B}(g_{I+a}^2 - g_{I+2a}^2). \tag{4.45}$$

By summing this over **I** we obtain $\delta_1 = -\delta_2$, for **k** odd this immediately implies that all anomalous dimensions vanish. From here:

$$- 2\gamma^I + \mathcal{B}(g_{I-a}^2 - g_I^2) = \mathcal{B}(g_{I+a}^2 - g_{I+2a}^2). \tag{4.46}$$

Summing this over $\mathcal{S}_{2a}^I$ we obtain that also for even **k** the anomalous dimensions have to vanish. And in particular we see that $(g_{I-a}^2 - g_I^2) = (g_{I+a}^2 - g_{I+2a}^2)$ and thus, if **k** is even we get two independent gauge couplings $(\mathbf{g}, \tilde{\mathbf{g}})$, and if **k** is odd we get only one independent gauge coupling.

By adding the first and the second equations of (4.44) we get:

$$h'^{(I+a)2} - h'^{I2} = \frac{\mathcal{C}}{\mathcal{A}}(p_{I-a}^2 - s_I^2) + \frac{\mathcal{B}}{\mathcal{A}}(g_{I+2a}^2 - g_{I+a}^2). \tag{4.47}$$

Thus we see that essentially we have here only $\alpha(=1)$ independent h'_Is, and all the others can be obtained from them. We see that we also get a constraint on s_I^2 and p_I^2. By summing over $\mathcal{S}_a^I$ the above equation we get that $\sum_{\mathcal{S}_a^I} s_I^2 = \sum_{\mathcal{S}_a^I} p_I^2$.

Now we write the non diagonal contributions:

$$\gamma_{12}^I \propto h_{123}^I h_{223}^{*I} + h_{132}^I h_{232}^{*I} + h_{113}^I h_{213}^{*I} + h_{131}^I h_{231}^{*I}$$

$$\downarrow$$

$$\gamma_{12}^I \propto h^I s^{*I} + h'^I s^{*I-a} + p^I h^{'*(I+a)} + p^{I-a} h^{*I-a},$$

$$\gamma_{21}^I = \gamma_{12}^{*I}. \tag{4.48}$$

Now let's assume that we are close to the orbifold theory, i.e. $h_I = h + \mathcal{O}(\epsilon), h'_I = -h + \mathcal{O}(\epsilon), p_I = \mathcal{O}(\epsilon), s_I = \mathcal{O}(\epsilon), g_I = g + \mathcal{O}(\epsilon)$ (where ϵ is some small parameter). In particular this implies in the leading order for the non diagonal elements above:

$$s^{*I} - s^{*I-a} - p^I + p^{I-a} = 0. \tag{4.49}$$

By using the $SU(2)^k$ global symmetry (see Appendix B) we can set all the h'_I's to zero.

We will use the notations of the even **k** case (the odd **k** case is obtained by simply putting $K_I = 0$). We saw that in this case we get two gauge couplings: $g_I \equiv g$ for even **I** and $g_I \equiv \tilde{g}$ for odd **I**. Define $K_{I+a} \equiv \frac{\mathcal{B}}{\mathcal{A}}(g^2_{I+2a} - g^2_{I+a})$, obviously $K_I = -K_{I+a}$. From (4.47) we obtain:

$$\frac{\mathcal{C}}{\mathcal{A}}(p^2_{I-a} - s^2_I) = -K_{I+a}, \tag{4.50}$$

So we can write the s_Is in terms of the p_Is.

Now from (4.48) we see:

$$\gamma^I_{12} \quad \propto \quad h^I s^{*I} + p^{I-a} h^{*I-a} = 0 \tag{4.51}$$

From here we see that:

$$h^2_I s^2_I = p^2_{I-a} h^2_{I-a} \tag{4.52}$$

So now from (4.42) we get $(G \equiv \mathcal{B}(g^2_I + g^2_{I+a}))$:

$$\mathcal{A}h^{I2} + \mathcal{C}(p^2_I + p^2_{I-a}) - G = 0 \tag{4.53}$$

From here we obtain: $\mathcal{A}h^{I2} = G - \mathcal{C}(p^2_I + p^2_{I-a})$. And by putting all our recent knowledge into (4.52) we get:

$$p^2_{I-a}(p^2_I - p^2_{I-2a}) = \frac{\mathcal{A}}{\mathcal{C}}K_{I+a}(\frac{1}{\mathcal{C}}G - p^2_I - p^2_{I-a}). \tag{4.54}$$

Obviously we have at least one solution because these equations are linearly dependent by summing over **I**. When we put the gauge couplings to be equal ($K_I = 0$), we get

$p_I^2 = p_{I+2a}^2$. and thus in **k** even case get **2** different p_Is and in **k** odd case get only one independent p_I.

The extension to higher loops is as follows. The solution will be parameterized by p_I, p_{I+a}, **g** as in the one-loop case. We notice that the solution we got necessarily has $\gamma_{ij}^I = \gamma_{ij}^{I+2a}$, thus we will now turn on interactions which will guarantee the existence of this property at any order of perturbation theory: we turn on $s_I = s_{I+2a}$, $h'_I = h'_{I+2a}$, $h_I = h_{I+2a}$. Note that at higher loops the anomalous dimensions don't necessarily vanish and so we can use the $SU(2)^k$ symmetry to make the anomalous dimensions diagonal as we did before, but now we can not take away h'_Is.

So we see that we have here **6** couplings, **2** parameters from the anomalous dimensions (δ_1, δ_2) and we have **8** different anomalous dimensions $(\gamma_{ii}^I, \gamma_{ii}^{I+a}, \gamma_{12}^I, \gamma_{12}^{I+a}) \rightarrow$ we have **8** equations for **8** variables thus in general we get a solution. The global $SU(2)^k$ symmetry was already used to get rid of h'_I in one loop and we can not use it again.

So to summarize, we get here one additional marginal deformation when all the gauge couplings are equal in the even **k** case. This extra operator cannot come from an operator in the $\mathcal{N} = 4$ theory because the marginal deformations which come from there are obviously **I** independent $\rightarrow$ this is a new kind of deformation existing in this theory. The $\mathcal{N} = 2$ case with k=2 is a special case of the type discussed here (see section (3.1.3)). The $\mathcal{N} = 1$ theory can be seen as an orbifold of the $\mathcal{N} = 2$ theory, thus the new *exactly* marginal deformations we get here can be seen as the descendants of the $\mathcal{N} = 2$ deformations.

We can make an interesting observation here: if we put all the h_Is and h'_Is to zero we are guaranteed to have diagonal γ matrices from the symmetry of the problem. The one loop expressions become:

$$
\begin{aligned}
\gamma_{Q_1^I} &= \mathcal{C}(p_I^2 + p_{I-a}^2) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma^I \\
\gamma_{Q_2^I} &= \mathcal{C}(s_I^2 + s_{I-a}^2) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma^I \\
\gamma_{Q_3^I} &= \mathcal{C}(s_{I-2a}^2 + p_{I-2a}^2) - \mathcal{B}(g_I^2 + g_{I-2a}^2) = -(\delta_1 + \delta_2).
\end{aligned}
$$

$$(4.55)$$

Further we get:

$$
\begin{aligned}
\mathcal{C}(p_I^2 + p_{I-a}^2) &= \mathcal{B}(g_I^2 + g_{I+a}^2) + \gamma^I \\
\mathcal{C}(p_I^2 + p_{I-a}^2) &= \mathcal{B}(g_{I+2a}^2 + g_{I-a}^2) - \gamma^I - 2(\delta_1 + \delta_2).
\end{aligned}
\tag{4.56}
$$

In the same way as before we obtain from here that the anomalous dimensions should vanish. Further:

$$
g_{I+a}^2 - g_{I-a}^2 = g_{I+2a}^2 - g_I^2,
\tag{4.57}
$$

and from here we see that all the gauge coupling should be equal (say to **g**).

From here we obtain that the solution is $p_I^2 = s_I^2 = g^2$. Thus we see that we get a one dimensional family of conformal theories. It can be easily proven that the solution extends to all orders of the perturbation series. This solution is $SU(2)^k$ equivalent to the solution without any p_Is or s_Is.

