Quantum mechanics in general quantum systems (III):
open system dynamics

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We investigate the exact solution, perturbation theory and master equation of open system dynamics based on our serial studies on quantum mechanics in general quantum systems [An Min Wang, quant-ph/0611216, quant-ph/0611217]. In a system-environment separated representation, a general and explicit solution of open system dynamics is obtained, and it is an exact solution since it includes all order approximations of perturbation. In terms of the cut-off approximation of perturbation and our improved scheme of perturbation theory, the improved form of the perturbed solution of open systems absorbing the partial contributions from the high order even all order approximations is deduced. Moreover, only under the factorizing initial condition, the exact master equation including all order approximations is proposed. Correspondingly, the perturbed master equation and its improved form different from the existed master equation are given. In special, the Redfield master equation is derived out without using Born-Markov approximation. The solution of open system dynamics in the Milburn model is also gained. As examples, Zurek model of two-state open system and its extension with two transverse fields are studied.

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I. INTRODUCTION

A realistic quantum system is never isolated, but is immersed in the surrounding environment (alias bath, reservoir) and interacts continuously with it. Such a system without ignorable coupling to the environment can be called open quantum system. Generally, the environment consists of a huge number of degrees of freedom, it is even whole outside world (universe) of the concerning open quantum system. In fact, we might not know the exact state of the outside world, having only some statistical information to describe it. However, we are really interested in a reliable and effective theory of open system dynamics under the influence of its environment.

The basic idea of quantum theory of open systems is thought of as an interesting open system and its surrounding environment form a total composite system, or vis versa, a composite system can be decomposed into an interesting open system and a surrounding environment. The key matters of quantum theory of open systems are to determine the interaction between the open system and its environment and build the physical models of the open system and its environment. Open system dynamics is just a law of this open quantum system how to evolute with time and its solution at any given time.

Open quantum system and its dynamics are very important for many interesting quantum theory branches such as quantum optics [1, 2], condensed matter theory, quantum information and computing [3, 4], more concretely, quantum decoherence, quantum measurement [5, 6], quantum dissipation [7, 8], quantum transport [9], quantum chaos [10] et. al. Study of open system dynamics is helpful for understanding some very essential problems in physics, for example, the transition from quantum to classical world.

A variety of different formal techniques have been developed and used in dealing with open quantum systems. From the above reviews and books, the interested readers may get them. Here, we intend to start with "the first principle" of quantum mechanics, that is, the Schrödinger equation or the von Neumann equation, and then try to build a theoretical formulism including the general and explicit forms of motion equation, dynamical solution, and perturbation theory of open systems.

It is clear that such a "first principle" scheme might be not suitable to the cases when one cannot clearly know the environment model and/or the system-environment coupling form since the environment is too huge and too complicated. However, our conclusions might be helpful for building the models of such some systems. Moreover, one of possible ways to avoid this difficulty is to use the Milburn dynamics [11]. That is, the environment is separated into near- and remote two parts, the Hamiltonians of the near environment (often with finite degree of freedoms) and the coupling to the interesting open system are assumed to be clearly known, but the influence of the remote

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environment on the interesting system is incarnated by an extra term in Milburn motion equation compared with the von Neumann motion equation. Similarly, we successfully obtain the general and explicit solution of Milburn dynamics of the interesting system according our scheme.

Because of the dissipative nature of open systems, we must turn to the density matrix for a proper description whatever the initial state is pure or mixed. Actually, we are interested in the properties of open systems only, it will be appropriate to study the reduced density matrix evolution with time or motion equation or its solution. Here, the reduced density matrix describing the open systems is obtained by tracing out the degrees of freedom of the environment from the total (system plus environment) density matrix.

Due to the system and environment being entangled generally in system evolution with time, directly solving Schrödinger equation or von Neumann equation of the total system is a formidable task by using the existed methods. Traditionally, this problem is studied by perturbation theory in system-environment coupling scheme. Ones often take the interaction between the system and its environment as a perturbed part and then use the interaction picture to derive out the master equation of open systems via some physical approximations such as Born-Markov ones and the others. If an open system is exactly solvable, the coupling H_{SE} is weak, the evolution time is short enough, and the used physical approximations are indeed appropriate, this has been proved to be an effective method. However, when the above conditions do not satisfied sufficiently, the problem gets complicated and perhaps leads to some difficulties, although some formal techniques have been developed and used in order to overcome some possible shortcomings. For generality and reliability in theory, we feel that we have to consider whether these approximations are necessary, if without these approximations, can we obtain the formulism of open system dynamics? The conclusions obtained here answer these problems.

In this paper, we will provide the amelioration of the existed scheme of open system dynamics and try to build a theoretical formulism using our recent investigations on quantum mechanics in general quantum systems [12, 13]. We first obtain the exact solution of open systems including all order approximations of perturbation and then give the improved form of perturbed solution of open systems absorbing the partial contributions from the high order even all order approximations of perturbation. Only under the factorizing initial condition, we derive out the exact master equation and its perturbed form via the standard cut-off approximation of perturbation. Moreover, we propose the improved form of perturbed master equation. In special, based on our master equation, we re-deduce the Redfield master equation without using Born-Markov approximation, and we point out the differences between our master equation and existed ones. We also obtain the solution of open system dynamics in the Milburn model. In order to illustrate our open system dynamics, we study Zurek model of two-state open system and its extension with two transverse fields. We are sure that our open system dynamics can be used to more open systems since its generality and clearness, and its calculations are simpler and more efficient, its results are more accurate and more reliable than the existed scheme.

This paper is arranged as follows: besides Sec. I is an introduction, in Sec. II, by virtue of a system-environment separated representation, we first obtain a general and explicit solution of open systems including all order approximations; in Sec. III, we gain the improved form of perturbed solution of open systems, which absorbs the partial contributions from the high order even all order approximations of perturbation; in Sec. IV we deduce the exact master equation of open systems only under the factorizing initial condition; in Sec. V we get the perturbed form of our master equation and its amelioration; in Sec. VI, based on our master equation, we re-deduce Redfield master equation without using Born-Markov approximation, and we point out the differences between our master equation and existed ones; in Sec. VII, we obtain the solution of open system dynamics in the Milburn model for the Milburn-type closed total-systems. This implies that our solution and methods are applicable to more general open systems; in Sec. VIII, we study Zurek model of two-state open system and its extension with two transverse fields; In Sec. IX, we summarize our conclusions and give some discussions.

II. GENERAL AND EXPLICIT SOLUTION OF OPEN SYSTEM DYNAMICS

In this section, we will derive out a general and explicit solution of open systems by using our recent work of exact solution in general quantum systems [12].

