Topological Strings and Quantum Curves

This work has been accomplished at the Institute for Theoretical Physics (ITFA) of the University of Amsterdam (UvA) and is financially supported by a Spinoza grant of the Netherlands Organisation for Scientific Research (NWO).





Cover illustration: Lotte Hollands Cover design: The DocWorkers

Lay-out: Lotte Hollands, typeset using \LaTeX

ISBN 978 90 8555 020 4

NUR 924

© L. Hollands / Pallas Publications — Amsterdam University Press, 2009

All rights reserved. Without limiting the rights under copyright reserved above, no part of this book may be reproduced, stored in or introduced into a retrieval system, or transmitted, in any form or by any means (electronic, mechanical, photocopying, recording or otherwise) without the written permission of both the copyright owner and the author of the book.

Topological Strings and Quantum Curves

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor

aan de Universiteit van Amsterdam

op gezag van de Rector Magnificus

prof. dr. D.C. van den Boom

ten overstaan van een door het college voor promoties

ingestelde commissie,

in het openbaar te verdedigen in de Agnietenkapel

op donderdag 3 september, te 14.00 uur

door

LOTTE HOLLANDS

geboren te Maasbree

PROMOTIECOMMISSIE

Promotor

prof. dr. R.H. Dijkgraaf

Overige leden

prof. dr. J. de Boer

prof. dr. A.O. Klemm

prof. dr. E.M. Opdam

dr. H.B. Posthuma

dr. S.J.G. Vandoren

prof. dr. E.P. Verlinde

Faculteit der Natuurwetenschappen, Wiskunde en Informatica

PUBLICATIONS

This thesis is based on the following publications:

R. Dijkgraaf, L. Hollands, P. Sułkowski and C. Vafa, Supersymmetric Gauge Theories, Intersecting Branes and Free Fermions, arXiv/0709.4446 [hep-th], JHEP 02 (2008) 106.

L. Hollands, J. Marsano, K. Papadodimas and M. Shigemori, *Nonsupersymmetric Flux Vacua and Perturbed N=2 Systems*, arXiv/0804.4006 [hep-th], JHEP 10 (2008) 102.

R. Dijkgraaf, L. Hollands and P. Sułkowski, *Quantum Curves and D-Modules*, arXiv/0810.4157 [hep-th], JHEP 11 (2009) 047.

M. Cheng and L. Hollands,

A Geometric Derivation of the Dyon Wall-Crossing Group,
arXiv/0901.1758 [hep-th], JHEP 04 (2009) 067.

Contents

1	Intr	oduction	1
	1.1	Fermions on Riemann surfaces	1
	1.2	Outline of this thesis	9
2	Cala	abi-Yau Geometry	15
	2.1	The Strominger-Yau-Zaslow conjecture	16
	2.2	The Fermat quintic	21
	2.3	Local Calabi-Yau threefolds	27
3	I-br	ane Perspective on Vafa-Witten Theory and WZW Models	31
	3.1	Instantons and branes	32
	3.2	Vafa-Witten twist on ALE spaces	42
	3.3	Free fermion realization	51
	3.4	Nakajima-Vafa-Witten correspondence	58
4	Тор	ological Strings, Free Fermions and Gauge Theory	69
	4.1	Curves in $\mathcal{N}=2$ gauge theories	70
	4.2	Geometric engineering	81
	4.3	Topological invariants of Calabi-Yau threefolds	89
	4.4	BPS states and free fermions	101
5	Qua	antum Integrable Hierarchies	113
	5.1	Semi-classical integrable hierarchies	114
	5.2	$\mathcal{D}\text{-modules}$ and quantum curves	119
	5.3	Fermionic states and quantum invariants	134

6	Quantum Curves in Matrix Models and Gauge Theory					
	6.1	Matrix model geometries	149			
	6.2	Conifold and $c=1$ string	163			
	6.3	Seiberg-Witten geometries	169			
	6.4	Discussion	185			
7	7 Dyons and Wall-Crossing					
	7.1	Compact curves of genus 1 and 2	190			
	7.2	Quarter BPS dyons	199			
	7.3	Wall-crossing in $\mathcal{N}=4$ theory	215			
8	8 Fluxes and Metastability					
	8.1	Ooguri-Ookouchi-Park formalism	237			
	8.2	Geometrically engineering the OOP formalism	238			
	8.3	Embedding in a larger Calabi-Yau	256			
	8.4	Concluding remarks	269			
A	A Level-rank duality 27					
Bibliography 2			277			
Sa	Samenvatting					
Ac	Acknowledgments 2					

Chapter 1

Introduction

The twentieth century has seen the birth of two influential theories of physics. On very small scales quantum mechanics is very successful, whereas general relativity rules our universe on large scales. Unfortunately, in regimes at small scale where gravity is nevertheless non-negligible—such as black holes or the big bang—neither of these two descriptions suffice. String theory is the best candidate for a unified theoretical description of nature to date.

However, strings live at even smaller scales than those governed by quantum mechanics. Hence string theory necessarily involves levels of energy that are so high that they cannot be simulated in a laboratory. Therefore, we need another way to arrive at valid predictions. Although string theory thus rests on physical arguments, it turns out that this framework carries rich mathematical structure. This means that a broad array of mathematical techniques can be deployed to explore string theory. Vice versa, physics can also benefit mathematics. For example, different physical perspectives can relate previously unconnected topics in mathematics.

In this thesis we are motivated by this fertile interaction. We both use string theory to find new directions in mathematics, and employ mathematics to discover novel structures in string theory. Let us illustrate this with an example.

1.1 Fermions on Riemann surfaces

This section briefly introduces the main ingredients of this thesis. We will meet all in much greater detail in the following chapters.

So-called Riemann surfaces play a prominent role in many of the fruitful interactions between mathematics and theoretical physics that have developed in

the second half of the twentieth century. Riemann surfaces are smooth twodimensional curved surfaces that have a number of holes, which is called their genus g. Fig. 1.1 shows an example.

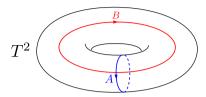


Figure 1.1: A compact Riemann surface with just one hole in it is called a 2-torus T^2 . We refer to its two 1-cycles as the A and the B-cycle.

Riemann surfaces additionally come equipped with a complex structure, so that any small region of the surface resembles the complex plane \mathbb{C} . An illustrative class of Riemann surfaces is defined by equations of the form

$$\Sigma: F(x,y) = 0$$
, where $x, y \in \mathbb{C}$.

Here, F can for instance be a polynomial in the complex variables x and y. A simple example is a (hyper)elliptic curve defined by

$$y^2 = p(x),$$

where p(x) is a polynomial. The curve is elliptic if p has degree 3 or 4, in which case its genus is g=1. For higher degrees of p it is hyperelliptic and has genus g>1.

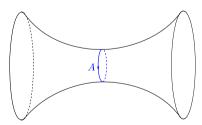


Figure 1.2: A simple example of a non-compact Riemann surface is defined by the equation $x^2 + y^2 = 1$ in the complex plane \mathbb{C}^2 .

The complex structure of a compact Riemann surface can be conveniently characterized by a period matrix. This is a symmetric square matrix τ_{ij} of complex numbers, whose rank equals the genus of the surface and whose imaginary part is strictly positive. The complex structure of a 2-torus, illustrated in Fig. 1.1, is

for example determined by one complex number τ that takes values in the upper half plane.

The period matrix encodes the contour integrals of the g independent holomorphic 1-forms over the 1-cycles on the surface. To this end one picks a canonical basis of A-cycles and B-cycles, where the only non-trivial intersection is $A_1 \cap B_j = \delta_{ij}$. The 1-forms ω_i may then be normalized by integrating them over the A-cycles, so that their integrals over the B-cycles determine the period matrix:

$$\int_{A_i} \omega_j = \delta_{ij}, \quad \int_{B_i} \omega_j = \tau_{ij}.$$

Conformal field theory

Riemann surfaces play a dominant role in the study of conformal field theories (CFT's). Quantum field theories can be defined on a space-time background M, which is usually a Riemannian manifold with a metric g. The quantum field theory is called conformal when it is invariant under arbitrary rescaling of the metric g. A CFT therefore depends only on the conformal class of the metric g.

The simplest quantum field theories study a bosonic scalar field ϕ on the background M. The contribution to the action for a massless scalar is

$$S_{\text{boson}} = \int_{M} \sqrt{g} g^{mn} \, \partial_{m} \phi \, \partial_{n} \phi,$$

yielding the Klein-Gordon equation $\partial_m \partial^m \phi = 0$ (for zero mass) as equation of motion. Note that, on the level of the classical action, this theory is clearly conformal in 2 dimensions; this still holds for the full quantum theory. Since a 2-dimensional conformal structure uniquely determines a complex structure, the free boson defines a CFT on any Riemann surface Σ .

Let us compactify the scalar field ϕ on a circle S^1 , so that it has winding modes along the 1-cycles of the Riemann surface. The classical part of the CFT partition function is determined by the solutions to the equation of motion. The holomorphic contribution to the classical partition function is well-known to be encoded in a Riemann theta function

$$\theta\left(\tau,\nu\right) = \sum_{p \in \mathbb{Z}^g} e^{2\pi i \left(\frac{1}{2} p^t \tau p + p^t \nu\right)},$$

where the integers p represent the momenta of ϕ that flow through the A-cycles of the 2-dimensional geometry.

Another basic example of a CFT is generated by a fermionic field ψ on a Riemann surface. Let us consider a chiral fermion $\psi(z)$. Mathematically, this field

transforms on the Riemann surface as a (1/2,0)-form, whence

$$\psi(z)\sqrt{dz} = \psi(z')\sqrt{dz'},$$

where z and z' are two complex coordinates. Such a chiral fermion contributes to the 2-dimensional action as

$$S_{\text{fermion}} = \int_{\Sigma} \psi(z) \, \overline{\partial}_A \psi(z),$$

where $\partial_A = \partial + A$, and A is a connection 1-form on Σ . This action determines the Dirac equation $\overline{\partial}_A \psi(z) = 0$ as its equation of motion. The total partition function of the fermion field $\psi(z)$ is computed as the determinant

$$Z_{\text{fermion}} = \det(\overline{\partial}_A).$$

Remarkably, this partition function is also proportional to a Riemann theta function. The fact that this is not just a coincidence, but a reflection of a deeper symmetry between 2-dimensional bosons and chiral fermions, is known as the boson-fermion correspondence.

Integrable hierarchy

Fermions on Riemann surfaces are also familiar from the perspective of integrable systems. Traditionally, integrable systems are Hamiltonian systems

$$\dot{x}_i = \{x_i, H\} = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = \{p_i, H\} = \frac{\partial H}{\partial x_i},$$

with coordinates x_i , momenta p_i and a Hamiltonian H, for which there exists an equal amount of integrals of motion I_i such that

$$\dot{I}_i = \{I_i, H\} = 0, \quad \{I_i, I_j\} = 0.$$

The left equation requires the integrals I_i to be constants of motion whereas the right one forces them to commute among each other.

A simple example is the real two-dimensional plane \mathbb{R}^2 with polar coordinates r and ϕ , see Fig. 1.3. Take the Hamiltonian H to be the radius r. Then H=r is itself an integral of motion, so that the system is integrable. Notice that H is constant on the flow generated by the differential $\partial/\partial\phi$.

A characteristic example of an integrable system that is intimately related to Riemann surfaces and theta-functions is the Korteweg de Vries (KdV) hierarchy. Although this system was already studied by Korteweg and de Vries at the end

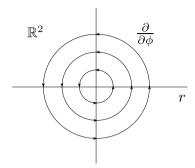


Figure 1.3: The two-dimensional plane \mathbb{R}^2 seen as an integrable system.

of the 19th century as a non-linear partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x},$$

it was only realized later that it is part of a very rich geometric and algebraic structure. It is impossible to do justice to this beautiful story in this short introduction. Instead, let us just touch on the aspects that are relevant for this thesis.

Geometrically, a special class of solutions to the KdV differential equation yields linear flows over a 2g-dimensional torus that is associated to a Riemann surface Σ . This torus is called the Jacobian. Its complex structure is determined by the period matrix τ_{ij} of Σ . The Riemann surface Σ is called the spectral curve of the KdV hierarchy.

The KdV spectral curve is an elliptic curve

$$Q^2 = P^3 - g_2 P - g_3,$$

where g_2 and g_3 are the Weierstrass invariants. The coordinates P and Q on this elliptic curve can best be described as commuting differential operators

$$[P,Q]=0,$$

that arise naturally when the KdV differential equation is written in the form of a Hamiltonian system.

As a side remark we notice that the KdV system is closely related to the Hitchin integrable system, which studies certain holomorphic bundles over a complex curve C. In this integrable system, too, the dynamics can be expressed in terms of a linear flow on the Jacobian associated to a spectral cover Σ of C.

We go one step beyond the geometrical structure of the KdV system when intro-

ducing chiral fermions on the spectral curve Σ . Their partition function $\det(\overline{\partial})$ transforms as a section of the determinant line bundle over the moduli space of the integrable hierarchy. It is known as the tau-function. The tau-function is proportional to the theta-function associated to the period matrix τ_{ij} .

Gauge theory and random matrices

Similar integrable structures have been found in wide variety of physical theories recently. An important ingredient of this thesis is the appearance of an auxiliary Riemann surface in 4-dimensional gauge theories. The basic field in a gauge theory is a gauge field A, which is mathematically a connection 1-form of a principal G-bundle over the 4-manifold M. Let us take G=U(1). In that situation the classical equations of motion reproduce the Maxwell equations. Once more, the holomorphic part of the gauge theory partition function is essentially a theta-function

$$\int DA \ e^{\int_{M} \frac{1}{2} \tau F_{+} \wedge F_{+}} \sim \sum_{n_{+}} e^{\frac{1}{2} \tau p_{+}^{2}},$$

where τ is the complex coupling constant of the gauge theory, and p_+ are the fluxes of the self-dual field strength F_+ through the 2-cycles of M. It turns out to be useful to think of τ as the complex structure parameter of an auxiliary elliptic curve. In other gauge theories a similar auxiliary curve is known to play a significant role as well. We will explain this in detail in the main body of this thesis.

Another interesting application is the theory of random matrices. This is a 0-dimensional quantum field theory based on a Hermitean N-by-N-matrix X. The simplest matrix model is the so-called Gaussian one, whose action reads

$$S_{
m matrix} = -rac{1}{2\lambda}{
m Tr}X^2,$$

where λ is a coupling constant. Other matrix models are obtained by adding higher order interactions to the matrix model potential $W(X) = X^2/2$. Such a matrix model can be conveniently evaluated by diagonalizing the matrices X. This reduces the path integral to a standard integral over eigenvalues x_i , yet adds the extra term

$$\frac{1}{N^2} \sum_{i,j} \log|x_i - x_j|$$

to the matrix model action. Depending on the values of the parameters N and λ the eigenvalues will either localize on the minima of the potential W(X) or, oppositely, spread over a larger interval. When one lets N tend to infinity while

keeping the 't Hooft parameter $\mu=N\lambda$ fixed, the eigenvalues can be seen to form a smooth distribution. For instance, in the Gaussian matrix model, the density of eigenvalues is given by the Wigner-Dyson semi-circle

$$\rho(x) \sim \sqrt{4\mu - x^2}$$

leading to the algebraic curve

$$x^2 + y^2 = 4\mu.$$

Likewise, the eigenvalue distribution of a more general matrix model takes the form of a hyperelliptic curve in the limit $N \to \infty$ with fixed 't Hooft parameter. Many properties of the matrix model are captured in this spectral curve.

Topological string theory

String theory provides a unifying framework to discuss all these models. In particular, topological string theory studies embeddings of a Riemann surface C—which shouldn't be confused with the spectral curve—into 6-dimensional target manifolds X of a certain kind. These target manifolds are called Calabi-Yau manifolds. The Riemann surface C is the worldvolume swept out in time by a 1-dimensional string. The topological string partition function is a series expansion

$$Z_{\text{top}}(\lambda) = \exp\left(\sum_{g} \lambda^{2g-2} \mathcal{F}_g\right),$$

where λ is called the topological string coupling constant. Each \mathcal{F}_g contains the contribution of curves C of genus g to the partition function.

Calabi-Yau manifolds are not well understood in general, and studying topological string theory is very difficult task. Nevertheless, certain classes of noncompact Calabi-Yau manifolds are much easier to analyse, and contain precisely the relevant backgrounds to study the above integrable structures.

Consider an equation of the form

$$X_{\Sigma}: \quad uv - F(x,y) = 0,$$

where u and v are \mathbb{C} -coordinates and

$$\Sigma: F(x,y) = 0$$

defines the spectral curve Σ in the complex (x,y)-plane. The 6-dimensional manifold X_{Σ} may be regarded as a \mathbb{C}^* -fibration over the (x,y)-plane that degenerates over the Riemann surface F(x,y)=0.

Indeed, the fiber over a point (x,y) is defined by uv = F(x,y). Over a generic point in the base this fiber is a hyperboloid. However, when $F(x,y) \to 0$ the hyperboloid degenerates to a cone. This is illustrated in Fig. 1.4. The zero locus of the polynomial F(x,y) thus determines the degeneration locus of the fibration. We therefore refer to the variety X_{Σ} as a non-compact Calabi-Yau threefold modeled on a Riemann surface.

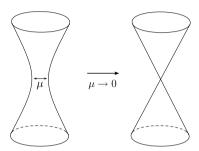


Figure 1.4: A local Calabi-Yau threefold defined by an equation of the form uv - F(x, y) = 0 can be viewed as a \mathbb{C}^* -fibration over the base parametrized by x and y. The fibers are hyperboloids over general points in the base, but degenerate when $\mu = F(x, y) \to 0$.

When the Riemann surface Σ equals the auxiliary Riemann surface that emerges in the study of gauge theories, the topological string partition function is known to capture important properties of the gauge theory. Furthermore, when it equals the spectral curve in the matrix model, it is known to capture the full matrix model partition function.

Quantum curves and \mathcal{D} -modules

However, topological string computations go well beyond computing a fermion determinant on Σ . In fact, it is known that the fermion determinant appears as the genus 1 part \mathcal{F}_1 of the free energy. We call this the semi-classical part of the free energy, as it remains finite when $\lambda \to 0$. Similarly, \mathcal{F}_0 is the classical contribution that becomes very large in this limit, and all higher \mathcal{F}_g 's encode the quantum corrections. In other words, we may interpret the loop parameter λ as a quantization of the semi-classical tau-function. But what does this mean in terms of the underlying integrable system? Answering this question is one of the goals in this thesis.

In both the gauge theory and the matrix model several hints have been obtained. In the context of gauge theories the higher loop corrections \mathcal{F}_g are known to correspond to couplings of the gauge theory to gravity. Mathematically, these appear by making the theory equivariant with respect to an SU(2)-action on the underlying 4-dimensional background. N. Nekrasov and A. Okounkov have

shown that the partition function of this theory is the tau-function of an integrable hierarchy.

In the theory of random matrices, λ -corrections correspond to finite N corrections. In this theory, too, it has been found that the partition function behaves as the tau-function of an integrable hierarchy. However, like in the gauge theory examples, there is no interpretation of the full partition function as a fermion determinant. Instead, a new perspective arises, in which the spectral curve is be replaced by a non-commutative spectral curve. In matrix models that relate to the KdV integrable system, it is found that the KdV differential operators P and Q do not commute anymore when the quantum corrections are taken into account:

$$[P,Q] = \lambda.$$

Similar hints have been found in the study of topological string theory on general local Calabi-Yau's modeled on a Riemann surface.

This thesis introduces a new physical perspective to study topological string theory, clarifying the interpretation of the parameter λ as a "non-commutativity parameter". We show that λ quantizes the Riemann surface Σ , and that fermions on the curve are sections of a so-called \mathcal{D} -module instead. This attributes Fourier-like transformation properties to the fermions on Σ :

$$\psi(y) = \int dy \, e^{xy/\lambda} \, \psi(x).$$

1.2 Outline of this thesis

Chapter 2 starts with an introduction to the geometry of Calabi-Yau threefolds. They serve as an important class of backgrounds in string theory, that can be used to make contact with the 4-dimensional world we perceive. Specifically, we explain the idea of a Calabi-Yau compactification, and introduce certain non-compact Calabi-Yau backgrounds that appear in many guises in this thesis.

Topological string theory on these non-compact Calabi-Yau backgrounds is of physical relevance in the description of 4-dimensional supersymmetric gauge theories and supersymmetric black holes. For example, certain holomorphic corrections to supersymmetric gauge theories are known to have an elegant description in terms of an auxiliary Riemann surface Σ . A compactification of string theory on a local Calabi-Yau that is modeled on this Riemann surface "geometrically engineers" the corresponding gauge theory in four dimensions. Topological string amplitudes capture the holomorphic corrections.

In Chapter 3 and 4 we elucidate these relations by introducing a web of dualities

in string theory. This web is given in full detail in Fig. 1.6. Most important are its outer boxes, that are illustrated pictorially in Fig. 1.5. They correspond to string theory embeddings of the theories we mentioned above. The upper right box is a string theory embedding of supersymmetric gauge theory, whereas the lowest box is a string theory embedding of topological string theory.

The main objective of the duality web is to relate both string frames to the upper left box, that describes a string theory configuration of *intersecting D-branes*. The most relevant feature of this intersecting brane configuration is that the D-branes intersect over a Riemann surface Σ , which is the same Riemann surface that underlies the gauge theory and appears in topological string theory. We explain that the duality chain gives a dual description of topological string theory in terms of a 2-dimensional quantum field theory of free fermions that live on the 2-dimensional intersecting brane, the so-called I-brane, wrapping Σ .

Chapter 3 studies 4-dimensional gauge theories that preserve a maximal amount of supersymmetry. They are called $\mathcal{N}=4$ supersymmetric Yang-Mills theories. Instead of analyzing this theory on flat \mathbb{R}^4 , we consider more general ALE backgrounds. These non-compact manifolds are resolved singularities of the type

$$\mathbb{C}^2/\Gamma$$
,

where Γ is a finite subgroup of SU(2) that acts linearly on \mathbb{C}^2 . In the midnineties it has been discovered that the gauge theory partition function on an ALE space computes 2-dimensional CFT characters. In Chapter 3 we introduce a string theoretic set-up that explains this duality between 4-dimensional gauge theories and 2-dimensional CFT's from a higher standpoint.

In Chapter 4 we extend the duality between supersymmetric gauge theories and intersecting brane configurations to the full duality web in Fig. 1.6. We start out with 4-dimensional $\mathcal{N}=2$ gauge theories, that preserve half of the supersymmetry of the above $\mathcal{N}=4$ theories, and line out their relation to the intersecting brane configuration and to local Calabi-Yau compactifications. Furthermore, we relate several types of objects that play an important role in the duality sequence, and propose a partition function for the intersecting brane configuration.

The I-brane configuration emphasizes the key role of the auxiliary Riemann surface Σ in 4-dimensional gauge theory and topological string theory. All the relevant physical modes are localized on the intersecting brane that wraps Σ . These modes turn out to be free fermions. In Chapter 5 and 6 we describe the resulting 2-dimensional quantum field on Σ in the language of integrable hierarchies.

We connect local Calabi-Yau compactifications to the Kadomtsev-Petviashvili, or shortly, KP integrable hierarchy, which is closely related to 2-dimensional free fermion conformal field theories. To any semi-classical free fermion system on a curve Σ it associates a solution of the integrable hierarchy. This is well-known

as a Krichever solution.

In Chapter 5 we explain how to interpret the topological string partition function on a local Calabi-Yau X_{Σ} as a quantum deformation of a Krichever solution. We discover that the curve Σ should be replaced by a quantum curve, defined as a differential operator P(x) obtained by quantizing $y=\lambda\partial/\partial x$. Mathematically, we are led to a novel description of topological string theory in terms of \mathcal{D} -modules.

Chapter 6 illustrates this formalism with several examples related to matrix models and gauge theory. In all these examples there is a canonical way to quantize the curve as a differential operator. Moreover, we can simply read off the known partition functions from the associated \mathcal{D} -modules. This tests our proposal.

The topological string partition function has several interpretations in terms of topological invariants on a Calabi-Yau threefold. Moreover, its usual expansion in the topological string coupling constant λ is only valid in a certain regime of the background Calabi-Yau parameters. Although the topological invariants stay invariant under small deformations in these background parameters, there are so-called *walls of marginal stability*, where the index of these invariants may jump. This phenomenon has recently attracted much attention among both physicists and mathematicians.

Chapter 7 studies wall-crossing in string theory compactifications that preserve $\mathcal{N}=4$ supersymmetry. The invariants in this theory that are sensitive to wall-crossing are called quarter BPS states. They have been studied extensively in the last years. In Chapter 7 we work out in detail the relation of these supersymmetric states to a genus 2 surface Σ . Moreover, we find that wall-crossing in this theory has a simple and elegant interpretation in terms of the genus 2 surface.

Finally, Chapter 8 focuses on supersymmetry breaking in Calabi-Yau compactifications. Since supersymmetry is not an exact symmetry of nature, we need to go beyond Calabi-Yau compactifications to find a more accurate description of nature. One of the possibilities is that supersymmetry is broken at a lower scale than the compactification scale. In Chapter 8 we study such a supersymmetry breaking mechanism for $\mathcal{N}=2$ supersymmetric gauge theories. These gauge theories are related to type II string compactifications on a local Calabi-Yau X_{Σ} . By turning on non-standard fluxes (in the form of generalized gauge fields) on the underlying Riemann surface Σ we find a potential on the moduli space of the theory. We show that this potential generically has non-supersymmetric minima.

Chapter 3 and 4 explain the ideas of the publication [1], whereas Chapter 5 and 6 contain the results of [2]. The first part of Chapter 7 includes examples from [1], whereas the second part is based on the article [3]. Finally, Chapter 8 is a slightly shortened version of the work [4].

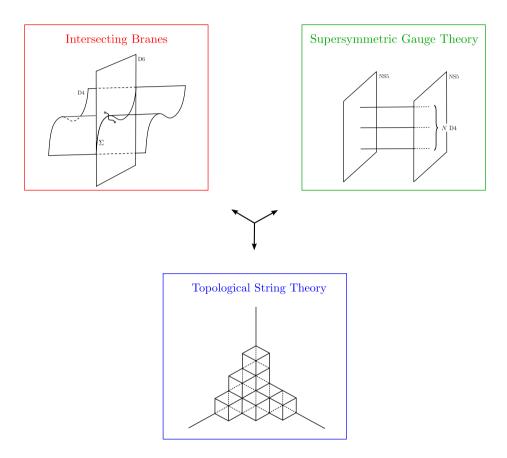


Figure 1.5: The web of dualities relates 4-dimensional supersymmetric gauge theory to topological string theory and to an intersecting brane configuration.

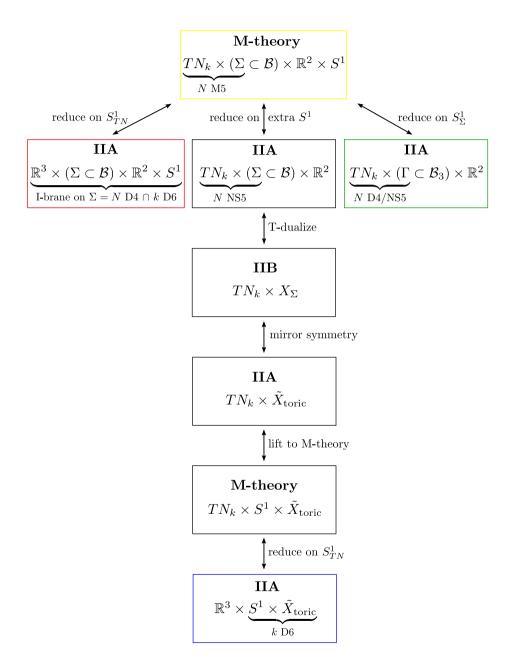


Figure 1.6: Web of dualities.

Chapter 2

Calabi-Yau Geometry

Although string theory lives on 10-dimensional backgrounds, it is possible to make contact with the 4-dimensional world we perceive by shrinking 6 out of the 10 dimensions to very small scales. Remarkably, this also yields many important examples of interesting interactions between physics and mathematics. The simplest way to compactify a 10-dimensional string background to 4 dimensions is to start with a compactification of the form $\mathbb{R}^4 \times X$, where X is a compact 6-dimensional Riemannian manifold and \mathbb{R}^4 represents Minkowski space-time.

Apart from the metric, string theory is based on several more quantum fields. Amongst them are generalized Ramond-Ramond (RR) gauge fields, the analogues of the famous Yang-Mills field in four dimensions. In a compactification to four dimensions these fields can be integrated over any cycle of the internal manifold X. This results in e.g. scalars and gauge fields, which are the building blocks of the standard model. This is illustrated in Fig. 2.1. Four-dimensional symmetries, like gauge symmetries, can thus be directly related to geometrical symmetries of the internal space. This geometrization of physics is a main theme in string theory, and we will see many examples in this thesis.

Although the topological properties of the compactification manifold X are restricted by the model that we try to engineer in four dimensions, other moduli of X, such as its size and shape, are a priori allowed to fluctuate. This leads to the so-called landscape of string theory, that parametrizes all the possible vacua. A way to truncate the possibilities into a discrete number of vacua is to introduce extra fluxes of some higher dimensional gauge fields. We will come back to this at the end of this thesis, in Chapter 8.

The best studied compactifications are those that preserve supersymmetry in four dimensions. This yields severe constraints on the internal manifold. Not only should it be provided with a complex structure, but also its metric should

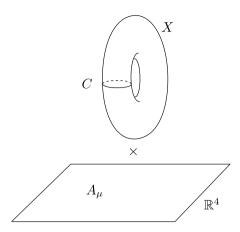


Figure 2.1: Compactifying string theory over an internal space X geometrizes 4-dimensional physics. Here we represented the 6-dimensional internal space as a 2-torus. The Yang-Mills gauge field A_{μ} on \mathbb{R}^4 is obtained by integrating a 10-dimensional RR 4-form over a 3-cycle C.

be of a special form. They are known as *Calabi-Yau* manifolds. Remarkably, this class of manifolds is also very rich from a mathematical perspective.

This chapter is meant to acquaint the reader with Calabi-Yau manifolds. The aim of Section 2.1 is to describe *compact* real 6-dimensional Calabi-Yau manifolds in terms of real 3-dimensional geometry. Section 2.2 illustrates this with the prime example of a compact Calabi-Yau manifold, the Fermat quintic. We find that its 3-dimensional representation is characterized by a Riemann surface. In Section 2.3 we explain how this places the simpler *local* or *non-compact* Calabi-Yau compactifications into context. These are studied in the main body of this thesis.

2.1 The Strominger-Yau-Zaslow conjecture

Calabi-Yau manifolds X are complex Riemannian manifolds with some additional structures. These structures can be characterized in a few equivalent ways. One approach is to combine the metric g and the complex structure J into a 2-form $k=g\circ J$. For Calabi-Yau manifolds this 2-form needs to be closed dk=0. It is called the $K\ddot{a}hler\ form$ and equips the Calabi-Yau with a Käher structure.

Moreover, a Calabi-Yau manifold must have a trivializable canonical bundle $K_X = \bigwedge^3 T^*X$, where T^*X is the holomorphic cotangent space of X. The canonical bundle is a line bundle over X. When it is trivializable the Calabi-Yau manifold contains a non-vanishing *holomorphic 3-form* Ω . Together, the Kähler

form k and the holomorphic 3-form Ω determine a Calabi-Yau manifold.

The only non-trivial cohomology of a Calabi-Yau X is contained in $H^{1,1}(X)$ and $H^{2,1}(X)$. These cohomology classes parametrize its moduli. The elements in $H^{1,1}(X)$ can change the Kähler structure of the Calabi-Yau infinitesimally, and are therefore called *Kähler moduli*. In string theory we usually complexify these moduli. On the other hand, $H^{2,1}(X)$ parametrizes the *complex structure moduli* of the Calabi-Yau. It is related to the choice of the holomorphic 3-form Ω , since contracting (2,1)-forms with Ω determines an isomorphism with the cohomology class $H^1_{\bar{\partial}}(X)$, that is well-known to characterize infinitesimal complex structure deformations.

Celebrated work of E. Calabi and S.-T. Yau shows that the Kähler metric g can be tuned within its cohomology class to a unique Kähler metric that satisfies $R_{mn}=0$. Calabi-Yau manifolds can therefore also be characterized by a unique Ricci-flat Kähler metric.

A tantalizing question is how to visualize a Calabi-Yau threefold. The above definition in terms of a Kähler form k and a holomorphic 3-form Ω is rather abstract. Is there a more concrete way to picture a Calabi-Yau manifold?

The SYZ conjecture

From the string theory perspective there are many relations between different string theory set-ups. These are known as *dualities*, and relate the different incarnations of string theory (type I, type IIA/B, heterotic, M-theory and F-theory) and different background parameters. One of the famous dualities in string theory connects type IIB string theory compactified on some Calabi-Yau X with type IIA string theory compactified on another Calabi-Yau X, where both Calabi-Yau manifolds are related by swapping their Kähler and complex structure parameters. This duality is called *mirror symmetry*. It suggests that any Calabi-Yau threefold X has a mirror X such that

$$H^{1,1}(X) \cong H^{2,1}(\tilde{X})$$
 and $H^{2,1}(X) \cong H^{1,1}(\tilde{X})$.

Although the mirror conjecture doesn't hold exactly as it is stated above, many mirror pairs have been found and much intuition has been obtained about the underlying Calabi-Yau geometry.

One of the examples that illustrates this best is known as the *Strominger-Yau-Zaslow (SYZ) conjecture* [5]. The starting point for this argument is mirror symmetry in type II string theory. Without explaining in detail what type II string theory is, we just note that an important role in this theory is played by so-called D-branes. These branes can be considered mathematically as submanifolds of the Calabi-Yau manifold with certain U(1) bundles on them.

The submanifolds are even (odd) dimensional in type IIA (type IIB) string theory, and they are restricted by supersymmetry. Even-dimensional cycles need to be holomorphic, whereas 3-dimensional cycles have to be of *special Lagrangian* type. (Note that these are the only odd-dimensional cycles in case the first Betti number b_1 of the Calabi-Yau is zero.) A special Lagrangian 3-cycle C is defined by the requirement that

$$k|_C = 0$$
 as well as $\operatorname{Im} \Omega|_C = 0$.

Both the holomorphic and the special Lagrangian cycles minimize the volume in their homology class, and are therefore stable against small deformations. This turns them into supersymmetric cycles.

Mirror symmetry not only suggests that the backgrounds X and \tilde{X} are related, but that type IIA string theory compactified on \tilde{X} is equivalent to type IIB string theory on X. This implies in particular that moduli spaces of supersymmetric D-branes should be isomorphic. In type IIB theory in the background $\mathbb{R}^4 \times X$ the relevant D-branes are mathematically characterized by special Lagrangian submanifolds in X with flat U(1)-bundles on them. In type IIA theory there is one particularly simple type of D-branes: D0-branes that just wrap a point in \tilde{X} . Their moduli space is simply the space \tilde{X} itself. Since mirror symmetry maps D0-branes into D3-branes that wrap special Lagrangian cycles in X, the moduli space of these D3-branes, i.e. the moduli space of special Lagrangian submanifolds with a flat U(1) bundle on it, should be isomorphic to \tilde{X} .

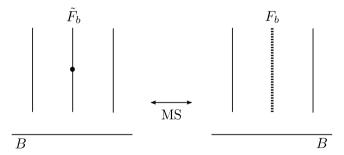


Figure 2.2: Schematic illustration of an SYZ fibration of \tilde{X} (on the left) and of X (on the right). Mirror symmetry (MS) acts by mapping each point in \tilde{X} to a special Lagrangian cycle of X. In the SYZ fibration of \tilde{X} this point is part of a fiber \tilde{F}_b over some $b \in B$. Mirror symmetry sends it to the total fiber F_b , over $b \in B$, of the SYZ fibration of X.

Special Lagrangian deformations of a special Lagrangian submanifold C are well-known to be characterized by the first cohomology group $H^1(C,\mathbb{R})$. On the other hand, flat U(1)-connections moduli gauge equivalence are parametrized by $H^1(C,\mathbb{R})/H^1(C,\mathbb{Z})$. The first Betti number of C therefore has to match the complex dimension of \tilde{X} , which is 3. So mirror symmetry implies that \tilde{X} must

admit a 3-dimensional fibration of 3-tori $T^3=(S^1)^3$ over a 3-dimensional base space that parametrizes the special Lagrangian submanifolds in X. Of course, the same argument proves the converse. We thus conclude that both X and \tilde{X} are the total space of a special Lagrangian fibration of 3-tori over some real three-dimensional base B.

Such a special Lagrangian fibration is called an SYZ fibration. It is illustrated in Fig. 2.2. Notice that generically this fibration is not smooth over the whole base B. We refer to the locus $\Gamma \subset B$ where the fibers degenerate as the singular, degenerate or discriminant locus of the SYZ fibration. Smooth fibers F_b and \tilde{F}_b , over some point $b \in B$, can be argued to be dual in the sense that $H^1(F_b, \mathbb{R}/\mathbb{Z}) = \tilde{F}_b$ and vice versa, since flat U(1) connections on a torus are parametrized by the dual torus.

Large complex structure

Mirror symmetry, as well as the SYZ conjecture, is not exactly true in the way as stated above. More precisely, it should be considered as a symmetry only around certain special points in the Kähler and complex structure moduli space. In the complex structure moduli space these are the large complex structure limit points, and in the Kähler moduli space the large Kähler structure limit points. Mirror symmetry will exchange a Calabi-Yau threefold near a large complex structure point with its mirror near a large Kähler structure limit point. Moreover, only in these regions the SYZ conjecture may be expected to be valid.

Topological aspects of type II string theory compactifications can be fully described in terms of either the Kähler structure deformations or the complex structure deformations of the internal Calabi-Yau manifold. In the following we will focus solely on describing the Kähler structure of an SYZ fibered Calabi-Yau threefold. Starting with this motivation we can freely go to a large complex structure point on the complex structure moduli space.

How should we think of such a large complex structure? Let us illustrate this with a 2-torus T^2 . This torus is characterized by one Kähler parameter t, which is the volume of the torus, and one complex structure parameter τ . When performing a mirror symmetry, these two parameters are exchanged. When we send the complex structure to infinity $\tau\mapsto i\infty$, the torus degenerates into a nodal torus. See Fig. 2.3. However, in this limit the volume also blows up. When we hold this volume fixed, by rescaling the Ricci flat metric $ds^2=\frac{t}{\tau_2}dz\otimes d\bar{z}$, the torus will collapse to a line. This line should be viewed as the base of the SYZ fibration.

For a general Calabi-Yau threefold a similar picture is thought to be true. When approaching a large complex structure point, the Calabi-Yau threefold should have a description as a special Lagrangian fibration over a 3-sphere whose T^3 -fibers get very small. The metric on these fibers is expected to become flat.

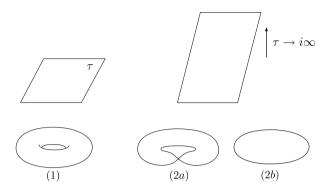


Figure 2.3: In picture (1) we see a regular torus with complex structure τ : below the torus itself and above its fundamental domain (that is obtained by removing the A and B-cycle from the torus). In the large complex structure limit τ is sent to $i\infty$. In picture (2a) this degeneration is illustrated topologically and in (2b) metrically: when we hold the volume fixed, the torus collapses to a circle.

Moreover, the singular locus of the fibration is thought to be of codimension two in the base in that limit, so that it defines a 1-dimensional graph on the base of the SYZ fibration.

In recent years much progress has been made in the mathematical study of the SYZ conjecture. Comprehensive reviews can be found in e.g. [6, 7, 8]. Especially the topological approach of M. Gross and others, with recent connections to tropical geometry, seems very promising. The main idea in this program is to characterize the SYZ fibration by a natural affine structure on its base B.

This affine structure is found as follows. Once we pick a zero-section of the (in particular) Lagrangian fibration, we can contract the inverse symplectic form with a one-form on the base B to find a vector field along the fiber. An integral affine structure can then be defined on B by an integral affine function f whose derivative df defines a time-one flow along the fiber that equals the identity. The affine structure may be visualized as a lattice on the base B.

Although this affine structure will be trivial in a smooth open region of the fibration, it contains important information near the singular locus. Especially the monodromy of the T^3 -fibers is encoded in it. In the large complex structure limit all non-trivial symplectic information of the SYZ fibration is thought to be captured in the monodromy of the T^3 -fibers around the discriminant locus of the fibration [9, 7]. The SYZ conjecture thus leads to a beautiful picture of how a Calabi-Yau threefold may be visualized as an affine real 3-dimensional space with an additional integral monodromy structure around an affine 1-dimensional degeneration locus (see also [10]).

One of the concrete results that have been accomplished is an SYZ description

of elliptic K3-surfaces [11] and Calabi-Yau hypersurfaces in toric varieties (in a series of papers by W.-D. Ruan starting with [12]). Let us illustrate this with the prime example of a compact Calabi-Yau threefold, the Fermat quintic. This is also the first Calabi-Yau for which a mirror pair was found [13]. On a first reading it is not necessary to go through all of the formulas in this example; looking at the pictures should be sufficient.

2.2 The Fermat quintic

The Fermat quintic is defined by an equation of degree 5 in projective space \mathbb{P}^4 :

$$X_{\mu}: \sum_{k=1}^{5} z_k^5 - 5\mu \prod_{k=1}^{5} z_k = 0,$$

where $[z] = [z_1 : \ldots : z_5]$ are projective coordinates on \mathbb{P}^4 . Here μ denotes the complex structure of the threefold.

Pulling back the Kähler form of \mathbb{P}^4 provides the Fermat quintic with a Kähler structure. Moreover, the so-called adjunction formula shows that X_μ has a trivial canonical bundle. So the Fermat quintic X_μ is a Calabi-Yau threefold. Note as well that \mathbb{P}^4 admits an action of T^4 that is parametrized by five angles θ_k with $\sum_k \theta_k = 0$:

$$T^4: [z_1:\ldots:z_5] \mapsto [e^{i\theta_1}z_1:\ldots:e^{i\theta_5}z_5]$$

Such a variety is called toric. The quintic is thus embedded as a hypersurface in a toric variety.

When we take $\mu \to \infty$ we reach the large complex structure limit point

$$X_{\infty}: \quad \prod_{k=1}^{5} z_k = 0.$$

This is just a union of five \mathbb{P}^3 's, that each inherit a T^3 -action from the above toric action on \mathbb{P}^4 . From this observation it simply follows that X_∞ can be seen as a SYZ fibration. We can make this explicit by considering the fibration $\pi:\mathbb{P}^4\to\mathbb{R}^4$ given by

$$\pi([z]) = \sum_{k=1}^{5} \frac{|z_k|^2}{\sum_{l=1}^{5} |z_l|^2} p_k,$$

where the p_k are five generic points in \mathbb{R}^4 . The image of F is a 4-simplex Δ spanned by the five points p_k in \mathbb{R}^4 and X_{∞} is naturally fibered over the bound-

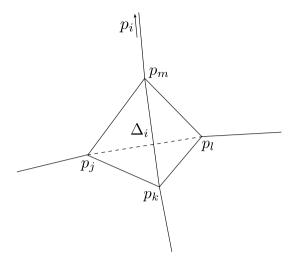


Figure 2.4: The base of the SYZ fibration of the Fermat quintic X_{μ} is the boundary $\partial \Delta$ of a 4-simplex Δ . To be able to draw this base we have placed the vertex p_i at infinity. The tetrahedron Δ_i is the projection of the subset $\{z_i = 0\} \subset \mathbb{P}^4$. The boundary $\partial \Delta$ consists of 5 such tetrahedrons.

ary $\partial \Delta$ with generic fiber being T^3 . The base of the SYZ fibration is thus topologically a 3-sphere S^3 . Similar to the large complex structure limit of a 2-torus, the Ricci-flat metric on X_{∞} is degenerate and the SYZ fibers are very small.

Let us introduce some notation to refer to different patches of \mathbb{P}^4 and Δ . Call D_{i_1,\ldots,i_n} the closed part of \mathbb{P}^4 where z_{i_1} up to z_{i_n} vanish, and denote its projection to \mathbb{R}^4 by Δ_{i_1,\ldots,i_n} . The ten faces of the boundary $\partial\Delta$ are thus labeled by Δ_{ij} , with $1 \leq i,j \leq 5$.

In the following it is also useful to introduce a notation for the S^1 -cycles that are fibered over the base Δ . So define the circles

$$\gamma_i^k = \{ z_i = 0, |\frac{z_k}{z_j}| = a_1, \frac{z_l}{z_j} = a_2, \frac{z_m}{z_j} = a_3 \},$$

where the indices $\{i,j,k,l,m\}$ are a permutation of $\{1,\ldots,5\}$ and the numbers $a_1,\ a_2,\ a_3$ are determined by choosing a base point [z] on the circle. Since a different choice for the index j leaves the circle invariant, it is not included as a label in the name. Which circle shrinks to zero-size over each cell of the boundary $\partial\Delta$ is summarized in Fig. 2.5. Notice that the only non-vanishing circle in the fibers over the triple intersection Δ_{ijk} is the circle γ_i^l (which may also be denoted as $\gamma_i^l, \gamma_k^l, -\gamma_i^m, -\gamma_i^m \text{ or } -\gamma_k^m$).

W.-D. Ruan works out how to generate a Lagrangian fibration for any X_{μ} where μ is large [12]. His idea is to use a gradient flow that deforms the Lagrangian fibration of X_{∞} into a Lagrangian fibration of X_{μ} . Let us consider the region D_i

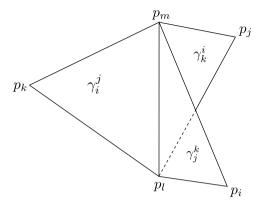


Figure 2.5: The boundary of each tetrahedron is a union of four triangles. The T^3 -fibration of the Fermat quintic degenerates over these triangles. This figure illustrates which cycle in the T^3 -fiber vanishes over which triangle in $\partial \Delta$: γ_i^j degenerates over the triangle Δ_{ij} , γ_j^k over Δ_{jk} , etc.

and choose $z_j \neq 0$ for some $j \neq i$. In this patch $x_k = z_k/z_j$ are local coordinates. It turns out that the flow of the vector field

$$V = \operatorname{Re} \Bigl(\frac{\sum_{k \neq i,j} x_k^5 + 1}{\prod_{k \neq i,j} x_k} \frac{\partial}{\partial x_i} \Bigr)$$

produces a Lagrangian fibration of X_{μ} over $\partial \Delta$.

All the smooth points of X_{∞} , i.e. those for which only one of the coordinates z_i vanishes, will be transformed into regular points of the Lagrangian fibration. Only the points in the intersection of X_{μ} with the singular locus of X_{∞} won't move with the flow. These can be shown to form the complete singular locus of the Lagrangian fibration of X_{μ} . Let us call this singular locus Σ and denote

$$\Sigma_{ij} = D_{ij} \cap \Sigma = \{ [z] \mid z_i = z_j = 0, z_k^5 + z_l^5 + z_m^5 = 0 \}.$$

The singular locus Σ_{ij} is thus a projective curve which has genus 6. It intersects D_{ijk} at the five points. The image of Σ_{ij} under the projection π is a deformed triangle in Δ_{ij} that intersects the boundary lines of Δ_{ij} once. This is illustrated in Fig. 2.6.

In the neighborhood of an inverse image of the intersection point $\frac{1}{2}(p_l+p_m)$ in D_{ij} the coordinate z_k gets very small. This implies that if we write $z_k=r_ke^{i\phi_k}$, it is the circle parametrized by ϕ_k that wraps the leg of the pair of pants in the limit $r_k\to 0$. In the notation we introduced before this circle is $\gamma_j^k=\gamma_i^k$.

The study of the cycles in the fibration reveals the structure of the singular locus. It is built out of two kinds of 3-vertices. We call a 3-vertex whose center lies at an edge D_{ijk} a plus-vertex and a 3-vertex that lies in the interior of some 2-cell

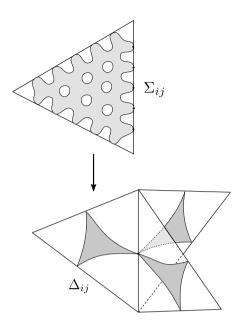


Figure 2.6: The shaded area in this figure illustrates the discriminant locus of the Lagrangian fibration of the quintic X_{μ} , for large μ . In the total space it has the shape of a genus 6 Riemann surface Σ_{ij} over each 2-cell Δ_{ij} . The five dots on the Riemann surface project to a single dot on the base.

 D_{ij} a minus-vertex. The plus-vertex is described by a different degenerating cycle at each of the three legs. Together they sum up to zero. The minus-vertex is characterized by a single vanishing cycle. The precise topological picture is shown in Fig. 2.7.

So in the large complex structure limit $\mu \to \infty$ the quintic has an elegant structure in terms of ten transversely intersecting genus 6 Riemann surfaces or equivalently in terms of 50 plus-vertices and 250 minus-vertices.

Two kinds of vertices

The structure of the singular locus of the Fermat quintic in the above example, ten genus 6 Riemann surfaces that intersect each other transversely, is very elegant. It is remarkable that it may be described by just two types of 3-vertices. This immediately raises the question whether this is accidental or a generic feature of SYZ fibrations of Calabi-Yau threefolds. In fact, M. Gross shows that under reasonable assumptions there are just a few possibilities for the topological structure in the neighborhood of the discriminant locus [9] (see also [7]). Indeed only two types of trivalent vertices can occur. Both are characterized by the monodromy that acts on three generators γ_i of the homology $H_1(F,\mathbb{Z})$ of the T^3 -fiber F when we transport them around each single leg of the vertex. Let

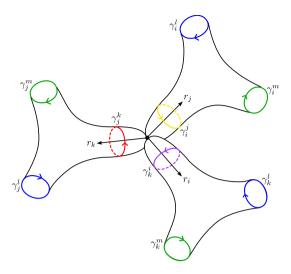


Figure 2.7: This picture shows one plus-vertex surrounded by three minus-vertices. It illustrates the discriminant locus in the neighborhood of an inverse image of the center of Δ_{ijk} , marked by a dot in the center of the plus-vertex. At this point the three genus 6 surfaces Σ_{ij} , Σ_{jk} and Σ_{ki} meet transversely. Notice that e.g. $\gamma_j^k + \gamma_j^l + \gamma_j^m = 0$ and $\gamma_i^j + \gamma_j^k + \gamma_k^i = 0$.

us summarize this monodromy in three matrices M_{α} such that

$$\gamma_i \mapsto \gamma_j(M_\alpha)_{ji}$$

when we encircle the α th leg of the vertex.

The first type of trivalent vertex is described by the three monodromy matrices

$$M_+: \left(egin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(egin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right), \left(egin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array} \right).$$

This vertex is characterized by a a distinguished $T^2 \subset T^3$ that is generated by γ_1 and γ_2 and stays invariant under the monodromy. Only the element γ_3 picks up a different cycle at each leg. This latter cycle must therefore be a vanishing cycle at the corresponding leg.

The legs of this vertex can thus be labeled by the cycles γ_1 , γ_2 and $-\gamma_1 - \gamma_2$ respectively, so that the vertex is conveniently represented as in Fig. 2.8. Note that this is precisely the topological structure of the plus-vertex in the Lagrangian fibration of the Fermat quintic.

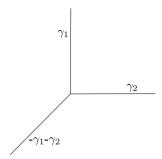


Figure 2.8: The plus-vertex is illustrated as a 1-dimensional graph. Its legs are labeled by the cycle in $T^2 \subset T^3$ that vanishes there. The cycle γ_3 picks up the monodromy.

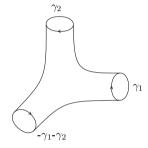


Figure 2.9: The minus-vertex is illustrated as a pair of pants. It is characterized by a single vanishing cycle γ_3 . Furthermore, its legs are labeled by the cycle that wraps it.

The monodromy of the second type of vertex is summarized by the matrices

$$M_{-}:$$

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix},$$

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1
\end{pmatrix},$$

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1
\end{pmatrix}.$$

In contrast to the plus-vertex these monodromy matrices single out a unique 1-cycle γ_3 that degenerates at all three legs of the vertex. Instead of labeling the vertex by this vanishing cycle, it is now more convenient to label the legs with the non-vanishing cycle that does not pick up any monodromy. Like in the example of the Fermat quintic these cycles topologically form a pair of pants. This is illustrated in Fig. 2.9.

Notice that both sets of monodromy matrices are related by simple duality $(M_+)^{-t} = M_-$. Since the mirror of an SYZ fibered Calabi-Yau may be obtained by dualizing the T^3 -fibration, the above vertices must be related by mirror symmetry as well. Obviously these vertices will become important when we describe string compactifications on Calabi-Yau threefolds.

2.3 Local Calabi-Yau threefolds

Topological string theory captures topological aspects of type II string theory compactifications; Kähler aspects of type IIA compactifications and complex structure aspects of type IIB compactification. Mirror symmetry relates them. The topological string partition function $Z_{\rm top}$ can be written as a generating function of either symplectic or complex structure invariants of the underlying Calabi-Yau manifold. It contains for example a series in the number of genus zero curves that are embedded in the Calabi-Yau threefold. Using mirror symmetry it is possible to go well beyond the classical computations of these numbers. A famous result is the computation of the whole series of these genus zero invariants for the Fermat quintic: 2875 different lines, 609250 conics, 317206375 cubics, etc. [14].

It is much more difficult to find the complete topological string partition function, which also contains information about higher genus curves in the Calabi-Yau threefold. The state of the art for the quintic is the computation of these invariant up to g=51 [15]. Although this is an impressive result, it is far from computing the total partition function. In contrast, the all-genus partition function has been found for a simpler type of Calabi-Yau manifolds, that are non-compact. What kind of spaces are these? And why is it so much easier to compute their partition function?

The simplest Calabi-Yau threefold is plain \mathbb{C}^3 with complex coordinates z_i . It admits a Kähler form

$$k = \sum_{i=1}^{3} dz_i \wedge d\bar{z}_i,$$

and a non-vanishing holomorphic 3-form

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3.$$

A special Lagrangian $T^2 \times \mathbb{R}$ fibration of \mathbb{C}^3 over \mathbb{R}^3 has been known for a long time [16]. It is defined by the map

$$(z_1, z_2, z_3) \mapsto (\operatorname{Im} z_1 z_2 z_3, |z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2).$$

Notice that its degeneration locus is a 3-vertex with legs (0,t,0), (0,0,t) and (0,-t,-t), for $t\in\mathbb{R}_{\geq 0}$. Over all these legs some cycle in the T^2 -fiber shrinks to zero-size. We can name these cycles γ_1 , γ_2 and $-\gamma_1-\gamma_2$ since they add up to zero. The degenerate fiber over each leg is a pinched cylinder times S^1 . It is thus the noncompact cycle ($\cong \mathbb{R}$) in the fiber that picks up monodromy when we move around one of the toric legs. This 3-vertex clearly has the same topological structure as the plus-vertex.

Many more non-compact Calabi-Yau's can be constructed by gluing \mathbb{C}^3 -pieces together. In fact, these constitute all non-compact toric Calabi-Yau threefolds. Their degeneration locus can be drawn in \mathbb{R}^2 as a 1-dimensional trivalent graph. The fact that their singular graph is simply planar, as opposed to actually 3-dimensional (as for the Fermat quintic), makes the computation of the partition function on such non-compact toric Calabi-Yau's much simpler. String theorists have managed to find the full topological partition function Z_{top} [17] (using a duality with a 3-dimensional topological theory, called Chern-Simons theory [18]). The recipe to compute the partition function involves cutting the graph in basic 3-vertices. To generalize this for compact Calabi-Yau's it seems one would need to find a way to glue the partition function for a plus-vertex with that of a minus-vertex.

Since the partition function is fully known, topological string theory on these toric manifolds is the ideal playground to learn more about its underlying structure. This has revealed many interesting mathematical and physical connections, for example to several algebraic invariants such as Donaldson-Thomas invariants [19, 20, 21] and Gopakumar-Vafa invariants [22, 23], to knot theory [24, 25, 26, 27], and to a duality with crystal melting [28, 29, 30].

To illustrate the last duality, let us write down the plain \mathbb{C}^3 partition function:

$$Z_{\text{top}}(\mathbb{C}^3) = \prod_{n>0} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + \dots$$

This q-expansion is well-known to be generating function of 3-dimensional partitions; it is called the MacMahon function [31]. The 3-dimensional partition can be visualized as boxes that are stacked in the positive octant of \mathbb{R}^3 . Three of the sides of each box must either touch the walls or another box. This is pictured in Fig. 2.10.

$$Z_{\text{top}}(\mathbb{C}^3) = + q + \text{etc.} + \text{etc.}$$

$$+ q^3 + \text{etc.} + \text{etc.} + \text{etc.}$$

Figure 2.10: Interpretation of the first terms in the expansion of $Z_{top}(\mathbb{C}^3)$ in terms of a three-dimensional crystal in the positive octant of \mathbb{R}^3 .

Since $q=e^{\lambda}$, where λ is the coupling constant of topological string theory, the boxes naturally have length λ . Whereas the regime with λ finite is described by a discrete quantum structure, in the limit $\lambda \to 0$ surprisingly a smooth Calabi-Yau geometry emerges. In a duality with statistical mechanics this corresponds

to the shape of a melting crystal. These observations have led to deep insights in the quantum description of space and time [32, 33].

Remarkably, it has been shown that the emergent smooth geometry of the crystal can be identified with the mirror of \mathbb{C}^3 . How does this limit shape look like? Using local mirror symmetry the equation for the mirror of \mathbb{C}^3 was found in [34]. It is given by

$$uv - x - y + 1 = 0,$$

where $u,v \in \mathbb{C}$ and $x,y \in \mathbb{C}^*$. Remember that the topological structure of the mirror of a plus-vertex should be that of a minus-vertex. Viewing the mirror as a (u,v)-fibration over a complex plane spanned by x and y confirms this:

The degeneration locus of the fibration equals the zero-locus x+y-1=0. Parametrizing this curve by x, it is easily seen that this is a 2-sphere with three punctures at the points x=0, 1 and ∞ . We can equivalently represent this curve as a pair of pants, by cutting off a disc at each boundary $|\tilde{x}|=1$, where \tilde{x} is a local coordinate that vanishes at the corresponding puncture. This realizes the mirror of \mathbb{C}^3 topologically as the minus-vertex in Fig. 2.9.

Mirrors of general non-compact toric manifolds are of the same form

$$X_{\Sigma}: uv - H(x,y) = 0,$$

where the equation H(x,y)=0 now defines a generic Riemann surface Σ embedded in $(\mathbb{C}^*)^2$. This surface is a thickening of the 1-dimensional degeneration graph Γ of its mirror. Its non-vanishing holomorphic 3-form is proportional to

$$\Omega = \frac{du}{u} \wedge dx \wedge dy.$$

These geometries allow a Ricci-flat metric that is conical at infinity [35, 36, 37, 38]. We refer to the threefold X_Σ as the local Calabi-Yau threefold modeled on Σ . The curve Σ plays a central role in this thesis. We study several set-ups in string theory whose common denominator is the relevance of the Riemann surface Σ . In particular, we study the melting crystal picture from the mirror perspective. One of our main results is a simple representation of topological string theory in terms a quantum Riemann surface, that reduces to the smooth Riemann surface Σ in the limit $\lambda \to 0$.

Chapter 3

I-brane Perspective on Vafa-Witten Theory and WZW Models

In the last decades enormous progress has been made in analyzing 4-dimensional supersymmetric gauge theories. Partition functions and correlation functions of some theories have been computed, spectra of BPS operators have been discovered and many other structures have been revealed. Most fascinating to us is that many exact results can be related to two-dimensional geometries.

Since 4-dimensional supersymmetric gauge theories appear in several contexts in string theory, much of this progress is strongly influenced by string theory. Often, string theory tools can be used to compute important quantities in supersymmetric gauge theories. Moreover, in many cases string theory provides a key understanding of new results. For example, when auxiliary structures in the gauge theory can be realized geometrically in string theory and when symmetries in the gauge theory can be understood as stringy dualities.

In this chapter we study a remarkable correspondence between 4-dimensional gauge theories and 2-dimensional conformal field theories. This correspondence connects a "twisted" version of supersymmetric Yang-Mills theory to a so-called Wess-Zumino-Witten model. In particular, generating functions of SU(N) gauge instantons on the 4-manifold $\mathbb{C}^2/\mathbb{Z}_k$ are related to characters of the affine Kac-Moody algebra $\widehat{su}(k)$ at level N. This connection was originally discovered by Nakajima [39], and further analyzed by Vafa and Witten [40]. The goal of this chapter is to make it more transparent. Once again, we find that string theory offers the right perspective.

32

We have strived to make this chapter self-contained by starting in Section 3.1 with a short introduction in gauge and string theory. We review how supersymmetric gauge theories show up as low energy world-volume theories on D-branes and how they naturally get twisted. Twisting emphasizes the role of topological contributions to the theory. Furthermore, we introduce fundamental string dualities as T-duality and S-duality.

In Section 3.2 we introduce Vafa-Witten theory as an example of a twisted 4-dimensional gauge theory, and study it on non-compact 4-manifolds that are asymptotically Euclidean. In Section 3.3 we show that Vafa-Witten theory on such a 4-manifold is embedded in string theory as a D4-D6 brane intersection over a torus T^2 . We refer to the intersecting brane wrapping T^2 as the *I-brane*. Since the open 4-6 strings introduce chiral fermions on the I-brane, we find a duality between Vafa-Witten theory and a CFT of free fermions on T^2 .

In Section 3.4 we show that the full I-brane partition function is simply given by a fermionic character, and reduces to the Nakajima-Vafa-Witten results after taking a decoupling limit. The I-brane thus elucidates the Nakajima-Vafa-Witten correspondence from a string theoretic perspective. Moreover, we gain more insights in level-rank duality and the McKay correspondence from this stringy point of view.

3.1 Instantons and branes

A four-dimensional gauge theory with gauge group G on a Euclidean 4-manifold M is mathematically formulated in terms of a G-bundle $E \to M$. A gauge field A is part of a local connection D = d + A of this bundle, whose curvature is the electro-magnetic field strength

$$F = dA + A \wedge A$$
.

If we denote the electro-magnetic gauge coupling by e and call * the Hodge star operator in four dimensions, the Yang-Mills path integral is

$$Z = \int_{\mathcal{A}/G} DA \exp\left(-\frac{1}{e^2} \int_M d^4x \operatorname{Tr} F \wedge *F\right),$$

This path integral, over the moduli space of connections $\mathcal A$ modulo gauge invariance, defines quantum corrections to the classical Yang-Mills equation D*F=0. When the gauge group is abelian, G=U(1), the equation of motion plus Bianchi identity combine into the familiar Maxwell equations

$$d * F = 0, \qquad dF = 0.$$

Topological terms

Topologically non-trivial configurations of the gauge field are measured by characteristic classes. If G is connected and simply-connected the gauge bundle E is characterized topologically by the *instanton charge*

$$ch_2(F) = \operatorname{Tr}\left[\frac{F \wedge F}{8\pi^2}\right] \in H^4(M, \mathbb{Z}).$$
 (3.1)

Instanton configuration are included in the Yang-Mills formalism by adding a topological term to the Yang-Mills Lagrangian

$$\mathcal{L} = -\frac{1}{e^2} F \wedge *F + \frac{i\theta}{8\pi^2} F \wedge F \tag{3.2}$$

Note that this doesn't change the equations of motion. The path integral is invariant under $\theta \to \theta + 2\pi$, and the parameter θ is therefore called the θ -angle. The total Yang-Mills Lagrangian can be rewritten as

$$\mathcal{L} = \frac{i\tau}{4\pi} F_+ \wedge F_+ + \frac{i\bar{\tau}}{4\pi} F_- \wedge F_-, \tag{3.3}$$

where $F_{\pm} = \frac{1}{2} (F \pm *F)$ are the (anti-)selfdual field strengths while

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \tag{3.4}$$

is the *complexified gauge coupling constant* . When G is not simply-connected magnetic fluxes on 2-cycles in M are detected by the first Chern class

$$c_1(F) = \operatorname{Tr}\left[\frac{F}{2\pi}\right] \in H^2(M, \mathbb{Z}).$$
 (3.5)

Electro-magnetic duality

The Maxwell equations are clearly invariant under the transformation $F \leftrightarrow *F$ that exchanges the electric and the magnetic field. To see that this is even a symmetry at the quantum level, we introduce a Lagrange multiplier field A_D in the U(1) Yang-Mills path integral that explicitly imposes dF=0:

$$\int DADA_D \exp \int_{\mathcal{M}} \left(\frac{i\tau}{4\pi} F_+ \wedge F_+ + \frac{i\bar{\tau}}{4\pi} F_- \wedge F_- + \frac{1}{2\pi} F \wedge *dA_D \right).$$

Integrating out A yields the dual path integral

$$\int DA_D \exp \int_M \left(\frac{i}{4\pi\tau} F_+^D \wedge F_+^D + \frac{i}{4\pi\bar{\tau}} F_-^D \wedge F_-^D \right).$$

So electric-magnetic duality is a strong-weak coupling duality, that sends the complexified gauge coupling $\tau \mapsto -1/\tau$. Moreover, this argument suggests an important role for the *modular group* $Sl(2,\mathbb{Z})$. This group acts on τ as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \text{for} \quad \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in Sl(2, \mathbb{Z}).$$

and is generated by

$$S = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \quad \text{and} \quad T = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right).$$

Hence, S is the generator of electro-magnetic duality (later also called S-duality) and T the generator of shifts in the θ -angle. The gauge coupling τ is thus part of the fundamental domain of $Sl(2,\mathbb{Z})$ in the upper-half plane, as shown in Fig. 3.1.

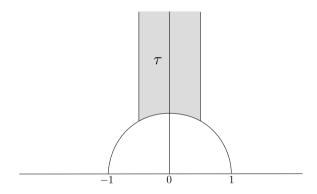


Figure 3.1: The fundamental domain of the modular group $Sl(2,\mathbb{Z})$ in the upper half plane.

Montonen and Olive [41] where pioneers in conjecturing that electro-magnetic duality is an exact non-abelian symmetry, that exchanges the opposite roles of electric and magnetic particles in 4-dimensional gauge theories. This involves replacing the gauge group G by the dual group \hat{G} (whose weight lattice is dual to that of G). The first important tests of S-duality have been performed in supersymmetric gauge theories.

For U(1) theories the partition function $Z^{U(1)}$ can be explicitly computed [42, 43]. The classical contribution to the partition function is given by integral fluxes $p \in H^2(M,\mathbb{Z})$, as in equation (3.5), to the Langrangian (3.3). Decomposing the flux p into a self-dual and anti-selfdual contribution yields the generalized theta-function

$$\theta_{\Gamma}(q,\bar{q}) = \sum_{(p_+,p_-)\in\Gamma} q^{\frac{1}{2}p_+^2} \bar{q}^{\frac{1}{2}p_-^2}$$
(3.6)

with $q=\exp(2\pi i \tau)$, whereas $\Gamma=H^2(M,\mathbb{Z})$ is the intersection lattice of M and $p^2=\int_M p\wedge p$. The total U(1) partition function is found by adding quantum corrections to the above result, which are captured by some determinants [42]. Instead of transforming as a modular invariant, $Z^{U(1)}$ transforms as a modular form under $Sl(2,\mathbb{Z})$ -transformation of τ

$$Z^{U(1)}\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{u/2}(c\overline{\tau}+d)^{v/2}Z^{U(1)}\left(\tau\right),\quad \left(\begin{array}{cc}a&b\\c&d\end{array}\right)\in Sl(2,\mathbb{Z}).$$

We will come back on this in Section 3.2.

3.1.1 Supersymmetry

In *supersymmetric* theories quantum corrections are much better under control, so that much more can be learned about non-perturbative properties of the theory. We will soon discuss such elegant results, but let us first introduce supersymmetric gauge theories.

The field content of the simplest supersymmetric gauge theories just consists of a bosonic gauge field A and a fermionic gaugino field λ . Supersymmetry relates the gauge field A to its superpartner χ . In any supersymmetric theory the number of physical bosonic degrees of freedom must be the same as the number of physical fermionic degrees of freedom. This constraints supersymmetric Yang-Mills theories to dimension $d \leq 10$.

The Lagrangian of a minimal supersymmetric gauge theory is

$$\mathcal{L} = -\frac{1}{4} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{i}{2} \bar{\chi} \Gamma^{\mu} D_{\mu} \chi,$$

and supersymmetry variations of the fields A and χ are generated by a spinor ϵ

$$\delta A_{\mu} = \frac{i}{2} \bar{\epsilon} \Gamma_{\mu} \chi, \qquad \delta \chi = \frac{1}{4} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon. \tag{3.7}$$

The number of supersymmetries equals the number of components of $\epsilon.$

Dimensionally reducing the above minimal $\mathcal{N}=1$ susy gauge theories to lowerdimensional space-times yield $\mathcal{N}=2,4$ and possibly $\mathcal{N}=8$ susy gauge theories. Their supersymmetry variations are determined by an extended supersymmetry algebra. In four dimensions this is a unique extension of the Poincaré algebra generated by the supercharges Q_{α}^{A} and $Q_{A\dot{\alpha}}$, with $A\in\{1,\ldots,\mathcal{N}\}$ and $\alpha,\dot{\alpha}\in\{1,2\}$ are indices in the 4-dimensional spin group $\mathfrak{su}(2)_{L}\times\mathfrak{su}(2)_{R}$. Nonvanishing anti-commutation relations are given by

$$\begin{split} \{Q_{\alpha}^{A}, \overline{Q}_{B\dot{\beta}}\} &= 2(\sigma^{\mu})_{\alpha\dot{\beta}} P_{\mu} \delta_{B}^{A} \\ \{Q_{\alpha}^{A}, Q_{\beta}^{B}\} &= \epsilon_{\alpha\beta} \mathcal{Z}^{AB} \\ \{\overline{Q}_{A\dot{\alpha}}, \overline{Q}_{B\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} \mathcal{Z}_{AB}^{\dagger} \end{split}$$

where \mathcal{Z}^{AB} and its Hermitean conjugate are the central charges. The automorphism group of this algebra, that acts on the supercharges, is known as the R-symmetry group.

BPS states

A special role in extended supersymmetric theories is played by supersymmetric *BPS states* [44]. They are annihilated by a some of the supersymmetry generators, e.g. quarter BPS states satisfy

$$Q^A|\text{BPS}\rangle = 0,$$

for 1/4 $\mathcal N$ indices $A \in \{1,\dots,\mathcal N\}$. BPS states saturate the bound $M^2 \leq |\mathcal Z|^2$ and form "small" representations of the above supersymmetry algebra. This implies that supersymmetry protects them against quantum corrections: a small deformation won't just change the dimension of the representation.

Twisting

Supersymmetric Yang-Mills requires a covariantly constant spinor ϵ in the rigid supersymmetry variations (3.7). Since these are impossible to find on a generic manifold M, the concept of *twisting* has been invented. Twisting makes use of the fact that supersymmetric gauge theories are invariant under a non-trivial internal symmetry, the R-symmetry group. By choosing a homomorphism from the space-time symmetry group into this internal global symmetry group, the spinor representations change and often contain a representation that transforms as a scalar under the new Lorentz group.

Such an odd scalar Q_T can be argued to obey $Q_T^2=0$. It is a topological supercharge that turns the theory into a *cohomological* quantum field theory. Observables \mathcal{O} can be identified with the cohomology generated by Q_T , and correlation functions are independent of continuous deformations of the metric

$$\frac{\partial}{\partial q_{\mu\nu}} \langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle = 0 \tag{3.8}$$

These correlation functions can thus be computed by going to short distances. This yields techniques to study the dynamics of these theories non-perturbatively.

For these topological theories it is sometimes possible to compute the partition function and other correlators. Witten initiated twisting in the context of $\mathcal{N}=2$ supersymmetric Yang-Mills [45]. He showed that correlators in the so-called Donaldson-Witten twist compute the famous Donaldson invariants.

A general theme in cohomological field theories is *localization*. Unlike in general physical theories, in these topological theories the saddle point approximation is actually exact. The path integral only receives contributions from fixed point locus \mathcal{M} of the scalar supercharge Q_T . Since the kinetic part of the action (that contains all metric-dependent terms) is Q_T -exact, the only non-trivial contribution to the path integral is given by topological terms:

$$Z_{\text{cohTFT}} = \int D\mathcal{X} \exp\left(-\frac{1}{e^2}S_{\text{kin}}(\mathcal{X}) + S_{\text{top}}(\mathcal{X})\right) \to \int_{\mathcal{M}} D\mathcal{X} \exp\left(S_{\text{top}}(\mathcal{X})\right).$$

Here \mathcal{X} represents a general field content. An elegant example in this respect is 2-dimensional gauge theory [46]. Extensive reviews of localization are [47, 48]. We will encounter localization on quite a few occasions, starting with Vafa-Witten theory in Section 3.2.

3.1.2 Extended objects

Whereas Yang-Mills theory is formulated in terms of a single gauge potential A, string theory is equipped with a whole set of higher-form gauge fields. Instead of coupling to electro-magnetic particles they couple to *extended objects*, such as D-branes. This is analogous to the coupling of a particle of electric charge q to the Maxwell gauge field A

$$q \int_{\mathcal{W}} A = q \int_{\mathcal{W}} A_{\mu} \frac{\partial x^{\mu}}{\partial t} dt,$$

where \mathcal{W} is the worldline of the particle. Notice that we need to pull-back the space-time gauge field A in the first term before integrating it over the worldline. We often don't write down the pull-back explicitly to simplify notation. D-branes and other extended objects appear all over this thesis. Let us therefore give a very brief account of the properties that are relevant for us.

Couplings and branes in type II

Gauge potentials in type II theory either belong to the so-called RR or the NS-NS sector. The RR potentials couple to D-branes, whereas the NS-NS potential couples to the fundamental string (which is often denoted by F1) and the NS5-brane. Let us discuss these sectors in a little more detail.

The only NS-NS gauge field is the 2-form B. The B-field plays a crucial role in Chapter 5. Aside from the B-field the NS-NS sector contains the dilaton field ϕ

and the space-time metric g_{mn} . Together these NS-NS fields combine into the sigma model action

$$S_{\sigma\text{-model}} = -\frac{1}{2\pi\alpha'} \int_{\Sigma} g_{mn} dx^m \wedge *dx^n + iB_{mn} dx^m \wedge dx^n + \alpha' \phi R. \tag{3.9}$$

This action describes a string that wraps the Riemann surface Σ and is embedded in a space-time with coordinates x^m . In particular, it follows that the B-field couples to a (fundamental) string F1.

Remember that the 1-form dx^m refers to the pull-back $\partial_{\alpha}x^m d\sigma^{\alpha}$ to the world-sheet Σ with coordinates σ^{α} . Furthermore, the symbol * stands for the 2-dimensio-nal worldsheet Hodge star operator and R is the worldsheet curvature 2-form.

This formula requires some more explanation though. The symbol $\sqrt{\alpha'}=l_s$ sets the string length, since α' is inversely proportional to its tension. Since the Ricci scalar of a Riemann surface equals its Euler number, the last term in the action contributes 2g-2 powers of

$$q_s = e^{\phi}$$

to a stringy g-loop diagram; g_s is therefore called the string coupling constant.

The extended object to which the B-field couples magnetically is called the NS5-brane. It can wrap any 6-dimensional geometry in the full 10-dimensional string background, but its presence will deform the transverse geometry. In the transverse directions the dilaton field ϕ is non-constant, and there is a flux H=dB of the B-field through the boundary of the transerve 4-dimensional space. The tension of a NS5-brane is proportional to $1/g_s^2$ so that it is a very heavy object when $g_s \to 0$. Moreover, unlike for D-branes open strings cannot end on it. This makes it quite a mysterious object.

For the RR-sector it makes a difference whether we are in type IIA or in type IIB theory: type IIA contains all odd-degree RR forms and type IIB the even ones. In particular, the gauge potentials C_1 and C_4 are known as the graviphoton fields for type IIA and type IIB, respectively, and the RR potential C_0 is the axion field. Any RR potential C_{p+1} couples electrically to a Dp-brane. This is a p-dimensional extended object that sweeps out a (p+1)-dimensional world-volume Σ_{p+1} . Type IIA thus contains Dp-branes with p even, whereas in type IIB p is odd.

The electric $\mathrm{D}p$ -brane coupling to C_{p+1} introduces the term

$$T_p \int_{\Sigma_{p+1}} C_{p+1}$$
 (3.10)

in the 10-dimensional string theory action, where $1/T_p=(2\pi)^p\sqrt{\alpha'}^{p+1}g_s$ is the inverse tension of the Dp-brane. Magnetically, the RR-potential C_{p+1} couples to

a D(6 – p)-brane that wraps a (7 - p)-dimensional submanifold Σ_{7-p} .

Calibrated cycles

D-branes are solitonic states as their tension T_p is proportional to $1/g_s$. To be stable against decay the brane needs to wrap a submanifold that preserves some supersymmetries. Geometrically, such configurations are defined by a *calibration* [49]. A calibration form is a closed form Φ such that $\Phi \leq \text{vol}$ at any point of the background. A submanifold Σ that is calibrated satisfies

$$\int_{\Sigma} \Phi = \int_{\Sigma} \text{vol},$$

and minimizes the volume in its holomogy class. On a Kähler manifold a calibration is given by the Kähler form t, and the calibrated submanifolds are complex submanifolds. On a Calabi-Yau threefold the holomorphic threeform Ω provides a calibration, whose calibrated submanifolds are special Lagrangians. Calibrated submanifolds support covariantly constant spinors, and therefore preserve some supersymmetry. D-branes wrapping them are supersymmetric BPS states.

Worldvolume theory

D-branes have a perturbative description in terms of open strings that end on them. The massless modes of these open strings recombine in a Yang-Mills gauge field A. When the D-brane worldvolume is flat the corresponding field theory on the p-dimensional brane is a reduction of $\mathcal{N}=1$ susy Yang-Mills from 10 dimensions to p+1. The 9-p scalar fields in this theory correspond to the transverse D-brane excitations. When N D-branes coincide the wordvolume theory is a U(N) supersymmetric Yang-Mills theory.

For more general calibrated submanifolds the low energy gauge theory is a twisted topological gauge theory [50], which we introduced in Section 3.1.1. Which particular twist is realized, can be argued by determining the normal bundel to the submanifold. Sections of the normal bundel fix the transverse bosonic excitations of the gauge theory, and should correspond to the bosonic field content of the twisted theory.

I-branes and bound states

Branes can intersect each others such that they preserve some amount of supersymmetry. This is called an *I-brane* configuration. In such a set-up there are more degrees of freedom than the ones (we described above) that reside on the individual branes. These extra degrees of freedom are given by the modes of open strings that stretch between the branes. In stringy constructions of the standard model on a set of branes they often provide the chiral fermions. Chiral fermions are intimately connected with quantum anomalies, and brane intersections likewise. To cure all possible anomalies in an I-brane system, a topological Chern-Simons term has to be added to the string action

$$S_{CS} = T_p \int_{\Sigma_{p+1}} \text{Tr } \exp\left(\frac{F}{2\pi}\right) \wedge \sum_i C_i \wedge \sqrt{\hat{A}(R)}.$$
 (3.11)

This term is derived through a so-called anomaly inflow analysis [51, 52]. The last piece contains the A-roof genus for the 10-dimensional curvature 2-form R pulled back to Σ_{p+1} . It may be expanded as

$$\hat{A}(R) = 1 - \frac{p_1(R)}{24} + \frac{7p_1(R)^2 - 4p_2(R)}{5760} + \dots,$$

where $p_k(R)$ is a Pontryagin class. For example, the Chern-Simons term (3.11) includes a factor

$$T_p \int_{\Sigma_{p+1}} \operatorname{Tr}\left(\frac{F}{2\pi}\right) \wedge C_{p-1}$$

when a gauge field F on the worldvolume Σ_{p+1} is turned on. It describes an induced $\mathrm{D}(p-2)$ brane wrapping the Poincare dual of $[F/2\pi]$ in Σ_{p+1} .

Vice versa, a bound state of a D(p-2)-brane on a Dp-brane may be interpreted as turning on a field strength F on the Dp-brane. Analogously, instantons (3.1) in a 4-dimensional gauge theory, say of rank zero and second Chern class n, have an interpretation in type IIA theory bound states of n D0-branes on a D4-brane. More generally, topological excitations in the worldvolume theory of a brane are often caused by other extended objects that end on it [53].

3.1.3 String dualities

The different appearances of string theory, type I, II, heterotic and M-theory, are connected through a zoo of dualities. Let us briefly introduce T-duality and S-duality in type II. There are many more dualities, some of which we will meet on our way.

T-duality

T-duality originates in the worldsheet description of type II theory in terms of open and closed strings. T-duality on an S^1 in the background interchanges the Dirichlet and Neumann boundary conditions of the open strings on that circle, and thereby maps branes that wrap this S^1 into branes that don't wrap it (and vice versa). It thus interchanges type IIA and type IIB theory.

Similar to electro-magnetic duality (see Section 3.1), T-duality follows from a path integral argument [54]. The sigma model action for a fundamental string

is based on the term

$$-\frac{1}{2\pi\alpha'}\int_{\Sigma}g_{mn}dx^m\wedge *dx^n,$$

in equation (3.9) when we forget the B-field for simplicity. Let us suppose that the metric is diagonal in the coordinate x that parametrizes the T-duality circle S^1 . Then we can add a Lagrange multiplier field dy to the relevant part of the action

$$\int DxDy \exp \int \left(-\frac{1}{2\pi\alpha'}dx \wedge *dx + \frac{i}{\pi}dx \wedge *dy\right).$$

On the one hand the Lagrange multiplier field dy forces d(dx)=0, which locally says that dx is exact. On the other hand integrating out dx yields

$$\int Dy \exp \int \left(-\frac{\alpha'}{2\pi}dy \wedge *dy\right)$$

So T-duality exchanges

$$\alpha' \leftrightarrow \frac{1}{\alpha'}$$

and is therefore a strong-weak coupling on the worldsheet. More precisely, one should also take into account the B-field coupling (3.9), which is related to non-diagonal terms is the space-time metric. This leads to the well-known Buscher rules [55]

Since the differential dx may be identified with a component of the gauge field A on the brane, and dy with a normal 1-form to the brane, the reduction of supersymmetric Yang-Mills from ten to 10-d dimensions can be understood as applying T-duality d times.

S-duality

Since $\mathcal{N}=4$ supersymmetric Yang-Mills is realized as the low-energy effective theory on a D3-brane wrapping \mathbb{R}^4 . Since electro-magnetic duality in this theory is an exact symmetry, it should have a string theoretic realization in type IIB theory as well. Indeed it does, and this symmetry is known as S-duality. In type IIB theory S-duality is a (space-time) strong-weak coupling duality that maps $g_s \leftrightarrow 1/g_s$. Analogous to Yang-Mills theory the complete symmetry group is $Sl(2,\mathbb{Z})$, where the complex coupling constant τ (3.4) is realized as

$$\tau = C_0 + ie^{-\phi}.$$

Since the ratio of the tensions of the fundamental string F1 and the D1-brane is equal to g_s , S-duality exchanges these objects as well as the B-field and the

 C_2 -field they couple to. Likewise, it exchanges the NS5-brane with the D5-brane.

M-theory

Type IIA is not invariant under S-duality. Instead, in the strong coupling limit another dimension of size $g_s l_s$ opens up. This eleven dimensional theory is called M-theory. The field content of M-theory contains a 3-form gauge field C_3 with 4-form flux $G_4=dC_3$ that couples to an M2-brane. It magnetic dual is an M5-brane. Other extended objects include the KK-modes and Taub-NUT space. We will introduce them in more detail later. For now, let us just note that a reduction over the M-theory circle consistently reproduces all the fields and objects of type IIA theory. (An extensive review can for instance be found in [56].)

3.2 Vafa-Witten twist on ALE spaces

The maximal amount of supersymmetry in 4-dimensional gauge theories is $\mathcal{N}=4$ supersymmetry. This gauge theory preserves so many supercharges that it has a few very special properties. Its beta function is argued to vanish non-perturbatively, making the theory exactly finite and conformally invariant [57]. It is also, not unrelated, the only 4-dimensional gauge theory where electromagnetic and $Sl(2,\mathbb{Z})$ duality are conjectured to hold at all energy scales.

In this section we apply the techniques of the previous section to study a twisted version of $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions. It is called the Vafa-Witten twist. We study Vafa-Witten theory on ALE (asymptotically locally Euclidean) spaces, which are defined as hyper-Kähler resolutions of the singularity

$$\mathbb{C}^2/\Gamma$$
,

where Γ is a finite subgroup of SU(2). ALE spaces are intimately connected to ADE Lie algebras.

The Vafa-Witten twist is an example of a topological gauge theory. It localizes on anti-selfdual instantons that are defined by the vanishing of the selfdual component F_+ of the field strength. The Vafa-Witten partition function is therefore a generating function that counts anti-selfdual instantons. On an ALE space this partition function turns out to compute the character of an affine ADE algebra.

This section starts off with the Vafa-Witten twist and ALE spaces. We discuss Vafa-Witten theory on ALE spaces, and embed the gauge theory into string theory as a worldvolume theory on top of D4-branes. This is the first step in finding a deeper explanation for the duality between $\mathcal{N}=4$ supersymmetric gauge theories and 2-dimensional conformal field theories.

3.2.1 Vafa-Witten twist

C. Vafa and E. Witten have performed an important S-duality check of $\mathcal{N}=4$ supersymmetric gauge theory by computing a twisted partition function on certain 4-manifolds M [40].

In total three inequivalent twists of $\mathcal{N}=4$ Yang-Mills theory are possible. These are characterized by an embedding of the rotation group $SO(4)\cong SU(2)_L\times SU(2)_R$ in the R-symmetry group $SU(4)_R$ of the supersymmetric gauge theory. The Vafa-Witten twist considers the branching

$$SU(4)_R \to SU(2)_A \times SU(2)_B$$

and twists either $SU(2)_L$ or $SU(2)_R$ with $SU(2)_A$. Both twists are related by changing the orientation of the 4-fold M and at the same time changing τ with $\bar{\tau}$. Let us choose the left-twist here. This results in a bosonic field content consisting of a gauge field A, an anti-selfdual 2-form and three scalars.

The twisted theory is a cohomological gauge theory with $\mathcal{N}_T = 2$ equivariant topological supercharges Q_{\pm} , whose Lagrangian can be written in the form

$$\mathcal{L} = \frac{i\tau}{4\pi} F \wedge F - \frac{2}{e^2} F_- \wedge *F_- + \dots$$

$$= \frac{i\tau}{4\pi} F \wedge F + Q_+ Q_- \mathcal{F}, \tag{3.12}$$

where \mathcal{F} is called the action potential [58]. In the spirit of our discussion in Section 3.1.1 this implies that the path integral localizes onto the critical points of the potential \mathcal{F} modulo gauge equivalence. On Kähler manifolds this critical locus is characterized by the vanishing of the anti-selfdual 2-form and the three scalars, whereas the gauge field obeys

$$F_{-} = 0$$
.

