### The relations between generalized fields and superfields formalisms of the Batalin-Vilkovisky method of quantization

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A general solution of the Batalin–Vilkovisky master equation was formulated in terms of generalized fields. Recently, a superfields approach of obtaining solutions of the Batalin–Vilkovisky master equation is also established. Superfields formalism is usually applied to topological quantum field theories. However, generalized fields method is suitable to find solutions of the Batalin–Vilkovisky master equation either for topological quantum field theories or the usual gauge theories like Yang–Mills theory. We show that by truncating some components of superfields with appropriate actions, generalized fields formalism of the usual gauge theories result. We demonstrate that for some topological quantum field theories and the relativistic particle both of the methods possess the same field contents and yield similar results. Inspired by the observed relations we give the solution of the BV–master equation for on–shell N=1 supersymmetric Yang–Mills theory utilizing superfields.

### 1 Introduction

The Batalin-Vilkovisky (BV) method offers a systematic procedure of finding actions which can be used in related path integrals respecting symmetries like Lorentz invariance of classical field theories possessing gauge invariance which may be reducible[1]. Some ad hoc definitions of the BV method of quantization were derived analytically introducing an odd time formulation[2] which is recently utilized to formulate some aspects of BV method on a geometrical setting[3] (for another approach see [4]). Odd time formalism inspired the generalized fields method of solving the BV-master equation [5]. In this approach one begins with a gauge theory whose action can be written as first order in derivatives. Although this seems to restrict the applicability of the method drastically, it was shown that BV-quantization of a vast class of gauge theories can be obtained by this method[5],[6]. Exceptions are theories like the relativistic superparticle where kinetic terms of anticommuting variables include at least three variables. After studying its minimal ghost content and antifields one introduces generalized fields which are defined utilizing differential form degree and BRST grading. Substitution of original fields with generalized ones in the original gauge theory action yields the desired solution of the BV-master equation. A similar approach was also given in [7].

The BV method of quantization is also studied introducing superfields[8], to reveal its geometrical aspects[9]. This method is usually applied to find BV-master actions (solutions of the BV-master equation) of topological quantum field theories. Superfield algorithm is used to discuss general solutions of the BV-master equation in [10]. In terms of this method first order systems[11] and deformations of some gauge theories[12] are also studied (for another superfield approach see [13] and the references given therein). In superfields formalism instead of specifying a gauge invariant classical action one starts with an action which can be used as a BV-master action. Underlying gauge invariant classical action can be deduced by setting ghost fields and antifields equal to zero.

Generalized fields formalism as well as superfields method yield general solutions of the BV-master equation. Purpose of this work is to discover the relations between these approaches. We show that superfields method leads to generalized fields solutions of the BV-master equation of the usual gauge theories like Yang-Mills theory if some components of superfields are truncated in a consistent manner. Different truncations with appropriate actions yield different gauge theories. However, when one deals with topological quantum field theories without any truncation, the superfields contents coincide with generalized fields components. This relation is not surprising: Actions of the both methods possess the same form. The difference can

only be in field contents. Indeed, one gets rid of this difference by the consistent truncations. Understanding these relations aggregates powers of generalized fields method and superfields approach. This may shed light on understanding of different aspects of gauge theories. Moreover, we show that once these relations are discovered they inspire derivation of BV-master actions of some other theories, like supersymmetric Yang-Mills theory, in terms of superfields method. Here we discuss some cases which are useful to illustrate these relations although there are many other gauge theories which can be studied in terms of both methods.

In the next two sections we briefly review generalized fields and superfields formulations of solving the BV–master equation to obtain BV–master actions. Then we show how truncations of superfields result in generalized fields which lead to the BV–master actions of Yang–Mills theory and the self interacting antisymmetric tensor field in 4 dimensions. Truncation of superfields are also shown to be applicable to spinor fields and thus to on–shell N=1 supersymmetric Yang–Mills theory in 4 dimensions. The relations between these approaches in other dimensions is discussed briefly, considering the relativistic particle, Yang–Mills theory in 2 dimensions and a topological quantum field theory in 5 dimensions. Some other theories which we can apply both methods are also mentioned. In the last section we discuss the results obtained.

## 2 Generalized fields method

To obtain BV formalism of a gauge theory, one introduces ghost and ghost of ghost fields if necessary inspecting properties of gauge transformations. Then, an antifield is assigned to each field[1]. These fields can be grouped together by extending the exterior derivative d to include the BRST transformation  $\delta_B$  as[14]

$$\tilde{d} \equiv d + \delta_B,\tag{1}$$

which is defined to be nilpotent:  $\tilde{d}^2 = 0$ . Thus, we can gather the differential form degree  $N_d$  and the ghost number  $N_g$  as the total degree

$$\mathcal{N} \equiv N_d + N_g. \tag{2}$$

We deal with systems whose actions can be put into the first order form,

$$S_0(A,B) = BdA + V(A,B)$$
(3)

and invariant under the gauge transformations

$$\delta^{(0)}(A,B) = R^{(0)}(A,B)\Lambda,\tag{4}$$

where  $\Lambda$  is gauge parameter. We suppress integration over space—time variables and all indices.

