# SOLUTION OF THE MASTER EQUATION IN TERMS OF THE ODD TIME FORMULATION

## Ömer F. DAYI

University of Istanbul, Faculty of Science,

Department of Physics, Vezneciler, 34459 Istanbul, Turkey\*

#### Abstract

A systematic way of formulating the Batalin-Vilkovisky method of quantization was obtained in terms of the "odd time" formulation. We show that in a class of gauge theories it is possible to find an "odd time lagrangian" yielding, by a Legendre transformation, an "odd time hamiltonian" which is the minimal solution of the master equation. This constitutes a very simple method of finding the minimal solution of the master equation which is usually a tedious task. To clarify the general procedure we discussed its application to Yang-Mills theory, massive (abelian) theory in Stueckelberg formalism, relativistic particle and the self-interacting antisymmetric tensor field.

<sup>\*</sup>Address after 1 August 1992: ICTP, P.O.Box 586, 34100-Trieste, Italy; bitnet: dayi@itsictp

The formulation of the Batalin-Vilkovisky (BV) method of quantization[1] in terms of a Grassmann odd parameter behaving as time ("odd time") was helpful to derive systematically the "ad hoc" definitions of Batalin and Vilkovisky[2]. In Ref.[2] existence of an appropriate lagrangian for the odd time formulation ("odd time lagrangian") was assumed. Although it is possible to write an odd time lagrangian by using the hamiltonian formalism of the BV method[3], it does not give new hints about finding a solution of the master equation of the lagrangian formalism. In the hamiltonian formalism a general solution of the master equation is available in terms of the Becchi-Rouet-Stora-Tyutin (BRST)-charge which gives a vanishing generalized Poisson bracket with itself.

Recently, the extended form method developed in Ref.[4] was utilized to write actions for topological quantum field theories which lead to solutions of the master equation[5]. In fact, those actions are nothing but odd time lagrangians of the related theories. Inspired by these, we showed that for a class of gauge theories it is possible to write an odd time lagrangian which directly leads to the minimal solution of the master equation in terms of "odd canonical formulation".

First we recall the basic concepts of the odd time formulation of the BV method and show that odd canonical formalism is similar to the normal one. After giving the rules of finding odd time lagrangian beginning from the initial gauge theory action, we discuss the conditions which they should satisfy in order to yield a solution of the master equation. We illustrate the method by its application to Yang-Mills theory, massive (abelian) theory in Stueckelberg formalism, relativistic particle and the self-interacting antisymmetric tensor field.

When we deal with a gauge theory we can introduce the odd time  $\tau_0$  (a parameter possessing odd Grassmann parity), such that the change of a function f by the BRST-charge  $\Omega_{BRST}$ , is written symbolically as

$$\Omega_{BRST} f = \frac{\partial f(\tau_0)}{\partial \tau_0}.$$

We assume that there exists an odd time lagrangian  $L(\phi(\tau_0), \phi(\tau_0))$ , which carries information about the BRST transformations.  $\phi$  includes the original fields of the started gauge theory, and the related ghost fields;  $\dot{\phi}(\tau_0) \equiv \partial \phi(\tau_0)/\partial \tau_0$ . The "odd time canonical momentum" which results from this

lagrangian is

$$\Pi(\tau_0) = \frac{\partial L(\phi(\tau_0), \dot{\phi}(\tau_0))}{\partial \dot{\phi}(\tau_0)}.$$
 (1)

On the cotangent bundle of a supermanifold an odd canonical two form is known to exist when it has an equal number of odd and even coordinates [6]. Thus we can define an "odd Poisson bracket" (antibracket)

$$(f,g) \equiv \frac{\partial_r f}{\partial \phi} \frac{\partial_l g}{\partial \Pi} - \frac{\partial_r f}{\partial \Pi} \frac{\partial_l g}{\partial \phi}, \tag{2}$$

where  $\partial_r$  and  $\partial_l$  indicate the right and the left derivatives. In this phase space odd time evolution is given by the Grassmann-even hamiltonian S:

$$\frac{\partial f}{\partial \tau_0} = (S, f). \tag{3}$$

Thus S must satisfy

$$\frac{\partial S}{\partial \tau_0} = (S, S) = 0. \tag{4}$$

This is the master equation of Batalin and Vilkovisky.

