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Integral representations of thermodynamic 1PI Green functions in the world-line formalism

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Abstract

The issue discussed is a thermodynamic version of the Bern-Kosower master amplitude formula, which contains all necessary one-loop Feynman diagrams. It is demonstrated how the master amplitude at finite values of temperature and chemical potential can be formulated within the framework of the world-line formalism. In particular we present an elegant method how to introduce a chemical potential for a loop in the master formula. Various useful integral formulae for the master amplitude are then obtained. The non-analytic property of the master formula is also derived in the zero temperature limit with the value of chemical potential kept finite.

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1. Introduction

Based on the field theory limit of string theory (with infinite string tension limit), a very elegant method was invented several years ago: the Bern-Kosower (BK) rules to obtain one-loop gluon scattering amplitudes in a compact form [1] (see also [2]). For example, five gluon scatterings are efficiently calculated by using these rules [3], and various field theory limits have been studied along the line of the BK formalism: perturbative gravity [4], super Yang-Mills theory [5], bosonic string theory approach [6], and multi-loop generalizations [7, 8, 9].

The most important and conspicuous point in the BK formalism is that all Feynman diagrams are included in a single master integrand. In this formalism, we hence do not compute the loop integrals and the Dirac traces of respective Feynman diagrams. These facts can also be achieved by another approach, called the world-line formalism, which reformulates Feynman (field theory) amplitudes similar to string theory amplitudes; i.e. the field theory amplitudes can generally be obtained as a path integral average of vertex operators [10]-[17].

In fact, there have been established many examples: ϕ^3 theory [10, 11], QED [12, 13], axial vector and pseudo-scalar couplings [14, 15], Yang-Mills theory [16, 17] (see also [18, 19]), and more references can be found in [20]. Both world-line and string theory methods help each other, and are useful to get an insight for solving mutually related problems in each own way; for example, this viewpoint has been very useful for specifying pinching limits and corners of string moduli in the multi-loop analysis [8]. It is very interesting that these two methods, which entirely differ from the conventional Feynman diagram technique, improve the computational efficiency for obtaining Feynman amplitudes.

However, compared to these developments, their thermodynamic versions have not been studied very much [22]-[26] from the viewpoint of general formulation for constructing a master amplitude formula. In particular, there has been no general and convenient method how to introduce a chemical potential for a loop in the world-line formalism. Before stepping in an unexplored calculation by using the formalism, it is important to establish a definite and universal foundation in the first place. In this paper, therefore as a basic step on the thermodynamic generalization of world-line field theory, we present a universal and fundamental prescription of one-loop N-point amplitudes at finite values of temperature and chemical potential without

help of any standard calculation. We shall study the amplitude of a particular form (the master formula), where the loop integration and the Matsubara summation are *a priori* finished and Feynman's parameter integrals are only left — laying emphasis on the point that we never mean the Feynman integrals as a single Feynman diagram, but as a sum of *all diagrams*. It is certainly nontrivial to introduce a chemical potential (as well as a temperature) with keeping this advantageous point pertaining to the master formula intact.

We address the following points in the matter of the thermodynamic generalization (at finite values of the temperature β^{-1} and the chemical potential μ for a loop). First, we present a formulation of the thermodynamic amplitudes along the same line as the non-thermodynamic world-line formalism. In particular, the way of introducing the chemical potential is a non-trivial problem. Since we do not introduce any idea of the continuous or discrete momentum integration/summation, we have to find an alternative to the shift of internal discrete momenta:

$$\omega_n \rightarrow \omega_n + i\mu$$
 (1.1)

This situation may be understood in the following way: In the standard method, the inverse temperature β is introduced by summing up all topological different S^1 paths along the zeroth component (imaginary time) direction. On the other hand in the world-line formalism, this procedure modifies the path integral of a corresponding periodic world-line field $x^0(\sigma)$; $0 \le \sigma \le 1$, into a summation of the path integrals with $x^0(\sigma)$ shifted by

$$x^0(\sigma) \rightarrow x^0(\sigma) + n\beta\sigma$$
 (1.2)

Although the radius β of S^1 and the world-line circumference (the unity) have nothing to do with each other, the shift (1.2) involves the world-line coordinate σ as well. In this sense, the way of prescribing the temperature and hence of the chemical potential become different from the standard Matsubara formalism. To introduce the chemical potential, one may of course transform a world-line amplitude formula to the Feynman-Matsubara form, and then resume the world-line form after some efforts to apply the shift (1.1). However, such a calculation does not utilize the merits of the master formula at all, and there should be a more direct and transparent method to introduce μ within the world-line formalism (without tracing back and referring any internal loop calculation).

To this end, we propose a new rule, instead of (1.1), to introduce the internal chemical potential. It is simply done by applying the new shift procedure

$$\bar{\omega} \rightarrow \bar{\omega} + i\mu$$
, (1.3)

where $\bar{\omega}$ is an average of the zeroth components of external continuous/discrete momenta k_j^0 ; $j=1,2,\cdots N$, with the summation weights σ_j (the local coordinates of the external legs on the closed loop). The prescription (1.3) is neither conceivable nor explicable from the standard method (1.1), because ω_n is the internal momentum while $\bar{\omega}$ concerns the external one. The parameter $\bar{\omega}$ will easily be identified in due course, if we adopt a statistical parameter to discern between boson and fermion loops when summing up the S^1 paths: This parameter dependence should vanish at the $\beta \to \infty$ limit as expected in the non-thermodynamic world-line formalism.

Another non-triviality in this formalism is how to derive a non-analytic property at $\beta = \infty$ with finite μ from the master amplitude formula. Since the non-analytic property can be derived from the $\beta = \infty$ limits of pure thermodynamic parts, we first have to separate a pure thermodynamic part $\tilde{\Gamma}_N^{\beta\mu}$ from a full thermodynamic amplitude $\Gamma_N^{\beta\mu}$. If the master integrand of $\Gamma_N^{\beta\mu}$ is composed only of the Jacobi Θ -function, the story is simple as expected. However, in a more complicated case like a photon scattering, the master integrand is not such a simple form but a product of a β - and μ -dependent operator $\mathcal{V}_{\beta\mu}$ and the Jacobi Θ -function part $\mathcal{K}_{\beta\mu}$. The pure thermodynamic parts of these quantities, $\tilde{\mathcal{V}}_{\beta\mu}$ and $\tilde{\mathcal{K}}_{\beta\mu}$, can easily be separated from their original full quantities, but we show that the pure thermodynamic part $\tilde{\Gamma}_N^{\beta\mu}$ is non-trivially given by $\mathcal{V}_{\beta\mu} \times \tilde{\mathcal{K}}_{\beta\mu}$ against naive expectation. (The simple arithmetical splittings indicate one more contribution, however we shall show that it vanishes). Separating the $\tilde{\Gamma}_N^{\beta\mu}$ in this way, we then analyze the non-analytic property of the master amplitude at zero temperature.

This paper is organized as follows. In Section 2, we explain our notations, definitions and the general structure of master amplitudes at zero temperature. In Section 3, we derive a set of general formulae for a master amplitude at finite β and μ parts by parts: path integral normalization and (scalar) kinematical factor in Section 3.1, and reduced kinematical factor in Section 3.2. In Section 3.3, we show a general master formula for the purely thermodynamic part of the full amplitude $\Gamma_N^{\beta\mu}$, and also prove the above statement, i.e., $\tilde{\Gamma}_N^{\beta\mu} \sim \mathcal{V}_{\beta\mu} \times \tilde{K}_{\beta\mu}$. The replacement rule (1.3) is verified in Appendices A-C from the viewpoints of both Feynman

rule's calculation and the world-line formalism. In Sections 4 and 5, for arbitrary N, we derive various integral formulae by analyzing two kinds of Θ -function representations, and check their consistency. Several explicit results are also presented up to N=5. In Section 6, we derive the non-analytic property of the master formula in the zero temperature limit with the chemical potential kept finite. Section 7 contains summary and conclusions.

2 Notations and definitions

First, we summarize the general structure of the one-loop N-point master amplitudes ¹ in the non-thermodynamic world-line formalism. We refer the reader to Refs. [12, 16, 20] for more details. The master amplitude of general form (for N external momenta k_j^{μ} ; $j=1,2,\cdots,N$; $\mu=0,1,\cdots,D-1$) is written in terms of the closed path integrals of bosonic $x^{\mu}(\sigma)$ and fermionic $\psi^{\mu}(\sigma)$ world-line fields as follows:

$$\Gamma_N = \frac{1}{2} \int_0^\infty \frac{ds}{s} \oint \mathcal{D}x^{\mu}(\sigma) \mathcal{D}\psi^{\mu}(\sigma) e^{-\int_0^1 \mathcal{L}(\sigma)d\sigma} \prod_{j=1}^N V_j$$
 (2.1)

with the world-line Lagrangian and the vertex operators

$$\mathcal{L}(\sigma) = \frac{1}{4s} \left(\frac{\partial x^{\mu}(\sigma)}{\partial \sigma} \right)^2 + \frac{1}{2} \psi^{\mu}(\sigma) \frac{\partial}{\partial \sigma} \psi_{\mu}(\sigma) + sm^2 , \qquad (2.2)$$

$$V_j = s \int_0^1 d\sigma \, v_j[x(\sigma), \psi(\sigma)] \, e^{ik_j \cdot x(\sigma)} ; \qquad j = 1, 2, \dots, N , \qquad (2.3)$$

where $k_j \cdot x$ stands for the Lorentz contraction, and we often omit the Lorentz indices as long as obvious. The zero mode integral of the bosonic path integral should be excluded [18]. The explicit form of v_j depends on what particle is inserted in the loop as an external leg; for example, $v_j = 1$ for ϕ^3 theory, and is in Eq.(C.2) for the photon vertex case. Note that our world-line coordinate σ is dimensionless, and is related to the standard notation [20] by scaling $\tau = s\sigma$. For the path integral average of a general quantity F,

$$\langle F(x,\psi) \rangle \equiv \mathcal{N}^{-1} \oint \mathcal{D}x \mathcal{D}\psi e^{-\int_0^1 \mathcal{L}(\sigma)d\sigma} F(x,\psi) ,$$
 (2.4)

one may use the Wick contractions with ²

$$\langle x^{\mu}(\sigma_1) x^{\nu}(\sigma_2) \rangle = -sg^{\mu\nu}G(\sigma_1, \sigma_2) , \qquad (2.5)$$

¹We assume that a particle change does not occur while circulating along the loop.