The Vafa-Witten twist thus localizes to the instanton moduli space

$$\mathcal{M} = \mathcal{W}/G$$
, $\mathcal{W} = \{A : F_{-}(A) = 0\}$.

of selfdual connections. 1 The moduli space ${\mathcal M}$ naturally decomposes in con-

¹Alternatively, the localization to F_- follows from the field equations. Since the field content of the right twist involves only anti-selfdual (instead of selfdual) fields, setting the fermion variations to zero forces $F_-=0$. Likewise, performing the left twist corresponds to changing τ with $\bar{\tau}$ as well changing F_- with F_+ in the Lagrangian (3.12). Ultimately, our conventions in equation (3.2) and (3.3) imply that the selfdual instantons receive contributions in τ . This choice is non-standard in the Vafa-Witten literature, but it fits better with the content of this thesis.

nected components \mathcal{M}_n that are labeled by the instanton number

$$n = \int_M \operatorname{Tr} \left[\frac{F \wedge F}{8\pi^2} \right].$$

The Vafa-Witten partition function computes the Euler characteristic of these components (without \pm signs). Up to possible holomorphic anomalies it is a holomorphic function of τ with a Fourier-expansion of the type

$$Z^{G}(\tau) \sim \sum_{n} d(n)q^{n}, \tag{3.13}$$

where the numbers d(n) represent the Euler characteristic of \mathcal{M}_n , whereas $q^n = \exp(2\pi i n \tau)$ denotes the contribution of the instantons to the topological term in the Lagrangian (3.12). The numbers d(n) are integers when G is connected and simply connected.

S-duality and modular forms

The Vafa-Witten partition function only transforms nicely under S-duality once local curvature corrections in the Euler characteristic χ and the signature σ of M have been added to the action [40]. Notice that this is justified since they do not change the untwisted theory on \mathbb{R}^4 . In particular, these additional terms introduce an extra exponent in (3.13)

$$Z^G(\tau) = q^{-c/24} \sum_n d(n) q^n,$$

where c is a number depending on χ and σ . The resulting Vafa-Witten partition function conjecturally transforms as a modular form of weight $w=-\chi(M)$ that exchanges G with its dual \hat{G}

$$Z^G(-1/\tau) \sim \tau^{w/2} Z^{\hat{G}}(\tau).$$

Since \hat{G} is often not simply connected (for example the dual of G = SU(2) is $\hat{G} = SO(3) = SU(2)/\mathbb{Z}_2$) one has to take into account magnetic fluxes $v \in H^2(M, \pi_1(\hat{G}))$. The components of the partition function Z_v mix under the S-duality transformation $\tau \to -1/\tau$. Furthermore, since for such \hat{G} instantons numbers are not integer, the vector valued partition function Z_v will only be covariant under a subgroup of $Sl(2, \mathbb{Z})$.

Characters of affine Lie algebras are examples of such vector valued modular forms. We will soon introduce them and see that they indeed appear as Vafa-Witten partition functions.

Unitary gauge group and Jacobi forms

In the following we will be especially interested in Vafa-Witten theory with gauge group U(N). This gauge group is not simply-connected, since it contains an abelian subgroup $U(1) \subset U(N)$. Instanton bundles are therefore not only characterized topologically by their second Chern class ch_2 , but also carry abelian fluxes measured by the first Chern class c_1 .

The U(N) Yang-Mills partition function gets extra contributions from these magnetic fluxes. The path integral can be performed by first taking care of the U(1) part of the field strength. This gives a contribution in the form of a Siegel theta function, precisely as explained in Section 3.1.

We can make these fluxes more explicit by introducing a topological coupling $v \in H^2(M, \mathbb{Z})$ in the original Yang-Mills Lagrangian:

$$\mathcal{L} = \frac{i\tau}{4\pi} \operatorname{Tr} F_{+} \wedge F_{+} + v \wedge \operatorname{Tr} F_{+} + \text{c.c.}.$$
 (3.14)

Here we define complex conjugation c.c. not only to change τ and v into their anti-holomorphic conjugates, but also to map the selfdual part F_+ of the field strength to the anti-selfdual part F_- . The v-dependence of the partition function is entirely captured by the U(1) factor of the field strength. It results in a Siegel theta-function of signature (b_2^+, b_2^-)

$$\theta_{\Gamma}(\tau, \bar{\tau}; v, \bar{v}) = \sum_{p \in \Gamma} e^{i\pi \left(\tau p_{+}^{2} - \bar{\tau} p_{-}^{2}\right)} e^{2\pi i (v \cdot p_{+} - \bar{v} \cdot p_{-})}.$$
 (3.15)

Here p and v are elements of $\Gamma = H^2(M, \mathbb{Z})$, so that $v \cdot p$ (and likewise p^2) refers to the intersection product $\int_M v \wedge p$.

In this chapter we focus on non-compact hyper-Kähler manifolds whose Betti number $b_2^+=0$. We change their orientation to find a non-trivial Vafa-Witten partition function. The U(1) contribution to their partition function is then purely holomorphic.

Because of S-duality the total U(N) partition function $Z(v,\tau)$ is expected to be given by a *Jacobi form* determined by the geometry M. That is, for

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2,\mathbb{Z}) \quad \text{and} \quad n,m \in H^2(M,\mathbb{Z}) \cong \mathbb{Z}^{b_2},$$

it should have the transformation properties

$$Z\left(\frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{w/2} e^{\frac{2\pi i \kappa c v^2}{c\tau+d}} Z(v,\tau)$$
$$Z(v+n\tau+m,\tau) = e^{-2\pi i \kappa (n^2\tau+2n\cdot v)} Z(v,\tau),$$

46

where w is the weight and κ is the index of the Jacobi form. Using the localization to instantons, the partition function has a Fourier expansion of the form

$$Z(v,\tau) = \sum_{m \in H^2(M), n > 0} d(m,n) y^m q^{n-c/24},$$
(3.16)

where $y=e^{2\pi i v}$, $q=e^{2\pi i \tau}$ and $c=N\chi$. The coefficients d(n,m) are roughly computed as the Euler number of the moduli space of U(N) instantons on M with total instanton numbers $c_1=m$ and $ch_2=n$.

3.2.2 M5-brane interpretation

In string theory U(N) Vafa-Witten theory is embedded as the topological subsector of $\mathcal{N}=4$ super Yang-Mills theory on N D4-branes that wrap a holomorphic 4-cycle $M\subset X$ in the IIA background

IIA:
$$X \times \mathbb{R}^4$$
. (3.17)

Topological excitations in the gauge theory amount to bound states of D0 and D2-branes on the D4-brane.

Let us now consider the 5-dimensional gauge theory on a Euclidean D4-brane wrapping $M \times S^1$. The partition function of this theory is given by a trace over its Hilbert space, whose components are labeled by the number $n = ch_2(F)$ of D0-branes and the number $m = c_1(F)$ of D2-branes. The coefficients d(n,m) in the Fourier expansion (3.16) thus have a direct interpretation as computing BPS invariants: the number of bound states of n D0-branes and m D2-branes on the D4-brane. For this reason they are believed to be integers in general. We can compute d(m,n) as the index²

$$d(m,n) = \operatorname{Tr}(-1)^F \in \mathbb{Z},$$

in the subsector of field configurations on M of given instanton numbers m, n.

From the string theory point of view the modular invariance of Z is explained naturally by lifting the D4-brane to M-theory, where it becomes an M5-brane on the product manifold

$$M \times T^2$$
.

The world-volume theory of the M5-brane is (in the low-energy limit) the rather mysterious 6-dimensional U(N) conformal field theory with (0,2) supersymmetry. The complexified gauge coupling τ can now be interpreted as the modulus of the elliptic curve T^2 , while the Wilson loops of the 3-form potential C_3 along

²Here and in the subsequent sections we assume that the two fermion zero modes associated to the center of mass movements of the D4-brane have been absorbed.

this curve are related to the couplings v, as we explain in more detail in Section 3.3.3. With this interpretation the action of modular group $SL(2,\mathbb{Z})$ on v and τ is the obvious geometric one.

Instead of compactifying over T^2 , we can also consider a compactification over M. We then find a 2-dimensional (0,8) CFT on the two-torus, whose moduli space consists of the solutions to the Vafa-Witten field equations on M. This duality motivates the appearance of CFT characters in Vafa-Witten theory. In Section 3.3 we will reach a deeper understanding.

\mathbb{R}^4 – Example

The simplest example is U(1) Vafa-Witten theory on \mathbb{R}^4 corresponding to a single D4-brane on \mathbb{R}^4 . Point-like instantons in this theory correspond to bound states with D0-branes and yield a non-trivial partition function

$$Z(\tau, v) = \frac{\theta_3(v, \tau)}{\eta(\tau)} = \frac{\sum_{p \in \mathbb{Z}} e^{\pi i \tau p^2 + 2\pi i v p}}{q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)}.$$

The Dedekind eta function $\eta(\tau)$ can be rewritten as a generating function

$$\frac{1}{\eta(\tau)} = q^{-1/24} \frac{1}{\prod_{n>0} (1-q^n)} = q^{-1/24} \sum_{n \geq 0} p(n) q^n,$$

of the number p(n) of partitions (n_1, \ldots, n_k) of n. We can identity each such a partition with a bosonic state

$$\alpha_{-n_1}\cdots\alpha_{-n_k}|p\rangle$$

in the Fock space \mathcal{H}_p of a chiral boson $\phi(x)$ with mode expansion

$$\partial \phi(x) = \sum_{n \in \mathbb{Z}} \alpha_n x^{-n-1}.$$

The state $|p\rangle$ is the vacuum whose Fermi level is raised by p units. The partition function $Z(\tau,v)$ is exactly reproduced by the $\widehat{u}(1)$ character

$$Z(\tau, v) = \operatorname{Tr}_{\mathcal{H}_p} \left(y^{J_0} q^{L_0 - c/24} \right) = \chi^{\widehat{u}(1)}(\tau, v),$$

where $L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n}\alpha_n$ measures the energy of the states and $J_0 = \alpha_0$ the U(1) charge. The instanton zero point energy c=1 now corresponds to the central charge for a single free boson.

3.2.3 Vafa-Witten theory on ALE spaces

So far we motivated that the Vafa-Witten partition function transforms under S-duality in a modular way. Furthermore, we have seen a simple example with $M=\mathbb{R}^4$ where the partition function equals a CFT-character. In this section we will see that this relation is more generally true for ALE spaces.

In the forthcoming sections we introduce quite a few notions from the theory of affine Lie algebras $\hat{\mathfrak{g}}$ and their appearance in WZW (Wess-Zumino-Witten) models. The classic reference for this subject is [59].

ALE spaces and geometric McKay correspondence

An ALE space M_{Γ} is a non-compact hyper-Kähler surface. It is obtained by resolving the singularity at the origin of \mathbb{C}^2/Γ ,

$$M_{\Gamma} \to \mathbb{C}^2/\Gamma$$
,

where Γ is a finite subgroup of SU(2) that acts linearly on \mathbb{C}^2 . These Kleinian singularities are classified into three families: the cyclic groups A_k , the dihedral groups D_k and the symmetries of the platonic solids E_k . For example, an A_{k-1} singularity is generated by the element

$$(z, w) \mapsto (e^{2\pi i/k} z, e^{-2\pi i/k} w).$$

of the cyclic subgroup $\Gamma = \mathbb{Z}_k$.

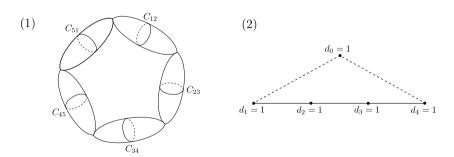


Figure 3.2: The left picture (1) illustrates an A_4 -singularity, which is a hyper-Kähler resolution of $\mathbb{C}^2/\mathbb{Z}_5$. Its homology is generated by 4 independent 2-cycles. They have self-intersection number -2 and intersect once with their neighbours. This Kleinian singularity is therefore dual to the Dynkin diagram of the Lie algebra $\mathfrak{su}(5)$, which is illustrated on the right in picture (2). The dotted lines complete this diagram into the Dynkin diagram of the extended Lie algebra $\widehat{\mathfrak{su}}(5)$. The labels are the dual Dynkin indices of the simple roots.

A hyper-Kähler resolution replaces the singularity at the origin with a number of

2-spheres. The (oriented) intersection product

$$(S_i^2, S_j^2) \mapsto S_i^2 \cup S_j^2$$

puts a lattice structure on the second homology. This turns out to be determined by the Cartan matrix of the corresponding ADE Lie algebra \mathfrak{g} , so that there is a bijection between a basis of 2-cycles and a choice of simple roots. A_{k-1} singularities correspond to the Lie group SU(k), D_k singularities lead to SO(2k) and E_k ones are related to one of the exceptional Lie groups E_6 , E_7 or E_8 .

The Dynkin diagram of each Lie algebra is thus realized geometrically in terms of intersections of 2-cyles in the resolution of the corresponding Kleinian singularity. This is the famous *geometric McKay correspondence* [60, 61]. We will encounter its string theoretic interpretation in the next chapter.

In this thesis we mainly consider A_{k-1} surface singularities, for which $\Gamma = \mathbb{Z}_k$. Let us work out this example in some more detail. A resolved A_{k-1} singularity M_k is defined by an equation of the form:

$$W_k = \prod_{i=1}^k (z - a_i) + u^2 + v^2 = 0,$$
 for $z, u, v \in \mathbb{C}$.

More precisely, this equation defines a family of A_{k-1} spaces that are parametrized by k complex numbers a_i . For any configuration with $a_1 \neq \ldots \neq a_k$ the surface M_k is smooth.

The 4-manifold M_k can be thought of $S^1 \times \mathbb{R}$ -fibration over the complex plane \mathbb{C} , where the fiber is defined by the equation $u^2 + v^2 = \mu = -\prod_{i=1}^k (z - a_i)$ over a point $z \in \mathbb{C}$. Notice, however, that the size μ of the circle S^1 becomes infinite when $z \to \infty$.

Over each of the points $z=a_i$ the fiber circle vanishes. Hence, non-trivial 2-cycles C_{ij} in the 4-manifold can be constructed as S^1 -fibrations over some line segment $[a_i,a_j]$ in the z-plane. In fact, the second homology of the 4-manifold M_k is spanned by k-1 of these two-spheres, say $C_{i(i+1)}$ for $1 \le i \le k-1$. This is illustrated in in Fig. 3.2.

Since the $(k-1) \times (k-1)$ intersection form on $H_2(M_k)$ in this basis

$$\begin{pmatrix} -2 & 1 & 0 & \cdots \\ 1 & -2 & 1 & \cdots \\ 0 & 1 & -2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

coincides with the Cartan form of the Lie algebra $\mathfrak{su}(k)$ (up to an overall minussign), the 2-cycles $C_{i(i+1)}$ generate the root lattice of A_{k-1} .

McKay-Nakajima correspondence

Let us now study Vafa-Witten theory on these ALE spaces. First of all we have to address the fact that the 4-manifold M_{Γ} is non-compact, so that we have to fix boundary conditions for the gauge field. The boundary at infinity is given by the Lens space S^3/Γ and here the U(N) gauge field should approach a flat connection. Up to gauge equivalence this flat connection is labeled by an N-dimensional representation of the quotient group Γ , that is, an element

$$\rho \in \text{Hom}(\Gamma, U(N))$$
.

If ρ_i label the irreducible representations of Γ (with ρ_0 the trivial representation), then ρ can be decomposed as

$$\rho = \bigoplus_{i=0}^{r} N_i \rho_i,$$

where the multiplicities N_i are non-negative integers satisfying the restriction

$$\sum_{i=0}^{r} N_i d_i = N, \qquad d_i = \dim \rho_i.$$

and r is the rank of the gauge group. Now the classic *McKay correspondence* (without the adjective "geometric") relates the irreducible representations ρ_i of the finite subgroup Γ to the nodes of the Dynkin diagram of the corresponding affine extension $\widehat{\mathfrak{g}}$, such that the dimensions d_i of these irreps can be identified with the dual Dynkin indices (see Fig. 3.2).

Furthermore, the non-negative integers N_i label a dominant weight of the affine algebra $\widehat{\mathfrak{g}}$ whose level is equal to N. Through the McKay correspondence each N-dimensional representation ρ of Γ thus determines an integrable highest-weight representation of $\widehat{\mathfrak{g}}_N$ at level N. We will denote this (infinite-dimensional) Lie algebra representation as V_ρ . For $\Gamma = \mathbb{Z}_k$, which is the case that we will mostly concentrate on, flat connections on S^3/\mathbb{Z}_k get identified with integrable representations of $\widehat{\mathfrak{su}}(k)_N$. In this particular case all Dynkin indices satisfy $d_i=1$.

With ρ labeling the boundary conditions of the gauge field at infinity, we will get a vector-valued partition function $Z_{\rho}(v,\tau)$. Formally the U(N) gauge theory partition function on the ALE manifold again has an expansion

$$Z_{\rho}(v,\tau) = \sum_{n,m} d(m,n) y^m q^{h_{\rho} + n - c/24},$$

where c = Nk with k the regularized Euler number of the A_{k-1} manifold [40]. The usual instanton numbers given by the second Chern class $n = ch_2$ in the

exponent are now shifted by a rational number h_{ρ} , which is related to the Chern-Simons invariant of the flat connection ρ . As we explain in Section 3.4.2, h_{ρ} gets mapped to the conformal dimension of the corresponding integrable weight in the affine Lie algebra $\widehat{\mathfrak{g}}$ related to Γ by the McKay correspondence. S-duality will act non-trivially on the boundary conditions ρ , and therefore $Z_{\rho}(v,\tau)$ will be a vector-valued Jacobi form [40].

For these ALE spaces the instanton computations can be explicitly performed, because there exists a generalized ADHM construction in which the instanton moduli space is represented as a quiver variety. The physical intuition underlying this formalism has been justified by the beautiful mathematical work of H. Nakajima [39, 62], who has proved that on the middle dimensional cohomology of the instanton moduli space one can actually realize the action of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}_N$ in terms of geometric operations. In fact, this work leads to the identification

$$Z_{\rho}(v,\tau) = \text{Tr}_{V_{\rho}} \left(y^{J_0} q^{L_0 - c/24} \right) = \chi_{\rho}(v,\tau),$$

with V_{ρ} the integrable highest-weight representation of $\widehat{\mathfrak{g}}_N$ and χ_{ρ} its affine character. Here c is the appropriate central charge of the corresponding WZW model. A remarkable fact is that, in the case of a U(N) gauge theory on a \mathbb{Z}_k singularity, we find an action of $\widehat{su}(k)_N$ and not of the gauge group SU(N). This is a important example of level-rank duality of affine Lie algebras. This setup has been studied from various perspectives in for instance [63, 64, 65].

Interestingly, I. Frenkel has suggested [66] that, if one works equivariantly for the action of the gauge group SU(N) at infinity (we ignore the U(1) part for the moment), there would similarly be an action of the $\widehat{su}(N)_k$ affine Lie algebra. Physically this means "ungauging" the SU(N) at infinity. In other words, we consider making the SU(N) into a global symmetry instead of a gauge symmetry at the boundary. This suggestion has recently been confirmed in [67]. So, depending on how we deal with the theory at infinity, there are reasons to expect both affine symmetry structures to appear and have a combined action of the Lie algebra

$$\widehat{su}(N)_K \times \widehat{su}(k)_N$$
.

We will now turn to a dual string theory realization, where this structure indeed becomes transparent.

3.3 Free fermion realization

In this section we discover a string theoretic set-up to study the correspondence between Vafa-Witten theory on ALE spaces and the holomorphic part of a WZW 52

model. We find that Vafa-Witten theory is dual to a system of intersecting D4 and D6-branes on a torus T^2 .

3.3.1 Taub-NUT geometry

To study Vafa-Witten theory on ALE spaces within string theory, we use a trick that proved to be very effectively in relating 4d and 5d black holes [68, 69, 70, 71, 72] and is in line with the duality between ALE spaces and 5-brane geometries [73]. We will replace the local A_{k-1} singularity with a Taub-NUT geometry. This can be best understood as an S^1 compactification of the singularity. The TN_k geometry is a hyper-Kähler manifold with metric [74, 75],

$$ds_{TN}^2 = R^2 \left[\frac{1}{V} (d\chi + \alpha)^2 + V d\vec{x}^2 \right],$$

with $\chi \in S^1$ (with period 4π) and $\vec{x} \in \mathbb{R}^3$. Here the function V and 1-form α are determined as

$$V(\vec{x}) = 1 + \sum_{a=1}^{k} \frac{1}{|\vec{x} - \vec{x}_a|}, \quad d\alpha = *_3 dV.$$

Just like a local ${\cal A}_{k-1}$ singularity, the Taub-NUT manifold may be seen as a circle fibration

$$\begin{array}{ccc} S^1_{TN} & \to & TN_k \\ & \downarrow \\ & & \mathbb{R}^3 \end{array}$$

where the size of the S^1_{TN} shrinks at the points $\vec{x}_1,\ldots,\vec{x}_k\in\mathbb{R}^3$, whose positions are the hyperkähler moduli of the space. The main difference with the (resolved) A_{k-1} singularity is that the Taub-NUT fiber stays of finite size R at infinity.

The total Taub-NUT manifold is perfectly smooth. At infinity it approximates the cylinder $\mathbb{R}^3 \times S^1_{TN}$, but is non-trivially fibered over the S^2 at infinity as a monopole bundle of charge (first Chern class) k

$$\int_{S^2} d\alpha = 2\pi k.$$

In the core, where we can ignore the constant 1 that appears in the expression for the potential $V(\vec{x})$, the Taub-NUT geometry can be approximated by the (resolved) A_{k-1} singularity.

The manifold TN_k has non-trivial 2-cycles $C_{a,b} \cong S^2$ that are fibered over the line segments joining the locations \vec{x}_a and \vec{x}_b in \mathbb{R}^3 . Only k-1 of these cycles are homologically independent. As a basis we can pick the cycles

$$C_a := C_{a,a+1}, \qquad a = 1, \dots, k-1.$$

The intersection matrix of these 2-cycles gives the Cartan matrix of A_{k-1} .

From a dual perspective, there are k independent normalizable harmonic 2-forms ω_a on TN_k , that can be chosen to be localized around the centers or NUTs \vec{x}_a . With

$$V_a = \frac{1}{|\vec{x} - \vec{x}_a|}, \qquad d\alpha_a = *dV_a,$$

they are given as

$$\omega_a = d\eta_a, \qquad \eta_a = \alpha_a - \frac{V_a}{V}(d\chi + \alpha).$$

Furthermore, these 2-forms satisfy

$$\int_{TN} \omega_a \wedge \omega_b = 16\pi^2 \delta_{ab},$$

and are dual to the cycles $C_{a,b}$

$$\int_{C_{a,b}} \omega_c = 4\pi (\delta_{ac} - \delta_{bc}).$$

A special role is played by the sum of these harmonic 2-forms

$$\omega_{TN} = \sum_{a} \omega_{a}. \tag{3.18}$$

This is the unique normalizable harmonic 2-form that is invariant under the triholomorphic U(1) isometry of TN. The form ω_{TN} has zero pairings with all the cycles C_{ab} . In the "decompactification limit", where TN_k gets replaced by A_{k-1} , the linear combination ω_{TN} becomes non-normalizable, while the k-1 two-forms orthogonal to it survive.

We will make convenient use of the following elegant interpretation of the two-form ω_{TN} . Consider the U(1) action on the TN_k manifold that rotates the S^1_{TN} fiber. It is generated by a Killing vector field ξ . Let η_{TN} be the corresponding dual one-form given as $(\eta_{TN})_{\mu} = g_{\mu\nu}\xi^{\nu}$, where we used the TN-metric to convert the vector field to a one-form. Up to an overall rescaling this gives

$$\eta_{TN} = \frac{1}{V}(d\chi + \alpha). \tag{3.19}$$

In terms of this one-form, ω_{TN} is given by $\omega_{TN}=d\eta_{TN}$.

3.3.2 The D4-D6 system

Our strategy will be that, since we consider the twisted partition function of the topological field theory, the answer will be formally independent of the radius

R of the Taub-NUT geometry. So we can take both the limit $R \to \infty$, where we recover the result for the ALE space $\mathbb{C}^2/\mathbb{Z}_k$, and the limit $R\to 0$, where the problem becomes essentially 3-dimensional.

Now, there are some subtleties with this argument, since a priori the partition function of the gauge theory on the TN manifold is not identical to that of the ALE space. In particular there are new topological configurations of the gauge field that can contribute. These can be thought of as monopoles going around the S^1 at infinity. We will come back to this subtle point later.³

In type IIA string theory, the partition function of the $\mathcal{N}=4$ SYM theory on the TN_k manifold can be obtained by considering a compactification of the form

(IIA)
$$TN \times S^1 \times \mathbb{R}^5$$
,

and wrapping N D4-branes on $TN \times S^1$. This is a special case of the situation presented in the box on the right-hand side in Fig. 1.6, with $\Gamma = S^1$, $\mathcal{B}_3 =$ $S^1 \times \mathbb{R}^2$, and S^1 decompactified. In the decoupling limit the partition function of this set of D-branes will reproduce the Vafa-Witten partition function on TN_k . This partition function can be also written as an index

$$Z(v,\tau) = \operatorname{Tr}\left((-1)^F e^{-\beta H} e^{in\theta} e^{2\pi imv}\right)$$

where $\beta = 2\pi R_9$ is the circumference of the "9th dimension" S^1 , and $m = c_1$, $n=ch_2$ are the Chern characters of the gauge bundle on the TN_k space. Here we can think of the theta angle θ as the Wilson loop for the graviphoton field C_1 along the S^1 . Similarly v is the Wilson loop for C_3 . The gauge coupling of the 4d gauge theory is now identified as

$$\frac{1}{g^2} = \frac{\beta}{g_s \ell_s}.$$

Because only BPS configurations contribute in this index, again only the holomorphic combination τ (3.4) will appear.

We can now further lift this configuration to M-theory with an additional S^1 of size $R_{11} = g_s l_s$, where we obtain the compactification

(M)
$$TN \times T^2 \times \mathbb{R}^5$$
,

now with N M5-branes wrapping the 6-manifold $TN_k \times T^2$. This corresponds to the top box in Fig. 1.6, with $\Sigma = T^2$. As we remarked earlier, after this lift the coupling constant au is interpreted as the geometric modulus of the elliptic curve

³Recently, instantons on Taub-NUT spaces have been studied extensively in [76, 77, 78]. In particular, [79] gives a closely related description of the duality between $\mathcal{N}=4$ supersymmetric gauge theory on Taub-NUT space and WZW models from the perspective of an M5-brane wrapping $\mathbb{R} \times S^1 \times TN$. It is called a geometric Langlands duality for surfaces.

 T^2 . In particular its imaginary part is given by the ratio R_9/R_{11} . Dimensionally reducing the 6-dimensional U(N) theory on the M5-brane world-volume over the Taub-NUT space gives a 2-dimensional (0,8) superconformal field theory, in which the gauge theory partition function is computed as the elliptic genus

$$Z = \operatorname{Tr}\left((-1)^F y^{J_0} q^{L_0 - c/24}\right).$$

In order to further analyze this system we switch to yet another duality frame by compactifying back to Type IIA theory, but now along the S^1 fiber in the Taub-NUT geometry. This is the familiar 9-11 exchange. In this fashion we end up with a IIA compactification on

(IIA)
$$\mathbb{R}^3 \times T^2 \times \mathbb{R}^5$$
,

with N D4-branes wrapping $\mathbb{R}^3 \times T^2$. However, because the circle fibration of the TN space has singular points, we have to include D6-branes as well. In fact, there will be k D6-branes that wrap $T^2 \times \mathbb{R}^5$ and are localized at the points $\vec{x}_1, \ldots, \vec{x}_k$ in the \mathbb{R}^3 . This situation is represented in the box on the left-hand side in Fig. 1.6.

Summarizing, we get a system of N D4-branes and k D6-branes intersecting along the T^2 . This intersection locus is called the I-brane. It is pictured in Fig. 3.3.2. We will now study this I-brane system in greater detail.

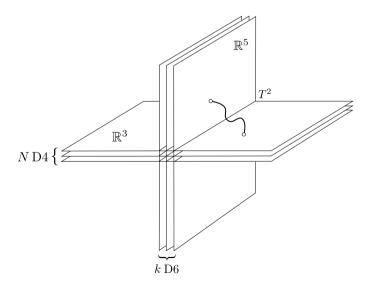


Figure 3.3: Configuration of intersecting D4 and D6-branes with one of the 4-6 open strings that gives rise to a chiral fermion localized on the I-brane.

3.3.3 Free fermions

A collection of D4-branes and D6-branes that intersect along two (flat) dimensions is a supersymmetric configuration. One way to see this is that after some T-dualities, it can be related to a D0-D8 or D1-D9 system. The supersymmetry in this case is of type (0,8). The massless modes of the 4-6 open strings stretching between the D4 and D6 branes reside entirely in the Ramond sector. All modes in the NS sector are massive. These massless modes are well-known to be chiral fermions on the 2-dimensional I-brane [51, 80, 81]. If we have N D4-branes and k D6-branes, the chiral fermions

$$\psi_{i,\overline{a}}(z), \ \psi_{i,a}^{\dagger}(z), \qquad i = 1, \dots, N, \ a = 1, \dots, k$$

transform in the bifundamental representations (N,\overline{k}) and (\overline{N},k) of $U(N)\times U(k)$. Since we are computing an index, we can take the $\alpha'\to 0$ limit, in which all massive modes decouple. In this limit we are just left with the chiral fermions. Their action is necessarily free and given by

$$S = \int d^2z \; \psi^{\dagger} \bar{\partial}_{A+\widetilde{A}} \psi,$$

where A and \widetilde{A} are the restrictions to the I-brane T^2 of the U(N) and U(k) gauge fields, that live on the worldvolumes of the D4-branes and the D6-branes respectively. (Here we have absorbed the overall coupling constant).

Under the two U(1)'s the fermions have charge (+1,-1). Therefore the overall (diagonal) U(1) decouples and the fermions effectively couple to the gauge group

$$U(1) \times SU(N) \times SU(k),$$

where the remaining U(1) is the anti-diagonal. At this point we ignore certain discrete identifications under the \mathbb{Z}_N and \mathbb{Z}_k centers, that we will return to later.

Zero modes

A special role is played by the zero-modes of the D-brane gauge fields. In the supersymmetric configuration we can have both a non-trivial flat U(N) and U(k) gauge field turned on along the T^2 . We will denote these moduli as u_i and v_a respectively. The partition function of the chiral fermions on the I-brane will be a function $Z(u,v,\tau)$ of both the flat connections u,v and the modulus τ . It will transform as a (generalized) Jacobi-form under the action of $SL(2,\mathbb{Z})$ on the two-torus.

The couplings u and v have straightforward identifications in the $\mathcal{N}=4$ gauge theory on the TN space. First of all, the parameters u_i are Wilson loops along the circle of the D4 compactified on $TN \times S^1$, and so in the 4-dimensional theory

they just describe the values of the scalar fields on the Higgs moduli space. That is, they parametrize the positions u_i of the N D4-branes along the S^1 . Clearly, we are not interested in describing these kind of configurations where the gauge group U(N) gets broken to $U(1)^N$ (or some intermediate case). Therefore we will in general put u=0.

The parameters v_a are the Wilson lines on the D6-branes and are directly related to fluxes along the non-trivial two-cycles of TN_k and (in the limiting case) on the A_{k-1} geometry. To see this, let us briefly review how the world-volume fields of the D6-branes are related to the TN geometry in the M-theory compactification.

The positions of the NUTs \vec{x}_a of the TN manifold are given by the vev's of the three scalar Higgs fields of the 6+1 dimensional gauge theory on the D6-brane. In a similar fashion the U(1) gauge fields \widetilde{A}_a on the D6-branes are obtained from the 3-form C_3 field in M-theory. More precisely, if ω_a are the k harmonic two-forms on TN_k introduced in Section 3.3.1, we have a decomposition

$$C_3 = \sum_a \omega_a \wedge \widetilde{A}_a. \tag{3.20}$$

We recall that the forms ω_a are localized around the centers \vec{x}_a of the TN geometry (the fixed points of the circle action). So in this fashion the bulk C_3 field gets replaced by k U(1) brane fields \widetilde{A}_a . This relation also holds for a single D6-brane, because the two-form ω is normalizable in the TN_1 geometry. Relation (3.20) holds in particular for a flat connection, in which case we get the M-theory background

$$C_3 = \sum_a v_a \,\omega_a \wedge dz + c.c.$$

Reducing this 3-form down to the type IIA configuration on $TN \times S^1$ gives a mixture of NS B fields and RR C_3 fields on the Taub-NUT geometry. Finally, in the $\mathcal{N}=4$ gauge theory this translates (for an instanton background) into a topological coupling

$$\int v \wedge \operatorname{Tr} F_+ + \bar{v} \wedge \operatorname{Tr} F_-,$$

with v the harmonic two-form

$$v = \sum_{a} v_a \omega_a.$$

The existence of this coupling can also be seen by recalling that the M5-brane action contains the term $\int H \wedge C_3$. On the manifold $M \times T^2$ the tensor field strength H reduces as $H = F_+ \wedge d\bar{z} + F_- \wedge dz$ and similarly one has $C_3 = v \wedge dz + \bar{v} \wedge d\bar{z}$, which gives the above result. If one thinks of the gauge theory in terms of a D3-brane, the couplings v, \bar{v} are the fluxes of the complexified 2-form combination $B_{RR} + \tau B_{NS}$.

Chiral anomaly

We should address another point: the chiral fermions on the I-brane are obviously anomalous. Under a gauge transformation of, say, the U(N) gauge field

$$\delta A = D\xi$$
,

the effective action of the fermions transforms as

$$k \int_{T^2} \operatorname{Tr}(\xi F_A).$$

A similar story holds for the U(k) gauge symmetry. Nonetheless, the overall theory including both the chiral fermions on the I-brane and the gauge fields in the bulk of the D-branes is consistent, due to the coupling between both systems. The consistency is ensured by Chern-Simons terms (3.11) in the D-brane actions, which cancel the anomaly through the process of anomaly inflow [51, 82]. For example, on the D4-brane there is a term coupling to the RR 2-form (graviphoton) field strength G_2 :

$$I_{CS} = \frac{1}{2\pi} \int_{T^2 \times \mathbb{R}^3} G_2 \wedge CS(A), \tag{3.21}$$

with Chern-Simons term

$$CS(A) = Tr(AdA + \frac{2}{3}A \wedge A \wedge A).$$

Because of the presence of the D6-branes, the 2-form G_2 is no longer closed, but satisfies instead

$$dG_2 = 2\pi k \cdot \delta_{T^2}.$$

Therefore under a gauge transformation $\delta A=D\xi$ the D4-brane action gives the required compensating term

$$\delta I_{CS} = \frac{1}{2\pi} \int G_2 \wedge d \operatorname{Tr}(\xi F_A) = -k \int_{T^2} \operatorname{Tr}(\xi F_A),$$

which makes the whole system gauge invariant.

3.4 Nakajima-Vafa-Witten correspondence

So far we have obtained a configuration of N D4-branes and k D6-branes that intersect transversely along a 2-torus. Moreover, the massless modes of the 4-6 open strings combine into Nk free fermions on this 2-torus. This already relates SU(N) Vafa-Witten theory on an A_k -singularity to a 2-dimensional conformal

field theory of free fermions. However, the I-brane system contains more information than just the Vafa-Witten partition function. In this section we analyze the I-brane system and extract the Nakajima-Vafa-Witten correspondence from the I-brane partition function.

3.4.1 Conformal embeddings and level-rank duality

The system of intersecting branes gives an elegant realization of the level-rank duality

$$\widehat{su}(N)_k \leftrightarrow \widehat{su}(k)_N$$

that is well-known in CFT and 3d topological field theory. The analysis has been conducted in [82] for a system of D5-D5 branes, which is of course T-dual to the D4-D6 system that we consider here. Hence we can follow this analysis to a large extent.

The system of Nk free fermions has central charge c=Nk and gives a realization of the $\widehat{u}(Nk)_1$ affine symmetry at level one. In terms of affine Kac-Moody Lie groups we have the embedding

$$\widehat{u}(1)_{Nk} \times \widehat{su}(N)_k \times \widehat{su}(k)_N \subset \widehat{u}(Nk)_1.$$
 (3.22)

This is a conformal embedding, in the sense that the central charges of the WZW models on both sides are equal. Indeed, using that the central charge of $\widehat{su}(N)_k$ is

$$c_{N,k} = \frac{k(N^2 - 1)}{k + N},$$

it is easily checked that

$$1 + c_{N,k} + c_{k,N} = Nk$$
.

The generators for these commuting subalgebras are bilinears constructed out of the fermions $\psi_{i,a}$ and their conjugates $\psi_{i,a}^{\dagger}$. In terms of these fields one can define the currents of the $\widehat{u}(N)_k$ and $\widehat{u}(k)_N$ subalgebras as respectively

$$J_{j\overline{k}}(z) = \sum_{a} \psi_{j}{}^{a} \psi_{\overline{k}a}^{\dagger},$$

and

$$J_{\overline{a}b}(z) = \sum_{j} \psi_{j,\overline{a}} \psi_b^{\dagger j}.$$

Now it is exactly the conformal embedding (3.22) that gives the most elegant explanation of level-rank duality. This correspondence should be considered as the affine version of the well-known Schur-Weyl duality for finite-dimensional Lie groups. Let us recall that the latter is obtained by considering the (commuting)

actions of the unitary group and symmetric group

$$U(N) \times S_k \subset U(Nk)$$

on the vector space \mathbb{C}^{Nk} , regarded as the k-th tensor product of the fundamental representation \mathbb{C}^N . Schur-Weyl duality is the statement that the corresponding group algebras are maximally commuting in $\operatorname{End}\left((\mathbb{C}^N)^{\otimes k}\right)$, in the sense that the two algebras are each other's commutants. Under these actions one obtains the decomposition

$$\mathbb{C}^{Nk} = \bigoplus_{\rho} V_{\rho} \otimes \widetilde{V}_{\rho},$$

with V_{ρ} and \widetilde{V}_{ρ} irreducible representations of u(N) and S_k respectively. Here ρ runs over all partitions of k with at most N parts. This duality gives the famous pairing between the representation theory of the unitary group and the symmetric group.

In the affine case we have a similar situation, where we now take the kth tensor product of the N free fermion Fock spaces, viewed as the fundamental representation of $\widehat{u}(N)_1$. The symmetric group S_k gets replaced by $\widehat{u}(k)_N$ (which reminds one of constructions in D-branes and matrix string theory, where the symmetry group appears as the Weyl group of a non-Abelian symmetry). The affine Lie algebras

$$\widehat{u}(1)_{Nk} \times \widehat{su}(N)_k \times \widehat{su}(k)_N$$

again have the property that they form maximally commuting subalgebras within $\widehat{u}(Nk)_1$. The total Fock space $\mathcal{F}^{\otimes Nk}$ of Nk free fermions now decomposes under the embedding (3.22) as

$$\mathcal{F}^{\otimes Nk} = \bigoplus_{\rho} U_{\|\rho\|} \otimes V_{\rho} \otimes \widetilde{V}_{\widetilde{\rho}}.$$
 (3.23)

Here $U_{\parallel\rho\parallel}$, V_{ρ} and $\widetilde{V}_{\widetilde{\rho}}$ denote irreducible integrable representations of $\widehat{u}(1)_{Nk}$, $\widehat{su}(k)_N$, and $\widehat{su}(N)_k$ respectively.

The precise formula for the decomposition (3.23) is a bit complicated, in particular due to the role of the overall U(1) symmetry, and is given in detail in Appendix A. But roughly it can be understood as follows: the irreducible representations of $\widehat{u}(N)_k$ are given by Young diagrams that fit into a box of size $N \times k$. Similarly, the representations of $\widehat{u}(k)_N$ fit in a reflected box of size $k \times N$. In this fashion level-rank duality relates a representation V_ρ of $\widehat{u}(N)_k$ to the representation $\widetilde{V}_{\widetilde{\rho}}$ of $\widehat{u}(k)_N$ labeled by the transposed Young diagram. If we factor out the $\widehat{u}(1)_{Nk}$ action, we get a representation of charge $\|\rho\|$, which is related to the total number of boxes $|\rho|$ in ρ (or equivalently $\widetilde{\rho}$).

At the level of the partition function we have a similar decomposition into char-

acters. To write this in more generality it is useful to add the Cartan generators. That is, we consider the characters for $\widehat{u}(N)_k$ that are given by

$$\chi_{\rho}^{\widehat{u}(N)_{k}}(u,\tau) = \text{Tr}_{V_{o}}\left(e^{2\pi i u_{j}J_{0}^{j}}q^{L_{0}-c_{N,k}/24}\right),$$

and similarly for $\widehat{u}(k)_N$ we have

$$\chi_{\widetilde{\rho}}^{\widehat{u}(k)_N}(v,\tau) = \mathrm{Tr}_{\widetilde{V}_{\widetilde{\rho}}}\left(e^{2\pi i v_a J_0^a} q^{L_0 - c_{k,N}/24}\right).$$

Here the diagonal currents

$$J_0^j = \oint \frac{dz}{2\pi i} J_{jj}(z), \qquad J_0^a = \oint \frac{dz}{2\pi i} J_{aa}(z)$$

generate the Cartan tori $U(1)^N \subset U(N)$ and $U(1)^k \subset U(k)$.

Including the Wilson lines u and v for the U(N) and U(k) gauge fields, the partition function of the I-brane system is given by the character of the fermion Fock space

$$Z_{I}(u, v, \tau) = \operatorname{Tr}_{\mathcal{F}} \left(e^{2\pi i (u_{j} J_{0}^{j} + v_{a} J_{0}^{a})} q^{L_{0} - \frac{Nk}{24}} \right)$$

$$= q^{-\frac{Nk}{24}} \prod_{\substack{j=1,\dots,N\\a=1,\dots,k}} \prod_{n\geq 0} \left(1 + e^{2\pi i (u_{j} + v_{a})} q^{n+1/2} \right) \left(1 + e^{-2\pi i (u_{j} + v_{a})} q^{n+1/2} \right).$$
(3.24)

Writing the decomposition (3.23) in terms of characters gives

$$Z_{I}(u,v,\tau) = \sum_{[\rho]\subset\mathcal{Y}_{N-1,k}} \sum_{j=0}^{N-1} \sum_{a=0}^{k-1} \chi_{|\rho|+jk+aN}^{\widehat{u}(1)_{Nk}}(N|u|+k|v|,\tau) \chi_{\sigma_{N}^{j}(\rho)}^{\widehat{su}(N)_{k}}(\overline{u},\tau) \chi_{\sigma_{k}^{a}(\overline{\rho})}^{\widehat{su}(k)_{N}}(\overline{v},\tau),$$

where the Young diagrams $\rho \in \mathcal{Y}_{N-1,k}$ of size $(N-1) \times k$ represent $\widehat{su}(N)_k$ integrable representations and σ denote generators of the outer automorphism groups \mathbb{Z}_N and \mathbb{Z}_k that connect the centers of SU(N) and SU(k) to the U(1) factor (see again Appendix A for notation and more details).

3.4.2 Deriving the McKay-Nakajima correspondence

In the intersecting D-brane configuration both the D4-branes and the D6-branes are non-compact. So, we can choose both the U(N) and U(k) gauge groups to be non-dynamical and freeze the background gauge fields A and \widetilde{A} . In fact, this set-up is entirely symmetric between the two gauge systems, which makes level-rank duality transparent.

However, in order to make contact with the $\mathcal{N}=4$ gauge theory computation, we will have to break this symmetry. Clearly, we want the U(N) gauge field to be dynamical — our starting point was to compute the partition function of the U(N) Yang-Mills theory. The U(k) symmetry should however not be dynamical, since we want to freeze the geometry of the Taub-NUT manifold. So, to derive the gauge theory result, we will have to integrate out the U(N) gauge field A on the I-brane. Particular attention has to be payed to the U(1) factor in the CFT on the I-brane. We will argue that in this string theory set-up we should not take that to be dynamical.

Therefore we are dealing with a partially gauged CFT or coset theory

$$\widehat{u}(Nk)_1/\widehat{su}(N)_k$$
.

In particular the $\widehat{su}(N)_k$ WZW model will be replaced by the corresponding G/G model. Gauging the model will reduce the characters. (Note that this only makes sense if the Coulomb parameters u are set to zero. If not, we can only gauge the residual gauge symmetry, which leads to fractionalization and a product structure.) In the gauged WZW model, which is a topological field theory, only the ground state remains in each irreducible integrable representation. So we have a reduction

$$\chi_{\rho}^{\widehat{su}(N)_k}(\overline{u}, \tau) \rightarrow q^{h_{\rho}-c/24},$$

with h_{ρ} the conformal dimension of the ground state representation ρ . Note that the choice of ρ corresponds exactly to the boundary condition for the gauge theory on the A_{k-1} manifold. We will explain this fact, that is crucial to the McKay correspondence, in a moment.

Gauging the full I-brane theory and restricting to the sector ρ finally gives

$$Z_I(u,v,\tau) \to Z_{\rho}^{N,k}(v,\tau) = q^{h_{\rho}-c/24} \sum_{a=0}^{k-1} \chi_{|\rho|+aN}^{\widehat{u}(1)_{Nk}}(k|v|,\tau) \chi_{\sigma_k^a(\widehat{\rho})}^{\widehat{su}(k)_N}(\overline{v},\tau).$$

Up to the $\chi^{\widehat{u}(1)_{Nk}}$ factor, this reproduces the results presented in [40, 39] for ALE spaces, which involve just $\widehat{su}(k)_N$ characters. This extra factor is is due to additional monopoles mentioned in Section 3.3.2. They are related to the finite radius S^1 at infinity of the Taub-NUT space and are absent in case of ALE geometries.

In fact, the extra U(1) factor can already be seen at the classical level, because the extra normalizable harmonic two-form ω in (3.18) disappears in the decompactification limit where TN_k degenerates into A_{k-1} . The lattice $H^2(TN_k, \mathbb{Z})$ is isomorphic to \mathbb{Z}^k with the standard inner product and contains the root lattice A_{k-1} as a sublattice given by $\sum_I n_I = 0$. Note also that the lattice \mathbb{Z}^k is not even, which explains why the I-brane partition function has a fermionic charac-

ter and only transforms under a subgroup of $SL(2,\mathbb{Z})$ that leaves invariant the spin structure on T^2 .

Relating the boundary conditions

By relating the original 4-dimensional gauge theory to the intersecting brane picture one can in fact derive the McKay correspondence directly. Moreover we can understand the appearance of characters of the WZW models (for both the SU(N) and the SU(k) symmetry) in a more natural way in this set-up. Recall that the SU(N) gauge theory on the A_{k-1} singularity or TN_k manifold is specified by a boundary state. This state is given by picking a flat connection on the boundary that is topologically S^3/\mathbb{Z}_k . If we think of this system in radial quantization near the boundary, where we consider a wave function for the time evolution along

$$S^3/\mathbb{Z}_k \times \mathbb{R}$$
,

we have a Hilbert space with one state $|\rho\rangle$ for each N-dimensional representation

$$\rho: \mathbb{Z}_k \to U(N).$$

After the duality to the I-brane system, we are dealing with a 5-dimensional SU(N) gauge theory on $\mathbb{R}^3 \times T^2$, with k D6-branes intersecting it along $\{p\} \times T^2$ where p is (say) the origin of \mathbb{R}^3 . Here the boundary of the D4-brane system is $S^2 \times T^2$. In other words, near the boundary the space-time geometry looks like $\mathbb{R} \times S^2 \times T^2$. We now ask ourselves what specifies the boundary states for this theory. Since we need a finite energy condition, this is equivalent to considering the IR limit of the theory. In M-theory the S^1 -bundle over S^2 carries a first Chern class k, which translates into the flux of the graviphoton field strength

$$\int_{S^2} G_2 = 2\pi k.$$

Therefore the term

$$\int_{S^2 \times T^2 \times \mathbb{R}} G_2 \wedge CS(A),$$

living on the D4 brane, leads upon reduction on S^2 (as is done in [82]) to the term

$$I_{CS} = 2\pi k \int_{T^2 \times \mathbb{R}} CS(A).$$

Hence we have learned that the boundary condition for the D4-brane requires specifying a state of the SU(N) Chern-Simons theory at level k living on T^2 . The Hilbert space for Chern-Simons theory on T^2 is well-known to have a state for each integrable representation of the $\widehat{u}(N)_k$ WZW model, which up to the level-rank duality described in the previous section, gives the McKay correspondence.

In fact, the full level-rank duality can be brought to life. Just as we discussed for the N D4-branes, a SU(k) gauge theory lives on the k D6-branes on $T^2 \times \mathbb{R}^5$. The boundary of the space is $S^4 \times T^2$. Furthermore, taking into account that the N D4-branes source the G_4 RR flux through S^4 , we get, as in the above, a SU(k) Chern-Simons theory at level N living on $T^2 \times \mathbb{R}$. Therefore the boundary condition should be specified by a state in the Hilbert space of the SU(k) Chern-Simons theory on T^2 . So we see three distinct ways to specify the boundary conditions: as a representation of \mathbb{Z}_k in SU(N), as a character of SU(N) at level k, and as a character of SU(k) at level N. Thus we have learned that, quite independently of the fermionic realization, there should be an equivalence between these objects.

To make the map more clearly we could try to show that the choice of the flat connection of the SU(N) theory on S^3/\mathbb{Z}_k gets mapped to the characters that we have discussed in the dual intersecting brane picture. To accomplish this, recall that the original SU(N) action on the A_{k-1} space leads to a boundary term (modulo an integer multiple of $2\pi i \tau$) given by the Chern-Simons invariant

$$\frac{\tau}{4\pi i} \int_{A_{k-1}} \operatorname{Tr} F \wedge F = \frac{\tau}{4\pi i} \int_{S^3/\mathbb{Z}_k} CS(A).$$

Restricting to a particular flat connection on S^3/\mathbb{Z}_k yields the value of the classical Chern-Simons action.

If we show that

$$S(\rho) = \frac{1}{8\pi^2} \int_{S^3/\mathbb{Z}_k} CS(A)$$

for the flat connection ρ on S^3/\mathbb{Z}_k gets mapped to the conformal dimension h_ρ of the corresponding state of the quantum Chern-Simons theory on T^2 , we would have completed a direct check of the map, because the gauge coupling constant τ above is nothing but the modulus of the torus in the dual description.

To see how this works, let us first consider the abelian case of N=1. In that case the flat connection ρ is given by a phase $e^{2\pi i n/k}$ with $n\in \mathbb{Z}/k\mathbb{Z}$. The corresponding CS term gives

$$S^{U(1)}(\rho) = \frac{n^2}{2k}.$$

This is the conformal dimension of a primary state of the U(1) WZW model at level k.

A general U(N) connection can always be diagonalized to $U(1)^N$, which therefore gives integers $n_1, \ldots, n_N \in \mathbb{Z}/k\mathbb{Z}$. The Chern-Simons action is therefore

given by

$$S^{U(N)}(\rho) = \sum_{i=1}^{N} \frac{n_i^2}{2k}.$$

On the other hand, a conformal dimension of a primary state in the corresponding WZW model is given by

$$h_{\rho} = \frac{C_2(\rho)}{2(k+N)},$$

where ρ is an irreducible integrable $\widehat{u}(N)_k$ weight. Such a weight can be encoded in a Young diagram with at most N rows of lengths R_i . There is a natural change of basis $n_i = R_i + \rho_i^{\text{Weyl}}$ where we shift by the Weyl vector ρ^{Weyl} . If we decompose U(N) into SU(N) and U(1), the basis n_i cannot be longer than k, which relates to the condition $n_i \in \mathbb{Z}_k$ on the Chern-Simons side. In this basis the second Casimir C_2 takes a simple form. Therefore the conformal dimension becomes

$$h_{\rho} = -\frac{N(N^2 - 1)}{24(k + N)} + \frac{1}{2(k + N)} \sum_{i=1}^{N} n_i^2.$$

The constant term combines nicely with the central charge contribution $-c_{N,k}/24$ to give an overall constant $(N^2-1)/24$. Apart from this term we see that h_ρ indeed matches the expression for $S^{U(N)}(\rho)$ given above, up to the usual quantum shift $k\to k+N$.

According to the McKay correspondence one might expect to find a relation between representations of \mathbb{Z}_k and $\widehat{u}(k)_N$ integrable weights. Instead, we have just shown how $\widehat{u}(N)_k$ weights ρ arise. Nonetheless, one can relate integrable weights of those algebras by a transposition of the corresponding Young diagrams. Then the conformal dimensions of $\widehat{u}(k)_N$ weights $\widetilde{\rho}$ are determined by the relation [83]

$$h_{\rho} + h_{\widetilde{\rho}} = \frac{|\rho|}{2} - \frac{|\rho|^2}{2Nk},$$

which is a consequence of the level-rank duality described in Appendix A. The above chain of arguments connects \mathbb{Z}_k representations and $\widehat{u}(k)_N$ integrable weights, thereby realizing the McKay correspondence.

3.4.3 Orientifolds and SO/Sp gauge groups

In this chapter we have considered a system of N D4-branes and k D6-branes intersecting along a torus, whose low energy theory is described by U(N) and U(k) gauge theories on each stack of branes, together with bifundamental fermions. We can reduce this system to orthogonal or symplectic gauge groups in a standard way by adding an orientifold plane. This construction can also be lifted to

M-theory. Let us recall that D6-branes in our system originated from a Taub-NUT solution in M-theory. The O6-orientifold can also be understood from M-theory perspective, and it corresponds to the Atiyah-Hitchin space [84]. Combining both ingredients, it is possible to construct the M-theory background for a collection of D6-branes with an O6-plane. The details of this construction are explained in [84].

Let us see what are the consequences of introducing the orientifold into our I-brane system. We start with a stack of k D6-branes. To get orthogonal or symplectic gauge groups one should add an orientifold O6-plane parallel to D6-branes [85], which induces an orientifold projection Ω which acts on the Chan-Paton factors via a matrix γ_{Ω} . Let us recall there are in fact two species O6[±] of such an orientifold. As the Ω must square to identity, this requires

$$\gamma_{\Omega}^t = \pm \gamma_{\Omega}$$

with the \pm sign corresponding to $O6^{\pm}$ -plane, which gives respectively SO(k) and Sp(2k) gauge group. In the former case k can be even or odd; k odd requires having *half-branes*, fixed to the orientifold plane (as explained *e.g.* in [86]).

Let us add now N D4-branes intersecting D6 along two directions. The presence of $O6^{\pm}$ -plane induces appropriate reduction of the D4 gauge group as well. The easiest way to argue what gauge group arises is as follows. We can perform a T-duality along three directions to get a system of D1-D9-branes, now with a spacetime-filling O9-plane. This is analogous to the D5-D9-09 system in [85], in which case the gauge groups on both stacks of branes must be different (either orthogonal on D5-branes and symplectic on D9-branes, or the other way round); the derivation of this fact is a consequence of having 4 possible mixed Neumann-Dirichlet boundary conditions for open strings stretched between branes. On the contrary, for D1-D9-O9 system there is twice as many possible mixed boundary conditions, which in consequence leads to the same gauge group on both stack of branes. By T-duality we also expect to get the same gauge groups in D4-D6 system under orientifold projection.

Let us explain now that the appearance of the same type of gauge groups is consistent with character decompositions resulting from consistent conformal embeddings or the existence of the so-called dual pairs of affine Lie algebras related to systems of free fermions. We have already come across one such consistent embedding in (3.22) for $\widehat{u}(Nk)_1$. A dual pair of affine algebras in this case is $(\widehat{su}(N)_k, \widehat{su}(k)_N)$. These two algebras are related by the level-rank duality discussed in Appendix A. As proved in [87, 88], all other consistent dual pairs are necessarily of one of the following forms

$$(\widehat{sp}(2N)_k, \widehat{sp}(2k)_N),$$

$$(\widehat{so}(2N+1)_{2k+1}, \widehat{so}(2k+1)_{2N+1}),$$

 $(\widehat{so}(2N)_{2k+1}, \widehat{so}(2k+1)_{2N}),$
 $(\widehat{so}(2N)_{2k}, \widehat{so}(2k)_{2N}).$

Corresponding expressions in terms of characters, analogous to (A.5), are also given in [88]. The crucial point is that both elements of those pair involve algebras of the same type, which confirms and agrees with the string theoretic orientifold analysis above.

Finally we wish to stress that the appearance of U, Sp and SO gauge groups which we considered so far in this paper is related to the fact that their respective affine Lie algebras can be realized in terms of free fermions, which arise on the I-brane from our perspective. It turns out there are other Lie groups G whose affine algebras have free fermion realization. There is a finite number of them, and fermionic realizations can be found only if there exists a symmetric space of the form G'/G for some other group G' [89]. It is an interesting question whether I-brane configurations can be engineered in string theory that support fermions realizing all these affine algebras.

From a geometric point of view we can remark the following. For ALE singularities of A-type and D-type a non-compact dimension can be compactified on a S^1 to give Taub-NUT geometries. For exceptional groups such manifolds do not exist. But one can compactify two directions on a T^2 to give an elliptic fibration. In this setting exotic singularities can appear as well. Such construction have a direct analogue in type IIB string theory where they correspond to a collection of (p,q) 7-branes [90, 91]. The I-brane is now generalized to the intersection of N D3-branes with this non-abelian 7-brane configuration [92]. However, there is in general no regime where all the 7-branes are weakly coupled, so it is not straightforward to write down the I-brane system.

Chapter 4

Topological Strings, Free Fermions and Gauge Theory

In Chapter 3 we found a stringy explanation for the appearance of CFT characters in $\mathcal{N}=4$ supersymmetric gauge theories. We discovered that these characters emerge from a free fermion system living on a torus T^2 .

Also in $\mathcal{N}=2$ supersymmetric gauge theories important exact quantities have turned out to be expressible in terms of an effective Riemann surface or complex curve Σ . In this chapter we find that the I-brane configuration can be generalized to this $\mathcal{N}=2$ setting by replacing the torus T^2 with the more general 2-dimensional topology Σ . Moreover, we find an extension of the duality chain to the complete web of dualities in Fig. 1.6. Interestingly, this makes it possible to compare local Calabi-Yau compactifications with intersecting brane configurations. This sheds new light on the presence of free fermions in those theories.

The theme of this chapter is the web of dualities in Fig. 1.6. The three keywords "topological strings", "free fermions" and "gauge theory" refer to the three cornerstones of the duality web. In all of these frames a holomorphic curve Σ plays a central role. The duality web relates the curves in all three settings, thereby giving a more fundamental understanding of the appearance of curves in $\mathcal{N}=2$ theories. The goal of this chapter is to introduce the three corners of the web and their relations. This yields a fruitful dual perspective on $\mathcal{N}=2$ supersymmetric gauge theory as well as topological string theory.

An instructive example of an $\mathcal{N}=2$ supersymmetric gauge theory is the celebrated Seiberg-Witten theory. In Section 4.1 we summarize how the low energy behaviour of SU(N) supersymmetric Yang-Mills is encoded in a Riemann surface Σ_{SW} of genus N-1, which is widely known as the Seiberg-Witten curve. In Section 4.1.4 we show that it is dual to an I-brane configuration of D4 and

D6 branes that intersect at the Seiberg-Witten curve Σ_{SW} . This set-up is easily generalized to more general $\mathcal{N}=2$ gauge theories.

In Section 4.2 we study non-compact Calabi-Yau threefolds that are modeled on a Riemann surface. Such 6-dimensional backgrounds *geometrically engineer* a supersymmetric gauge theory in the four transverse dimensions. In Section 4.2.3 we relate the I-brane configuration to such Calabi-Yau compactifications in a second chain of dualities.

This far we haven't discussed topological invariants in these duality frames. This is subject of Section 4.3. We review the most relevant aspects of Calabi-Yau compactifications and the way topological string theory enters. In Section 4.4 we introduce the several types of topological invariants that the topological string captures, and we show how they enter the web of dualities. Moreover, we discuss the relation of these invariants to the free fermions on the I-brane. As an application we write down a partition function that counts bound states of D0-D2-D4 branes on a D6 brane and argue that this computes the I-brane partition function.

Let us emphasize that novel results in this chapter may be found in Section 4.1.4, Section 4.2.3, Section 4.4.3 and Section 4.4.2.

4.1 Curves in $\mathcal{N}=2$ gauge theories

 $\mathcal{N}=2$ supersymmetric gauge theories (unlike their $\mathcal{N}=4$ relatives) are sensitive to quantum corrections and thus not conformally invariant. In particular, the SU(N) theory is asymptotically free: its complex gauge coupling constant τ depends on the energy scale μ such that $g_{YM}(\mu)$ decreases at high energies. This dependence can be argued to be of the form [57]

$$\tau_{\text{eff}}(\mu) = \tau_{\text{clas}} + \frac{i}{\pi} \log \frac{\mu^2}{\Lambda^2} + \sum_{k=1}^{\infty} c_k \left(\frac{\Lambda}{\mu}\right)^{4k}$$
(4.1)

for some to be determined constants c_k , where Λ is the scale at which the gauge coupling becomes strong. The second term on the right-hand side is the only perturbative contribution, which follows from a one-loop computation, and the third term captures all possible instanton contributions.

Surprisingly, N. Seiberg and E. Witten discovered that an elegant geometrical story is hidden behind the coefficients c_k [93]. They realized that many properties of $\mathcal{N}=2$ supersymmetric gauge theories have a geometrical interpretation in terms of an auxiliary Riemann surface, which is now called the Seiberg-Witten curve. One of the successes of string theory is the physical embedding of the Seiberg-Witten curve in a 10- or 11-dimensional geometry. This has deepened

the insight in supersymmetric gauge theories considerably.

In this first section of this chapter we explain how the Seiberg-Witten curve comes about, and which information it holds about the underlying gauge theory. Moreover, we explain its embedding in string theory as the rightmost diagram in the web of dualities in Fig. 1.6. All these preliminaries are needed to get to the main result of this section: the duality of $\mathcal{N}=2$ supersymmetric gauge theories with intersecting brane configurations of D4 and D6-branes wrapping the gauge theory curve Σ .

4.1.1 Low energy effective description

Let us start with the basics. Since $\mathcal{N}=2$ super Yang-Mills on \mathbb{R}^4 is a reduction of $\mathcal{N}=1$ super Yang-Mills in six dimensions, it follows immediately that its field content consists of a gauge field A_μ , a complex scalar field ϕ and two Weyl spinors λ_\pm . The last three fields transform in the adjoint representation of the gauge group. The bosonic part of the action follows likewise from this reduction

$$\mathcal{L} = -\frac{1}{e^2} \text{Tr} \left(F \wedge *F + 2D\phi \wedge *D\phi^{\dagger} + [\phi, \phi^{\dagger}]^2 \right), \tag{4.2}$$

where we could have added the topological term $\frac{i\theta}{8\pi^2} \text{Tr}(F \wedge F)$. Supersymmetric vacua are therefore found as solutions of

$$V(\phi) = \text{Tr}[\phi, \phi^{\dagger}]^2 = 0,$$

i.e. ϕ and ϕ^\dagger have to commute. Notice that this gives a continuum set of solutions, since ϕ has an expansion in the Cartan generators $\{h_i\}$ of the gauge group

$$\phi = \sum a_i h_i \qquad a_i \in \mathbb{C}.$$

The gauge group is thus generically broken to a number of U(1)-factors. Dividing out the residual Weyl symmetry, for SU(N) we find a moduli space \mathcal{M}_c of classical vacua that is parametrized by the symmetric polynomials

$$u_k = \text{Tr}\phi^k$$

in the parameters a_i .

Classically, there are singularities in this moduli space where W-particles become massless and the gauge symmetry is partially restored. To understand the theory fully, it is important to find out what happens to these singularities quantum-mechanically. This information is contained in the quantum metric on the moduli space, which is part of the *low energy effective action*.

Quantum moduli space

Let us explain this in some detail. The abelian low energy effective action is very much restricted by supersymmetry

$$\mathcal{L} = \operatorname{Im} \int d^4 \theta \operatorname{Tr} \mathcal{F}_0(\mathbf{\Psi}^i), \tag{4.3}$$

where \mathcal{F}_0 is any holomorphic function in the $\mathcal{N}=2$ abelian vector superfields Ψ^i . The holomorphic function \mathcal{F}_0 is known as the *prepotential*, whereas the supermultiplets Ψ^i form a representation of the $\mathcal{N}=2$ supersymmetry algebra and contain the U(1) fields A^i_{μ} , ϕ^i and λ^i_+ as physical degrees of freedom.

In $\mathcal{N}=1$ language Ψ is decomposed into two $\mathcal{N}=1$ chiral multiplets Φ , containing the scalar field ϕ and λ_- , and \mathbf{W}_{α} , which can be expanded in terms of λ_+ and the field strength $F_{\mu\nu}$. This results in the well-known low energy Lagrangian

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \, \mathcal{K}(\mathbf{\Phi}^k, \overline{\mathbf{\Phi}^k}) + \int d^2\theta \, \tau_{ij}(\mathbf{\Phi}^k) \mathbf{W}_{\alpha}^i \mathbf{W}^{\alpha j}, \tag{4.4}$$

with

$$\mathcal{K}(\boldsymbol{\Phi}^k, \overline{\boldsymbol{\Phi}^k}) = \operatorname{Im} \left[\frac{\partial \mathcal{F}_0(\boldsymbol{\Phi}^k)}{\partial \boldsymbol{\Phi}^i} \overline{\boldsymbol{\Phi}^i} \right] \quad \text{and} \quad \tau_{ij}(\boldsymbol{\Phi}^i) = \frac{\partial^2 \mathcal{F}_0(\boldsymbol{\Phi}^k)}{\partial \boldsymbol{\Phi}^i \partial \boldsymbol{\Phi}^j}$$

Important is that the first term (the so-called *D-term*) in this Lagrangian determines a Kähler metric $g_{i\bar{j}}$ on the quantum vacuum moduli space \mathcal{M}_q . Indeed, when written in terms of components, we find a sigma model action $\mathcal{L}=g_{i\bar{j}}\partial\phi^i\overline{\partial\phi^j}+\ldots$ for the scalar fields ϕ^i with Kähler metric

$$g_{i\bar{j}}(\phi^k, \overline{\phi^k}) = \frac{\partial^2 \mathcal{K}(\phi^k, \overline{\phi^k})}{\partial \phi^i \partial \overline{\phi^j}} = \operatorname{Im}\left(\frac{\partial^2 \mathcal{F}_0(\phi^k)}{\partial \phi^i \partial \phi^j}\right). \tag{4.5}$$

Furthermore, the second term in the $\mathcal{N}=2$ Lagrangian (the *F-term*) yields the familiar Yang-Mills action for the field strengths $F^i_{\mu\nu}$ with gauge coupling constants τ_{ij} . It captures the holomorphic dependence of the theory.

For the SU(2) theory, when $\phi=a\sigma_3$, the quantum prepotential \mathcal{F}_0 has an expansion

$$\mathcal{F}_{0} = \frac{1}{2}\tau_{0}a^{2} + \frac{i}{2\pi}a^{2}\log\frac{a^{2}}{\Lambda^{2}} + \sum_{k=1}^{\infty} \mathcal{F}_{0,k}\left(\frac{\Lambda}{a}\right)^{4k}a^{2},\tag{4.6}$$

whose second derivative determines the effective gauge coupling $\tau_{\rm eff}(a)$ in equation (4.1). But this expression cannot be valid all over the moduli space: the

resulting metric is harmonic, and thus cannot have a minimum, while it should be positive definite. So $\tau_{\rm eff}$ must have singularities and a cannot be a global coordinate on the quantum moduli space \mathcal{M}_q . Instead, one needs another local description in the strong coupling regions on the moduli space.

Let us introduce the magnetically dual coordinate

$$a_D = \frac{\partial \mathcal{F}_0}{\partial a}.$$

The idea of Seiberg and Witten is that the tuple (a_D, a) should be considered as a holomorphic section of a $Sl(2, \mathbb{Z}) = Sp(1, \mathbb{Z})$ -bundle over the moduli space \mathcal{M}_q . Indeed, the metric on \mathcal{M} may be rewritten as

$$ds^2 = \operatorname{Im} \tau_{\text{eff}} da \otimes d\bar{a} = \operatorname{Im} da_D \otimes d\bar{a}, \quad \text{with} \quad \tau_{\text{eff}} = \left(\frac{\partial a_D}{\partial a}\right)$$
 (4.7)

Since this tuple experiences a monodromy around the singularities of \mathcal{M}_q

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M \begin{pmatrix} a_D \\ a \end{pmatrix},$$

just finding these monodromies defines a Riemann-Hilbert problem whose solution determines the quantum metric. And this turns out to be feasible. Except for the monodromy M_{∞} around $u=\infty$, Seiberg and Witten find two other quantum singularities at $u=\pm\Lambda^2$ with monodromy matrices $M_{\pm\Lambda}$. They are shown in Fig. 4.1.

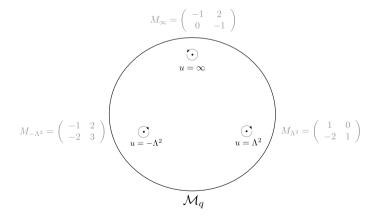


Figure 4.1: The quantum moduli space \mathcal{M}_q for the SU(2) Seiberg-Witten theory is a 2-sphere with three singularities at $u=\infty$ and $u=\pm\Lambda^2$. The low energy effective theory is described by an $Sp(1,\mathbb{Z})=Sl(2,\mathbb{Z})$ -bundle over \mathcal{M}_q with monodromies $M_{\infty,\pm\Lambda^2}$ around the singularities.

4.1.2 Seiberg-Witten curve

Mathematically, this solution has another interesting characterization. Notice that the metric in equation (4.7) equals that of an elliptic curve at each point $a \in \mathcal{M}_q$. Moreover, the monodromies $M_{\infty,\pm\Lambda^2}$ altogether generate a subgroup $\Gamma_0(2) \subset Sl(2,\mathbb{Z})$, which is exactly the moduli space for an elliptic curve Σ_{SW} . The singularity structure suggests that the moduli space is parametrized by the family

$$\Sigma_{SW}(u): \quad y^2 = (x^2 - u)^2 - \Lambda^4$$
 (4.8)

of Seiberg-Witten curves. This family of elliptic curves, illustrated in Fig. 4.2, has four branch points in the x-plane, which can be connected by two cuts running from $\pm \sqrt{u-\Lambda^2}$ to $\pm \sqrt{u+\Lambda^2}$.

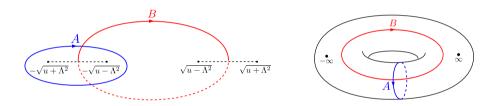


Figure 4.2: On the left we see a hyperelliptic representation of the SU(2) Seiberg-Witten curve defined in equation (4.8), together with a choice of A and B-cycle. On the right it is compactified by adding two points at infinity.

At the points $u\in\{\infty,\pm\Lambda^2\}$ some of these branch points come together so that the elliptic curve degenerates. Precisely which 1-cycle degenerates at a quantum singularity, labeled by the monodromy matrix M, can be found by solving the equation

$$\begin{pmatrix} p & q \end{pmatrix} M = 0.$$

The 1-cycle pB + qA vanishes at the corresponding singularity of \mathcal{M}_q .

Remember that the period matrix τ of an elliptic curve is defined by

$$\tau = \frac{\int_B \omega}{\int_A \omega},$$

where ω is a holomorphic 1-form on Σ_{SW} , and satisfies ${\rm Im}\tau>0$. This suggests

the identifications

$$\partial_u a_D = \int_B \omega, \quad \partial_u a = \int_A \omega$$

and leads to the introduction of a meromorphic Seiberg-Witten form η_{SW} that obeys

$$\partial_u \eta_{SW} = \omega.$$

The metric on the quantum moduli space \mathcal{M}_q can thus be given a geometric meaning in terms of an auxiliary Riemann surface Σ_{SW} together with a meromorphic 1-form η_{SW} .

Monodromies and BPS particles

Physically, a quantum singularity hints that certain BPS particles becomes massless. In $\mathcal{N}=2$ supersymmetric Yang-Mills BPS particles are characterized by their electro-magnetic charge $\gamma=(p,q)$. BPS particles with both types of charge are called BPS dyons. Their central charge Z is of the form

$$Z = qa + pa_D = q \int_A \eta_{SW} + p \int_B \eta_{SW},$$
 (4.9)

where on the righthand-side we have written the parameters a and a_D in terms of the geometric variables η_{SW} and the 1-cycles on the Seiberg-Witten curve. This formula implies that BPS dyons have a geometric interpretation as wrapping a combination of A and B-cycles of the Seiberg-Witten curve. This is shown in Fig. 4.2: magnetic particles wrap the B-cycle of the curve, whereas electric particles wrap the A-cycle.

The monodromies $M_{\infty,\pm\Lambda^2}$ can therefore indeed be explained in terms of BPS particles that become massless. Magnetic monopoles of charge (1,0) become very light in the neighbourhood of $u=\Lambda^2$, since they are associated with the vanishing of the B-cycle. On the other hand, compared to the electric gauge bosons W^\pm of charge $\pm(0,1)$, they become very heavy in the weak-coupling region of the moduli space. This has led to an understanding of confinement in $\mathcal{N}=2$ supersymmetric gauge theories [93].

Determining the full spectrum of the SU(2) Seiberg-Witten theory is more subtle. For example, because of the monodromy around the three singular points, the BPS charges are not determined uniquely. A careful analysis [94] reveals that there is a contour on \mathcal{M}_q going through the singular points $u=\pm\Lambda^2$, where BPS dyons may decay into other BPS dyons. This contour separates the strong and the weak coupling region and leads to a consistent BPS spectrum.

U(N) Seiberg-Witten curve

The discussion in Section 4.1.1 on the classical moduli space \mathcal{M}_c can easily be extended to other gauge groups. For gauge group U(N) the singularity structure on \mathcal{M}_c is encoded in the characteristic polynomial

$$P_N(x,\phi) = \det[x\mathbb{1} - \phi],$$

that defines coordinates $u_k = \text{Tr}\phi^k$ on the moduli space. When two or more a_i 's assume the same value, the gauge group is classically partially restored.

On the quantum level the low energy theory is captured by a section

$$(a_i, a_{D,i}) \in \Gamma(\mathcal{M}_q, \mathcal{H})$$

of an $Sp(N,\mathbb{Z})$ -bundle \mathcal{H} over \mathcal{M}_q . This section is related to the genus N-1 hyperelliptic curve

$$\Sigma_{SW}: \quad y^2 = P_N(x, u_k)^2 - \Lambda^{2N},$$

where u_k now stands for the quantum vacuum expectation value (vev) $u_k = \langle {\rm Tr} \phi^k \rangle$ of the scalar field ϕ . The extra constraint $u_1 = 0$ defines the Seiberg-Witten curve for gauge group SU(N). The curve Σ_{SW} can be represented by a two-sheeted x-plane with N cuts. Whenever two branch points coincide a quantum singularity arises where some BPS dyon becomes massless.

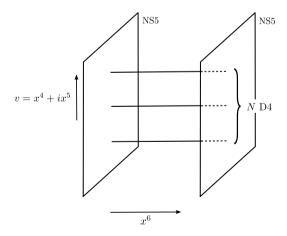


Figure 4.3: Configuration of N D4-branes stretched between two NS5-branes that realizes the SU(N) Seiberg-Witten gauge theory on the D4 worldvolume. This picture is only valid in the small g_s limit where the effect of the ending of the D4-branes on the NS5 branes may be neglected.

By applying a coordinate transformation $y = P_N(x; u_k) + \Lambda^N t$ we find

$$\Sigma_{SW}: \quad \Lambda^N(t+t^{-1}) + 2P_N(v; u_k) = 0.$$
 (4.10)

The Seiberg-Witten differential now takes a particularly simple form

$$\eta_{SW} = v(t, u_k) \frac{dt}{t}. (4.11)$$

Exactly this representation of the Seiberg-Witten curve and differential plays a crucial role in the following. It relates $\mathcal{N}=2$ gauge theories to integrable systems and makes it possible to embed them in string theory as configurations of D4 and NS5 branes. Note that the section $(a_i,a_{D,i})$ is recovered by the period integrals

$$a_i = \int_{A_i} \eta_{SW}, \quad a_{D,i} = \int_{B_i} \eta_{SW}.$$

4.1.3 Brane webs

In string theory $\mathcal{N}=2$ supersymmetric gauge theories can be studied on a configuration of D4, NS5, and D6 branes [95]. Pure U(N) (and SU(N)) Seiberg-Witten theory are embedded in type IIA theory on \mathbb{R}^{10} as a combination of two NS5-branes with N D4-branes stretched between them, see Fig. 4.3. The NS5-branes are located at some fixed classical value of x^6 and are parametrized by x^0,\ldots,x^5 , while the D4-branes are parametrized by x^0,\ldots,x^3 and stretch between the two NS5-branes in the x^6 -direction. Furthermore, we define a complex coordinate $v=x^4+ix^5$. Since the coordinate x^6 has to fulfill the Laplace equation $\nabla^2 x^6=0$ on the NS5-worldvolume, the NS5-branes are actually curved logarithmically at infinity

$$x_6 \sim \pm N \log |v|, \quad v \to \infty.$$
 (4.12)

The D4-branes cause dimples on the NS5-worldvolume.

The N abelian gauge fields on the N D4-branes realize a low energy description of $\mathcal{N}=2$ super Yang-Mills in the Minkowski directions x^0,\ldots,x^3 . The vev's for the complex scalar ϕ correspond to the positions of the D4-branes on the NS5-brane, and the effective gauge coupling is given by

$$\frac{1}{g_{\rm eff}^2(v)} = \frac{L(v)}{\lambda},$$

where L(v) is the x^6 -distance between both NS5-branes at position v. Hence v plays the role of mass scale in four dimensions. The corresponding logarithmic behaviour of the gauge coupling constant agrees with its one-loop correction

(see equation (4.1)).

Since the gauge coupling constant is naturally complexified, it is natural to introduce a new complex coordinate $s=x^6+ix^{10}$ and lift the configuration to M-theory, where x^{10} parametrizes the M-theory circle S^1 . In M-theory the N D4-branes and the two NS5-branes lift to a single M5-brane. Supersymmetry forces this M5-brane to wrap a holomorphic cycle in the complex 2-dimensional surface spanned by v and $t=e^{-s}$. Moreover, the classical IIA geometry forces this Riemann surface to have genus g=N-1.

By decomposing the self-dual 3-form field strength T on the M5-brane

$$T = F \wedge \Lambda + *F \wedge *\Lambda,$$

into a 2-form F on \mathbb{R}^4 and a 1-form Λ on the Seiberg-Witten curve, we recover the abelian gauge field strengths F^i in the 4-dimensional theory. The five-brane kinetic energy $\int T \wedge *T$ reduces to the 4-dimensional effective gauge Lagrangian

$$\mathcal{L} = \tau_{ij} F_+^i \wedge F_+^j, \tag{4.13}$$

where τ_{ij} is the period matrix of the M-theory curve [95]. So choosing the Seiberg-Witten curve

$$\Sigma_{SW}: \Lambda^{N}(t+t^{-1}) + 2P_{N}(v; u_{k}) = 0.$$

as M-theory curve, consistent with the boundary conditions (4.12), indeed engineers the U(N) (or SU(N)) gauge theory dynamics in 4 dimensions (depending on whether $u_1=0$).

4.1.4 I-brane configuration

This brings us to the most important paragraph in this section. We can generalize the I-brane configuration in Figure 3.3.2 to gauge theories with $\mathcal{N}=2$ supersymmetry. Like for the $\mathcal{N}=4$ gauge theories in Chapter 3, we study $\mathcal{N}=2$ theories on more general Taub-NUT backgrounds TN_k . Remember that TN_1 is related to \mathbb{R}^4 in the limit that the Taub-NUT circle becomes very large.

The first duality chain

The M5-brane configuration in Section 4.1.3 is in many ways the most elegant starting point to study supersymmetric gauge theories. For a pure $\mathcal{N}=2$ susy Yang-Mills theory it wraps the Seiberg-Witten curve Σ_{SW} we met in equation (4.10). We call \mathcal{B} the 2-dimensional complex surface in which this curve is embedded. So let us start with the M-theory compactification

(M)
$$TN \times \mathcal{B} \times \widetilde{\mathbb{R}}^3$$
,

corresponding to the top box in Fig. 1.6 (with S^1 decompactified). Here we have denoted the three non-compact directions as $\widetilde{\mathbb{R}}^3$ to distinguish them from the \mathbb{R}^3 in the base of TN. We further pick \mathcal{B} to be a flat complex surface that is topologically a T^4 or some decompactification of it. That is, in the most general case \mathcal{B} will be a product

$$\mathcal{B} = E \times E'$$

of two elliptic curves. But more often we will consider the degenerations $\mathcal{B}=\mathbb{C}^*\times\mathbb{C}^*$ and $\mathcal{B}=\mathbb{C}\times\mathbb{C}$, or any mixed combination. (In the relation with integrable hierarchies the cases \mathbb{C},\mathbb{C}^* , and E correspond to rational, trigonometric, and elliptic solutions respectively.) We will denote the affine coordinates on \mathcal{B} as $(x,y)\in\mathcal{B}$. The complex surface \mathcal{B} has a (2,0) holomorphic form

$$\omega = dx \wedge dy.$$

We will now pick a holomorphic curve Σ inside \mathcal{B} given by an equation

$$\Sigma: H(x,y) = 0,$$

and wrap a single M5-brane over $TN \times \Sigma$. Because Σ is holomorphically embedded this is a configuration with $\mathcal{N}=2$ supersymmetry in four dimensions.

There are two obvious reductions to type IIA string theory depending on whether we take the S^1 inside \mathcal{B} , or an S^1 in the Taub-NUT fibration. In the first case we will compactify \mathcal{B} along a S^1 down to a three dimensional base \mathcal{B}_3 . The curve Σ , and therefore also the M5-brane, will partially wrap this S^1 . Consequently, we arrive at a configuration of NS5-branes and D4-branes that are spanned between them [95]. In the classical situation discussed by Witten we take $\mathcal{B} = \mathbb{C} \times \mathbb{C}^*$ and end up with a IIA string theory on

(IIA)
$$TN \times \mathbb{R}^6$$

with a set of parallel NS 5-branes with D4-branes ending on them, exactly as we discussed in Section 4.1.3.

In the dual interpretation we switch to the other duality frame, by compactifying to Type IIA theory along the S^1 fiber in the Taub-NUT geometry. This is the familiar 9-11 exchange. In this fashion we end up with a IIA compactification on

(IIA)
$$\mathbb{R}^3 \times \mathcal{B} \times \widetilde{\mathbb{R}}^3$$
.

with N D4-branes wrapping $\mathbb{R}^3 \times \Sigma$. However, because the circle fibration of the TN space has singular points, we have to include D6-branes as well. In fact, there will be k D6-branes that wrap $\mathcal{B} \times \widetilde{\mathbb{R}}^3$ and are localized at the points $\vec{x}_1, \ldots, \vec{x}_k$ in the $\widetilde{\mathbb{R}}^3$. This situation is represented in the box on the left-hand side in Fig. 1.6 and illustrated in Fig. 4.4. Summarizing, we get a system of

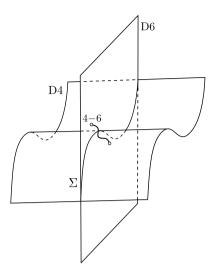


Figure 4.4: A more general configuration of D4 and D6-branes where the intersection locus is an affine holomorphic curve Σ .

N D4-branes and k D6-branes intersecting along the holomorphic curve Σ . As before we refer to this intersection locus as the I-brane.

Comparing partition functions

In this generalized geometry we should consider the free fermion system on a higher genus Riemann surface with action

$$I = \int_{\Sigma} \psi^{\dagger} \overline{\partial} \psi.$$

Let us compare the I-brane with the gauge theory computation. In the gauge theory we are computing two contributions. Firstly, there is a gauge coupling matrix τ_{ij} of the $U(1)^g$ fields F^i

$$\int_{TN} \frac{i}{4\pi} \tau_{ij} F_+^i \wedge F_+^j + v_i \wedge F_+^i$$

for a genus g curve Σ . Compared to equation (4.13) we added magnetic couplings v_i , like in equation (3.14). On the TN geometry the gauge field strengths F^i have fluxes in the lattice

$$[F^i/2\pi] = p^i \in H^2(TN, \mathbb{Z}).$$

Since the cohomology lattice $H^2(TN,\mathbb{Z}) \cong \mathbb{Z}^k$, these fluxes are labeled by integers p_a^i with $i=1,\ldots,g$ and $a=1,\ldots,k$.

Secondly, there is a gravitational coupling \mathcal{F}_i that appears in the term (we discuss this term in detail in Section 4.3 and Chapter 5)

$$\int_{TN} \mathcal{F}_1(\tau) \operatorname{Tr} R_+ \wedge R_+.$$

Since the regularized Euler number of TN_k equals k, combining these two terms yields the partition function

$$Z_{\text{gauge}} = \sum_{p_a^i \in \mathbb{Z}} e^{\pi i p_a^i \tau_{ij} p^{j,a} + 2\pi i v_i^a p_a^i} e^{k\mathcal{F}_1}.$$
 (4.14)

In the I-brane model the partition function $Z_{\rm gauge}$ is nothing but the determinant of the chiral Dirac operator acting on k free fermions living on the "spectral curve" Σ_g , coupled to the flat U(k) connection v on corresponding rank k vector bundle $\mathcal{E} \to \Sigma$. So we are led in a very direct way to

$$Z_{\text{gauge}} = \det \overline{\partial}_{\mathcal{E}}.$$
 (4.15)

The results (4.14) and (4.15) are just the usual bosonization formula, where the fermion determinant is equivalent to a sum over the lattice of momenta together with a boson determinant. Here we use the identification

$$\mathcal{F}_1 = -\frac{1}{2} \log \det \Delta_{\Sigma}.$$

To complete this map we need to show why the p^i are identified with fermion currents on the Riemann surface through the corresponding cycle, but this is relatively clear. Consider a cycle A_i on the Riemann surface and a disc ending on this cycle (which can always be done as Σ is contractible in the full CY). Then the statement that F^i is turned on corresponds to the fact that the integral of the corresponding flux over this disc is not zero. Since the fermions are charged under the U(k) gauge group, this means that they pick up a phase as they go along this cycle on the Riemann surface (the Aharanov-Bohm effect). Thus the holonomy of the fermions correlates with the p^i . Later (in Section 4.4.2), we provide an alternative view of the fluxes p^i : they also correspond to D4-branes, wrapping 4-cycles of the Calabi-Yau and bound to the D6-brane.

4.2 Geometric engineering

In Section 3.2 we considered $\mathcal{N}=4$ supersymmetric Yang-Mills in the background of an ALE space \mathbb{C}^2/Γ , and discovered an interesting relationship with two dimensional conformal field theory. We explained this by embedding the

gauge theory in type IIA theory on a D4-brane wrapping the ALE space. In the ten dimensions of string theory, however, much more is possible.

Remember that ALE spaces \mathbb{C}^2/Γ are characterized by their vanishing two-cycles (see Section 3.2.3), whose intersection matrix realizes the Dynkin diagram for the corresponding ADE Lie group. Resolving the singularity gives a finite volume ϵ to the vanishing cycles. Let us call the dual two-forms ω_i . The RR gauge field C_3 reduces to a set of r abelian gauge fields A^i

$$C_3 = \sum_{i=1}^{r} A^i \omega_i. {(4.16)}$$

When $\epsilon \to 0$ the gauge symmetry is enhanced to the corresponding ADE gauge, similarly (and in fact dual) to when D-branes approach each other [96]. This realizes the geometric McKay correspondence in string theory.