The initial fields, minimal set of ghost fields and their antifields can be used to define generalized fields by grouping them in terms of the general grading given by  $\tilde{d}$  as  $\tilde{A}$  and  $\tilde{B}$  satisfying

$$\mathcal{N}(\tilde{A}) = N_d(A) \; ; \; \mathcal{N}(\tilde{B}) = N_d(B). \tag{5}$$

Now, substitute the original fields A and B with the generalized ones  $\tilde{A}$  and  $\tilde{B}$  in the action (3):

$$S \equiv S_0(\tilde{A}, \tilde{B}) = \tilde{B}d\tilde{A} + V(\tilde{A}, \tilde{B}). \tag{6}$$

Multiplication is defined such that S is a zero ghost number scalar functional. The action (6) is invariant under the transformations

$$\delta_{\tilde{\Lambda}}(\tilde{A}, \tilde{B}) = \tilde{R}\tilde{\Lambda},\tag{7}$$

where the generators are

$$\tilde{R} \equiv R^{(0)}(\tilde{A}, \tilde{B}) \tag{8}$$

and  $\tilde{\Lambda}$  is an appropriate generalization of the original gauge parameter  $\Lambda$ . If (7) can be written as

$$\begin{pmatrix} \delta_{\tilde{\Lambda}} \tilde{A}_{i} \\ \delta_{\tilde{\Lambda}} \tilde{B}_{i} \end{pmatrix} = \begin{pmatrix} -\frac{\delta_{l} \delta_{r} S}{\delta \tilde{B}_{i} \delta \tilde{A}_{j}} & -\frac{\delta_{l} \delta_{r} S}{\delta \tilde{B}_{i} \delta \tilde{B}_{j}} \\ \frac{\delta_{l} \delta_{r} S}{\delta \tilde{A}_{i} \delta \tilde{A}_{j}} & \frac{\delta_{l} \delta_{r} S}{\delta \tilde{A}_{i} \delta \tilde{B}_{j}} \end{pmatrix} \begin{pmatrix} \tilde{\Lambda}_{1}^{j} \\ \tilde{\Lambda}_{2}^{j} \end{pmatrix}, \tag{9}$$

where  $\delta_r$  and  $\delta_l$  denote right and left functional derivatives, S given by (6) satisfies

$$(S,S) = 2\frac{\delta_r S}{\delta \tilde{B}_i} \frac{\delta_l S}{\delta \tilde{A}_i} = k$$

where k is a constant. We deal only with k = 0, otherwise it leads to non-consistency of equations of motion.

Here we consider the theories which can be written in the form

$$S = \tilde{B}d\tilde{A} + \alpha \tilde{B}\tilde{B} + \beta \tilde{A}\tilde{A} + \gamma \tilde{A}\tilde{A}\tilde{B}, \tag{10}$$

where either  $\alpha = 0$  or  $\beta = 0$  and the other constant  $\gamma$  is dictated by the original theory.

The transformations

$$\delta_B \tilde{A} = \frac{\delta_l S}{\delta \tilde{B}}, \ \delta_B \tilde{B} = -\frac{\delta_l S}{\delta \tilde{A}}, \tag{11}$$

can be written in terms of the covariant derivative  $\tilde{D}=d+\tilde{A}$  and the related curvature  $\tilde{F}$  when  $\beta=0$  as

$$\delta_B \tilde{A} = \tilde{F} - \tilde{B}, \ \delta_B \tilde{B} = -\tilde{D}\tilde{B} \tag{12}$$

and when  $\alpha = 0$  as

$$\delta_B \tilde{A} = \tilde{F}, \ \delta_B \tilde{B} = -\tilde{D}\tilde{B} + \tilde{A}. \tag{13}$$

The components of the right hand sides are restricted to possess the same form degree and one more ghost number of the components of the left hand sides. If these transformations are nilpotent:

$$\delta_B^2 \tilde{A} = 0, \ \delta_B^2 \tilde{B} = 0, \tag{14}$$

one can conclude that the BV-master equation is satisfied:

$$(S,S) = 0 (15)$$

In both of the cases (12), (13) it is shown that (14) are satisfied[5] due to the Bianchi identities  $\tilde{D} \cdot \tilde{F} = 0$  and the definition of the curvature  $\tilde{F} = \tilde{D} \cdot \tilde{D}$ . When  $\tilde{A} = \tilde{B}$  (Chern-Simons type) we have  $\alpha = 0$  and  $\beta = 0$  in (10) and  $\delta_B \tilde{A} = \tilde{F}$ , so that  $\delta_B^2 \tilde{A} = 0$  follows from the Bianchi identities.

This method is applied to some usual gauge theories[5], shown to be applicable to a generalized version of Chern–Simons theory and BRST field theory[6]. It is reviewed in [15] as a pedagogical approach to BV–method. Moreover, it gives an efficient formulation of consistent interactions[16].

