WHY IS HOMOLOGY SO POWERFUL?

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ABSTRACT. My short answer to this question is that homology is powerful because it computes invariants of higher categories. In this article we show how this true by taking a leisurely tour of the connection between category theory and homological algebra **Dependencies**: This article assumes familiarity with the basics of category theory and the basics of algebraic topology.

1. Extending Eckmann-Hilton

There are many reasons why homology is powerful and this article gives only one perspective. Most of my explanation boils down to the Eckmann-Hilton argument. This is the following theorem:

Theorem 1.1. *Let X be a set with two binary unital operations*

$$+\colon X \times X \to X \text{ and } \circ\colon X \times X \to X$$

such that $\cdot$ is a homomorphism of $+$, i.e.

$$(a+b)\circ(c+d) = a\circ c + b\circ d$$

Then $+$ and $\circ$ are the same operation and this operation is commutative.

The proof of this argument is a lot of fun.

Proof. Let 1 denote the unit for $\cdot$ and 0 denote the unit for $+$. First we will show that $1 = 0$. This follows from the chain of equations

$$\begin{aligned} f &= f + 0 \\ &= (1\circ f) + (0 \circ 1) \\ &= (1+0)\circ(f+1) \\ &= 1\circ(f+1) \\ &= (f+1) \end{aligned}$$

Therefore 0 and 1 are both units for the operation $+$. Because units must be unique, we have that $1 = 0$. Now we show that the two operations are the same

$$\begin{aligned} f\circ g &= (f+0)\circ(0+g) \\ &= (f+1)\circ(1+g) \\ &= (f\circ 1)+(1\circ g) \\ &= f+g \end{aligned}$$

Lastly we show that this operation is commutative

$$\begin{aligned} f \circ g &= (0 + f) \circ (g + 0) \\ &= (0 \circ g) + (f \circ 0) \\ &= (1 \circ g) + (f \circ 1) \\ &= g + f \\ &= g \circ f \end{aligned}$$

□

One way of thinking of a category is as a monoid whose operation is partial. For this reason, the Eckmann-Hilton argument bears on categories equipped with a binary operation. One way to equip categories with operations like this is through internalization.

Definition 1.2. Let V be a category with finite pullbacks. A category C internal to V is a graph in V

$$\mathrm{Mor}\, C \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Ob}\, C$$

along with an identity assigning morphism

$$i\colon \mathrm{Ob}\, C \to \mathrm{Mor}\, C$$

and composition morphism

$$\circ\colon \mathrm{Mor}\, C \times_{\mathrm{Ob}\, C} \mathrm{Mor}\, C \to \mathrm{Mor}\, C$$

commuting suitably with the source and target maps. These morphisms are required to satisfy the axioms of unitality and associativity expressed as commutative diagrams.

Let Vect be the category where objects are vector spaces over the real numbers and morphisms are linear transformations. Categories internal to Vect were first studied by Baez and Crans in *Higher Dimensional Algebra VI: Lie 2-Algebras* [BC04]. Here we explicitly describe what these internal categories are like.

Definition 1.3. A category C internal to Vect is a

- a vector space of objects C_0,
- a vector space of morphisms C_1,
- source and target linear transformations $s, t\colon C_1 \to C_0$,
- an identity assigning linear transformation $i\colon C_0 \to C_1$ and,
- a composition linear transformation $\circ\colon C_1 \times_{C_0} C_1 \to C_1$

satisfying the required axioms. A category internal to Vect is called a **2-vector space**.

For now lets assume that C_1 is $\mathbb{R}^2$. The geometry of the situation suggests a natural categorical structure. For a morphism $f\colon x \to y$ in C_1, we can define its **arrow part**, $\hat{f}$, by

$$\hat{f} = f - i(s(f))$$

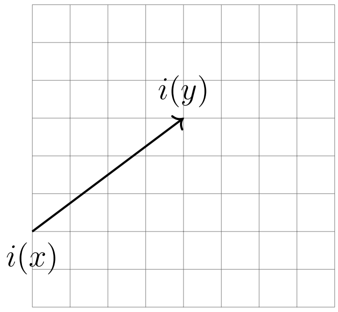
The idea is that this "translates f to 0". Now the source of $\hat{f}$ is 0