As is well-known, if assuming that the interesting open quantum system and its environment are taken as a closed (or isolated) larger composite system, that is, a total system, we can think that this total system obeys the Schrödinger equation or the von Neumann equation, respectively, for a pure state $|\Psi_{tot}(t)\rangle$ or a mixed state $\rho_{tot}(t)$, that is

$$-\,\mathrm{i}\frac{\partial}{\partial t}|\Psi_{tot}(t)\rangle \;=\; H_{tot}|\Psi_{tot}(t)\rangle, \tag{1}$$

$$\dot{\rho}_{tot}(t) \;=\; -\mathrm{i}\left[H_{tot}, \rho_{tot}(t)\right]. \tag{2}$$

where the total system Hamiltonian H_{tot} that we consider here is made of the sum of the interesting open system

Hamiltonian H_S and its surrounding environment Hamiltonian H_E plus an interaction H_{SE} between the system and the environment, that is

$$H_{tot} = H_S + H_E + H_{SE}. \tag{3}$$

Note that the total system Hilbert space $\mathcal{H}_{tot}$ is defined by the direct product $\mathcal{H}_S \otimes \mathcal{H}_E$ of open system Hilbert space $\mathcal{H}_S$ and its environment Hilbert space $\mathcal{H}_E$. Here and in the following, we will discuss time-independent Hamiltonian and we have taken $\hbar = 1$ for simplicity.

In an open system dynamics, a key difficulty to lead to the problem becomes intractable is that there is the interaction between the open system and its environment with huge degree of freedom. With time evolution, the open system inevitably entangles with its environment. Therefore, we starts from a system-environment separated representation (SESR). This representation is beneficial for obtaining the general and explicit solution of open system dynamics as well as proposing the improved scheme of perturbed theory [13], because in the SESR we can conveniently trace off the degree of freedom of environment. Introducing the SESR is a simple and natural idea, and we will see it is also very useful. To this purpose, we first divide the H_{tot} into two parts

$$H_{tot} = H_{tot0} + H_{tot1}, \tag{4}$$

and, without loss of generality, we denote

$$H_{tot0} = H_{S0} + H_{E0} + H_{SE0}, \quad H_{tot1} = H_{S1} + H_{E1} + H_{SE1}. \tag{5}$$

It is clear that

$$H_{S0} = h_{S0} \otimes I_E, \quad H_{E0} = I_S \otimes h_{E0}, \tag{6}$$

while we need the coupling Hamiltonian with the following form

$$H_{SE0} = \sum_{m,n} c_{mn} S_{m0} \otimes B_{n0}. \tag{7}$$

It is general enough if we do not restrict the forms of S_{m0} and B_{n0}. In the above expressions, the total Hilbert space is $\mathcal{H}_{tot} = \mathcal{H}_S \otimes \mathcal{H}_E$, I_S and I_E are, respectively, the identity operators in $\mathcal{H}_S$ and $\mathcal{H}_E$, and c_{mn} are coupling constants between the open system and its environment. Note that H_{S0} and H_{E0} are always hermitian as usual. In addition, we need H_{SE0} be also necessarily hermitian. In fact, because H_{S0} and H_{E0} commute, the SESR always exists. The aim to add H_{SE0} is to obtain better precision and to simplify the perturbed part when passing to perturbation theory. It must be pointed out that the general principle to divide H into two parts is to let the terms as more as possible belong to H_0 but the precondition is that there exist the commuting relations:

$$[h_{S0}, S_{m0}], \quad [h_{E0}, \sum_n c_{mn} B_{n0}] = 0, \quad \text{or} \quad [h_{S0}, \sum_m c_{mn} S_{m0}], \quad [h_{E0}, B_{n0}] = 0. \tag{8}$$

Moreover, that the eigenvalue problem of H_{tot0} is solvable. In fact, this solvability implies that h_{S0} and h_{E0} are solvable, then h_{S0} and S_m, h_{E0} and $\sum_n c_{mn} B_n$ have the common eigenvectors, or h_{S0} and $\sum_m c_{mn} S_m$, h_{E0} and B_n have the common eigenvectors i.e, we have, respectively,

$$h_{S0} |\phi^\gamma\rangle = E_\gamma |\phi^\gamma\rangle, \quad S_{m0} |\phi^\gamma\rangle = s_{m\gamma} |\phi^\gamma\rangle, \quad h_{E0} |\chi_v\rangle = \varepsilon_v |\chi_v\rangle, \quad \sum_n c_{mn} B_{n0} |\chi_v\rangle = r_{mv} |\chi_v\rangle \tag{9}$$

and

$$h_{S0} |\phi^\gamma\rangle = E_\gamma |\phi^\gamma\rangle, \quad \sum_m c_{mn} S_{m0} |\phi^\gamma\rangle = s_{n\gamma} |\phi^\gamma\rangle, \quad h_{E0} |\chi_v\rangle = \varepsilon_v |\chi_v\rangle, \quad B_{n0} |\chi_v\rangle = r_{nv} |\chi_v\rangle. \tag{10}$$

They indicate that the eigenvectors of H_{tot0}, or the common eigenvectors of H_{S0}, H_{E0} and H_{SE0} are

$$|\Phi^{\gamma v}\rangle = |\phi^\gamma\rangle \otimes |\chi^v\rangle, \tag{11}$$

which span a separate representation of the system and the environment, and it is clear that

$$H_{tot0} |\Phi^{\gamma v}\rangle = E_{\gamma v} |\Phi^{\gamma v}\rangle, \tag{12}$$

$$E_{\gamma v} = E_\gamma + \varepsilon_v + \sum_m s_{m\gamma} r_{mv} \quad \text{or} \quad E_{\gamma v} = E_\gamma + \varepsilon_v + \sum_n s_{n\gamma} r_{nv}. \tag{13}$$

It must be emphasized that the principle of Hamiltonian split is not just the best solvability in more general cases. If the cut-off approximation of perturbation is necessary, it requires that the off-diagonal elements of the perturbing Hamiltonian H_{tot1} matrix in the SESR is small enough compared with the diagonal elements of $H_{tot} = H_{tot0} + H_{tot1}$ matrix in the same representation according to our improved scheme of perturbation theory. In addition, if there are the degeneracies, the Hamiltonian split is also restricted by the condition that the degeneracies can be completely removed via the usual diagonalization procedure of the degenerate subspaces and our Hamiltonian redivision, or specially, if the remained degeneracies are allowed, it requires that the off-diagonal elements of the perturbing Hamiltonian matrix between any two degenerate levels are always vanishing, in order to let our improved scheme of perturbation theory work well [13]. As an example, it has been studied in Sec. VIII.