4.2.1 $SU(N = 3)$

In addition to the (a,b,-a-b) case $SU(N = 3)^k$ interactions we can have here also:

$$
\begin{aligned}
&\frac{\rho_{12}^I}{3!} \epsilon_{lmn} \epsilon^{abc} (Q_1^I)_a^l (Q_1^I)_b^m (Q_2^I)_c^n \\
&\frac{\rho_{21}^I}{3!} \epsilon_{lmn} \epsilon^{abc} (Q_2^I)_a^l (Q_2^I)_b^m (Q_1^I)_c^n.
\end{aligned}
\tag{4.58}
$$

These new interactions affect only $\gamma_{Q_1^I}$, $\gamma_{Q_2^I}$ and the mixing terms. There are no additional constraints that we have to impose. We use the $SU(2)^k$ symmetry in order to cancel **k** couplings out of the p_i, s_i. Now we can take the **k** gauge couplings, **2k** h_I and h_I', the remaining **k** out of the p_i, s_i and **k** ρ_{12}^I couplings as parameters $\to \rho_{21}^I$ will be set from the non diagonal anomalous dimensions and the ρ_i^I will be set from the diagonal ones. So we get here a **5k** dimensional manifold of fixed points.

From Leigh-Strassler analysis we get: here all anomalous dimensions have to vanish, we have here **3k+k** possibly non zero anomalous dimensions. We have **3k+2k+3k+2k=10k** couplings, **k** of the couplings can be set to zero by $SU(2)^k$ symmetry (see Appendix B) $\to$ have a naive expectation for a **5k** dimensional manifold of fixed points.

4.3 $(a, \frac{k}{3} - a, -\frac{k}{3})$, k=3k'

In this case we can have an additional operator of the form $h_{333}^I Q_3^I Q_3^{I+\frac{2k}{3}} Q_3^{I+\frac{4k}{3}}$. As we saw these theories are the same as $(a, \frac{k}{3}, -a - \frac{k}{3})$ and we will give an example of such a theory: the (1,2,3) $\mathbb{C}^3/\mathbb{Z}_6$ theory,
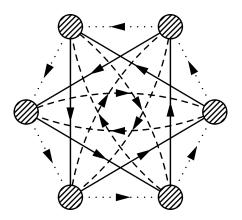

Figure 4.3: $\mathbb{C}^3/\mathbb{Z}_6$ (1,2,3) quiver diagram

From the redundancy condition (4.3) we see that:

$$
\begin{aligned}
h_{123}^I &= h_{231}^{I+a} = h_{312}^{I+k'} \\
h_{132}^I &= h_{321}^{I+a} = h_{213}^{I+a-k'} \\
h_{333}^I &= h_{333}^{I+2k'} = h_{333}^{I+k'}.
\end{aligned}
\tag{4.59}
$$

The general Leigh-Strassler analysis we did in the **(a,b,-a-b)** section is applicable here, only we'll have new β-functions for the new interactions:

$$
\beta_{h_{333}^I} \propto \gamma_3^I + \gamma_3^{I+k'} + \gamma_3^{I+2k'} = 0.
\tag{4.60}
$$

We see that this is essentially $\sum_{\mathcal{S}_{\frac{2k}{3}}^I} \gamma_3^I = 0$. Thus we will use the **(a,b,-a-b)** notations with the new constraint. We add **k'** couplings to the theory but also have **k'** new β-functions $\rightarrow$ we do not expect any new marginal directions.

The perturbation theory calculations:

$$\gamma_{Q_1^I} = \mathcal{A}(h_{123}^{I2} + h_{132}^{I2}) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma_1^I$$

$$\gamma_{Q_2^I} = \mathcal{A}(h_{231}^{I2} + h_{213}^{I2}) - \mathcal{B}(g_I^2 + g_{I+k'}^2) = \gamma_2^I$$

$$\gamma_{Q_3^I} = \mathcal{A}(h_{312}^{I2} + h_{321}^{I2}) + \mathcal{C}h_{333}^{I2} - \mathcal{B}(g_I^2 + g_{I-a-k'}^2) = \gamma_3^I. \tag{4.61}$$

And from here we get ($h_{333}^I \equiv s^I$, and h_I, h'_I as in previous sections):

$$\gamma_{Q_1^I} = \mathcal{A}(h_I^2 + h_I'^2) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma_1^I$$

$$\gamma_{Q_2^I} = \mathcal{A}(h_{I-a}^2 + h_{I+k'-a}'^2) - \mathcal{B}(g_I^2 + g_{I+k'-a}^2) = \gamma_2^I$$

$$\gamma_{Q_3^I} = \mathcal{A}(h_{I-k'}^2 + h_{I-a}'^2) + \mathcal{C}s^{I2} - \mathcal{B}(g_I^2 + g_{I-k'}^2) = \gamma_3^I. \tag{4.62}$$

Defining A_I, B_I, C_I, K_I as in the **(a,b,-a-b)** section, and $S_{I-a-b} \equiv \mathcal{C}s^{I2}$:

$$A_{I+a} + B_I = \gamma_1^I$$

$$A_I + B_{I+b} = \gamma_2^I$$

$$A_{I+a} + B_{I+b} + C_{I+a+b} + S_I = -(\gamma_1^I + \gamma_2^I + K_I). \tag{4.63}$$

By subtracting the first equation from the second and summing over $\mathcal{S}_{a+b}^I$ we get that:

$$\sum_{\mathcal{S}_{a+b}^I} \gamma_1^I = \sum_{\mathcal{S}_{a+b}^I} \gamma_2^I. \tag{4.64}$$

But from (4.60) we see that($a + b = \frac{k}{3}$):

$$\sum_{\mathcal{S}_{a+b}^I} \gamma_1^I + k'K_I = -\sum_{\mathcal{S}_{a+b}^I} \gamma_2^I. \tag{4.65}$$

And from here summing over $\mathcal{S}_b^I$ and using $\sum_{\mathcal{S}_b^I} K_I = 0$ we get:

$$\sum_I \gamma_1^I = -\sum_I \gamma_2^I. \tag{4.66}$$

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From here and from (4.64) we get that $\sum_I \gamma_1^I = \sum_I \gamma_2^I = 0$. By subtracting the first equation from the third in (4.63) and summing over **I** we get that:

$$\sum_I S_I = -\sum_I (\gamma_1^I + \gamma_2^I + K_I) = 0 \tag{4.67}$$

Because S_I is a positive definite quantity this implies that $S_I = 0$ and thus we can not turn on the new interaction at one loop. Of course the question that arises is about the possibility of turning it on at higher loops, like in section (17).

4.3.1 $SU(N = 3)$

This case is exactly like the **(a,b,-a-b)** case, the new interaction here does not add any new possibilities for the $SU(N = 3)^k$ interactions. But however lets look at the one loop analysis (ρ_i^{I2} is defined as in previous sections):

$$
\begin{aligned}
\gamma_{Q_1^I} &= \mathcal{A}(h_I^2 + h_I'^2) + \mathcal{C}\rho_1^{I2} - \mathcal{B}(g_I^2 + g_{I+a}^2) = 0 \\
\gamma_{Q_2^I} &= \mathcal{A}(h_{I-a}^2 + h_{I+k'-a}'^2) + \mathcal{C}\rho_2^{I2} - \mathcal{B}(g_I^2 + g_{I+k'-a}^2) = 0 \\
\gamma_{Q_3^I} &= \mathcal{A}(h_{I-k'}^2 + h_{I-a}'^2) + \mathcal{C}s^{I2} + \mathcal{C}\rho_3^{I2} - \mathcal{B}(g_I^2 + g_{I-k'}^2) = 0.
\end{aligned}
\tag{4.68}
$$

We see that we can parameterize a solution by h_I, h_I', g_I and by $s_I \rightarrow$ we get a **4k** dimensional manifold of fixed points.

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4.4 $(1,1,4)$

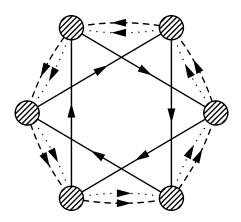The (1,1,4) case can be represented by the following diagram.

