A STUDY OF SOME SOLVABLE QUANTUM MECHANICAL MODELS
AND
THEIR SYMMETRIES

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By

BIKASHKALI   MIDYA

Physics and Applied Mathematics Unit
Indian Statistical Institute
Kolkata-700108, India.

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I Dedicate This Thesis
To
My Parents
For
Their Loving Understanding When I Spent Time On Research Instead Of
Being With Them....

Acknowledgement

Bikashkali Midya

Abstract

The importance of exactly solvable quantum mechanical systems are ubiquitous in modern physics. This is because, the qualitative understanding of a complicated realistic system can be acquired by analyzing exactly solvable simplified model that portrays the purport of a physical reality. On the other hand, discovery of new symmetry enlarges our ability and possibility to describe physical phenomena and to construct their mathematical formulations. In the present thesis, studies on some solvable quantum mechanical systems and the underlying symmetry of their solutions have been addressed.

The problems dealt with in the context of quantum systems characterized by position dependent mass (PDM) are: (A) The modified factorization technique based on excited state wave function(s) has been extended to these systems. It has been shown that the singular superpotential defined in terms of a mass function and an excited state wave function of a given PDM Hamiltonian can be used to construct non-singular isospectral Hamiltonians. (B) Some exactly solvable potentials in the PDM background are generated whose bound states are given in terms of recently discovered Laguerre- or Jacobi-type X_1 exceptional orthogonal polynomials. These potentials are shown, with the help of supersymmetric quantum mechanics, to possess shape invariant symmetry. (C) First and second-order intertwining approach to generate isospectral PDM Hamiltonians with pre-determined spectral characteristics, have been formulated in a systematic way. The proposed techniques allow one to generate isospectral potentials with the following spectral modifications: (i) to add new bound state(s), (ii) to remove bound state(s) and (iii) to leave the spectrum unaffected.

Quantum systems with non-Hermitian Hamiltonians are considered for the following problems: (A) The $\mathcal{PT}$-symmetric Hamiltonian with the periodic potential $V(x) = 4\cos^2 x + 4iV_0 \sin 2x$ has been shown to map into a Hermitian Hamiltonian for $V_0 < .5$, by a similarity transformation: $e^{-\theta p}\, x\, e^{\theta p} = x + i\theta$, $\theta \in \mathbb{R}$. The corresponding energy band structure has been studied using Floquet analysis. It is also shown that, in addition to the $\mathcal{PT}$-threshold at $V_0 = 0.5$, there exist a second critical point near $V_0^c \sim .888437$ after which no part of the eigenvalues and the band structure remains real. (B) Using the method of point canonical transformation, some exactly solvable rationally extended

quantum Hamiltonians which are non-Hermitian in nature and whose bound state wave functions are associated with Laguerre or Jacobi-type X_1 exceptional orthogonal polynomials, have been derived. These Hamiltonians are shown to be quasi Hermitian so that the corresponding energy spectra is entirely real. (C) The generalized Swanson Hamiltonian $H_{GS} = w(\tilde{a}\tilde{a}^\dagger + 1/2) + \alpha\tilde{a}^2 + \beta\tilde{a}^{\dagger^2}$ with $\tilde{a} = A(x)d/dx + B(x)$, has been shown to transform into the harmonic oscillator like Hamiltonian so long as $[\tilde{a}, \tilde{a}^\dagger] = $ constant. This reveals an intriguing result in the sense that in this case the resulting Hamiltonian does not always possess the spectrum of the harmonic oscillator. Reason for this anomaly is discussed in the frame work of position dependent mass models by choosing $A(x)$ as the inverse square root of the mass function.

The aspects of quantum nonlinear oscillator (QNLO) studied in this thesis are the following: (A) The coherent state of QNLO are constructed using the Gazeau-Klauder formalism. The details of the revival structures arising from different time scales underlying the quadratic energy spectrum of the above mentioned quantum system are investigated by the phase analysis of the autocorrelation function. (B) Various generalizations e.g, exactly solvable, quasi exactly solvable and non Hermitian PT-symmetric variants of the quantum nonlinear oscillator have been examined. Supersymmetric approach has been used to show that all the exactly solvable potentials for the QNLO, are shape invariant.

Publications

The thesis is based on the following published papers:

M1. B. Midya

Nonsingular potentials from excited state factorization of a quantum system with position dependent mass

Journal of Physics A44 (2011) 435306.

M2. B. Midya and B. Roy,

Exceptional orthogonal polynomials and exactly solvable potentials in position dependent mass Schrödinger Hamiltonians,

Physics Letters A373 (2009) 4117.

M3. B. Midya, B. Roy and R. Roychoudhury,

Position Dependent Mass Schrödinger Equation and Isospectral Potentials : Intertwining Operator approach ,

Journal of Mathematical Physics 51 (2010) 022109.

M4. B. Midya, B. Roy and R. Roychoudhury,

A note on the $\mathcal{PT}$-invariant periodic potential $V(x) = 4\cos^2 x + 4iV_0 \sin 2x$,

Physics Letters A374 (2010) 2605.

M5. B. Midya,

Quasi-Hermitian Hamiltonians associated with exceptional orthogonal polynomials,

Physics Letters A (2012) in press; arXiv: 1205.5860.

M6. B. Midya, P. Dube and R. Roychoudhury,

Non-isospectrality of the generalized Swanson Hamiltonian and harmonic oscillator

Journal of Physics A44 (2011) 062001.

M7. B. Midya, B. Roy and A. Biswas,

Coherent state of a nonlinear oscillator and its revival dynamics,

Physica Scripta 79 (2009) 065003.

M8. B. Midya and B. Roy,

A generalized quantum nonlinear oscillator ,

Journal of Physics A42 (2009) 285301.

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Chapter 1

Introduction

The search for exactly solvable and integrable models in quantum mechanics have great significance in modern physics. This is because, the qualitative understanding of a complicated realistic system can be acquired by analyzing exactly solvable simplified model that portrays the purport of a physical reality.

The study of such systems in non-relativistic quantum mechanics has a long and cherished history. By exactly solvable, one means that the spectral properties, that is, the eigenvalues and eigenfunctions of the Hamiltonian characterizing the quantum system under consideration, can be given in an explicit and closed form. The most important examples are the harmonic oscillator and the hydrogen atom. First unified venture in finding these systems was made by Schrödinger [1] who initiated the *factorization* method. The main idea of the factorization method is to replace a given Schrödinger equation, which is a second order differential equation, by an equivalent pair of first-order differential equations. This enables one to find the eigenvalues and the normalized eigenfunctions in a far easier manner than solving the original Schrödinger equation directly. As has been emphasized in the literature, the factorization method is a special case of Darboux construction [2]. Darboux's method has been generalized by Crum [3] to the case of an arbitrary number of eigenfunctions. Also, the Schrödinger factorization technique was generalized by Infeld and Hull [4] and Mielnik [5] to obtain a wide family of exactly solvable Hamiltonians. Another method for constructing exactly solvable systems is the method of integral transformation, which forms the basis of the inverse scattering approach [6, 7]. These transformations are derived from the solution of the Gelfand-Levitan equation [8] and are known as Abraham-Moses-Pursey transformations [9, 10].

The concept of solvability has also been extended to *quasi-exactly solvable* (QES) [11, 12] and *conditionally exactly solvable* (CES)[13, 14] models. Quantum mechanical Hamiltonians are said to be QES if a finite portion of the energy spectrum and associated eigenfunctions can be found exactly and in closed form. On the other hand, principal feature of CES Hamiltonians is that they have exact solutions only when certain conditions are satisfied by the potential parameters.

Besides their role in describing physical problems of the microscopic world, solvable quantum mechanical models also represent an interesting direction of research due to the mathematical elegance associated with the symmetries of the systems. Discovery of new symmetry enlarges our ability and possibility to describe new physical phenomena and to construct their mathematical formulations. The most widely known symmetries of quantum mechanical Hamiltonians are based on group theory (in particular Lie algebra). For example, Pauli [15] considered $\mathcal{SO}(4)$ symmetry in quantum mechanics and obtained hydrogen atom spectrum by only algebraic methods, Fock [16] considered the Schrödinger equation of the Kepler problem in momentum space and had shown that the spectrum has $\mathcal{SO}(4)$ symmetry in energy-momentum space. The usual approach [17, 18, 19] to the analysis of QES quantum systems is an algebraic one in which the Hamiltonian is expressed as a nonlinear combination of generators of a Lie algebra like $\mathcal{SL}(2, Q)$, where $Q = R, C$, etc.. The other familiar symmetries of quantum mechanical systems are based on *supersymmetry, $\mathcal{PT}$-symmetry, quasi* and *pseudo-Hermiticity*.

1.1 Supersymmetric quantum mechanics and its application

Supersymmetry: The concept of Supersymmetry (SUSY) owes its origin to the remarkable papers by Ramond [20], Neveu and Schwarz [21], Gelfand and Likhtam [22]. Subsequent to these works, various models embedding SUSY were proposed within a field theoretical framework [23, 24]. The basic composition rules of SUSY contain both commutators and anti-commutators which enable ones to circumvent the powerful "no-go" theorem of Coleman and Mandula [25]. It should be mentioned here that, if supersymmetry exists in nature then it must be spontaneously broken, because there is no evidence for degenerate boson-fermion pairs in nature, and this fact has generated interest in supersymmetry breaking mechanism. In an attempt to construct a theory of SUSY that is unbroken at the tree-level but could be broken by small non-perturbative corrections, Witten [26, 27] proposed a class of grand unified models within a field theoretical framework. Specifically, he considered models (in less than four dimensions) in which SUSY could be broken dynamically. This led to the remarkable discovery of supersymmetric quantum mechanics [28, 29, 30, 31, 32, 33, 34, 35] [1]. Historically, however, it was Nicolai [42] who sowed the seeds of SUSY in non-relativistic quantum mechanics.

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[1]There are some excellent books and review articles on Supersymmetric quantum mechanics e.g. the books [36, 37, 38, 39] and reviews [40, 41].

A quantum system, described by a Hamiltonian H and a set of N self-adjoint charge operators Q_i, is called supersymmetric if the following anti-commutation and commutation relations are valid for all $i, j = 1, \ldots, N$: $\{Q_i, Q_j\} = H\delta_{ij}$, $[H, Q_i] = 0$ [38, 41]. However, Witten's supersymmetry corresponds to $N = 2$ (referred to in the literature as SUSYQM[2]) in which the two complex supercharges $Q^{\pm} = (Q_1 \pm iQ_2)/\sqrt{2}$ satisfy the linear super-algebra [40, 36]:

$$(Q^{\pm})^2 = 0, \quad [H, Q^{\pm}] = 0, \quad \{Q^+, Q^-\} = H,$$

i.e. the charge operators are nilpotent and commute with the super Hamiltonian H. The one-dimensional realization of this super algebra is usually given in terms of 2×2 matrices:

$$Q^- = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} H^- & 0 \\ 0 & H^+ \end{pmatrix}.$$

This implies that the super Hamiltonian H consists of two Schrödinger type operators $H^{\pm}$, which are known as supersymmetric partner Hamiltonians. $H^{\pm}$ can be factorized as:

$$H^{\pm} = A^{\mp}A^{\pm} == -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^{\pm}(x), \quad V^{\pm}(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2m}}W'(x),$$

where the operators A^- and $A^+ = (A^-)^{\dagger}$ are given by $A^{\pm} = \mp\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$. $W(x)$ is known as the superpotential. The Hamiltonians $H^{\pm}$ are isospectral and differ at most in the ground-state energy level. An important aspect of SUSYQM is that a hierarchy of isospectral solvable Hamiltonians, with a relationship between the eigenvalues and eigenfunctions of the different members of the hierarchy, can be constructed by repeated factorization [43, 44, 45, 46]. SUSY is said to be unbroken when $Q^+|0\rangle = Q^-|0\rangle = 0$, where $|0\rangle = (\psi_0^- \quad \psi_0^+)^T$ is the ground state of H [36]. Thus the ground state energy must be zero in case of unbroken SUSY and non-zero for broken SUSY. For unbroken SUSY the energy eigenvalues $E_n^{\pm}$ and the eigenfunctions $\psi_n^{\pm}$ of $H^{\pm}$ are related in the following way:

$$E_n^+ = E_{n+1}^-, \quad E_0^- = 0, \quad n = 0, 1, 2...$$

$$\psi_n^+ = [E_{n+1}^-]^{-\frac{1}{2}}A^-\psi_{n+1}^-, \quad \psi_{n+1}^- = [E_n^+]^{-\frac{1}{2}}A^+\psi_n^+$$

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[2]Throughout the thesis, we use the abbreviation SUSYQM to refer $N = 2$ supersymmetric quantum mechanics.

Witten [27] proposed an index Δ_F, known as Witten index, to determine whether SUSY is broken or unbroken. This index, defined by the difference between the number of zero-energy states of the partner Hamiltonians, is non-zero (zero) for unbroken (broken) SUSY. However, certain supersymmetric periodic quantum systems [47] may produce a zero-energy doublet of the ground states, resulting in a completely isospectral pair of partner Hamiltonians. If this happens, one has $\Delta_F = 0$ even in the case of unbroken supersymmetry. Such special property is named as self-isospectrality in the literature [48, 49].

In a nonlinear generalization[3] of SUSYQM, two supercharges are higher-order ($n \geq 1$) differential operators, which satisfy the nonlinear super algebra [50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60]: $(Q^{\pm})^2 = 0$, $[H, Q^{\pm}] = 0$, $\{Q^+, Q^-\} = \mathcal{P}_n(H)$, where $\mathcal{P}_n(H)$ is a n-th degree polynomial of super-Hamiltonian H. The generalization of one-dimensional SUSYQM to higher space dimension $d \geq 2$ is made in ref.[61, 62, 63, 64]. The attempt to preserve the one-dimensional SUSYQM algebra but with superpotentials depending on multidimensional coordinate leads to potentials which are amenable to standard separation of variables [65]. Among multidimensional SUSY QM models, those most developed are the two-dimensional ones [67, 68, 69, 66]. In the two-dimensional case the supersymmetrization of a given Schrödinger Hamiltonian involves not only a second Hamiltonian but also a 2×2 matrix Schrödinger operator. The spectra of these three components of the super Hamiltonian are interrelated, and their wave functions are connected by the components of supercharges. Other generalization of the one-dimensional SUSYQM are para [70, 71, 72], ortho [73] and fractional [74, 75] supersymmetry.

Shape Invariance: A conceptual breakthrough in understanding the connection between solvable potentials and SUSY was made by Gendenshtein [76] who introduced a discrete re-parametrization invariance called *shape invariance* (SI). Two supersymmetric partner potentials $V^{\pm}$ are said to be shape invariant if they satisfy [40]:

$$V^+(x, a_0) = V^-(x, a_1) + R(a_0),$$

where the parameters a_0, $a_1 = f(a_0)$ represent characteristics of the potentials, such as its strength, depth or width. Gendenshtein showed that whenever the shape invariance

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[3]In the literature, there are several synonyms for the nonlinear SUSY: the higher order SUSY [60, 59], the polynomial SUSY [50, 52, 53] or the $\mathcal{N}-$fold SUSY [57, 58].

relationship is satisfied by the two supersymmetric potentials, the energy spectra can be determined in purely algebraic fashion. Specifically, energy eigenvalues are given by

$$E_n^- = \sum_{k=1}^{n} R(a_k), E_0^- = 0.$$

Subsequently, it has been shown [77, 78] that the wave functions $\psi_n^-(x; a_0)$ of the potential V^- can also be obtained in a straightforward way by consecutive application of ladder operator $A^+(x; a_k)$ as

$$\psi_n^-(x, a_0) = A^+(x; a_0)A^+(x; a_1)....A^+(x; a_{n-1})\psi_0^-(x; a_0).$$

A systematic study of the relationship between solvability and shape invariance was carried out in [79]. This study has revealed that most of the known solvable potentials (such as Coulomb, harmonic oscillator, Morse, Eckart, Poschl-Teller, Scarf, Rosen-Morse etc) are shape invariant. The explicit expressions of the wave functions of these known SI potentials are obtained by Dabrowska et al [80]. However, the exactly solvable Ginocchio class [81] of potentials and more generally the class of Natanzon potentials [82], for which the corresponding Schrödinger equation can be reduced to confluent or general hypergeometric differential equation, are in general not shape invariant. This implies that shape invariance integrability condition is a sufficient, but not necessary, condition for exact solvability. Mainly four classes of shape invariant potentials (SIP) are reported in the literature: (i) the translational class [79, 83], where the parameters a_0 and a_1 of the two supersymmetric partners are related to each other by translation $a_1 = a_0 + \alpha$; (ii) the scaling class [84, 85], where $a_1 = qa_0, (0 < q < 1)$; (iii) the cyclic class [86], characterized by the parameters which repeat after a cycle of p iterations i.e. $a_0 = a_p, a_1 = f(a_0) = a_{p+1}$ and (iv) the exotic class [85], where $a_1 = qa_0^p$ and $a_1 = qa_0/(1 + pa_0)$, $p = 2, 3....$ The potential algebra of these systems have also been identified [87, 88, 89, 90, 91, 92], which provides an alternate methods of finding exact solutions. The idea of SI has been extended to the more general concept of SI in two steps and even multi-steps [85, 93]. The two-step SI approach was utilized for dealing with problems with spontaneously broken SUSY [94, 95]. Very recently [96, 97, 98], it has been shown that all the additive SI potentials can be generated from the Euler equation of momentum conservation for inviscid fluid flow.

Application of SUSYQM and SI approach: The idea of of SUSYQM and shape invariance approach have profitably been applied to enlarge the families of exactly [99, 100, 101, 102, 83], quasi-exactly [103, 104, 105, 106, 107] and conditionally exactly [14, 108, 109] solvable potentials in non-relativistic quantum mechanics. SUSYQM allows one to derive potentials which provide the same phase shifts as a given potential, even if the number of bound states is modified [110, 111, 112, 113, 114]. Starting from a potential with a continuum of energy eigenstates, the methods of supersymmetric quantum mechanics has been used to generate families of potentials which have bound states embedded in the continuum [115, 116]. The relationships between reflection and transmission amplitudes of two partner potentials supporting scattering states are obtained by SUSYQM [117, 118]. In order for scattering to take place, it is necessary that both the partner potentials $V^\pm$ are finite as $x \to +\infty$ or as $x \to -\infty$ or both [40]. Using the formalism of supersymmetric quantum mechanics, a large number of new analytically solvable one-dimensional periodic potentials have been obtained and their band structures are studied [119, 120, 121, 122]. In particular the energy band structure of lame potential $V(x) = ma(a+1) \; sn^2(x, m)$ and associated Lame potential $V(x) = pm \; sn^2(x, m) + qm \; cn^2(x, m)/dn^2(x, m)$, both of which involve Jacobi elliptic function and modulation parameter m, are discussed. The supersymmetric partners of these potentials and corresponding band edge energies and the wave functions are also derived [119]. Application of irreducible second order SUSYQM to the Schrödinger operators, with Lame and periodic piece-wise transparent potentials, reveal that the pairs of factorization energies inside the same forbidden band can create new non-singular potentials with periodicity defects and bound states embedded into the spectral gaps [123].

Approximate methods based on SUSYQM, like large-N expansion [124, 125], δ-expansion for the superpotential [126], a variational method [127, 128, 129], and WKB approximation (SWKB) for the case of unbroken SUSY [130, 131, 132] have also been developed. The large-N expansion method has been formulated [124] in the framework of SUSYQM to study the spherically symmetric potentials, e.g. Hulthen, screened Coulomb potentials etc., in N spatial dimensions. It is shown that the supersymmetric partner potential of a given potential can be effectively treated as being in $N + 2$ dimensions. This fact is responsible for improving the convergence of large-N expansions. The δ expansion method has been used in ref.[126] to solve the Riccati equation for the superpotential involved in factorization of the Hamiltonian for the anharmonic oscillator with

$V(x) = \lambda x^{2+2\delta}$. Variational method based on the hierarchy of Hamiltonians [127] permits one to evaluate the excited states for one-dimensional systems like anharmonic [128] and Morse [129] oscillator. SWKB reveals several interesting features. Unlike the standard WKB method, the leading order SWKB formula yields exact analytic expressions for the energy eigenvalues. It is not only exact for large n (like WKB approximation) but by construction it is also exact for the ground state of the partner potentials [134]. Besides, it has been proved [77] that the lowest-order SWKB quantization relation reproduces the exact bound-state spectra for several analytically solvable potentials. The reason behind obtaining the exact analytic results for these potentials is that they satisfy the shape-invariance condition and for shape-invariant potentials (with translation), lowest-order SWKB is necessarily exact [77]. A systematic higher order SWKB expansion has also been developed to obtain an explicit expression for the quantization condition which contains all terms up to order $\hbar^6$, which vanishes for all known shape-invariant potentials [135]. Energy eigenvalue spectrum has also been obtained for several non-SIPs [136, 137] and it turns out that in many of the cases the SWKB gives better results than the usual WKB approximation. In ref.[138, 139], some attempts have been made to obtain the bound state eigenfunctions within SWKB formalism. The lowest order WKB quantization condition, in case of broken SUSY, has also been derived [140, 141]. It is worth mentioning here that the ideas of supersymmetric quantum mechanics are applied to the tunneling problem for both the symmetric [142] and asymmetric [143] double-well potentials. The tunneling time is evaluated by developing a systematic perturbation expansion whose leading term is an improvement over the standard WKB result.

Several aspects of the Dirac equation have been studied within SUSYQM formalism [144, 145, 146] as well. In particular, it has been shown [146] that corresponding to an analytically solvable one-dimensional Schrödinger problem with potential $V(x)$ there always exists a corresponding Dirac problem with static scalar potential $\phi(x)$ in (1+1) dimensions which is also analytically solvable, $V(x)$ and $\phi(x)$ are related by $V(x) = \phi^2(x) + \phi'(x)$. The existence of SUSY for massless Dirac equation in two as well as in four Euclidean dimensions has also been shown. The problem of the Dirac particle in a Coulomb field has been solved algebraically by using the concepts of SUSY and shape invariance [147]. The SUSY of the Dirac electron in the field of a magnetic monopole has been studied [148]. The level degeneracies of a Dirac electron in a constant magnetic field [149] and the accidental degeneracy of systems with spin-orbit coupling [150] have been interpreted using SUSYQM.

A supersymmetric analysis is presented for the d-dimensional Dirac equation with central potentials under spin-symmetric and pseudo spin-symmetric regimes [151]. In particular, such Dirac Hamiltonian with Coulomb and Kratzer potentials have been solved. Exactly solvable scalar and vector potentials for $(1 + 1)$ dimensional Klein-Gordon equation has been obtained using supersymmetric shape invariance approach [152]. Supersymmetry in quantum mechanics is also extended from square-integrable states to quasi-normal modes (which satisfy the outgoing-wave boundary condition), in a Klein-Gordon formulation [153, 154].

The supersymmetric generalization of the exactly solvable N-particle Calogero model was considered and related spectrum was found exactly in [155, 156, 157, 537]. The intimate connection between the soliton solutions of the KdV equation and SUSYQM was studied [159, 160]. Supersymmetry has been extended to the non-commutative plane [161, 162] as well. It has been shown [161] that the Supersymmetry algebra for Pauli equation on non-commutative plane holds for all orders of the noncommutative parameter θ provided the gyro-magnetic ratio is 2. The energy spectrum of this model is also obtained when the corresponding magnetic field is uniform. Supersymmetric quantum mechanics has also been formulated [163] on a two-dimensional noncommutative plane. This approach has been applied to factorize the non-commutative harmonic oscillator. It turns out that the supersymmetry is partially broken and the number of supercharges decreases in the presence of non-commutativity. Recently, supersymmetry is formulated for one-dimensional quantum mechanical systems with reflections [164, 165]. One of its characteristic features is that both a supersymmetric Hamiltonian and a supercharge component involve reflection operator $\mathcal{R}$ (which is defined by $\mathcal{R}f(x) = f(-x)$). An important aspect shown in ref. [164] is that exact eigenfunctions of such a system are expressed in terms of little -1 Jacobi polynomials which is one of the missing families of classical orthogonal polynomials.

Factorization based on Excited state: The factorization of Schrödinger Hamiltonian in SUSYQM, discussed earlier, gives rise a Riccati differential equation for the super-potential [36, 37]. The superpotential is usually defined in terms of ground state of a Hamiltonian. But the factorization is not unique in the sense that two different super-potentials can give rise to the same potential. One can try to construct new isospectral potentials exploiting non-uniqueness of factorization and obtain a one-parameter family of

potentials with the parameter arising as an integration constant [5, 166]. In the standard unbroken SUSYQM, it is impossible to use an excited state of the initial Hamiltonian and at the same time avoid creating singularities in the partner potential [167]. Singularities, which appear at the nodes of excited state wave function, are responsible for both the destruction of degeneracy of the spectrum of the partner Hamiltonians and creation of negative energy state(s) [168, 169, 170]. Physically, the destruction of degeneracy with a singular superpotential occurs because the singularity imposes additional boundary conditions on the wave functions and breaks the real axis into two or more disjoint intervals. As a result, some or all of the wave functions of the SUSY partner Hamiltonian may not belong to the same Hilbert space of square integrable functions. In that case, the usual proof of degeneracy between the excited states of the partner Hamiltonians does not hold, since the supersymmetric operators map square integrable functions to state outside the Hilbert space [171]. Very recently, the non-uniqueness of the factorization has been exploited in ref. [172, 173, 174] to construct nonsingular isospectral partner Hamiltonians by using excited state wave function of a given nonsingular Hamiltonian. The algorithm of this 'modified factorization' is independent of the choice of wave functions used to define the superpotential and removes the ambiguity about the non degeneracy between the spectrum of isospectral partners.

Exceptional Orthogonal Polynomials and SUSYQM: Recently, SUSYQM has also attracted some attention in the construction of exactly solvable rationally extended potentials [175, 176, 177, 178, 179, 180, 181, 182, 183, 184] whose bound state wave functions are given in terms of recently discovered exceptional orthogonal polynomials (EOPs). Exceptional X_l orthogonal polynomials are the infinite sequence of eigenfunctions, $\{\widehat{P}_n\}_{n \geq l}^{\infty}$, $l = 1, 2, 3...$, which satisfy a class of Sturm-Liouville problems:

$$p(x)\widehat{P}_n''(x) + q(x)\widehat{P}_n'(x) + r(x)\widehat{P}_n(x) = \lambda_n \widehat{P}_n(x),$$

where $p(x), q(x)$ and $r(x)$ are rational functions of x [185]. In contrast to the families of (Jacobi, Laguerre and Hermite) classical orthogonal polynomials, which start with a constant, the X_l EOP families begin with polynomial of degree $l \geq 1$, but still form complete orthogonal sets with respect to some positive-definite measure. The first examples of EOP, the so-called Laguerre- and Jacobi-type X_1 families, corresponding to $l = 1$, are proposed in [185, 186]. These X_1 EOPs are then shown to appear in the bound state wave functions of rational extensions of the radial oscillator and Scarf I potential respectively

[175]. Such rational extensions are also re-constructed in the framework of SUSYQM, where they give rise to a new class of translationally shape invariant potentials [187]. The most general X_l, $l = 1, 2, 3...$, Laguerre and Jacobi type exceptional polynomials and associated shape invariant potentials are obtained by Sasaki and Odake [179] (the case of $l = 2$ was also discussed in Ref.[176]). Some important properties of the X_l EOPs, such as Gram-Schmidt orthonormalization, Rodrigues formulas, generating functions, the actions of the forward and backward shift operators on these polynomials [188] and structure of their zeros [189] are studied. The X_l EOP and associated shape invariant potentials are also shown to be obtainable through several equivalent approaches to the SUSYQM procedure, such as the Darboux-Crum transformation [186, 190], the Darboux-Bäcklund transformation [177] and the pre-potential approach [191, 192]. EOPs are generalized to multi-indexed families by making use of multi-step Darboux transformations [193], the Crum-Adler mechanism [194], and higher-order SUSYQM [181]. Exceptional X_l Laguerre polynomials are shown to constitute the main part of the solutions to the Dirac equation coupled minimally and non-minimally with some external fields and also to the Fokker-Planck equation [195]. Conditionally exactly solvable partners of the radial and linear oscillators, with broken as well as unbroken supersymmetry, are shown to be associated with exceptional Laguerre and exceptional Hermite polynomials [196]. Also the EOPs have been studied comprehensively within the framework of the supersymmetric WKB quantization condition [197], quasi-exactly solvable problems and $\mathcal{N}$-fold supersymmetry [198]. Recently, EOPs are studied in connection with Wilson, Askey-Wilson, Racah and q-Racah polynomials used in the framework of discrete quantum mechanics [199, 180, 200].

Spectral design via SUSYQM: There is a growing interest nowadays to design systems whose Hamiltonians have predetermined spectral characteristics. This is useful mostly in producing artificial low dimensional structures with controllable physical properties e.g., quantum dots, quantum wells [201], traps in atomic physics [202], spatially confined quantum systems [203] etc. In the quantum well profile optimization isospectral potentials are generated through SUSYQM [204]. These are necessary because, for instance, intersubband optical transitions in a quantum well, may be grossly enhanced by achieving the resonance conditions i.e. appropriate spacings between the most relevant states and also by tailoring the wave functions so that the matrix elements relevant for this particular effect are maximized [205]. In this context, the idea of designing potentials with prescribed quantum spectra is worth investigating. Some progress in this area has

been made by restricting the construction of potentials isospectral to a given initial one except for a few energy values through the usage of SUSYQM, Darboux transformation, factorization method [5, 166, 34, 206, 207, 208, 209, 210, 211, 212, 213] and Abraham-Moses-Pursey procedure [9, 10]. The underlying idea of most of these procedures has been summarized in an algebraic scheme known as *intertwining* approach. In the n-th order intertwining approach, two different Hamiltonians H_0 and H_1, with nearly same spectrum, are intertwined by a nth order linear differential operator $\mathcal{L}$ as [214, 215]:

$$\mathcal{L}H_0 = H_1\mathcal{L}.$$

$\mathcal{L}$ is known as the intertwiner. This intertwining relation allows one to construct a new Hamiltonian H_1 isospectral to the initial Hamiltonian H_0. Moreover, if ψ is an eigenfunction of H_0 with eigenvalue E then $\phi = \mathcal{L}\psi$ is also an eigenfunction of H_1 with the same eigenvalue E. When H_0 and H_1 are Hermitian operators, $\mathcal{L}^\dagger$ acts as another intertwiner which intertwines in the reverse way: $H_0\mathcal{L}^\dagger = \mathcal{L}^\dagger H_1$ and this in turn implies the existence of a hidden symmetry $[H_0, \mathcal{L}^\dagger\mathcal{L}] = 0 = [\mathcal{L}\mathcal{L}^\dagger, H_1]$ [216]. The ingredients to implement the first order intertwining, which is equivalent to first order SUSYQM, are nodeless seed solutions of H_0 associated to factorization energies less than or equal to the ground state energy [59, 217]. Thus first order intertwining allows one to modify only the ground-state energy level of the initial Hamiltonian H_0 to obtain new non-singular partner Hamiltonian H_1. The second order intertwining, which is equivalent to second order SUSYQM [218, 121, 219], has been proved to be very powerful technique to build a new family of isospectral Hamiltonians with desired spectral modifications. In this approach one has the freedom of choosing any seed solution not necessarily nodeless, as in the case of first order intertwining, to generate non-singular SUSY partners. Second order intertwining [121, 219] can be implemented by either (i) iteration of first order Darbaux transformation twice or (ii) looking for the intertwining operator directly as a second order differential operator. The latter option offers several interesting possibilities of spectral manipulation e.g. [220]:

– creation of two new levels between two neighboring energies,

– deletion of two consecutive bound states

– leave the spectrum unchanged.

It is possible to generate families of real isospectral partner potentials by using two com-

plex factorization energies as well [218]. The higher order differential operators for the intertwiners leads in a natural way to the nonlinear [212] or higher order SUSY QM [60] (see also the references cited there in). The method of intertwining has also been extended to higher dimensional time independent [221] and time dependent [222, 223] Schrödinger equation. For the latter case, the conditions for the existence of a Darboux transformation in $(n + 1)$ dimensions are analyzed and compared to their $(1 + 1)$ dimensional counterparts. A complete solution of these conditions is given for $(2 + 1)$ dimensions. The differential-matrix intertwining operator has been used to construct one-dimensional electric potentials or one-dimensional external scalar fields for which the Dirac equation is exactly solvable [224, 225, 226]. Quite recently, the method of intertwining has been used [227] to generate families of isospectral position dependent mass Hamiltonians.

Other areas where the method of SUSYQM has been extended and used are classical mechanics [37, 228, 229, 230], condensed matter [231, 232], statistical [38, 233], nuclear [234, 235, 236, 237, 238] and atomic [239] physics.

1.2 $\mathcal{PT}$-symmetry and Pseudo-Hermiticity in Quantum mechanics

$\mathcal{PT}$-**symmetry**: The interest in non-Hermitian Hamiltonians was stepped up by a conjecture of Bender and Boettcher [240] that some $\mathcal{PT}$-symmetric Hamiltonians could possess real bound-state eigenvalues. A non-Hermitian Hamiltonian H is said to be $\mathcal{PT}$-symmetric if it is invariant under the combined transformation of parity $(\mathcal{P} : \hat{x}, \hat{p} \rightarrow -\hat{x}, -\hat{p})$ and time-reversal $(\mathcal{T} : \hat{x}, \hat{p}, i \rightarrow \hat{x}, -\hat{p}, -i)$ operators i.e.

$$H^{\mathcal{PT}} = \mathcal{PT}H\mathcal{PT} = H.$$

Because the operator $\mathcal{PT}$ is not linear, $[H, \mathcal{PT}] = 0$ does not always signify that the Hamiltonian operator and $\mathcal{PT}$ operator has simultaneous eigenstates. If every eigenfunction of a $\mathcal{PT}$-symmetric Hamiltonian is also an eigenfunction of the $\mathcal{PT}$ operator, it is said that the $\mathcal{PT}$-symmetry of H is unbroken. Conversely, if some of the eigenfunctions of a $\mathcal{PT}$-symmetric Hamiltonian are not simultaneously eigenfunctions of the $\mathcal{PT}$ operator, the$\mathcal{PT}$-symmetry of H is said to be broken [241, 242]. Thus to establish that the eigenfunctions of a particular $\mathcal{PT}$-symmetric Hamiltonian are real, it is necessary to prove

that the $\mathcal{PT}$-symmetry of the Hamiltonian is unbroken. This is difficult to show but a complete and rigorous proof has been given in [243].

A $\mathcal{PT}$-symmetric Hamiltonian need not be Hermitian, thus it is essential to have a fully consistent quantum theory whose dynamics is described by a non-Hermitian Hamiltonian. In the case of a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian it is crucial that the boundary conditions be imposed properly to solve the corresponding Schrödinger eigenvalue problem. Boundary conditions $\psi(x)_{|x|\to\infty} = 0$ in this context, are in general located within wedges bounded by Stokes lines in the complex x-plane [242]. The eigenfunction $\psi(x)$ vanishes most rapidly at the center of the wedges. The natural choice of the inner product suitable for $\mathcal{PT}$-symmetric quantum mechanics is the $\mathcal{PT}$-inner product which is defined by

$$\langle \phi|\psi\rangle^{\mathcal{PT}} = \int_\Gamma \ [\phi(x)]^{\mathcal{PT}}\psi(x) \ dx = \int_\Gamma \ [\phi(-x)]^*\psi(x) \ dx,$$

Γ being the contour in the Stokes wedges. However, this guess for an inner product is not acceptable for formulating a valid quantum theory because the norm of a state is not always positive [244]. One of the ways to circumvent this difficulty is to introduce $\mathcal{CPT}$ inner product [245, 246]:

$$\langle \phi|\psi\rangle^{\mathcal{CPT}} = \int_\Gamma \ [\phi(x)]^{\mathcal{CPT}}\phi(x) \ dx,$$

where $\mathcal{C}$ is the charge conjugation operator which commute with both H and $\mathcal{PT}$. The $\mathcal{C}$ operator is defined in co-ordinate space as a sum over the $\mathcal{PT}$ normalized eigenfunctions ϕ_n of the Hamiltonian: $\mathcal{C}(x,y) = \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y)$. The $\mathcal{C}$ operator has eigenvalues ± 1 and it measures the sign of the $\mathcal{PT}$-norm of an eigenstate. As a result this inner product is positive definite, as states with negative norm are multiplied by -1 when acted on by the $\mathcal{C}$ operator. It is not always easy to obtain a closed form expression of the $\mathcal{C}(x,y)$ operator and often one has to rely on various approximation techniques [247, 248, 249]. In some of the exactly solvable $\mathcal{PT}$-symmetric systems, like Scarf I, the closed form expression of the operator $\mathcal{C}(x,y)$ has been constructed algebraically [250]. The completeness relation for $\mathcal{PT}$-symmetric system has been derived in terms of the $\mathcal{CPT}$ conjugate [251]:

$$\sum_{n=0}^{\infty} \phi_n(x)[\mathcal{CPT}\phi_n(y)] = \delta(x-y).$$

For a $\mathcal{PT}$-symmetric Hamiltonian the potential satisfies $V^*(-x) = V(x)$. This allows one to derive the modified continuity equation for $\mathcal{PT}$-symmetric quantum mechanics which is given by [252, 253] $\rho_t(x,t) + j_x(x,t) = 0$, where $\rho(x,t) = \psi^*(-x,t)\psi(x,t)$ is the probability density and $j(x,t) = i\psi_x^*(-x,t)\psi(x,t) - i\psi^*(-x,t)\psi_x(x,t)$ is the current density. The $\mathcal{PT}$-symmetric extension of quantum mechanics is recently shown [254] to exhibit a correspondence to complex classical mechanics [255] via complex correspondence principle. Other mathematical developments of the theory of $\mathcal{PT}$-symmetry can be found in [256, 257, 258, 259]. All these studies on $\mathcal{PT}$-symmetric quantum mechanics make it clear that the requirement of Dirac Hermiticity, $H = H^\dagger$, for a Hamiltonian to possess real eigenvalues may be relaxed. However,$\mathcal{PT}$-symmetry of a non-Hermitian Hamiltonian is neither a necessary nor a sufficient condition for the reality of its spectrum, only unbroken $\mathcal{PT}$-symmetry is a sufficient condition.

Pseudo-Hermiticity: In a parallel development [260, 261, 262, 263], it has been clarified that energy spectrum of a non-Hermitian Hamiltonian H acting on the Hilbert space $\mathcal{H}$, is real if and only if the following three equivalent conditions (i), (ii) and (iii) holds:

(i) there exist a positive definite, Hermitian operator $\eta : \mathcal{H} \to \mathcal{H}$ that fulfills $H^\dagger = \eta H \eta^{-1}$ i.e. H is *pseudo-Hermitian*,

(ii) H is Hermitian with respect to some positive-definite inner product $\langle .|.\rangle_+$ on $\mathcal{H}$ (which is usually different from the standard inner product $\langle .|.\rangle$). A specific choice for $\langle .|.\rangle_+$ is $\langle \psi|\phi\rangle_\eta = \langle \psi|\eta|\phi\rangle$,

(iii) H may be mapped to a Hermitian Hamiltonian h by a similarity transformation: $h = \rho H \rho^{-1}$, where ρ is linear, invertible and unitary operator. This in turn implies that H is *quasi-Hermitian* [264, 265, 266].

One can relax H to be weak pseudo-Hermitian [267] by not restricting η to be Hermitian. It has been shown [268] that weak pseudo-Hermiticity is not more general than pseudo-Hermiticity, in fact these are essentially complementary concepts. A pseudo-Hermitian quantum system is defined by a (quasi-Hermitian) Hamiltonian operator and an associated metric operator η. So the construction of η is important from a physical point of view. There are various methods of calculating a metric operator e.g. spectral method [269, 270], perturbation technique [271, 272] etc. For potentials with unbroken $\mathcal{PT}$-symmetry it has been shown that the operator $\mathcal{P}$ plays the role of η [260]. In ref.[273], the Hermitian linear

automorphism $\eta = e^{-\theta p}$ is introduced as a pseudo-Hermitian operator which affects an imaginary shift of the co-ordinate:

$$e^{-\theta p} \; x \; e^{\theta p} = x + i\theta.$$

For real potential $V(x)$ and real β, the non-Hermitian Hamiltonians $H = [p + i\beta\nu(x)]^2 + V(x)$ are shown [274] to be pseudo-Hermitian under the gauge like transformation $e^{-2\beta \int \nu(x) dx} [p + i\beta\nu(x)] e^{2\beta \int \nu(x) dx} = p - i\beta\nu(x)$. In an interesting paper [275], it has been shown that every pair of diagonalizable (not necessarily Hermitian) Hamiltonians with discrete spectra possessing real or complex-conjugate pairs of eigenvalues are isospectral and have identical degeneracy structure except perhaps for the zero eigenvalue if and only if they are pseudo-supersymmetric partners. These pseudo-supersymmetric partner Hamiltonians $H_\pm$ may be factorized as $H_+ = LL^\sharp$ and $H_- = L^\sharp L$. The operator $L^\sharp = \eta^{-1} L^\dagger \eta$ is the pseudo-adjoint of the operator L. In particular cases, this factorization applies to $\mathcal{PT}$-symmetric and Hermitian Hamiltonians as well. Spontaneous $\mathcal{PT}$-symmetry breaking is shown to be accompanied by the explicit breaking of pseudo-supersymmetry [276].

The spectrum of non-Hermitian Hamiltonians differ from the spectrum of Hermitian ones by two essential features. These are the possible presence of spectral singularities [277, 278, 279, 280] and exceptional points [281, 282, 283, 284]. In the case of continuous spectra the lack of digonalizability of the Hamiltonian is responsible for the presence of spectral singularity. At the spectral singularity both the reflection and transmission coefficient diverge. Physically, a spectral singularity is the energy of a scattering state that behaves exactly like a zero-width resonance [285, 280]. This resonance phenomena may be used as a method for amplifying guided electromagnetic waves [286]. The spectral singularity of a one-dimensional non-Hermitian potential has been discussed in connection with the supersymmetric quantum mechanics [278] and with the completeness of the bi-orthogonal basis [287]. The existence of a spectral singularity in the case of a delta function potential with complex coupling [279], complex barrier potentials [288] and complex Scarf II potential [289], has been shown. On the other hand, exceptional points (EP) are branch point singularities of the spectrum of a Hamiltonian. When a physical system is characterized by $H = H_0 + \lambda H_1$ with λ being a strength parameter, the spectrum E_n and eigenfunctions ψ_n are in general analytic function of λ. At certain points in the complex λ-plane two energy levels coalesce. Such points are called exceptional points [290, 281, 282, 283]. Physical systems in which such a phenomenon occur are open quan-

tum systems with decaying unbound states [291], atomic spectra [292], $\mathcal{PT}$-symmetric waveguides [293] and chaotic optical microcavity [294] etc. Quasi exactly solvable complex $\mathcal{PT}$-symmetric π periodic potential $V(x) = (i\xi \sin 2x + N)^2$ where ξ is real and N is a positive integer, is shown to have exceptional points for odd values of $N \geq 3$ [295].

Application of $\mathcal{PT}$-symmetry and Pseudo-Hermiticity: The first example of $\mathcal{PT}$-symmetric potential $V(x) = x^2(ix)^\epsilon$, $\epsilon \in \mathbf{R}$, possesses real energy eigenvalues for $\epsilon \geq 0$, which however, turns into complex conjugate pairs as the potential parameter ϵ exceeds a certain critical value at which $\mathcal{PT}$-symmetry is spontaneously broken [240]. This study was performed using numerical technique and WKB approximation. Subsequently, some of the exactly solvable $\mathcal{PT}$-symmetric potentials are obtained by simply setting the coupling constant of the well known potential to imaginary values, while in some other cases the co-ordinate x is shifted to $x - i\epsilon$. Among the one-dimensional exactly solvable $\mathcal{PT}$-symmetric potentials, harmonic oscillator [296], anharmonic oscillator [298], Morse potential [297], Eckert, Poschl-Teller and Hulthen potentials [299], square well potential [300, 301], double well potential with point-interactions [302], Scarf I, II [303, 304, 305, 306] and Rosen-Morse [307, 308] potentials are worth to be quoted. The $\mathcal{PT}$-symmetric versions of shape-invariant potentials have been systematically constructed and conditions have been formulated for having real [309] or complex [310] energy eigenvalues in their spectra. These results follows from the method of variable transformations in which imaginary shift of the co-ordinate appears in a natural way. The Coulomb-harmonic oscillator correspondence in $\mathcal{PT}$-symmetric quantum mechanics has been shown in [311]. Exactly solvable models with $\mathcal{PT}$-symmetry with an asymmetric coupling of channels [312] as well as the coupled channel version of $\mathcal{PT}$-symmetric square well potential [313] are also studied. Group theoretical formalism of potential algebras has been extended to non-Hermitian Hamiltonians by complexifying the algebras $so(2, 1)$, thereby getting the algebra $sl(2, C)$ [314]. The algebra $sl(2, C)$ enables one to construct both $\mathcal{PT}$-symmetric and non $\mathcal{PT}$-symmetric Hamiltonians with real eigenvalues. $\mathcal{PT}$-symmetric potentials like, complex quartic potential [315], quartic potentials with centrifugal and Coulombic terms [316], complex sextic potential [317, 318] and the Khare-Mandal potential $V(x) = -(z \cosh 2x - iM)^2$ [319] are shown to be QES. The QES solvable $\mathcal{PT}$-symmetric partner potential of the Khare-Mandal potential has been obtained in [320]. It has been pointed out recently that even after suitably perturbing either a Hermitian or a complex $\mathcal{PT}$-invariant QES potential, one can still obtain another Hermitian or complex, $\mathcal{PT}$-

invariant, QES potential [321].