² The Euclidean metric is given by $g^{\mu\nu} = -\delta^{\mu\nu}$.

$$\langle \psi^{\mu}(\sigma_1)\psi^{\nu}(\sigma_2)\rangle = \frac{1}{2}g^{\mu\nu}\operatorname{sign}(\sigma_1 - \sigma_2),$$
 (2.6)

where \mathcal{N} is the path integral normalization

$$\mathcal{N} = \oint \mathcal{D}x \mathcal{D}\psi e^{-\int_0^1 \mathcal{L}(\sigma)d\sigma} = e^{-sm^2} (4\pi s)^{-D/2} , \qquad (2.7)$$

and G the bosonic world-line correlator

$$G(\sigma_i, \sigma_j) = |\sigma_i - \sigma_j| - (\sigma_i - \sigma_j)^2 \equiv G_{ij} . \tag{2.8}$$

Performing the path integrals (or Wick contractions), we arrive at the following master amplitude formula:

$$\Gamma_N = c \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \cdots \int_0^1 d\sigma_N \int_0^\infty ds \, s^{N-1} \mathcal{V} \times \mathcal{K} , \qquad (2.9)$$

where the constant c depends on a theory ³ (particle circulating in a loop); For example,

$$c = \begin{cases} \frac{1}{2} & \text{for a (neutral) scalar loop} \\ -\frac{1}{2} \text{tr}[1] & \text{for a fermion loop,} \end{cases}$$
 (2.10)

where the tr[1] expresses the trace of a unit matrix in the D-dimensional gamma matrix space. For a gluon loop, c is no longer a simple constant [16, 17]. The quantity K, which we shall call the *kinematical factor* (multiplied by the normalization N), is defined by the world-line path integral average of N scalar vertex operators [10, 16]

$$\mathcal{K} \equiv \mathcal{N} < \prod_{j=1}^{N} e^{ik_j x(\sigma_j)} > = \oint \mathcal{D}x^{\mu} \mathcal{D}\psi^{\mu} \left(\prod_{j=1}^{N} e^{ik_j \cdot x(\sigma_j)} \right) e^{-\int_0^1 \mathcal{L}(\sigma) d\sigma} . \tag{2.11}$$

The \mathcal{V} stands for an effective vertex function, which is obtained by

$$\mathcal{V} = \frac{\langle \prod_{j=1}^{N} v_j \exp[ik_j \cdot x(\sigma_j)] \rangle}{\langle \prod_{j=1}^{N} \exp[ik_j \cdot x(\sigma_j)] \rangle}.$$
 (2.12)

We shall refer to this quantity as the vertex structure function/operator, which corresponds to the quantities called the reduced kinematical factor [1] or the generating kinematical factor [16]. At this stage, the \mathcal{V} is still a function of s and σ_j $(j = 1, 2, \dots, N)$, however it will be generalized to an operator in the thermodynamic case.

³ It is also related to an over-counting factor of σ -integration regions [21].

If the vertex structure function can be expanded in the form

$$\mathcal{V} = \sum_{l \in \mathbf{Z}} a_l \, s^{l-N} \, \exp[-sb_l] \,, \tag{2.13}$$

where the coefficients a_l and b_l may not depend on s, but on σ_j , the amplitude (2.9) is obtained as the sum of the 'partial' amplitudes

$$\mathcal{A}_{l} \stackrel{\text{def.}}{=} \int_{0}^{\infty} ds s^{l-1} \mathcal{K} = \int_{0}^{\infty} ds s^{l-1} \mathcal{N} < \prod_{j=1}^{N} e^{ik_{j} \cdot x(\sigma_{j})} > , \qquad (2.14)$$

with shifting m^2 to $m^2 + b_l$.

In the case of finite β , we assume the zeroth components of the external momenta k_j^{μ} to be the bosonic Matsubara frequencies ⁴

$$k_j^0 = \omega_{k_j} = \frac{2\pi}{\beta} n_j , \qquad (2.15)$$

as well as

$$k_j \cdot x \equiv \omega_{k_j} x^0 - \vec{k}_j \cdot \vec{x} \ . \tag{2.16}$$

We use the notations for the counterparts of Eqs. (2.9), (2.11), (2.7), (2.12), (2.14)

$$\Gamma_N^{\beta}, \quad \mathcal{K}_{\beta}, \quad \mathcal{N}_{\beta}, \quad \mathcal{V}_{\beta}, \quad \mathcal{A}_l^{\beta}$$
 (2.17)

In the case of finite μ and β , we denote

$$\Gamma_N^{\beta\mu}, \quad \mathcal{K}_{\beta\mu}, \quad \mathcal{N}_{\beta\mu}, \quad \mathcal{V}_{\beta\mu}, \quad \mathcal{A}_I^{\beta\mu}$$
 (2.18)

We explain how to define these quantities in the next section.

3 The thermodynamic generalization

3.1 The kinematical factor $\mathcal{K}_{\beta\mu}$ and the normalization $\mathcal{N}_{\beta\mu}$

We take a two-step procedure to introduce the temperature and the chemical potential. As the first step, we consider the $\mu=0$ case. Basically we follow the same method as discussed in

⁴ We do not consider the fermionic external states, since the bosonic external states are only well-formulated in the world-line formalism. However one may formally generalize to the fermionic states.

Refs. [23]-[26], and we start with the following slightly general definition:

$$\Gamma_N^{\beta} \stackrel{\text{def.}}{=} \sum_{n=-\infty}^{\infty} e^{\frac{2n\pi i}{\epsilon}} \Gamma_N \Big|_{x^0(\sigma) \to x^0(\sigma) + n\beta\sigma} , \qquad (3.1)$$

where ϵ is a constant related to the statistics of the loop. For example, the $\epsilon = 2$ case corresponds to a fermion loop, and the $\epsilon = \infty$ case to a bosonic loop. Without specifying the value of ϵ , we deal with both cases simultaneously (formally fractional statistics as well). The Γ_N on the right-hand side (r.h.s.) of the above formula denotes the path integral representation (2.1), and the replacement (1.2) should be applied to the x^0 fields for all variables σ_j . In a nutshell, we have only to replace the bosonic path integral in the following way:

$$\oint \mathcal{D}x^{\mu} \quad \to \quad \oint_{\beta} \mathcal{D}x^{\mu} \equiv \sum_{n=-\infty}^{\infty} e^{\frac{2n\pi i}{\epsilon}} \oint_{x^{0}(\sigma) \to x^{0}(\sigma) + n\beta\sigma} \quad .$$
(3.2)

For simplicity, let us consider the $\mathcal{V}=1$ case, or the 'partial' amplitude \mathcal{A}_N (the l=N term in (2.13)). In this case, we have only to generalize the kinematical factor \mathcal{K} to the finite temperature version \mathcal{K}_{β} ;

$$\mathcal{K}_{\beta} \stackrel{\text{def.}}{=} \sum_{n=-\infty}^{\infty} e^{\frac{2n\pi i}{\epsilon}} \mathcal{N} < \prod_{j=1}^{N} e^{ik_{j} \cdot x(\sigma_{j})} > \Big|_{x^{0}(\sigma) \to x^{0}(\sigma) + n\beta\sigma}$$
(3.3)

$$= (4\pi s)^{\frac{-D}{2}} e^{-sM^2} \Theta_3(\frac{1}{\epsilon} + \frac{\beta \bar{\omega}}{2\pi}, i \frac{\beta^2}{4\pi s}) , \qquad (3.4)$$

where the definition of $\Theta_3(v,\tau)$ is

$$\Theta_3(v,\tau) = \sum_{n=-\infty}^{\infty} e^{n^2 \tau \pi i} e^{2nv\pi i} , \qquad (3.5)$$

and we have introduced the following two s-independent quantities:

$$M^2 = m^2 - \sum_{i < j,=1}^{N} k_i \cdot k_j G_{ij} , \qquad (3.6)$$

$$\bar{\omega} = \sum_{j=1}^{N} \sigma_j \omega_{k_j} . \tag{3.7}$$

Here the two-point correlator (world-line Green function) G_{ij} , given by Eq.(2.8), looks different from the one used in Ref.[23], however the value of M^2 does not differ under the condition of momentum conservation regarding the external legs. The sign of M^2 seems not always to be positive in spite that $0 \le G_{ij} \le 1/2$. We then assume $M^2 \ge 0$ in the following discussion, choosing the off-shell symmetric point (satisfying the momentum conservation constraint)

$$k_i k_j = \delta_{ij} k^2 + (\delta_{ij} - 1) \frac{1}{N - 1} k^2 . {(3.8)}$$

For the $\mathcal{V} \neq 1$ case, we have to apply Eq.(3.2) to the \mathcal{V} and \mathcal{K} parts at the same time, however the things are rather straightforward. We refer the reader to Appendix C for an example.

The second step is to introduce the chemical potential for the internal loop. One may do it with applying the Jacobi transformation to the quantity \mathcal{K}_{β} ; i.e., rewriting Eq.(3.4) in such a way to revive the discrete summation over internal Matsubara frequencies ω_n , one may perform the replacement (1.1) (see [25] for more details). However, this is a roundabout way, since one has to make the Matsubara summation re-appear in spite of dealing with the world-line formulation, where the integration and summation of the loop are already performed. Instead, we propose a much simpler and direct alternative method to steer clear of this problem. It can be promptly done by the replacement (1.3),

$$\bar{\omega} \rightarrow \Omega \equiv \bar{\omega} + i\mu \ . \tag{3.9}$$

This shift looks similar to the one (1.1), however we stress that our shifting parameter is not the internal frequency but an average of external ones (cf. Eq. (3.7)). In this sense, this prescription is nontrivial. For a rigorous reader, we put a justification of this replacement in Appendix A (the case of $\mathcal{V}=1$), and also refer to Appendix B for more complicated case (the $\mathcal{V}\neq 1$ case). After all, the general path integral average for finite μ can be obtained in a couple of simple replacements:

$$\oint \mathcal{D}x^{\mu} \quad \to \quad \oint_{\beta} \mathcal{D}x^{\mu} \quad \to \quad \oint_{\beta\mu} \mathcal{D}x^{\mu} \equiv \oint_{\beta} \mathcal{D}x^{\mu} \Big|_{\bar{\omega} \to \Omega} .$$
(3.10)

Note that the fermion path integral does not change by any means.

Let us write down the thermodynamic kinematical factor $\mathcal{K}_{\beta\mu}$ and the normalization factor $\mathcal{N}_{\beta\mu}$. Applying the above replacement to Eq.(3.4), we thereby yield the desired thermodynamic kinematical factor for finite β and μ :

$$\mathcal{K}_{\beta\mu} \stackrel{\text{def.}}{=} \operatorname{Eq.}(3.4) \Big|_{\bar{\omega} \to \Omega}$$

$$= (4\pi s)^{\frac{-D}{2}} e^{-sM^2} \Theta_3 (i \frac{\beta \bar{\mu}}{2\pi}, i \frac{\beta^2}{4\pi s}) , \qquad (3.12)$$

$$= (4\pi s)^{\frac{-D}{2}} e^{-sM^2} \Theta_3(i\frac{\beta\bar{\mu}}{2\pi}, i\frac{\beta^2}{4\pi s}) , \qquad (3.12)$$

where we have introduced the shorthand notation $\bar{\mu}$ by reason of analogy to the vacuum amplitude of a scalar loop

$$\bar{\mu} \equiv \mu - i\bar{\omega} - \frac{2\pi i}{\epsilon \beta} = -i(\Omega + \frac{2\pi}{\epsilon \beta}) \ .$$
 (3.13)

We refer to the representation (3.12) as the *first representation*. This representation with $\bar{\mu} = 0$ ($\mu = \bar{\omega} = 0$, $\epsilon = \infty$) is studied in Ref. [23]. We also obtain another expression (which we shall call the *second representation*) through the Jacobi transformation,

$$\mathcal{K}_{\beta\mu} = \frac{1}{\beta} (4\pi s)^{\frac{1-D}{2}} e^{-s(M^2 - \bar{\mu}^2)} \Theta_3(\frac{2s\bar{\mu}}{\beta}, i\frac{4\pi s}{\beta^2}) . \tag{3.14}$$

Note that for a fermion loop ($\epsilon = 2$), the above first and the second Θ_3 representations become the Θ_4 and Θ_2 representations respectively.