Let us suppose that one can perform a Legendre transformation to find the odd time hamiltonian S, from the lagrangian L. L is taken to be in first-order form (linear in  $\dot{\phi}$ ), and due to the fact that

$$\Pi \equiv \phi^{\star}$$

where "\*" denotes the conjugation mapping fields into antifields and vice versa, one can see that there are two candidates:

$$i) \quad L = \sigma \dot{\phi} - H_{01}(\phi, \sigma), \tag{5}$$

*ii*) 
$$L = \phi \dot{\phi} - H_{02}(\phi)$$
. (6)

Multiplication of fields is defined such that the product possesses a definite ghost number, e.g. ghost number of L must be 0. On the other hand when we deal with the components  $\phi_i$ ,

$$\phi_i \dot{\phi}_j \equiv \phi_j \dot{\phi}_i.$$

The case (6) is similar to the case (5): in terms of the components  $\phi_i$ ,  $\phi\dot{\phi}$  which is defined to have ghost number 0, yields

$$\phi_i \dot{\phi}_j, \ i \neq j.$$

Then it is sufficent to deal only with the former case (5). We will proceed with odd time canonical formalism following the analysis of Dirac for the constrained hamiltonian systems[7]. In the case i) there are two constraints

$$\psi_1 \equiv \Pi_{\phi} - \sigma = 0, \ \psi_2 \equiv \Pi_{\sigma} = 0.$$

The only non-vanishing antibracket of the constraints is

$$(\psi_1, \psi_2) = -1.$$

Define the extended hamiltonian as

$$H_{e1} \equiv H_{01} + \psi_i \lambda_i$$

so that  $\dot{\psi}_i|_{\psi=0}=0$  will lead to

$$\lambda_1 = -\frac{\partial_l H_{01}}{\partial \phi}, \ \lambda_2 = \frac{\partial_l H_{01}}{\partial \sigma}.$$

Therefore we can eliminate the constraints  $\psi_i$ , and the odd time hamiltonian reads  $H_{01}(\phi, \Pi_{\phi})$ . Thus, we conclude that as in the normal canonical formulation one can directly read from the first order lagrangian, the non-vanishing odd Poisson brackets and the related hamiltonian.

Now, we would like to give a method to find the odd time lagrangian, which is in the form discussed above.

To gather the original fields and the ghosts one can extend differential forms to include also the ghost number. This can be achieved by generalizing the exterior derivative as [4]

$$d \to \tilde{d} \equiv d + \partial/\partial \tau_0. \tag{7}$$

In order to utilize this generalization of d, as well as the odd time canonical formulation discussed above, to find the minimal solution of the master equation we should follow the following procedure.

- i) If the started gauge theory is not already first order in d and the terms containing d are not bilinear in fields, one should find an equivalent formulation of it possessing these properties.
- ii) Perform the change given in (7) and generalize the original fields to include also the ghosts and antifields which possess the same order with them in terms of  $\tilde{d}$ . (The ghost content of the theory should be found by analyzing the related gauge invariance and the proper solution condition of Batalin and Vilkovisky.)
- iii) In terms of the Legendre transformation find the related odd time hamiltonian.

Of course, the crucial point is to find the conditions which should be satisfied such that the odd time hamiltonian following from this procedure is a solution of the master equation.

To have a unified notation let us deal with the case given in (6). Thus  $(\phi, \phi) = 1$ , and the master equation can be written as

$$(S,S) = \frac{\partial S}{\partial \phi_i} \frac{\partial S}{\partial \phi_j} g_{ij} = 0,$$

with an appropriate metric  $g_{ij}$ .