The energy band formations due to one-dimensional $\mathcal{PT}$-symmetric periodic potentials $V(x) = i(\sin x)^{2N+1}$ [322] and infinite chain of delta functions with different couplings [323], have been calculated numerically. It has appeared from the graph of the discriminant $\Delta(E)$, of the energy momentum $(E - k)$ dispersion relation, that the band structure of complex periodic potentials appear and disappear under perturbation [322]. Later it was shown that [324] the the appearance and disappearance of such energy bands imply the existence of non real spectra due to complex deformation of real intervals. The energy-momentum dispersion relation of another periodic potential, e.g. complex Kroning-Penney type potential, reveals unusual rounded (continuous) double band (without band gap) structure near Brillouin zone boundaries [325, 326]. Complex $\mathcal{PT}$-invariant periodic potentials have been obtained from the associated Lame potentials using the anti-isospectral transformation $x \to ix + \beta$, where β is any non-zero real number [327, 328]. These periodic potentials are analytically solvable and have finite number of band gaps, when potential parameters are integers. Explicit expressions for the band edges and dispersion relation of some of these potentials are also given.

A general formalism is worked out for the description of one-dimensional scattering in non-hermitian quantum mechanics and constraints on transmission and reflection coefficients are derived in the cases of $\mathcal{P}$, $\mathcal{T}$ or $\mathcal{PT}$ invariance of the Hamiltonian [329]. For a non-Hermitian (complex) scattering potential the probability of reflection (R) and transmission (T) do not always add to 1 and one instead has $R + T + A = 1$, where A is the probability of absorption [330]. Also it has been proved that $R_{right} \neq R_{left}, T_{left} = T_{right}$. The reflectivity turns out to be symmetric i.e. $R_{right} = R_{left}$, if the complex potential is spatially symmetric. One-dimensional scattering by Complex Coulomb potential [331], complex square well [329], complex delta potential [270, 332] and complex Scarf II potential [289] are studied in details. Recently, it has been conjectured that all these $\mathcal{PT}$-symmetric scattering potentials yields the following invariance: $R_{left}(-k) = R_{right}(k), T(-k) = T(k)$ [289].

The idea of supersymmetric quantum mechanics with some modifications can be utilized to study non-Hermitian Hamiltonians having $\mathcal{PT}$-symmetry [333, 334, 335, 243, 336]. The key feature of $\mathcal{PT}$-symmetric SUSYQM is the existence of a complex superpotential

$W(x)$. $W(x)$ has to be broken into real and imaginary parts which are subsequently splitted into even and odd components. The PT-symmetric requirement $V^*(-x) = V(x)$ then imposes conditions on these components of $W(x)$, which are expressed in terms of a system of first-order linear differential equations. This equation is homogeneous when the factorization energy is real and inhomogeneous when it is complex [337]. Realizations of $\mathcal{PT}$-symmetry in terms of second-order SUSYQM and para-supersymmetric quantum mechanics have been constructed [334] and a quasi-parity-dependent factorization energy has been introduced in order to account for the dual structure of the energy levels [338]. SUSYQM has also been used to explain the spontaneous $\mathcal{PT}$-symmetry breaking [243, 301]. Spontaneous $\mathcal{PT}$-symmetry breaking has been studied in the case of exactly solvable $\mathcal{PT}$-symmetric square-well [339], Scarf II [304, 340], purely imaginary delta function potential [341], Coulomb potential [342], as well as for the quasi exactly solvable Khare-Mandal potential [343] and sextic potential [318]. $\mathcal{PT}$-symmetric quantum brachistochrone problem has been studied in refs.[344, 345, 346]. It has been shown [347] that the minimum time, required to reach the final quantum state from an initial state, can be made arbitrarily small without violating time-energy uncertainty principle. The idea of $\mathcal{PT}$-symmetry has been applied to relativistic quantum mechanics, where the time-reversal operator $\mathcal{T}$ reverses the sign of the time operator x^0 [348]. Apart from the above mentioned $\mathcal{PT}$-symmetric models, the other solvable models discussed in the literature are multi-dimensional $\mathcal{PT}$-symmetric models [349, 350, 351], $\mathcal{PT}$-symmetric deformation of Calogero model [352, 353], non-Hermitian quantum graphs [354], quantum lattice models [355, 356].

A major impetus for the study of non-Hermitian Hamiltonians has come from recent experimental observation of $\mathcal{PT}$-symmetry breaking in structured optical waveguides [357, 358], where the complex index of refraction is manipulated by introducing loss and gain terms. These experiments of optical waveguides make use of the quantum-optical analogy that the wave equation for the transverse electric field mode is formally equivalent to the one-dimensional Schrödinger equation. $\mathcal{PT}$-symmetry demands that the complex refractive index obeys the condition $n(\vec{r}) = n^*(-\vec{r})$, in other words, the real part of the refractive index should be an even function of position, whereas the imaginary part must be odd. This analogy has led to several studies on $\mathcal{PT}$-symmetric optical lattices [359, 360, 361] in order to synthesize a new class of materials with properties and characteristics that have no analog in Hermitian gain/loss structures. The stationary

properties of such $\mathcal{PT}$-symmetric optical lattices are theoretically studied in both ID and 2D spatial dimensions. In particular, beam dynamics [359, 362] and Bloch oscillations [363] in these new type of optical structures reveal some of the exotic structures associated with these periodic potentials, such as skewness of the associated Floquet-Bloch modes, non-reciprocity of light propagation, band merging, double refraction, abrupt phase transition [359], power oscillations [358, 359], unidirectional invisibility near the spontaneous $\mathcal{PT}$-symmetry breaking point [364, 365, 366] etc. Quantum scenarios analogous to the waveguide experiments in optics have been reported in ref.[367] where a Bose-Einstein condensate is placed either in a double well potential [293] or into two $\mathcal{PT}$-symmetric δ-function traps [367] and the effects of the non-linearity, in the Gross-Pitaevskii equation, on the $\mathcal{PT}$-properties of the condensate were studied. In the nonlinear domain, one and two dimensional soliton solutions were found to exist below and above the phase transition point in nonlinear $\mathcal{PT}$-lattices [359, 361]. The interplay of the Kerr nonlinearity with the $\mathcal{PT}$-threshold was analyzed [361] and analytical periodic solutions were also derived [368]. Also there exists a vast amount of literature on solitons and Bose-Einstein condensates in periodic optical and nonlinear lattices with $\mathcal{PT}$-symmetry and their nonlinear optical analogs [369, 370, 371, 372] (and references cited there in). The notion of non-Hermiticity in quantum mechanics has also been discussed in quantum field theory [373, 374, 375, 376, 377], cosmology [378, 379], classical mechanics [255, 380], graphene [381], classical optics [382], spin chain [383], many particle quantum systmes [384, 385, 386] and in nonlinear dynamical systems [387, 388] etc.

1.3 Position-dependent mass Schrödinger equation (PDMSE)

The concept of position dependent mass (PDM) comes from the effective mass approximation [389] which is an useful tool for studying the motion of carrier electrons in pure crystals [390] and the virtual-crystal approximation in the treatment of homogeneous alloys as well as in semiconductors. The attention to the effective mass approach stems from the extraordinary development in crystallographic growth techniques like molecular beam epitaxy [391] which allow the production of non uniform semiconductor specimen with abrupt heterojunctions [392]. The presence of position dependent mass in quantum mechanical problems gives rise to the following basic problems:

- Ordering ambiguity [393, 394] arises in defining a unique Hermitian Hamiltonian operator, due to non-commutativity between the momentum operator p and the

mass function $m(x)$. There are many prescriptions for the suitable kinetic energy operator in this scenario, e.g. Gora and Williams [395], Bastard [396] ($r = -1, s = t = 0$), Zhu and Kroemer [397] ($r = t = -\frac{1}{2}, s = 0$), Li and Kuhn [398] ($r = 0, s = t = -\frac{1}{2}$) etc, all of which are reducible from a general form of kinetic energy operator given by Von Roos [399]:

$$H = -\frac{1}{4} \left(m(x)^r p m(x)^s p m(x)^t + m(x)^t p m(x)^s p m(x)^r \right) + V(x),$$

where the ambiguity parameters r, s, t are related by $r + s + t = -1$. For an another choice $r = t = 0, s = -1$, the Von-Roos Hamiltonian reduces to $H = -\frac{1}{2} \left(p \frac{1}{m} p \right) + V$. This form of the Hamiltonian had been first proposed by Ben-Daniel and Duke [400] and later it has been reconsidered by Levy-Leblond [401] in connection with the concept of instantaneous Galilean invariance of the problem of particle with position-dependent mass.

- The presence of discontinuity in the mass function gives rise to the problem of determining appropriate matching condition, on the effective-mass wave function ψ and its spatial derivative, for which the probability current is continuous across the the heterojunction. The matching condition in one-dimension requires that both $m^r \psi$ and $m^{r+t} \frac{d\psi}{dx}$ must be continuous across the junction [402]. The matching conditions to be obeyed by a time-dependent wave function, at a potential and mass discontinuity when the particle interacts with an oscillating electric field, has recently been investigated [403].

- Apart from the usual square integrability condition on the bound state wave function, an additional restriction $|\psi(x)|^2/m(x) \to 0$ at the end points of $\psi(x)$, has to be imposed for the physical systems with position dependent mass which vanishes at one or both end points [404, 405].

Much attention has been given to seek exact, quasi-exact or conditionally exact solutions of Schrödinger equations with PDM due to their usefulness in physical applications. The one dimensional PDMSE with smooth mass and potential steps was solved exactly by Dekar et al.[406, 407]. The wave functions of this system was obtained in terms of Heun's function. Dutra and Almeida [408] discussed the relationship between exact solvability of PDMSE and the ordering ambiguity. In the context of discussing the exact

solvability of Morse potential in PDMSE, it is shown [409] how the map from the full real line $(-\infty, \infty)$ to the half line $(0, \infty)$ connects an exactly solvable Morse potential into an exactly solvable Coulomb one. In this context it is appropriate to mention that quantum nonlinear oscillator (QNLO) can be considered as a system with position dependent mass $m(x) = (1 + \lambda x^2)^{-1}$. The Schrödinger equation for QNLO is exactly solved as a Sturm-Liouville problem and the eigenfunctions are obtained in terms of λ-deformed Hermite polynomials [410, 411]. The formalism of supersymmetric quantum mechanics was extended to include position dependent mass [412, 413, 414, 415, 416, 417, 418] and it has been shown that any one-dimensional quantum system with a given PDM and a potential has a supersymmetric partner characterized by the same PDM but with a new potential. The form of this supersymmetric partner potential depends on the form of the original potential and the form of the position dependent mass [412]. A procedure has been given for the generation of isospectral combinations of the potential and the effective mass variations by supersymmetric quantum mechanics, accompanied by the co-ordinate transformation method [420]. The framework of $\mathcal{N}$-fold supersymmetry in one body quantum mechanical systems has been extended to position dependent mass scenario [421]. Complete classification and general form of effective potentials for type-A $\mathcal{N}$-fold SUSY in the PDM case are given. The effect of position dependent mass profiles on dynamical breaking of $\mathcal{N}$-fold SUSY in several type-B and X_2 models has been investigated in [419]. The SI integrability condition in the PDM scenario leads to certain relationship between the superpotential and the mass function [412]. The method of deformed shape invariance [404], which is based on the equivalence between the PDMSE and the deformed canonical commutation relation [405], has been used to generate pairs of potential for which the PDMSE is exactly solvable. This approach not only gives the bound-state energy spectrum, but also the ground-state and the first few excited-state wave functions. The determination of general wave functions, of these systems, requires to solve a differential-difference equation, which can be avoided by combining the deformed shape-invariance technique with the point canonical transformation method (PCT) [422].

The method of PCT [83, 423], which consists in transforming a PDMSE into a second-order differential equation of some special functions, has been used to obtain wide class of both exactly and quasi-exactly solvable PDM systems [424, 425, 422, 426]. It is shown that [425] the nature of the potentials, namely exact or quasi-exact solvability, depend on the relationship between the mass function and the change of variable used in

point canonical transformation. The interest in quadratic algebra approach to PDMSE is highlighted by constructing spectrum generating algebras for a class of d-dimensional radial harmonic oscillators with $d \geq 2$ [427]. The d-dimensional generalization of the point canonical transformation [428, 429] and first order intertwining [427] for a quantum particle endowed with a position-dependent mass is also described. Quasi-exactly solvable PDM systems like, sextic oscillator, QES Coulomb and QES Morse potentials, whose solutions are given in terms of orthogonal polynomials satisfying mass dependent recurrence relation, are constructed using $sl(2, R)$ Lie algebra [430]. The transmission probabilities of the scattering problem with various position dependent mass are calculated using variable phase method [431] and co-ordinate transformation [432]. Other methods which have been used to study one-dimensional systems with non-constant mass are Green's function approach [433], path integral approach [434], Darboux transformation [435, 436] and Lie algebraic approach [437, 438]. The last one is particularly interesting since it produces not only the solutions but also reveals the symmetry of the problem. In this context, it may be mentioned that for a class of PDMSE the shape invariance condition is equivalent to a potential symmetry algebra [439]. The investigations concerning PDM problems have also been extended to relativistic [440] and non-Hermitian $\mathcal{PT}$-symmetric or more generally pseudo-Hermitian [441, 442] systems as well.

1.4   Coherent States and revival dynamics

<u>**Coherent state**</u>: Another important line of research related to exactly solvable quantum mechanical models is the construction of coherent states (CS). Coherent states were first proposed by Schrödinger [443] in 1926 in connection with the most classical states of the quantum harmonic oscillator. However, CS became widely recognized after the work of Glauber [444] and Sudarshan [445] in 1963. Glauber [444], constructed the eigenstates of the annihilation operator of the harmonic oscillator in order to study the correlation function of the electromagnetic field, a subject of great importance in quantum optics. Sudarshan [445] showed that in the coherent states basis $\{|\alpha\rangle\}$, it is always possible to express the density operator in the diagonal form: $\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha$, where $P(\alpha)$ is known as P-representation of the phase space distribution of the density matrix. Roughly at the same time as Glauber and Sudarshan, Klauder [446] developed a set of continuous states in which the basic ideas of coherent states for arbitrary Lie groups were contained. Coherent states are usually defined in one of the the following three equivalent ways [447]:

- The coherent states $|\alpha\rangle$ are eigenstates of the annihilation operator a with eigenvalue α: $a|\alpha\rangle = \alpha|\alpha\rangle$, where α is a complex number,

- The coherent states $|\alpha\rangle$ can be obtained by applying a displacement operator $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ on the vacuum state of the harmonic oscillator, i.e. $|\alpha\rangle = D(\alpha)|0\rangle$,

- These coherent states saturate the Heisenberg's uncertainity relation i.e. $(\Delta p)(\Delta x) = \frac{\hbar}{2}$, where $\Delta p = [\langle p^2 \rangle - \langle p \rangle^2]^{1/2}$.

From a group theoretic point of view, the harmonic oscillator coherent states arise in systems whose dynamical symmetry group is associated with the Heisenberg-Weyl algebra. Gilmore [448, 449] and Perelomov [450, 451] defined a set of generalized coherent states for any semi-simple group G. Barut and Girardello [452], constructed a set of generalized coherent states for the group $SU(1,1)$ as eigenstates of the non-compact operator of $SU(1,1)$. Coherent states for general potentials were constructed in a series of papers by Nieto and co-authors [453, 454, 455, 456]. Their results are valid only to the lowest order of $\hbar$, i.e., they are semi-classical in nature. Indeed, it is not always easy to obtain a closed form expressions for coherent states for arbitrary potentials[4].

For quantum mechanical systems with known discrete spectrum a very elegant method of constructing coherent states has been suggested by Klauder [458]. Later, this method has been extended to systems with discrete as well as continuous spectrum energy by Gazaeu and Klauder [459]. The Gazaeu-Klauder (G-K) coherent states of a Hamiltonian H are a two parameter set of coherent states, $\{|j, \gamma\rangle : j \geq 0, -\infty \leq \gamma \leq \infty\}$, which satisfy the following requirements:

- Continuity of labelling– $(j', \gamma') \to (j, \gamma) \Rightarrow |j', \gamma'\rangle \to |j, \gamma\rangle$

- Resolution of unity– $\mathbf{I} = \int |j, \gamma\rangle \langle j, \gamma| \, d\mu(j, \gamma)$

- Temporal stability– $e^{-iHt} |j, \gamma\rangle = |j, \gamma + \omega t\rangle$, $\omega =$ constant.

- Action identity– $\langle j, \gamma|H|j, \gamma\rangle = \omega j$.

The G-K coherent states have been studied for a variety of quantum systems: Poschl-Teller [460, 461] and Morse potentials [462, 463], trigonometric Rosen-Morse [464], anharmonic oscillator [465], pseudo-harmonic oscillators [466], quantum particles confined by

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[4]A large body of work on the coherent states has by now appeared. This vast literature was exhaustively collected, catalogued and classified by Klauder and Skagerstam in the book 'Coherent states: Applications in Physics and Mathematical Physics' [457].

a double-well potential [467], and non-Hermitian $\mathcal{PT}$-symmetric Scarf I potential [468]. The Morse and anharmonic oscillator coherent states possess both classical like feature (approximate minimum uncertainty property) as well as nonclassical feature (squeezing property) in different parameter regimes. In ref. [469], G-K coherent states for one mode periodic potentials have been constructed and compared with the canonical (Glauber) coherent state. The Mandel parameter of the G-K coherent states corresponding to a class of shape invariant potentials (like Eckart and Rosen-Morse II) are shown [470] to be non-negative, so that the corresponding weighting distribution of these states describe either the Poissonian or the sub-Poissonian statistics. Generalized coherent state which minimize the Robertson-Schrödinger uncertainty relation are constructed [471]. G-K coherent states are shown to be a special case of these generalized coherent states. An extension of the G-K coherent states for higher dimension was made in [472], where the formalism was applied to a two-dimensional fermion gas in a constant magnetic field. The vector coherent states of the G-K type, have also been constructed and some physical applications of these have been addressed [473]. It is worth mentioning that, nowadays, coherent states are being widely used in quantum information processing and quantum computing with continuous variables [474, 475, 476].

Revival dynamics: In addition to exhibiting non-classical features such as squeezing, Poissonian and/or sub-Poissonian statistics etc, coherent states of systems possessing non-linear energy spectra also exhibit revival and fractional revival, leading to the formation of Schrodinger cat-like states (a quantum superposition of macroscopically distinguishable states). The phenomenon of wave packet revivals [477], arises when a well-localized wave packet is produced and initially exhibits a short-term time evolution with almost classical periodicity (T_{cl}) and then spreads significantly after a number of orbits, entering a so-called collapsed phase when the probability is spread (not uniformly) around the classical trajectory. On a much longer time scale after the initial excitation, however, called the revival time (with $T_{rev} >> T_{cl}$), the packet relocalizes, in the form of a quantum revival, in which the spreading reverses itself and the classical periodicity is once again apparent. Even more interestingly, an additional temporal structures with smaller periodicities (fractions of T_{cl}) are found at times equal to rational fractions of the revival time (pT_{rev}/q). These structures have been interpreted as the temporary formation of a number of mini-packets or clones of the original packet, exhibiting local periodicities T_{cl}/q, and have come to be known as fractional revivals [478, 479, 480, 481].

For a general localized wave packet formed as a one-dimensional superposition of energy eigenstates $|n\rangle$:

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n e^{-iE_n t} |n\rangle$$

with $\sum_{n\geq 0} |c_n|^2 = 1$, the concept of revival arises from the weighting probabilities $|c_n|^2$. The weighting factors c_n in the superposition are related to the initial wave function as $c_n = \langle n|\psi(x,0)\rangle$. Assuming the wave packet is strongly weighted around the mean value $\langle n \rangle = \bar{n}$ and the spread $\Delta n = [\langle n^2 \rangle - \langle n \rangle^2]^{1/2}$ is small compared to $\bar{n}$, the energy E_n can be expanded in Taylor series [482]:

$$E_n \simeq E_{\bar{n}} + (n - \bar{n})E'_{\bar{n}} + \frac{1}{2}(n - \bar{n})^2 E''_{\bar{n}} + \frac{1}{6}(n - \bar{n})^3 E'''_{\bar{n}} + ...,$$

where each prime on $E_{\bar{n}}$ denotes a derivative with respect to quantum number n. These derivatives define distinct time scales that depend on n. The first time scale, $T_{cl} = \frac{2\pi}{|E'_{\bar{n}}|}$ is the classical period. It controls the initial behavior of the packet. The second time scale, $T_{rev} = \frac{4\pi}{|E''_{\bar{n}}|}$ is the revival time, which governs the appearance of the fractional and full revival. The behavior of the wave packet at a time larger than T_{rev} is controlled by the third time scale $T_{sr} = \frac{12\pi}{|E'''_{\bar{n}}|}$ and is called super-revival time. The revival of wave packets may be studied[5] by examining the absolute square of the autocorrelation function [478, 483]

$$|A(t)|^2 = |\sum_{n=0}^{\infty} |a_n|^2 e^{-iE_n t}|^2.$$

The quantity $|A(t)|^2$ measures the overlap of the initial state wave packet $\psi(x,0)$ with the state $\psi(x,t)$ at later time t. Numerically, $|A(t)|^2$ varies between 0 and 1. The maximum value 1 is reached when $\psi(x,t)$ exactly matches the initial wave packet $\psi(x,0)$, and minimum value 0 corresponds to $\psi(x,t)$ whose shape is distinct from initial wave packet. On the other hand, fractional revival and super-revival appear when periodic peaks in $|A(t)|^2$ occur at the rational fractions of the classical period and revival time respectively.

Recently, the sum of Shannon information entropies in position and momentum spaces has been shown [484] to provide a useful tool for describing fractional revivals, complementary to the usual approach in terms of the above mentioned autocorrelation

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[5] One of the early papers to point out the possibility of revival behavior in the context of infinite well was by Segre and Sullival [486]. They studied the motion of the bound state wave packets in such a system using a simple "sum over energy eigenstates" method.

function. The underlying idea is that the position-space Shannon entropy measures the uncertainty in the localization of the particle in space. The lower is this entropy, the more concentrated is the probability density $|\psi(x)|^2$. This in turn implies that the uncertainity is smaller and the accuracy in predicting the localizatiobn of the particle is higher. Equivalently, the momentum-space entropy measures the uncertainty in predicting the momentum of the particle. Thus, the sum of Shannon entropies in position and momentum spaces gives an account of the spreading (high entropy values) and the regeneration (low entropy values) of initially well localized wave packets during the time evolution. The temporary formation of fractional revivals of the wave packet correspond to the relative minima of the sum of these Shannon information entropies. Similar studies associated with Renyi entropy has also been reported [485].

It has been shown that the phenomena of full and fractional revival occur in the wave packet dynamics of various atomic, molecular, optical and condensed matter systems such as Rydberg atom [479, 487, 480], Jaynes-Cummings model [481, 488, 489], molecular vibrational states [490], atom bouncing vertically in a gravitational cavity [491], graphene [492], Morse like anharmonic system [493], and system with position dependent mass [494]. The simplest quantum systems for which one would see fractional and full revivals are those for which the energy spectrum depends as n^2, e.g., the potential well [482, 495, 496], the Poschl-Teller potential [460, 497] and the rigid rotator in two dimensions (a quantum system closely related to the infinite square well) [498]. Revival phenomena have also been observed for systems with energy spectra depending on two quantum numbers [499, 500]. These phenomena has been experimentally observed in both atomic [501] and molecular systems [502].

1.5   Outline of the Thesis

In the light of previously mentioned developments, we have studied some solvable quantum systems and explored the underlying symmetry of their solutions. The result of these studies are discussed in the remaining part of the thesis in a systematic way.

Chapter 2 deals with the quantum systems which are characterized by position dependent mass Schrödinger Hamiltonians. The modified factorization technique for a PDM Hamiltonian to construct nonsingular isospectral Hamiltonians have been proposed in section 2.1. It should be mentioned here that due to the presence of position dependent mass, the algorithm is different and more complicated compared to the constant mass case.

In section 2.2, the method of point canonical transformation has been used to generate some exactly solvable shape invariant PDM Hamiltonians associated with Laguerre- or Jacobi-type X_1 EOPs. The results are analyzed in the light of SUSYQM approach.

A complete scheme for the first and second order intertwining approach have been then developed, in section 2.3, to construct isospectral Hamiltonians with several possible spectral modifications e.g. deletion of bound state(s), creation of bound state(s) and leave the spectrum unchanged. The methods have also been explained with suitable examples.

In **Chapter 3**, the quantum systems with some non-Hermitian Hamiltonians are considered. It is shown that the $\mathcal{PT}$-symmetric Hamiltonian with the optical potential $V(x) = 4\cos^2 x + 4iV_0 \sin 2x$ can be mapped into a Hermitian Hamiltonian for $V_0 < .5$, by a similarity transformation. The energy band structure of this Hamiltonian has been studied using the Floquet analysis. Existence of a second critical point ($V_0^c \sim .888437$) of the potential $V(x)$, apart from the known critical point $V_0 = 0.5$, has also been identified. All these results are reported in section 3.1.

In the section 3.2 , we have used the method of point canonical transformation to construct some non-Hermitian Hamiltonians whose bound state wave functions are given in terms of exceptional X_1 Laguerre- or Jacobi- type polynomials. These Hamiltonians are shown to be both quasi- and pseudo-Hermitian.

In the last part of this chapter, the generalized Swanson Hamiltonian $H_{GS} = w(\tilde{a}\tilde{a}^\dagger + 1/2) + \alpha\tilde{a}^2 + \beta\tilde{a}^{\dagger 2}$ with $\tilde{a} = A(x)d/dx + B(x)$, $\alpha, \beta, w \in \mathbb{R}$, has been considered. It is shown that the equivalent Hermitian Hamiltonian of H_{GS} can be transformed into the

harmonic oscillator type Hamiltonian so long as $[\tilde{a}, \tilde{a}^\dagger] = $ constant. It has also been shown that, though the commutator of $\tilde{a}$ and $\tilde{a}^\dagger$ is constant, the generalized Swanson Hamiltonian is not necessarily isospectral to the harmonic oscillator. Reason for this anomaly is discussed in the frame work of position dependent mass models by choosing $A(x)$ as the inverse square root of the mass profile.

Studies of quantum nonlinear oscillator has been addressed in **Chapter 4**. In the first part of this chapter, the coherent state of the nonlinear oscillator is constructed using the Gazeau-Klauder formalism. The nature of the weighting distribution and the Mandel parameter are discussed. The details of the revival structure arising from different time scales underlying the quadratic energy spectrum are also investigated by the phase analysis of the autocorrelation function. These studies

In section 4.2, various generalizations e.g, exactly solvable Hermitian and non-Hermitian variants of the quantum nonlinear oscillator are discussed. SUSYQM has also been used to show that these generalized potentials possess shape invariance symmetry.

Finally in **Chapter 5**, some future issues related to the works presented in this thesis have been mentioned.

Chapter 2

Quantum Systems with Position Dependent Mass [1]

The one-dimensional time-independent Schrödinger equation associated with a particle endowed with a position-dependent effective mass reads [400, 401]:

$$H\psi(x) = \left[-\frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx} + V(x) \right]\psi(x) = E\psi(x) \tag{2.1}$$

where $M(x)$ is the dimensionless form of the particle's effective mass $m(x) = m_0 M(x)$ (we have considered $\hbar = 2m_0 = 1$), $V(x)$ denotes the potential, $\psi(x)$ is the particles wave function corresponding to the energy E.

In this chapter, we have studied several aspects of the quantum systems characterized by the above mentioned position-dependent mass Schrodinger equation (PDMSE). First, in section 2.1 we have extended the notion of modified factorization technique based on excited state wave function to the PDMSE. Section 2.2 deals with the PDM Hamiltonians associated with exceptional orthogonal polynomials. Finally in section 2.3, first and second order intertwining method have been studied systematically to generate isospectral PDM Hamiltonians with several spectral modifications.

2.1 Modified factorization of quantum system with position dependent mass

The modified factorization technique for constant mass Schrödinger Hamiltonian [173, 172] is based on the non-uniqueness of the superpotential used to factorize the Hamiltonian. In this technique, excited state wave functions of a given Hamiltonian can be used to construct non-singular isospectral partner potentials. In this section, we extend the modified factorization technique with suitable examples to handle cases with position-dependent mass. It is to be noted here that due the position dependent mass function, the modified

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[1]This chapter is based on the following three papers:
(i) B. Midya, " Nonsingular potentials from excited state factorization of a quantum system with position-dependent mass" J. Phys.A 44 (2011) 435306; (ii) B Midya and B. Roy,"Exceptional orthogonal polynomials and exactly solvable potentials in position dependent mass Schroedinger Hamiltonians", Phys. Lett. A 373 (2009) 4117-4122; (iii) B. Midya, B. Roy and R. Roychoudhury, "Position Dependent Mass Schroedinger Equation and Isospectral Potentials : Intertwining Operator approach", J. Math. Phys. 51 (2010) 022109.

factorization technique of the PDMSE is different and more involved than constant mass case. In order to make this section self contained we have first briefly recalled the notion of usual factorization technique based on ground state wave function of PDM Hamiltonian.

2.1.1   Ground state factorization

The minimal version of PDM SUSYQM is based on ground state wave function and the charge operators are represented by $Q^+ = A_0^+ \sigma_-$ and $Q^- = A_0^- \sigma_+$, where $\sigma_\pm = \sigma_1 \pm i\sigma_2$ are the combinations of pauli matrices $\sigma_{1,2}$ and the associated two first order linear operators $A_0^\pm$ are given by [412]

$$A_0^\pm = \pm \frac{1}{\sqrt{M}} \frac{d}{dx} - \left(\frac{1 \mp 1}{2} \right) \left(\frac{1}{\sqrt{M}} \right)' + W_0. \tag{2.2}$$

where 'prime' denotes the differentiation with respect to x. The components of the supersymmetric PDM Hamiltonian $\mathbf{H} = diag(H^-, H^+)$ can be factorized in terms of A_0^+ and A_0^- to obtain

$$H_0^\mp = A_0^\mp A_0^\pm = -\frac{1}{M} \frac{d^2}{dx^2} + \frac{M'}{M^2} \frac{d}{dx} + V_0^\mp(x) \tag{2.3}$$

where without loss of generality we have considered zero ground state energy of H_0^- and the two partner potentials $V_0^\mp$ can be written in terms of superpotential W_0 and mass function, as

$$V_0^\mp(x) = W_0^2 \mp \left(\frac{W_0}{\sqrt{M}} \right)' + \left(\frac{1 \mp 1}{2} \right) \left[\frac{W_0 M'}{M^{\frac{3}{2}}} - \frac{1}{\sqrt{M}} \left(\frac{1}{\sqrt{M}} \right)'' \right]. \tag{2.4}$$

Clearly, $V_0^+ = V_0^- + \frac{2W_0'}{\sqrt{M}} - \frac{1}{\sqrt{M}} \left(\frac{1}{\sqrt{M}} \right)''$. The ground state wave function ψ_0^- of the Hamiltonian H_0^- determines the superpotential

$$W_0(x) = -\frac{\psi_0^{-'}}{\sqrt{M} \psi_0^-} \, , \tag{2.5}$$

which implies that for a physically acceptable mass function[2] the superpotential and hence two partner potentials $V_0^\mp$ are free from singularity (provided the given potential V_0^- is

---

[2]The mass function should be positive and without any singularity through out its domain of definition.

without any singularity). It is well established that when SUSY is unbroken then the spectrum of the two PDM partner Hamiltonians $H_0^{\mp}$ are degenerate except for the lowest energy of H_0^-. Moreover, like constant mass case, the energy eigenvalues and normalized eigenfunctions of the two PDM SUSY partner Hamiltonians $H_0^{\pm}$ are related by [412]

$$
\begin{aligned}
& E_n^{(0)+} = E_{n+1}^{(0)-}, \quad E_0^{(0)-} = 0 \\
& \psi_n^+ = (E_{n+1}^{(0)-})^{-\frac{1}{2}} \, A_0^+ \psi_{n+1}^- \\
& \psi_{n+1}^- = (E_n^{(0)+})^{-\frac{1}{2}} \, A_0^- \psi_n^+, \; n = 0, 1, 2...
\end{aligned}
\tag{2.6}
$$

For a well behaved mass function the usual factorization technique for PDMSE allows us to construct nonsingular isospectral potentials of a given nonsingular potential using equations (2.5) and (2.4). It should be noted here that the factorization technique mentioned above is not unique and can be extended to use excited state wave functions. This generalization is mathematically straightforward but physically very nontrivial since it yields singular partner potentials. In the following we shall show that the excited state of a given position dependent mass Hamiltonian can be used to construct nonsingular isospectral potentials.

2.1.2   Excited state factorization and its modification

In this case the superpotential is defined using the excited state wave functions, as

$$
W_n(x) = -\frac{\psi_n^{-\prime}}{\sqrt{M}\psi_n^-} \quad , n > 0
\tag{2.7}
$$

where ψ_n^- is the n-th excited state wave function of an exactly solvable nonsingular Hamiltonian H_0^-. We subtracting the energy $E_n^{(0)-}$ of the excited state from the Hamiltonian H_n^- so that the resulting Hamiltonian can be factorized. In this case two partner potentials $V_n^{\pm}$ are given by

$$
\begin{aligned}
& V_n^-(x) = W_n^2 - \left(\frac{W_n}{\sqrt{M}}\right)' = V_0^-(x) - E_n^{(0)-} \\
& V_n^+(x) = V_n^-(x) + \frac{2W_n'}{\sqrt{M}} - \frac{1}{\sqrt{M}}\left(\frac{1}{\sqrt{M}}\right)''
\end{aligned}
\tag{2.8}
$$

The singular superpotential W_n contributes nothing new to V_n^- except a constant energy shift to V_0^-. In this case energy eigenvalues of the Hamiltonians H_n^- and H_0^- are related

by $E_k^{(n)-} = E_k^{(0)-} - E_n^{(0)-}, k = 0, 1, 2...$ (k denotes the energy level and n refers to the nth eigenfunction of H_n^- used in factorization). But the partner potential V_n^+ becomes singular at the node(s) of the wave function ψ_n^-. This singularity is responsible for both the destruction of degeneracy of the spectrum and creation of negative energy state(s). The singularity breaks the real axis into more than one disjoint intervals and imposes additional boundary conditions on the wave functions. In this case, the supersymmetry operators map square integrable functions to states outside the Hilbert space. As a result, some or all of the wave functions of the SUSY partner Hamiltonians may not belong to the same Hilbert space of square integrable functions and usual proof of degeneracy between the excited states of the partner Hamiltonians does not hold [168, 169]. This warrants the following modification of the excited state factorization.

We define the operators [3] $\tilde{A}_n^\pm$ as

$$\tilde{A}_n^\pm = \pm \frac{1}{\sqrt{M}} \frac{d}{dx} - \left(\frac{1 \mp 1}{2} \right) \left(\frac{1}{\sqrt{M}} \right)' + W_n + f_n \tag{2.9}$$

where W_n is given by equation (2.7) and the unknown function $f_n(x)$ will be evaluated shortly. Using this operators $\tilde{A}_n^\pm$ we first obtain

$$\tilde{H}_n^+ = \tilde{A}_n^+ \tilde{A}_n^- = -\frac{1}{M} \frac{d^2}{dx^2} + \frac{M'}{M^2} \frac{d}{dx} + \tilde{V}_n^+(x) \tag{2.10}$$

where

$$\tilde{V}_n^+ = V_n^+ + \frac{f_n'}{\sqrt{M}} + \left(2W_n + \frac{M'}{2M^{\frac{3}{2}}} \right) f_n + f_n^2. \tag{2.11}$$

At this point we assume

$$\frac{f_n'}{\sqrt{M}} + \left(2W_n + \frac{M'}{2M^{\frac{3}{2}}} \right) f_n + f_n^2 = \beta, \quad \beta \in \mathbb{R} \tag{2.12}$$

so that $\tilde{H}_n^+ = H_n^+ + \beta$ is solvable. This assumption also ensures the non uniqueness of the factorization viz.

$$A_n^+ A_n^- + \beta = \tilde{A}_n^+ \tilde{A}_n^-. \tag{2.13}$$

For $\beta = 0$, the Riccati equation (2.12) reduces to the Bernoulli equation whose solution

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[3]$A_n^\pm$ are obtained from equation (2.2) after replacing W_0 by W_n.

is given by

$$f_n(x) = \frac{(\psi_n^-)^2}{\sqrt{M}\left(\lambda + \int(\psi_n^-)^2 dx\right)}, \qquad \lambda \in \mathbb{R}. \tag{2.14}$$

For nonzero β the Riccati equation (2.12) is not always solvable but it can be transformed into the following second order linear differential equation

$$-\frac{1}{M}\psi^{+''} + \frac{M'}{M^2}\psi^{+'} + \left[W_n^2 + \left(\frac{W_n}{\sqrt{M}}\right)' + \frac{W_n M'}{M^{\frac{3}{2}}} - \frac{1}{\sqrt{M}}\left(\frac{1}{\sqrt{M}}\right)''\right]\psi^+ = -\beta\psi^+ \tag{2.15}$$

with the help of following transformation

$$f_n(x) = \frac{1}{\sqrt{M}}(log\chi_n)', \qquad \chi_n = \frac{e^{-\int\sqrt{M}W_n dx}}{\sqrt{M}}\psi^+. \tag{2.16}$$

Using the relation (2.8) it is clear that the equation (2.15) is (nearly) isospectral to the following solvable equation

$$-\frac{1}{M}\psi^{-''} + \frac{M'}{M^2}\psi^{-'} + V_0^-(x)\psi^- = (E_n^{(0)-} - \beta)\psi^- \tag{2.17}$$

where the solutions $\psi^{\pm}(x)$ corresponding to two equations (2.15) and (2.17) are related by

$$\psi^+ \sim A_n^+\psi^-. \tag{2.18}$$

By changing the order of operations between $\tilde{A}_n^+$ and $\tilde{A}_n^-$ and using relations (2.8), (2.12) we obtain the partner Hamiltonian $\tilde{H}_n^-$ of $\tilde{H}_n^+$ as

$$\tilde{H}_n^- = \tilde{A}_n^-\tilde{A}_n^+ = -\frac{1}{M}\frac{d^2}{dx^2} + \frac{M'}{M^2}\frac{d}{dx} + \tilde{V}_n^-(x) \tag{2.19}$$

where

$$\tilde{V}_n^- = V_n^- - \frac{2f_n'}{\sqrt{M}} + \beta = V_0^- - \frac{2f_n'}{\sqrt{M}} - E_n^{(0)-} + \beta. \tag{2.20}$$

At this point few comments are worth mentioning. For a given nonsingular potential V_0^- and a mass function the new potential $\tilde{V}_n^+(= V_n^+ + \beta)$ is singular but the potential $\tilde{V}_n^-$ has no singularity if $f_n(x)$ is nonsingular in which case the corresponding spectrum of the
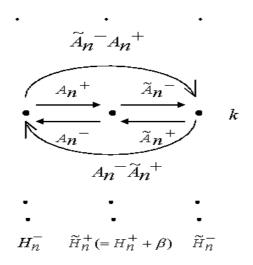
Figure 2.1: Energy levels of two partner Hamiltonians H_n^-, $\tilde{H}_n^-$ and the intermediate Hamiltonian $\tilde{H}_n^+$. Combined actions of the operator $\tilde{A}_n^- A_n^+$ and $A_n^- \tilde{A}_n^+$ are shown.

two isospectral Hamiltonians $\tilde{H}_n^-$ and H_n^- are related by

$$\tilde{E}_k^{(n)-} = E_k^{(n)-} + \beta, \quad k = 0, 1, 2.... \tag{2.21}$$

For $\beta = 0$, the singularity of f_n given in (2.14) can be controlled by the arbitrary constant λ and in all other cases it depends on the nature of solution of the equation (2.17). Moreover, the eigenfunctions of the two nonsingular Hamiltonians H_n^- and $\tilde{H}_n^-$ are related by

$$\begin{aligned}
\tilde{\psi}_k^- &\sim \tilde{A}_n^- A_n^+ \psi_k^- \\
\psi_k^- &\sim A_n^- \tilde{A}_n^+ \tilde{\psi}_k^-
\end{aligned} \tag{2.22}$$

except for the zero energy state of $\tilde{H}_n^-$.

The proof of the result 2.22 is as follows: we start with

$$\begin{aligned}
\tilde{H}_n^-(\tilde{A}_n^- A_n^+ \psi_k^-) &= \tilde{A}_n^-(\tilde{A}_n^+ \tilde{A}_n^-)A_n^+ \psi_k^- \\
&= \tilde{A}_n^-(A_n^+ A_n^- + \beta)A_n^+ \psi_k^- \\
&= \tilde{A}_n^- A_n^+(A_n^- A_n^+)\psi_k^- + \beta \tilde{A}_n^- A_n^+ \psi_k^- \\
&= \tilde{A}_n^- A_n^+(H_n^- \psi_k^-) + \beta \tilde{A}_n^- A_n^+ \psi_k^- = (E_k^{(n)-} + \beta)\tilde{A}_n^- A_n^+ \psi_k^-.
\end{aligned} \tag{2.23}$$

In the second step we have used the non-uniqueness of the factorization of H_n^+ and in the last step we have used the PDMSE $H_n^- \psi_k^- = E_k^{(n)-} \psi_k^-$. Hence, if ψ_k^- is a eigenfunction of H_n^- corresponding to eigenvalue $E_k^{(n)-}$ then $\tilde{A}_n^- A_n^+ \psi_k^-$ is a eigenfunction of $\tilde{H}_n^-$ with

eigenvalue $(E_k^{(n)-} + \beta)$. The second relation of equation (2.22) can be proved in a analogous way. The wave function $\tilde{\psi}_{k'}^-$ (say) corresponding to zero energy of the Hamiltonian $\tilde{H}_n^-$ can be obtained by directly solving $\tilde{A}_n^+ \tilde{\psi}_{k'}^- = 0$.

Hence, excited state wave function of a given position dependent mass Hamiltonian can be used to construct nonsingular partner Hamiltonian $\tilde{H}_n^-$ using equation (2.20) provided the function f_n is nonsingular. This has been made possible in two step factorization. In the first step, an intermediate singular Hamiltonian H_n^+ has been created and in the second step non-uniqueness of the factorization technique has been used to remove the singularities. The operator $-\tilde{A}_n^- A_n^+$ is similar to the second order reducible intertwining operator $\mathcal{L} = \frac{1}{M(x)} \frac{d^2}{dx^2} + \eta(x) \frac{d}{dx} + \gamma(x)$ mentioned in ref.[503] which intertwines two Hamiltonians H_n^- and $\tilde{H}_n^-$

$$\tilde{H}_n^- \tilde{A}_n^- A_n^+ = \tilde{A}_n^- A_n^+ H_n^-, \tag{2.24}$$

if one identifies $-\eta(x) = \frac{f_n}{\sqrt{M}} + \frac{M'}{M^2}$ and $-\gamma(x) = V_n^- + f_n W_n$. In this regard the present algorithm is equivalent to the confluent second order SUSY transformation [504] in constant mass case but the relation between the two isospectral partner potentials is different in the present approach due to the position dependence of the mass function. Moreover, the energy levels of the two nonsingular isospectral partner Hamiltonians H_n^- and $\tilde{H}_n^-$ are degenerate. In figure 2.1, we have shown schematically the combined action of the operators $\tilde{A}_n^- A_n^+$ and $A_n^- \tilde{A}_n^+$ on the wave function of two isospectral partner Hamiltonians H_n^- and $\tilde{H}_n^-$ respectively. The operators A_n^+ or $\tilde{A}_n^+$ destroy a node but $\tilde{A}_n^-$ or A_n^- create the same in the eigenfunctions so that the overall number of nodes remains the same.

2.1.3   Examples of modified factorization

It is worth mentioning here that the proposed technique mentioned in the preceding section is more general and can be applied to any solvable PDM Hamiltonian to obtain some acceptable isospectral Hamiltonians which might be useful in various fields of condensed matter physics. For illustration purpose we are considering here two exactly solvable position dependent mass Hamiltonians. In example 1.1, we have considered $\beta = 0$ while non zero β has been considered in example 1.2.