We can extract the thermodynamic quantity $\mathcal{N}_{\beta\mu}$ corresponding to the path integral normalization (2.7) from the kinematical factor $\mathcal{K}_{\beta\mu}$. If we rewrite Eq.(3.12) as

$$\mathcal{K}_{\beta\mu} = e^{-sm^2} (4\pi s)^{-D/2} \Theta_3(i\frac{\beta\bar{\mu}}{2\pi}, i\frac{\beta^2}{4\pi s}) \exp\left[s \sum_{i < j}^N k_i \cdot k_j G_{ij}\right]
= \frac{1}{\beta} (4\pi s)^{\frac{1-D}{2}} e^{-s(m^2 - \bar{\mu}^2)} \Theta_3(\frac{2s\bar{\mu}}{\beta}, \frac{4\pi is}{\beta^2}) < \prod_{j=1}^N e^{ik_j \cdot x(\sigma_j)} > ,$$
(3.15)

the non-correlator part can be regarded as an overall normalization to the N-point correlator of zero temperature type (the zeroth components k_j^0 are formally regarded as continuous variables here). Paraphrasing this fact in analogy to Eq.(2.11)

$$\mathcal{K}_{\beta\mu} \equiv \mathcal{N}_{\beta\mu} < \prod_{j=1}^{N} e^{ik_j \cdot x(\sigma_j)} > = \oint_{\beta\mu} \mathcal{D}x \mathcal{D}\psi e^{-\int_0^1 \mathcal{L}(\sigma)d\sigma} \prod_{j=1}^{N} e^{ik_j \cdot x(\sigma_j)} , \qquad (3.16)$$

the thermodynamic version of the path integral normalization (for finite β and μ) is found to be

$$\mathcal{N}_{\beta\mu} = \frac{1}{\beta} (4\pi s)^{\frac{1-D}{2}} e^{-s(m^2 - \bar{\mu}^2)} \Theta_3(\frac{2s\bar{\mu}}{\beta}, \frac{4\pi is}{\beta^2}) . \tag{3.17}$$

This is exactly the same normalization as assumed in Ref.[22] (taking a mass term inclusion into account).

3.2 The vertex structure operator $\mathcal{V}_{\beta\mu}$

In this subsection, we discuss the part of vertex structure ($\mathcal{V} \neq 1$). Because of diversities of explicit forms of \mathcal{V} , there is no concrete formula such as Eq.(3.12). However, we can derive a

general property of the pure thermodynamic part $\tilde{\mathcal{V}}_{\beta\mu}$. To explain this, we classify \mathcal{V} into two categories by the criterion whether or not \mathcal{V} contains local world-line variables σ_i .

First, let us start with the first category, the σ -independent \mathcal{V} case. Apparently, $\mathcal{V}=1$ is the case. A nontrivial example of this category is the $\pi^0 \to 2\gamma$ decay ($\epsilon=2, D=4$) without background field [25]:

$$c\mathcal{V} = -tr[1]m\lambda e^2 \epsilon_{\mu\nu\rho\sigma} \epsilon_1^{\mu} \epsilon_2^{\nu} k_1^{\rho} k_2^{\sigma} , \qquad (3.18)$$

where λ is the pseudo-scalar coupling, m and e the (space-time) fermion mass and the QED coupling constant. Another nontrivial example is the effective potential of a fermion loop $(\epsilon = 2, N = 0)$ in a constant magnetic field in D dimensions [22]. In this case, \mathcal{V} depends on the integration variable s,

$$cV = -\frac{1}{2} \text{tr}[1] sB \coth(sB) , \qquad (3.19)$$

and this is the case of the expansion (2.13), if we use

$$\coth(sB) = 1 + 2\sum_{n=1}^{\infty} e^{-2nsB} . {(3.20)}$$

With the shift $M^2 \to M^2 + 2lB$, this case is essentially described by the 'partial' amplitude \mathcal{A}_1 defined by Eq.(2.14). Now, at finite values of β and μ , we can, in principle, express any thermodynamic vertex structure function $\mathcal{V}_{\beta\mu}$ as

$$\mathcal{V}_{\beta\mu} = \mathcal{V} + \tilde{\mathcal{V}}_{\beta\mu} \,\,, \tag{3.21}$$

where $\tilde{\mathcal{V}}_{\beta\mu}$ denotes the purely thermodynamic part. However, in this category, as can be seen from the above examples, we simply have

$$V_{\beta\mu} = V$$
, $\tilde{V}_{\beta\mu} = 0$ (for σ -independent V), (3.22)

because the V's of this category do not contain the local world-line variables σ_j , strictly speaking the G_{ij} which are generated by the bosonic field correlation from v_j — Recall that the β dependence only appears through the shift $x^0 \to x^0 + n\beta\sigma$ in the formula (3.1). The β dependence can not be created from the quantity which does not contain a x field correlation.

Second, let us consider the other category, the σ -dependent \mathcal{V} case. In this category, we obtain a nonzero structure function $\tilde{\mathcal{V}}_{\beta\mu}$. We here consider the photon polarization case by way

of example. In this case, the V is given by Eq.(C.5) at zero temperature [16]:

$$\mathcal{V} = \epsilon_1 \cdot \epsilon_2 \frac{1}{s} \ddot{G}_{12} + \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 (\dot{G}_{12})^2 + \epsilon_1 \cdot \epsilon_2 k_1 \cdot k_2 - \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 , \qquad (3.23)$$

where dots on G_{12} means the first and the second derivatives w.r.t. the first argument of G_{12} . The thermodynamic generalization can be done by applying the formulae (3.1) and (3.9). For ease of presentation, we put the computational details in Appendix C, and a comparison with the Feynman diagram technique is also in Appendix B. For finite β , we derive Eq.(C.17):

$$\mathcal{V}_{\beta} = \mathcal{V} - \epsilon_0^1 \epsilon_0^2 (\frac{1}{s} \frac{\partial}{\partial \bar{\omega}})^2 - \frac{1}{s} \frac{\partial}{\partial \bar{\omega}} (\epsilon_0^1 \epsilon_2 \cdot k_1 - \epsilon_0^2 \epsilon_1 \cdot k_2) \dot{G}_{12} , \qquad (3.24)$$

and then shifting $\bar{\omega} \to \Omega$, we acquire the operator part given by Eq.(C.19):

$$\tilde{\mathcal{V}}_{\beta\mu} = -\epsilon_0^1 \epsilon_0^2 (\frac{1}{s} \frac{\partial}{\partial \Omega})^2 - \frac{1}{s} \frac{\partial}{\partial \Omega} (\epsilon_0^1 \epsilon_2 \cdot k_1 - \epsilon_0^2 \epsilon_1 \cdot k_2) \dot{G}_{12} . \tag{3.25}$$

Note that the origin of $\partial/\partial\Omega$ is the Wick contractions of the bosonic world-line fields furnished in the v_j parts of photon vertex operators (see Eqs.(C.2) and (C.10)), and it always happens if a vertex operator comprises a bosonic field in such a way.

We therefore conclude that if \mathcal{V} includes G_{ij} or its derivatives, $\tilde{\mathcal{V}}_{\beta\mu}$ gives rise to a differential polynomial in $\partial/\partial\Omega$, and otherwise $\tilde{\mathcal{V}}_{\beta\mu}=0$. An important result followed from this fact is

$$\tilde{\mathcal{V}}_{\beta\mu} \times \mathcal{K} = 0$$
 (for all \mathcal{V}). (3.26)

This is a model-independent result, and we shall make use of this relation in order to decouple the pure thermodynamic part $\tilde{\Gamma}_N^{\beta\mu}$ from the total N-point amplitude $\Gamma_N^{\beta\mu}$.

3.3 The master formulae $\tilde{\Gamma}_N^{\beta\mu}$ and $\tilde{\mathcal{A}}_l^{\beta\mu}$

Gathering the formulae obtained in the above sections, we compose the purely thermodynamic master amplitude $\tilde{\Gamma}_N^{\beta\mu}$, and define the thermodynamic 'partial' amplitudes $\mathcal{A}_l^{\beta\mu}$. Applying the following formula to the first representation (3.12)

$$\Theta_3(v,\tau) = 1 + 2\sum_{n=1}^{\infty} e^{n^2 \tau \pi i} \cos(2n\pi v) ,$$
(3.27)

we separate the pure thermodynamic part $\tilde{\mathcal{K}}_{\beta\mu}$ from $\mathcal{K}_{\beta\mu}$ as

$$\mathcal{K}_{\beta\mu} = \mathcal{K} + \tilde{\mathcal{K}}_{\beta\mu} , \qquad (3.28)$$

where

$$\tilde{\mathcal{K}}_{\beta\mu} = 2(4\pi s)^{\frac{-D}{2}} e^{-sM^2} \sum_{n=1}^{\infty} e^{-\frac{n^2 \beta^2}{4s}} \cosh(n\beta\bar{\mu}) , \qquad (3.29)$$

or for the second representation (3.14)

$$\tilde{\mathcal{K}}_{\beta\mu} = \frac{2}{\beta} (4\pi s)^{\frac{1-D}{2}} e^{-s(M^2 - \bar{\mu}^2)} \sum_{n=1}^{\infty} e^{-s(\frac{2n\pi}{\beta})^2} \cos(\frac{4n\pi s\bar{\mu}}{\beta}) . \tag{3.30}$$

For a given $\mathcal{V}_{\beta\mu}$, the thermodynamic master amplitude $\Gamma_N^{\beta\mu}$ is calculated as

$$\Gamma_N^{\beta\mu} \stackrel{\text{def.}}{=} c \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \cdots \int_0^1 d\sigma_N \int_0^\infty ds s^{N-1} \mathcal{V}_{\beta\mu} \times \mathcal{K}_{\beta\mu} . \tag{3.31}$$

Using the decompositions (3.21) and (3.28) with the general formula (3.26), we find

$$\Gamma_N^{\beta\mu} = \Gamma_N + \tilde{\Gamma}_N^{\beta\mu} \,, \tag{3.32}$$

where

$$\tilde{\Gamma}_{N}^{\beta\mu} \stackrel{\text{def.}}{=} c \int_{0}^{1} d\sigma_{1} \int_{0}^{1} d\sigma_{2} \cdots \int_{0}^{1} d\sigma_{N} \int_{0}^{\infty} ds s^{N-1} \mathcal{V}_{\beta\mu} \times \tilde{\mathcal{K}}_{\beta\mu} . \tag{3.33}$$

Therefore we have only to separate the pure thermodynamic part of $\mathcal{K}_{\beta\mu}$ in order to obtain the purely thermodynamic part $\tilde{\Gamma}_N^{\beta\mu}$: no separation in the $\mathcal{V}_{\beta\mu}$ part at all (Note the difference between Eqs.(3.31) and (3.33)).