From the formal solution of the von Neumann equation of the total system

$$\rho_{tot}(t) = \mathrm{e}^{-\mathrm{i}H_{tot}t}\rho_{tot}(0)\mathrm{e}^{\mathrm{i}H_{tot}t}, \tag{14}$$

and our expression of the time evolution operator [12]

$$\mathrm{e}^{-\mathrm{i}H_{tot}t} = \sum_{l=0}^\infty \mathcal{A}_l(t), \tag{15}$$

it immediately follows that the solution of total system density matrix with time evolution is

$$\rho_{tot}(t) = \sum_{k,l=0}^\infty \mathcal{A}_k(t)\rho_{tot}(0)\mathcal{A}_l(-t) = \sum_{k,l=0}^\infty \mathcal{A}_k(t)\rho_{tot}(0)\mathcal{A}_l^\dagger(t). \tag{16}$$

In the SESR, we have

$$\rho_{tot}(t) = \sum_{\beta,u,\beta',u'} \sum_{\gamma,v,\gamma',v'} \sum_{k,l=1}^\infty A_k^{\gamma\beta}(t)\rho^{\beta u,\beta'u'}(0)A_l^{\beta'u',\gamma'v'}(-t)|\Phi^{\gamma v}\rangle\langle\Phi^{\gamma'v'}| \tag{17}$$

$$= \sum_{\beta,u,\beta',u'} \sum_{\gamma,v,\gamma',v'} \sum_{k,l=1}^\infty A_k^{\gamma\beta}(t)\rho^{\beta u,\beta'u'}(0)A_l^{\beta'u',\gamma'v'}(-t)\left[|\phi^\gamma\rangle\langle\phi^{\gamma'}|\right] \otimes \left[|\chi^v\rangle\langle\chi^{v'}|\right], \tag{18}$$

where

$$A_l^{\gamma v,\gamma'v'}(t) = \langle\Phi^{\gamma v}|\mathcal{A}_l(t)|\Phi^{\gamma'v'}\rangle, \tag{19}$$

$$\rho_{tot}^{\gamma v,\gamma'v'}(0) = \langle\Phi^{\gamma v}|\rho_{tot}(0)|\Phi^{\gamma'v'}\rangle. \tag{20}$$

In Ref. [12], we have found the explicit forms of $\mathcal{A}_l(t)$. In the SESR, they read

$$A_0^{\gamma v,\gamma'v'}(t) = \mathrm{e}^{-\mathrm{i}E_{\gamma v}t}\delta_{\gamma\gamma'}\delta_{vv'}, \tag{21}$$

$$A_l^{\gamma v,\gamma'v'}(t) = \sum_{\gamma_1,\cdots,\gamma_{l+1}} \sum_{v_{\gamma_1},\cdots,v_{\gamma_{l+1}}} \left[\sum_{i=1}^{l+1}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i v_i}t}}{d_i(E[\gamma v,l])}\right]$$

$$\times \prod_{j=1}^l H_{tot1}^{\gamma_j v_j,\gamma_{j+1}v_{j+1}}\delta_{\gamma_1\gamma}\delta_{v_1 v}\delta_{\gamma_{l+1}\gamma'}\delta_{v_{l+1}v'}, \tag{22}$$

and all $H_{tot1}^{\gamma_j v_{\gamma_j},\gamma_{j+1}v_j} = \langle\Phi^{\gamma_j v_j}|H_{tot1}|\Phi^{\gamma_{j+1}v_{j+1}}\rangle$ form so-called "perturbing Hamiltonian matrix", that is, the representation matrix of the perturbing Hamiltonian in the unperturbed Hamiltonian representation (SESR). While

$$d_1(E[\gamma v,l]) = \prod_{i=1}^l \left(E_{\gamma_1 v_1} - E_{\gamma_{i+1}v_{j=i}}\right), \tag{23}$$

$$d_i(E[\gamma v,l]) = \prod_{j=1}^{i-1}\left(E_{\gamma_j v_j} - E_{\gamma_i v_i}\right)\prod_{k=i+1}^{l+1}\left(E_{\gamma_i v_i} - E_{\gamma_k v_k}\right), \tag{24}$$

$$d_{l+1}(E[\gamma v,l]) = \prod_{i=1}^l\left(E_{\gamma_i v_i} - E_{\gamma_{l+1}v_{l+1}}\right). \tag{25}$$

By tracing off the degree of freedom of environment space, we obtain the explicit expression of time evolution of reduced density matrix of open system

$$\rho_{\mathrm{S}}(t) = \sum_{k,l=0}^{\infty} \sum_{\beta,u,\beta',u'} \sum_{\gamma v,\gamma'} A_k^{\gamma v,\beta u}(t) \rho_{tot}^{\beta u,\beta' u'}(0) A_l^{\beta' u',\gamma' v}(-t) |\phi^\gamma\rangle \langle \phi^{\gamma'}|, \qquad (26)$$

where we have used the fact $\mathrm{Tr}_{\mathrm{E}}\left(|\Phi^{\gamma v}\rangle\langle\Phi^{\gamma' v'}|\right) = |\psi^\gamma\rangle\langle\psi^{\gamma'}|\delta_{vv'}$, which is an advantage of the SESR.

It is clear that in the above expression, we need to know the concrete forms of $|\phi^\gamma\rangle$ and $|\chi^v\rangle$ in order to obtain the explicit expressions of $A_k^{\gamma v,\beta u}(t)$. In fact, this is a physical reason why we take the form of H_{tot0} as Eq.(5) so that the eigenvectors and eigenvalues of H_{tot0} are obtainable.

Note that there are apparent divergences in the above exact solution. For the tidiness in form, we keep these apparent divergences in our expressions, but we can completely eliminate them by the limit process [13]. In other words, our exact solution of open systems should be understood in the limitation sense.

Just as pointed out above, there is, at least, an inherent SESR (ISESR) in the total system if taking $H_{tot0} = H_{\mathrm{S}} + H_{\mathrm{E}}$. We will be able to obtain the similar solution as Eq. (26). However, the ISESR is not unique in general because, in principle, a part of H_{S} and/or a part of H_{E} can be absorbed to H_{SE} if I_{S} and I_{E} are thought of as, respectively, the system operator and the environment operator. In this sense, the difference between the SESR and the ISESR is that the SESR allows to contain a part of $H_{\mathrm{SE0}} = \sum_{m,n} c_{mn} S_{m0} \otimes B_{n0}$, in which, $S_{m0} \neq I_{\mathrm{S}}$ and $\sum_n c_{mn} B_{n0} \neq I_{\mathrm{E}}$ for all m. Of course, if the cut-off approximation of perturbation is necessary, it requires that the absorbed parts from H_{S} and H_{E} are small enough. Such an example is discussed in Sec. VIII. In addition, one of the reasons to introduce the SESR is to make the Hamiltonian redivision and absorbing the perturbing parts of H_{S} and H_{E} to the perturbing Hamiltonian of the total system look like more natural.

Different from the general and explicit solution (14), the coefficients of our above solution (26) of open system dynamics are c-number functions whose forms are expressed clearly. Because $A_k^{\gamma v,\beta u}(t)$ include all of order approximations, this solution is, in fact, exact although it is an infinite series. Our solution in form is general enough, and it is able to applied to the cases that H_{S} and/or H_{E} are not exactly solvable. It is clear that we do not use the accustomed approximations such as the Born-Markov approximation, the factorization assumption for the initial state et. al. Hence, it should be more general and more reliable in theory. Moreover, by virtue of the improved scheme of perturbation theory proposed by us, we can obtain the improved perturbed solution of open system dynamics with better precision and higher efficiency because the contributions from the high order even all order approximations of perturbation are absorbed into the lower order approximations.