Example 1.1 ($\beta = 0$): We consider the following mass function and potential possessing

harmonic oscillator like spectra [505]:

$$M(x) = \frac{1}{1 + \alpha^2 x^2} \tag{2.25}$$

$$V_0^-(x) = \left(\frac{\sinh^{-1}(\alpha x)}{\alpha}\right)^2 - \frac{\alpha^2}{4}\left(\frac{2 + \alpha^2 x^2}{1 + \alpha^2 x^2}\right), \quad x \in (-\infty, \infty). \tag{2.26}$$

The bound state solution are given by

$$\psi_k^- = \sqrt{\frac{1}{2^k k!}}\left(\frac{1}{\pi}\right)^{\frac{1}{4}} \frac{e^{-\frac{1}{2}\left(\frac{\sinh^{-1}(\alpha x)}{\alpha}\right)^2}}{(1 + \alpha^2 x^2)^{1/4}} H_k\left(\frac{\sinh^{-1}(\alpha x)}{\alpha}\right) \tag{2.27}$$

$$E_k^{(0)-} = 2k + 1, \quad k = 0, 1, 2... \tag{2.28}$$

where $H_k(x)$ denotes the Hermite polynomial. Now we consider here first excited state $\psi_1^- = \frac{2\sinh^{-1} x \; e^{-\frac{1}{2}\left(\sinh^{-1} x\right)^2}}{(1 + x^2)^{1/4}}$, $\beta = 0$, and $\alpha = 1$ for which the partner potential V_1^- and $\tilde{V}_1^-$ are obtained using equation (2.8), (2.14) and (2.20) as

$$V_1^- = \left(\sinh^{-1}(x)\right)^2 - \frac{1}{4}\left(\frac{2 + x^2}{1 + x^2}\right) - 3 \tag{2.29}$$

$$\tilde{V}_1^- = -\frac{2 + x^2}{4 + 4x^2} + (\sinh^{-1} x)^2 - 3 + 2\sqrt{(1 + x^2)}\frac{d}{dx}\left[\frac{4\left(\sinh^{-1} x\right)^2}{\sqrt{\pi}e^{(\sinh^{-1} x)^2}(2\lambda + \mathrm{Erf}(\sinh^{-1} x)) - 2\sinh^{-1} x}\right] \tag{2.30}$$

respectively. The energy eigenvalues of the Hamiltonian $\tilde{H}_1^-$ are given by $\tilde{E}_k^{(1)-} = 2k - 2$, $k = 0, 1, 2...$ The normalized first excited state wave function corresponding to zero energy of $\tilde{H}_1^-$ is evaluated by solving $\tilde{A}_1^+ \tilde{\psi}_1^- = 0$ as

$$\tilde{\psi}_1^- = \frac{\pi^{\frac{1}{4}}\sqrt{8\lambda^2 - 2} \; e^{\frac{(\sinh^{-1} x)^2}{2}} \sinh^{-1} x}{(1 + x^2)^{\frac{1}{4}}\left[2\sqrt{\pi}\lambda e^{(\sinh^{-1} x)^2} - 2\sinh^{-1} x + \sqrt{\pi}e^{(\sinh^{-1} x)^2} \mathrm{Erf}(\sinh^{-1} x)\right]}. \tag{2.31}$$

Hence, the new potential $\tilde{V}_1^{(1)-}$ is nonsingular for $\lambda > 1/2$. The lowest energy wave
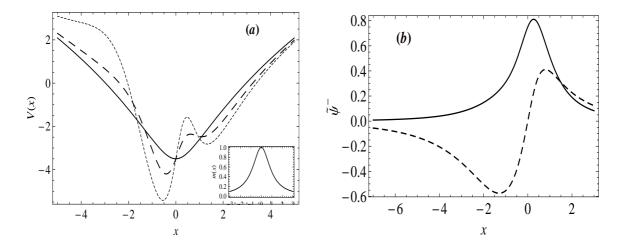
Figure 2.2: (a) Plots of the given potentials V_1^- (solid line) and $\tilde{V}_1^-$ for $\lambda = 1$ (dashed line), $\lambda = .7$ (dotted line) and the mass function $m(x)$. (b) The lowest (solid line) and first excited state (dashed line) wave functions of the Hamiltonian $\tilde{H}_1^-$ for $\lambda = 1$.

function corresponding to negative energy -2 is obtained through $\tilde{A}_1^- A_1^+ \psi_0^-$ as

$$\tilde{\psi}_0^- = \frac{\pi^{\frac{1}{4}}[2\lambda + \mathrm{Erf}(\sinh^{-1} x)]e^{\frac{(\sinh^{-1} x)^2}{2}}}{(1+x^2)^{\frac{1}{4}} \left[2\sqrt{\pi}\lambda e^{(\sinh^{-1} x)^2} - 2\sinh^{-1} x + \sqrt{\pi}\,\mathrm{Erf}(\sinh^{-1} x)e^{(\sinh^{-1} x)^2}\right]}. \quad (2.32)$$

In figure 2.2(a), we have plotted the potential $V_1^-(x)$, its isospectral partner $\tilde{V}_1^-$ for two parameter values $\lambda = 1, .7$ and the mass function. In figure 2.2(b), lowest and first excited state wave functions of the Hamiltonian $\tilde{H}_1^-$ for $\lambda = 1$ have been plotted.

Example 1.2 ($\beta \neq 0$):   Here we consider the following mass function and the potential

$$M(x) = \frac{1}{4}\,\mathrm{sech}^2\,\left(\frac{x}{2}\right) \quad (2.33)$$

$$V_0(x) = \frac{(a+b-c)^2 - 1}{4}e^x + \frac{c(c-2)}{4}e^{-x}, \quad c > \frac{1}{2}, a+b-c+\frac{1}{2} > 0, \ x \in (-\infty, \infty), \quad (2.34)$$

for which one linearly independent solution of the equation (2.17) is known to be (see appendix of ref.[503])

$$\psi^-(x) = C_1 \, e^{cx/2}(1+e^x)^{\frac{P-1}{2}} \ {}_2F_1\left(\frac{a+b+P}{2}, \frac{-a-b+2c+P}{2}, c, e^{-x}\right), \quad (2.35)$$

where

$$P^2 = (a+b)^2 - 2c(a+b-c+1) + 4(E_n^{(0)-} - \beta),$$
$$E_n^{(0)-} = n^2 + n(a+b) + \frac{c(a+b-c+1)}{2}. \tag{2.36}$$

The bound state wave functions of the Hamiltonian H_0^- are given [503], in terms of Jacobi polynomial $P_n^{(\sigma,\delta)}(x)$, as

$$\psi_n^- \sim \frac{e^{cx/2}}{(1+e^x)^{(a+b+1)/2}} \; P_n^{(c-1,a+b-c)}\left(\frac{1-e^x}{1+e^x}\right), \quad n = 0, 1, 2... \tag{2.37}$$

Now we consider first excited state i.e. $n = 1$ for which the function f_n can be obtained using equations (2.16) and (2.18) as

$$f_n = \frac{1}{\sqrt{M}}\frac{d}{dx}[\log(A_1^+\psi^-(x))] + \frac{\cosh\left(\frac{x}{2}\right)}{4}\left[\frac{2c}{(a+b-c+1)e^x - c} + \frac{a+b+3}{1+e^x} + c - a - b - 1\right] + \sinh\left(\frac{x}{2}\right) \tag{2.38}$$

where $A_1^+ = \frac{1}{\sqrt{M(x)}}\left(\frac{d}{dx} - \frac{\psi_1^{-\prime}}{\psi_1^-}\right)$, $M(x), \psi^-$ and ψ_1^- are given by equations (2.33), (2.35) and (2.37) respectively. In order to obtain new nonsingular potential $\tilde{V}_1^-$ we have to choose a, b, c, β such that f_n has no singularity. The analytical expression of f_n is too involved so it is very difficult to find the range of parameters values for which f_n is nonsingular. In figure 2.3(a), we have plotted the potential $V_1^-(x) = V_0^- - E_1^{(0)-}$ and its nonsingular partner $\tilde{V}_1^- = V_1^- - 2f_n/\sqrt{M} - E_1^{(0)-} + \beta$ for particular values of $a = 1, b = 5, c = 4$ and $\beta = 1$. For the same set of parameter values, isospectral partner potential $\tilde{V}_2^-$ has been drawn in figure 2.3(b) by considering second excited state factorization i.e. $n = 2$. The energy eigenvalues of the new Hamiltonians $\tilde{H}_1^-$ and $\tilde{H}_2^-$ are given by $\tilde{E}_k^{(1)-} = k^2 + 6k - 6$ and $\tilde{E}_k^{(2)-} = k^2 + 6k - 15$ respectively. In both cases the wave functions can be obtained using equations (2.22) and (2.37).
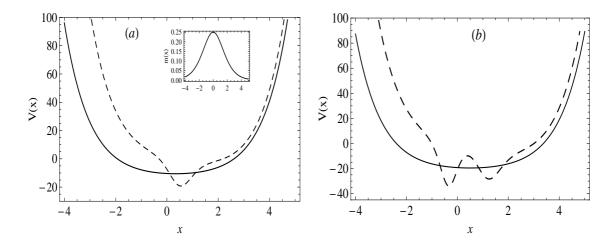
Figure 2.3: Plots of (a) the potential $V_1^- = V_0^- - 13$ (solid line), its nonsingular partner $\tilde{V}_1^-$ (dashed line) and the mass function $m(x)$, (b) the potential $V_2^- = V_0^- - 22$ (solid line) and its partner $\tilde{V}_2^-$ (dashed line) for $a = 1, b = 5, c = 4, \beta = 1$.

2.2 New exactly solvable PDM Hamiltonians associated with exceptional X_1 orthogonal polynomials

Here, some new exactly solvable potentials, whose bound state solutions are written in terms of recently discovered [185, 186] exceptional X_1 Laguerre or Jacobi type orthogonal polynomials, are obtained using the method of point canonical transformation (PCT). In the constant-mass case, i.e., for $M(x) = 1$, this procedure has been thoroughly investigated [423, 83]. A similar study in the PDM context is more involved as has been discussed in the below.

2.2.1 The method of PCT in PDM background

In PCT approach [83, 423], we search for the solution of equation (2.1) in the form

$$\psi(x) = f(x) \, F\left(g(x)\right) \tag{2.39}$$

where $F(g)$ is some special function which satisfies the second order differential equation

$$\frac{d^2F}{dg^2} + Q(g)\frac{dF}{dg} + R(g)F = 0. \tag{2.40}$$

and $f(x)$ and $g(x)$ are unknown functions of x to be determined. It should be mentioned here that in order to be physically acceptable, $\psi(x)$ has to satisfy the following two

conditions:

(i) Like quantum systems with constant mass it will be square integrable over domain of definition **D** of $M(x)$ and $\psi(x)$ i.e.,

$$\int_D |\psi_n(x)|^2 dx < \infty$$

(ii) The Hermiticity of the Hamiltonian in the Hilbert space spanned by the eigenfunctions of the potential $V(x)$ is ensured by the following extra condition [404]

$$\frac{|\psi_n(x)|^2}{\sqrt{M(x)}} \to 0$$

at the end points of the interval where $V(x)$ and $\psi_n(x)$ are defined. This condition imposes an additional restriction whenever the mass function $M(x)$ vanishes at any one or both the end points of $\mathcal{D}$.

Now, substituting $\psi(x)$ from equation (2.39) in equation (2.1) and comparing the coefficients of different order derivatives appear in the resulting equation, one obtains

$$Q(g(x)) = \frac{g''}{g'^2} + \frac{2f'}{fg'} - \frac{M'}{Mg'}, \quad R(g(x)) = \frac{f''}{fg'^2} + (E - V)\frac{M}{g'^2} - \frac{M'f'}{Mfg'^2} \quad (2.41)$$

After some algebraic manipulations we obtain explicit expressions for $f(x)$ and $E - V(x)$ as

$$f(x) \propto \sqrt{\frac{M}{g'}} \quad \exp\left(\frac{1}{2}\int^{g(x)} Q(t)dt\right) \quad (2.42)$$

and

$$E - V = \frac{g'''}{2Mg'} - \frac{3}{4M}\left(\frac{g''}{g'}\right)^2 + \frac{g'^2}{M}\left(R - \frac{1}{2}\frac{dQ}{dg} - \frac{Q^2}{4}\right) - \frac{M''}{2M^2} + \frac{3M'^2}{4M^3} \quad (2.43)$$

respectively. For equation (2.43) to be satisfied, we need to find some functions $M(x)$, $g(x)$ ensuring the presence of a constant term on the right hand side of equation (2.43) to compensate E on its left hand side and giving rise to an potential $V(x)$ with well behaved wave functions involving special function $F(g)$. In PCT approach there are many options for choosing $M(x)$, for example $M(x) = \lambda g'^2(x)$ [506, 424], $M = \lambda g'(x)$, $M = \dfrac{\lambda}{g'(x)}$

[425], λ being a constant. Some of these choices give rise to exactly solvable potentials while some other correspond to quasi-solvable potentials. Here, we choose $M(x) = \lambda g'(x)$ so that the expression $E - V(x)$ reduces to

$$E - V = \frac{g'}{\lambda} \left(R - \frac{1}{2} \frac{dQ}{dg} - \frac{Q^2}{4} \right) \tag{2.44}$$

and we are now in a position to choose the form of $F(g)$, $Q(g)$ and $R(g)$ which satisfy equation (2.40). It is to be noted here that different choices for the function $F(g)$ will give rise to different solvable Hamiltonians whose solutions are given by (2.39) and (2.42).

2.2.2   Hamiltonian associated with exceptional X_1 Laguerre polynomial

We consider the special function $F_n(g) \propto \hat{L}_n^{(\alpha)}$, where $\hat{L}_n^{(\alpha)}$, $n = 1, 2, 3, ...$, $\alpha > 0$ is the Laguerre type X_1 exceptional orthogonal polynomials [185] for which $Q(g) = -\frac{(g-\alpha)(g+\alpha+1)}{g(g+\alpha)}$ and $R(g) = \frac{n-2}{g} + \frac{2}{g+\alpha}$. For these values of $Q(g)$ and $R(g)$ the equation (2.44) reeds

$$E - V = \frac{g'}{\lambda g} \left(\frac{2\alpha n + \alpha^2 - \alpha + 2}{2\alpha} \right) - \frac{g'}{g^2} \left(\frac{\alpha^2 - 1}{4\lambda} \right) - \frac{2g'}{\lambda(g+\alpha)^2} - \frac{g'}{\lambda\alpha(g+\alpha)} - \frac{g'}{4\lambda}. \tag{2.45}$$

Taking $\frac{g'}{\lambda g} = $ Constant (say C), a constant term can be created on the right-hand side of the above equation (C must be positive in order to get increasing energy eigenvalues for successive n values). Consequently we have the expression of mass function, energy eigenvalues and potential as

$$M(x) = e^{-bx}, \qquad -\infty < x < \infty \tag{2.46}$$

$$E_m = b^2 \left(m + \frac{\alpha + 1}{2} \right) + \frac{b^2}{\alpha} + \bar{V}_0 \tag{2.47}$$

$$V(x) = \left[\frac{b^2}{4} \left((\alpha^2 - 1)e^{bx} + e^{-bx} \right) \right] + \frac{b^2}{4} \left(\frac{4}{\alpha(1 + \alpha e^{bx})} + \frac{8e^{bx}}{(1 + \alpha e^{bx})^2} \right) + \bar{V}_0 \tag{2.48}$$

respectively, where $C\lambda = -b, b > 0$ and $\bar{V}_0$ are constants. In the expression (2.47) we have reset the quantum number $n = m + 1$. The normalized eigenfunctions of the Hamiltonian
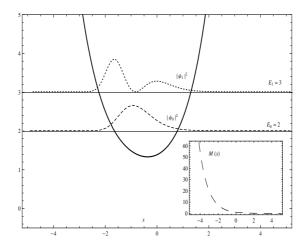
Figure 2.4: Plot of the potential V (solid line) given in equation (2.48), square of its first two bound state wave functions $|\psi_o(x)|^2$ (dashed line) and $|\psi_1(x)|^2$ (dotted line), for the mass function $M(x)$ (long dashed line) given in equation (2.46). Here we have considered $b = 1, \alpha = 2$.

are obtained using equations (2.39) and (2.42) as

$$\psi_m(x) = \left(\frac{b \, m!}{(m + \alpha + 1)\Gamma(m + \alpha)} \right)^{1/2} \frac{exp\left[-\frac{1}{2}\{(\alpha + 1)bx + e^{-bx}\} \right]}{\alpha + e^{-bx}} \, \hat{L}^{(\alpha)}_{m+1}(e^{-bx}), \quad m = 0, 1, 2...$$

(2.49)

It is to be noted that the square integrability condition (i) stated earlier does not impose any additional restriction on α but for condition (ii) to be satisfied, α should be greater than -1. In figure 2.4, we have plotted the mass function $M(x)$ given in (2.46), potential $V(x)$ given in (2.48) and square of first two bound state wave functions.

2.2.3   Hamiltonian associated with exceptional X_1 Jacobi polynomial

Next we choose $F_n(g) \propto \widehat{P}^{(\alpha,\beta)}_n(x)$, where $\widehat{P}^{(\alpha,\beta)}_n$, $n = 1, 2, 3, ... \cdots$, $\alpha, \beta > -1$, $\alpha \neq \beta$ is the Jacobi type X_1 exceptional orthogonal polynomial, for which we have [185]

$$\begin{aligned} Q(g) &= -\frac{(\alpha + \beta + 2)g - (\beta - \alpha)}{1 - g^2} - \frac{2(\beta - \alpha)}{(\beta - \alpha)g - (\beta + \alpha)}, \\ R(g) &= -\frac{(\beta - \alpha)g - (n - 1)(n + \alpha + \beta)}{1 - g^2} - \frac{(\beta - \alpha)^2}{(\beta - \alpha)g - (\beta + \alpha)}. \end{aligned}$$

(2.50)

For these $Q(g)$ and $R(g)$ equation (2.43) becomes

$$E - V = \frac{g'}{\lambda} \left[\frac{A_1 g + A_2}{1 - g^2} + \frac{A_3 g + A_4}{(1 - g^2)^2} + \frac{A_5}{(\beta - \alpha)g - (\beta + \alpha)} + \frac{A_6}{[(\beta - \alpha)g - (\beta + \alpha)]^2} \right]$$

(2.51)

where

$$A_1 = \frac{\beta^2 - \alpha^2}{2\alpha\beta} , \quad A_2 = n^2 + (\beta + \alpha - 1)n + \frac{1}{4}\{(\beta + \alpha)^2 - 2(\beta + \alpha) - 4\} + \frac{\beta^2 + \alpha^2}{2\alpha\beta}$$

$$A_3 = \frac{\beta^2 - \alpha^2}{2} , \quad A_4 = -\frac{\beta^2 + \alpha^2 - 2}{2} , \quad A_5 = \frac{(\beta + \alpha)(\beta - \alpha)^2}{2\alpha\beta} , \quad A_6 = -2(\beta - \alpha)^2 .$$

A constant term on the right-hand side of (2.51) can be generated by making $\frac{g'}{\lambda(1-g^2)} = $ constant (say $C_1 > 0$). Consequently we have

$$g(x) = \tanh(ax) \quad , \qquad M(x) = \mathrm{sech}^2(ax) , \qquad -\infty < x < \infty.$$

(2.52)

For this choice of $g(x)$ and mass function we obtain the potential

$$V(x) = \left[\frac{a^2}{4} \left((\alpha^2 - 1)e^{2ax} + (\beta^2 - 1)e^{-2ax} \right) \right] + \frac{a^2}{4} \left(\frac{4(\alpha - \beta)(\alpha - 3\beta)}{\alpha(\beta + \alpha e^{2ax})} - \frac{8\beta(\alpha - \beta)^2}{\alpha(\beta + \alpha e^{2ax})^2} \right) + \hat{V}_0,$$

(2.53)

corresponding bound state energy eigenvalues and eigenfunctions are given by

$$E_m = a^2 \left(m + \frac{\alpha + \beta}{2} \right) \left(m + \frac{\alpha + \beta + 2}{2} \right) + a^2 \left(\frac{\beta}{\alpha} - \frac{\alpha^2 + \beta^2 - 2}{4} \right) + \hat{V}_0$$

(2.54)

and

$$\psi_m(x) = \left[\frac{a(\alpha - \beta)^2 \, m!(2m + \alpha + \beta + 1)\Gamma(m + \alpha + \beta + 1)}{2^{\alpha+\beta-1}(m + \alpha + 1)(m + \beta + 1)\Gamma(m + \alpha)\Gamma(m + \beta)} \right]^{1/2}$$

(2.55)

$$\times \frac{(1 - \tanh(ax))^{\frac{\alpha+1}{2}}(1 + \tanh(ax))^{\frac{\beta+1}{2}}}{\alpha + \beta + (\alpha - \beta)\tanh(ax)} \, \widehat{P}_{m+1}^{(\alpha,\beta)}(\tanh(ax)) \quad m = 0, 1, 2, ...$$

respectively, where $\hat{V}_0$ and $C_1 \lambda = a > 0$ are constants. In expressions (2.54) and (2.55) we have reset the quantum number $n = m + 1$. An additional restriction $\alpha, \beta > -1/2$ is to be imposed to satisfy condition (ii) stated before whereas the square integrability
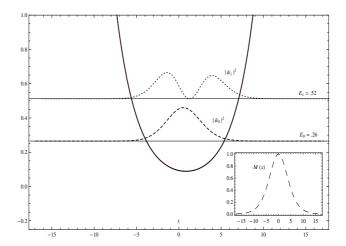
Figure 2.5: Plot of the potential V (solid line) given in equation (2.53), square of its first two bound state wave functions $|\psi_o(x)|^2$ (dashed line) and $|\psi_1(x)|^2$ (dotted line), for the mass function $M(x)$ (long dashed line) given in equation (2.52). We have considered here $a = .2, \alpha = 2, \beta = 2.5$.

condition does not require any extra restriction on the parameters. In figure 2.5 we have plotted the mass function given in (2.52) and potential $V(x)$ given in (2.53) and square of its first two bound state wave functions.

It should be mention here that the new potentials (2.48) and (2.53) are isospectral to the potentials previously obtained in [425] whose bound state wave functions are given in terms of classical Laguerre and Jacobi orthogonal polynomials respectively. More interestingly in both cases the expression within the square bracket coincides with the potentials associated with classical Laguerre and Jacobi orthogonal polynomials. So the new potentials obtained here can be considered as rationally extended version of the standard potentials, of PDM Hamiltonians, which are associated with classical Laguerre or Jacobi polynomials.

2.2.4   Supersymmetry and Hamiltonians associated with EOP

Here we combine the results of Point canonical transformation with supersymmetric quantum mechanics (SUSY QM) method described in section 2.1.1. The superpotentials $W_0(x)$, corresponding to the potentials obtained in (2.48) and (2.53), are determined by the equation (2.5) and the ground state wave functions of both the potentials, as

$$W_0(x) = \frac{b}{2}\left[(\alpha+1)e^{\frac{bx}{2}} - e^{-\frac{bx}{2}}\right] - \frac{b\, e^{\frac{3}{2}bx}}{\alpha(\alpha+1)e^{2bx} + (2\alpha+1)e^{bx} + 1} \tag{2.56}$$

and

$$W_0(x) = \frac{a}{2} \left[(\alpha - \beta)\cosh(ax) + (\alpha + \beta + 2)\sinh(ax)\right]$$
$$+ \frac{2a(\alpha - \beta)}{[(\alpha + \beta)\cosh(ax) + (\alpha - \beta)\sinh(ax)][(\alpha + \beta + 2)\cosh(ax) + (\alpha - \beta)\sinh(ax)]}$$
$$(2.57)$$

respectively. Using equations (2.4) and (2.56) one obtains

$$V_0^-(x) = \frac{b^2}{4}\left[(\alpha^2 - 1)e^{bx} + e^{-bx} + \frac{4}{\alpha(1 + \alpha e^{bx})} + \frac{8e^{bx}}{(1 + \alpha e^{bx})^2}\right] - b^2\left(\frac{\alpha + 1}{2} + \frac{1}{\alpha}\right) \quad (2.58)$$

and

$$V_0^+ = \frac{b^2}{4}\left[\alpha(\alpha + 2)e^{bx} + e^{-bx} + \frac{4}{(\alpha + 1)(1 + (\alpha + 1)e^{bx})} + \frac{8e^{bx}}{(1 + (\alpha + 1)e^{bx})^2}\right] - \frac{b^2}{2}\left(\frac{\alpha^2 + \alpha + 2}{\alpha + 1}\right). \quad (2.59)$$

Also using equations (2.4) and (2.57) we have

$$V_0^- = \frac{a^2}{4}\left[(\alpha^2 - 1)e^{2ax} + (\beta^2 - 1)e^{-2ax} + \frac{4(\alpha - \beta)(\alpha - 3\beta)}{\alpha(\beta + \alpha e^{2ax})} - \frac{8\beta(\alpha - \beta)^2}{\alpha(\beta + \alpha e^{2ax})^2}\right]$$
$$(2.60)$$
$$- \frac{a^2}{4}\left(2\alpha\beta + 2\alpha + 2\beta + 2 + \frac{4\beta}{\alpha}\right)$$

$$V_0^+ = \frac{a^2}{4}\left[(\alpha(\alpha + 2))e^{2ax} + (\beta(\beta + 2))e^{-2ax} + \frac{4(\alpha - \beta)(\alpha - 3\beta - 2)}{(\alpha + 1)((\alpha + 1)e^{2ax} + \beta + 1)}\right]$$
$$(2.61)$$
$$- \frac{a^2}{4}\left(\frac{8(\beta + 1)(\alpha - \beta)^2}{(\alpha + 1)((\alpha + 1)e^{2ax} + \beta + 1)^2} + 2\alpha\beta + \frac{4(\beta + 1)}{\alpha + 1}\right).$$

The potentials V_0^- obtained in (2.58) and (2.60) are same with the potentials given in (2.48) and (2.53) for $\bar{V}_0 = -b^2(\frac{\alpha+1}{2} + \frac{1}{\alpha})$ and $\widehat{V}_0 = -\frac{a^2}{4}\left(2\alpha\beta + 2\alpha + 2\beta + 2 + \frac{4\beta}{\alpha}\right)$ respectively. From the equations (2.58) and (2.59), we observe that the the potential V_0^- and it's supersymmetric partner potentials V_0^+ satisfy the following relation

$$V_0^+(x, \alpha) = V_0^-(x, \alpha + 1) + b^2. \quad (2.62)$$

Also from equation (2.60) and (2.61) we have the relation

$$V_0^+(x, \alpha, \beta) = V_0^-(x, \alpha + 1, \beta + 1) + a^2(\alpha + \beta + 2). \tag{2.63}$$

Hence, we conclude that the new potentials obtained here have translational shape invariant symmetry.

2.3   Intertwining operator approach to spectral design of a PDM Hamiltonian

In this section, systematic procedures of first and second order intertwining operator approach to generate the isospectral Hamiltonians in PDM scenario have been studied with the following spectral modifications: (i) to add new bound state(s) below the ground state (ii) to remove lowest bound state(s) and (iii) to leave the spectrum unaffected. Second order intertwiner for PDM Hamiltonian has been constructed earlier [227] by iterating two first order intertwiner. But this approach failed to perform some of the above mentioned spectral modifications. This shortcoming can be removed by considering directly a second order linear differential operator as the intertwiner. In the following we have first discussed the first order intertwining which uses a first order differential operator as intertwiner.

2.3.1   First-order intertwining

We consider the following two one-dimensional PDM Schrödinger Hamiltonians with same spectrum but with different potential

$$H_0\psi(x) = E^{(0)}\psi(x) , \quad H_0 \equiv -\left[\frac{d}{dx}\left(\frac{1}{M(x)}\right)\frac{d}{dx}\right] + V_0(x) \tag{2.64}$$

and

$$H_1\phi(x) = E^{(1)}\phi(x) , \quad H_1 \equiv -\left[\frac{d}{dx}\left(\frac{1}{M(x)}\right)\frac{d}{dx}\right] + V_1(x). \tag{2.65}$$

The solution of one of the above equation (say (2.64)) is known. We connect above two Hamiltonians (2.64) and (2.65) by means of the intertwining relation

$$\mathcal{L}H_0 = H_1\mathcal{L}, \qquad \phi(x) = \mathcal{L}\psi(x). \tag{2.66}$$

Without loss of generality let us consider the first order intertwining operator as

$$\mathcal{L} = \frac{1}{\sqrt{M(x)}}\frac{d}{dx} + A(x) \tag{2.67}$$

Now using the intertwining relation (2.66) and equating the coefficients of like order of derivatives we obtain

$$V_1 = V_0 + \frac{2A'}{\sqrt{M}} - \frac{3M'^2}{4M^3} + \frac{M''}{2M^2} \tag{2.68}$$

and

$$A(V_1 - V_0) = -\frac{A'M'}{M^2} + \frac{V_0'}{\sqrt{M}} + \frac{A''}{M} \tag{2.69}$$

where 'prime' denotes differentiation with respect to x. Using (2.68), equation (2.69) reduces to

$$\frac{A''}{\sqrt{M}} - \frac{A'M'}{M^{\frac{3}{2}}} - \frac{AM''}{2M^{\frac{3}{2}}} + \frac{3AM'^2}{4M^{\frac{5}{2}}} - 2AA' + V_0' = 0 \tag{2.70}$$

Integrating equation (2.70) we get

$$\frac{A'}{\sqrt{M}} - \frac{AM'}{2M^{\frac{3}{2}}} - A^2 + V_0 = \mu \tag{2.71}$$

where μ is a constant of integration. After substituting

$$A(x) = -\frac{K}{\sqrt{M}}, \tag{2.72}$$

$K = K(x)$ being an auxiliary function, in equation (2.71) we obtain the following Riccati differential equation

$$-\frac{K'}{M} + \frac{KM'}{M^2} - \frac{K^2}{M} + V_0 = \mu \tag{2.73}$$

The equation (2.73) can be linearized by the assumption $K(x) = \frac{\mathcal{U}'(x)}{\mathcal{U}(x)}$. Substituting this value of $K(x)$ in equations (2.72) and (2.73) we get

$$A(x) = -\frac{\mathcal{U}'}{\sqrt{M}\mathcal{U}} \tag{2.74}$$

and

$$-\frac{1}{M}\mathcal{U}'' - \left(\frac{1}{M}\right)' \mathcal{U}' + V_0\,\mathcal{U} = \mu\,\mathcal{U} \tag{2.75}$$

respectively. The equation (2.75) is similar to equation (2.64) with $E^{(0)} = \mu$. μ is the factorization energy and $\mathcal{U}(x)$ is the seed solution. It should be noted here that $\mathcal{U}(x)$ need not be normalizable solution of (2.75). However, for an acceptable $V_1(x)$ (without singularity), $\mathcal{U}(x)$ must not have any node on the real line. For this, we shall restrict ourself on $\mu \leq E_0^{(0)}$ throughout, $E_0^{(0)}$ being the ground state energy eigenvalues of the Hamiltonian H_0. For a given potential $V_0(x)$ the first order isospectral partner potential $V_1(x)$ and the solution of the corresponding Hamiltonian H_1 can be obtained using equations (2.66). The solution of H_1 at the factorization energy μ can not be obtained with the help of (2.66) because $\mathcal{L}\mathcal{U} = 0$. It can be obtained by directly solving $\mathcal{L}^\dagger \phi_\mu = 0$ and is given by [227]

$$\phi_\mu \sim \frac{\sqrt{M(x)}}{\mathcal{U}(x)}. \tag{2.76}$$

At this point following conditions of spectral modification are worth mentioning.

○ Condition for deletion of ground state : For a suitably chosen mass function $M(x)$, if $\mathcal{U}(x)$ corresponds to the ground state wave function of H_0 then the potential $V_1(x)$ has no new singularity, except the singularity due to $V_0(x)$ and the solution $\phi_\mu(x)$ defined in (2.76) is not normalized so that μ does not belong to the bound state spectrum of H_1.

○ Condition for isospectral transformation : If $\mathcal{U}(x)$ is nodeless and unbounded at one end point of the domain of definition of $V_0(x)$ then $\phi_\mu(x)$ is not normalized. So we can generate H_1 strictly isospectral to H_0.

○ Condition for creation of bound state below the ground state : If $\mathcal{U}(x)$ is nodeless and unbounded at the both end points then $\phi_\mu(x)$ defined in (2.76) is normalizable,

so that μ can be included in the bound state spectrum of H_0 to generate H_1. In this case maximal set of bound state wave functions of H_1 are given by $\{\phi_\mu, \mathcal{L}_1\psi\}$.

2.3.2 Examples of First order Intertwining

It may be emphasized that the results mentioned in section 2.3.1 are most general and valid for any potential $V_0(x)$. However to illustrate the above procedure with the help of an example we shall need non-normalizable solutions of equation (2.75) corresponding to a particular mass function $M(x)$. The general solution of this equation can be obtained using PCT approach mentioned in section 2.2.1 by assuming $F(g) \sim {}_2F_1(a,b,c,g)$, where ${}_2F_1(a,b,c,g)$ is the Hypergeometric function. In such case, the potential, mass function and the general non-normalized solutions of the PDMSE (2.75) are given by (detail calculation has been given in the appendix of ref.[503], we have considered $p = \lambda = 1$ for simplicity)

$$V_0(x) = \frac{[(a+b-c)^2 - 1]}{4}e^x + \frac{c(c-2)}{4}e^{-x} \tag{2.77}$$

$$M(x) = \frac{1}{4}\operatorname{sech}^2\left(\frac{1}{2}x\right) \tag{2.78}$$

$$\begin{aligned}\mathcal{U}(x) = {} & \alpha\frac{e^{\frac{c}{2}x}}{(1+e^x)^{\frac{a+b+1}{2}}} \, {}_2F_1\left(a,b,c,\frac{e^x}{1+e^x}\right) \\ & + \beta\frac{e^{(1-\frac{c}{2})x}}{(1+e^x)^{\frac{a+b-2c+3}{2}}} \, {}_2F_1\left(a-c+1,b-c+1,2-c,\frac{e^x}{1+e^x}\right)\end{aligned} \tag{2.79}$$

$$\mu = -ab + \frac{(a+b+1)c}{2} - \frac{c^2}{2} \tag{2.80}$$

respectively. For the potential $V_0(x)$, the bound state solutions and eigenstates of the equation (2.64) are obtained as [503]

$$\psi_n(x) = \left(\frac{(2n+\sigma+\delta+1)n!\,\Gamma(n+\sigma+\delta+1)}{\Gamma(n+\sigma+1)\Gamma(n+\delta+1)}\right)^{1/2}\frac{e^{\frac{(\sigma+1)}{2}x}}{(1+e^x)^{\frac{\sigma+\delta+2}{2}}}P_n^{(\sigma,\delta)}\left(\frac{1-e^x}{1+e^x}\right) \tag{2.81}$$

and

$$E_n^{(0)} = n^2 + n(\sigma+\delta+1) + \frac{(\sigma+1)(\delta+1)}{2}, \quad n = 0,1,2,... \tag{2.82}$$

respectively, where $b = 1 - a + \sigma + \delta$, $c = 1 + \sigma$ with $c > \frac{1}{2}$ and $a + b - c + \frac{1}{2} > 0$ and $P_n^{(\sigma, \delta)}$ is the Jacobi polynomial. The asymptotic behavior of the solution $\mathcal{U}(x)$ given in (2.79), at both end points $\pm\infty$ are given by [507]

$$\mathcal{U}(x) \sim (A_1\alpha + B_1\beta)e^{-\frac{a+b-c+1}{2}x} + (A_2\alpha + B_2\beta)e^{-\frac{c-a-b+1}{2}x} \quad \text{as } x \to \infty \tag{2.83}$$

where

$$A_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}, \quad B_1 = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}$$

$$A_2 = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \quad B_2 = \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}$$

and

$$\mathcal{U}(x) \sim \alpha e^{\frac{c}{2}x} + \beta\, e^{(1-\frac{c}{2})x} \quad \text{as } x \to -\infty \tag{2.84}$$

From these asymptotic behaviors it is clear that $\mathcal{U}(x)$ will unbounded at $x \to \infty$ if $|a + b - c| > 1$ and it is unbounded at $x \to -\infty$ if $c < 0$ or $c > 2$. Therefore $\mathcal{U}(x)$ will nodeless at the finite part of the x axis if $A_1\alpha + B_1\beta$, $A_2\alpha + B_2\beta$,α and β are all positive and $|a + b - c| > 1$, $c < 0$ or $c > 2$.

Now we are going to generate isospectral potentials of the potential (2.77) with various possible spectral modifications.

Deletion of the initial ground state : In this case the factorization energy μ is equal to the ground state energy E_0 giving a and/or $b = 0$ and $\mathcal{U}(x)$ becomes the ground state wave function $\psi_0(x)$ which is obtained from (2.81) as

$$\mathcal{U}(x) = \psi_0(x) \propto \frac{e^{\frac{c}{2}x}}{(1+e^x)^{\frac{a+b+1}{2}}} \tag{2.85}$$

The isospectral partner of $V_0(x)$ given in equation (2.45), is obtained using equation (2.68), (2.77), (2.78) and (2.85) and is given by

$$V_1(x) = \frac{c^2-1}{4}e^{-x} + \frac{(a+b-c)(2+a+b-c)}{4}e^x + \frac{a+b}{2} \tag{2.86}$$

The above potential (2.86) can also be obtained from the initial potential (2.77) by making the changes $a \to a+1, b \to b+1, c \to c+1$. This in turn imply that the potentials $V_0(a)$
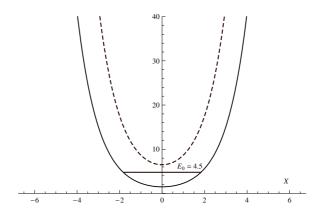
Figure 2.6: Plot of the potential $V_0(x)$ (solid line) given in (2.77) and its first order isospectral partners $V_1(x)$ (dashed line) given in (2.86) by deleting the ground state $E_0^{(0)} = 4.5$, we have considered here $a = 5, b = 0, c = 3, \alpha = 1, \beta = 0$.

and $V_1(x)$ are shape invariant.

Since $\psi_0(x)$ is a bounded solution, $\phi_\mu(x) = \frac{\sqrt{M}}{\psi_0}$ is unbounded at $x \to \pm\infty$ so that ϕ_μ can not be the ground state wave function of V_1. Thus we delete the ground state of V_0 to obtain V_1. Therefore the eigenvalues of H_1 are given by

$$E_n^{(0)} = E_{n+1}^{(1)} = (n+1)^2 + (n+1)(a+b) + \frac{c(a+b-c+1)}{2} , \quad n = 0, 1, 2... \quad (2.87)$$

Corresponding bound state wave functions of $V_1(x)$ are obtained using equations (2.66) and (2.81) as

$$\phi_n(x) \propto \frac{e^{(1+\frac{c}{2})x}}{(1+e^x)^{(\frac{a+b+5}{2})} \operatorname{sech}(\frac{x}{2})} P_n^{(c,a+b-c+1)}\left(-\tanh\frac{x}{2}\right) , \quad n = 0, 1, 2... \quad (2.88)$$

We have plotted the potentials $V_0(x)$ given in (2.77) and $V_1(x)$ given in (2.86) for $a = 5, b = 0, c = 3, \alpha = 1, \beta = 0$ in figure 2.6.

Strictly isospectral potentials : The strictly (strict in the sense that the spectrum of the initial potential and its isospectral potential are exactly the same) isospectral potentials can be generated with the help of those seed solutions which vanish at one of the ends of the x-domain. Now for $\beta = 0$ and $\alpha > 0$, it is seen from (2.83) that $\mathcal{U}$ is unbounded at $x \to \infty$ if $|a+b-c| > 1$. But the solution (2.81) become unbounded for $a+b-c < -1$. So we must take $a+b-c > 1$. On the other hand from (2.84) it is observed that $\mathcal{U}(x) \to 0$ at $x \to -\infty$ if $c < 2$ or $c > 0$ but $\psi_n(x)$ are not normalizable for the values of $c < 2$ so we must take $c > 0$. So $\mathcal{U}(x)$ vanishes at $x \to -\infty$ and unbounded at $x \to \infty$ if
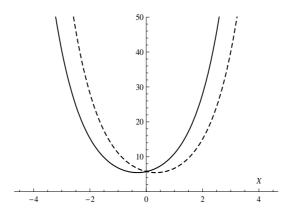
Figure 2.7: Plot of the potential $V_0(x)$ (solid line) and its first order isospectral partner (dashed line) given in (2.90).

$a + b - c > 1$ and $c > 0$. In this case the spectrum of the isospectral potential as well as original potential are identical i.e. $E_n^{(0)} = E_n^{(1)}$, $n = 0, 1, 2...$ Considering the seed solution as

$$\mathcal{U}(x) = \frac{e^{\frac{c}{2}x}}{(1 + e^x)^{\frac{a+b+1}{2}}} \, {}_2F_1\left(a, b, c, \frac{e^x}{1 + e^x}\right), \qquad a + b - c > 1, c > 0$$

we have calculated the explicit form of the partner potential using equations (2.45), (2.68), (2.77), and (2.78)

$$V_1(x) = \frac{1}{8}[2c(c-2)e^{-x} + 2((a+b-c)^2 - 1)e^x$$

$$- \frac{\text{sech}^2(\frac{x}{2})}{c^2(1+c) \, {}_2F_1\left(a,b,c,\frac{e^x}{1+e^x}\right)} \Big\{ -4a^2b^2(1+c)\left({}_2F_1\left(1+a, 1+b, 1+c, \frac{e^x}{1+e^x}\right)\right)^2 + 4abc \, {}_2F_1\left(a, b, c, \frac{e^x}{1+e^x}\right)$$

$$\left((a+1)(b+1)\left({}_2F_1\left(2+a, 2+b, 2+c, \frac{e^x}{1+e^x}\right)\right)^2 - (1+c)\sinh x \left({}_2F_1\left(a+1, b+1, c+1, \frac{e^x}{1+e^x}\right)\right)^2\right)$$

$$- 8c^2(1+c)\cosh^3 x \left({}_2F_1\left(a, b, c, \frac{e^x}{1+e^x}\right)\right)^2 \left((a+b)\cosh(\frac{x}{2}) + (1+a+b-2c)\sinh(\frac{x}{2})\right)\Big\}$$

$$(2.89)$$

In particular for $a = 3, b = 5, c = 4, \alpha = 1, \beta = 0$ we have

$$V_0(x) = \frac{1}{4}(23\cosh x + 7\sinh x)$$
$$V_1(x) = \frac{15e^{-x}}{4} + 2e^x + \frac{4 + e^x - 3e^{2x}}{(4 + 3e^x)^2}. \tag{2.90}$$

These potentials are plotted in figure 2.7. In this case eigenfunctions and eigenvalues of

the above partner potential $V_1(x)$ are given by

$$
\phi_n(x) \propto \frac{e^{3x}\left[(2+e^x)(n+7)P_{n-1}^{(4,4)}\left(-\tanh\left(\frac{x}{2}\right)\right) + (1+e^x)(5+3e^x)P_n^{(3,3)}(-\tanh(\frac{x}{2}))\right]}{(1+e^x)^6(2+e^x)\operatorname{sech}\left(\frac{x}{2}\right)}
$$
$$
E_n^{(1)} = n^2 + 8n + 10 , \qquad n = 0,1,2...
$$

(2.91)

respectively.

Creation of a new ground state : In this case we consider $\mu < E_0$. The new state can be created below the ground state of the initial potential with the help of those seed solutions which satisfies the following two conditions: (i) it should be nodeless throughout the x-domain and (ii) it should be unbounded at both the end points of the domain of definition of the given potential $V_0(x)$. From the asymptotic behaviors of the seed solution $\mathcal{U}$, given in equations (2.83) and (2.84) we have, for $|a+b-c| > 1$ together with either $c < 0$ or $c > 2$, the above two conditions are satisfied. But to get $\psi_n(x)$ as physically acceptable, we shall take $c > 2$ and $a + b - c > 1$. In this case the spectrum of the partner potential is $\{\mu, E_n^{(0)}, n = 0,1,2...\}$, $E_n^{(0)}$ being the energy eigenvalues of the original potential $V_0(x)$ given in (2.77). Corresponding bound state wave functions are $\{\phi_\mu(x), \phi_n(x), n = 0,1,2,...\}$. For $a + b - c > 1, c > 2$ and the seed solution $\mathcal{U}$ given in (2.79), the general expression of the isospectral potential becomes too involved so instead of giving the explicit expression of the partner potential we have plotted in figure 2.8 the original potential $V_0(x)$ given in (2.77) and its partner potential $V_1(x)$ (which is obtained using (2.68)) considering the particular values $a = 2.8, b = 20, c = 4.4$ and $\alpha = \beta = 1$. In this case the energy eigenvalues of $V_1(x)$ are given by

$$
E_n^{(1)} = \{-13.32, E_n^{(0)}, n = 0,1,2...\} = \{-13.32, n^2 + 22.8n + 42.68, n = 0,1,2...\} \quad (2.92)
$$
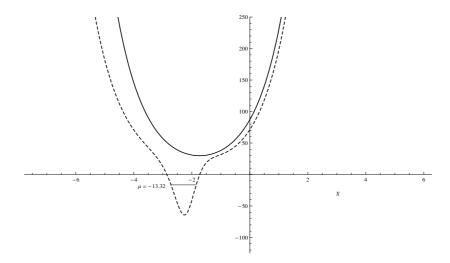
Figure 2.8: Plot of the potential $V_0(x)$ (solid line) given in (2.77) and its first order isospectral partner V_1 (dashed line) by inserting the state $\mu = -13.32$. We have considered here $a = 2.8, b = 20, c = 4.4, \alpha = \beta = 1$.