The form of the odd time hamiltonian  $H_0$ , is the same with the original lagrangian  $L_0$ , which is gauge invariant

$$\frac{\partial L_0}{\partial \Phi_a} R_a^{\alpha} = 0,$$

where  $\Phi_a$  indicate the original fields;  $\phi = \Phi + \cdots$ , and  $R_a^{\alpha}$  are the gauge generators. Hence the odd time hamiltonian  $H_0$ , will satisfy

$$\frac{\partial H_0}{\partial \phi_i} \delta \phi_i = 0, \tag{8}$$

where

$$\delta \phi_i = \tilde{R}_i^j \phi_i.$$

 $\tilde{R}$  is the generalization of R. If it is possible to write

$$\tilde{R}_i^j \phi_j = g_{ij} \frac{\partial H_0}{\partial \phi_j},\tag{9}$$

one can conclude that the odd time hamiltonian is the minimal solution of the master equation

$$H_0 = S. (10)$$

Of course, there may be some other theories whose odd time hamiltonian satisfies (10), because of some other conditions.

To clarify the procedure outlined above, let us see some applications of it.

## 1) Yang-Mills Theory

It is defined in terms of the second order action (we suppress Tr)

$$L_0 = \int d^4x \ F_{\mu\nu} F^{\mu\nu}, \tag{11}$$

where  $F = d \wedge A + (1/2)A \wedge A$ . The theory given by

$$L = \int d^4x \ (B_{\mu\nu}F^{\mu\nu} - \frac{1}{2}B_{\mu\nu}B^{\mu\nu}), \tag{12}$$

is equivalent to (11) on mass-shell, and moreover it is first order in d. (12) is invariant under the infinitesimal gauge transformations

$$\delta A_{\mu} = D_{\mu} \Lambda , \ \delta B_{\mu\nu} = [B_{\mu\nu}, \Lambda],$$

where D = d + [A, ] is the covariant derivative. They are irreducible, so that for the covariant quantization we need only (in the minimal sector) the ghost field  $\eta$ , which possesses ghost number 1.

Upon performing the change (7) and generalizing the fields of (12) as

$$A \to \tilde{A}, B \to \tilde{B},$$

one obtains

$$\tilde{L} = \int d^4x \ [\tilde{B}(d\tilde{A} + \partial \tilde{A}/\partial \tau_0 + \tilde{A}\tilde{A}) - \frac{1}{2}\tilde{B}\tilde{B}], \tag{13}$$

which is defined to possess 0 ghost number. Order of the extended forms  $\tilde{A}$  and  $\tilde{B}$ , respectively, are 1 and 2, and their first components are  $A_{\mu}$ ,  $B_{\mu\nu}$ . The definition of odd time canonical momentum yields

$$\Pi_{\tilde{A}} = \tilde{B}, \ \Pi_{\tilde{B}} = 0.$$

Thus odd time canonical hamiltonian is

$$H = \int d^4x \left[ -\Pi_{\tilde{A}}(d\tilde{A} + \tilde{A}\tilde{A}) + \frac{1}{2}\Pi_{\tilde{A}}\Pi_{\tilde{A}} \right]. \tag{14}$$

By using the fact that

$$N_{ah}(\phi) + N_{ah}(\phi^*) = -1,$$

where  $N_{gh}$  denotes the ghost number, we write the generalized fields as

$$\tilde{A} = A_{(1+0)} + \eta_{(0+1)} + B_{(2-1)}^{\star},$$
  
 $\Pi_{\tilde{A}} = B_{(2+0)} + A_{(3-1)}^{\star} + \eta_{(4-2)}^{\star},$ 

where the first number in paranthesis is the order of d-forms and the second is the ghost number. Here " $\star$ " indicates the antifields as well as the Hodge-map. Substitution of these in (12) and using the property of the multiplication that the product is different from zero only when its ghost number vanishes, we get

$$H = -\int d^4x \left( B_{\mu\nu} F^{\mu\nu} + B^{\mu\nu} [\eta, B^{\star}_{\mu\nu}] + A^{\star}_{\mu} D^{\mu} \eta + \eta^{\star} [\eta, \eta] - \frac{1}{2} B_{\mu\nu} B^{\mu\nu} \right). \tag{15}$$