III. IMPROVED PERTURBED SOLUTION OF OPEN SYSTEM DYNAMICS

Traditional scheme of perturbation theory has been successfully used to solve many systems. However, in our point of view, it is still improvable, even it has a flaw because it introduce the perturbing parameter too early so that the contributions from the high order even all order approximations of the diagonal and off-diagonal elements of the perturbing Hamiltonian matrix are, respectively, inappropriately dropped and prematurely cut off. For some systems, the influences on the calculational precision because of this flaw can be not neglectable with the evolution time increasing. Actually, the traditional scheme of perturbation theory does not give a general term form of expanding coefficient evolution with time for any order approximation and does not explicitly express the general term as an obvious c-number function. Thus, it is necessary to find the perturbed solution (or perturbed energy and perturbed state vector) from the low to the high order approximation step by step up to some order approximation for a needed precision. Recently, in our work, we proposed an improved scheme of perturbation theory based on the general and explicit form of our exact solution [12, 13]. In our improved scheme, we introduce the approximation as late as possible, and consider subtly and systemically the affection of high-order approximation to the low-order one by the dynamical rearrangement and summation method. This finally results in the improved form of perturbed solution, and its expansion coefficients contain reasonably the high-order energy improvement. In this section, we will apply our improved scheme of perturbation theory to open systems.

It must be emphasized that before applying our improved form of perturbed solution, we have to first carried out the digonalization of degenerate subspaces if there is degeneracy and do the Hamiltonian redivision when H_{tot1} has the diagonal elements, in order to completely removed possible degeneracies by this procedure. When the remained degeneracies are allowed, it requires that the off-diagonal elements of the perturbing Hamiltonian matrix between any two degenerate levels are always vanishing. For more complicated cases, we will study in the near further.

Therefore, up to the three order improved approximation, we have

$$\rho_{\mathrm{S}}(t) = \sum_{\substack{l,k=0 \\ k+l\leq 3}}^{3} \sum_{\beta,u,\beta',u'} \sum_{\gamma v,\gamma'} A_{\mathrm{I}l}^{\gamma v,\beta u}(t)\rho_{tot}^{\beta u,\beta'u'}(0)A_{\mathrm{I}k}^{\beta'u',\gamma'v}(-t)|\phi^\gamma\rangle\langle\phi^{\gamma'}| + \mathcal{O}\left(H_1^4\right), \qquad (27)$$

where

$$A_{\mathrm{I}0}^{\gamma v,\gamma'v'}(t) = \mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma v}t}\delta_{\gamma\gamma'}\delta_{vv'}, \qquad (28)$$

$$A_{\mathrm{I}1}^{\gamma v,\gamma'v'}(t) = \left[\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma v}t}}{E_{\gamma v}-E_{\gamma'v'}} - \frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma'v'}t}}{E_{\gamma v}-E_{\gamma'v'}}\right]g_1^{\gamma v,\gamma'v'}, \qquad (29)$$

$$A_{\mathrm{I}2}^{\gamma v,\gamma'v'}(t) = \sum_{\gamma_1,v1}\left\{-\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma v}t}-\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma_1v_1}t}}{\left(E_{\gamma v}-E_{\gamma_1v_1}\right)^2}g_1^{\gamma v,\gamma_1v_1}g_1^{\gamma_1v_1,\gamma v}\delta_{\gamma\gamma'}\delta_{vv'} + \left[\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma v}t}}{\left(E_{\gamma v}-E_{\gamma_1v_1}\right)\left(E_{\gamma v}-E_{\gamma'v'}\right)}\right.\right.$$
$$\left.\left.-\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma_1v_1}t}}{\left(E_{\gamma v}-E_{\gamma_1v_1}\right)\left(E_{\gamma_1v_1}-E_{\gamma'v'}\right)}+\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma'v'}t}}{\left(E_{\gamma v}-E_{\gamma'v'}\right)\left(E_{\gamma_1v_1}-E_{\gamma'v'}\right)}\right]g_1^{\gamma v,\gamma_1v_1}g_1^{\gamma_1v_1,\gamma'v'}\eta_{\gamma v,\gamma'v'}\right\}, \qquad (30)$$

$$A_{\mathrm{I}3}^{\gamma v,\gamma'v'}(t) = \sum_{\gamma_1v_1,\gamma_2v_2}\left[-\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma v}t}}{\left(E_{\gamma v}-E_{\gamma_1v_1}\right)\left(E_{\gamma v}-E_{\gamma_2v_2}\right)^2} - \frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma v}t}}{\left(E_{\gamma v}-E_{\gamma_1v_1}\right)^2\left(E_{\gamma v}-E_{\gamma_2v_2}\right)}\right.$$
$$\left.+\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma_1v_1}t}}{\left(E_{\gamma v}-E_{\gamma_1v_1}\right)^2\left(E_{\gamma_1v_1}-E_{\gamma_2v_2}\right)} - \frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma_2v_2}t}}{\left(E_{\gamma v}-E_{\gamma_2v_2}\right)^2\left(E_{\gamma_1v_1}-E_{\gamma_2v_2}\right)}\right]g_1^{\gamma v,\gamma_1v_1}g_1^{\gamma_1v_1\gamma_2v_2}g_1^{\gamma_2v_2,\gamma v}\delta_{\gamma\gamma'}\delta_{vv'}$$
$$-\sum_{\gamma_1}\left[\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma v}t}}{\left(E_{\gamma v}-E_{\gamma_1v_1}\right)\left(E_{\gamma v}-E_{\gamma'v'}\right)^2}+\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma v}t}}{\left(E_{\gamma v}-E_{\gamma_1v_1}\right)^2\left(E_{\gamma v}-E_{\gamma'v'}\right)}\right]g_1^{\gamma v,\gamma_1v_1}g_1^{\gamma_1v_1,\gamma v}g_1^{\gamma v\gamma'v'}$$
$$+\sum_{\gamma_1,\gamma_2}\left[\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma v}t}\eta_{\gamma v,\gamma_2v_2}}{\left(E_{\gamma v}-E_{\gamma_1v_1}\right)\left(E_{\gamma v}-E_{\gamma_2v_2}\right)\left(E_{\gamma v}-E_{\gamma'v'}\right)} - \frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma_1v_1}t}\eta_{\gamma_1v_1,\gamma'v'}}{\left(E_{\gamma v}-E_{\gamma_1v_1}\right)\left(E_{\gamma_1v_1}-E_{\gamma_2v_2}\right)\left(E_{\gamma_1v_1}-E_{\gamma'v'}\right)}\right.$$
$$\left.+\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma_2v_2}t}\eta_{\gamma v,\gamma_2v_2}}{\left(E_{\gamma v}-E_{\gamma_2v_2}\right)\left(E_{\gamma_1v_1}-E_{\gamma_2v_2}\right)\left(E_{\gamma_2v_2}-E_{\gamma'v'}\right)}\right]g_1^{\gamma v,\gamma_1v_1}g_1^{\gamma_1v_1,\gamma_2v_2}g_1^{\gamma_2v_2,\gamma'v'}\eta_{\gamma v,\gamma'v'}, \qquad (31)$$