2.3.3   Second-order intertwining

Here we have constructed the second intertwiner directly by taking the following second-order differential operator

$$\mathcal{L}_2 \equiv \frac{1}{M}\frac{d^2}{dx^2} + \eta(x)\frac{d}{dx} + \gamma(x) \tag{2.93}$$

where $\eta(x)$ and $\gamma(x)$ are unknown functions of x. Substitution of this intertwiner in the intertwining relation (2.66) and comparing the coefficients of like order derivatives leads to a set of equations involving $V_0(x), V_2(x)^4, \eta(x), \gamma(x)$ and their derivatives

$$V_2 = V_0 + 2\eta' + \frac{M'}{M}\eta - \frac{3M'^2}{M^3} + \frac{2M''}{M^2} \tag{2.94}$$

$$(V_2 - V_0)\eta = \frac{2V_0'}{M} + \frac{2\gamma'}{M} + \frac{\eta''}{M} - \frac{\eta'M'}{M^2} + \frac{M''\eta}{M^2} - \frac{2M'^2\eta}{M^3} + \frac{6M'^3}{M^5} - \frac{6M'M''}{M^4} + \frac{M'''}{M^3} \tag{2.95}$$

$$(V_2 - V_0)\gamma = \frac{V_0''}{M} + V_0'\eta + \frac{\gamma''}{M} - \frac{M'\gamma'}{M^2} \tag{2.96}$$

---

[4]In second order intertwining we assume H_1 to be H_2 with $V_1(x)$ as $V_2(x)$ and $E_n^{(1)}$ as $E_n^{(2)}$.

Now using (2.94) the equations (2.95) and (2.96) reads

$$2\eta\eta' + \frac{M'\eta^2}{M} - \frac{3\eta M'^2}{M^3} + \frac{2M''\eta}{M^2} - \frac{2\gamma\,'}{M} + \frac{M'\eta'}{M^2} + \frac{2M'^2\eta}{M^3} - \frac{\eta''}{M} - \frac{M''\eta}{M^2} - \frac{2V_0'}{M} - \frac{6M'^3}{M^5} + \frac{6M'M''}{M^4} - \frac{M'''}{M^3} = 0$$

(2.97)

and

$$\gamma\left(2\eta' + \frac{M'\eta}{M} - \frac{3M'^2}{M^3} + \frac{2M''}{M^2}\right) + \frac{M'\gamma\,'}{M^2} - \frac{\gamma\,''}{M} - \eta V_0' - \frac{V_0''}{M} = 0$$

(2.98)

respectively. Equation (2.97) can be integrated to obtain

$$\gamma = \frac{M\eta^2}{2} + \frac{M'\eta}{2M} - \frac{\eta'}{2} - V_0 + \frac{M'^2}{M^3} - \frac{M''}{2M^2} + C_1$$

(2.99)

where C_1 is an arbitrary constant. Using (2.99) in (2.98) we obtain

$$\frac{\eta'''}{2M} + M\eta'\eta^2 - \frac{\eta\eta'M'}{2M} - 2\eta'^2 - 2\eta'V_0 + \frac{5\eta'M'^2}{M^3} - \frac{3\eta'M''}{M^2} + \frac{\eta^3M'}{2} - \frac{\eta^2M'^2}{2M^2} - \frac{\eta V_0 M'}{M} - \frac{2\eta M'^3}{M^4}$$

$$+ \frac{\eta^2M''}{2M} + \frac{5\eta M'M''}{2M^3} - \frac{\eta''M'}{M^2} - \eta\eta'' - \eta V' - \frac{\eta M'''}{2M^2} + 2C_1\eta' + \frac{C_1\eta M'}{M} + \frac{3V M'^2}{M^3} - \frac{18M'^4}{M^6}$$

$$+ \frac{49M'^2M''}{2M^5} - \frac{3C_1M'^2}{M^3} - \frac{2V_0M''}{M^2} - \frac{4M''^2}{M^4} + \frac{2C_1M''}{M^2} - \frac{V_0'M'}{M^2} - \frac{9M'M'''}{2M^4} + \frac{M''''}{2M^3} = 0$$

(2.100)

Multiplying by $\left(\eta M + \frac{M'}{M}\right)$, above equation (2.100) can be integrated to obtain

$$\frac{\eta\eta''}{2} - \frac{\eta'^2}{4} - \eta'\eta^2 M + \frac{\eta^4 M^2}{4} - M\eta^2 V_0 + C_1 M\eta^2 + \frac{C_1 M'^2}{M^3} + \frac{2C_1 M'\eta}{M} - \frac{2M'V_0\eta}{M} - \frac{M'^2 V_0}{M^3}$$

$$- \frac{M''\eta^2}{2M} + \frac{M'\eta^3}{2} + \frac{5M'^3\eta}{m^4} - \frac{2M'\eta\eta'}{M} + \frac{5M'^2\eta^2}{4M^2} + \frac{M'\eta''}{2M^2} - \frac{M''\eta'}{2M^2} + \frac{M'''\eta}{2M^2} - \frac{4M'M''\eta}{M^3} + \frac{3M'^4}{M^6}$$

$$- \frac{5M'^2 M''}{2M^5} - \frac{M''^2}{4M^4} + \frac{M'M'''}{2M^4} + C_2 = 0,$$

(2.101)

where C_2 is the constant of integration. For a given potential $V_0(x)$, the new potential $V_2(x)$ and $\gamma(x)$ can be obtained from (2.94) and (2.99) if the solution $\eta(x)$ of (2.101) is

known. To obtain $\eta(x)$ we take the Ansätz

$$\eta' = M\eta^2 + 2\left(\eta + \frac{M'}{M^2}\right)\tau + \frac{M'}{M}\eta + \frac{2M'^2}{M^3} - \frac{M''}{M^2} + \xi \tag{2.102}$$

where ξ is a constant to be determined and τ is a function of x. Using above ansätz in equation (2.101) we obtain the following equation

$$M\left(\frac{\tau'}{M} + \frac{\tau^2}{M} - \frac{M'\tau}{M^2} - V_0 + C_1 - \frac{\xi}{2}\right)\eta^2 + \frac{2M'}{M}\left(\frac{\tau'}{M} + \frac{\tau^2}{M} - \frac{M'\tau}{M^2} - V_0 + C_1 - \frac{\xi}{2}\right)\eta$$

$$+ \frac{M'^2}{M^3}\left(\frac{\tau'}{M} + \frac{\tau^2}{M} - \frac{M'\tau}{M^2} - V_0 + C_1 - \frac{\xi}{2}\right) + \left(C_2 - \frac{\xi^2}{4}\right) = 0 \tag{2.103}$$

Since equation (2.103) is valid for arbitrary η, the coefficients of each power of η must vanish, which give $\xi^2 = 4C_2$ and

$$\frac{\tau'}{M} + \frac{\tau^2}{M} - \frac{M'\tau}{M^2} - V_0 + C_1 - \frac{\xi}{2} = 0 \tag{2.104}$$

Now defining $\mu = C_1 - \frac{\xi}{2}$, the above equation can be written as

$$\frac{\tau'}{M} + \frac{\tau^2}{M} - \frac{M'\tau}{M^2} = V_0 - \mu , \quad \mu = C_1 - \frac{\xi}{2} = C_1 \mp \sqrt{C_2} \tag{2.105}$$

The equation (2.105) is a Riccati equation which can be linearized by defining $\tau = \frac{\mathcal{U}'}{\mathcal{U}}$. Making this change in equation (2.105) we obtain

$$-\frac{1}{M}\mathcal{U}'' - \left(\frac{1}{M}\right)'\mathcal{U}' + V_0\,\mathcal{U} = \mu\,\mathcal{U} \tag{2.106}$$

Depending on whether C_2 is zero or not, ξ vanishes or takes two different values $\pm\sqrt{C_2}$. If $C_2 = 0$, we need to solve one equation of the form (2.105) and then the equation (2.102) for $\eta(x)$. If $C_2 \neq 0$, there will be two different equations of type (2.105) for two factorization energies $\mu_{1,2} = C_1 \mp \sqrt{C_2}$. Once we solve them, it is possible to construct algebraically a common solution $\eta(x)$ of the corresponding pair of equations (2.102). There is an obvious difference between the (A) real case with $C_2 > 0$ and (B) complex case $C_2 < 0$; thus there follows a natural scheme of classification for the solutions $\eta(x)$ based on the sign of C_2. Here, we shall not discuss the case $C_2 = 0$.

(A) Real Case $(C_2 > 0)$: Here we have $\mu_{1,2} \in \mathbb{R}, \mu_1 \neq \mu_2$. Let the corresponding solutions of the Riccati equation (2.105) be denoted by $\tau_{1,2}(x)$. Now the associated pair of equations (2.102) become

$$\eta' = M\eta^2 + 2\left(\eta + \frac{M'}{M^2}\right)\tau_1 + \frac{M'}{M}\eta + \frac{2M'^2}{M^3} - \frac{M''}{M^2} + \mu_2 - \mu_1 \tag{2.107}$$

and

$$\eta' = M\eta^2 + 2\left(\eta + \frac{M'}{M^2}\right)\tau_2 + \frac{M'}{M}\eta + \frac{2M'^2}{M^3} - \frac{M''}{M^2} + \mu_1 - \mu_2 \tag{2.108}$$

respectively. Subtracting (2.107) from (2.108) and using (2.106) we obtain $\eta(x)$ as

$$\eta(x) = \frac{\mu_1 - \mu_2}{\tau_1 - \tau_2} - \frac{M'}{M^2} = -\frac{W'(\mathcal{U}_1, \mathcal{U}_2)}{MW(\mathcal{U}_1, \mathcal{U}_2)} \tag{2.109}$$

where $\mathcal{U}_1, \mathcal{U}_2$ are the seed solutions of the equation (2.106) corresponding to the factorization energy μ_1 and μ_2 respectively and $W(\mathcal{U}_1, \mathcal{U}_2) = \mathcal{U}_1\mathcal{U}_2' - \mathcal{U}_1'\mathcal{U}_2$, is the Wronskian of $\mathcal{U}_1$ and $\mathcal{U}_2$. It is clear from (2.94) and (2.109) that mass function $M(x)$ is nonsingular and does not vanish at the finite part of the x-domain, so that the new potential $V_2(x)$ has no extra singularities (i.e. the number of singularities in V_0 and V_2 remains the same) if $W(\mathcal{U}_1, \mathcal{U}_2)$ is nodeless there. The spectrum of H_2 depends on whether or not its two eigenfunctions $\phi_{\mu_{1,2}}$ which belongs as well to the kernel of $\mathcal{L}^\dagger$ can be normalized [59], namely

$$\mathcal{L}^\dagger\phi_{\mu_j} = 0 \quad \text{and} \quad H_2\phi_{\mu_j} = \mu_j\phi_{\mu_j}, \quad j = 1, 2 \tag{2.110}$$

where $\mathcal{L}^\dagger$ is the adjoint of $\mathcal{L}$ and is given by

$$\mathcal{L}^\dagger = \frac{1}{M}\frac{d^2}{dx^2} - \left(\eta + \frac{2M'}{M^2}\right)\frac{d}{dx} + \left(\frac{2M'^2}{M^3} - \frac{M''}{M^2} - \eta' + \gamma\right)$$

For $j = 1$ the explicit expression of the two equation mentioned in (2.110) are

$$\frac{1}{M}\frac{d^2\phi_{\mu_1}}{dx^2} - \left(\eta + \frac{2M'}{M^2}\right)\frac{d\phi_{\mu_1}}{dx} + \left(\frac{2M'^2}{M^3} - \frac{M''}{M^2} - \eta' + \gamma\right)\phi_{\mu_1} = 0 \tag{2.111}$$

and

$$-\frac{1}{M}\phi_{\mu_1}'' - \left(\frac{1}{M}\right)' \phi_{\mu_1}' + (V_2 - \mu_1)\phi_{\mu_1} = 0 \tag{2.112}$$

respectively. Adding (2.111) from (2.112) we obtain

$$-\left(\frac{M'}{M^2} + \eta\right)\frac{d\phi_{\mu_1}}{dx} + \left(V_2 - \mu_1 + \frac{2M'^2}{M^3} - \eta' + \gamma - \frac{M''}{M^2}\right)\phi_{\mu_1} = 0 \tag{2.113}$$

Substituting the values of V_2 and γ from (2.94) and (2.99) with $2C_1 = \mu_1 + \mu_2$, in the above equation (2.113), we get

$$\frac{d}{dx}\left(log\phi_{\mu_1}\right) = \frac{\eta' + 3\eta\frac{M'}{M} + \frac{M''}{M^2} + M\eta^2 + 2(C_1 - \mu_1)}{2(\eta + \frac{M'}{M^2})} \tag{2.114}$$

Now using our ansätz (2.102) in (2.114) and then integrating we obtain

$$\phi_{\mu_1} \propto \frac{M\left(\eta + \frac{M'}{M^2}\right)}{\mathcal{U}_1} \propto \frac{M\mathcal{U}_2}{W(\mathcal{U}_1, \mathcal{U}_2)} \tag{2.115}$$

Above procedure can be applied to obtain ϕ_{μ_2} as

$$\phi_{\mu_2} \propto \frac{\eta M + \frac{M'}{M}}{\mathcal{U}_2} \propto \frac{M\mathcal{U}_1}{W(\mathcal{U}_1, \mathcal{U}_2)} \tag{2.116}$$

If both $\phi_{\mu_{1,2}}$ are normalizable then we get the maximal set of eigenfunctions of H_2 as $\{\phi_{\mu_1}, \phi_{\mu_2}, \phi_n \propto \mathcal{L}\psi_n\}$.

Among the several spectral modifications which can be achieved through the real second order intertwining of PDM Hamiltonians, some cases are worth to be mentioned.

- Condition of deletion of first two energy levels : For $\mu_1 = E_0^{(0)}$ and $\mu_2 = E_1^{(0)}$ the two solutions of equation (2.106) are the normalizable solutions of equation (2.64) i.e, $\mathcal{U}_1(x) = \psi_0(x)$ and $\mathcal{U}_2(x) = \psi_1(x)$ respectively. It turns out that the Wronskian is nodeless in the finite part of the real line and vanishes at the boundary of the domain of $V_0(x)$ but two solutions ϕ_{μ_1} and ϕ_{μ_2} are non-normalizable which is clear from the relation (2.115). Thus $Sp(H_2) = Sp(H_0) - \{E_0^{(0)}, E_1^{(0)}\} = \{E_2^{(0)}, E_3^{(0)}, ...\}$.

- Condition of isospectral transformations : If we take $\mu_2 < \mu_1 < E_0^{(0)}$ and choose two seed solutions $\mathcal{U}_1(x)$ and $\mathcal{U}_2(x)$ such way that either $\mathcal{U}_{1,2}(x_l) = 0$ or $\mathcal{U}_{1,2}(x_r) = 0$, x_l and x_r being the end points of the domain of definition of $V_0(x)$, then the Wronskian

$W(\mathcal{U}_1, \mathcal{U}_2)$ vanishes at x_l or x_r. Hence ϕ_{μ_1} and ϕ_{μ_2} become non-normalizable so that $Sp(H_0) = Sp(H_2)$.

○ Condition of creation of two new levels below the ground state : For $\mu_2 < \mu_1 < E_0^{(0)}$ and choosing $\mathcal{U}_1(x)$ and $\mathcal{U}_2(x)$ in such way that $\mathcal{U}_2(x)$ has exactly one node and $\mathcal{U}_1(x)$ is nodeless on the domain of $V_0(x)$ then the Wronskian $W(\mathcal{U}_1, \mathcal{U}_2)$ becomes nodeless and unbounded at the boundary so that two wave functions ϕ_{μ_1} and ϕ_{μ_2} become normalizable. Therefore the spectrum of H_2 becomes $Sp(H_2) = Sp(H_0) \bigcup \{\mu_1, \mu_2\} = \{\mu_1, \mu_2, E_n^{(0)}, n = 0, 1, 2...\}$.

(B) Complex case $(C_2 < 0)$: For $C_2 < 0$ the two factorization energies μ_1 and μ_2 become complex. In order to construct real $\bar{V}$ we shall choose μ_1 and μ_2 as complex conjugate to each other i.e, $\mu_1 = \mu \in \mathbb{C}$ and $\mu_2 = \bar{\mu}$. For the same reason we shall take $\tau_1(x) = \tau(x)$ and $\tau_2(x) = \bar{\tau}(x)$. Hence the real solution $\eta(x)$ of (2.102) generated from the complex $\tau(x)$ of (2.105) becomes

$$\eta(x) = \frac{\mu - \bar{\mu}}{\tau - \bar{\tau}} - \frac{M'}{M^2} = \frac{Im(\mu)}{Im(\tau)} - \frac{M'}{M^2} = -\frac{W'(\mathcal{U}, \bar{\mathcal{U}})}{MW(\mathcal{U}, \bar{\mathcal{U}})} \tag{2.117}$$

Defining $w(x) = \frac{W(\mathcal{U}, \bar{\mathcal{U}})}{M(\mu - \bar{\mu})}$, $\eta(x)$ becomes

$$\eta(x) = -\frac{w'}{Mw} - \frac{M'}{M^2} \tag{2.118}$$

For the factorization energies μ and $\bar{\mu}$ the equation (2.105) becomes

$$-\frac{1}{M}\mathcal{U}'' - \left(\frac{1}{M}\right)' \mathcal{U}' + V_0\mathcal{U} = \mu\,\mathcal{U} \quad \text{and} \quad -\frac{1}{M}\bar{\mathcal{U}}'' - \left(\frac{1}{M}\right)' \bar{\mathcal{U}}' + V_0\bar{\mathcal{U}} = \bar{\mu}\,\bar{\mathcal{U}}$$

Multiplying first equation by $\bar{\mathcal{U}}$ and second equation by $\mathcal{U}$ and then subtracting we obtain

$$\frac{W'(\mathcal{U}, \bar{\mathcal{U}})}{M(\mu - \bar{\mu})} - \frac{M'W(\mathcal{U}, \bar{\mathcal{U}})}{M^2(\mu - \bar{\mu})} = |\mathcal{U}|^2 \tag{2.119}$$

Using above relation (2.119) we have

$$w'(x) = \frac{W(\mathcal{U}, \bar{\mathcal{U}})}{M(\mu - \bar{\mu})} - \frac{M'W(\mathcal{U}, \bar{\mathcal{U}})}{M^2(\mu - \bar{\mu})} = |\mathcal{U}|^2 \tag{2.120}$$

which implies that $w(x)$ is a non-decreasing function. So it is sufficient to choose

$$\lim_{x \to x_l} \mathcal{U} = 0 \quad \text{or} \quad \lim_{x \to x_r} \mathcal{U} = 0 \tag{2.121}$$

for the Wronskian W to be nodeless. It is to be noted here that in this case we can only construct potentials which are strictly isospectral with the initial potential.

2.3.4   Examples of second order intertwining

To illustrate the second order intertwining with the help of an example we shall need non-normalizable solutions of (2.106) (which is similar to equation (2.75) but with two factorization energies) corresponding to a particular mass function $M(x)$. To illustrate the second order intertwining with an example we have considered the potential (2.77) as an initial potential. Corresponding seed solution and the factorization energy $\mu = \mu_1$ are given by [503]

$$\mu_1 = -ab + \frac{(a+b+1)c}{2} - \frac{c^2}{2} \tag{2.122}$$

$$\mathcal{U}_1(x) = \alpha \frac{e^{\frac{c}{2}x}}{(1+e^x)^{\frac{a+b+1}{2}}} \; {}_2F_1\left(a, b, c, \frac{e^x}{1+e^x}\right)$$
$$+ \beta \frac{e^{\left(1-\frac{c}{2}\right)x}}{(1+e^x)^{\frac{a+b-2c+3}{2}}} \; {}_2F_1\left(a-c+1, b-c+1, 2-c, \frac{e^x}{1+e^x}\right) \tag{2.123}$$

We notice that the potential (2.77) and corresponding Hamiltonian are invariant under the transformation $a \to a + \nu$ and $b \to b - \nu$, $\nu \in \mathbb{R} - \{0\}$. But the solution (2.79) of the corresponding Schrödinger equation changes to

$$\mathcal{U}_2(x) = \alpha \frac{e^{\frac{c}{2}x}}{(1+e^x)^{\frac{a+b+1}{2}}} \; {}_2F_1\left(a+\nu, b-\nu, c, \frac{e^x}{1+e^x}\right)$$
$$+ \beta \frac{e^{\left(1-\frac{c}{2}\right)x}}{(1+e^x)^{\frac{a+b-2c+3}{2}}} \; {}_2F_1\left(a+\nu-c+1, b-\nu-c+1, 2-c, \frac{e^x}{1+e^x}\right) \tag{2.124}$$

and the corresponding factorization energy is given by

$$\mu_2 = -ab + \frac{(a+b+1)c}{2} - \frac{c^2}{2} + \nu(a-b) + \nu^2 \tag{2.125}$$
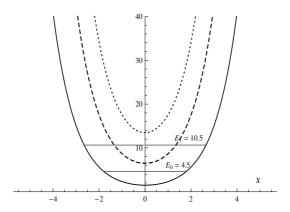
Figure 2.9:   Plot of the original potential (solid line) for $a = 5, b = 0, c = 3$ and its first-order SUSY partner (dashed line) by deleting the ground state $E_0 = 4.5$ and second-order SUSY partner (dotted line) by deleting two successive states $E_0 = 4.5, E_1 = 10.5$.

Thus the general solutions of the equation (2.106) for the two factorization energies μ_1 and μ_2, are given by (2.123) and (2.124)respectively. The asymptotic behaviors of the seed solution $\mathcal{U}_2$ remains same as $\mathcal{U}_1$, which are given in (2.83) and (2.84). In the following we have constructed the second order isospectral partner potentials of the potential (2.77) with various possible spectral modifications.

Deletion of first two energy levels : Let us take $\mu_1 = E_0^{(0)}$ and $\mu_2 = E_1^{(0)}, \mathcal{U}_1 = \psi_0(x)$ and $\mathcal{U}_2 = \psi_1(x)$ which are given in (2.81). The Wronskian $W(\mathcal{U}_1, \mathcal{U}_2)$ is given by

$$W(\mathcal{U}_1, \mathcal{U}_2) \propto \frac{e^{(c+1)x}}{(1 + e^x)^{a+b+3}} \tag{2.126}$$

which is nodeless and bounded in $(-\infty, \infty)$ as $c > -\frac{1}{2}$ and $a + b - c + \frac{1}{2} > 0$. The second-order SUSY partner of $V_0(x)$ is obtained using equation (2.94) and is given by

$$V_2(x) = \frac{1}{4} \left[\left(c^2 - 2c(a + b + 2) + (a + b + 1)(a + b + 3) \right) e^x + c(c + 2)e^{-x} + 4(a + b + 1) \right] \tag{2.127}$$

Here, the eigenfunctions $\phi_{\mu_1} \propto \frac{M\mathcal{U}_2}{W(\mathcal{U}_1, \mathcal{U}_2)}$ and $\phi_{\mu_2} \propto \frac{M\mathcal{U}_1}{W(\mathcal{U}_1, \mathcal{U}_2)}$ of H_2 associated to $\mu_1 = E_0^{(0)}$ and $\mu_2 = E_1^{(0)}$ are not normalizable since $\lim_{x \to -\infty, \infty} \bar{\psi}_{\mu_{1,2}}(x) = \infty$. Thus $Sp(H_2) = Sp(H_0) - \{E_0^{(0)}, E_1^{(0)}\} = \{E_2^{(0)}, E_3^{(0)}...\}$.

In particular taking $a = 5, b = 0, c = 3$ we have plotted the potential $V_0(x)$ and its second-order SUSY partner $V_2(x)$ given in (2.77) and (2.127) respectively, in figure 2.9.

<u>Strictly isospectral potentials</u> : The strictly isospectral partner potentials can be constructed by creating two new energy levels in the limit when each seed vanishes at one of the ends of the x-domain. Now from the asymptotic behaviors of the seed solutions, we note that both the seed solutions vanish at $x \to -\infty$ for $\beta = 0, \alpha > 0$ if $a + b - c > 1$ and $c > 0$. Considering $\beta = 0, \alpha = 1$ in (2.123) and (2.124) we take two seed solution as

$$\mathcal{U}_1(x) = \frac{e^{\frac{c}{2}x}}{(1 + e^x)^{\frac{a+b+1}{2}}} \, _2F_1\left(a, b, c, \frac{e^x}{1 + e^x}\right) \tag{2.128}$$

and

$$\mathcal{U}_2(x) = \frac{e^{\frac{c}{2}x}}{(1 + e^x)^{\frac{a+b+1}{2}}} \, _2F_1\left(a + \nu, b - \nu, c, \frac{e^x}{1 + e^x}\right) \tag{2.129}$$

Since $\mathcal{U}_{1,2}(x) \to 0$ at $x \to -\infty$, from the expressions (2.115) and (2.116) we can conclude that $\lim_{x \to -\infty} \phi_{\mu_{1,2}}(x) = \infty$ which implies that $\mu_{1,2}$ does not belongs to $Sp(H_2)$ i.e. $V_2(x)$ is strictly isospectral to $V_0(x)$. Here the general expression of the partner potential is too involved so instead of giving the explicit expression we have considered particular values $a = 3, b = 5, c = 4, \nu = 1, \alpha = 1, \beta = 0$. Corresponding expression of the partner potential and its energy spectrum are

$$V_2(x) = 1 + \frac{3}{4}(9 \cosh x - 7 \sinh x), \qquad E_n^{(2)} = E_n^{(0)} = n^2 + 8n + 10, \quad n = 0, 1, 2...$$

respectively. In figure 2.10, we have plotted the initial potential, its first and second-order strictly isospectral partner potentials for the parameter values $a = 3, b = 5, c = 4, \alpha = 1, \beta = 0, \nu = 1$.
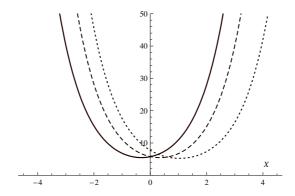
Figure 2.10:   Plot of the original potential (solid line) and its first-order (dashed line) and second-order (dotted line) SUSY partner by making the isospectral transformation for $a = 3, b = 5, c = 4, \alpha = 1, \beta = 0, \nu = 1$.
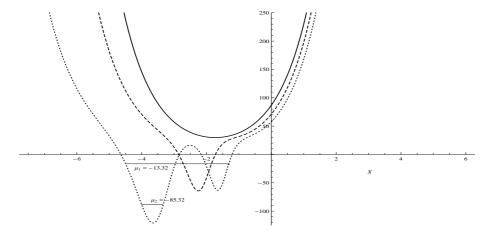

Figure 2.11: Plot of the original potential (solid line) for $a = 2.8, b = 20, c = 4.4$ and its first-order SUSY partner (dashed line) by creating a new level $\mu_1 = -13.32$ and second-order SUSY partner (dotted line) by creating two new levels $\mu_1 = -13.32, \mu_2 = -85.32$

Creation of two new levels below the ground state : Two energy levels can be created taking $\mu_2 < \mu_1 < E_0$ and using those seed solutions $\mathcal{U}_1$ and $\mathcal{U}_2$ for which the Wronskian become nodeless. In this case the expressions of the Wronskian contains several Hypergeometric function, so it is very difficult to mention the range of a, b, c and ν for which it is nodeless. In particular for $a = 2.8, b = 20, c = 4.4, \alpha = 1, \beta = 1$ we have the Wronskian is found to be nodeless. For the same values of a, b, c we have plotted the potential and its second order partner in figure 2.11. The second-order isospectral partner is obtained using equation (2.94).

Example of strictly isospectral transformation for complex factorization energies : The complex factorization energy μ_1 and μ_2 given by equation (2.122) and (2.125), can be made conjugate to each other in several ways. One of the way is by making following restrictions on a, b, c, ν : $c \in \mathbb{R}$, $Im(a) = -Im(b)$, $\nu = Re(b) - Re(a)$. But in order to keep the initial potential real we have to made two more restrictions e.g. $Re(a) + Re(b) - c > 1$ and $c > 2$. In particular taking $a = 6.1 - 5\,i, b = 8 + 5\,i, c = 4.1, \nu = 1.9$ we have two factorization energy $\mu_1(=\mu) = -51.25 + 9.5\,i$ and $\mu_2(=\bar{\mu}) = -51.25 - 9.5\,i$. For these values of a, b, c, ν and $\alpha = 1, \beta = 0$ the seed solution $\mathcal{U}$ becomes

$$\mathcal{U}(x) = \frac{e^{2.05x}}{(1 + e^x)^{7.55}} \, {}_2F_1\left(6.1 - 5\,i, 8 + 5\,i, 4.1, \frac{e^x}{1 + e^x}\right) \tag{2.130}$$

Clearly $\mathcal{U}(-\infty) = 0$ and $|\mathcal{U}| \to \infty$ as $x \to \infty$ so this seed $\mathcal{U}$ and its conjugate $\bar{\mathcal{U}}$ can be used to obtain the second-order SUSY partner potential $V_2(x)$ with the help of equations (2.94) and (2.117). In figure 2.12 we have plotted the initial potential $V_0(x)$ given in (2.77) and its isospectral partner $V_2(x)$ for the parameter values mentioned earlier.
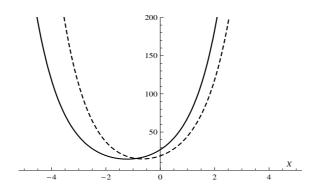
Figure 2.12: Plot of the original potential (solid line) for $a = 6.1 - 5\,i, b = 8 + 5\,i, c = 4.1, \nu = 1.9, \mu_1 = -51.25 + 9.5\,i, \mu_2 = -51.25 - 9.5\,i$. and its second-order isospectral partner (dashed line).

2.4   Summary

To summarize, we point out the following main results :

▶ We have generalized the modified factorization technique of a Schrödinger Hamiltonian suggested in ref.[172, 173] to a quantum system characterized by position-dependent mass Hamiltonian. This generalization is done in such a way that a given mass function and the excited state wave function of a PDM Hamiltonian can be used to generate new solvable physical Hamiltonians. This method is applied to a number of exactly solvable PDM Hamiltonians which leads to nontrivial isospectral Hamiltonians.

▶ The method of point canonical transformation has been used to generate some exactly solvable shape invariant potentials for position dependent (effective) mass Schrödinger equation whose bound state solutions are given in terms of Laguerre or Jacobi type X_1 exceptional orthogonal polynomials. The obtained potentials are in fact generalizations (by some rational functions) of the potentials whose bound state solutions involve classical Laguerre or Jacobi orthogonal polynomials. We have also shown that these two type of potentials, one is associated with exceptional orthogonal polynomials and other is to classical orthogonal polynomials, are isospectral. The method discussed here can be used for other choices of the function $g(x)$ in order to generate other type of exactly and quasi exactly solvable potentials for the one dimensional Schrödinger equation with position dependent mass.

▶ We have discussed the possibilities for designing quantum spectra of position dependent mass Hamiltonians offered by the first and second order intertwining technique. To generate spectral modifications by first order intertwining, we have used solutions to the position dependent mass Schrödinger equation corresponding to factorization energy (not belonging to the physical spectrum of the initial problem) less than or equal the ground state energy in order to avoid singularity in the isospectral partner potential. Thus it is possible to generate isospectral partner potentials (i) with the ground state of the given potential deleted (ii) with a new state created below the ground state of the given potential (iii) with the spectrum of the given potential unaffected. In the case of second order intertwining, instead of using the iterative method used in [227], the second order intertwiner is constructed directly by taking it as second order linear differential operator with unknown coefficients. The main

advantage of this construction is that one can generate isospectral partner potentials directly from the initial potential without generating first-order partner potentials. The apparently intricate system of equations arising from the second-order intertwining relationship is solved for the coefficients by taking an ansatz. In this case the spectral modifications are done by taking appropriately chosen factorization energies which may be real or complex. For real unequal factorization energies, it is possible to generate potentials (i) with deletion of first two energy levels (ii) with two new levels embedded below the ground state of the original potential (iii) with identical spectrum as of the original potential. For complex factorization energies, it is shown how to obtain strictly isospectral potentials.

Chapter 3

Quantum Systems with Non-Hermitian Hamiltonian[1]

In this chapter, we shall study some non-Hermitian Hamiltonians: (A) Hamiltonian corresponding to a complex periodic potential (3.3); (B) Hamiltonians which are associated with exceptonal X_1 orthogonal polynomials and (C) the non-Hermitian oscillator known as Swanson Hamiltonian (3.37).

3.1 Non Hermitian Hamiltonian with $V(x) = 4\cos^2 x + 4iV_0 \sin 2x$

Quite recently, the prospect of realizing complex $\mathcal{PT}$-symmetric periodic potentials within the framework of optics has been suggested [357, 358, 361, 359, 363, 556, 362]. The $\mathcal{PT}$-symmetric optical beam evolution is governed by the nonlinear Schrödinger (NLS)-like equation [368, 359]

$$i\frac{\partial \phi}{\partial z} + \frac{\partial^2 \phi}{\partial x^2} + V(x)\phi + g|\phi|^2\phi = 0 \tag{3.1}$$

where $\phi(x, z)$ is proportional to the electric field envelope, z is a scaled propagation distance and $g = \pm 1$ correspond to self-focusing or defocusing nonlinearity. The optical potential $V(x) = V_R(x) + iV_I(x)$ plays the role of a complex refractive index, where $V_R(-x) = V_R(x)$ is the index guiding and $V_I(-x) = -V_I(x)$ is the gain/loss distribution. The nonlinear stationary solution of equation (3.1) are sought in the form $\phi(x, z) = e^{i\beta z}\psi(x)$ where $\psi(x)$ satisfies

$$\frac{d^2\psi}{dx^2} + V(x)\psi(x) + g|\psi|^2\psi = \beta\psi. \tag{3.2}$$

As mentioned in chapter 1, many interesting results have already been obtained by considering the above equations with some $\mathcal{PT}$-symmetric optical potentials. Here we consider

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[1]This chapter is based on the following three papers:
(i) B. Midya, B Roy and R. Roychoudhury, "A note on $\mathcal{PT}$ invariant periodic potential $V(x) = 4\cos^2 x + 4iV_0 \sin 2x$", Phys. Lett. A 374 (2010) 2605. (ii) B. Midya, "Quasi-Hermitian Hamiltonians associated with exceptional orthogonal polynomials", Phys. Lett. A (2012) in press; arXiv:1205.5860. (iii) B Midya, P Dube and R Roychoudhury," Non-isospectrality of the generalized Swanson Hamiltonian and harmonic oscillator", J. Phys. A 44 (2011) 062001.

the $\mathcal{PT}$ invariant periodic potential

$$V(x) = 4\cos^2 x + 4iV_0 \sin 2x, \quad V_0 \in \mathbb{R} \tag{3.3}$$

where $V(x + \pi) = V(x)$. The reason for considering this particular potential (3.3) lies in the fact that many physically interesting results have been obtained while considering this potential in optical lattice. Specifically, it has been identified that the $\mathcal{PT}$ threshold occurs at $V_0 = 0.5$ [361, 359]. Below this threshold all the eigenvalues for every band structures are real and all the forbidden gaps are open whereas at the threshold, the spectrum is whole real line and there are no band gap. On the other hand, when $V_0 > 0.5$ the first two bands (starting from the lowest band) start to merge together and in doing so they form oval-like structure with a related complex spectrum. The soliton solution of the nonlinear Schrödinger equation corresponding to this potential has also been studied [368]. But in all these studies some numerical techniques have been used. Therefore it is interesting to reconsider this potential so that the notion of pseudo-Hermiticity [263] can be used to obtain some interesting (except the existing one) results analytically. Here, we shall argue that one can anticipate by analytical argument that when $V_0 < .5$ the $\mathcal{PT}$-symmetry of the potential is unbroken. We shall first derive the equivalent Hermitian Hamiltonian of the corresponding non-Hermitian Hamiltonian with the potential (3.3) for $V_0 < 0.5$. It will be shown that there exist another critical point $V_0^c \sim 0.888437$ after which no part of the band structure remains real. The corresponding band structure will be obtained using the Floquet analysis.

3.1.1 Equivalent Hermitian analogue of $V(x) = 4\cos^2 x + 4iV_0 \sin 2x$

We consider the following linear eigenvalue problem that correspond to the linearized version of equation (3.2)

$$H\psi(x) = \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = \beta\psi(x) \tag{3.4}$$

where the potential $V(x)$ is given in equation (3.3). In order to find the equivalent Hermitian analogue of the above Hamiltonian, we re-write potential $V(x)$ given in (3.3)

as

$$V(x) = \begin{cases} 2 + 2\sqrt{1 - 4V_0^2} \ \cos\left(2x - i \ \tanh^{-1} 2V_0\right), & V_0 < .5 \\ 2 + 2e^{2ix}, & V_0 = .5 \\ 2 + 2i\sqrt{4V_0^2 - 1} \ \sin\left(2x - i \ \tanh^{-1} \frac{1}{2V_0}\right), & V_0 > .5 \end{cases} \tag{3.5}$$

Due to the fact that for a $\theta \in \mathbb{R}$ the Hermitian linear automorphism $\eta = e^{-\theta p}$ affects an imaginary shift of the co-ordinate [273]: $e^{-\theta p}xe^{\theta p} = x + i\theta$, we have for $V_0 < 0.5$ the Hamiltonian defined in (3.4) reduced to a equivalent Hermitian Hamiltonian h relations

$$h = \rho H \rho^{-1} \tag{3.6}$$

where

$$\rho = e^{-\frac{\theta}{2}p}, \qquad \theta = \tanh^{-1} 2V_0, \ p = -i\frac{d}{dx} \tag{3.7}$$

is a unitary operator. The Hermitian Hamiltonian h is given by

$$h \equiv \frac{d^2}{dx^2} + 2 + 2\sqrt{1 - 4V_0^2} \ \cos \ 2x. \tag{3.8}$$

Defining $\eta = \rho^2 = e^{-\theta p}$, a positive definite operator, we have $H^\dagger = \eta H \eta^{-1}$. Which implies that the Hamiltonian considered here is pseudo-Hermitian with respect to a positive definite metric operator η, so that the spectrum of H is entirely real for $V_0 < 0.5$. For $V_0 = .5$, it was shown that [508] the spectrum of this potential $V(x) = 2 + 2e^{2ix}$ is purely continuous and fills the semi axis $[0, \infty)$. While for $V_0 > .5$, we have $\eta V(x)\eta^{-1} = V(x + i\theta) \neq V^*(x)$, where $\eta = e^{-\theta p}, \theta = \tanh^{-1} \frac{1}{2V_0}$ which indicates that in this case the Hamiltonian H is not η pseudo-Hermitian so the reality of the eigenvalues is not ensured.

Hence, we propose that $V_0 = V_0^{th} = 0.5$ is the transition point after which the eigenvalues of equation (3.4) become complex and spontaneous breaking of $\mathcal{PT}$ symmetry occurs. In the next section, we show that this proposition is indeed true.

3.1.2   Band structure of the potential (3.3) and existence of second critical point

After making the change of variable $z = x - \frac{i}{2}\tanh^{-1}\frac{1}{2V_0}$ in case of $V_0 > .5$, the equation (3.4) transforms into

$$\frac{d^2\psi(z;\beta,V_0)}{dz^2} + \left(2 - \beta + 2i\sqrt{4V_0^2 - 1}\,\sin 2z\right)\psi(z;\beta,V_0) = 0. \tag{3.9}$$

Under simple coordinate translation $z \to \frac{\pi}{4} - z$ the equation (3.9) reduces to Mathieu equation $\frac{d^2\psi}{dz^2} + (a - 2iq\cos 2z)\psi = 0$ [507], with characteristic value $a = 2 - \beta$ and $q = \sqrt{4V_0^2 - 1}$. Tabular values from an old paper of Mulholland and Goldstein [509] and Bouwkamp [510] reflect that the characteristic values q would be both real and complex conjugate: the transition occurring in the neighborhood of $q = q_0 = 1.46876852$. The same critical(exceptional) point was obtained in ref.[295], while considering the large N limit of the quasi exactly solvable $\mathcal{PT}$-symmetric potential $V(x) = -(i\xi\sin 2x + N)^2$. Hence it follows that there is a second critical point in the neighborhood of $V_0 = V_0^c \sim .888437$ after which all the eigenvalues of the equation (3.9) becomes complex and occur in complex conjugate pairs. Consequently no part of the band structure remain purely real.

Now to determine the eigenvalues of the equation (3.9) and correspondingly the band structure of the potential given in (3.3), we proceed as follows: as the coefficient of the eq.(3.9) are π periodic, so according to Floquet-Bloch theorem [512] the eigenfunctions are of the form $\psi(z;\beta,V_0) = \phi_\nu(z;\beta,V_0)e^{i\nu z}$, where $\phi_\nu(z+\pi) = \phi_\nu(z)$ and ν stands for the real Bloch momentum. Since $\phi_\nu(z)$ is periodic so the entire solution of equation (3.9) can be expanded in the Fourier series

$$\psi(z,\beta) = \sum_{k\in\mathbb{Z}} c_{2k}(\nu;\beta,V_0)e^{i(\nu+2k)z} \tag{3.10}$$

which, inserted into eq.(3.9), gives the recurrence relation

$$c_{2k} + \zeta_{2k}[c_{2(k-1)} - c_{2(k+1)}] = 0, \quad \zeta_{2k} = \frac{-\sqrt{4V_0^2 - 1}}{[(\nu + 2k)^2 + \beta - 2]}, \quad \forall k \in \mathbb{Z} \tag{3.11}$$

for the coefficients $c_{2k}(\beta,\nu)$. For any finite(truncated) upper limit $n \in \mathbb{Z}$, eq. (3.11) can

(a)   (b)
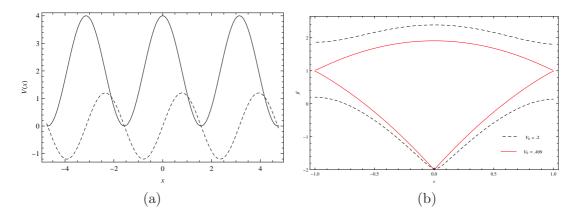
Figure 3.1: (a) Real (solid line) and imaginary (dashed) part of the potential $V(x) = 4\cos^2 x + 4iV_0 \sin 2x$ for $V_0 = 0.3$. (b) The band structure corresponding to $V_0 = 0.2$ (dashed line) and $V_0 = .499$ (solid red line).

be written as a matrix equation

$$\mathcal{M}_n(\nu; \beta, V_0)\mathcal{C}_n^t = 0 \tag{3.12}$$

where $\mathcal{C}_n = (c_{-2n}, ..., c_{-2}, c_0, c_2, ..., c_{2n})$ and $\mathcal{M}_n(\nu; \beta, V_0)$ is a $(2n+1) \times (2n+1)$ matrix given by

$$\mathcal{M}_n(\nu; \beta, V_0) = \begin{pmatrix} 1 & -\zeta_{2n} & & & & & & \\ \zeta_{2n-2} & . & . & . & . & . & . & \\ & . & \zeta_2 & 1 & -\zeta_2 & 0 & 0 & . \\ & . & 0 & \zeta_0 & 1 & -\zeta_0 & 0 & . \\ & . & 0 & 0 & \zeta_{-2} & 1 & -\zeta_{-2} & . \\ & . & . & . & . & . & . & -\zeta_{-2n+2} \\ & & & & & \zeta_{-2n} & 1 \end{pmatrix}. \tag{3.13}$$

For the finite dimensional case, finding the non-trivial solutions of the homogeneous system of eqn.(3.12) is equivalent to solve

$$det(\mathcal{M}_n) = 0. \tag{3.14}$$

The solution of the above eqn.(3.14) gives the relation between the energy eigenvalues β and bloch momentum ν. Figures 3.1b, 3.2 and 3.3 have been plotted taking $n = 3$. Figure 3.1a shows the real and imaginary part of the complex periodic potential given in
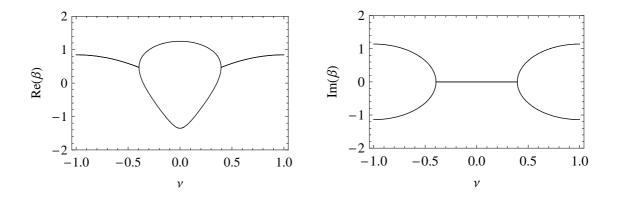
Figure 3.2: Band structure for the $\mathcal{PT}$ invariant potential $V(x) = 4\cos^2 x + 4iV_0 \sin 2x$ for $V_0 = .75$

(3.3). In figure 3.1b, the band structure of the potential $V(x)$ for $V_0 = .2$ and $.499$ have been shown. In figures 3.2 and 3.3, we have plotted the band structures for $V_0 = 0.75$ and near the value of second critical point $V_0^c \sim .888437$ respectively. It is found that below the value $V_0 = 0.5$ all bands are real and all the forbidden band gaps are open, whereas after this point the bands become complex and start to merge together and form oval like structure. After the second critical point this oval like structure vanishes consequently no part of the band structure remains real. Hence, we conclude that $V_0 = V_0^{th} = 0.5$ is the $\mathcal{PT}$ threshold and $V_0 = V_0^c \sim .888437$ is the second critical point.
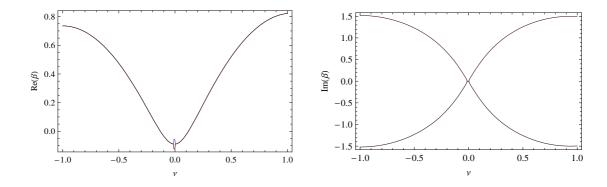
Figure 3.3: Band structure for the $\mathcal{PT}$-invariant potential $V(x) = 4\cos^2 x + 4iV_0 \sin 2x$ for $V_0 = .8885$

3.2 Quasi-Hermitian Hamiltonians associated with Laguerre or Jacobi type X_1 exceptional orthogonal polynomials

Here we use the method of point canonical transformation (PCT) to derive some exactly solvable non-Hermitian Hamiltonians whose bound state wave functions are associated with Laguerre or Jacobi type X_1 exceptional orthogonal polynomials. For this we first briefly recall the method of point canonical transformation.