We may perform a partial gauge fixing  $B^* = 0$ , and then use the equations of motion with respect to  $B_{\mu\nu}$  to obtain

$$H \to S = -\int d^4x \ (F_{\mu\nu}F^{\mu\nu} + A^*_{\mu}D^{\mu}\eta + \eta^*[\eta, \eta]),$$

which is the minimal solution of the master equation for Yang-Mills theory.

#### 2) Massive Abelian Theory in Stueckelberg Formalism

It is defined in terms of the second order lagrangian

$$L_0 = \int d^4x \left[ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + m^2 (A_\mu - m^{-1} \partial_\mu v) (A^\mu - m^{-1} \partial^\mu v) \right], \tag{16}$$

where  $F = d \wedge A$ . The action linear in d, and equivalent to (16) on mass-shell is

$$L = \int d^4x \left[ \frac{1}{2} B_{\mu\nu} (d \wedge A)^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + m(A_{\mu} - m^{-1} \partial_{\mu} v) K^{\mu} - \frac{1}{2} K_{\mu} K^{\mu} \right]. \tag{17}$$

It is invariant under the gauge transformations

$$\delta A_{\mu} = \partial_{\mu} \Lambda, \quad \delta B_{\mu\nu} = 0,$$
  
 $\delta v = m\Lambda, \quad \delta K_{\mu} = 0,$ 

which are irreducible, so that in the minimal sector there is only one ghost:  $\eta$ . By performing the change (7) and replacing the fields with their generalized ones, we can see that

$$\Pi_{\tilde{A}} = \frac{1}{2}\tilde{B}, \ \Pi_{\tilde{v}} = -\tilde{K}, \ \Pi_{\tilde{B}} = 0, \ \Pi_{\tilde{K}} = 0.$$
(18)

Following the general procedure outlined above we find the odd time hamiltonian as

$$H = \int d^4x \left[ -\Pi_{\tilde{A}} d\tilde{A} + \Pi_{\tilde{A}} \Pi_{\tilde{A}} + m(\tilde{A} - m^{-1} d\tilde{v}) \Pi_{\tilde{v}} + \frac{1}{2} \Pi_{\tilde{v}} \Pi_{\tilde{v}} \right].$$
 (19)

The generalized fields are

$$\tilde{A} = A_{(1+0)} + \eta_{(0+1)} + B_{(2-1)}^{\star}, 
\Pi_{\tilde{A}} = B_{(2+0)} + A_{(3-1)}^{\star} + \eta_{(4-2)}^{\star}, 
\tilde{v} = v_{(0+0)} + K_{(1-1)}, 
\Pi_{\tilde{v}} = K_{(3+0)} + v_{(4-1)}^{\star}.$$

By respecting the rules of multiplication one finds in components

$$H = -L - \int d^4x \ [A^{\star}_{\mu} \partial^{\mu} \eta - m \eta v^{\star}].$$

We can eliminate B and K by using their equations of motion to obtain

$$H \to S = \int d^4x \ [-\frac{1}{2}F_{\mu\nu}^2 - m^2(A_{\mu} - m^{-1}\partial_{\mu}v)^2 - A_{\mu}^{\star}\partial^{\mu}\eta + m\eta v^{\star}].$$

Indeed, this is the minimal solution of the master equation for the theory given by (16).

#### 3) Relativistic Particle

In terms of the canonical variables satisfying the Poisson bracket relation  $\{p_{\mu}, q^{\nu}\} = \delta^{\nu}_{\mu}$ , relativistic particle is given by

$$L_0 = -\int (p \cdot dq - \frac{1}{2}ep \cdot p), \tag{20}$$

where  $dq^{\mu} = \partial_t q^{\mu} dt$ . A variable possesses two different grading: one of them is due to one dimensional manifold of t and the other one is related to spacetime manifold.