# 3 Superfields method

Let us deal with a superspace with n commuting and n anticommuting variables:  $x_{\mu}, \tau_{\mu}; \mu = 1, 2, \dots, n$ , and the action

$$\Sigma[U^{(n)}, V^{(n)}] = \int d^n x \ d^n \tau \ \left( (-1)^{\epsilon_U} V^{(n)}(x, \tau) D U^{(n)}(x, \tau) - \mathcal{V}(U^{(n)}, V^{(n)}) \right). \tag{16}$$

Summation index is suppressed. The fields V and U possess, respectively, Grassmann parity  $\epsilon_U + 1$  and  $\epsilon_U$  in even dimensions and the same Grassmann parity  $\epsilon_U$  for odd dimensions[10]. D is the Grassmann odd, nilpotent differential operator

$$D = \tau^{\mu} \frac{\partial}{\partial x^{\mu}}.$$
 (17)

Antibracket of arbitrary superfields X, Y is defined as

$$(X,Y) = \frac{\delta_r X}{\delta V^{(n)}} \frac{\delta_l Y}{\delta U^{(n)}} - \frac{\delta_r X}{\delta U^{(n)}} \frac{\delta_l Y}{\delta V^{(n)}}.$$

Boundary conditions and  $\mathcal{V}$  are chosen such that the BV-master equation is satisfied

$$(\Sigma, \Sigma) = 0 \tag{18}$$

at the classical level.

 $\tau$  dependence of any field  $X_A$  can explicitly be written in terms of the components  $X_k^{\mu_1\cdots\mu_k}(x)$  as

$$X(x,\tau) = \sum_{k=0}^{n} \tau_{\mu_1} \cdots \tau_{\mu_k} X_k^{\mu_1 \cdots \mu_k}(x).$$
 (19)

The unique non-vanishing integral on Grassmann variables is normalized as

$$\int d^n \tau \tau^{\mu_1} \cdots \tau^{\mu_n} = \epsilon^{\mu_1 \cdots \mu_n},$$

in terms of the totally antisymmetric tensor in n-dimensions  $e^{\mu_1 \cdots \mu_n}$ .

Grassmann odd coordinates  $\tau^{\mu}$  are defined to possess ghost number:

$$N_g(\tau_\mu) = 1. (20)$$

One demands that

$$N_g(\Sigma) = 0. (21)$$

Thus, the superfields  $U^{(n)}$ ,  $V^{(n)}$  should satisfy

$$N_a(U^{(n)}) + N_a(V^{(n)}) = n - 1. (22)$$

Obviously, the components (19) possess

$$N_g(X_k^{\mu_1\cdots\mu_k}) = N_g(X) - k.$$
 (23)

Without implementing any restriction one can deal with the cases

$$N_g(V) \ge N_g(U). \tag{24}$$

Classical gauge theory which leads to the BV-master action  $\Sigma$  can be derived from it by setting to zero the fields possessing nonzero ghost number. Components of a superfield which is defined to have negative ghost number cannot contain vanishing ghost number. Their components related to the underlying gauge theory can only be antifields of some Lagrange multipliers of the underlying classical action. Thus, as far as we do not deal with gauge theories containing Lagrange multipliers it is sufficient to consider only positive ghost number superfields.

# 4 The relations between generalized fields and superfields methods

The action in terms of generalized fields (6) as well as the one written by super-fields (16) are first order in space—time derivatives and both of them are defined to satisfy the BV—master equation. However, their field contents can be different. We will illustrate the relations between these formalisms by focusing on some examples which are chosen because they represent some basic gauge systems treated by these methods.

(16) is mainly used in topological quantum field theories for which it seems that in general field contents of the both formalisms coincide, although we demonstrate it for some specific cases. However, when we consider the usual gauge systems like Yang-Mills theory, their field contents in general do not coincide. Nevertheless, by truncating some components of superfields consistently, such that remaining ones still lead to a proper solution of the BV-master equation, one obtains the generalized fields. Truncation is performed by keeping some positive ghost number components of fields and setting the others equal to zero. Obviously consistent truncations depend on the chosen  $\mathcal{V}$ . This may give the impression that a general receipt for consistent truncations is missing. However, this is not the case: Superfields contain all possible ghost number fields available in the dimension which one considers. Moreover, action of the superfield formalism is appropriately chosen which coincides with the form of action of the generalized fields formalism. Thus, consistent truncation is to demand that some components of superfields possessing ghost number different from zero are defined to be vanishing such that related antifields are also vanishing.

To discuss actions like (10) we need to introduce dual of a superfield. In general  $\Phi_D$  dual of a superfield  $\Phi$  in n dimensions is defined to satisfy

$$N_g(\Phi_D) = n - N_g(\Phi). \tag{25}$$

The components of  $\Phi_D$  should be chosen such that they are related to components of  $\Phi$  with the correct ghost number attribution and they may also be Hodge duals of them. The components which cannot fulfill these conditions should be eliminated by setting to zero as it will be clarified in examples.