where $\delta_{\gamma\gamma'}$ and $\delta_{vv'}$ are the usual discrete delta functions, while $\eta_{\gamma\gamma'} = 1 - \delta_{\gamma\gamma'}$, $\eta_{vv'} = 1 - \delta_{vv'}$, and $\eta_{\gamma v,\gamma'v'} = \eta_{\gamma\gamma'}+\delta_{\gamma\gamma'}\eta_{vv'} = \eta_{vv'}+\eta_{\gamma\gamma'}\delta_{vv'}$. Moreover, we have defined so-called improved form of perturbed energy by

$$\widetilde{E}_{\gamma v} = E_{\gamma v} + G_{\gamma v}^{(1)} + G_{\gamma v}^{(2)} + G_{\gamma v}^{(3)} + G_{\gamma v}^{(4)} + G_{\gamma v}^{(5)} + \cdots, \qquad (32)$$

where, $G_{\gamma v}^{(1)} = h_1^{\gamma v}$ are diagonal elements of H_{tot1} and $g_1^{\gamma_iv_i,\gamma_jv_j}$ are off-diagonal elements of H_{tot1} in the representation of H_{tot0}. In addition, $h_1^{\gamma v}$ include the diagonal elements after the diagonalization of degenerate subspaces. While

$$G_{\gamma v}^{(2)} = \sum_{\gamma_1,v_1}\frac{1}{E_{\gamma v}-E_{\gamma_1v_1}}g_1^{\gamma v,\gamma_1v_1}g_1^{\gamma_1v_1,\gamma v}, \qquad (33)$$

$$G_{\gamma v}^{(3)} = \sum_{\gamma_1,v_1,\gamma_2,v_2}\frac{1}{(E_{\gamma v}-E_{\gamma_1v_1})(E_{\gamma v}-E_{\gamma_2v_2})}g_1^{\gamma v,\gamma_1v_1}g_1^{\gamma_1v_1,\gamma_2v_2}g_1^{\gamma_2v_2,\gamma v}, \qquad (34)$$

$$G_{\gamma v}^{(4)} = \sum_{\gamma_1,\gamma_2,\gamma_3}\sum_{v_1,v_2,v_3}\frac{g_1^{\gamma v,\gamma_1v_1}g_1^{\gamma_1v_1,\gamma_2v_2}g_1^{\gamma_2v_2,\gamma_3v_3}g_1^{\gamma_3v_3,\gamma v}\eta_{\gamma v,\gamma_2v_2}}{(E_{\gamma v}-E_{\gamma_1v_1})(E_{\gamma v}-E_{\gamma_2v_2})(E_{\gamma v}-E_{\gamma_3v_3})}$$
$$-\sum_{\gamma_1,\gamma_2}\sum_{v_1,v_2}\frac{g_1^{\gamma v,\gamma_1v_1}g_1^{\gamma_1v_1,\gamma v}g_1^{\gamma v,\gamma_2v_2}g_1^{\gamma_2v_2,\gamma v}}{(E_{\gamma v}-E_{\gamma_1v_1})^2(E_{\gamma v}-E_{\gamma_2v_2})}, \qquad (35)$$

$$G_\gamma^{(5)} = \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} \sum_{v_1,v_2,v_3,v_4} \frac{g_1^{\gamma v,\gamma_1 v_1} g_1^{\gamma_1 v_1,\gamma_2 v_2} g_1^{\gamma_2 v_2,\gamma_3 v_3} g_1^{\gamma_3 v_3,\gamma_4 v_4} g_1^{\gamma_4 v_4,\gamma v} \eta_{\gamma v,\gamma_2 v_2} \eta_{\gamma v,\gamma_3 v_3}}{(E_{\gamma v} - E_{\gamma_1 v_1})(E_{\gamma v} - E_{\gamma_2 v_2})(E_{\gamma v} - E_{\gamma_3 v_3})(E_{\gamma v} - E_{\gamma_4 v_4})}$$

$$- \sum_{\gamma_1,\gamma_2,\gamma_3} \sum_{v_1,v_2,v_3} \left[\frac{g_1^{\gamma v,\gamma_1 v_1} g_1^{\gamma_1 v_1,\gamma v} g_1^{\gamma v,\gamma_2 v_2} g_1^{\gamma_2 v_2,\gamma_3 v_3} g_1^{\gamma_3 v_3,\gamma v}}{(E_{\gamma v} - E_{\gamma_1 v_1})^2 (E_{\gamma v} - E_{\gamma_2 v_2})(E_{\gamma v} - E_{\gamma_3 v_3})} \right.$$

$$\left. + \frac{g_1^{\gamma v,\gamma_1 v_1} g_1^{\gamma_1 v_1,\gamma v} g_1^{\gamma v,\gamma_2 v_2} g_1^{\gamma_2 v_2,\gamma_3 v_3} g_1^{\gamma_3 v_3,\gamma v}}{(E_{\gamma v} - E_{\gamma_1 v_1})(E_{\gamma v} - E_{\gamma_2 v_2})^2 (E_{\gamma v} - E_{\gamma_3 v_3})} + \frac{g_1^{\gamma v,\gamma_1 v_1} g_1^{\gamma_1 v_1,\gamma v} g_1^{\gamma v,\gamma_2 v_2} g_1^{\gamma_2 v2,\gamma_3 v_3} g_1^{\gamma_3 v_3,\gamma v}}{(E_{\gamma v} - E_{\gamma_1 v_1})(E_{\gamma v} - E_{\gamma_2 v_2})(E_{\gamma v} - E_{\gamma_3 v_3})^2} \right]. \quad (36)$$

It must be emphasized that if only based on our calculations that was completed in Ref. [13], the improved perturbed energy in the exponential powers of A_{I1}, A_{I2} and A_{I3} will be cut off, respectively, to $G_{\gamma_i v_i}^{(4)}$, $G_{\gamma_i v_i}^{(3)}$ and $G_{\gamma_i v_i}^{(2)}$. However, according to our conjecture, we think that they can congruously written as the definition (32).

Our improved perturbed solution inherits some features from our exact solutions, for example, it is an explicit c-number function, easy to calculate, does not need the extra approximations. In principle, we should can calculate to any order of improved approximation. It must be emphasized that our improved form of perturbed solution absorbs the partial contributions from the high order even all order approximations of perturbation. This means that our solution has better precision and higher efficiency. In fact, these advantages have been seen in our recent work [12, 13].

IV.   MASTER EQUATION OF OPEN SYSTEMS INCLUDING ALL ORDER APPROXIMATIONS

Because we have obtained the general and explicit solution of the open system dynamics when the Hamiltonians of the system, its environment and the interaction between them are known, it is unnecessary to derive out the dynamical equation of open systems. However, in order to understand the affection from the environment, compare our solution with the existed motion equations and reveal the improvement of our method, we would like to discuss the motion equation and master equation in this section.