(20) is invariant under

$$\delta q^{\mu} = p^{\mu} \Lambda, \ \delta p = 0, \ \delta e = \partial_t \Lambda.$$

Now, we perform the change (7) and generalize the fields as

$$q, e \to \tilde{q}; p \to \tilde{p}.$$

q and e are treated on the same footing due to the fact that there is not de term in (20). The first component of  $\Pi_{\tilde{e}}$  would be vanishing, so that it behaves like a component of a field, i.e.  $\tilde{q}$ . Hence the odd time lagrangian is

$$L = -\int [\tilde{p}d\tilde{q} + \tilde{p}\partial\tilde{q}/\partial\tau_0 - \frac{1}{2}\tilde{q}\tilde{p}^2], \qquad (21)$$

where

$$\begin{array}{rcl} \tilde{q} & = & q^{\mu}_{(1+0+0)} + e_{(0+1+0)} + \eta_{(0+0+1)} + p^{\star \mu}_{(1+1-1)}, \\ \tilde{p} & = & p_{\mu(d-1+0+0)} + q^{\star}_{\mu(d-1+1-1)} + e^{\star}_{(d+0-1)} + \eta^{\star}_{(d+1-2)}. \end{array}$$

The numbers in the paranthesis indicate, respectively, grading due to spacetime, grading due to 1-dimensinal manifold and ghost number.

By calculating the product in components one can show that the odd time hamiltonian yields

$$H = \int dt \ [p \cdot \partial_t q + e^* \partial_t \eta - \frac{1}{2} e p^2 + q^* \cdot p \eta],$$

which is the minimal solution of the master equation for the relativistic particle.

#### 4) The Self-interacting Antisymmetric Tensor Field

The action[8] (we suppress Tr)

$$L_0 = \int d^4x \ [B_{\mu\nu}(d \wedge A + \frac{1}{2}A \wedge A)^{\mu\nu} - \frac{1}{2}A_{\mu}A^{\mu}], \tag{22}$$

is invariant under the transformations

$$\delta B_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} D^{\rho} \Lambda^{\sigma}, \ \delta A_{\mu} = 0,$$

is analysed in terms of the BRST methods in Ref.[9]. If we set  $\Lambda_{\mu} = D_{\mu}\alpha$ , the gauge transformation vanishes on shell  $\delta B|_{F=0} = 0$ . This is a first-stage reducible theory, hence we need to introduce the ghost fields

$$C_0^{\mu}$$
,  $C_1$ ;  $N_{qh}(C_0^{\mu}) = 1$ ,  $N_{qh}(C_1) = 2$ .

By following the general procedure we find the odd time lagrangian

$$L = \int d^4x \left[ \tilde{B}(d\tilde{A} + \partial \tilde{A}/\partial \tau_0 + \tilde{A}\tilde{A}) - \frac{1}{2}\tilde{A}\tilde{A} \right], \tag{23}$$

where the generalized fields are

$$\begin{split} \tilde{A} &= A_{(1+0)} + B_{(2-1)}^{\star} + C_{0(3-2)}^{\star} + C_{1(4-3)}^{\star}, \\ \tilde{B} &= B_{(2+0)} + A_{(3-1)}^{\star} + C_{0(1+1)} + C_{1(0+2)}. \end{split}$$

In terms of the components one can see that the odd time hamiltonian is

$$H = -\int d^4x \left\{ B_{\mu\nu} F^{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} C_0^{\mu} D^{\nu} B^{\star\rho\sigma} + C_1 D^{\mu} C_{0\mu}^{\star} + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} C_1 [B_{\mu\nu}^{\star}, B_{\rho\sigma}^{\star}] - \frac{1}{2} A_{\mu} A^{\mu} \right\}.$$
(24)

This is the minimal solution of the master equation of the theory defined by (22)[9].

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