As can be inferred from the examples in Section 3.2, a general expansion form of $\mathcal{V}_{\beta\mu}$ is

$$\mathcal{V}_{\beta\mu} = \sum_{l,n \in \mathbf{Z}} a_{ln} \, s^{l-N} \, \frac{\partial^n}{\partial \Omega^n} \, \exp[-sb_l] \, . \tag{3.34}$$

where the coefficients a_{ln} and b_l may not depend on s, but on σ_j $(j = 1, 2, \dots, N)$. In analogy to Eq.(2.14), the relevant 'partial' amplitudes for Eqs.(3.31) and (3.33) are defined as

$$\mathcal{A}_{l}^{\beta\mu} \stackrel{\text{def.}}{=} \int_{0}^{\infty} ds \, s^{l-1} \mathcal{K}_{\beta\mu} \Big|_{m^{2} \to m^{2} + b_{l}} , \qquad (3.35)$$

$$\tilde{\mathcal{A}}_{l}^{\beta\mu} \stackrel{\text{def.}}{=} \int_{0}^{\infty} ds \, s^{l-1} \tilde{\mathcal{K}}_{\beta\mu} \Big|_{m^{2} \to m^{2} + b_{l}} , \qquad (3.36)$$

and of course the following relation holds:

$$\mathcal{A}_{l}^{\beta\mu} = \mathcal{A}_{l} + \tilde{\mathcal{A}}_{l}^{\beta\mu} \ . \tag{3.37}$$

In the following sections, we present various integral representations for the pure thermodynamic parts $\tilde{A}_l^{\beta\mu}$ of the N-point 'partial' amplitudes without specifying any values of N, D and ϵ . These parts are the essential quantities to analyze the zero temperature limits with revealing the non-analyticity on μ .

4 The integral formulae from the first representation

In this section, we derive various integral formulae for $\tilde{\mathcal{A}}_l^{\beta\mu}$ based on the first representation (3.12). This representation is examined in the special case l=1 (with N=0 and $\epsilon=2$), and actually $(4\pi)^{D/2}\tilde{\mathcal{A}}_1^{\beta\mu}$ is the function \mathcal{O}_{β} analyzed in Ref.[22]:

$$\mathcal{O}_{\beta}(m) = 4 \sum_{n=1}^{\infty} (-1)^n \cosh(n\beta\mu) (\frac{n\beta}{2m})^{1-D/2} K_{D/2-1}(n\beta m) . \tag{4.1}$$

We want to generalize this function to more generic l and ϵ cases, and it is convenient to mimic the computational technique of Ref.[22], introducing the parallel notation

$$\mathcal{O}_{\beta}^{(k)}(M) \stackrel{\text{def.}}{=} (4\pi)^{\frac{D}{2}} \tilde{\mathcal{A}}_{l}^{\beta\mu} ; \qquad k \equiv 2l + 1 - D . \tag{4.2}$$

Hereafter, for simplicity we set

$$b_l = 0 (4.3)$$

and for later convenience, we also define

$$d \equiv 3 - k = D + 2 - 2l \ . \tag{4.4}$$

From Eqs.(3.29) and (3.36), the pure thermodynamic part $\tilde{\mathcal{A}}_l^{\beta\mu}$ is now given by

$$\mathcal{O}_{\beta}^{(k)}(M) = 2(4\pi)^{\frac{D}{2}} \int_{0}^{\infty} ds s^{l-1} (4\pi s)^{\frac{-D}{2}} e^{-sM^{2}} \sum_{n=1}^{\infty} e^{-\frac{n^{2}\beta^{2}}{4s}} \cosh(n\beta\bar{\mu}) . \tag{4.5}$$

Performing the s-integration, we obtain

$$\mathcal{O}_{\beta}^{(k)}(M) = 4\sum_{n=1}^{\infty} \cosh(n\beta\bar{\mu}) (\frac{n\beta}{2M})^{1-d/2} K_{d/2-1}(n\beta M) , \qquad (4.6)$$

where $K_{\nu}(z)$ is the modified Bessel function of second kind. This is a generalized version (for arbitrary ϵ and l) of the above function \mathcal{O}_{β} . The aim is to obtain integral representations with performing the summation in Eq.(4.6) (for generic k or d). To this end, we first apply the following formula to Eq.(4.6):

$$K_{\nu}(z) = \frac{\sqrt{\pi}(z/2)^{\nu}}{\Gamma(\nu + \frac{1}{2})} \int_{1}^{\infty} e^{-zt} (t^{2} - 1)^{\nu - \frac{1}{2}} dt \; ; \quad \operatorname{Re} \nu > -\frac{1}{2} \; , \operatorname{Re} z > 0 \; , \tag{4.7}$$

and note that the k < 2 (d > 1) and $k \ge 2$ $(d \le 1)$ cases involve different calculations. Here we put a few remarks. For the convergence of the summation over n, we have to assume

$$\mu < M \tag{4.8}$$

in order to satisfy the condition

$$|e^{-\beta(Mt\pm\bar{\mu})}| = e^{-\beta(Mt\pm\mu)} < 1 ; \qquad t > 1 .$$
 (4.9)

The k = 1, 0, -1 (d = 2, 3, 4) cases are calculated in Ref.[22], and the k = 3 (d = 0) case corresponds to Ref.[25], although they did not discuss this representation (Note that their argument belongs to our second representation).

(i) In the first case, k < 2 (d > 1), the condition on ν in the formula (4.7) is satisfied as it is; i.e., $\nu = \frac{d}{2} - 1 > -\frac{1}{2}$, and the calculation is almost parallel to Ref.[22]. We then just write down the result

$$\mathcal{O}_{\beta}^{(k)}(M) = \frac{-2\sqrt{\pi}}{\Gamma(\frac{d-1}{2})} \left[\int_0^{\infty} \frac{(u^2 + 2Mu)^{\frac{d-3}{2}}}{1 - e^{\beta(u+M+\bar{\mu})}} du + (\bar{\mu} \to -\bar{\mu}) \right], \tag{4.10}$$

where note that the minus sign of the denominator differs from the previous formula (Eq.(3.13) in Ref.[22]). For later convenience, let us derive another formula from this result. Performing the change of variable $u = M(\sqrt{p^2 + 1} - 1)$, and applying the formula $\Gamma(z)\Gamma(1 - z) = \pi/\sin \pi z$ with z = (d-1)/2, we derive

$$\mathcal{O}_{\beta}^{(k)}(M) = \frac{2}{\sqrt{\pi}} \Gamma(\frac{k}{2}) M^{1-k} \sin(\frac{2-k}{2}\pi) \int_0^\infty \frac{p^{1-k}(p^2+1)^{-1/2}}{e^{\beta X_+(p)} - 1} dp + (\bar{\mu} \to -\bar{\mu}) , \qquad (4.11)$$

where we have introduced the compact notation

$$X_{\pm}(p) = M\sqrt{p^2 + 1} \pm \bar{\mu} \ . \tag{4.12}$$

(ii) In the second case, $2 \le k$ ($d \le 1$), the calculation is similar to the case (i), excepting the point that we apply the formula (4.7) for $\nu = 1 - \frac{d}{2}$ instead of $\nu = \frac{d}{2} - 1$ owing to the relation $K_{\nu}(z) = K_{-\nu}(z)$. Then the summation over n in Eq.(4.6) leads to the Lerch transcendental function:

$$\mathcal{O}_{\beta}^{(k)}(M) = \alpha \int_{1}^{\infty} dt (t^{2} - 1)^{\frac{1-d}{2}} e^{-\beta(Mt + \bar{\mu})} \Phi(e^{-\beta(Mt + \bar{\mu})}, d - 2, 1) + (\bar{\mu} \to -\bar{\mu})
= \frac{\alpha \Gamma(3 - d)}{2\pi i} \int_{1}^{\infty} dt (t^{2} - 1)^{\frac{1-d}{2}} \int_{\infty}^{(0+)} \frac{(-z)^{d-3} dz}{1 - e^{z + \beta(Mt + \bar{\mu})}} + (\bar{\mu} \to -\bar{\mu}), \quad (4.13)$$

where just for conciseness we have defined the coefficient

$$\alpha = \frac{\sqrt{4\pi}}{\Gamma(\frac{3-d}{2})} (\frac{\beta}{2})^{2-d} = \frac{\sqrt{4\pi}}{\Gamma(\frac{k}{2})} (\frac{\beta}{2})^{k-1} . \tag{4.14}$$

The z-integrand (4.13) possesses poles at z=0 and $z=-\beta Mt\pm\beta\bar{\mu}+2n\pi i$ $(n\in\mathbf{Z})$. Since $-\beta(Mt\pm\mu)$ is a negative value, there is no pole in the positive region on the real axis other than z=0 as long as $\beta\bar{\omega}+\frac{2\pi}{\epsilon}$ is not an integer. With the change of variable $t=\sqrt{p^2+1}$, and a replacement of one derivative $\partial/\partial z$ by $\partial/\partial p$, we have

$$\mathcal{O}_{\beta}^{(k)}(M) = \alpha \frac{(-1)^{-k} \Gamma(k)}{M \beta(k-1)!} \frac{\partial^{k-2}}{\partial z^{k-2}} \int_{0}^{\infty} p^{k-2} \frac{\partial}{\partial p} \left(\frac{1}{1 - e^{z + \beta X_{+}}} \right) dp \Big|_{z=0} + (\bar{\mu} \to -\bar{\mu}) . \tag{4.15}$$

Because of $k \geq 2$, the surface terms from a partial integral vanish in Eq.(4.15), and finally we find

$$\mathcal{O}_{\beta}^{(k)}(M) = 2\Gamma(\frac{k+1}{2}) \frac{(-1)^{1-k}\beta^{k-2}(k-2)}{M(k-1)!} \frac{\partial^{k-2}}{\partial z^{k-2}} \int_{0}^{\infty} \frac{p^{k-3}}{1 - e^{z+\beta X_{+}}} dp \Big|_{z=0} + (\bar{\mu} \to -\bar{\mu}) . \quad (4.16)$$

More explicitly, we present the results for k = 2, 3, 4 as follows:

$$\mathcal{O}_{\beta}^{(2)}(M) = -\frac{\sqrt{\pi}}{M} \left[\frac{1}{1 - e^{\beta(M + \bar{\mu})}} + (\bar{\mu} \to -\bar{\mu}) \right], \tag{4.17}$$

$$\mathcal{O}_{\beta}^{(3)}(M) = \frac{\beta}{M} \int_{0}^{\infty} \frac{e^{\beta X_{+}}}{(1 - e^{\beta X_{+}})^{2}} dp + (\bar{\mu} \to -\bar{\mu}) , \qquad (4.18)$$

$$\mathcal{O}_{\beta}^{(4)}(M) = -\frac{\sqrt{\pi}}{2M^3} \left[\frac{1}{1 - e^{\beta(M + \bar{\mu})}} - \frac{\beta M e^{\beta(M + \bar{\mu})}}{(1 - e^{\beta(M + \bar{\mu})})^2} + (\bar{\mu} \to -\bar{\mu}) \right]. \tag{4.19}$$

For even values of k, one may directly perform the summation (4.6) without using the integral representation (4.7); e.g., using $K_{1/2}(z) = \sqrt{\pi/2z}e^{-z}$ and $K_{3/2}(z) = \sqrt{\pi/2z}(1+z^{-1})e^{-z}$ for k=2 and 4. We will see in the next section that the results (4.11) and (4.17)-(4.19) can be reproduced from the second representation as well.

