A note on the Ramanujan's master theorem

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Abstract

In this note, it is shown that the Ramanujan's Master Theorem (RMT) when n is a positive integer can be obtained, as a special case, from a new integral formula. Furthermore, we give a simple proof of the RMT when n is not an integer.

Keywords: Cauchy-Frullani integral, Ramanujan's master theorem, Euler integral, Gaussian integral.

1 Introduction

In this note, we prove a new integral formula for the evaluation of definite integrals and show that the Ramanujan's Master Theorem (RMT) [1, 2] when n is a positive integer can be easily derived, as a special case, from this integral formula. This formula can be used to quickly evaluate certain integrals not expressible in terms of elementary functions. For n is not an integer, we shall also give a simple proof of the RMT.

2 Main result

To clarify the procedure, we begin by considering the following Cauchy-Frullani integral [3]:

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Lemma 1 Let f be a continuous function and assume that both $f(\infty)$ and f(0) exist. Then

$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x} dx = (f(\infty) - f(0)) \ln \frac{\alpha}{\beta}, \ \alpha, \ \beta > 0.$$
 (2.1)

This formula was first published by Cauchy in 1823, and more completely in 1827 with a beautiful proof.

The following lemma is a new helpful tool in the proof of the Ramanujan's Master Theorem [1, 2] and other integrals.

Lemma 2 Let $f \in \mathbb{C}^n([0,\infty))$ such that both $f(\infty)$ and f(0) exist. Then

$$\int_0^\infty x^{n-1} f^{(n)}(x) dx = (-1)^{n-1} \left[f(\infty) - f(0) \right] \Gamma(n), \ \Gamma(n) = (n-1)!. \quad (2.2)$$

Proof. Differentiating both sides of Eq.(2.1) in Lemma 1 n-times with respect to α , and using the chain rule $\frac{d}{d\alpha}f(\alpha x) = \frac{d}{d(\alpha x)}[f(\alpha x)] \times \frac{d(\alpha x)}{d\alpha}$, we obtain

$$\int_0^\infty x^{n-1} \frac{d^n}{d(\alpha x)^n} [f(\alpha x)] dx = (-1)^{n-1} [f(\infty) - f(0)] \frac{(n-1)!}{\alpha^n}, \ \alpha > 0.$$
(2.3)

The change of variable $t = \alpha x$ in the LHS of (2.3) yields

$$\frac{1}{\alpha^n} \int_0^\infty t^{n-1} \frac{d^n f(t)}{dt^n} dt = (-1)^{n-1} \left[f(\infty) - f(0) \right] \frac{(n-1)!}{\alpha^n}, \ \alpha > 0.$$
 (2.4)

The proof is complete.

3 Applications

3.1 The Ramanujan's Master Theorem

The Ramanujan's Master Theorem [1, 2] states that

Theorem 3 If F(x) is defined through the series expansion $F(x) = \sum_{k=0}^{\infty} \phi(k) \frac{(-x)^k}{k!}$, with $\phi(0) \neq 0$. Then

$$\int_{0}^{\infty} x^{n-1} \sum_{k=0}^{\infty} \phi(k) \frac{(-x)^{k}}{k!} dx = \Gamma(n)\phi(-n), \tag{3.1}$$

where n is a positive integer.

It was widely used by the indian mathematician Srinivasa Ramanujan (1887-1920) to calculate definite integrals and infinite series.

Ramanujan asserts that his proof is legitimate with just simple assumptions [1, 2]: (1) F(x) can be expanded in a Maclaurin series; (2) F(x) is continuous on $(0, \infty)$; (3) n > 0; and (4) $x^n F(x)$ tends to 0 as x tends to ∞ .

We note below that the Ramanujan's Master Theorem can be derived as a special case from (2.2) when n is a positive integer.

Proof. (Using (2.2)) Assume that f(x) is expanded in a Maclaurin series $f(x) = \sum_{k=0}^{\infty} \psi(k) \frac{(-x)^k}{k!}$, where $f(0) = \psi(0) \neq 0$ and f(x) tends to 0 as x tends to ∞ .