We first study some examples in 4 dimensions which reveal the main properties of the truncation of superfields. We also present a formulation of on–shell N=1 supersymmetric Yang–Mills theory by superfields, inspired by the observed relations. Then we will discuss some other dimensions which clarify the relation of the

generalized fields and superfields methods for the usual gauge theories and also for topological quantum field theories.

### 4.1 Examples in 4 dimensions

In 4 dimensions the superfields  $U^{(4)}$  and  $V^{(4)}$  should satisfy

$$N_g(U_A^{(4)}) + N_g(V_A^{(4)}) = 3. (26)$$

Therefore, considering only positive ghost numbers and the condition (24), there are two possible choices:

I. 
$$N_g(U_I^{(4)}) = 1, N_g(V_I^{(4)}) = 2,$$
 (27)

II. 
$$N_g(U_{II}^{(4)}) = 0, N_g(V_{II}^{(4)}) = 3.$$
 (28)

Moreover, in each case one should choose one of the superfields to be Grassmann even and the other to be Grassmann odd. Though all fields are taking values in a Lie algebra, we suppress trace over group elements.

Let us deal with case I calling  $U_I$  and  $V_I$  as a and b which can be written in terms of components depending only on x as

$$U_I^{(4)} \equiv a(x,\tau) = a_0 + \tau_\mu a_1^\mu + \tau_\mu \tau_\nu a_2^{\mu\nu} + \tau_\mu \tau_\nu \tau_\rho a_3^{\mu\nu\rho} + \tau_\mu \tau_\nu \tau_\rho \tau_\sigma a_4^{\mu\nu\rho\sigma}, \quad (29)$$

$$V_I^{(4)} \equiv b(x,\tau) = b_0 + \tau_\mu b_1^\mu + \tau_\mu \tau_\nu b_2^{\mu\nu} + \tau_\mu \tau_\nu \tau_\rho b_3^{\mu\nu\rho} + \tau_\mu \tau_\nu \tau_\rho \tau_\sigma b_4^{\mu\nu\rho\sigma}. \tag{30}$$

Ghost number and Grassmann parity of the components are

The action is

$$S(a,b) = -\int d^4x d^4\tau \left(bDa + \mathcal{V}(a,b)\right). \tag{31}$$

Analogous to the form of the action in terms of generalized fields (10) we would like to consider for V the following choices:

$$i. \quad \mathcal{V}_1 = \frac{1}{2}b_D b + \frac{1}{2}b[a, a],$$
 (32)

*ii.* 
$$V_2 = \frac{1}{2}a_D a + \frac{1}{2}b[a, a],$$
 (33)

where  $a_D$  and  $b_D$  are duals of a and b satisfying  $N_g(a_D) = 3$ ,  $N_g(b_D) = 2$ . Commutator denotes antisymmetrization of the Lie algebra generators.

When we deal with  $V_1$  a consistent truncation is to keep  $N_g = 1$  component of a and set  $b_0 = b_1 = 0$ . Thus antifields of them should also be taken as  $a_3 = a_4 = 0$ . Now, by renaming the field components we write the truncated superfields

$$a_{t1} = \eta + \tau^{\mu} A_{\mu} + \frac{1}{2} \tau^{\mu} \tau^{\nu} B_{\mu\nu}^{\star},$$
 (34)

$$b_{t1} = \frac{1}{4} \tau^{\mu} \tau^{\nu} \epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma} + \frac{1}{3!} \tau^{\mu} \tau^{\nu} \tau^{\rho} \epsilon_{\mu\nu\rho\sigma} A^{\star\sigma} + \frac{1}{4!} \tau^{\mu} \tau^{\nu} \tau^{\rho} \tau^{\sigma} \epsilon_{\mu\nu\rho\sigma} \eta^{\star}, \tag{35}$$

where as usual  $\star$  indicates antifields. The dual superfield can be written as

$$b_{t1D} = \frac{1}{2} \tau^{\mu} \tau^{\nu} B_{\mu\nu} + \frac{1}{3!} \tau^{\mu} \tau^{\nu} \tau^{\rho} \epsilon_{\mu\nu\rho\sigma} A^{\star\sigma} + \frac{1}{4!} \tau^{\mu} \tau^{\nu} \tau^{\rho} \tau^{\sigma} \epsilon_{\mu\nu\rho\sigma} \eta^{\star}. \tag{36}$$

Substituting the superfields  $a, b, b_D$  with the truncated fields (34–36) in (31) with  $V_1$  one obtains

$$S_{4,YM} \equiv -\int d^4x d^4\tau [b_{t1}Da_{t1} + \mathcal{V}_1(a_{t1}, b_{t1})] = -\int d^4x \left(\frac{1}{2}B_{\mu\nu}F^{\mu\nu}\right) -B^{\mu\nu}\eta B^{\star}_{\mu\nu} + A^{\star}_{\mu}D^{\mu}\eta + \frac{1}{2}\eta^{\star}[\eta, \eta] - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\right),$$
(37)

where  $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + [A_{\mu}, A_{\nu}]$  and  $D_{\mu}\eta = \partial_{\mu}\eta + [A_{\mu}, \eta]$ .