In PCT approach [423, 83], the general solution of the Schrödinger equation (with $\hbar = 2m = 1$)

$$H\psi(x) = -\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \tag{3.15}$$

can be assumed as

$$\psi(x) \sim f(x)F(g(x)) \tag{3.16}$$

where $F(g)$ satisfies the second order linear differential equation of a special function

$$\frac{d^2F}{dg^2} + Q(g)\frac{dF}{dg} + R(g)F(g) = 0. \tag{3.17}$$

Substituting the assumed solution $\psi(x)$ in equation (3.15) and comparing the resulting equation with the equation (3.17) one obtains the following two equations for $Q(g(x))$ and $R(g(x))$

$$Q(g) = \frac{g''}{g'^2} + \frac{2f'}{fg'} \tag{3.18a}$$

$$R(g) = \frac{E - V(x)}{g'^2} + \frac{f''}{fg'^2} \tag{3.18b}$$

respectively. After some algebraic manipulations above two equations reduces to

$$f(x) \approx g'(x)^{-1/2}\, e^{\frac{1}{2}\int Q(g)dg}, \tag{3.19a}$$

$$E - V(x) = \frac{g'''}{2g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + g'^2\left(R - \frac{1}{2}\frac{dQ}{dg} - \frac{1}{4}Q^2\right). \tag{3.19b}$$

Now we are in a position to choose the special function $F(g)$ (consequently $Q(g)$ and $R(g)$). The equation (3.19b) becomes meaningful for a proper choice of $g(x)$ ensuring the presence of a constant term in the right-hand side which connects the energy in the left-hand side. The remaining part of equation (3.19b) gives the potential. Corresponding bound state wave functions involving the special function $F(g)$ are obtained with the help of equations (3.16) and (3.19a), as

$$\psi(x) \sim g'(x)^{-1/2} \; e^{\frac{1}{2} \int Q(g)dg} \; F(g(x)).$$ (3.20)

Here, we choose the special function to be the exceptional X_1 Laguerre polynomial viz, $F(g) \propto \hat{L}_n^{(a)}(x)$. For real $a > 0$ and $n = 1, 2, 3...$, these polynomials $\hat{L}_n^{(a)}(x)$ satisfy the differential equation [185, 186]

$$x\frac{d^2y}{dx^2} - \frac{(x - a)(x + a + 1)}{x + a}\frac{dy}{dx} + \left(\frac{x - a}{x + a} + n - 1\right)y = 0.$$ (3.21)

The polynomial $\hat{L}_n^{(a)}(x)$ has one zero in $(-\infty, -a)$, remaining $n - 1$ zeros lie in $(0, \infty)$. Moreover, these polynomials are orthonormal [185] with respect to the rational weight $\widehat{W} = \frac{e^{-x}x^a}{(x+a)^2}$

$$\int_0^\infty \frac{e^{-x}x^a}{(x + a)^2}\hat{L}_n^{(a)}(x)\hat{L}_m^{(a)}(x)dx = \frac{(a + n)\Gamma(a + n - 1)}{(n - 1)!}\delta_{nm}.$$ (3.22)

The expressions for $Q(g)$ and $R(g)$, corresponding to the choice $F(g) = \hat{L}_n^{(a)}(x)$, are given by

$$Q(g) = -\frac{(g - a)(g + a + 1)}{g(g + a)}, \quad R(g) = \frac{g - a}{g(g + a)} + \frac{n - 1}{g}.$$ (3.23)

Using them in equation (3.19b), we have the expression for $E - V(x)$ as

$$E - V(x) = \frac{g'''}{2g'} - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + \frac{(2na + a^2 - a + 2)g'^2}{2ag} - \frac{g'^2}{a(g + a)} - \frac{(a^2 - 1)g'^2}{4g^2} - \frac{2g'^2}{(g + a)^2} - \frac{g'^2}{4}$$ (3.24)

At this point we choose $g'^2/g = \text{constant} = k^2, k \in \mathbb{R} - \{0\}$, which is satisfied by

$$g(x) = \frac{1}{4}(kx + d)^2,$$ (3.25)

where d is an arbitrary constant of integration. Here two cases may arise, namely, $d = 0$ and $d \neq 0$. Without loss of generality we can choose, for the moment, $d = 0$. For this choice, substituting $g(x)$ in equation (3.24) and separating out the potential and the energy, we have

$$V(x) = \tfrac{k^4}{16}x^2 + \left(a^2 - \tfrac{1}{4}\right)\tfrac{1}{x^2} + \tfrac{4k^2}{k^2x^2+4a} - \tfrac{32ak^2}{(k^2x^2+4a)^2},$$

$$(3.26)$$

$$E_n = \tfrac{k^2(2n+a-1)}{2}.$$

The potential $V(x)$ is singularity free in the interval $0 < x < \infty$. The same potential has earlier been reported in ref.[175]. It has been shown that the potential $V(x)$ is the extension of the standard radial oscillator by addition of last two rational terms. Such terms do not change the behavior of the potential for large values of x, while small values of x produce some drastic effect on the minima of the potential. The normalized wave functions corresponding to the potential can be determined, in terms of Laguerre X_1 EOPs, using equations (3.20), (3.22) and (3.23), as

$$\psi_n(x) = \left(\frac{(n-1)!\ k^{2a+2}}{2^{2a-3}(a+n)\Gamma(a+n-1)}\right)^{\frac{1}{2}} \frac{x^{a+\frac{1}{2}}}{k^2x^2+4a}\ e^{-\frac{k^2x^2}{8}}\ \hat{L}_n^{(a)}\left(\frac{k^2x^2}{4}\right),\quad n=1,2,3...$$

$$(3.27)$$

It is worth mentioning here that the choice $d = 0$ in equation (3.25) always gives rise to Hermitian potential. Nonzero real values of d do not make any significant difference in the potential and its solutions. The non-Hermiticity can be invoked into the potential only if d is purely imaginary. We set $d = i\epsilon$, $\epsilon \in \mathbb{R} - \{0\}$, and $g(x) = \tfrac{1}{4}(kx + i\epsilon)^2$ for which the potential reduces to

$$\widetilde{V}(x) = \frac{k^2(kx+i\epsilon)^2}{16} + \frac{k^2(a^2-\tfrac{1}{4})}{(kx+i\epsilon)^2} + \frac{4k^2}{(kx+i\epsilon)^2+4a} - \frac{32ak^2}{[(kx+i\epsilon)^2+4a]^2} \qquad (3.28)$$

The non-Hermitian potential (3.28) is free from singularity through out the whole real x axis. Since the energy E_n has no dependence on d, the non-Hermitian Potential $\widetilde{V}(x)$ also shares the same real energy spectrum of $V(x)$. This requires further explanation. In the following, we show that the potential $\widetilde{V}(x)$ is actually quasi-Hermitian. For this we

(a)   (b)
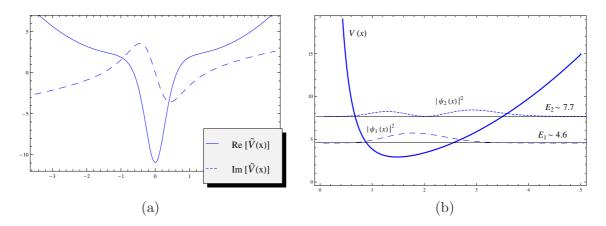
Figure 3.4: (a) Plot of the real (solid line) and imaginary (dashed line) parts of the quasi-Hermitian Potential $\widetilde{V}(x)$ associated with X_1 Laguerre EOPs.   (b). Plot of the corresponding equivalent Hermitian potential $V(x)$ (thick line) and square of the absolute value of its lowest two wave functions. Here we have considered $\epsilon = 1.2, a = 2, k = 1.75$.

define the operator

$$\rho = e^{\frac{\epsilon}{k}p}, \quad p = -i\frac{d}{dx} \tag{3.29}$$

which has the following properties

$$\rho x \rho^{-1} = x - \frac{i\epsilon}{k}, \quad \rho p \rho^{-1} = p, \quad \rho f(x)\rho^{-1} = f\left(x - \frac{i\epsilon}{k}\right). \tag{3.30}$$

In other words, the operator ρ has an effect of shifting the coordinate x to $x - \frac{i\epsilon}{k}$. For the proof of the results (3.30), follow the reference [273]. For this operator we have the following similarity transformation

$$\rho\widetilde{V}(x)\rho^{-1} = \widetilde{V}\left(x - \frac{i\epsilon}{k}\right) = V(x). \tag{3.31}$$

This ensures that the non-Hermitian Hamiltonian corresponding to the potential $\widetilde{V}(x)$ is quasi-Hermitian. The equivalent Hermitian potential $V(x)$, which corresponds to $d = 0$, is given in equation (3.26). It is very easy to show that the positive definite operator $\eta = \rho^2$ satisfies $\eta\widetilde{V}(x)\eta^{-1} = \widetilde{V}^\dagger(x)$ ensuring the potential to be pseudo-Hermitian. The potential $\widetilde{V}(x)$ also satisfies $\widetilde{V}^*(-x) = \widetilde{V}(x)$ and hence is $\mathcal{PT}$-symmetric. The wave functions of the potential $\widetilde{V}(x)$ can be determined by $\widetilde{\psi}_n(x) = \rho^{-1}\psi_n(x) = \psi_n\left(x + \frac{i\epsilon}{k}\right)$.

In figure 3.4a, we have shown the real and imaginary parts of the potential $\widetilde{V}(x)$ given in (3.28), while figure 3.4b shows its equivalent Hermitian analogue $V(x)$ given

in (3.26). Using first two members of exceptional X_1 Laguerre polynomials $\hat{L}_1^{(a)}(x) = -x - a - 1$, $\hat{L}_2^{(a)}(x) = x^2 - a(a+2)$, we have also plotted in figure 3.4b the absolute value of first two wave functions given in (3.27).

Next we choose $F(g)$ to be Jacobi-type X_1 EOP, $\hat{P}_n^{(a,b)}$, which is defined for real $a, b > -1$, $a \neq b$ and $n = 1, 2, 3...$ In this case the expression for $Q(g)$ and $R(g)$ are given by [185]

$$
\begin{aligned}
Q(g) &= -\frac{(a+b+2)g + a - b}{1 - g^2} - \frac{2(b-a)}{(b-a)g - b - a}, \\
R(g) &= -\frac{(b-a)g - (n+a+b)(n-1)}{1 - g^2} - \frac{(a-b)^2}{(b-a)g - b - a}
\end{aligned}
\tag{3.32}
$$

Using these expressions in (3.19b) and choosing $g'^2/(1 - g^2) = \text{constant} = k^2 (k \neq 0)$, we have

$$
g(x) = \sin(kx + d).
\tag{3.33}
$$

Like the exceptional Laguerre polynomials, the choice $d = 0$ gives rise to the potential, energies and corresponding bound state wave functions, as

$$
V(x) = \frac{k^2(2a^2 + 2b^2 - 1)}{4} \sec^2 kx - \frac{k^2(b^2 - a^2)}{2} \sec kx \tan kx + \frac{2k^2(a+b)}{a+b-(b-a)\sin kx} - \frac{8k^2 ab}{[a+b-(b-a)\sin kx]^2},
$$

$$
E_n = \frac{k^2}{4}(2n + a + b - 1)^2,
\tag{3.34}
$$

$$
\psi_n(x) \approx \frac{(1 - \sin kx)^{\frac{a}{2} + \frac{1}{4}}(1 + \sin kx)^{\frac{b}{2} + \frac{1}{4}}}{a + b - (b-a)\sin kx} \hat{P}_n^{(a,b)}(\sin kx), \qquad n = 1, 2, 3...
\tag{3.35}
$$

respectively. The above periodic potential $V(x)$, which is free from singularity in the interval $-\frac{\pi}{2k} < x < \frac{\pi}{2k}$, can be interpreted [175] as the rational extension of the standard trigonometric scarf potential which is associated with classical Jacobi polynomials. The wave functions in equation (3.35) are regular iff $a, b > -1/2$.
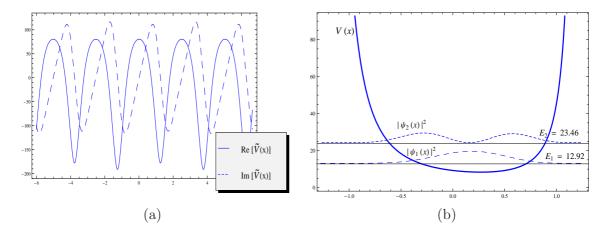
Figure 3.5: (a) Plot of the real (solid line) and imaginary (dashed line) parts of the quasi-Hermitian Potential $\widetilde{V}(x)$ associated with X_1 Jacobi EOPs.　(b). Plot of the corresponding equivalent Hermitian potential $V(x)$ (thick line) and square of the absolute value of its lowest two wave functions. Here we have considered $a = 1.75, b = 3, k = 1.25, \epsilon = 1$.

Here, the non-Hermitian potential corresponding to the choice $d = i\epsilon$ is obtained as

$$
\widetilde{V}(x) = \frac{k^2(2a^2 + 2b^2 - 1)}{4} \sec^2(kx + i\epsilon) - \frac{k^2(b^2 - a^2)}{2} \sec(kx + i\epsilon) \tan(kx + i\epsilon)
$$

$$
+ \frac{2k^2(a + b)}{a + b - (b - a)\sin(kx + i\epsilon)} + \frac{2k^2[(a - b)^2 - 4ab]}{[a + b - (b - a)\sin(kx + i\epsilon)]^2}
\tag{3.36}
$$

This potential $\widetilde{V}(x)$, which is defined on whole real line, also shares the same real eigenvalues of the potential given in (3.34). Like the rationally extended radial oscillator the above non-Hermitian potential is also quasi-Hermitian under the the operator ρ defined in (3.29). The corresponding equivalent analogue is the one given in equation (3.34) which corresponds to the choice $d = 0$. The potential $\widetilde{V}(x)$ also fulfills the requirement of $\mathcal{PT}$-symmetry i.e. $\widetilde{V}^*(-x) = \widetilde{V}(x)$, only if $a = b$. However, if we consider the other solution $g(x) = \cos(kx + d)$ of $g'^2/(1 - g^2) = k^2$, the corresponding potential $\widetilde{V}(x)$ becomes $\mathcal{PT}$-symmetric for all real values of a and b. The wave functions of $\widetilde{V}(x)$ can be determined by operating ρ^{-1} on ψ_n given in (3.35). Here, we have not considered the complex values of a, b because this will give rise to the exceptional Jacobi polynomials with complex indices and complex arguments. The orthogonality properties for such complex polynomials may depend on the interplay between integration contour and parameter values.

In figure 3.5a, we have plotted the real and imaginary parts of the potential $\widetilde{V}(x)$. The corresponding equivalent Hermitian analogue $V(x)$ and square of its first two wave

functions are plotted in figure 3.5b. We have used the expression of first two members of Jacobi type X_1 EOPs, $\widehat{P}_1^{(a,b)} = -\frac{x}{2} - \frac{2+a+b}{2(a-b)}$ and $\widehat{P}_2^{(a,b)} = -\frac{a+b+2}{4}x^2 - \frac{a^2+b^2+2(a+b)}{2(a-b)}x - \frac{a+b+2}{4}$ to plot the square of the wave functions.

3.3   Generalized Swanson Hamiltonian

3.3.1   Generalized Swanson Hamiltonian and its equivalent Hermitian analogue

The harmonic oscillator Hamiltonian augmented by a non-Hermitian $\mathcal{PT}$-symmetric part is one of the important example of pseudo-Hermitian Hamiltonian. This non-Hermitian oscillator was first discussed by Swanson [513] who considered the Hamiltonian

$$H_S = w \left(a^\dagger a + \frac{1}{2} \right) + \alpha a^2 + \beta a^{\dagger^2}, \quad w, \alpha, \beta \in \mathbf{R} \tag{3.37}$$

where $a, a^\dagger$ are bosonic harmonic oscillator annihilation and creation operators satisfying usual commutation relationship $[a, a^\dagger] = 1$. If $\alpha \neq \beta$, H_S is non-Hermitian $\mathcal{PT}$-symmetric ($\mathcal{P}$-pseudo-Hermitian). Swanson, with the help of a Bogoliubov type transformation, showed that the transformed Hamiltonian has the eigenvalue of a harmonic oscillator system with frequency $\sqrt{w^2 - 4\alpha\beta}$ so long as $w > \alpha + \beta$ and the eigenfunctions can be derived from the eigenfunctions of the harmonic oscillator. Transition probabilities governed by the H_S are also shown to be manifestly unitary [513]. Consequently, many properties and generalizations of this Hamiltonian have been reported [271, 514, 515, 516, 517, 518]. In particular, it has been shown [271] that the non-Hermitian oscillator (3.37) can be mapped to harmonic oscillator with the help of following similarity transformation,

$$h = \rho_{(\alpha,\beta)} \, H_S \, \rho_{(\alpha,\beta)}^{-1} = -\frac{1}{2}(w - \alpha - \beta)\frac{d^2}{dx^2} + \frac{1}{2}\frac{w^2 - 4\alpha\beta}{w - \alpha - \beta}x^2 \tag{3.38}$$

with $\rho_{(\alpha,\beta)} = \exp[-\frac{1}{2}\frac{\alpha-\beta}{w-\alpha-\beta}x^2]$. However, this choice for metric operator ρ is not unique. A family of metric operators for the Hamiltonian H_S have been constructed [515] and they are re-examined in the light of $su(1,1)$ [516]. In ref.[514], the authors have explained the hidden symmetry of the Hamiltonian H_S and found the pseudo-Hermitian operators η in the form $\eta = \rho_{(\alpha,\beta)}^{-1}\rho_{(\beta,\alpha)}$. The positivity of η is provided by the property satisfied by ρ viz., $\rho_{(\alpha,\beta)} = \rho_{(\beta,\alpha)}^{-1}$. It should be put emphasize here that if one writes $\eta = \rho_{(\alpha,\beta)}^{-1}\rho_{(\beta,\alpha)}$ then the boundedness of the operator η can not be guaranteed. If η is not bounded metric operator an alternative construction of ρ can be made which is not based on the metric operator [265]. The Hamiltonian given in (3.37) is generalized many ways, for example, a general first order differential form of the annihilation and creation operators $\tilde{a}$ and $\tilde{a}^\dagger$

are proposed [514]

$$\tilde{a} = A(x)\frac{d}{dx} + B(x) \quad \tilde{a}^\dagger = -A(x)\frac{d}{dx} + B(x) - A'(x) \qquad A(x), B(x) \in \mathbb{R}, \quad (3.39)$$

where 'prime' denotes derivative with respect to x and $[\tilde{a}, \tilde{a}^\dagger] = 2AB' - AA'' \ (\neq 1$ in general). After replacing $a, a^\dagger$ by $\tilde{a}$ and $\tilde{a}^\dagger$ in equation (3.37) we have the the generalized Swanson Hamiltonian [514]

$$H_{GS} = w\left(\tilde{a}^\dagger \tilde{a} + \frac{1}{2}\right) + \alpha\tilde{a}^2 + \beta\tilde{a}^{\dagger 2}, \qquad w, \alpha, \beta \in \mathbb{R}. \quad (3.40)$$

The Hamiltonian H_{GS} is $\mathcal{PT}$-symmetric if $A(x)$ is an odd function and $B(x)$ is an even function of x. For the general choice of $\tilde{a}, \tilde{a}^\dagger$ and $w - \alpha - \beta = 1$, the Hermitian equivalent form of the Hamiltonian H_{GS} is obtained by a similarity transformation [514]

$$\tilde{h} = \rho_{(\alpha,\beta)} H_{GS} \, \rho_{(\alpha,\beta)}^{-1} = -\frac{d}{dx}A(x)^2\frac{d}{dx} + V_{eff}(x) \quad (3.41)$$

where

$$\tilde{\rho}_{(\alpha,\beta)} = A(x)^{\frac{\alpha-\beta}{2}} \exp\left(-(\alpha-\beta)\int^x \frac{B(x)}{A(x)}dx'\right), \quad (3.42)$$

$$V_{eff}(x) = \frac{(\alpha+\beta)AA''}{2} + \left[\frac{\alpha+\beta}{2} + \frac{(\alpha-\beta)^2}{4}\right]A'^2 - 4\tilde{w}^2 A'B$$
$$+ 4\tilde{w}^2 B^2 - (\alpha+\beta+1)AB' + \frac{\alpha+\beta+1}{2} \quad (3.43)$$

$$\tilde{w} = \frac{\sqrt{1 + 2(\alpha+\beta) + (\alpha-\beta)^2}}{2}. \quad (3.44)$$

There is an one to one correspondence between the energy eigenvalues of $\tilde{h}$ given in (3.41) and H_{GS}. If $\psi_n(x)$ are the wave functions of the equivalent Hermitian Hamiltonian $\tilde{h}$ then the wave functions of the Hamiltonian H_{GS} are given by $\rho_{(\alpha,\beta)}^{-1}\psi_n(x)$.

Here our aim is to show that the equivalent Hermitian Hamiltonian (3.41) of the Hamiltonian (3.40) can be further transformed into a harmonic oscillator like Hamiltonian. It will also be shown that though the commutator $[\tilde{a}, \tilde{a}^\dagger] = 1$, the generalized Swanson Hamiltonian is not always isospectral to the harmonic oscillator. This anomaly will be explained in the frame work of position dependent mass scenario.

3.3.2 Reduction of equivalent Hermitian Hamiltonian of H_{GS} into harmonic oscillator

We consider the commutator of $\tilde{a}$ and $\tilde{a}^\dagger$ as

$$[\tilde{a}, \tilde{a}^\dagger] = 2A(x)B'(x) - A(x)A''(x) = 1. \tag{3.45}$$

On setting

$$z(x) = \int^x \frac{dx'}{A(x')} \qquad B(x) = \frac{z''}{2z'^2} + \frac{1}{2}z, \tag{3.46}$$

(for the sake of simplicity we assume integration constant to be zero), the Hamiltonian $\tilde{h}$ given in eq.(3.41) becomes

$$\tilde{h} = -\frac{d}{dx}A^2\frac{d}{dx} + V_{eff}(x), \qquad V_{eff}(x) = \frac{1}{2}\frac{z'''}{z'^3} - \frac{5}{4}\frac{z''^2}{z'^4} + \tilde{w}^2z^2. \tag{3.47}$$

For the change of variable (3.46) we have

$$A' = \frac{\dot{A}}{A}, \qquad A'' = \frac{\ddot{A}}{A^2} - \frac{\dot{A}^2}{A^3} \qquad \text{etc.}$$

where 'dot' represents derivative with respect to z. Consequently the Schrödinger eigenvalue equation for the Hamiltonian $\tilde{h}$ reduces to

$$\left[-\frac{d^2}{dz^2} - \frac{\dot{A}}{A}\frac{d}{dz} + \left(\frac{\dot{A}^2}{4A^2} - \frac{\ddot{A}}{2A} + \tilde{w}^2z^2 \right) \right] \psi(z) = E\psi(z) \tag{3.48}$$

In order to eliminate the first derivative term from equation (3.48) we assume

$$\psi(z) = A(z)^{-\frac{1}{2}}\phi(z). \tag{3.49}$$

For this assumption equation (3.48) reduces to

$$\left[-\frac{d^2}{dz^2} + \tilde{w}^2z^2 \right] \phi(z) = E\phi(z) \tag{3.50}$$

This equation (3.50) is the Schrödinger equation with harmonic oscillator like potential $V(z) = \tilde{w}^2z^2$. The important point is to note here that the co-ordinate transformation

given by the first of the equation (3.46) has certain peculiarities. The domain of $z(x)$ depends on the functional form of $A(x)$ and may not be the same as that of x. This may lead to non-isospectrality of the original Hamiltonian $\tilde{h}$ and the transformed Hamiltonian. Now the equation (3.50) can be transformed into confluent Hypergeometric equation

$$y\frac{d^2\chi}{dy^2} + \left(\frac{1}{2} - y\right)\frac{d\chi}{dy} + \left(\frac{E}{4\tilde{w}} - \frac{1}{4}\right)\chi = 0 \tag{3.51}$$

by the following transformations

$$y = \tilde{w}z^2 \qquad\qquad \phi(y) = e^{-\frac{y}{2}}\chi(y). \tag{3.52}$$

Hence the general solution of the equation (3.50) are given by

$$\phi_e(z) \sim e^{-\frac{\tilde{w}}{2}z^2}\,_1F_1\left(\frac{1}{4} - \frac{E}{4\tilde{w}}, \frac{1}{2}, \tilde{w}z^2\right) \tag{3.53}$$

$$\phi_o(z) \sim ze^{-\frac{\tilde{w}}{2}z^2}\,_1F_1\left(\frac{3}{4} - \frac{E}{4\tilde{w}}, \frac{3}{2}, \tilde{w}z^2\right), \tag{3.54}$$

where ϕ_e and ϕ_o denote the even and odd solutions. The eigenfunctions of the Hamiltonian $\tilde{h}$ are given by

$$\psi(x) \sim A(z)^{-\frac{1}{2}}\phi(z), \tag{3.55}$$

where $z(x)$ is given by equation (3.46).

It should be mentioned here that though the equivalent Hermitian analogue of H_{GS} is transformed into harmonic oscillator like Hamiltonian, they may not share the same eigenvalues. This apparent anomaly occurs because the domain of x and the domain of the variable z used in co-ordinate transformation to obtain the harmonic oscillator are not necessarily same. In the next section, we determine the condition of isospectrality between H_{GS} and harmonic oscillator.

3.3.3   Non-isospectrality between generalized Swansan Hamiltonian and harmonic oscillator

The condition of isospectrality between the the equivalent Hermitian Hamiltonian $\tilde{h}$ and harmonic oscillator can be obtained by analyzing the behaviors of the eigenfunctions

(3.53) and (3.54) which determine the eigenvalues of the equation (3.50). If the domain of argument $\tilde{w}z^2$ is unbounded then $\phi_e(z)$ and $\phi_o(z)$ will not in general, square integrable because of the asymptotic behavior of the confluent Hypergeometric function viz. [507],

$$_1F_1(a, b, y) = \frac{\Gamma(b)}{\Gamma(a)} e^y y^{a-b} [1 + O(y^{-1})], \quad Re(y) > 0. \tag{3.56}$$

So in order to make the eigenfunctions square integrable one must take $a = -m$ ($m = 0, 1, 2, 3...$) in which case $_1F_1(a, b, y)$ reduces to a polynomial. So from (3.53) $1/4 - E/(4\tilde{w}) = -m$ or equivalently

$$E_{2m} = 2\tilde{w}\left(2m + \frac{1}{2}\right) \tag{3.57}$$

and from (3.54) $3/4 - E/(4\tilde{w}) = -m$ or equivalently

$$E_{2m+1} = 2\tilde{w}\left(2m + \frac{3}{2}\right) \tag{3.58}$$

Combining (3.57) and (3.58) we have

$$E_n = 2\tilde{w}\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2... \tag{3.59}$$

Hence in this case the Hamiltonian $\tilde{h}$ (as well as H_{GS}) has the spectrum of a harmonic oscillator.

However, if the domain of $\tilde{w}z^2$ is finite then the required boundary condition to be satisfied by the eigenfunctions (3.53) and (3.54) is that they must vanish at the end points of the domain of z and the eigenvalues are given by the zeroes of the confluent hypergeometric functions when the arguments attain their end points. The first approximation of the m'th ($m = 1, 2...$) positive zero X_0 of $_1F_1(a, b, y)$ is given by [507]

$$X_0 = \frac{\pi^2\left(m + \frac{b}{2} - \frac{3}{4}\right)^2}{2b - 4a}\left[1 + O\left(\frac{1}{(\frac{b}{2} - a)^2}\right)\right], \quad m = 1, 2... \tag{3.60}$$

Proceeding in a similar way, as discussed just above, we have from eqns.(3.53) and (3.54)

$$E_{2m} \approx \frac{\tilde{w}\pi^2}{4z_\pm^2}(2m - 1)^2, \quad E_{2m-1} \approx \frac{\tilde{w}\pi^2}{z_\pm^2}m^2 \tag{3.61}$$

respectively. Combining these two one can write

$$E_n \approx \frac{\tilde{w}\pi^2}{4z_\pm^2}(n-1)^2, \quad n = 1, 2, 3...$$ (3.62)

where $z_\pm$ are the end points of the domain of definition of $V(z)$. From the above fact it follows that, in this case the Hamiltonian $\tilde{h}$ (as well as H_{GS}) is not isospectral to the harmonic oscillator.

3.3.4   Examples

Here we give some concrete examples in favor of our obtained results.

The generalization of the Swanson model enables us to connect those physical systems which are describable by a position dependent mass by choosing $A(x) = M(x)^{-1/2}$ which is a strictly positive function. For this choice of $A(x)$ the equation (3.41) reduces to the time independent PDMSE

$$\left(-\frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx} + V_{eff}(x)\right)\psi(x) = E\psi(x)$$ (3.63)

with $V_{eff}(x)$ is given by equation (3.47). At this point we consider different mass profiles $M(x)$ to illustrate that the Hamiltonian given in (3.41) is not always isospectral to harmonic oscillator.

Example 1: Isospectral case

Let us consider the following mass function

$$M(x) = \frac{1}{1+x^2} , \quad x \in (-\infty, \infty)$$ (3.64)

which has been considered in the study of quantum nonlinear oscillator [411]. For this choice of mass function, $z(x)$ is given by

$$z(x) = \sinh^{-1}(x).$$ (3.65)

It is clear that $z(x) \to \pm\infty$ as $x \to \pm\infty$. So according to the condition obtained in section

, we have the Hamiltonian $\tilde{h}$ with the effective potential $V_{eff}(x)$

$$V_{eff}(x) = -\frac{2+x^2}{4(1+x^2)} + \tilde{w}^2(\sinh^{-1} x)^2 \tag{3.66}$$

is isospectral to harmonic oscillator. In table 3.1, we have given a list of physically interesting mass functions, $V_{eff}(x)$, and eigen energies, for which the corresponding position dependent mass Hamiltonians are isospectral to the harmonic oscillator.

$M(x)$	$z(x)$	$V_{eff}(x)$
$\frac{1}{(1+x^2)}$	$\sinh^{-1} x$	$-\frac{2+x^2}{4(1+x^2)} + \tilde{w}^2(\sinh^{-1} x)^2$
$\cosh^2 x$	$\sinh x$	$\frac{1}{8}(7 - 3\cosh 2x)sech^4 x + \tilde{w}^2 \sinh^2 x$
$\left(\frac{\gamma+x^2}{1+x^2}\right)^2$	$x + (\gamma-1)\tan^{-1} x$	$\frac{(\gamma-1)(3x^4-2(a-2)x^2-a)}{(x^2+\gamma)^4} + \tilde{w}^2[x + (\gamma-1)\tan^{-1} x]^2$
$e^{2x} sech^2 x$	$\log(1+e^{2x})$	$-\frac{3}{4}e^{-4x} - \frac{1}{2}e^{-2x} + \tilde{w}^2[\log(1+e^{2x})]^2$
e^{-x}	$-2e^{-\frac{x}{2}}$	$-\frac{3}{16}e^x + 4\tilde{w}^2 e^{-x}$

Table 3.1: Some physically interesting mass functions and corresponding effective potentials for which the corresponding Hamiltonians $\tilde{h}$ are isospectral to harmonic oscillator with $E_n = 2\tilde{w}\left(n + \frac{1}{2}\right), n = 0, 1, 2....$ In all these cases $z(x)$ are unbounded as $x \to \pm\infty$.

Example 2: Non-isospectral case

Now let us choose

$$M(x) = \text{sech}^2(x), \quad x \in (-\infty, \infty) \tag{3.67}$$

which depicts the solitonic profile [519]. For this choice, the function $z(x)$ reeds

$$z(x) = \tan^{-1}(\sinh x). \tag{3.68}$$

Here as $x \to \pm\infty, z(x) \to \pm\pi/2$, so the eigenvalues of the Hamiltonian $\tilde{h}$ with

$$V_{eff}(x) = \frac{1}{4} - \frac{3}{4}\cosh^2 x + \tilde{w}^2\left(\tan^{-1}(\sinh x)\right)^2, \tag{3.69}$$

will be given by zeroes of the functions given in eqns.(3.53) and (3.54) at $z = \pm\pi/2$. First approximate value of the energy eigenvalues are given by equation (3.62)

$$E_n \approx n^2, \ n = 1, 2, 3... \tag{3.70}$$

In table 3.2, we have given a list of physically interesting mass functions, $V_{eff}(x)$, and eigen energies, for which the corresponding Hamiltonians $\tilde{h}$ are not isospectral to the harmonic oscillator.

$M(x)$	$z(x)$	$V_{eff}(x)$	$E_n \approx$
$\text{sech}^2 x$	$\tan^{-1}(\sinh\ x)$	$\frac{1}{4} - \frac{3}{4}\cosh^2 x + \tilde{w}^2(\tan^{-1}(\sinh\ x))^2$	n^2
e^{-2x^2}	$\frac{\sqrt{\pi}}{2}\text{Erf}\ x$	$-(1+3x^2)e^{2x^2} + \frac{\pi\tilde{w}^2}{4}(\text{Erf}\ x)^2$	πn^2
$\frac{1}{(1+x^2)^2}$	$\tan^{-1}\ x$	$-(1+2x^2) + \tilde{w}^2(\tan^{-1}\ x)^2$	n^2

Table 3.2: Some physically interesting mass functions and corresponding effective potentials for which the corresponding Hamiltonians $\tilde{h}$ are not isospectral to harmonic oscillator. In all these cases $z(x)$ are finite as $x \to \pm\infty$. Here n takes the values $1, 2, 3...$

3.4   Summary

To summarize, we point out the following main results :

▶ We have shown that for a positive definite operator η the potential (3.3) is η pseudo-Hermitian for $V_0 < .5$ which ensures that all the band edges are real. It has also been that the eigenvalue problem (3.4) reduces to solving the Mathieu equation for all real V_0 . The band structure of the same potential has been studied using Floquet analysis. It is also shown that, in addition to the $\mathcal{PT}$-threshold at $V_0 = 0.5$, there exist a second critical point near $V_0^c \sim .888437$ after which no part of the eigenvalues and the band structure remains real.

▶ Some exactly solvable non-Hermitian Hamiltonians whose bound state wave functions are associated with Laguerre and Jacobi-type X_1 exceptional orthogonal polynomials have been generated using the method of point canonical transformation. The Hamiltonians are shown to be quasi-Hermitian in nature so that the energy spectrum is real. It is to be noted here that the other choices of $g(x)$ in the expression $E - V(x)$ associated with Laguerre and Jacobi EOPs give rise to the several other exactly solvable Hermitian as well as quasi-Hermitian extended potentials. But in all these cases we have to redefine the parameters carefully so that n dependent term appears only in the constant energy.

▶ We have studied a class of non-Hermitian Hamiltonians of the form $H_{GS} = w(\tilde{a}\tilde{a}^\dagger + 1/2) + \alpha\tilde{a}^2 + \beta\tilde{a}^{\dagger^2}$, where w, α, β are real constants and $\tilde{a}, \tilde{a}^\dagger$ are generalized anni-

hilation and creation operators. When $\tilde{a}$ and $\tilde{a}^\dagger$ satisfy the commutation relation $[\tilde{a}, \tilde{a}^\dagger] =$ constant, the Hamiltonian H_{GS} has been shown to be transformed into a harmonic oscillator like Hamiltonian. This reveals an intriguing result in the sense that in this case the resulting Hamiltonian does not always possess the spectrum of the harmonic oscillator. Reason for this anomaly is discussed in the frame work of position dependent mass models.

Chapter 4

Quantum Nonlinear Oscillator [1]

The nonlinear differential equation

$$(1 + \lambda x^2)\ddot{x} - (\lambda x)\dot{x}^2 + \alpha^2 x = 0, \quad \lambda > 0 \tag{4.1}$$

was first studied by Mathews and Lakshmanan [520] as an example of a non-linear oscillator and it was shown that the solution of (4.1) is

$$x = A \, \sin(\omega t + \phi) \tag{4.2}$$

with the following additional restriction linking frequency and amplitude $\omega^2 = \frac{\alpha^2}{1+\lambda A^2}$. Furthermore equation (4.1) can be obtained from the Lagrangian [520, 521]

$$L = \frac{1}{2} \frac{1}{(1 + \lambda x^2)} (\dot{x}^2 - \alpha^2 x^2) \tag{4.3}$$

so that both the kinetic and the potential energy terms depend on the same parameter λ. This λ-dependent system can be considered as:

(i) a deformation of the standard harmonic oscillator in the sense that for $\lambda \to 0$ all the characteristics of the linear oscillator are recovered

(ii) a particular case of a system with a position dependent effective mass of the form $m(x) = \frac{1}{1+\lambda x^2}$.

Recently, this particular nonlinear system has been generalized to the higher dimensions and various properties of this system have been studied [410, 522]. The classical Hamiltonian corresponding to the λ-dependent oscillator is given by [520, 521]

$$H_{cl} = \left(\frac{1}{2m} \right) P_x^2 + \left(\frac{1}{2} \right) g \left(\frac{x^2}{1 + \lambda x^2} \right), \qquad P_x = \sqrt{1 + \lambda x^2} p_x, \;\; g = m\alpha^2, \tag{4.4}$$

---

[1]This chapter is based on following two papers:
(i) B. Midya, B. Roy and A. Biswas, "Coherent state of nonlinear oscillator and its revival dynamics", Physica Scripta 79 (2009) 065003; (ii) B. Midya and B. Roy, "A generalized quantum nonlinear oscillator", J. Phys. A42 (2009) 285301.

where p_x is the canonically conjugate momentum defined by $p_x = \frac{\partial L}{\partial \dot{x}}$, L being the Lagrangian and m is the mass. It has been shown [411] that in the space $\mathcal{L}^2(\Re, d\mu)$ where $d\mu = \left(\frac{1}{\sqrt{1+\lambda x^2}}\right) dx$, the differential operator $\sqrt{1+\lambda x^2}\frac{d}{dx}$ is skew self adjoint. Therefore, contrary to the naive expectation of ordering ambiguities, the transition from the classical system to the quantum one is given by defining the momentum operator $P_x = -i\sqrt{1+\lambda x^2}\frac{d}{dx}$ so that

$$(1+\lambda x^2)p_x^2 \rightarrow -\left(\sqrt{1+\lambda x^2}\frac{d}{dx}\right)\left(\sqrt{1+\lambda x^2}\frac{d}{dx}\right).$$

Therefore the quantum version of the Hamiltonian (4.4) with $\hbar = 1$ becomes [411]

$$H = -\frac{1}{2m}(1+\lambda x^2)\frac{d^2}{dx^2} - \left(\frac{1}{2m}\right)\lambda x\frac{d}{dx} + \frac{1}{2}g\left(\frac{1}{1+\lambda x^2}\right) \tag{4.5}$$

where $g = \alpha(m\alpha + \lambda)$. It is to be noted here that in reference [411] the value of the parameter g has been slightly modified from that given in equation (4.4). After introducing adimensional variables (y, Λ) as was done in [411]

$$x = \sqrt{\frac{1}{m\alpha}}y \; ; \quad \lambda = m\alpha\Lambda, \tag{4.6}$$

the Schrödinger equation $H\psi = E\psi$ reduces to

$$\left[(1+\Lambda y^2)\frac{d^2}{dy^2} + \Lambda y\frac{d}{dy} - (1+\Lambda)\frac{y^2}{(1+\Lambda y^2)} + 2e\right]\psi = 0 \tag{4.7}$$

where $E = e(\alpha)$. The eigenvalues and eigenfunctions for $\Lambda < 0$ are given by [411]

$$\begin{aligned}
\psi_n(y,\Lambda) &= (1-|\Lambda|y^2)^{\frac{1}{(2|\Lambda|)}} \mathcal{H}_n(y,\Lambda) \\
\epsilon_n &= (n+\tfrac{1}{2}) - \tfrac{1}{2}n^2\Lambda \quad, \quad n = 0,1,2,...
\end{aligned} \tag{4.8}$$

where $\mathcal{H}_n(y,\Lambda)$ is Λ-deformed Hermite polynomial. For $\Lambda > 0$,

$$\begin{aligned}
\psi_n(y,\Lambda) &= (1+\Lambda y^2)^{-\frac{1}{2\Lambda}} \mathcal{H}_n(y,\Lambda) \\
\epsilon_n &= n+\tfrac{1}{2}) - \tfrac{1}{2}n^2\Lambda \quad, \quad n = 0,1,2\cdots,N_\Lambda
\end{aligned} \tag{4.9}$$

where N_Λ denotes the greatest integer lower than $n_\Lambda(=\frac{1}{\Lambda})$.

Here we have first studied, in the next section, the coherent state of the above mentioned quantum nonlinear oscillator (QNLO) and its revival dynamics. In section 4.2, we have examined various Hermitian and non-Hermitian generalizations of QNLO. In doing so a relationship between Λ-deformed Hermite polynomial and Jacobi polynomial has been established in section 4.3.

4.1　Gazeau-Klauder coherent state for nonlinear oscillator

We have first constructed the Gazeau-Klauder coherent states [458, 459] of the QNLO and then studied their revival dynamics to demonstrate the effect of mass parameter λ on the temporal evaluation of these states. The motivation comes from the following two facts (i) the temporal evolution of the coherent states of systems, like Pöschl-Teller, Morse and Rosen-Morse, possessing nonlinear energy spectra can lead to revival and fractional revival, leading to Schrödinger cat and cat-like states, (ii) the study of temporal evolution of a free wave-packet with position dependent mass inside an infinite well [494] has revealed that revival and partial revivals are not only different from the constant mass case but also they are very much dependent on the mass function $m(x)$. It should be noted here that, the mass parameter λ of QNLO may be positive or negative. However for $\lambda > 0$ the discrete energy spectrum of QNLO are finite and consequently for completeness property the continuum has to be taken into account. On the other hand for $\Lambda < 0$ there are only discrete energy states and henceforth we shall consider this choice.