Figure 4.4: $\mathbb{C}^3/\mathbb{Z}_6$ (1,1,4) quiver diagram

Essentially this case is in the intersection of $(a, \frac{k}{3} - a, -\frac{k}{3})$ and (a,a,-2a). Thus we can treat this case as (a,a,-2a) with the additional operator $h_{333}^I Q_3^I Q_3^{I-2a} Q_3^{I-4a}$. As in the previous section we have here $\sum_{2a} \gamma_3^I = 0$, from the **(a,a,-2a)** case we know that $\gamma_3^I = -(\delta_1 + \delta_2)$ and thus $\delta_1 = -\delta_2 \to \gamma_3^I = 0$, $\gamma_I = -\gamma_{I+a}$. From section (4.2) we get($h_{333}^I \equiv t^I$):

$$
\begin{aligned}
\gamma_{Q_1^I} &= \mathcal{A}(h_I^2 + h_I'^2) + \mathcal{C}(p_I^2 + p_{I-a}^2) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma^I \\
\gamma_{Q_2^I} &= \mathcal{A}(h_{I-a}^2 + h_{I+a}'^2) + \mathcal{C}(s_I^2 + s_{I-a}^2) - \mathcal{B}(g_I^2 + g_{I+a}^2) = \gamma^I \\
\gamma_{Q_3^I} &= \mathcal{A}(h_{I-2a}^2 + h_{I-a}'^2) + \mathcal{C}(s_{I-2a}^2 + p_{I-2a}^2) + \mathcal{D}t_I^2 - \mathcal{B}(g_I^2 + g_{I-2a}^2) = 0.
\end{aligned}
$$

$$(4.69)$$

From here $(\gamma_I \to \frac{1}{\mathcal{A}}\gamma_I)$:

$$
\begin{aligned}
h_I^2 &= -h_I'^2 - \frac{1}{\mathcal{A}}(\mathcal{C}(p_I^2 + p_{I-a}^2) - \mathcal{B}(g_I^2 + g_{I+a}^2)) + \gamma^I \\
h_I^2 &= -h_{I+2a}'^2 - \frac{1}{\mathcal{A}}(\mathcal{C}(s_{I+a}^2 + s_I^2) - \mathcal{B}(g_{I+2a}^2 + g_{I+a}^2)) + \gamma^{I+a} \\
h_I^2 &= -h_{I+a}'^2 - \frac{1}{\mathcal{A}}(\mathcal{C}(s_I^2 + p_I^2) - \mathcal{D}t_{I+2a}^2 - \mathcal{B}(g_{I+2a}^2 + g_I^2)).
\end{aligned}
$$

$$(4.70)$$

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From here we can write:

$$
\begin{aligned}
h_I'^2 \quad &+\frac{1}{\mathcal{A}}(\mathcal{C}(p_I^2 + p_{I-a}^2) - \mathcal{B}g_I^2) - \gamma_I = h_{I+2a}'^2 + \frac{1}{\mathcal{A}}(\mathcal{C}(s_{I+a}^2 + s_I^2) - \mathcal{B}g_{I+2a}^2) - \gamma_{I+a} \\
h_{I+2a}'^2 \quad &+\frac{1}{\mathcal{A}}(\mathcal{C}s_{I+a}^2 - \mathcal{B}g_{I+a}^2) - \gamma_{I+a} = h_{I+a}'^2 + \frac{1}{\mathcal{A}}(\mathcal{D}t_{I+2a}^2 + \mathcal{C}p_I^2 - \mathcal{B}g_I^2) \\
h_{I+a}'^2 \quad &+\frac{1}{\mathcal{A}}(\mathcal{D}t_{I+2a}^2 + \mathcal{C}s_I^2 - \mathcal{B}g_{I+2a}^2) = h_I'^2 + \frac{1}{\mathcal{A}}(\mathcal{C}p_{I-a}^2 - \mathcal{B}g_{I+a}^2) - \gamma_I.
\end{aligned}
$$

$$(4.71)$$

Obviously the last two equations imply the first one. Now from the last two equations we obtain:

$$
\mathcal{D}t_{I+a}^2 - \mathcal{B}(g_{I-a}^2 - g_I^2) = -\mathcal{D}t_{I+2a}^2 - \mathcal{B}(g_{I+a}^2 - g_{I+2a}^2) - 2\gamma_I.
$$

$$(4.72)$$

Again by summing both sides over $\mathbf{I}$ we get $\sum_I t_I^2 = -\sum_I t_I^2 \rightarrow t_I = 0$. Thus we see that we are essentially back to the $(\mathbf{a,a,-2a})$ case.

4.4.1 $SU(N = 3)$

Here again exactly as in the $(\mathbf{a,a,-2a})$ case we can have a solution parameterized by h_I, h_I', $\mathbf{k}$ of the p_I and s_I, the gauge couplings, ρ_{12}^I and now also by the $t_I \rightarrow$ giving a total of a $\mathbf{6k{=}36}$ dimensional manifold of fixed points.

4.5 (1,1,1)

This case is the richest one - all the interactions we discussed in previous sections can be turned on here. This theory can be depicted by the following quiver diagram:
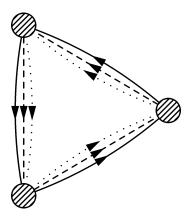

Figure 4.5: $\mathbb{C}^3/\mathbb{Z}_3$ (1,1,1) quiver diagram

Here the different lines represent the three sectors we have in our theory.

The (1,1,1) is the only $\mathbb{C}^3/\mathbb{Z}_3$ orbifold to have $\mathcal{N} = 1$ SUSY. We get a $U(N)^3$ gauge group with matter content:

$$3 \times ((N, \bar{N}, 1) \oplus (1, N, \bar{N}) \oplus (\bar{N}, 1, N)) \tag{4.73}$$

This theory was treated in [18]. It was argued there that the theory has a single marginal direction corresponding to the gauge coupling. However in that analysis the cases were classified by the global S_3 symmetry of the three complex space-time coordinates. This overlooks the symmetries leading to the additional operators that we saw in previous sections. So we will now look for *exactly* marginal deformations of this theory.

Obviously this case is a special case of the cases we considered above, in particular of the **(a,a,-2a)** case. There we found a three dimensional manifold of fixed points, so here we expect at least this dimensionality.

The most general marginal deformation here is[8]:

$$h^I_{ijk} Q^I_i Q^{I+1}_j Q^{I+2}_k \qquad (4.74)$$

We have here a global $SU(3)^3$ symmetry rotating Q^1_is, Q^2_is and Q^3_is. By these rotations we can assume that the anomalous dimensions are diagonal, and then:

$$\beta_{h^I_{ijk}} \propto \gamma_{Q^I_i} + \gamma_{Q^{I+1}_j} + \gamma_{Q^{I+2}_k} \qquad (4.75)$$

So we see that in general we have to put all the anomalous dimensions to zero. In general we have $\mathbf{3k} \times \mathbf{2}$ anomalous dimensions and $\mathbf{3k} \times \mathbf{3} + \mathbf{3}$ couplings, $\mathbf{3k}$ of which can be set to zero by the global symmetry, and thus expect for three dimensional manifold of *exactly* marginal deformations. However if we restrict ourselves to cases where we will be guaranteed to have anomalous dimensions proportional to the identity matrix (by turning on only $h^I_{123} = h^I_{231} = h^I_{312}$, $h^I_{213} = h^I_{132} = h^I_{321}$ and $h^I_{ijk} \propto \delta_{ij}\delta_{jk}$ interactions, and keeping all the gauge couplings equal) we will get only $\mathbf{1}$ independent anomalous dimension, and $\mathbf{4}$ couplings $\rightarrow$ we expect a $\mathbf{3}$ dimensional manifold of fixed points and thus we can deal only with this restricted case[9] .

The anomalous dimensions still have to vanish, because $\beta_{h^I_{lll}} \propto \gamma^I_l$.