We may perform a partial gauge fixing  $B^* = 0$  and then use the equations of motion with respect to  $B_{\mu\nu}$  to obtain

$$S_{4,YM} = -\int d^4x \; (\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A^{\star}_{\mu} D^{\mu} \eta + \frac{1}{2} \eta^{\star} [\eta, \eta]),$$

which is the minimal solution of the master equation for Yang-Mills theory. Indeed, the components of the fields (34) and (35) are the same with the components of the generalized fields given in [5].

For  $V_2$  there is another consistent truncation:  $a_0 = 0$  and its antifield  $b_4 = 0$ . We rename the components to write the truncated superfields as

$$a_{t2} = \tau^{\mu} A_{\mu} + \frac{1}{2} \tau^{\mu} \tau^{\nu} B_{\mu\nu}^{\star} + \frac{1}{3!} \tau^{\mu} \tau^{\nu} \tau^{\rho} \epsilon_{\mu\nu\rho\sigma} C_{0}^{\star\sigma} + \frac{1}{4!} \tau^{\mu} \tau^{\nu} \tau^{\rho} \tau^{\sigma} \epsilon_{\mu\nu\rho\sigma} C_{1}^{\star}, \quad (38)$$

$$b_{t2} = C_1 + \tau^{\mu} C_{0\mu} + \frac{1}{4} \tau^{\mu} \tau^{\nu} \epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma} + \frac{1}{3!} \tau^{\mu} \tau^{\nu} \tau^{\rho} \epsilon_{\mu\nu\rho\sigma} A^{\star\sigma}. \tag{39}$$

In general the dual superfield  $a_D$  possesses components satisfying

Components of  $a_D$  should be chosen such that they are related to components of a or their Hodge duals with correct ghost number. The components which cannot fulfill these conditions should be defined to vanish. Thus, for the truncation (38) the only non-vanishing component is  $a_{t2D3}$ . Indeed, we define

$$a_{t2D} = \frac{1}{3!} \tau^{\mu} \tau^{\nu} \tau^{\rho} \epsilon_{\mu\nu\rho\sigma} A^{\sigma}.$$

Thus, one obtains

$$S_{4,AS} \equiv -\int d^4x d^4\tau [b_{t2}Da_{t2} + \mathcal{V}_2(a_{t2}, b_{t2})] = -\int d^4x \left\{ \frac{1}{2} B_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} C_0^{\mu} D^{\nu} B^{\star\rho\sigma} + C_1 D^{\mu} C_{0\mu}^{\star} + \epsilon^{\mu\nu\rho\sigma} C_1 B_{\mu\nu}^{\star} B_{\rho\sigma}^{\star} - \frac{1}{4} A_{\mu} A^{\mu} \right\}.$$
(40)

This is the BV-master action of the self dual antisymmetric tensor field which was obtained in terms of the generalized fields whose components coincide with the components of the superfields (38, 39).

Let us deal with case II (28). Let us denote the fields  $U_{II}$  and  $V_{II}$  as  $\Psi^{\alpha}$  and  $\bar{\Psi}_{\alpha}$ , where  $\alpha$  is an index which will be specified below and supposed to be lowered with an appropriate metric. Ghost numbers of the components of  $\Psi$  and  $\bar{\Psi}$  follow from the definition (28) and their Grassmann parity are chosen as

A consistent truncation is to set all the ghost fields (positive ghost number carrying fields) equal to zero, i.e. no gauge invariance

$$\Psi_{t\alpha} = \Psi_{0\alpha} + \tau_{\mu} \Psi^{\mu}_{1\alpha}, \tag{41}$$

$$\bar{\Psi}_{t\alpha} = \tau_{\mu}\tau_{\nu}\tau_{\rho}\bar{\Psi}_{3\alpha}^{\mu\nu\rho} + \tau_{\mu}\tau_{\nu}\tau_{\rho}\tau_{\sigma}\bar{\Psi}_{4\alpha}^{\mu\nu\rho\sigma}. \tag{42}$$

We define

$$\bar{\Psi}_{3\alpha}^{\mu\nu\rho} = \frac{1}{3!} \epsilon^{\mu\nu\rho\sigma} \bar{\Psi}_{\alpha\sigma} \tag{43}$$

$$\bar{\Psi}_{4\alpha}^{\mu\nu\rho\sigma} = \frac{1}{4!} \epsilon^{\mu\nu\rho\sigma} \Psi_{0\alpha}^{\star}, \tag{44}$$

$$\Psi_{1\alpha}^{\mu} = \bar{\Psi}_{\alpha}^{\star\mu}, \tag{45}$$

where star indicates antifields as usual. By substituting the superfields (28) with the truncated superfields (41, 42) in the action (16) one finds its kinetic part as

$$S_{\Psi} = \int d^4x d^4\tau \bar{\Psi}_{t\alpha} D\Psi_t^{\alpha} = \int d^4x \bar{\Psi}_{\alpha\mu} \partial^{\mu} \Psi_0^{\alpha}. \tag{46}$$