5 The integral formulae from the second representation

In this section, based on the second representation (3.14) case, we derive some more formulae on $\mathcal{O}_{\beta}^{(k)}(M)$, with re-deriving the results of the previous section. From Eqs.(3.30), (3.36), and (4.2), we have the following form of the pure thermodynamic part to start with:

$$\mathcal{O}_{\beta}^{(k)}(M) = \frac{(4\pi)^{1/2}}{\beta} \int_{0}^{\infty} ds s^{\frac{k-2}{2}} e^{-s(M^2 - \bar{\mu}^2)} \sum_{n \in \mathbf{Z}, \neq 0} e^{-s(\frac{2n\pi}{\beta})^2} \cos(4n\pi s \bar{\mu}/\beta) . \tag{5.1}$$

The summation can be converted to the integrals on the contour C_{μ} (see Fig.1a):

$$\mathcal{O}_{\beta}^{(k)}(M) = \frac{1}{2} (4\pi)^{\frac{1}{2}} \int_{0}^{\infty} ds s^{\frac{k-2}{2}} e^{-sM^{2}} \frac{1}{2\pi i} \int_{C_{\mu}} \frac{e^{sz^{2}}}{e^{\beta(z-\bar{\mu})} - 1} dz + (\bar{\mu} \to -\bar{\mu}) . \tag{5.2}$$

After performing the s-integration, one may deform the contour C_{μ} into another one C_{δ} (Fig.1b) in the same way as Ref.[25], thus obtaining

$$\mathcal{O}_{\beta}^{(k)}(M) = \sqrt{\pi}\Gamma(\frac{k}{2}) \frac{1}{2\pi i} \int_{\infty}^{(M+)} \frac{(M^2 - z^2)^{-k/2}}{e^{\beta(z-\bar{\mu})} - 1} dz + (\bar{\mu} \to -\bar{\mu}) . \tag{5.3}$$

Parametrizing the circular part of C_{δ} (centered at M) by $z = M(1 - \delta e^{i\phi})$, and using $p^2 = z^2 - 1$ for the remaining parts of the contour, we arrive at

$$\mathcal{O}_{\beta}^{(k)}(M) = \frac{2}{\sqrt{\pi}} \Gamma(\frac{k}{2}) M^{1-k} \sin(\frac{2-k}{2}\pi) \left[\int_{\sqrt{2\delta}}^{\infty} \frac{p^{1-k}(p^2+1)^{-1/2}}{e^{\beta X_{+}} - 1} dp + \frac{\frac{1}{2-k}(2\delta)^{1-\frac{k}{2}}}{e^{\beta (M-\bar{\mu})} - 1} + (\bar{\mu} \to -\bar{\mu}) \right]. \tag{5.4}$$

Substituting k=3 and $\epsilon=2$, Eq.(5.4) reproduces the result (without background field) of Ref.[25].

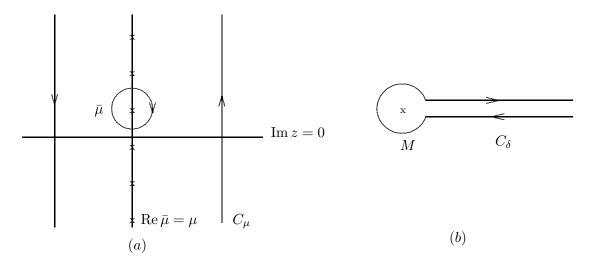


Figure 1: The contours C_{μ} and C_{δ} . (a) The C_{μ} impounds an infinite number of poles at $z = \bar{\mu} + 2n\pi i$, where n denotes nonzero integers. (b) Coming from $z = \infty + i\epsilon$ and going to $z = \infty - i\epsilon$, the C_{δ} impounds the pole at z = M.

Here are a few remarks on the computational difference between Ref.[25] and ours. We have extracted the pure thermodynamic part (5.1) from the beginning, and this fact is expressed by the circle at $\bar{\mu}$ in C_{μ} . On the other hand in the method of Ref.[25], one needs an algebra to separate the non-thermodynamic part by using

$$\frac{-1}{e^{\beta(z-\bar{\mu})}-1} = 1 + \frac{1}{e^{-\beta(z-\bar{\mu})}-1} \ . \tag{5.5}$$

The first term on the r.h.s. of Eq.(5.5) corresponds to the non-thermodynamic part, and the second term to the following expression instead of Eq.(5.2):

$$\mathcal{O}_{\beta\mu}^{(k)}(M) = (4\pi)^{\frac{1}{2}} \int_0^\infty ds s^{\frac{k-2}{2}} e^{-sM^2} \frac{1}{2\pi i} \left\{ \int_{C_+} \frac{e^{s(z-i\bar{\omega})^2}}{e^{\beta(z-\bar{\mu}-i\bar{\omega})} - 1} + \int_{C_-} \frac{e^{s(z-i\bar{\omega})^2}}{e^{-\beta(z-\bar{\mu}-i\bar{\omega})} - 1} \right\} dz , \quad (5.6)$$

where the contours C_{\pm} run from $\mu \pm o - i\infty$ to $\mu \pm o + i\infty$ for a small value of o. Although one can of course obtain the same result (5.4) from Eq.(5.6), it is not obvious at first glance that the C_{-} integral term corresponds to the $\bar{\mu} \to -\bar{\mu}$ term of Eq.(5.3).

Now, further results depend on the cases (i) and (ii). In the case (i), k < 2, the second term in the square brackets on the r.h.s. of Eq.(5.4) is proportional to $\delta^{1-k/2}$, and vanishes as $\delta \to 0$. Therefore Eq.(5.4) reproduces the first representation result (4.11).

In the case (ii), $k \geq 2$, the $\delta^{1-k/2}$ term diverges as $\delta \to 0$. We expect that it can be canceled with the lower surface term from the first integral in the square brackets on the r.h.s. of Eq.(5.4). This can be seen by a partial integral with applying the formula

$$\frac{d}{dp}(\frac{\sqrt{p^2+1}}{p}) = \frac{1}{p^2\sqrt{p^2+1}} \tag{5.7}$$

to the p-integration in Eq.(5.4) as follows:

$$\int_{\sqrt{2\delta}}^{\infty} \frac{p^{1-k}(p^2+1)^{-1/2}}{e^{\beta X_{+}} - 1} dp = \frac{\frac{1}{k-2}(2\delta)^{1-k/2}}{e^{\beta (M-\bar{\mu})} - 1} + \frac{1}{2-k} \int_{0}^{\infty} p^{3-k} \left\{ \frac{(p^2+1)^{-3/2}}{e^{\beta X_{+}} - 1} + \frac{M\beta(p^2+1)^{-1}e^{\beta X_{+}}}{(e^{\beta X_{+}} - 1)^2} \right\} dp .$$
(5.8)

The first term on the r.h.s. of Eq.(5.8) cancels the divergence as expected, and we derive

$$\mathcal{O}_{\beta}^{(k)}(M) = \sqrt{4\pi} \Gamma(\frac{k}{2}) M^{1-k} \frac{\sin(\frac{2-k}{2}\pi)}{(2-k)\pi}$$

$$\times \left[\int_{0}^{\infty} p^{3-k} \left\{ \frac{(p^2+1)^{-3/2}}{e^{\beta X_{+}} - 1} + \frac{M\beta(p^2+1)^{-1}e^{\beta X_{+}}}{(e^{\beta X_{+}} - 1)^2} \right\} dp + (\bar{\mu} \to -\bar{\mu}) \right].$$
(5.9)

Putting $t^2 = p^2 + 1$ in Eq.(5.9), we finally obtain the following concise expression:

$$\mathcal{O}_{\beta}^{(k)}(M) = \sqrt{4\pi} \Gamma(\frac{k}{2}) M^{1-k} \frac{\sin(\frac{2-k}{2}\pi)}{(2-k)\pi} \int_{1}^{\infty} \sqrt{t^2 - 1}^{2-k} \frac{\partial}{\partial t} \left[\frac{t^{-1}}{1 - e^{\beta(Mt + \bar{\mu})}} \right] dt + (\bar{\mu} \to -\bar{\mu}) , \quad (5.10)$$

which is an alternative representation of Eq.(4.16). As a result, we have derived two expressions in each case: Eqs.(4.10) and (4.11) in the case (i), and Eqs.(4.16) and (5.10) in the case (ii). In

Eq.(4.16) we have to perform k-2 derivatives (after one integration), while just one integration in Eq.(5.10), whose integrand thus contains a common Boltzmann factor for all $k \geq 2$.

Let us check the consistency of Eq.(5.10) with the previous result (4.16) for k = 2, 3, 4. The k = 2 case (4.17) is immediate from the representation (5.10), however let us handle these three cases simultaneously. Using $(p^2 + 1)^{-1} = 1 - p^2(p^2 + 1)^{-1}$, we first divide the second term in the curly brackets in Eq.(5.9), and then perform a partial integral on the r.h.s. in the following quantity:

$$-\int_0^\infty p^{3-k} \frac{M\beta p^2 (p^2+1)^{-1} e^{\beta X_+}}{(e^{\beta X_+} - 1)^2} dp = \int_0^\infty p^{3-k} \frac{p}{\sqrt{p^2+1}} \frac{\partial}{\partial p} (\frac{1}{e^{\beta X_+} - 1}) dp . \tag{5.11}$$

Changing the integration variable by $t^2 = p^2 + 1$, we arrive at

$$\mathcal{O}_{\beta}^{(k)}(M) = \sqrt{4\pi}\Gamma(\frac{k}{2})M^{1-k}\frac{\sin(\frac{2-k}{2}\pi)}{(2-k)\pi}\Big[I_k + (\bar{\mu} \to -\bar{\mu})\Big], \qquad (5.12)$$

where

$$I_{k} = \int_{1}^{\infty} \sqrt{t^{2} - 1}^{2-k} t \left[\frac{k - 3}{t} N(t) - N'(t) \right] dt$$
 (5.13)

and $N'(t) \equiv \partial_t N(t)$ means the derivative of the function

$$N(t) = \frac{1}{e^{\beta(Mt - \bar{\mu})} - 1} \ . \tag{5.14}$$

For the k = 2 and 3 cases, this expression is sufficient to see the consistency in each. However, for the k = 4 case, notice that we have to extract a singularity, which cancels a zero from the sine function in Eq.(5.10), and we hence perform a partial integral once more:

$$I_{k} = -\int_{1}^{\infty} \frac{1}{4-k} \sqrt{t^{2}-1}^{4-k} \frac{\partial}{\partial t} \left[(k-3) \frac{N(t)}{t} - N'(t) \right] dt \; ; \qquad 2 \le k < 4 \; , \tag{5.15}$$

where the surface terms can be dropped only when k < 4 (also in Eq.(5.11)), and this is the reason why this formula is valid for $2 \le k < 4$ ($-1 < d \le 1$). Substituting $k = 2, 3, 4 - \epsilon$ (with $\epsilon \to 0$), we verify that these expressions for I_k reproduce Eqs.(4.17)-(4.19). In this sense, Eq.(5.10) is equivalent to Eq.(4.16).