From equation (4.8) the eigenvalues of the Hamiltonian $H^1 = H - \frac{\alpha}{2}$ are given by

$$E_n^1 = E_n - \frac{\alpha}{2} = \hbar\alpha\left[n + \frac{1}{2}n^2|\Lambda|\right] = \alpha\frac{n(n+\mu)}{\mu} = \alpha e_n^1 \qquad (4.10)$$

where $\mu = \frac{2}{|\Lambda|}$. The Gazeau-Klauder coherent state [459] for this system is given by

$$|J,\gamma\rangle = \frac{1}{N(J)}\sum_n \frac{(J)^{\frac{n}{2}}exp(-i\gamma e_n^1)}{\sqrt{\rho_n}}|n\rangle \qquad (4.11)$$

where $\gamma = \alpha t$. The normalization constant $N(J)$ is given by

$$N(J) = \left[\sum_n \frac{J^n}{\rho_n}\right]^{\frac{1}{2}} \quad , 0 < J < R = \limsup_{n\to+\infty} \rho_n^{\frac{1}{n}} \qquad (4.12)$$

where R denotes the radius of convergence and ρ_n denotes the moments of a probability distribution $\rho(x)$

$$\rho_n = \int_0^R x^n \rho(x) dx = \prod_{i=1}^n e_i^1 \quad ; \quad \rho_0 = 1 \tag{4.13}$$

For the coherent state (4.11)

$$\rho_n = \prod_{i=1}^n \frac{i(i+\mu)}{\mu} = \frac{\Gamma(n+1)\Gamma(n+1+\mu)}{\mu^n \Gamma(1+\mu)} \quad , \rho_0 = 1 \tag{4.14}$$

so that R is infinite and

$$\rho(J) = \frac{2\mu(J\mu)^{\frac{\mu}{2}}}{\Gamma(1+\mu)} K_\mu(2\sqrt{J\mu}), \tag{4.15}$$

$K_\nu(cx)$ being modified Bessel function [507]. Also

$$N(J)^2 = \frac{\Gamma(1+\mu)}{(J\mu)^{\frac{\mu}{2}}} I_\mu(2\sqrt{J\mu}) \tag{4.16}$$

where $I_\mu(cx)$ is the modified Bessel function [507]. So the coherent state (4.11) finally becomes

$$|J,\gamma\rangle = \frac{\sqrt{\Gamma(1+\mu)}}{N(J)} \sum_{n=0}^\infty \frac{(J\mu)^{\frac{n}{2}} e^{-i\alpha(n+\frac{n^2}{\mu})t}}{\sqrt{n!\Gamma(n+1+\mu)}} |n\rangle \tag{4.17}$$

Below we shall see that the coherent state (4.17) satisfies following four conditions:

1. Continuity of labeling: From the definition (4.11) it is obvious that

$$(J'\gamma') \to (J,\gamma) \Rightarrow |J',\gamma'\rangle \to |J,\gamma\rangle.$$

2. Resolution of unity:

$$\int |J,\gamma\rangle\langle J,\gamma| d\mu(J,\gamma) = \int |J,\gamma\rangle\langle J,\gamma| d\mu(J,\gamma) = \frac{1}{2\pi} \int_{-\pi}^\pi d\gamma \int_0^\infty k(J) |J,\gamma\rangle\langle J,\gamma| \ dJ$$

where $k(J)$ is defined by

$$
\begin{aligned}
k(J) &= N(J)^2 \rho(J) \geq 0, \quad 0 \leq J < R \\
&= \rho(J) \equiv 0, \quad J > R.
\end{aligned}
\tag{4.18}
$$

For the coherent state (4.17)

$$
k(J) = 2\mu I_\mu(2\sqrt{J\mu}) K_\mu(2\sqrt{J\mu})
\tag{4.19}
$$

so that the resolution of unity is satisfied

$$
\int |J,\gamma\rangle\langle J,\gamma| d\mu(J,\gamma) = 1.
$$

3. Temporal stability: $e^{-iH^1 t}|J,\gamma\rangle = |J,\gamma + \alpha t\rangle.$
4. Action identity: $\langle J,\gamma|H^1|J,\gamma\rangle = \alpha J.$

The overlapping of two coherent states is given by

$$
\langle J',\gamma'|J,\gamma\rangle = \frac{\Gamma(\mu+1)}{N(J)N(J')} \sum_{n=0}^{\infty} \frac{(JJ'\mu^2)^{\frac{n}{2}} e^{i(\gamma'-\gamma)e_n^1}}{n!\Gamma(n+1+\mu)}.
\tag{4.20}
$$

If $\gamma = \gamma'$, the overlapping is reduced to

$$
\langle J',\gamma|J,\gamma\rangle = \frac{1}{\sqrt{I_\mu(2\sqrt{J\mu})I_\mu(2\sqrt{J'\mu})}} I_\mu\left(2(JJ'\mu^2)^{\frac{1}{4}}\right).
\tag{4.21}
$$

4.1.1   Revival dynamics

In this section we shall study the revival dynamics of the coherent states (4.17) which are constructed in the previous section. To demonstrate the role of the mass parameter λ on the revival dynamics, all the figures below are drawn for a fixed value of $J = 10$ and two different values of μ which is inversely proportional to the mass parameter Λ ($\mu = \frac{2}{|\Lambda|}$). For a general wave packet of the form

$$
|\psi(t)\rangle = \sum_{n \geq 0} c_n e^{-iE_n t}|n\rangle
\tag{4.22}
$$
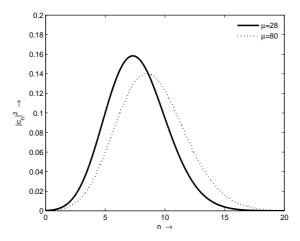
Figure 4.1: Plot of the weighted distribution $|c_n|^2$ given in (4.23) for $J = 10$ and $\mu = 28, 80$.

with $\sum_{n \geq 0} |c_n|^2 = 1$, the concept of revival arises from the weighting probabilities $|c_n|^2$. For the coherent state (4.17), the weighting distribution is given by

$$|c_n|^2 = \frac{(J\mu)^{n+\frac{\mu}{2}}}{n!\Gamma(n+1+\mu)I_\mu(2\sqrt{J\mu})} \tag{4.23}$$

Since the weighting distribution $|c_n|^2$ is crucial for understanding the temporal behavior of the coherent state (4.17), we show the curves of $|c_n|^2$ for $J = 10$ and different μ in figure 4.1.

The Mandel parameter Q is defined by

$$Q = \frac{(\Delta n)^2}{\langle n \rangle} - 1 \tag{4.24}$$

where

$$\begin{aligned}
\langle n \rangle &= \sum_{n=0}^{\infty} \frac{nJ^n}{N(J)^2 \rho_n} \\
\Delta n &= [\langle n^2 \rangle - \langle n \rangle^2]^{\frac{1}{2}}.
\end{aligned} \tag{4.25}$$

The Mandel parameter determines the nature of the weighting distribution function $|c_n|^2$. The case of $Q = 0$ coincides with the Poissonian weighting distribution $\dfrac{\langle n \rangle^n e^{-\langle n \rangle}}{n!}$ while the cases of $Q > 0$ and $Q < 0$ correspond to the super-Poissonian or sub-Poissonian
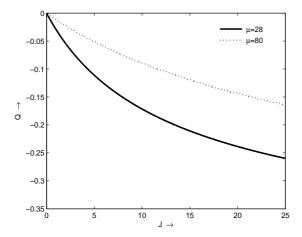
Figure 4.2: Plot of the Mandel parameter Q given in (4.26) for $J = 10$ and $\mu = 28, 80$.

statistics respectively. For the coherent state (4.17),

$$Q = \sqrt{J\mu}\left[\frac{I_{\mu+2}(2\sqrt{J\mu})}{I_{\mu+1}(2\sqrt{J\mu})} - \frac{I_{\mu+1}(2\sqrt{J\mu})}{I_\mu(2\sqrt{J\mu})}\right] \tag{4.26}$$

where

$$\langle n \rangle = \sqrt{J\mu}\frac{I_{\mu+1}(2\sqrt{J\mu})}{I_\mu(2\sqrt{J\mu})}. \tag{4.27}$$

In figure 4.2, we plot the Mandel parameter Q for $J = 10$ and $\mu = 28, 80$. It is evident from the figure that the Mandel parameter is sub-Poissonian and it has been observed that it remains so for all values μ.

Now assuming that the spread $\Delta n = [\langle n^2 \rangle - \langle n \rangle^2]^{\frac{1}{2}}$ is small compared to $\langle n \rangle \approx \bar{n}$, we expand the energy E_n^1 in a Taylor series in n around the centrally excited value $\bar{n}$:

$$E_n^1 \approx E_{\bar{n}}^1 + E_{\bar{n}}^{1\prime}(n - \bar{n}) + \frac{1}{2}E_{\bar{n}}^{1\prime\prime}(n - \bar{n})^2 + \frac{1}{6}E_{\bar{n}}^{1\prime\prime\prime}(n - \bar{n})^3 + \cdots \tag{4.28}$$

where each prime on $E_{\bar{n}}^1$ denotes a derivative with respect to n. These derivatives define distinct time scales, namely the classical period $T_{cl} = \dfrac{2\pi}{|E_{\bar{n}}^{1\prime}|}$, the revival time $t_{rev} = \dfrac{2\pi}{\frac{1}{2}|E_{\bar{n}}^{1\prime\prime}|}$ and so on. For E_n^1 in Eqn.(4.10), $T_{cl} = \dfrac{2\pi}{\alpha(2\bar{n} + \mu)}$ and $t_{rev} = \dfrac{2\pi\mu}{\alpha}$. There is no superrevival time here because the energy is a quadratic function in n. It is convenient
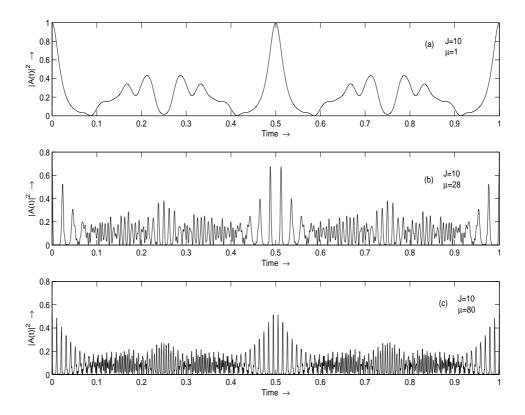
Figure 4.3: Plot of the square modulus of the autocorrelation function $|A(t)|^2$ given in (4.30) for $(a) J = 10, \mu = 1, \ (b) J = 10, \mu = 28, (c) J = 10, \mu = 80$.

to describe the wave packet dynamics by an autocorrelation function

$$A(t) = \langle \psi(x,0) | \psi(x,t) \rangle = \sum_{n \geq 0} |c_n|^2 e^{-iE_n^1 t} \tag{4.29}$$

For the coherent state (4.17) the autocorrelation function (4.29) is given by

$$A(t) = \langle J, 0 | J, \alpha t \rangle = \frac{\Gamma(1+\mu)}{N(J)^2} \sum_{n \geq 0} \frac{(J\mu)^n}{n! \Gamma(n+1+\mu)} e^{-i\alpha(n+\frac{n^2}{\mu})t}. \tag{4.30}$$

From (4.30) it follows that $|A(t - t_{rev})|^2 = |A(t)|^2$ so that the autocorrelation function is symmetric about $\frac{t_{rev}}{2}$. In other words whatever happens in $[0, \frac{t_{rev}}{2}]$ is repeated subsequently. We note that $0 \leq |A(t)|^2 \leq 1$ and it denotes the overlap between the coherent state at $t = 0$ and the one at time t. So, the larger the value of $|A(t)|^2$, the greater the coherent state at time t will resemble the initial one. In figures 4.3(a), (b) and (c) we plot

the squared modulus of the autocorrelation function (4.30) in units of t_{rev} for $J = 10$ and $\mu = 1, 28, 80$ respectively. Figure 4.3(a) clearly shows the full revival. The sharp peaks in Figure 4.3(b) and 4.3(c) arise due to fractional revivals. We can see from figures 4.3(a) to 4.3(c) that different values of μ, which is inversely proportional to the mass parameter Λ, lead to qualitatively different types of motion of the coherent wave packet.

Now we shall study the mechanism of the fractional revival by phase analysis [523, 524]. For this, we use the time scale determined by the period of the complete revival $t_{rev} = \frac{2\pi\mu}{\alpha} = 1$. So the phase of the n-th stationary state is

$$\phi_n(t) = 2\pi(\mu n + n^2)t \tag{4.31}$$

where $\mu = \frac{2}{|\Lambda|}$ is assumed to be an integer. At arbitrary moments of time, the phases (4.31) of individual components of the packet are uniformly mixed so that it is not possible to make any definite conclusion about the value of the autocorrelation function. However, at specific moments, the distribution of phases can gain some order; for example, the phases can split into several groups of nearly constant values. The fractional revival of q-th order is defined as the time interval during which the phases are distributed among q groups of nearly constant values. To consider fractional revival of order q, in the vicinity of time $t = \frac{1}{q}$, it is convenient to write n as $n = kq + \Delta$, where $k = 0, 1, 2, \cdots$ and $\Delta = 0, 1, 2, \cdots q - 1$. Then the autocorrelation function can be written as

$$A(t) = \sum_{\Delta=0}^{q-1} P_\Delta(t) \tag{4.32}$$

where

$$P_\Delta(t) = \sum_k c_{kq+\Delta} e^{-i\phi_{kq+\Delta}(t)}, \tag{4.33}$$

$$c_{kq+\Delta} = \frac{(J\mu)^{kq+\Delta+\frac{\mu}{2}}}{(kq + \Delta)!\Gamma(kq + \Delta + 1 + \mu)I_\mu(2\sqrt{J\mu})} \tag{4.34}$$

and $\phi_{kq+\Delta}(t)$ is given by (4.31) replacing n by $kq + \Delta$. For quantum numbers that are
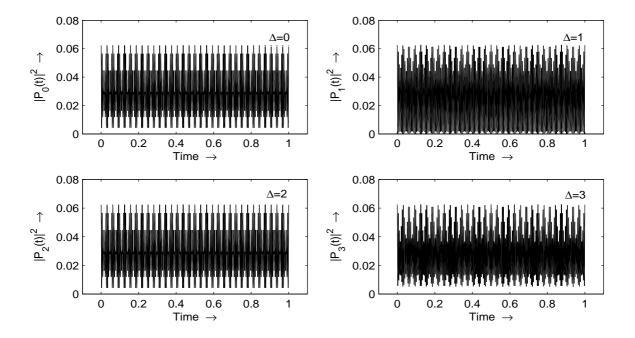
Figure 4.4: Plots of the Survival functions $|P_\Delta(t)|^2$ given in (4.34) for $\Delta = 0, 1, 2, 3$ and $J = 10, \mu = 28$.

multiples of the revival order i.e.$n = kq$, we have

$$
\begin{aligned}
\phi_{kq}(\tfrac{1}{q}) &= 2\pi(k\mu + k^2 q) = 0 \ (mod 2\pi) \\
P_0(\tfrac{1}{q}) &= \sum_k c_{kq}
\end{aligned}
\tag{4.35}
$$

When $\Delta \neq 0$, the phases of the corresponding states are

$$
\begin{aligned}
\phi_{kq+\Delta}(\tfrac{1}{q}) &= 2\pi(\mu\Delta + \Delta^2)/q \\
P_\Delta(\tfrac{1}{q}) &= exp[-2\pi i(\mu\Delta q^{-1} + \Delta^2 q^{-1}) \sum_k c_{kq+\Delta}
\end{aligned}
\tag{4.36}
$$

where in obtaining (4.34) and (4.35) we have made use of the result

$$
2\pi(k(\mu + 2\Delta) + k^2 q) = 0(mod 2\pi)
\tag{4.37}
$$

Thus from (4.35) and (4.36) it follows that, around time $t = \frac{1}{q}$ the Gazeau-Klauder coherent state (10) splits into q packet fractions such that the $\Delta = 0$ packet fraction has zero phase while relative to this the other packet fractions have a constant phase $(\mu\Delta q^{-1} + q^{-1}\Delta^2)$.

It must be mentioned that the way the packet is divided into fractions is determined
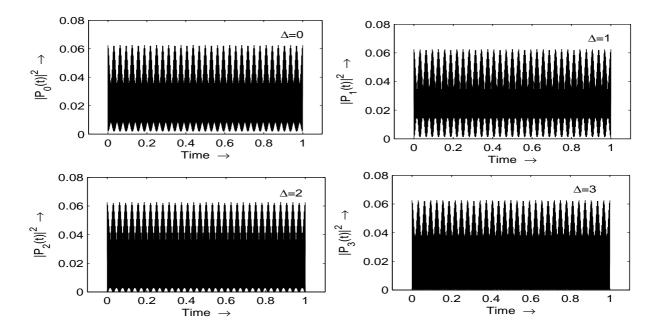
Figure 4.5: Plots of the Survival functions $|P_\Delta(t)|^2$ given in (4.34) for $\Delta = 0, 1, 2, 3$ and $J = 10, \mu = 80$.

only by the order of the revival analyzed. Figure 4.4 and Figure 4.5 give the time dependence of the survival functions $|P_\Delta(t)|^2$ in units t_{rev} for $\Delta = 0, 1, 2, 3$ during the whole period of the complete revival for fixed $J = 10$ and $\mu = 28, 80$ respectively.

The intensity of the fractional revival can be defined as the value of the wave packet survival function at the moment of revival. This value is composed of two terms: the sum of survival functions for the packet fractions (the diagonal term) and the interference term describing the interaction of the packet fractions

$$S\left(\frac{1}{q}\right) = \sum_{\Delta=0}^{q-1} |P_\Delta\left(\frac{1}{q}\right)|^2 + \sum_{\Delta=0}^{q-1} \sum_{\Gamma \neq \Delta} P_\Delta\left(\frac{1}{q}\right) P_\Gamma^*\left(\frac{1}{q}\right) \tag{4.38}$$
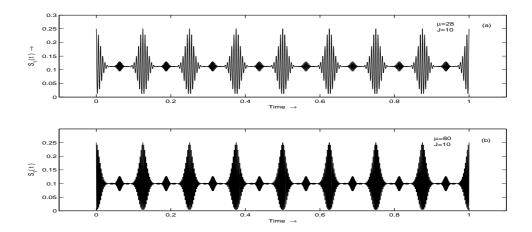
Figure 4.6: Plots of the diagonal term $S_1(t) = \Sigma_{\Delta=0}^3 |P_\Delta(t)|^2$ of the survival function for (a)$J = 10, \mu = 28$ and (b) $J = 10, \mu = 80$.

In Figure 4.6(a) and Figure 4.6(b) we present the time dependence of the diagonal terms in units of t_{rev} for fixed $J = 10$ and $\mu = 28, 80$ respectively. It is seen from the figures that the diagonal term related to the individual packet fractions takes positive values only near the moments of fractional revival. Figure 4.7(a) and Figure 4.7(b) show the time dependence of the interference terms in units of t_{rev} for fixed $J = 10$ and $\mu = 28, 80$ respectively. It is seen from the figures that the interference term plays a constructive, destructive or indifferent role near the moments of fractional revival.
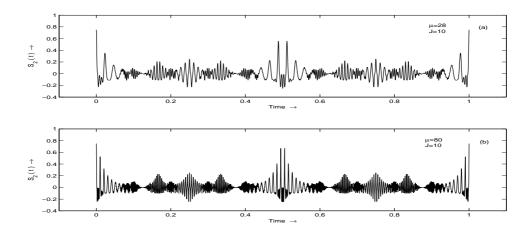
Figure 4.7: Plots of the interference term $S_2(t) = \Sigma_{\Delta=0}^{3}\Sigma_{\Gamma\neq\Delta}^{3} P_\Delta(t)P_\Gamma^*(t)$ of the survival function.(a) $J = 10, \mu = 28$; (b) $J = 10, \mu = 80$.

4.2   Generalization of quantum nonlinear oscillator

Here we shall obtain a number of exactly solvable Hermitian, non-Hermitian $\mathcal{PT}$-symmetric and $\mathcal{PT}$-symmetric quasi exactly solvable potentials corresponding to the QNLO. In all these cases the same mass function $m(x) = \frac{1}{1+\lambda x^2}$ has been considered.

4.2.1   Hermitian generalization

We have the Schrödinger equation corresponding to the Hamiltonian given in Eqn.(4.5) with $m = 1$ and $\lambda > 0$ as

$$\left[-(1 + \lambda x^2)\frac{d^2\psi}{dx^2} - \lambda x \frac{d\psi}{dx} - \frac{g}{\lambda}\left(\frac{1}{1 + \lambda x^2}\right)\right]\psi = E\psi \tag{4.39}$$

$$E = 2e - \frac{g}{\lambda} \tag{4.40}$$

where e is the energy of the Hamiltonian (4.5). Now expanding $(1 + \lambda x^2)^{-1}$ for $|x| < \frac{1}{\sqrt{\lambda}}$ we can write the potential of equation (4.39) as

$$V(x) = -\frac{g}{\lambda} + gx^2 - \lambda \, O(x^3) \tag{4.41}$$

It is clear from (4.41) that the term $(-\frac{g}{\lambda})$ in equation (4.40) cancels from both sides of the equation (4.39), so that the new eigenvalues (4.40) are actually the old eigenvalues e

of the Hamiltonian (4.5).

Now generalizing the potential of the equation (4.39) as below, the corresponding Schrödinger equation now reads

$$-(1+\lambda x^2)\frac{d^2\psi}{dx^2} - \lambda x\frac{d\psi}{dx} + \left[\frac{B^2 - A^2 - A\sqrt{\lambda}}{1+\lambda x^2} + B(2A+\sqrt{\lambda})\left(\frac{\sqrt{\lambda}x}{1+\lambda x^2}\right) + A^2\right]\psi = E\psi$$

(4.42)

It is seen from (4.42) that if we put $B = 0$ then the potential reduces to that of the nonlinear oscillator with $\frac{g}{\lambda} = A^2 + A\sqrt{\lambda}$. This particular generalization is made so that it corresponds to the hyperbolic Scarf II potential [40] in the constant mass case. In order to solve (4.42), we now perform a transformation involving change of variable given by [511]

$$z = \int \frac{dx}{\sqrt{F(x)}} = \frac{1}{\sqrt{\lambda}} \sinh^{-1}(\sqrt{\lambda}x)$$

(4.43)

where

$$F(x) = 1 + \lambda x^2 , \quad \lambda > 0$$

(4.44)

Under the transformation (4.43), Eqn.(4.42) reduces to a Schrödinger equation

$$-\frac{d^2\psi}{dz^2} + V(z)\psi(z) = E\psi(z)$$

(4.45)

where the potential $V(z)$ is given by

$$V(z) = (B^2 - A^2 - A\sqrt{\lambda})\operatorname{sech}^2\left(z\sqrt{\lambda}\right) + B(2A+\sqrt{\lambda})\tanh\left(z\sqrt{\lambda}\right)\operatorname{sech}\left(z\sqrt{\lambda}\right) + A^2 \quad (4.46)$$

The potential (4.46) is a standard solvable potential and the solutions are given by [40]

$$\psi_n(z) = N_n i^n \left(1 + \sinh^2(z\sqrt{\lambda})\right)^{-\frac{s}{2}} e^{-r\tan^{-1}\left(\sinh(z\sqrt{\lambda})\right)} P_n^{(ir-s-\frac{1}{2}, -ir-s-\frac{1}{2})}\left(i\,\sinh(z\sqrt{\lambda})\right)$$

(4.47)

where N_n is the normalization constant , $s = \frac{A}{\sqrt{\lambda}}$, $r = \frac{B}{\sqrt{\lambda}}$ and $P_n^{(\alpha,\beta)}(x)$ is the Jacobi

Polynomial [507]. The normalization constants $N_n, n = 0, 1, 2, ...$ are given by [525],

$$N_n = \left[\frac{\sqrt{\lambda}\, n!\, (s-n)\Gamma(s-ir-n+\frac{1}{2})\Gamma(s+ir-n+\frac{1}{2})}{\pi\, 2^{-2s}\Gamma(2s-n+1)} \right]^{1/2} \tag{4.48}$$

The eigenvalues E_n are given by

$$E_n = n\sqrt{\lambda}(2A - n\sqrt{\lambda}), \qquad n = 0, 1, 2, ... < s \tag{4.49}$$

Subsequently by performing the inverse of the transformation (4.43) we find the solution of eqn.(4.42) as

$$\begin{aligned} \psi_n(x) &= \left[\frac{\sqrt{\lambda}\, n!\, (s-n)\Gamma(s-ir-n+\frac{1}{2})\Gamma(s+ir-n+\frac{1}{2})}{\pi\, 2^{-2s}\Gamma(2s-n+1)} \right]^{1/2} \\ &\quad i^n(1+\lambda x^2)^{-\frac{s}{2}}\, e^{-r\tan^{-1}(x\sqrt{\lambda})}\, P_n^{(ir-s-\frac{1}{2},\,-ir-s-\frac{1}{2})}(ix\sqrt{\lambda}), \\ &\quad n = 0, 1, 2, \cdots < s \ (= \frac{A}{\sqrt{\lambda}}) \end{aligned} \tag{4.50}$$

In the following, we show that the above mentioned generalization of QNLO produces all the results of a linear harmonic oscillator in the limit $\lambda \to 0$. For $B = 0$, $A = \frac{\alpha}{\sqrt{\lambda}}$ the potential of equation (4.42) and it's energy eigenvalues (4.49) reduces to

$$V(x) = \left(-\frac{\alpha^2}{\lambda} - \alpha \right)(1 + \lambda x^2)^{-1} + \frac{\alpha^2}{\lambda} \tag{4.51}$$

$$E_n = 2n\alpha - n\lambda. \tag{4.52}$$

respectively. For $|x| < \frac{1}{\sqrt{\lambda}}$, the potential (4.51) can be written as

$$\begin{aligned} V(x) &= \left(-\frac{\alpha^2}{\lambda} - \alpha \right)(1 - \lambda x^2 + \lambda^2 x^4 - \lambda^3 x^6 + ...) + \frac{\alpha^2}{\lambda} \\ &= \alpha^2 x^2 - \lambda(\alpha^2 x^4 - \lambda\alpha^2 x^6 + ...) + \lambda(\alpha x^2 - \lambda\alpha x^4 + ...) - \alpha \end{aligned} \tag{4.53}$$

For $\lambda \to 0$ the potential reduces to

$$V(x) = \alpha^2 x^2 - \alpha \tag{4.54}$$

It is clear from (4.54) and (4.52) that for $\lambda \to 0$ the potential of the equation (4.42) and

the energy eigenvalues (4.49) reduces to those of a simple harmonic oscillator. Now, for $A = \frac{\alpha}{\sqrt{\lambda}}, B = 0$ and using the relation (4.99) the expression for the wavefunction (4.50) is reduced to

$$\psi_n(x) = N'_n (1 + \lambda x^2)^{-\frac{2\alpha}{\lambda}} \; \mathcal{H}_n \left(\sqrt{\alpha} x, \frac{\lambda}{\alpha} \right) \tag{4.55}$$

where

$$N'_n = \frac{1}{2^n n!} \left(\frac{\alpha}{\lambda} \right)^{\frac{n}{2}} N_n \;\; = \left[\frac{\alpha^n (\frac{\alpha}{\lambda} - n) \Gamma(\frac{\alpha}{\lambda} - n + \frac{1}{2}) \; \Gamma(\frac{\alpha}{\lambda} - n + \frac{1}{2})}{\pi n! \; 2^{2n - \frac{2\alpha}{\lambda}} \lambda^{n - \frac{1}{2}} \Gamma(\frac{2\alpha}{\lambda} - n + 1)} \right]^{1/2} \tag{4.56}$$

When $\lambda \to 0$ the λ-deformed Hermite polynomial becomes the conventional Hermite polynomial H_n [411]. Consequently at $\lambda \to 0$ limit the unnormalized wave function given in equation (4.55) reduces to

$$\psi_n(x) \propto e^{-\frac{\alpha x^2}{2}} H_n(\sqrt{\alpha} x) \tag{4.57}$$

Using the asymptotic formula $\Gamma(az + b) \sim \sqrt{2\pi} e^{-az} (az)^{az + b - \frac{1}{2}}$ (see 6.1.39 of the ref. [507]) in (4.56) we have

$$N'_n = \left(\frac{\sqrt{\alpha} - \frac{n\lambda}{\sqrt{\alpha}}}{\sqrt{\pi} 2^n n!} \right)^{1/2} . \tag{4.58}$$

Therefore from equations (4.57) and (4.58) it follows that for $\lambda \to 0$ the wave function given in equation (4.50) reduce to that of simple harmonic oscillator.

At this point it is natural to ask the following question : Are there other solvable potentials for the QNLO corresponding to the mass function $m(x) = \left(\frac{1}{1 + \lambda x^2} \right)$? The answer to this question is in the affirmative. The procedure to obtain these potentials is similar to the one already described and so instead of treating each case separately we have presented the potentials and the corresponding solutions in Table 4.1. The first two and the last two potentials in Table 4.1 are actually the generalizations of the nonlinear oscillator potential. Although the other two potentials in the Table are not generalizations of the nonlinear oscillator potential, nevertheless they are exactly solvable potentials with the same mass function. Incidentally all the potentials in table 4.1 are shape invariant as shown in section 4.3 and so the energy eigenvalues and eigenfunctions can also be obtained

algebraically.

4.2.2   Non-Hermitian $\mathcal{PT}$-symmetric generalization

Here we shall find some complex potentials for which the QNLO is exactly solvable. For a constant mass Schrödinger Hamiltonian the condition for $\mathcal{PT}$-symmetry reduces to $V(x) = V^*(-x)$. However, in the case of position dependent mass an additional condition is required. To see this, we note that in the present case the Hamiltonian is of the form

$$H = -\frac{1}{2m(x)}\frac{d^2}{dx^2} - \frac{m'(x)}{2m^2(x)}\frac{d}{dx} + V(x), \tag{4.59}$$

with $m(x) = \frac{1}{1+\lambda x^2}$. The conditions for the Hamiltonian (4.59) to be $\mathcal{PT}$-symmetric are given by

$$m(x) = m(-x) \quad , \quad V(x) = V^*(-x). \tag{4.60}$$

It may be pointed out that here we are working with a mass profile $m(x) = (1 + \lambda x^2)^{-1}$ which is an even function and consequently satisfies the first condition of (4.60). To generate non-Hermitian interaction in the present case we introduce complex coupling constant. As an example let us first consider the potential appearing in (4.42). It can be seen from (4.49) that the energy for this potential does not depend on one of the potential parameters, namely B. Thus we make the generalized nonlinear oscillator potential of the equation (4.42) by replacing B in terms of iB

$$V(x) = \left[\frac{B^2 - A^2 - A\sqrt{\lambda}}{1 + \lambda x^2} + iB(2A + \sqrt{\lambda})\left(\frac{\sqrt{\lambda}x}{1 + \lambda x^2}\right) + A^2 \right] \tag{4.61}$$

From (4.61) it can be easily verified that $V(x) = V^*(-x)$ so that the Hamiltonian (4.59) with this potential is $\mathcal{PT}$-symmetric. In this case the spectrum is real and given by (4.49). Proceeding in a similar way we have obtained the complex $\mathcal{PT}$-symmetric version of the potentials given in table 4.1. These complex potentials for which the QNLO is exactly solvable are given in tabular form in the table 4.2.

$V(x)$	$W(x)$	E_n	$\psi_n(x)$	$a_i, i = 0,1,..$	$R(a_i)$
$\frac{B^2-A^2-A\sqrt{\lambda}}{1+\lambda x^2} + B(2A+\sqrt{\lambda})\frac{\sqrt{\lambda}x}{1+\lambda x^2} + A^2$	$A\frac{\sqrt{\lambda}x}{\sqrt{1+\lambda x^2}} + B\frac{1}{\sqrt{1+\lambda x^2}}$	$n\sqrt{\lambda}(2A - n\sqrt{\lambda})$	$i^n(1+\lambda x^2)^{-\frac{s}{2}}e^{-r\,tan^{-1}(x\sqrt{\lambda})}$ $P_n^{(ir-s-\frac{1}{2},-ir-s-\frac{1}{2})}(ix\sqrt{\lambda})$	$(A - i\sqrt{\lambda}, B)$	$\sqrt{\lambda}\left[2A - (2i+1)\sqrt{\lambda}\right]$
$A^2 + \frac{B^2}{A^2} - \frac{A(A+\sqrt{\lambda})}{1+\lambda x^2} + 2B\frac{\sqrt{\lambda}x}{\sqrt{1+\lambda x^2}}$, $B < A^2$	$A\frac{\sqrt{\lambda}x}{\sqrt{1+\lambda x^2}} + \frac{B}{A}$	$A^2 + \frac{B^2}{A^2} - (A - n\sqrt{\lambda})^2 - \frac{B^2}{(A-n\sqrt{\lambda})^2}$	$\left(1 - \frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)^{\frac{s_1}{2}}\left(1 + \frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)^{\frac{s_2}{2}}$ $P_n^{(s_1,s_2)}\left(\frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)$	$(A - i\sqrt{\lambda}, B)$	$A^2 - \left[A - (i+1)\sqrt{\lambda}\right]^2$ $+\frac{B^2}{A^2} - \frac{B^2}{\left[A-(i+1)\sqrt{\lambda}\right]^2}$
$A^2 + \frac{B^2}{A^2} - 2B\frac{\sqrt{1+\lambda x^2}}{\sqrt{\lambda}x} + \frac{A(A-\sqrt{\lambda})}{\lambda x^2}$, $B > A^2$ $0 \le x\sqrt{\lambda} \le \infty$	$\frac{B}{A} - A\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}}$	$A^2 + \frac{B^2}{A^2} - (A + n\sqrt{\lambda})^2 - \frac{B^2}{(A+n\sqrt{\lambda})^2}$	$\left(\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}} - 1\right)^{\frac{s_3}{2}}\left(\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}} + 1\right)^{\frac{s_4}{2}}$ $P_n^{(s_3,s_4)}\left(\frac{1+\lambda x^2}{x\sqrt{\lambda}}\right)$	$(A + i\sqrt{\lambda}, B)$	$A^2 - \left[A + (i+1)\sqrt{\lambda}\right]^2$ $+\frac{B^2}{A^2} - \frac{B^2}{\left[A+(i+1)\sqrt{\lambda}\right]^2}$
$\frac{A^2+B^2+A\sqrt{\lambda}}{\lambda x^2} - B(2A+\lambda)\frac{\sqrt{1+\lambda x^2}}{\lambda x^2} + A^2$, $A < B, \ 0 \le x\sqrt{\lambda} \le \infty$	$A\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}} - B\frac{1}{x\sqrt{\lambda}}$	$n\sqrt{\lambda}(2A - n\sqrt{\lambda})$	$(\sqrt{1+\lambda x^2} - 1)^{(\frac{r-s}{2})}(\sqrt{1+\lambda x^2} + 1)^{-(\frac{r+s}{2})}$ $P_n^{(r-s-\frac{1}{2},-r-s-\frac{1}{2})}\left(\sqrt{1+\lambda x^2}\right)$	$(A - i\sqrt{\lambda}, B)$	$\sqrt{\lambda}\left[2A - (2i+1)\sqrt{\lambda}\right]$
$\frac{A^2+B^2-A\sqrt{|\lambda|}}{1+\lambda x^2} - B(2A - \sqrt{|\lambda|})\frac{x\sqrt{|\lambda|}}{1+\lambda x^2} - A^2$ $\frac{-1}{\sqrt{|\lambda|}} \le x \le \frac{1}{\sqrt{|\lambda|}}$	$A\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}} - B\frac{1}{\sqrt{1+\lambda x^2}}$	$n\sqrt{|\lambda|}(2A + n\sqrt{|\lambda|})$	$(1 - x\sqrt{|\lambda|})^{(\frac{s'-r'}{2})}(1 + x\sqrt{|\lambda|})^{(\frac{r'+s'}{2})}$ $P_n^{(s'-r'-\frac{1}{2},s'+r'-\frac{1}{2})}\left(x\sqrt{|\lambda|}\right)$	$(A + i\sqrt{|\lambda|}, B)$	$\sqrt{|\lambda|}\left[2A + (2i+1)\sqrt{|\lambda|}\right]$
$\frac{A(A-\sqrt{|\lambda|})}{1+\lambda x^2} - 2B\frac{x\sqrt{|\lambda|}}{1+\lambda x^2} - A^2 + \frac{B^2}{A^2}$ $\frac{-1}{\sqrt{|\lambda|}} \le x \le \frac{1}{\sqrt{|\lambda|}}$	$A\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}} - \frac{B}{A}$	$\frac{B^2}{A^2} - A^2 + (A + n\sqrt{|\lambda|})^2 - \frac{B^2}{(A+n\sqrt{|\lambda|})^2}$	$\left(\frac{\lambda x^2}{1+\lambda x^2} - 1\right)^{-(\frac{s'+n}{2})}e^{-a\sqrt{|\lambda|}x}$ $P_n^{(-s'-n-ia,-s'-n+ia)}\left(-i\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}}\right)$	$(A + i\sqrt{|\lambda|}, B)$	$\frac{B^2}{A^2} - \frac{B^2}{\left[A+(i+1)\sqrt{|\lambda|}\right]^2}$ $-A^2 + \left[A + (i+1)\sqrt{|\lambda|}\right]^2$

Table 4.1: Shape invariant potentials $V(x)$ for which the QNLO is exactly solvable. Corresponding, superpotential $W(x)$, energy eigenvalue E_n and wave functions $\psi_n(x)$ are also given. Here $s = \frac{A}{\sqrt{\lambda}}, r = \frac{B}{\sqrt{\lambda}}, r_1 = \frac{B}{\lambda}, a = \frac{r_1}{s-n}, s_1 = s - n + a, s_2 = s - n - a, s_3 = a - n - s, s_4 = -(s+n+a), s' = \frac{A}{\sqrt{|\lambda|}}$ & $r' = \frac{B}{\sqrt{|\lambda|}}$.The first four entries correspond to $\lambda > 0$ and the last two correspond to $\lambda < 0$.

$V(x)$	$W(x)$	E_n	$\psi_n(x)$
$\frac{-B^2-A^2-A\sqrt{\lambda}}{1+\lambda x^2}+iB(2A+\sqrt{\lambda})\frac{\sqrt{\lambda}x}{1+\lambda x^2}+A^2$	$A\frac{\sqrt{\lambda}x}{\sqrt{1+\lambda x^2}}+iB\frac{1}{\sqrt{1+\lambda x^2}}$	$n\sqrt{\lambda}(2A-n\sqrt{\lambda})$	$i^n(1+\lambda x^2)^{-\frac{s}{2}}e^{-r\,tan^{-1}(x\sqrt{\lambda})}$ $P_n^{(ir-s-\frac{1}{2},-ir-s-\frac{1}{2})}\left(ix\sqrt{\lambda}\right)$
$A^2-\frac{B^2}{A^2}-\frac{A(A+\sqrt{\lambda})}{1+\lambda x^2}+i2B\frac{\sqrt{\lambda}x}{\sqrt{1+\lambda x^2}}\ ,\ B<A^2$	$A\frac{\sqrt{\lambda}x}{\sqrt{1+\lambda x^2}}+i\frac{B}{A}$	$A^2-\frac{B^2}{A^2}-(A-n\sqrt{\lambda})^2+\frac{B^2}{(A-n\sqrt{\lambda})^2}$	$\left(1-\frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)^{\frac{s_1}{2}}\left(1+\frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)^{\frac{s_2}{2}}$ $P_n^{(s_1,s_2)}\left(\frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)$
$A^2-\frac{B^2}{A^2}-2iB\frac{\sqrt{1+\lambda x^2}}{\sqrt{\lambda}x}+\frac{A(A-\sqrt{\lambda})}{\lambda x^2}\ ,\ B>A^2$ $0\le x\sqrt{\lambda}\le\infty$	$i\frac{B}{A}-A\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}}$	$A^2-\frac{B^2}{A^2}-(A+n\sqrt{\lambda})^2+\frac{B^2}{(A+n\sqrt{\lambda})^2}$	$\left(\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}}-1\right)^{\frac{s_3}{2}}\left(\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}}+1\right)^{\frac{s_4}{2}}$ $P_n^{(s_3,s_4)}\left(\frac{1+\lambda x^2}{x\sqrt{\lambda}}\right)$
$\frac{A^2-B^2+A\sqrt{\lambda}}{\lambda x^2}-iB(2A+\lambda)\frac{\sqrt{1+\lambda x^2}}{\lambda x^2}A^2\ ,\ A<B$ $0\le x\sqrt{\lambda}\le\infty$	$A\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}}-iB\frac{1}{x\sqrt{\lambda}}$	$n\sqrt{\lambda}(2A-n\sqrt{\lambda})$	$(\sqrt{1+\lambda x^2}-1)^{(\frac{r-s}{2})}(\sqrt{1+\lambda x^2}+1)^{-(\frac{r+s}{2})}$ $P_n^{(r-s-\frac{1}{2},-r-s-\frac{1}{2})}\left(\sqrt{1+\lambda x^2}\right)$
$\frac{A^2-B^2-A\sqrt{|\lambda|}}{1+\lambda x^2}-iB(2A-\sqrt{|\lambda|})\frac{x\sqrt{|\lambda|}}{1+\lambda x^2}-A^2$ $\frac{-1}{\sqrt{|\lambda|}}\le x\le\frac{1}{\sqrt{|\lambda|}}$	$A\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}}-iB\frac{1}{\sqrt{1+\lambda x^2}}$	$n\sqrt{\lambda}(2A+n\sqrt{|\lambda|})$	$(1-x\sqrt{|\lambda|})^{(\frac{s'-r'}{2})}(1+x\sqrt{|\lambda|})^{(\frac{r'+s'}{2})}$ $P_n^{(s'-r'-\frac{1}{2},s'+r'-\frac{1}{2})}\left(x\sqrt{|\lambda|}\right)$
$\frac{A(A-\sqrt{|\lambda|})}{1+\lambda x^2}-2iB\frac{x\sqrt{|\lambda|}}{1+\lambda x^2}-A^2-\frac{B^2}{A^2}$ $\frac{-1}{\sqrt{|\lambda|}}\le x\le\frac{1}{\sqrt{|\lambda|}}$	$A\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}}-i\frac{B}{A}$	$-\frac{B^2}{A^2}-A^2+(A+n\sqrt{|\lambda|})^2+\frac{B^2}{(A+n\sqrt{|\lambda|})^2}$	$\left(\frac{\lambda x^2}{1+\lambda x^2}-1\right)^{-(\frac{s'+n}{2})}e^{-a\sqrt{|\lambda|}x}$ $P_n^{(-s'-n-ia,-s'-n+ia)}\left(-i\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}}\right)$

Table 4.2: Non-Hermitian $\mathcal{PT}$-symmetric potentials corresponding to QNLO. Here, $s=\frac{A}{\sqrt{\lambda}}, r=i\frac{B}{\sqrt{\lambda}}, r_1=i\frac{B}{\lambda}, a=\frac{r_1}{s-n}, s_1=s-n+a, s_2=s-n-a, s_3=a-n-s, s_4=-(s+n+a), \quad s'=\frac{A}{\sqrt{|\lambda|}}$ & $r'=i\frac{B}{\sqrt{|\lambda|}}$. The first four entries correspond to $\lambda>0$ and the last two correspond to $\lambda<0$.

4.2.3 Quasi exactly solvable $\mathcal{PT}$-symmetric generalization

The quasi-exactly solvable complex sextic potential in the constant mass Schrödinger equation has been discussed in ref.[526]. By using the transformations (4.43) for $\lambda > 0$ we obtain the corresponding quasi exactly solvable potentials of the QNLO.

For $\lambda > 0$, we consider the potential

$$V(x) = \sum_{k=1}^{6} \frac{c_k}{\lambda^{\frac{k}{2}}} \left(\sinh(x\sqrt{\lambda}) \right)^{-k} \tag{4.62}$$

where $V(x)$ is $\mathcal{PT}$-symmetric if c_1, c_3, c_5 are purely imaginary and c_2, c_4, c_6 are real. Following ref.[526], the ansatz for the wave function is taken as

$$\psi(x) = f(x) \, exp \left(- \sum_{j=1}^{4} \frac{b_j}{\lambda^{\frac{j}{2}}} (\sinh(x\sqrt{\lambda}))^{-j} \right) \tag{4.63}$$

where $f(x)$ is some polynomial function of x. We focus on the following choices of $f(x)$:

$$(a) \quad f(x) = 1$$

$$(b) \quad f(x) = \frac{\left[\sinh(x\sqrt{\lambda}) \right]^{-1}}{\sqrt{\lambda}} + a_0$$

$$(c) \quad f(x) = \frac{\left[\sinh(x\sqrt{\lambda}) \right]^{-2}}{\lambda} + a_1 \frac{\left[\sinh(x\sqrt{\lambda}) \right]^{-1}}{\sqrt{\lambda}} + a_0$$

For complex potentials, a_0 is purely imaginary in (b), but in (c) a_1 is purely imaginary, but a_0 is real. Without going into the details of calculation (which are done in ref.[526] for constant mass case), which are quite straightforward, let us summarize our results.

Case 1: $f(x) = 1$
In this case the relation between the parameters c_i and b_i are found to be $c_1 = -3b_3 + 2b_1 b_2$, $c_2 = -6b_4 + 3b_1 b_3 + 2b_2^2$, $c_3 = 4b_1 b_4 + 6b_2 b_3$, $c_4 = 8b_2 b_4 + \frac{9}{2} b_3^2$, $c_5 = 12b_3 b_4$, $c_6 = 8b_4^2$ and energy is given by

$$E = b_2 - \frac{1}{2} b_1^2 \tag{4.64}$$

Without loss of generality, we can choose $c_6 = \frac{1}{2}$ which fixes the leading coefficient of

$V(x)$. It gives $b_4 = \pm\frac{1}{4}$. Taking the positive sign to ensure the normalizability of the wave function we obtain

$$\psi(x) = \exp\left(-\frac{b_1\left[\sinh(x\sqrt{\lambda})\right]^{-1}}{\sqrt{\lambda}} - \frac{b_2\left[\sinh(x\sqrt{\lambda})\right]^{-2}}{\lambda} - \frac{b_3\left[\sinh(x\sqrt{\lambda})\right]^{-3}}{\lambda\sqrt{\lambda}} - \frac{\left[\sinh(x\sqrt{\lambda})\right]^{-4}}{4\lambda^2}\right)$$

(4.65)

Now if b_1 and b_3 are purely imaginary then c_1, c_3, c_5 are also purely imaginary. In that case $V(x)$ in Eqn.(4.62) and $\psi(x)$ in Eqn.(4.63) are $\mathcal{PT}$-symmetric and E is real.

Case 2: $f(x) = \dfrac{\left[\sinh(x\sqrt{\lambda})\right]^{-1}}{\sqrt{\lambda}} + a_0$, where a_0 is purely imaginary.

In this case wave function is of the form

$$\psi(x) = \left(\frac{\left[\sinh(x\sqrt{\lambda})\right]^{-1}}{\sqrt{\lambda}} + a_0\right)$$
$$\exp\left(-\frac{b_1\left[\sinh(x\sqrt{\lambda})\right]^{-1}}{\sqrt{\lambda}} - \frac{b_2\left[\sinh(x\sqrt{\lambda})\right]^{-2}}{\lambda} - \frac{b_3\left[\sinh(x\sqrt{\lambda})\right]^{-3}}{\lambda\sqrt{\lambda}} - \frac{\left[\sinh(x\sqrt{\lambda})\right]^{-1}}{4\lambda^2}\right)$$

(4.66)

In this case the relation between the parameters are given by

$$c_1 = -6b_3 + 2b_1b_2 + a_0 \ , \ \ c_2 = -\frac{5}{2} + 3b_1b_3 + 2b_2^2 \ , \ \ c_3 = b_1 + 6b_2b_3c_4 = 2b_2 + \frac{9}{2}b_3^2 \ , \ \ c_5 = 3b_3 \ , \ \ c_6 = \frac{1}{2}$$

and a_0 satisfies the condition

$$a_0^3 - 3b_3a_0^2 + 2b_2a_0 - b_1 = 0 \tag{4.67}$$

The corresponding energy is given by

$$E = -\frac{1}{2}b_1^2 + 3b_2 - 3a_0b_3 + a_0^2 \tag{4.68}$$

We now consider following two special cases.