Wrapping a D2-brane over any of these two-cycles yields a BPS particle in the six transverse directions to the ALE space, whose mass is proportional to the volume ϵ of the two-cycle it is wrapping. Since these 6-dimensional BPS particles transform as vectors and are charged under the Cartan of the ADE-group, they are the W-bosons corresponding to the breaking of the ADE gauge group to its Cartan subalgebra.

A simple example is given by the ALE-fibration

$$(z - a)(z + a) + u2 + v2 = 0.$$

Its only two-cycle is spanned in between z=-a and z=a. Two-branes can wrap this two-cycle with two possible orientations, generating a W^+ and W^- boson with masses proportional to a. So this engineers a broken SU(2) gauge group for a generic value for a that is enhanced to SU(2) when a=0. Note that this matches with the classical Seiberg-Witten moduli space \mathcal{M}_c (see Section 4.1.1). Indeed, the ALE fibration only breaks half of the supersymmetries, which amounts to $\mathcal{N}=4$ supersymmetry in four dimensions. As we pointed out before, in $\mathcal{N}=4$ theories classical results are exact.

We can turn this example into a string theoretic setting that studies 4-dimensional $\mathcal{N}=2$ Yang-Mills by fibering the ALE space over a genus zero curve. The genus zero curve breaks the supersymmetry from $\mathcal{N}=4$ to $\mathcal{N}=2$, without introducing extra particles. This idea of looking for a string theory set-up that engineers a particular supersymmetric gauge theory in string theory is called geometric engineering [97, 98, 99, 100].

In Section 4.2.2 we will see how the results of Seiberg and Witten can be elegantly embedded in string theory in the language of string compactifications.

4.2.1 Non-compact Calabi-Yau threefolds

Let us return to the local Calabi-Yau threefolds that we introduced in Chapter 2, and give some explicit examples that we will meet later-on in this thesis. We start with one of the simplest Calabi-Yau threefolds that is modeled on an affine curve Σ : the deformed conifold.

Deformed conifold

The deformed conifold X_{μ} is defined by the equation

$$X_{\mu}: \quad xy - uv = \mu, \quad (x, y, u, v) \in \mathbb{C}^4.$$

More precisely, the parameter $\mu \in \mathbb{C}$ parametrizes a family of Calabi-Yau three-folds X_{μ} that becomes singular at $\mu = 0$. This singularity is called the conifold singularity. The non-vanishing holomorphic three-form Ω equals

$$\Omega = \frac{du}{u} \wedge dx \wedge dy. \tag{4.17}$$

The threefold X_{μ} just contains one compact cycle: a 3-cycle with topology S^3 that shrinks to zero-size when $\mu \to 0$. This is particularly easy to see after a change of variables

$$X_{\mu}: \quad z^2 + w^2 + \tilde{u}^2 + \tilde{v}^2 = \mu, \quad (z, w, \tilde{u}, \tilde{v}) \in \mathbb{C}^4.$$

So μ parametrizes a family of T^*S^3 's.

When we view X_μ as a (u,v)-fibration over the complex plane spanned by z and w, its degeneration locus is

$$\Sigma_{\mu}: \quad z^2 + w^2 = \mu$$

in the (z,w)-plane. The S^3 -cycle in X_μ may then be viewed as an S^1 -fibration over a disk D in the (z,w)-plane, that is bounded by the curve Σ_μ . Since the (u,v)-fibration degenerates at the locus $z^2+w^2=\mu$ the resulting 3-cycle has topology S^3 . This is illustrated in Fig. 4.5.

Since the S^3 -cycle is special Lagrangian it is a supersymmetric cycle. In type IIB we can wrap a D3-brane around it and find a vector BPS particle in the transverse four dimensions, whose mass is proportional to the complex structure modulus X. With Cauchy's theorem we can reduce this formula to the 1-cycle ∂D of Σ_{μ}

$$\int_{S^3} \Omega = \int_D dz \wedge dw = \int_{\partial D} \eta, \tag{4.18}$$

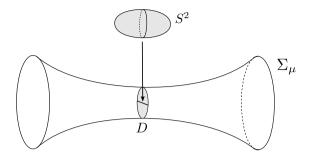


Figure 4.5: The deformed conifold is a non-compact Calabi-Yau threefold that is modeled on the curve Σ_{μ} defined by $z^2 + w^2 = \mu$. Its only compact 3-cycle is S^1 -fibered over the disk D: Over each line segment in D with endpoints on ∂D the S^1 fibration forms a 2-sphere. Moving the line segment over the disk D shows that the compact 3-cycle is homeomorphic to S^3 .

where $\eta=wdz$ is a meromorphic 1-form on the (z,w) plane. The 4-dimensional BPS particle couples to a U(1) gauge field

$$A_{\mu} = \int_{S^3} C^4,$$

that is obtained as a reduction of the RR 4-form over the S^3 -cycle.

This bijective correspondence between 3-cycles in a local Calabi-Yau threefold X_{Σ} modeled on a curve Σ and 1-cycles on Σ holds in general. The 3-cycles may be constructed by filling in a disk D whose boundary ∂D is a 1-cycle on Σ . If one of the variables in the complex surface $\mathcal B$ is $\mathbb C^*$ -valued, the disk D will be punctured. In such a situation differences of 1-cycles have to be considered. We will see an example of this shortly.

Resolved conifold and toric Calabi-Yau's

Instead of deforming the conifold singularity we can also resolve the singularity. This is described by \mathbb{C}^4 parametrized by (x,y,u,v) together with the identification

$$(x, y, z, w) \sim (k^{-1}x, k^{-1}y, kz, kw), \qquad k \in \mathbb{C}^*.$$

The first two complex coordinates parametrize a sphere \mathbb{CP}^1 and the last two coordinates two line-bundle over it. Altogether this gives the total space of the line bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$. The resolved conifold is an example of a local toric Calabi-Yau, just like \mathbb{C}^3 in Chapter 2.

Like any local toric Calabi-Yau, the resolved conifold can be obtained by glueing a few copies of \mathbb{C}^3 such that its singular locus is a linear trivalent graph in \mathbb{R}^2 .

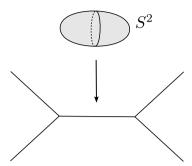


Figure 4.6: The resolved conifold is a non-compact toric Calabi-Yau threefold. The $T^2 \times \mathbb{R}$ -fibration degenerates over a trivalent graph consisting of two 3-vertices. The S^1 -fibration over the inner leg of this graph forms a 2-sphere.

The toric diagram of the resolved conifold is shown in Fig 4.6. It consists of two copies of \mathbb{C}^3 with coordinates (x, z, w) and (y, z, w) respectively. Since both 1-cycles in T^2 shrink at the vertices, the middle toric leg represents the \mathbb{CP}^1 -cycle.

4.2.2 Geometrically engineering Seiberg-Witten theory

Supersymmetric gauge theories with any gauge group and matter content may be engineered by a local Calabi-Yau compactification. Pure $\mathcal{N}=2$ Seiberg-Witten theory with an ADE gauge group is embedded string theory as a local K3-fibration over \mathbb{CP}^1 . Zooming in on the ALE-singularities of the K3 reveals an ADE gauge group in four dimensions, which is broken by the Higgs mechanism when some of the two-cycles gain a non-zero mass.

Let us consider SU(2) Seiberg-Witten theory in some detail. This may be engineered in type IIA by any local Hirzebruch surface, which is the total space of the canonical bundle over a Hirzebruch surface. For example, take the simplest Hirzebruch surface $\mathbb{CP}^1_b \times \mathbb{CP}^1_f$, where \mathbb{CP}^1_b denotes the base sphere, and \mathbb{CP}^1_f is the only two-cycle in the resolved A_1 -singularity. This non-compact Calabi-Yau manifold is toric, and its toric diagram is shown in Fig. 4.7. The W_\pm bosons correspond to D2-branes that are wrapped around the \mathbb{CP}^1_f with opposite orientations.

To go to the field theory limit we should take the string scale to infinity, while keeping the masses of the W-bosons fixed. This corresponds to letting the size t_b of \mathbb{CP}^1_b to infinity, while taking the size t_f of \mathbb{CP}^1_f proportional to the mass a of the W-bosons, as in

$$\exp\left(-t_{b}\right) = \exp\left(-1/g_{YM}^{2}\right) = \left(\frac{\beta\Lambda}{2}\right)^{4}, \quad \exp\left(-t_{f}\right) = \exp\left(-\beta a\right) \tag{4.19}$$

when $\beta \rightarrow 0$. Without this decoupling limit we end up with a 5-dimensional

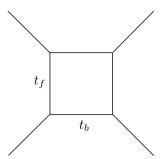


Figure 4.7: SU(2) Seiberg-Witten theory can be engineered as a local Calabi-Yau compactification, based on a local Hirzebruch surface. The 4-dimensional field theory results are recovered when taking the limit $t_b \to \infty$, as in equation (4.19).

theory on $S^1 \times \mathbb{R}^4$, where the size of the S^1 is given by β .

The Seiberg-Witten prepotential \mathcal{F}_0 is reproduced by stringy instanton corrections to the toric Calabi-Yau geometry. In this geometry the instantons may wrap the base n times and the fiber m times. This gives a contribution to the so-called type IIA prepotential (4.34), as we explain in much more detail in Section 4.3. In the 4-dimensional field theory limit only the fiber worldsheet instantons remain. They recombine into the Seiberg-Witten prepotential (4.6).

The mirror map translates this type IIA configuration into a type IIB configuration where we can see the Seiberg-Witten curve explicitly in the limit $\beta \to 0$. We find a non-compact Calabi-Yau X_{SW} based on the Seiberg-Witten curve (4.10)

$$H_{SW}(t,v) = \Lambda(t+t^{-1}) - 2(v^2 - a^2) = 0.$$
 (4.20)

Recall that $t\in\mathbb{C}^*$ while $v\in\mathbb{C}$. The holomorphic three-form thus reduces to the Seiberg-Witten form

$$\eta_{SW} = v \frac{dt}{t}. (4.21)$$

A-cycles on Σ_{SW} are not contractible on the (t,v)-plane. Instead, compact A-cycles in the noncompact Calabi-Yau threefold will reduce to a difference of A-cycles on Σ_{SW} . Indeed, notice that a point on the 1-cycle A^i and one on another 1-cycle $-A^j$, with opposite orientation, are connected by a \mathbb{CP}^1 in the Calabi-Yau. The resulting 3-cycle therefore has the topology of $S^2 \times S^1$. For the B-cycles this subtlety does not arise, and compact B-cycles in the Calabi-Yau have S^3 topology and reduce to compact 1-cycles connecting the two hyperelliptic planes. See Figure 4.8.

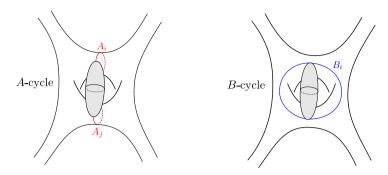


Figure 4.8: The relation between 3-cycles in the Calabi-Yau X_{SW} and 1-cycles on the Riemann surface Σ_{SW} for the Seiberg-Witten geometry. For the A-cycle, by fibering S^1 over the line segment whose endpoints are at a point on A_i and a point on $-A_j$, one obtains S^2 . By moving the endpoints over A_i and $-A_j$, one obtains $S^2 \times S^1$. For the B-cycle, similarly moving the S^2 ending on S^2 , one obtains S^3 .

Self-dual strings

Wrapping D3-branes around a 3-cyle $\Gamma=qA+pB$ of X_{SW} , introduces a BPS dyon in four dimensions whose mass is the absolute value of

$$Z \sim q \int_A \Omega + p \int_B \Omega.$$

This reproduces the mass-formula (4.9). On the Seiberg-Witten curve Σ_{SW} the D3-branes reduce to strings, whose tension is proportional to to the size of the 2-cycle above it. These strings are not fundamental strings, but instead non-critical. They live in six dimensions and couple to the B-field that is part of a 6-dimensional tensor multiplet. Since the field strength of B is self-dual, they are called *selfdual strings* [101].

The crucial difference between $\mathcal{N}=2$ and $\mathcal{N}=4$ gauge theories is that the metric on the SU(2) Seiberg-Witten curve is not just the usual flat metric, as it is on the $\mathcal{N}=4$ torus. The Seiberg-Witten metric can be derived from the metric on the D3-branes. Since these branes wrap supersymmetric 3-cycles they are calibrated by Re Ω . The pull-back of Ω to their worldvolume is proportional to the volume element on the brane [102]. On the space part of the self-dual string, parametrized by x, we thus find

$$\frac{\partial}{\partial x} \left(\eta_{SW}^t(t) \frac{\partial \log t}{\partial x} \right) = 0,$$

where $\eta_{SW} = \eta_{SW}^t dt/t$. This is the geodesic equation for a metric

$$g_{t\bar{t}} = \eta_{SW}^t \overline{\eta^t}_{SW}. \tag{4.22}$$

Since η_{SW} is meromorphic, this metric has poles. Its geodesics are therefore curved, so that not all BPS particles are stable. Studying these BPS trajectories gives a powerful method to determine quantum BPS dyon spectra. In particular this leads to the correct SU(2) Seiberg-Witten spectrum [101].

Notice that when we scale the Seiberg-Witten form η_{SW} as

$$\eta_{SW} \to \frac{\eta_{SW}}{\lambda}$$
(4.23)

and take the limit $\lambda \to 0$, the Seiberg-Witten metric reduces to the usual $\mathcal{N}=4$ flat metric. This is a preview on Section 4.3, where we study gravitational corrections to the prepotential \mathcal{F}_0 that scale the prepotential to \mathcal{F}_0/λ^2 . In the limit $\lambda \to 0$ we recover $\mathcal{N}=4$ results out of the $\mathcal{N}=2$ data.

4.2.3 Completing the web of dualities

Now that we have seen the intimate connection between non-compact Calabi-Yau compactifications and $\mathcal{N}=2$ supersymmetric gauge theories, it is time to link them to our first chain of dualities in Section 4.1.4. We accomplish this by going through another chain of dualities, represented by the vertical sequence of boxes in Fig. 1.6.

The second duality chain

Let us consider a slightly more general compactification of M-theory, namely

(M)
$$TN \times \mathcal{B} \times \mathbb{R}^2 \times S^1$$
,

corresponding to the top box in Fig. 1.6. The extra S^1 does not really influence the earlier results, since the D6-branes remain non-compact and therefore the gauge theory that they support stays non-dynamical. Hence the I-brane configuration remains the same. In the other compactification with an ensemble of NS5-branes and D4-branes we can now perform a T-duality on S^1 to give a web of (p,q) 5-branes in Type IIB, which is another familiar and convenient realization of the $\mathcal{N}=2$ system.

However, in this situation there is an obvious third possible compactification to type IIA, by just reducing on the extra S^1 that we have introduced. This will give IIA on

(IIA)
$$TN \times \mathcal{B} \times \mathbb{R}^2$$

with N NS5-branes wrapping $TN \times \Sigma$. We haven't gained much in this step, but now we can T-dualize the NS5-brane away to remain with a purely geometric situation [73, 101]. In general a T-duality *transverse* to a set of N NS5-branes

produces a local A_{N-1} singularity of the form

$$uv = z^N. (4.24)$$

In the case of a single 5-brane N=1 this gives an A_0 "singularity". We recall that world-sheet instanton effects are important to understand this very non-trivial duality [73, 103, 104].

Applying this T-duality in the present set-up gives us a type IIB compactification of the form

(IIB)
$$TN \times X$$
,

where X is a non-compact Calabi-Yau geometry of the form

$$X: uv + H(x, y) = 0,$$

modeled on the affine Riemann surface Σ that is defined by H(x,y)=0. This is just an application of (4.24) in the case N=1, where z is the local coordinate transverse to the curve Σ .

Let us emphasize that the resulting web 1.6 can be used to study any non-compact Calabi-Yau threefold X that is modeled on a Riemann surface Σ . In particular, the Calabi-Yau doesn't need to have a toric mirror. Examples of such Calabi-Yau's are the Dijkgraaf-Vafa geometries that we will encounter in Chapter 6 as well as Chapter 8.

Moreover, notice that we restricted ourselves to N=1, corresponding to a single M5-brane in the upper node of the duality web 1.6. There is no problem in generalizing this to arbitrary number N of M5-branes though, we just find non-compact Calabi-Yau threefolds whose fiber over the affine curve Σ is given by a more complicated local singularity.

4.3 Topological invariants of Calabi-Yau threefolds

So far we paid attention to some geometrical aspects of the Calabi-Yau background itself in a Calabi-Yau compactification. For associating topological invariants to Calabi-Yau threefolds, we need to go one step deeper and study the moduli spaces of the fields in a Calabi-Yau compactification. That is the theme of this section.

A string compactification is a generalization of the old idea of Kaluza and Klein [105, 106] to unify electromagnetics and gravity in four dimensions by introducing an extra dimension. A reduction of the 5-dimensional metric over a circle S^1 yields the 4-dimensional metric $g_{\mu\nu}$ and gauge field $A_{\mu}=\int_{S^1}g_{\mu 5}$.

In this thesis we will mostly consider compactifications of type II to four dimensions, with a Calabi-Yau threefold X as compactification manifold. The low energy effective theory in four dimensions is determined by reducing the massless fields of type II theory. These are given by a metric g_{MN} , the B-field B_{MN} and the dilaton ϕ , together with the R-R gauge potentials and their superpartners.

String compactifications are characterized by the amount of supersymmetry that is preserved. This is proportional to the number of covariantly constant spinors on the compactification manifold, or equivalently, to the amount of reduced holonomy. For example, complex Calabi-Yau threefolds with SU(3) holonomy preserve a quarter of the 10-dimensional supersymmetry. This results in a 4-dimensional theory with $\mathcal{N}=2$ supersymmetry. Supersymmetry dictates that the resulting massless fields can be combined into 4-dimensional $\mathcal{N}=2$ multiplets. E.g. the 4-dimensional metric $g_{\mu\nu}$ combines with the graviphoton and two gravitinos in a gravity multiplet, and yields for example the 4-dimensional Einstein gravity Lagrangian

$$\int d^4x \, \sqrt{-g} R.$$

But there are more multiplets that play a role in the resulting 4-dimensional low energy effective theory. The scalar components of these multiplets all have an interpretation in terms of the internal Calabi-Yau metric g_{mn} . As this internal metric may vary over $x \in \mathbb{R}^4$, though preserving the Calabi-Yau condition $R_{mn}=0$, their moduli appear in the 4-dimension theory as fields $\phi(x)$. A reduction of the 10-dimensional Einstein gravity Lagrangian shows that they are part of a 4-dimensional sigma model action

$$\int d^4x \ G_{\alpha\beta}(\phi)d\phi^{\alpha} \wedge *d\phi^{\beta}, \tag{4.25}$$

where $G_{\alpha\beta}(\phi)$ is the metric on the moduli space of the Calabi-Yau. Now we already learned in Chapter 2 that this metric splits into two pieces, the first describing $h^{2,1}$ complex structure moduli X^i and the second $h^{1,1}$ complexified Kähler moduli t^j , which combine the Kähler metric g and the B-field.

In the language of $\mathcal{N}=2$ supersymmetry the type IIB low energy effective theory is captured by a $\mathcal{N}=2$ supergravity theory in 4d coupled to $h^{2,1}$ vector multiplets, whose scalar components parametrize the complex structure deformations of X, and to $h^{1,1}+1$ hypermultiplets, whose scalar components describe deformations of the complexified Kähler moduli of X plus the axion-dilaton $C+ie^{-\phi}$. In the context of IIA compactifications these identifications are reversed: there are $h^{1,1}$ vector multiplets and $h^{2,1}+1$ hypermultiplets.

As we will see in detail in a moment the metric on the scalar moduli spaces is captured by classical geometry, up to such quantum corrections in g_s in full string

theory and in α' at world-sheet level. The fact that these two sets of multiplets are decoupled in the 4-dimensional theory is important in constraining these corrections. For example, the metric on the IIB moduli space \mathcal{M}_V doesn't receive any string loop corrections nor instanton corrections, since the dilaton as well as the Kähler moduli (which control the size of the Calabi-Yau) are part of the hypermultiplets. On the other hand, the metric on the IIA moduli space \mathcal{M}_V is sensitive to α' corrections.

4.3.1 Special geometry

Supersymmetry imposes strong constraints on the geometry of the moduli spaces of the scalar fields. It forces the metric $G_{i\bar{j}}$ on any $\mathcal{N}=2$ vector multiplet moduli space \mathcal{M}_V to be of the special Kähler form [107, 108]

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K},$$
 (4.26)

with Kähler potential \mathcal{K} . This Kähler potential can be expressed in terms of a prepotential \mathcal{F}_0 , similar to our discussion in Section 4.1.1. Let us first show how the prepotential \mathcal{F}_0 is related to the Calabi-Yau geometry in both type IIA and IIB compactifications, and in Section 4.3.1 we explain the relation to the quantum moduli space of $\mathcal{N}=2$ supersymetric gauge theories.

Type IIB vector moduli space

In a type IIB Calabi-Yau compactification the Kähler potential on \mathcal{M}_V is given by

$$\mathcal{K} = -\log i \left(\int_{X} \Omega \wedge \overline{\Omega} \right). \tag{4.27}$$

Here Ω is the nonvanishing holomorphic three-form of X, locally determined up to multiplication by a nonvanishing holomorphic function. The transformation $\Omega \to e^{f(z)}\Omega$ changes the Kähler potential into

$$\mathcal{K}(z,\bar{z}) \to \mathcal{K}(z,\bar{z}) - f(z) - \overline{f(z)},$$

so that the Kähler metric (4.26) indeed remains invariant. Notice that (4.27) corresponds precisely to the Weil-Petersson metric on the moduli space of complex structures of the Calabi-Yau manifold X.

The fact that the Kähler potential K is captured by a single function \mathcal{F}_0 is known as special geometry. To see this let us start by choosing a basis of three-cycles $\{A_i, B_i\}$ of X, with $i, j = 0, 1, \ldots, h^{2,1}$, such that

$$A^i \cap B_j = \delta^i_j, \quad A^i \cap A^j = B_i \cap B_j = 0.$$

As such a basis is preserved under the symplectic group $Sp(b^3, \mathbb{Z})^1$, it is called a symplectic basis of X.

The complex structure periods of X with respect to this basis are defined as

$$X^{i} = \int_{A^{i}} \Omega, \quad \mathcal{F}_{0,i} = \int_{B_{i}} \Omega. \tag{4.28}$$

The tuple $(X^i, \mathcal{F}_{0,i})$ forms a local section of an $Sp(b^3, \mathbb{Z})$ -bundle over the moduli space \mathcal{M}_V , whereas the quotients

$$Z^i = \frac{X^i}{X^0}, \qquad i = 1, \dots, h^{2,1}$$

yield a set of local coordinates on \mathcal{M}_V .

Since the variation of a (3,0)-form just contains a (3,0) piece and a (2,1) piece, $\int_X \Omega \wedge \partial_i \Omega = 0$. This implies that the functions $\mathcal{F}_i(X)$ are in fact first derivatives

$$\mathcal{F}_{0,i}(X) = \partial_i \mathcal{F}_0(X), \quad \mathcal{F}_0(X^i) = \frac{1}{2} X^i \mathcal{F}_{0,i}(X)$$

of a homogeneous function \mathcal{F}_0 of degree 2 in the complex moduli $X = \{X^i\}$. This is the prepotential. Geometrically, this means that the image of the period map, which sends

$$\Omega \mapsto (X^i, \mathcal{F}_{0,i}),$$

is a Lagrangian submanifold of $H^3(X,\mathbb{C})$.

The Riemann bilinear identity $\int_X \alpha \wedge \beta = \sum_i \left(\int_{A^i} \alpha \int_{B_i} \beta - \int_{A^i} \beta \int_{B_i} \alpha \right)$ immediately shows that the Kähler potential is fully determined by the prepotential \mathcal{F}_0 :

$$\mathcal{K}(X, \overline{X}) = -\log i \left(X^{i} \overline{\mathcal{F}}_{0,i} - \overline{X}^{i} \mathcal{F}_{0,i} \right). \tag{4.29}$$

Decoupling gravity

In general, to make contact with the 4-dimensional world around us, we would like to consider compactifications with a relatively small Calabi-Yau. The Planck scale in four dimensions is then proportional to the size of the threefold (and inversely related to its curvature).

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{such that} \quad g^t J g = J \quad \text{with} \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where a, b, c and d are $N \times N$ matrices.

 $^{^1\}mathrm{Recall}$ that $Sp(N,\mathbb{Z})$ is the group consisting of $2N\times 2N$ integer-valued matrices

However, when we are purely interested in the gauge dynamics of the reduced theory, we want to *decouple* gravity in 4d. This is commonly done by zooming in on the part of the complex moduli space where the Calabi-Yau develops a singularity. In such a region of the moduli space a particular set of cycles becomes very small. (Think of the singularities on the quantum SU(2) Seiberg-Witten moduli space \mathcal{M}_q where 1-cycles degenerate.)

In four dimensions this results in a new lower energy scale, governing the dynamics of the fields that correspond to the singularity. Zooming in close to the singularity, we can forget about the rest of the Calabi-Yau so that we are effectively studying non-compact Calabi-Yau compactifications. Depending on the type of the compactification, wrapping branes around vanishing 2, 3 and 4-cycles yield massless BPS vector and hyperparticles, that can engineer any gauge symmetry and matter content that one is interested in.

In this non-compact limit the local special geometry of $\mathcal{N}=2$ supergravity reduces to rigid special geometry, relevant for the low energy dynamics of $\mathcal{N}=2$ gauge theories (more details e.g. [93, 109, 110]). In particular, the Kähler potential reduces to

$$\mathcal{K} = i \int_{X} \Omega \wedge \overline{\Omega}. \tag{4.30}$$

so that the metric is given by

$$g_{i\bar{j}} = \operatorname{Im}\left(\frac{\partial^2 \mathcal{F}_0}{\partial X^i \partial X^j}\right). \tag{4.31}$$

This is indeed in agreement with the Seiberg-Witten expression (4.5).

Normalizable and non-normalizable modes

An important point is the distinction between normalizable and non-normalizable complex structure moduli in the case of noncompact Calabi-Yau manifolds. To be more precise let us consider the local Calabi-Yau modeled on a Riemann surface Σ . The coefficients of the polynomial H(x,y) characterize the complex structure of X, so that they are the complex structure moduli of X. However not all of them are dynamical. Some of them control the complex structure of 3-cycles which are localized in the "interior" of the singularity and are dynamical, while others describe how the singularity is embedded in the bigger Calabi-Yau and become frozen when we take the decoupling limit.

To determine if a specific complex structure modulus X is dynamical or not, one has to compute the corresponding Kähler metric

$$g_{X\overline{X}} = \partial_X \overline{\partial_X} K = i \partial_X \overline{\partial_X} \int \Omega \wedge \overline{\Omega}. \tag{4.32}$$

If this expression is finite, then the modulus X is dynamical, otherwise it is decoupled and should be treated as a parameter of the theory. We will refer to the first set of moduli as *normalizable* and to the second as *non-normalizable*.

Type IIA vector moduli space

Similar to how the complex structure moduli enter in the IIB vector moduli space, the Kähler moduli parametrize the IIA vector moduli space. It is convenient to enlarge the moduli space to all complex even-dimensional forms on the Calalabi-Yau threefold \widetilde{X} . The Kähler periods of \widetilde{X} are then found with respect to the element

$$\Omega = \exp(t) \in H^{2*}(\widetilde{X}, \mathbb{C}), \tag{4.33}$$

where t=B+iJ denote the complexified Kähler form on the Calabi-Yau. When integrated over a basis of two-cycles C^i this yields the Käher parameters $t^i=\int_{C^i}t$, for $i=1,\ldots,h^{1,1}$. More precisely, these are the normalized complex Kähler parameters $t^i=X^i/X^0$.

The Kähler potential is given by

$$\mathcal{K} = -\log i \left(\int \Omega \wedge \tilde{\overline{\Omega}} \right),$$

where $\tilde{\Omega}$ reminds us that we should include a minus sign for components in $H^{1,1}$ and $H^{3,3}$ (to define a Hermitean inner product). The Kähler potential can then be written in terms of a holomorphic function $\mathcal{F}_{0,\text{clas}}$ as in equation (4.29). It equals the triple intersection number

$$\mathcal{F}_{0,\mathrm{clas}}(t) = \frac{1}{6} \int_{\widetilde{X}} t \wedge t \wedge t.$$

in the normalized coordinates t. As we already suggested in our notation, in a type IIA compactification this only determines the full prepotential \mathcal{F}_0 up to non-perturbative corrections in α' .

Both type II vector multiplet moduli spaces are related by *mirror symmetry*, which exchanges a type IIB compactification on a Calabi-Yau threefold X, with Hodge numbers $h^{1,1}$ and $h^{2,1}$, with a type IIA compactification on its mirror \widetilde{X} , whose Hodge numbers are flipped. There is a precise mirror map between the complex structure moduli Z^i on X and the Kähler moduli t^i on its mirror \widetilde{X} , that determines the full IIA prepotential in the large radius limit to be of the form

$$\mathcal{F}_{0}(t) = \frac{1}{6} \int_{\tilde{X}} t \wedge t \wedge t - i \frac{\chi}{2(2\pi)^{3}} \zeta(3) + \sum_{d \in H_{2}(\tilde{X})} GW_{0}(d) \ e^{-\frac{1}{\alpha'} \sum_{i} d_{i} t^{i}}$$
(4.34)

where $\mathrm{GW}_0(d)$ are the so-called genus zero Gromov-Witten invariants. They count the number of worldsheet instantons in the Kähler class $d = \sum_i d_i C^i$. Gromov-Witten invariants are one type of topological invariants on a Calabi-Yau threefold. They give a mathematical definition of what is called the A-model in topological string theory. This is the topic of Section 4.3.2.

4.3.2 Topological string theory

Topological strings were introduced by Witten in [111] as a twisted version of a supersymmetric 2-dimensional sigma model

$$Z_{\sigma}(X;t) = \int Dx \, Dg_C \, \exp\left(-t \int_C g_{mn} dx^m \wedge *dx^n + \dots\right),$$

that integrates all maps x of a worldsheet C into the target manifold X with metric g_{mn} . The maps are weighted by the volume of x(C). The extra integral over all metrics g_C on C couples the sigma model to gravity. When X is a Calabi-Yau threefold the sigma model is conformally invariant and $\mathcal{N}=(2,2)$ supersymmetry on the worldsheet is preserved.

Similarly as in Section 3.1.1, the sigma model is twisted by choosing an embedding of the worldsheet Lorentz group U(1) into the R-symmetry group $U(1)_R$ of the supersymmetry algebra. Essentially this gives two possibilities, which are called the A-model and the B-model. Precisely as in Section 3.1.1 both models are characterized by their BRST operators, which make sure the amplitudes are independent of the metric g_{mn} of X. In fact, one finds that the A-model only depends on the $h^{1,1}$ Kähler moduli of X, while the B-model is just based on the $h^{2,1}$ complex structure moduli of X. Mirror symmetry equates the B-model on the Calabi-Yau X and the A-model on its mirror \tilde{X} .

There are many excellent reviews on topological string theory and mirror symmetry, see for instance [112, 113, 114, 115, 116].

4.3.3 Gravitational corrections

How do topological string amplitudes pop up in the physical string? In Section 4.1 we wrote down the most general Lagrangian (4.3) one can build out of any number of $\mathcal{N}=2$ vector multiplets \mathbf{X}^i (which contain the scalars X^i as lowest components). A Calabi-Yau compactification furthermore produces an $\mathcal{N}=2$ gravity multiplet, whose components may be recombined in the so-called

 $^{^2}$ Be careful not to confuse the worldsheet C in the sigma-model with the target space curve Σ in the local Calabi-Yau X_{Σ} .

Weyl multiplet $W_{\mu\nu}$. This multiplet is expanded as

$$W_{\mu\nu} = F_{\mu\nu}^+ - R_{\mu\nu\lambda\rho}^+ \theta \sigma^{\lambda\rho} \theta + \dots,$$

where F^+ is the self-dual part of graviphoton field strength and R^+ the self-dual Riemann tensor. In the 4-dimensional low energy effective action $W_{\mu\nu}$ appears together with the \mathbf{Z}^{i} 's in F-terms of the form

$$\int d^4x \int d^4\theta \, \mathcal{F}_g(\mathbf{X}^i)(\mathcal{W}^2)^g,$$

which clearly generalize the Lagrangian (4.3) that only captures the prepotential \mathcal{F}_0 . Expanding the above Lagrangian in components yields the terms

$$\int d^4x \, \mathcal{F}_g(X^i) \, R_+^2 \, F_+^{2g-2}. \tag{4.35}$$

In other words, the F_g 's are coefficients in a gravitational correction to the amplitude for the scattering of 2g-2 graviphotons. In [117, 118] it was shown that these coefficients may be computed in topological string theory. The vector multiplets in a IIB compactification parametrize complex structure moduli, so that the above F-terms compute B-model free energies \mathcal{F}_g , whereas in a type IIA compactification the F-terms generate A-model invariants.

These gravitational couplings are crucial in the study of supersymmetric black holes [22, 23, 119, 120, 121, 122]. But even when we decouple gravity by zooming in on the singularity of a Calabi-Yau (the topic of the next paragraph), they appear as meaningful holomorphic corrections to the effective 4-dimensional gauge theory [123, 124, 125].

A-model and Gromov-Witten invariants

The A-model enumerates degree d holomorphic maps $x:C_g\to \tilde X$ from a genus g worldsheet C_g into the Calabi-Yau $\tilde X$. These are captured by the *Gromov-Witten invariants*

$$GW_g(d) = \int_{\overline{\mathcal{M}}_g(\tilde{X},d)]^{\mathrm{vir}}} 1 \in \mathbb{Q},$$

that "count" the number of points in their compactified moduli space $\overline{\mathcal{M}}_g(\tilde{X},d)$ (which is rigorously defined in [126, 127]). In a large radius limit the partition function of the A-model has a genus expansion

$$Z_{GW}(t;\lambda) = \exp\left(\mathcal{F}(t;\lambda)\right); \qquad \mathcal{F}(t;\lambda) = \sum_{g\geq 0} \lambda^{2g-2} \mathcal{F}_g(t),$$
 (4.36)

where we introduced the topological string coupling constant λ that keeps track of the worldsheet genus g, and where we absorbed the coupling constant t in the complexified Kähler class t. The \mathcal{F}_g 's can be expressed in terms of the Gromov-Witten invariants

$$\mathcal{F}_{g}(t) = \sum_{d} GW_{g}(d)e^{d \cdot t} + \begin{cases} \frac{1}{6} \int_{X} t \wedge t \wedge t & (g = 0) \\ \frac{1}{24} \int_{\tilde{X}} c_{2}(\tilde{X}) \wedge t & (g = 1) \end{cases}, \tag{4.37}$$

with the notation $d \cdot t = \int_d x^*(t)$. In particular, the genus zero contribution \mathcal{F}_0 equals the prepotential (4.34).

A-model on toric Calabi-Yau threefolds

In [17] the A-model on a non-compact toric Calabi-Yau threefold is solved by taking advantage of the large N open-closed string duality between Chern-Simons theory on S^3 and the A-model on the resolved conifold [18]. It was discovered that the all-genus free energy can be computed by splitting the toric target manifold in \mathbb{C}^3 -patches, as discussed in Section 4.2.1. On each \mathbb{C}^3 -patch the worldsheet instantons localize onto the trivalent vertex in the base of the $T^2 \times \mathbb{R}$ -fibration, where at least one of the cycles of the T^2 -fibration degenerates. In fact, the topological string amplitude on such a patch is an open string amplitude.

The boundary conditions are specified by so-called A-branes that sit at the end points of the trivalent graphs. Supersymmetry conditions determine these A-branes to wrap special Lagrangian cycles in the Calabi-Yau threefold. In noncompact toric Calabi-Yau's such cycles end on the degeneration locus and wrap the T^2 -fiber together with a half-line in the base transverse to the trivalent graph. Since one of the T^2 -cycles degenerates at the graph the topology of the A-brane is $S^1 \times \mathbb{C}$. It can only move up and down a toric edge. Open strings may wrap the S^1 of each A-brane an arbitrary number of times.

In total there can be three stacks of A-branes at the toric legs of the trivalent vertex, which the open string wraps. With the Frobenius relation this winding information can be rewritten in terms of representations of the Lie group $U(\infty)$. The open topological string amplitude on each \mathbb{C}^3 -patch yields the *topological vertex* expression

$$C_{R_1 R_2 R_3} = \sum_{Q_1, Q_3} N_{Q_1 Q_3^t}^{R_1 R_3^t} q^{\kappa_{R_2}/2 + \kappa_{R_3}/2} \frac{W_{R_2^t Q_1} W_{R_2 Q_3^t}}{W_{R_2 0}},$$
(4.38)

where the R_i are $U(\infty)$ representations, the numbers N are integers that count the number of ways that the representations Q_1 and Q_3^t go into R_1 and R_3^t , and the W's are link invariants for the $U(\infty)$ Hopf link. For more details we refer to reference [17].

The full closed topological string amplitude is found by multiplying the topological vertices. When two \mathbb{C}^3 -patches are glued along a toric edge, one has to sum over the attached representation R with weight $\exp\left(-tl(R)\right)$, where t is the Kähler parameter of the toric edge and l(R) the number of boxes in the Young diagram corresponding to R. We will see examples of such computations in Chapter 6 and Chapter 7.

B-model and Kodaira-Spencer theory

The free energy of the B-model has a similar genus expansion as in (4.36) in a large complex structure limit, that is related to the A-model expansion by the mirror map. But in contrast to the A-model, the B-model is best understood from a target space point of view, where it is described as a *Kodaira-Spencer* path integral over complex structure deformations of X [117]

$$Z = \int D\phi \exp\left(\int \partial\phi \wedge \overline{\partial}\phi + \lambda A^3\right).$$

Here, the complex structure field $A=X+\partial\phi\in H^{2,1}$ is expanded around a background complex structure specified by $Z\in H^{2,1}(X)$, so that $\phi\in H^{1,1}(X)$ describes cohomologically trivial complex structure fluctuations. The interaction term schematically denoted by A^3 is more carefully equal to $\Omega\cdot (A'\wedge A')\wedge A$, where $A'\in H^{0,1}(T_X)$ is uniquely determined $\Omega\cdot A'=A$.

This partition function quantizes complex structures as its equation of motion is the Kodaira-Spencer equation

$$\overline{\partial}A' + \frac{1}{2}[A', A'] = 0,$$

which is condition for a complex structure $\overline{\partial} + A' \cdot \partial$ to be integral. At tree level the free energy is captured by special geometry in terms of the periods

$$X^{i} = \int_{A^{i}} \Omega, \qquad \frac{\partial \mathcal{F}_{0}}{\partial X^{i}} = \int_{B_{i}} \Omega$$

as described in Section 4.3.1. At one loop level the free energy computes the logarithm of the *analytic (Ray-Singer) torsion* of X [117]

$$\mathcal{F}_1 = \sum_{p,q=0}^{3} \frac{1}{2} pq(-)^{p+q} \log \det' \Delta_{p,q}, \tag{4.39}$$

where the prime ' refers to subtracting the zero modes. Higher \mathcal{F}_g 's don't have such a neat interpretation; they just give quantum corrections to special geometry.

B-model on local Calabi-Yau modeled on a curve

Kodaira-Spencer theory on a local Calabi-Yau X_{Σ} reduces to a 2-dimensional quantum field theory on the curve Σ that quantizes the complex structure deformations of Σ [17, 128]. These complex structure variations are controlled by a bosonic chiral field ϕ such that

$$\eta = \partial \phi(x),\tag{4.40}$$

Let us assume for a moment that the complex structure of Σ can only be changed at its k asymptotic infinities (so Σ is a genus zero curve), and that these regions are described by a local coordinate x_i such that $x_i \to \infty$ at the boundary. Near each boundary $\partial \Sigma_i$ the chiral boson ϕ admits a mode expansion

$$\partial \phi(x) = \sum_{n \in \mathbb{Z}} \alpha_n x_i^{-n-1}, \qquad [\alpha_n \cdot \alpha_m] = n\lambda^2 \delta_{n+m,0}$$

(When x_i is a periodic coordinate, the series should be expanded in e^{nx_i} .)

Denote the Hilbert space of this free boson by \mathcal{H} . The B-model free energy sweeps out a state

$$|\mathcal{W}\rangle \in \mathcal{H}^{\otimes k}$$
. (4.41)

At each boundary $\partial \Sigma_i$ we can introduce a coherent state

$$|t^i\rangle = \exp\left(\sum_{n>0} t_n^i \alpha_{-n}\right) |0\rangle,$$

where the t_n^i 's represent the complex structure deformations at the corresponding asymptotic infinity. Roughly, t_n^i corresponds to complex structure variations $y_i \sim x_i^{n-1}$ at the ith boundary when n>0. It follows that the B-model partition function can be written as the correlator

$$\exp \mathcal{F}(t^1, \dots, t^k) = \langle t^1 | \otimes \dots \otimes \langle t^k | \mathcal{W} \rangle. \tag{4.42}$$

In [128] it is discovered that there is a large symmetry group that acts on this theory, whose broken symmetries generate an infinite sequence of Ward identities that are strong enough to determine the free energy $\mathcal F$ as a function of the deformation parameters t_n^i . In three complex dimensions these symmetries are the global diffeomorphisms that preserve Ω . In two complex dimensions they reduce to diffeomorphisms that preserve the complex symplectic form $dy \wedge dx$ on $\mathcal B$. Obviously, these symmetries are broken by Σ . This gives ϕ an interpretation as Goldstone boson.

Geometrically, the Lie algebra of infinitesimal tranformations is generated by Hamiltonian vector fields

$$\delta x_i = \frac{\partial f(y_i, x_i)}{\partial y_i}, \quad \delta y_i = -\frac{\partial f(y_i, x_i)}{\partial x_i},$$

with the Hamiltonian f(y,x) being any polynomial in y and x. This is known as the $\mathcal{W}_{1+\infty}$ -algebra. On the quantum level any change of local coordinates $\delta x_i = \epsilon(x_i)$ is implemented by the operator

$$\oint \epsilon(x_i)T(x_i)dx_i$$

acting on the Hilbert space, where $T(x)=(\partial\phi)^2/2$ is the energy momentum tensor of the free boson CFT. Similarly, when $f(y_i,x_i)=y_i^nx_i^m$ at the ith asymptotic infinity, we find a quantum generator

$$W_m^{n+1}(x_i) \sim \oint_{\partial \Sigma_i} x_i^m \frac{(\partial \phi)^{n+1}}{n+1}.$$

These generators are the modes of the W-current $W^{n+1}(x) \sim (\partial \phi)^{n+1}/(n+1)$, that includes the Virasoro current for n=1.

To write down the Ward identities, corresponding to the broken $\mathcal{W}_{1+\infty}$ -symmetry, we first need to see how the complex structure deformations in the various asymptotic regions are related. It turns out that this is highly non-trivial, and can be best understood in terms of the topological B-branes in the geometry X_{Σ} .

Topological B-branes wrap holomorphic cycles of the Calabi-Yau geometry. In the local geometry X_σ complex 1-dimensional branes are parametrized by Σ . These branes wrap the non-compact fiber uv=0 over the curve Σ . More precisely, such a fiber is degenerated into two parts v=0 and u=0, which are wrapped by a brane and an anti-brane resp. Inserting such a brane at position p on an asymptotic infinity deforms the meromorphic 1-form η by a small amount $\oint_p \eta = \lambda$, where the contour encircles the point p on Σ . Since we identified η with $\partial \phi$ in (4.40), we find the equality

$$\langle \cdots \oint_{p} \partial \phi(x_{i}) \ \psi(p) \cdots \rangle = \lambda \ \langle \cdots \psi(p) \cdots \rangle.$$

Now the OPE : $\psi^*(x_i)\psi(p)$: $\sim 1/(x_i-p)$, when $x_i \to p$, shows that the B-brane corresponds to the fermion ψ by the familiar bosonization rules

$$: \psi(x_i)\psi^*(x_i) := \partial \phi(x_i)/\lambda, \quad \psi(x_i) = \exp(\phi(x_i)/\lambda).$$

Similarly, an anti-brane corresponds to its conjugate $\psi^* = \exp(-\phi/\lambda)$.

This has some very interesting consequences. It is argued in [128] that the brane partition function $\Psi(x_i) = \langle \psi(x_i) \rangle$, for a B-brane inserted at a point x_i on the *i*th asymptotic patch, transforms to another asymptotic region as a *wave function* (similar to the closed partition function in [129], see also [130, 131]). As a result the fermion fields in both patches are related in a Fourier-like way

$$\psi(x_j) = \int dx_i \ e^{S(x_i, x_j)/\lambda} \psi(x_i), \tag{4.43}$$

where S is a gauge transformation that relates the symplectic coordinates (y,x) in both patches

$$y_i dx_i - y_j dx_j = dS(x_i, x_j).$$

Notice that when the local coordinates are related by a symplectic transformation $Sp(1,\mathbb{Z})\cong SL(2,\mathbb{Z})$, the exponent $S(x_i,x_j)$ is a homogeneous quadratic function in x_i and x_j , so that the wave-function transforms in the metaplectic representation.

Putting everything together we find the Ward identities [128]

$$\oint_{\partial \Sigma_i} \psi^*(x_i) x_i^m y_i^n \psi(x_i) | \mathcal{W} \rangle =$$

$$- \sum_{j \neq i} \oint_{C_j} \psi(x_j)^* x_i(x_j, y_j)^m y_i(x_j, y_j)^n \psi(x_j) | \mathcal{W} \rangle.$$

By bosonizing the fermionic fields and contracting the above state with a coherent state $\langle t|$, this yields a set of differential equations that can be solved recursively in the worldsheet genus g.

Let us recapitulate: Topological B-branes that end on Σ behave as free fermions on Σ . However, their transformation properties from patch to patch are rather unusual. In this thesis we examine these free fermions from the I-brane perspective. In Section 4.4.2 we relate the B-model fermions to the ones on the I-brane, whereas Chapter 5 is devoted to explaining their characteristics from the I-brane point of view.

4.4 BPS states and free fermions

It is time to take another look at the web of dualities in Fig. 1.6. The lowest 4 boxes in the vertical sequence of the web are all related to Calabi-Yau compactification down to a 4-dimensional Taub-NUT manifold. The middle ones correspond to topological string theory, in the sense that the holomorphic F-terms on

the Taub-NUT compute topological string amplitudes. The lowest ones are also familiar as string theory realizations of other types of topological invariants on Calabi-Yau threefolds, called Gopakumar-Vafa (GV) invariants and Donaldson-Thomas (DT) invariants. We are particularly interested in the DT perspective, as this leads to a natural I-brane partition function, see Section 4.4.3.

4.4.1 BPS states in the 9-11 flip

Let us start by going through the 9-11 flip in the vertical sequence once again. This time we pay attention to the topological invariants that are associated to these duality frames [71].

Donaldson-Thomas theory and D0-D2-D6 bound states

Instead of studying holomorphic embeddings ϕ of a worldsheet Σ_g into the Calabi-Yau threefold \tilde{X} , one can also think of its image $\phi_*([\Sigma_g]) \subset H_2(\tilde{X})$ as the zero-locus of a finite set of holomorphic functions on \tilde{X} . Mathematically this is rigorously defined in terms of rank 1 ideal sheaves \mathcal{E} . On a Calabi-Yau \tilde{X} ideal sheaves \mathcal{E} with $ch_2=d$ and $c_3=n$ can be counted as the *Donaldson-Thomas invariants*

$$\mathrm{DT}_{n,d} = \int_{[\mathcal{E}_n(ilde{X},d)]^{\mathrm{vir}}} 1 \in \mathbb{Z},$$

since their moduli space $[\mathcal{E}_n(\tilde{X},d)]^{\text{vir}}$ has virtual dimension zero [132, 133]. Their generating function is

$$Z_{\text{DT}}(t, e^{\lambda}) = \sum_{d,n} \text{DT}_{d,n} e^{-d \cdot t} e^{n\lambda}$$
(4.44)

The degree zero Gromov-Witten and Donaldson-Thomas partition functions are conjectured to be related as $Z_0^{\mathrm{DT}}(t,e^\lambda)=\left(Z_0^{\mathrm{GW}}(t,\lambda)\right)^2$, whereas their reduced parts $Z'=Z/Z_0$ are conjectured mathematically to be equal

$$Z'_{\mathrm{DT}}(t,e^{\lambda}) = Z'_{\mathrm{GW}}(t,\lambda).$$

This has been verified in some important cases, such as toric Calabi-Yau three-folds [134, 135, 136].

Physically, these invariants can be thought of as capturing D0-D2 bound states on a single D6 brane. In the maximally supersymmetric U(1) worldvolume theory on the D6-brane these bound states correspond to non-trivial topological sectors of the rank 1 gauge bundle \mathcal{E} : the D2 charge d equals $ch_2(E)$, while the D0 charge n equals $ch_3(E)$. So $\mathrm{DT}_{d,n}$ should correspond to the number of bound

states of a D6 brane with D2-branes that are wrapped around a cycle $d \in H_2(\tilde{X})$ and n added D0-branes.

When we wrap the D6-brane around a Calabi-Yau threefold \tilde{X} the worldvolume gauge theory is naturally twisted. On $\tilde{X} \times S^1$ its partition function is given by the Witten index

$$Tr[(-1)^F e^{-\beta H}]$$

where β is the radius of S^1 . Since the theory is topological we can take the limit $\beta \to 0$ to find a 6-dimensional maximally supersymmetric U(1) gauge theory. The D0-D2 bound states are captured by the topological terms

$$S = \lambda \int_{\tilde{X}} c_3(\mathcal{E}) + \int_{\tilde{X}} t \wedge ch_2(\mathcal{E}), \tag{4.45}$$

where t denotes the Kähler class of \tilde{X} . Since the only relevant contributions in this topological gauge theory are given by the above topological terms, we find the Donaldson-Thomas partition function (4.44). The mathematical relationship between GW and DT invariants corresponds physically to a gauge-gravity correspondence between the 6-dimensional U(1) gauge theory and topological string theory [32].

In string theory this configuration is embedded in the type IIA background

IIA :
$$\tilde{X} \times \mathbb{R}^3 \times S^1_{\beta}$$

with a single D6-brane wrapped on $\tilde{X} \times S^1_{\beta}$ bound to D2 and D0-branes.

Notice that in our duality web the Taub-NUT space TN may have an arbitrary number k of nuts. In the remainder of this thesis we restrict to k=1 just because that makes it easier to write down generating functions for BPS states. It is in fact a striking question how to use the I-brane web of dualities to write down a partition function that holds for any k.

The 9-11 flip and Gopakumar-Vafa invariants

As the D6-brane dissolves into a non-trivial Taub-NUT geometry, this configuration lifts to M-theory as

$$M: \quad \tilde{X} \times TN \times S^1_{\beta}, \tag{4.46}$$

accompanied by M2-branes and KK momenta along the Taub-NUT circle S^1_{TN} . Note that instead of reducing over S^1_{TN} to go down to IIA, it is also possible to

³The obstacle in writing down higher-rank Donaldson-Thomas invariants is the definition of a nice moduli space. Lately, this has been studied in [137]. It would be interesting to find out how this generalized Donaldson-Thomas theory is related to our web of dualities.

shrink the thermal circle S^1_{β} . This yields the background $\tilde{X} \times TN$ with fundamental strings wrapping 2-cycles of the Calabi-Yau X. Since the 5-dimensional angular momentum around S^1_{β} couples to the graviphoton field, the Witten index will in this configuration compute the A-model partition function $Z_{\rm GW}(t,\lambda)$ [71].

Moreover, in [71] it is argued that this S^1_{TN} - S^1_{β} flip (9-11 flip) encodes yet another interpretation of the topological string partition function in terms of the *Gopakumar-Vafa invariants*. These integer invariants

$$\mathsf{GV}_d^m \in \mathbb{Z}$$

count the number of M2-branes wrapped over a 2-cycle d in the Calabi-Yau \tilde{X} in a compactification of M-theory on \tilde{X} to five dimensions [22, 23, 119].⁴ The integer m stands for the left spin content of the corresponding 5-dimensional BPS particle. More accurately, its spin $(m_L, m_R) = (2j_L^3, 2j_R^3)$ takes value in the 5-dimensional rotation group $SO(4) = SU(2)_L \times SU(2)_R$, but to find a proper complex structure invariant one should sum over the right spin content. Compactifying the fifth dimension gives the BPS particles an extra angular momentum.

Let us go back to the M-theory set-up (4.46). In the strong coupling limit in which the radius R of the Taub-NUT circle S^1_{TN} is large, we effectively zoom in on the center of the Taub-NUT. Near this center the geometry cannot be distinguished from \mathbb{R}^4 , so that the M2-branes form a free gas of spinning BPS particles in \mathbb{R}^4 . Their spin receives a internal contribution m from the $U(1)\subset SU(2)_L$ isometry of the Taub-NUT plus an orbital contribution n that originates from the KK-momenta along the S^1_{TN} . This motivates the Gopakumar-Vafa partition function

$$Z_{\text{GV}}(t,\lambda) = \prod_{d \in H_2} \prod_{m \in \mathbb{Z}} \prod_{k=-m}^{m} \prod_{n=1}^{\infty} (1 - e^{\lambda(k+n)} e^{t \cdot d})^{(-1)^{m+1} n \, \text{GV}_d^m}, \tag{4.47}$$

as a second quantized product over the spinning BPS states [71, 140, 22]. The equivalence of the GV partition function $Z_{\text{GV}}(t,\lambda)$ to the GW partition function $Z_{\text{GW}}(t,\lambda)$ conjectures the integrality of the Gopakumar-Vafa invariants GV_d^m .

So although the Gromov-Witten invariants are not integer, and thus cannot be made sense of as counting BPS states in string theory, their generating function does allow a reformulation in terms of the integer-valued Donaldson-Thomas invariants and Gopakumar-Vafa invariants, which do have a BPS-interpretation in string theory and are connected by this 9-11 flip.

⁴Mathematically it is not so easy to define Gopakumar-Vafa invariants rigorously. Recent attempts can be found in [138, 139].

Degree zero example

Let us give a simple example of these dualities (more can be found in e.g. [114, 141]). In Gromov-Witten theory the constant map contributions are captured by the generating function

$$\mathrm{GW}_g(0) = \frac{(-1)^g \chi(\tilde{X})}{2} \int_{\overline{\mathcal{M}}_g} c_{g-1}^3 = \frac{(-1)^g \chi(\tilde{X})}{2} \frac{|B_{2g}| |B_{2g-2}|}{2g(2g-2)(2g-2)!}$$

where c_{g-1}^3 is (g-1)th Chern class of the Hodge bundle over \mathcal{M}_g (the moduli space of Riemann surfaces of genus g) and B_n are the Bernouilli numbers. The corresponding partition function $Z_{\text{GW},0}$ can be rewritten as a Gopakumar-Vafa sum

$$Z_{\text{GV},0} = \prod_{n>0} (1 - e^{\lambda n})^{-n\chi(\tilde{X})} = \prod_{n>0} (1 - e^{\lambda n})^{n\text{GV}_{0,0}}.$$

This generating function is a power of the MacMahon function

$$M(q) = \prod_{n \ge 1} (1 - q^n)^{-n},$$

which counts 3-dimensional partitions, and as such appears in the localization of the 6-dimensional gauge theory and the computation of the Donaldson-Thomas invariants [32, 28, 142].

4.4.2 Fermions and BPS states

Let us compare the different types of objects that play a role in the remainder of the duality web. In the I-brane frame these are the free fermions themselves and the fluxes through the a-cycles of the Riemann surface that they couple to. In the B-model these are BPS states that are obtained by wrapping D3-branes around A-cycles and the fluxes through these 3-cycles. These B-model invariants can be translated to analogous charges in the Gopakumar-Vafa M-theory setting or the Donaldson-Thomas frame in type IIA. Let us look at this closer.

Relating the I-brane fermions to BPS states

Perhaps it is clarifying to compare the chiral fermions, that appear so naturally in the intersecting brane system, to the usual BPS states. This is most naturally done in the M-theory picture, where we consider an M5-brane with topology $TN_k \times \Sigma$.

First of all, the chiral fermions are given by the open fundamental strings that stretch between D4-branes and D6-branes. On the Riemann surface Σ they wrap

an A-cycle, which is their time trajectory, times an interval I. At one end of the interval the open string ends on a D4-brane, whereas on the other end it ends on a D6-brane. These fundamental strings are lifted to open M2-branes in M-theory. Since the Taub-NUT circle shrinks at the D6-endpoint, the topology of the membranes is $S^1 \times D^2$, where S^1 is the time trajectory and D^2 is a 2-dimensional disc whose boundary ∂D^2 lies on the M5-brane. This boundary encircles the S^1_{TN} of the Taub-NUT geometry. Another way to see this is that the BPS mass of these open M2-brane states is given by

$$\mathcal{Z} = \int_{D^2} \omega_{TN} = \oint_{\partial D^2} \eta_{TN},$$

where η_{TN} is the one-form (3.19) on TN. This mass goes to zero exactly when the M2-brane approaches one of the NUTs of the Taub-NUT geometry. There the chiral fermions appear.

So, if we compute a fermion one-loop diagram, this is represented in M-theory in terms of open M2-brane world-volumes with geometry $D^2 \times S^1$ and boundary $T^2 = S^1 \times S^1$ on the M5-brane. Note that the S^1 on the TN factor is filled in by a disc. This is illustrated in Fig. 4.9.

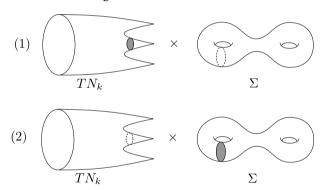


Figure 4.9: Two kinds of open M2-brane instantons that contribute to a M5-brane with geometry $TN_k \times \Sigma$: (1) the free fermions, massless at the NUTs of TN_k ; (2) the usual BPS states that become massless when the Riemann surface Σ pinches.

On the other hand we have the traditional BPS states in Calabi-Yau compactifications, that we discussed in Section 4.2. In the IIB compactification on $TN_k \times X$ these will be given by D3-branes that wrap special Lagrangian 3-cycles in X. Let us consider the D3-branes that wrap the A-cycles of the local Calabi-Yau, and thus fill in a disc D that ends on the Riemann surface Σ . After the dualities that map this configuration to an M5-brane in M-theory, these states become open M2-branes with the topology of a disc as well. But now the boundary S^1 of these discs will lie on the surface Σ . The time trajectory will be along the space-time TN_k . More invariantly, we have again M2-branes with world-volumes $D^2 \times S^1$

and boundary $S^1 \times S^1$, but now the S^1 on the Riemann surface Σ is filled in. This makes sense, since the mass of the BPS states, given by

$$\mathcal{Z} = \oint y dx,$$

goes to zero exactly when the surface Σ is pinched or forms a long neck. These states are the M-theory interpretation of the well-known massless monopoles of Seiberg and Witten [95, 143, 144].

In a full quantum theory of the M5-brane, both kinds of open M2-branes should contribute to the partition function. In fact, the boundaries of these M2-branes are the celebrated self-dual strings (see Section 4.2.2) that should describe the M5-brane world-volume theory [145, 101]. Clearly the corresponding massless states are contributing in different regimes. In that sense the relation between the free fermions and the usual BPS states can be considered as a strong-weak coupling duality.

Fermion charges on the I-brane

It might be good as well to follow the fermion numbers p^1,\ldots,p^g (through the handles of the Riemann surface) and the dual U(1) holonomies v_1,\ldots,v_g (that couple to these fluxes) through this chain of dualities. We pick k=1 for simplicity. First we remark that we have to choose a basis of a-cycles on Σ to define these quantities.

In the IIB compactification on $TN \times X$ the quantum numbers p^i appear as fluxes of the RR field G_5 through a basis of 3-cycles of the Calabi-Yau $X.^5$ In reduction of G_5 to the 4-dimensional low-energy theory on TN this gives the U(1) gauge fields F^i of the vector multiplets that appear in the gauge coupling

$$\int_{TN} \tau_{ij} F_+^i \wedge F_+^j,$$

analogous to the reduction in equation (4.13).

After mapping this configuration to IIA theory on $\mathbb{R}^3 \times S^1 \times \widetilde{X}$ with a D6-brane wrapping $S^1 \times \widetilde{X}$, the flux p is carried by the U(1) gauge field strength F on the world-volume of the D6-brane, $[F^i/2\pi] = p^i \in H^2(\widetilde{X}, \mathbb{Z})$. This can be interpreted as a bound state of a single D6-brane with p D4-branes. In a similar fashion the Wilson lines v_i are mapped to potentials for the D4-branes. So in this duality frame we can simply shift the fermion number by adding D4-branes.

 $^{^5}$ Indeed, the fermion flux originates from a 2-form field strength on the D6-brane. This lifts in M-theory to a 4-form flux G_4 and arises as a G_5 -flux in type IIB. We will explain these last steps in more detail in Section 5.2.1.

B-model branes compared to I-brane fermions

Another natural question to ask is whether these fermions in the topological B-model, see Section 4.3.2, are the same as the physical fermions we encountered on the I-brane. An important property of the fermions in the context of topological vertex is that the insertions of the operators $\psi(x_i)$, that create fermions (which is of course not the same as the quanta of the corresponding field), change the geometry of the curve. The fermions will produce extra poles in the meromorphic differential ydx which encodes the embedding of the curve as H(x,y)=0. We have the identification $y=\partial\phi(x)$, and by bosonization the operator product with a fermion insertion gives a single pole at each fermion insertion

$$\partial \phi(x) \cdot \psi(x_i) \sim \frac{1}{x - x_i} \psi(x_i).$$

In the superstring such a correlator

$$\langle \psi(x_1) \cdots \psi(x_n) \rangle$$

of fermion creation operators corresponds to the insertion of D5-branes in type IIB compactification. These 5-branes all have topology $\mathbb{R}^4 \times \mathbb{C}$, where in the Calabi-Yau uv+H(x,y) they are located at specific points x_i of the curve H(x,y)=0 along the line v=0 (so they are parametrized by the remaining coordinate u). Having this extra pole for y means that the Riemann surface has extra tubes attached to it at $x=x_i$.

If we T-dualize this geometry to replace the Calabi-Yau X by an NS5-brane wrapping $\mathbb{R}^4 \times \Sigma$, the D5-branes, which are all transverse to Σ , will become D4-branes. So we get an NS5-brane with a bunch of D4-branes attached, that all end on the NS5-brane. This configuration can be lifted to M-theory to give a single irreducible M5-brane, now with "spikes" at the positions x_1,\ldots,x_n . So we indeed see that the two kinds of fermions (or at least their sources) are directly related and have the same effect on the geometry of the Riemann surface.

4.4.3 I-brane partition function

In Type II compactifications on Calabi-Yau geometries four-dimensional F-terms can be computed using topological string theory techniques. We now want to see how these kind of computations can be mapped to the I-brane model.

Starting point will be the end of the chain of dualities in Figure 1.6: the IIA compactification on $\mathbb{R}^3 \times S^1 \times \widetilde{X}$ with \widetilde{X} a (non-compact) Calabi-Yau geometry, where we wrap a D6-brane along $S^1 \times \widetilde{X}$. This is the set-up of Donaldson-Thomas theory. With the right background values of the moduli turned on, the topological string partition function can be reproduced as an index that counts

BPS states degeneracies of D-branes in this configuration.

More precisely, the topological string theory partition function in the A-model naturally splits in a classical and a quantum (or instanton) contribution

$$Z_{\text{top}}(t,\lambda) = \exp\left(-\frac{t^3}{6\lambda^2} - \frac{1}{24}t \wedge c_2\right) Z_{\text{qu}}(t,\lambda). \tag{4.48}$$

Here $t \in H^2(\widetilde{X})$ is the complexified Kähler class, λ the topological string coupling constant, and $c_2 = c_2(\widetilde{X})$. The classical contributions should be read as integrals over the Calabi-Yau \widetilde{X} . The quantum contribution is decomposed as

$$Z_{\mathrm{qu}}(t,\lambda) = \exp\left(\sum_{g\geq 0} \lambda^{2g-2} \mathcal{F}_g(t)\right),$$

with the genus g free energy expressed in terms of the Gromov-Witten invariants of degree d as

$$\mathcal{F}_g(t) = \sum_{d} \mathsf{GW}_g(d) e^{d \cdot t}. \tag{4.49}$$

We can now use the fact that Z_{qu} has a dual interpretation as the Donaldson-Thomas partition function counting D0-D2-D6 bound states (here we ignore a subtlety with the degree zero maps)

$$Z_{\text{qu}}(t,\lambda) = \sum_{n,d} \text{DT}(n,d) e^{-n\lambda} e^{d \cdot t}.$$
 (4.50)

In this sum $n \in H_0(\widetilde{X},\mathbb{Z}) \cong \mathbb{Z}$ and $d \in H_2(\widetilde{X},\mathbb{Z})$ are the numbers of D0-branes and D2-branes. As before $d \cdot t$ stands for $\int_d t$. The integers $\mathrm{DT}(n,d)$ are the Donaldson-Thomas invariants of the ideal sheaves with these characteristic classes. From the BPS counting perspective it is also natural to add the exponential cubic prefactor in (4.48), since this is nothing but the tension of a single D6-brane (including the geometrically induced D2-brane charge). Remember that this tension is measured by integrating $\Omega = \exp t$ over the submanifold.

In type IIA string theory set-up the complex parameters λ and t can be expressed in terms of geometric moduli of the S^1 and the Calabi-Yau \widetilde{X} , and the Wilson loops of the flat RR fields C_1 and C_3 . In particular, we can write

$$\lambda = \frac{\beta}{\ell_s q_s} + i\theta,\tag{4.51}$$

with

$$\beta = 2\pi R_9 = \oint_{S^1} ds$$

the length of the Euclidean time circle S^1 , and θ the Wilson loop

$$\theta = \oint_{S^1} C_1.$$

That is, λ can be written as the holonomy of the complexified one-form ds/g_s+iC_1 (in string units). An important remark is that, as expressed in equation (4.50), for BPS states θ and β only appear in the *holomorphic* combination (4.51). In the same way the parameter t is given by the integral of the complex 3-form

$$t = \oint_{C_1} \frac{k \wedge ds}{q_s} + iC_3.$$

It is rather trivial to also include the coupling to D4-branes in this BPS sum. As explained before, such a bound state of p D4-branes to a D6-brane is given by a flux of the U(1) gauge field on the D6-brane. We can think of this as a non-trivial first Chern class of the line bundle over the D6-brane that wraps \widetilde{X} , so that the D4-brane has charge $p=\mathrm{Tr}[F/2\pi]$. Tensoring with the extra line bundle will not change the BPS degeneracy, since the moduli space of such twisted sheaves is isomorphic to that of the untwisted one. The only part that changes are the induced D0 and D2 charges, that shift as

$$d \to d - \frac{1}{2}p^2 - \frac{1}{24}c_2$$

$$n \to n + d \wedge p + \frac{1}{6}p^3 + \frac{1}{24}p \wedge c_2.$$
(4.52)

Here we also wrote the charges $d \in H^4(\tilde{X})$ and $n \in H^6(\tilde{X})$ as de Rham forms by using Poincaré duality. The induced charges follow from applying equation (3.11). For example, the D2-brane charge d gets a contribution from the couplings $1/2\operatorname{Tr}[F/2\pi]^2$ and $c_2/24$ on the D6-brane worldvolume.

So, if we also include a sum over the number of D4-branes, weighted by a potential $v \in H^4(\tilde{X})$, we get a generalized partition function

$$Z(v,t,\lambda) = \sum_{p \in H^{2}(\widetilde{X},\mathbb{Z})} e^{p(v-t^{2}/2\lambda)} e^{-(p^{2}/2+c_{2}/24)t} e^{-(p^{3}/6+pc_{2}/24)\lambda} e^{-t^{3}/6\lambda^{2}} Z_{qu}(t+p\lambda,\lambda)$$

$$= \sum_{p \in H^{2}(\widetilde{X},\mathbb{Z})} e^{pv} Z_{top}(t+p\lambda,\lambda). \tag{4.53}$$

Apart from adding the D6-brane tension $-t^3/6\lambda^2$, we have also added the tension $-\int_p t^2/2\lambda$ of the D4-branes. Remember that the D4-branes translate into fermion fluxes on the I-brane. From this fermionic (I-brane) point of view it is natural to sum of these fermion numbers. The structure (4.53) is therefore

the object that we want to identify with the I-brane partition function and that should be computable in terms of free fermions. It should be remarked that the partition function (4.53) was also found in [146], where a dual object was studied: a NS5-brane wrapping \widetilde{X} . Also in [147], a partition function of this type was considered and directly related to fermionic expressions. Moreover, recently it was even conjectured to capture non-perturbative aspects of the topological string [148]. We will comment more on this in Section 6.4.

Let us finish this chapter by noting that the above expression for $Z(v,t,\lambda)$ has an interesting limit for $\lambda\to 0$, where only genus zero and one contribute. In that case we have

$$Z_{\text{top}}(t+p\lambda) \sim \exp\left[\frac{1}{\lambda^2}\mathcal{F}_0(t) + \frac{1}{\lambda}p^i\partial_i\mathcal{F}_0 + \frac{1}{2}p^ip^j\tau_{ij} + \mathcal{F}_1(t) + \mathcal{O}(\lambda)\right]. \quad (4.54)$$

If we now subtract the singular terms (which have a straightforward interpretation as we shall see in a moment), we are left with the familiar $\lambda = 0$ answer

$$Z(v,t) = \sum_{p} e^{\frac{1}{2}p \cdot \tau \cdot p + p \cdot v} e^{\mathcal{F}_1}.$$

Chapter 5

Quantum Integrable Hierarchies

The string theory embedding of topological string theory in terms of a Calabi-Yau compactification to a Taub-NUT space, suggests a duality of the topological string partition function and the I-brane partition function. We thus expect that the topological string partition function on a non-compact Calabi-Yau threefold X_{Σ} , that is modeled on a holomorphic curve Σ , has an interpretation in terms of *physical* chiral fermions on Σ . However, we discover that these fermions don't just transforms as sections of the square-root of the canonical bundle on Σ (just like the topological B-branes in the B-model). Instead, the topological string coupling constant translates into a non-trivial flux on Σ that quantizes the curve into a non-commutative object. Fermions should in this context be mathematically interpreted as elements of a holomorphic \mathcal{D} -module supported on Σ . The goal of this chapter is to explain the topological string partition function as a tau-function of a quantum or non-commutative integrable hierarchy.

In Section 5.1 we start with (semi-classical) Krichever solutions of the Kadomtsev-Petviashvili (KP) hierarchy. We explain how the semi-classical contribution \mathcal{F}_1 to the free energy can be interpreted in terms of such a Krichever solution. In Section 5.2 we study the effect of the topological string coupling constant λ on the I-brane. We introduce \mathcal{D} -modules and show in detail how the resulting I-brane configuration should be understood mathematically in terms of a quantum integrable hierarchy. In Section 5.3 we use our framework to assign a fermionic state to a non-commutative curve corresponding to Σ . In analogy with the semi-classical recipe the determinant of such a state yields a tau-function. We conjecture that this *quantum* tau-function is equal to the all-genus topological string partition function. The advantage of this line of thought is that the total topological string partition function on a local Calabi-Yau threefold acquires a very sim-

ple interpretation as a fermion determinant on an underlying non-commutative curve. In Chapter 6 we work out some insightful examples.

5.1 Semi-classical integrable hierarchies

An important class of conformal field theories is that of free chiral fermions on a Riemann surface Σ . Already in the mid-eighties it was discovered that these CFT's are intimitely related to a certain integrable system, well-known as the Kadomtsev-Petviashvili (KP) hierarchy. The CFT partition function appears as a so-called tau-function of this integrable hierarchy. It computes the determinant of the Cauchy-Riemann operator $\bar{\partial}$ on Σ . Let us explain this in more detail.

5.1.1 KP integrable hierarchy

The KP integrable system originates from the non-linear partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x}.$$
 (5.1)

This equation was written down around 1900 by Kortweg and de Vries as a description of non-linear waves in shallow water. To see how this differential equation is included in an integrable hierarchy, let us write down a pseudo-differential operator

$$L = \partial + \sum_{i=0}^{\infty} u_i(x)\partial^{-i},$$

where $\partial = \partial/\partial x$. This operator L is known as a Lax operator. The negative powers in its expansion are defined through the Leibnitz rule

$$\partial^i f = \sum_{k=0}^{\infty} {i \choose k} (\partial^k f) \partial^{i-k}, \quad \text{for } i \in \mathbb{Z}$$

with $\binom{i}{k}=i\cdots(i-k+1)/k!$. So ∂^{-1} should be interpreted as partial integration. The Lax operator may depend on an infinite set of times $x=t_1,t_2,\ldots$ and satisfies the KP flow equations

$$\frac{\partial L}{\partial t_n} = [H_n, L], \quad \text{for } n \ge 1$$
 (5.2)

where the Hamiltonians $H_n = (L^n)_+$ are powers of the Lax operator L. The subscript in $(L^n)_+$ denotes the restriction to the positive powers of the pseudo-

differential operator L^n . The Hamiltonians obey the Zakharov-Shabat relations

$$\frac{\partial}{\partial t_n} H_m - \frac{\partial}{\partial t_m} H_n = [H_n, H_m].$$

They make sure that the defining KP equations (5.2) line out a commuting flow.

We can just as well write down the KP integrable hierarchy (5.2) for a power $P = L^p$ of the Lax operator L. When P is actually a differential operator, the resulting integrable hierarchy is of so-called KdV type. For

$$P = \partial^2 + u(x), \tag{5.3}$$

we easily recover the original KdV equation (5.1).

In the last decades many complementary perspectives on the KP integrable hierarchy have been developed (starting with [149, 150, 151]). In this section we are mostly interested in its geometric interpretation in terms of (possible singular) Riemann surfaces. This relation, discovered by Krichever and illustrated in Fig. 5.1, comes about as follows. Look at the algebra of differential operators that commute with a given differential operator P. This turns out to be a commutative algebra that we denote by \mathcal{A}_P . In the KdV example \mathcal{A}_P is a polynomial ring generated by the degree two KdV operator P from equation (5.3) and a degree three operator Q

$$\mathcal{A}_P = \mathbb{C}[P,Q]/(Q^2 = P^3 - g_2P - g_3),$$

in terms of the Weierstrass invariants g_2 and g_3 . Note that this forms the algebra of functions on an elliptic curve. Any (pointed) curve that is obtained via a commuting ring of differential operators is called a *Krichever* curve. It is parametrized by the eigenvalues of the commuting operators in \mathcal{A}_P , and therefore also referred to as a *spectral curve* of the KP system corresponding to P.

According to the above map many differential operators P will correspond to the same spectral curve Σ : in fact, for most of them the algebra \mathcal{A}_p will be generated by P only, so that the Krichever curve is just a pointed projective plane. However, when we fix a line bundle¹ \mathcal{L} on the spectral curve Σ , together with a point $p \in \Sigma$ and trivializations of Σ and \mathcal{L} in the neighbourhood of p, there is a natural correspondence with a subset of solutions to the KP hierarchy. This is called the *Krichever correspondence*.

So let us start with a spectral curve Σ together with a line bundle \mathcal{L} , that comes equipped with a connection A. Both from the physical and mathematical perspective it is natural to consider a chiral fermion field $\psi(z)$ on Σ that couples to the gauge field A. The free fermion partition function computes the determinant

¹For singular curve we need to consider rank 1 torsion free sheafs.

of the twisted Dirac operator $\overline{\partial}_A$ coupled to the line bundle \mathcal{L} .

It has been known for a long time that these chiral determinants are closely related to integrable hierarchies of KP-type. If one picks a point $p \in \Sigma$ on the curve together with a local trivialization e^t of the line bundle around p, the ratio with respect to a reference connection A_0

$$\tau(t) = \frac{\det \overline{\partial}_A}{\det \overline{\partial}_{A_0}}$$

equals a so-called *tau-function* of the KP integrable hierarchy. This determinant has many interesting properties, e.g. the dependence of this determinant on the connection A is captured by a Jacobi theta-function

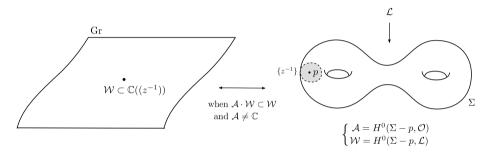


Figure 5.1: The Krichever correspondence assigns a geometric configuration $(\Sigma - p, z^{-1}, \mathcal{L})$ to a dense subset of points \mathcal{W} in the Grassmannian Gr, where z^{-1} is a local coordinate in the neighborhood p on the curve Σ . Such a geometric interpretation only exists when the stabilizer \mathcal{A} of \mathcal{W} is non-trivial. Under the Krichever map it turns into the algebra of holomorphic functions on $\Sigma - p$.

How is the above tau-function related to the KP hierarchy (5.2) that we started with? This becomes clear when we study the system from a Hamiltonian perspective. In this formulation the partition function $Z(S^1)$ is an element of a Hilbert space $\mathcal H$ assigned to a circle S^1 surrounding p. This Hilbert space consists of functionals on all possible boundary conditions of the fields at the circle around p. In this example that Hilbert space is just the Fock space $\mathcal F$ that is constructed out of the modes ψ_n and ψ_n^* of the fermion field $\psi(z)$. So we find a state

$$|\mathcal{W}\rangle \in \mathcal{F}$$
.

In fact, we can describe the state $|\mathcal{W}\rangle$ explicitly. It is the semi-infinite wedge product of a basis that spans the space \mathcal{W} of holomorphic sections

$$\mathcal{W} = H^0(\Sigma - p, \mathcal{L}).$$

These are all meromorphic sections of the line bundle \mathcal{L} that are only allowed to have poles at p. Notice that \mathcal{W} is a module for the algebra $\mathcal{A} = H^0(\Sigma - p, \mathcal{O})$ consisting of holomorphic functions on $\Sigma - p$.

Let us choose a coordinate z^{-1} in a neighbourhood around p. Since a holomorphic section of $\mathcal L$ on $\Sigma-p$ can at most have a (finite order) pole at p, the subspace $\mathcal W$ is a subset of $\mathbb C((z^{-1}))=\mathbb C[z]\oplus\mathbb C[[z^{-1}]]$. The set of all subspaces $\mathcal W$ of $\mathbb C((z^{-1}))$ whose projection to $\mathbb C[z]$ has a finite index μ , with respect to the projection map $\mathbb C((z^{-1}))\mapsto\mathbb C[z]$, has the structure of a Grassmannian $Gr(\mu)$. The Krichever map assigns a geometric set-up to an element $\mathcal W$ of the Grassmannian that has a non-trivial $(\mathcal A\neq\mathbb C)$ stabilizer²

$$A \cdot W \subset W$$

The infinite wedge products $|\mathcal{W}\rangle$ form a line bundle over the Grassmannian. Pulling this line bundle back to the family of configurations (Σ, P, \mathcal{L}) reconstructs the determinant line bundle corresponding to the $\overline{\partial}$ -operator on Σ .

The big-cell of the index zero Grassmannian is the subset that is comparable to the vacuum $\mathbb{C}[z^{-1}]$ (both the kernel and the cokernel of the projection map have dimension zero). To compute the tau-function on the big cell, we first associate a coherent state $|t\rangle$ to the local trivialisation around P, corresponding to the chosen boundary conditions. Then, combining the above ingredients, the tau-function can be written as

$$\tau(t) = \langle t | \mathcal{W} \rangle. \tag{5.4}$$

It is even possible to reconstruct the differential operator P. Notice that any subspace \mathcal{W} in the big cell contains a unique element of the form

$$\eta(z,t) = 1 + \sum_{i=1}^{\infty} a_i(t)z^{-i}.$$

such that the Baker function

$$\psi(t,z) = \eta(z,t)g(z), \quad g(z) = \exp\left(\sum_{n} t_n z^n\right)$$

is an element of $\mathcal W$ for all times t. Substituting $z \leftrightarrow \partial$ in the expression for η defines a pseudo-differential operator $K(\partial,t)$, such that $P=K\partial^p K^{-1}$ solves the differential equation

$$P\psi(t,z) = z^p \psi(t,z).$$

²More carefully, the rank of $\mathcal W$ over $\mathcal A$ determines the rank of the bundle $\mathcal L$ over Σ .

We conclude that this differential operator P is fixed by the state \mathcal{W} and thus by the geometrical configuration (Σ, P, L) together with trivializations.

Multiplying the subspace W by the function g(z) defines an action on the Grassmannian, that leaves the differential operator P fixed. The flow parametrized by t_n is equivalent to the nth KP flow (5.2). Geometrically it leaves the spectral curve Σ invariant and multiplies $\mathcal L$ by a flat line bundle, so that it corresponds to a straight line motion on the Jacobian of Σ .

5.1.2 Topological strings at 1-loop

The formalism of the last section can be easily extended to more general geometric configurations, consisting of a genus g Riemann surface Σ with an arbitrary number k of punctures. A free fermionic field $\psi(z)$ coupled to a line bundle $\mathcal L$ over Σ determines a state $|\mathcal W\rangle$ in the kth fold tensor product $\mathcal H^{\otimes k}$. This has been exploited in the late eighties to understand the Polyakov string in the operator formalism (see e.g. [152, 153, 154]). The KP formalism determines that the state $|\mathcal W\rangle$ is a Bogoliubov transformation

$$|\mathcal{W}\rangle = \exp\left(\sum_{i,j=1}^{k} \sum_{r,s>0} a_{rs}^{ij} \psi_{-r}^{i} \psi_{-s}^{j*}\right) |0\rangle$$
 (5.5)

of the vacuum.

The most elegant feature of this formalism is that the Riemann surface can be split up in elementary building stones. The total partition function can be reconstructed by glueing these using a natural inner product on \mathcal{H} . One of such elementary pieces is the three-punctured sphere

$$|V\rangle \in \mathcal{H}^{\otimes 3}$$
. (5.6)

Its coefficients may be determined by the CFT equality

$$\langle \phi_1, \phi_2, \phi_3 | V \rangle = \langle \phi_1(0) | \phi_2(1) | \phi_3(\infty) \rangle,$$

that follows since the three punctures on the sphere may be fixed at $0, 1, \infty$.

Returning to the topological B-model on X_{Σ} , the \mathcal{W}_{∞} -constraints seem to suggest that also the dual topological string partition function Z_{top}^D can be written as a Bogoliubov state (5.5) [17, 128]. This implies that also the topological string is governed by the KP hierarchy. However, it is known that Z_{top}^D is generically not of the Krichever form, *i.e.* it does not correspond to regular free fermions on a Riemann surface. The goal of this chapter is to show how the B-model partition function should be interpreted geometrically. We will be motivated by

the fact that the fermion fields that play a role in the B-model do not obey the usual transformation rules.

But before going into this, let us remark that in the limit $\lambda \to 0$ the B-model fermions do transform semi-classically. To see this we expand the fermion field $\psi(x)$

$$\psi(x) = e^{\phi_{cl}/\lambda} \psi_{qu}(x)$$

in a classical piece plus quantum corrections. It is shown in [17, 128] that in the limit of small quantum corrections the Fourier-like transformation (4.43) reduces to the transformation property

$$\psi_{\text{qu}}(x_j)dx_j^{1/2} = \psi_{\text{qu}}(x_i)dx_i^{1/2}$$

for the quantum field $\psi_{\rm qu}$. In other words, the semiclassical approximation of the fermionic field $\psi(x)$ transforms in the classic way as a spin 1/2 field on Σ .

Relatedly, let us turn to the Ray-Singer determinant (4.39) that describes the (worldsheet) genus one part of the free energy \mathcal{F} . When the Calabi-Yau threefold is non-compact and modeled on a Riemann surface Σ , the Ray-Singer torsion reduces to the bosonic determinant

$$\mathcal{F}_1 = -\frac{1}{2} log \det \Delta_{\Sigma}.$$

By the boson-fermion correspondence this determinant is equal to the logarithm of the determinant of the $\overline{\partial}$ -operator on Σ . In other words, the KP tau function computes the 1-loop semi-classical partition function $\exp(\mathcal{F}_1)$ on a non-compact Calabi-Yau background X_{Σ} . In particular, the topological vertex reduces to the CFT vertex (5.6) in the limit that $\lambda \to 0$ [17].

We will see this explicitly in some examples in Chapter 7. In these examples we compute $\exp(\mathcal{F}_1)$ for two threefolds X_{Σ} in which Σ is a compact genus one and a compact genus two Riemann surface. Other instances can be found in the correspondence between the B-model and matrix models [155].

5.2 \mathcal{D} -modules and quantum curves

Let us now release the constraint $\lambda=0$ and turn our attention to the full free energy

$$\mathcal{F} = \sum_{q} \lambda^{2g-2} \mathcal{F}_{g}.$$

As we discussed in Chapter 4 the topological string coupling constant λ is embedded in a type IIA Calabi-Yau compactification as the graviphoton field strength

 C_1 . Let us therefore chase the graviphoton field through the duality web to find out its implications on the I-brane.

5.2.1 Chasing the string coupling constant

We start with the purely geometric M-theory compactification on $TN \times S^1 \times \widetilde{X}$. Asymptotically this geometry has a T^2 fibration over \mathbb{R}^3 . The ratio of the radii of the two circles of this two-torus is given by

$$\frac{\beta}{2\pi q_s \ell_s} = \frac{R_9}{R_{11}},$$

whereas the complex modulus of this T^2 is

$$\frac{\lambda}{2\pi i} = \frac{\theta}{2\pi} - \frac{\beta}{2\pi q_s l_s} i.$$

(The last equality sign follows from equation (4.51).) Indeed, remember that in general when we compactify an M-theory circle, the off-diagonal components $g_{11,i}$ combine into the type IIA graviphoton field C_1 . In the M-theory background described above, the only off-diagonal components in the T^2 -metric are parametrized by the real part of the T^2 complex structure. The real part of the complex modulus should therefore be proportional to the graviphoton $\theta = \oint_{S^1} C_1$. The asymptotic torus T^2 is illustrated in Fig. 5.2.

When we exchange the roles of the 9th and 11th dimension, we will perform a modular transformation or S-duality

$$\lambda \to 4\pi^2/\lambda$$
.

In the dual IIA compactification on $TN \times \widetilde{X}$ the asymptotic values for the radius of the circle fibration and the graviphoton Wilson line are thus proportional to $-1/\lambda$. In the somewhat singular limit $\beta \to 0$ the complex modulus λ limits to $i\theta$. So after the flip we find that the new graviphoton field becomes proportional to $1/\lambda$.

After the flip the Wilson loop only makes sense asymptotically, since S^1_{β} is contractible in TN. In fact, the graviphoton gauge field is given by

$$C_1 = \frac{1}{\lambda}\eta,$$

where the one-form η is our friend (3.19). This field is not flat but has curvature

$$G_2 = dC_1 = \frac{1}{\lambda}\omega. \tag{5.7}$$

Here ω is the unique harmonic 2-form that is invariant under the tri-holomorphic circle action (3.18). Note that this is a natural choice, since in the case of TN_1 this form reduces to the usual constant self-dual two-form in the center \mathbb{R}^4 (up to hyper-Kähler rotations).