For further values of k, one should repeat the similar calculation for each interval between zero points of the sine function. For example, for k = 6 and 8, we obtain

$$\mathcal{O}_{\beta}^{(6)}(M) = \frac{\pi}{16M^5} \Big[3N(1) - 3N'(1) + N''(1) \Big] , \qquad (5.16)$$

$$\mathcal{O}_{\beta}^{(8)}(M) = \frac{\sqrt{\pi}}{8M^7} \left[15N(1) - 15N'(1) + 6N''(1) - N'''(1) \right]. \tag{5.17}$$

These pure thermodynamic 'partial' amplitudes exactly correspond to the N=4 and 5 pure thermodynamic amplitudes in D=3 when $\mathcal{V}_{\beta\mu}=\mathcal{V}$ (the first category). In D=4, odd k integers are the similar cases. For the second category; for example, the N-photon amplitude in D=4, we need to combine the quantities from k=-1 to 2N-5. These cases can not get rid of non-integrated quantities such as Eq.(4.18). After all, Eqs.(4.16) and (5.10) describe the general amplitude formulae which contain all Feynman diagrams.

6 The $\beta \to \infty$ limit

In the above arguments, we have assumed $\mu < M$, and all $\beta \to \infty$ limits vanish because of it:

$$\mathcal{O}_{\beta}^{(k)} \stackrel{\beta \to \infty}{\longrightarrow} 0. \tag{6.1}$$

To obtain a nontrivial (nonzero) limit, we should remove this condition after all. In this sense, the μ -dependence of the master amplitudes is non-analytic. To see this, we need to transform the function $\mathcal{O}_{\beta}^{(k)}$ in a form indicating a Bose (or a Fermi) distribution, and we already derived this kind of representations in Sections 4 and 5.

Let us begin with the case (i) k < 2; in particular the case of k = 0 (which is also an odd dimensional case D = 2l + 1 by the way). From Eq.(4.10) with changing the variable E = u + M, we have

$$\mathcal{O}_{\beta}^{(0)}(M) = -2\sqrt{\pi} \int_{M}^{\infty} \frac{dE}{1 \pm e^{\beta(E-\mu')}} + (\mu' \to -\mu') , \qquad (6.2)$$

where the plus/minus signs correspond to the Fermi/Bose statistics, and we put

$$\mu' = \mu - i\bar{\omega} \ . \tag{6.3}$$

Eq.(6.2) can be interpreted an effective action or total energy density with the chemical potential μ' and the mass M ($<\mu'$, E). Now, we understand in Eq.(6.2) that the $\mathcal{O}_{\beta}^{(0)}$ becomes zero as $\beta \to \infty$ if $\mu' < E$, while the $\mathcal{O}_{\beta}^{(0)}$ takes a nonzero value if $E < \mu'$. The similar arguments apply to the case of generic k value in the following way. Since we observe

$$e^{\beta(M-\bar{\mu}+u)} = e^{\beta(u+M-\mu)}e^{i\beta(\bar{\omega}+\frac{2\pi}{\epsilon\beta})} \stackrel{\beta\to\infty}{\longrightarrow} 0 \quad \text{for } u<\mu-M$$
, (6.4)

we have the finite upper boundary on the integral (4.10) at $u = \mu - M$, thus obtaining

$$\mathcal{O}_{\infty}^{(k)}(M) \equiv \lim_{\beta \to \infty} \mathcal{O}_{\beta}^{(k)}(M) = \frac{-2\sqrt{\pi}}{\Gamma(\frac{2-k}{2})} \int_{0}^{\mu-M} (u^{2} + 2Mu)^{-k/2} du \, \theta(\mu - M)$$

$$= \frac{-2\sqrt{\pi}}{\Gamma(\frac{4-k}{2})} (2M)^{-k/2} (\mu - M)^{\frac{2-k}{2}} F(\frac{k}{2}, \frac{2-k}{2}; \frac{4-k}{2}; \frac{M-\mu}{2M}) \, \theta(\mu - M) .$$
(6.5)

The explicit results for k = -1, 0, 1 are

$$\mathcal{O}_{\infty}^{(1)}(M) = -2\operatorname{arccosh}(\mu/M) \theta(\mu - M) , \qquad (6.6)$$

$$\mathcal{O}_{\infty}^{(0)}(M) = -2\sqrt{\pi}(\mu - M)\,\theta(\mu - M)\,,$$
(6.7)

$$\mathcal{O}_{\infty}^{(-1)}(M) = \left[-2\mu\sqrt{\mu^2 - M^2} + 2M^2 \operatorname{arccosh}(\mu/M) \right] \theta(\mu - M) .$$
 (6.8)

In one of the k=0 cases $(D=3, N=0, l=1, \epsilon=2)$ [22], Eq.(6.7) coincides with an exact result in [27].

The case (ii) $k \ge 2$ can also be estimated in the same way. Applying the similar treatment as above to the Boltzmann factor in Eq.(5.10), we then obtain the nonzero limit given by

$$\mathcal{O}_{\infty}^{(k)}(M) = -\sqrt{4\pi}\Gamma(\frac{k}{2})M^{1-k}\frac{\sin(\frac{2-k}{2}\pi)}{(2-k)\pi}\int_{1}^{\mu/M}(t^{2}-1)^{\frac{2-k}{2}}\frac{dt}{t^{2}}\theta(\mu-M) \ . \tag{6.9}$$

Performing the t-integration, some of explicit results (the k = 2, 3, 4 cases) can be shown as follows:

$$\mathcal{O}_{\infty}^{(2)}(M) = \frac{\sqrt{\pi}}{M} (\frac{M}{\mu} - 1) \theta(\mu - M) ,$$
 (6.10)

$$\mathcal{O}_{\infty}^{(3)}(M) = -\frac{1}{\mu M} \sqrt{(\frac{\mu}{M})^2 - 1} \,\theta(\mu - M) ,$$
 (6.11)

$$\mathcal{O}_{\infty}^{(4)}(M) = -\frac{\sqrt{\pi}}{2M^3}\theta(\mu - M)$$
 (6.12)

The calculations are rather straightforward for further k values, and the general formula (6.9) suffices. The ϵ dependence is gone away from the $\mathcal{O}_{\infty}^{(k)}$. This is natural because there is no difference in the kinematical factors (hence in 'partial' amplitudes) between boson and fermion loops at zero temperature.

7 Conclusions

In this paper, we studied the thermodynamic generalization of the one-loop amplitudes which can be cast into the BK master formula (2.9) at zero temperature (with zero chemical potential). We followed the two-step procedure: first, calculating the path integral S^1 summation (3.1) to introduce the temperature, and then performing the shift manipulation (1.3) to insert the chemical potential in a loop. This procedure has been applied parts by parts to the kinematical factor \mathcal{K} , the normalization factor \mathcal{N} and the vertex structure function (reduced kinematical factor) \mathcal{V} , and we have derived the general formulae for these parts. From Eqs.(3.31), (3.34) and (3.35), the thermodynamic N-point amplitude of general form is thus summarized as

$$\Gamma_N^{\beta\mu} = c \sum_{l n \in \mathbb{Z}} a_{ln} \frac{\partial^n}{\partial \Omega^n} \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \cdots \int_0^1 d\sigma_N \mathcal{A}_l^{\beta\mu} . \tag{7.1}$$

One can check the validity of this master formula in various cases; the free effective potentials and the photon polarization in Appendix B, the effective potential in a constant magnetic field in Ref.[22], and the $\pi^0 \to 2\gamma$ decay in Ref.[25] etc.

The detail analyses on the building blocks $V_{\beta\mu}$ and $K_{\beta\mu}$ have given us useful information. We have realized that the total thermodynamic kinematical factor $K_{\beta\mu}$ behaves as the thermodynamically generalized normalization to the zero-temperature type correlator shown in Eq.(3.17). This is certainly a useful result and makes calculations for large N much simpler than ever; namely we have only to attach the new normalization $\mathcal{N}_{\beta\mu}$ to the N-point scalar correlator (with switching $k_j^0 \to \omega_{k_j}$). Another interesting point is that the pure thermodynamic part of a vertex structure function can be expressed in terms of either vanishing or Ω -differential operators, and this fact makes us possible to decouple the master formula (3.31) into $\Gamma_N^{\beta\mu} = \Gamma_N + \tilde{\Gamma}_N^{\beta\mu}$ in a nontrivial way (cf. Eqs.(3.31) and (3.33)). It is worth noticing that we do not decouple the vertex structure function but the kinematical factor only. From Eqs.(3.33), (3.34) and (3.36), we then conclude that the pure thermodynamic master amplitude is simply given by

$$\tilde{\Gamma}_{N}^{\beta\mu} = c \sum_{l,n \in \mathbb{Z}} a_{ln} \frac{\partial^{n}}{\partial \Omega^{n}} \int_{0}^{1} d\sigma_{1} \int_{0}^{1} d\sigma_{2} \cdots \int_{0}^{1} d\sigma_{N} \tilde{\mathcal{A}}_{l}^{\beta\mu} . \tag{7.2}$$

Apart from Γ_N , the pure thermodynamic part is renormalization free, and hence Eq.(7.2) is the final form to apply our integral formulae derived in Sections 4-6. More explicitly, we have (from Eqs.(3.13) and (4.2))

$$\tilde{\Gamma}_N^{\beta\mu} = c(4\pi)^{\frac{-D}{2}} \sum_{l,n \in \mathbb{Z}} a_{ln} (-i)^n \frac{\partial^n}{\partial \bar{\mu}^n} \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \cdots \int_0^1 d\sigma_N \mathcal{O}_{\beta}^{(2l+1-D)}(M) \Big|_{m^2 \to m^2 + b_l} . \tag{7.3}$$

and in particular for the $\beta \to \infty$ limit with $\mu \neq 0$

$$\lim_{\beta \to \infty} \tilde{\Gamma}_{N}^{\beta \mu} = c(4\pi)^{\frac{-D}{2}} \sum_{l \in \mathbf{Z}} a_{l0} \int_{0}^{1} d\sigma_{1} \int_{0}^{1} d\sigma_{2} \cdots \int_{0}^{1} d\sigma_{N} \mathcal{O}_{\infty}^{(2l+1-D)}(M) \Big|_{m^{2} \to m^{2} + b_{l}} . \tag{7.4}$$

The thermodynamic master amplitudes (7.3) and (7.4) are provided with Eqs.(4.16), (5.10) and (6.9), and several explicit 'partial amplitudes' $\mathcal{O}_{\beta}^{(k)}$ are also given up to k = 8.