$$s(\hat{f}) = s(f - i(s(f)) = s(f) - s(i(s(f))) = s(f) - s(f) = 0$$

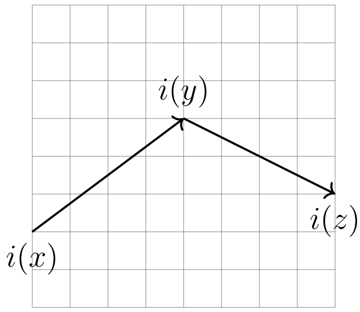
and the target of $\hat{(f)}$ is now given by

$$t(f - i(s(f))) = t(f) - t(i(s(f))) = t(f) - s(f) = y - x$$

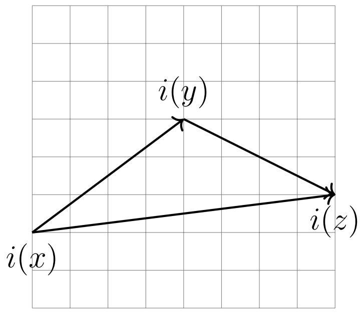
This allows us to think of $f\colon x \to y$ in C is as the vector $\hat{f}$ in the plane pointing from $i(x)$ to $i(y)$.



Note that the arrow part of $i(x)$ and $i(y)$ are 0 because the the source and target maps commute with the identity map. This justifies the lack of an arrow attached to $i(x)$ and $i(y)$ in the above picture. Given another morphism $g\colon y \to z$



we can compose to get a morphism $g \diamond f\colon x \to z$



formally, the composite $g \diamond f$ is given by

$$g \diamond f = \hat{f} + \hat{g} + i(s(f))$$

The arrow part of $g \circ f$ is now given by the sum of the arrow parts of f and of g. The source of $g \diamond f$ is

$$\begin{aligned} s(g \diamond f) &= s(\hat{f} + \hat{g} + i(x)) \\ &= 0 + 0 + s(i(x)) \\ &= x \end{aligned}$$

and the target of $g \circ f$ is given by

$$\begin{aligned} t(g \circ f) &= t(\hat{f} + \hat{g} + t(i(s(f)) \\ &= t(\hat{f}) + t(\hat{g}) + s(f) \\ &= y - x + z - y + x \\ &= z \end{aligned}$$

Note that the last step of this computation requires commutativity of vector sum. If the sum was not commutative, then the above composition would not form the structure of a category on the underlying reflexive graph of C. i does assign elements to their identity morphism under $\diamond$. Composing on the right gives

$$\begin{aligned} f \diamond i(x) &= 0 + \hat{f} + i(s(i(x))) \\ &= (f - i(s(f))) + i(x) \\ &= f \end{aligned}$$

Similarly, composing on the left gives

$$\begin{aligned} i(y) \diamond f &= \hat{i(y)} + \hat{f} + i(x) \\ &= i(y) - i(s(i(y)) + \hat{f} + i(x) \\ &= i(y) - i(y) + \hat{f} + i(x) \\ &= \hat{f} + i(x) \\ &= f - i(x) + i(x) \\ &= f \end{aligned}$$

Therefore the composition rule $\diamond$ defines a composition rule for a category internal to Vect. What's really surprising is that this is the only way to define composition in a 2-vector space.

Proposition 1.4. *Every 2-vector space has composition defined as above.*

Proof. Let C be a 2-vector space whose underlying reflexive graph is

$$C_1 \underset{t}{\overset{s}{\underset{\leftarrow i -}{\rightrightarrows}}} C_0$$