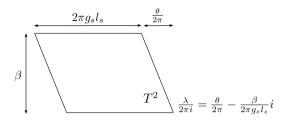


Figure 5.2: The above torus T^2 is present asymptotically in the M-theory geometry. In a type IIA reduction over the Taub-NUT circle S_{TN}^1 , of size $R_9 = \beta$, the real part of the T^2 complex modulus $\lambda/2\pi i$ is proportional to the graviphoton field C_1 integrated over S^1 .

Summarizing, the topological string partition function will be reproduced by a type IIA compactification on $TN \times \widetilde{X}$ with graviphoton flux given by (5.7). Note that in this case the graviphoton is *inversely* proportional to the topological string coupling!

It is now straightforward to follow this flux further through the duality chain. In the type IIB compactification on $TN \times X$ the self-dual 5-form RR field G_5 is given by

$$G_5 = \frac{1}{\lambda}\omega \wedge \Omega.$$

with Ω the holomorphic (3,0) form on the Calabi-Yau X.

We will now T-dualize this configuration to the IIA background that includes a NS5-brane. In that case there is 4-form RR-flux

$$G_4 = \frac{1}{\lambda} \omega \wedge dx \wedge dy. \tag{5.8}$$

Here $dx \wedge dy$ is the (2,0) form on the complex surface \mathcal{B} . This 4-form flux can be directly lifted to M theory, where we have the geometry

$$TN \times \mathcal{B} \times \mathbb{R}^3$$
.

B-field

Now we have to discuss what the interpretation of this flux is, if we reduce to IIA theory along the S^1 inside TN to produce our system of intersecting branes. In that case we will have an extra set of D6-branes with geometry $\mathcal{B} \times \mathbb{R}^3$. We want to argue that the G_4 flux becomes a constant NS B-field on their world-volumes.

As a preparation, let us recall again how the world-volume fields of the D6-branes are related to the TN geometry in the M-theory compactification. First of all, the centers \vec{x}_a of the TN manifold are given by the vev's of the three scalar Higgs fields of the 6+1 dimensional gauge theory on the D6-brane. In a similar fashion the U(1) gauge fields A_a on the D6-branes are obtained from the 3-form C_3 field in M-theory. More precisely, if ω_a are the k harmonic two-forms on TN_k introduced in section 2, we have a decomposition

$$C_3 = \sum_a \omega_a \wedge A_a.$$

We recall that the forms ω_a are localized around the centers \vec{x}_a of the TN geometry. So in this fashion a bulk field gets replaced by a brane field. This relation also holds for a single D6-brane, because ω is normalizable in TN_1 .

As a direct consequence of this, the reduction of the 4-form field strength $G_4=dC_3$ can be identified with the curvature of the gauge field

$$G_4 = \sum_a \omega_a \wedge F_a.$$

Combining this relation with the presence of the flux (5.8), we find that in the I-brane configuration the D6-branes support a constant flux

$$\sum_{a} F_a = \frac{1}{\lambda} dx \wedge dy.$$

There is simple and equivalent way to induce such a constant magnetic field on all of the D6-branes: turn on a NS B-field in the IIA background³. We can therefore conclude that in the presence of the background flux (5.8) translates into a constant B_{NS} field⁴ induced on the surface \mathcal{B}

$$B_{NS} = \frac{1}{\lambda} dx \wedge dy. \tag{5.9}$$

In the next section we will discuss the full consequences of this.

 $^{^3}$ This can be most easily understood from the worldsheet point of view. The coupling of the B-field to the fundamental string in equation (3.9) is not invariant under gauge transformations $B\mapsto B+d\Lambda$ when the worldsheet has boundaries. This can be repaired by a simultaneous gauge transformation $A\mapsto A-\Lambda/2\pi\alpha'$ of the U(1) worldvolume gauge field A on the brane which the string is ending on. This implies that only $B+2\pi\alpha' F$ is a gauge-invariant combination on the brane.

⁴Note that the above B-field is holomorphic because we started out with a holomorphic graviphoton field strength. This is unconventional: string theory forces the B-field to be real. More precisely, the full topological content of the intersecting brane configuration is captured by a holomorphic and an anti-holomorphic piece that couple to the real B-field $B+\bar{B}$.

Classical term \mathcal{F}_0

But at this point we first want to point out one immediate and more elementary consequence of the B-field. The presence of the flux induces a U(1) gauge field on the D6-brane

$$A = \frac{1}{\lambda} y dx. ag{5.10}$$

For the 4-6 strings we have to restrict A to Σ . Therefore the chiral fermions are coupled to a non-zero U(1) gauge field. This background gauge field gives a contribution to the effective action on Σ of the form

$$\mathcal{F} = \frac{1}{2} \sum_{i} \oint_{a^{i}} A \oint_{b_{i}} A.$$

Here (a^i, b_i) is a canonical basis of $H_1(\Sigma, \mathbb{Z})$. Plugging the expression for A we obtain precisely the (genus zero) prepotential of topological string theory

$$\mathcal{F} = \frac{1}{\lambda^2} \mathcal{F}_0,\tag{5.11}$$

where, as always,

$$X^{i} = \oint_{a^{i}} y dx, \qquad \partial_{i} \mathcal{F}_{0} = \oint_{b_{i}} y dx.$$

In fact, we can also include a non-trivial flux p^i through the cycles a^i . This will give a second contribution to the free energy given by

$$p^{i} \oint_{b_{i}} A = \frac{1}{\lambda} p^{i} \partial_{i} \mathcal{F}_{0}. \tag{5.12}$$

We recognize the contributions (5.11) and (5.12) as the genus zero contributions in the expansion for small λ of the general expression (4.54).

5.2.2 D-Modules

Let us now explain how \mathcal{D} -modules naturally appear in the I-brane set-up. (See also [156] for a much more involved setting.) First of all, by very general arguments the algebra \mathcal{A} of open string fields on the D6-brane is naturally non-commutative. This is a consequence of the fact that the Riemann surface that describes the interaction

$$\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$$

only has a cyclic symmetry (see Fig. 5.3).

This non-commutativity is particularly clear if one includes a B-field as in (5.9). One simple way to see this is, that in the presence of such a B-field a gauge field A is induced that couples to the open strings. The gauge field satisfies dA = B

and can be chosen as

$$A = \frac{1}{\lambda} y dx.$$

Therefore, on the 6-6 strings there is a boundary term

$$\int A = \int dt \, \frac{1}{\lambda} y \dot{x}.$$

If we reduce this term to the zero-modes of the strings we get the usual term of quantum mechanics, where λ plays the role of \hbar . (Indeed, notice that the momentum coordinate is given by $\partial L/\partial \dot{x}=y$.) Therefore the coordinates x and y become non-commutative operators

$$[\hat{x}, \hat{y}] = \lambda.$$

So, the zero-slope limit of the algebra \mathcal{A} becomes the Heisenberg algebra, generated by the variables \hat{x}, \hat{y} . This is also known as the non-commutative or quantum plane. In the case where $\mathcal{B} = \mathbb{C}^2$ we can identify this algebra with the algebra of differential operators on \mathbb{C}

$$\mathcal{A}\cong\mathcal{D}_{\mathbb{C}}.$$

The algebra $\mathcal{D}_{\mathbb{C}}$ consists of all operators

$$D = \sum_{i} a_i(x)\partial^i, \qquad \partial = \frac{\partial}{\partial x}.$$

Here we have identified $\hat{y} = -\lambda \partial$.

Now suppose that we add some other brane to this system, in our case a D4-brane localized on Σ . We pick x as our local coordinate, so that, once restricted to the curve, the variable y is given by a function y = p(x). Now the space of

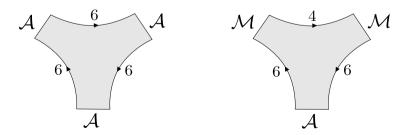


Figure 5.3: The 6-6 strings naturally form a non-commutative algebra A: glueing the inner endpoints of two 6-6 yields another 6-6 string. On the other hand, 4-6 strings are a module \mathcal{M} for the algebra A: glueing the right endpoint 4-6 to the left one of a 6-6 string yields another 4-6 string.

4-6 open strings, that we will denote as \mathcal{M} , is by definition a module for the algebra \mathcal{A} of 6-6 strings. This is a simple consequence of the fact that glueing a 6-6 string to a 4-6 string produces again a 4-6 string, as illustrated in Fig. 5.3. Therefore there is an action

$$A \times M \to M$$
.

(More completely, \mathcal{M} is a \mathcal{A} - \mathcal{B} bimodule, where \mathcal{B} is the algebra of 4-4 open strings.)

Modules \mathcal{M} for the algebra of differential operators are called \mathcal{D} -modules. In this case we are interested in \mathcal{D} -modules that in the semi-classical limit reduce to curves or equivalently Lagrangians. Such \mathcal{D} -modules are called holonomic.

So we can draw the following conclusion: in the presence of a background flux, the chiral fermions on the I-brane should no longer be regarded as local fields or sections of the spin bundle $K^{1/2}$. Instead they should be interpreted as sections of a non-commutative \mathcal{D} -module.

Notice that if Σ is a non-compact curve, having marked points at infinity, the symplectic form $dx \wedge dy$ becomes very large at the asymptotic legs. This can be see by using the appropriate variables at infinity x'=1/x and y'=1/y. In these variables the B-field becomes singular which means that the non-commutativity goes to zero. This explains why it makes sense to speak about the usual free chiral fermions at infinity, and to discuss their nontrivial transformation properties from patch to patch as in [128]. Considering compact spectral curves seems to be much more involved from this perspective.

In the context of string theory it is worth stressing the range of parameters α' and λ in which \mathcal{D} -module description is valid. The string coupling λ , which enters in the B-field flux as $B=\frac{1}{\lambda}dz\wedge dw$, plays an important role as quantization parameter. From the \mathcal{D} -module point of view there seems to be no restriction on λ , so one might hope that the \mathcal{D} -module even captures non-perturbative information. However, in a particular system under consideration some restrictions on the values of λ could arise that are related to the radius of convergence of the partition function. Although we do make some additional remarks in Chapter 6, we do not study these issues in this thesis.

On the other hand, the string scale α' does not play a fundamental role in the \mathcal{D} -module. The \mathcal{D} -module describes the topological sector of the intersecting brane configuration, which is realized in terms of massless modes of the I-brane system. Therefore the \mathcal{D} -module description is valid only in the regime where α' is small (so that no massive modes interfere with our description). The most interesting case is of course when it is non-zero, as it provides a normalization factor for the worldsheet instanton contributions to the open 4-6 strings in the I-brane partition function (4.53). Section 6.3 clarifies this with an example.

\mathcal{D} -modules and differential operators

The theory of \mathcal{D} -modules was introduced and developed, among others, by I. Bernstein, M. Kashiwara, T. Kawai and M. Sato, to study linear partial differential equations from an algebraic perspective [157, 158, 159, 160]. Currently this is a very active field, with connections and applications to many other branches of mathematics.

As already mentioned above, \mathcal{D} -modules are defined as modules for the algebra of differential operators \mathcal{D} . In general, in a local \mathbb{C}^n patch with complex coordinates (z_1,\ldots,z_n) , the operators z_i and ∂_{z_i} represent the nth Weyl algebra. The operators $P \in \mathcal{D}$ are of the form

$$P = \sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} \partial_{z_{i_1}} \cdots \partial_{z_{i_n}}.$$

With a set of operators $P_1, \dots, P_m \in \mathcal{D}$ one can associate a system of differential equations

$$P_1\Psi = \dots = P_m\Psi = 0, (5.13)$$

where Ψ takes values in some function space \mathcal{V} . An algebraic description of solutions to such a system can be given in terms of a \mathcal{D} -module \mathcal{M} determined by the ideal generated by $P_1, \ldots, P_m \in \mathcal{D}$

$$\mathcal{M} = \frac{\mathcal{D}}{\mathcal{D} \cdot \langle P_1, \dots, P_m \rangle}.$$
 (5.14)

The advantage of considering such a \mathcal{D} -module is, firstly, that it captures the solutions to the above system of differential equations independently of the form in which this system is written. Secondly, it is also independent of the function space \mathcal{V} – be it the space of square-integrable functions, the space of distributions, the space of holomorphic functions, etc.

Suppose that one would want to extract solutions Ψ from \mathcal{M} that take value in the function space \mathcal{V} . Such a space \mathcal{V} is itself a \mathcal{D} -module. Namely, \hat{x} and \hat{y} are realized as multiplication by x and the differential $-\lambda \partial_x$. One of the important properties of \mathcal{D} -modules is that the space of solutions to the system of differential equations (5.13) in \mathcal{V} is given by algebra homomorphism

$$\operatorname{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{V}).$$
 (5.15)

An important notion is a dimension of a \mathcal{D} -module. The so-called Bernstein inequality asserts that a non-zero \mathcal{D} -module \mathcal{M} over the n^{th} Weyl algebra has a dimension $2n \geq \dim \mathcal{M} \geq n$. In particular, \mathcal{D} considered itself as a \mathcal{D} -module has a dimension 2n. On the other hand, $\dim \mathbb{C}[x_1,\ldots,x_n]=n$. For a non-zero $P \in \mathcal{D}$, $\dim \mathcal{D}/\mathcal{D}P=2n-1$.

A special role in theory of \mathcal{D} -modules is played by the so-called holonomic \mathcal{D} -modules, which by definition have a minimal dimension n. In particular they are cyclic, which means they are determined by a single element $\Psi \in \mathcal{M}$ called a generator.

In the context of the I-brane in \mathbb{C}^2 we are just interested in the 1st Weyl algebra $\langle x,\lambda\partial_x\rangle$ of dimension 2. In this case we immediately conclude that the module $\mathcal{D}/\mathcal{D}P$ has a dimension n=1 for any non-zero P, and is thus holonomic and hence cyclic. It can be realized as

$$\mathcal{M} = \{ D\Psi : D \in \mathcal{D} \},\tag{5.16}$$

where the generator Ψ is a solution to the differential equation $P\Psi=0$. Such \mathcal{D} -modules reduce to curves or equivalently complex Lagrangians in the semi-classical limit, which is of course the main reason to study them in the context of I-branes.

Rank 1 example

Let us explain this structure in more detail with a simple example of a $\mathcal D$ -module. We start with the commuting case, in which the algebra $\mathcal A$ is given by the ring $\mathcal O$ of functions on the plane. If the spectral curve Σ is given by P=0, then we can write $\mathcal M=\mathcal O_\Sigma$ as the quotient

$$\mathcal{M} = \mathcal{O}/\mathcal{I}_{\Sigma},$$

where $\mathcal{I}_{\Sigma} = \mathcal{O} \cdot P$ is the ideal of functions vanishing on Σ .

Now suppose that P is not a polynomial, but a differential operator

$$P \in \mathcal{D}$$
.

Then we can similarly define a \mathcal{D} -module as an equivalence class of differential operators

$$\mathcal{M} = \frac{\mathcal{D}}{\mathcal{D} \cdot P}.$$

One way to think about such a module is in terms of a formal solution to the equation

$$P\Psi = 0$$
.

Mathematically, such a solution $\Psi \in \mathcal{V}$ is captured by the \mathcal{D} -module homomor-

phism (5.15)

$$\mathcal{M} = \frac{\mathcal{D}}{\mathcal{D} \cdot P} \rightarrow \mathcal{V},$$

Indeed, the map that sends the element

$$[1] \in \mathcal{M} \mapsto \Psi(z) \in \mathcal{V}$$

is well-defined because every element $P' \in \mathcal{D}P$ is mapped to zero (remember that Ψ fulfills $P\Psi=0$), and it is a bijection; conversely, any map \mathcal{M} to \mathcal{V} is determined by a holomorphic solution to the differential equation $P\Psi=0$. So \mathcal{M} can also be realized as the vector space of expressions of the form

$$\mathcal{M} = \{D\Psi; D \in \mathcal{D}\}.$$

The \mathcal{D} -module \mathcal{M} and the corresponding differential operator P should be considered as the non-commutative generalization of the classical curve Σ . This is the *quantum spectral curve* from the theory of quantization of integrable systems, as is known from the geometric Langlands perspective [156].

Within the context of string theory it is clear that there should be a *unique* \mathcal{D} -module that corresponds to the curve Σ . This prescription should fix possible normal ordering ambiguities in P. It would be interesting to understand this directly from the mathematical formalism.

D-modules and flat connections

More generally, \mathcal{D} -modules are defined as differential sheaves on any variety X. The sections of the sheaf \mathcal{D}_X over an open neighbourhood U are given by linear differential operators on U. Therefore, both the structure sheaf \mathcal{O}_X (of holomorphic functions) as well as the tangent sheaf T_X (whose local sections are vector fields) may be embedded in \mathcal{D}_X .

$$\mathcal{O}_{\mathcal{X}} \hookrightarrow \mathcal{D}_{\mathcal{X}} \hookleftarrow T_{\mathcal{X}}$$

In fact, \mathcal{D}_X is generated by these inclusions.

A sheaf \mathcal{M} on X is defined to be a left module for \mathcal{D}_X when $v \cdot s \in \mathcal{M}$, for any $v \in \mathcal{D}_X$ and $s \in \mathcal{M}$. Furthermore, it has to fulfill

$$v \cdot (fs) = v(f)s + f(v \cdot s)$$

$$[v, w] \cdot s = v \cdot (w \cdot s) - w \cdot (v \cdot s)$$

for any $v \in \mathcal{D}_X$, $f \in \mathcal{O}_X$ and $s \in \mathcal{M}$. Suppose that \mathcal{M} is a left \mathcal{D}_X -module whose sections are the local sections of some vector bundle V (this encomprises

all \mathcal{D}_X -modules that are finitely generated as \mathcal{O}_X -modules). Then the action of \mathcal{D}_X defines a connection on V as

$$\nabla_v(s) = v \cdot s,\tag{5.17}$$

whose curvature is zero. So a \mathcal{D} -module structure on the sheaf of sections of a vector bundle V defines a flat connection on this vector bundle. And conversely, any module consisting of sections of a vector bundle V with flat connection ∇_A , has an interpretation as a \mathcal{D} -module defined through the action of the flat connection. Therefore, a \mathcal{D} -module is in general just a system of linear differential equations, changing from patch to patch on X. This is known as a local system. In the main part of this paper X is just \mathbb{C} or \mathbb{C}^* .

Examples

1) Take a linear partial differential operator on \mathbb{C} , for example

$$P = \lambda z \partial_z - 1. \tag{5.18}$$

The differential equation $P\psi=0$ is solved by $\psi(z)=z^{1/\lambda}$, so according to (5.16) the corresponding \mathcal{D} -module can be represented as

$$\mathcal{M} = \langle z, \lambda \partial_z \rangle z^{1/\lambda}.$$

There are many equivalent ways of writing this module. For example, introducing $\widetilde{\Psi}=z\Psi$, the above differential equation is transformed into $\widetilde{P}\widetilde{\Psi}=0$ with

$$\widetilde{P} = \lambda z \partial_z - \lambda - 1,$$

This follows simply from the relation $(\lambda z \partial_z - \lambda - 1)x = x(\lambda z \partial_z - 1)$. This new operator, as well as the solution to the new equation $\widetilde{\Psi} = z^{1+1/\lambda}$ look different than before. Nonetheless, they represent the same \mathcal{D} -module

$$\mathcal{M} = \langle z, \lambda \partial_z \rangle z^{1+1/\lambda} = \langle z, \lambda \partial_z \rangle z^{1/\lambda}.$$

This simple example indeed shows that the formlism of \mathcal{D} -modules allows to study solutions to partial differential equations independently of the way in which the differential equation is written.

The flat connection corresponding to P is determined by the action of ∂_z on the elements of the \mathcal{D} -module, as in (5.17). It is given by

$$\nabla_A = \partial_z dz - \frac{1}{\lambda z} dz,$$

determining $\psi(z)$ as a local flat section.

2) All the modules that we will study in this paper are over \mathbb{C} or \mathbb{C}^* . It is important that they may be of any rank though. Let us therefore also give a rank two example on the complex plane \mathbb{C} . The second order differential equation

$$P\psi = (\lambda^2 \partial_z^2 - z)\psi \tag{5.19}$$

can be written equivalently as a rank two differential system

$$P_{ij}\psi_j=0, \quad \text{with } P_{ij}=\left(egin{array}{cc} \lambda\partial_z & 0 \\ 0 & \lambda\partial_z \end{array}
ight)-\left(egin{array}{cc} 0 & 1 \\ z & 0 \end{array}
ight).$$

Holomorphic solutions of this linear system are captured by the map

$$\mathcal{M} = rac{\mathcal{D}^{\oplus 2}}{\mathcal{D}^{\oplus 2}\,P_{ij}}
ightarrow \mathcal{O}_{\mathbb{C}}^{\oplus 2}$$

that sends the two generators $[(1,0)^t]$ and $[(0,1)^t]$ to two independent (2-vector) solutions of $P\psi = 0$. The corresponding flat connection reads

$$\nabla_A = \partial_z dz - \frac{1}{\lambda} \left(\begin{array}{cc} 0 & 1 \\ z & 0 \end{array} \right) dz$$

and turns the two solutions into a locally flat frame.

5.2.3 Quantum curve

The B-field quantizes the I-brane configuration into a \mathcal{D} -module. Last subsection we introduced these objects and saw that they represent solutions to a linear system of differential equations. However, the I-brane setup doesn't provide us with just any \mathcal{D} -module: this \mathcal{D} -module must represent a quantization of the curve Σ together with a meromorphic 1-form τ , obeying $d\tau = \omega$, on it. In this article we focus on smooth curves that are given by an equation of the form

$$\Sigma: \quad H(z,w) = w^n + u_{n-1}(z)w^{n-1} + \ldots + u_0(z) = 0, \tag{5.20}$$

where $z \in \mathbb{C}$ (or \mathbb{C}^*) and $w \in \mathbb{C}$. These play a prominent role in integrable systems as spectral curves. We will describe \mathcal{D} -modules corresponding to these curves.

Semi-classical geometry

The spectral curve Σ in (5.20) is a degree n cover over \mathbb{C} (or \mathbb{C}^*)

$$\begin{array}{ccc} \Sigma & \subset & T^*\mathbb{C} \\ \downarrow \pi & \\ \mathbb{C} \end{array}$$

with possible branch points (from now on we restrict to $z \in \mathbb{C}$ for simplicity in notation). The curve Σ is imbedded in \mathbb{C}^2 and equipped with the (meromorphic) 1-form

$$\tau = \frac{1}{\lambda} w dz|_{\Sigma}.$$

Furthermore, fermions on Σ transform as holomorphic sections of a line bundle $\mathcal{L}\otimes K^{1/2}_\Sigma$, provided by the D6-brane. The tuple (\mathcal{L},τ) on Σ pushes forward to a couple

$$\pi_*: (\mathcal{L}, \tau) \mapsto (V = \pi_* \mathcal{L}, \phi = \pi_* \tau) \tag{5.21}$$

on $\mathbb C$ under the projection map $\pi:\Sigma\to\mathbb C$. So V is a rank n vector bundle on $\mathbb C$, whereas $\phi\in H^0(\mathbb C,K_C\otimes\mathrm{End}V)$ is a meromorphic 1-form valued in gl(n). Such an object is called a Higgs field. It endows V with the structure of a Higgs bundle. Setting the characteristic polynomial

$$\det(\tau - \phi(z)) = 0$$

returns the equation for the spectral curve. The push-forward map π_* sets up a bijection between spectral data and (stable) Higgs pairs

$$(\Sigma, \mathcal{L}) \leftrightarrow (V, \phi).$$
 (5.22)

When the base $\mathbb C$ is a compact curve C instead, the moduli space of stable Higgs pairs forms an algebraically completely integrable system, the Hitchin integrable system. The Hitchin map

$$H^0(C, \operatorname{End} V \otimes K_C) \to \bigoplus_{i=1}^n H^0(C, K_C^i)$$

 $\phi \mapsto \det(w - \phi(z)),$

provides this moduli space with the structure of a Lagrangian fibration, whose fibers are generically Lagrangian tori. Any point in the image of the Hitchin map can be identified with a spectral curve Σ , whereas the fiber of the Hitchin map equals the moduli space of line bundles on Σ of a certain degree, which is iso-

morphic to the Jacobian $\operatorname{Jac}(\Sigma)$. This verifies that the Hitchin fiber is generically given by a torus. Multiplying $\mathcal L$ by a flat bundle on Σ defines a linear flow over the Hitchin fiber, exactly as in our discussion of the Krichever correspondence in Section 5.1.1.

Quantum geometry

The \mathcal{D} -modules generated by the B-field (5.9), as well as those considered in the last subsection, depend on λ (where λ is a formal variable $\lambda \in \mathbb{C}[[\lambda]]$). These are known as \mathcal{D}_{λ} -modules [161]. As a linear differential system such a \mathcal{D}_{λ} -module corresponds to a λ -connection ∇_{λ}

$$\nabla_{\lambda} = \lambda \partial_z - A(z). \tag{5.23}$$

that is defined through the Leibnitz rule $\nabla_{\lambda}(fs) = f\nabla_{\lambda}(s) + \lambda s \otimes df$ for any function f and section s. Since all the \mathcal{D} -modules and connections we consider are \mathcal{D}_{λ} and λ -connections, we often omit the subscript λ .

Semi-classically, a λ -connection ∇_{λ} reduces to a 1-form $\nabla_0(z)$ with values in gl(n)

$$\nabla_{\lambda} \mapsto \nabla_{0}, \quad (\lambda \to 0).$$
 (5.24)

We just encountered this object as a Higgs field ϕ . Moreover, we explained with (5.21) that a Higgs (V,ϕ) and spectral data (Σ,\mathcal{L}) provide equivalent information. In particular, the spectral curve can be recovered by the determinant of the Higgs field. This implies that λ -connections quantize spectral data.⁵

It tells us exactly which requirements a \mathcal{D} -module quantizing the I-brane configuration has to satisfy. Fermions on a degree n spectral curve have to transform under a rank n λ -connection ∇_{λ} on \mathbb{C} , whose semi-classical $\lambda \to 0$ limit is given by the Higgs field

$$\nabla_0 = \pi_*(\tau).$$

A simple example of a λ -connection is given by

$$\nabla = \lambda \partial_z - A(z),$$

with $A(z) = \pi_*(\tau)$. Its determinant is a degree n differential equation that canonically quantizes the defining equation for Σ .

⁵These λ -connections are also known as λ -opers, and play an important role in the quantum integrable system of Beilinson and Drinfeld [162, 163].

Examples

In the last subsection we gave two examples of \mathcal{D}_{λ} -modules: one of rank 1 and another one of rank 2. Let us examine them here in the light of deformation quantization.

1) In the first example we found the λ -connection

$$\nabla_{\lambda} = \lambda \partial_z - \frac{1}{z} \tag{5.25}$$

on the z-plane. The semi-classical spectral data is given by the degree 1 spectral cover

$$\Sigma: \quad w = \frac{1}{z},$$

with $z, w \in \mathbb{C}^*$, together with the (meromorphic) 1-form

$$A = \frac{1}{z}dz.$$

Equivalently, the \mathcal{D} -module \mathcal{M} corresponding to ∇_{λ} is generated by

$$\mathcal{M} = \langle z, \lambda \partial_z \rangle z^{1/\lambda},$$

which is clearly invariant under the algebra $\mathcal A$ of functions on Σ at $\lambda=0$. It is therefore a quantization of Σ . This example enters string theory as the deformed conifold geometry describing the c=1 string. It is described as a $\mathcal D$ -module in [1]. We will come back to it in Section 5.3.3.

2) In the second example we considered a \mathcal{D}_{λ} -module generated by a single second order generator

$$P = \lambda^2 \partial_z^2 - z. {(5.26)}$$

The corresponding rank 2 λ -connection

$$\nabla_{\lambda} = \left(\begin{array}{cc} \lambda \partial_z & 0 \\ 0 & \lambda \partial_z \end{array} \right) - \left(\begin{array}{cc} 0 & 1 \\ z & 0 \end{array} \right)$$

is a λ -deformation of the degree 2 spectral cover (illustrated in Fig. 5.4)

$$\Sigma: \ w^2 = z,$$

with meromorphic 1-form $au=wdz|_{\Sigma}.$ Note that this one-form pushes forward

to the connection 1-form, or Higgs field,

$$A = \left(\begin{array}{cc} 0 & 1\\ z & 0 \end{array}\right) dz$$

in the basis $\{dz,wdz\}$ of ramification 1-forms on the z-plane. Indeed, local sections of $\mathcal L$ push forward to local sections generated by 1 and w on the z-plane. Now, $wdz\cdot 1=wdz$ and $wdz\cdot w=w^2dz=zdz$ on Σ .

We discuss the string theory interpretation of this \mathcal{D} -module in Section 6.1.

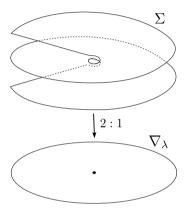


Figure 5.4: A second order differential operator P in $\lambda \partial_z$ defines a rank 2 λ -connection ∇_{λ} . The determinant of ∇_0 determines a degree 2 cover over $\mathbb C$ which is called the spectral curve Σ .

5.3 Fermionic states and quantum invariants

In the last section of this chapter we quantify the quantum integrable system that we described in the last section. Just like the Krichever map embeds a geometric set-up consisting of a line bundle over a curve into the KP Grassmannian, there is a straightforward way in which a quantum curve or \mathcal{D} -module \mathcal{M} gives rise to a solution of the KP-hierarchy. By definition \mathcal{M} carries an action of both x and ∂_x . However we are free to ignore the second action, which leaves us with the structure of an \mathcal{O} -module, \mathcal{O} being the algebra of functions in x. By applying the infinite-wedge construction to the \mathcal{O} -module \mathcal{W} we obtain in the usual way a state $|\mathcal{W}\rangle$ in the fermion Fock space. Roughly speaking, \mathcal{W} can be considered as the space of local sections that can be continued as sections of a (non-commutative) \mathcal{D} -module, instead of sections of a line bundle over a curve. We explain this in Section 5.3.1. In Section 5.3.2 and Section 5.3.3 we continue

to discuss how to assign quantum invariants to quantum curves. Also this is inspired on the KP-hierarchy.

5.3.1 Fermionic state

In this section we introduce an infinite dimensional Grassmannian and its description in terms of the second quantized fermion field (we learned this material e.g. from [151, 164, 165, 166, 167]). Later on we will extend this formalism in a way which associates fermionic states to \mathcal{D} -modules, thereby providing a fermionic description of the I-brane system. This technology is required as pre-knowledge to Chapter 6.

Grassmannian and second quantized fermions

The space $\mathcal{H}=\mathbb{C}((z^{-1}))$ of all formal Laurent series in z^{-1} can be given an interpretation of a Hilbert space. Basis vectors z^n , for $n\in\mathbb{Z}$, correspond to one particle states of energy n associated to the Hamiltonian $z\partial_z$. This Hilbert space has a decomposition

$$\mathcal{H} = \mathcal{H}_{+} \oplus \mathcal{H}_{-}$$

such that the first factor $\mathcal{H}_+ = \mathbb{C}[z]$ is a subspace generated by z^0, z^1, z^2, \ldots , while \mathcal{H}_- is generated by negative powers z^{-1}, z^{-2}, \ldots Consider now a subspace \mathcal{W} of \mathcal{H} with a basis $\{w_k(z)\}_{k\in\mathbb{N}}$. We say it is comparable to \mathcal{H}_+ , if in the projection onto positive and negative modes

$$w_k = \sum_{j>0} (w_+)_{ij} z^j + \sum_{j>0} (w_-)_{ij} z^{-j}$$

the matrix w_+ is invertible. The Grassmannian Gr_0 is the set of all subspaces $\mathcal{W} \subset \mathbb{C}((z^{-1}))$ which are comparable to \mathcal{H}_+ .

In what follows we take much advantage of the correspondence between Gr_0 and the charge zero sector of the second quantized fermion Fock space \mathcal{F}_0 . In this correspondence the subspace \mathcal{H}_+ is quantized as the Dirac vacuum

$$|0\rangle = z^0 \wedge z^1 \wedge z^2 \wedge \dots, \tag{5.27}$$

with all positive energy states filled. The fermionic state associated to the subspace W with basis $w_0(z)$, $w_1(z)$, $w_2(z)$, ... is represented by the semi-infinite wedge⁶

$$|\mathcal{W}\rangle = w_0 \wedge w_1 \wedge w_2 \wedge \dots \tag{5.28}$$

⁶Actually, we have to tensor with \sqrt{z} to make the state fermionic.

which is an element of the fiber of a determinant line bundle over the element $W \in Gr$ (and therefore determined up a complex scalar c)

To make contact with the usual formulation of the second quantized fermion Fock space, we can identify the differentiation and wedging operators with the fermionic modes

$$\psi_{n+\frac{1}{2}} = \frac{\partial}{\partial z^{-n}} \qquad \psi_{n+\frac{1}{2}}^* = z^n \wedge .$$

These half-integer modes are annihilation and creation operators which arise from a decomposition of the fermion field $\psi(z)$ and its conjugate $\psi^*(z)$

$$\psi(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r z^{-r - \frac{1}{2}} \qquad \psi^*(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r^* z^{-r - \frac{1}{2}}, \tag{5.29}$$

and they obey the anti-commutation relations $\{\psi_r, \psi_{-s}^*\} = \delta_{r,s}$.

For subspaces $W \in Gr_0$ the determinant of the projection onto \mathcal{H}_+ is well defined and can be expressed as

$$\det w_+ = \langle 0 | \mathcal{W} \rangle.$$

More generally, one can consider the Fock space $\mathcal F$ which splits into subspaces of charge p

$$\mathcal{F} = \bigoplus_{p \in \mathbb{Z}} \mathcal{F}_p.$$

Each subspace \mathcal{F}_p is built by acting with creation and annihilation operators on a vacuum

$$|p\rangle = z^p \wedge z^{p+1} \wedge z^{p+2} \wedge \dots,$$

with the property

$$\psi_r|p\rangle = 0$$
 for $r > p$,
 $\psi_r^*|p\rangle = 0$ for $r > -p$.

The Fermi level of the vacuum $|p\rangle$ is shifted by p units with respect to the Dirac vacuum $|0\rangle$. This fermion charge is measured by the U(1) current

$$J(z) =: \psi(z)\psi^*(z) := \sum_n \alpha_n z^{-n-1},$$

whose components $\alpha_n = \sum_k : \psi_r \psi_{n-r}^*$ satisfy the bosonic commutation rela-

tions

$$[\alpha_m, \alpha_{-n}] = m\delta_{m,n}.$$

With each subspace $\mathcal{W}\subset\mathbb{C}((z))$ comparable to the one generated by $(z^k)_{k\geq p}$ one can associate a state $|\mathcal{W}\rangle\in\mathcal{F}$ of charge p. This charge is equal to the index of the projection operator $pr_+:\mathcal{W}\to\mathcal{H}_+$.

A state in the Fock space \mathcal{F} has also a simple representation in terms of the so-called Maya diagram (see Fig. 5.5). Black boxes in such a diagram represent excitations, whereas white boxes are gaps in the energy spectrum of the fermion. The charge of a state is given by the number of excitations minus the number of gaps. Fermionic states or Maya diagrams of a fixed charge p can also be associated to two-dimensional partitions. In particular in p=0 sector the state

$$|R\rangle = \prod_{i=1}^{d} \psi_{-a_i - \frac{1}{2}}^* \psi_{-b_i - \frac{1}{2}} |0\rangle$$

corresponds to the partition $R = (R_1, \dots, R_l)$ such that

$$a_i = R_i - i, \qquad b_i = R_i^t - i.$$

In what follows a state corresponding to a partition R of charge p is denoted as $|p,R\rangle$.

Flow on the Grassmannian

Multiplying a basis vector $w_k(z)$ of \mathcal{W} by a power series $f(z) = \sum_{n>0} f_n z^n$, that vanishes at z=0, defines an action on the Grassmannian:

$$f(z)|\mathcal{W}\rangle = \sum_{k} w_0 \wedge \dots \omega_{k-1} \wedge f \cdot w_k \wedge w_{k+1} \dots$$

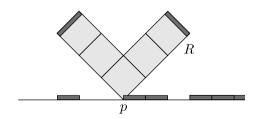


Figure 5.5: Elements of the Fock space \mathcal{F} are in a bijective correspondence with Maya diagrams. The bottom line represent a Maya diagram corresponding to a fermionic state with charge p. As illustrated it is characterized by a two-dimensional partitions R located at position p. We therefore denote the state as $|p,R\rangle \in \mathcal{F}$.

When we write $w_k(z)$ in terms of the basis $(z^l)_{l\in\mathbb{Z}}$ this action is encoded by the multiplication by an infinite matrix in gl_{∞} , whose $(i,j)^{th}$ entry is given by f_{i-j} . On the fermionic state $|\mathcal{W}\rangle$ a multiplication by z^n translates into a commutator with the bosonic mode α_n , since α_n increases the fermionic mode number by

$$[\alpha_n, \psi_r] = \psi_{r+n}.$$

Multiplication by a power series f(z) therefore translates to the operator

$$f = \sum_{n} f_n[\alpha_n, \bullet] \in gl_{\infty}.$$

on the Fock space.

Exponentiating the action of gl_{∞} yields the group Gl_{∞} . An element $g(z) = \exp(f(z))$ of this group acts on $|\mathcal{W}\rangle$ by multiplying all its basis vectors

$$g(z)|\mathcal{W}\rangle = g \cdot w_0 \wedge \ldots \wedge g \cdot w_k \wedge \ldots$$

From the fermionic point of view this action is given by conjugating each basis vector w_k with the element

$$g = \exp\left(\sum f_n \alpha_n\right) = \exp\left(\oint dz \ f(z)J(z)\right) \in Gl_{\infty}.$$

We call Γ the group of exponentials $g(z): S^1 \to \mathbb{C}^*$. An important subgroup of Γ is the group Γ_+ of functions $g_0: S^1 \to \mathbb{C}^*$ that extend holomorphically over the disk $D_0 = \{z: |z| \le 1\}$:

$$\Gamma_+ = \{g_0 : D_0 \to \mathbb{C}^* : g_0(0) = 1\}.$$

Another subgroup is the group Γ_- of functions $g_\infty: S^1 \to \mathbb{C}^*$ that extend over the disk $D_\infty = \{z \in \mathbb{C} \cup \{\infty\} : |z| \leq 1\}$:

$$\Gamma_{-} = \{g_{\infty} : D_{\infty} \to \mathbb{C}^* : g_{\infty}(\infty) = 1\}.$$

Any $g \in \Gamma$ can be written as an exponential $\exp(f)$. When $g \in \Gamma_+$ the function f vanishes at z = 0, and when $g \in \Gamma_-$ it vanishes at $z = \infty$.

 Γ_+ and Γ_- have different properties when acting on Grassmannian. The action of Γ_- is free, since any $\mathcal{W} \in Gr$ has only a finite number of excitations. On the contrary, Γ_+ acts trivially on a vacuum state $|p\rangle$. Although the action of the groups Γ_+ and Γ_- on a subspace \mathcal{W} is commutative, as it is just given by multiplication, as operators on the fermionic state $|\mathcal{W}\rangle$ it matters which element is applied first. This introduces normal ordering ambiguities.

An element

$$g(t,z) = \exp\left(\sum_{k\geq 1} t_k z^k\right) = \exp\left(f(t,z)\right) \in \Gamma_+,\tag{5.30}$$

defines a linear flow over the Grassmannian Gr. On the Fock space it acts as an evolution operator

$$U(t) = \exp\left(\oint \frac{dz}{2\pi i} f(t,z)J(z)\right).$$

The determinant $det(W)_+$ is not equivariant with respect to the action of Γ_+ . The difference is measured by the so-called tau-function

$$\tau_{\mathcal{W}}(g) = \frac{\det(g^{-1}w)_{+}}{g^{-1}\det w_{+}} = \frac{\langle 0|U(t)|\mathcal{W}\rangle}{g^{-1}\langle 0|\mathcal{W}\rangle},\tag{5.31}$$

which yields a holomorphic function $\tau:\Gamma_+\to\mathbb{C}$. This can be regarded as a wave function of $|\mathcal{W}\rangle$.

Blending

So far we considered the Hilbert space $\mathcal{H} \equiv \mathcal{H}^{(1)}$ of functions with values in \mathbb{C} . More generally, one can consider a Hilbert space $\mathcal{H}^{(n)}$ of functions with values in \mathbb{C}^n . Let $(\epsilon_i)_{i=1,\dots,n}$ denote a basis of \mathbb{C}^n . For each n there is an isomorphism between $\mathcal{H}^{(n)}$ and \mathcal{H} given by the lexicographical identification of the basis

$$\epsilon_i z^k \mapsto z^{nk+i-1}$$
.

This isomorphism is called blending.

In the fermionic language the Hilbert space $\mathcal{H}^{(n)}$ lifts to the Fock space of n fermions $\psi^{(i)}$, $i=1,\ldots,n$, each one with the expansion (5.29) and such that

$$\{\psi_r^{(i)}, \psi_s^{*(j)}\} = \delta_{i,j}\delta_{r,-s}.$$

Now blending translates to the following redefinitions of these n fermions into a single fermion ψ

$$\psi_{n(r+\rho_i)} = \psi_r^{(i)}, \qquad \psi_{n(r-\rho_i)}^* = \psi_r^{*(i)},$$

where

$$\rho_i = \frac{2i - n - 1}{2n}. ag{5.32}$$

Blending can also be expressed in terms of two-dimensional partitions introduced above. Consider n partitions $R_{(i)}$ of charges p_i , with $\sum_i p_i = p$, corre-

sponding to states in n independent Hilbert spaces of fermions $\psi^{(i)}$. Associating with each such partition a state of a chiral fermion $|p_i,R_{(i)}\rangle$, we have a decomposition

$$|p, \mathbf{R}\rangle = \bigotimes_{i=1}^{n} |p_i, R_{(i)}\rangle,$$

and the blended partition \mathbf{R} of charge p, corresponding to a state in the Hilbert space of the blended fermion Ψ , is defined as

$$\{n(p_i + R_{(i),m} - m) + i - 1 \mid m \in \mathbb{N}\} = \{p + \mathbf{R}_K - K \mid K \in \mathbb{N}\}.$$
 (5.33)

A simple example: rational curves

Let us illustrate how this formalism can be used to describe quantum curves. Take a very simple example: the curve y=0. Topologically $\Sigma=\mathbb{C}$ can be viewed as a disc and in the fermion CFT it will correspond to a state in the Fock space \mathcal{F} . In this case the corresponding \mathcal{D} -module just consists of the polynomials in x

$$\mathcal{M} = \mathbb{C}[x]$$

and \hat{y} is realized as $-\lambda \partial_x$. The one-form ydx vanishes identically. The free fermion theory based on this module consists of the usual vacuum state $|0\rangle$.

But now we can make a small variation, by picking the curve

$$y = p(x),$$

with p(x) some function. In this case the (meromorphic) one-form is p(x)dx. The corresponding \mathcal{D} -module is still isomorphic to $\mathbb{C}[x]$ as a vector space, but now the operator \hat{y} is represented as

$$\hat{y} = -\lambda \partial_x + p(x).$$

Of course, there is an obvious map between these two modules: we simply multiply the functions $\psi(x) \in \mathbb{C}[x]$ as

$$\psi(x) \to e^{-S(x)/\lambda} \psi(x), \qquad \partial S(x) = p(x) dx.$$

In the quantum field theory, where $\psi(x)$ becomes an operator acting on the Fock space \mathcal{F} , this correspondence is represented by a linear map U such that

$$U(t) \cdot \psi(x) \cdot U(t)^{-1} = e^{-S(x)/\lambda} \psi(x).$$

If $S(x) = \sum_{k} t_k x^k$ such a map is given by

$$U(t) = \exp \sum_{k} \frac{1}{\lambda} t_k \alpha_k$$

where $\sum_k \alpha_k x^{-k-1} = \partial \phi(x) =: \psi^{\dagger} \psi(x)$: is the usual mode expansion of the U(1) current. Here we use that $[\alpha_k, \psi(x)] = x^k \psi(x)$. The corresponding state in the Sato Grassmannian is given by $U(t)|0\rangle$. Since $\alpha_k|0\rangle = 0$ for $k \geq 0$, this state is only different from the vacuum if the function S(x) (or p(x)) has poles.

5.3.2 Second quantizing \mathcal{D} -modules

Finally, we have all the ingredients to associate a fermionic state to a given \mathcal{D} -module on \mathbb{C} . The \mathcal{D} -module \mathcal{M} encodes holomorphic solutions to a system of differential equations. In particular it is an $\mathcal{O}_{\mathbb{C}}$ -module, and forms a subspace of $\mathbb{C}((z^{-1}))$ that we will name \mathcal{W} . Second quantization turns this subspace into a fermionic state $|\mathcal{W}\rangle$. Since the I-brane configuration in fact provides a \mathcal{D}_{λ} -module (in contrast to a \mathcal{D} -module) the resulting I-brane fermionic state $|\mathcal{W}\rangle$ is a λ -deformation as well.

In Section 5.2.3 we learned that a \mathcal{D}_{λ} -module on \mathbb{C} when $\lambda \to 0$ reduces to a spectral tuple (Σ, τ) . We will argue here that the fermionic state $|\mathcal{W}\rangle$, associated to the \mathcal{D} -module \mathcal{W} , reduces to the semi-classical Krichever state corresponding to this spectral data in the same limit. The tau-function (5.31) corresponding to this Krichever state computes the determinant of the $\overline{\partial}$ -operator, which is just the genus one part \mathcal{F}_1 of the all-genus free energy. The \mathcal{D}_{λ} -module gives a λ -deformation of this solution that computes the full I-brane partition function (4.53).

In this section we motivate and explain the second quantization of the \mathcal{D}_{λ} -module. But first we summarize the semi-classical correspondence.

Semi-classical state

Let us remind ourselves shortly how we associate a semi-classical fermionic state to a pointed curve $\Sigma-z_{\infty}$, together with a line bundle \mathcal{L} . As we explained in Section 5.1 the traditional way [151] is to form the space of global holomorphic sections of \mathcal{L} on $\Sigma-z_{\infty}$

$$\mathcal{W} = H^0(\Sigma - z_{\infty}, \mathcal{L}).$$

Recall that W is a an A-module for

$$\mathcal{A} = H^0(\Sigma - z_{\infty}, \mathcal{O}_{\Sigma}).$$

Tensoring the line bundle \mathcal{L} with a square root $K_{\Sigma}^{1/2}$ turns sections of \mathcal{L} into fermionic sections: \mathcal{W} thus corresponds to a subspace that is being swept out in time by a free fermion field $\psi(z)$ living on $\Sigma - z_{\infty}$.

When we denote the semi-infinite set of generators W by $w_k(z)$, a second quantization turns the subspace W into the fermionic state

$$|\mathcal{W}\rangle = w_0 \wedge w_1 \wedge w_2 \wedge \dots$$

The map that assigns a subspace W to the couple (Σ, \mathcal{L}) is the inverse of the Krichever map. It yields a geometric solution in the Grassmannian Gr.

When π is an (n:1) projection of Σ onto some other curve C, while \mathcal{L} pushes forward to a rank n vector bundle $V = \pi_* \mathcal{L}$ over C, it is equivalent to look at the subspace of global sections of V on $C - \pi(z_\infty)$. This yields a fermionic state in the n-component Grassmannian Gr_n [168, 169]. As we described in the previous subsection blending (or the lexiographical ordening) recovers the state $|\mathcal{W}\rangle$ that is part of Gr_1 .

The flow over the Grassmannian, generated by Γ_+ , obtains a geometric interpretation as a linear flow over the Jacobian of Σ , and therefore relates such geometric solutions to the Hitchin integrable system in Section 5.2.3 [170].

The tau-function associated to a fermionic state $|\mathcal{W}\rangle$, that is found as a Krichever solution, equals the determinant of the $\overline{\partial}$ -operator on Σ , and thus computes the partition function of free fermions on Σ . This is exactly the contribution to the free energy that we expect from the I-brane set-up when $\lambda \to 0$, so that commutativity is restored.

For all spectral curves in \mathbb{C}^2 or $\mathbb{C}^* \times \mathbb{C}$ that we consider later in this thesis, the line bundle \mathcal{L} is almost trivial, so that the choice of trivialization of \mathcal{L} at z_{∞} is the only non-trivial piece of data. Picking such a frame is equivalent to choosing a flat connection on $\Sigma - z_{\infty}$. This corresponds to an element of Γ_+ . However, we noticed in the previous subsection that this acts trivially on the fermionic state. So the Krichever map just yields the Dirac vacuum.

Rank 1 quantum state

To find a non-trivial fermionic state $|\mathcal{W}\rangle$, as an expansion in λ , we start with a \mathcal{D}_{λ} -module. Let us first explain the rank 1 case, with a \mathcal{D}_{λ} -module on \mathbb{C} specified by the (meromorphic) connection

$$\nabla_A = \partial_z - \frac{1}{\lambda} A(z)$$

that may be trivialized as

$$\nabla_A = \partial_z - g_\lambda(z)^{-1} (\partial_z g_\lambda(z)).$$

When $g_{\lambda}(z) \in \Gamma_+$ this represents a pure gauge transformation on the disk (so that ∇_A corresponds to a regular flat connection on \mathbb{C}).

For any $g_{\lambda}(z)$ a fermionic section $\psi(z)$ of $\mathcal{L} \otimes K^{1/2}$ may be written as

$$\psi(z) = g_{\lambda}(z)\xi(z),$$

where $\xi(z)$ is a section of $\mathcal{L} \otimes K^{1/2}$ with trivial connection ∂_z . Flat sections $\Psi(z)$ are defined by the differential equation

$$\left(\partial_z - \frac{1}{\lambda}A(z)\right)\Psi(z) = 0.$$

They define a local trivialization of the bundle \mathcal{L} with connection ∇_A , and we will use them to translate the geometric configuration into a quantum state.

A flat section for the trivial connection ∂_z is given by $\Xi(z)=1$. We associate the tuple (\mathbb{C},∂_z) to the ground state

$$|0\rangle = z^0 \wedge z^1 \wedge z^2 \wedge \dots$$

The gauge transformation $g_{\lambda}(z)$ maps the trivial solution $\Xi(z)=1$ to $\Psi(z)=g_{\lambda}(z)$, which transforms the vacuum into the fermionic state

$$|\mathcal{W}\rangle = g_{\lambda}|0\rangle,$$

where g_{λ} acts as a Gl_{∞} transformation on the fermionic modes

$$|\mathcal{W}\rangle = g_{\lambda}(z)z^0 \wedge g_{\lambda}(z)z^1 \wedge g_{\lambda}(z)z^2 \wedge \dots$$

as we explained in the last subsection.

In other words, we build the quantum state by acting with the \mathcal{D} -module generator $\Psi(z)=g_{\lambda}(z)$ on the vacuum

$$\mathcal{W} = \mathcal{D}_{\lambda} \cdot \Psi(z).$$

The state $|\mathcal{W}\rangle$ is just second quantization of the the \mathcal{D}_{λ} -module \mathcal{W} . Notice that this state is non-trivial only when $g_{\lambda}(z)$ is not a pure gauge transformation (which would correspond to a Krichever solution). This implies that the flat sections will diverge near z=0, corresponding to a distorted geometry in this region.

Rank n quantum state

A degree n spectral curve Σ is quantized as a λ -connection of rank n. This is equivalent to a \mathcal{D}_{λ} -module \mathcal{M} that is generated by a single degree n differential operator P. As an $\mathcal{O}_{\mathbb{C}}$ -module, though, \mathcal{M} is generated by an n-tuple

$$(\Psi(z), \partial_z \Psi(z), \dots, \partial_z^{n-1} \Psi(z)),$$

where $\Psi(z)$ is a solution of the differential equation $P\Psi=0$. In other words, this blends an n-vector of solutions to the linear differential system that the λ -connection defines. We will name this $\mathcal{O}_{\mathbb{C}}$ -module

$$\mathcal{W} = \mathcal{O}_{\mathbb{C}} \cdot (\Psi(z), \partial_z \Psi(z), \dots, \partial_z^{n-1} \Psi(z)) \subset \mathbb{C}((z^{-1})).$$

(of course it contains the same elements as \mathcal{M}) This is the subspace we want to second quantize into a fermionic state $|\mathcal{W}\rangle$.

Now P has n independent solutions Ψ_i , that differ in their behaviour at infinity. These solutions have an asymptotic expansion around $z=\infty$ that contains a WKB-piece plus an asymptotic expansion in λ , and should thus be interpreted as perturbative solutions that live on the spectral cover. We suggest that the asymptotic expansion of any solution can be turned into a fermionic state that captures the all-genus I-brane partition function. This partition function thus depends on the choice of boundary conditions near z_∞ .

Some of the WKB-factors will be exponentially suppressed near z_{∞} , while others exponentially grow. This depends on the specific region in this neighbourhood. The lines that characterize the changing behaviour of the solutions Ψ_i are called Stokes and anti-Stokes rays. Boundary conditions at infinity specify the solution up to a Stokes matrix: a solution that decays in that region can be added at no cost.

This implies that the perturbative fermionic state we assign to a \mathcal{D} -module depends on the choice of boundary conditions. On the other hand, the \mathcal{D} -module itself is independent of any of these choices and thus in some sense contains non-perturbative information and goes beyond the all-genus I-brane partition function. This agrees with the discussion in [171]. Nonentheless, the focus in this paper is on the perturbative information a \mathcal{D} -module provides.

5.3.3 Quantum invariants and λ -deformed CFT's

Up to here we have explained how a spectral curve may be quantized into a \mathcal{D}_{λ} -module, and how this translates into a fermionic state $|\mathcal{W}\rangle$. Most importantly, this allows us to associate a quantum invariant to a spectral curve.

Spectral curves embedded in \mathbb{C}^2 are characterized by a single region near infin-

ity. The all-genus I-brane partition function is (like in the semiclassical Krichever setting) given by the determinant of $|\mathcal{W}\rangle$. When Σ is part of $\mathbb{C}^* \times \mathbb{C}$, it contains two regions at infinity. In this situation the I-brane partition function is computed as a correlation function, by contracting two fermionic states.

Likewise, we expect that it is important for any I-brane curve to consider all regions where the curve approaches infinity. In such a patch the B-field becomes singular so that the non-commutativity parameter λ tends to zero. Hence we can associate a perturbative fermionic state $|\mathcal{W}\rangle$ to this neighbourhood, once a choice of local symplectic coordinates (z,w) is made. The result is a quantized module that is invariant under the action of the Weyl algebra \mathcal{D}_{λ} . Any other choice of local coordinates amounts to a combination of the following two transformations

$$z\mapsto z+f(w) \quad \text{and} \quad w\mapsto w,$$
 $z\mapsto w \quad \text{and} \quad w\mapsto -z.$ (5.34)

These generate automorphisms of the Weyl algebra, but they change the \mathcal{D}_{λ} -module associated to the asymptotic neighbourhood.

Moving from one patch at infinity to another changes the canonical coordinates by such a complex symplectic transformation as well. Correspondingly the \mathcal{D}_{λ} -module, and thus the quantum fermion field, transforms in the metaplectic representation [128]. Indeed, in the $\lambda \to 0$ limit one recovers a semi-classical fermionic section of $K^{1/2}$.

This suggests that the complete I-brane partition function may be found by writing down the correct fermionic state for each point at infinity and then glueing them by using a symplectic identification of coordinates. Although the \mathcal{D}_{λ} -modules associated to the regions near infinity depend on the choice of local coordinates, this quantum invariant should be independent of the chosen parametrizations. This is a claim that we cannot yet justify, except that from the philosophy of string theory there should be a unique such quantum invariant. A simple supporting example is found in the Chapter 6.

One of the motivations of Chapter 6 is to see in practise how quantum curves lead to I-brane partition functions. We study several well-known examples of spectral curves in string theory, and determine the \mathcal{D}_{λ} -module that underlies their partition function. The first set of examples treats matrix model spectral curves embedded in \mathbb{C}^2 with just one region at infinity. In the second set we study Seiberg-Witten geometries embedded in $\mathbb{C}^* \times \mathbb{C}$.

Chapter 6

Quantum Curves in Matrix Models and Gauge Theory

This chapter illustrates the \mathcal{D} -module formalism of Chapter 5 from a string theory perspective, with examples from the theory of random matrices, minimal (non-critical) string theory, supersymmetric gauge theory and topological strings. As a result we connect these familiar ingredients in a common framework centered around \mathcal{D} -modules. String theory provides solutions to integrable hierarchies of the KP type. This was first noted in the context of non-critical ($c \neq 26$) bosonic string theory, which has a dual formulation in terms of Hermitean random matrices. The matrix model partition function

$$Z_{\rm mm}(\lambda) = \frac{1}{{\rm vol}(U(N))} \int DM \ e^{-\frac{1}{\lambda} {\rm Tr} \ W(M)} \tag{6.1}$$

is known to be a tau-function of the KP integrable system. Although an algebraic curve Σ emerges in the limit that the size N of the square matrix M tends to infinity, these matrix model solutions do not correspond to geometric Krichever solutions. In particular, the relevant Fock space state $|\mathcal{W}\rangle$ does not have a purely geometric interpretation as being swept out by regular free fermions living on the matrix model spectral curve Σ .

The matrix model partition function admits a formal expansion

$$Z_{\mathrm{mm}}(\lambda) = \exp \sum_{q} \lambda^{2g-2} \mathcal{F}_{g}$$

in the string coupling constant λ . In the 't Hooft limit N is sent to infinity while the product of N with λ is held fixed, so that the geometric curve Σ equivalently emerges in the classical limit $\lambda \to 0$. This suggests that λ should be

interpreted as some form of non-commutative deformation of the underlying algebraic curve. In fact, there have been many indications that this is the right point of view.

In the simplest matrix models Σ appears as an affine rational curve given by a relation of the form

$$H(x,y) = 0$$

in the complex two-plane \mathbb{C}^2 , with a (local) parametrization x=p(z) and y=q(z), with p,q polynomials. Of course, p and q commute: [p,q]=0. However, the string-type solutions with $\lambda \neq 0$ are characterized by quantities P and Q that no longer commute but instead satisfy the canonical commutation relation

$$[P,Q] = \lambda.$$

In this case clearly P, Q cannot be polynomials, but are represented as differential operators, *i.*e. polynomials in z and ∂_z .

As we will point out in Section 6.1 these solutions are naturally captured by a \mathcal{D} -module. Instead of classical curve in the (x,y)-plane, we should think of a quantum curve as its analogue in the non-commutative plane $[x,y]=\lambda$. If we interpret

$$y = -\lambda \frac{\partial}{\partial x},$$

one can identify such a quantum curve as a holonomic \mathcal{D} -module \mathcal{W} for the algebra \mathcal{D} of differential operators in x. Roughly speaking, \mathcal{W} can be considered as the space of local sections that can be continued as sections of a (noncommutative) \mathcal{D} -module, instead of sections of a line bundle over a curve.

One other important instance of integrable hierarchies in string theory is in four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories. In Chapter 4 we have seen that the low energy effective description of Seiberg-Witten theories is determined by a twice-punctured algebraic curve Σ_{SW} , defined by an equation of the form

$$H(t,v) = 0$$

with $t \in \mathbb{C}^*$ and $v \in \mathbb{C}$, that appears as a spectral curve of a Hitchin integrable system. In Chapter 4 we geometrically engineered Seiberg-Witten theory as a Calabi-Yau compactification and encountered additional gravitational corrections \mathcal{F}_q to the effective action.

Like in the matrix model setting these \mathcal{F}_g -terms are multiplied by some power of the string coupling constant λ , suggesting that the full genus free energy $\mathcal{F} = \sum_g \lambda^{2g-2} \mathcal{F}_g$ has an interpretation in terms of a quantum Seiberg-Witten curve. This motivates us to quantize the Seiberg-Witten surface Σ_{SW} in Section 6.3.

Again, we see that a \mathcal{D} -module underlies the structure of the total partition function.

6.1 Matrix model geometries

Hermitian one-matrix models with an algebraic potential

$$W(M) = \sum_{j=0}^{d+1} u_j M^j$$

are defined through the matrix integral (6.1). In the large N limit the distribution of the eigenvalues λ_i of M on the real axis becomes continuous and defines a hyperelliptic curve

$$\Sigma_{\rm mm}: \quad y^2 - W'(x)^2 + f(x) = 0,$$
 (6.2)

called the (matrix model) spectral curve. The polynomial $f(x)=4\mu\sum_{j=0}^{d-1}b_jx^j$ is determined as

$$f(x) = \frac{4\mu}{N} \sum_{i=1}^{N} \frac{W'(x) - W'(\lambda_i)}{x - \lambda_i},$$

with $\mu=N\lambda$. The potential W(x) determines the positions of the cuts of the hyperelliptic curve, and contains the non-normalizable moduli. On the other hand, the size of the cuts is determined by the polynomial f(x), that comprises the normalizable moduli b_0,\ldots,b_{d-2} and the log-normalizable modulus b_{d-1} .

R. Dijkgraaf and C. Vafa discovered that this matrix model has a dual description in string theory. In the 't Hooft limit $N\to\infty$ (with μ fixed) it is equivalent to the topological B-model on a Calabi-Yau geometry X_Σ modeled on the matrix model spectral curve $\Sigma_{\rm mm}$ [123, 124, 125]. A good review is [172]. This duality may be generalized by starting with multi-matrix models, whose spectral curve is a generic (in contrast to hyperelliptic) algebraic curve in the variables x and y.

The I-brane picture suggests that the full B-model partition function on these Calabi-Yau geometries can be understood in terms of \mathcal{D} -modules. Even better, we will find that finite N matrix models are determined by an underlying \mathcal{D} -module structure.

In the past, as well as recently, Hermitean matrix models have been studied in great detail in many contexts. Already in [173, 174] an attempt has been made to understand the string equation $[P,Q]=\lambda$ in terms of a quantum curve in

terms of the expansion in the parameter λ . In G. Moore's approach this surface seemed to emerge from an interpretation of the string equation as isomonodromy equations.

More recently, quantum curves have appeared in the description of branes in a dual string model. In topological string theory as well as non-critical string theory a dominant role is played by holomorphic branes: either topological B-branes [128] or FZZT branes [175, 176, 177, 171]. Their moduli space equals the spectral curve, whereas the branes themselves may be interpreted as fermions on the quantized spectral curve. As was reviewed in Chapter 4, in these theories it is possible to compute correlation functions using a $W_{1+\infty}$ -algebra that implements complex symplectomorphisms of the complex plane $\mathcal B$ – as in (5.34) – in quantum theory as Ward identities [128, 178, 179].

These advances strongly hint at a fundamental appearance of \mathcal{D} -modules in the theory of matrix models. Indeed, this section unifies recent developments in matrix models in the framework of Chapter 5. Firstly, after a self-contained introduction in double scaled models we uncover the \mathcal{D} -module underlying the double-scaled (p,1)-models. In the second part of this section we shift our focus to general Hermitian multi-matrix models, and unravel their \mathcal{D} -module structure.

6.1.1 Double scaled matrix models and the KdV hierarchy

Our first goal is to find the \mathcal{D} -modules that explain the quantum structure of double scaled Hermitean matrix models. This double scaling limit is a large N limit in which one also fine-tunes the parameters to find the right critical behaviour of the multi-matrix model potential. Geometrically the double scaling limit zooms in on some branch points of the spectral curve that move close together. Spectral curves of double scaled matrix models are therefore of genus zero and parametrized as

$$\Sigma_{p,q}: \quad y^p + x^q + \ldots = 0.$$

The one-matrix model only generates hyperelliptic spectral curves, whereas the two-matrix model includes all possible combinations of p and q. These double scaled multi-matrix models are known to describe non-critical (c<1) bosonic string theory based on the (p,q) minimal model coupled to two-dimensional gravity [180, 181, 182, 183, 184, 185] (reviewed extensively in e.g. [186, 187]). This field is called minimal string theory.

Zooming in on a single branch point yields the geometry

$$\Sigma_{p,1}: \quad y^p = x,$$

corresponding to the (p,1) topological minimal model. This model is strictly not a well-defined conformal field theory, but does make sense as 2d topological field theory. For p=2 it is known as topological gravity [188, 189, 190, 165].

All (p,q) minimal models turn out to be governed by two differential operators

$$P = (\lambda \partial_x)^p + u_{p-2}(x)(\lambda \partial_x)^{p-2} + \dots + u_0(x),$$

$$Q = (\lambda \partial_x)^q + v_{q-2}(x)(\lambda \partial_x)^{q-2} + \dots + v_0(x),$$

of degree p and q respectively, which obey the string (or Douglas) equation

$$[P,Q] = \lambda.$$

P and Q depend on an infinite set of times $t = (t_1, t_2, t_3, \ldots)$, which are closed string couplings in minimal string theory, and evolve in these times as

$$\begin{split} &\lambda \frac{\partial}{\partial t_j} P = [(P^{j/p})_+, P], \\ &\lambda \frac{\partial}{\partial t_j} Q = [(P^{j/p})_+, Q], \end{split}$$

The fractional powers of P define a basis of commuting Hamiltonians. This integrable system defines the p-th KdV hierarchy and the above evolution equations are the KdV flows.

The differential operator Q is completely determined as a function of fractional powers of the Lax operator P and the times t

$$Q = -\sum_{\substack{j \ge 1 \\ p \ne 0 \mod p}} \left(1 + \frac{j}{p}\right) t_{j+p} P_+^{j/p},$$

This implies that when we turn off all the KdV times except for $t_1 = x$ and fix t_{p+1} to be constant we find $Q = \lambda \partial_x$. This defines the (p,1)-models

$$P = (\lambda \partial_x)^p - x, \quad Q = \lambda \partial_x. \tag{6.3}$$

One can reach any other (p,q) model by flowing in the times t.

The partition function of the p-th KdV hierarchy is a tau-function as in equation (5.31). The associated subspace $\mathcal{W} \in Gr$ may be found by studying the eigenfunctions $\psi(t,z)$ of the Lax operator P

$$P\psi(t,z) = z^p \psi(t,z).$$

The Baker function $\psi_{\lambda}(t,z)$ represents the fermionic field that sweeps out the subspace \mathcal{W} in the times t.

If we restrict to the (p,1)-models the Baker function $\psi(x,z)$ can be expanded in a Taylor series

$$\psi(x,z) = \sum_{k=0}^{\infty} v_k(z) \frac{x^k}{k!}.$$

Since $\psi(x,z)$ is an element of \mathcal{W} for all times, this defines a basis $\{v_k(z)\}_{k\geq 0}$ of the subspace \mathcal{W} . In fact, it is not hard to see that the (p,1) Baker function is given by the generalized Airy function

$$\psi(x,z) = e^{\frac{pz^{p+1}}{(p+1)\lambda}} \sqrt{z^{p-1}} \int dw \ e^{\frac{(-1)^{1/p+1}(x+z^p)w}{\lambda^{p/p+1}} + \frac{w^{p+1}}{p+1}}, \tag{6.4}$$

which is normalized such that its Taylor components $v_k(z)$ can be expanded as

$$v_k(z) = z^k (1 + \mathcal{O}(\lambda/z^{p+1})).$$

The (p, 1) model thus determines the fermionic state

$$|\mathcal{W}\rangle = v_0 \wedge v_1 \wedge v_2 \wedge \dots, \tag{6.5}$$

where the $v_k(z)$ can be written explicitly in terms of Airy-like integrals (see [165] for a nice review). The invariance under

$$z^p \cdot \mathcal{W} \subset \mathcal{W}$$

characterizes this state as coming from a p-th KdV hierarchy. In the other direction, the state $|\mathcal{W}\rangle$ determines the Baker function (and thus the Lax operator) as the one-point function

$$\psi(t,z) = \langle t | \psi(z) | \mathcal{W} \rangle.$$

In the dispersionless limit $\lambda \to 0$ the derivative $\lambda \partial_x$ is replaced by a variable d, and the Dirac commutators by Poisson brackets in x and d. The leading order contribution to the string equation is given by the Poisson bracket

$$\{P_0, Q_0\} = 1,$$

where P_0 and Q_0 equal P and Q at $\lambda = 0$. The solution to this equation is

$$P_0(d;t) = x$$
$$Q_0(d;t) = y(x;t)$$

and recovers the genus zero spectral curve $\Sigma_{p,q}$ of the double scaled matrix model, parametrized by d. The KdV flows deform this surface such a way that

its singularities are preserved. (See the appendix of [171] for a detailed discussion.)

Note that $\Sigma_{p,q}$ is not a spectral curve for the Krichever map. The Krichever curve is instead found as the space of simultaneous eigenvalues of the differential operators

$$[P,Q] = 0,$$

that is *preserved* by the KdV flow as a straight-line flow along its Jacobian. In fact, there is no such Krichever spectral curve corresponding to the doubled scaled matrix model solutions.

Wrapping an I-brane around $\Sigma_{p,q}$ quantizes the semi-classical fermions on the spectral curve $\Sigma_{p,q}$. The only point at infinity on $\Sigma_{p,q}$ is given by $x\to\infty$. The KdV tau-function should thus be the fermionic determinant of the quantum state $|\mathcal{W}\rangle$ that corresponds to this \mathcal{D} -module. In the next subsection we write down the \mathcal{D} -module describing the (p,1) model and show precisely how this reproduces the tau-function using the formalism developed in Chapter 5.

6.1.2 \mathcal{D} -module for topological gravity

We are ready to reconstruct the \mathcal{D} -module that yields the fermionic state $|\mathcal{W}\rangle$ in equation (6.5). For simplicity we study the (2,1)-model, associated to an I-brane wrapping the curve

$$\Sigma_{(2,1)}: \qquad y^2 = x \qquad \text{with } x, y \in \mathbb{C}.$$

Notice that this is an 2:1 cover over the x-plane. It contains just one asymptotic region, where $x\to\infty$. Fermions on this cover will therefore sweep out a subspace $\mathcal W$ in the Hilbert space

$$\mathcal{W} \subset \mathcal{H}(S^1) = \mathbb{C}((y^{-1})),$$

the space of formal Laurent series in y^{-1} . The fermionic vacuum $|0\rangle\subset \mathcal{H}(S^1)$ corresponds to the subspace

$$|0\rangle = y^{1/2} \wedge y^{3/2} \wedge y^{5/2} \wedge \dots,$$

which encodes the algebra of functions on the disk parametrized by y and with boundary at $y=\infty$. Exponentials in y^{-1} represent non-trivial behaviour near the origin and therefore act non-trivially on the vacuum state. In contrast, exponentials in y are holomorphic on the disk and thus act trivially on the vacuum.

The $B\text{-field }B=\frac{1}{\lambda}dx\wedge dy$ quantizes the algebra of functions on \mathbb{C}^2 into the

differential algebra

$$\mathcal{D}_{\lambda} = \langle x, \lambda \partial_x \rangle.$$

Furthermore, it introduces a holomorphic connection 1-form $A=\frac{1}{\lambda}ydx$ on $\Sigma_{(2,1)}$, which pushes forward to the rank two λ -connection

$$\nabla_A = \lambda \partial_x - \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \tag{6.6}$$

on the base \mathbb{C} , parametrized by x. We claim that the corresponding \mathcal{D}_{λ} -module \mathcal{M} , generated by

$$P = (\lambda \partial_x)^2 - x,$$

describes the (2,1) model. Let us verify this.

Trivializing the λ -connection ∇_A in (6.6) implies finding a rank two matrix g(x) such that

$$\nabla_A = \lambda \partial_x - g'(x) \circ g^{-1}(x).$$

The columns of g define a basis of solutions $\Psi(x)$ to the differential equation $\nabla_A \Psi(x) = 0$. They are meromorphic flat sections for ∇_A that determine a trivialization of the bundle near $x = \infty$. As the connection ∇_A is pushed forward from the cover, $\Psi(x)$ is of the form

$$\Psi(x) = \left(\begin{array}{c} \psi(x) \\ \psi'(x) \end{array}\right).$$

Independent solutions have different asymptotics in the semi-classical regime where $x\to\infty$. In the (2,1)-model the two independent solutions $\psi_\pm(x)$ solve the differential equation

$$P\psi_{\pm}(x) = ((\lambda \partial_x)^2 - x)\psi_{\pm}(x) = 0.$$

Hence these are the functions $\psi_+(x)={\rm Ai}(x)$ and $\psi_-(x)={\rm Bi}(x)$, that correspond semi-classically to the two saddles

$$w_{\pm} = \pm \sqrt{x}/\lambda^{1/3}$$

of the Airy integral

$$\psi(x) = \frac{1}{2\pi i} \int dw \ e^{-\frac{xw}{\lambda^{2/3}} + \frac{w^3}{3}}.$$

The \mathcal{D} -module \mathcal{M} can be quantized into a fermionic state for any choice of boundary conditions. Depending on this choice we find an $\mathcal{O}(x)$ -module \mathcal{W}_{\pm} spanned by linear combinations of $\psi_{\pm}(x)$ and of $\psi'_{\pm}(x)$. The fermionic state is generated by asymptotic expansions in the parameter λ of these elements.

The saddle-point approximation around the saddle $w_{\pm}=\pm\sqrt{x}/\lambda^{1/3}$ yields

$$\psi_{\pm}(x) \sim y^{-1/2} e^{\mp \frac{2y^3}{3\lambda}} \left(1 + \sum_{n \ge 1} c_n \lambda^n (\pm y)^{-3n} \right)$$
$$\sim y^{-1/2} e^{\mp \frac{2y^3}{3\lambda}} v_0(\pm y).$$

To see the last step just recall the definition of $v_0(z)$ as being equal to the Baker function $\psi(x,z)$ evaluated at $x=0.^1$ A similar expansion can be made for $\psi'(x)$ with the result

$$\psi'_{\pm}(x) \sim y^{1/2} e^{\mp \frac{2y^3}{3\lambda}} v_1(\pm y).$$

Note that both expansions in λ are functions in the coordinate y on the cover. They contain a classical term (the exponential in $1/\lambda$), a 1-loop piece and a quantum expansion in λy^{-3} . When we restrict to the saddle $w=\sqrt{x}/\lambda^{1/3}$, these series blend the into the fermionic state

$$|\mathcal{W}_{+}\rangle = \psi_{+}(y) \wedge \psi'_{+}(y) \wedge y^{2}\psi_{+}(y) \wedge y^{2}\psi'_{+}(y) \wedge \dots$$

Does this agree with the well-known result (6.5)?

First of all, notice that the basis vectors $x^k\psi(x)$ and $x^k\psi'(x)$, with k>0, contain in their expansions the function $v_k(y)$ plus a sum of lower order terms in $v_l(y)$ (with l< k). The wedge product obviously eliminates all these lower order terms. Secondly, the extra factor $y^{-1/2}$ factors just reminds us that we have written down a fermionic state.

Furthermore, the WKB exponentials are exponentials in y and thus elements of Γ_+ , whereas the expansions $v_k(y)/y^k$ are part of Γ_- . Up to normal ordening ambiguities this shows that the WKB part gives a trivial contribution. In fact, the tau-function even cancels these ambiguities.

This shows that

$$|\mathcal{W}_{+}\rangle = v_0(y) \wedge v_1(y) \wedge v_2(y) \wedge \dots,$$

which is indeed the same as in equation (6.5), when we change variables from z to y in that equation. Of course, this doesn't change the tau-function.

¹Remark that x and z^2 appear equivalently in $\psi(x,z)$ in equation (6.4), while $\psi(x)$ and $\psi(x,z)$ only differ in the normalization term in z.

So our conclusion is that the \mathcal{D} -module underlying topological gravity is the canonical \mathcal{D} -module

$$\mathcal{M} = \frac{\mathcal{D}_{\lambda}}{\mathcal{D}_{\lambda}((\lambda \partial_{x})^{2} - x)}.$$

This \mathcal{D} -module gives the definition of the quantum curve corresponding to the (2,1) model and defines its quantum partition function in an expansion around λ . Exactly the same reasoning holds for the (p,1)-model, where we find a canonical rank p connection on the base. It would be good to be able to write down a \mathcal{D} -module for general (p,q)-models as well.

6.1.3 *D*-module for Hermitean matrix models

D-modules continue to play an important role in any Hermitean matrix model. In this subsection we are guided by [191] and [192, 193] of Bertola, Eynard and Harnad.

We first summarize how the partition function for a 1-matrix model defines a tau-function for the KP hierarchy. As we saw before, such a tau-function corresponds to a fermionic state $|\mathcal{W}\rangle$, whose basis elements we will write down. Following [191] we discover a rank two differential structure in this basis, whose determinant reduces to the spectral curve in the semi-classical limit. This \mathcal{D} -module structure is somewhat more complicated then the \mathcal{D} -module we just found describing double scaled matrix models.

We continue with 2-matrix models, based on [193]. Instead of one differential equation, these models determine a group of four differential equations, that characterize the \mathcal{D} -module in the local coordinates z and w at infinity. The matrix model partition function may of course be computed in either frame.

1-matrix model

Let us start with the 1-matrix model partition function

$$Z_N = \frac{1}{\text{vol}(U(N))} \int DM \ e^{-\frac{1}{\lambda} \text{Tr } W(M)}. \tag{6.7}$$

By diagonalizing the matrix M the matrix integral may be reduced to an integral over the eigenvalues λ_i

$$Z_N = \frac{1}{N!} \int \prod_{i=1}^N \frac{d\lambda_i}{2\pi} \ \Delta(\lambda)^2 \ e^{-\frac{1}{\lambda} \sum_i W(\lambda_i)},$$

with the Vandermonde determinant $\Delta(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j) = \det(\lambda_i^{j-1})$. The method of orthogonal polynomials solves this integral by introducing an infinite

set of polynomials $p_k(x)$, defined by the properties

$$p_k(x) = x^k (1 + \mathcal{O}(x^{-1})),$$

$$\int dx \ p_k(x) \ p_l(x) \ e^{-\frac{1}{\lambda}W(x)} = 2\pi h_k \delta_{k,l}.$$

The normalization of their leading term determines the coefficients $h_n \in \mathbb{C}$. Since the Vandermonde determinant $\Delta(x)$ is not sensitive to exchanging its entries x_i^{j-1} for $p_{j-1}(x_i)$, substituting $\Delta(x) = \det(p_{j-1}(x_i))$ turns the partition function into a product of coefficients

$$Z_N = \prod_{k=0}^{N-1} h_k.$$

With the help of orthogonal polynomials the large N behaviour of Z_N may be studied, while keeping track of 1/N corrections.

The orthogonal polynomials are crucial since they build up a basis for the fermionic KP state. In an appendix of [191] it is shown that one should start at t=0 with a state $|\mathcal{W}_0\rangle$ generated by the polynomials $p_k(x)$ for $k \geq N$

$$|\mathcal{W}_0\rangle = p_N(x) \wedge p_{N+1}(x) \wedge p_{N+2}(x) \wedge \dots$$

Notice that the vector $p_N(x)$ thus corresponds to the Fermi level and defines the Baker function in the double scaling limit. Acting on them with the commuting flow generated by

$$\Gamma_{+} = \left\{ g(t) = e^{\sum_{n \ge 1} \frac{1}{n} t_n x^n} \right\}$$

defines a state $|\mathcal{W}_t\rangle = |g(t)\mathcal{W}_0\rangle$ at time t, which allows to compute a tau-function at time t. If the coefficients u_j in the potential W(x) are taken to be $u_j = u_j^{(0)} + t_j$, this τ -function equals the ratio of the matrix model partition function Z_N at time t divided by that at t=0.

Multiplying the orthogonal polynomials by $\exp(-\frac{1}{2\lambda}W(x))$ doesn't change the fermionic state $\mathcal{W}=\mathcal{W}_0$ in a relevant way, since this factor is an element of Γ_+ . To find the right \mathcal{D} -module structure, it is necessary to proceed with the quasi-polynomials

$$\psi_k(x) = \frac{1}{\sqrt{h_k}} p_k e^{-\frac{1}{2\lambda}W(x)},$$

which form an orthonormal basis with respect to the bilinear form

$$(\psi_k, \psi_l) = \int dx \, \psi_k \psi_l. \tag{6.8}$$

It is possible to express both multiplication by x and differentiation with respect to x in terms of the basis of ψ_m 's. The Weyl algebra $\langle x, \lambda \partial_x \rangle$ acts on these (quasi)-polynomials by two matrices Q and P

$$x\psi_k(x) = \sum_{l=0}^{\infty} Q_{kl}\psi_l$$
$$\lambda \partial_x \psi_k(x) = \sum_{l=0}^{\infty} P_{kl}\psi_l(x),$$

and the space of quasi-polynomials ψ_k is thus a \mathcal{D}_{λ} -module.

Notice that we anticipate that the \mathcal{D} -module possesses a rank two structure, since we started with a flat connection $A=\frac{1}{\lambda}ydx$ on an I-brane wrapped on a hyperelliptic curve.

Now, the matrices Q and P only contain non-zero entries in a finite band around the diagonal. The action of ∂_x on the semi-infinite set of $\psi_k(x)$'s can therefore indeed be summarized in a rank two differential system ([191] and references therein)

$$\lambda \partial_x \left[\begin{array}{c} \psi_N(x) \\ \psi_{N-1}(x) \end{array} \right] = A_N(x) \left[\begin{array}{c} \psi_N(x) \\ \psi_{N-1}(x) \end{array} \right], \tag{6.9}$$

where $A_N(x)$ is a rather complicated 2×2 -matrix involving the derivative W' of the potential and the infinite matrix Q:

$$A_N(x) = \frac{1}{2}W'(x) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \gamma_N \begin{bmatrix} -\widetilde{W}'(Q,x)_{N,N-1} & \widetilde{W}'(Q,x)_{N,N} \\ -\widetilde{W}'(Q,x)_{N-1,N-1} & \widetilde{W}'(Q,x)_{N-1,N} \end{bmatrix},$$

with

$$\widetilde{W}'(Q,x) = \left(\frac{W'(Q) - W'(x)}{Q - x}\right) \quad \text{and} \quad \gamma_N = \sqrt{\frac{h_N}{h_{N-1}}}.$$

Equation (6.9) is thus the rank two λ -connection defining the \mathcal{D}_{λ} -module structure on \mathcal{W} that we were searching for! As a check, the determinant of this connection reduces to the spectral curve in the semiclassical, or dispersionless, limit [191]:

$$\begin{split} \Sigma_N: \quad 0 &= \det \big(y \mathbf{1}_{2 \times 2} - A_N(x) \big) \\ &= y^2 - W'(x)^2 + 4\lambda \sum_{j=0}^{N-1} \left(\frac{W'(Q) - W'(x)}{Q - x} \right)_{jj} \end{split}$$

(To make the coefficients in the above equation agree with (6.2), we rescaled

 $y\mapsto y/2$.) In conclusion we found the \mathcal{D} -module structure underlying Hermitean 1-matrix models.

Remark that in the $N \to \infty$ limit we expect that the hyperelliptic curve defining the B-model Calabi-Yau (6.2) emerges from Σ_N . Indeed, in the 't Hooft limit Q corresponds classically to the coordinate x on the curve, whereas quantum-mechanically it is an operator whose spectrum is described by the eigenvalues λ_i of the infinite matrix M. In the large N limit we can therefore replace the matrix Q_{ij} in the definition for Σ_N by $\lambda_i \delta_{ij}$.

We can rewrite the rank two connection for the vector $(\psi_N, \psi'_N)^t$ as

$$\lambda \partial_x \left[\begin{array}{c} \psi_N(x) \\ \psi_N'(x) \end{array} \right] = \left[\begin{array}{cc} 0 & 1 \\ -\det(A_N(x)) + \lambda Y & \lambda Z \end{array} \right] \left[\begin{array}{c} \psi_N(x) \\ \psi_N'(x) \end{array} \right],$$

at least when $\operatorname{tr}(A_N(x))=0$, with Y and Z some derivatives of entries of $A_N(x)$. This brings the λ -connection in the familiar form of Chapter 5. In the next subsection we clarify the differential structure in a simple example.

2-matrix model

Let us first say a few words on the \mathcal{D} -module structure underlying multi-matrix models, which capture spectral curves of any degree in x and y [192, 193]. The partition function for a two-matrix model, with two rank N matrices M_1 and M_2 , is

$$Z_N = \frac{1}{\text{Vol}(U(N))^2} \int DM_1 DM_2 \ e^{-\frac{1}{\lambda} \text{Tr}(W_1(M_1) + W_2(M_2) - M_1 M_2)},$$

where W_1 and W_2 are two potentials of degree d_1+1 and d_2+1 . Choosing W_2 to be Gaussian reduces the 2-matrix model to a 1-matrix model. The 2-matrix model is solved by introducing two sets of orthogonal polynomials $\pi_k(x)$ and $\sigma_k(y)$. Again it is convenient to turn them into quasi-polynomials

$$\psi_k(x) = \pi_k(x)e^{-\frac{1}{\lambda}W_1(x)}, \quad \phi_k(y) = \sigma_k(y)e^{-\frac{1}{\lambda}W_2(y)}.$$

obeying the orthogonality relations

$$\int dx dy \, \psi_k(x)\phi_l(y)e^{\frac{xy}{\lambda}} = h_k \delta_{kl}. \tag{6.10}$$

Multiplying with or taking a derivative with respect to either x or y yields (just) two operators Q and P (and their transposes because of (6.10)), that form a representation of string equation [P,Q]=0. Since Q is only non-zero in a band around the diagonal of size d_2+1 and P of size d_1+1 , the quasi-polynomials

may be folded into the vectors

$$\vec{\psi} = [\psi_N, \dots, \psi_{N-d_2}]^t, \quad \vec{\phi} = [\phi_N, \dots, \phi_{N-d_1}]^t.$$

Any other quasi-polynomial can be expressed as a sum of entrees of these vectors, with coefficients in the polynomials in x and y. These vectors are called windows. The differential operators $\lambda \partial_x$ and $\lambda \partial_y$ respect them, so that their action is summarized in a rank d_2+1 resp. rank d_1+1 λ -connection

$$\lambda \partial_x \vec{\psi}(x) = A_1(x) \vec{\psi}(x), \quad \lambda \partial_y \vec{\phi}(y) = A_2(y) \vec{\phi}(x).$$
 (6.11)

This we interpret as two representations of the \mathcal{D}_{λ} -module underlying 2-matrix models. Indeed, [192] proves that the determinant of both differential systems equals the same spectral curve Σ , in the limit $\lambda \to 0$ when we replace $\lambda \partial_x \to y$ and $\lambda \partial_y \to x$. The defining equation of Σ is of degree $d_1 + 1$ in x and of degree $d_2 + 1$ in y.

In fact, it is useful to introduce two more semi-infinite sets of quasi-polynomials $\underline{\psi}_k(y)$ and $\underline{\phi}_k(x)$, as the Fourier transforms of $\psi_k(x)$ and $\phi_k(y)$ respectively. The action of the Weyl algebra on them may be encoded as the transpose of the above linear systems. The full system can therefore be summarized by (compare to (6.22))

$$\begin{split} x\text{-axis}: \quad &\{\psi_k(x),\ \underline{\phi}_k(x)\}, \quad \nabla_\lambda = \lambda \partial_x - A_1(x), \\ y\text{-axis}: \quad &\{\phi_k(y),\ \psi_k(y)\}, \quad \nabla_\lambda = \lambda \partial_y - A_2(y). \end{split}$$

Moreover, the matrix model partition function can be rewritten as a fermionic correlator in either local coordinate

$$Z_N \propto \frac{1}{N!} \int \prod_i d\lambda_i^1 d\lambda_i^2 \ \Delta(\lambda^1) \Delta(\lambda^2) \ e^{-\frac{1}{\lambda} \sum_i W_1(\lambda_i^1) + W_2(\lambda_i^2) - \lambda_i^1 \lambda_i^2}$$
$$= \prod_{k=0}^{N-1} \langle \psi_k(x) | \underline{\phi}_k(x) \rangle = \prod_{k=0}^{N-1} \langle \phi_k(y) | \underline{\psi}_k(y) \rangle$$

with respect to the bilinear form in (6.8).