In the master formulae (7.2) and (7.3), the 'partial' amplitudes $\tilde{\mathcal{A}}_{l}^{\beta\mu}$ or $\mathcal{O}_{\beta}^{(k)}$ (s-integrals of the pure thermodynamic kinematical factor) are the fundamental computational blocks in our formalism, and in Sections 4 and 5, we focused on some mathematical aspects and techniques how to calculate these integrals. We derived various representations and formulae on these integrals for arbitrary values of N, D and ϵ . It is obvious that a variety of equivalent formulae makes easy to find some relation and consistency with the results obtained by different methods. These should generally be related by analytic continuation, and a different analytic expression is certainly useful by definition; i.e., another formula can cover the region which can not be reached in an original representation, and it also overcomes some defect of an inconvenient representation. For example in Ref. [22], a dual rotation is used to obtain an electric gap equation from magnetic one in a four-fermion model. We expect such kind of utility when our formulae are applied in more explicit stages.

Although we could have possibly simplified our results furthermore through a certain technique [23], our formulas were sufficiently convenient to extract the nonzero values of zero temperature limit with μ kept finite. For this purpose, it is necessary to have the integral representation, such as Eqs.(4.10) and (5.10), which clearly indicates a non-analytic cut in its integrand at $\beta = \infty$. We also had to go beyond the condition $\mu < M$ in these representations, however this might be justified by analytic continuation, and at least we know that the k = 0 case coincides with the exact result.

We have examined the model independent structure of the thermodynamic BK master formula with consulting several simple examples. The model dependence appears in the vertex structure functions, and hence one has to evaluate the vertex structure in the first place for explicit calculations in each model. This task is case by case and will become more lengthy as N increasing for the cases involving more bosonic x-fields in the vertex operator v_j parts such as photon, gluon and pseudo-scalar particle. However, our systematic prescription is certainly

promising to obtain the (pure) thermodynamic N-point amplitudes in a straightforward way as long as the $V_{\beta\mu}$ belongs to the general form (3.34). Finally, we should not forget the advantage that the world-line formula encapsulates all necessary Feynman diagrams in a single integrand.

Appendix A. Insertion of chemical potential

In this appendix, we give a brief explanation of the shift procedure (3.9). First consider the vacuum case $\bar{\omega} = 0$. In this case, $M^2 = m^2$ and

$$-\operatorname{Trln}(\partial^2 + m^2) = \frac{1}{\beta} \sum_{n} \int \frac{d^D p}{(2\pi)^{D-1}} \int_0^\infty \frac{ds}{s} e^{-s(p^2 + m^2)} . \tag{A.1}$$

We usually perform the insertion of chemical potential in terms of the shift

$$p^2 = p_0^2 + \vec{p}^2 \rightarrow (\omega_n + i\mu)^2 + \vec{p}^2$$
 (A.2)

with the internal Matsubara frequencies

$$\omega_n = \frac{2\pi}{\beta} (n + \frac{1}{\epsilon}) , \qquad (A.3)$$

and we have

$$\sum_{n} e^{-s(\omega_n + i\mu)^2} = e^{s\bar{\mu}_0^2} \Theta_3(\frac{2s\bar{\mu}_0}{\beta}, is\frac{4\pi}{\beta^2}) , \qquad (A.4)$$

where we have defined

$$\bar{\mu}_0 = \mu - \frac{2\pi i}{\epsilon \beta} \ . \tag{A.5}$$

Thus, we prove Eq.(3.14) for N = 0:

$$Eq.(A.1) = \frac{1}{\beta} \int_0^\infty \frac{ds}{s} (4\pi s)^{\frac{1-D}{2}} e^{s(\bar{\mu}_0^2 - m^2)} \Theta_3(\frac{2s\bar{\mu}_0}{\beta}, is \frac{4\pi}{\beta^2}) . \tag{A.6}$$

For further nonzero values of N, the proofs are straightforward, and we shall not explain the details anymore. For example, one can find the N=2 case in Appendix B.

Instead, we add a supplemental interpretation. Applying the transformation

$$\frac{\sqrt{4\pi s}}{\beta} e^{s\bar{\mu}_0^2} \Theta_3(\frac{2s\bar{\mu}_0}{\beta}, is\frac{4\pi}{\beta^2}) = \Theta_3(\frac{i\beta\bar{\mu}_0}{2\pi}, i\frac{\beta^2}{4\pi s}) , \qquad (A.7)$$

we rewrite Eq.(A.6) as

Eq.(A.6) =
$$\mathcal{A}_0^{\beta\mu} = \int_0^\infty \frac{ds}{s} (4\pi s)^{\frac{-D}{2}} e^{-sm^2} \Theta_3(i\frac{\beta\bar{\mu}_0}{2\pi}, i\frac{\beta^2}{4\pi s})$$
. (A.8)

Here $\bar{\mu}_0$ now appears only in the first argument of the Θ_3 , and remember that $\bar{\omega}$ also appears in the same place as $\bar{\mu}_0$ does for $N \neq 0$. Taking account of exponent's additivity in the first argument of Θ_3 (q.v. Eq.(3.5)), one can imagine that the final result is given by the replacement of $\bar{\mu}_0$ with $\bar{\mu}_0 - i\bar{\omega}$ ($= \bar{\mu}$). This result coincides with the $\mathcal{K}_{\beta\mu}$ obtained by the shift manipulation (3.9) in \mathcal{K}_{β} .

Appendix B. Photon self-energy part from Feynman rule

In this appendix, we rearrange the N=2 Feynman amplitude of photon scattering into the world-line formula at finite β and μ . The photon self-energy part (in D=4) by the Feynman diagram technique is written in the form

$$\Pi_{\beta} = -\frac{1}{\beta} \sum_{n} \int \frac{d^{3}p}{(2\pi)^{3}} \frac{\text{Tr}[\not e_{1}(\not p - \not k_{1})\not e_{2}\not p]}{p^{2}(p-k)^{2}} , \qquad (B.1)$$

where $p^{\mu} = (\omega_n, \vec{p})$, $k_i^{\mu} = (\omega_{k_i}, \vec{k_i})$, and $\epsilon_i \equiv \epsilon_{\mu}^i$, i = 1, 2. We also make use of $k^{\mu} = k_1^{\mu} = -k_2^{\mu}$. The QED coupling e is set to be the unity for simplicity. For convenience, we decompose the Π_{β} into the following three parts:

$$\Pi_i = \frac{1}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} W_i , \qquad (B.2)$$

in terms of

$$-\operatorname{Tr}[\mathscr{E}_{1}(\mathscr{p}-\mathscr{k})\mathscr{E}_{2}\mathscr{p}] = W_{1} + W_{2} + W_{3} , \qquad (B.3)$$

where

$$W_1 = 2\epsilon_1 \cdot \epsilon_2 (p^2 + (p - k)^2) , \qquad (B.4)$$

$$W_2 = 2(\epsilon_1 \cdot \epsilon_2 k_1 \cdot k_2 - \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1) , \qquad (B.5)$$

$$W_3 = -2\epsilon_1 \cdot (2p - k_1)\epsilon_2 \cdot (2p - k_1)$$
 (B.6)

Applying the Feynman integral formula

$$\frac{1}{p^2(p-k)^2} = \int_0^\infty ds \, s \int_0^1 du e^{-sk^2(u-u^2)} e^{-s(p-(1-u)k)^2} , \qquad (B.7)$$

we easily rewrite Π_1 and Π_2 in the following forms:

$$\Pi_1 = \frac{4\epsilon_1 \cdot \epsilon_2}{\beta} \sum_n \int_0^\infty \frac{dss}{(4\pi s)^{3/2}} \int_0^1 du e^{-sk^2(u-u^2)} e^{-s(\omega_n + a)^2} \frac{1}{s} \delta(1 - u) , \qquad (B.8)$$

$$\Pi_2 = (\epsilon_1 \cdot \epsilon_2 k_1 \cdot k_2 - \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1) \frac{2}{\beta} \sum_n \int_0^\infty \frac{dss}{(4\pi s)^{3/2}} \int_0^1 du e^{-sk^2(u-u^2)} e^{-s(\omega_n + a)^2} , \qquad (B.9)$$

where

$$a = (1 - u)\omega_k . (B.10)$$

The Π_3 can be rewritten in the similar way by using Eq.(B.7) and shifting $\vec{p} \to \vec{p} + (1-u)\vec{k}$ in the *p*-integral. After some algebra, we have

$$\Pi_{3} = \frac{2}{\beta} \sum_{n=0}^{\infty} \int_{0}^{\infty} \frac{ds}{(4\pi s)^{3/2}} s \int_{0}^{1} du e^{-sk^{2}(u-u^{2})} e^{-s(\omega_{n}+a)^{2}}
\times \left[\frac{2}{s} (\epsilon_{0}^{1} \epsilon_{0}^{2} - \epsilon_{1} \cdot \epsilon_{2}) - \left\{ 2\epsilon_{0}^{1}(\omega_{n} + a) - (1 - 2u)\epsilon_{1} \cdot k \right\} \left\{ 2\epsilon_{0}^{2}(\omega_{n} + a) - (1 - 2u)\epsilon_{2} \cdot k \right\} \right].$$
(B.11)

Gathering Eqs.(B.8), (B.9) and (B.11), we obtain

$$\Pi_{\beta} = \frac{2}{\beta} \sum_{n} \int_{0}^{\infty} \frac{dss}{(4\pi s)^{3/2}} \int_{0}^{1} du e^{-sk^{2}(u-u^{2})} e^{-s(\omega_{n}+a)^{2}} \\
\times \left[\frac{2}{s} \epsilon_{1} \cdot \epsilon_{2} \{ \delta(1-u) - 1 \} + (\epsilon_{1} \cdot \epsilon_{2}k_{1} \cdot k_{2} - \epsilon_{1} \cdot k_{2}\epsilon_{2} \cdot k_{1}) \right] \\
- (1-2u)^{2} \epsilon_{1} \cdot k\epsilon_{2} \cdot k - 4\epsilon_{0}^{1} \epsilon_{0}^{2} \{ (\omega_{n}+a)^{2} - \frac{1}{2s} \} \\
+ 2(1-2u)(\epsilon_{0}^{1} \epsilon_{2} \cdot k + \epsilon_{0}^{2} \epsilon_{1} \cdot k)(\omega_{n}+a) \right].$$
(B.12)

Now, we can eliminate the forms $(\omega_n + a)^m$; m = 1, 2, in the summand of Eq.(B.12) due to the following operations:

$$-\frac{1}{2s}\partial_a \sum_n e^{-s(\omega_n + a)^2} = \sum_n (\omega_n + a)e^{-s(\omega_n + a)^2},$$
 (B.13)

$$\left(\frac{1}{2s}\partial_{a}\right)^{2}\sum_{n}e^{-s(\omega_{n}+a)^{2}} = \sum_{n}\left[(\omega_{n}+a)^{2} - \frac{1}{2s}\right]e^{-s(\omega_{n}+a)^{2}}.$$
 (B.14)