As shown above, the underlying reflexive graph of C can be turned into a category via the rule

$$g \diamond f = \hat{f} + \hat{g} + i(s(f))$$

where $\hat{}$ denotes the *arrow part* of a morphism. Because $\circ$ is a linear transformation it satisfies the law

$$(g \circ f) + (g' \circ f') = (g + g') \circ (f + f')$$

where $f \colon x \to y$, $g \colon y \to z$, $f' \colon x' \to y'$ and $g' \colon y' \to z'$. This equation is called the **interchange law**. The interchange law allows us to apply a version of the Eckmann-Hilton argument to the operations $\diamond$ and $\circ$. Indeed for $f \colon x \to y$ and $g \colon y \to z$ we have that

$$\begin{aligned} g \circ f + 1_y &= g \circ f + 1_y \circ 1_y \\ &= (g + 1_y) \circ (f + 1_y) \end{aligned}$$

by the interchange law. Using commutativity and the interchange law again we get that

$$\begin{aligned}(g+1_y)\circ(f+1_y) &= (1_y+g)\circ(f+1_y)\\ &= f\circ 1_y + 1_y\circ g\\ &= f+g\end{aligned}$$

However, we can decompose this using the arrow parts of f and g to get the $\diamond$ composition:

$$\begin{aligned}f+g &= 1_x+\hat{f}+1_y+\hat{g}\\ &= (1_x+\hat{f}+\hat{g})+1_y\\ &= g\diamond f+1_y\end{aligned}$$

Setting $y=0$ gives that $1_y=0$ as well because i is a linear transformation. Therefore, when $y=0$, the above sequence of equations gives that $g\circ f = g\diamond f$. □

Proposition 1.4 extends to the following equivalence of categories:

Proposition 1.5. *There is an equivalence of categories*

$$2\text{-}\mathsf{Vect} \cong \mathrm{RGraph}(\mathsf{Vect})$$

where $\mathrm{RGraph}(\mathsf{Vect})$ *is the category of reflexive graphs internal to* Vect.

Proof. the left inverse sends reflexive graphs to the category with composition rule given by $\diamond$ and sends morphisms of reflexive graphs to the unique functor which respects this composition rule. The right inverse sends Vect-categories to their underlying reflexive graph and Vect-functors to their underlying morphisms of reflexive graphs. A detailed proof of this proposition can be found in Crans' thesis [Cra04]. □

2. From Categories to Chain Complexes

When I first learned about chain-complexes I didn't understand what sort of thing they were trying to describe. What I was looking for was some down-to-earth explanation of their motivation.

I got a clue about this when I learned the definition of a homotopy between chain maps.

Definition 2.1. Given chain maps $f, g\colon C_{\cdot} \to D_{\cdot}$, a **homotopy** $\alpha_{\cdot}\colon f \Rightarrow g$ is a family of functions α_n of the following form:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}-g_{n+1}"'] & C_n \arrow[r, "\delta_n"] \arrow[dl, "\alpha_n"'] \arrow[d, "f_n-g_n"] & C_{n-1} \arrow[r] \arrow[dl, "\alpha_{n-1}"] \arrow[d, "f_{n-1}-g_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\delta_{n+1}"'] & D_n \arrow[r, "\delta_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

However, the above triangles do not commute. Instead they satisfy the equations

$$f_n - g_n = \delta_{n+1}\circ\alpha_n + \alpha_{n-1}\circ\delta_n$$

I noticed that it reminded me of the definition of natural transformation.

Definition 2.2. Given functors $F, G\colon C \to D$ a **natural transformation** $\alpha\colon F \Rightarrow G$, is a function of the form:

$$\begin{tikzcd}
\mathrm{Mor}\, C \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, shift right, "G_1"'] \arrow[d, shift left, "F_1"] & \mathrm{Ob}\, C \arrow[dl, "\alpha"] \arrow[d, shift right, "G_o"'] \arrow[d, shift left, "F_0"] \\
\mathrm{Mor}\, D \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathrm{Ob}\, D
\end{tikzcd}$$

Not all the triangles here commute, but we do have the equations

$$s \circ \alpha = F_0$$
$$t \circ \alpha = G_0$$

expressing that α offers comparison morphisms between the images of F and G. α must also satisfy a naturality condition expressing compatibility with composition in C and D.

Both of these definitions consist of maps going diagonally and up a dimension and it turns out that the equations they must satisfy are related as well. This relationship is part of a larger story which relates higher categories to chain complexes in order to reason about them more effectively. To see how this works, we need to understand three things:

(1) How categories can be turned into simplicial sets,
(2) How simplical sets can be turned into simplicial vector spaces and,
(3) How simplicial vector spaces can be turned into chain complexes.