Now, let  $\alpha$  be the spinor index and

$$\bar{\Psi}^{\mu}_{\alpha} = \frac{1}{2} \bar{\Psi}_{0\beta} \gamma^{\mu\beta}_{\alpha}, \Psi^{\alpha}_{1\mu} = \gamma^{\alpha}_{\mu\beta} \bar{\Psi}^{\star\beta}_{0}. \tag{47}$$

Thus, the action (46) becomes

$$S_{\Psi} = -\frac{1}{2} \int d^4x \bar{\Psi}_{0\alpha} \gamma^{\alpha}_{\mu\beta} \partial^{\mu} \Psi^{\beta}_{0}. \tag{48}$$

Obviously, one can also deal with a theory which is a mixture of case I and II. Thus, we can couple this theory to the Yang-Mills theory by

$$S_{int} \equiv \int d^4x d^4\bar{\Psi}_{t\alpha}[a_{t1}, \Psi_t^{\alpha}] = -\int d^4x \left[ \frac{1}{2} \bar{\Psi}_0 \gamma^{\mu} [A_{\mu}, \Psi_0] - [\bar{\Psi}_0, \eta] \bar{\Psi}_0^{\star} - \Psi_0^{\star} [\eta, \Psi_0] \right].$$

Thus the 4–dimensional N=1 on–shell supersymmetric Yang–Mills theory can be written in terms of the superfields as

$$S_{SYM} = S_{4YM} + S_{\Psi} + S_{int}. \tag{49}$$

Supersymmetric case was not discussed in terms of generalized fields method. Here it follows as a natural consequence of truncating superfields to obtain generalized fields.

Truncations which we studied do not exhaust all of the possibilities. We studied examples which are illustrating the procedure of truncation for some basic gauge theories.

## 4.2 Examples in other dimensions

Equipped with the detailed knowledge of how generalized fields and superfields approaches are related for 4–dimensional theories we can discuss their relation in other dimensions without focusing on the details.

In one dimension we have shown that generalized fields method leads to BV-formulation of first order Lagrangians for some constrained systems[5]. By using the

formulation given in [17] for phase space variables one can easily observe that both methods lead to the same conclusions. To clarify it let us deal with the superfields

$$Q_{\mu} = q_{\mu} + \tau q_{1\mu}, \quad P_{\mu} = p_{\mu} + \tau p_{1\mu}, \quad \epsilon(Q) = \epsilon(P) = 0;$$
  
 $E = e_0 + \tau e_1, \quad F = f_0 + \tau f_1, \quad \epsilon(E) = \epsilon(F) = 1,$ 

where  $\mu = 1, \dots, d$ . These will be shown to be suitable for the relativistic particle. Let us attribute the ghost numbers

$$N_q(Q) = 0, \quad N_q(P) = 0,$$
 (50)

$$N_g(E) = 1, \quad N_g(F) = -1.$$
 (51)

We permit a negative ghost number because we need a Lagrange multiplier. Let us deal with the action

$$S_{p} = \int dt d\tau [P_{\mu}DQ^{\mu} + EDF - \frac{1}{2}EP_{\mu}P^{\mu}]. \tag{52}$$

By renaming the components as

$$q_{1\mu} = -p_{\mu}^{\star}, \ p_{1\mu} = q_{\mu}^{\star}, \ e_0 = \eta, \ e_1 = e, \ f_0 = e^{\star}, \ f_1 = \eta^{\star},$$

where  $\eta$  is ghost field and as usual star indicates antifields, the action (52) reads

$$S = \int dt \ [p \cdot \frac{\partial q}{\partial t} + e^* \frac{\partial \eta}{\partial t} - \frac{1}{2} e p^2 + q^* \cdot p \eta],$$

which is the minimal solution of the BV–master equation for the relativistic particle. Let us clarify relation between the two approaches. In generalized fields method the fields are grouped as

$$\tilde{q} = q_{(1+0+0)}^{\mu} + e_{(0+1+0)} + \eta_{(0+0+1)} - p_{(1+1-1)}^{\star \mu}, \tag{53}$$

$$\tilde{p} = p_{\mu(d-1+0+0)} + q_{\mu(d-1+1-1)}^{\star} + e_{(d+0-1)}^{\star} + \eta_{(d+1-2)}^{\star},$$
 (54)

where the numbers in the parenthesis indicate, respectively, grading due to spacetime, grading due to 1–dimensional manifold and ghost number. The relevant action is

$$S_p = \int dt \left[ \tilde{p} \frac{\partial \tilde{q}}{\partial t} + \frac{1}{2} \tilde{q} \tilde{p} \tilde{p} \right]. \tag{55}$$

Here multiplication is defined such that all three of the gradings of the action  $S_p$  vanish. This property leads to the fact that indeed the generalized fields (53, 54)

can be imagined as composed of two objects carrying different indices. Thus, the field contents of both methods are the same.