It is more convenient to derive out the master equation in the inherent SESR (ISESR) of open systems. That is, we take $H_{tot} = H_S + H_E$. In fact, it make us more easily compare our results with the existed ones. Obviously, the bases of ISESR are $|\psi^\gamma\rangle \otimes |\omega^v\rangle$, that is

$$H_S|\psi^\gamma\rangle \otimes |\omega^v\rangle = E_{S\gamma}|\psi^\gamma\rangle \otimes |\omega^v\rangle, \quad (37)$$

$$H_E|\psi^\gamma\rangle \otimes |\omega^v\rangle = \varepsilon_{Ev}|\psi^\gamma\rangle \otimes |\omega^v\rangle. \quad (38)$$

Similar to the way in Sec. II, we can obtain the exact solutions $\rho_{tot}(t)$ and $\rho_S(t)$. All we need to do is to change $|\phi^\gamma\rangle \otimes |\chi^v\rangle$ as $|\psi^\gamma\rangle \otimes |\omega^v\rangle$ and define the all matrix elements in the ISESR, for example, $A_k^{\beta' u',\gamma' v'}(t) = \langle\psi^\gamma \omega^v|\mathcal{A}_k(t)\big|\psi^{\gamma'}\omega^{v'}\big\rangle$. Therefore,

$$\rho_{tot}(t) = \sum_{k,l=0}^{\infty} \sum_{\beta,u,\beta',u'} \sum_{\gamma,v,\gamma',v'} A_k^{\gamma v,\beta u}(t) \varrho_{tot}^{\beta u,\beta' u'}(0) A_l^{\beta' u',\gamma' v'}(-t)|\psi^\gamma\rangle\big\langle\psi^{\gamma'}\big| \otimes |\omega^v\rangle\big\langle\omega^{v'}\big|. \quad (39)$$

$$\rho_S(t) = \sum_{\gamma v,\gamma'} \sum_{k,l=0}^{\infty} \sum_{\beta,u,\beta',u'} A_k^{\gamma v,\beta u}(t) \varrho_{tot}^{\beta u,\beta' u'}(0) A_l^{\beta' u',\gamma' v}(-t)|\psi^\gamma\rangle\big\langle\psi^{\gamma'}\big|. \quad (40)$$

From the solution (39), it is easy to get that $\text{Tr}_E\{[H_S,\rho_{tot}(t)]\} = [h_S,\rho_S(t)]$ and $\text{Tr}_E\{[H_E,\rho_{tot}(t)]\} = 0$. Hence,

$$\dot{\rho}_S(t) = \text{Tr}_E\dot{\rho}_{tot}(t) = -\text{i}\text{Tr}_E\{[H_{tot},\rho_{tot}(t)]\} = -\text{i}[h_S,\rho_S(t)] - \text{i}\text{Tr}_E\{[H_{SE},\rho_{tot}(t)]\}, \quad (41)$$

where $h_S = \text{Tr}_E H_S$. Denoting system operators by S_m and bath operators by B'_n, the most general form of H_{SE} is

$$H_{tot1} = H_{SE} = \sum_{m,n} c_{mn} S_m \otimes B'_n = \sum_m S_m \otimes B_m, \quad (42)$$

where $B_m = \sum_n c_{mn} B'_n$. Substituting the above relation into Eq. (41), we obtain the motion equation of open systems

$$\dot{\rho}_S(t) = -\text{i}[h_S,\rho_S(t)] - \text{i}\sum_m [S_m, \text{Tr}_E\{(I_S \otimes B_m)\rho_{tot}(t)\}]. \quad (43)$$

The second term of its right side represents the influence of the environment on the system.

In order to express the motion equation (43) of the open systems in the explicit matrix form, we introduce so-called factorizing initial state assumption, that is, the system and its environment are uncorrelated initially such that the total density matrix is a direct product of the system and its environment density matrices,

$$\rho_{tot}(0) = \rho_S(0) \otimes \rho_E(0). \tag{44}$$

Its advantage is to make us easily consider the actions of the operators on, respectively, the open system space and its environment space, and finally we can easily trace off the degree of freedom of environment space. In order to use this advantage, we introduce two new operators

$$\mathcal{A}_L^{(k)}(t) = \mathcal{A}_k(t)\mathcal{A}_0^{-1}(t) = \sum_{\beta,\beta'} \mathcal{P}_S(\beta,\beta') \otimes \mathcal{A}_{EL}^{(k)}(t,\beta,\beta'), \tag{45}$$

$$\mathcal{A}_R^{(l)}(-t) = \mathcal{A}_0^{-1}(-t)\mathcal{A}_l(-t) = \sum_{\gamma,\gamma'} \mathcal{P}_S(\gamma,\gamma') \otimes \mathcal{A}_{ER}^{(l)}(-t,\gamma,\gamma'), \tag{46}$$

where $\mathcal{P}_S(\beta,\beta') = \left|\psi^\beta\right\rangle\left\langle\psi^{\beta'}\right|$ are the basis operators of the system Hilbert space $\mathcal{H}_S$, while the operators $\mathcal{A}_{EL}^{(k)}(t,\beta,\beta')$ and $\mathcal{A}_{ER}^{(k)}(-t,\gamma,\gamma')$ are defined in environment Hilbert space $\mathcal{H}_E$ by

$$\mathcal{A}_{EL}^{(k)}(t,\beta,\beta') = \sum_{u,u'} A_k^{\beta u,\beta' u'}(t) e^{iE_{\beta' u'}t}|\omega^u\rangle\langle\omega^{u'}|, \tag{47}$$

$$\mathcal{A}_{ER}^{(l)}(-t,\gamma,\gamma') = \sum_{v,v'} e^{-iE_{\gamma v}t} A_l^{\gamma v,\gamma' v'}(-t)|\omega^v\rangle\langle\omega^{v'}|. \tag{48}$$

Thus, we see that $\mathcal{A}_L^{(k)}(t)$ and $\mathcal{A}_R^{(l)}(-t)$ are decomposed as the summations whose every terms with the form that the open system parts and its environment parts are separate. Hence, we obtain

$$\rho_{tot}(t) = \sum_{\beta,\beta',\gamma,\gamma'} [\mathcal{P}_S(\beta,\beta')\varrho_S(t)\mathcal{P}_S(\gamma,\gamma')] \otimes \left[\mathcal{A}_{EL}^{(k)}(t,\beta,\beta')\varrho_E(t)\mathcal{A}_{ER}^{(l)}(-t,\gamma,\gamma')\right], \tag{49}$$

where

$$\varrho_S(t) = e^{-ih_S t}\rho_S(0)e^{-ih_S t}, \tag{50}$$
$$\varrho_E(t) = e^{-ih_E t}\rho_E(0)e^{-ih_E t}, \tag{51}$$

and then

$$\varrho_{tot}(t) = \mathcal{A}_0(t)\rho_{tot}(0)\mathcal{A}_0(-t) = \varrho_S(t) \otimes \varrho_E(t). \tag{52}$$