(a) $b_1 = b_3 = 0$ and $a_0^2 < 0$.

In this case c_1 is purely imaginary and $c_3 = c_5 = 0$. Moreover, $c_1 = a_0 = \pm i\sqrt{2b_2}$. So we get two different complex potentials corresponding to above two values of c_1 with same

real energy eigenvalues. The potential, energy values and the eigenfunctions are given by

$$
\begin{aligned}
V(x) &= \tfrac{1}{2}\frac{[\sinh(x\sqrt{\lambda})]^{-6}}{\lambda^3} + \frac{2b_2}{\lambda^2}[\sinh(x\sqrt{\lambda})]^{-4} + \frac{(2b_2^2-\frac{5}{2})}{\lambda}[\sinh(x\sqrt{\lambda})]^{-2} \pm \frac{i\sqrt{2b_2}}{\sqrt{\lambda}}\left[\sinh(x\sqrt{\lambda})\right]^{-1} \\
E &= b_2 > 0 \\
\psi(x) &= \left(\frac{[\sinh(x\sqrt{\lambda})]^{-1}}{\sqrt{\lambda}} \pm i\sqrt{2b_2}\right)\exp\left(-\frac{b_2}{\lambda}\left[\sinh(x\sqrt{\lambda})\right]^{-2} - \frac{1}{4\lambda^2}\left[\sinh(x\sqrt{\lambda})\right]^{-4}\right)
\end{aligned}
$$

$$(4.69)$$

It can be easily seen from the above equations that the potential is $\mathcal{PT}$-symmetric, while the wave function is odd under $\mathcal{PT}$-symmetry.

(b) $b_1 = 0, b_3 \neq 0$

Then from (4.67) we get

$$
a_0 = \frac{1}{2}(3b_3 \pm \sqrt{9b_3^2 - 8b_2}) \tag{4.70}
$$

So in order to make a_0 imaginary we must have $9b_3^2 - 8b_2 < 0$. or $b_3^2 = -|b_3|^2 \leq \frac{8}{9}b_2$.
In this case also there exist two different complex potentials corresponding to two values of b_3 with the same real energy eigenvalues $E = 3b_2 - 3a_0b_3 + a_0^2$.

Case 3: $f(x) = \frac{\left[\sinh(x\sqrt{\lambda})\right]^{-2}}{\lambda} + a_1\frac{\left[\sinh(x\sqrt{\lambda})\right]^{-1}}{\sqrt{\lambda}} + a_0$, where a_1 is imaginary and a_0 is real.
In this case the relation between the parameters is given by

$$
a_1 = 2b_3 \quad , \quad a_0 = \frac{1}{2}\left(2b_2 - b_3^2 \pm \sqrt{(2b_2 - 3b_3^2)^2 + 2}\right) \tag{4.71}
$$

The wave function, energy and the potential are of the form

$$
\begin{aligned}
\psi_\pm(x) = &\left[\frac{\left[\sinh(x\sqrt{\lambda})\right]^{-2}}{\lambda} + 2b_3\frac{\left[\sinh(x\sqrt{\lambda})\right]^{-1}}{\sqrt{\lambda}} + \tfrac{1}{2}(2b_2 - b_3^2 \pm \sqrt{(2b_2 - 3b_3^2)^2 + 2})\right] \\
&\exp\left(-2b_3(b_2-b_3^2)\frac{\left[\sinh(x\sqrt{\lambda})\right]^{-1}}{\sqrt{\lambda}} - b_2\frac{\left[\sinh(x\sqrt{\lambda})\right]^{-2}}{\lambda} - b_3\frac{\left[\sinh(x\sqrt{\lambda})\right]^{-3}}{\lambda\sqrt{\lambda}} - \frac{1}{4}\frac{\left[\sinh(x\sqrt{\lambda})\right]^{-4}}{\lambda^2}\right),
\end{aligned}
$$

$$(4.72a)$$

$$
E_\pm = -2b_3^2(b_2 - b_3^2)^2 + 3b_2 - b_3^2 \pm \sqrt{(2b_2 - 3b_3^2)^2 + 2}, \tag{4.72b}
$$

$$V(x) = \frac{1}{2\lambda^3}[\sinh(x\sqrt{\lambda})]^{-6} + \frac{3b_3}{\lambda^2\sqrt{\lambda}}[\sinh(x\sqrt{\lambda})]^{-5} + \frac{(2b_2 + \frac{9}{2})}{\lambda^2}[\sinh(x\sqrt{\lambda})]^{-4}$$
$$+ \frac{2b_3}{\lambda\sqrt{\lambda}}(4b_2 - b_3^2)[\sinh(x\sqrt{\lambda})]^{-3} + \frac{[2(b_2^2 + 3b_2b_3^2 - 3b_3^4) - \frac{7}{2}]}{\lambda}[\sinh(x\sqrt{\lambda})]^{-2} \quad \text{(4.72c)}$$
$$+ \frac{b_3(4b_2^2 - 4b_2b_3^2 - 7)}{\sqrt{\lambda}}[\sinh(x\sqrt{\lambda})]^{-1},$$

respectively. The result (4.71) to (4.72c) are valid both for real and purely imaginary b_i. When b_i are purely imaginary the potential and wave function are $\mathcal{PT}$-symmetric while for real b_i $\mathcal{PT}$-symmetry is broken. In particular when b_3 is purely imaginary we have a complex $\mathcal{PT}$-symmetric two parameter family of potentials corresponding to two values of a_0 with two distinct real eigenvalues.

4.3   Supersymmetric Shape invariance approach to generalized QNLO

4.3.1   Unbroken supersymmetry

We consider two first order operators of the form

$$A = P_x - iW(x), \quad A^\dagger = P_x + iW(x), \quad P_x = \frac{1}{\sqrt{m(x)}}\left(-i\frac{d}{dx}\right) \quad \text{(4.73)}$$

to factorize the non-linear oscillator Hamiltonians $H_\pm$ as

$$H_\pm = -\frac{1}{m(x)}\frac{d^2}{dx^2} + \left(\frac{m'}{2m^2}\right)\frac{d}{dx} + W^2 \pm \frac{W'}{\sqrt{m}} \quad \text{(4.74)}$$

For unbroken supersymmetry (SUSY), the ground state of H_- has zero energy, $E_0^{(-)} = 0$, which implies that the ground state wave function $\psi_0^{(-)}(x)$ given by $(A\psi_0^{(-)} = 0)$

$$\psi_0^{(-)}(x) = N_0 \, \exp\left[-\int^x \sqrt{m(y)}\, W(y)dy\right] \quad \text{(4.75)}$$

is normalizable. Hence, the superpotential $W(x)$ could be generated from the ground state solution of H_- i.e. $W(x) = -\frac{\psi_0^{(-)}{}'}{\sqrt{m}\psi_0^{(-)}}$. For the generalized nonlinear oscillator potential

appears in equation(4.42) the superpotential is given by

$$W = A\frac{\sqrt{\lambda}x}{\sqrt{1 + \lambda x^2}} + B\frac{1}{\sqrt{1 + \lambda x^2}} \tag{4.76}$$

Therefore the Hamiltonians $H_\pm$ given in (4.74) reduced to

$$
\begin{aligned}
H_- &= A^\dagger A \\
&= -(1 + \lambda x^2)\frac{d^2}{dx^2} - \lambda x\frac{d}{dx} + \frac{B^2 - A^2 - A\sqrt{\lambda}}{1 + \lambda x^2} + B(2A + \sqrt{\lambda})(\frac{\sqrt{\lambda}x}{1 + \lambda x^2}) + A^2 \\
H_+ &= AA^\dagger \\
&= -(1 + \lambda x^2)\frac{d^2}{dx^2} - \lambda x\frac{d}{dx} + \frac{B^2 - A^2 + A\sqrt{\lambda}}{1 + \lambda x^2} + B(2A - \sqrt{\lambda})(\frac{\sqrt{\lambda}x}{1 + \lambda x^2}) + A^2
\end{aligned}
\tag{4.77}
$$

The Hamiltonian H_- is the Hamiltonian of the generalized QNLO with the potential of the Schrödinger equation (4.42). Also these two Hamiltonians $H_\pm$ are related by

$$H_+(x; A, B) = H_-(x; A - \sqrt{\lambda}, B) + \sqrt{\lambda}(2A - \sqrt{\lambda}) \tag{4.78}$$

so that they satisfy shape invariance condition

$$H_+(x, a_0) = H_-(x, a_1) + R(a_0) \tag{4.79}$$

where $\{a_0\} = (A, B), \{a_1\} = (A - \sqrt{\lambda}, B)$ and $R(a_0) = \sqrt{\lambda}(2A - \sqrt{\lambda})$. The zero energy ground state $\psi_0(x, a_0)$ of the Hamiltonian H_- is found by solving $A\psi_0(x, a_0) = 0$ i.e.

$$H_-(x, a_0)\psi_0(x, a_0) = 0 \tag{4.80}$$

Now using (4.79) we can see that $\psi_0(x, a_1)$ is an eigenstate of H_+ with the energy $E_1 = R(a_0)$, because

$$
\begin{aligned}
H_+(x, a_0)\psi_0(x, a_1) &= H_-(x, a_1)\psi_0(x, a_1) + R(a_0)\psi_0(x, a_1) \\
&= R(a_0)\psi_0(x, a_1) , \quad [\text{using (4.80)}]
\end{aligned}
\tag{4.81}
$$

Next, using the intertwining relation $H_-(x, a_0)A^\dagger(x, a_0) = A^\dagger(x, a_0)H_+(x, a_0)$ and equation (4.79), we see that

$$H_-(x, a_0)A^\dagger(x, a_0)\psi_0(x, a_1) = A^\dagger(x, a_0)H_+(x, a_0)\psi_0(x, a_1) = A^\dagger[H_-(x, a_1) + R(a_0)]\psi_0(x, a_1)$$

and hence using (4.80) we arrive at

$$H_-(x, a_0)A^\dagger(x, a_0)\psi_0(x, a_1) = R(a_1)A^\dagger(x, a_0)\psi_0(x, a_1) \tag{4.82}$$

This indicates that $A^\dagger(x, a_0)\psi_0(x, a_1)$ is an eigenstate of H_- with an energy $E_1 = R(a_0)$. Now iterating this process we will find the sequence of energies for H_- as

$$E_n^{(-)} = \sum_{i=0}^{n-1} R(a_i) = n\sqrt{\lambda}(2A - n\sqrt{\lambda}) , \quad E_0^{(-)} = 0 \tag{4.83}$$

and corresponding eigenfunctions being

$$\psi_n(x, a_0) = A^\dagger(x, a_0)A^\dagger(x, a_1)...A^\dagger(x, a_{n-1})\psi_0(x, a_n) \tag{4.84}$$

where $a_i = f(a_{i-1}) = \underbrace{f(f(....(f(a_0)))}_{i \text{ times}} = \left(A - i\sqrt{\lambda}, B\right)$ and $R(a_i) = \sqrt{\lambda}\left[2\left(A - i\sqrt{\lambda}\right) - \sqrt{\lambda}\right]$.

In a similar way it can be shown that other potentials of table 4.1 are also shape invariant. For all these potentials the energy, wave functions and other parameters related to shape invariance property are given in the same table.

4.3.2   Broken supersymmetry

When supersymmetry is broken neither of the wave functions $\psi_0^{(\pm)}(x) \approx exp[\pm \int^x \sqrt{m(y)}W(y)dy]$ are normalizable and in this case all the energy values are degenerate i.e, H_+ and H_- have identical energy eigenvalues [36]

$$E_n^{(-)} = E_n^{(+)} \tag{4.85}$$

with ground state energies greater than zero. So far as we know, little attention has been paid till now to study problems involving broken SUSY in the case of PDMSE. Broken supersymmetric shape invariant systems in the case of constant mass Schrödinger equation has been discussed in ref.[38, 37]. Below we illustrate the two step procedure discussed in [94, ?] for obtaining the energy spectra of generalized QNLO when the SUSY is broken. For this, we consider the superpotential as

$$W(x, A, B) = A\sqrt{|\lambda|}\frac{x}{\sqrt{1 + \lambda x^2}} - \frac{B}{\sqrt{|\lambda|}}\frac{\sqrt{1 + \lambda x^2}}{x}, \quad 0 < x < \frac{1}{\sqrt{|\lambda|}}, \quad \lambda < 0. \tag{4.86}$$

Then the supersymmetric partner potentials of the Hamiltonians $H_\pm$ are obtained using (??) as

$$
\begin{aligned}
V_-(x, A, B) &= \frac{A(A - \sqrt{|\lambda|})}{1 + \lambda x^2} - \frac{B(B - \sqrt{|\lambda|})}{\lambda x^2} - (A + B)^2 \\
V_+(x, A, B) &= \frac{A(A + \sqrt{|\lambda|})}{1 + \lambda x^2} - \frac{B(B + \sqrt{|\lambda|})}{\lambda x^2} - (A + B)^2
\end{aligned}
\tag{4.87}
$$

The ground state wave function is obtained from (4.75) as

$$
\psi_0^{(-)} \sim x^{\frac{B}{\sqrt{|\lambda|}}} \left(1 + \lambda x^2\right)^{\frac{A}{2\sqrt{|\lambda|}}}
\tag{4.88}
$$

For $A > 0$, $B > 0$ the ground state wave function $\psi_0^{(-)}$ is normalizable which means the SUSY is unbroken. But for $A > 0$, $B < 0$ and $A < 0$, $B > 0$, neither of $\psi_0^{(\pm)}$ are normalizable. Hence SUSY is broken in both cases.

For $A > 0$, $B < 0$, the eigenstates of $V_\pm(x, A, B)$ are related by

$$
\begin{aligned}
\psi_n^{(+)}(x, a_0) &= A(x, a_0)\psi_n^{(-)}(x, a_0) \\
\psi_n^{(-)}(x, a_0) &= A^\dagger(x, a_0)\psi_n^{(+)}(x, a_0), \\
E_n^{(-)}(a_0) &= E_n^{(+)}(a_0)
\end{aligned}
\tag{4.89}
$$

Now we can show that the potentials in equation (4.87) are shape invariant by two different relations between the parameters.

Step 1

The potentials of equation (4.87) are shape invariant if we change $A \to A + \sqrt{|\lambda|}$ and $B \to B + \sqrt{|\lambda|}$. The shape invariant condition is given by

$$
V_+(x, A, B) = V_- \left(x, A + \sqrt{|\lambda|}, B + \sqrt{|\lambda|}\right) + \left(A + B + 2\sqrt{|\lambda|}\right)^2 - (A + B)^2 \tag{4.90}
$$

Now for $B < -\frac{1}{\sqrt{|\lambda|}}$ it is seen that the superpotential (4.86) resulting from change of parameters as above falls in the class of broken SUSY problem for which $E_0^{(-)} \neq 0$. Though the potentials of equation (4.87) are shape invariant but we are unable to determine the spectra for these potentials because of the absence of zero energy ground state.

Another way of parameterizations $A \to A + \sqrt{|\lambda|}$ and $B \to -B$ gives us

$$
V_+(x, A, B) = V_- \left(x, A + \sqrt{|\lambda|}, -B\right) + \left(A - B + \sqrt{|\lambda|}\right)^2 - (A + B)^2 \tag{4.91}
$$

which shows that V_- and V_+ are shape invariant. This change of parameters ($A \to A + \sqrt{|\lambda|}$ and $B \to -B$) leads to a system with unbroken SUSY since the parameter B changes sign. Hence the ground state energy of the potential $V_-(x, A + \sqrt{|\lambda|}, -B)$ is zero. From the relation (4.91) we observe that $V_+(x, A, B)$ and $V_-\left(A + \sqrt{|\lambda|}, -B\right)$ differ only by a constant, hence we have

$$\begin{aligned} \psi_+(x, A, B) &= \psi_-(x, A + \sqrt{|\lambda|}, -B) \\ E_n^{(+)}(A, B) &= E_n^{(-)}(x, A + \sqrt{|\lambda|}, -B) + \left(A - B + \sqrt{|\lambda|}\right)^2 - (A + B)^2 \end{aligned} \tag{4.92}$$

Thus, if we can evaluate the spectrum and energy eigenfunctions of unbroken SUSY $H_-(x, A + \sqrt{|\lambda|}, -B)$, then we can determine the spectrum and eigenfunctions $H_+(x, A, B)$ with broken SUSY. In the 2nd step we will do this.

Step 2

With the help of shape invariant formalism in case of unbroken SUSY discussed earlier, we obtain spectrum and eigenfunctions for $V_-(x, A + \sqrt{|\lambda|}, -B)$ as

$$\begin{aligned} E_n^{(-)}(A + \sqrt{|\lambda|}, -B) &= \left(A - B + \sqrt{|\lambda|} + 2n\sqrt{|\lambda|}\right)^2 - \left(A - B + \sqrt{|\lambda|}\right)^2 \\ \psi_n^{(-)}(x, A + \sqrt{|\lambda|}, -B) &\propto x^{\frac{B}{\sqrt{|\lambda|}}} (1 + \lambda x^2)^{\frac{A}{2\sqrt{|\lambda|}}} P_n^{\left(\frac{B}{\sqrt{|\lambda|}} - \frac{1}{2}, \frac{A}{\sqrt{|\lambda|}} - \frac{1}{2}\right)}(1 + 2\lambda x^2) \end{aligned} \tag{4.93}$$

Now using (4.93), (4.92) and (4.89) we obtain spectrum and eigenfunctions for $V^-(x, A, B)$ with broken SUSY as

$$\begin{aligned} E_n^{(-)}(A, B) &= \left(A - B + \sqrt{|\lambda|} + 2n\sqrt{|\lambda|}\right)^2 - (A + B)^2 \\ \psi_n^{(-)}(x, A, B) &\propto x^{\frac{1-B}{\sqrt{|\lambda|}}} (1 + \lambda x^2)^{\frac{A}{2\sqrt{|\lambda|}}} P_n^{\left(\frac{1}{2} - \frac{B}{\sqrt{|\lambda|}}, \frac{A}{\sqrt{|\lambda|}} - \frac{1}{2}\right)}(1 + 2\lambda x^2) \end{aligned} \tag{4.94}$$

Similar approach can be applied in case of $A < 0$ and $B > 0$. In this case we change (A, B) into $(-A, B + \sqrt{|\lambda|})$ and the shape invariance condition is

$$V_+(x, A, B) = V_-\left(x, -A, B + \sqrt{|\lambda|}\right) + \left(B - A + \sqrt{|\lambda|}\right)^2 - (A + B)^2 \tag{4.95}$$

And

$$\begin{aligned} E_n^{(-)}(A, B) &= \left[B - A + \sqrt{|\lambda|} + 2n\sqrt{|\lambda|}\right]^2 - (A + B)^2 \\ \psi_n^{(-)}(x, A, B) &\propto x^{\frac{1-A}{\sqrt{|\lambda|}}} (1 + \lambda x^2)^{\frac{B}{2\sqrt{|\lambda|}}} P_n^{\left(\frac{B}{\sqrt{|\lambda|}} - \frac{1}{2}, \frac{A}{\sqrt{|\lambda|}} - \frac{1}{2}\right)}(1 + 2\lambda x^2) \end{aligned} \tag{4.96}$$

4.4   Relation between Λ-deformed Hermite polynomial and Jacobi polynomial

Here we shall obtain a relationship between the Λ-deformed Hermite polynomials [411] and Jacobi polynomials. We recall that the solutions of the Hamiltonian for nonlinear oscillator (4.5) is given by the equations (4.8) and (4.9) in terms of Λ-deformed Hermite polynomial $\mathcal{H}_m(y, \Lambda)$ whose Rodrigues formula and generating function are given in (4.100). On the other hand, putting $B = 0$ and $A = \frac{\alpha}{\sqrt{\lambda}}$ in the solution (4.50) of Eqn.(4.42), the eigenfunctions of the QNLO (equation (4.7)) can be written in terms of Jacobi polynomial as

$$\psi_n(y) = N_n(1 + \Lambda y^2)^{-\frac{1}{2\Lambda}} P_n^{(-\frac{1}{2} - \frac{1}{\Lambda}, -\frac{1}{2} - \frac{1}{\Lambda})}(iy\sqrt{\Lambda}) \ , \quad n = 0, 1, 2 \cdots < \frac{1}{\Lambda} \quad (\Lambda > 0) \ \ (4.97)$$

For $\Lambda < 0$, putting $B = 0, A = \frac{\alpha}{\sqrt{|\lambda|}}$ in the wavefunction of the 5th entry of Table 4.1 and using (4.6) we obtain

$$\psi_n(y) = N_n(1 + \Lambda y^2)^{-\frac{1}{2\Lambda}} P_n^{(-\frac{1}{2} - \frac{1}{\Lambda}, -\frac{1}{2} - \frac{1}{\Lambda})}(y\sqrt{|\Lambda|}) \ , \quad n = 0, 1, 2 \cdots \quad (\Lambda < 0) \quad (4.98)$$

Comparing Eqns.(4.9) and (4.98) and also Eqns.(4.8) and (4.97), it is possible to derive a relation between Λ-deformed Hermite polynomial $\mathcal{H}_n(y, \Lambda)$ and Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$ as

$$P_n^{(-\frac{1}{2} - \frac{1}{\Lambda}, -\frac{1}{2} - \frac{1}{\Lambda})}(iy\sqrt{\Lambda}) = \frac{1}{n!} \left(\frac{1}{2i\sqrt{\Lambda}} \right)^n \mathcal{H}_n(y, \Lambda), \qquad \forall \ \Lambda \quad\quad (4.99)$$

The Rodrigues formula and the generating function for the Λ-deformed Hermite polynomial $\mathcal{H}_n(y, \Lambda)$ were given by [411]

$$
\begin{aligned}
\mathcal{H}_n(y, \Lambda) &= (-1)^n z_y^{\frac{1}{\Lambda} + \frac{1}{2}} \frac{d^n}{dy^n} \left[z_y^n z_y^{-(\frac{1}{\Lambda} + \frac{1}{2})} \right], \quad z_y = 1 + \Lambda y^2 \\
\mathcal{F}(t, y, \Lambda) &= (1 + \Lambda(2ty - t^2))^{\frac{1}{\Lambda}}
\end{aligned}
\quad (4.100)
$$

It was shown [411] that the polynomials obtained from the generating function $\mathcal{F}(t, y, \Lambda)$ with those obtained from Rodrigues formula are essentially the same and only differ in the values of the global multiplicative coefficients. We have observed that if the generating

function $\mathcal{F}(t, y, \Lambda)$ is taken as

$$(1 + \Lambda(2ty - t^2))^{\frac{1}{\Lambda}} = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\left(-\frac{1}{\Lambda}\right)_n}{\left(\frac{1}{2} - \frac{1}{\Lambda}\right)_n} \mathcal{H}_n(y, \Lambda) \frac{t^n}{n!} \tag{4.101}$$

where $(a)_n$ represents Pöchhammer symbol given by $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ then the polynomials obtained from the above relation are exactly same with those obtained from Rodrigues formula given in Eqn.(4.100). Correspondingly the recursion relations are obtained as

$$(\Lambda(2n+1)-2) \left[2(1-n\Lambda)y\mathcal{H}_n(y,\Lambda)+(\Lambda(2n-1)-2)n\mathcal{H}_{n-1}(y,\Lambda)\right] = (n\Lambda-2)\mathcal{H}_{n+1}(y,\Lambda) \tag{4.102}$$

and

$$(\Lambda(n-2)-2) \left[2(\Lambda(2n-1)-2)n\mathcal{H}_n(y,\Lambda) - (\Lambda(n-1)-2)\mathcal{H}'_n(y,\Lambda)\right]$$
$$= n\Lambda(\Lambda(2n-1)-2) \left[2(\Lambda(n-2)-2)y\mathcal{H}'_{n-1}(y,\Lambda) - (n-1)(\Lambda(2n-3)-2)\mathcal{H}'_{n-2}(y,\Lambda)\right] \tag{4.103}$$

where 'prime' denotes differentiation with respect to y. For $\Lambda \to 0$ Eqns.(4.102) and (4.103) give the recursion relations for Hermite polynomial.

4.5   Summary

To summarize, we point out the following main results :

▶ We have constructed the coherent states for nonlinear oscillator via Gazeau-Klauder formalism. These coherent states are shown to satisfy the requirements of continuity of labeling, resolution of unity, temporal stability and action identity. The plots of the weighting distribution for these coherent states are almost Gaussian in nature. The Mandel parameter Q is sub-Poissonian which indicates that the coherent states (4.11) exhibit squeezing for all values of μ. The fractional revivals of the coherent states are evident from the figures 4.3(b) and 4.3(c) depicting squared modulus of the autocorrelation function. This is in contrast to the results obtained in ref [494] where the wave packet revival in an infinite well for the Schrödinger equation with position dependent mass was studied. In ref.[494], it was found that though full revival takes place, there is no fractional revival in the usual sense. Instead, a very narrow wave packet is located near one wall of the well, when the mass is higher. In the present paper the spectral compositions and the intensities of the fractional revivals are determined by phase analysis.

▶ We have studied various exactly solvable Hermitian, non-Hermitian $\mathcal{PT}$-symmetric and quasi-exactly solvable generalizations of the quantum nonlinear oscillator with the mass function $\left(\frac{1}{1+\lambda x^2}\right)$. The eigenfunctions of the quantum nonlinear oscillator are previously obtained [411] in terms of Λ-deformed Hermite polynomials. These Λ-deformed Hermite polynomials are actually related to classical Jacobi Polynomials as has been shown in equation (4.99). In the case of unbroken supersymmetry, it has been shown that all the generalizations of the quantum nonlinear oscillator listed in tables 4.1 and table 4.2 are shape invariant. In this context, the case of broken supersymmetry has also been discussed.

Chapter 5

Future Directions

Here we point out some future issues related to the work presented in chapters 2, 3 and 4.

▶ In section 2.1, the modified factorization approach based on unbroken supersymmetry, has been extended to quantum systems with position dependent mass. It would be interesting to formulate such algorithm when supersymmetry is broken.

▶ Some exactly solvable position-dependent mass Hamiltonians and quasi-Hermitian Hamiltonians have been obtained in section 2.2 and section 3.2 respectively. The associated bound state wave functions are given in terms of X_1 Laguerre or Jacobi type EOPs. To look for solvable position dependent mass and quasi-Hermitian Hamiltonians associated with exceptional orthogonal polynomials of higher co-dimension as well as multi-indexed ones will be worthwhile.

▶ Formulation of higher order intertwining method for spectral modification of a position-dependent mass Hamiltonian, by considering an n-th ($n \geq 3$) order differential operator as an intertwiner, is worth investigating in future. Also one can study the other possibilities of spectral modifications apart form the ones reported in section 2.3.

▶ In ref.[359, 361], the authors have shown that it is possible to find solitons (with real eigenvalues) of the nonlinear Schrödinger equation $\frac{d^2\psi}{dx^2} + [4\cos^2 x + 4iV_0 \sin 2x]\psi(x) + |\psi|^2\psi(x) = E\psi(x)$ above the $\mathcal{PT}$-threshold $V_0^{th} = 0.5$. This happens because the corresponding band structure remains real for some values of bloch momentum ν even above the $\mathcal{PT}$-threshold. Stability analysis revealed that these solitons are unstable. However, we have shown, in section 3.1.2, that no part of the band structure remains real beyond the second critical point $V_0^c \sim 0.888437$. Therefore an important direction of future work would be to study the existence as well as the stability of solitons (if any) beyond the second critical point V_0^c. Also it will be interesting to study other related properties of a $\mathcal{PT}$-optical lattice (e.g. power oscillation, double refraction, non-reciprocity) near this second critical point.

▶ Generalized Swanson Hamiltonian is shown, in section 3.3, to be non-isospectral to the harmonic oscillator despite of $[\tilde{a}, \tilde{a}^\dagger] = $ Constant. The reason behind this

anomaly is not clear till now. This requires further studies.

▶ The study of the revival dynamics of the quantum nonlinear oscillator using Renyi and Shanon information entropies and compare the results with the auto-correlation based analysis already done in section 4.1.1, will be worth pursuing.

▶ The classical analogs of the Hamiltonians with generalized quantum nonlinear oscillator potentials presented in tables 4.1 and 4.2 is another area of future investigation.

✳ ✳ ✳

Bibliography

[1] E. Schrödinger, *Proc. R. Irish Acad.* A46 (1940) 9; A46 (1940) 183.

[2] M. G. Darboux, *C. R. Acad. Sci. Paris* 94 (1882) 1456.

[3] M. Crum, *Quart. J. Math.* 6 (1955) 121.

[4] L. Infeld and T. D. Hull, *Rev. Mod. Phys.* 23 (1951) 21.

[5] B. Mielnikh, *J. Math. Phys.*, 25 (1984) 3387.

[6] M.M. Nieto, *Phys. Lett.* B 145 (1984) 208.

[7] C.V. Sukumar *J.Phys. A* 18 (1985) 2937.

[8] I.M. Gelfand and B.M. Levitan, *Am. Math. Soc. Transl.* 1 (1951) 253.

[9] P.B. Abraham and H.E. Moses, *Phys. Rev.* A22 (1980) 1333.

[10] D.L. Pursey, Phys. Rev. D33 (1985) 1048.

[11] G.P. Flessas, *Phys. Lett.* A72 (1979) 289.

[12] M. Razavy, *Am. J. Phys.* 48 (1980) 285.

[13] A. de Souza Dutra, *Phys. Rev.* A47 (1993) R2437.

[14] G. Junker and P. Roy, *Ann. Phys.* 270 (1998) 155.

[15] W. Pauli, *Zeitschrift fur Fizik* 36 (1926) 336.

[16] V. Fock, *Zeitschrift fur Fizik* 98 (1935) 145.

[17] A. V. Turbiner, *Commun. Math. Phys.* 118 (1988) 467.

[18] M. A. Shifman, *Int. J. Mod. Phys.* A126 (1989) 2897.

[19] A.G. Ushveridze, *Quasi-Exactly Solvable Models in Quantum Mechanics* (IOP, Bristol), 1993.

[20] P. Ramond, *Phys. Rev.* D3 (1971) 2415.

[21] A. Neveu and J. Schwarz, *Nucl. Phys.* B31, (1971) 86.

[22] Y. A. Gelfand and E. P. Likhtman, *JETP Lett.* 13 (1971) 323.

[23] D. Volkov and V. Akulov, *Phys. Lett.* B46 (1973) 109.

[24] J. Wess and B. Zumino, *Nucl. Phys.* B70 (1974) 39.

[25] S. Coleman and J. Mandula, *Phys. Rev.* 159 (1967) 5.

[26] E. Witten, *Nucl. Phys.* B185 (1981) 513.

[27] E. Witten, *Nucl. Phys.* B202 (1982) 253.

[28] M. de Crumbrugghe and V. Rittenberg, *Ann. Phys.* 151, (1983) 99.

[29] F. Cooper and B. Freedman, *Ann. Phys.* 146 (1983) 262.

[30] V.A. Kostelecky, M.M. Nieto and D.R. Truax, *Phys. Rev.* D32 (1985) 2627.

[31] M.F. Sohnius, *Phys. Rep.* 128 (1985) 39.

[32] R.W. Haymaker and A.R.P. Rau, *Am.J.Phys* 54 (1984) 928.

[33] A. R. P. Rau, *Phys. Rev. Lett.* 56, (1986) 95.

[34] C.V. Sukumar, J. Phys. A18 (1985) 2917.

[35] A.A. Andrianov, N.V. Borisov and M.V. Ioffe, *Phys. Lett.* A 105 (1984) 19.

[36] F.Cooper, A.Khare and U.Sukhatme, *Supersymmetry in Quantum Mechanics*, (World Scientific, 2000).

[37] B. Bagchi, *Supersymmetry in Quantum and Classical Mechanics* (Chapman and Hall /CRC Monographs and Surveys in Pure and Applied mathematics (2000)).

[38] G. Junker,*Supersymmetric methods in quantum mechanics and statistical physics*, Springer, 1996.

[39] A. Gangopadhyay, J.V. Mallow and C. Rasinariu, Supersymmetric quantum mechanics, (World SCintific, 2011).

[40] F. Cooper, A Khare and U. Sukhatme, *Phys. Rep.* 251 (1995) 268.

[41] R.L. Rodrigues, The quantum mechanics SUSY algebra: An introductory review, arXiv: 0205017.

[42] H. Nicolai, *J.Phys.* A9 (1976) 1497.

[43] C.V. Sukumar, *J. Phys.* A19 (1986) 2297.

[44] A. Andrianov, N. Borisov, M. Eides and M. Ioffe, *Phys. Lett.* A109 (1985) 143.

[45] C.V. Sukumar, *J. Phys.* A20 (1987) 2461.

[46] C. V. Sukumar, *J. Phys.* A18 (1985) L57.

[47] G. Dunne and J. Feinberg, *Phys. Rev.* D 57 (1998) 1271.

[48] F. Correa, V. Jakubsky, L.M. Nieto and M. Plyushchay, *Phys. Rev. Lett.* 101 (2008) 030403.

[49] F. Correa, V. Jakubsky, and M. Plyushchay *J.Phys. A* 41 (2008) 485303.

[50] A.A. Andrianov, M.V. Ioffe, F. Cannata, and J.P. Dedonder, *Int. J. Modern Phys. A10* (1995) 2683.

[51] A.A. Andrianov and A.V. Sokolov, *Nucl. Phys.* B660 (2003) 25.

[52] A.A. Andrianov, M.V. Iofe and V.P. Spiridonov, *Phys. Lett.* A174 (1993) 273.

[53] A.A. Andrianov, M.V. Ioffe, and D.N. Nishnianidze, *Phys. Lett.* A201 (1995) 103.

[54] F. Correa, L.M. Nieto and M. Plyushchay, *Phys. Lett. B* 644 (2007) 94.

[55] M.Plyushchay, *J.Phys. A* 37 (2004) 10375.

[56] S.M. Klishevich and M.S. Plyushchay, *Nucl. Phys.* B606 (2001) 583.

[57] H. Aoyama, M. Sato and T. Tanaka, *Nucl. Phys.* B619 (2001) 105.

[58] H. Aoyama, M. Sato and T. Tanaka, *Phys. Lett.* B503 (2001) 423.

[59] D.J. Fernandez and N.F. Garcia, *AIP Conf. Proc.* 744 (2005) 236.

[60] D.J. Fernandez, *AIP Conf.Proc.* 1287 (2010) 3.

[61] A.A. Andrianov et al., *Theor. Math. Phys.* 61 (1985) 965.

[62] A.A. Andrianov et al., *Phys. Lett.* A105 (1984) 19.

[63] A.A. Andrianov, M.V. Ioffe, D.N. Nishnianidze, *Theor. Math. Phys.* 104 (1995) 1129.

[64] M.V. Ioffe, J.Mateos Guilarte and P.A. Valinevich , *Ann. Phys.* 321 (2006) 2552.

[65] S. Kuru, A. Tegmen and A. Vercin, *J. Math. Phys.* 42 (2001) 3344.

[66] M.V. Ioffe, *J. Phys.* A37 (204) 10363.

[67] A. A. Anderianov and M.V. Ioffe, *Phys. Lett.* B205 (1986) 507.

[68] F. Cannata et al., *J. Phys. A* 37 (2004) 10339.

[69] M.V. Ioffe and A.I. Neelov, *J. Phys* A36 (2003) 2493.

[70] A. Khare, *J. Math. Phys.* 34 (1993) 1277.

[71] S. Durand and L. Vinet, *J. Phys.* A23 (1990) 3661.

[72] Y.R. Huang and W.C. Su, *J. Phys.* A43 (2010) 115302.

[73] A. Khare, A.K. Mishra and G. Rajasekara, *Int. J. Mod. Phys.* A8 (1993) 1245.

[74] S. Durand, *Phys. Lett.* B312 (1993) 115.

[75] C. Quesne, *Mod. Phys. Lett.* A18 (2003) 515.

[76] L. Gendenshtein, *JETP Lett.* 38 (1983) 356.

[77] R. Dutt, A. Khare and U.P. Sukhatme, *Phys. Lett.* B181 (1986) 295.

[78] R. Dutt, A. Khare and U.P. Sukhatme, *Am. J. Phys.* 56 (1988) 163.

[79] F. Cooper, J.N. Ginocchio, and A. Khare, *Phys. Rev.* D36 (1987) 2458.

[80] J.W. Dabrowska, A. Khare and U. Sukhatme, *J. Phys.* A21 (1988) L195.

[81] J.N. Ginocchio, *Ann. Phys.* 152 (1984) 203.

[82] G. A. Natanzon, *Theor. Mat. Fiz.* 38 (1979) 146.

[83] G. Levai, *J.Phys.* A 22 (1989) 689.

[84] A. Khare, U.P. Sukhatme, *J. Phys. A* 26 (1993) L901.

[85] D. Barclay et al., *Phys. Rev.* A48 (1993) 2786.

[86] U.P. Sukhatme, C. Rasinariu, A. Khare, *Phys. Lett. A* 234 (1997) 401.

[87] M.J. Englefield and C. Quesne, *J.Phys.* A24 (1987)3557.

[88] J. Wu and Y. Alhassid, *J. Math. Phys.* 31 (1990) 557.

[89] A.B. Balantekin, *Phys. Rev.* D58 (1998) 013001.

[90] S. Chaturvedi et al., *Phys. Lett.* A248 (1998) 109.

[91] A. Gangopadhyaaya, C. Rasinariu and U. Sukhatme, *Phys. Rev.* A60 (1999) 3482.

[92] A.B. Balantekin, M.A. Ribeiro and A.N.F. Aleixo, *J.Phys.* A32 (1999) 2785.

[93] W.C. Su, *J.Phys. A* 41 (2008) 255307; *J. Phys. A* 42 (2009) 435301.

[94] R. Dutt et al., *Phys. Lett.* A174 (1993) 363.

[95] A. Gangopadhyaya, J.V. Mallow anmd U.P. Sukhatme, *Phys. Lett.* A283 (2001) 279.

[96] J. Bougie, A. Gangopadhyaya and J.V. Mallow, Phys.Rev.Lett. 105(2010)210402.

[97] J. Bougie, A. Gangopadhyaya and J.V. Mallow, *J. Phys.* A44 (2011) 275307.

[98] A. Ramos, *J. Phys. A* 44 (2011) 342001.

[99] G. Levai, *J. Phys. A* 25 (1992) L521.

[100] A. Gangopadhyaya, J.V. Mallow, C. Rasinariu and U.P. Sukhatme, *Theo. Math. Phys.* 118 (1999) 285.

[101] G. Levai, *Czech J. Phys.* 54 (2004) 1121.

[102] V.M. Tkachuk and T.V. Fityo, *J. Phys.* A 34 (2001) 8673.

[103] V.M. Tkachuk, *J. Phys. A* 32 (1999) 1291.

[104] C. Quesne and V.M. Tkachuk, *J. Phys. A* 36 (2003) 10373.

[105] V.M. Tkachuk, *J. Phys.* A34 (2001) 6339.

[106] V.M. Tkachuk and O. Vozhyak, *Phys. Lett.* A301 (2002) 177.

[107] A. Khare and U. Sukhatme, *J. Math. Phys.* 40 (1999) 5473.

[108] G. Levai and P. Roy, *Phys. Lett.* A264 (1999) 117.

[109] N. Nag, R. Roychoudhury and Y.P. Varshini, *Phys. Rev.* A49 (1994) 5098.

[110] R.D. Amado, F. Cannata and J. Dedonder, *Phys. Rev.* A38 (1988) 3797.

[111] D. Baye, *Phys. Rev. Lett.* 58 (1987) 2738.

[112] D. Baye, *J. Phys.* A20 (1987) 5529.

[113] A. Khare and U. Sukhatme, *J. Phys.* A22 (1989) 2847.

[114] J.M. Sparenberg and D. Baye, *Phys. Rev. Lett.* 79 (1997) 3802.

[115] J. Pappademos, U. Sukhatme and A. Pagnamenta, *Phys. Rev.* A48 (1999) 3535.

[116] A.A. Stahlhofen, *Phys. Rev.* A51 (1995) 934.

[117] A. Khare and U.P. Sukhatme, *J. Phys.* A21 (1988) L501

[118] A.A. Andrianov, F. Cannata, J.P. Dedonder, M.V. Ioffe, *Int.J.Mod.Phys.* A10 (1995) 2683.

[119] A. Khare, and U. Sukhatme, *J. Phys. A* 37 (2004) 10037.

[120] M.V. Ioffe, J.M. Guitarte, P.A. Valinevich, *Nucl. Phys.* B790 (2008) 414.

[121] D.J. Fernandez, J. Negro, L.M. Nieto *Phys Lett A* 275 (2000) 338.

[122] U. Sukhatme and A Khare, *Phys. Atom. Nucl.* 65 (2002) 1122.

[123] D.J. Fernandez et al, *J. Phys.* A 35 (2002) 4279.

[124] T. Imbo and U. Sukhatme, *Phys. Rev. Lett.* 54 (1985) 2184.

[125] R. Roychoudhury and Y.P. Varshni, *Phys. Rev.* A37 (1988) 2309.

[126] F. Cooper and P. Roy, *Phys. Lett.* A143 (1990) 202.

[127] E. Gozzi, M. Reuter and W.D. Thacker, *Phys. Lett.* A183 (1993) 29.

[128] F. Cooper, J. Dawson and H. Shepard, *Phys. Lett.* A187 (1994) 140.

[129] E.D. Filho and R.M. Ricotta, *Phys. Lett.* A269 (2000) 269.

[130] A. Khare, *Phys. Lett.* B161 (1988) 131.

[131] R. Dutt, A. Khare and U. Sukhatme, *Am. J. Phys.* 59 (1991) 723.

[132] M. Hruska, W. Keung and U. Sukhatme, *Phys. Rev.* A 55 (1997) 3345.

[133] G. Chen, Z. Chen and P. Xuan, *Phys. Lett.* A 352 (2006) 317.

[134] D.T. Barclay and C.J. Maxwell, it Phys. Lett. A157 (1991) 357.

[135] R. Adhikary et al., *Phys. Rev.* A38 (1986) 1679.

[136] R. Dutt, A. Khare and Y.V. Varshni, *Phys. Lett.* A123 (1987) 375.

[137] Y.P. Varshni, *J.Phys. A* 25 (1992) 5761.

[138] S. Fricke et al., *Phys. Rev.* A37 (1988) 1686.

[139] U. Sukhatme and a. Pagnamenta, *Phys. Lett.* A 151 (1990) 7.

[140] R. Dutt et al., *Phys. Rev.* A48 (1999) 1845.

[141] A. Inomata and G. Junker, *Phys. Rev.* A50 (1994) 3638.

[142] W. Keung, E. Kovacs and U.P. Sukhatme, *Phys. Rev. Lett.* 60 (1988) 41.

[143] A. Gangopadhaya, P.K. Panigrahi and U.P. Sukhatme, *Phys. Rev.* A47 (1993) 2720.

[144] R.J. Hughes, V.A. Kostelecky and M.M. Nieto, *Phys. Rev.* D34 (1986) 1100.

[145] Y. Nogami and F.M. Toyama, *Phys. Rev.* A 47 (1993) 1708.

[146] F. Cooper et al, *Ann. Phys.* 187 (1988) 1.

[147] C. V. Sukumar, *J. Phys.* A18 (1985) L697.

[148] E. DHoker and L.Vinet *Phys. Lett.* B137 (1984) 72.

[149] A. Khare and J. Maharana, *Nucl. Phys.* B244 (1984) 409.

[150] H. Ui, *Prog. Theor. Phys.* 72 (1984) 813.

[151] R.L. Hall and O.Yesiltas, *Int. J. Mod. Phys.* E 19 (2010) 1923.

[152] T. Jana and P. Roy, *Phys. Lett.* A 361 (2007) 55.

[153] P.T. Leung et al, *J. Math. Phys.* 42 (2001) 4802.

[154] T.K. Jana and P. Roy, *J. Phys.* A40 (2007) 5865.

[155] D.Z. Freedman and P.F. Mende, *Nucl. Phys.* B 344 (1990) 317.

[156] M.V. Ioffe and A.I. Neelov, *J. Phys. A* 35 (2002) 7613.

[157] P.K. Ghosh, *J.Phys.* A45 (2012) 183001.

[158] P.K. Ghosh, A. Khare and M. Shivakumar, *Phys. Rev.* A 58 (1998) 821.

[159] C.V. Sukumar and P. Guha, *J.Math. Phys.* 36 (1995) 321.

[160] A.K. Grant and J.L. Rosen, *J. Math. Phys.* 35 (1994) 2142.

[161] E. Harikumar, V. Sunil Kumar and A. Khare, *Phys. Lett.* B589 (2004) 155.