In 2 dimensions generally one deals with theories where there is no need of any truncation to obtain the similar results in either generalized or superfield formalism. The superfields  $U^{(2)}$  and  $V^{(2)}$  can only have the ghost numbers  $N_g(V^{(2)}) = 0$ ,  $N_g(U^{(2)}) = 1$ , when we restrict the ghost number to be positive. We choose  $U^{(2)}$  to be Grassmann odd and  $V^{(2)}$  to be Grassmann even. Now, each superfield has three components  $^1$ :

$$U^{(2)} \equiv u = u_0 + \tau_\mu u_1^\mu + \tau_\mu \tau_\nu u_2^{\mu\nu}, \tag{56}$$

$$V^{(2)} \equiv v = v_0 + \tau_\mu v_1^\mu + \tau_\mu \tau_\nu v_2^{\mu\nu}. \tag{57}$$

We deal with the action

$$S_{(2)} = -\int d^2x d^2\tau \ (vDu + \frac{1}{2}v_Dv + \frac{1}{2}v[u, u]), \tag{58}$$

where due to ghost number constraint  $N_g(v_D) = 2$ , two of the components of  $v_D$  should vanish and we write

$$v_D = \tau_\mu \tau_\nu \epsilon^{\mu\nu} v_0. \tag{59}$$

Observe that one obtains the BV-master action of 2-dimensional Yang-Mills theory in first order formalism when we use (56, 57, 59) in the action  $S_{(2)}$ . In generalized fields method one has the same field content with the action in the form of (58).

When ghost numbers of superfields are restricted to be positive, in 3 dimensions we have two possibilities:

1. 
$$N_g(U_1^{(3)}) = 1, N_g(V_1^{(3)}) = 1,$$
 (60)

2. 
$$N_q(U_2^{(3)}) = 0, N_q(V_2^{(3)}) = 2.$$
 (61)

Let us briefly discuss these cases. In case 1 we can take both of the fields to be the same and Grassmann odd:  $U_c$ . By taking the action

$$S_c = \int d^3x d^3\tau (U_c D U_c + \frac{1}{3} U_c [U_c, U_c])$$

the BV-master action of Chern-Simons theory follows[10]. In fact generalized fields method yields the same field content and the same form of the action[6]. However,

<sup>&</sup>lt;sup>1</sup>Here and in the following fields are Lie algebra valued but we suppress traces.

we can take two different fields  $U_1$  and  $V_1$  possessing the same ghost number. In this case with the appropriate action

$$S_{3YM} = \int d^3x d^3\tau (V_1 D U_1 + \frac{1}{2} V_{1D} V_1 + \frac{1}{2} V_1 [U_1, U_1]),$$

3-dimensional Yang-Mills theory can be found by truncating the fields in accordance with 4-dimensional formalism. Case 2 can also be treated similar to 4-dimensional case in terms of spinor fields with an appropriate truncation and action.

In 5 dimensions we have the choices:

$$N_g(V^{(5)}) \quad N_g(U^{(5)}) \ (i). \qquad 2 \qquad 2 \ (ii). \qquad 3 \qquad 1 \ (iii). \qquad 4 \qquad 0$$

for positive ghost numbers. Let us discuss some possibilities. Case (i) can be discussed by setting  $U_1^{(5)} \equiv V_1^{(5)} = U$  and taking the action

$$S_1 = \int d^5x d^5\tau (UDU + \frac{1}{3}U[U, U]).$$

An example to this case is studied below. Case (ii) with the action

$$S_2 = \int d^5x d^5\tau \left( V_2^{(5)} D U_2^{(5)} + \frac{1}{2} V_{2D}^{(5)} V_2^{(5)} + \frac{1}{2} V_2^{(5)} [U_2^{(5)}, U_2^{(5)}] \right)$$

yields Yang–Mills theory in 5 dimensions after an appropriate truncation of ghost fields according to its general fields formulation. Similarly case (iii) can be written in terms of spinor fields with appropriate action and truncation of superfields. Obviously, by some other choices of actions and truncations one can obtain different theories.