Once we understand these three things, we will have a 2-functor

$$\mathrm{Ch}_\bullet\colon \mathsf{2\text{-}Vect} \to \mathrm{Ch}_\bullet(\mathsf{Vect})$$

which connects the disparate worlds of higher category theory and homological algebra. These three things will be addressed by the proceeding three subsections.

2.1. **The Nerve Construction.** An intimidating slogan is that simplical sets, satisfying a certain property, are models of $(\infty, 1)$-categories. A less intimidating version of this is that n-coskeletal simplicial sets, simplicial sets which have interesting simplices only for dimension $k \leq n$, are models of $n-1$-categories [nLab]. An even less intimidating version of this fact is that 2-coskeletal simplicial sets correspond to regular old categories. The nerve construction makes this precise by providing a full and faithful embedding from Cat to sSet whose essential image is the category 2-coskeletal simplicial sets. The idea is that categories correspond to simplicial sets which only have interesting 0-simplices, 1-simplicies, and 2-simplices.

Definition 2.3. For a category C, its nerve is a simplicial set

$$N(C)\colon \Delta^{op} \to \mathsf{Set}$$

with

$$N(C)[n] = \mathsf{Cat}([n], C)$$

i.e. the set of functors from the poset $[n]$ to the category C [nLaa]. In other words, this is the set of composable n-chains of morphisms in C. The boundary map $d_i\colon N(C)[n+1] \to N(C)[n]$ comes in two cases:

- If $i = 0$ or n then it sends an n-chain to $n-1$ chain which forgets the first and the last morphism in the chain respectively.

- Otherwise d_i acts by composing i-th morphism with the $i+1$-th morphism to get an n-chain.

The degeneracy maps

$$s_i \colon N(c)[n] \to N(c)[n+1]$$

turn n-chains into $n+1$-chains by inserting and identity in the i-th spot.

At first this definition seems too intuitive to be the right thing. John Baez said this about the nerve:

> When I first heard of this idea I cracked up. It seemed like an insane sort of joke. Turning a category into a kind of geometrical object built of simplices? What nerve! What use could this possibly be? [Bae98]

Category theory is a field of math where you not only guess answers to questions but also the questions and definitions. With enough experience, certain definitions in category theory will feel inevitable, as the only natural way it could be defined. The following definition feels that way:

Definition 2.4. For a functor between categories $F\colon C \to D$, there is a natural transformation

$$N(F)\colon N(C) \to N(D)$$

defined on 0-cells by the object component of F. For higher dimensional simplices, the map

$$N(F)[n]\colon N(C)[n] \to N(D)[n]$$

sends a commuting n-chain

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n$$

to its image under F

$$F(x_0) \xrightarrow{F(f_0)} F(x_1) \xrightarrow{F(f_1)} \dots \xrightarrow{F(f_{n-1})} F(x_{n-1}) \xrightarrow{F(f_n)} F(x_n)$$

Note that $N(F)$ is well defined because every functor sends commuting diagrams to commuting diagrams. It's a surprising and incredible fact that only the 0, 1 and 2-chains contain all the necessary data of your category.

Theorem 2.5. *The nerve construction*

$$N\colon \mathsf{Cat} \hookrightarrow \mathsf{sSet}$$

is a full and faithful functor whose essential image is the category of 2-coskeletal simplical sets.

To understand this theorem, it will be useful to unpack the definition of the nerve. For a category C, the 0-simplices of $N(C)$ are given by

$$N(C)[0] = \operatorname{Ob} C$$

and the 1-simplices of $N(C)$ are given by

$$N(C)[1] = \operatorname{Mor} C.$$

The 2-cells are more interesting. They can be thought of as commuting triangles

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & x_2 \end{tikzcd}$$

These encode the relations between morphisms. Just like how groups and other algebraic gadgets can be described using generators and relations, categories can be described with two things

- its data i.e. objects and morphisms and,
- its relations, i.e. equations between morphisms and their compositions.