Now the anomalous dimensions for the most general case are at one-loop:

$$\gamma^I_{ij} = \mathcal{A} h^I_{ilm} h^{*I}_{jlm} - \mathcal{B}(g^2_I + g^2_{I+1})\delta_{ij} \qquad (4.76)$$

We see that we will have solutions to $\gamma = 0$ only when $h^I_{ilm} h^{*I}_{jlm}$ is proportional to the identity matrix, up to the global $SU(3)^3$ rotations this implies the restrictions above.

We now look at the one loop expressions in the restricted case (h_Is, h'_Is defined as in the previous sections and $y_I \equiv h^I_{111} = h^I_{222} = h^I_{333}$, in our cases all the $\mathbf{h}$s are equal and all the $\mathbf{h}$'s are equal):

---

[8]Here $a_l + a_m + a_n = 0 (mod\,\mathbf{k})$ for any choice (l,m,n).

[9]Obviously we have other choices of the couplings satisfying this, but they all are related by the global $SU(3)^3$ symmetry.

$$\gamma_I = \mathcal{A}(h^2 + h'^2) + \mathcal{C}y_I^2 - 2\mathcal{B}g^2. \tag{4.77}$$

From here we easily see that all the y_Is have to be equal, and we get one condition for **4** couplings $\to$ we get a **3** dimensional manifold of fixed points.

As in previous sections, it is easy to construct the solutions in all order of the perturbation series.

4.5.1 $SU(N = 3)$

If we take the gauge group to be SU(3) there is a much larger class of marginal operators one can add to the theory. All the operators of the following form are possible:

$$\frac{\rho_{ijk}^I}{3!}\epsilon_{lmn}\epsilon^{abc}(Q_i^I)_a^l(Q_j^I)_b^m(Q_k^I)_c^n. \tag{4.78}$$

Obviously ρ_{ijk}^I has to be symmetric in the lower indices. The Leigh-Strassler analysis teaches us in this case: we have 6×3 anomalous dimensions and $3 \times 3 \times 3 + 10 \times 3 + 3$ couplings, have global $SU(3)^3$ symmetry with which we can fix 3×3 parameters $\to$ naively we expect a **11k** dimensional manifold of fixed points.

Look now at the general one loop expression:

$$\gamma_{ij}^I = \mathcal{A}h_{ilm}^I h_{jlm}^{*I} + \mathcal{C}\rho_{ilm}^I \rho_{jlm}^{*I} - \mathcal{B}(g_I^2 + g_{I+1}^2)\delta_{ij}. \tag{4.79}$$

We will have a solution when $\mathcal{A}h_{ilm}^I h_{jlm}^{*I} + \mathcal{C}\rho_{ilm}^I \rho_{jlm}^{*I}$ is proportional to the identity matrix.

ρ_{ijk}^I is symmetric we have only $10 \times k$ ρs. We will take all the h_{ijk}^Is and gauge couplings to be parameters. Also will take $\rho_{iii}^I, \rho_{123}^I$ as parameters $\to$ we have $3 \times 3 \times 3 + 3 + 3 + 3 \times 3 = 3 \times 14$ parameters, and so 6×3 of the couplings are still undetermined. We have 6×3 anomalous dimensions with which we determine the yet undetermined couplings $\to$ we get 11×3 dimensional manifold of fixed points (as expected) after dividing by the $SU(3)^3$ global symmetry.

Chapter 5

Summary and Discussion

First we summarize the results:

- $\mathcal{N} = 4$

  We conclude that the only supersymmetric *exactly* marginal deformations of $\mathcal{N} = 4$
  SYM are the superpotentials (other than changing the gauge coupling):

$$
\frac{i\delta\lambda\sqrt{2}}{3!}\epsilon_{ijk}Tr(\Phi^i[\Phi^j,\Phi^k])
$$
$$
\sum_i \frac{h}{3!}Tr(\Phi^i\left\{\Phi^i,\Phi^i\right\})
$$
$$
\frac{h_{123}}{3!}Tr(\Phi^1\left\{\Phi^2,\Phi^3\right\}), \tag{5.1}
$$

  with one relation relating λ, h_{123}, h and the gauge coupling. These fixed points are
  IR stable. We saw that this theory is not asymptotically free for any choice of the
  coupling constants nor does it have (in perturbation theory) UV fixed points. These
  exactly marginal deformations can be mapped to the strong coupling region by the
  S-duality transformation.

  In the strong coupling limit all our calculations are done with the assumption that
  we are close to the $\mathcal{N} = 4$ line, because only there we can trust the S-duality trans-
  formation. An interesting question is whether there is a UV fixed point at strong
  coupling "far" from the $\mathcal{N} = 4$ line. If there is such a UV fixed point then we can

talk about the marginal deformations above not just at the conformal fixed point but at a larger set of points.

- $\mathcal{N} = 2$

  We conclude that for the $\mathbb{C}^3/\mathbb{Z}_k$ orbifold theories we obtain:

  - General k

    Here we are able to show the existence of one *exactly* marginal direction (in addition to the gauge couplings) parameterized by a parameter **X**. This direction can be seen at any order of perturbation theory. The X=0 case has $\mathcal{N} = 2$ SUSY and all γs vanish, however if $X \neq 0$ we can in principle have nonzero γs. From Leigh and Strassler analysis we expect here to have another **k-1** *exactly* marginal directions which appear by turning on $Tr(\Phi_i\Phi_i\Phi_i)$ operators. We don't see these marginal directions up to three loops. This however does not necessarily prevent them from appearing at higher loops. The fate of these fixed points is in the hands of a linear combination of the γ_{Q_i} and the γ_{Φ_i} from the case without the operators above: if it is positive or zero then these marginal directions are ruled out and if it is negative then we can have them. In any case the total number of *exactly* marginal directions is at least **k+1**.

  - k=3

    In this case we have the **X** direction like in general **k**, and we can also have another **3** *exactly* marginal directions. Here the result agrees with Leigh/Strassler analysis and we see all the marginal deformations already at one loop. So the total number of *exactly* marginal directions here is **7**.

  - SU(N=3)

    Here we have yet a larger space of deformations: we get **2k-1** additional deformations, which give for general **k** a total of **3k** and for k=3 **3k+2=11** *exactly* marginal directions. Again we see all the deformations already at one-loop and they agree with the Leigh/Strassler type analysis.

- $\mathcal{N} = 1$

  Here for general $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$, $\mathbb{Z}_k$, we show that the number of *exactly* marginal directions is (we denote the largest common divisor of a_i with $\mathbf{k}$ by α_i) $\sum_i \alpha_i - 1$. In the case where two of the a_is are equal and $\mathbf{k}$ is even we get additional *exactly* marginal directions. In the special case of SU(N=3) we get much larger manifolds of fixed points, ranging from dimension $\mathbf{3k}$ in the most general $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$ $\mathbb{Z}_k$ theory to $\mathbf{11k}$ in the k=3 case.

  There is also a special case here of $\mathbb{Z}_3$ in which we can turn on any interactions which we can turn on in principle in an $\mathcal{N} = 1$ orbifold theory.

There is a simple generalization of the results above. We are looking at a $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$ $\mathbb{Z}_k$ orbifold theory. Thus we have in our theory, as we saw, three sectors: denote fields from each sector by Q_i. The number of marginal operators of the form $Q_1 Q_2 Q_3$ is $\sum_i \alpha_i - 1$.

First we can consider the $\mathcal{N} = 4$ theory as a $(0,0,0)$ orbifold theory. Then, it corresponds to the following quiver diagram:
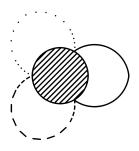

Figure 5.1: $\mathcal{N} = 4$ (0,0,0) quiver diagram

We found that there is always $\mathbf{1}$ additional marginal direction when we put all our gauge couplings equal (the $\mathbf{X}$ solution in the $\mathcal{N} = 2$ case and the $\mathbf{h}$ solution in the $\mathcal{N} = 1$ case). It is easy to understand its origin from the "mother" $\mathcal{N} = 4$ theory: it is the $\mathbb{Z}_k$ projection of the $\Phi_1 \Phi_2 \Phi_3$ deformation appearing there. For general $\mathbf{k}$ the other Φ^3 deformation does not survive the orbifolding, however for the special $\mathbf{k=3}$ case it does and we indeed see it in the reduced SUSY cases. From the analysis of [21, 22] we know that

the marginal deformations which survive the orbifold projection are known to be *exactly* marginal at far as only the planar diagrams are concerned, however what we find is that even the non-planar diagrams don't prevent the surviving deformations from being *exactly* marginal.