A simple computation leads to $f^{(n)}(x) = (-1)^n \sum_{k=0}^{\infty} \psi(n+k) \frac{(-x)^k}{k!}$. Substituting into (2.2), we obtain

$$\int_0^\infty x^{n-1} \sum_{k=0}^\infty \psi(n+k) \frac{(-x)^k}{k!} dx = f(0)\Gamma(n) = \psi(0)\Gamma(n).$$
 (3.2)

We see that, in the notation of the Ramanujan's Master Theorem, $\phi(k) = \psi(n+k), \ k=0,1,...$ and hence $\phi(-n)=\psi(0), \ n\in\mathbb{N}.$

This is precisely formula (3.1), and the proof is complete. \blacksquare

3.2 Other integrals involving special functions

3.2.1 The Euler integral

An immediate consequence of (2.2) is the evaluation of the following integral.

$$\int_0^\infty x^{n-1} e^{-ax} dx = a^{-n} \Gamma(n), \ a > 0.$$
 (3.3)

This integral is known as the Euler integral representation of the gamma function. It was considered by Euler in 1729 and 1730 [3].

This follows simply by letting $f(x) = e^{-ax}$, f(0) = 1, $f(\infty) = 0$ and $f^{(n)}(x) = (-a)^n e^{-ax}$ in (2.2).

3.3 Integral representation of the beta function

The beta function B(n, m) is defined by [3]

$$B(n,m) = \int_0^\infty x^{n-1} \frac{1}{(1+x)^{n+m}} dx = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}, \ m, n = 1, 2, ...,.$$
 (3.4)

This follows simply by letting $f(x) = \frac{1}{(1+x)^m}$, $f(\infty) = 0$, f(0) = 1 and $f^{(n)}(x) = (-1)^n m(m+1) \dots (m+n-1) \frac{1}{(1+x)^{n+m}}$, $n = 1, 2, \dots$ in (2.2), and using the above property of the gamma function.

3.3.1 Gaussian integral

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.\tag{3.5}$$

This follows simply by letting f(x) = erf(x), $f'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$, $erf(\infty) = 1$, n = 1 and erf(0) = 0 in (2.2).

3.3.2 Integral involving Hermite polynomials $H_n(x)$

$$\int_{0}^{\infty} x^{n-1} H_{n-1}(x) e^{-x^{2}} dx = \frac{\sqrt{\pi}}{2} \Gamma(n).$$
 (3.6)

This follows simply by letting f(x) = erf(x) in (2.2) and using the Rodrigues formula for the Hermite polynomials:

$$\frac{d^n f(x)}{dx^n} \left[erf(x) \right] = (-1)^{n-1} \frac{2}{\sqrt{\pi}} H_{n-1}(x) e^{-x^2}. \tag{3.7}$$

3.3.3 Integral involving Laguerre polynomials $L_n(x)$

$$\int_0^\infty x^{n-1} L_n(x) e^{-x} dx = 0. \tag{3.8}$$

where $L_{n-1}(x)$ are Laguerre polynomials. This follows simply by letting $f(x) = x^n e^{-x}$, $f(\infty) = 0 = f(0)$ in (2.2) and using the Rodrigues formula for the Laguerre polynomials:

$$\frac{d^n f(x)}{dx^n} \left[x^n e^{-x} \right] = n! L_n(x) e^{-x}. \tag{3.9}$$

4 A simple proof of the RMT when n is not an integer

We now give a simple proof of the RMT when n is not an integer. We recall that the Mellin transform is the integral transform defined by

$$\mathscr{M}\lbrace f(t), \ s\rbrace = \int_0^\infty t^{s-1} f(t) dt, \tag{4.1}$$

where s is a complex number.