[162] P.K. Ghosh, *Eur. J. Phys.* C 42 (2005) 355.

[163] J. Ben Geloun and F.G. Scholtz, *J. Phys.* A42 (2009) 165206.

[164] S. Post, L. Vinet and A. Zhedanov, *J. Phys.* A 44 (2011) 435301.

[165] T. Tanaka, *Int. J. Mod. Phys.* A27 (2012) 1250102.

[166] D.J. Fernandez, *Lett. Math. Phys.* 8 (1984) 337.

[167] M.Robnik, *J. Phys.* A30 (1997) 1287.

[168] A. Jevicki and J.P. Rodrigues, *Phys. Lett.* B146 (1984) 55.

[169] J. Casahorran, *Physica* A217 (1995) 429.

[170] A. Das and S.A. Pernice, *Nucl.Phys.* B561 (1999) 357.

[171] P.K. Panigrahi and U.P. Sukhatme, *Phys. Lett.* A 178 (1993) 1993.

[172] M.S. Berger and N.S. Ussembayev, *Phys. Rev.* A 82 (2010) 022121.

[173] M.S. Berger and N.S. Ussembayev, *J. Phys.* A43 (2010) 385309.

[174] D. Datta and P. Roy, *Phys. Rev.* A 83 (2011) 054102.

[175] C. Quesne, *J. Phys.* A41 (2008) 392001.

[176] C. Quesne, *SIGMA* 5 (2009) 084.

[177] Y. Grandati, *Ann. Phys.* 326 (2011) 2074.

[178] Y. Grandati, *J. Math. Phys.* 52 (2011) 103505.

[179] S. Odake and R. Sasaki, *Phys. Lett.* B679 (2009) 414.

[180] S. Odake and R. Sasaki, *J. Phys. A* 43 (2010) 335201.

[181] C. Quesne, *Mod. Phys. Lett.* A26 (2011) 1843.

[182] B. Bagchi and C. Quesne, *J. Phys. A* 43 (2010) 305301.

[183] S. Odake and R. Sasaki, *J. Math. Phys.* 51 (2010) 053513.

[184] C. Quesne, *Int. J. Mod. Phys. A* 26 (2011) 5337.

[185] D. Gomez-Ullate, N. Kamran and R. Milson, *J. Math. Anal. Appl.* 359 (2009) 352.

[186] D. Gomez-Ullate, N. Kamran and R. Milson, *J. Approx. Theory* 162 (2010) 987.

[187] B. Bagchi, C. Quesne and R. Roychoudhury, *Pramana: J. Phys.* 73 (2009) 337.

[188] C-L. Ho, S. Odake and R. Sasaki, *SIGMA* 7 (2011) 107.

[189] C-L. Ho and R. Sasaki, *ISRN Mathematical Physics*, 2012 (2012) 920475.

[190] R. Sasaki, S. Tsujimoto and A. Zhedanov, *J. Phys.* A43 (2010) 315204.

[191] C.L. Ho, *J. Math. Phys.* 52 (2011) 122107.

[192] C.L. Ho, *Prog. Theor. Phys.* 126 (2011) 185.

[193] D. Gomez-Ullate, N. Kamran and R. Milson, *J. Math. Anal. Appl.* 387 (2012) 410.

[194] S. Odake and R. Sasaki, *Phys. Lett.* B702 (2011) 164.

[195] C.L. Ho, *Ann. Phys.* 326 (2011) 797.

[196] D. Dutta and P. Roy, J. Math. Phys. 51 (2010) 042101.

[197] S. Sree Ranjani et al, *J. Phys. A* 45 (2012) 055210.

[198] T. Tanaka, *J. Math. Phys.* 51 (2010) 032101.

[199] S. Odake and R. Sasaki, *Phys. Lett.* B682 (2009) 130.

[200] S. Odake and R. Sasaki, *Prog. Theor. Phys.* 125 (2011) 851.

[201] K. Goser, P. Glösekötter and J. Dienstuhl, Nanoelectronics and Nanosystems. From Transistors to Molecular and Quantum Devices, Springer-Verlag, Berlin 2004.

[202] W.D. Phillips *Rev. Mod. Phys.* 70 (1998) 721.

[203] C. Zicovich-Wilson, W. Jaskilski and J.H. Planelles,*Int . J. Quant. Chem.* 54 (1995) 61.

[204] G. Todorovic et al., *Phys. Lett.* A279 (2001) 268.

[205] D. Indjin et al., *SuperLattices and Microstructures*, 28 (2000) 143.

[206] V.G. Bagrov and B.F. Samsonov, *Theo. Math. Phys.* 104 (1995) 1051.

[207] V.G. Bagrov and B.F. Samsonov, *Phys. Part. Nucl.* 28 (1997) 374.

[208] A. S. Halberg, *J. Math. Phys.* 52 (2011) 083505.

[209] A. S. Halberg, *J. Math. Phys.* 51 (2010) 033521.

[210] D. Baye, Phys. rev. A48 (1993) 2040.

[211] R.D. Amado, *Phys. Rev.* A37 (1988) 2277.

[212] A.A. Andrianov and F. Canata, *J. Phys.* A37 (2004) 10297.

[213] A. Khare and U.P. Sukhatme, Phys. Rev. A40 (1989) 6185.

[214] M.B. Levitan and I.S. Sargsian, *Introduction to Spectral Theory* (AMS) 1975.

[215] J.F. Carinena, A. Ramos and D.J. Fernandez, *Ann. Phys.* 292 (201) 42.

[216] F. Cannata et al., *J. Phys.* A32 (1999) 3583.

[217] J.O. Rosas-Ortiz, *J. Phys.* A31 (1998) 10163.

[218] D.J. Fernandez, R. Munoz and A. Ramos, *Phys Lett* A 308 (2003) 11.

[219] A.C. Astorga and D.J. Fernandez, J. Phys. A41 (2008) 475303.

[220] D.J. Fernandez, E. Hernandez, *Phys.Lett.A* 338 (2005) 13.

[221] S. Kuru, A. Tegmen, and A. Vercin, *J. Math. Phys.* 42 (2001) 3344.

[222] A.S. Halberg, *J. Math. Phys.* 51 (2010) 033521.

[223] A. Schulze-Halberg, E. Pozdeeva and A. Suzko, *J. Phys.* A42 (2009) 115211.

[224] A. Anderson, *Phys. Rev. A* 43 (1991) 4602.

[225] B.F. Samsonov and A.A. Pecheritsin, *Rus. Phys. J.* 43 (2000) 48.

[226] L.M. Nieto, A.A. Pecheritsin and B.F. Samsonov, *Ann. Phys.* 305 (2003) 151.

[227] A. A. Suzko and A. S. Halberg, *Phys. Lett.* A372 (2008) 5865.

[228] G. Junker and S. Matthiesen, *J. Phys.* A 27 (1984) L751.

[229] W.M. Suen, C.W. Wong and K. Young, *Phys. Lett.* A271 (2000) 54.

[230] S. Kuru and J. Negro, *Ann. Phys.* 323 (208) 413.

[231] H.W. Lee and D.S. Novikov, *Phys. Rev. B* 68 (2003) 155402.

[232] V. Jakubsky and M. Plyushchay , *Phys. Rev. D* 85 (2012) 045035.

[233] J. Morales and J.J. Pena, *Phys. Scr.* 74 (2006) 71.

[234] D. Baye, J-M. Sparenberg, *Phys. Rev. Lett.* 73 (1994) 2789.

[235] B.F. Samsonov, J-M. Sparenberg and D. Baye, *J. Phys.* A40 (2007) 4225.

[236] H. Leeb, S.A. Sofianos, J.M.. Sparenberg and D. Baye, *Phys. Rev.* C62 (2000) 064003.

[237] A.B. Balantekin, M.A. Candido Ribeiro and A. N. Aleixo, *J. Phys.* A32 (1999) 2785.

[238] A.N. Aleixo, A.B. Balantekin and M.A. Candido Ribeiro *J. Phys.* A33 (2000) 1503.

[239] A. Kostelecky and M.M. Nieto, *Phys. Rev. Lett.* 53, (1984) 2285.

[240] C.M. Bender and S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243.

[241] C. M. Bender, *Contemp. Phys.* 46 (2005) 277.

[242] C. M. Bender, *Rep. Prog. Phys.* 70 (2007) 947.

[243] P. Dorey, C. Dunning, and R. Tateo, *J. Phys.* A34 (2001) 5679.

[244] C.M. Bender, D.C. Brody, and H.F. Jones, *Phys. Rev. Lett.* 89 (2002) 270401.

[245] C.M. Bender, D.C. Brody and H.F. Jones, *Am. J. Phys.* 71 (2003) 1095.

[246] C.M. Bender, J. Brod, A. Refig and M. Reuter, *J. Phys.* A37 (2004) 10139.

[247] C.M. Bender, P.N. Meisinger and Q. Wang, *J. Phys.* A36 (2003) 1973.

[248] C. M. Bender and H. Jones, *Phys. Lett.* A 328 (2004) 102

[249] C.M. Bender and B. Tan *J. Phys.* A39 (2006) 1945.

[250] R. Roychoudhury and P. Roy, *J. Phys.* A40 (2007) F617.

[251] S. Weigert, *Phys. Rev.* A 68 (2003) 062111.

[252] B. Bagchi, C. Quesne and M. Znojil, *Mod. Phys.* A 16 (2001) 2047.

[253] C.M. Bender et al., *Ann. Phys.* 325 (2010) 2332.

[254] C.M. Bender et al., *Phys. Rev. Lett.* 104 (2010) 061601.

[255] C.M. Bender, S. Boettcher, and P.N. Meisinger, *J. Math. Phys.* 40 (1999) 2201.

[256] K. C. Shin, *J. Math. Phys.* 42 (2001) 2513; *Commun. Math. Phys.* 229 (2002) 543; *J. Math. Phys.* 46 (2005) 082110.

[257] F. Pham and E. Delabaere, *Phys. Lett.* A250 (1998) 25; E. Delabaere and F. Pham, *Phys. Lett.* A250 (1998) 29; E. Delabaere and D. T. Trinh, *J. Phys.* A33 (2000) 8771.

[258] S. Weigert, *J. Opt.* B5 (2003) S416; *J. Phys.* A39 (2006) 235; *J. Phys.* A39 (2006) 10239.

[259] F.G. Scholtz and H.B. Geyer, *Phys. Lett.* B634 (2006) 84; *J. Phys.* A39 (2006) 10189.

[260] A. Mostafazadeh, *J. Math. Phys.* 43 (2002) 205.

[261] A. Mostafazadeh *J. Math. Phys.* 43 (2002) 2814

[262] A. Mostafazadeh 2002 *J. Math. Phys.* **43** 3944

[263] A. Mostafazadeh, *Int. J. Geom. Meth. Mod. Phys.* 7 (2010) 1191.

[264] F.G. Scholtz, H.B. Geyer and J.W. Hahne, *Ann. Phys.*, 213 (1992) 74.

[265] Kretschmer R and Szymanowski L 2004 *Phys. Lett. A* **325** 112.

[266] A. Mostafazadeh and A. Batal, *J.Phys.* A37 (2004) 11645.

[267] L. Solombrino, *J. Math. Phys.* 43 (2002) 5439.

[268] B. Bagchi and C. Quesne, *Phys. Lett.* A 301 (2002) 173.

[269] A. Mostafazadeh, *J. Math. Phys.* 46 (2005) 102108.

[270] A. Mostafazadeh, *J. Phys.* A39 (2006) 13506.

[271] H.F. Jones, *J. Phys.* A 38 (2005) 1741.

[272] A. Mostafazadeh, *J. Phys.* A 38 (2005) 3213.

[273] Z. Ahmed, *Phys. Lett.* A290 (2001) 19.

[274] Z. Ahmed, *Phys. Lett.* A294 (2002) 287.

[275] A. Mostafazadeh, *Nucl. Phys.* B640 (2002) 419.

[276] A. Sinha and P. Roy, *J. Phys.* A39 (2007) L377.

[277] M.A. Naimark, *Proc. Mosc. Math. Soc.* 3 (1954) 181.

[278] B.F. Samsonov, *J. Phys.* A38 (2005) L571.

[279] A. Mostafazadeh, *J. Phys.* A44 (2011) 375302.

[280] Z. Ahmed, *J. Phys.*, A42 (2009) 472005.

[281] W.D. Heiss, *J. Phys.* A 37 (2004) 2455.

[282] M.V. Berry, *Czech.J.Phys.* 54 (2004) 1039.

[283] A.V. Smilga, *J. Phys. A* 42 (2009) 095301.

[284] M. Muller and I. Rotter *J. Phys.* A41 (2008) 244018.

[285] A. Mostafazadeh, *Phys. Rev. Lett.* 102 (2009) 220402; Phys. Rev. A80 (2009) 032711.

[286] A. Ruschhaupt, F. Delgado, and J. G. Muga, *J. Phys.* A38 (2005) L171.

[287] A. Mostafazadeh and H.M. Dehnavi, *J. Phys.* A42(2009)125303.

[288] A. Mostafazadeh and M. Sarisman, *Phys. Lett.* A375(2011) 3387.

[289] Z. Ahmed, *J. Phys.* A45 (2012) 032004.

[290] T. Kato, *Perturbation Theory of Linear Operators* (Berlin, Springer, 1966)

[291] I. Rotter, *J. Phys.* A42 (2009) 153001.

[292] H. Cartarius, J. Main, and G. Wunner, *Phys. Rev. Lett.* 99 (2007) 173003.

[293] S. Klaiman, U. Gunther and N. Moiseyev, *Phys. Rev. Lett.* 101 (2008) 080402.

[294] S. Leev et al., *Phys. Rev. Lett.* 103 (2009) 134101.

[295] B. Bagchi, C. Quesne and R. Roychoudhury, *J. Phys.* A41 (2008) 022001.

[296] M. Znojil, Phys. Lett. A259 (1999) 220.

[297] M. Znojil, *Phys. Lett.* A264 (1999) 108.

[298] A. Mostafazadeh, *J. Phys.* A38 (2005) 6557.

[299] M. Znojil, *J. Phys.* A33(2000) 4561.

[300] M. Znojil, *Phys. Lett.* A285 (2001) 7.

[301] B. Bagchi, S. Mallik and C. Quesne, *Mod. Phys. Lett.* A17 (2002) 1651.

[302] M. Znojil, V. Jakubsky, *J. Phys.* A38 (2005) 5041.

[303] B. Bagchi and R. Roychoudhury *J. Phys.* A33 (2000) L1.

[304] Z. Ahmed, *Phys. Lett.* A 282 (2001) 343.

[305] G. Levai, *J. Phys.* A39 (2006) 10161.

[306] G. Levai, *Czech. J. Phys.* 56 (2006) 453.

[307] G. Levai, *Phys. Lett.* A372 (2008) 6484.

[308] G. Levai and E. Magyari, *J. Phys.* A39 (2009) 195302.

[309] G. Levai and M. Znojil, *J. Phys.* A33 (2000) 7165.

[310] G. Levai and M. Znojil, *Mod. Phys. Lett.* A 30 (2001) 1973.

[311] M. Znojil and G. Levai, *Phys. Lett.* A 271 (2000) 327.

[312] M. Znojil, *J. Phys.* A 39 (2006) 4047.

[313] M. Znojil, *J. Phys.* A39 (2006) 441.

[314] B. Bagchi and C. Quesne, *Phys. Lett.* A 273 (2000) 285.

[315] C.M. Bender and S. Boettcher, *J. Phys.* A 31 (1998) L273.

[316] M. Znojil, J. Phys. A33 (2000) 4203.

[317] B. Bagchi, F. Cannata and C. Quesne, *Phys. Lett.* A269 (2000) 79.

[318] C.M. Bender, and M. Monou, *J. Phys.* A38 (2005) 2179.

[319] A. Khare and B.P. Mandal, *Phys. Lett.* A272 (2000) 53.

[320] B. Bagchi et al., *Phys. Lett.* A 289 (2000) 34.

[321] A. Khare and B.P. Mandal, *Pramanna- J. Phys.* 73 (2009) 387.

[322] C. M. Bender, G. V. Dunne and P. N. Meisinger, *Phys. Lett.* A 252 (1999) 272.

[323] J.M. Cervero, *Phys. Lett.* A317 (2003) 26.

[324] K. C. Shin, *J. Phys.* A37 (2004) 8287.

[325] H.F. Jones, *Phys. Lett.* A 262 (1999) 242.

[326] Z. Ahmed, *Phys. Lett.* A286 (2001) 231.

[327] A. Khare and U. Sukhatme, *J. Phys.* A39 (2006) 10133.

[328] A. Khare and U. Sukhatme, *Phys. Lett.* A324(2004)406.

[329] F. Cannata, J.P. Dedonder and A. Ventura, *Annals of Physics* 322 (2007) 397.

[330] Z. Ahmed, *Phys. Rev.* A64 (2001) 042716.

[331] G. Levai, P. Siegl and M. Znojil, *J. Phys.* A42 (2009) 295201.

[332] H.F. Jones, *Phys. Rev.* D 78(2008)065032.

[333] M. Znojil, *Czech. J. Phys.* 51 (2001) 420.

[334] A.A. Andrianov et al., *Int. J. Mod. Phys.* A14 (1999) 2675.

[335] M. Znojil, *J. Phys.* A35 (2002) 2341.

[336] M. Znojil, F. Cannata, B. Bagchi and R. Roychoudhury,*Phys. Lett.* B483 (2000) 284.

[337] G. Levai, *J. Phys.* A37 (2004) 10179.

[338] G. Levai and M. Znojil, *J. Phys.* A35 (2002) 8793.

[339] M. Znojil and G. Levai, *Mod. Phys. Lett.* A16 (2001) 2273.

[340] G. Levai, F. Cannata and A. Ventura, *Phys. Lett.* A300 (2002) 271.

[341] V. Jakubsky and M. Znojil, *Czech. J. Phys.* 55 (2005) 1113.

[342] G. Levai, *Pramana-J. Phys.* 73 (2009) 329.

[343] B.P. Mandol and A. Khare, *Phys. Lett.* A 272 (2000) 53.

[344] A. Mostafazadeh, *Phys. Rev. Lett.* 99 (2007) 130502.

[345] P.E. Assis and A. Fring, *J. Phys.* A41 (2008) 244002.

[346] U. Gunther and B.F. Samsonov, *Phys. Rev. Lett.* 101 (2008) 230404.

[347] C.M. Bender, D.C. Brody, H.F. Jones and B.K. Meister, *Phys. Rev. Lett.* 98 (2007) 040403.

[348] C.M. Bender, P.D. Mannheim, *Phys. Rev.* D84 (2011) 105038.

[349] A. Nanayakkara, *Phys. Lett.* A304(2002)67.

[350] G. Levai, *J. Phys.* A40 (2007) F273.

[351] C.M. Bender, C.V. Dunne, P.N. Meisinger and M. Simsek, *Phys. Lett.* A311(2001)281.

[352] M. Znojil and M. Tater, *J. Phys.* A34 (2001) 1793.

[353] A. Fring and M. Znojil, *J. Phys.* A41 (2009) 194010.

[354] M. Znojil, *Phys. Rev.* D80 (2009) 105004; PT-symmetry and Quantum graph, arXiv: 1205.5211 (2012).

[355] M. Znojil, *Phys. Lett.* A375 (2011) 3435.

[356] M. Znojil, *Ann. Phys.* 327 (2012) 893.

[357] A. Guo et al., *Phys. Rev. Lett.* 103 (2009) 093902.

[358] C.E. Ruter et al., *Nature Phys.* 6 (2010) 192.

[359] K.G. Makris et al, *Phys. Rev. Lett.* 100 (2008) 103904.

[360] M. V. Berry, *J. Phys.* A41 (2008) 244007.

[361] Z.H. Musslimani et.al., *Phys. Rev. Lett.* 100 (2008) 030402.

[362] K.G. Makris et al, *Phys. Rev.* A81 (2010) 063807.

[363] S. Longhi, *Phys. Rev. Lett.* 103 (2009) 123601.

[364] Z. Lin et al, *Phys. Rev. Lett.* 106 (2011) 213901.

[365] H.F. Jones, *J. Phys.* A45 (2012) 135306.

[366] S. Longhi, *J. Phys.* A 44(2011)485302.

[367] H. Cartarius, G. Wunner, *Phys. Rev.* A86 (2012) 013612.

[368] Z.H. Musslimani et. al., *J. Phys.* A41 (2008) 244019.

[369] A.A. Abdullaev, A. Abdumalikov and R. Galimzyanov, *Phys. Lett.* A 367(2007)149.

[370] Y.V. Bludov and V.V. Konotop, *Phys. Rev.* A 81(2010)013625.

[371] R. Driben and B.A. Malomed, *Optics Letters* 36 (2011)4323.

[372] E. M. Graefe, H.J. Korsch and A.E. Niederle, *Phys. REv. Lett.* 101 (2008) 150408; *Phys. Rev.* A82(2010) 013629.

[373] C.F.M. Faria and A. Fring, *Laser Physics* 17 (2007) 424.

[374] C.M Bender, I.C.Pelaez, K.A. Milton and K.V.Shajesh, *Phys. Lett.* B613 (2005) 97.

[375] C.M. Bender, H.F. Jones and R.J. Rivers,*Physics Letters* B625 (2005) 333.

[376] C. M. Bender et al, *Physical Review* D71 (2005) 025014.

[377] C.M. Bender and P.D. Mannheim, *Phys. Rev. Lett.* 100(2008)110402.

[378] J.W. Moffat, *Phys. Lett.* B627 (2005) 9.

[379] A. Mostafazadeh, *Ann. Phys.* 309 (2004) 1.

[380] A. Nanayakkara, *J. Phys.* A37 (2004) 4321.

[381] M. Fagoti et al, *Phys. Rev.* B83 (2011) 241406.

[382] T. Kottos, *Nature Physics* 6(2010)166.

[383] O.C. Alvaredo and A. Fring, *J. Phys.* A42 (2009) 465211.

[384] B. Basu-Mallick and B.P. Mandal, *Phys. Lett.* A 284 (2001) 231.

[385] B. Basu-Mallick, *Int. J. Mod. Phys.* B16 (2002) 1875.

[386] B. Basu-Mallick et al., *Czech. J. Phys.* 54 (2004) 5.

[387] C.M. Bender, D.D. Holm and D.W. Hook, *J. Phys.* A40 (2007) F793.

[388] A. Cavagilla, A. Fring and B. Bagchi, *J. Phys.* A44 (2011) 325201.

[389] W. Trzeciakowski, *Phys. Rev.* B38 (1988) 12493.

[390] M. R. Geller and W. Kohn, *Phys. Rev. Lett.* 70 (1993) 3103.

[391] B.A. Joyce, *Rep. Prog. Phys.* 48 (1985) 1637.

[392] G. Bastard, *Wave Mechanics Applied to semiconductors Heterostructure* (Les Editions de Physique, Les Ulis, France 1988).

[393] G.T. Einevoll, P.C. Hemmer and J. Thomsen, *Phys. Rev.* B42 (1990) 3485.

[394] J.R. Shewell, *Am. J. Phys.* 27 (1959) 16.

[395] T. Gora and F. Williams, *Phys. Rev.* 177 (1969) 1179.

[396] G. Bastard, *Phys. Rev.* B24 (1981) 5693.

[397] Q.G. Zhu and H. Kroemer, *Phys. Rev.* B27 (1983) 3519.

[398] T. Li and K.J. Kuhn, *Phys. Rev.* B27 (1983) 12760.

[399] O. Von Roos, *Phys. Rev.* B27 (1983) 7547.

[400] D.J. BenDaniel and B.C. Duke, *Phys. Rev.* B152 (1966) 683.

[401] J.M. Levy-Leblond, *Phys. Rev.* A52 (1995) 1845.

[402] R.A. Morrow and K. R.Brownstein, *Phys. Rev.* B30 (1984) 678.

[403] N. Moiseyev and R. Lefebvre, *Phys. Rev.* A 64 (2001) 052711.

[404] B. Bagchi, A. Banerjee, C. Quesne and V.M. Tkachuk, *J. Phys.* A38 (2005) 2929.

[405] C. Quesne and V.M. Tkachuk, *J. Phys.* A 37 (2004) 4267.

[406] L. Dekar, L. Chetouani and T. F. Hammann, *Phys.Rev.* A59 (1999) 107.

[407] L. Dekar, L. Chetouani and T. F. Hammann, *J. Math. Phys.* 39 (1998) 2551.

[408] A. de Souza Dutra and C.A.S. Almeida, *Phys. Lett.* A275 (2000) 25.

[409] B. Bagchi, P.S. Gorain and C. Quesne, *Mod. Phys. Lett* A21 (2006) 2703.

[410] J.F. Carinena et al, *Nonlinearity* 17 (2004) 1941.

[411] J.F. Carinena et al, *Ann.Phys.* 322 (2007) 434.

[412] A. R. Plastino et al., *Phys.Rev.* A60 (1999) 4398.

[413] A. de Souza Dutra, M. Hott, C.A.S. Almeida, *Europhys. Lett.* 62 (2003) 8.

[414] B. Gönül, O. Özer, B. Gönül, and F. Üzgün, *Mod.Phys.Lett.* A17 (2002) 2453.

[415] B. Gönül, B. Gönül, D. Tutcu and O. Özer, *Mod.Phys.Lett.* A17 (2002) 2057.

[416] A. Ganguly and L. M. Nieto, *J.Phys.* A40 (2007) 7265.

[417] B. Bagchi, A. Ganguly and A. Sinha, *Phys. Lett.* A 374 (2010) 2397.

[418] K. Samani and F. Loran, Shape Invariant Potentials for Effective Mass Schrödinger Equation, arXive:quant-ph/0302191.

[419] B. Midya, B. Roy and T. Tanaka, *J. Phys.* A45 (2012) 205303.

[420] V. Milanovic and Z. Ikonic, *J.Phys.* A32 (1999) 7001.

[421] T. Tanaka, *J. Phys.* A39 (2006) 219.

[422] C. Quesne, *SIGMA* 5 (2009) 046.

[423] A. Bhattacharjee and E. C. G. Sudarshan, *Nuovo Cimento* 25 (1962) 864.

[424] A. D. Alhaidari, *Phys.Rev.* A66 (2002) 042116.

[425] B. Bagchi, P. Gorain, C. Quesne and R. Roychoudhury, *Europhys. Lett.* 72 (2005) 155.

[426] A. Ganguly, M.V. Ioffe and L.M. Nieto, *J. Phys. A* 39 (2006) 14659.

[427] C. Quesne, *SIGMA* 3 (2007) 067.

[428] G. Chen and Z. Chen, *Phys. Lett.* A331 (2004) 312.

[429] O. Mustafa and S.H. Mazharimousavi, *J. Phys.* A 39 (2006) 10537.

[430] R. Kock, E. Korcuk and M. Koca, *J. Phys.* A 35 (2002) L527.

[431] M.S. de Bianchi and M.D. Ventra, *Eur. J. Phys.* 16 (1995) 260.

[432] R. Koc, G. Sahinoglu and M. Koca, *Eur. Phys. J.* B48 (2005) 583.

[433] L. Chetouani, L. Dekar and T.F. Hammann, *Phys. Rev.* A52 (1995) 82.

[434] K.C. Yung and J.H. Yee, *Phys. Rev.* A50 (1994) 104.

[435] A.S. Halberg, *Int. J. Mod. Phys.* A23 (2008) 537.

[436] H. Cobian and A.S¿ Halberg, *J. Phys.* A44 (2011) 285301.

[437] B. Roy and P. Roy, *J.Phys.* A35 (2002) 3961.

[438] B. Roy and P. Roy, *Phys. Lett.* A 340 (2005) 70.

[439] T.K. Jana and P. Roy, *Eurphys. Lett.* 87 (2009) 3003.

[440] A. D. Alhaidary et al., *Phys. Rev.* A75 (2007) 062711.

[441] L. Jiyang, L. Yi and C. Jia, *Phys. Lett.* A345 (2005) 279.

[442] B. Bagchi, C. Quesne and R. Roychoudhury, *J. Phys.* A39 (2006) L127.

[443] E. Schrödinger, *Naturwissenschaften* 14 (1926) 664.

[444] R.J. Glauber, *Phys. Rev.* 130 (1963) 2529; *Phys. Rev. Lett.* 10 (1963) 84.

[445] E.C.G. Sudarshan, *Phys. Rev. Lett.* 10 (1963) 227.

[446] J.R. Klauder, *J. Math. Phys.* 4 (1963) 1055; *J. Math. Phys.* 4 (1963) 1058.

[447] W.M. Zhang et al, Rev.Mod.Phys. **62** 867 (1990).

[448] R. Gilmore, *Ann. Phys.* 74 (1972) 391.

[449] R. Gilmore, *Rev. Mev. de Fisica* 23 (1974) 142.

[450] A.M. Perelomov, *Commun. Math. Phys.* 26 (1972) 222.

[451] A.M. Perelomov, *Generalized Coherent States and Their Application*, (Springer-Verlag, Berlin,1986).

[452] A.O. Barut and L. Girardello, *Commun.Math.Phys.* 21 (1971) 41.

[453] M.M. Nieto and L.M. Simmons, *Phys. Rev. Lett.* 41 (1978) 207.

[454] M.M. Nieto and L.M. Simmons, *Phys. Rev.* D20 (1979) 1321.

[455] V.P. Gutschick, M.M. Nieto and L.M. Simmons, *Phys. Lett.* A76 (1980) 15.

[456] V.P. Gutschick, M.M. Nieto, F. Baker, *Am. J. Phys.* 47 (1979) 755.

[457] J. R. Klauder and B. S. Skagerstam, (1985)*Coherent states: Applications in Physics and Mathematical Physics* (World Scientific, Singapore).

[458] J.R.Klauder, *J.Phys.* A29 (1996) L293.

[459] J.P.Gazeau and J.R.Klauder, *J.Phys.* A32 (1999) 123.

[460] J.P.Antoine et al, *J.Math.Phys.* 42 (2001) 2349.

[461] A.H. EI Kinani and M. Daoud, *J. Math. Phys.* 43 (2002) 714.

[462] B. Roy and P. Roy, *Phys. Lett.* A296 (2002) 187.

[463] D. Popov, *Phys. Lett.* A316 (2003) 369.

[464] A. Chenaghlou and O. Faizy, *J. Math. Phys.* 49 (2008) 022014.

[465] P. Roy, *Opt. Comm.* 221 (2003) 145.

[466] D. Popov, V. Sajfert and I. Zaharie, *Physica* A387 (2008) 4459.

[467] M. Novaes, M.A.M. Aguiar and J.E.M. Hornos, *J. Phys.* A36 (2003) 5773.

[468] B. Roy and P. Roy, *Phys. Lett.* A 359 (2006) 110.

[469] J.M. Hollingworth et al, *J.Phys.A* 34 (2001) 9463.

[470] A. Chenaghlou and H. Fakhri, *Mod. Phys. Lett.* A17 (202) 1701.

[471] A.H. EI Kinani and M. Daoud, *J. Phys.* A34 (2001) 5373.

[472] M. Novaes and J.P. Gazeau, *J. Phys.* A36 (2003) 199.

[473] S. Twareque Ali and F. Bagarello, *J. Math. Phys.* 46 (2005) 053518; *J. Math. Phys.* 49 (2008) 032110.

[474] H. Jeong, M.S. Kim and J. Lee, *Phys. Rev.* A64 (2001) 052308.

[475] S.L. Braunstein and P. van Loock, *Rev. Mod. Phys.* 77 (2005) 513.

[476] B. Sanders, *J. Phys.* A 45 (2012) 244002.

[477] R.W. Robinett , *Phys. Rep.* 392 (2004) 1.

[478] I. Sh. Averbukh and N. F. Perelman, *Phys. Lett.* A139 (1989)449.

[479] J.A. Yeazell, M. Mallalieu and C.R. Stroud, *Phys. Rev. Lett.* 64 (1990) 2007.

[480] J.A. Yeazell, C.R. Stroud, *Phys. Rev.* A43 (1991) 5153.

[481] I.Sh. Averbukh, *Phys. Rev.* A46 (1992) R2205.

[482] D.L. Aronstein and C.R. Stroud, *Phys. Rev.* A55 (1997) 4526.

[483] M. Nauenberg, *J. Phys.* B23 (1990) L385.

[484] E. Romera and F. de los Santos, *Phys. Rev. Lett.* 99 (2007) 263601.

[485] E. Romera and F. de los Santos, *Phys. Rev.* A79 (2008) 013837.

[486] C.U. Segre and J.D. Sullival, *Am. J. Phys.* 44 (1976) 729.

[487] R. Bluhm, V.A. Kostelecky and B. Tudose, *Phys. Rev.* A52 (1995) 2234.

[488] R.R. Puri and G.S. Agarwal, *Phys. Rev.* A33 (1986) 3610.

[489] V. Buzek, H. Moya-Cessa and P.L. Knight, *Phys. Rev.* A45 (1992) 8190.

[490] V.V. Eryomin, S.I. Vetchinkin and I.M. Umanskii, *J. Chem. Phys.* 222 (1994) 394.

[491] W.Y. Chen and G.J. Milburn, *Phys. Rev.* A51 (1995) 2328.

[492] V. Krueckl and T. Kramer, *New. J. Phys.* 11 (2009) 093010.

[493] S.L.Vetchinkin and V.V. Eryomin, *Chem. Phys. Lett.* 222 (1994) 394.

[494] A. Schmidt, *Phys. Lett.* A 353 (2006) 459.

[495] R.W. Robinett, *Am. J. Phys.* 68 (2000) 410.

[496] R.W. Robinett and S. Heppelmann, *Phys. Rev.* A65 (2002) 062103.

[497] U. Roy, J. Banerji and P. K Panigrahi, *J. Phys.* A38 (2005) 9115.

[498] R. Bluhm, V.A. Kostelecky and J.F. Porter, *Am. J. Phys.* 64 (1996) 944.

[499] J. Banerji and G.S. Agarwal, *Optics Express* 5 (1999) 220.

[500] R. Bluhm, V.A. Kostelecky and B. Tudose, *Phys. Rev.* A55 (1999) 819.

[501] J. Wals et al., Phys. Rev. Lett. 72 (1994) 3783.

[502] M.J. Vrakking, D.M. Villeneuve and A. Stolow, *Phys. Rev.* A54 (1996) R37.

[503] B. Midya, B. Roy and R. Roychoudhury, *J. Math. Phys.* 51 (2010) 022109.

[504] B. Mielnik, L.M. Nieto and O. Rosas-Ortiz, *Phys. Lett.* **A269**(2000) 70.

[505] J.J. Pena et al., *Int. J. Quant. Chem.* 108 (2008) 2906.

[506] B. Bagchi, P. Gorain, C. Quesne and R. Roychoudhury, *Mod.Phy.Lett.* A19 (2004) 2765.

[507] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, (Dover publivcations, New York, 1965).

[508] M.G. Gasymov, *Funct. Anal. Appl.* 14 (1980) 11.

[509] H. P. Mulholland and S. Goldstein, *Phil. Mag.* 8 (1929)834.

[510] C.J. Bouwkamp, *Kon. Nederl. Akad. Wetensch. Proc.* 51 (1948) 891.

[511] N. Kamran and P.J. Olver, *J. Math. Ana. App.* 145 (1990) 342.

[512] G. Floquet, *Annales Scientifiques de l'E.N.S.* 12 (1883) 47.

[513] M.S. Swanson, *J. Math. Phys.* 45 (2004) 585.

[514] B.Bagchi, C.Quesne and R.Roychoudhury *J. Phys.* A38 (2005) L647.

[515] D.P. Musumbu, H.B. Geyer and W.D. Heiss,*J. Phys. A: Math. Theor.* 40 (2007) F75.
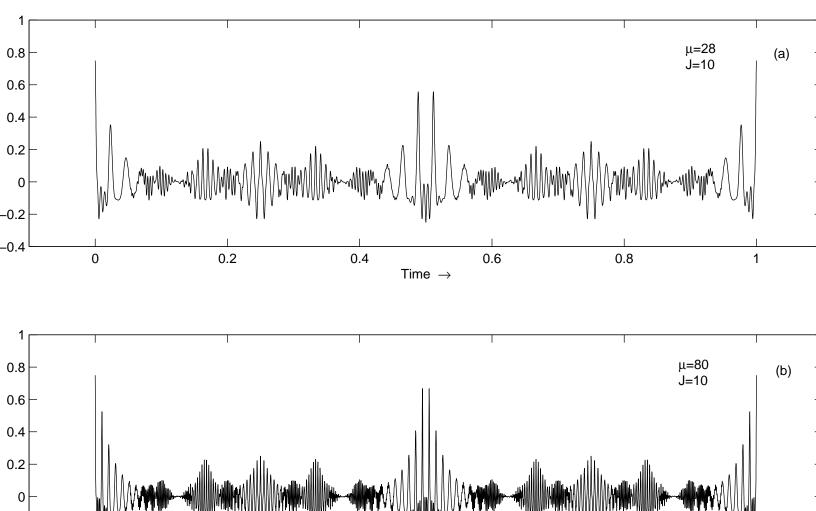
[516] C. Quesne, *J. Phys.* A40 (2007) F745.

[517] B. Bagchi and T. Tanaka, *Phys. Lett.* A372 (2008) 5390.

[518] A. Sinha and P. Roy, *J. Phys. A: Math. Theor.* 40 (2007) 10599.

[519] B. Bagchi, *J. Phys.* A 40 (2007) F1041.

[520] P.M. Mathews and M. Lakshmanan, *Quart.Appl.Maths.* 32 (1974) 215.

[521] M. Lakshmanan and S. Rajasekar, *Nonlinear dynamics,Integrability, Chaos and Patterns* (Advanced Texts in Physics, Springer, Berlin), 2003.

[522] J.F. Carinena et al., *Regul. Caot. Dyn.* 10 (2005) 423.

[523] S.I. Vetchinkin et al., *Chem. Phys. Let.* 215 (1993) 11.

[524] V.V. Eryomin et al., *J. Chem. Phys.* 101 (1994) 10730.

[525] G. Levai, *Czech. J. Phys.* 51 (2001) 1.

[526] B. Bagchi et al., *Phys. Lett.* A 269 (2000) 79.

[527] C. Quesne, *Int. J. Mod. Phys. A* 6 (1991) 1567.

[528] V.M. Tkachuk, S.I. Vakarchuk, *Phys. Lett. A* 228 (1997) 141.

[529] M. Znojil, *J.Phys.A* 37 (2004) 9557.

[530] C.L. Ho, *Ann.Phys.* 321 (2006) 2170.

[531] A. Metz et al, *Phys. Rev. Lett.* 83(1999) 1542.

[532] L. F. Urrutia and E. Hernandez, *Phys. Rev. Lett.* 51, (1983) 755.

[533] A. Gangopadhyay, J.V. Mallow and U.P. Sukhatme, *Phys. Rev.* A58 (1998) 4287.

[534] A.N. Aleixo, A.B. Balantekin and M.A. Candido Ribeiro, *J. Phys.* A35 (2002) 9063.

[535] A.N. Aleixo, A.B. Balantekin and M.A. Candido Ribeiro, J. Phys. A 33 (2000) 3173.

[536] A.N. Aleixo, A.B. Balantekin and M.A. Candido Ribeiro, *J. Phys.* A34 (2001) 1109.

[537] P.K. Ghosh, A. Khare, and M. Sivakumar, *Phys. Rev. A* 58 (1998) 821.

[538] F. Cannata, M.V. Ioffe, and D. N. Nishnianidz, *J. Math. Phys.* 52 (2011) 022106.

[539] D.J. Fernandez, *Int. J. Mod. Phys.* A12 (1997) 171.

[540] D.J. Fernandez and E. Salinas-Hernandez *J. Phys.* A36 (2003) 2537.

[541] B. Mielnik B and O. Rosas-Ortiz, *J. Phys.* A37 (2004) 10007.

[542] A.A. Suzko, *Phys. Lett.* A335 (2005) 88.

[543] L.M. Nieto, B.F. Samsanov and A.A. Suzko, *J. Phys.* A36 (2003) 12293.

[544] B. Roy, *Europhys. Lett.* 72 (2005) 1.

[545] A.S. Halberg, *Int. J. Mod. Phys.* A21 (2006) 1359.

[546] K.C. Shin, *J. Math. Phys.* 42 (2001) 2513.

[547] Z. Ahmed, Phys. Lett. A310 (2003) 139.

[548] C.M. Bender, M.V. Berry and A. Mandilara, *J. Phys.* A35 (2002) L467.

[549] Z. Ahmed, C.M. Bender and M.V. Berry, *J. Phys.* A38 (2005) L627.

[550] C.M. Bender and P.D. Mannheim, *Phys. Lett.* A374 (2010) 1616.

[551] A. Mostafazadeh, *J.Phys.* A36 (2003) 7081.

[552] U. Guenther, F. Stefani, and M. Znojil, *J. Math. Phys.* 46 (2005) 063504.

[553] M. Znojil, *Phys. Lett.* A342 (2005) 36.

[554] M. Znojil, *SIGMA* 7 (2011), 018.

[555] M. Znojil and V. Jakubsky, *Pramana- J. of Phys.* 73 (2009) 397.

[556] S. Longhi, *Phys. Rev.* B 80 (2009) 165125.

[557] C.T. West, T. Kottos and T. Prosen, *Phys. Rev. Lett.* 104 (2010) 054102.

[558] T. Hertog and G.T. Horowitz, *JHEP* 04 (2005) 005.

[559] J.B. Gong and Q. Wang, *Phys. Rev.* A 82 (2010) 012103.

[560] G.S. Agarwall and K. Tara, *Phys. Rev.* A43 (1991) 492.

[561] R.L. de Matos Filho and W. Vogel, *Phys. Rev.* A54 (1996) 4560.

[562] L. Mandel, *Phys. Rev. Lett.* 49 (1982) 136.

[563] L. Mandel and E. Wolf, *Rev. Mod. Phys.* 37 (1965) 231.

[564] H.J. Kimble and L. Mandel, *Phys. Rev.* A15 (1977) 689.

[565] R. Loudon, *Rep. Prog. Phys.* 43 (1980) 913.

[566] T.Shreecharan et al, *Phys.Rev.* A69 (2004) 012102.

[567] A.N.F. Aleixo and A.B. Balantekin, *J. Phys. A* 37 (2004) 8513.

[568] J. Parker and C.R. Stroud, *Phys. Rev. Lett.* 56 (1986) 716.

[569] T Baumert et al, *Chem. Phys. Lett.* 191 (1992) 639.

[570] M. Spanner, E.A. Shapiro, and M. Ivanov, *Phys. Rev. Lett.* 92 (2004) 093001.

[571] F. Cannata, G. Junker, J. Trost, *Phys. lett.* A246 (1998) 219.

[572] J. E. Strang, *Acad. Roy. Belg. Bull. Cl. Sci.* 16 (2005) 269.

[573] A. de Souza Dutra, M.B. Hott and V.G.C. Santos, *Eurphys. Lett.* 71 (2005) 166.

[574] C. Yuce, *Phys. Lett.* A 336(2004)290.

[575] C.M. Bender and H.F. Jones, *Phys. Rev.* A 34(2011)032103.

[576] B. Roy and R. Roychoudhury, *J. Phys.* A 40 (2007)8479.

[577] G. Levai, A. Sinha and P. Roy, *J. Phys.* A36 (2003)7611.

[578] M. Znojil, *J. Phys.* A32(2001)7419.

[579] F.M. Fernandez et al., *J.Phys.* A32(1999)3105.

[580] A. Sinha and P. Roy, *J. Phys.* A 37(2004)2509.

[581] N. Moiseyev, *Non-Hermitian Quantum Mechanics*, (Cambridge University press, 2011).

[582] M. Znojil, *J. Phys.* A 36 (2003) 7639.

[583] U. Guenther, I. Rotter and B.F. Samsonov, *J. Phys.* A 40 (2007) 8815.

[584] C.M. Bender and H.F. Jones, *J. Phys.* A41 (2008) 244006.

[585] H.F. Jones, *Phys. Rev.* D 76 (2007) 125003.

[586] F. Correa, V. Jakubsky and M.S. Plyushchay, *Ann. Phys.* 324 (2009) 1078.

[587] F. Correa and M.S. Plyushchay, *J. Phys.* A 40 (2007) 14403.

[588] P.Roy, R. Roychoudhury and Y.P. Varshni, *J. Phys.* A21 (1988) 1589.

[589] T. Shreecharan, P.K. Panigrahi and J. Banerji, *Phys. Rev.* A69 (2004) 012102.

[590] M. Daoud and A.H. EI Kinani, *J. Phys.* A35 (2002) 2639.

[591] P.F. Gora and C. Jedrzejek, *Phys. Rev.* A58 (1993) 3291.

[592] A.I. Solomon, *Phys. Lett.* A196 (1994) 29.

[593] L. Mandel, *Opt. Lett.* 4 (1979) 205.

[594] J. Banerji and G.S. Agarwal, *Phys. Rev.* A59 (1999) 4777.

[595] Z. Zhang and H. Fan, *Phys. Lett.* A165 (1992) 14.

[596] V.V. Dodonov et al., *Quant. Semiclassic. Opt* 8(1996) 413.

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(a) μ=28, J=10 — waveform plotted against Time →

(b) μ=80, J=10 — waveform plotted against Time →