Furthermore, Bertola, Eynard and Harnad study the dependence on the parameters $u_j^{(1)}$ and $u_j^{(2)}$ appearing in the potentials W_1 and W_2 . Deformations in these parameters leave the two sets of quasi-polynomials invariant as well. On $\vec{\psi}$ and $\vec{\phi}$ they act as matrices $U_j^{(1)}$ and $U_j^{(2)}$. This yields the 2-Toda system

$$\begin{split} \partial_{u_j^{(1)}} Q &= -[Q, U_j^{(1)}] \qquad \partial_{u_j^{(1)}} P = -[P, U_j^{(1)}] \\ \partial_{u_j^{(2)}} Q &= [Q, U_j^{(2)}] \qquad \partial_{u_j^{(2)}} P = [P, U_j^{(1)}]. \end{split}$$

In [192] it is proved that the linear differential systems (6.11) are compatible with these deformations, so that the parameters $u_j^{(1)}$ and $u_j^{(2)}$ in fact generate isomonodromic deformations. This shows precisely how the non-normalizable parameters in the potential respect the central role of the \mathcal{D}_{λ} -module (6.11) in the 2-matrix model.

6.1.4 Gaussian example

Let us consider the Gaussian 1-matrix model with quadratic potential

$$W(x) = \frac{x^2}{2}, (6.12)$$

that is associated to the spectral curve

$$y^2 = x^2 - 4\mu \tag{6.13}$$

in the large N limit. In the Dijkgraaf-Vafa correspondence this matrix model is thus dual to the topological B-model on the deformed conifold geometry (see Fig. 4.5).

The Hermite functions

$$\begin{split} \psi_k^\lambda(x) &= \frac{1}{\sqrt{h_k}} e^{-\frac{x^2}{4\lambda}} H_k^\lambda(x), \quad \text{with} \\ H_k^\lambda(x) &= \lambda^{k/2} H_k\left(\frac{x}{\sqrt{\lambda}}\right) = x^k \left(1 + \mathcal{O}\left(\frac{\sqrt{\lambda}}{x}\right)\right), \end{split}$$

form an orthogonal basis for this model. Their inner product is given by

$$\int \frac{dx}{2\pi} \, \psi_k^{\lambda}(x) \psi_l^{\lambda}(x) = \lambda^k k! \sqrt{\frac{\lambda}{2\pi}} \delta_{kl} \quad \Rightarrow \quad h_k = \lambda^k k! \sqrt{\frac{\lambda}{2\pi}}.$$

The partition function of the Gaussian matrix model can be computed as a product of the normalization constants h_k . Using the asymptotic expansion of the Barnes function $G_2(z)$, that is defined by $G_2(z+1) = \Gamma(z)G_2(z)$, the free energy can be expanded in powers of λ

$$\mathcal{F}_{N} = \log \prod_{k=1}^{N-1} h_{k} = \log \left(G_{2}(N+1) \frac{\lambda^{N^{2}/2}}{(2\pi)^{N/2}} \right)$$

$$= \frac{1}{2} \left(\frac{\mu}{\lambda} \right)^{2} \left(\log \mu - \frac{3}{2} \right) - \frac{1}{12} \log \mu + \zeta'(-1) + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g-2)} \left(\frac{\lambda}{\mu} \right)^{2g-2},$$
(6.14)

where B_{2g} are the Bernoulli numbers and $\mu = N\lambda$.

The derivatives of the Hermite functions are related as

$$\lambda \frac{d}{dx} \left[\begin{array}{c} \psi_k^{\lambda}(x) \\ \psi_{k-1}^{\lambda}(x) \end{array} \right] = \left[\begin{array}{cc} -x/2 & \sqrt{k\lambda} \\ -\sqrt{k\lambda} & x/2 \end{array} \right] \left[\begin{array}{c} \psi_k^{\lambda}(x) \\ \psi_{k-1}^{\lambda}(x) \end{array} \right].$$

So, according to the previous discussion, the \mathcal{D}_{λ} -module connection is given by

$$\lambda \frac{d}{dx} - A_N(x) = \lambda \frac{d}{dx} + \begin{bmatrix} x/2 & -\sqrt{N\lambda} \\ \sqrt{N\lambda} & -x/2 \end{bmatrix}. \tag{6.15}$$

Here we choose $\vec{\psi} = [\psi_N, \psi_{N-1}]^t$ as window. In the large N limit the determinant of this rank two differential system indeed yields the spectral curve (6.13) with $\mu = N\lambda$.

Instead of using ψ_k^{λ} and ψ_{k-1}^{λ} as a basis, we can also write down the differential system for ψ_k^{λ} and its derivative ${\psi'}_k^{\lambda}(x) = \lambda \partial_x \psi_k^{\lambda}(x)$. Since this derivative is a linear combination of ψ_{k-1}^{λ} and $x\psi_k^{\lambda}(x)$ (as we saw above), it is equivalent to use this basis to generate the fermionic state \mathcal{W} . We compute that

$$\lambda \frac{d}{dx} \left[\begin{array}{c} \psi_N^{\lambda}(x) \\ {\psi'}_N^{\lambda}(x) \end{array} \right] = \left[\begin{array}{cc} 0 & 1 \\ x^2 - \lambda N - \lambda/2 & 0 \end{array} \right] \left[\begin{array}{c} \psi_N^{\lambda}(x) \\ {\psi'}_N^{\lambda}(x) \end{array} \right].$$

The spectral curve in the large N limit hasn't changed. Notice that in this form it is clear that the rank 2 connection is the push-forward of the connection $A=\frac{1}{3}ydx$ on the spectral curve $y^2=x^2-4\mu$ to the $\mathbb C$ -plane, up to some λ -corrections.

In the double scaling limit the limits $N\to\infty$ and $\lambda\to 0$ are not independent as in the 't Hooft limit, but correlated, such that the higher genus contributions to the partition function are taken into account. In terms of the Gaussian spectral curve this limit implies that one zooms in onto one of the endpoints of the cuts. The orthogonal function $\psi_N^\lambda(x)$ turns into the Baker function $\psi(x)$ of the double scaled state $\mathcal W$.

In the Gaussian matrix model this is implemented by letting $x \to \sqrt{\mu} + \epsilon x$, where ϵ is a small parameter. So the double scaled spectral curve reads

$$y^2 = x$$

while the differential system reduces to

$$\lambda \frac{d}{dx} \left[\begin{array}{c} \psi(x) \\ \psi'(x) \end{array} \right] = \left[\begin{array}{cc} 0 & 1 \\ x & 0 \end{array} \right] \left[\begin{array}{c} \psi(x) \\ \psi'(x) \end{array} \right].$$

This is indeed the \mathcal{D} -module corresponding to the (2,1)-model.

6.2 Conifold and c = 1 string

The free energy (6.14) of the Gaussian matrix model pops up in the theory of bosonic c=1 strings. This c=1 string theory is formulated in terms of a single bosonic coordinate X, that is compactified on a circle of radius r in the Euclidean theory. A critical bosonic string theory (with c=26) is obtained by coupling the above CFT to a Liouville field ϕ . The Liouville field corresponds to the non-decoupled conformal mode of the worldsheet metric. The local worldsheet action reads

$$\frac{1}{4\pi} \int d^2\sigma \left(\frac{1}{2} (\partial X)^2 + (\partial \phi)^2 + \mu e^{\sqrt{2}\phi} + \sqrt{2}\phi R \right),$$

where the coupling μ is seen as the worldsheet cosmological constant. In the Euclidean model there are only two sets of operators, that describe the winding and momenta modes of the field X. These vertex and vortex operators can be added to the action as marginal deformations with coefficients t_n and \tilde{t}_n .

Just like in c<1 minimal string theories (the (p,q)-models of last section), the partition function of the c=1 string is first computed using a dual matrix model description [194]. At the self-dual radius r=1 it agrees with the Gaussian matrix model partition function in equation (6.14), where λ now plays the role of the c=1 string coupling constant.

The matrix model dual to the c=1 string is called matrix quantum mechanics. This duality is reviewed in much detail in e.g. [195, 196, 197]. Matrix quantum mechanics is described by a gauge field A and a scalar field M that are both $N \times N$ Hermitean matrices. The momentum modes of the c=1 string correspond to excitations of M, whereas the winding modes are excitations of A. If we focus on the momentum modes, the (double scaled) matrix model is governed by the Hamiltonian

$$H = \frac{1}{2} \operatorname{Tr} \left(-\lambda^2 \frac{\partial^2}{\partial M^2} - M^2 \right).$$

Let us focus on solutions that depend purely on the eigenvalues λ_i of M. The Hamiltonian may be rewritten in terms of the eigenvalues as

$$H = \frac{1}{2} \Delta^{-1}(\lambda) \sum_i \left(-\lambda^2 \frac{\partial^2}{\partial \lambda_i^2} - \lambda_i^2 \right) \Delta(\lambda),$$

where $\Delta(\lambda)$ is Vandermonde determinant. It is convenient to absorb the factor Δ in the wavefunction solutions, making them anti-symmetric. Hence, the singlet sector of matrix quantum mechanics describes a system of N free fermions in an upside-down Gaussian potential.

To describe the partition function of the c=1 model it is convenient to move over to light-cone coordinates $\lambda_{\pm}=\lambda\pm p$, so that elementary excitations of the c=1 model are represented as collective excitations of free fermions near the Fermi level

$$\lambda_{+}\lambda_{-} = \mu. \tag{6.16}$$

When we restrict to $\lambda_{\pm}>0$, scattering amplitudes can be computed by preparing asymptotic free fermionic states $\langle \tilde{t}|$ and $|t\rangle$ at the regions where one of λ_{\pm} becomes very large.

In this picture the generating function of scattering amplitudes has a particularly simple form. It can be formulated as a fermionic correlator [198]

$$Z = \langle t|S|\tilde{t}\rangle,\tag{6.17}$$

where the fermionic scattering matrix $S \in GL(\infty, \mathbb{C})$ was first computed in [199]. Moreover, in [200] (see also Chapter V of [197]) and later in [128] it is noticed that S just equals the Fourier transformation

$$(S\psi)(\lambda_{-}) = \int d\lambda_{+} e^{\frac{1}{\lambda}\lambda_{-}\lambda_{+}} \psi(\lambda_{+}). \tag{6.18}$$

In the next section we show that this follows naturally from the perspective of \mathcal{D} -modules.

The result (6.17) shows that c=1 string theory is an integrable system, just like the (p,q)-models in the last section. Since it depends on two sets of times this integrable system is not a KP system. Instead, the above expression defines a tau function of a 2-Toda hierarchy.

Notice that the Fermi level (6.16) is a real cycle on the complex curve

$$\Sigma: \quad zw = \mu, \tag{6.19}$$

which is a different parametrization of the spectral curve $y^2=x^2-\mu$ of the Gaussian 1-matrix model. In the revival of this subject a few years ago, a number of other matrix model interpretations have been found. This includes a duality with the Hermitean 2-matrix model, which makes the 2-Toda structure manifest [201], a Kontsevich-type model [202, 203] at the self-dual radius, and a so-called normal matrix model [204, 205], that parametrizes the dual real cycle on the complex curve Σ . Let us also mention that the well-known duality of the c=1 string with the topological B-model on the deformed conifold [206], that follows, with a detour, from the more general Dijkgraaf-Vafa correspondence.

\mathcal{D} -module description of the c=1 string

This paragraph reproduces the c=1 partition function (6.17) from a \mathcal{D} -module point of view. The discussion continues the line of thought in Section 5.5 of [128].

As we have just seen, the c=1 string is geometrically characterized by the presence of a holomorphic curve in $\mathbb{C} \times \mathbb{C}$ defined by

$$\Sigma_{c=1}: zw = \mu.$$

Let us consider an I-brane wrapping the curve $\Sigma_{c=1}$. When we assume z as local coordinate the curve quantizes into the differential operator

$$P = -\lambda z \partial_z - \mu. \tag{6.20}$$

It is amusing that the differential operator P appears as a canonical example in the theory of \mathcal{D} -modules (see *e.g.* [159]) in the same way as the c=1 string is an elementary example of a string theory.

We recognize this example from Chapter 5, where a \mathcal{D} -module was associated to the differential operator P. However, now it is important not to forget that there are two asymptotic points z_{∞} and w_{∞} . Let us call their local neighbourhoods U_z and U_w , as local coordinates are z and w respectively. At both asymptotic points the I-brane fermions will sweep out an asymptotic state. The quantum partition function should therefore be constructed from two quantum states.

Before constructing these states for general λ , let us first consider the semi-classical limit $\lambda \to 0$. In this limit the I-brane degrees of freedom are just conventional chiral fermions on $\Sigma_{c=1}$. The genus 1 part \mathcal{F}_1 of the free energy is obtained as the partition function of these semi-classical fermions. It can be computed by assigning the Dirac vacuum

$$|0\rangle_z = z^{1/2} \wedge z^{3/2} \wedge z^{5/2} \wedge \dots$$

to U_z and likewise the conjugate state

$$|0\rangle_w = w^{1/2} \wedge w^{3/2} \wedge w^{5/2} \wedge \dots$$

to U_w . To compare these states, we need an operator S that relates z to 1/z. The semi-classical partition can then be computed as a fermionic correlator ${}_w\langle 0|S|0\rangle_z$, with the result that

$$e^{\mathcal{F}_1} = {}_{w}\langle 0|S|0\rangle_z = \prod_{k>0} \mu^{k+1/2}.$$
 (6.21)

Using ζ -function regularization we find that this expression yields the familiar

answer $\mathcal{F}_1 = -\frac{1}{12} \log \mu$.

In order to go beyond 1-loop, we should think in terms of \mathcal{D} -modules. Let us for a moment not represent their elements in terms of differential operators yet. In both asymptotic regions we then find the \mathcal{D} -modules

$$U_z: \mathcal{M} = \mathcal{D}/\mathcal{D}P, \text{ with } P = \hat{z}\hat{w} - \mu,$$

 $U_w: \mathcal{M} = \mathcal{D}/\mathcal{D}\underline{P}, \text{ with } \underline{P} = \hat{w}\hat{z} - \mu + \lambda.$

Notice that the Weyl algebra $\mathcal{D} = \langle \hat{z}, \hat{w} \rangle$, with the relation $[\hat{z}, \hat{w}] = \lambda$, acts on monomials z^k and w^k in the module \mathcal{M} as

$$\begin{split} \hat{z}(z^k) &= z^{k+1} \qquad \hat{z}(w^k) = \left(\lambda \partial_w + \frac{\mu - \lambda}{w}\right) w^k \\ \hat{w}(z^k) &= \left(-\lambda \partial_z + \frac{\mu}{z}\right) z^k \qquad \hat{w}(w^k) = w^{k+1}. \end{split}$$

Here, we just used the relation $\mathcal{D}P \equiv 0$ and wrote the elements in the basis $\{z^k, w^k \mid k \in \mathbb{Z}\}$ of \mathcal{M} . A basis of a representation of \mathcal{M} on which \hat{z} and \hat{w} just act by multiplication by z resp. differentiation with respect to z is given by

$$\begin{split} v_k^z(z) &= z^k \cdot z^{-\mu/\lambda}, \\ v_k^w(z) &= \int dw \; e^{-zw/\lambda} \; w^{k-1} \cdot w^{\mu/\lambda}. \end{split}$$

Indeed, differentiation with respect to z clearly gives the same result as applying \hat{w} . Moreover, multiplying v_k^w by z gives

$$z \cdot v_k^w(z) = \lambda \int dw \ e^{-zw/\lambda} \frac{\partial}{\partial w} \left(w^{k-1+\mu/\lambda} \right) = (\mu + \lambda(k-1)) v_{k-1}^w.$$

Similarly, in the module M one can verify that

$$\hat{w}(w^k) = w^{k+1} \qquad \hat{w}(z^k) = \left(-\lambda \partial_w + \frac{\mu}{w}\right) w^k$$

$$\hat{z}(w^m) = \left(\lambda \partial_z + \frac{\mu - \lambda}{z}\right) z^k \qquad \hat{z}(z^k) = z^{k+1}.$$

Hence in the representation of M defined by

$$\begin{split} &\underline{v}_k^w(w) = w^{k-1} \cdot w^{\mu/\lambda}, \\ &\underline{v}_k^z(w) = \int dz \; e^{zw/\lambda} \; z^k \cdot z^{-\mu/\lambda}, \end{split}$$

w and ∂_w act in the usual way.

Since we moved over to representations of the \mathcal{D} -module where the differential operator acts as we are used to, the S transformation, that connects the U_z and the U_w patch and thereby exchanges \hat{z} and \hat{w} , must be a Fourier transformation. This is clear from the expressions for the basis elements w and \tilde{w} : S interchanges $v_k^z(z)$ with $\underline{v}_k^z(w)$, and $v_k^w(z)$ with $\underline{v}_k^w(w)$. In total we thus find the \mathcal{D} -module elements

$$U_z: v_k^z, v_k^w$$

$$U_w: v_k^w, v_k^z$$
(6.22)

Representing the \mathcal{D} -module in terms of differential operators of course gives the same result. A fundamental solution of $P\Psi(z)=0$ is $\Psi(z)=z^{-\mu/\lambda}$, so that acting with $\mathcal{D}=\langle z,\partial_z\rangle$ on $\Psi(z)$ gives the elements v_k^z in \mathcal{M} . Likewise, we reconstruct the elements \underline{v}_k^w from the fundamental solution of $\underline{P\Psi}(w)=0$. Since $\mathcal{D}=\langle z,\partial_z\rangle$ and $\underline{\mathcal{D}}=\langle w,\partial_w\rangle$ are related by a Fourier transform, an element v_k of the \mathcal{D} -module in one asymptotic region is represented by its Fourier transform in the opposite region. This reproduces all elements in (6.22).

A λ -expansion of the \mathcal{D} -module element \underline{v}_k^z , using for example the stationary phase approximation, yields as zeroth order contribution

$$e^{\mu/\lambda} \left(\frac{\mu}{w}\right)^{k-\mu/\lambda},$$

while the subdominant contribution is given by

$$\sqrt{-\frac{2\pi\lambda\mu}{w^2}}.$$

So in total we find that

$$\underline{v}_k^z(w) = \sqrt{-2\pi\lambda}\; (\mu/e)^{-\mu/\lambda}\; w^{\mu/\lambda}\; \mu^{k+1/2}\; w^{-k-1}\; \psi_{\mathrm{qu}}\left(\frac{\mu}{w}\right).$$

This summarizes the contributions that we found before: the genus zero $w^{\mu/\lambda}$ and genus one $\mu^{k+1/2}w^{-k-1}$ results, plus the higher order contributions that are collected in $\psi_{\rm qu}$.

The all-genus partition function Z of this I-brane system can be easily computed exactly. Schematically it equals the correlation function

$$Z_{c=1} = \langle \mathcal{W}_w | S_\mu | \mathcal{W}_z \rangle,$$

where the S-matrix implements the Fourier transform between the two asymptotic patches. Similar to the arguments in (the appendices of) [200] and [128]²

²The argument presented in the appendix of [128] is not fully correct. The proper argument (as

we find that the result reproduces the perturbative expansion of the free energy as in equation (6.14). For completeness let us review the argument by comparing $\underline{v}_k^z(w)$ with $\underline{v}_k^w(w)$.

Notice that $\underline{v}_k^z(w)$ almost equals the gamma-function $\Gamma(z)=\int_0^\infty dt\ e^{-t}\ t^{z-1}$. Indeed, let us take the integration contour from $-i\infty$ to $i\infty$ and choose the cut of the logarithm to run from 0 to ∞ . Then

$$\begin{split} &\underline{v}_k^z(w) = \left(\frac{\lambda}{w}\right) \int_{-i\infty}^{i\infty} dz' \, e^{z'} \, \left(\frac{\lambda z'}{w}\right)^{k-\frac{\mu}{\lambda}} \\ &= \left(\frac{i\lambda}{w}\right)^{k+1-\frac{\mu}{\lambda}} \left[\int_{-\infty}^{0} dz' \, e^{iz'} \, e^{(k-\frac{\mu}{\lambda})\log z'} + \int_{0}^{\infty} dz' \, e^{iz'} \, e^{(k-\frac{\mu}{\lambda})\log z'}\right] \\ &= \left(\frac{i\lambda}{w}\right)^{k+1-\frac{\mu}{\lambda}} \left[\int_{i\infty}^{0} dz' \, e^{iz'} \, e^{(k-\frac{\mu}{\lambda})\log z'} + \int_{0}^{i\infty} dz' \, e^{iz'} \, e^{(k-\frac{\mu}{\lambda})\log z'}\right], \end{split}$$

where we moved the contour along the positive imaginary axis. A change of variables and using that $\log(iz'-\epsilon) = \log z' - 3i\pi/2$ and $\log(iz'+\epsilon) = \log z' + i\pi/2$, for ϵ small and real, then yields

$$\underline{v}_k^z(w) = \left(\frac{i\lambda}{w}\right)^{k+1-\frac{\mu}{\lambda}} \left[e^{\pi i(k+1-\frac{\mu}{\lambda})/2} - e^{-3\pi i(k+1-\frac{\mu}{\lambda})/2}\right] \Gamma\left(k+1-\frac{\mu}{\lambda}\right).$$

which is the same as the theory of type II result in the appendix of [200]. Ignoring the exponential factor (which will only play a role non-perturbatively), we find that the free energy $\mathcal F$ equals the sum

$$\mathcal{F}\left(\lambda,\,\mu\right) = \sum_{k\geq 0} \left(k+1-\frac{\mu}{\lambda}\right) \log \lambda + \log \Gamma\left(k+1-\frac{\mu}{\lambda}\right).$$

It obeys the recursion relation

$$\mathcal{F}\left(\lambda, \, \mu + \frac{\lambda}{2}\right) - \mathcal{F}\left(\lambda, \, \mu - \frac{\lambda}{2}\right) = \left(\frac{1}{2} - \frac{\mu}{\lambda}\right) \log \lambda + \log \Gamma\left(\frac{1}{2} - \frac{\mu}{\lambda}\right).$$

which is known to be fulfilled by the c=1 string (see for example Appendix A in [147]), up to a term $-\frac{1}{2}\log(2\pi\lambda)$ that can be taken care of by normalizing the functions \underline{v}_k . The same result is found when analyzing the function v_k .

This concludes our discussion of the c=1 string. It is the first \mathcal{D} -module example where we see how to handle curves with two punctures. The physical interpretation of the I-brane set-up furthermore provides a check of our formalism. Moreover, this example agrees with the claim that the \mathcal{D} -module partition function should be invariant under different parametrizations. Both the repre-

shown below) recovers a slightly different prefactor in front of the Gamma-function, related to the doubling in the appendix of [200].

sentation as c=1 curve, $\Sigma_{c=1}:zw=\mu$, and that as a Gaussian matrix model spectral curve, $\Sigma_{mm}:y^2=x^2+\mu$, yield the same partition function.

6.3 Seiberg-Witten geometries

More than once $\mathcal{N}=2$ supersymmetric gauge theories have proved to provide an important theoretical framework to test new ideas in physics. The most important advances in this context are the solution of Seiberg and Witten in terms of a family of hyperelliptic curves, as well as the explicit solution of Nekrasov and Okounkov in terms of two-dimensional partitions. In what follows we will provide a novel perspective on these results, by wrapping an I-brane around a Seiberg-Witten curve. The B-field on the I-brane quantizes the curve, and a fermionic state is obtained from the corresponding \mathcal{D} -module. As we will see, this state sums over all possible fermion fluxes through the Seiberg-Witten geometry, and may be interpreted as a sum over geometries. First we briefly review the Seiberg-Witten and Nekrasov-Okounkov approaches.

The solution of the U(N) Seiberg-Witten theory is encoded in its partition function $Z(a_i,\lambda,\Lambda)$, which is a function of the scale Λ , the coupling λ and boundary conditions for the Higgs field denoted by a_i for $i=1,\ldots,N$ (with $\sum_i a_i=0$ for the SU(N) theory). The partition function is related to the free energy $\mathcal F$ as

$$Z(a_i, \lambda, \Lambda) = e^{\mathcal{F}} = e^{\sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g(a_i, \Lambda)}.$$

In the above expansion \mathcal{F}_0 is the prepotential which contains in particular an instanton expansion in powers of Λ^{2N} , while higher \mathcal{F}_g 's encode gravitational corrections. The U(N) Seiberg-Witten solution identifies the a_i 's and the derivatives of the prepotential $\frac{1}{2\pi i}\frac{\partial \mathcal{F}_0}{\partial a_i}$ as the A_i and B_i periods of the meromorphic differential

$$\eta_{SW} = \frac{1}{2\pi i} v \frac{dt}{t}$$

on the hyperelliptic curve (4.10)

$$\Sigma_{SW}: \quad \Lambda^N(t+t^{-1}) = P_N(v) = \prod_{i=1}^N (v-\alpha_i).$$
 (6.23)

Despite great conceptual advantages, extracting the instanton expansion of the prepotential from this description is a non-trivial task. However, an explicit formula for the partition function, encoding not only the full prepotential but also entire expansion in higher \mathcal{F}_g terms, was postulated by Nekrasov in [207]. Subsequently this formula was derived rigorously jointly by him and Okounkov

in [147] and independently by Nakajima and Yoshioka in [208, 209]. For U(N) theory this partition function is given by a sum over N partitions $\vec{R}=(R_{(1)},\ldots,R_{(N)})$

$$Z(a_i, \lambda, \Lambda) = Z^{pert}(a_i, \lambda) \sum_{\vec{R}} \Lambda^{2N|\vec{R}|} \mu_{\vec{R}}^2(a_i, \lambda), \tag{6.24}$$

where

$$\mu_{\vec{R}}^2(a_i, \lambda) = \prod_{(i,m) \neq (j,n)} \frac{a_i - a_j + \lambda(R_{(i),m} - R_{(j),n} + n - m)}{a_i - a_j + \lambda(n - m)},$$
(6.25)

and

$$Z^{pert}(a_i, \lambda) = \exp\left(\sum_{i,j} \gamma_{\lambda}(a_i - a_j, \Lambda)\right). \tag{6.26}$$

The function $\gamma_{\lambda}(x,\Lambda)$ is related to the free energy of the topological string theory on the conifold, and its various representations and properties are discussed extensively in [147] in Appendix A. The vevs a_i are quantized in terms of λ , so that for $p_i \in \mathbb{Z}$,

$$a_i = \lambda(p_i + \rho_i),$$
 $\rho_i = \frac{2i - N + 1}{2N}.$

The approach of [210] is based on the localization technique in presence of the so-called Ω -background. In general this background provides a two-parameter generalization of the prepotential: the coupling λ is replaced by two geometric parameters ϵ_1 and ϵ_2 . The prepotential, as given above, is recovered for $\lambda=\epsilon_1=-\epsilon_2$. By the duality web Fig. 1.6 supersymmetric gauge theories are related to intersecting brane configurations. The Nekrasov-Okounkov solution must therefore have an interpretation in terms of a quantum Seiberg-Witten curve, where λ plays the role of the non-commutativity parameter.

6.3.1 Dual partition functions and fermionic correlators

For a relation to the I-brane partition function (4.53), it is necessary to consider the dual of the partition function (6.24). This is introduced in [147] as the Legendre dual

$$Z^{D}(\xi, p, \lambda, \Lambda) = \sum_{\sum_{i} p_{i} = p} Z(\lambda(p_{i} + \rho_{i}), \lambda, \Lambda) e^{\frac{i}{\lambda} \sum_{j} p_{j} \xi_{j}}.$$
 (6.27)

An important observation of Nekrasov and Okounkov is that this dual partition function can be elegantly written as a free fermion correlator. This is a consequence of the correspondence between fermionic states and two-dimensional partitions described in Section 5.3.1. For U(1) there is no difference between the partition function and its dual and both can be written as

$$Z_{U(1)}^{D}(p,\lambda,\Lambda) = \langle p|e^{-\frac{1}{\lambda}\alpha_1}\Lambda^{2L_0}e^{\frac{1}{\lambda}\alpha_{-1}}|p\rangle, \tag{6.28}$$

where $|p\rangle$ is the fermionic vacuum whose Fermi level is raised by $p=a/\lambda$ units and L_0 measures the energy of the state. A version of the boson-fermi correspondence implies the following decomposition

$$e^{\frac{1}{\lambda}\alpha_{-1}}|p\rangle = \sum_{R} \frac{\mu_R}{\lambda^{|R|}}|p;R\rangle \tag{6.29}$$

in terms of partitions R, where μ_R is the Plancherel measure

$$\mu_R = \prod_{1 \le m < n < \infty} \frac{R_m - R_n + n - m}{n - m} = \prod_{\square \in R} \frac{1}{h(\square)}$$

which can be written equivalently as a product over hook lengths $h(\Box)$.

For general N the dual partition function (6.27) looks very similar

$$Z_{U(N)}^{D}(\xi_i; p, \lambda, \Lambda) = \langle p|e^{-\frac{1}{\lambda}\alpha_1}e^{H_{\xi_i}}\Lambda^{2L_0}e^{\frac{1}{\lambda}\alpha_{-1}}|p\rangle, \tag{6.30}$$

however, now this expression is obtained by blending N free fermions $\psi^{(i)}$ into a single fermion ψ , as explained in Section 5.3.1. In particular

$$H_{\xi_i} = \frac{1}{\lambda} \sum_r \xi_{(r+1/2) \mod N} \psi_r \psi_{-r}^{\dagger},$$

while the bosonic mode α_{-1} arises from the bosonization of the single blended fermion ψ . In formula (6.29) the Plancherel measure of a blended partition ${\bf R}$ can be decomposed into N constituent partitions as

$$\mu_{\mathbf{R}} = \sqrt{Z^{pert}(a_i, \lambda)} \,\mu_{\vec{R}}(a_i, \lambda),\tag{6.31}$$

with $\mu_{\vec{R}}$ and Z^{pert} given in (6.25) and (6.26). When read in terms of the N twisted fermions $\psi^{(i)}$, the correlator (6.30) involves a sum over the individual fermion charges p_i .

Our aim in this section is to derive the above fermionic expressions for the dual partition function from the \mathcal{D} -module perspective. In the next subsections we will see how first quantizing the Seiberg-Witten curve in terms of a \mathcal{D} -module elegantly reproduces to the fermionic correlators (6.28) and (6.30).

6.3.2 Fermionic correlators as \mathcal{D} -modules

In this section we compute the I-brane partition function for U(N) Seiberg-Witten geometries. We start with the simpler U(1) and U(2) examples and then generalize this to U(N). As a first principal step we notice that the U(N) Seiberg-Witten geometry

$$\Sigma_{SW}: \quad \Lambda^N(t+t^{-1}) = P_N(v) = \prod_{i=1}^N (v - \alpha_i),$$
 (6.32)

can be rewritten as

$$(P_N(v) - \Lambda^N t)(P_N(v) - \Lambda^N t^{-1}) = \Lambda^{2N}.$$

This shows that the Seiberg-Witten surface may be seen as a transverse intersection of a left and a right half-geometry defined by

$$\Sigma_L: \Lambda^N t = P_N(v)$$
 resp. $\Sigma_R: \Lambda^N t^{-1} = P_N(v)$,

which are connected by a tube of size Λ^{2N} . The left geometry parametrizes the asymptotic region where both $t\to\infty$ and $v\to\infty$, whereas the right geometry describes the region where $v\to\infty$ while $t\to0$. This is illustrated in Fig. 6.1.

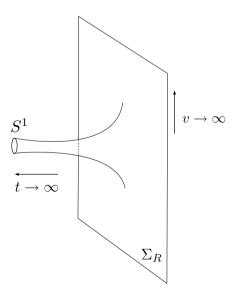


Figure 6.1: The right-half Seiberg-Witten geometry is distorted around the asymptotic point $(t \to 0, v \to \infty)$. A fermion field on the quantized curve can be described as an element of a \mathcal{D} -module, and sweeps out a state $|\mathcal{W}\rangle$ at the S^1 -boundary where $t \to \infty$.

Next we wish to associate a subspace in the Grassmannian to both half Seiberg-Witten geometries. This will be swept out by a fermion field on the curve that couples to the holomorphic part of the *B*-field

$$B = \frac{1}{\lambda} ds \wedge dv$$

Since this *B*-field quantizes the coordinate v into the differential operator $\lambda \partial_s$, any subspace in this section is a \mathcal{D} -module for the differential algebra

$$D_{\mathbb{C}^*} = \langle t, \lambda \partial_s \rangle.$$

The free fermions on the Seiberg-Witten curves couple to the gauge field $A=\frac{1}{\lambda}\eta_{SW}$. This determines their flux through the A_i cycles of the Seiberg-Witten geometry as

$$p_i = \frac{1}{\lambda} \int_{A_i} \eta_{SW}.$$

The flux leaking through infinity is $p=\sum_{i=1}^N p_i$, which is zero for SU(N). A fermion field with fermion flux p at infinity, will sweep out a fermionic state in the pth Fock space. The parameters $\xi_i=\int_{B_i}\eta_{SW}$ are dual to the fermion fluxes. Notice that in the perturbative regime p_i can be written as a λ -expansion

$$\lambda p_i = \alpha_i + \mathcal{O}(\lambda).$$

Since both half Seiberg-Witten geometries are distorted near $v = \infty$ (see Fig. 6.1), while a fermionic subspace can be read off in the neighbourhood where v is finite, both half-geometries parametrize a subspace of $\mathbb{C}((v))$:

$$\mathcal{W}_L, \ \mathcal{W}_R \subset \mathbb{C}((v)).$$

The trivial geometry corresponds to a disk with origin at $v=\infty$, whereas its boundary encloses the point v=0. The vacuum state is therefore given by

$$|0\rangle = v^0 \wedge v^{-1} \wedge v^{-2} \wedge \dots \tag{6.33}$$

Exponentials in v^{-1} act trivially (as pure gauge transformations in Γ_+) on this state, whereas exponentials in v transform the vacuum into a non-trivial fermionic state.

Finally, the partition function is recovered by contracting the left and the right fermionic state. Note that $s=-\log t$ is a local spatial coordinate on both half Seiberg-Witten geometries, which tends to $-\infty$ on the left and to $+\infty$ on the right. This makes a huge difference with the c=1 geometry discussed in Sec-

tion 6.2, where the local coordinate is the exponentiated coordinate, which on the left is the inverse of that on the right. While in that example a non-trivial S-matrix is required to identify the left and right half-geometries, here we can just glue the fermionic states using the classic Hamiltonian L_0 . Let us now find these quantum states.

U(1) theory

The U(1) Seiberg-Witten curve is embedded in $\mathbb{C}^* \times \mathbb{C}$ as

$$\Lambda(t+t^{-1}) = v - \alpha, \qquad (t = e^s \in \mathbb{C}^*, \ v \in \mathbb{C})$$

where $\alpha \in \mathbb{C}$ is a normalizable mode. This geometry may be factorized into a left and a right geometry

$$\Sigma_L: v = \Lambda t + \alpha$$
 and $\Sigma_R: v = \Lambda t^{-1} + \alpha$,

that intersect transversely with degeneration parameter Λ^2 .

The symplectic form $B=\frac{1}{\lambda}ds\wedge dv$ quantizes both half geometries into \mathcal{D}_{λ} -modules on a punctured disc \mathbb{C}_t^* , parametrized by t. We claim that these are characterized by the U(1) λ -connections

$$\nabla_L = -\lambda t \partial_t + \Lambda t + \lambda p$$
 and $\nabla_R = \lambda t \partial_t + \Lambda t^{-1} + \lambda p$.

These are just the canonical quantizations of the classical Seiberg-Witten geometries, where additionally u is quantized into λp , with $p \in \mathbb{Z}$. They yield the linear differential equations

$$P_L \psi_L^{\lambda}(t; p) = (-\lambda t \partial_t + \Lambda t + \lambda p) \,\psi_L^{\lambda}(t; p) = 0,$$

$$P_R \psi_R^{\lambda}(t; p) = (\lambda t \partial_t + \Lambda t^{-1} + \lambda p) \,\psi_R^{\lambda}(t^{-1}; p) = 0.$$
(6.34)

The \mathcal{D}_{λ} -modules are of the canonical form

$$\mathcal{M}_{L/R} = \frac{\mathcal{D}_{\lambda}}{\mathcal{D}_{\lambda} \cdot P_{L/R}},$$

and are generated by the solutions

$$\psi_L^{\lambda}(t;p) = t^p e^{\frac{\Lambda}{\lambda}t}$$
 and $\psi_R^{\lambda}(t;p) = t^{-p} e^{\frac{\Lambda}{\lambda}t^{-1}}.$

From the discussion in Section 5.3.1 it follows that the factor t^{-p} acts on the right Dirac vacuum by raising the Fermi level into $|p\rangle$, while the exponent of t^{-1} translates to the exponentiated α_{-1} operator. With an analogous statement for

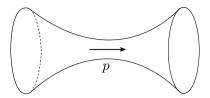


Figure 6.2: Contracting two Seiberg-Witten half-geometries yields the Nekrasov-Okounkov partition corresponding to a fermion flux p through the surface.

the left state, the modules $\mathcal{M}_{L/R}$ translate into the Bogoliubov states

$$\langle \mathcal{W}_L | = \langle p | e^{\frac{\Lambda}{\lambda} \alpha_1}$$
 and $| \mathcal{W}_R \rangle = e^{\frac{\Lambda}{\lambda} \alpha_{-1}} | p \rangle$. (6.35)

The U(1) Nekrasov-Okounkov partition function with fermion flux p (see Figure 6.2) is found by contracting the above fermion states

$$Z_{NO}^{\lambda}(p;\Lambda) = \langle p|e^{\frac{\Lambda}{\lambda}\alpha_1}e^{\frac{\Lambda}{\lambda}\alpha_{-1}}|p\rangle.$$

The factors Λ can be pulled out of the exponentials by using the commutator $[L_0, \alpha_{\pm 1}] = \alpha_{\pm 1}$. Up to an extra factor $\Lambda^{-p^2/2}$ we find that

$$Z_{NO}^{\lambda}(p;\Lambda) \sim \langle p|e^{\frac{\alpha_1}{\lambda}}\Lambda^{2L_0}e^{\frac{\alpha_{-1}}{\lambda}}|p\rangle.$$

This has a nice geometrical explanation, since the left and right half geometries are connected by a tube of size Λ^2 as in the factorized form of the complete U(1) geometry. The factor Λ^{2L_0} is the Hamiltonian that describes the propagation of the fermion field along the tube. There is no need to generalize this standard-CFT factor, since both patches are described by the same space-coordinate s.

We also note that, as consistent with [128], the solution $\psi_R^\lambda(t;u)$ to $P_R\psi=0$ equals the one-point-function

$$\langle p-1|\psi(t)|\mathcal{W}_R\rangle = \sum_n t^{-p-n}\langle p; R_n|\mathcal{W}_R\rangle = t^{-p}e^{\frac{\lambda}{\Lambda}t^{-1}} = \psi_R^{\lambda}(t;u),$$

where R_n represents a Young tableau consisting of just one row of n boxes.

U(2) theory

We apply now the above strategy for the U(2) geometry. We split the corresponding curve into a left and a right half geometry, and for brevity focus just on the right part defined by

$$\Sigma_R: \quad \Lambda^2 t^{-1} = (v - \alpha_2)(v - \alpha_1).$$
 (6.36)

The B-field quantizes this equation into the second order differential equation

$$P_R \psi(t) = \left\{ \lambda^2 (t\partial_t - p_2)(t\partial_t - p_1) - \Lambda^2 t^{-1} \right\} \psi(s) = 0.$$
 (6.37)

A change of variables $z=2t^{-1/2}$ followed by the ansatz $\psi(z)=z^{-(p_1+p_2)}\phi(z)$ and the rescaling $z\mapsto (\lambda/\Lambda)z$ transforms this differential equation into the familiar Bessel equation

$$(z^2 \partial_z^2 + z \partial_z - \nu^2 - z^2) \phi(z) = 0$$
, with $\nu^2 = (p_1 - p_2)^2$,

whose linearly independent solutions are given by modified Bessel functions $I_{\nu}(z)$ and $K_{\nu}(z)$ of the first kind. The total solution in the original t-coordinate is therefore a linear combination of

$$\psi_R^{\lambda}(t; p_1, p_2) = \begin{cases} t^{\frac{p}{2}} I_{\nu} \left(\frac{2\Lambda}{\lambda \sqrt{t}} \right), \\ t^{\frac{p}{2}} K_{\nu} \left(\frac{2\Lambda}{\lambda \sqrt{t}} \right), \end{cases}$$
(6.38)

where $p = p_1 + p_2$. These modified Bessel functions have different asymptotics at infinity and relate to each other by going around the punctured disc \mathbb{C}_t^* .

The second order differential operator P_R defines the \mathcal{D}_{λ} -module

$$\mathcal{M}_R = \frac{\mathcal{D}_{\lambda}}{\mathcal{D}_{\lambda} \cdot P_R},$$

which we claim represents fermions on the quantum SU(2) Seiberg-Witten geometry. To check this statement, we have to find the fermionic state corresponding to \mathcal{M}_R . So we asymptotically expand of the modified Bessel functions around t=0 in λ :

$$I_{\nu}\left(\frac{2\Lambda}{\lambda\sqrt{t}}\right) \sim t^{1/4} \exp\left(\frac{2\Lambda}{\lambda\sqrt{t}}\right) \left\{1 - \frac{(\mu - 1)}{8} \frac{\lambda\sqrt{t}}{2\Lambda} + \frac{(\mu - 1)(\mu - 9)}{2! \cdot 8^2} \frac{\lambda^2 t}{4\Lambda^2} + \dots\right\}$$
$$K_{\nu}\left(\frac{2\Lambda}{\lambda\sqrt{t}}\right) \sim t^{1/4} \exp\left(-\frac{2\Lambda}{\lambda\sqrt{t}}\right) \left\{1 + \frac{(\mu - 1)}{8} \frac{\lambda\sqrt{t}}{2\Lambda} + \frac{(\mu - 1)(\mu - 9)}{2! \cdot 8^2} \frac{\lambda^2 t}{4\Lambda^2} + \dots\right\},$$

with $\mu = 4\nu^2$.

Recall that equation (6.33) implies that any exponential function in the local coordinate $v^{-1}=\sqrt{t}$ near the puncture acts trivially on the vacuum state. Equivalently, this is true for any asymptotic series in \sqrt{t} that assumes the value 1 at $\sqrt{t}=0$. In other words, we can forget about the complete expansion in \sqrt{t} ! Only the WKB pieces

$$t^{1/4} \exp\left(\pm \frac{2\Lambda}{\lambda\sqrt{t}}\right)$$

are relevant in writing down the fermionic state. This is exactly opposite to the matrix model examples, where the WKB-piece can be neglected and the perturbative series in λ defines the fermionic state.

The derivatives of the above solutions have one term proportional to $\psi(s)$ (which we may forget about), and a term proportional to the derivative of the Bessel functions. The latter may be expanded as

$$\partial_{s}I_{\nu}(t) \sim t^{-1/4} \exp\left(\frac{2\Lambda}{\lambda\sqrt{t}}\right) \left\{ 1 - \frac{(\mu+3)}{8} \frac{\lambda\sqrt{t}}{2\Lambda} + \frac{(\mu-1)(\mu+15)}{2! \cdot 8^{2}} \frac{\lambda^{2}t}{4\Lambda^{2}} + \dots \right\}$$

$$\partial_{s}K_{\nu}(t) \sim t^{-1/4} \exp\left(\frac{2\Lambda}{\lambda\sqrt{t}}\right) \left\{ 1 + \frac{(\mu+3)}{8} \frac{\lambda\sqrt{t}}{2\Lambda} + \frac{(\mu-1)(\mu+15)}{2! \cdot 8^{2}} \frac{\lambda^{2}t}{4\Lambda^{2}} + \dots \right\}$$

around $\sqrt{t}=0$. Again with the same reasoning only the WKB piece is necessary to write down the quantum state. Taking into account the extra factor $t^{\frac{p}{2}}$ in (6.38) the subspace \mathcal{W}_{R}^{+} is thus generated by the $\mathcal{O}(t)$ -module

$$t^{\frac{p}{2}} \left(\begin{array}{c} t^{\frac{1}{4}} \exp\left(\frac{2\Lambda}{\lambda\sqrt{t}}\right) \\ t^{-\frac{1}{4}} \exp\left(\frac{2\Lambda}{\lambda\sqrt{t}}\right) \end{array} \right) \mathcal{O}(t),$$

and blends (via the lexicographical ordening) into the fermionic state

$$|\mathcal{W}_{R}^{+}\rangle = v^{-p} e^{\frac{\Lambda}{\bar{\lambda}}v} \left(v^{0} \wedge v^{-1} \wedge v^{-2} \wedge v^{-3} \wedge \ldots\right)$$

on the cover. Here we used a cover coordinate v^{-1} obeying $v^{-2}=t$, and rescaled the topological string coupling as $\tilde{\lambda}=\lambda/2$. \mathcal{W}_R^+ is thus simply generated by a single function

$$\psi^{\lambda}(v) = v^{-p} e^{\frac{\Lambda}{\lambda}v}$$

Hence the fermions blend into the Bogoliubov state

$$|\mathcal{W}_{R}^{+}\rangle = e^{\frac{\Lambda}{\bar{\lambda}}\alpha_{-1}}|p\rangle,$$
 (6.39)

when p is an integer.

Note that the only modulus that appears in this expression is p. This represents the diagonal U(1), denoting the total fermion flux through the geometry. The moduli p_1 and p_2 measures the fermion flux through an internal cycle and are not visible in the result, because the final state sums over all internal momenta. In general any SU(2) Seiberg-Witten geometry with the same quantized p yields the same fermionic state.

The fermionic (or dual) partition function is found by contracting the left and the right states, similarly as in the U(1) example above. The left state is just the

complex conjugate of the right one, so we find

$$Z_{NO}^{D}(p;\lambda,\Lambda) = \langle p|e^{\frac{\Lambda}{\bar{\lambda}}\alpha_{1}}e^{\frac{\Lambda}{\bar{\lambda}}\alpha_{-1}}|p\rangle \sim \langle p|e^{\frac{1}{\bar{\lambda}}\alpha_{1}}\Lambda^{2L_{0}}e^{\frac{1}{\bar{\lambda}}\alpha_{-1}}|p\rangle.$$

The result is very similar to the U(1) example, up to the shift $\lambda \mapsto \lambda/2$. But notice that this fermionic state is written in terms of a single blended fermion. Decomposing this fermion into two twisted fermions makes it natural to insert an extra operator in the middle of the correlator, that measures the momenta of the two fermions through the A-cycles of the SW geometry. Weighting these momenta with a potential ξ_i , for i=1,2, yields

$$Z_{NO}^{D}(\xi_{i}, p; \lambda, \Lambda) \sim \langle p|e^{\frac{1}{\bar{\lambda}}\alpha_{1}}e^{H_{\xi_{i}}}\Lambda^{2L_{0}}e^{\frac{1}{\bar{\lambda}}\alpha_{-1}}|p\rangle,$$

where $H_{\xi_i}=\frac{1}{\lambda}\sum_r \xi_{(r+1/2) \bmod 2} \psi_r \psi_{-r}^\dagger=\frac{1}{\lambda}(p_1\xi_1+p_2\xi_2)$. This is the answer conjectured by Nekrasov and Okounkov in [147].

U(N) theory

It is not difficult to extend this discussion to the U(N) theory (6.32), whose corresponding right half geometry we write as

$$\Sigma_N: \Lambda^N t^{-1} = \prod_{i=1}^N (v - \alpha_i).$$
 (6.40)

Canonically quantizing this geometry and changing the coordinates $z = \left(\frac{\Lambda}{\lambda}\right)^N t^{-1}$, brings us to the degree N differential equation

$$P_N \psi(z) = \left(\prod_{i=1}^N (z\partial_z - p_i) - z\right) \psi(z) = 0.$$
 (6.41)

It turns out that a solution to the above equation is given by a particular Meijer G-function, denoted $G_{p,q}^{m,n}(z)$. The Meijer G-function is a complicated special function which was introduced in order to unify a number of standard special function [211, 212, 213], and is defined in terms of a complex integral

$$G_{p,q}^{m,n}\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array} | z\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j-t) \prod_{j=1}^n \Gamma(1-a_j+t) z^t}{\prod_{j=m+1}^q \Gamma(1-b_j+t) \prod_{j=n+1}^p \Gamma(a_j-t)} dt,$$

where L is a contour which goes from $-i\infty$ to $+i\infty$ and separates the poles of $\Gamma(b_j-t)$, for $j=1,\ldots,m$, from those of $\Gamma(1-a_i+t)$, for $i=1,\ldots,n$.

It can be shown that the Meijer G-function solves the differential equation

$$\left(\prod_{i=1}^{q} (z\partial_z - b_i) + (-1)^{p-m-n+1} z \prod_{j=1}^{p} (z\partial_z - a_j + 1)\right) G(z) = 0.$$
 (6.42)

So, indeed the Seiberg-Witten differential equation (6.41) is a special case of Meijer differential equation (6.42) with p=n=0 and q=N. Therefore the differential equation (6.41) is solved by

$$\psi(z) = G_{0,N}^{0,0} \begin{pmatrix} \emptyset & | z \\ p_1, p_2, \dots, p_N & | z \end{pmatrix}.$$

Similarly as before we claim that the \mathcal{D} -module corresponding to U(N) Seiberg-Witten curve is generated by P_N . A subspace \mathcal{W} corresponding to this \mathcal{D} -module is this generated by a solution $\psi(t)$ and its derivatives in $t\partial_t$.

For p < q the Meyer differential equation (6.42) has a regular singularity at z=0 and an irregular one for $z=\infty$. To extract the I-brane fermionic state, we are interested in the behaviour around the irregular singularity, where $t\to 0$. It turns out that one of the independent solutions of the Seiberg-Witten differential equation (6.41) has the asymptotic expansion [211, 212, 213]

$$\psi(v) \sim e^{\frac{\Lambda}{\lambda/N}v} v^{\frac{1-N}{2}} v^p \sum_{j=0}^{\infty} k_j v^{-j},$$

around this singularity, which is conveniently written in the cover coordinate $(-v)^N=t^{-1}=\left(\frac{\lambda}{\Lambda}\right)^Nz$. The other solutions are found by multiplying the coordinate v by N-th roots of unity, and thus behave distinctly at infinity. As before, $p=\sum_{i=1}^N p_i$.

To find the fermionic state corresponding to the U(N) Seiberg-Witten curve, we act with $\psi(v)$ on the Dirac vacuum. The positive power of v in the exponent of $\psi(v)$ corresponds in the operator language to α_{-1} , whereas v^p lifts the Fermi level. The remaining series just contains negative powers of v which translate to a trivial action on the vacuum in the operator formalism. Therefore, the above asymptotic solution and its derivatives (in $t\partial_t$) blend into the state

$$|\mathcal{W}_{R}\rangle = e^{\frac{\Lambda}{\lambda}\alpha_{-1}}|p\rangle,\tag{6.43}$$

with rescaled topological string coupling $\tilde{\lambda} = \lambda/N$. Like for the U(2) Seiberg-Witten geometry the dependence on the individual moduli p_i has dropped out.

Similarly as in U(1) and U(2), in the present case we also find the U(N) Nekrasov-

Okounkov dual partition function

$$Z_{NO}^{D}(\xi_{i};\lambda,\Lambda) = \langle p|e^{\frac{1}{\lambda}\alpha_{1}}e^{H_{\xi_{i}}}\Lambda^{2L_{0}}e^{\frac{1}{\lambda}\alpha_{-1}}|p\rangle.$$

$$\tag{6.44}$$

This fermionic correlator is indeed the one postulated in [147]. For N=1 or N=2 the Meijer G-function specializes respectively to the exponent and Bessel functions, which reproduces the results derived in previous subsections.

Although the normalizable moduli p_i disappear in the final I-brane partition function, they reappear when the state is unblended in terms of N single fermions

$$e^{\frac{1}{\tilde{\lambda}}\alpha_{-1}}|p\rangle = \sum_{R} \frac{\mu_{R}}{\tilde{\lambda}^{|R|}}|p,R\rangle = \sum_{\sum p_{i}=p} \sum_{R_{(i)}} \sqrt{Z^{pert}(p)} \frac{\mu_{\vec{R}}(p,\tilde{\lambda})}{\tilde{\lambda}^{|R|}} \bigotimes_{l=1}^{N} |p_{i},R_{(i)}\rangle, \tag{6.45}$$

as may be seen from (6.29) and (6.31). The charges p_i have an interpretation as the fermion fluxes through the N tubes of the Seiberg-Witten geometry we started with.

Actually, we find the same fermionic state when starting with any other Seiberg-Witten geometry whose fermion flux at infinity is p. Hence one microstate in the total sum (6.45) can be interpreted as a fermion flux through an infinite set of geometries. This gives the state (6.45) as well as the partition function (6.27) the interpretation of a sum over geometries.

6.3.3 Topological string theory and quantum groups

Nekrasov and Okounkov also derive a partition function for the 5-dimensional U(N) Seiberg-Witten theory compactified on the circle of circumference β [210, 147, 209] . It is given by a K-theoretic generalization of the 4-dimensional formula in equation (6.24).

This 5-dimensional theory is closely related to the topological string theory by geometric engineering on a toric Calabi-Yau background [214, 215]. Namely, the partition function of the topological string theory on an A_N -singularity fibered over \mathbb{P}^1 (whose toric diagram consists of N-1 meshes as in Fig. 6.3) is equal to the partition function of the 5-dimensional gauge theory given above, when the Kähler sizes of the internal legs are (see Section 4.2.2)

$$Q_{F_i} = e^{\beta(a_{i+1} - a_i)}, \qquad Q_B = \left(\frac{\beta\Lambda}{2}\right)^{2N}, \tag{6.46}$$

where F_i labels the vertical legs and B the horizontal ones. In the so-called gauge theory limit, when $\beta \to 0$, the topological string partition function reduces to the 4-dimensional Seiberg-Witten partition function. The corresponding B-

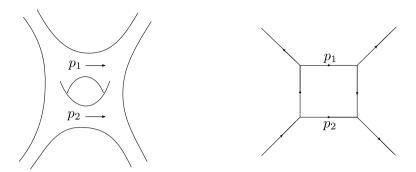


Figure 6.3: On the left we see the five-dimensional U(2) Seiberg-Witten surface with fermion fluxes through its A-cycles, and on the right a corresponding toric diagram. The fermion flux deforms the Kähler lengths of the toric diagram as in equation (6.46).

model mirror geometry is of the form

$$X_{SW}: xy - H(t,v) = 0,$$

where H(t,v)=0 represents a Riemann surface of genus N-1. In the gauge theory limit this surface becomes the Seiberg-Witten curve Σ_{SW} , parametrized as in the equation (6.23).

In topological string theory it is natural as well to write down a dual partition function [128]. In a local B-model this allows the possibility of arbitrary fermion fluxes through the handles of the Riemann surface. In this setting it has been argued before that turning on a fermion flux is equivalent to deforming the geometry. More precisely, fermion flux parametrized by $\mathcal{P}=p_iB_i$ changes the integral of the holomorphic 3-form over any linking 3-cycle A_i^3 , and thereby shifts the complex structure moduli $S_i=\int_{A_i}\Omega$ as

$$S_i \mapsto S_i + \lambda p_i$$

In the A-model fermion flux translates into wrapping D4 branes around 4-cycles, and thereby deforms the Kähler moduli. The I-brane partition function thus equals the dual topological string partition function.

Because the Seiberg-Witten surface is embedded in $\mathbb{C} \times \mathbb{C}^*$, A^3 and B^3 -cycles in the toric threefold will have topologies $S^1 \times S^2$ and S^3 , respectively (see Fig. 4.8). In particular, a basis of A_i^3 -cycles can be chosen to reduce to the surface as the combination of 1-cycles $A_i^1 - A_{i+1}^1$. Now notice that the 3-cycle A_i^3 with topology $S^1 \times S^2$ is mirror to the vertical 2-cycle F_i that connects the i-th and the i+1-th horizontal leg. So turning on a fermion flux p_i through the i-th leg of the Seiberg-Witten geometry changes the complex structure parameter S_i by an amount proportional to $a_i - a_{i+1}$. This explains the Kähler size Q_{F_i}

in (6.46) in terms of fermionic fluxes through the Seiberg-Witten curve, and in reverse why (6.45) may be interpreted as a sum over Seiberg-Witten geometries, or equivalently toric diagrams. So we conclude that the fermionic interpretation in 4d of Nekrasov and Okounkov is dual in 6d to the fermionic interpretation of the topological string, and has a deeper interpretation in terms of \mathcal{D} -modules.

Topological vertex

An important step to understand Seiberg-Witten curves (as well as other local Calabi-Yau geometries) is the topological vertex, introduced in Section 4.2.1. Recall that its mirror is a genus zero curve with three punctures given by the equation

$$x + y - 1 = 0 ag{6.47}$$

in $\mathbb{C}^* \times \mathbb{C}^*$. In this case the symplectic form is given by $du \wedge dv$ where u, v are logarithmic coordinates: $x = e^u$ and $y = e^v$. The corresponding \mathcal{D} -module is now given by the operator [128]

$$P = e^u + e^{-\lambda \partial_u} - 1. ag{6.48}$$

P is actually a difference operator, instead of a differential operator, so we have to generalize the notion of a \mathcal{D} -module somewhat. This is a well-known procedure in the field of quantum groups. These quantum groups appear because in the \mathbb{C}^* case the operators \hat{x} and \hat{y} now satisfy the Weyl algebra or q-commutation relation

$$\hat{x}\,\hat{y} = q\,\hat{y}\,\hat{x}, \qquad q = e^{\lambda}.$$

The fundamental solution to $P\Psi = 0$ is the quantum dilogarithm

$$\Psi(u) = \prod_{n=1}^{\infty} (1 - e^u q^n).$$

The corresponding module \mathcal{M} for the Weyl algebra can again be written in terms of the coordinate u or in terms of the dual variable v. There is another unitary map U that implements this transformation on the free fermion fields. Because of the hidden cyclic symmetry of the vertex, this can be made transparent by writing it as

$$e^{u_1} + e^{u_2} + e^{u_3} = 0.$$

Up to an overall rescaling of the three variables u_i , the map U satisfies $U^3 = 1$. This line of reasoning leads one directly to the formalism of [128], but we will

not pursue this here in more detail. We reach the important conclusion that the notion of a quantum curve, as expressed in the concept of a (generalized) \mathcal{D} -module, is the right framework to derive the complicated transformations of [128]. We will later use this correspondence in two concrete examples of compact curves, but first make a few remarks about five-dimensional U(1) Seiberg-Witten theory.

Five-dimensional U(1) theory

Quantizing any five-dimensional Seiberg-Witten geometry yields a difference (instead of differential) equation. Working out \mathcal{D} -modules for these geometries we leave for future work. Let us treat one example in detail though. The five-dimensional right-half U(1) Seiberg-Witten half-geometry

$$\Sigma_R^{5d}: \quad \beta \Lambda e^{-\beta \lambda} t^{-1} + e^{-\beta v} - 1 = 0$$
 (6.49)

is isomorphic to the topological vertex (6.47) and may be drawn as a pair of pants. In the field theory limit $\beta \to 0$ it reduces to the familiar equation $\Lambda t^{-1} = v$ for the right-half Seiberg-Witten geometry (with u=0).

In the B-model the most general state assigned to a local pair of pants geometry is given by a Bogoliubov state [128]

$$|\mathcal{W}\rangle = \exp\left[\sum_{i,j} \sum_{m,n=0}^{\infty} a_{mn}^{ij} \psi_{-m-1/2}^{i} \psi_{-n-1/2}^{*j}\right] |0\rangle,$$
 (6.50)

where the index i=1,2,3 describes the fermion field on the three asymptotic regions of the pair of pants, and the coefficients are determined by a comparison with the A-model topological vertex. This exponent can be expanded as a sum over states (see Fig. 6.4)

$$|p_1,R_1\rangle\otimes|p_2,R_2\rangle\otimes|p_3,R_3\rangle,$$

where the fermion flux is conserved: $p_1 + p_2 + p_3 = 0$. To describe the 5d Seiberg-Witten U(1) geometry we won't need this state in full generality.

The B-field quantizes this geometry into the difference equation

$$P(t)\Psi(t) = \left(\beta\Lambda e^{-\beta\lambda}t^{-1} + e^{\beta\lambda t\partial_t} - 1\right)\Psi(t) = 0.$$
(6.51)

Like for the topological vertex its fundamental solution is the quantum dilogarithm

$$\Psi(t) = \exp \sum_{n>0} \frac{(\beta \Lambda)^n t^{-n}}{n(1 - e^{\beta \lambda n})}.$$

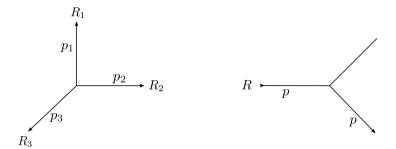


Figure 6.4: The B-model vertex (on the left) may be expanded as a sum over fermionic states $|p_1, R_1\rangle \otimes |p_2, R_2\rangle \otimes |p_3, R_3\rangle$, with $p_1 + p_2 + p_3 = 0$, corresponding to a conserved fermion flux through the pair of pants. The five-dimensional right-half Seiberg-Witten geometry (on the right) with charge p only has one partition $R \neq 0$.

As an intermezzo, notice that quantizing the equation

$$\beta v = -\log\left(1 - \beta \Lambda e^{-\beta \lambda} t^{-1}\right),\,$$

which is just a rewriting of equation (6.49) for Σ_R^{5d} , we find a differential equation which may be interpreted as the WKB approximation of difference equation (6.51). A fundamental solution of the differential equation is given by the genus 0 disc amplitude

$$\Psi_0(u) = \exp \sum_{n>0} \frac{(\beta \Lambda)^n t^{-n}}{\lambda n^2 e^{\beta \lambda n}}.$$

Acting with the five-dimensional dilogarithm on the Dirac vacuum state yields the fermionic state

$$|\mathcal{W}\rangle_{U(1)}^{5d} = \exp\sum_{n>0} \frac{(\beta\Lambda)^n \alpha_{-n}}{n(1 - e^{\beta\lambda n})} |0\rangle.$$

This describes a subset of $|\mathcal{W}\rangle$ where only the quantum number R_1 is non-trivial. Summing over all external states of the form

$$|-p,R\rangle\otimes|p,\bullet\rangle\otimes|0,\bullet\rangle,$$

incorporates a fermion flux p through the pair of pants. In the field theory limit $\beta \to 0$ the resulting state reduces to the familiar four-dimensional state

$$\exp(\alpha_{-1}/\lambda)|p\rangle\otimes|p,\bullet\rangle\otimes|0,\bullet\rangle.$$

The partition function is found as the contraction of the left and right 5d halfgeometries. (Or equivalently in the topological B-model by inserting a propagator [128].) This yields the fermionic correlator

$$\langle 0|\tilde{\Gamma}_{+}\tilde{\Gamma}_{-}|0\rangle = \langle 0|\Gamma_{+}(\beta\Lambda)^{2L_{0}}\Gamma_{-}|0\rangle,$$

with

$$\tilde{\Gamma}_{\pm} = \exp \sum_{\pm n > 0} \frac{(\beta \Lambda)^{|n|} \alpha_n}{|n|(1 - e^{\beta \lambda n})} \quad \text{and} \quad \Gamma_{\pm} = \exp \sum_{\pm n > 0} \frac{\alpha_n}{|n|(1 - e^{\beta \lambda n})}.$$

Indeed, the result equals the five-dimensional U(1) partition function

$$Z^{U(1)}_{5d}(\lambda,\Lambda,\beta) = \exp \sum_{n=1}^{\infty} \frac{(\beta\Lambda)^{2n}}{4n \sinh^2(\beta\lambda n/2)},$$

that was found by Nekrasov and Okounkov in [147].

6.4 Discussion

In this chapter we argued that the fundamental objects underlying various matters in theoretical physics are chiral fermions living on quantum curves. In our formulation the quantum curve is defined, similarly to an affine classical curve, in terms of an equation P(z,w)=0. Its crucial feature, however, is the noncommutative character of the coordinates z,w. It thereby generalizes the classical curve that comes up in the standard formulation of a given topic. Examples of such classical curves are spectral curves in matrix models, c=1 string theory, Seiberg-Witten theory, and more generally in topological string theory. Semiclassically their (genus one) free energy is computed as a fermionic determinant on the classical curve. In our approach chiral fermions on the quantum curve generate the all-genus expansion of the free energy with respect to the noncommutativity parameter λ .

As we explained in Chapter 5, fermions on a non-commutative curve can be realized physically within string theory as massless states of open strings on an I-brane in the presence of the B-field. In this chapter we have exploited this system in a few important examples. At the same time we stressed the fundamental importance of \mathcal{D} -modules, which are the appropriate mathematical structures describing non-commutative holomorphic curves. First of all we showed, while reinterpreting the results in [191], that I-branes and \mathcal{D} -modules provide an insightful formulation of matrix models. This quite general statement is also appealing when certain matrix model limits are considered, such as a double scaling limits. In this case one recovers an I-brane formulation of minimal string theory, topological gravity and c=1 string theory.

The I-brane configuration can be related to topological string theory and to Seiberg-Witten theory via a sequence of string dualities Fig. 1.6. In the last part of this chapter we focused on supersymmetric gauge theories. Using the \mathcal{D} -module formalism we derived the fermionic expression for the U(N) partition function of the pure $\mathcal{N}=2$ gauge theory, reproducing the dual all-genus partition function introduced in [147]. We considered mainly 4-dimensional Seiberg-Witten theories with unitary gauge groups, though, and explained only the simplest U(1) example of 5-dimensional theory. It would be insightful to extend these results to other gauge groups and include matter content. It is clear that this should be possible, as these aspects of the 5-dimensional Seiberg-Witten theory are captured by topological strings on toric manifolds. The latter system can be solved in a fermionic B-model formulation of the topological vertex [128] which is equivalent to the I-brane fermions. Nonetheless, finding the quantum I-brane curve representing such configurations appears to be a nontrivial task.

In the process of unraveling the \mathcal{D} -module structure in both sets of examples, we noticed some crucial differences. While the WKB piece of the \mathcal{D} -module generator can be ignored in finding the matrix model partition function, we discovered that it plays an eminent role for the Seiberg-Witten geometries. Another distinction is the difference in (non-)normalizable modes. While the potential W parametrizes non-normalizable modes that appear in the \mathcal{D} -module as parameters, in contrast, the normalizable modes in the Seiberg-Witten geometries are eaten by the \mathcal{D} -module, and only visible as a sum over internal fermion fluxes in the geometry. On the other hand, varying the \mathcal{D} -module with respect to the non-normalizable modes yields differential equations which relate to isomonodromy and the Stokes phenomenon.

While in this chapter our focus has been to associate a λ -perturbative quantum state to a spectral curve, we noticed that \mathcal{D} -modules in fact contain non-perturbative information. These bits get lost when we turn the \mathcal{D} -module in a fermionic state by making an asymptotic expansion of the \mathcal{D} -module generators in λ . This is in line with the discussion on non-perturbative aspects of minimal string theory in [171], where it is argued that non-perturbative effects drastically modify the non-trivial target space curve into a plain complex plane.

It also agrees with more recent studies of non-perturbative effects in matrix models [148, 216, 217, 218]. These articles revealed that a series of instantons in the matrix model can be summarized in a non-perturbative partition function that sums over all possible filling fractions $p_i = \frac{1}{\lambda} \oint_{a_i} \eta$ as

$$Z_{\text{non-pert}}(\mu,\nu) = \sum_{p \in \mathbb{Z}^g} Z_{\text{pert}}(\lambda(p+\mu)) e^{2\pi i p \nu}.$$

In this formula (μ, ν) is a choice of characteristics on the matrix model spectral curve, that encodes the choice of integration contour in the matrix model. The

integer g is the genus of the spectral curve. Interestingly, this partition function turns out to have very nice properties. $Z_{\text{non-pert}}(\mu,\nu)$ is not only holomorphic, but also transforms in a modular fashion under the symplectic group $Sp(g,\mathbb{Z})$. Moreover, it satisfies the Hirota equations and can thus be interpreted in terms of a twisted fermion field on Σ with twists (μ,ν) .

The partition function $Z_{\text{non-pert}}(\mu, \nu)$ is obviously closely related to the I-brane partition function, that is defined in equation 4.53 and studied from several angles in this chapter. A choice of saddle in the λ -expansion of our \mathcal{D} -module partition function corresponds to a choice of characteristics in $Z_{\text{non-pert}}(\mu, \nu)$.

How do these latter matrix model results and our \mathcal{D} -module insights fit in with other developments that have taken place the last years in the area of topological string theory? Let us start with the observation that the (perturbative) topological string partition function is known to suffer from background dependence [129]. As a result the free energy does not transform as a proper modular form. The modularity can be restored, however, but then the resulting function is not holomorphic anymore [219]. Instead it obeys the holomorphic anomaly equations [117]. It is natural to suggest that a partition function which is both holomorphic and modular, is a candidate for a non-perturbative completion of topological string theory. This is argued in [148].

The claim is strengthened by the following discoveries. In a sequence of papers [220, 221, 131] the Dijkgraaf-Vafa correspondence between matrix models and topological string theory has been extended to arbitrary local Calabi-Yau geometries modeled on a Riemann surface Σ . As a result topological string amplitudes can be computed in terms of a simple recursion relation that originates from the theory of matrix models [222]. The Eynard-Orantin formalism is closely related to the Kodaira-Spencer formulation of the B-model, and may be viewed as the bosonized version of our fermionic formulation [223]. It would be valuable to understand this non-commutative version of the familiar boson/fermion correspondence and its interpretation in terms of \mathcal{D} -modules in more detail.

Moreover, our formalism seems to be closely related to a non-commutative extension of the Eynard-Orantin formalism, that is studied in [224]. The resulting non-commutative invariants depend on two deformation parameters: The first deformation parameter is the usual topological string coupling constant λ , whereas the second one is an independent non-commutative deformation. The connection to our \mathcal{D} -module formalism should arise when we only turn on this second deformation. Turning on either of the deformation parameters is possibly equivalent.

Let us also note that while we mainly studied the web of dualities Fig. 1.6 in the large radius regime, where the topological string partition function has an expansion in terms of the usual Gromov-Witten and Donaldson-Thomas invariants, the \mathcal{D} -module formalism suggests a relation to invariants in other regimes. Since

we need to make a choice of boundary conditions, when we turn a \mathcal{D} -module into a quantum state, the final state is troubled by the Stokes effect: Solutions that decay faster can be added at no cost and the state changes when one crosses certain lines in the moduli space. This suggests that the \mathcal{D} -modules we studied may be helpful in understanding the phenomenon of wall-crossing in $\mathcal{N}=2$ theories [225, 226]. We discuss wall-crossing in $\mathcal{N}=4$ theories extensively in Chapter 7.

More mathematically, \mathcal{D} -modules play an important role in the geometric Langlands program [227, 156, 228, 229]. In the physical description of this program \mathcal{D} -modules enter in the description of eigenbranes of the magnetic 't Hooft operator in a reduction of 4-dimensional $\mathcal{N}=4$ gauge theory down to a 2-dimensional sigma model. However, in this sigma model (which is not yet coupled to gravity) the \mathcal{D} -modules describe coisotropic A-branes. This is in contrast to their physical appearance in our intersecting brane configuration.

There seems to be a deeper connection of our formalism to quantum integrable systems as they are studied in for example [162, 163, 230]. Quantum curves feature in these quantum systems as so-called opers, that parametrize the base of the integrable system, in the same way that spectral curves parametrize the base of the Hitchin integrable system. It would be enlightening to find out whether the fermions on the quantum curve can be described in a similar way in terms of the quantum integrable system as holds in the semi-classical limit. Does this lead to a better description of the quantum fermion CFT on the quantum curve? Is our set-up related to WZW models based on opers in the geometric Langlands program [227]? Most importantly, we would like to be able to write down a 2-dimensional action for the quantum fermion theory. In Section 7.1 we succeed in writing down the action for a propagator in the I-brane geometry, but not yet for a 3-vertex.

Chapter 7

Dyons and Wall-Crossing

In this chapter we study examples of local Calabi-Yau threefolds that are modeled on a compact Riemann surface. We start in Section 7.1 with a family of Calabi-Yau's that are built on a 2-torus, and then consider threefolds based on a genus 2 curve. In both examples we will discover nice automorphic structures and confirm the relation of the exponent of \mathcal{F}_1 to the fermionic determinant det $\bar{\partial}$.

Notice that these compact curves have genera g>0, and no asymptotic endpoints at infinity. To compute their I-brane partition function we thus have to cut the curves in affine pieces. In the genus 1 example we employ the \mathcal{D} -module techniques to glue the end-points of a cylinder. In the genus 2 example we use the topological vertex formalism to compute the contribution of a pair of pants.

In Section 7.2 and Section 7.3 we focus on the semi-classical contribution to the partition function on the genus 2 Calabi-Yau. This is summarized in an automorphic invariant that surprisingly turns out to count the number of non-perturbative BPS dyons in $\mathcal{N}=4$ theories. We make this relation explicit in Section 7.2. Since the generating function of these BPS invariants merely corresponds to $\exp \mathcal{F}_1$ in topological string theory, it is relatively easy to study additional structures, that go beyond properties of the partition function in the large radius regime.

Wall-crossing is one such topic. The charge of a BPS state varies over the moduli space of the theory. When it aligns with the charge of another BPS state, these BPS states can form a bound state. This gives a complication in the counting of BPS states. There are so-called walls of marginal stability in the moduli space, where the number of BPS states may jump. This is an important issue in $\mathcal{N}=2$ theories, but it also plays a major role in the counting of the above non-perturbative BPS states in 4-dimensional $\mathcal{N}=4$ string theory.

One may wonder what happens to the generating function of such invariants

under a crossing of a wall of marginal stability, and whether this phenomenon has an interpretation in terms of the underlying Riemann surface. In Section 7.3 we will answer these questions for quarter BPS dyons in $\mathcal{N}=4$ theories.

7.1 Compact curves of genus 1 and 2

Let us begin with finding Calabi-Yau threefolds that are modeled on compact curves of genus 1 and 2. The goal of this section is to analyze their all-genus partition functions in the framework of I-branes and \mathcal{D} -modules.

7.1.1 Elliptic curve

A well-studied example is the geometry mirror to the total space of a rank two bundle over a 2-torus

$$\widetilde{X}: \mathcal{O}(-r) \oplus \mathcal{O}(r) \to T^2.$$
 (7.1)

The latter has a description in toric geometry as gluing the toric propagator to itself with a framing factor r. This factor changes the intersection of $[T^2]$ with the 4-cycles that project onto T^2 into $\pm r$ [121]. Here we show how one can use the free fermionic system living on the boundary of the non-commutative plane to completely solve this model and recover the existing results for the all genus topological string amplitudes for this background.

The local T^2 model (7.1) has a simple interpretation in the B-model obtained after mirror symmetry. Note that we can write this geometry as a global quotient of $\mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$. If we pick toric coordinates (e^u, e^v, e^w) , the identification is

$$(u, v, w) \sim (u + t, v + ru, w - ru).$$

This transformation is an affine transformation consisting of a shift (t,0,0) and a linear map

$$A = \begin{pmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ -r & 0 & 1 \end{pmatrix} \in SL(3, \mathbb{Z}).$$

The linear transformation A is the monodromy of the fiber, when we view this non-compact CY as a T^3 fibration. Mirror symmetry will now replace the torus fibers with their duals, and the monodromy A with the dual monodromy A^{-T} . So the B-model can be described as a quotient of the dual coordinates given by

$$(u, v, w) \sim (u + t - rv + rw, v, w).$$

In order to map this B-model to the NS5-brane and finally the I-brane, we have

to perform one more T-duality on the combination v+w. That coordinate is not touched by the action of the framing and it will be subsequently ignored. If we relabel the coordinates as

$$x = u, \qquad y = v - w,$$

we see that this gives indeed a T^2 curve, embedded as the zero section y=0 in the geometry $\mathcal B$ defined as the quotient of $\mathbb C^*\times\mathbb C$ by

$$(x,y) \sim (x+t-ry,y).$$
 (7.2)

The A-model topological string partition function is computed as [121, 231]

$$Z_{\rm top}(t,\lambda) = e^{-t^3/6r^2\lambda^2} Q^{-1/24} \sum_{R} Q^{|R|} q^{r\kappa_R/2},$$

where $Q=e^{-t}$ with t the Kähler parameter of the torus and $q=e^{-\lambda}$, whereas |R| is the number of boxes of the Young tableau R and $\kappa_R=2\sum_{\square\in R}i(\square)-j(\square)$. After the mirror transformation t becomes the modulus of the elliptic curve T^2 . The instanton part of Z_{top} can be rewritten in the form

$$Z_{\text{qu}}(t,\lambda) = \tag{7.3}$$

$$= \oint \frac{dy}{2\pi i y} \prod_{n=0}^{\infty} \left(1 + y Q^{n+1/2} q^{r(n+1/2)^2/2} \right) \left(1 + y^{-1} Q^{n+1/2} q^{-r(n+1/2)^2/2} \right)$$

which is familiar from [232, 233] in the case r=1. In this model the genus zero answer does not have instanton contributions and so is given entirely by the classical cubic form $\mathcal{F}_0(t) = -\frac{1}{6r^2}t^3$, while at genus one the classical and quantum contributions combine into

$$\mathcal{F}_1(t) = -\log \eta(Q).$$

The g-loop contributions \mathcal{F}_g , for g>1 and r>0, incorporate only quantum effects. They are quasi-modular forms of weight 6g-6 that can be expressed as polynomials in the Eisenstein series $E_2(q)$, $E_4(q)$ and $E_6(q)$, where

$$E_k(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1}q^n}{1 - q^n}$$

and B_n are the Bernoulli numbers.

In fact, it is well-known that this answer is reproduced by a chiral fermion field with action [234]

$$S = \frac{1}{\pi} \int_{T^2} d^2 x \; \psi^{\dagger} \left(\bar{\partial} - r \lambda \partial^2 \right) \psi. \tag{7.4}$$

We will re-derive this same answer from the fermionic perspective we have developed in this thesis below. For now note that this action can be bosonized into [233, 235]

$$S = \frac{1}{\pi} \int_{T^2} d^2x \left(\frac{1}{2} \partial \phi \bar{\partial} \phi - \frac{r\lambda}{6} (\partial \phi)^3 \right), \tag{7.5}$$

which is closely related to the Kodaira-Spencer field theory on the Calabi-Yau manifold. This Kodaira-Spencer theory reduces to a free boson ϕ on a cylinder, while the framing quantizes into an action of the zero mode of the W^3 operator [128]

$$W_0^3 = \oint dx \frac{(\partial \phi)^3}{3}.$$

This implies that W_0^3 defines how to glue the torus quantum mechanically,

$$Z_{\rm top} = {\rm Tr} \exp \left(-\frac{r\lambda}{2} \ W_0^3 \right),$$

explaining (7.5). The action of W_0^3 is quadratic in the fermions and therefore acts on the single fermion states.

The topological string partition function (7.3) is obtained as the fermion number zero sector. Including a sum over the U(1) flux gives the full fermion partition function that corresponds to the I-brane. This can be thought of as a generalized Jacobi triple formula [236]. Adding the classical contributions we obtain

$$\begin{split} &Z(v,t,\lambda) = \\ &= e^{-t^3/6r^2\lambda^2}Q^{-1/24}\prod_{n=0}^{\infty}(1+yQ^{n+1/2}q^{r(n+1/2)^2/2})(1+y^{-1}Q^{n+1/2}q^{-r(n+1/2)^2/2}) \\ &= \sum_{p=-\infty}^{\infty}y^pe^{-t^3/6r^2\lambda^2}e^{-pt^2/2r\lambda}Q^{p^2/2-1/24}q^{rp^3/6-rp/24}Z_{\mathrm{qu}}(t+rp\lambda,\lambda) \\ &= \sum_{p=-\infty}^{\infty}y^pZ_{\mathrm{top}}(t+rp\lambda,\lambda). \end{split}$$

In the second line we have extracted a factor $e^{-t^2/2r\lambda}$ out of y. This is the result of turning on flux in the I-brane set-up, and corresponds to the D4-brane tension on the BPS side. Notice that the combination $rp \in r\mathbb{Z}$. This is because rp is the Poincaré dual of the four-cycle having intersection number $\pm r$ with $[T^2]$. Hence this indeed reproduces formula (7.23) with an appropriate choice of cubic form. For r=0 this result reduces to the standard Jacobi triple formula

$$Z_{r=0} = \frac{\theta_3(y, Q)}{\eta(Q)} = \sum_{p \in \mathbb{Z}} \frac{Q^{n^2/2} y^n}{\eta(Q)}.$$
 (7.6)

We now come to deriving (7.4) from the perspective of this thesis. From the considerations in the last chapters it is clear that we have a free fermion system living on T^2 with the *standard* action. The only subtlety has to do with the fact that T^2 is at the boundary of a non-commutative plane and as we will see this is crucial in recovering (7.4). From (7.2) we see that $x \sim x + t - ry$. If we treated y as commuting with x we could set it to y = 0 and we have a copy of the torus. But here we know that y does not commute with x. So we have a free fermion on a torus where the modulus is changed from

$$t \to t - ry$$
.

The variation of t can be absorbed into the fermionic action by the usual Beltrami differential $\mu_{\bar{z}}^z = \delta t$:

$$S = \frac{1}{\pi} \int_{T^2} d^2 x \; \psi^{\dagger} \left(\bar{\partial} + \mu \partial \right) \psi.$$

Here we need to substitute $\mu=\delta t=-ry$. In the classical case where y is commuting, this would give $\mu=0$ and we get the same system as the usual fermions. However, since x and y do not commute, we should view $y=\lambda\partial_x$ leading to $\mu=-ry=-r\lambda\partial_x$. Substituting this operator for μ in the above action reproduces (7.4). We have thus re-derived the known result for the topological string in this background from our framework.

7.1.2 Genus two curve

An interesting generalization of the elliptic curve example is given by a local Calabi-Yau geometry containing a genus two curve. Its mirror model can be constructed using the topological vertex technology of Section 4.3.3. Although the vertex technology is able to deal with arbitrary toric curves, it is instructive to see this explicit case in more detail.

Let us start in the A-model with the toric diagram of the resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$ (see Section 4.2.1) and identify the two pairs of parallel external legs, as shown in Fig. 7.1. In this section we refer to this geometry as \tilde{X} . The B-model geometry corresponding to \tilde{X} is a locally elliptic Calabi-Yau X, described by an equation of the form uv=H(x,y), where H vanishes on a compact genus two Riemann surface Σ .

This B-model geometry is well-studied in [140] as an example of an elliptic threefold geometrically engineering a 6-dimensional gauge theory on $\mathbb{R}^4 \times T^2$. The prepotential of this gauge theory is computed as the A-model partition function of \tilde{X} . Since this is a topological vertex calculation, the all-genus partition function is known. Moreover, instanton calculus in the 6-dimensional gauge theory shows that it can be elegantly rewritten in terms of the equivariant elliptic

genus of an instanton moduli space. The equivariant parameter q equals $e^{-\lambda}$ on the A-model side.

Explicitly, the A-model on \tilde{X} can be expressed in topological vertices as

$$Z_{\mathrm{qu}}(Q_1,Q_2,Q_3) = \sum_{R_1,R_2,R_3} Q_1^{R_1} Q_2^{R_2} Q_3^{R_3} (-)^{l_{R_1}+l_{R_2}+l_{R_3}} C_{R_1R_2R_3} C_{R_1^t R_2^t R_3^t},$$

where $Q_i = \exp(-t_i)$ represent the exponentiated Kähler classes of the legs with attached U(N) representations R_i , and where $C_{R_1R_2R_3}$ is the topological vertex (see Section 4.3.3). Notice that $C_{R_1R_2R_3}$ is symmetric under permutations of the R_i , while in terms of the toric graph it is more natural to use the variables

$$Q_{\sigma} := Q_1 Q_3, \qquad Q_{\rho} := Q_1 Q_2, \qquad Q_{\nu} = Q_1, \tag{7.7}$$

that exhibit the \mathbb{Z}_2 symmetry between Q_{σ} and Q_{ϱ} . Using these definitions

$$\begin{split} Z_{\text{qu}}(q,\rho,\sigma,\nu) &= \sum_{R} Q_{\rho}^{l_R} \prod_{\square \in R} \frac{(1 - Q_{\nu} q^{h(\square)})(1 - Q_{\nu}^{-1} q^{h(\square)})}{(1 - q^{h(\square)})^2} \\ &\times \prod_{k=1}^{\infty} \frac{(1 - Q_{\sigma}^k Q_{\nu} q^{h(\square)})(1 - Q_{\sigma}^k Q_{\nu} q^{-h(\square)})(1 - Q_{\sigma}^k Q_{\nu}^{-1} q^{h(\square)})(1 - Q_{\sigma}^k Q_{\nu}^{-1} q^{-h(\square)})}{(1 - Q_{\sigma}^k q^{h(\square)})^2(1 - Q_{\sigma}^k q^{-h(\square)})^2(1 - Q_{\sigma}^k)}. \end{split}$$

And this may be rewritten as [237]

$$Z_{\text{qu}}(q, \rho, \sigma, \nu) = \sum_{k \geq 0} Q_{\rho}^{k} \chi((\mathbb{C}^{2})^{[k]}; Q_{\sigma}, Q_{\nu})(q, q^{-1}) =$$

$$\prod_{\substack{k, a \geq 0, \ b = -j, \ l > 0, \ c \in \mathbb{Z}}} \prod_{\substack{l = -j, \ l > 0, \ c \in \mathbb{Z}}} \left(\frac{(1 - Q_{\rho}^{l} Q_{\sigma}^{a} Q_{\nu}^{c-1} q^{2b+k+1})(1 - Q_{\rho}^{l} Q_{\sigma}^{a} Q_{\nu}^{c+1} q^{2b+k+1})}{(1 - Q_{\rho}^{l} Q_{\sigma}^{a} Q_{\nu}^{c} q^{2b+k+2})(1 - Q_{\rho}^{l} Q_{\sigma}^{a} Q_{\nu}^{c} q^{2b+k})} \right)^{(k+1)C(la,j,c)}$$

with $b=-j,-j+1,\ldots,j-1,j$ and $q=e^{-\lambda}$, whereas the coefficients C(a,j,c) are related to the equivariant elliptic genus of \mathbb{C}^2 in the following way

$$\chi(\mathbb{C}^2, y, p, q) = \prod_{n \ge 1} \frac{(1 - yp^n q)(1 - y^{-1}p^n q^{-1})(1 - yp^n q^{-1})(1 - y^{-1}p^n q)}{(1 - p^n q)(1 - p^n q^{-1})(1 - p^n q^{-1})(1 - p^n q)}$$
$$= \sum_{a, 2j \ge 0} \sum_{c \in \mathbb{Z}} C(a, j, c)p^a (q^{2j} + q^{2(j-1)} + \dots + q^{-2j})y^c.$$

Starting with the IIA background $TN_1 \times \tilde{X}$ and going backwards through the duality chain, we find ourselves in the I-brane set-up on $\mathbb{R}^3 \times T^4 \times \mathbb{R}^2 \times S^1$. The genus two curve Σ is holomorphically embedded in the abelian surface T^4 by the Abel-Jacobi map. The I-brane is the intersection of a D4-brane wrapping $\mathbb{R}^3 \times \Sigma$ and a D6-brane wrapping $T^4 \times \mathbb{R}^2 \times S^1$. The aim of this section is to give

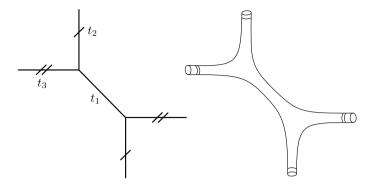


Figure 7.1: The resolved conifold with identified legs (left) and its mirror (right). The parameters t_1 , t_2 and t_3 parametrize the Kähler lengths of the toric diagram on the left.

an interpretation of the above A-model result on \tilde{X} in the I-brane picture.