This statement can be justified by the fact that Cat is the category of algebras for the "free-category on a directed graph" monad. The relevant consequence of this is that every category can be described as a graph homomorphism

$$A\colon F(X) \to X$$

where $F(X)$ is the underlying graph of the free category on some graph X. Here, the map A picks out relations between arbitrary compositions of morphisms in a category whose underlying graph is given by X. So, because the 0,1, and 2-simplices of $N(C)$ contain the objects, morphisms, and relations of C, it makes at least intuitive sense that these simplices capture all the essential information of C. A 3-simplex of $N(C)$ is a commuting square

$$\begin{tikzcd} x_0 \arrow[r, "f"] \arrow[d, "k"'] & x_1 \arrow[d, "g"] \\ x_3 & x_2 \arrow[l, "h"] \end{tikzcd}$$

However, instead we could write this as two commuting triangles

$$\begin{tikzcd} & x_1 \arrow[dr, "h"] & \\ x_0 \arrow[ur, "g\circ f"] \arrow[rr, "k"'] & & x_2 \end{tikzcd} \qquad \begin{tikzcd} & x_1 \arrow[dr, "h\circ g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "k"'] & & x_2 \end{tikzcd}$$

Actually, these two triangles are the two inner boundaries of the above 3-simplex according to the nerve construction. In this way, every 3-simplex is redundant because the data it represents is already contained as 2-simplices. Note that this phenomenon already occurs with groups. It is the fact that every product of three elements $x_1 x_2 x_3$ can be turned into two different products of two elements, $(x_1 x_2)x_3$ and $x_1(x_2 x_3)$, by adding parentheses. Therefore groups only need a binary operation rather than an n-ary operation for every natural number n.

2.2. **Simplicial Vector Spaces.** Simplicial vector spaces are just like simplicial sets, except that the n-simplices form a vector space rather than a set.

Definition 2.6. A **simplicial vector space** is a functor

$$\Delta^{op} \to \mathsf{Vect}$$

To turn a simplicial set

$$\Delta^{op} \to \mathsf{Set}$$

we will compose it with a reasonable functor

$$F\colon \mathsf{Set} \to \mathsf{Vect}$$

The functor F that we use will be the following:

Proposition 2.7. *Let*

$$U\colon \mathsf{Vect} \to \mathsf{Set}$$

be the forgetful functor which sends every vector space to its underlying set. Then U has a left adjoint

$$F\colon \mathsf{Set} \to \mathsf{Vect}$$

called the **free vector space functor**. *For a set X,*

$$F(X) = \mathbb{R}^X$$

and for a function $f\colon X \to Y$,

$$F(f)\colon \mathbb{R}^X \to \mathbb{R}^Y$$

is the unique linear transformation which extends f.

Roughly, F is a reasonable functor to choose because we want it to preserve the information in each simplicial set as faithfully as possible. For a set X, $F(X)$ is a vector space which

- includes the elements of X and,
- only includes other elements if they are necessary to make $F(X)$ into a vector space. These include all sums and scalar multiples of elements in X without any relations.

This is perfect because we're not doing anything too fancy. To summarize:

Definition 2.8. There is a functor

$$(-) \circ F\colon \mathsf{sSet} \to \mathsf{sVect}$$

which composes every simplicial set with the free vector space on a set functor. For a natural transformation of simplicial sets $\alpha\colon X \to Y$, this functor whiskers the natural transformation with the functor F.

2.3. **The Dold-Kan Correspondence.** To complete our quest of turning categories into chain complexes, we have to turn simplical sets into chain complexes. This is done by taking an alternating sum of the boundary maps.

Definition 2.9. Given a simplicial vector space $X\colon \Delta^{op} \to \mathsf{Vect}$, the **alternating face map chain complex** of X is chain complex

$$\dots \xrightarrow{\delta_{n+1}} X([n]) \xrightarrow{\delta_n} X([n-1]) \xrightarrow{\delta_{n-1}} \dots$$

The boundary maps are defined by

$$\delta_n := \sum_{i=1}^{n} (-1)^i d_i$$

where the $d_i\colon X([n]) \to X([n-1])$ are the face maps of X.