The more delicate point is with the marginal operators of the form Φ_i^3 in the general **k** case, we have to deal with them in each case when they appear independently. These deformations are marginal and from Leigh-Strassler analysis some of them are expected to be *exactly* marginal. However we saw that they are prevented from appearing at one loop (see sections (3.1.1, 4.3). Their fate has to be decided from higher loop calculations.

Another interesting observation is that in the $\mathcal{N} = 2$ case the orbifold keeps a direction in $\mathbb{C}^3$ fixed, thus because the orbifold acts on the S^5 factor of the $\mathbf{AdS_5 \times S^5}$ as it acts on the angular coordinates of the $\mathbb{R}^6 \sim \mathbb{C}^3$, we will get fixed points on the sphere (actually a fixed circle). This enables the appearance of massless twisted sector states which can correspond to some *exactly* marginal operators on the field theory side. And we see these twisted states $\rightarrow$ the **(k-1)** blow up modes. Another case of *exactly* marginal operators coming from the twisted sector are for example the **2** additional operators we get in the $\mathbb{Z}_3$ case.

In the $\mathcal{N} = 1$ case the only fixed point of the $\mathbb{Z}_k$ action is the origin of the $\mathbb{C}^3$. However we still can have massless twisted sector states. Remember that twisted sector strings are defined as strings which are closed up to the action of the twisting group Γ ($\Gamma = \mathbb{Z}_k$ in our case). There are $\|\Gamma\|$ twisted sectors defined by every element of the twisting group. In our case the action of the orbifold is defined by vector $\overrightarrow{a}$:

$$\wp \equiv \begin{pmatrix} e^{\frac{2\pi i}{k}a_1} & 0 & 0 \\ 0 & e^{\frac{2\pi i}{k}a_2} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{k}a_3} \end{pmatrix} \tag{5.2}$$

Now, assume a_1 has a largest common divisor larger than one (say α) with **k**. Then, if we start with vector the (1,0,0) we will get back to our starting point after $\frac{k}{\alpha}$ applications of $\wp$. So, the $\wp^{\frac{k}{\alpha}}$th twisted sector (which is not the identity $\equiv$ the untwisted sector) has fixed points and could include massless states. There are $\alpha - 1$ twisted sectors with massless

states, and we find that each contributes one state which corresponds to an exactly marginal operator. Thus over all we have $\sum_i \alpha_i - 3$ massless states from the twisted sector.

Thus, we conclude that in the $\mathcal{N} = 1$ case we can get massless states from the twisted sectors when the a_i have non trivial largest common divisors with $\mathbf{k} \rightarrow$ and this is what we get. We got $\alpha + \beta + \gamma - 3$ marginal deformations that don't come from the $\mathcal{N} = 4$ theory $\rightarrow$ they have to come from the twisted sectors and we see that this is possible.

We see that the marginal deformations that we get are in agreement with the string theory. The marginal deformations can be divided to deformations that originate from the twisted sector ($\rightarrow$ there are always at least $\sum_i \alpha_i - 3$ such deformations) and to deformations which come from restrictions of the $\mathcal{N} = 4$ marginal deformations to orbifold group invariant parts. All this is true for general SU($\mathbf{N}$) gauge group. However in SU($\mathbf{N}$=3) we get a much larger space of deformations. These deformations can not be related directly to the string theory, because the field$\leftrightarrow$string theory correspondence is well understood only in the large $\mathbf{N}$ limit, if at all it is true for finite $\mathbf{N}$.

The $\sum_i \alpha_i - 3$ *exactly* marginal deformations coming from the twisted sector are related to the $\mathcal{N} = 2$ blow up modes. However we got several *exactly* marginal deformations which come from the twisted sector and are not related to blow up modes: the **2** extra deformations of $\mathcal{N} = 2$ $\mathbb{Z}_3$ and the $(\mathbf{a}, \mathbf{a}, -\mathbf{2a})$ extra deformations for instance. These *exactly* marginal deformations are predictions of our analysis and their counterparts on the string theory side have to be found. As was mentioned above the other predictions are the SU($\mathbf{N}$=3) extra deformations, however they are hard to see in the string theory.

We see that in each of the cases above we find a rich "zoo" of marginal deformations, giving a large set of conformal theories. From the **AdS/CFT** correspondence, the *exactly* marginal deformations should correspond to some fields on the string theory side, which are moduli of the theory with the SO(2,4) symmetry. The exact details of this correspondence have to be explored.

The behavior of the marginal operators in the reduced SUSY cases under S-duality is another interesting question. We believe that there is an S-duality acting on these theories because we know that on the string theory side we have a type **IIB** superstring on some background, which is believed to be self dual under S-duality.

Appendix A

Three-loop calculation of γ

We calculate here the three-loop contribution to the γ parameter as a function of g^2 and **X**, which determine α_i, δ_i at leading order according to (3.14). We saw explicitly that at one loop order γ is forced to be zero. The one-loop solution assures us of having a two loop finite theory, thus also at two loops γ is zero. It is easy to convince oneself of this. The only non finite two loop diagram, not including one loop subdiagrams and containing Yukawa type interactions, is:

$$(A.1)$$

This diagram is proportional to $(\frac{N^2-1}{N})^2(\alpha_i^2 + \delta_i^2)(g_i^2 + g_{i+1}^2)$ for the Q propagator and to $N^2(N^2 - 1)(\alpha_i^2 + \delta_{i-1}^2)g_i^2$ for the Φ propagator. Both of these expressions don't contain the **X** parameter and thus they are the same as in the $\mathcal{N} = 2$ case, and there we know that the two-loop γ-function vanishes.

So the first non zero contribution to γ is expected to appear at three loops. Three loops is also the first order of appearance of non-planar diagrams. It was argued in [21], [22] that the planar diagrams in the orbifold theory are the same as in the $\mathcal{N} = 4$ theory. So

we are tempted to assume that also when considering marginal deformations the planar diagrams in both theories will be the same (if the coupling of the different gauge groups are the same). [1] We can assume that the γ parameter is proportional to some power of the **X** parameter, because when X=0 we are in the $\mathcal{N} = 2$ case and we know that the anomalous dimensions vanish. Thus we are not interested in diagrams consisting of only gauge interactions. Also at three loops we can have diagrams with four gauge vertices and two matter vertices, these diagrams are proportional to $g_i^2 g_{i+1}^2 \alpha_i^2$ for instance.

We argue that these diagrams don't contribute to the γ parameter. First we look at the Q propagator. At three loops the Q propagator superspace diagrams are of a general structure:
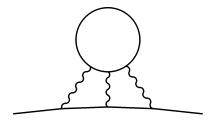

Figure A.1: A typical Q propagator

Here the horizontal line is a line of Q and $\tilde{Q}$ propagators and the curled lines are the fields in adjoint (Φs and gauge fields). By assumption we have only one Φ propagator and assume it's Φ_i. Then in case all six interactions happen on the Q line we have a product of six matrices as the gauge factor. Now, change Q_i to Q_{i-1} and V_{i+1} to V_{i-1}. The order of the product will be reversed because the fundamental representation will be exchanged with

the antifundamental, every gauge vertex will receive a minus (we have four such vertices, so the overall sign won't change), and the diagram will have a factor of $\delta_{i-1}^2 g_{i,i-1}^2 g_{i,i-1}^2$ if the original diagram had $\alpha_i^2 g_{i,i+1}^2 g_{i,i+1}^2$. Now, if the diagram is symmetric the reverse order of the product won't be important, and so all the factors of these two diagrams are the same, and their sum is proportional to $\delta_{i-1}^2 g_{i,i-1}^2 g_{i,i-1}^2 + \alpha_i^2 g_{i,i+1}^2 g_{i,i+1}^2$, and thus after summation over **i** the **X** dependence vanishes because α_i^2 has $+\mathbf{X}$ and δ_i^2 has $-\mathbf{X}$. If the diagram is not left right symmetric, there is also its mirror image and we couple our new diagram with the mirror image of the original one and obtain the same result.