Also, the change of variable $t = e^{-x}$ transforms $\mathcal{M}\{f(t), s\}$ into the twosided Laplace transform of $f(e^{-x})$. This can be written as

$$\mathcal{M}{f(t), s} = \mathcal{L}{f(e^{-x}), s} = \int_{-\infty}^{\infty} e^{-sx} f(e^{-x}) dx.$$
 (4.2)

1. Let

$$f(x) = \begin{cases} \sum_{k=0}^{\infty} \frac{\phi(k)}{k!} (-x)^k, & x \ge 0, \\ 0, & x < 0. \end{cases}$$
 (4.3)

Thus

$$\int_0^\infty x^{s-1} \sum_{k=0}^\infty \phi(k) \frac{(-x)^k}{k!} dx = \int_0^\infty e^{-sx} \sum_{k=0}^\infty \phi(k) \frac{(-1)^k}{k!} e^{-kx} dx. \quad (4.4)$$

Since $\mathcal{L}\lbrace e^{-kx}, s\rbrace = \frac{1}{s+k}, \Re(s) > -k$. Therefore,

$$\int_0^\infty x^{s-1} \sum_{k=0}^\infty \phi(k) \frac{(-x)^k}{k!} dx = \sum_{k=0}^\infty \phi(k) \frac{(-1)^k}{k!} \frac{1}{s+k}.$$
 (4.5)

We recall that from the well-known functional equation $\Gamma(s+1) =$ $s\Gamma(s)$, we have

$$\Gamma(s) = \frac{\Gamma(s+m+1)}{s(s+1)...(s+m)}.$$
(4.6)

Thus $\Gamma(s)$ has poles at $s=-m, \ m=0,1,2,...$ Thus $\lim_{s\to -m}(s+m)\Gamma(s)=\frac{(-1)^m}{m!}$ as $s\to -m$. Hence $\Gamma(s)\sim \frac{(-1)^m}{m!}\frac{1}{s+m}$. Consequently,

$$\phi(-s)\Gamma(s) \sim \phi(m) \frac{(-1)^m}{m!} \frac{1}{s+m} \text{ as } s \to -m.$$
 (4.7)

This means that $\phi(m) \frac{(-1)^m}{m!} \frac{1}{s+m}$ is a singular element of the function $\phi(-s)\Gamma(s)$ at s=-m. From the definition of the singular expansion of $\phi(-s)\Gamma(s)$, we obtain

$$\phi(-s)\Gamma(s) \approx \sum_{k=0}^{\infty} \phi(k) \frac{(-1)^k}{k!} \frac{1}{s+k}$$
(4.8)

and the proof of Ramanujan's Master Theorem is complete.

2. Let

$$f(x) = \begin{cases} \sum_{k=0}^{\infty} \phi(k)(-x)^k, & x > 0, \\ 0, & x \le 0. \end{cases}$$
 (4.9)

Thus

$$\int_0^\infty x^{s-1} \sum_{k=0}^\infty \phi(k) (-x)^k dx = \sum_{k=0}^\infty (-1)^k \phi(k) \frac{1}{s+k}.$$
 (4.10)

Proceeding as before, we have

$$\phi(-s)(-s)!\Gamma(s) \sim \phi(m)(-1)^m \frac{1}{s+m} \text{ as } s \to -m.$$
 (4.11)

This means that $\phi(m)(-1)^m \frac{1}{s+m}$ is a singular element of the function $\phi(-s)(-s)!\Gamma(s)$. From the definition of the singular expansion of $\phi(-s)(-s)!\Gamma(s)$, we obtain

$$\phi(-s)(-s)!\Gamma(s) \approx \sum_{k=0}^{\infty} (-1)^k \phi(k) \frac{1}{s+k}.$$
 (4.12)

Using the well-known property $(-z)!\Gamma(z) = \frac{\pi}{\sin \pi z}, \ z \neq 0, \pm 1, \pm 2,$, we get

$$\int_0^\infty x^{s-1} \sum_{k=0}^\infty \phi(k) (-x)^k dx = \frac{\pi}{\sin \pi s} \phi(-s), \tag{4.13}$$

which is the Hardy's version of the the RMT (Theorem (Hardy))[2].

References

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