The case $\lambda = 0$

As a result of the duality chain, we expect that the 1-loop free energy $\mathcal{F}_{1,\text{top}}$ of the topological A-model equals the free energy $\mathcal{F}_{1,\text{boson}} = -\frac{1}{2}\log\det\Delta_{\Sigma}$ of a chiral boson on Σ . Another sum over the lattice of momenta should then result in the chiral fermion determinant. Since not only the A-model partition function, but also the partition function of chiral bosons on a genus two surface is known, we can perform an explicit check of these conjectures.

Singling out the λ^0 -part of the A-model partition function (7.8), yields the sum

$$\mathcal{F}_{1,\text{qu}}(\rho,\sigma,\nu) = \tilde{c}(kl,m) \prod_{k,l,m} \log \left(1 - e^{2\pi i(k\rho + l\sigma + m\nu)} \right),$$

where the coefficients $\tilde{c}(kl,m)$ are related to the Fourier coefficients C(a,j,c) as

$$\tilde{c}(kl,m) = -\sum_{j \in (\mathbb{Z}/2)_{\geq 0}} \sum_{b=-j}^{j} \left[\left(2b^2 - \frac{1}{12} \right) \left(C(kl,j,m+1) + \right. \right.$$

$$\left. + C(kl,j,m-1) \right) - \left(4b^2 + \frac{5}{6} \right) C(kl,j,m) \right].$$
(7.9)

Remarkably, the same relation can be found by rewriting the elliptic genus of a K3 surface, which is the unique weak Jacobi form of index 1 and weight 0. This elliptic genus has an expansion

$$\chi(K3,\tau,z) = \sum_{h \ge 0, m \in \mathbb{Z}} 24 c (4h - m^2) e^{2\pi i (h\tau + mz)},$$

and can be represented as an integral over the equivariant elliptic genus of \mathbb{C}^2 :

$$\chi(K3, y, p) = -y^{-1} \int_{K3} x^{2} \prod_{n \ge 1} \frac{(1 - yp^{n-1}q^{-1})(1 - yp^{n-1}q)(1 - y^{-1}p^{n}q^{-1})(1 - y^{-1}p^{n}q)}{(1 - p^{n-1}q^{-1})(1 - p^{n-1}q)(1 - p^{n}q^{-1})(1 - p^{n}q)}
= -\int_{K3} x^{2} \left(\frac{y + y^{-1} - q - q^{-1}}{A(x)A(-x)} \right) \chi(\mathbb{C}^{2}, y, p, q),$$
(7.10)

with $q=e^x$ and $A(x):=\sum_{k\geq 0}\frac{x^k}{(k+1)!}$. Writing out this equation in terms of the K3 and \mathbb{C}^2 coefficients, reveals exactly equation (7.9) where $\tilde{c}(kl,m)$ is exchanged with $c(kl,m)=c(4kl-m^2)$.

This identification implies that $Z_{1,qu} = \exp(\mathcal{F}_{1,qu})$ corresponds to the 24th root of the generating function of the elliptic genera of symmetric products of K3's:

$$(Z_{1,qu}(\rho,\sigma,\nu))^{24} = \sum_{N} e^{2\pi i N\sigma} \chi_{\rho,\nu} \left((K3)^{N} / S_{N} \right) =$$

$$= \prod_{\substack{k>0,l \geq 0,\\ m \in \mathbb{Z}}} \left(1 - e^{2\pi i (k\rho + l\sigma + m\nu)} \right)^{-24c(4kl - m^{2})}. \tag{7.11}$$

When the K3 surface is realized as an elliptic fibration, with 24 points on the elliptic base where the fibration degenerates in the simplest possible way, we can think about the K3 surface as consisting of 24 local TN_1 -spaces. The above result then motivates us to relate $Z_{1,\mathrm{qu}}$ to one such TN_1 -factor.

Furthermore, $Z_{1,qu}$ is closely related to the generating function

$$e^{-\pi i(\rho+\sigma+v)/12} \prod_{(k,l,m)>0} \left(1 - e^{2\pi i(k\rho+l\sigma+m\nu)}\right)^{-c(4kl-m^2)},$$
 (7.12)

where (k,l,m)>0 means $k,l\geq 0, m\in \mathbb{Z}$, but m<0 when k=l=0. The first terms on the second line of equation (7.11) have a clear interpretation as classical contributions to the genus 1 topological string amplitude (proportional to the Kähler class $t=\rho+\sigma+\nu$). We loosely refer to the above generating function as the total genus 1 partition function $Z_{1,\text{top}}$ for the genus 2 Calabi-Yau.

The 24th power of this topological partition function

$$Z_{1,\mathrm{top}}(\Omega)^{24} = \frac{1}{\Phi_{10}(\Omega)}, \quad \mathrm{with} \quad \Omega = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix}$$

is well-known to both mathematicians and physicists. Mathematically, it is characterized by its nice transformation properties under the symplectic group $Sp(2,\mathbb{Z})$. The form $\Phi_{10}(\Omega)$ transforms as

$$\Phi_{10}\left(g(\Omega)\right) = \left(\det(C\Omega + D)\right)^{10}\Phi_{10}\left(\Omega\right)$$

under $Sp(2,\mathbb{Z})$ -transformations. It is thus $Sp(2,\mathbb{Z})$ -automorphic of weight 10. In fact, it is the unique form with this property [238, 239]. In relation to the topological string amplitude, we should of course interpret Ω as the period matrix of the genus 2 surface.

The automorphic form $1/\Phi_{10}$ is familiar in string theory as the partition function of 24 chiral bosons [240, 241]. More precisely, it appears as the holomorphic part of the (worldsheet) genus 2 bosonic string partition function, or equivalently, as the left-moving partition function of the heterotic string wrapping a genus 2 surface. Both partition functions describe 26 chiral bosons in the lightcone gauge. Effectively, their partition function thus captures 24 chiral bosons.

So after adding the classical contributions to $\mathcal{F}_{1,\mathrm{qu}}(\tilde{X})$, we conclude that the total 1-loop partition function $Z_{1,\mathrm{top}} = \exp(\mathcal{F}_{1,\mathrm{top}})$ of the B-model topological string on X equals the partition function Z_{boson} of a single boson:

$$Z_{\text{boson}}(\rho, \sigma, \nu) = e^{\mathcal{F}_{1,\text{top}}(\rho, \sigma, \nu)}.$$

In order to find the contribution to the all-genus partition function for small λ , we have to consider $\mathcal{F}_{0,\text{top}}$ as well. In the B-model on X its second derivative has a simple interpretation: it is just the period matrix Ω_{ij} of the genus two curve Σ . In terms of the mirror map, these periods will have classical contributions linear in ρ , σ and τ , and quantum corrections determined by $Z_{\text{qu}}(\tilde{X})$. We will write these down in the next paragraph. Right now, let us conclude with

$$Z_{\text{fermion}}(\rho, \sigma, \nu) = \sum_{p_1, p_2 \in \mathbb{Z}} e^{\pi i p_i \Omega_{ij} p_j} e^{\mathcal{F}_{1, \text{top}}}(\rho, \sigma, \nu).$$

Automorphic properties

Knowing the full instanton partition function (7.8) makes it possible to examine the λ -corrections to $\mathcal{F}_{1,\text{top}}$ explicitly. In fact, let us start more generally with the Gopakumar-Vafa partition function (4.47)

$$Z_{\text{qu}} = \prod_{d \in H_2} \prod_{m \in \mathbb{Z}} \prod_{k=-m}^{m} \prod_{n=1}^{\infty} (1 - q^{k+n} Q^d)^{(-1)^{m+1} n \text{GV}_d^m}.$$

 $^{^1}$ There is a subtlety here. Performed computations of this partition function show that, unlike for the partition function $1/\eta^{24}$ of a heterotic string wrapping a 2-torus, the ghost determinants and the light-cone directions do not completely cancel out in the genus 2 configuration (see the remarks in [242]). However, we expect that there exists a gauge choice in which there is a clean interpretation in terms of 24 bosons. We thank E. Verlinde for a discussion on this point.

In order to get the g-loop free energies we note that

$$\begin{split} & \log \prod_{k \geq 1} (1 - Yq^{k+l})^k = \\ & = -\frac{1}{\lambda^2} \text{Li}_3(Y) + \frac{1}{2} (l^2 - \frac{1}{6}) \log(1 - Y) - \lambda^2 \left(\frac{1}{240} - \frac{l^2}{24} + \frac{l^4}{24} \right) \text{Li}_{-1}(Y) - \dots \\ & = : -\sum_{g \geq 0} \lambda^{2g-2} P_{2g}(l) \sum_{n \geq 1} n^{2g-3} Y^n, \end{split}$$

where the degree 2g polynomials $P_{2g}(l)$ are defined through the last equality. Hence

$$\mathcal{F}_{\text{qu}} = -\sum_{g \geq 0} \lambda^{2g-2} \sum_{d \in H_2} \sum_{m \in \mathbb{Z}} \sum_{k=-m}^m (-1)^{m+1} P_{2g}(k) \, \mathrm{GV}_d^m \, \sum_{n=1}^\infty n^{2g-3} \left(Q^d \right)^n.$$

Making this expansion for the genus two Calabi-Yau X reveals that the coefficients

$$c_g = \sum_{m \in \mathbb{Z}} \sum_{k=-m}^{m} (-1)^{m+1} P_{2g}(k) \, GV_d^m$$

are the Fourier coefficients of Jacobi forms $J_g(q,y) = \sum_{k,l} c_g(k,l) q^k y^l$ of weight 2g-2 and index 1. More precisely, we can write the \mathcal{F}_q 's as

$$\begin{split} \mathcal{F}_g(\lambda;Q_{\rho},Q_{\sigma},Q_{\nu}) &= -\sum_{k,l,m} c_g(kl,m) \sum_{n \geq 1} n^{2g-3} (Q_{\rho}^k Q_{\sigma}^l Q_{\nu}^m)^n \\ &= -\sum_{N > 0} Q_{\rho}^N \sum_{kn = N} n^{2g-3} \sum_{l \geq 0,m} c_g(kl,m) Q_{\sigma}^{ln} Q_{\nu}^{mn} \\ &= -\sum_{N > 0} Q_{\rho}^N \sum_{kn = N} N^{2g-3} \sum_{b = 0}^{k-1} k^{2-2g} J_g \left(\frac{n\sigma + b}{k}, n\nu \right) \\ &= -\sum_{N > 0} Q_{\rho}^N T_{g,N}(J_g), \end{split}$$

where $T_{g,N}$ are Hecke operators acting on Jacobi forms of weight 2g-2. This implies that all $\mathcal{F}_{g,\text{top}}$'s are lifts of Jacobi forms, and therefore almost automorphic forms of $O(3,2,\mathbb{Z})=Sp(4,\mathbb{Z})$ [243].

Interpretation in the duality chain

First of all, notice that the partition function of \tilde{X} can be build out of topological vertices, and as such is known to have an interpretation in terms of chiral bosons and fermions [17, 128, 131]. The duality chain elucidates these observations: the chiral fermions can be identified with the intersecting brane fermions. Moreover, the B-field on the D6-brane makes it necessary to treat these fermions as

non-commutative objects, which gives an explanation for the nontrivial transformation properties in [128].

In terms of the gauge theory picture we can just refer to [140]. Here it is shown that the six dimensional gauge theory on $TN_1 \times T^2$ can be engineered with matrix model techniques, revealing Σ as the Seiberg-Witten curve, whose period matrix equals the second derivative of $\mathcal{F}_{0,\text{top}}$.

Finally, the automorphic properties of \mathcal{F}_{top} seems to fit in best in the M5-brane frame of the duality chain. Recall that S-duality relates IIB on $TN_1 \times X$ to a NS5-brane wrapping around $TN_1 \times \Sigma$ in the background $TN_1 \times T^4 \times \mathbb{R}^2$. This lifts to a M5-brane in M-theory on $TN_1 \times T^4 \times \mathbb{R}^2 \times S^1$. Since the M5-brane partition function is expected to be an automorphic form of $O(3,2,\mathbb{Z})$ [244], this perspective offers a physical reason for the automorpic properties.

Actually, we know exactly which Jacobi forms enter: $J_0=\phi_{-2,1}$ is the unique Jacobi form of weight -2 and index 1, $J_1=-\frac{1}{12}\phi_{0,1}=-\frac{1}{24}\chi(K3,q,y)$ as we encountered before, $J_2=\frac{1}{240}E_4\phi_{-2,1}$, $J_3=-\frac{1}{6048}E_6\phi_{-2,1}$ and $J_4=\frac{1}{172800}E_4^2\phi_{-2,1}$ etc. Interestingly, these can all be defined as twisted elliptic genera of TN_1 in the sense that (compare with (7.10))

$$J_g(q,y) = -y^{-1} \int_{TN_1} x^{4-2g} \prod_{n>1} \frac{(1-yp^{n-1}q^{-1})(1-yp^{n-1}q)(1-y^{-1}p^nq^{-1})(1-y^{-1}p^nq)}{(1-p^{n-1}q^{-1})(1-p^{n-1}q)(1-p^nq^{-1})(1-p^nq)},$$

coinciding with the M5-brane point of view. This results longs for a two dimensional conformal field theory interpretation.

7.2 Quarter BPS dyons

The next few sections center on the weight 10 Igusa cusp form

$$\Phi_{10}(\rho, \sigma, \nu) = pqy \prod_{(k,l,m)>0} (1 - p^k q^l y^m)^{24c(4kl - m^2)},$$

with $p=\exp(2\pi i\rho)$, $q=\exp(2\pi i\sigma)$ and $y=\exp(2\pi i\nu)$, that we encountered in equation (7.11) when analyzing the semi-classical contribution to the topological partition function on the genus 2 Calabi-Yau. Let us explain why it plays an important role in the counting of quarter BPS states in an $\mathcal{N}=4$ compactification of string theory down to 4 dimensions.

One perspective to study 4-dimensional $\mathcal{N}=4$ string theory is in the framework of type II theory as a $K3\times T^2$ compactification. Another duality frame is heterotic string theory compactified on a 6-torus. The heterotic compactification is

described by an $\mathcal{N}=4$ supergravity multiplet coupled to 22 additional vector multiplets. In this compactification a vector multiplet consists of a single vector field and six real scalars, while the supergravity multiplet contains the metric, the heterotic axion-dilaton field and six graviphotons. Altogether there are 28 U(1) gauge fields, and hence 28 electric and 28 magnetic conserved charges. Both the magnetic charge vector P and the electric charge vector Q are elements in the lattice $\Gamma^{22,6}$. They may be combined into a charge matrix

$$\begin{pmatrix} Q \\ P \end{pmatrix} \in \Gamma^{22,6} \oplus \Gamma^{22,6}, \tag{7.13}$$

on which heterotic S-duality acts as a $SL(2,\mathbb{Z})$ transformation. From the type IIB point of view the heterotic S-duality group is visible as a geometric T-duality group that acts as the mapping class group on the T^2 -factor.

Half BPS states in $\mathcal{N}=4$ string theory are relatively easy to describe. They either carry a purely electric or magnetic charge, so that they can be counted in the perturbative regime of heterotic string theory in terms of 24 bosonic oscillators in the left-moving sector. This yields

$$d(Q) = \oint d\sigma \frac{e^{-\pi i Q^2 \sigma}}{\eta^{24}(\sigma)} \tag{7.14}$$

states with electric charge Q, and an analogous formula for the number d(P) of magnetic states with charge P (now using ρ as integration variable). These formulas suggest that counting half BPS states in $\mathcal{N}=4$ string theory is in some way related to 24 free bosons on a genus 1 surface (compare for example with formula (7.6) and with the K3 and the \mathbb{R}^4 example in Chapter 3).

Quarter BPS states are more complicated to analyze. As observed in [245], it is natural to introduce the genus 2 period matrix

$$\Omega = \begin{pmatrix} \sigma & \nu \\ \nu & \rho \end{pmatrix}$$

as this decouples into two genus 1 period matrices when $\nu \to 0$. The complex structure parameters of the resulting tori describe the half-BPS electric and magnetic states. The degeneracy formula

$$d(Q, P) = (-1)^{Q \cdot P + 1} \oint d\Omega \frac{e^{-\pi i(Q^2 \sigma + 2Q \cdot P\nu + P^2 \rho)}}{\Phi_{10}(\Omega)}$$
(7.15)

indeed reduces to a product of the electric with the magnetic half-BPS formula (7.14) when the limit $\nu \to 0$ is taken. Notice that the degeneracies d(Q,P) should be invariant under the symmetries of the charge-lattice. Indeed, they are functions $d(1/2\,Q^2,Q\cdot P,1/2\,P^2)$ in terms of the integer invariants $1/2\,Q^2$,

 $1/2\,P^2$ and $Q\cdot P$ on the even lattice $\Gamma^{22,6}\oplus\Gamma^{22,6}$. The physical relation of quarter BPS states to the genus 2 surface with period matrix Ω is not immediately clear at all, though. As we explain in a little bit, this relation can be understood through a number of string dualities.

Recent years have seen a lot of progress in the understanding of the BPS spectrum of 4-dimensional $\mathcal{N}=4$ string theory. For example, the above proposal for the generating function of BPS states has been extended to so-called CHL orbifolds [246]. In the heterotic string perspective such a CHL compactification is obtained by orbifolding $T^4 \times S^1 \times \tilde{S}^1$ by a \mathbb{Z}_N symmetry, that is generated by a product of an internal symmetry together with an order N translation along the circle \tilde{S}^1 . The original Dijkgraaf-Verlinde-Verlinde partition function as well as its CHL extension are now fairly well-understood. They have passed all consistency checks performed so far, such as $SL(2,\mathbb{Z})$ invariance (on which we come back in Section 7.3) and agreement with black hole physics [247, 248, 249].²

Schematically, the genus two surface occurs in the dyon counting problem in the following way, illustrated in Fig. 7.2 [250, 242]. Consider type IIB string theory compactified on $K3 \times T^2$. Half BPS states (that are pointlike in the noncompactified directions in \mathbb{R}^4) in this frame are represented as bound states of D5/NS5-branes that wrap the K3-fold, with D3-branes that wrap some 2cycles in the K3 and with D1/F1-branes. The total bound states furthermore wraps one cycle of the T^2 , depending on whether the half BPS state is electric or magnetic. A quarter BPS state carries both electric and magnetic charges. It can be represented by a network of D5 and NS5-brane bound states that is embedded in the 2-torus. An example of such a network is shown in Fig. 7.2. It consists of two three-vertices that are known as three-string junctions. Compactifying the time direction, in order to calculate a partition function, and lifting into Mtheory we obtain Euclidean M-theory compactified on $K3 \times T^4$. The BPS dyon is now represented as an M5 brane wrapping K3 times a genus two Riemann surface Σ that is holomorphically embedded in T^4 . Electric BPS states wrap one A-cycle on Σ while magnetic BPS states wrap the other one. Since an M5-brane that wraps a K3-surface is dual to a heterotic string, we furthermore recover the partition function $1/\Phi_{10}$ of 24 chiral bosons on a genus two surface (although there are some unsolved subtleties [242, 260]).

This is very similar to the M5-brane description of topological string theory on a non-compact Calabi-Yau threefold as alluded to in Section 7.1. In fact, when we view the K3-surface as 24 copies of a TN_1 -space and the M5-brane wrapping this K3-surface as 24 M5-branes each wrapping a copy of TN_1 , the set-up reduces in type IIA to the intersecting brane background

IIA:
$$\mathbb{R}^3 \times (\Sigma \subset T^4) \times \mathbb{R}^2$$
 (7.16)

²See also [70, 250, 251, 252, 253, 242, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263].

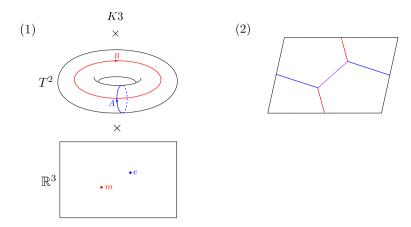


Figure 7.2: (1): BPS dyons in an $\mathcal{N}=4$ compactification of string theory on $K3\times T^2$ correspond to D4/NS5 brane wrappings over the internal K3-manifold times a circle in T^2 . Depending on which 1-cycle in the T^2 is wrapped the resulting 4-dimensional BPS particle will carry electric or magnetic charge. (2): When both 1-cycles in the T^2 are wrapped it is more efficient for the branes to recombine into a network wrapping the T^2 .

with 24 D4-branes wrapping $\mathbb{R}^3 \times \Sigma$ and a D6-brane wrapping $T^4 \times \mathbb{R}^2$, that intersect on the genus 2 surface Σ . This relates the appearance of the automorphic form Φ_{10} as the semi-classical amplitude \mathcal{F}_1 in topological string theory and as a dyon index in 4-dimensional $\mathcal{N}=4$ string theory physically.

In this first section about quarter BPS dyons we explain the duality between the $\mathcal{N}=4$ dyons, string webs and the genus two surface in detail. We pay extra attention to the dependence on background moduli, as this will be important in the description of wall-crossing in the next section. In Section 7.2.1 we review and extend the results of A. Sen in [264] in detail and discuss how the 1/4-BPS dyons are realized as a periodic network of effective strings in type IIB frame at arbitrary moduli. In Section 7.2.2 we review and extend the results of S. Banerjee et al. in [260] by going to Euclidean M-theory and analyze the Riemann surface wrapped by the M5 brane that makes up the 1/4-BPS dyon. In particular we analyze the complex structure of the surface and its relation to the stability of the dyon states. On general grounds and from earlier results we expect the Riemann surface to degenerate in a certain way when the moduli cross a wall of marginal stability [254, 256]. We analyze wall-crossing in great detail in Section 7.3.

7.2.1 The five-brane network

Following S. Banerjee, A. Sen and Y. Srivastava [260], in this subsection we consider 1/4-BPS dyons made up from a type IIB T^2 -compactified network of

effective strings which are bound states of (p,q) strings and K3-wrapped fivebranes. Working in the limit of large K3 and thus heavy five-branes and using the supersymmetry condition of the network [264], we will write down explicit expressions for the shape and size of the network with a given range of values of the IIB axion-dilaton field λ and the complex structure τ of the 2-torus. After that we briefly discuss how the network is realized at generic values of moduli, while leaving the details to Section 7.3.1.

First consider type IIB string theory compactified on the product of a K3 manifold and a 2-torus which we shall call $T^2_{\rm (IIB)}$, and two effective strings wrapping the two homological cycles of the torus. Each effective string is a bound state of F1 and D1 string together with NS5 and D5 branes wrapped on K3.

To be more specific, let's consider the following charges. Suppose we have the Q effective string, which is a bound string of a (n_1,n_2) string together with a K3-wrapped (q_1,q_2) five-brane, wrapping the A-cycle of the $T^2_{({\rm IIB})}$. Wrapping the B-cycle is what we call the P effective string, which is a bound string of a (m_1,m_2) string together with a K3-wrapped (p_1,p_2) five-brane. The three T-duality invariants corresponding to this charge configuration are given by

$$Q^{2} = 2\sum_{i=1}^{2} n_{i}q_{i}, \quad P^{2} = 2\sum_{i=1}^{2} m_{i}p_{i}, \quad Q \cdot P = \sum_{i=1}^{2} (m_{i}q_{i} + n_{i}p_{i}).$$
 (7.17)

In the limit of large K3, the tension of the Q- and P- string are given by

$$T_Q = q_1 - \bar{\lambda}q_2, \quad T_P = p_1 - \bar{\lambda}p_2$$
 (7.18)

rescaled by a factor of the volume of K3 in 10-dimensional Planck unit $V_{K3}^{(\mathrm{P})} = V_{K3}\lambda_2$. Here V_{K3} denotes the the volume of K3 in string unit, and $-\bar{\lambda} = -\lambda_1 + i\lambda_2$ is the axion-dilaton of the type IIB theory. In particular, the string coupling is given by $g_s = \lambda_2^{-1}$. Similarly, we will denote by $-\bar{\tau} = -\tau_1 + i\tau_2$ and $R_B^2\tau_2$ the complex structure and the area of the type IIB torus $T_{(\mathrm{IIB})}^2$ respectively.

Using the above convention, the $SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$ symmetry of the theory acts as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \ \begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \ ,$$

and independently

$$\lambda \to \frac{a'\lambda + b'}{c'\lambda + d'}, \quad \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \to \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}, \quad \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2, \mathbb{Z}) \quad (7.19)$$

for all (Γ_1, Γ_2) strings or five-branes. The second symmetry is the type IIB S-

duality, while the first symmetry is the modular transformation of the type IIB torus $T_{\rm (IIB)}^2$, which is mapped to the S-duality of the heterotic string under string duality.

It will turn out to be useful to organize the above complex structure of the torus $T_{({\rm IIB})}^2$ and the type IIB axion-dilaton field in terms of the following 2×2 symmetric real matrices

$$\mathcal{M}_{\tau} = \frac{1}{\tau_2} \begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix}, \quad \mathcal{M}_{\lambda} = \frac{1}{\lambda_2} \begin{pmatrix} |\lambda|^2 & \lambda_1 \\ \lambda_1 & 1 \end{pmatrix} \; ,$$

which transforms as $\mathcal{M}_{\tau} \to \gamma \mathcal{M}_{\tau} \gamma^T$ and $\mathcal{M}_{\lambda} \to \gamma' \mathcal{M}_{\lambda} \gamma'^T$ under the above $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ transformation. Furthermore, we will use the following standard metric on the space of 2×2 symmetric real matrices X

$$||X||^2 = \det X \,, \tag{7.20}$$

such that both \mathcal{M}_{τ} , \mathcal{M}_{λ} have unit space-like length.

To make the analysis more explicit, let us assume a certain orientation of the string network, given by $q_1p_2-p_1q_2>0$. To ensure the irreducibility of the string network made of the (q_1,q_2) and the (p_1,p_2) five-branes, we will further require $q_1p_2-q_2p_1=1$, namely that the corresponding 2×2 matrix

$$\Gamma = \begin{pmatrix} q_1 & q_2 \\ p_1 & p_2 \end{pmatrix}$$

is an $SL(2,\mathbb{Z})$ matrix [242]. The generalization to the charges with $\Gamma \in GL(2,\mathbb{Z})$, including the opposite orientation of the string network with $q_1p_2-p_1q_2=-1$, is a straightforward modification of the following discussion and will not be separately discussed here³.

Simple kinematic consideration, or relatedly supersymmetry, requires that the three lines meeting at a vertex satisfy the following constraints [264]. The angles formed by the three legs meeting at a vertex in the periodic string network must be the same as the angles formed by the three tension vectors (7.18) of the corresponding charges in a complex plane. Two examples are shown in Fig. 7.3.

As we shall see shortly, how the supersymmetric network will be realized depends on the background moduli of the theory. For the time being, let us focus on the one specific case depicted in the first figure in Fig. 7.3. In this case the statement about the angles simply means the following. If we view the compactification torus as $\mathbb{C}/R_B(\mathbb{Z}-\bar{\tau}\mathbb{Z})$ and draw the network on the same complex

³ It simply involves exchanging τ and $\bar{\tau}$ in equations (7.22) - (7.23), (7.24), (7.31), (7.34), (7.39), (7.42).

plane, the three vectors $\ell_{1,2,3} \in \mathbb{C}$ in this periodic network are given by

$$\ell_1 = t_1(T_Q + T_P), \quad \ell_2 = t_2 T_P, \quad \ell_3 = t_3 T_Q,$$
 (7.21)

where the tension vectors $T_{Q,P}$ are given in (7.18) and $t_{1,2,3} \in \mathbb{R}_+$ are the length parameters given by the background moduli in a way we will now describe.

The fact that this network fits in the geometric torus $T^2_{(IIB)}$ means the length parameters satisfy

$$\begin{pmatrix} t_1 + t_3 & t_1 \\ t_1 & t_1 + t_2 \end{pmatrix} \begin{pmatrix} T_Q \\ T_P \end{pmatrix} = \begin{pmatrix} t_1 + t_3 & t_1 \\ t_1 & t_1 + t_2 \end{pmatrix} \Gamma \begin{pmatrix} 1 \\ -\bar{\lambda} \end{pmatrix} = e^{i\theta} R_B \begin{pmatrix} 1 \\ -\bar{\tau} \end{pmatrix}$$
(7.22)

for some angle θ as shown in Fig. 7.3. The obvious fact that

$$(T_Q + T_P)\bar{\ell}_1 + T_P\bar{\ell}_2 + T_Q\bar{\ell}_3 \in \mathbb{R}_+$$

then gives

$$\theta = \operatorname{Arg}(T_O - \tau T_P). \tag{7.23}$$

The mass of the string network, which is given by the sum of the product of the length of the legs in the type IIB torus and their respective tension, is then given by

$$M_{\text{IIB}} = (T_Q + T_P)\bar{\ell}_1 + T_Q\bar{\ell}_2 + T_P\bar{\ell}_3 = R_B V_{K3} \lambda_2 | T_Q - \tau T_P |.$$
 (7.24)

Furthermore, by first solving (7.22) for the simplest case with $\Gamma=\mathbb{1}_{2\times 2}$ and considering other solutions related to it by a type IIB S-duality (7.19), we obtain the expression for the lengths of the three different legs in the string network

$$\begin{pmatrix} t_1+t_3 & t_1 \\ t_1 & t_1+t_2 \end{pmatrix} = \sqrt{\frac{R_B^2\tau_2}{\lambda_2}} \, \frac{\mathcal{M}_\tau^{-1} + (\Gamma^{-1})^T \mathcal{M}_\lambda \Gamma^{-1}}{\|\mathcal{M}_\tau^{-1} + (\Gamma^{-1})^T \mathcal{M}_\lambda \Gamma^{-1}\|}.$$

Like in equation (7.7) this shows that the lengths of the string network are naturally expressed in terms of variables t_1 , $t_1 + t_2$ and $t_1 + t_3$. As will become clear in equation (7.40), the above matrix corresponds as well to the period matrix of the genus 2 surface. The string network is thus equivalent to the toric graph in Section 7.1.2.

While the quantity on the right-hand side depends on our specific choice among charges lying on the same T-duality orbit and furthermore its derivation is only valid in the part of the moduli space with $V_{K3} \gg 1$, in what follows we shall see how this quantity can naturally be written as an T-duality invariant expression which is well-defined for general values of moduli.

From the 4-dimensional macroscopic analysis we know the BPS mass of a dyon

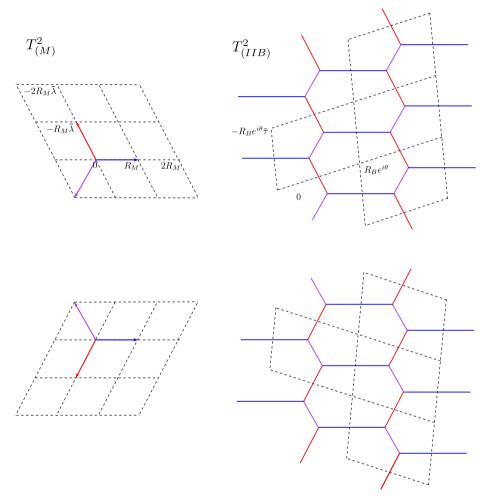


Figure 7.3: An example of the effective string network, with $(q_1, q_2) = (1, 0)$ and $(p_1, p_2) = (0, 1)$. Which one of the network is realized depends on the sign of the off-diagonal component of the moduli vector Z.

should be expressed in terms of the charges and the moduli in a specific way [265, 266, 259]. Especially, in the heterotic frame it depends on the right-moving charges only, which can be combined into the following T-duality invariant matrix

$$\Lambda_{Q_R,P_R} = \begin{pmatrix} Q_R \cdot Q_R & Q_R \cdot P_R \\ Q_R \cdot P_R & P_R \cdot P_R \end{pmatrix}$$

and further combined with the heterotic axion-dilaton into the matrix

$$Z = \frac{1}{\tau_2} \begin{pmatrix} 1 & -\tau_1 \\ -\tau_1 & |\tau|^2 \end{pmatrix} + \frac{1}{\|\Lambda_{Q_R, P_R}\|} \begin{pmatrix} P_R \cdot P_R & -Q_R \cdot P_R \\ -Q_R \cdot P_R & Q_R \cdot Q_R \end{pmatrix}, \tag{7.25}$$

which is again invariant under T-duality transformation.

In terms of these 2×2 matrices, the mass in string frame is given by

$$M_{\text{IIB}}^{2} = V_{K3} R_{B}^{2} \lambda_{2}^{2} \left(|Q_{R} - \bar{\tau} P_{R}|^{2} + 2\tau_{2} \|\Lambda_{Q_{R}, P_{R}}\| \right)$$

$$= V_{K3} R_{B}^{2} \tau_{2} \lambda_{2}^{2} \|\Lambda_{Q_{R}, P_{R}}\| \|Z\|^{2}.$$
(7.26)

Comparing with the mass formula for the string network (7.24), we can read out the expression for Q_R, P_R

$$\|\Lambda_{Q_R,P_R}\| = V_{K3}\lambda_2 = V_{K3}^{(P)}, \quad \frac{\Lambda_{Q_R,P_R}}{\|\Lambda_{Q_R,P_R}\|} = \Gamma \mathcal{M}_{\lambda}^{-1}\Gamma^T,$$

and thus

$$Z = \mathcal{M}_{\tau}^{-1} + (\Gamma^{-1})^T \mathcal{M}_{\lambda} \Gamma^{-1}.$$

From this we see that the moduli vector Z has the following two physical roles in the type IIB supersymmetric string network. First its length gives the mass of the network as in (7.26). Furthermore its direction dictates the relation between the lengths of various legs of the network by

$$\begin{pmatrix} t_1 + t_3 & t_1 \\ t_1 & t_1 + t_2 \end{pmatrix} = \sqrt{\frac{R_B^2 \tau_2}{\lambda_2}} \frac{Z}{\|Z\|}.$$
 (7.27)

But there is clearly a problem with this formula. As the reader might have noticed, the above formula is devoid of a geometric meaning when one or more of the length parameters t_i is negative. To take the simplest example, while the diagonal terms of the matrix Z are manifestly positive (7.25), the off-diagonal term can be of either sign. It means that when the entries of Z fail to be all positive, for example, the network we have just described cannot exist.

The solution to this problem is the following. As we have mentioned earlier, there are more than just one possible way to realize a supersymmetric string network with given 4D charges. For illustration let's now consider the following example. Writing $Z=\begin{pmatrix}z_1&z\\z_2\end{pmatrix}$ and assume $z_1,z_2>-z>0$ such that the network we discussed above does not exist, we will now see that the network is realized as a periodic honeycomb network with three legs given by

$$\ell_1 = t_1(T_Q - T_P), \quad \ell_2 = -t_2T_P, \quad \ell_3 = t_3T_Q.$$
 (7.28)

Repeating the same analysis as before we obtain the same expression for the angle θ which measures the "tilt" of the network (7.23) and the mass of the network (7.24), but now the length parameters are given instead by

$$\begin{pmatrix} t_1 + t_3 & -t_1 \\ -t_1 & t_1 + t_2 \end{pmatrix} = \sqrt{\frac{R_B^2 \tau_2}{\lambda_2}} \frac{Z}{\|\mathcal{Z}\|}.$$
 (7.29)

It is then easy to see that the above network (7.28), shown in the second figure in Fig. 7.3, does exist for the range of moduli space $z_1, z_2 > -z > 0$.

In general, as will be discussed in detail in Section 7.3.1, for any arbitrary point in the moduli space, exactly one network which is given by effective strings with charges aQ+bP and cQ+dP wrapping the cycles dA-cB and -bA+aB, will be realized. Here we again use A and B to denote the A- and B-cycle of the compactification torus $T^2_{\rm (IIB)}$. And the integers

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$$

are determined by the value of moduli, which is given by the values of λ, τ in the five-brane system we consider. Recall that the requirement that the inverse of an element in $GL(2,\mathbb{Z})$ is again an element of the same group means that the matrix γ must have determinant ± 1 .

In more details, the periodic network will consist of three legs given by

$$\ell_1 = t_1 \left((a+c)T_Q + (b+d)T_P \right), \quad \ell_2 = t_2 \left(cT_Q + dT_P \right), \quad \ell_3 = t_3 \left(aT_Q + bT_P \right)$$

with length parameters given by

$$\begin{pmatrix} t_1 + t_3 & t_1 \\ tau_1 & t_1 + t_2 \end{pmatrix} = \sqrt{\frac{R_B^2 \tau_2}{\lambda_2}} \frac{(\gamma^{-1})^T Z \gamma^{-1}}{\|Z\|}.$$
 (7.30)

As will be explained in more details in Section 7.3.1, for a given point in the moduli space, the integral matrix γ has to satisfy the requirement that the above equation has a solution with $t_{1,2,3} \in \mathbb{R}_+$.

7.2.2 The Riemann surface

Following the idea of [250] and adopting the approach of [260], in this subsection we study the holomorphic embedding of a Riemann surface wrapped by the M5 brane in Euclidean M-theory which makes up the 1/4-BPS dyons of the theory. In particular, following [260] we write down the period matrix of such a surface for generic values of the moduli of the theory, and discuss the relationship between the degeneration of the surface and the crossing of walls of marginal stability where some dyon states might become unstable.

In order to compute the dyon partition function of the compactified type IIB theory discussed in the previous Subsection, it is necessary to go to the Euclidean spacetime with a Euclidean time circle. Now recall that type IIB compactified on a circle is equivalent to M-theory compactified on a torus, which we will refer to as the "M-theory torus" $T_{(\mathrm{M})}^2$, by a T-duality transformation followed by a lift to eleven dimensions. In particular, letting the eleventh-dimension circle to have asymptotic radius R_M , the complex moduli and the area of the M-theory torus $T_{(\mathrm{M})}^2$ are given by the type IIB axion-dilaton as $-\bar{\lambda}$ and $R_M^2\lambda_2$.

In other words, in order to discuss the dyon partition function we consider M-theory compactified down to \mathbb{R}^3 on the internal manifold $K3 \times T_{\mathrm{M}}^2 \times T_{(\mathrm{IIB})}^2$. Since the configuration we will be considering is the M5 brane wrapping the whole K3, we will now focus on the $T_{(\mathrm{M})}^2 \times T_{(\mathrm{IIB})}^2$ factor whose moduli play the most important role in the rest of the Chapter. Clearly, it can be thought of as a space of the form \mathbb{C}^2/Λ , where the two complex planes can be taken to be the complex planes associated with the tori $T_{(\mathrm{M})}^2$ and $T_{(\mathrm{IIB})}^2$ respectively. Writing the coordinate of the two complex planes as $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the lattice Λ is generated by the following four vectors in \mathbb{R}^4 parametrized by (x_1,y_1,x_2,y_2) :

$$e_{1} = R_{M} (1, 0, 0, 0)$$

$$e_{2} = R_{M} (-\operatorname{Re}\bar{\lambda}, -\operatorname{Im}\bar{\lambda}, 0, 0)$$

$$e_{3} = R_{B} (0, 0, \operatorname{Re} e^{i\theta}, \operatorname{Im} e^{i\theta})$$

$$e_{4} = R_{B} (0, 0, -\operatorname{Re} e^{i\theta}\bar{\tau}, -\operatorname{Im} e^{i\theta}\bar{\tau}).$$
(7.31)

For convenience we have chosen the coordinates of \mathbb{R}^4 such that the Q-string lies along the x_2 -axis. See Fig. 7.3.

A priori there is no reason to require the two tori $T_{(\mathrm{M})}^2$ and $T_{(\mathrm{IIB})}^2$ be orthogonal to each other. A non-zero inner product in \mathbb{R}^4 between the vectors $\{e_1,e_2\}$ and $\{e_3,e_4\}$ (7.31) corresponds to turning on time-like Wilson lines for the B- and C- two-form fields along the A- and B-cycles of compactification torus $T_{(\mathrm{IIB})}^2$ in the original type IIB theory. But since they are absent in the Lorentzian type IIB theory we started with, in most of the following discussion we will assume

that such a cross-term is absent.

After describing the M-theory set-up we now turn to the dyons in the theory. The type IIB effective string network discussed in Equation (7.17) now becomes a genus two Riemann surface Σ inside T^4 upon compactifying the temporal direction and going to the M-theory frame, which has the effect of fattening the network in Fig. 7.3. As usual, we would like to choose a canonical basis for the homology cycles of the Riemann surface Σ such that the A- and B-cycles have the following canonical intersections:

$$A_a \cap B_b = \delta_{ab}, \quad A_a \cap A_b = B_a \cap B_b = 0, \quad a, b = 1, 2.$$
 (7.32)

We now choose the basis cycles $A_{1,2}$ and $B_{1,2}$ as shown in Fig. 7.4. Beware that they are not directly related to the A- and B-cycles of the tori $T^2_{(IIB)}$ and $T^2_{(M)}$.

From the charges of the network, which translate in the geometry into the homology classes of the two-cycle in T^4 wrapped by the M5 brane, we see that the Riemann surface Σ defines a lattice inside \mathbb{R}^4 , with generators related to those of Λ in the following way

$$\begin{pmatrix} \oint_{A_1} dX \\ \oint_{A_2} dX \end{pmatrix} = \Gamma \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad \begin{pmatrix} \oint_{B_1} dX \\ \oint_{B_2} dX \end{pmatrix} = \begin{pmatrix} e_3 \\ e_4 \end{pmatrix}. \tag{7.33}$$

In the above formula, $dX=(dx_1,dy_1,dx_2,dy_2)$ is the pullback on the Riemann surface \varSigma of the one-forms on \mathbb{R}^4 in which \varSigma is embedded⁴. It is easy to see that the this lattice is identical to the lattice Λ (7.31) generated by $e_{1,\cdots,4}$ which defines the spacetime four-torus in \mathbb{R}^4 , as long as we restrict to the M5 brane charges with $|\text{det}\Gamma|=g.c.d.(Q\wedge P)=1$. We shall say more about the role of this lattice for the Riemann surface \varSigma shortly, but for that we will first need to discuss the complex structure of this surface.

The spacetime supersymmetry requires that the genus two Riemann surface to be holomorphically embedded in the spacetime T^4 . To find the period matrix of the Riemann surface, we are interested in finding the complex structure of \mathbb{R}^4 which is compatible with the holomorphicity of Σ . By definition this complex structure will then determine the complex structure of the Riemann surface. Using the natural flat metric on \mathbb{R}^4 , its volume form is given by

$$vol = dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2,$$

and the space of self-dual two-forms in \mathbb{R}^4 will then be spanned by the following

⁴For convenience and given that there's little room for confusion, here and elsewhere in this section we will not distinguish in our notation for a form in \mathbb{R}^4 and its pullback along the embedding map (7.37) onto the Riemann surface.

three two-forms

$$f_1 = dx_1 \wedge dy_1 - dx_2 \wedge dy_2$$

$$f_2 = dx_1 \wedge dy_2 + dx_2 \wedge dy_1$$

$$f_3 = dx_1 \wedge dx_2 + dy_1 \wedge dy_2.$$

Recall that this 3-dimensional space corresponds to the S^2 worth of complex structures of the hyper-Kähler space \mathbb{R}^4 in the following way. For a given complex structure two-form Υ , the space of self-dual two-forms is spanned by the (2,0), (1,1) and (0,2) form $\Upsilon=\Upsilon_1+i\Upsilon_2$, J and $\bar{\Upsilon}=\Upsilon_1-i\Upsilon_2$, where J is the Kähler form. From

$$\Upsilon \wedge \bar{\Upsilon} = J \wedge J = \text{vol}$$

 $\Upsilon \wedge \Upsilon = \Upsilon \wedge J = 0$

we conclude that $J, \Upsilon_1, \Upsilon_2$ are mutually perpendicular in the pairing $\frac{\cdot \wedge \cdot}{\text{vol}}$ for two-forms and $\Upsilon_1 \wedge \Upsilon_1 = \Upsilon_2 \wedge \Upsilon_2 = \frac{1}{2}J \wedge J$.

If the Riemann surface Σ is holomorphically embedded in \mathbb{R}^4 with respect to the complex structure Υ , the following condition is satisfied

$$\int_{\Sigma} \Upsilon = 0.$$

To find the complex structure Υ compatible with the holomorphicity of Σ we therefore have to find a vector J in the 3-dimensional space of self-dual two-forms, such that the plane normal to it is the plane of all two-forms f satisfying $\int_{\Sigma} f = 0$. This plane will then be the plane spanned by Υ_1 and Υ_2 . From (7.33) we can compute the value of $f_{1,2,3}$ integrated over the surface Σ using the Riemann bilinear relation. From the results

$$\int_{\Sigma} f_{1} = 0$$

$$\int_{\Sigma} f_{2} = -R_{B}R_{M} \operatorname{Im} \left(e^{-i\theta} \left((q_{1} - \bar{\lambda}q_{2}) - \tau(p_{1} - \bar{\lambda}p_{2}) \right) \right) = 0$$

$$\int_{\Sigma} f_{3} = R_{B}R_{M} \operatorname{Re} \left(e^{-i\theta} \left((q_{1} - \bar{\lambda}q_{2}) - \tau(p_{1} - \bar{\lambda}p_{2}) \right) \right)$$

$$= R_{B}R_{M} \left| (q_{1} - \bar{\lambda}q_{2}) - \tau(p_{1} - \bar{\lambda}p_{2}) \right|, \tag{7.34}$$

we see that the correct complex structure of \mathbb{R}^4 that gives the holomorphic embedding of the surface Σ is as follows

$$\Upsilon = f_1 + if_2 = w_1 \wedge w_2, \quad w_1 = dx_1 + idx_2, \quad w_2 = dy_1 + idy_2$$

$$J = f_3. \tag{7.35}$$

In particular, the above one-forms w_1 , w_2 form a basis of the holomorphic one-forms on the Riemann surface when pulled back along the embedding map. Notice that, although the above expression for the complex structure Υ seems to be independent of the charges and moduli, this is not quite true since we have hidden the dependence in our choice of coordinates $x_{1,2}, y_{1,2}$ of \mathbb{R}^4 (7.31). More explicitly, one can view the complex structure as charge- and moduli-dependent through our definition of the angle θ (7.23).

Now we are ready to discuss the embedding of Σ into the spacetime tori $T^2_{(\mathrm{IIB})} \times T^2_{(\mathrm{IIB})}$. Recall that the Jacobian variety of a genus g Riemann surface $\Sigma^{(\mathrm{g})}$ is given by the complex torus $\mathcal{J}(\Sigma^{(\mathrm{g})}) = \mathbb{C}^g/\Lambda(\Sigma^{(\mathrm{g})})$, where $\Lambda(\Sigma^{(\mathrm{g})})$ is the lattice generated by the 2g vectors

$$(\oint_{A_1} w_1, \cdots, \oint_{A_1} w_g)$$

$$\vdots$$

$$(\oint_{A_g} w_1, \cdots, \oint_{A_g} w_g)$$

$$(\oint_{B_1} w_1, \cdots, \oint_{B_1} w_g)$$

$$\vdots$$

$$(\oint_{B_g} w_1, \cdots, \oint_{B_g} w_g)$$

$$(7.36)$$

and $\{w_1,\cdots,w_g\}$ is a basis of one-forms on the Riemann surface which are holomorphic with respect to its given complex structure. The following map, the so-called Abel-Jacobi map, then gives a holomorphic embedding of the Riemann surface $\Sigma^{(\mathrm{g})}$ into its Jacobian $\Lambda(\Sigma^{(\mathrm{g})})$:

$$\varphi: \Sigma^{(g)} \to \mathcal{J}(\Sigma^{(g)}), \quad \varphi(P) = \left(\int_{P_0}^P w_1, \quad \cdots, \int_{P_0}^P w_g\right),$$
 (7.37)

where P_0 is a given arbitrary point on $\Sigma^{(\mathrm{g})}$. Notice that the Jacobian is defined in such a way that the above map is well-defined, namely that the images are independent of the path of integration. In the case of our genus two surface Σ , using the holomorphic one-forms w_1, w_2 given in (7.35), from (7.33) we see that $\Lambda = \Lambda(\Sigma)$, and therefore the Jacobian of the surface $\mathcal{J}(\Sigma)$ is naturally identified with the spacetime T^4 . The Abel-Jacobi map (7.37) therefore provides us with an explicit holomorphic embedding of the M5 brane Riemann surface Σ into the spacetime torus, as was suggested in [250].

After discussing the complex structure and the embedding of the surface, now we are ready to compute its normalized period matrix Ω . Consider two holomorphic one-forms $(\hat{w}_1 \ \hat{w}_2) = (w_1 \ w_2)V$, where V is a real 2×2 matrix, such that

$$\begin{pmatrix} \oint_{A_1} \hat{w}_1 & \oint_{A_1} \hat{w}_2 \\ \oint_{A_2} \hat{w}_1 & \oint_{A_2} \hat{w}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.38}$$

The (normalized) period matrix $\Omega={\rm Re}\Omega+i{\rm Im}\Omega$ is then the symmetric 2×2 matrix given by

$$\Omega = \begin{pmatrix} \oint_{B_1} \hat{w}_1 & \oint_{B_1} \hat{w}_2 \\ \oint_{B_2} \hat{w}_1 & \oint_{B_2} \hat{w}_2 \end{pmatrix} = \begin{pmatrix} \rho & \nu \\ \nu & \sigma \end{pmatrix}, \quad \rho, \sigma, \nu \in \mathbb{C} \ .$$

Comparing (7.38) and the first part of (7.33) one can easily obtain the explicit solution for the real matrix V. Integrating the resulting $\hat{w}_{1,2}$ over the B-cycles then gives $\mathrm{Re}\Omega$ satisfies

$$\operatorname{Im}\Omega\Gamma\begin{pmatrix}1\\-\bar{\lambda}\end{pmatrix} = e^{i\theta}\frac{R_B}{R_M}\begin{pmatrix}1\\-\bar{\tau}\end{pmatrix}. \tag{7.39}$$

Up to a multiplicative factor involving the M-theory radius, this is exactly the same equation (7.22) that the matrix of the length parameters $t_{1,2,3}$ of the type IIB string network satisfies. We therefore conclude that the period matrix of the genus two curve wrapped by the supersymmetric M5 brane configuration is given by

$$\operatorname{Im}\Omega = \sqrt{\frac{R_B^2 \tau_2}{R_M^2 \lambda_2}} \frac{Z}{\|Z\|}, \quad \operatorname{Re}\Omega = 0.$$
 (7.40)

Note that the direction of the above vector in $\mathbb{R}^{2,1}$ is given by the moduli vector Z (7.25), while the length is given by the ratio of the area of the two spacetime tori. And the requirement $\|\mathrm{Im}\Omega\|\gg 1$ for rapid convergence of the partition function is the physical requirement that we work in the low temperature limit in the type IIB frame in which $R_B^2\tau_2\gg R_M^2\lambda_2$.

The fact that the period matrix is purely imaginary is really a consequence of the fact that our two spacetime tori $T_{\rm (M)}^2$ and $T_{\rm (IIB)}^2$ are orthogonal to each other, which in turn reflects the absence of temporal Wilson lines in the original type IIB setup. If these Wilson lines are turned on, the real part of the period matrix will instead be

$$\operatorname{Re}\Omega = \begin{pmatrix} C_{t1} & B_{t1} \\ C_{t2} & B_{t2} \end{pmatrix} \Gamma^{-1} = (\Gamma^{-1})^T \begin{pmatrix} C_{t1} & C_{t2} \\ B_{t1} & B_{t2} \end{pmatrix}, \tag{7.41}$$

where $B_{t1}, B_{t2}, C_{t1}, C_{t2}$ denote the background two-form B- and C-fields along the A- and B-cycles of the torus $T_{(\text{IIB})}^2$ and the temporal circle in type IIB. The extra condition on these Wilson lines $\text{Re}\Omega = (\text{Re}\Omega)^T$ could be thought of as a part of the supersymmetry condition, since if the Wilson lines do not satisfy this condition, the holomorphic embedding of the M5 brane world volume into the spacetime four-torus is not possible with respect to the given complex structure Υ (7.35). Put in another way, turning on the temporal Wilson lines for the two-form fields will generically change the complex structure of the surface Σ ,

with exception when (7.41) is satisfied. But as mentioned before, we will not consider this possibility further.

Finally we would like to comment on the fact that the surface area of the holomorphically embedded genus two surface Σ is simply given by

$$A_{\Sigma} = \int_{\Sigma} J = \frac{i}{2} \int_{\Sigma} (w_1 \wedge \bar{w}_1 + w_2 \wedge \bar{w}_2) = R_M R_B \left| T_Q - \tau T_P \right|$$
 (7.42)

as already computed in (7.34). As expected, the surface area is related to the mass of the BPS object in the following simple way

$$A_{\Sigma} = \frac{R_M}{V_{K3}\lambda_2} M_{\text{IIB}} = \frac{R_M}{V_{K3}^{(M)}} M^{(M)}$$

where the quantities with the superscript (M) denote the quantities in the M-theory unit.

This relation between the mass and the area of the corresponding Riemann surface suggests a geometric way of understanding the walls of marginal stability, defined as the subspace in the moduli space where the BPS masses of the components of a potential bound state sum up to the BPS mass of the total charges. When the Riemann surface degenerates in such a way that it falls apart into different component surfaces which are simultaneously holomorphic, the area of the combined surface clearly equals to the sum of the area of each component surface. Upon using the above relation between the area and the BPS mass, this then directly translates into an expected correspondence between the wall of marginal stability and wall of degeneration of the surface Σ .

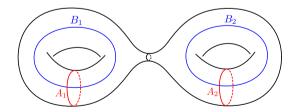


Figure 7.4: The degeneration of the genus two surface described in equation (7.43).

One simplest example of the above-mentioned phenomenon is when the genus two curve \varSigma degenerates in such a way that it splits from the middle and falls apart into two tori as shown in Fig. 7.4. In this simple case, one can indeed check explicitly that the criterion on the period matrix for such a degeneration to happen is exactly the criterion that the mass, or the surface area, becomes the

sum of the contribution of the two components

$$\Omega = \begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix} \quad \Leftrightarrow \quad A_{\Sigma_1} + A_{\Sigma_2} = A_{\Sigma} \tag{7.43}$$

where $A_{\Sigma_1}=|q_1-\bar{\lambda}q_2|,~A_{\Sigma_2}=|-\tau(p_1-\bar{\lambda}p_2)|.$ In other words, the above wall of marginal stability is the co-dimension one subspace of the moduli space such that the two tori defined by $\oint_{A_1} dX,~\oint_{B_1} dX$ and $\oint_{A_2} dX,~\oint_{B_2} dX$ respectively (7.33), are simultaneously holomorphic with respect to the complex structure Υ .

To obtain a geometric understanding of the physics of crossing the walls of marginal stability, in the following subsection we will study the degeneration of the genus two Riemann surfaces of this kind in detail. As we shall see, this geometric consideration will lead to a construction of a group of crossing the walls of marginal stability and therefore provides a geometric derivation of the group of dyon wall-crossing observed in [259].

7.3 Wall-crossing in $\mathcal{N}=4$ theory

The $Sp(2,\mathbb{Z})$ automorphic form Φ_{10} is part of an even richer algebraic structure than we have described so far (see e.g. [267, 268] for a mathematical treatment and [245, 259] for its importance in 4-dimensional $\mathcal{N}=4$ string theory). Rewriting it in the form

$$\Phi_{10}(\Omega) = e^{-2\pi i (\rho, \Omega')} \prod_{\alpha \in R_{\perp}} \left(1 - e^{-\pi i (\alpha, \Omega')} \right)^{c(-||\alpha||^2/2)}, \tag{7.44}$$

where Ω' has coefficient $-\nu$ instead of ν , makes an underlying generalized Lie algebra structure apparent, that we reveal while explaining the symbols used in this formula. The lattice

$$R_+ = \{ \mathbb{Z}_+ \alpha_1 + \mathbb{Z}_+ \alpha_2 + \mathbb{Z}_+ \alpha_3 \}$$

is spanned by three elements, that may be represented as the 2×2 matrices

$$\alpha_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$
 (7.45)

These matrices can be interpreted as the three simple roots of a Lie algebra, with Weyl vector

$$\rho = \frac{1}{2} \sum_{i=1}^{3} \alpha_i.$$

The bilinear form (.,.) on this Lie algebra determines the equivalence of Φ_{10} with the right-hand side of (7.44). For any two symmetric real 2×2 matrices X and Y it is given by

$$(X,Y) = -\det Y \operatorname{Tr}(XY^{-1}).$$

Notice the factor -2 difference with equation (7.20); this is needed to bring the Cartan matrix in a standard form. Indeed, we now find the Cartan matrix

$$(\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}. \tag{7.46}$$

This matrix is not positive-definite, as it has one negative eigenvalue. Therefore, the simple roots α_i are part of a Kac-Moody algebra.

The Weyl group W of this algebra is generated by the reflections

$$s_i: X \mapsto X - 2\frac{(X,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i =: w_i(X)$$

with respect to the simple roots α_i , where $w_i(X) = w_i X w_i^T$ and w_i is a symmetric real 2×2 matrix. Acting with the Weyl group on the simple roots α_i generates all the roots of the Kac-Moody algebra. Half of these roots are positive; by definition these are positive combinations of the simple roots and thus part of R_+ . Remark that these positive roots have length 2, they are real or space-like, and thus contribute to the product formula (7.44) with a power c(-1)=2.

Note that no other space-like elements in R_+ show up in the infinite product (7.44). However, there are many more contributions from vectors in R_+ that have a non-positive length, for instance 2ρ whose length is -6. To describe the full algebraic structure underlying Φ_{10} we should therefore extend our notion of a Kac-Moody algebra. An extended Kac-Moody algebra that incorporates imaginary roots, which are either light-like or time-like, is called a Borcherds Kac-Moody algebra.

Finally, the coefficients $c(-||\alpha||^2/2)$ can be either positive or negative. This property hints at the presence of both bosonic and fermionic roots, that contribute respectively to the numerator and denominator of an index formula. The full algebraic structure behind (7.44) is thus a *Borcherds Kac-Moody superalgebra*. Indeed, the subset $\Delta_+ \subset R_+$ that contributes to Φ_{10} may be interpreted as the total set of roots of such an "automorphic form corrected" superalgebra. The infinite product (7.44) is then called a *denominator formula* for this algebra, where the powers $c(-||\alpha||^2/2)$ determine the multiplicities of the real and imaginary roots.

As a side-remark we notice that the half-BPS partition function $\eta^{24}(\sigma)$ can likewise be written in the algebraic form

$$\eta^{24}(\sigma) = e^{-\pi i(\beta, \Omega')} \prod_{\tilde{R}_{\perp}} \left(1 - e^{-\pi i(\beta, \Omega')} \right)^{c(-||\beta||^2/2)}$$

where the degenerated lattice \tilde{R}_{+} is one-dimensional and generated by

$$\beta = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

In [259] it was found that the Weyl group W has an elegant interpretation in terms of the wall-crossing of quarter BPS dyons. Using the $\mathcal{N}=4$ central charge matrix

$$\mathcal{Z} = \frac{1}{\sqrt{\tau_2}} (P_R - \tau Q_R)^m \Gamma_m,$$

one finds that the central charges of two BPS dyons can align on all $PGL(2,\mathbb{Z})$ -images of the wall

$$\frac{\tau_1}{\tau_2} + \frac{(P_R \cdot Q_R)}{|P_R \wedge Q_R|} = 0$$

in the moduli space parametrized by τ and the left-moving charges P_L , Q_L . In terms of the moduli vector Z in equation (7.25) and the roots α_i these walls are parametrized by

$$\left(\frac{Z}{||Z||}\,,\,\alpha\right)=0,$$

where α can be any $PGL(2,\mathbb{Z})$ -image of α_1 , i.e. any positive real root. These walls are straight lines in the upper-half plane, and become arcs of circles on the Poincaré disc. The simple roots α_i form a triangle in the Poincaré disc, whose vertices lie on the boundary of the Poincaré disc. All the other $PGL(2,\mathbb{Z})$ images can be obtained from this fundamental domain by a Weyl reflection. This yields a tessellation of the Poincaré disc in triangles, whose vertices all end at infinity, and realizes each fundamental domain as a hyperbolic Weyl chamber (see e.g. the cover of [269]).

Due to the presence of walls of marginal stability, where BPS-states could (dis)appear, the graded degeneracies of the BPS states are generically only piecewise constant functions of the values of moduli at spatial infinity. In particular they typically jump when a wall of marginal stability is crossed.

In the $\mathcal{N}=4$ set-up wall-crossing has a geometrical interpretation of the under-

lying genus two surface with period matrix $\Omega \sim Z/||Z||$. At any wall of marginal stability this surface degenerates into a transverse intersection of two tori, so that $\Phi_{10}(\Omega)$ develops a pole.⁵ This raises the question whether Φ_{10} really gives a good description of the quarter BPS dyons in any chamber of the moduli space. This has been analyzed in two (equivalent) ways [254, 255, 256].

First of all, the BPS degeneracies are dependent on a choice of contour

$$d(P,Q) = (-1)^{P \cdot Q + 1} \oint_C d\Omega \frac{e^{-\pi i(\rho P^2 + \sigma Q^2 + 2\nu P \cdot Q)}}{\Phi_{10}(\Omega)}.$$

An $SL(2,\mathbb{Z})$ transformation of the moduli and charges in d(P,Q) encodes this degeneracy in another Weyl chamber. Since all factors in the integrand are invariant under such an $SL(2,\mathbb{Z})$ transformation, exactly same answer would be obtained when the transformed contour C' is equivalent to C. However, the poles of Φ_{10} prevent this. They give an extra contribution to the degeneracy corresponding to half-BPS states on the intersecting tori. Still, it is possible to avoid this factor by changing the charges along.

Secondly, the expansion of the partition function in the degeneracies

$$\frac{1}{\Phi_{10}(\Omega)} = \sum_{P^2,Q^2,P\cdot Q} (-1)^{P\cdot Q+1} d(P,Q) e^{\pi i (\rho P^2 + \sigma Q^2 + 2\nu P\cdot Q)},$$

is dependent on the choice of expansion parameters. Instead of the above representation one could for example have exchanged ν on the right-hand side for $-\nu$, corresponding to a crossing of the wall at $\nu=0$. This changes the degeneracies d(P,Q). Only when one transforms the charges P and Q accordingly the resulting degeneracies remain the same. So if one expands the automorphic form using moduli-dependent expansion parameters that are appropriate for a specific Weyl chamber, the partition function $1/\Phi_{10}$ does encode the BPS degeneracies at all points in the moduli space.

These unexpected properties of the BPS degeneracies certainly hint at deeper structures of the theories yet to be fully uncovered. Specifically, while the properties pertaining to the intricate moduli dependence of the BPS index mentioned above have been observed within the framework of $\mathcal{N}=4, d=4$ supergravity, a microscopic understanding of these properties is clearly desirable. In particular, we would like to understand why the different indices at different points in the moduli space can be extracted from the same generating function. More explicitly, from the fact that the group of wall-crossing is a subgroup of the $(\mathbb{Z}_2$ -parity-extended) S-duality group, when the moduli cross a wall of marginal

⁵Note that there are more walls of marginal stability than the ones we just described, since a degeneration of the genus 2 surface is labeled by an element of $Sp(2,\mathbb{Z})\supset SL(2,\mathbb{Z})$. Recently, these $Sp(2,\mathbb{Z})$ -poles have been given a supergravity interpretation as well [263].

stability, the change of the BPS index can be summarized by a change of the "effective charges" by a Weyl reflection [259]. We would like to understand why the index should change in such a simple way.

Carrying the analysis of the previous section one step further, we study the change of the surface when it goes through such a degeneration, and find that it is equivalent to a particular change of the homological cycles of the surface. Using the relation between the homology class in the spacetime T^4 of the Riemann surface wrapped by the M5 brane and the conserved charges, we see how the change of the BPS index when crossing the wall of marginal stability under consideration amounts to a change of the "effective charges" by acting by a certain element of the hyperbolic reflection group W. Following such a strategy and using essentially only the supersymmetry condition, we derive the specific group structure underlying the wall-crossing of the theory, and the fact that the BPS degeneracies at different moduli are given by the same partition function. In particular, we see how the moduli space and its partitioning by the walls of marginal stability can be identified with the dual graph of the type IIB (p,q) 5brane network compactified on the spacetime torus, with the symmetry group of the network identified with the symmetry group of the fundamental domain of the group of wall-crossing. We hope that this microscopic derivation of the Weyl group will be a first step towards an understanding of the microscopic origin of the Borcherds-Kac-Moody algebra in the dyon spectrum.

In Section 7.3.1 we focus on one specific degeneration of the surface and analyze the change of the homology cycles under such degeneration by using a hyperelliptic model of the genus two surface. In this way we derive one of the elements of the reflection group W. In Section 7.3.1 we study the symmetry of the hyperelliptic surface, or equivalently the symmetry of the periodic network of effective strings in the type IIB frame. In this way we obtain the other generators of the group W. Using these results, in Section 7.3.1 we discuss how the moduli space and its partitioning by the walls of marginal stability, or equivalently the walls of degenerations of the Riemann surface, can be understood simply as being the dual graph of the periodic effective string network. We also discuss the implication of these results for the counting of BPS dyonic states, and in particular why the index simply changes by an appropriate change of the "effective charges" when the moduli cross a wall of marginal stability. In Section 7.3.2 we conclude by a discussion, in particular we discuss what we cannot derive by such a simple analysis and sketch an analogous treatment for the case of the CHL models.

7.3.1 Deriving the group of discrete attractor flow

In this subsection we first study a specific degeneration of the Riemann surface and show how the effect of going through such a degeneration boils down to a change of the homology cycles. This then in turn gets translated into a change of the "effective charges" of the system under the identification between the homology classes of the cycles of the surface in the internal space and the conserved charges of the system. Secondly we study the symmetry of the system and thereby recover the full hyperbolic reflection group underlying the structure of wall-crossing of the present theory. Thirdly, we discuss the implication of these results to the problem of enumerating supersymmetric dyonic states, and show how it leads to the prescription proposed in [259] of retrieving BPS indices at different points in the moduli space from the same partition function (see also [254, 256] for earlier discussions).

The First Degeneration

First we will study what happens to the Riemann surface when the moduli change such that the surface goes through a degeneration mentioned at the end of the previous Subsection. To remain in the open moduli space of the genus two Riemann surface, we study the change of the Riemann surface Σ when its period matrix Ω changes as

$$\begin{pmatrix}
\rho & -\nu \\
-\nu & \sigma
\end{pmatrix} \rightarrow \begin{pmatrix}
\rho & \nu \\
\nu & \sigma
\end{pmatrix}

(7.47)$$

following the path depicted in Fig. 7.5. Clearly, the two end points of the path are on the different sides of the wall of marginal stability (7.43) considered earlier.

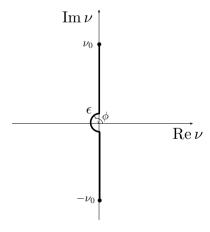


Figure 7.5: In Section 7.3.1 we study the change of the Riemann surface Σ when its period matrix changes as (7.47) following the above path, where $\epsilon \to 0_+$ and ρ and σ are held fixed at values satisfying $\text{Im}\rho\,\text{Im}\sigma\gg(\text{Im}\nu_0)^2$.

To focus on what happens to the surface when the wall is crossed, we will further

zoom into the part of the path in Fig. 7.5 that is a half-circle with vanishing size:

$$\Omega = \begin{pmatrix} \rho & \epsilon e^{i\phi} \\ \epsilon e^{i\phi} & \sigma \end{pmatrix}, \quad \epsilon \to 0_+, \quad \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] . \tag{7.48}$$

First recall that, every Riemann surface of genus two can be represented as a hyperelliptic surface with six branch points $b_{1,\dots,6}$

$$y^{2} = x(x-1)(x-b_{1})(x-b_{2})(x-b_{3}), (7.49)$$

where we have used the conformal invariance to fix b_4, b_5, b_6 to be $\infty, 0, 1$ respectively. In other words, we represent the genus two Riemann surface Σ as a two-sheet cover of \mathbb{CP}^1 with six branch points $b_{1,\dots,6}$ and three branch cuts between b_{2i-1} and b_{2i} for all i=1,2,3, as shown in Fig. 7.6.

To analyze the change of the surface, in particular the homology cycles of the surface, after the imaginary part of ν changes sign, we would like to determine the normalized basis $\hat{w}_{1,2}$, satisfying (7.38), in terms of the local coordinate x of \mathbb{CP}^1 .

It is a familiar fact about hyperelliptic curves that the two one-forms

$$\frac{dx}{u}$$
, $\frac{x\,dx}{u}$

form a basis of the holomorphic one-forms on the genus two surface Σ given by (7.49), see for example [270]. To achieve our goal we need to compute the integral of the above one-forms along the A_1 , A_2 cycles. First we observe that, with the choice of cycles as in Fig. 7.6, the integrals of a holomorphic one-form w along the A-cycles are given by the so-called "half-period"

$$\frac{1}{2} \oint_{A_1} w = \int_0^1 w, \quad \frac{1}{2} \oint_{A_2} w = \int_{b_1}^{b_2} w \tag{7.50}$$

on the upper sheet of the hyperelliptic surface.

To obtain an expression for these quantities in terms of the period matrix Ω and in particular in terms of the angle ϕ (7.48), we recall that the locations of the branch points $b_{1,2,3}$ are uniquely determined by the genus two Riemann theta functions up to theta function identities [270]. Explicitly, we have [271]

$$b_1 = \frac{\theta^2 \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right] \theta^2 \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right]}{\theta^2 \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right] \theta^2 \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} \right]} (0, \Omega)$$

$$b_2 = \frac{\theta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \theta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}{\theta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \theta^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} (0, \Omega)$$

$$b_3 = \frac{\theta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \theta^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}{\theta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \theta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} (0, \Omega),$$

where $\theta[{\varepsilon_1^{\varepsilon_1} \varepsilon_2^{\varepsilon_2}}](\zeta,\Omega)$ is the genus-two Riemann theta functions, defined as

$$\theta[\begin{smallmatrix} \varepsilon_1 & \varepsilon_2 \\ \varepsilon_1' & \varepsilon_2' \end{smallmatrix}](\zeta,\Omega) = \sum_{n_1,n_2 \in \mathbb{Z}} e^{2\pi i \left(\frac{1}{2}(n+\frac{1}{2}\varepsilon)^T \cdot \Omega \cdot (n+\frac{1}{2}\varepsilon) + (n+\frac{1}{2}\varepsilon)^T \cdot (\zeta+\frac{1}{2}\varepsilon')\right)},$$

where the "." denotes matrix multiplication.

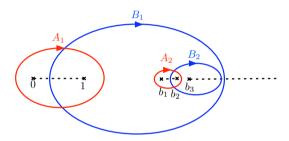


Figure 7.6: Hyperelliptic representation of the genus two surface Σ together with a choice of its A_i and B_i -cycles. A degeneration corresponding to the one shown in Fig. 7.4 corresponds to coalescing the branch points b_1 , b_2 and b_3 . Note that when we set the background twoform fields B and C along the time-like direction to zero, so that $\text{Re}\Omega=0$ (7.41), all branch points are collinear.

While the details of these formulas are not so important for us, there are a few important immediate consequences of these expressions that we can draw. First of all, due to the fact that the genus two theta functions are a product of two genus one theta functions at leading order in ν when $\nu \to 0$:

$$\theta[\begin{smallmatrix} \varepsilon_1 & \varepsilon_2 \\ \varepsilon_1' & \varepsilon_2' \end{smallmatrix}](0, \left(\begin{smallmatrix} \rho & \nu \\ \nu & \sigma \end{smallmatrix}\right)) = \theta[\begin{smallmatrix} \varepsilon_1 \\ \varepsilon_1' \end{smallmatrix}](0, \rho) \theta[\begin{smallmatrix} \varepsilon_2 \\ \varepsilon_2' \end{smallmatrix}](0, \sigma) \left(1 + \mathcal{O}(\nu^2)\right),$$

the three branch points coalesce when $\nu \to 0$

$$b_1, b_2, b_3 \to b_0 = \left(\frac{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \rho)}{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0, \rho)}\right)^4.$$
 (7.51)

Furthermore, from the definition of the genus two theta functions we see that

$$\frac{\partial}{\partial \nu} b_i \big|_{\nu=0} = 0, \quad i = 1, 2, 3.$$

Therefore, for the period matrix on the half-circle given by (7.48) and in Fig. 7.5,

we have

$$b_i = b_0 + \epsilon^2 e^{2i\phi} k_i + \mathcal{O}(\epsilon^4), \quad k_i = \frac{1}{2} \frac{\partial^2}{\partial \nu^2} b_i \big|_{\nu=0} \in \mathbb{C}, \quad i = 1, 2, 3.$$

In particular, the branch points go through a 2π rotation under a change $\phi \to \phi + \pi$. In other words, the branch points return to themselves while the period matrix undergoes a change $\nu \to -\nu$.

Now we can use the above expression for the branch points near the degeneration point and (7.50) to compute the periods along the A_i -cycles of the holomorphic one-forms $\frac{dx}{y}$, and $\frac{xdx}{y}$, and obtain the following expression for the normalized holomorphic one-forms satisfying (7.38)

$$\hat{w}_1 = \frac{-1}{2(\alpha b_0 - 1)} \frac{(x - b_0)dx}{y} \left(1 + \mathcal{O}(\epsilon^2) \right)$$
 (7.52)

$$\hat{w}_2 = \epsilon e^{i\phi} \frac{1}{2\gamma(\alpha b_0 - 1)} \frac{(\alpha x - 1)dx}{y} \left(1 + \mathcal{O}(\epsilon^2)\right), \tag{7.53}$$

where α, β, γ are ϕ -independent, order one constants

$$\alpha = \int_0^1 \frac{dx}{\sqrt{x(x-1)(x-b_0)^3}}$$

$$\beta = \int_0^1 \frac{xdx}{\sqrt{x(x-1)(x-b_0)^3}}$$

$$\gamma = \frac{1}{\sqrt{b_0(b_0-1)(k_2-k_1)}} \int_0^1 \frac{dx}{\sqrt{x(x-1)(x-\frac{k_3-k_1}{k_2-k_1})}}.$$

While the precise values of these constants are not important for us, the above expression (7.52) immediately shows that, when ${\rm Im}\nu$ changes sign by a ϕ to $\phi+\pi$ rotation, the normalized holomorphic one-forms change like

$$\begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \end{pmatrix} \to \begin{pmatrix} \hat{w}_1 \\ -\hat{w}_2 \end{pmatrix}$$

as linear combinations of the holomorphic one-forms $\frac{dx}{y}$ and $\frac{xdx}{y}$, despite of the fact that the three coalescing branch points $b_{1,2,3}$ simply return to the original locations after a 2π rotation.

This suggests that, in a representation of the hyperelliptic surface in which the holomorphic one-forms are held fixed, the homology cycles go through the following transformation

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \rightarrow \begin{pmatrix} A_1 \\ -A_2 \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \rightarrow \begin{pmatrix} B_1 \\ -B_2 \end{pmatrix} . \tag{7.54}$$

Indeed, it is not difficult to check that the periods of any holomorphic one-form w along A_2 and B_2 cycles

$$\frac{1}{2} \oint_{A_2} w = \int_{b_1}^{b_2} w, \quad \frac{1}{2} \oint_{B_2} w = \int_{b_2}^{b_3} w$$

change sign under $\phi \rightarrow \phi + \pi$.

Another way to understand this change of homology basis is the following. From the expression of the normalized holomorphic one-forms (7.52) we see that, to the leading order in ϵ we have the two separated genus one surfaces described by

$$y'^2 = x(x-1)(x-b_0), \quad y''^2 = (x-b_1)(x-b_2)(x-b_3).$$
 (7.55)

Indeed, from the following relationship between the cross-ratio of the four branch points $\mathfrak{b}_{1,2,3,4}$ of a genus one surface and the torus complex moduli $\tilde{\tau}$ [270]

$$\frac{(\mathfrak{b}_3 - \mathfrak{b}_1)(\mathfrak{b}_4 - \mathfrak{b}_2)}{(\mathfrak{b}_2 - \mathfrak{b}_1)(\mathfrak{b}_4 - \mathfrak{b}_3)} = \left(\frac{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(0, \tilde{\theta})}{\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}(0, \tilde{\theta})}\right)^4$$

one can check that the following two genus-one curves have complex moduli equal to ρ and σ respectively. From the above expression (7.55) it is manifest that, when b_i 's go through a 2π rotation around their common converging point b_0 , nothing happens to the first genus one surface while the second one goes through a sheet exchange (or "hyperelliptic involution") $y'' \to -y''$ corresponding to the monodromy

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} \rightarrow \begin{pmatrix} -A_2 \\ -B_2 \end{pmatrix}.$$

The latter can be explicitly seen by substituting

$$x = b_0 + e^{2i\phi}\tilde{x}, \quad y'' = e^{3i\phi}\tilde{y}$$

in the second equation of (7.55).

In general, when we change the basis such that the A_i -cycles are changed to

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}), \tag{7.56}$$

the corresponding change of the B_i -cycles is then fixed by the canonical intersection (7.32) to be

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \to \pm \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = (\gamma^T)^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where the \pm signs are taken when $ad-bc=\pm 1$. Under this transformation, the period matrix transforms as

$$\Omega \to \gamma(\Omega) \equiv (\gamma^{-1})^T \Omega \gamma^{-1}. \tag{7.57}$$

Without changing the Riemann surface, such a change of basis has an interpretation as performing a physical S-duality in the heterotic frame, extended with the \mathbb{Z}_2 spacetime parity exchange. To see this, first inspect the expression (7.33) for the vectors defining the Jacobian of the surface. The effect of the above change of basis on these vectors is equivalent to the following heterotic S-duality transformation, or equivalently the modular transformation of the torus in the type IIB frame

$$\begin{pmatrix} Q \\ P \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \quad \tau \rightarrow \frac{a\tau + b}{c\tau + d} \text{ or } \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \text{ for } ad - bc = \pm 1$$
 (7.58)

with the corresponding change of R_B such that the area of the type IIB torus remains invariant. In particular, the fact that the moduli vector Z transforms as $Z \to (\gamma^{-1})^T Z \gamma^{-1}$ under the above S-duality transformation is then consistent with the transformation of the period matrix under a change of homology basis.

Now let's go back to the evolution (7.48) of the Riemann surface through the degeneration wall $\nu = 0$, due to the corresponding change of the moduli vector Z (7.40).

What we have seen can be summarized as follows: when the moduli change across the wall of marginal stability following the path corresponding to an angle- π rotation of the phase of ν (7.48), the holomorphic one-forms of the surface Σ change in such a way that all their A_2 , B_2 periods change signs while their periods along the A_1 , B_1 cycles remain the same. This is equivalent to keeping the surface unchanged but change the basis for the homology cycles in the way (7.56) given by the following element in $GL(2,\mathbb{Z})$

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.59}$$

In other words, the process of keeping the charge fixed while varying the moduli across the wall of marginal stability following (7.48) is equivalent to keeping the moduli vector Z unchanged but changing the charges

$$\begin{pmatrix} Q \\ P \end{pmatrix} \to R \begin{pmatrix} Q \\ P \end{pmatrix}. \tag{7.60}$$

This observation has the following implication for the counting of the BPS states.

Consider the partition function of the theory

$$Z(\Omega) = \sum_{P^2, Q^2, P \cdot Q} (-1)^{P \cdot Q + 1} d(P, Q) e^{i\pi(\rho P^2 + \sigma Q^2 + 2\nu P \cdot Q)},$$

which is a path integral computed on the Riemann surface Σ . It clearly depends on its period matrix Ω and therefore on the moduli vector Z through its relation to the period matrix (7.40). When the moduli change in such a way that the Riemann surface goes through a degeneration described in (7.43), from the above reasoning we see that the partition function remains unchanged while a transformation of "effective charges" given in (7.60) has to be performed. This corresponds to the change of the highest weight of the Verma module as described in [259].

It is also easy to understand the nature of this degeneration in the type IIB fivebrane network picture. From the relation between the period matrix of the genus two curve in M-theory and the length parameters for the periodic string network (7.30,7.40), we see that the degeneration of the Riemann surface characterized by $\nu=0$ corresponds to the degeneration of the string network characterized by $t_1=0$. For example, starting from a region in the moduli space with $Z=\begin{pmatrix}z_1&z\\z&z_2\end{pmatrix}$ with $z_1,z_2>z>0$, what happens when $t_1=0$ is a transition from the network described by (7.21), or the first figure in Fig. 7.3, to the network described by (7.28), or the second figure in Fig. 7.3. The above claim that the final surface has the same period matrix under a change of homology basis corresponding to (7.60), is then reflected by the fact that the two defining equations (7.21), (7.28) transform into each other under the transformation of the charges (7.60).

The symmetry of the Weyl chamber

In the previous paragraph we have studied in detail a particular degeneration of the Riemann surface and what it implies for the index counting the BPS states under the crossing of the corresponding wall of marginal stability. In this paragraph we will turn to studying the symmetry of the system and see how it will help us to uncover the full structure of the group of wall-crossing of the theory.

First we note that, under our convention that the two A-cycles of the surface are chosen to circle two of the three pairs of branch points $\{b_{2i-1},b_{2i}\}$, the choice shown in Fig. 7.6 is not quite unique. In other words, from all the possible change of basis of the form (7.56), the exchange and permutation of the cycles $A_1,A_2,-A_1-A_2$ correspond to a symmetry of our hyperelliptic model (7.49) in that we do not need to change the set of branch points $\{b_1,\cdots,b_6\}$ in order for the new A_i -cycles to again circle the cuts joining the pairs $\{b'_{2i-1},b'_{2i}\}$ of the new branch points. We therefore conclude that there is a symmetry group with six elements acting on the hyperelliptic surface (Fig. 7.6), corresponding to six ways of associating the three cuts joining $\{b_{2i-1},b_{2i}\}$, i=1,2,3, to the three

homology cycles $(A_1, A_2, -A_1 - A_2)$. From the above discussion we see that this group $D_3 \subset GL(2,\mathbb{Z})$ is the same as the symmetry group of a regular triangle, generated by the order two element which acts on the period matrix as

$$\Omega \to RS(\Omega) \tag{7.61}$$

and which corresponds to $(A_1, A_2, -A_1 - A_2) \rightarrow (A_2, A_1, -A_1 - A_2)$, together with the order-three element which acts as

$$\Omega \to ST(\Omega)$$
 (7.62)

and corresponds to $(A_1,A_2,-A_1-A_2) \to (A_2,-A_1-A_2,A_1)$, where T and S denotes the usual T- and S- transformation matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, while R was already given in (7.59). Also here we have used the shorthand notation introduced in (7.57).

The existence of this symmetry is even more apparent in the type IIB picture of five-brane network. For concreteness of the discussion we will now assume that $Z = \begin{pmatrix} z_1 & z \\ z & z_2 \end{pmatrix}$ satisfies $z_1, z_2 > z > 0$, so that the network shown in the first figure in Fig. 7.3 is realized. But, suppose that we are given this periodic network given by (7.21), there is actually more than one way to fit it into a parallelogram tessellation of the plane. In other words, there is in fact more than one torus compactification of the network possible. From the fact that the vertices of the parallelogram lie at the center of the honeycomb lattice, we conclude that there are three such parallelogram tessellations possible, as shown in Fig. 7.7 as resembling the three sides of a 3-dimensional cube. These three parallelograms then give six possible tori (for each parallelogram we have two ways of choosing the A- and B-cycle), corresponding to 3! = 6 ways of placing the charge labels (Q, P, -Q-P) to the three legs of the network. This singles out a six-element subgroup of the extended type IIB modular group (or the heterotic S-duality group) $GL(2,\mathbb{Z})$ (7.58). Not surprisingly, this is exactly the same D_3 we discovered earlier as the symmetry group of the hyperelliptic representation of the Riemann surface Σ . This correspondence is to be expected from the identification between the change of basis of the homology cycles on Σ and the heterotic S-duality discussed in the previous sub-subsection (7.56-7.58).

More explicitly, from (7.61,7.62) and the relation between the period matrix and the length parameters of the network (7.27,7.40) we see that the length parameters indeed transform as

$$RS: (t_1, t_2, t_3) \to (t_1, t_3, t_2), \quad ST: (t_1, t_2, t_3) \to (t_2, t_3, t_1)$$

under the action of the order two and three generators of the symmetry group D_3 .

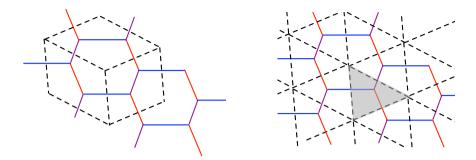


Figure 7.7: (i) Different possible ways of compactifying the periodic network on a torus. **(ii)** The moduli space as the dual graph of the five-brane network.

As mentioned earlier, this symmetry group is the symmetry group of a equilateral triangle. Geometrically, the relevant equilateral triangle here cannot be literally the triangle in the dual graph of the five-brane network as shown in Fig. 7.7, since in general there is no reason to expect them to be equilateral using the flat metric on the plane. This inspires us to take a closer look into the matrix of length parameters, which can be written in a way which makes the D_3 symmetry manifest

$$R_M \operatorname{Im} \Omega = \begin{pmatrix} t_1 + t_3 & t_1 \\ t_1 & t_1 + t_2 \end{pmatrix} = \frac{t_2 + t_3}{2} \alpha_1 + \frac{t_1 + t_3}{2} \alpha_2 + \frac{t_1 + t_2}{2} \alpha_3$$

where

$$\alpha_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$
 (7.63)

Remember that this natural basis $\{\alpha_{1,2,3}\}$ has the following matrix of inner products using the standard $GL(2,\mathbb{Z})$ -invariant Lorentzian metric (7.20)

$$-2(\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$
 (7.64)

and therefore forms a equilateral triangle in the hyperbolic space $\mathbb{R}^{2,1}$. The group D_3 which permutes $\alpha_{1,2,3}$ can therefore be thought of the symmetry group of this equilateral triangle. We note that the basis $\{\alpha_{1,2,3}\}$ we used above is exactly the basis (7.45) for the roots of the Borcherds-Kac-Moody algebra adopted in [259], and in particular the matrix of inner products in (7.64) is simply the real part of the Cartan matrix (7.46) of the algebra.

To summarize, we have found a six-element symmetry group D_3 of the dyon system at a given point in the moduli space which is evident in both the M-

theory Riemann surface picture as well as the type IIB network picture. In the following paragraph we will use this symmetry to find all the generators of the hyperbolic reflection group W playing the role of the group of wall-crossing in the $\mathcal{N}=4$ theory we discussed, and subsequently derive the full group structure of the dyon BPS index.