If one of the gauge interactions happens not on the Q line we get an extra factor of f_{abc}. Now reversing the product changes the sign of the expression, but we also now have an odd number of gauge interactions on the Q line and so we have another sign flip $\rightarrow$ again we find overall no X dependence. We can proceed in this way by taking out the gauge vertices from the Q line and we always get the same result.

Now for the γ_{Φ_i} diagrams, here we have not a Q line but a Q loop and thus instead of a product of matrices we have a trace of such product, but these changes don't effect our considerations above because we always can take the trace in the end.

Thus we conclude that the γ parameter is proportional at least to the second power of **X**, and comes from diagrams consisting of at most two gauge vertices.

Now there are only four diagrams giving contributions that satisfy the conditions above (see [23]):

$$(A.2)$$

$$(A.3)$$

Where:

$$(A.4)$$

We begin with the first diagram.

A calculation of a diagram consists of two main things: calculation of the integrals and the symmetry factors and the calculation of the algebra (gauge algebra, couplings etc.). The first part here is the same for the Q propagator and for the Φ propagator and we will neglect it. From the algebraic point of view: We calculate $\sum_i \gamma_{Q_i} - \frac{N^2-1}{N^2}\gamma_{\Phi_i}$, thus while calculating $\sum_i \gamma_{Q_i}$ we can close the external legs of the diagram and multiply by N^2 (which is the gauge factor) and get the desired quantity. When calculating $\sum_i \gamma_{\Phi_i}$ we can also close the external legs, but now the gauge factor is N^2-1. Essentially we see that when we calculate $\sum_i \gamma_{\Phi_i}$ or $\sum_i \gamma_{Q_i}$ for the first diagram we calculate the same thing (when viewed as closed diagrams in the sense above). Thus, we get that $N^2 \sum_i \gamma_{Q_i} = (N^2-1)\sum_i \gamma_{\Phi_i}$ and the expression we want to calculate is zero.

What remains to calculate now is the second and third diagrams from (A.2) and the diagram (A.3). The second and third diagrams from (A.2) happen to be finite, thus we are left with the (A.3)diagram. This diagram, although naively planar, is not necessarily such when considered in the double line notation. We will calculate it using the general three loop results given in [10](see also [24]).

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In the notations of [10] we have a superpotential $\frac{1}{6}Y_{ijk}\Theta_i\Theta_j\Theta_k$ and we define $C(R)^i_j \equiv (T^aT^a)^i_j$ where the T^a are matrices of the representation $\mathbf{R}$ of the gauge group. In our case we have k different gauge groups, but in the diagrams we are interested in, only one gauge group appears in each diagram, so we can use the results of [10] and simply sum over all gauge groups. In our case we have couplings of the form:

$$Y\begin{pmatrix} a \\ i \end{pmatrix}\begin{pmatrix} \alpha;l \\ i \end{pmatrix}\begin{pmatrix} l;\gamma \\ i \end{pmatrix} \quad \Phi^a_i \quad Q^l_\alpha \tilde{Q}^\gamma_l$$

$$\downarrow \qquad\qquad\qquad\qquad (A.5)$$

$$Y\begin{pmatrix} a \\ i \end{pmatrix}\begin{pmatrix} \alpha;l \\ i \end{pmatrix}\begin{pmatrix} l;\gamma \\ i \end{pmatrix} \quad \equiv \quad \alpha_i(T^a)^\alpha_\gamma$$

$$Y\begin{pmatrix} a \\ i+1 \end{pmatrix}\begin{pmatrix} \alpha;l \\ i \end{pmatrix}\begin{pmatrix} m;\alpha \\ i \end{pmatrix} \quad \equiv \quad \delta_i(T^a)^m_l$$

All other Y's vanish.

$$(C(R)_Q)^i_j = \frac{1}{2}\frac{N^2-1}{N}\delta^i_j$$
$$(C(R)_\Phi)^i_j = N\delta^i_j. \qquad\qquad (A.6)$$

In these notations in a one-loop and two-loop finite theory the three loop contribution to γ, proportional to g^2 is $\frac{1}{(16\pi^2)^3}\Delta$, where:

$$\Delta^i_j \equiv \kappa g^2 (YS_1Y)^i_j$$
$$S^i_{1j} \equiv Y^{imn}C(R)^p_m Y_{jpn} \qquad\qquad (A.7)$$
$$(YS_1Y)^i_j \equiv Y^{imn}S^p_{1m}Y_{jpn}$$
$$\kappa \equiv 6\zeta(3).$$

Here the indices represent each set of indices in (A.5). Thus we should calculate the quantities above. We calculate for gauge group i and then sum over all gauge groups. First S_1 for a Φ index :

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$$(S_1)^{\Phi_i^a}_{\Phi_i^b} = 2Y\begin{pmatrix} a \\ i \end{pmatrix}\begin{pmatrix} \alpha;l \\ i \end{pmatrix}\begin{pmatrix} l;\gamma \\ i \end{pmatrix} C(R)_Q Y\begin{pmatrix} b \\ i \end{pmatrix}\begin{pmatrix} \alpha;l \\ i \end{pmatrix}\begin{pmatrix} l;\gamma \\ i \end{pmatrix} +$$

$$+ \; 2Y\begin{pmatrix} a \\ i \end{pmatrix}\begin{pmatrix} \alpha;l \\ i-1 \end{pmatrix}\begin{pmatrix} m;\alpha \\ i-1 \end{pmatrix} C(R)_Q Y\begin{pmatrix} b \\ i \end{pmatrix}\begin{pmatrix} \alpha;l \\ i-1 \end{pmatrix}\begin{pmatrix} m;\alpha \\ i-1 \end{pmatrix} =$$

$$= \; 2\frac{1}{2}\frac{N^2-1}{N}(\sum_l \alpha_i^2 TrT^aT^b + \sum_\alpha \delta_{i-1}^2 TrT^aT^b) =$$

$$= \; \frac{1}{2}(N^2-1)(\alpha_i^2 + \delta_{i-1}^2)\delta_b^a \qquad\qquad\qquad\qquad (A.8)$$

$$(S_1)^{\Phi_{i+1}^a}_{\Phi_{i+1}^b} = 2Y\begin{pmatrix} a \\ i+1 \end{pmatrix}\begin{pmatrix} \alpha;l \\ i \end{pmatrix}\begin{pmatrix} m;\alpha \\ i \end{pmatrix} C(R)_Q Y\begin{pmatrix} b \\ i+1 \end{pmatrix}\begin{pmatrix} \alpha;l \\ i \end{pmatrix}\begin{pmatrix} m;\alpha \\ i \end{pmatrix} =$$

$$= \; 2\frac{1}{2}\frac{N^2-1}{N}\sum_\alpha \delta_i^2 TrT^aT^b = \frac{1}{2}(N^2-1)\delta_i^2\delta_b^a \qquad\qquad\qquad (A.9)$$

$$(S_1)^{\Phi_{i-1}^a}_{\Phi_{i-1}^b} = 2Y\begin{pmatrix} a \\ i-1 \end{pmatrix}\begin{pmatrix} \alpha;l \\ i-1 \end{pmatrix}\begin{pmatrix} l;\gamma \\ i-1 \end{pmatrix} C(R)_Q Y\begin{pmatrix} b \\ i-1 \end{pmatrix}\begin{pmatrix} \alpha;l \\ i-1 \end{pmatrix}\begin{pmatrix} l;\gamma \\ i-1 \end{pmatrix} =$$

$$= \; 2\frac{1}{2}\frac{N^2-1}{N}\sum_l \alpha_{i-1}^2 TrT^aT^b = \frac{1}{2}(N^2-1)\alpha_{i-1}^2\delta_b^a \qquad\qquad (A.10)$$