This gives a functor

$$A\colon \mathsf{sVect} \to \mathrm{Ch}_\bullet(\mathsf{Vect})$$

in a natural way. For a natural transformation $\alpha\colon X \to Y$ between simplicial vector spaces, there is a chain map

$$A(\alpha)\colon A(X) \to A(Y)$$

whose n-th component is given by $\alpha_{X[n]}$. Usually people don't stop here. The alternating face map chain complex can be **normalized** by quotienting each vector space of n-chains by the subspace of degenerate simplices. The composition of normalization and alternating face map chain complex is called the Dold-Kan correspondence. It is famous because it forms an equivalence of categories between simplical vector spaces and chain complexes.

2.4. **Natural Transformations to Homotopies.** Now let's put all this together. Given a category

$$C = C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$$

we can form a simplical vector space $N(C)\colon \Delta^{op} \to \mathsf{Vect}$ with

$$N(C)([0]) = F(C_1)$$

and

$$N(C)([1]) = F(C_0).$$

This can be turned into a chain complex

$$\ldots \longrightarrow F(C_1) \xrightarrow{F(s)-F(t)} F(C_0) \xrightarrow{0} 0 \longrightarrow \ldots$$

Although we didn't say it, everything here is 2-functorial, i.e. it defines a 2-functor

$$\mathrm{Ch}\colon \mathsf{Cat} \to \mathrm{Ch}.(\mathsf{Vect}).$$

This means that for a natural transformation

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, shift right, "G_1"'] \arrow[d, shift left, "F_1"] & C_0 \arrow[d, shift right, "G_o"'] \arrow[d, shift left, "F_0"] \arrow[dl, "\alpha"] \\
D_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & D_0
\end{tikzcd}$$

we get a homotopy between chain maps as follows:

$$\begin{tikzcd}
\ldots \arrow[r] & N(C)[2] \arrow[r, "\delta_2"] \arrow[d, dashed, "N(F)_{[2]}-N(G)_{[2]}"'] & F(C_1) \arrow[r, "s-t"] \arrow[d, dashed, "F_1-G_1"] \arrow[dl, "\alpha_2"'] & F(C_0) \arrow[r, "0"] \arrow[d, dashed, "F_0-G_0"] \arrow[dl, "\alpha_1"'] & 0 \arrow[r] \arrow[d, "0"] \arrow[dl, "0"'] & \ldots \\
\ldots \arrow[r] & N(D)[2] \arrow[r, "\delta_2"'] & F(D_1) \arrow[r, "s'-t'"'] & F(D_0) \arrow[r, "0"'] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

α_1 is defined to be the natural transformation α. Recall that because α is a natural transformation, it satisfies the equations

$$s \circ \alpha = F_0 \text{ and } t \circ \alpha = G_0$$

Therefore,

$$\begin{aligned}(s'-t')\circ\alpha+0\circ 0&=(s'-t')\circ\alpha\\&=s'\circ\alpha-t'\circ\alpha\\&=F_0-G_0\end{aligned}$$

so these two squares do indeed satisfy the equations for a homotopy of chain maps. This fact boils down to the fact that components of a natural transformations have the right source and target.

The homotopy condition for the two squares on the left expresses the fact that natural transformations respect composition. For a morphism $f\colon x\to y$ in C, the naturality square is a 3-simplex in $N(D)$

$$\begin{tikzcd}F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha_x"'] & F(y) \arrow[d, "\alpha_y"] \\ G(x) \arrow[r, "G(f)"'] & G(y)\end{tikzcd}$$

The problem is that α_1 must send this to a sum of triangular 2-simplices. Luckily,the above square has two nice triangles as boundaries given by collapsing the edges with composition. These are

$$A=\begin{tikzcd}F(x) \arrow[r, "F(f)"] \arrow[dr, "G(f)\circ\alpha_x"'] & F(y) \arrow[d, "\alpha_y"] \\ & G(y)\end{tikzcd}$$

and

$$B=\begin{tikzcd}F(x) \arrow[d, "\alpha_x"'] \arrow[dr, "\alpha_y\circ F(f)"] & \\ G(x) \arrow[r, "G(f)"'] & G(y)\end{tikzcd}$$