Moduli space as the dual graph

In Section 7.3.1 we have studied in detail the change of the Riemann surface across a degeneration point (7.48) where the surface falls into two separate tori as depicted in Fig. 7.4. From the above discussion about the symmetry of the system, we see that there are two more natural degenerations of the genus two Riemann surface Σ we should consider. The corresponding transformation of the period matrix is simply the transformation (7.47) of the first degeneration we have studied in Section 7.3.1, now conjugated with elements of the symmetry group D_3 . From (7.61,7.62), we see that apart from the group generator $w_1 = R$ (7.60), we should also consider the generators

$$w_2 = (ST)^{-2}R(ST)^2$$
 and $w_3 = (ST)^{-1}R(ST)$.

Together they generate a non-compact reflection group, which we will denote by W. Furthermore, it is not difficult to show [272, 259] that the extended S-duality group $PGL(2,\mathbb{Z})^6$ is a semi-direct product

$$PGL(2,\mathbb{Z}) = W \times D_3$$

of the reflection group W and the symmetry group D_3 .

It is clear what these three degenerations correspond to in the type IIB and as well as in the M-theory picture. In the former case they are the three ways in which the five-brane network can disintegrate, namely letting one of the three legs having vanishing length. In the latter case, on the other hand, they correspond to coalescing the branch points $b_{3,4,5}$, $b_{5,6,1}$ or $b_{1,2,3}$. Remember that coalescing three out of the total of six branch points is equivalent to coalescing the complementary set of three branch points.

More explicitly, these three generators $w_{1,2,3}$ of the group W correspond to the following change of the network

$$w_i: t_i \to -t_i, \quad t_i + t_j \text{ invariant for } j \neq i.$$

⁶Recall that $PGL(2,\mathbb{Z})$ is obtained from $GL(2,\mathbb{Z})$ by identifying the elements γ and $-\mathbb{1}\cdot\gamma\in GL(2,\mathbb{Z})$. In the heterotic frame this corresponds to identifying two systems with the same value of axion-dilaton τ and charges which are related to each other by a charge conjugation $\binom{Q}{P}\to\binom{-Q}{-P}$. In the type IIB frame this corresponds to a trivial change of basis for the homology cycles $\binom{A}{beta}\to\binom{-A}{-B}$ of the compactification torus $T^2_{(\mathrm{IIB})}$.

Equivalently, they can also be represented as the following reflections in the (2+1)-dimensional Minkowski space in which the period matrix ${\rm Im}\Omega$ takes its value

$$w_i: \Omega \to \Omega - 2 \frac{(\alpha_i, \Omega)}{(\alpha_i, \alpha_i)} \alpha_i,$$

where the basis vectors α_i 's are defined in (7.63). Recall that we have chosen the M-theory background such that the period matrix Ω is purely imaginary and therefore directly related to the length parameters of the five-brane network.

Applying the same analysis as in Section 7.3.1 to the other two degenerations corresponding to w_2 and w_3 , one can conclude that going through such a degeneration wall has the following effect on the counting of BPS states. Upon applying a suitable change of basis analogous to (7.54), after the degeneration we regain the original partition function but now with a different effective charges related to the original charges by

$$\begin{pmatrix} Q \\ P \end{pmatrix} \to w_i \begin{pmatrix} Q \\ P \end{pmatrix}. \tag{7.65}$$

From the above consideration, we arrive at a picture of the moduli space with its partitioning by the walls of marginal stability given by the the dual graph of the honeycomb lattice representing the five-brane network. This is shown in Fig. 7.7. To understand this better, let's start in one of the dual triangles, let's say the gray triangle which denotes the part of the moduli space with $Z = \begin{pmatrix} z_1 & z \\ z & z_2 \end{pmatrix}$, $z_1, z_2 > z > 0$, such that the network (7.21) as shown in the first figure in Fig. 7.3 is realized. We shall choose it to be our "fundamental domain" \mathcal{W} , a name that will be justified shortly.

A degeneration happens when one of the length parameters t_i 's goes to zero. As we discussed at the end of Section 7.2.2, this corresponds to crossing a physical wall of marginal stability. When this happens we move to the neighboring triangle, divided from the fundamental triangle $\mathcal W$ by the side of the triangle intersecting the leg of the network whose length parameter has just goes through a zero. In this new triangle, the effective charges are related to the original one by the corresponding group element (7.65). For instance, associated to the triangle that shares one side with $\mathcal W$ which intersects the leg whose length parameter is denoted by t_1 are the effective charges (Q, -P) and the region in the moduli space with $Z = {z_1 \choose z}$, where $z_1, z_2 > -z > 0$.

Given the original charges, this procedure can then be iterated. We thus conclude that each triangle of the dual graph has the effective charges (Q_v,P_v) associated to it, where v labels the vertices in the hexagonal lattice, or equivalently the triangles (the faces) of the dual graph, which represent the corresponding

regions in the moduli space. Furthermore, in this way each of the triangles can be identified with the fundamental domain of the group W, generated by the three elements $w_{1,2,3}$ (7.65), which by construction plays the role of crossing the walls of marginal stability of the theory.

Now that each triangle has a set of charges (Q_v, P_v) associated to it, while the period matrix of the genus two Riemann surface Σ and therefore the partition function remains the same for each triangle, generically we conclude that there is also a different BPS index $D(Q,P)|_v=D(Q_v,P_v)$ associated with each triangle. The difference between $D(Q_v,P_v)$ with different v has been calculated in [254, 259] for the present theory and was shown to be consistent with the macroscopic wall-crossing formula.

To sum up, we have derived the following one-to-one correspondence

```
vertex v of string network \leftrightarrow a triangle in dual graph (7.66)

\leftrightarrow effective charges (Q_v, P_v) \leftrightarrow BPS index D_v = D(Q_v, P_v)

\leftrightarrow an element w_v \in W \leftrightarrow a region in the moduli space Z \in w_v(W).
```

The property of the BPS dyon index of the present $\mathcal{N}=4$ theory that the indices in different parts of the moduli space are given by the same partition function and have the form $D(Q,P)|_v=D(Q_v,P_v)$ was observed in [254, 256] based on the macroscopic prediction for the change of index upon crossing a wall of marginal stability. And the fact that these different regions of the moduli space with different BPS indices are in one-to-one correspondence with elements of a hyperbolic reflection group W is later observed in [259] based on a 4-dimensional macroscopic analysis. What we have seen is how these properties can be understood as the consequence of the simple consideration of the supersymmetry of the effective string network, or equivalently the holomorphicity of the M5 brane world-volume, when the limit of decoupled 4D gravity is taken.

7.3.2 Discussion

In this section we worked in the decompactification limit $(V_{K3}\gg 1)$ and showed how the group structure underlying the moduli dependence of the dyon BPS index of the $\mathcal{N}=4$ $K3\times T^2$ compactification of type II theory can be understood as simply a consequence of the supersymmetry of the dyonic states. From the other point of view, this group structure is simply the consequence of the fact that the BPS spectrum of the theory is given by the appropriate representation of a Borcherds-Kac-Moody algebra. The Weyl group of the algebra, which is a symmetry group of the root system of the algebra, then plays the role of the group of wall-crossing for the physical degeneracy of the dyonic states [259]. Therefore, we hope that the microscopic derivation of the Weyl group presented

in this section will be the first step towards an understanding of the microscopic origin of the Borcherds-Kac-Moody algebra in the dyon spectrum.

For this purpose, it is important to be clear about what we do not derive from the simple analysis of this section. First of all, while we assume that the partition function is a functional integral on the genus two Riemann surface Σ and therefore depends only on the period matrix of the surface, justified by the fact that an M5 brane wrapping the surface and the K3 manifold in the Euclidean spacetime is equivalent to a fundamental heterotic string whose world volume is the genus two surface [273], we have not derived the partition function itself from our simple consideration. A discussion about the subtleties of computing the partition function in a very similar context can be found in [260] and will therefore not be repeated here. Relatedly, the presence of the Borcherds-Kac-Moody algebra [245, 259] is far from evident from our simple analysis. It seems likely that the physical interpretation of this complete algebra can be found in terms of the chiral fermions on the genus 2 surface in type IIA, or equivalently, in terms of fluctuations on the M5-brane wrapping the genus 2 surface in M-theory.

Furthermore, we have not commented on the role of the group W as the group of a discretized version of attractor flows. As discussed in detail in [259], this interpretation naturally arises due to the existence of a natural ordering among the elements of the hyperbolic reflection group W, and the fact that for given total charges, there is a unique endpoint of this ordering, corresponding to the attractor point of these charges. From the point of view of the Borcherds-Kac-Moody algebra, the Verma module relevant for the BPS index is the smallest one when the moduli are at their attractor value. By working in the limit that the type IIB five-branes are all much heavier than all the (p,q) strings, we have no way of telling which of the triangles in the dual graph in Fig. 7.7 contains the attractor point. This is of course consistent with the fact that our analysis in the text is independent of the values of the T-duality invariants Q^2 , P^2 , $Q \cdot P$ (7.17), due to the decompactification limit we are taking. But this can easily be cured by going to the next leading order in $\mathcal{O}(V_{K3}^{-1})$. See also [260]. By minimizing the surface area of the genus two surface (7.42) with the volume of the two tori $R_B^2 \tau_2, R_M^2 \lambda_2$ held fixed, with now the next leading corrections included, one indeed obtains the attractor equation

$$\mathcal{M}_{\tau} = \frac{\Lambda_{Q_R,P_R}}{\|\Lambda_{Q_R,P_R}\|} = \frac{1}{\sqrt{Q^2 P^2 - (Q \cdot P)^2}} \begin{pmatrix} Q \cdot Q & Q \cdot P \\ Q \cdot P & P \cdot P \end{pmatrix} ,$$

as expected. This extra piece of information will then single out a triangle in the dual graph as the attractor region and completes the interpretation of the hyperbolic reflection group derived in this section as the group underlying the macroscopic attractor flow of the theory.

Moreover, we would like to comment on the cases of other $\mathcal{N}=4$ string theo-

ries. Here we have focused on the $\mathcal{N}=4$ theory of $K3\times T^2$ compactified type II theory, while from the analysis in [261] we expect very similar group structures to be present also in the \mathbb{Z}_n -orbifolded theories, the so-called CHL models [274, 275, 276], for n < 4. For the \mathbb{Z}_2 -orbifold theory, considering the double cover of the genus two Riemann surface relevant for the computation of the partition function [242, 277] and the corresponding string network, a similar analysis can be employed to understand the group structure in that case. The situation of other orbifolded theories is much less clear. In particular, in [261] it was discovered that a similar group structure and an underlying Borcherds-Kac-Moody algebra cease to exist when n > 4. In particular, macroscopic analysis showed that the symmetry group of a fundamental region bounded by walls of marginal stability has infinitely many elements when n > 4. From the analysis of this section, this symmetry group is expected to be the symmetry group of the dyonic string network/Riemann surface. One might thus suspect that the corresponding dyon network does not exist in the $\mathbb{Z}_{n>4}$ theories. It would be interesting to understand the group structure of the dyon degeneracies of other orbifolded $\mathcal{N}=4$ theories.

Finally, we would like to see our analysis of 1/4 BPS states in $\mathcal{N}=4$ theories as a starting point to study wall-crossing of 1/2 BPS states on local Calabi-Yau manifolds in $\mathcal{N}=2$ theories. That these topics are closely related follows from the duality of the dyon background with the intersecting brane background (7.16). Do notice that this duality only maps the dyon generating function to the semiclassical piece \mathcal{F}_1 of the topological string partition function, since the I-brane background doesn't include a graviphoton field strength. To describe $\mathcal{N}=2$ wall-crossing fully we should thus go beyond deformations of the Riemann surface Σ . This naturally raises the question whether we can formulate $\mathcal{N}=2$ wall-crossing in terms of the underlying quantum curve that we studied in Chapter 5 and Chapter 6. We leave this for future work.

Another relation between $\mathcal{N}=2$ and $\mathcal{N}=4$ string theories is that both the Gopakumar-Vafa partition function (4.47) and the dyon partition function (7.11) can be formulated as an infinite product expansion. The wall-crossing examples that are worked out in $\mathcal{N}=2$ context, can actually be formulated most elegantly in this representation: depending on the region in the moduli space additional factors have to be added to the partition function⁷. One might wonder whether also in this setting wall-crossing can be described in terms of a universal generating function.

Let us point out one interesting observation in this respect. The quantum McKay correspondence formulated by J. Bryan and A. Gholampour in [283, 284] relates the positive roots of an ADE Lie algebra $\mathfrak g$ to 1/2 BPS states of a Calabi-Yau resolution of the quotient singularity $\mathbb C^3/\widetilde\Gamma$, where $\widetilde\Gamma$ is the corresponding fi-

⁷Recent developments can be found in e.g. [225, 21, 278, 279, 280, 281, 226, 282].

nite subgroup of SO(3). (These Calabi-Yau singularities are closely related to the hyper-Kähler surface singularities \mathbb{C}^2/Γ discussed in Chapter 3 through the double cover of SU(2) over SO(3) [285].) More precisely, they prove that

$$Z'_{GW}(t.\lambda) = \prod_{\alpha \in R_+} \prod_{k=1}^{\infty} (1 - e^{t \cdot \alpha} q^k)^{k/2}.$$

Here, α is a positive real root that is mapped to a curve in Y by the complex 3-dimensional McKay correspondence (roughly it is the push-forward of the map $\mathbb{C}^2/\Gamma \to \mathbb{C}^3/\widetilde{\Gamma}$), where the sum doesn't include positive roots that are mapped to zero in homology. Up to the factor 1/2 in the overall power (which is due to non-compactness), this product equals the part of the Gopakumar-Vafa partition function (4.47) corresponding to the genus 0 curves in the resolution of the singularity. In particular, each curve contributes a factor

$$\prod_{\alpha \in R_+} (1 - e^{t \cdot \alpha})^{1/2}$$

semi-classically. From this point of view the dyon counting formula (7.44) seems to be a generalization of the (quantum) McKay correspondence, where the Cartan matrix (7.46) corresponds geometrically to the transverse intersection of three genus 0 curves in two points. This is illustrated in Fig. 7.8. Indeed, the corresponding toric diagram (or string web) exactly represents this configuration of curves. It is interesting to find out whether this locus can indeed appear as a resolved Calabi-Yau singularity. Of course, one might also wonder whether such an algebra can be extracted from other toric Calabi-Yau's that are modeled on a number of compact spheres, and about the relation to wall-crossing.

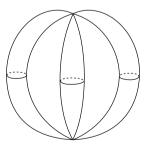


Figure 7.8: The intersection matrix of a configuration of three 2-spheres, that have self-intersection number -2 and that intersect each other 2-sphere transversely in two points, equals the Cartan matrix (7.46).

Chapter 8

Fluxes and Metastability

Over the last years much progress has been made in studying Calabi-Yau compactifications where one turns on extra fluxes, so-called flux compactifications. Turning on fluxes is a method to lift part of the large degeneracy in the four-dimensional scalar moduli fields. By now there is strong evidence that there is a huge number of supersymmetric vacua with negative cosmological constant in which all scalar moduli are stabilized, the so-called landscape of string theory. Typical constructions start with a warped Calabi-Yau compactification of type IIB string theory to four dimensions. (In such a compactification the four-dimensional scale is dependent on the coordinates of the internal space.) Some of the scalar moduli are stabilized by the addition of fluxes through the compact cycles of the internal manifold and others by various quantum effects.

Since supersymmetry is broken in the real world, it is necessary to extend the previous constructions to non-supersymmetric (meta)stable vacua with small positive cosmological constant to make contact with phenomenology. For this we need to understand the mechanism of supersymmetry breaking in string theory. So far several methods of supersymmetry breaking for string vacua have been proposed, such as the introduction of anti-branes and the existence of metastable points of the flux-induced potential. The main drawback of these constructions is that, in most cases, they are not under complete quantitative control.

While the question of supersymmetry breaking should be ultimately understood in an honest compactification, that is in a theory including gravity in four dimensions, it is technically easier to study simpler systems where the gravitational dynamics has been decoupled from the gauge theory degrees of freedom. This typically happens in the limit where a local singularity develops in the Calabi-Yau manifold. In such a situation all the interesting dynamics related to the degrees of freedom of the singularity takes place at energy scales much lower than the four dimensional Planck scale. Assuming that supersymmetry breaking is re-

lated to these light degrees of freedom, it is then possible to zoom in towards the singularity and forget about the rest of the Calabi-Yau.

Recently K. Intriligator, N. Seiberg and D. Shih discovered that even simple supersymmetric gauge theories can exhibit dynamical supersymmetry breaking in metastable vacua [286]. From a phenomenological point of view this possibility is quite attractive. A certain class of gauge theories where supersymmetry breaking in metastable vacua can be studied with good control is that of $\mathcal{N}=2$ gauge theories perturbed by a small superpotential, initiated by H. Ooguri, Y. Ookouchi and C.-S. Park in [287]. In such theories the exact Kähler metric on the moduli space is known. This makes it possible to compute the scalar potential that is produced by the perturbation of the theory by a small superpotential exactly to first order in the perturbation. It was shown that generically there are metastable supersymmetry breaking vacua generated by appropriate superpotentials. We will refer to this as the *OOP mechanism* for supersymmetry breaking in $\mathcal{N}=2$ theories.

String theory in a local Calabi-Yau singularity realizes geometric aspects of supersymmetric gauge theories, see Chapter 4. So also supersymmetry breaking in these two systems should be related. The first goal in this chapter is to make this connection more precise by proposing a geometric realization of the OOP supersymmetry breaking mechanism in type II theory on a local Calabi-Yau singularity. Starting from a geometrically engineered type II compactification, in Section 8.2 we introduce the appropriate superpotential as a non-conventional 3-form flux in the non-compact Calabi-Yau threefold. This flux does not pierce the compact cycles of the local geometry, but instead has support at infinity. In Section 8.2.2 we exemplify this with the study of local Calabi-Yau geometries modeled on a Riemann surface.

In Section 8.3 we turn to the second goal of this chapter: Finding a "natural" way to generate the supersymmetry breaking flux configurations described above while starting from a more standard setup. In this process, we also clarify the meaning of flux which has support at infinity and the various subtleties related to it. The natural interpretation of the flux described in the previous paragraph emerges once we embed the previous supersymmetry-breaking local singularity into a bigger IIB compactification with standard flux of compact support. Section 8.3.2 supplements this general discussion by providing an explicit demonstration of the factorization limit in the class of local geometries studied in Section 8.2.2. Matrix model techniques can be used to compute the prepotential in the factorization limit.² Finally, we finish in Section 8.4 with some concluding remarks concerning the generalization of our story to other $\mathcal{N}=2$

¹In [288] a related set-up is studied from a different perspective.

²This chapter is a shortened version of [4]. In particular, it doesn't include the Appendices of [4] where the prepotential for the Cachazo-Intriligator-Vafa/Dijkgraaf-Vafa geometry is studied using the dual matrix model.

contexts, such as M and F-theory compactifications.

Let us start by introducing superpotentials, scalar potentials, metastable vacua and the OOP mechanism.

8.1 Ooguri-Ookouchi-Park formalism

 $\mathcal{N}=2$ supersymmetry in four-dimensional gauge theories, as introduced in Section 4.1, can be broken to $\mathcal{N}=1$ supersymmetry by adding a superpotential $W(\phi)$ to the low energy effective action. The superpotential generates a scalar potential $V(\phi)$ on the complex structure moduli space \mathcal{M}_q of the $\mathcal{N}=2$ gauge theory. When the gauge theory is perturbed by just a small superpotential, the scalar potential is computed by

$$V(\phi) = G^{i\bar{j}} \partial_i W \overline{\partial_j W},$$

to lowest order in the perturbation. As an implication not all the vacua parametrized by \mathcal{M}_q are equivalent anymore. Furthermore, the value of ϕ will be stabilized in a local minimum of the scalar potential. So adding a superpotential lifts the degeneracy in the moduli of ϕ .

Notice that since the quantum metric $G_{i\bar{j}}$ is positive definite, the scalar potential $V(\phi) \geq 0$. Now, since the Hamiltonian of a supersymmetric theory is an anti-commutator of the supersymmetry generators, $H = \{Q^\dagger, Q\}$, the potential V should vanish for a four-dimensional vacuum that preserves supersymmetry. This implies that other local minima of V correspond to non-supersymmetric four-dimensional vacua. When the potential has more than one (local) minimum, there is a probability of quantum tunneling. Such non-supersymmetric vacua are therefore M

For phenomenological reasons it is very interesting to study the above perturbed $\mathcal{N}=2$ gauge theories. In particular, it is important to find out whether there are choices for the superpotential W such that the scalar potential V admits non-supersymmetric vacua. This was addressed in [289, 290, 287] and affirmed in the latter two articles. H. Ooguri, Y. Ookouchi and C.-S. Park observed in [287] that it is even possible to create a metastable vacuum at any generic point of the complex structure moduli space of an $\mathcal{N}=2$ supersymmetric gauge theory by turning on a suitable small superpotential deformation. Their proof just depends on the sectional curvature of the quantum moduli space. To create a metastable vacuum at $p \in \mathcal{M}_q$ the sectional curvature at p should be positive semi-definite. This means that for any holomorphic vector field $w \in T\mathcal{M}_q$ the curvature R satisfies

$$\langle w, R(v,v) w \rangle \geq 0,$$

for all $v \in (T\mathcal{M}_q)_p$. The curvature of special Kähler manifolds, of which the complex structure moduli space is an example, is indeed semi-positive [287].

If we want to realize a nonsupersymmetric minimum at some point p in the moduli space, the OOP procedure tells us to first construct holomorphic normal coordinates around p. Choose any local coordinates X^i near p. Then the holomorphic Kähler normal coordinates are defined as

$$Z^{i} = X^{'i} + \tilde{G}^{i\bar{j}} \sum_{n=2}^{\infty} \frac{1}{n!} \partial_{i_{3}} \dots \partial_{i_{n}} \tilde{\Gamma}_{ji_{1}i_{2}} X^{'i_{1}} X^{'i_{2}} \dots X^{'i_{n}}, \tag{8.1}$$

where $X^{'i}=X^i-X^i_0$ and $X^i_0=X^i(p)$ [291, 292, 293]. Furthermore, Γ is a Christoffel symbol and the tilde $\tilde{\ }$ means evaluation at $X=X_0$. We then choose the superpotential

$$W = k_i Z^i, (8.2)$$

with $k_i \in \mathbb{C}$, consisting of a linear combination of the Z^i . Stability can be demonstrated by expanding V near p

$$V(Z^{i}) = k_{i}\bar{k}_{\bar{j}}\tilde{G}^{i\bar{j}} + k_{i}\bar{k}_{\bar{j}}\tilde{R}^{i\bar{j}}_{k\bar{l}}Z^{k}\bar{Z}^{\bar{l}} + \mathcal{O}(Z^{3}).$$

Since R is positive definite at generic points $p \in \mathcal{M}_q$, the quadratic term in the above expansion is generically positive. In that case the vacuum at $Z^i = 0$ is naturally metastable.

As a result, any potential that agrees with (8.2) near X_0^i to cubic order will engineer a nontrivial vacuum at X_0^i . For non-generic X_0 , the curvature may have a zero eigenvalue in which case higher order agreement with (8.2) is required. More remarks about the OOP formalism can be found in [294].

8.2 Geometrically engineering the OOP formalism

In string compactifications one can generate scalar potentials by for example turning on higher dimensional gauge fields across cycles of the internal manifold, inserting D-branes and/or non-perturbative effects. Recent reviews include [295, 296]. In this section we geometrically engineer the OOP formalism by turning on fluxes in a type IIB local Calabi-Yau compactification. These fluxes will however be non-conventionally supported at infinity.

8.2.1 Flux at infinity

Before introducing these fluxes that grow large at infinity, let us review some general aspects of type IIB flux compactifications.

In Section 4.3 we summarized how a type IIB Calabi-Yau compactification leads to an $\mathcal{N}=2$ supergravity theory in 4d coupled to $h^{2,1}$ vector multiplets and $h^{1,1}+1$ hypermultiplets. We discussed how the vector multiplets play an important role in the geometrical engineering of $\mathcal{N}=2$ supersymmetric gauge theories. In this chapter we concentrate as well on the dynamics of the vector multiplets.

Denote a symplectic basis of 3-cycles on the compact Calabi-Yau threefold X by $\{A^i,B_j\}$ with $i,j=0,1,\ldots,h^{2,1}$, and denote the periods of the nowhere vanishing holomorphic (3,0)-form Ω by X^i and F_j . Remember that the metric on the complex structure moduli space is special Kähler and the Kähler potential is given by

$$K = -\log\left(i\int\Omega\wedge\overline{\Omega}\right),\tag{8.3}$$

which is an exact result which does not receive any α' or g_s corrections.

The easiest way to lift the phenomenologically unrealistic moduli space of these Calabi-Yau compactifications is to turn on fluxes through the compact cycles of the Calabi-Yau. In type IIB theory we can turn on RR and NS-NS 3-form flux F_3 and H_3 through the 3-cycles of the threefold. This generates a Gukov-Vafa-Witten superpotential for the complex structure moduli [297, 298, 299] given by

$$W = \int G_3 \wedge \Omega, \tag{8.4}$$

where $G_3 = F_3 - \tau H_3$ and $\tau = C_0 + i/g_s$. The scalar potential is computed by the standard $\mathcal{N} = 1$ supergravity expression³

$$V = e^{\widetilde{K}} \left(G^{a\overline{b}} D_a W \overline{D_b W} - 3|W|^2 \right),$$

where $G_{a\overline{b}}$ is the metric on the moduli space derived from the Kähler potential \widetilde{K} , and where we have introduced the Kähler covariant derivative $D_aW=\partial_aW+(\partial_a\widetilde{K})W$.

The F_3 and H_3 fluxes generate charge for the F_5 -form via a Chern-Simons coupling in the 10d IIB supergravity action. The F_5 flux has nowhere to end, so we are lead to the tadpole cancellation condition for IIB compactifications

$$\frac{1}{l_s^4} \int F_3 \wedge H_3 + Q_{D3} = 0,$$

where Q_{D3} receives positive contribution from probe D3 branes and negative contribution from induced charge on D7 and orientifold planes.

 $^{^3}$ In this expression the indices a,b run over complex structure moduli, Kähler moduli and the axion-dilaton. We denote by \widetilde{K} the total Kähler potential for all moduli and by K, as in (8.3), the one for the complex structure moduli alone.

Local limit and flux at infinity

In the limit in which we zoom in to some singularity locus of a compact Calabi-Yau (see Section 4.3.1) the structure of special geometry described above reduces to rigid special geometry, which is relevant for the low energy dynamics of $\mathcal{N}=2$ gauge theories. In this case the Kähler potential reduces to

$$K = i \int \Omega \wedge \overline{\Omega}, \tag{8.5}$$

and the Kähler covariant derivative D_i reduces to the ordinary derivative ∂_i .

As in the compact case, the addition of fluxes to the local Calabi-Yau introduces a superpotential for the moduli. The dynamics of the Kähler moduli and the dilaton decouple, and we can concentrate on the normalizable complex structure moduli. The superpotential is still given by (8.4), but now the scalar potential is computed by the rigid $\mathcal{N}=2$ expression

$$V = G^{i\overline{j}} \partial_i W \overline{\partial_j W} = \int G_3 \wedge * \overline{G_3}.$$

Since we are in a noncompact Calabi-Yau it is not necessary to impose the tadpole cancellation condition. Instead, the quantity

$$\int F_3 \wedge H_3$$

represents the F_5 flux going off to infinity and remains constant as we vary the moduli. We will use this to simplify the potential in the next section.

In most treatments of fluxes in noncompact Calabi-Yau manifolds the assumption is made that the flux is threading the compact cycles of the singularity and is going to zero at infinity. As we explained in the introduction the goal of this chapter is to study the dynamics in the case where the flux is actually coming in from infinity and is not supported on the compact three-cycles. Of course, in a local singularity inside a bigger compact Calabi-Yau, what is meant by infinity is the rest of the Calabi-Yau and we should think of flux coming from infinity as flux leaking towards the singularity from the other compact cycles.

More precisely, in a noncompact Calabi-Yau threefold we consider the vector space $H^3(X)$ of harmonic 3-forms which do not necessarily have compact support, so they can grow at infinity. The harmonic 3-forms of compact support form a linear subspace $H^3_{\mathrm{cpct}}(X) \subset H^3(X)$. There is a natural way to define the complement subspace $H^3_\infty(X) \subset H^3(X)$ as the harmonic forms with vanishing

integrals on the compact 3-cycles⁴. Then we have the decomposition

$$H^3(X) = H^3_{\infty}(X) \oplus H^3_{\text{cpct}}(X). \tag{8.6}$$

We will also refer to the forms in $H^3_{\mathrm{cpct}}(X)$ as harmonic 3-forms with compact support and to those in $H^3_\infty(X)$ as 3-forms with support at infinity.

Now we want to consider the case where the 3-form field strength that we have turned on has support at infinity

$$G_3 \in H^3_\infty(X)$$
,

which means that G_3 has zero flux through the compact cycles

$$\int_{A^i} G_3 = \int_{B_i} G_3 = 0.$$

The intuitive picture that one should keep in mind, is that this flux at infinity represents usual flux piercing other 3-cycles which are very far away from the singularity in the big Calabi-Yau. As we will see in more detail in the next section, in this case and if one zooms into the local singularity it is a good approximation to treat the flux from the distant 3-cycles as flux which "diverges" at infinity. In other words both $H^3_\infty(X)$ and $H^3_{\mathrm{cpct}}(X)$ correspond to the usual $H^3_{\mathrm{cpct}}(\widetilde{X})$ of the bigger Calabi-Yau \widetilde{X} in which the singularity X develops.

Flux potential

What is maybe more surprising is that the 3-form flux G_3 with support at infinity generates a potential for the complex structure moduli of the singularity X, even though it is not directly piercing the compact cycles of X, as can be seen from (8.2.1). Our starting point for the computation of this potential is the energy stored in the 3-form field

$$\widetilde{V} = \int G_3 \wedge *\overline{G_3}. \tag{8.7}$$

Since G_3 has noncompact support, this is a divergent integral meaning that the energy of the flux is infinite. This was to be expected and is not really a problem, since we are interested in the *changes* of this energy as we vary the sizes of the 3-cycles in the neighborhood of the singularity. We would like to throw away the

⁴We should clarify that we are not interested in the most general harmonic 3-form with noncompact support, but only in a restricted subset characterized by 3-forms which grow in a "controlled" way at infinity. This means that we want to consider forms which have at most a "pole" of finite order at infinity, and not essential singularities. This statement has a nice interpretation in the example where we have a local Calabi-Yau based on a Riemann surface that we will study later. Another way to state this restriction is that we will consider harmonic 3-forms on a local Calabi-Yau which do have a lift to the original Calabi-Yau that we started with before we took the local limit near its singularity.

divergent, moduli independent piece of this quantity and keep the finite, moduli dependent one. A nice way to achieve this is to use the fact that the net F_5 form flux leaking off at infinity, being a topological quantity, has to be kept constant as we vary the moduli. It is easy to show that we can write

$$\int G_3 \wedge \overline{G_3} = (\tau - \overline{\tau}) \int F_3 \wedge H_3,$$

and the left hand side must be constant for the reason we explained. Since it is a constant we can subtract it from the potential and define

$$V \equiv \int G_3 \wedge *\overline{G_3} - \int G_3 \wedge \overline{G_3}.$$

It is easy to show that this is equal to

$$V = \int G_3^- \wedge * \overline{G_3^-}, \tag{8.8}$$

where ${\cal G}_3^-$ is the imaginary anti-self dual part of the ${\cal G}_3$ flux

$$*G_3^- = -iG_3^-.$$

The expression (8.8) is the finite and moduli dependent piece of the potential (8.7).

Simplifying the potential

In this subsection we simplify the expression (8.8) for the potential. In general we have the following relation between the Hodge decomposition and the \ast operator on a threefold

$$\begin{split} *\,H^{3,0} &= -iH^{3,0}, & *H^{1,2} &= -iH^{1,2}, \\ *\,H^{2,1} &= iH^{2,1}, & *H^{0,3} &= iH^{0,3}. \end{split}$$

Before we proceed we would like to analyze the relation between the decomposition (8.6) and the Hodge decomposition. In general we have the following decomposition⁵

$$H^{3}(X) = H^{3,0}_{\infty} \oplus H^{3,0}_{\text{cpct}} \oplus H^{2,1}_{\infty} \oplus H^{2,1}_{\text{cpct}} \oplus \{c.c.\}.$$

Harmonic forms in $H^{p,q}_{\mathrm{cpct}}$ have compact support, while those in $H^{p,q}_{\infty}$ do not, and are chosen to have vanishing \mathcal{A} -periods on the compact cycles. (Notice that a harmonic (p,q)-form cannot have vanishing periods on all compact cycles

⁵Again, we are only considering a certain subset of all harmonic 3-forms with noncompact support, as explained in footnote 4.

unless it is identically zero.) Since we do not want to break supersymmetry explicitly by the boundary conditions of the system, we want our configuration to be supersymmetric at infinity, which means that the flux at infinity has to be imaginary self dual so

$$G_3 \in H^{2,1}_{\infty} \oplus H^{2,1}_{\text{cpct}} \oplus H^{1,2}_{\text{cpct}}.$$
 (8.9)

where the subscript ∞ means that we have to consider the elements of the cohomology with noncompact support. We pick a basis

$$\Xi_m \in H^{2,1}_{\infty}, \qquad \Omega_i \in H^{2,1}_{\text{cpct}}$$

with the following periods

$$\int_{A^{i}} \Xi_{m} = 0, \qquad \int_{A^{i}} \Omega_{j} = \delta_{j}^{i},
\int_{B_{i}} \Xi_{m} = K_{im}, \qquad \int_{B_{i}} \Omega_{j} = \tau_{ij},$$
(8.10)

where τ_{ij} is the period matrix of the Calabi-Yau, and K_{im} are holomorphic functions of the normalizable-complex structure moduli.

The flux has an expansion of the form

$$G_3 = T^m \Xi_m + h^i \Omega_i + \overline{l^i} \, \overline{\Omega_i}. \tag{8.11}$$

The parameters T^m are fixed by the boundary conditions and have to be kept constant as we vary the normalizable moduli. We have also assumed that

$$\int_{A^i} G_3 = \int_{B_i} G_3 = 0. \tag{8.12}$$

which means

$$T^{m} \int_{A^{i}} \Xi_{m} + h^{j} \int_{A^{i}} \Omega_{j} + \overline{l^{j}} \int_{A^{i}} \overline{\Omega_{j}} = 0$$

$$T^{m} \int_{B_{i}} \Xi_{m} + h^{j} \int_{B_{i}} \Omega_{j} + \overline{l^{j}} \int_{B_{i}} \overline{\Omega_{j}} = 0.$$
(8.13)

The first equation of (8.13) implies that

$$\overline{l^j} = -h^j.$$

and the second

$$h^{i} = -\frac{1}{2i} \left(\frac{1}{\operatorname{Im} \tau} \right)^{ij} \left(K_{jm} T^{m} \right).$$

As we explained before, only the imaginary anti-self dual part of the flux $G_3^-=$

 $\overline{l^i}\,\overline{\Omega_i}$ contributes to the regularized potential and we have

$$V = \int G_3^- \wedge \overline{G_3^-}$$

$$= \frac{1}{4} \overline{(K_{im}T^m)} \left(\frac{1}{\text{Im}\tau}\right)^{ij} (K_{jn}T^n). \tag{8.14}$$

In this final expression the period matrix τ^{ij} and K_{im} are functions of the normalizable complex structure moduli, while T^m 's have to be considered as constants which play the role of external parameters. This potential is in general very complicated and can have local nonsupersymmetric minima for appropriate choices of the parameters T^m as we will explain later.

Although we do not discuss this here, from the viewpoint of flux compactification it is a natural generalization to consider fluxes through the compact 3-cycles, relaxing the condition (8.12). Such flux will make additional contribution to the superpotential of the form $N^iF_i-\alpha_iX^i$, $\alpha_i=\int_{B_i}\Omega$, which cannot be controlled by external parameters and makes realization of OOP-like vacua more difficult.

Recovering the OOP potential

The potential (8.14) looks familiar. It shares the same basic structure as the scalar potential that arises when one adds a small superpotential to Seiberg-Witten theory. This connection can be made even more transparent by noting that K_{im} can in general be written as a total derivative with respect to the special coordinates X^i .

$$K_{im} = \frac{\partial}{\partial X^i} \kappa_m(X^j), \qquad X^i = \oint_{\Delta^i} \Omega.$$

One quick way to see this is to use the identity $\int_X \Xi_m \wedge \partial_i \overline{\Omega} = 0$ to derive

$$K_{im} \sim \int_{\partial X} \Lambda_m \wedge \partial_i \overline{\Omega}$$

for a 2-form Λ_m satisfying $d\Lambda_m = \Xi_m$ on the boundary (at infinity) of X. Because the divergent contributions to Λ_m at infinity can be chosen independent of the dynamical moduli, we can pull the derivative outside of everything.

With this notation, (8.14) takes the standard form

$$V = \frac{1}{4} \overline{\left(\frac{\partial W_{\text{eff}}(X^k)}{\partial X^i}\right)} \left(\frac{1}{\text{Im}\tau}\right)^{ij} \left(\frac{\partial W_{\text{eff}}(X^k)}{\partial X^j}\right), \tag{8.15}$$

where

$$W_{\rm eff}(X^k) = T^m \kappa_m(X^k)$$

is in fact proportional to the Gukov-Vafa-Witten superpotential induced by the flux G_3 . These equations make manifest the relation between our flux-induced potential (8.14) and that which arises in deformed Seiberg-Witten theory and allows us to utilize the OOP technology of Section 8.1 to engineer supersymmetry-breaking vacua.

Lifetime of supersymmetry-breaking vacua

Because we have managed to achieve supersymmetry-breaking vacua while freezing all non-normalizable moduli, the energies V_0 will in general be finite and independent of the cutoff scale Λ_0 that we use to regulate the local geometry. This means that our vacua are truly metastable, even within this local model, and can decay to any of the supersymmetric vacua that exist in these models. Because the number the supersymmetric vacua is potentially large and their properties quite model-dependent, it is difficult to make general statements about the lifetime of our OOP vacua. Nevertheless, we recall here one observation from [287], namely that the decay rates will in general scale like

$$e^{-S}$$
 with $S \sim \frac{(\Delta Z)^4}{V_+}$, (8.16)

where ΔZ is the distance in field space between the initial and final vacuum state and V_+ is the difference in their energies. By simultaneously scaling all T^m by a common factor, $T^m \to \epsilon T^m$, we can retain our supersymmetry-breaking vacua while decreasing V_+ by the same factor, $V_+ \to \epsilon V_+$. In this manner, we see that, just as with OOP vacua in deformed Seiberg-Witten theory, these OOP flux vacua can be made arbitrarily long-lived. Because we should really think of the local Calabi-Yau as sitting inside some larger compact geometry, one important caveat to this statement of longevity is that the noncompact fluxes T^m in reality derive from a suitable set of compact fluxes in the full Calabi-Yau. This means that there will be a series of quantization conditions that must be imposed that may affect the degree to which they may be tuned.

8.2.2 Local Calabi-Yau examples

In the previous section, we saw that, starting from a compact Calabi-Yau and taking a decoupling limit, one ends up with a local Calabi-Yau with noncompact flux with support at infinity, which is nothing but the flux leaking from the rest of the full Calabi-Yau that have been decoupled, towards "our" local Calabi-Yau. Furthermore, this noncompact flux induces potential (8.14) for the complex structure moduli in the local Calabi-Yau. Depending on the noncompact flux, this potential can be very complicated and create nonsupersymmetric metastable vacua in the local Calabi-Yau; the OOP mechanism [287] reviewed in Section 8.1 tells us exactly how this can be done.

In this section we study specific examples of our formalism. Let us start with a non-compact Calabi-Yau modeled on a Riemann surface, defined by

$$X_{\Sigma}: uv - H(x, y) = 0,$$
 (8.17)

where x,y can both be variables in $\mathbb C$ or $\mathbb C^*$. Recall that the holomorphic 3-form of X_{Σ} is given, e.g. for $x,y\in\mathbb C$, by

$$\Omega = \frac{du \wedge dx \wedge dy}{\partial H/\partial v} = \frac{du}{u} \wedge dx \wedge dy.$$

Many important properties of the noncompact Calabi-Yau threefold X_{Σ} have an interpretation in terms of the underlying Riemann surface Σ . For example, the compact 3-cycles $\{A^i,B_j\}$ in X_{Σ} are lifts of compact 1-cycles on Σ , which we denote by $\{a^i,b_j\}$ here. This one-to-one correspondence between 3- and 1-cycles shows an equivalence between the complex structure moduli on X_{Σ} and Σ .

A basis of (2,1)-forms with compact support on X_Σ is given by derivatives of Ω with respect to the normalizable complex structure moduli: $\{\Omega_i = \partial_i \Omega\}$. If X_Σ were compact, these derivatives ∂_i would be Kähler covariant derivatives D_i on the moduli space. Being noncompact instead, the moduli space is described by rigid special geometry and, as we saw before, the covariant derivatives simplify into partial derivatives. Another reduction over the compact 3-cycles in the Calabi-Yau shows that all these compactly supported (2,1)-forms Ω_i reduce to a basis of holomorphic 1-forms ω_i on Σ . Similarly, (1,2)-forms $\overline{\partial_i \Omega}$ in X_Σ reduce to antiholomorphic 1-forms $\overline{\omega}_i$ on Σ . The ω_i satisfy the relations

$$\frac{1}{2\pi i} \int_{a^i} \omega_j = \delta_j^i, \qquad \frac{1}{2\pi i} \int_{b_i} \omega_j = \tau_{ij}, \tag{8.18}$$

where τ_{ij} is the period matrix of Σ .

The relation between the 3-cycles/3-forms on X_Σ and the 1-cycles/1-forms on Σ through the trivial uv-fibration being understood, we can rewrite the various relations in Section 8.2 in terms of the Riemann surface Σ . First of all, the holomorphic 3-form Ω of X_Σ is easily seen to reduce to a meromorphic 1-form $\eta = y\,dx$ on the Riemann surface in this case [101, 128]. The special coordinates parametrizing complex structure moduli are

$$X^{i} = \frac{1}{2\pi i} \int_{a^{i}} \eta, \qquad F_{i} = \frac{1}{2\pi i} \int_{b_{i}} \eta,$$
 (8.19)

and the Kähler potential (8.5) is given by

$$K = i \int_{\Sigma} \eta \wedge \overline{\eta}. \tag{8.20}$$

Recall that, in the special coordinates $\{X^i\}$, the moduli space metric takes a particularly simple form:

$$ds^{2} = \left(\frac{\partial^{2} K}{\partial X^{i} \partial \overline{X^{j}}}\right) dX^{i} d\overline{X^{j}} = (\operatorname{Im}\tau)_{ij} dX^{i} d\overline{X^{j}}, \tag{8.21}$$

as can be shown using $\partial_i \eta = \omega_i$ and the Riemann bilinear relation.

Let us now consider a very small deformation of the system breaking supersymmetry to $\mathcal{N}=1$, thus generating a potential V for the moduli. As we saw before, this can be accomplished by turning on 3-form flux G_3 with support at infinity in the local Calabi-Yau. This flux can be thought of as leaking from the other part of the full compact Calabi-Yau, which has been frozen in the decoupling limit. We assume that the decoupling limit is taken consistently with the elliptic fibration structure; namely, we assume that the noncompact flux is supported at the asymptotic infinities of Σ , while being compact in the direction of the uv-fibers.

The basis of (2,1)-forms with noncompact support, $\{\Xi_m\}$, in the Calabi-Yau X_{Σ} descend to meromorphic 1-forms $\{\xi_m\}$ on the Riemann surface Σ , satisfying the relations

$$\int_{a^{i}} \xi_{m} = 0, \qquad \int_{b_{i}} \xi_{m} = K_{im}, \tag{8.22}$$

which are reductions of (8.10). Therefore, the 3-form flux G_3 with noncompact support on X_{Σ} , as given in (8.11), descends to a harmonic 1-form flux

$$g = g_H + \overline{g_A},$$

$$g_H = T^m \xi_m + h^i \omega_i, \qquad \overline{g_A} = \overline{l^i \omega_i},$$
(8.23)

which will have poles at the punctures (or asymptotic legs) of Σ . The 3-form flux G_3 in X_{Σ} induces superpotential (8.4), which reduces to an integral on Σ :

$$W = \int_{\Sigma} g \wedge \eta,$$

while the associated scalar potential (8.8) reduces to an integral on Σ :

$$V = \int_{\Sigma} g_A \wedge \overline{g_A}. \tag{8.24}$$

If we require the condition (8.12) that the flux (8.23) is zero through compact 3-cycles of $X_{\rm SW}$, which translates into

$$\int_{a^i} g = \int_{b_i} g = 0, \tag{8.25}$$

then by reducing the argument we made for general Calabi-Yau's in the previous section to the Riemann surface Σ (or simply by borrowing the result (8.14)), we can rewrite (8.24) in terms of periods on Σ :

$$V = \frac{1}{4} \overline{(K_{im} T^m)} \left(\frac{1}{\text{Im}\tau} \right)^{ij} K_{jn} T^n.$$
 (8.26)

Let us study this potential in more detail for Seiberg-Witten and Dijkgraaf-Vafa geometries, and make remarks on the gauge theory interpretation of the physics of these geometries.

Differentials on a hyperelliptic surface

When the underlying Riemann surface is hyperelliptic, say

$$\Sigma: \quad y^2 = f(x)$$

where f(x) is a polynomial of degree 2N, there are convenient representations for $\{\xi_m\}$ and $\{\omega_i\}$. Since we will need them later, let us briefly review them here.

A basis of holomorphic differentials ω_i can be constructed by

$$\omega_i = \frac{Q_i(x)}{y} dx = \frac{Q_i(x)}{\sqrt{f(x)}} dx, \tag{8.27}$$

where $Q_i(x)$ is a polynomial of degree up to N-2 chosen so that (8.18) holds. Note that this ω_i asymptots to $\mathcal{O}(x^{-2})dx$ when $x\to\infty,\widetilde{\infty}$. This means that it is regular at $x=\infty,\widetilde{\infty}$.

On a hyperelliptic surface it is convenient to take the meromorphic differentials of the second kind, ξ_m , as

$$\xi_m = \frac{R_m(x)}{y} dx = \frac{R_m(x)}{\sqrt{f(x)}} dx, \qquad m \ge 1.$$
 (8.28)

Here, $R_m(x) = mx^{m+N-1} + \dots$ is a polynomial and the coefficients of x^{m+N-2} , ..., x^{N-1} are chosen so that

$$\xi_m = \pm \left[mx^{m-1} + \mathcal{O}(x^{-2}) \right] dx, \qquad x \sim \infty, \widetilde{\infty}$$
 (8.29)

is satisfied. Note that this ξ_m has poles at two points, $x=\infty,\widetilde{\infty}$, instead of one. The coefficients of x^{N-2},\ldots,x^0 are chosen so that (8.22) is satisfied.

The meromorphic differential of the third kind, ξ_0 , can be defined likewise using

a polynomial $R_0(x) = x^{n-1} + \dots$, where the coefficients are chosen so that

$$\xi_0 = \frac{R_0(x)}{y} dx = \pm \left[\frac{1}{x} + \mathcal{O}(x^{-2}) \right] dx, \qquad x \sim \infty, \widetilde{\infty}$$

holds and (8.22) is satisfied.

Let us derive a formula that will be useful. By expanding the right hand side of the trivial identity $0 = \int_{\Sigma} \omega_i \wedge \xi_m$ by the Riemann bilinear identity, one finds

$$0 = \sum_{j} \left(\int_{a^{j}} \omega_{i} \int_{b_{j}} \xi_{m} - \int_{a^{j}} \xi_{m} \int_{b_{j}} \omega_{i} \right) + \sum_{p=\infty, \infty} \oint_{p} \omega_{i} d^{-1} \xi_{m}$$
$$= K_{im} + \sum_{p=\infty, \infty} \oint_{p} \omega_{i} d^{-1} \xi_{m}.$$

Because the behaviors of ω_i and ξ_m at $x=\infty$ is the same as those at $x=\widetilde{\infty}$ up to a sign, we find that

$$K_{im} = -\sum_{p=\infty,\widetilde{\infty}} \oint_p \omega_i \, d^{-1} \xi_m = -2 \oint_{\infty} \omega_i \, d^{-1} \xi_m = -2 \oint_{\infty} x^m \omega_i. \tag{8.30}$$

Seiberg-Witten geometries

The SU(N) Seiberg-Witten geometry

$$X_{SW}: uv - H_{SW}(v,t) = 0,$$
 (8.31)

with $v\in\mathbb{C}$ and $t\in\mathbb{C}^*$, is an illustrative example of a local Calabi-Yau threefold. The underlying Riemann surface Σ_{SW} is a hyperelliptic curve

$$\Sigma_{SW}: H_{SW}(v,t) = \Lambda^N \left(t + \frac{1}{t} \right) - P_N(v) = 0$$
 (8.32)

where $P_N(v) = \prod_{i=1}^N (v-\alpha_i)$ is a polynomial of degree N with the coefficient of v^{N-1} being zero. The coefficients of $P_N(v)$ are normalizable moduli, while Λ is a fixed parameter.

The holomorphic 3-form on $X_{\rm SW}$ is $\Omega_{\rm SW}=\frac{du}{u}\wedge dv\wedge \frac{dt}{t}$ and reduces to $\eta_{\rm SW}=v\frac{dt}{t}$ on the Riemann surface $\Sigma_{\rm SW}$. For a description of the bijection between 3-cycles on the local Calabi-Yau and 1-cycles on the Seiberg-Witten curve, we refer to Section 4.2.2.

The complex structure moduli space is conveniently parametrized by the special coordinates (8.19), which are conventionally denoted by a_i , i = 1, ..., N-1 in

the Seiberg-Witten case

$$a_i = \frac{1}{2\pi i} \int_{A_i^1} \eta_{SW} = \frac{1}{2\pi i} \int_{A_i^1} v \frac{dt}{t}.$$
 (8.33)

(To avoid confusion we denote the 1-cyles in the geometry with capital letters.) As in (8.21), the moduli space metric takes the special form for these:

$$ds^2 = (\operatorname{Im}\tau_{ij}) \, da_i d\overline{a_j}. \tag{8.34}$$

Using a_i , the normalized basis of holomorphic 1-forms ω_i can be obtained as follows. Differentiating (8.33) with respect to a^j ,

$$\delta^i_j = \frac{1}{2\pi i} \frac{\partial}{\partial a_j} \int_{A^i_1} v \frac{dt}{t}.$$

Comparing with the first equation in (8.18), this means that

$$\omega_i = \frac{\partial}{\partial a_i} \left(v \frac{dt}{t} + d\sigma \right), \tag{8.35}$$

where the total derivative term $d\sigma$ is fixed by requiring that $\omega_i = \mathcal{O}(v^{-2})dv$ as $v \to \infty$. Specifically, this leads to $d\sigma = d(-v \log t)$ and ω_i is given by

$$\omega_i = \frac{\partial}{\partial a_i} (-\log t \, dv) = -\frac{\partial P_N(v)/\partial a_i}{\sqrt{P_N(v)^2 - 4\Lambda^{2N}}} dv.$$

Although $\log t$ may appear problematic because it is not single-valued on the Riemann surface, its a_i derivative is single-valued and does not cause any problem.

As we discussed in Section 8.2.1, turning on noncompact flux breaks $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$ by inducing a superpotential. As in (8.23), the 3-form flux in $X_{\rm SW}$ reduces to a harmonic 1-form

$$g = \sum_{m \ge 1} T^m \xi_m + \sum_{i=1}^{N-1} h^i \omega_i + \sum_{i=1}^{N-1} \overline{l^i} \overline{\omega_i}.$$
 (8.36)

on Σ_{SW} . Under the condition that the compact flux vanishes (8.25), this leads to the scalar potential (8.26).

We can write the superpotential we are adding to the system in a form that will be useful later. By manipulating the quantity $K_{jn}T^n$ appearing in (8.26),

$$K_{jn}T^n = T^n \oint_{B^1_j} \xi_n = -2T^n \oint_{\infty} v^n \omega_j = 2T^n \frac{\partial}{\partial a_j} \left(\oint_{\infty} v^n \log t \, dv \right)$$

$$= -\frac{2T^n}{N+1}\frac{\partial}{\partial a_j}\left(\oint_{\infty} v^{n+1}\frac{dt}{t}\right).$$

Here we used (8.22), (8.30) and (8.35). By examining (8.26), and (8.34) one sees that the superpotential is given by

$$W_{SW} = \sum_{m} T^{m} u_{m+1}, \tag{8.37}$$

where we defined

$$u_m \equiv \frac{1}{2\pi i m} \oint_{\infty} v^{m-1} \eta_{\text{SW}} = \frac{1}{2\pi i m} \oint_{\infty} v^m \frac{dt}{t}.$$
 (8.38)

So far everything was about geometry. Now let us turn to the gauge theory interpretation of these. As we mentioned above, the local CY geometry (8.31) without flux realizes $\mathcal{N}=2$ Seiberg-Witten theory, with the hyperelliptic curve (8.32) identified with the $\mathcal{N}=2$ curve of gauge theory. The special coordinates a_i defined in (8.33) correspond to the U(1) adjoint scalars in the IR and parametrize the Coulomb moduli space. The superpotential (8.37) also has a simple gauge theory interpretation. To see it, we need the relation between the vev of the adjoint scalar Φ and the curve Σ_{SW} , given by [300, 301]:

$$\langle \operatorname{Tr} \frac{dv}{v - \Phi} \rangle = \frac{dt}{t} = \frac{P_N'(v)}{\sqrt{P_N(v)^2 - 4\Lambda^{2N}}} dv.$$

In other words, u_m defined geometrically in (8.38) has an interpretation in gauge theory as follows:

$$u_m = \frac{1}{m} \langle \operatorname{Tr} \Phi^m \rangle.$$

From this, one immediately sees that the superpotential (8.37) can be written as

$$W_{SW} = \sum_{m \ge 1} \frac{T^m}{m+1} \text{Tr } \Phi^{m+1} = \text{Tr } [W(\Phi)],$$
 (8.39)

where we defined

$$W(x) = \sum_{m} \frac{T^m}{m+1} v^{m+1}.$$

In (8.39) Φ is understood as the chiral superfield whose lowest component is the adjoint scalar.

Therefore, the $\mathcal{N}=2$ gauge theory perturbed by the single-trace superpotential

(8.39) corresponds to the geometry (8.31) with the flux g obeying the following asymptotic boundary condition:

$$g \sim \sum_{m} mT^{m}v^{m-1} dv = W''(v)dv,$$
 (8.40)

where we used (8.29). The perturbed $\mathcal{N}=2$ theory is precisely the system which was shown in [287, 290] to have nonsupersymmetric metastable vacua if the superpotential is chosen appropriately. (It was shown in [287] to be possible to create metastable vacua by a single-trace superpotential of the form (8.39) at any point in the Coulomb moduli space for SU(2) and at least at the origin of the moduli space for SU(N).) Therefore, it tautologically follows that the IIB Seiberg-Witten geometry with flux at infinity also has metastable vacua, if we tune the parameters T^m appropriately.

As we mentioned above, the IIB Seiberg-Witten geometry is dual to a IIA brane configuration of NS5-branes and D4-branes which can be lifted to an M5-brane configuration. In [302] it was shown that a superpotential perturbation corresponds in the M-theory setup to "curving" the $\mathcal{N}=2$ configuration of the M5-brane at infinity in a way specified by the superpotential. The metastable gauge theory configuration of [287, 290] was realized as a metastable M5-brane configuration and its local stability was given a geometrical interpretation. The above proof of (8.39) is exactly in parallel to the one given in [302] for the Mtheory system. In passing, it is also worth mentioning that the M-theory analysis of [302] revealed that at strong coupling the nonsupersymmetric configuration "backreacts" on the boundary condition and it is no longer consistent to impose a holomorphic boundary condition specified by a holomorphic superpotential, which is in accord with [303]. Therefore, also in the IIB flux setting, it is expected that if we go beyond the approximation that the flux does not backreact on the background metric, nonsupersymmetric flux configurations will backreact and it will be impossible to impose a holomorphic boundary condition of the type (8.36).

Although we do not discuss this here, from the viewpoint of flux compactification, it is a natural generalization to consider fluxes through the compact 3-cycles. Such flux will make additional contribution to the superpotential of the form $e_i a^i + m^i F_i$. On the gauge theory side, in the Seiberg-Witten theory, this can be interpreted as perturbation one adds at the far IR, but its UV interpretation is not clear [294].

Dijkgraaf-Vafa geometries

Another example of geometries of the type (8.17) is type IIB on

$$X_{\rm DV}: \quad uv - H_{\rm DV}(x, y) = 0, \quad x, y \in \mathbb{C},$$
 (8.41)

where the underlying Riemann surface Σ_{DV} is a hyperelliptic curve

$$\Sigma_{\text{DV}}: \qquad H_{\text{DV}}(x,y) \equiv y^2 - \left[P_n(x)^2 - f_{n-1}(x) \right] = 0$$
 (8.42)

and $P_n(x)$ and $f_{n-1}(x)$ are polynomials of degree n and n-1, respectively. If we write

$$f_{n-1}(x) = \sum_{i=1}^{n-1} b_i x^i,$$

then the coefficients of $P_n(x)$ as well as b_{n-1} are non-normalizable and fixed, while b_i , $i=0,\ldots,n-2$ are normalizable complex structure moduli. The holomorphic 3-form is $\Omega_{\rm DV}=\frac{du}{u}\wedge dx\wedge dy$ which reduces to

$$\eta_{\rm DV} = x \, dy$$

on the Riemann surface $\Sigma_{\rm DV}$. The geometry (8.41) was studied by Cachazo, Intriligator and Vafa (CIV) [304] (see also [305]) in the context of large N transition [18, 306] and further generalized in [307, 308]. The Dijkgraaf-Vafa (DV) conjecture [123, 124, 125] was also based on the same geometry. We will refer to this geometry as the CIV-DV geometry (8.41) or as the Dijkgraaf-Vafa geometry henceforth.

The structure of the underlying hyperelliptic Riemann surface $\Sigma_{\rm DV}$ (8.42) is similar to the Seiberg-Witten case (8.32); $\Sigma_{\rm DV}$ is a genus n-1 surface with two punctures at infinity. If we represent $\Sigma_{\rm DV}$ as a two-sheeted x-plane branched over 2n points, those infinities correspond to $x=\infty$ on the two sheets. The coefficients of $P_n(x)$, which are nonnormalizable, determine the position of the n cuts on the x-plane, while the coefficients of $f_{n-1}(x)$, which are normalizable, are related to the sizes of the cuts.

The first homology $H_1(\Sigma_{\rm DV})$ is spanned by n-1 pairs of compact a- and b-cycles $(a^i,b_j),\,i,j=1,\ldots,n-1$ with in addition a closed cycle a^∞ around one of the infinities which is dual to the noncompact b-cycle b_∞ connecting two infinities. Because $x,y\in\mathbb{C}$, compact a- and b-cycles on $\Sigma_{\rm DV}$ are all contractible in the x,y-plane and hence all compact 1-cycles on $\Sigma_{\rm DV}$ lifts to 3-cycles in $X_{\rm DV}$ with S^3 topology.

The special coordinates (8.19) in this case is conventionally denoted by S^i , Π_i :

$$S^{i} = \frac{1}{2\pi i} \int_{a^{i}} \eta_{\text{DV}}, \qquad \Pi_{i} = \frac{1}{2\pi i} \int_{b_{i}} \eta_{\text{DV}},$$
 (8.43)

for which, as in (8.21), the moduli space metric takes the special form:

$$ds^2 = (\operatorname{Im}\tau_{ij}) dS^i d\overline{S^j}.$$

We can write the basis of holomorphic 1-forms ω_i using S^i as:

$$\omega_i = \frac{\partial}{\partial S^i} (-y \, dx) = \frac{\partial f_{n-1}(x) / \partial S^i}{2\sqrt{P_n(x)^2 - f_{n-1}(x)}} \, dx.$$

Adding flux at infinity works the same as in the Seiberg-Witten case. The Riemann surface $\Sigma_{\rm DV}$ is hyperelliptic and we take $\{\xi_m\}$ and $\{\omega_i\}$ to be the ones given in (8.27) and (8.28). Just like (8.23) and (8.36), the 3-form flux in $X_{\rm DV}$ reduces to a harmonic 1-form on $\Sigma_{\rm DV}$

$$g = \sum_{m>1} T^m \xi_m + \sum_{i=1}^{N-1} h^i \omega_i + \sum_{i=1}^{N-1} \overline{l^i \omega_i}.$$

Under the condition that the compact flux vanishes (8.25), the 1-form g leads to the scalar potential (8.26) which, just as we derived (8.37), can be shown to correspond to the following superpotential:

$$W_{\rm DV} = \sum_{m} T^m \Sigma_{m+1},\tag{8.44}$$

where we defined

$$\Sigma_m \equiv \frac{1}{2\pi i m} \oint_{\infty} x^{m-1} \eta_{\text{DV}} = \frac{1}{2\pi i m} \oint_{\infty} x^m \, dy. \tag{8.45}$$

The 1-form $\eta_{\rm DV}$ depends on the complex structure moduli S^i of the Riemann surface (8.42). Therefore, by changing the parameters T^m , we can generate a superpotential which is a quite general function of S^i 's. The OOP mechanism [287] states that, if one tunes superpotential appropriately, one can create a metastable vacuum at any point of the $\mathcal{N}=2$ moduli space. Therefore, also for this Dijkgraaf-Vafa geometry, we expect to be able to create metastable vacua by appropriately tuning T^m , i.e., flux at infinity. Indeed, at the end of this section we will demonstrate the existence of metastable vacua in a simple example.

We have been focusing on the case where there is flux at infinity but there is no flux through compact cycles. However, let us digress a little while and think about the case where there is flux through compact cycles but there is no flux at infinity. In this case, the IIB system has a standard interpretation [304, 305, 123, 124, 125] as describing the IR dynamics of $\mathcal{N}=2$ SU(N) theory broken to $\mathcal{N}=1$ by a superpotential $W=\mathrm{Tr}[W_n(\Phi)],\ W_n'(x)=P_n(x),$ with the moduli S^i identified with glueball fields. More precisely, if there are N^i units of flux through the cycle a^i , where $N=\sum_i N^i$, then the system corresponds to the supersymmetric ground state of SU(N) gauge theory broken to $\left[\prod_i SU(N^i)\right] \times U(1)^{n-1}$. It is important to note that this equivalence between the Dijkgraaf-Vafa flux geometry and gauge theory is guaranteed to work only

for holomorphic dynamics, or for the F-term. On the geometry side, one is considering the underlying geometry (8.41) determined by $P_n(x)$ and small flux perturbation on it. On the gauge theory side, this corresponds to the limit of large superpotential, where one has no control of the D-term. Therefore, there is no a priori reason to expect that the D-term of the Dijkgraaf-Vafa geometry, which governs e.g. existence of nonsupersymmetric vacua, and that of gauge theory are the same, even qualitatively. After all, two systems are different theories and it is only the holomorphic dynamics that is shared by the two.

Despite such subtlety, it is interesting to ask what is the gauge theory interpretation of adding flux at infinity, in addition to flux through compact cycles. It is known that the curve (8.42) is related to the vev in gauge theory as [123, 124, 125, 300, 309]:

$$-\frac{1}{32\pi^2} \langle \operatorname{Tr} \frac{\mathcal{W}^2}{x-\Phi} \rangle dx = y \, dx = \sqrt{P_n(x)^2 - f_{n-1}(x)} \, dx.$$

where $\mathcal{W}^2 = \mathcal{W}_{\alpha}\mathcal{W}^{\alpha}$ and \mathcal{W}_{α} is the gaugino field. Comparing this with (8.45), one finds that the quantity Σ_m defined geometrically in (8.45) has the following interpretation:

$$\Sigma_m = \frac{1}{32\pi^2} \langle \operatorname{Tr} \mathcal{W}^2 \Phi^{m-1} \rangle.$$

Therefore the superpotential (8.44) can be written as

$$W_{\rm DV} = \frac{1}{32\pi^2} \sum_{m} T^m \text{Tr} \left[W^2 \Phi^m \right] = \frac{1}{32\pi^2} \text{Tr} \left[W^2 M(\Phi) \right],$$
 (8.46)

where we defined

$$M(x) = \sum_{m} T^m x^m.$$

So flux at infinity of the asymptotic form

$$g \sim \sum_{m} mT^{m} x^{m-1} dx = M'(x) dx,$$

corresponds in gauge theory to adding a novel superpotential of the form (8.46). Again, this correspondence must be taken with a grain of salt, since it holds only for holomorphic physics.

Note also that flux through compact cycles will induce glueball superpotential [304] of the form $\alpha_i S^i + N^i \Pi_i(S)$ added to (8.44). Because this part does not contain tunable parameters such as T^m that can be made very small, it is difficult, if not possible, to use the OOP mechanism to produce metastable vacua

in that case.

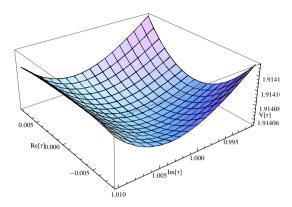


Figure 8.1: Plot of $V(\tau)$ in the neighborhood of our engineered OOP minimum at $\tau = i$.

In the situation where there is no flux through compact cycles, we do not have an interpretation of the system as such an SU(N) theory described above, simply because $N = \sum_i N^i = 0$.

In Section 3.2 of [4] we work out a simple example, based on the SU(2) Dijkgraaf-Vafa geometry

$$H_{DV}(x,y) = y^2 - (P_2(x)^2 - b_0) = 0$$
, with $P_2(x) = x^2 - \frac{\Delta^2}{4}$,

with flux at infinity and no flux through compact cycles, where we demonstrate that we can truly realize metastable vacua in type IIB using the OOP mechanism outlined in the previous section by simply adjusting the parameters T^m . Choosing $\tau=i$ for convenience, we find a metastable vacuum by turning on

$$T^2 = \frac{885}{8}, \qquad T^6 = -\frac{5832}{5\Delta^4}, \qquad T^{10} = \frac{2400}{\Delta^8}.$$

The corresponding local minimum of the potential is plotted in Fig. 8.1.

8.3 Embedding in a larger Calabi-Yau

In the previous sections we described how we can generate a supersymmetry breaking potential for the complex structure moduli of a local Calabi-Yau singularity by the introduction of 3-form flux which has support at infinity. Allowing flux with noncompact support may lead to various conceptual difficulties, such as the divergence of the total energy density. To clarify these difficulties we

would like to sketch how such a system can be interpreted as an approximation of a larger Calabi-Yau threefold with flux of compact support in a certain factorization limit.

As shown in figure 8.2, the physical idea is to start with a Calabi-Yau manifold with a set of three-cycles which are isolated from the other three-cycles by a large distance. We turn on 3-form flux on all cycles except for the isolated set. While the flux that we have turned on is not piercing the isolated cycles, it does leak into their region. (This means that the 3-form field strength is nonzero in the region around the isolated set of 3-cycles, but once integrated over one of these 3-cycles the integral is zero.) It produces a potential for their complex structure moduli.

In the limit where the distance between the two sets of cycles of the Calabi-Yau becomes very large, which we will refer to as the *factorization limit*, the flux leaking towards the isolated set will start to look like the flux coming from "infinity". In this sense, we manage to embed the scenario considered in the previous section as a small part of a larger Calabi-Yau with compactly supported flux.

8.3.1 Factorization

In this section we would like to understand this embedding into a bigger Calabi-Yau in more detail. Our goal is to see how the potential (8.14) arises starting from the standard Gukov-Vafa-Witten superpotential for 3-form flux in the larger Calabi-Yau. For simplicity we will work with a noncompact Calabi-Yau X,

$$X: \quad uv - H(x, y) = 0,$$

which is based on a Riemann surface Σ given by H(x,y)=0.

As we explained before the complex parameters entering the defining equation of the Riemann surface correspond to complex structure moduli of the Calabi-Yau. Some of them are non-normalizable and can be considered as external parameters. We want to tune these parameters to approach the limit where the surface Σ factorizes into two surfaces Σ_L and Σ_R connected by long tubes. This factorization lifts to the entire Calabi-Yau X and divides it into two regions X_L and X_R that are widely separated. We introduce 3-form flux G_3 of compact support on the 3-cycles of X_R . The superpotential and scalar potential are given by

$$W = \int G_3 \wedge \Omega$$
 and $V = G^{I\overline{J}} \partial_I W \overline{\partial_J W},$ (8.47)

where the indices I,J run over all complex structure moduli of the total three-fold X. Using the properties of the Kähler metric $G_{I\overline{J}}$ in the factorization limit

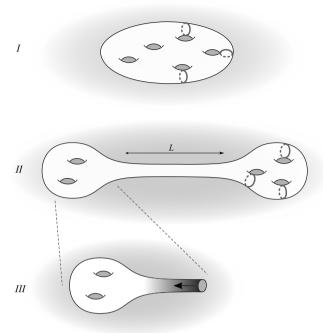


Figure 8.2: Factorization idea: In I we start with a generic Calabi-Yau with flux piercing through some of its 3-cycles, while making the distance between the cycles with and without flux very large in II. This is seen as flux from infinity in the left sector without compact flux in III, and generates an OOP-like potential in that sector.

we show that the part of the potential (8.47) which depends on the complex structure moduli of L is of the form (8.14). Furthermore, we find an understanding of the effective value of the parameters T^m .

Geometry of factorization

Let us study the degeneration of a Riemann surface Σ into two components Σ_L and Σ_R , depicted in figure 8.3.⁶ In this factorization data of the full Riemann surface is expressed in terms of the complex structure of the individual surfaces. It is well known that in the limit where the length of the tubes $L=1/\epsilon$ goes to infinity the period matrix of the full surface becomes block diagonal

$$\tau = \begin{pmatrix} \tau^{LL} & 0\\ 0 & \tau^{RR} \end{pmatrix} + \mathcal{O}\left(\epsilon\right). \tag{8.48}$$

⁶In general these components could be connected in a non-trivial way. We restrict our computations in this section to the case in which they are linked by just one long tube. These should be easily extendible to more general cases.

While the off-diagonal components au^{LR} go to zero in the factorization limit, their subleading behavior is quite important in our analysis since it expresses the weak interaction between the two sectors. The period matrix au^{LR} can be computed systematically in an expansion in ϵ from data on each of the two surfaces as we explain below.



Figure 8.3: Two conformally equivalent ways of viewing the factorization of a Riemann surface into two parts. Physically though, we should distinguish both points of view, since particle masses depend on the size of the cycles. Because in our situation no new massless appears in the factorization limit, the left diagram represents our point of view best.

Technically, we describe the factorization of the Riemann surface with the *plumbing fixture* method [310]. So consider two Riemann surfaces Σ_L and Σ_R of genus g_L and g_R respectively. On the left surface Σ_L we have g_L holomorphic differentials ω_i , while on the right surface Σ_R similarly g_R holomorphic differentials $\omega_{i'}$. The complex structure of the left surface is determined by the periods of the holomorphic differentials

$$\frac{1}{2\pi i} \int_{a^i} \omega_j = \delta^i_j, \qquad \frac{1}{2\pi i} \int_{b_i} \omega_j = \tau^{LL}_{ij},$$

where τ_{ij}^{LL} is the period matrix of Σ_L , and we choose our definitions similarly for the right surface.

The plumbing fixture method works after choosing a puncture P on Σ_L and P' on Σ_R . It connects the two surfaces by a long tube of length L which is glued onto neighborhoods of the punctures P and P'. More precisely, we pick a local holomorphic coordinate z around the puncture P such that z(P)=0 and a holomorphic coordinate z' near P' with z'(P')=0. Then we identify points in these neighborhoods as

$$zz' = \epsilon$$
.

Now we want to compute the period matrix of the full Riemann surface in terms of complex structure data of the two surfaces. For this we need to understand how the differentials ω_i and $\omega_{i'}$ extend to well-defined holomorphic differentials on the full surface $\Sigma = \Sigma_L \cup \Sigma_R / \sim$, where \sim is the above identification. Let us first consider how to lift the differential ω_i . Around the puncture P it may be expanded as

$$\omega_i = \sum_{m=1}^{\infty} K_{im}^P z^{m-1} dz,$$

where the functions K^P_{im} are defined in equation (8.22). Once we write this in terms of z' we observe that, as seen from the right surface, the differential has a Laurent expansion. So ω_i will be written as a linear combination of the meromorphic differentials $\xi_m^{P'}$ of the right surface. A meromorphic differential has the following expansion around the puncture

$$\xi_m^P = \left(\frac{m}{z^{m+1}} + \sum_{n=1}^{\infty} h_{mn}^P z^{n-1}\right) dz.$$

Here we have introduced the functions h_{mn}^P , which depend on the complex structure moduli of the surface and the position of P. So in general the differential ω_i will lift to a differential $\widetilde{\omega_i}$ on the full surface which can be written as

$$\widetilde{\omega_i} = \left\{ \begin{array}{ll} \omega_i + \sum_{m=1}^{\infty} x_{im} \xi_m^P & \text{on} \quad \Sigma_L, \\ \sum_{m=1}^{\infty} y_{im} \xi_m^{P'} & \text{on} \quad \Sigma_R. \end{array} \right.$$

for some coefficients x_{im} and y_{im} . Matching the differential on the two sides we find the following conditions

$$x_{im} = -\frac{\epsilon^m}{m} \sum_{n=1}^{\infty} y_{in} h_{nm}^{P'}, \qquad y_{im} = -\frac{\epsilon^m}{m} \left(K_{im}^P + \sum_{n=1}^{\infty} x_{in} h_{nm}^P \right).$$

This allows us to compute the cross-period matrix as

$$\tau_{ij'}^{LR} = \int_{b_{j'}} \omega_i = \sum_{m=1}^{\infty} K_{j'm}^{P'} y_{im} = -\sum_{m,n=1}^{\infty} \frac{\epsilon^n}{n} K_{im}^P G_{mn}^{-1} K_{j'n}^{P'},$$

$$G_{mn} \equiv \delta_{mn} - \sum_{l=1}^{\infty} \frac{\epsilon^{n+l}}{nl} h'_{ml} h_{ln}.$$
(8.49)

From this equation we can read off all order ϵ -corrections to the off-diagonal piece of the period matrix when a surface Σ degenerates.

Also, this procedure gives a clear understanding of the term "flux at infinity". We see that the flux at infinity is generated by regular forms on the degenerated surface, and therefore will at most have finite order poles at the punctures.

Notice that for a Calabi-Yau threefold that is based on a Riemann surface, the factorization region is described by the deformed conifold geometry

$$uv + x^2 + y^2 = \epsilon$$
, or equivalently $uv + zz' = \epsilon$.

Usually, this is described as a 3-sphere shrinking to zero-size when $\epsilon \to 0$. How-

ever, as for the complex 1-dimensional plumbing fixture case we want the two sectors to be far apart from each other. Therefore we consider the conformally equivalent setup where the 3-sphere is scaled to be of finite size, while the transverse directions are made very large. The finite size three-sphere reduces to the cross-section of the tube on the left in figure 8.3, whereas the transverse directions reduce to the tube-length.

To describe the left and right neighborhoods of the degeneration, we can fix $x=\sqrt{\epsilon-y^2-uv}$ on the left and $x=-\sqrt{\epsilon-y^2-uv}$ on the right. In the limit that $\epsilon\to 0$ these neighborhoods will not just intersect in a point, but in the divisor $uv+y^2=0$. This is the region where regular forms on the total threefold will develop poles when the degeneration starts.

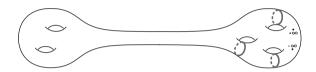


Figure 8.4: Turning on flux on the right part of the factorized Calabi-Yau.

Dynamics

Now we consider turning on flux on the threefold. For simplicity we again take a Calabi-Yau (8.3.1) that is based on a factorized Riemann surface. We turn on 3-form flux $G_3 = F_3 - \tau H_3$ which is only piercing the set of A-cycles corresponding to Σ_R , as can be seen in figure 8.4, and write down the corresponding (super) potential. For regularization issues later, we take two more punctures on the right surface labeled by $\pm \infty$ and turn on some flux α through the noncompact B_∞ cycle running from $+\infty$ to $-\infty$.

A basis of A and B cycles is given by the compact 3-cycles on the left and the right, together with the lift A^{∞} of the A-cycle enclosing $+\infty$ and B_{∞} . So the flux is determined by

$$\begin{split} & \int_{A^i} G_3 = 0, \qquad \int_{A^{i'}} G_3 = N^{i'}, \qquad \int_{A^{\infty}} G_3 = 0, \\ & \int_{B_i} G_3 = 0, \qquad \int_{B'_i} G_3 = 0, \qquad \int_{B_{\infty}} G_3 = \alpha. \end{split}$$

Let us denote the complex structure moduli and their duals by X^I and F_I , which are the A^I resp. B_I periods of the holomorphic 3-form Ω . Here we use the capital indices $I = \{i, i', \infty\}$ to run over both the left and the right sides. Then the GVW

superpotential for the complex structure moduli is given by

$$W = \int G_3 \wedge \Omega = \alpha X^{\infty} + \sum_{i'} N^{i'} F_{i'}^R,$$

and the corresponding scalar potential by

$$V = \sum_{I,J} G^{I\overline{J}} \partial_I W \overline{\partial_J W}.$$

Since X^∞ corresponds to a log-normalizable period and the derivatives in the above potential just correspond to normalizable modes, the α -factor decouples. This shows that

$$V = \sum_{i,j,k',l'} \left(N^{k'} \tau_{k'i}^{LR} \right) \left(\frac{1}{\text{Im}\tau} \right)_{LL}^{ij} \overline{\left(N^{l'} \tau_{l'j}^{LR} \right)}$$

$$+ \sum_{i,j',k',l'} \text{Re} \left[\left(N^{k'} \tau_{k'i}^{LR} \right) \left(\frac{1}{\text{Im}\tau} \right)_{LR}^{ij'} \overline{\left(N^{l'} \tau_{l'j'}^{RR} \right)} \right]$$

$$+ \sum_{i',j',k',l'} \left(N^{k'} \tau_{k'i'}^{RR} \right) \left(\frac{1}{\text{Im}\tau} \right)_{RR}^{i'j'} \overline{\left(N^{l'} \tau_{l'j'}^{RR} \right)}.$$
(8.50)

Thus the total potential is the sum of three terms, which we denote in the obvious way by $V = V_1 + V_2 + V_3$.

Next we consider what happens in the limit where the distance L between the two sets of 3-cycles gets very large. As explained before the period matrices τ^{LL} and τ^{RR} remain of order one in this limit and become almost independent of the moduli X_R and X_L , respectively.

On the other hand, τ^{LR} goes to zero which would make the first term V_1 in the potential vanish in the limit that $\epsilon \to 0$, at least if we don't scale the fluxes $N^{i'}$ appropriately. Since V_1 describes the interaction between the two sides of the Calabi-Yau, we really want to scale the fluxes $N^{i'}$ to go to infinity in such a way that the term V_1 remains finite.

Then it becomes clear that the term V_3 of the potential dominates over the other two contribution to V. This implies that in the limit $\epsilon \to 0$ the term V_3 should be minimized first, *i.e.*,

$$\sum_{k'} N^{k'} \tau^{RR}_{k'i'} = 0, \quad \forall i',$$

which is a set of n_R equations for the n_R moduli $x^{j'}$. The solutions of this system correspond to supersymmetric vacua for the 3-cycles on the right side. Once we have fixed all $X^{j'}$ to their supersymmetric values $\widehat{X}^{j'}$, we can consider the effect of the backreaction of the right side to the left. This is purely expressed through

the potential V_1 , since the term V_2 vanishes as well at the supersymmetric point. So effectively the potential for the complex structure moduli X_L^i of the left surface is

$$V_{1} = \sum_{i,j,k',l'} \left(N^{k'} \tau_{k'i}^{LR} \right) \left(\frac{1}{\text{Im}\tau} \right)_{LL}^{ij} \overline{\left(N^{l'} \tau_{l'j}^{LR} \right)}. \tag{8.51}$$

This may be written as $V_1 = \sum_{i,j} \partial_i W_{\rm eff} (1/{\rm Im}\tau)^{ij}_{LL} \overline{\partial_j W_{\rm eff}}$, where we define the effective "superpotential" for the left complex structure moduli as

$$\partial_i W_{\rm eff} \equiv \sum_{k'} N^{k'} \tau^{LR}_{k'i}.$$

Comparing with expression (8.49) it is clear that the fluxes on the right should be scaled in such a way that the coefficients

$$T^{m} = \epsilon^{m} \sum_{k'} N^{k'} K'_{k'm} \tag{8.52}$$

remain constant. In that situation the effective superpotential is

$$\partial_i W_{\text{eff}} = \sum_m T^m K_{im} \tag{8.53}$$

to leading order in ϵ , which is precisely of the form (8.14).

Genericity of Potential and Metastable Vacua

Let us summarize what we have demonstrated so far. We started with a large Calabi-Yau that consists of two parts X_L and X_R separated by a large distance, and turned on a large 3-form flux on one of the sides, say X_R . This flux generates a large potential for the complex structure moduli of X_R , which are therefore set to their supersymmetric minima. The flux on X_R is also weakly backreacting to the other side X_L , inducing a small superpotential for the complex structure moduli of X_L . We computed this superpotential in equations (8.3.1) and (8.53) and found that it is of the form (8.14). The main point is that the side X_L only knows about X_R via the parameters T^m given by (8.52).

In this section we discuss two questions. The first is to which degree we can tune the parameters T^m independently. And the second is whether these T^m 's can be chosen to realize an OOP supersymmetry breaking superpotential.

As we can see from (8.52), the values of the parameters T^m depend on the fluxes $N^{l'}$ on the cycles of X_R and also on the value of the (generalized) period matrix $K'_{l'm}$. The last one depends on the choice of the supersymmetric vacuum $\widehat{X}^{j'}$ on the right side. For given large fluxes $N^{l'}$ there is a huge number of supersymmetric vacua, or solutions of equation (8.3.1), with different values of

 $\widehat{X}^{j'}$ and consequently of $K'_{l'm}$. The density of such supersymmetric vacua over the complex structure moduli space of X_R has been studied before [311, 312, 313, 314, 315], and it is believed that the vacua become dense in the moduli space in the limit where the fluxes are very large.

The coefficients $K'_{l'm}$ are holomorphic functions over the complex structure moduli space of X_R . So naively one would conclude that when the dimension of this moduli space is large enough, meaning that the number of 3-cycles in X_R is large, we can always find supersymmetric points where the $K'_{l'm}$'s have the desired values. However the functions $K'_{l'm}$ are not "generic" and there may be relations between them which affect the naive counting. We have not analyzed this problem in detail but we think the following statement is true. Any number of the T^m 's in the superpotential (8.53) can be tuned by considering a Calabi-Yau whose right side X_R has a sufficiently large number of 3-cycles, and there will be some supersymmetric vacua with right values of $K'_{l'm}$ to reproduce the desired T^m 's to good accuracy.

This claim is made more intuitive by the following physical interpretation of equation (8.52). Start by turning on fluxes $N^{l'}$ on the cycles of X_R , which is based on the Riemann surface Σ_R . When reduced on the Riemann surface the flux looks like the electric field produced by a charge in two dimensions. The set of fluxes $N^{l'}$ resembles a charge distribution on the cycles of the Riemann surface. To compute the field produced by these charges in the distant region of the other set of cycles Σ_L , one has to consider a multipole expansion. Since the matrix $K'_{l'm}$ computes the mth multipole expansion of a charge distributed along the l'th cycle, the coefficients T^m are exactly the multipole moments of the charge distribution. In this formulation our first question reads whether we can arrange a charged distribution to have the desired multipole moments given by the coefficients T^m . We expect that the answer is positive.

The second question is more subtle. To realize a metastable nonsupersymmetric vacuum via the OOP mechanism, one has to tune the superpotential in a way which is determined by properties of the Kähler metric at that point. As we saw in Section 8.1 one has to tune the coefficients of the effective superpotential only up to cubic order in an expansion around the candidate metastable point. Since we have a very large number of parameters T^m at our disposal it seems that generically we should be able to tune them to generate metastable vacua at most points on the moduli space. However we do not have a proof of this statement and it is possible that various relations between the period matrices and the Kähler metric invalidate the naive counting⁷.

 $^{^7}$ This question is similar to whether one can realize the OOP mechanism with a single trace superpotential for the adjoint scalar in an SU(N) gauge theory. In [287] it was demonstrated that for SU(2) a metastable vacuum can be generated anywhere on the moduli space by a single trace superpotential, and for SU(N) at the center of the moduli space. It was not fully analyzed whether this is possible in generality.

8.3.2 Example of factorization

In the previous section, we argued, based on the factorization of the Riemann surface and the Calabi-Yau, that it is possible to embed the nonsupersymmetric metastable vacua we found in Section 8.2.2 in a "larger" Calabi-Yau, the idea being that the flux threading compact cycles on one side of the Calabi-Yau looks like flux coming from infinity from the viewpoint of the other side of the Calabi-Yau. In this section, we will discuss the Dijkgraaf-Vafa geometries

$$\Sigma_{\text{DV}}: \quad y^2 = P_n(x)^2 - f_{n-1}(x), \qquad P_n(x) = \prod_{I=1}^n (x - \alpha_I),$$
 (8.54)

as an example where our proposal can in principle be implemented, and make some steps towards actually confirming our proposal.

Factorization in practice

Remember that the α_I 's are non-normalizable parameters which represent the positions of the cuts on the x-plane, while the coefficients in $f_{n-1}(x)$, or equivalently variables S^I defined in (8.43), are normalizable (or at least log-normalizable) and hence are dynamical variables describing the size of those cuts. Therefore, in this Dijkgraaf-Vafa case (8.54), α_I are the parameters we want to adjust in order to approach the factorization limit where $\Sigma_{\rm DV}$ degenerates into two subsectors.

So, what we should do is clear: we divide the n cuts into two parts as $n=n_L+n_R$, the ones on the left indexed by i and on the right by i', and send these two groups apart from each other by a large factor $L=1/\epsilon$ so that

$$\alpha_i - \alpha_{i'} = \mathcal{O}(L)$$
 (when $L \to \infty$).

In the $L\to\infty$ limit, the left and right sides will be very far apart and the factorization we discussed in the previous section must be achieved. For example, the period matrix of the total Riemann surface must diagonalize as in (8.48) up to 1/L correction.

There is one thing we should be careful about when taking the $L \to \infty$ limit. If we try to separate the two sets of cuts by naively taking the typical difference between α_i and $\alpha_{i'}$ to be of order L while keeping the size of the cuts fixed, then a simple estimate of the scaling of $S_i^L, S_{i'}^R$ using (8.43) shows that the physical size of the 3-cycles in the Calabi-Yau blows up. What we want instead is to end up with two sets of 3-cycles of finite size, separated by a large distance, so that we are left with nontrivial dynamics of $S_i^L, S_{i'}^R$. To achieve this we must also scale the size of the cuts, as we send $L \to \infty$. Let x_L and x_R be local coordinates

in the left and right sectors, respectively, and set

$$\widetilde{x}_L = L^r x_L, \qquad \widetilde{x}_R = L^{r'} x_R, \tag{8.55}$$

where

$$r = \frac{n_R}{n_L + 1}, \qquad r' = \frac{n_L}{n_R + 1}.$$

Then, from (8.43), it is not difficult to see that we can keep $S_i^L, S_{i'}^R$ finite if we keep \widetilde{x}_L , \widetilde{x}_R finite while taking the $L \to \infty$ limit. A similar rescaling of local coordinates must be also necessary when taking a factorization limit in any other examples than (8.54).