There are two Qs coupled to gauge group i: Q_i and Q_{i-1}, so we calculate S_1 for each one separately. First for Q_i:

$$(S_1)^{(Q_i)^\alpha_l}_{(Q_i)^\beta_l} = Y\begin{pmatrix} \alpha;l \\ i \end{pmatrix}\begin{pmatrix} a \\ i \end{pmatrix}\begin{pmatrix} l;\gamma \\ i \end{pmatrix} (C(R)_Q + C(R)_\Phi)Y\begin{pmatrix} \beta;l \\ i \end{pmatrix}\begin{pmatrix} a \\ i \end{pmatrix}\begin{pmatrix} l;\gamma \\ i \end{pmatrix} +$$

$$+ \; Y\begin{pmatrix} \alpha;l \\ i \end{pmatrix}\begin{pmatrix} a \\ i+1 \end{pmatrix}\begin{pmatrix} m;\alpha \\ i \end{pmatrix} C(R)_Q Y\begin{pmatrix} \beta;l \\ i \end{pmatrix}\begin{pmatrix} a \\ i+1 \end{pmatrix}\begin{pmatrix} m;\alpha \\ i \end{pmatrix} =$$

$$= \; ((N + \frac{1}{2}\frac{N^2-1}{N})\alpha_i^2 T^aT^a + \frac{1}{2}\frac{N^2-1}{N}\delta_i^2 T^aT^a)\delta_\beta^\alpha =$$

$$= \; ((\frac{1}{2}\frac{N^2-1}{N})^2(\alpha_i^2 + \delta_i^2) + \frac{1}{2}(N^2-1)\alpha_i^2)\delta_\beta^\alpha \qquad\qquad (A.11)$$

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And the same for Q_{i-1}:

$$(S_1)^{(Q_{i-1})_l^\alpha}_{(Q_{i-1})_l^\beta} = ((\frac{1}{2}\frac{N^2-1}{N})^2(\alpha^2_{i-1} + \delta^2_{i-1}) + \frac{1}{2}(N^2-1)\delta^2_{i-1})\delta^\alpha_\beta \qquad (A.12)$$

Now we calculate Y^*S_1Y. First for Φ (We have contributions for the i'th gauge group for $\gamma_{\Phi_i}, \gamma_{\Phi_{i+1}}, \gamma_{\Phi_{i-1}}$):

$$(Y^*S_1Y)^{\Phi^a_i}_{\Phi^b_i} = 2Y\begin{pmatrix} a \\ i \end{pmatrix}\begin{pmatrix} \alpha; l \\ i \end{pmatrix}\begin{pmatrix} l; \gamma \\ i \end{pmatrix} S^{Q_i}_1 Y\begin{pmatrix} b \\ i \end{pmatrix}\begin{pmatrix} \alpha; l \\ i \end{pmatrix}\begin{pmatrix} l; \gamma \\ i \end{pmatrix} +$$

$$+ \ 2Y\begin{pmatrix} a \\ i \end{pmatrix}\begin{pmatrix} \alpha; l \\ i-1 \end{pmatrix}\begin{pmatrix} m; \alpha \\ i-1 \end{pmatrix} S^{Q_{i-1}}_1 Y\begin{pmatrix} b \\ i \end{pmatrix}\begin{pmatrix} \alpha; l \\ i-1 \end{pmatrix}\begin{pmatrix} m; \alpha \\ i-1 \end{pmatrix} =$$

$$= \ ((\frac{1}{2}\frac{N^2-1}{N})^2(\alpha^2_i + \delta^2_i) + \frac{1}{2}(N^2-1)\alpha^2_i)N\alpha^2_i\delta^a_b +$$

$$+ \ ((\frac{1}{2}\frac{N^2-1}{N})^2(\alpha^2_{i-1} + \delta^2_{i-1}) + \frac{1}{2}(N^2-1)\delta^2_{i-1})N\delta^2_{i-1}\delta^a_b \qquad (A.13)$$

$$(Y^*S_1Y)^{\Phi^a_{i+1}}_{\Phi^b_{i+1}} = 2Y\begin{pmatrix} a \\ i+1 \end{pmatrix}\begin{pmatrix} \alpha; l \\ i \end{pmatrix}\begin{pmatrix} m; \alpha \\ i \end{pmatrix} S^{Q_i}_1 Y\begin{pmatrix} b \\ i+1 \end{pmatrix}\begin{pmatrix} \alpha; l \\ i \end{pmatrix}\begin{pmatrix} m; \alpha \\ i \end{pmatrix} =$$

$$= \ ((\frac{1}{2}\frac{N^2-1}{N})^2(\alpha^2_i + \delta^2_i) + \frac{1}{2}(N^2-1)\alpha^2_i)N\delta^2_i\delta^a_b \qquad (A.14)$$

$$(Y^*S_1Y)^{\Phi^a_{i-1}}_{\Phi^b_{i-1}} = 2Y\begin{pmatrix} a \\ i-1 \end{pmatrix}\begin{pmatrix} \alpha; l \\ i-1 \end{pmatrix}\begin{pmatrix} l; \gamma \\ i-1 \end{pmatrix} S^{Q_{i-1}}_1 Y\begin{pmatrix} b \\ i-1 \end{pmatrix}\begin{pmatrix} \alpha; l \\ i-1 \end{pmatrix}\begin{pmatrix} l; \gamma \\ i-1 \end{pmatrix} =$$

$$= \ ((\frac{1}{2}\frac{N^2-1}{N})^2(\alpha^2_{i-1} + \delta^2_{i-1}) + \frac{1}{2}(N^2-1)\delta^2_{i-1})N\alpha^2_{i-1}\delta^a_b. \qquad (A.15)$$

From here we calculate the contribution of such diagrams to $\sum_i \gamma_{\Phi_i}$, by multiplying the expression above by g^2_i and summing over $\mathbf{i}$, and because we are interested only in the part proportional to X^2 and X we insert the expressions (3.14). We get:

$$\sum_i \gamma_{\Phi_i}|_{X^2} \propto N(N^2-1)(X^2 - X^2)\sum_i g^2_i = 0 \qquad (A.16)$$

$$\sum_i \gamma_{\Phi_i}|_X \ \propto \ \sum_i ((\frac{1}{2}\frac{N^2-1}{N})^2((\alpha_i^2+\delta_i^2)(\alpha_i^2 g_i^2+\delta_i^2 g_{i+1}^2)+$$

$$+ \ (\alpha_i^2+\delta_i^2)(\alpha_i^2 g_{i+1}^2+\delta_i^2 g_i^2))+\frac{1}{2}N(N^2-1)\alpha_i^2\delta_i^2(g_{i+1}^2+g_i^2))=$$

$$\propto \ \sum_i ((\frac{1}{2}\frac{N^2-1}{N})^2(g_{i+1}^2+g_i^2)(X(g_i^2-g_{i+1}^2)+g_i^2 g_{i+1}^2(X-X))+$$

$$+ \ \frac{1}{2}N(N^2-1)(X(g_{i+1}^2+g_i^2)(g_{i+1}^2-g_i^2)))=0. \tag{A.17}$$

Essentially the vanishing of $\sum_i \gamma_{\Phi_i}|_X$ can be concluded by a similar argument to the one we presented in the beginning of the section, so this calculation can be seen as an explicit check of that argument.