Substituting Eq. (49) into the motion equation (43) it immediately follows that

$$\dot{\rho}_S(t) = -i[h_S,\rho_S(t)] - i\sum_m \sum_{k,l=0}^{\infty} \sum_{\beta,\beta',\gamma,\gamma'} C_{\beta\beta',\gamma\gamma'}^{m,kl}(t)[S_m,\mathcal{P}_S(\beta,\beta')\varrho_S(t)\mathcal{P}(\gamma,\gamma')], \tag{53}$$

where we have used the fact that

$$C_{\beta\beta',\gamma\gamma'}^{m,kl}(t) = \text{Tr}_E\left[B_m\mathcal{A}_{EL}^{(k)}(t,\beta,\beta')\varrho_E(t)\mathcal{A}_{ER}^{(l)}(-t,\gamma,\gamma')\right]. \tag{54}$$

Further deduction needs us to obtain the motion equation of $\varrho_S(t)$ that is expressed by $\rho_S(t)$. In fact, based on Eq. (49), we have

$$\varrho_S(t) = \rho_S(t) - \sum_{\substack{k,l=0\\k+l>0}} \sum_{\beta,\beta',\gamma,\gamma'} K_{\beta\beta',\gamma\gamma'}^{kl}(t)[\mathcal{P}_S(\beta,\beta')\varrho_S(t)\mathcal{P}_S(\gamma,\gamma')], \tag{55}$$

where we define the coefficients

$$K_{\beta\beta',\gamma\gamma'}^{kl}(t) = \text{Tr}_E\left[\mathcal{A}_{EL}^{(k)}(t,\beta,\beta')\varrho_E(t)\mathcal{A}_{ER}^{(l)}(-t,\gamma,\gamma')\right]. \tag{56}$$

Therefore, we can use the iterative method to rewrite it as

$$
\varrho_S(t) = \rho_S(t) + \sum_{M=1}^{\infty} (-1)^M \left[\prod_{m=1}^{M} \sum_{\substack{k_m,l_m=0 \\ k_m+l_m>0}} \sum_{\beta_m,\beta_m',\gamma_m,\gamma_m'} K^{k_m l_m}_{\beta_m\beta_m',\gamma_m\gamma_m'}(t) \right]
$$
$$
\times \left[\prod_{i=1}^{M} \mathcal{P}_S(\beta_i,\beta_i') \right] \rho_S(t) \left[\prod_{j=1}^{M} \mathcal{P}_S(\gamma_j,\gamma_j') \right]. \tag{57}
$$

Substituting it into Eq. (53), we obtain

$$
\dot{\rho}_S(t) = -i[h_S,\rho_S(t)] - i \sum_m \sum_{k,l=0}^{\infty} \sum_{\beta,\beta',\gamma,\gamma'} C^{m,kl}_{\beta\beta',\gamma\gamma'}(t) [S_m,\mathcal{P}_S(\beta,\beta')\rho_S(t)\mathcal{P}(\gamma,\gamma')]
$$
$$
-i \sum_m \sum_{k,l=0}^{\infty} \sum_{\beta,\beta',\gamma,\gamma'} C^{m,kl}_{\beta\beta',\gamma\gamma'}(t) \sum_{N=1}^{\infty} (-1)^N \left(\prod_{n=1}^{N} \sum_{\substack{k_n,l_n=0 \\ k_n+l_n>0}} \sum_{\beta_n,\beta_n',\gamma_n,\gamma_n'} K^{k_n l_n}_{\beta_n\beta_n',\gamma_n\gamma_n'}(t) \right)
$$
$$
\times \left[S_m, \mathcal{P}_S(\beta,\beta') \left(\prod_{i=1}^{N} \mathcal{P}_S(\beta_i,\beta_i') \right) \rho_S(t) \left(\prod_{j=1}^{N} \mathcal{P}_S(\gamma_j,\gamma_j') \right) \mathcal{P}_S(\gamma,\gamma') \right]. \tag{58}
$$

Up to now, we have not introduced any approximation except for the factorization assumption for the initial state. Since our master equation (58) including all order approximations, we can say it is an exact master equation of open systems.

V. PERTURBED MASTER EQUATION OF OPEN SYSTEMS

In the most cases, the interaction between the open system and its environment is weak. We can cut off the above exact master equation to some given order approximation. It is clear that since we absorb the coupling coefficients into B_m, we known $C^{m,nl}_{\beta\beta',\gamma\gamma'}(t)$ is a quantity of the $(n+l+1)$th order approximation, $K^{nl}_{\beta\beta',\gamma\gamma'}(t)$ is a quantity of the $(n+l)$th order approximation from their definitions. Although we can obtain any given order approximation of master equation based on our exact master equation (58), in most cases, we only are interested in up to the second order approximation. Because

$$
C^{m,00}_{\beta\beta',\gamma\gamma'} = \text{Tr}_E\left[B_m\varrho_E(t)\right]\delta_{\beta\beta'}\delta_{\gamma\gamma'}, \tag{59}
$$

$$
C^{m,0l}_{\beta\beta',\gamma\gamma'} = \delta_{\beta\beta'}\text{Tr}_E\left[B_m\varrho_E(t)\mathcal{A}^{(l)}_{ER}(-t,\gamma,\gamma')\right], \tag{60}
$$
$$
C^{m,k0}_{\beta\beta',\gamma\gamma'} = \text{Tr}_E\left[\mathcal{A}^{(k)}_{EL}(t,\beta,\beta')\varrho_E(t)B_m\right]\delta_{\gamma\gamma'}. \tag{61}
$$

$$
K^{0l}_{\beta\beta',\gamma\gamma'} = \delta_{\beta\beta'}\text{Tr}_E\left[\varrho_E(t)\mathcal{A}^{(l)}_{ER}(-t,\gamma,\gamma')\right], \tag{62}
$$
$$
K^{k0}_{\beta\beta',\gamma\gamma'} = \text{Tr}_E\left[\mathcal{A}^{(k)}_{EL}(t,\beta,\beta')\varrho_E(t)\right]\delta_{\gamma\gamma'}, \tag{63}
$$

we have

$$
\dot{\rho}_S(t) = -i[h_S,\rho_S(t)] - i[J(t),\rho_S(t)] - i\sum_m [S_m,\rho_S(t)C^{(1)}_{Rm}(t) + C^{(1)}_{Lm}(t)\rho_S(t)]
$$
$$
+i[J(t),\rho_S(t)R^{(1)}(t) + L^{(1)}(t)\rho_S(t)] + \mathcal{O}(H_{tot1}^3), \tag{64}
$$