The a-dependence of the summand exponential can be transformed into a linear exponent form by using the Jacobi transformation:

$$\sum_{n} e^{-s(\omega_{n}+a)^{2}} = e^{-sa^{2}}\Theta_{2}(\frac{2sia}{\beta}, \frac{is4\pi}{\beta^{2}}) = \frac{\beta}{\sqrt{4\pi s}}\Theta_{4}(\frac{\beta a}{2\pi}, \frac{i\beta^{2}}{4\pi s})$$

$$= \frac{\beta}{\sqrt{4\pi s}}\sum_{n} (-1)^{n} e^{-\frac{n^{2}\beta^{2}}{4s}} e^{in\beta a}.$$
(B.15)

Then we are allowed to perform the following replacement in the transformed summands

$$\frac{\partial}{\partial a} \rightarrow in\beta$$
, (B.16)

and we finally derive the formula

$$\Pi_{\beta} = 2 \int_{0}^{\infty} \frac{dss}{(4\pi s)^{4/2}} \int_{0}^{1} du e^{-sk^{2}(u-u^{2})} \sum_{n} (-1)^{n} e^{-\frac{n^{2}\beta^{2}}{4s}} e^{in\beta a} \\
\times \left[\epsilon_{0}^{1} \epsilon_{0}^{2} \frac{n^{2}\beta^{2}}{s^{2}} + \frac{in\beta}{s} (2u-1)(\epsilon_{0}^{1} \epsilon_{2} \cdot k_{1} - \epsilon_{0}^{2} \epsilon_{1} \cdot k_{2}) \right. \\
+ \left. \frac{2}{s} \epsilon_{1} \cdot \epsilon_{2} \left\{ \delta(1-u) - 1 \right\} + (\epsilon_{1} \cdot \epsilon_{2} k_{1} \cdot k_{2} - \epsilon_{1} \cdot k_{2} \epsilon_{2} \cdot k_{1}) \\
+ \left. (1-2u)^{2} \epsilon_{1} \cdot k_{2} \epsilon_{2} \cdot k_{1} \right]. \tag{B.17}$$

For $\mu \neq 0$, we have only to make the standard shift (1.1), and this causes the following change of the a defined in Eq.(B.10):

$$a \rightarrow (1-u)\omega_k + i\mu$$
 (B.18)

This modification exactly corresponds to the shift (3.9). Note that the a coincides with $\bar{\omega}$ with fixing $\sigma_1 = 1$.

Appendix C. World-line method for photon self-energy part

In this appendix, we illustrate how we obtain the thermodynamic version of the vertex structure function in the case of N=2 photon scattering. In the world-line formalism, the photon self-energy part at zero temperature can be obtained from the formula

$$\Gamma_2 = -\frac{1}{2} 2^{\frac{D}{2}} \int_0^\infty \frac{ds}{s} e^{-sm^2} \oint \mathcal{D}x \mathcal{D}\psi \exp\left[-\int_0^s (\frac{1}{4}\dot{x}(\tau) + \frac{1}{2}\psi\dot{\psi})d\tau\right] V_1 V_2 , \qquad (C.1)$$

where V_j , j = 1, 2, are the photon vertex operators

$$V_j = -ie \int_0^s d\tau_j (\epsilon_j \cdot \dot{x} + 2i\psi \cdot \epsilon_j \psi \cdot k_j)(\tau_j) e^{ik_j \cdot x(\tau_j)} , \qquad (C.2)$$

and $\dot{x} = \partial_{\tau} x$ etc. Here we follow the standard world-line notation τ , which is related to the main text notation σ by

$$\tau_i = s\sigma_i$$
 (C.3)

Eq.(C.1) is known to become ⁵

$$\Gamma_2 = -\frac{1}{2} 2^{\frac{D}{2}} \int_0^\infty \frac{ds}{s} \int_0^s d\tau_1 \int_0^s d\tau_2 \, \mathcal{V} \times \mathcal{K}$$
 (C.4)

⁵We set the QED coupling e = 1 in the following (as in Appendix B).

with the vertex structure function

$$\mathcal{V} = \epsilon_1 \cdot \epsilon_2 \ddot{G}_R^{12} + \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 (\dot{G}_R^{12})^2 + (\epsilon_1 \cdot \epsilon_2 k_1 \cdot k_2 - \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1) (G_F^{12})^2 , \qquad (C.5)$$

and the kinematical factor

$$\mathcal{K} = \mathcal{N} \langle e^{ik_1 \cdot x(\tau_1)} e^{ik_2 \cdot x(\tau_2)} \rangle = e^{-sm^2} \oint \mathcal{D} x e^{-\int_0^s \frac{1}{4} \dot{x}^2 d\tau} \prod_{j=1}^2 e^{ik_j \cdot x(\tau_j)}$$

$$= (4\pi s)^{-D/2} e^{-sm^2} e^{k_1 \cdot k_2 G_B^{12}}, \qquad (C.6)$$

where

$$G_B^{12} = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{s}, \qquad G_F^{12} = \operatorname{sign}(\tau_1 - \tau_2) .$$
 (C.7)

The path integral normalizations are chosen to be

$$\oint \mathcal{D}x e^{-\int_0^s \frac{1}{4}\dot{x}^2 d\tau} = (4\pi s)^{-D/2} , \qquad \oint \mathcal{D}\psi e^{-\int_0^s \psi \cdot \dot{\psi} d\tau} = 1 .$$
(C.8)

It can be said that the kinematical factor is defined by the insertions of the ϕ^3 scalar vertex operators

$$V_j = \int_0^s d\tau_j \exp[ik_j \cdot x(\tau_j)] \; ; \qquad j = 1, 2 \; .$$
 (C.9)

According to the program presented in Section 3, we are led to calculate the following path integral at finite temperature:

$$\Gamma_{2}^{\beta} = -\frac{1}{2} 2^{\frac{D}{2}} (ie)^{2} \int_{0}^{\infty} \frac{ds}{s} e^{-sm^{2}} \oint \mathcal{D}x \mathcal{D}\psi \exp\left[-\int_{0}^{s} (\frac{1}{4}\dot{x}^{2} + \frac{1}{2}\psi \cdot \dot{\psi})d\tau\right]
\times \left(\prod_{i=1}^{2} \int_{0}^{s} d\tau_{i} e^{ik_{i}x(\tau_{i})}\right) \sum_{n} (-1)^{n} e^{-\frac{n^{2}\beta^{2}}{4s}} e^{in\frac{\beta}{s}(\tau_{1} - \tau_{2})\omega_{k}}
\times \left(\epsilon_{0}^{1} \frac{n\beta}{s} + \epsilon^{1} \cdot \dot{x} + 2i\psi \cdot \epsilon_{1}\psi \cdot k_{1}\right) (\tau_{1}) (\epsilon_{0}^{2} \frac{n\beta}{s} + \epsilon^{2} \cdot \dot{x} + 2i\psi \cdot \epsilon_{2}\psi \cdot k_{2}) (\tau_{2}) . \quad (C.10)$$

Using the Wick contraction method with the correlators

$$\langle x^{\mu}(\tau_1)x^{\nu}(\tau_2)\rangle = -g^{\mu\nu}G_B^{12}, \qquad \langle \psi^{\mu}(\tau_1)\psi^{\nu}(\tau_2)\rangle = \frac{1}{2}g^{\mu\nu}G_F^{12}, \qquad (C.11)$$

one can verify the coincidence of the Γ_2^{β} with the Π_{β} derived in Appendix B; we have arrived at the form

$$\Pi_{\beta} = \frac{1}{2} 2^{\frac{D}{2}} \int_0^{\infty} \frac{ds}{s} e^{-sm^2} \left(\prod_{i=1}^2 \int_0^s d\tau_i \right) \sum_n (-1)^n \mathcal{K}_{\beta}^{(n)} (\mathcal{V} + \tilde{\mathcal{V}}) , \qquad (C.12)$$

where

$$\tilde{\mathcal{V}} = \epsilon_0^1 \epsilon_0^2 \frac{n^2 \beta^2}{s^2} + \frac{i n \beta}{s} (\epsilon_0^1 \epsilon_2 \cdot k_1 - \epsilon_0^2 \epsilon_1 \cdot k_2) \dot{G}_B^{12} , \qquad (C.13)$$

and the $\mathcal{K}_{\beta}^{(n)}$ is the nth mode of the bosonic two point function at finite β defined by

$$\mathcal{K}_{\beta}^{(n)} = \oint \mathcal{D}x e^{-\int_{0}^{s} \frac{1}{4}\dot{x}^{2}} \prod_{j=1}^{2} e^{ik_{j} \cdot x(\tau_{j})} \Big|_{x^{0} \to x^{0} + n\beta\tau/s}$$

$$= e^{-\frac{n^{2}\beta^{2}}{4s}} e^{in\beta(\tau_{1} - \tau_{2})\omega_{k}/s} (4\pi s)^{-D/2} e^{-k^{2}G_{B}^{12}}.$$
(C.14)

Here Eq.(C.12) with (C.13) and (C.14) reproduces the Feynman rule result (B.17) with rescaling $\tau_i = s\sigma_i$ and fixing $\sigma_1 = 1$ with $\sigma_2 = u$. Note that \dot{G}_B^{12} behaves as 2u - 1 in this respect.

Let us further rewrite the above $\tilde{\mathcal{V}}$ in an operator form suitable to Eq.(3.31). In the following, we shall not employ the fixing $\sigma_1 = 1$. As shown in Appendix B, we can replace $in\beta$ with $\partial/\partial\bar{\omega}$, where

$$\bar{\omega} = (\sigma_1 - \sigma_2)\omega_k , \qquad (C.15)$$

and we therefore find

$$\Gamma_2^{\beta} = \frac{1}{2} 2^{\frac{D}{2}} \int_0^{\infty} dss \left(\prod_{i=1}^2 \int_0^1 d\sigma_i \right) \mathcal{V}_{\beta} \times \mathcal{K}_{\beta}$$
 (C.16)

with having

$$V_{\beta} = V - \epsilon_0^1 \epsilon_0^2 (\frac{1}{s} \frac{\partial}{\partial \bar{\omega}})^2 - \frac{1}{s} \frac{\partial}{\partial \bar{\omega}} (\epsilon_0^1 \epsilon_2 \cdot k_1 - \epsilon_0^2 \epsilon_1 \cdot k_2) \dot{G}_B^{12}$$
 (C.17)

and

$$\mathcal{K}_{\beta} = \sum_{n} (-1)^{n} \mathcal{K}_{\beta}^{(n)} e^{-sm^{2}} ,$$
 (C.18)

where the change of variables $\tau_i = s\sigma_i$ in G_B^{12} should be understood. Since we already justified the shift (3.9) in the end of Appendix B, we can use the following parts for the finite μ case

$$\mathcal{V}_{\beta\mu} = \mathcal{V}_{\beta} \left[\frac{\partial}{\partial \bar{\omega}} \to \frac{\partial}{\partial \Omega} \right], \qquad \mathcal{K}_{\beta\mu} = \mathcal{K}_{\beta} \left[\bar{\omega} \to \Omega \right].$$
 (C.19)

Substituting these in Eq.(C.16), we obtain the two-point function $\Gamma_2^{\beta\mu}$, which therefore coincides with the corresponding Feynman rule result suggested in the end of Appendix B.

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