Now doing the same for the Q part we obtain:

$$(Y^*S_1Y)^{(Q_i)_l^\alpha}_{(Q_i)_m^\beta} = Y\begin{pmatrix}\alpha;l\\i\end{pmatrix}\begin{pmatrix}a\\i\end{pmatrix}\begin{pmatrix}l;\gamma\\i\end{pmatrix}(S_1^{Q_i}+S_1^{\Phi_i})Y\begin{pmatrix}\beta;l\\i\end{pmatrix}\begin{pmatrix}a\\i\end{pmatrix}\begin{pmatrix}l;\gamma\\i\end{pmatrix}+$$

$$+ \ Y\begin{pmatrix}\alpha;l\\i\end{pmatrix}\begin{pmatrix}a\\i+1\end{pmatrix}\begin{pmatrix}m;\alpha\\i\end{pmatrix}(S_1^{Q_i}+S_1^{\Phi_{i+1}})Y\begin{pmatrix}\beta;l\\i\end{pmatrix}\begin{pmatrix}a\\i+1\end{pmatrix}\begin{pmatrix}m;\alpha\\i\end{pmatrix}=$$

$$= \ (\frac{1}{2}\frac{N^2-1}{N})^3(\alpha_i^2+\delta_i^2)^2\delta_m^l\delta_\beta^\alpha+N((\frac{1}{2}\frac{N^2-1}{N})^2(\alpha_i^2(2\alpha_i^2+\delta_i^2+\delta_{i-1}^2)+\delta_i^4)\delta_m^l\delta_\beta^\alpha$$

$$\tag{A.18}$$

And similarly for $Q_{i-1},Q_{i-2},,Q_{i+1}$:

$$(Y^*S_1Y)^{(Q_{i-1})_l^\alpha}_{(Q_{i-1})_m^\beta} = \ (\frac{1}{2}\frac{N^2-1}{N})^3(\alpha_{i-1}^2+\delta_{i-1}^2)^2\delta_m^l\delta_\beta^\alpha+$$

$$+ \ N((\frac{1}{2}\frac{N^2-1}{N})^2(\delta_{i-1}^2(\alpha_i^2+2\delta_{i-1}^2+\alpha_{i-1}^2)+\alpha_{i-1}^4)\delta_m^l\delta_\beta^\alpha \tag{A.19}$$

$$(Y^*S_1Y)^{(Q_{i-2})_l^\alpha}_{(Q_{i-2})_m^\beta} = \ N(\frac{1}{2}\frac{N^2-1}{N})^2\delta_{i-2}^2\alpha_{i-1}^2\delta_m^l\delta_\beta^\alpha \tag{A.20}$$

$$(Y^*S_1Y)^{(Q_{i+1})_l^\alpha}_{(Q_{i+1})_m^\beta} = \ N(\frac{1}{2}\frac{N^2-1}{N})^2\delta_i^2\alpha_{i+1}^2\delta_m^l\delta_\beta^\alpha \tag{A.21}$$

Again we calculate $\sum_i \gamma_{Q_i}$ taking only the X^2 term:

$$
\begin{aligned}
\sum_i \gamma_{Q_i}|_{X^2} \quad &\propto \quad N(\frac{N^2-1}{N})^2 \sum_i g_i^2(\alpha_i^2(2\alpha_i^2 + \delta_i^2 + \delta_{i-1}^2) + \\
&+ \quad \delta_{i-1}^2(\alpha_i^2 + 2\delta_{i-1}^2 + \alpha_{i-1}^2) + \alpha_{i-1}^4 + \delta_i^4 + \delta_{i-2}^2\alpha_{i-1}^2 + \delta_i^2\alpha_{i+1}^2) = \\
&= \quad N(\frac{N^2-1}{N})^2 \sum_i (2(g_i^2\alpha_i^4 + g_{i+1}^2\delta_i^4) + 2g_i^2\alpha_i^2\delta_{i-1}^2 + \\
&+ \quad (g_i^2 + g_{i+1}^2)\alpha_i^2\delta_i^2 + g_{i+1}^2\alpha_i^4 + g_i^2\delta_i^4 + g_i^2\delta_{i-2}^2\alpha_{i-1}^2 + g_i^2\delta_i^2\alpha_{i+1}^2) = \\
&\rightarrow \quad N(\frac{N^2-1}{N})^2 X^2 \sum_i g_i^2(6-6) = 0
\end{aligned}
\tag{A.22}
$$

The same calculation for the term proportional to $\mathbf{X}$ also gives zero. Thus we conclude that for these diagrams and at three loop order:

$$
\sum_i \gamma_{Q_i} - \frac{N^2-1}{N^2}\gamma_{\Phi_i} = 0.
\tag{A.23}
$$

So the γ parameter is zero up to three loops and we can not turn on any non zero h_is.

Appendix B

$SU(2)^k$ global symmetry

We will show here how one can use the $SU(2)^k$ global symmetry of the **(a,a,-2a)** orbifold to get rid of the h'_Is.

First we notice that the interactions we have can be written as:

$$\sum_I Q_3^{I+2a}(h^I Q_1^I Q_2^{I+a} + h'^{I+a} Q_2^I Q_1^{I+a} + p^I Q_1^I Q_1^{I+a} + s^I Q_2^I Q_2^{I+a}).$$ (B.1)

This can be rewritten as:

$$\sum_I Q_3^{I+2a} \begin{pmatrix} Q_1^I & Q_2^I \end{pmatrix} \begin{pmatrix} p^I & h^I \\ h'^{I+a} & s^I \end{pmatrix} \begin{pmatrix} Q_1^{I+a} \\ Q_2^{I+a} \end{pmatrix}.$$ (B.2)

By the global $SU(2)^k$ symmetry we can rotate Q_1^I and Q_2^I one into the other. Let's define:

$$H^I \equiv \begin{pmatrix} p^I & h^I \\ h'^{I+a} & s^I \end{pmatrix}.$$ (B.3)

We will rotate the **I**th set of Q's with a unitary matrix U^I. Then the coupling matrix H^I transforms as $(U^I)^{-1}H^I U^{I+a}$.

Our goal is to get rid of the h'_Is, obviously we can get rid of at least **k** such couplings. Now let's look at:

$$(U^{I_0})^{-1}H^{I_0}U^{I_0+a}(U^{I_0+a})^{-1}H^{I_0+a}U^{I_0+2a}\cdots(U^{I_0+(k-1)a})^{-1}H^{I_0+(k-1)a}U^{I_0} = (U^{I_0})^{-1}\prod_{l=0}^{k-1}H^{I_0+la}U^{I_0}$$

$$(B.4)$$

Define: $U^{I_0+la} \equiv U_{l+1}$, $H^{I_0+la} \equiv H^l$, $p_{I_0+la} \equiv p_l$, $s_{I_0+la} \equiv s_l$.

Let's set U_1 so that:

$$U_1^{-1}\prod_{l=0}^{k-1}H^{I+la}U_1 = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$(B.5)$$

(It does not matter what the stars represent.) It is easy to convince oneself that if we choose to be close to the orbifold theory (see section (4.2)) we get that in the even **k** case:

$$U_1^{-1} \ \prod_{l=0}^{k-1} H^{I+la}U_1 =$$
$$= \ U_1^{-1}(-1)^{\frac{k}{2}} \begin{pmatrix} h^k + \mathcal{O}(\epsilon) & h^{k-1}(\sum_{odd}p_l + \sum_{even}s_l) + \mathcal{O}(\epsilon^2) \\ -h^{k-1}(\sum_{odd}s_l + \sum_{even}p_l) + \mathcal{O}(\epsilon^2) & h^k + \mathcal{O}(\epsilon) \end{pmatrix} U_1$$

And in the odd **k** case:

$$U_1^{-1} \ \prod_{l=0}^{k-1} H^{I+la}U_1 =$$
$$= \ U_1^{-1}(-1)^{\frac{k-1}{2}} \begin{pmatrix} h^{k-1}(\sum_{odd}p_l + \sum_{even}s_l) + \mathcal{O}(\epsilon^2) & h^k + \mathcal{O}(\epsilon) \\ -h^k + \mathcal{O}(\epsilon) & h^{k-1}(\sum_{odd}s_l + \sum_{even}p_l) + \mathcal{O}(\epsilon^2) \end{pmatrix} U_1$$

We see that the determinant of this expression is obviously non zero ($h \gg s, p$), and thus on the right hand side of (B.5) we don't get zeros on the diagonal.

We will choose U_{l+1} for $1 \leq l < k$ so that:

$$U_l^{-1}H^lU_{l+1} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

$$(B.6)$$

Here the point is that:

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \tag{B.7}$$

So that (B.4) can be written as:

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} U_k^{-1} H_k U_1 = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \tag{B.8}$$

And from here we easily see that:

$$U_k^{-1} H_k U_1 = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \tag{B.9}$$

And thus we succeeded in getting rid of all h'_l.

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