We would like to discuss a theory which elucidates the essential points of the relation between the two methods for topological quantum field theories which is also an example to case (i). There exists a generalized Chern–Simons theory[18] in any odd dimension d = 2n + 1, which was considered in terms of generalized fields method[6] yielding the BV–master action

$$S_d = \frac{1}{2} \int_{M_d} \left( \tilde{A} d\tilde{A} + \frac{2}{3} \tilde{A}^3 \right), \tag{62}$$

where  $\tilde{A} = \tilde{\phi} + \tilde{\psi}$  defined as<sup>2</sup>

$$\tilde{\phi} = \sum_{i=0}^{n-1} \left[ \phi_{(2i+1,0)} + \sum_{j=1}^{2i+1} \eta_{(2i+1-j,j)} + \phi_{(2i+2,-1)}^* + \sum_{j=-2n+2i}^{-2} \eta_{(2i+1-j,j)}^* \right]$$

$$\tilde{\psi} = \sum_{i=0}^{n} \psi_{(2i,0)} + \sum_{i=1}^{n} \sum_{j=1}^{2i} \kappa_{(2i-j,j)} + \sum_{i=0}^{n} \psi_{(2i+1,-1)}^* + \sum_{i=0}^{n-1} \sum_{j=-2n+2i-1}^{-2} \kappa_{(2i-j,j)}^*.$$
(63)

The antifield of the field  $a_{(k,l)}$  is defined as  $a_{(2n+1-k,-l-1)}^*$ . In terms of  $\tilde{\phi}$  and  $\tilde{\psi}$  one can write (62) as

$$S_d = \frac{1}{2} \int_{\mathcal{M}_d} \left( \tilde{\phi} d\tilde{\phi} + \frac{1}{3} \tilde{\phi} [\tilde{\phi}, \tilde{\phi}] + \tilde{\psi} d\tilde{\psi} + \tilde{\psi} [\tilde{\phi}, \tilde{\psi}] \right). \tag{64}$$

Let us discuss this theory in 5 dimensions to illustrate how generalized fields and superfields are related for topological quantum field theories. In terms of superfields whose ghost numbers and Grassmann parities are

one can write the action

$$S_{s5} = \int_{M_5} d^5 \tau [\Phi_1 D \Phi_3 + \Psi_0 D \Psi_4 + \frac{1}{2} \Psi_2 D \Psi_2 + \frac{1}{3} \Phi_3 [\Phi_1, \Phi_1] + \Psi_0 [\Phi_1, \Psi_4]$$

$$+ \Psi_0 [\Phi_3, \Psi_2] + \frac{1}{2} \Psi_2 [\Phi_1, \Psi_2]], \qquad (65)$$

Components of these superfields and the components of generalized fields (63) are in one to one correspondence. Indeed, superfields can be written as

$$\Phi_1 = \eta_{(0,1)} + \tau \phi_{(1,0)} + \tau^2 \phi_{(2,-1)}^{\star} + \tau^3 \eta_{(3,-2)}^{\star} + \tau^4 \eta_{(4,-3)}^{\star} + \tau^5 \eta_{(5,-4)}^{\star}, \tag{66}$$

$$\Phi_3 = \eta_{(0,3)} + \tau \eta_{(1,2)} + \tau^2 \eta_{(2,1)} + \tau^3 \phi_{(3,0)} + \tau^4 \phi_{(4,-1)}^{\star} + \tau^5 \eta_{(5,-2)}^{\star}, \tag{67}$$

$$\Psi_0 = \psi_{(0,0)} + \tau^1 \psi_{(1,-1)}^{\star} + \tau^2 \kappa_{(2,-2)}^{\star} + \tau^3 \kappa_{(3,-3)}^{\star} + \tau^4 \kappa_{(4,-4)}^{\star} + \tau^5 \kappa_{(5,-5)}^{\star}, \quad (68)$$

$$\Psi_2 = \kappa_{(0,2)} + \tau^1 \kappa_{(1,1)} + \tau^2 \psi_{(2,0)} + \tau^3 \psi_{(3,-1)}^{\star} + \tau^4 \kappa_{(4,-2)}^{\star} + \tau^5 \kappa_{(5,-3)}^{\star}, \tag{69}$$

$$\Psi_4 = \kappa_{(0,4)} + \tau^1 \kappa_{(1,3)} + \tau^2 \kappa_{(2,2)} + \tau^3 \kappa_{(3,1)} + \tau^4 \psi_{(4,0)} + \tau^5 \psi_{(5,-1)}^{\star}, \tag{70}$$

<sup>&</sup>lt;sup>2</sup>There are some typos in [6] whose corrected versions are given in (63).

where a shorthand notation is used for the odd coordinates  $\tau^i \equiv \tau_{\mu_1} \cdots \tau_{\mu_i}$ . One can observe that by these definitions (64) in 5 dimensions and (65) coincide:

$$S_5 = S_{s5}$$
.

Higher dimensions can be studied in a similar manner.

### 5 Discussions

Generalized fields and superfields formulation of finding solutions of the BV-master equation are shown to be related by consistent truncations of the latter for the usual gauge theories. For topological quantum field theories they yield the same field contents without any truncation. Though the latter relation is shown for specific cases, we believe that it is a general conclusion. Generalized fields method is based on a classical gauge theory action. However, superfields approach begin with a BV-master action from which one can read the underlying classical gauge theory by eliminating ghost variables and antifields. Having a connection between these theories we can formulate different kind of gauge theories in terms of superfields as it is illustrated for supersymmetric Yang-Mills theories. Moreover, this relation can give some hints to discover some other theories which can be discussed either in terms of generalized fields or superfield algorithms and suitable to describe some physical systems.

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