Computation of Period Matrix

In the Dijkgraaf-Vafa geometry (8.54), the period matrix is given by

$$\tau_{IJ} = \frac{\partial^2 \mathcal{F}_0}{\partial S^I \partial S^J},\tag{8.56}$$

Here, \mathcal{F}_0 is the B-model prepotential, which by the Dijkgraaf-Vafa relation [123, 316] is related to matrix models. The precise way to scale various quantities to take the factorization limit being understood, it is in principle possible to confirm our proposal for the Dijkgraaf-Vafa geometry using (8.56). For doing that, it is important to be able to compute the prepotential \mathcal{F}_0 for a large number of cuts n. The results from Section 8.2.2 show that generating a metastable vacuum requires quite a lot of coefficients T^m . Since we roughly need the same number of cuts on the right as the number of tuned Σ_m 's on the left, the total Riemann surface must have quite a large number of cuts. So, in this subsection we will explain the way to compute \mathcal{F}_0 and thus τ_{IJ} for an arbitrary n.

For Dijkgraaf-Vafa geometries (8.54) the prepotential \mathcal{F}_0 may in fact be computed for any number of cuts n in a number of ways. The most direct way is evaluating the period integrals on the hyperelliptic curve. This has been done up to cubic order in S^I in [317]. Duality with a U(N) matrix model [123, 316]

$$Z = \exp\left[\sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g(S)\right] = \int d^{N^2} \Phi \, \exp\left[\frac{1}{g_s} \text{Tr} W(\Phi)\right],$$

where the matrix model action is given by

$$W'(x) = P_n(x) = \prod_{I=1}^{n} (x - \alpha_I)$$

makes this computation quite a bit simpler. Let us quickly show this argument [316].

The field Φ is an $N \times N$ matrix. Say N^I eigenvalues of Φ are placed at the critical point $x = \alpha_I$ and divide the matrix Φ into $N^I \times N^J$ blocks Φ_{IJ} , where $\sum_{I=1}^n N^I = N$. One can go to the gauge $\Phi_{IJ} = 0$ for $I \neq J$ by introducing fermionic ghosts in the matrix model action. This produces the following extra term in the action, where $\Phi_I \equiv \Phi_{II}$:

$$W_{\rm ghost} = \sum_{I \neq J} {\rm Tr} (B_{JI} \Phi_I C_{IJ} + C_{JI} \Phi_I B_{IJ}). \label{eq:wghost}$$

To write down Feynman diagrams, we expand Φ_I around $x=\alpha_I$ as $\Phi_I=\alpha_I+\phi_I$. A Taylor series of $W(\Phi_I)=W(\alpha_I+\phi_I)$ around α_I yields the propagator and p-vertices for ϕ_I . In particular, this shows that the propagator for ϕ_I is given by

$$\langle \phi_I \phi_I \rangle = \frac{1}{W''(\alpha_I)} = \frac{1}{\Delta_I},$$

where $\Delta_I = W''(\alpha_I) = \prod_{J \neq I}^n \alpha_{IJ}$. Moreover, expanding the ghost action determines the ghost propagator to be

$$\langle B_{JI}C_{IJ}\rangle = \frac{1}{\alpha_{IJ}},$$

and gives the Yukawa interactions between ϕ_I , B_{JI} and C_{IJ} .

The contribution to the prepotential \mathcal{F}_0 of order three in the S^I 's is given by planar diagrams with three holes, see Fig. 8.5. Writing down the expressions $g_{I,3}$ and $g_{I,4}$ in terms of α 's and Δ 's shows that

$$\mathcal{F}_{0,3} = \sum_{I=1}^{n} u_{I} S_{I}^{3} + \sum_{I \neq J}^{n} u_{I;J} S_{I}^{2} S_{J} + \sum_{I < J < K}^{n} u_{IJK} S_{I} S_{J} S_{K},$$

where

$$\begin{split} u_I &= \frac{2}{3} \bigg(-\sum_{J \neq I} \frac{1}{\alpha_{IJ}^2 \Delta_J} + \frac{1}{4\Delta_I} \sum_{\substack{J < K \\ J,K \neq i}} \frac{1}{\alpha_{IJ} \alpha_{IK}} \bigg), \\ u_{I;J} &= -\frac{3}{\alpha_{IJ}^2 \Delta_I} + \frac{2}{\alpha_{IJ}^2 \Delta_J} - \frac{2}{\alpha_{IJ} \Delta_I} \sum_{K \neq I,J} \frac{1}{\alpha_{IK}} \quad \text{and} \\ u_{IJK} &= 4 \left(\frac{1}{\alpha_{IJ} \alpha_{IK} \Delta_I} + \frac{1}{\alpha_{JI} \alpha_{JK} \Delta_J} + \frac{1}{\alpha_{KI} \alpha_{KJ} \Delta_K} \right). \end{split}$$

In appendix D of [4] we discuss the generalization of this result to higher order in S^I . In particular, we compute \mathcal{F}_0 up to S^5 terms.

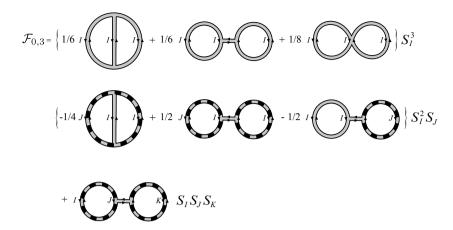


Figure 8.5: The contribution to $\mathcal{F}_{0,3}$ given in terms of matrix diagrams. Gray double lines represent ϕ_I fields, while black-and-gray double lines represent BC ghosts.

Scaling of Period Matrix

The method explained in Section 8.3.2 allows one in principle to compute the period matrix to any order in S^I for general Dijkgraaf-Vafa curves (8.54). Then the factorization limit can be achieved simply by taking the $L \to \infty$ limit of the result and one can start looking for metastable vacua. In this subsection, as a step towards it, let us pursue a more modest goal of seeing the factorized behavior of the period matrix, (8.48).

The form of the scaling can be elegantly derived for any possible contributing matrix model diagram to \mathcal{F}_0 . First note that Δ_i scales as L^{2r} as $L \to \infty$, and $\Delta_{i'}$ as $L^{2r'}$. All propagators with indices from either side of the surface have an expansion in terms of α_{IJ} 's and Δ_{I} 's, and thus a scaling in L which is easy to determine. The total scaling of a planar diagram with an arbitrary number of these elements turns out to depend just on the number of ghost vertices that connect the left side to the right side. It is given by

$$\frac{1}{L^{(1+r)N_{ii'}+(1+r')N_{i'i}}},$$
(8.57)

where $N_{ii'}$ is the number of ghost vertices with external ghost lines indexed by (i,i') and the external ϕ -line by (i,i). Note that in deriving this we assumed the scaling (8.55) and thus $S_L^i, S_R^{i'}$ are of order one.

This shows that a diagram with only indices on the left (or on the right) will be of order 1 in L. Since such diagrams contribute to the period matrix τ_{ij} (or $\tau_{i'j'}$), this shows that the period matrix is of order 1 in L, with corrections in 1/L from diagrams that contain at least two loops indexed by i and j. On the other hand, the off-diagonal pieces of the period matrix $\tau_{ii'}$ and $\tau_{i'i}$ contain at least

one ghost cross-vertex with indices i and i'. These parts will therefore scale at least as 1/L. In particular, for large L the properties of the full Riemann surface Σ are determined by those of the two factors Σ_L, Σ_R , and the period matrix τ_{IJ} indeed diagonalizes as in (8.48).

Having checked the diagonalization (8.48), the problem of actually finding an example of a metastable vacuum then just amounts to solving equation (8.52) together with (8.3.1) using the data from matrix model, for T^m giving a metastable vacuum. Solving these equations is nontrivial, since the relation between the flux parameters $N^{i'}$ on the right and the coefficients in the superpotential T^m we want on the left are non-linear, although we expect that the solutions do exist by the multipole argument we gave in Section 8.3. We leave matrix model computations up to requisite orders as well as finding the actual metastable vacua by solving those equations for the future work.

8.4 Concluding remarks

Summarizing, we found that turning on flux with support at infinity in local Calabi-Yau in type IIB induces a superpotential for the moduli in the local Calabi-Yau, thus breaking $\mathcal{N}=2$ of the Calabi-Yau compactification down to $\mathcal{N}=2$. Then we demonstrated that one can create metastable vacua by tuning the flux at infinity using the OOP mechanism, using a Dijkgraaf-Vafa (CIV-DV) geometry as a primary example. The metastable vacua known to exist [287, 290] in perturbed Seiberg-Witten theory can also be understood in terms of metastable flux configuration.

Flux diverging at infinity may appear problematic, but in reality a local Calabi-Yau must be regarded as a local approximation of a larger compact Calabi-Yau and the flux at infinity has a natural interpretation there; there is flux floating around in the rest of the Calabi-Yau, which "leaks" into our local Calabi-Yau and just appear to be coming in from infinity. This, furthermore, motivates a more natural setting to realize metastable flux vacua: in a part, say on the right side, of the full Calabi-Yau X, there are some 3-cycles threaded by flux (and possibly O-planes to cancel net charge if X is compact) and on the left side there are some 3-cycles without flux through them. If the distance between the left and right sectors is large, the full Calabi-Yau X factorizes into an almost decoupled system of X_L and X_R , and the flux in X_R appears to be flux at infinity from the viewpoint of X_L and induces superpotential in X_L . By adjusting the number of fluxes in X_R , we can tune the superpotential and generate metastable vacua in X_L . This is a very well controlled setting to analyze flux vacua, which may shed light on the structure of the nonsupersymmetric landscape of string vacua. We also made some steps toward actually embedding metastable vacua in a larger Calabi-Yau as sketched above in the case of Dijkgraaf-Vafa geometry by computing certain matrix model amplitudes. Actually finding explicit vacua along that line is an interesting open problem.

Note that we needed just two main ingredients to achieve this result: Firstly, the OOP mechanism requires that the complex structure moduli space is special Kähler. Secondly, it is important that the generated Gukov-Vafa-Witten superpotential is very much controllable by tuning the flux. This means that we can generalize the above story to any setting which fulfills these two requirements. Other possibilities therefore include M-theory and F-theory on Calabi-Yau fourfolds [318, 298]. Let us finish by saying a few words on these two setups.

Compactifying M-theory on a Calabi-Yau fourfold X_4 with fluxes yields a three-dimensional low energy theory with 4 supercharges. The complex structure moduli of the Calabi-Yau are part of the chiral supermultiplets and are described by variations of the holomorphic (4,0)-form Ω . In the local limit where the fourfold becomes noncompact, the Kähler potential on the moduli space is given by

$$K = \int_{X_4} \Omega \wedge \overline{\Omega}, \tag{8.58}$$

so that the metric on the moduli space is indeed special Kähler. Moreover, it is well-known that the complex moduli may be stabilized by turning on 4-form flux F_4 , which introduces the superpotential

$$W = \int_{X_4} F_4 \wedge \Omega.$$

The condition for unbroken supersymmetry is W=dW=0, so that F_4 has to be a (2,2)-form. Stabilizing the Kähler moduli as well requires that the flux is primitive under the Lefschetz decomposition (and in particular self-dual). Turning on primitive (2,2) flux on some compact 4-cycles, we can now follow an equivalent procedure as in IIB.

M-theory compactified on X_4 is equivalent to compactifying F-theory on $X_4 \times S^1$, at least if X_4 is an elliptically fibered Calabi-Yau. This leads to a four-dimensional space-time with 4 supercharges. So again, the Kähler potential is given by (8.58), and the flux F_4 is a primitive (2,2)-form. The relation with IIB consistently reduces F_4 to a harmonic (2,1)-flux G_3 . The extra seven-branes that must be inserted in IIB when reducing over a singular T^2 do not contribute to the superpotential and thus don't play an important role here.

In particular, consider as an example the local Calabi-Yau fourfold

$$u^2 + v^2 + w^2 + H(x, y) = 0,$$

where all variables are $\mathbb C$ (or $\mathbb C^*$) valued, and F(x,y) defines a smooth curve in

the x, y-plane. Its holomorphic four-form is given by

$$\Omega = \frac{du \wedge dv}{w} \wedge dx \wedge dy.$$

The u,v,w-fiber defines a two-sphere over each point in the x,y-plane, which shrinks to zero-size over the curve H(x,y)=0. (Like in the Calabi-Yau threefold case, the real part of H(x,y) changes sign when crossing the Riemann surface. This flop changes the parametrization of the compact S^2 in the T^*S^2 -fiber from a "real" S^2 into an "imaginary" S^2 .) Four-cycles can be constructed as an S^2 fibration over some disk D ending on the curve and have the topology of a four-sphere (when x and $y \in \mathbb{C}$). Notice that the intersection lattice is symmetric now and not simply symplectic anymore, so that the bilinear identity takes a more complicated form. However, like in the threefold case all relevant quantities reduce to the Riemann surface, and the analysis is similar as before.

Appendix A

Level-rank duality

The affine algebras $\widehat{su}(N)_k$ and $\widehat{su}(k)_N$ are related by the so-called level-rank duality [66, 87, 319, 83, 88], which maps to each other orbits of their irreducible integrable representations under outer automorphism groups. Let us explain this in more detail. The Dynkin diagram of $\widehat{su}(N)_k$ consists of N nodes permuted in a cyclic order by the outer automorphism group \mathbb{Z}_N . This also induces an action on affine irreducible integrable representations. There are

$$\frac{(N+k-1)!}{(N-1)!\,k!} \tag{A.1}$$

such representations of $\widehat{su}(N)_k$, which can be identified in a standard way with Young diagrams ρ with at most N-1 rows and at most k columns. We denote the set of such diagrams by $\mathcal{Y}_{N-1,k}$. In particular, the generator of the outer automorphism group σ_N , the so-called basic outer automorphism, has a simple realization in terms of a Young diagram $\rho=(\rho_1,\ldots,\rho_{N-1})$ corresponding to a given integrable representation. The action of σ_N amounts to adding a row of length k as a first row of ρ , and then reducing the diagram, i.e. removing ρ_{N-1} columns which acquired a length N (so that indeed $\sigma_N(\rho) \in \mathcal{Y}_{N-1,k}$),

$$\sigma_N(\rho_1, \dots, \rho_{N-1}) = (k - \rho_{N-1}, \rho_1 - \rho_{N-1}, \dots, \rho_{N-2} - \rho_{N-1}).$$
 (A.2)

It follows that $\sigma_N^N(\rho)=\rho$, as expected for \mathbb{Z}_N symmetry. All N irreducible integrable representations related by an action of σ_N constitute an orbit denoted as $[\rho]\subset\mathcal{Y}_{N-1,k}$. As an example, the \mathbb{Z}_4 orbit generated from $\widehat{su}(4)_3$ irreducible integrable representation corresponding to a diagram $\rho=(2,1)\in\mathcal{Y}_{3,3}$ is given by

The number of such \mathbb{Z}_N orbits is given by (A.1) divided by N. For both $\widehat{su}(N)_k$ and $\widehat{su}(k)_N$ this number is the same, therefore a bijection between orbits of their integrable irreducible representations exists. The level-rank duality is a statement that for $\widehat{su}(N)_k$ orbit represented by a diagram $\rho \in \mathcal{Y}_{N-1,k}$ there is a canonical bijection realized as

$$\mathcal{Y}_{N-1,k} \supset [\rho] = \{ \sigma_N^j(\rho) \mid j = 0, \dots, N-1 \} \mapsto \{ \sigma_k^a(\rho^t) \mid a = 0, \dots, k-1 \} = [\rho^t] \subset \mathcal{Y}_{k-1,N}, \quad (A.3)$$

where t denotes a transposition and a diagram ρ^t should be reduced (i.e. all columns of length k should be removed if ρ_1 was equal to k).

The level-rank duality can also be formulated in terms of the embedding

$$\widehat{u}(1)_{Nk} \times \widehat{su}(N)_k \times \widehat{su}(k)_N \subset \widehat{u}(Nk)_1.$$

The $\widehat{u}(Nk)_1$ affine Lie algebra can be realized in terms of Nk free fermions, so that their total Fock space $\mathcal{F}^{\otimes Nk}$ decomposes under this embedding as

$$\mathcal{F}^{\otimes Nk} = \bigoplus_{\rho} U_{\|\rho\|} \otimes V_{\rho} \otimes \widetilde{V}_{\widetilde{\rho}}, \tag{A.4}$$

where $U_{\parallel\rho\parallel}$, V_{ρ} and $\widetilde{V}_{\widetilde{\rho}}$ denote irreducible integrable representations of $\widehat{u}(1)_{Nk}$, $\widehat{su}(k)_N$, and $\widehat{su}(N)_k$ respectively. In the above decomposition only those pairs $(\rho,\widetilde{\rho})$ arise, which represent orbits mapped to each other by the duality (A.3). For a given $\widehat{su}(N)_k$ orbit $[\rho]$ represented by ρ , these pairs are therefore of the form $(\sigma_N^j(\rho),\sigma_k^a(\rho^t))$, where σ_N and σ_k are appropriate outer automorphism groups. The U(1) charge corresponding to such a pair is $\|\rho\|=(|\rho|+jk+aN)$ mod Nk, where $|\rho|$ is the number of boxes in the Young diagram ρ . With such identifications, the decomposition (A.4) can be written in terms of characters as [88]

$$\chi^{\widehat{u}(Nk)_{1}}(u, v, \tau) = \sum_{[\varrho] \subset \mathcal{V}_{N-1, k}} \sum_{j=0}^{N-1} \sum_{a=0}^{k-1} \chi^{\widehat{u}(1)_{Nk}}_{|\rho|+jk+aN}(N|u|+k|v|, \tau) \chi^{\widehat{su}(N)_{k}}_{\sigma_{N}^{j}(\rho)}(\overline{u}, \tau) \chi^{\widehat{su}(k)_{N}}_{\sigma_{k}^{a}(\rho^{t})}(\overline{v}, \tau).$$
(A.5)

Here $u=(u_j)_{j=1...N}$ are elements of the Cartan subalgebra of u(N), $|u|=\sum_j u_j$ and \overline{u} denotes the traceless part. Similarly $v=(v_a)_{a=1...k}$ are elements of Cartan subalgebra of u(k). $\chi_{\rho}^{\widehat{su}(N)_k}(\overline{u},\tau)$ are characters of $\widehat{su}(N)_k$ at level k for an integrable irreducible representation specified by a Young diagram ρ , and $\chi_j^{\widehat{u}(1)_N}$ characters are defined as

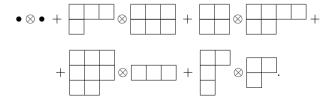
$$\chi_j^{\widehat{u}(1)_N}(z,\tau) = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} q^{\frac{N}{2}(n+j/N)^2} e^{2\pi i z(n+j/N)}$$

for $q = e^{2\pi i \tau}$.

As an example of a decomposition (A.4) let us consider the case of $\widehat{u}(1)_{12} \times \widehat{su}(4)_3 \times \widehat{su}(3)_4 \subset \widehat{u}(12)_1$, with N=4 and k=3. From (A.1) we deduce there are 5 orbits of outer automorphism groups \mathbb{Z}_4 and \mathbb{Z}_3 . Let us consider $\widehat{su}(4)_3$ integrable representation related to a diagram $\rho=\square$, and the corresponding $\widehat{su}(3)_4$ diagram $\rho^t=\square$. The two orbits under σ_4 and σ_3 are shown respectively in the first row and column of a table below. All 12 pairs of representations appear in the decomposition (A.4) with $\widehat{u}(1)_{12}$ charges given in the table. Note that acting with σ_4 takes us to another pair of weights given by a step to the right in the table, and increases $\widehat{u}(1)_{12}$ charge by 3 (modulo 12). The action of σ_3 takes us a step to the bottom in the table and increases $\widehat{u}(1)_{12}$ charge by 4 (modulo 12). Of course the same table is generated when we build it starting from any other element of these two orbits.

	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$		→ <u> </u>	
	1	4	7	10
	5	8	11	2
<u> </u>	9	0	3	6

Pairs of $\widehat{su}(4)_3 \times \widehat{su}(3)_4$ integrable weights with the same fixed $\widehat{u}(1)_{12}$ charge, arising in the decomposition of $\widehat{u}(12)_1$, are easily found if all 5 such tables of orbits are drawn. For example for charge 0 we then get



Bibliography

- [1] R. Dijkgraaf, L. Hollands, P. Sulkowski, and C. Vafa, Supersymmetric Gauge Theories, Intersecting Branes and Free Fermions, JHEP **02** (2008) 106, [0709.4446].
- [2] R. Dijkgraaf, L. Hollands, and P. Sulkowski, Quantum Curves and D-Modules, 0810.4157.
- [3] M. C. N. Cheng and L. Hollands, A Geometric Derivation of the Dyon Wall-Crossing Group, JHEP **04** (2009) 067, [0901.1758].
- [4] L. Hollands, J. Marsano, K. Papadodimas, and M. Shigemori, *Nonsupersymmetric Flux Vacua and Perturbed N=2 Systems, JHEP* **10** (2008) 102, [0804.4006].
- [5] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, *Nucl. Phys.* **B479** (1996) 243–259, [hep-th/9606040].
- [6] R. P. Thomas, The geometry of mirror symmetry, math/0512412.
- [7] M. Gross, The Strominger-Yau-Zaslow conjecture: From torus fibrations to degenerations, 0802.3407.
- [8] Joyce, Dominic, Calabi-Yau manifolds and Related Geometries, ch. Lectures on special Lagrangian geometry. Springer, 2003. math/0108088.
- [9] M. Gross, Topological mirror symmetry, math/9909015.
- [10] Kontsevich, Maxim and Soibelman, Yan, Affine structures and non-archimedean analytic spaces, math/0406564.
- [11] Gross, Mark and Wilson, P. M. H., Large Complex Structure Limits of K3 Surfaces, math/0008018.
- [12] Wei-Dong Ruan, Lagrangian torus fibration of quintic Calabi-Yau hypersurfaces I: Fermat quintic case, in Winter school in mirror symmetry, vector bundles and Lagrangian submanifolds (S. Yau and C. Vafa, eds.), AMS and International Press, 1999. math.DG/9904012.
- [13] B. R. Greene and M. R. Plesser, Duality in Calabi-Yau moduli space, Nucl. Phys. B338 (1990) 15–37.
- [14] P. Candelas, X. C. De La Ossa, P. S. Green, and L. Parkes, A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory, Nucl. Phys. B359 (1991) 21–74.
- [15] M.-x. Huang, A. Klemm, and S. Quackenbush, Topological String Theory on Compact Calabi-Yau: Modularity and Boundary Conditions, Lect. Notes Phys. 757 (2009) 45–102, [hep-th/0612125].

- [16] R. Harvey and J. Lawson, H. B., Calibrated geometries, Acta Math. 148 (1982) 47.
- [17] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, *The topological vertex, Commun. Math. Phys.* **254** (2005) 425–478, [hep-th/0305132].
- [18] R. Gopakumar and C. Vafa, On the gauge theory/geometry correspondence, Adv. Theor. Math. Phys. 3 (1999) 1415–1443, [hep-th/9811131].
- [19] D. Maulik and N. Nekrasov and A. Okounkov and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory I*, math.AG/0312059.
- [20] D. Maulik and N. Nekrasov and A. Okounkov and R. Pandharipande, Gromov-Witten theory and Donaldson-Thomas theory II, math.AG/0406092.
- [21] B. Szendroi, Non-commutative Donaldson-Thomas theory and the conifold, Geom. Topol. 12 (2008) 1171–1202, [0705.3419].
- [22] R. Gopakumar and C. Vafa, *M-theory and topological strings*. *I*, hep-th/9809187.
- [23] R. Gopakumar and C. Vafa, M-theory and topological strings. II, hep-th/9812127.
- [24] J. M. F. Labastida, M. Marino, and C. Vafa, *Knots, links and branes at large N, JHEP* **11** (2000) 007, [hep-th/0010102].
- [25] S. Gukov, A. S. Schwarz, and C. Vafa, Khovanov-Rozansky homology and topological strings, Lett. Math. Phys. 74 (2005) 53–74, [hep-th/0412243].
- [26] S. Gukov, A. Iqbal, C. Kozcaz, and C. Vafa, *Link homologies and the refined topological vertex*, 0705.1368.
- [27] R. Dijkgraaf and H. Fuji, The Volume Conjecture and Topological Strings, 0903.2084.
- [28] A. Okounkov, N. Reshetikhin, and C. Vafa, *Quantum Calabi-Yau and classical crystals*, hep-th/0309208.
- [29] Dijkgraaf, Robbert and Orlando, Domenico and Reffert, Susanne, *Quantum Crystals and Spin Chains*, *Nuclear Physics B* **811** (2008) 463, [0803.1927].
- [30] H. Ooguri and M. Yamazaki, Crystal Melting and Toric Calabi-Yau Manifolds, 0811.2801.
- [31] Percy A. Macmahon, Combinatory Analysis. Cambridge University Press, 1915.
- [32] A. Iqbal, N. Nekrasov, A. Okounkov, and C. Vafa, *Quantum foam and topological strings*, *JHEP* **04** (2008) 011, [hep-th/0312022].
- [33] H. Ooguri and M. Yamazaki, Emergent Calabi-Yau Geometry, 0902.3996.
- [34] K. Hori and C. Vafa, Mirror symmetry, hep-th/0002222.
- [35] Calabi E, Métriques Kählérienes et fibrés holomorphes, Ann. Sci. de ´. N. S. (1979), no. 12 266292.
- [36] G. Tian and S. T. Yau, Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry, in Mathematical Aspects of String Theory (S. Yau, ed.), World Scientific Publishing Co., Singapore, 1987.
- [37] Tian, Gang and Yau, Shing-Tung, Complete Khler manifolds with zero Ricci curvature I, Amer. Math. Soc. 3 (3) (1990).
- [38] Tian, Gang and Yau, Shing-Tung, Complete Khler manifolds with zero Ricci curvature II, Invent. Math. 106 (1) (1991).
- [39] H. Nakajima, Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras, Duke Math. **76** (1994) 365–416.

- [40] C. Vafa and E. Witten, *A strong coupling test of S duality*, *Nucl. Phys.* **B431** (1994) 3–77, [hep-th/9408074].
- [41] C. Montonen and D. I. Olive, Magnetic Monopoles as Gauge Particles?, Phys. Lett. B72 (1977) 117.
- [42] E. Witten, On S duality in Abelian gauge theory, Selecta Math. 1 (1995) 383, [hep-th/9505186].
- [43] E. P. Verlinde, Global aspects of electric magnetic duality, Nucl. Phys. **B455** (1995) 211–228, [hep-th/9506011].
- [44] E. Witten and D. I. Olive, Supersymmetry Algebras That Include Topological Charges, Phys. Lett. **B78** (1978) 97.
- [45] E. Witten, Supersymmetric Yang-Mills theory on a four manifold, J. Math. Phys. **35** (1994) 5101–5135, [hep-th/9403195].
- [46] E. Witten, Two-dimensional gauge theories revisited, J. Geom. Phys. 9 (1992) 303–368, [hep-th/9204083].
- [47] M. Blau and G. Thompson, Localization and diagonalization: A review of functional integral techniques for low dimensional gauge theories and topological field theories, J. Math. Phys. **36** (1995) 2192–2236, [hep-th/9501075].
- [48] S. Cordes, G. W. Moore, and S. Ramgoolam, Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories, Nucl. Phys. Proc. Suppl. 41 (1995) 184–244, [hep-th/9411210].
- [49] R. Harvey and H.B. Lawson, Calibrated geometry, Acta Math. 148 (1982) 47–157.
- [50] M. Bershadsky, C. Vafa, and V. Sadov, *D-Branes and Topological Field Theories*, *Nucl. Phys.* B463 (1996) 420–434, [hep-th/9511222].
- [51] M. B. Green, J. A. Harvey, and G. W. Moore, I-brane inflow and anomalous couplings on D-branes, Class. Quant. Grav. 14 (1997) 47–52, [hep-th/9605033].
- [52] Y.-K. E. Cheung and Z. Yin, Anomalies, branes, and currents, Nucl. Phys. B517 (1998) 69–91, [hep-th/9710206].
- [53] A. Strominger, Open p-branes, Phys. Lett. **B383** (1996) 44–47, [hep-th/9512059].
- [54] M. Rocek and E. P. Verlinde, Duality, quotients, and currents, Nucl. Phys. B373 (1992) 630–646, [hep-th/9110053].
- [55] T. H. Buscher, Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models, Phys. Lett. B201 (1988) 466.
- [56] N. A. Obers and B. Pioline, *U-duality and M-theory*, *Phys. Rept.* **318** (1999) 113–225, [hep-th/9809039].
- [57] N. Seiberg, Supersymmetry and Nonperturbative beta Functions, Phys. Lett. B206 (1988) 75.
- [58] R. Dijkgraaf and G. W. Moore, Balanced topological field theories, Commun. Math. Phys. 185 (1997) 411–440, [hep-th/9608169].
- [59] di Franscesco, Philippe and Mathieu, Pierre and Sénéchal, David, Conformal Field Theory. Springer, 1997.
- [60] McKay, John, Graphs, singularities, and finite groups, in The Santa Cruz Conference on Finite Groups, vol. 37 of Proc. Symp. Pure Math., pp. 183–186, Amer. Math. Soc., 1980.

- [61] P. Slodowy, Platonic solids, Kleinian singularities, and Lie groups in Algebraic geometry, in Algebraic Geometry (J. Dolgachev, ed.), vol. 1008 of Lecture Notes in Mathematics, pp. 102–138. Springer-Verlag, 1983.
- [62] H. Nakajima, Instantons and affine Lie algebras, Nucl. Phys. Proc. Suppl. 46 (1996) 154–161, [alg-geom/9510003].
- [63] M. Bianchi, F. Fucito, G. Rossi, and M. Martellini, Explicit Construction of Yang-Mills Instantons on ALE Spaces, Nucl. Phys. B473 (1996) 367–404, [hep-th/9601162].
- [64] F. Fucito, J. F. Morales, and R. Poghossian, *Multi instanton calculus on ALE spaces*, *Nucl. Phys.* **B703** (2004) 518–536, [hep-th/0406243].
- [65] F. Fucito, J. F. Morales, and R. Poghossian, *Instanton on toric singularities and black hole countings*, *JHEP* **12** (2006) 073, [hep-th/0610154].
- [66] I. Frenkel, *Lie Algebras and Related Topics*, vol. 933, ch. Representations of affine Lie algebras, Hecke modular forms and Korteweg-De Vries type equations, p. 71. Springer, 1982.
- [67] A. Licata, Framed rank r torsion-free sheaves on CP^2 and representations of the affine Lie algebra $\widehat{gl(r)}$, math.RT/0607690.
- [68] D. Gaiotto, A. Strominger, and X. Yin, New Connections Between 4D and 5D Black Holes, JHEP 02 (2006) 024, [hep-th/0503217].
- [69] D. Gaiotto, A. Strominger, and X. Yin, 5D black rings and 4D black holes, JHEP 02 (2006) 023, [hep-th/0504126].
- [70] D. Shih, A. Strominger, and X. Yin, *Recounting dyons in N* = 4 *string theory*, *JHEP* **10** (2006) 087, [hep-th/0505094].
- [71] R. Dijkgraaf, C. Vafa, and E. Verlinde, M-theory and a topological string duality, hep-th/0602087.
- [72] I. Bena, D.-E. Diaconescu, and B. Florea, *Black string entropy and Fourier-Mukai transform*, *JHEP* **04** (2007) 045, [hep-th/0610068].
- [73] H. Ooguri and C. Vafa, Two-Dimensional Black Hole and Singularities of CY Manifolds, Nucl. Phys. B463 (1996) 55–72, [hep-th/9511164].
- [74] P. J. Ruback, The motion of Kaluza-Klein monopoles, Commun. Math. Phys. 107 (1986) 93–102.
- [75] A. Sen, Dynamics of multiple Kaluza-Klein monopoles in M and string theory, Adv. Theor. Math. Phys. 1 (1998) 115–126, [hep-th/9707042].
- [76] S. A. Cherkis, Moduli Spaces of Instantons on the Taub-NUT Space, 0805.1245.
- [77] E. Witten, Branes, Instantons, And Taub-NUT Spaces, 0902.0948.
- [78] S. A. Cherkis, Instantons on the Taub-NUT Space, 0902.4724.
- [79] E. Witten, Geometric Langlands From Six Dimensions, 0905.2720.
- [80] C. P. Bachas, M. B. Green, and A. Schwimmer, (8,0) quantum mechanics and symmetry enhancement in type I' superstrings, JHEP **01** (1998) 006, [hep-th/9712086].
- [81] L.-Y. Hung, Comments on I1-branes, JHEP 05 (2007) 076, [hep-th/0612207].
- [82] N. Itzhaki, D. Kutasov, and N. Seiberg, *I-brane dynamics*, *JHEP* 01 (2006) 119, [hep-th/0508025].

- [83] S. G. Naculich, H. A. Riggs, and H. J. Schnitzer, *Group level duality in wzw models and Chern-Simons theory*, *Phys. Lett.* **B246** (1990) 417–422.
- [84] A. Sen, A note on enhanced gauge symmetries in M- and string theory, JHEP **09** (1997) 001, [hep-th/9707123].
- [85] E. G. Gimon and J. Polchinski, *Consistency Conditions for Orientifolds and D-Manifolds, Phys. Rev.* **D54** (1996) 1667–1676, [hep-th/9601038].
- [86] N. J. Evans, C. V. Johnson, and A. D. Shapere, *Orientifolds, branes, and duality of 4D gauge theories*, *Nucl. Phys.* **B505** (1997) 251–271, [hep-th/9703210].
- [87] M. Jimbo, T. Miwa, On a duality of branching rules for affine Lie algebras, Adv. Stud. Pure Math. 6 (1985) 17–65.
- [88] K. Hasegawa, Spin module versions of weyl's reciprocity theorem for classical kac-moody lie algebras an application to branching rule duality, RIMS, Kyoto Univ. 25 (1989).
- [89] P. Goddard, W. Nahm, and D. I. Olive, Symmetric Spaces, Sugawara's Energy Momentum Tensor in Two-Dimensions and Free Fermions, Phys. Lett. B160 (1985) 111.
- [90] D. R. Morrison and C. Vafa, Compactifications of F-Theory on Calabi-Yau Threefolds I, Nucl. Phys. **B473** (1996) 74–92, [hep-th/9602114].
- [91] D. R. Morrison and C. Vafa, Compactifications of F-Theory on Calabi-Yau Threefolds II, Nucl. Phys. **B476** (1996) 437–469, [hep-th/9603161].
- [92] J. A. Harvey and A. B. Royston, Localized Modes at a D-brane–O-plane Intersection and Heterotic Alice Strings, JHEP 04 (2008) 018, [0709.1482].
- [93] N. Seiberg and E. Witten, *Monopole Condensation, And Confinement In N=2 Supersymmetric Yang-Mills Theory*, *Nucl. Phys.* **B426** (1994) 19–52, [hep-th/9407087].
- [94] F. Ferrari and A. Bilal, The Strong-Coupling Spectrum of the Seiberg-Witten Theory, Nucl. Phys. B469 (1996) 387–402, [hep-th/9602082].
- [95] E. Witten, Solutions of four-dimensional field theories via M- theory, Nucl. Phys. **B500** (1997) 3–42, [hep-th/9703166].
- [96] M. Bershadsky, C. Vafa, and V. Sadov, D-Strings on D-Manifolds, Nucl. Phys. B463 (1996) 398–414, [hep-th/9510225].
- [97] S. H. Katz and C. Vafa, Matter from geometry, Nucl. Phys. B497 (1997) 146–154, [hep-th/9606086].
- [98] S. H. Katz, A. Klemm, and C. Vafa, Geometric engineering of quantum field theories, Nucl. Phys. **B497** (1997) 173–195, [hep-th/9609239].
- [99] S. H. Katz and C. Vafa, Geometric engineering of N=1 quantum field theories, Nucl. Phys. **B497** (1997) 196–204, [hep-th/9611090].
- [100] S. Katz, P. Mayr, and C. Vafa, Mirror symmetry and exact solution of 4D N = 2 gauge theories. I, Adv. Theor. Math. Phys. 1 (1998) 53–114, [hep-th/9706110].
- [101] A. Klemm, W. Lerche, P. Mayr, C. Vafa, and N. P. Warner, Self-Dual Strings and N=2 Supersymmetric Field Theory, Nucl. Phys. B477 (1996) 746–766, [hep-th/9604034].
- [102] K. Becker, M. Becker, and A. Strominger, Five-branes, membranes and nonperturbative string theory, Nucl. Phys. B456 (1995) 130–152, [hep-th/9507158].

- [103] D. Tong, NS5-branes, T-duality and worldsheet instantons, JHEP 07 (2002) 013, [hep-th/0204186].
- [104] J. A. Harvey and S. Jensen, Worldsheet instanton corrections to the Kaluza-Klein monopole, JHEP 10 (2005) 028, [hep-th/0507204].
- [105] Kaluza, Theodor, Zum Unitätsproblem in der Physik, Sitzungsber. Preuss. Akad. Wiss. Berlin. (Math. Phys.) (1921) 966–972.
- [106] Klein, Oskar, Quantentheorie und fünfdimensionale Relativitätstheorie, Zeitschrift für Physik a Hadrons and Nuclei 37 (12) (1926) 895906.
- [107] B. de Wit, P. G. Lauwers, and A. Van Proeyen, Lagrangians of N=2 Supergravity Matter Systems, Nucl. Phys. **B255** (1985) 569.
- [108] A. Strominger, Special geometry, Commun. Math. Phys. 133 (1990) 163–180.
- [109] B. Craps, F. Roose, W. Troost, and A. Van Proeyen, *What is special Kaehler geometry?*, *Nucl. Phys.* **B503** (1997) 565–613, [hep-th/9703082].
- [110] M. Billo, F. Denef, P. Fre, I. Pesando, W. Troost, A. Van Proeyen, and D. Zano, *The rigid limit in special Kaehler geometry: From K3- fibrations to special Riemann surfaces: A detailed case study, Class. Quant. Grav.* **15** (1998) 2083–2152, [hep-th/9803228].
- [111] E. Witten, Topological Sigma Models, Commun. Math. Phys. 118 (1988) 411.
- [112] M. Vonk, A mini-course on topological strings, hep-th/0504147.
- [113] A. Neitzke and C. Vafa, Topological strings and their physical applications, hep-th/0410178.
- [114] A. Klemm, "Introduction to Topological String Theory on Calabi-Yau manifolds." http://www.math.ist.utl.pt/ strings/AGTS/topstrings.pdf, 2003.
- [115] K. Hori et al., *Mirror Symmetry*, vol. 1 of *Clay Mathematics Monographs*. American Mathematical Society, 2003.
- [116] D. A. Cox and S. Katz, *Mirror symmetry and Algebraic Geometry*, vol. 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1999.
- [117] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Commun. Math. Phys. 165 (1994) 311–428, [hep-th/9309140].
- [118] I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor, *Topological amplitudes in string theory*, *Nucl. Phys.* **B413** (1994) 162–184, [hep-th/9307158].
- [119] S. H. Katz, A. Klemm, and C. Vafa, *M-theory, topological strings and spinning black holes, Adv. Theor. Math. Phys.* **3** (1999) 1445–1537, [hep-th/9910181].
- [120] H. Ooguri, A. Strominger, and C. Vafa, *Black hole attractors and the topological string, Phys. Rev.* **D70** (2004) 106007, [hep-th/0405146].
- [121] C. Vafa, Two dimensional Yang-Mills, black holes and topological strings, hep-th/0406058.
- [122] M. Aganagic, H. Ooguri, N. Saulina, and C. Vafa, *Black holes, q-deformed 2d Yang-Mills, and non- perturbative topological strings, Nucl. Phys.* **B715** (2005) 304–348, [hep-th/0411280].
- [123] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings, and supersymmetric gauge theories*, *Nucl. Phys.* **B644** (2002) 3–20, [hep-th/0206255].

- [124] R. Dijkgraaf and C. Vafa, On geometry and matrix models, Nucl. Phys. B644 (2002) 21–39, [hep-th/0207106].
- [125] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.
- [126] K. Behrend, Gromov-Witten invariants in algebraic geometry, Invent. Math. 127 (1997) 601–617.
- [127] J. Li and G. Tian, Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties, JAMS 11 (1998) 119174.
- [128] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, and C. Vafa, *Topological strings and integrable hierarchies*, Commun. Math. Phys. 261 (2006) 451–516, [hep-th/0312085].
- [129] E. Witten, Quantum background independence in string theory, hep-th/9306122.
- [130] A.-K. Kashani-Poor, The Wave Function Behavior of the Open Topological String Partition Function on the Conifold, JHEP 04 (2007) 004, [hep-th/0606112].
- [131] V. Bouchard, A. Klemm, M. Marino, and S. Pasquetti, *Remodeling the B-model*, 0709.1453.
- [132] S. K. Donaldson and R. P. Thomas, *Gauge theory in higher dimensions*, in *The geometric universe; science, geometry and the work of Roger Penrose* (S. A. Huggett et al., ed.), vol. 62, pp. 31–47. Oxford University Press, 1998.
- [133] R. P. Thomas, A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations, J. Diff. Geom. 54 (2000) 367–438.
- [134] D. Maulik and N. Nekrasov and A. Okounkov and R. Pandharipande, Gromov-Witten theory and Donaldson-Thomas theory, I, Compos. Math. 142 (2006) 1263–1285, [math/0312059].
- [135] D. Maulik and N. Nekrasov and A. Okounkov and R. Pandharipande, Gromov-Witten theory and Donaldson-Thomas theory, II, Compos. Math. 142 (2006) 1286–1304, [math/0406092].
- [136] D. Maulik and A. Oblomkov and A. Okounkov and R. Pandharipande, *Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds*, 0809.3976.
- [137] Joyce, Dominic and Song, Yinan, A theory of generalized Donaldson-Thomas invariants. I. An invariant counting stable pairs, 0810.5645.
- [138] S. Katz, Genus zero Gopakumar-Vafa invariants of contractible curves, J. Diff. Geom. **79(2)** (2008) 185–195, [0601193].
- [139] R. Pandharipande and R. P. Thomas, Stable pairs and BPS invariants, 07011.3899.
- [140] T. J. Hollowood, A. Iqbal, and C. Vafa, Matrix Models, Geometric Engineering and Elliptic Genera, JHEP 03 (2008) 069, [hep-th/0310272].
- [141] S. Katz, Gromov-Witten, Gopakumar-Vafa, and Donaldson-Thomas invariants of Calabi-Yau threefolds, math/0408266.
- [142] M. Cirafici, A. Sinkovics, and R. J. Szabo, Cohomological gauge theory, quiver matrix models and Donaldson-Thomas theory, Nucl. Phys. B809 (2009) 452–518, [0803.4188].
- [143] M. Henningson and P. Yi, Four-dimensional BPS-spectra via M-theory, Phys. Rev. D57 (1998) 1291–1298, [hep-th/9707251].
- [144] N. D. Lambert and P. C. West, *Monopole dynamics from the M-fivebrane*, *Nucl. Phys.* **B556** (1999) 177–196, [hep-th/9811025].

- [145] M. J. Duff and J. X. Lu, Black and super p-branes in diverse dimensions, Nucl. Phys. B416 (1994) 301–334, [hep-th/9306052].
- [146] R. Dijkgraaf, E. P. Verlinde, and M. Vonk, On the partition sum of the NS five-brane, hep-th/0205281.
- [147] N. Nekrasov and A. Okounkov, Seiberg-Witten theory and random partitions, hep-th/0306238.
- [148] B. Eynard and M. Marino, A holomorphic and background independent partition function for matrix models and topological strings, 0810.4273.
- [149] M. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, RIMS Kokyuroku **439** (1981) 30–46.
- [150] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, *Transformation groups for soliton equations*, in *RIMS Symp. Nonlinear integrable systems classical theory and quantum theory*, 1983. RIMS-394.
- [151] G. Segal and G. Wilson, Loop groups and equations of KdV type, Inst. Hautes Études Sci. Publ. Math. **61** (1985) 5–65.
- [152] N. Ishibashi, Y. Matsuo, and H. Ooguri, Soliton equations and free fermions on Riemann surfaces, Mod. Phys. Lett. A2 (1987) 119.
- [153] C. Vafa, Operator Formulation on Riemann Surfaces, Phys. Lett. B190 (1987) 47.
- [154] L. Alvarez-Gaume, C. Gomez, G. W. Moore, and C. Vafa, Strings in the Operator Formalism, Nucl. Phys. B303 (1988) 455.
- [155] R. Dijkgraaf, A. Sinkovics, and M. Temurhan, *Matrix models and gravitational corrections*, *Adv. Theor. Math. Phys.* **7** (2004) 1155–1176, [hep-th/0211241].
- [156] A. Kapustin and E. Witten, Electric-magnetic duality and the geometric Langlands program, hep-th/0604151.
- [157] S.C. Coutinho, *A primer of algebraic D-modules*. Cambridge University Press, 1995.
- [158] J. Bjork, Rings of differential operators. North-Holland Publishing Company, 1979.
- [159] M. Kashiwara, *D-modules and micolocal calculus*. No. 217 in Transl. Math. Monographs. AMS, 2000.
- [160] J. Bernstein, *Algebraic theory of D-modules*, http://www.math.uchicago.edu/~arinkin/langlands/Bernstein/.
- [161] D. Arinkin, Moduli of connections with a small parameter on a curve, math/0409373.
- [162] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, http://www.math.uchicago.edu/~mitya/langlands.html.
- [163] D. Arinkin, On quasiclassical limit of Langlands correspondence, talk at the KITP workshop "Gauge theory and Langlands duality" (2008).
- [164] M. Sato, The KP hierarchy and infinite-dimensional Grassmann manifolds, Theta functions Bowdoin 1987 Part 1, Proc. Sympos. Pure Math. 49 (1989).
- [165] R. Dijkgraaf, Intersection theory, integrable hierarchies and topological field theory, hep-th/9201003.
- [166] M. Mulase, Algebraic theory of KP equations, Perspectives in Mathematical Physics (1994) 157–223.

- [167] V. G. Kac and J. W. van de Leur, *The n component of KP hierarchy and representation theory*, *J. Math. Phys.* **44** (2003) 3245–3293, [hep-th/9308137].
- [168] M. R. Adams and M. J. Bergvelt, *The Krichever map, vector bundles over algebraic curves, and Heisenberg algebras, Commun. Math. Phys.* **154** (1993).
- [169] Y. Li and M. Mulase, Prym varieties and integrable systems, Commun. Anal. Geom. 5 (1997).
- [170] A. Hodge and M. Mulase, *Hitchin integrable systems, deformations of spectral curves, and KP-type equations*, math.AG/0801.0015.
- [171] J. M. Maldacena, G. W. Moore, N. Seiberg, and D. Shih, *Exact vs. semiclassical target space of the minimal string*, *JHEP* **10** (2004) 020, [hep-th/0408039].
- [172] M. Marino, Les Houches lectures on matrix models and topological strings, hep-th/0410165.
- [173] G. W. Moore, Geometry of the string equations, Commun. Math. Phys. 133 (1990) 261–304.
- [174] G. W. Moore, Matrix models of 2-D gravity and isomonodromic deformation, Prog. Theor. Phys. Suppl. 102 (1990) 255–286.
- [175] N. Seiberg and D. Shih, *Branes, rings and matrix models in minimal (super)string theory*, *JHEP* **02** (2004) 021, [hep-th/0312170].
- [176] V. A. Kazakov and I. K. Kostov, *Instantons in non-critical strings from the two-matrix model*, hep-th/0403152.
- [177] D. Kutasov, K. Okuyama, J.-w. Park, N. Seiberg, and D. Shih, *Annulus amplitudes and ZZ branes in minimal string theory*, *JHEP* **08** (2004) 026, [hep-th/0406030].
- [178] M. Fukuma, H. Kawai, and R. Nakayama, Infinite dimensional Grassmannian structure of two- dimensional quantum gravity, Commun. Math. Phys. 143 (1992) 371–404.
- [179] M. Fukuma, H. Irie, and Y. Matsuo, *Notes on the algebraic curves in (p,q) minimal string theory*, *JHEP* **09** (2006) 075, [hep-th/0602274].
- [180] M. R. Douglas, Strings in less than one dimension and the generalized kdv hierarchies, Phys. Lett. **B238** (1990) 176.
- [181] M. R. Douglas and S. H. Shenker, Strings in Less Than One-Dimension, Nucl. Phys. B335 (1990) 635.
- [182] D. J. Gross and A. A. Migdal, Nonperturbative Two-Dimensional Quantum Gravity, Phys. Rev. Lett. 64 (1990) 127.
- [183] E. Brezin and V. A. Kazakov, Exactly solvable field theories of closed strings, Phys. Lett. B236 (1990) 144–150.
- [184] R. Dijkgraaf, H. L. Verlinde, and E. P. Verlinde, Loop equations and Virasoro constraints in nonperturbative 2-D quantum gravity, Nucl. Phys. B348 (1991) 435–456.
- [185] J. M. Daul, V. A. Kazakov, and I. K. Kostov, Rational theories of 2-D gravity from the two matrix model, Nucl. Phys. B409 (1993) 311–338, [hep-th/9303093].
- [186] P. H. Ginsparg and G. W. Moore, Lectures on 2-D gravity and 2-D string theory, hep-th/9304011.
- [187] P. Di Francesco, P. H. Ginsparg, and J. Zinn-Justin, 2-D Gravity and random matrices, Phys. Rept. **254** (1995) 1–133, [hep-th/9306153].

- [188] E. Witten, On the structure of the topological phase of two-dimensional gravity, Nucl. Phys. B340 (1990) 281–332.
- [189] E. Witten, Two-dimensional gravity and intersection theory on moduli space, Surveys Diff. Geom. 1 (1991) 243–310.
- [190] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Commun. Math. Phys. 147 (1992) 1–23.
- [191] M. Bertola and B. Eynard and J. Harnad, *Partition functions for matrix models and isomonodromic tau functions*, *J. Phys. A* **36** (2003) 3067, [nlin/0204054].
- [192] M. Bertola, B. Eynard, and J. P. Harnad, Duality, Biorthogonal Polynomials and Multi-Matrix Models, Commun. Math. Phys. 229 (2002) 73–120, [nlin/0108049].
- [193] M. Bertola, B. Eynard, and J. Harnad, *Differential systems for biorthogonal polynomials appearing in 2-matrix models and the associated Riemann-Hilbert problem, Comm. Math. Phys.* **243** (2003) 193–240, [nlin/0208002].
- [194] D. J. Gross and I. R. Klebanov, One-dimensional string theory on a circle, Nucl. Phys. B344 (1990) 475–498.
- [195] I. R. Klebanov, String theory in two-dimensions, hep-th/9108019.
- [196] J. Polchinski, What is string theory?, hep-th/9411028.
- [197] S. Alexandrov, Matrix quantum mechanics and two-dimensional string theory in non-trivial backgrounds, hep-th/0311273.
- [198] R. Dijkgraaf, G. W. Moore, and R. Plesser, *The Partition function of 2-D string theory*, *Nucl. Phys.* **B394** (1993) 356–382, [hep-th/9208031].
- [199] G. W. Moore, M. R. Plesser, and S. Ramgoolam, *Exact S matrix for 2-D string theory*, *Nucl. Phys.* **B377** (1992) 143–190, [hep-th/9111035].
- [200] S. Y. Alexandrov, V. A. Kazakov, and I. K. Kostov, *Time-dependent backgrounds of 2D string theory*, *Nucl. Phys.* **B640** (2002) 119–144, [hep-th/0205079].
- [201] L. Bonora and C. S. Xiong, Extended Toda lattice hierarchy, extended two matrix model and c = 1 string theory, Nucl. Phys. B434 (1995) 408–444, [hep-th/9407141].
- [202] J. Distler and C. Vafa, A Critical matrix model at c = 1, Mod. Phys. Lett. A6 (1991) 259–270.
- [203] C. Imbimbo and S. Mukhi, *The Topological matrix model of c = 1 string*, *Nucl. Phys.* **B449** (1995) 553–568, [hep-th/9505127].
- [204] S. Y. Alexandrov, V. A. Kazakov, and I. K. Kostov, 2D string theory as normal matrix model, Nucl. Phys. B667 (2003) 90–110, [hep-th/0302106].
- [205] A. Mukherjee and S. Mukhi, c = 1 matrix models: Equivalences and open-closed string duality, *JHEP* **10** (2005) 099, [hep-th/0505180].
- [206] D. Ghoshal and C. Vafa, C = 1 string as the topological theory of the conifold, Nucl. *Phys.* **B453** (1995) 121–128, [hep-th/9506122].
- [207] N. A. Nekrasov, Seiberg-Witten Prepotential From Instanton Counting, Adv. Theor. Math. Phys. 7 (2004) 831–864, [hep-th/0206161].
- [208] H. Nakajima and K. Yoshioka, *Instanton counting on blowup. I*, math/0306198.
- [209] H. Nakajima and K. Yoshioka, *Instanton counting on blowup. II: K-theoretic partition function*, math/0505553.

- [210] N. A. Nekrasov, Seiberg-Witten prepotential from instanton counting, hep-th/0306211.
- [211] C. Meijer, On the g-function, Nederl. Akad. Wetensch. Proc. Ser. A 49 (1946) 344–356.
- [212] Y. Luke, The special functions and their approximations, vol. I. Academic Press, New York, 1969.
- [213] J. Fields, The asymptotic expansion of the meijer g-function, Mathematics of Computation **26** (1972) 757–765.
- [214] A. Iqbal and A.-K. Kashani-Poor, Instanton counting and Chern-Simons theory, Adv. Theor. Math. Phys. 7 (2004) 457–497, [hep-th/0212279].
- [215] A. Iqbal and A.-K. Kashani-Poor, SU(N) geometries and topological string amplitudes, Adv. Theor. Math. Phys. 10 (2006) 1–32, [hep-th/0306032].
- [216] B. Eynard, Large N expansion of convergent matrix integrals, holomorphic anomalies, and background independence, JHEP **0903** (2009) 003, [0802.1788].
- [217] M. Marino, R. Schiappa, and M. Weiss, Nonperturbative Effects and the Large-Order Behavior of Matrix Models and Topological Strings, 0711.1954.
- [218] M. Marino, Nonperturbative effects and nonperturbative definitions in matrix models and topological strings, 0805.3033.
- [219] M. Aganagic, V. Bouchard, and A. Klemm, Topological Strings and (Almost) Modular Forms, Commun. Math. Phys. 277 (2008) 771–819, [hep-th/0607100].
- [220] M. Marino, Open string amplitudes and large order behavior in topological string theory, JHEP **03** (2008) 060, [hep-th/0612127].
- [221] B. Eynard, M. Marino, and N. Orantin, Holomorphic anomaly and matrix models, JHEP 06 (2007) 058, [hep-th/0702110].
- [222] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, math-ph/0702045.
- [223] R. Dijkgraaf and C. Vafa, Two Dimensional Kodaira-Spencer Theory and Three Dimensional Chern-Simons Gravity, 0711.1932.
- [224] B. Eynard and O. Marchal, Topological expansion of the Bethe ansatz, and non-commutative algebraic geometry, 0809.3367.
- [225] F. Denef and G. W. Moore, Split states, entropy enigmas, holes and halos, hep-th/0702146.
- [226] D. Gaiotto, G. W. Moore, and A. Neitzke, Four-dimensional wall-crossing via three-dimensional field theory, 0807.4723.
- [227] E. Frenkel, Lectures on the Langlands program and conformal field theory, hep-th/0512172.
- [228] E. Witten, Gauge Theory And Wild Ramification, 0710.0631.
- [229] S. Gukov and E. Witten, Branes and Quantization, 0809.0305.
- [230] A. Chervov and D. Talalaev, Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence, hep-th/0604128.
- [231] J. Bryan and R. Pandharipande, The local Gromov-Witten theory of curves, math/0411037.
- [232] R. Dijkgraaf, Mirror symmetry and elliptic curves, in The moduli space of curves, vol. 129 of Progress in Mathematics, Birkhauser, 1995.

- [233] M. R. Douglas, Conformal field theory techniques in large N Yang-Mills theory, hep-th/9311130.
- [234] R. Dijkgraaf, Chiral deformations of conformal field theories, Nucl. Phys. **B493** (1997) 588–612, [hep-th/9609022].
- [235] D. J. Gross and I. R. Klebanov, Fermionic string field theory of c = 1 two-dimensional quantum gravity, Nucl. Phys. **B352** (1991) 671–688.
- [236] M. Kaneko and D. Zagier, A generalized Jacobi theta function and quasimodular forms, in The moduli space of curves, vol. 129 of Progress in Mathematics, Birkhauser, 1995.
- [237] J. Li, K. Liu, and J. Zhou, Topological string partition functions as equivariant indices, math/0412089.
- [238] J.-L. Igusa, On Siegel modular forms of genus two, Amer. J. Math 84 (1962) 175–200.
- [239] J.-L. Igusa, On Siegel modular forms of genus two (2), Amer. J. Math **86** (1964) 164–412.
- [240] G. W. Moore, Modular forms and two loop string physics, Phys. Lett. **B176** (1986) 369.
- [241] A. A. Belavin, V. Knizhnik, A. Morozov, and A. Perelomov, *Two and three loop amplitudes in the bosonic string theory*, *JETP Lett.* **43** (1986) 411.
- [242] A. Dabholkar and D. Gaiotto, Spectrum of CHL dyons from genus-two partition function, JHEP 12 (2007) 087, [hep-th/0612011].
- [243] R. Borcherds, Automorphic forms on $O_{s+2,2}(R)$ and infinite products, Invent. Math. 120 (1995) 161–213.
- [244] R. Dijkgraaf, *The mathematics of five-branes*, hep-th/9810157.
- [245] R. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, *Counting Dyons in N=4 String Theory*, *Nucl. Phys.* **B484** (1997) 543–561, [hep-th/9607026].
- [246] D. P. Jatkar and A. Sen, *Dyon spectrum in CHL models*, *JHEP* **04** (2006) 018, [hep-th/0510147].
- [247] G. Lopes Cardoso, B. de Wit, J. Kappeli, and T. Mohaupt, *Asymptotic degeneracy of dyonic N* = 4 string states and black hole entropy, *JHEP* **12** (2004) 075, [hep-th/0412287].
- [248] J. R. David and A. Sen, *CHL dyons and statistical entropy function from D1-D5 system*, *JHEP* **11** (2006) 072, [hep-th/0605210].
- [249] S. Banerjee, A. Sen, and Y. K. Srivastava, Partition Functions of Torsion > 1 Dyons in Heterotic String Theory on T^6 , JHEP **05** (2008) 098, [0802.1556].
- [250] D. Gaiotto, Re-recounting dyons in N = 4 string theory, hep-th/0506249.
- [251] J. R. David, D. P. Jatkar, and A. Sen, Product representation of dyon partition function in CHL models, JHEP 06 (2006) 064, [hep-th/0602254].
- [252] J. R. David, D. P. Jatkar, and A. Sen, *Dyon spectrum in N = 4 supersymmetric type II string theories*, *JHEP* 11 (2006) 073, [hep-th/0607155].
- [253] J. R. David, D. P. Jatkar, and A. Sen, Dyon spectrum in generic N=4 supersymmetric Z(N) orbifolds, JHEP **01** (2007) 016, [hep-th/0609109].
- [254] A. Sen, Walls of Marginal Stability and Dyon Spectrum in N=4 Supersymmetric String Theories, JHEP **05** (2007) 039, [hep-th/0702141].

- [255] A. Dabholkar, D. Gaiotto, and S. Nampuri, *Comments on the spectrum of CHL dyons*, *JHEP* **01** (2008) 023, [hep-th/0702150].
- [256] M. C. N. Cheng and E. Verlinde, *Dying Dyons Don't Count*, *JHEP* **09** (2007) 070, [0706.2363].
- [257] A. Mukherjee, S. Mukhi, and R. Nigam, *Dyon Death Eaters*, *JHEP* **10** (2007) 037, [0707.3035].
- [258] A. Dabholkar, J. Gomes, and S. Murthy, Counting all dyons in N=4 string theory, 0803.2692.
- [259] M. C. N. Cheng and E. P. Verlinde, Wall Crossing, Discrete Attractor Flow, and Borcherds Algebra, SIGMA 4 (2008) 068, [0806.2337].
- [260] S. Banerjee, A. Sen, and Y. K. Srivastava, Genus Two Surface and Quarter BPS Dyons: The Contour Prescription, 0808.1746.
- [261] M. C. N. Cheng and A. Dabholkar, Borcherds-Kac-Moody Symmetry of N=4 Dyons, 0809.4258.
- [262] N. Banerjee, D. P. Jatkar, and A. Sen, *Asymptotic Expansion of the N=4 Dyon Degeneracy*, 0810.3472.
- [263] S. Murthy and B. Pioline, A Farey tale for N=4 dyons, 0904.4253.
- [264] A. Sen, String network, JHEP 03 (1998) 005, [hep-th/9711130].
- [265] M. Cvetic and A. A. Tseytlin, Solitonic strings and BPS saturated dyonic black holes, Phys. Rev. D53 (1996) 5619–5633, [hep-th/9512031].
- [266] M. Cvetic and D. Youm, *Dyonic BPS saturated black holes of heterotic string on a six torus*, *Phys. Rev.* **D53** (1996) 584–588, [hep-th/9507090].
- [267] V. A. Gritsenko and V. V. Nikulin, Siegel automorphic form corrections of some Lorentzian Kac–Moody Lie algebras, 9504006.
- [268] R. Borcherds, Automorphic forms with singularities on Grassmannians, Inventiones mathematicae 132 (1998) 491–562, [9609022].
- [269] M. C. N. Cheng, The Spectra of Supersymmetric States in String Theory, 0807.3099.
- [270] David Mumford, Tata Lectures on Theta II Jacobian Theta Functions and Differential Equations. Birkhäuser, 1984.
- [271] Aaron Lebowitz, Degeneration of a Compact Riemann Surface of Genus 2, Israel Journal of Mathematics 12 (1972).
- [272] A. J. Feingold and I. B. Frenekel, A hyperbolic Kac-Moody Algebra and the theory of Siegel modular forms of genus 2, Math. Ann. 263 (1983) 87.
- [273] E. Witten, String theory dynamics in various dimensions, Nucl. Phys. **B443** (1995) 85–126, [hep-th/9503124].
- [274] S. Chaudhuri, G. Hockney, and J. D. Lykken, *Maximally supersymmetric string theories in D i 10, Phys. Rev. Lett.* **75** (1995) 2264–2267, [hep-th/9505054].
- [275] S. Chaudhuri and J. Polchinski, Moduli space of CHL strings, Phys. Rev. D52 (1995) 7168–7173, [hep-th/9506048].
- [276] S. Chaudhuri and D. A. Lowe, Type IIA heterotic duals with maximal supersymmetry, Nucl. Phys. B459 (1996) 113–124, [hep-th/9508144].
- [277] R. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, C = 1 Conformal Field Theories on Riemann Surfaces, Commun. Math. Phys. 115 (1988) 649–690.

- [278] M. Kontsevich and Y. Soibelman, Stability structures, motivic Donaldson-Thomas invariants and cluster transformations, 0811.2435.
- [279] D. L. Jafferis and G. W. Moore, Wall crossing in local Calabi Yau manifolds, 0810.4909.
- [280] W.-y. Chuang and D. L. Jafferis, Wall Crossing of BPS States on the Conifold from Seiberg Duality and Pyramid Partitions, 0810.5072.
- [281] S. Mozgovoy and M. Reineke, On the noncommutative Donaldson-Thomas invariants arising from brane tilings, 0809.0117.
- [282] T. Dimofte and S. Gukov, Refined, Motivic, and Quantum, 0904.1420.
- [283] J. Bryan and A. Gholampour, *The Quantum McKay Correspondence for polyhedral singularities*, 0803.3766.
- [284] J. Bryan and A. Gholampour, *BPS invariants for resolutions of polyhedral singularities*, 0905.0537.
- [285] S. Boissiere and A. Sarti, Contraction of excess fibres between the McKay correspondences in dimension two and three, Ann. Inst. Fourier **57** (2007), no. 6 1839–1861, [0504360].
- [286] K. A. Intriligator, N. Seiberg, and D. Shih, *Dynamical SUSY breaking in meta-stable vacua*, *JHEP* **04** (2006) 021, [hep-th/0602239].
- [287] H. Ooguri, Y. Ookouchi, and C.-S. Park, Metastable Vacua in Perturbed Seiberg-Witten Theories, 0704.3613.
- [288] M. Aganagic, C. Beem, J. Seo, and C. Vafa, Extended Supersymmetric Moduli Space and a SUSY/Non-SUSY Duality, 0804.2489.
- [289] K. A. Intriligator, N. Seiberg, and D. Shih, Supersymmetry Breaking, R-Symmetry Breaking and Metastable Vacua, JHEP 07 (2007) 017, [hep-th/0703281].
- [290] G. Pastras, Non supersymmetric metastable vacua in N=2 SYM softly broken to N=1,0705.0505.
- [291] L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, The Background Field Method and the Ultraviolet Structure of the Supersymmetric Nonlinear Sigma Model, Ann. Phys. 134 (1981) 85.
- [292] C. M. Hull, A. Karlhede, U. Lindstrom, and M. Rocek, *Nonlinear sigma models and their gauging in and out of superspace*, *Nucl. Phys.* **B266** (1986) 1.
- [293] K. Higashijima and M. Nitta, Kaehler normal coordinate expansion in supersymmetric theories, Prog. Theor. Phys. 105 (2001) 243–260, [hep-th/0006027].
- [294] J. Marsano, H. Ooguri, Y. Ookouchi, and C.-S. Park, Metastable Vacua in Perturbed Seiberg-Witten Theories, Part 2: Fayet-Iliopoulos Terms and Káhler Normal Coordinates, Nucl. Phys. B798 (2008) 17–35, [0712.3305].
- [295] F. Denef, Les Houches Lectures on Constructing String Vacua, 0803.1194.
- [296] M. R. Douglas and S. Kachru, *Flux compactification, Rev. Mod. Phys.* **79** (2007) 733–796, [hep-th/0610102].
- [297] J. Michelson, Compactifications of type IIB strings to four dimensions with non-trivial classical potential, Nucl. Phys. B495 (1997) 127–148, [hep-th/9610151].
- [298] S. Gukov, C. Vafa, and E. Witten, *CFT's from Calabi-Yau four-folds, Nucl. Phys.* **B584** (2000) 69–108, [hep-th/9906070].

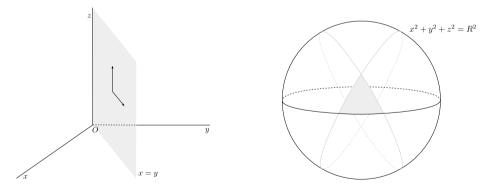
- [299] T. R. Taylor and C. Vafa, RR flux on Calabi-Yau and partial supersymmetry breaking, Phys. Lett. **B474** (2000) 130–137, [hep-th/9912152].
- [300] F. Cachazo, M. R. Douglas, N. Seiberg, and E. Witten, *Chiral Rings and Anomalies in Supersymmetric Gauge Theory*, *JHEP* **12** (2002) 071, [hep-th/0211170].
- [301] F. Cachazo, N. Seiberg, and E. Witten, *Phases of N = 1 supersymmetric gauge theories and matrices*, *JHEP* **02** (2003) 042, [hep-th/0301006].
- [302] J. Marsano, K. Papadodimas, and M. Shigemori, Off-shell M5 Brane, Perturbed Seiberg-Witten Theory, and Metastable Vacua, Nucl. Phys. B804 (2008) 19–69, [0801.2154].
- [303] I. Bena, E. Gorbatov, S. Hellerman, N. Seiberg, and D. Shih, *A note on (meta)stable brane configurations in MQCD, JHEP* **11** (2006) 088, [hep-th/0608157].
- [304] F. Cachazo, K. A. Intriligator, and C. Vafa, A large N duality via a geometric transition, Nucl. Phys. **B603** (2001) 3–41, [hep-th/0103067].
- [305] F. Cachazo and C. Vafa, N = 1 and N = 2 geometry from fluxes, hep-th/0206017.
- [306] C. Vafa, Superstrings and topological strings at large N, J. Math. Phys. **42** (2001) 2798–2817, [hep-th/0008142].
- [307] F. Cachazo, S. Katz, and C. Vafa, Geometric transitions and N=1 quiver theories, hep-th/0108120.
- [308] F. Cachazo, B. Fiol, K. A. Intriligator, S. Katz, and C. Vafa, A geometric unification of dualities, Nucl. Phys. **B628** (2002) 3–78, [hep-th/0110028].
- [309] J. de Boer and S. de Haro, *The off-shell M5-brane and non-perturbative gauge theory*, *Nucl. Phys.* **B696** (2004) 174–204, [hep-th/0403035].
- [310] C. Vafa, Conformal theories and punctured surfaces, Phys. Lett. **B199** (1987) 195.
- [311] M. R. Douglas, *The statistics of string/M theory vacua*, *JHEP* **05** (2003) 046, [hep-th/0303194].
- [312] S. Ashok and M. R. Douglas, *Counting flux vacua*, *JHEP* **01** (2004) 060, [hep-th/0307049].
- [313] F. Denef and M. R. Douglas, *Distributions of nonsupersymmetric flux vacua*, *JHEP* **03** (2005) 061, [hep-th/0411183].
- [314] F. Denef and M. R. Douglas, *Distributions of flux vacua*, *JHEP* **05** (2004) 072, [hep-th/0404116].
- [315] G. Torroba, Finiteness of flux vacua from geometric transitions, JHEP **02** (2007) 061, [hep-th/0611002].
- [316] R. Dijkgraaf, S. Gukov, V. A. Kazakov, and C. Vafa, *Perturbative analysis of gauged matrix models*, *Phys. Rev.* **D68** (2003) 045007, [hep-th/0210238].
- [317] H. Itoyama and A. Morozov, *Calculating gluino condensate prepotential, Prog. Theor. Phys.* **109** (2003) 433–463, [hep-th/0212032].
- [318] K. Becker and M. Becker, *M-Theory on Eight-Manifolds*, *Nucl. Phys.* **B477** (1996) 155–167, [hep-th/9605053].
- [319] T. Nakanishi and A. Tsuchiya, Level-rank duality of WZW models in conformal field theory, Commun. Math. Phys. 144 (1992) 351–372.

Samenvatting

In de Euclidische meetkunde bekijkt men rechte oppervlakken zoals die links in Fig. 1.1. Sinds de 19e eeuw bestuderen wiskundigen echter ook gekromde oppervlakken, zoals de bol, geïllustreerd rechts in deze figuur. Meetkunde op de bol is anders dan op een vlak: de hoeken van een driehoek tellen niet op tot 180 graden, maar meer! Dit is een teken dat de ruimte niet vlak is, maar gekromd.

Algemenere gekromde oppervlakken zijn als eerste bestudeerd door B. Riemann en worden *Riemannoppervlakken* genoemd. Door te doen alsof zo'n oppervlak van rubber is, en geleidelijke vervormingen toe te staan, kan een Riemannoppervlak beschreven worden door het aantal handvaten in het oppervlak. De bol heeft bijvoorbeeld geen handvaten. Fig. 1.2 toont twee Riemannoppervlakken met handvaten, die veelvuldig voorkomen in dit proefschrift.

In de natuur vinden we veel van zulke gekromde ruimten, zoals het oppervlak



Figuur 1.1: Links een meetkundig vlak, gedefinieerd door de lineaire vergelijking x=y. Op elk punt in dit vlak kun je twee richtingen uit. Het vlak is dus een 2-dimensionaal deel van het 3-dimensionale (x,y,z)-assenstelsel. Bovendien zijn die twee richtingen overal op het vlak hetzelfde.

Rechts een bol, gedefinieerd door de kwadratische vergelijking $x^2 + y^2 + z^2 = R^2$, waarbij R de straal van de bol is. De hoeken van de gearceerde driehoek tellen op tot iets meer dan 180 graden. Op ieder punt van de bol kun je twee richtingen uit, zodat ook de bol 2-dimensionaal is. Die twee richtingen veranderen echter als je naar een ander punt op de bol beweegt.

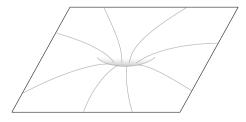


Figuur 1.2: Twee voorbeelden van Riemannoppervlakken: links een Riemannoppervlak in de vorm van een donut — dit wordt een torus genoemd — en rechts een oppervlak met twee handvaten.

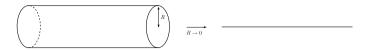
van de aarde. Sterker nog, A. Einstein ontdekte dat de ruimte waarin we leven in een diepere zin gekromd is. Om dit te beschrijven bracht hij ruimte en tijd onder één noemer. Grofweg wordt deze zogenaamde *ruimte-tijd* gegeven door een 4-dimensionaal assenstelsel dat verandert als we van de ene naar de andere plaats reizen. Net als op de bol. De kromming is het grootst in de buurt van grote massa's zoals sterren en zwarte gaten. Fig. 1.3 is hier een illustratie van.

Snaartheorie probeert eigenschappen van ons universum te beschrijven door naar nog hoger-dimensionale gekromde ruimten te kijken. Die ruimten hebben meestal 10 dimensies, waarvan 1 tijd-richting. Oftewel, je kunt in 9 verschillende richtingen reizen die allemaal loodrecht op elkaar staan. De relatie met onze wereld kan dan gemaakt worden door zes van die dimensies heel klein te maken, zodat je ze bijna niet ziet. Zie Fig. 1.4 voor een illustratie. Alhoewel het moeilijk is die kleine dimensies te meten, zijn ze wel degelijk belangrijk voor de natuurkunde in onze wereld. Ze worden bestudeerd om oplossingen te vinden voor allerlei raadsels waarmee natuurkundigen geconfronteerd worden.

Een van de grootste raadsels is het vinden van een goede beschrijving van zwaartekracht op heel kleine lengteschaal. Sinds het begin van de vorige eeuw is bekend dat op afstanden kleiner dan de grootte van een atoom de kwantummechanica een rol speelt. In deze theorie kunnen afstand en snelheid niet gelijktijdig exact bepaald worden, en zijn grootheden die continu lijken, zoals energie, opgebouwd uit discrete pakketjes, de kwanta. Om de zwaartekracht op kleine schaal te kunnen begrijpen, hebben we een kwantummechanische beschrijving



Figuur 1.3: Een 2-dimensionale voorstelling van de kromming van de ruimte-tijd nabij een zwaar object in de ruimte.



Figuur 1.4: Als de straal R van een cylinder heel klein wordt, zijn de cylinder en een rechte lijn moeilijk van elkaar te onderscheiden.

van Einsteins theorie nodig. Dit blijkt echter erg moeilijk te zijn.

Een andere formulering van dit probleem is dat we een beschrijving van de natuur proberen te vinden waarin we alle vier de fundamentele krachten—electromagnetische kracht, sterke en zwakke kernkracht en zwaartekracht—verenigen. Het zogeheten standaardmodel unificeert de eerste drie van deze krachten, in het kader van de kwantummechanica. Maar de vierde kracht, gravitatie, wil niet zo meewerken. Dit staat een beschrijving in de weg van de meest fundamentele vraagstukken in het heelal, bijvoorbeeld de vraag naar het ontstaan van het heelal.

Snaartheorie is een van de beste kandidaten om inzicht te verkrijgen in deze fundamentele vraagstukken. Vermoedelijk beschrijft deze theorie alle vier de krachten. Maar tegelijkertijd is ze veelomvattend en ingewikkeld. Hoewel we al ontzettend veel over snaartheorie weten, is dat nog lang niet genoeg om de hele theorie te doorgronden. Dit proefschrift zet een paar kleine stapjes in deze richting.

Belangrijk om te weten is dat we nog niet kunnen meten aan snaartheorie. De theorie is dus volledig gebouwd op fysische argumenten en een heleboel wiskunde. Dit heeft als voordeel dat er een actieve interactie is met allerlei takken van de wiskunde. Snaartheorie blijkt interessante wiskundige vermoedens te genereren en nieuwe verbindingen tussen verschillende subdisciplines te leggen. Dit proefschrift speelt ook daar op in.

In dit proefschrift bestuderen we snaartheorie op een 10-dimensionale ruimte die we onderverdelen in de 4-dimensionale ruimte-tijd en een 6-dimensionale interne ruimte. Om precies te zijn bestuderen we interne ruimten die zogenaamde Calabi-Yau variëteiten vormen, en wel Calabi-Yau variëteiten die gemodelleerd zijn in termen van een Riemannoppervlak. Dit Riemannoppervlak vormt een rode draad door het proefschrift.

Eigenlijk hadden we iets nauwkeuriger moeten zijn bij het definiëren van Riemannoppervlakken: het zijn een speciaal soort oppervlakken die er lokaal uitzien als het complexe vlak, met coördinaten x+iy, waarbij $i^2=-1$. Een Riemannoppervlak wordt daarom ook wel een complexe of algebraïsche kromme genoemd.

Op Calabi-Yau variëteiten die gemodelleerd zijn op een Riemannoppervlak, kunnen we met behulp van snaartheorie partitiefuncties uitrekenen. Een voorbeeld van een relatief eenvoudige, maar toch interessante partitiefunctie, die we aan-

duiden met $Z(\tau)$, is

$$Z(\tau) = e^{-\pi i \tau / 12} \prod_{n>1}^{\infty} \frac{1}{(1 - e^{2\pi i \tau n})}.$$

De variabele τ neemt waarden aan in de bovenste helft van het complexe vlak. In de wiskunde staat de functie $Z(\tau)$ vooral bekend om zijn mooie eigenschappen onder transformaties van zijn argument τ . De functiewaarde blijft namelijk bijna hetzelfde wanneer τ naar $-1/\tau$ gestuurd wordt:

$$Z(-1/\tau) = \frac{1}{\sqrt{-i\tau}} \ Z(\tau)$$

Deze symmetrieën hebben tevens een interessante fysische interpretatie. Om dit kort uit te leggen, beginnen we met een reeks-ontwikkeling van de partitiefunctie $Z(\tau)$. Oftewel, we schrijven het product als een oneindige som:

$$Z(\tau) = q^{-1/24} \left(1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots \right),$$

Hier hebben we voor de bondigheid $q=e^{2\pi i\tau}$ gedefinieerd. De puntjes verwijzen naar alle termen q^k met een macht k groter dan 4. Het grappige is dat deze reeks grafisch geïnterpreteerd kan worden, zie Fig. 1.5.

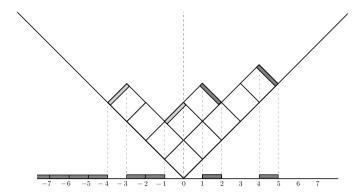
$$Z(\tau) = q^{-1/24} \left(1 + \diamondsuit q + \left(\diamondsuit + \diamondsuit \right) q^2 + \left(\diamondsuit + \diamondsuit + \diamondsuit \right) q^3 + \left(\diamondsuit + \diamondsuit + \diamondsuit + \diamondsuit \right) q^4 + \dots \right)$$

Figuur 1.5: In deze reeks voor de partitiefunctie $Z(\tau)$ geeft het aantal diagrammen met k vierkantjes de numerieke factor voor de term q^k aan. Dat is er dus eentje voor q, twee voor q^2 , drie voor q^3 , vijf voor q^4 , enzovoorts.

Deze diagrammen met vierkantjes moeten aan een aantal regels voldoen. Als je een rechte hoek met het midden op de grond zet, zoals in Fig. 1.6, moeten ze te verkrijgen zijn door vierkantjes in deze wig te laten vallen.

De fysica achter deze reeks heeft te maken met de kleinst mogelijke deeltjes. Die zijn er namelijk maar in twee soorten: bosonen en fermionen. Bosonen willen zich graag in dezelfde toestand bevinden, terwijl de fermionen dit nooit zullen doen. Een van de simpelste kwantum-systemen beschrijft fermionen op een cirkel. Hun energieën nemen discrete (oftewel kwantum-) waarden aan.

Een bepaalde toestand van dit systeem wordt dan beschreven door aan te geven welke energietoestanden bezet zijn door een fermion. Omdat de fermionen in



Figuur 1.6: Deze figuur laat een iets algemener diagram zien in de reeksontwikkeling van $Z(\tau)$ in Fig. 1.5; omdat dit diagram uit tien vierkantjes bestaat komt het pas tevoorschijn bij de term q^{10} . Elk zo'n diagram kun je op een unieke manier afbeelden naar een toestand van fermionen. In de bovenstaande figuur bepalen de donkere en lichte rechthoekjes in het diagram welke energietoestanden op de getallenlijn wel of resp. niet bezet zijn. Op de getallenlijn indiceren de donker gekleurde rechthoekjes de ingenomen energietoestanden.

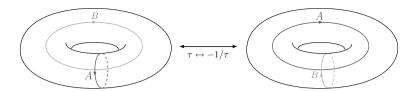
dit systeem nooit dezelfde energie zullen hebben, is zo'n energietoestand óf onbezet, óf bezet door een enkel fermion. De totale toestand van het systeem kan daarom gevisualiseerd worden door een getallenlijn, waarop bij elk geheel getal wordt aangegeven of er wel of geen fermion zit.

Nu blijken alle diagrammen in Fig. 1.5 op een unieke manier af te beelden naar zo'n fermionische toestand, zoals geïllustreerd in Fig. 1.6. De partitiefunctie $Z(\tau)$ codeert dus alle mogelijke toestanden van het fermionische systeem!

In de snaartheorie vinden we deze specifieke partitiefunctie $Z(\tau)$ voor een relatief eenvoudige Calabi-Yau variëteit die gebaseerd is op de torus uit Fig. 1.2. De parameter τ karakteriseert in die interpretatie de vorm van de torus. De fermionen leven op de torus: één opspannende cirkel van de torus kan gezien worden als de coördinaat-cirkel, en de ander als de tijd. (Preciezer gezegd is $Z(\tau)$ de partitiefunctie van chirale, oftewel holomorfe, fermionen op de torus.)

Onder de afbeelding $\tau\mapsto -1/\tau$ verandert de torus zodat de ruimte- en tijdcirkels worden omgewisseld, zie Fig. 1.7. De vorm van de torus blijft echter dezelfde. Dit verklaart waarom de partitiefunctie $Z(\tau)$ zo goed als invariant blijft onder deze transformatie.

Interessant is dat de partitiefunctie $Z(\tau)$ niet alleen een betekenis heeft in de interne 6-dimensionale ruimte, maar ook in de 4-dimensionale ruimte-tijd. In deze ruimte-tijd is er een duale theorie die de electromagnetische interacties beschrijft. De symmetrie $\tau \mapsto -1/\tau$ blijkt hierin elektrische en magnetische deeltjes te verwisselen. Zo verkrijgen we ook inzicht in de beschrijving van natuurkunde in onze wereld: er is een dieper liggende symmetrie tussen de



Figuur 1.7: De parameter τ karakteriseert de vorm van de torus. Onder de afbeelding $\tau \mapsto -1/\tau$ verwisselen de twee opspannende cirkels op de torus.

electrische en magnetische krachten!

Deze wisselwerking tussen enerzijds de wiskunde van interne 6-dimensionale ruimten en anderzijds de fysica in de 4-dimensionale ruimte-tijd speelt een grote rol in dit proefschrift. Ruwweg heet het vakgebied waarin die wiskunde van 6-dimensionale Calabi-Yau ruimten wordt ontwikkeld *topologische snaartheorie*. In dit proefschrift generalizeren we de bovenstaande duale beschrijving van partitiefuncties in termen van fermionen.

In het bijzonder zien we dat een algemene partitiefunctie in de topologische snaartheorie kan worden gezien als een partitiefunctie van fermionen op een vreemd soort Riemannoppervlak: de coördinaten van dit Riemannoppervlak (ofwel algebraïsche kromme) gedragen zich niet klassiek, maar kwantummechanisch. We noemen dit dan ook een *kwantum kromme*. Dit verheldert de naam van het proefschrift: "Topologische Snaren en Kwantum Krommen".

Acknowledgments

Let me finally take the opportunity to thank some people who have been invaluable for the completion of this thesis.

First and foremost, I would like to express my gratitude to my supervisor Robbert Dijkgraaf. It must have been at some point in my fourth year in Utrecht that I stumbled upon his website and was immediately struck by the elegant combination of mathematics and physics. In these last years Robbert has guided me through this subject and we have worked together on several interesting projects. Unfortunately his increasingly busy schedule didn't always allow time for me. However, our meetings were always inspiring and I have greatly benefited from his wisdom and creativity. Most important for this thesis is that Robbert has been a driving force behind both of our articles. I would like to thank him very much for this.

Secondly, I am indebted to my other co-authors Miranda Cheng, Kyriakos Papadodimas, Masaki Shigemori, Piotr Sułkowski and Cumrun Vafa for sharing their insights and ideas. Only in these collaborations I really learned what doing research encompasses, and I enjoyed it very much. I would also like to thank many other colleagues I met during the last years for the stimulating discussions and the pleasant times in Amsterdam and on all those trips abroad. Special thanks go to Chris Beasley, Jan de Boer, Sheer El-Showk, Sergei Gukov, Amir Kashani-Poor, Albrecht Klemm, Johan van de Leur, Jan Manschot, Marcos Marino, Ilies Messamah, Andy Neitzke, Hessel Posthuma, Ani Sinkovics, Balázs Szendröi, Stefan Vandoren and Erik Verlinde. I'm looking forward to elaborating on our ideas.

Furthermore I am grateful to all my (former) colleagues in Amsterdam for the friendly and animate atmosphere in the institute, in particular to Amir, Asad, Ben, Masaki, Tomeu and especially Kyriakos for being keen to answer all my questions. I also mustn't forget to mention Bianca and Yocklang for their administrative support. Moreover, the times that I shared an office with Jan, Manuela and Piotr bring back pleasant memories. Balt, Ilies, Ingmar, Johannes, Joost, Leo, Meindert, Paul, Sheer, Xerxes and many others made lunches and cookie

times, too, very lively with interesting discussions. Also, many thanks to the members of the theoretical physics and mathematics institutes in Utrecht for welcoming me with open arms on each of my visits.

In addition to academic pursuits, I have very much enjoyed all after-hours fun, and would like to thank everyone who contributed to this. Let me mention here my highschool friends Casper & Sabine, Joost & Monique, Jules, Lieselotte & Thijs, Martijn, Thijs & Jiska and Vera & Constantijn, my former study-buddies Gerben, Jaap, Jan & Afke, as well as Alex, Erik & Judith, Hendrik, Josien, Luuk, Marielle, Rob, Rogier, Rudy & Magda, Sheer & Hanna, Sylvain and Thijs & Hieke. Also thanks to Ana & Harald and Ichiro & Kumiko for the great holidays in Austria and Japan. I appreciate it particularly that Gerben and Sheer are willing to be my paranymfs.

Let me close by adding some words of appreciation to all my family (in-law). Especially, many thanks to my brother Ramon & Anneke for their frequent company. I am sure they are going to love our apartment as much as we did. Moreover I am very grateful to my parents, for always being there and supporting me. But most of all I would like to thank Chris, for just everything.

Amsterdam, July 2009