To avoid choosing one triangle we take the second one and subtract the first. Indeed the map

$$\bar{\alpha}\colon \mathrm{Mor}\, C\to N(D)[2]$$

sends a morphism $f\colon x\to y$ in C to the difference $B-A$. It's a fun exercise to verify that this satisfies the chain homotopy condition on f. Before reading this computation it may help to recall the definition of δ_2^D using Definitions 2.3 and 2.9.

$$\begin{aligned}\delta_2^D\circ\alpha_2(f)+\alpha_1\circ(t-s)(f)&=\delta_2^D\circ\alpha_2(f)+\alpha_1(y)-\alpha_1(x)\\&=\delta_2^D(B-A)+\alpha_1(y)-\alpha_1(x)\end{aligned}$$

Using the definition δ_2^D we get that

$$\begin{aligned}(\alpha_x-\alpha_y\circ F(f)+G(f))-(F(f)-G(f)\circ\alpha_x+\alpha_y)+\alpha_y-\alpha_x&\\=G(f)-F(f)+G(f)\circ\alpha_x-\alpha_y\circ F(f)&\end{aligned}$$

However, because the naturality square for f commutes we have that

$$G(f)\circ\alpha_x=\alpha_y\circ F(f)$$

and

$$G(f)\circ\alpha_x-\alpha_y\circ F(f)=0$$

Applying this to the above equation gives that

$$\delta_2^D \circ \alpha_2(f) + \alpha_1 \circ (t-s)(f) = G(f) - F(f)$$

and this is the homotopy condition for chain maps.

3. Going Backwards

So far we have seen a nice way of interpreting categories as chain complexes using the 2-functor

$$N \circ D \circ \bar{F} \colon \mathsf{Cat} \to \mathrm{Ch}_\bullet(\mathsf{Vect})$$

Mathematicians are very suggestable. A way of interpreting A's as B's suggests that B's can be thought of as a generalization of the A's. With this logic, chain complexes generalize categories. In what way do they do this? The chain complexes which come from categories contain interesting data in only the 0, 1, and 2-simplices.

Theorem 3.1. *The homology groups of a category are trivial for* $n \geq 2$

Sketch. This follows from the fact that the nerve of a category is coskeletal. This means that that every $(n-1)$-simplicex which forms the boundary of an n-simplex is uniquely filled by an n-simplex. Because every set of n-simplices is uniquely filled for $n > 2$, it is always the boundary of an $n+1$-simplex and therefore always equal to 0 in the n-th homology. □

Therefore chain complexes generalize categories by containing higher dimensional content, i.e. nontrivial homology for $n > 2$. There is another notion of category with "higher dimensional content" called n-categories for which a good exposition can be found in [Bae97]. n-categories are notoriously complex. Todd trimble in 1996 drafted a 51-page definition of a weak 4-category [Tri06]

> In 1995, at Ross Street's request, I gave a very explicit description of weak 4-categories, or tetracategories as I called them then, in terms of nuts-and-bolts pasting diagrams, taking advantage of methods I was trying to develop then into a working definition of weak n-category. Over the years various people have expressed interest in seeing what these diagrams look like – for a while they achieved a certain notoriety among the few people who have actually laid eyes on them (Ross Street and John Power may still have copies of my diagrams, and on occasion have pulled them out for visitors to look at, mostly for entertainment I think).

This quote is referring to weak n-categories, strict n-categories have much simpler axioms as every composition operation is associative strictly. Regardless, classifying and understanding n-categories is a large and arduous mathematical quest which is relevant to many subjects in math. For example n-categories can be used to do rewriting theory in a more sophisticated way [FM18].

It is a theorem of the heart that Proposition 1.5 extends as follows. Maybe it has been proved somewhere but I do not know where.

Hypothesis 3.2. There is a suitable equivalence

$$\mathsf{nGraph}(\mathsf{Vect}) \cong \mathsf{nCat}(\mathsf{Vect})$$

between n-dimensional graphs internal to Vect and n-categories internal to Vect.

An n-dimensional graph should be something like a graph with edges between edges, and edges between those edges ad infinitum until you get n-levels deep. What this equivalence would say is that every n-dimensional graph already has an intricate network of interacting composition operations built into it in a unique way. Homology is so powerful because it allows you to reason about these complicated networks of composition just by thinking about vector spaces and linear maps.

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