where

$$
J(t) = \sum_m S_m(t)\text{Tr}_E\left(B_m\varrho_E(t)\right), \tag{65}
$$

$$C_{Lm}^{(1)}(t) = \mathrm{Tr}_{\mathrm{E}}\left\{[I_{\mathrm{S}} \otimes B_m]\,\mathcal{A}_1(t)\mathrm{e}^{\mathrm{i}H_{tot0}t}[I_{\mathrm{S}} \otimes \varrho_{\mathrm{E}}(t)]\right\}, \tag{66}$$

$$C_{Rm}^{(1)}(t) = \mathrm{Tr}_{\mathrm{E}}\left\{[I_{\mathrm{S}} \otimes \varrho_{\mathrm{E}}(t)]\,\mathrm{e}^{-\mathrm{i}H_{tot0}t}\mathcal{A}_1(-t)[I_{\mathrm{S}} \otimes B_m]\right\}, \tag{67}$$

$$L^{(1)}(t) = \mathrm{Tr}_{\mathrm{E}}\left\{\mathcal{A}_1(t)\mathrm{e}^{\mathrm{i}H_{tot0}t}[I_{\mathrm{S}} \otimes \varrho_{\mathrm{E}}(t)]\right\}, \tag{68}$$

$$R^{(1)}(t) = \mathrm{Tr}_{\mathrm{E}}\left\{[I_{\mathrm{S}} \otimes \varrho_{\mathrm{E}}(t)]\,\mathrm{e}^{-\mathrm{i}H_{tot0}t}\mathcal{A}_1(-t)\right\}, \tag{69}$$

while

$$\mathcal{A}_1(t) = \sum_{\beta,\gamma,u,v}\frac{\mathrm{e}^{-\mathrm{i}E_{\beta u}t}-\mathrm{e}^{-\mathrm{i}E_{\gamma v}t}}{E_{\beta u}-E_{\gamma v}}\left(\sum_m S_m^{\beta\gamma}B_m^{uv}\right)|\psi^\beta\rangle\langle\psi^\gamma| \otimes |\omega^u\rangle\langle\omega^v| \tag{70}$$

$$= \sum_{\beta,\gamma,u,v}\mathcal{A}_1^{\beta u,\gamma v}(t)|\psi^\beta\rangle\langle\psi^\gamma| \otimes |\omega^u\rangle\langle\omega^v|. \tag{71}$$

We can see that the Redfield master equation will be obtained from our this master equation without using Born-Markov approximation in next section.

In order to absorbing the partial contributions from the high order even all order approximations into the lower order approximations, we can use our improved scheme of perturbation theory. In similar way used above, we have

$$\begin{aligned}\dot{\rho}_{\mathrm{S}}(t) = {}&-\mathrm{i}[h_{\mathrm{S}},\rho_{\mathrm{S}}(t)]+\mathrm{i}[J(t),\rho_{\mathrm{S}}(t)]-\mathrm{i}\sum_{a=0}^{1}\sum_m\left[S_m,\rho_{\mathrm{S}}(t)C_{\mathrm{I}Rm}^{(a)}(t)+C_{\mathrm{I}Lm}^{(a)}(t)\rho_{\mathrm{S}}(t)\right]\\
&-\mathrm{i}\left[J(t),\rho_{\mathrm{S}}(t)R_{\mathrm{I}}^{(1)}(t)+L_{\mathrm{I}}^{(1)}(t)\rho_{\mathrm{S}}(t)\right]+\mathrm{i}\sum_m\left[S_m,C_{\mathrm{I}Lm}^{(0)}(t)\rho_{\mathrm{S}}(t)R_{\mathrm{I}}^{(1)}(t)+C_{\mathrm{I}Lm}^{(0)}(t)L_{\mathrm{I}}^{(1)}(t)\rho_{\mathrm{S}}(t)\right.\\
&\left.+\rho_{\mathrm{S}}(t)R_{\mathrm{I}}^{(1)}(t)C_{\mathrm{I}Rm}^{(0)}(t)+L_{\mathrm{I}}^{(1)}(t)\rho_{\mathrm{S}}(t)C_{\mathrm{I}Rm}^{(0)}(t)\right]+\mathcal{O}(H_{tot1}^3),\end{aligned} \tag{72}$$

where we have defined

$$C_{\mathrm{I}Lm}^{(k)}(t) = \mathrm{Tr}_{\mathrm{E}}\left\{[I_{\mathrm{S}} \otimes B_m]\,\mathcal{A}_{\mathrm{I}k}(t)\mathrm{e}^{\mathrm{i}H_{tot0}t}[I_{\mathrm{S}} \otimes \varrho_{\mathrm{E}}(t)]\right\}, \tag{73}$$

$$C_{\mathrm{I}Rm}^{(l)}(t) = \mathrm{Tr}_{\mathrm{E}}\left\{[I_{\mathrm{S}} \otimes \varrho_{\mathrm{E}}(t)]\,\mathrm{e}^{-\mathrm{i}H_{tot0}t}\mathcal{A}_{\mathrm{I}l}(-t)[I_{\mathrm{S}} \otimes B_m]\right\}, \tag{74}$$

$$L_{\mathrm{I}}^{(k)}(t) = \mathrm{Tr}_{\mathrm{E}}\left\{\mathcal{A}_{\mathrm{I}k}(t)\mathrm{e}^{\mathrm{i}H_{tot0}t}[I_{\mathrm{S}} \otimes \varrho_{\mathrm{E}}(t)]\right\}, \tag{75}$$

$$R_{\mathrm{I}}^{(l)}(t) = \mathrm{Tr}_{\mathrm{E}}\left\{[I_{\mathrm{S}} \otimes \varrho_{\mathrm{E}}(t)]\,\mathrm{e}^{-\mathrm{i}H_{tot0}t}\mathcal{A}_{\mathrm{I}l}(-t)\right\}, \tag{76}$$

while

$$\mathcal{A}_{\mathrm{I}1}(t) = \sum_{\beta,\gamma,u,v}\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\beta u}t}-\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma v}t}}{E'_{\beta u}-E'_{\gamma v}}H'^{\beta u,\gamma v}_{tot1}\left(1-\delta_{\beta\gamma}\delta_{uv}\right)|\psi^\beta\rangle\langle\psi^\gamma| \otimes |\omega^u\rangle\langle\omega^v| \tag{77}$$

$$= \sum_{\beta,\gamma,u,v}\mathcal{A}_{\mathrm{I}1}^{\beta u,\gamma v}(t)|\psi^\beta\rangle\langle\psi^\gamma| \otimes |\omega^u\rangle\langle\omega^v|. \tag{78}$$

Here, $\widetilde{E}_{\gamma v}=E_{\mathrm{S}\gamma}+\varepsilon_{\mathrm{E}v}+h_{\gamma v}+G_{\gamma v}^{(2)}+G_{\gamma v}^{(3)}+G_{\gamma v}^{(4)}+\cdots$, $E'_{\gamma v}=E_\gamma+\varepsilon_v+h_{\gamma v}$, $h_{\gamma v}$ are diagonal elements of H_{tot1}, and the perturbed part of Hamiltonian in $G_{\gamma v}^{(i)}$ has be redivided as $H'_{tot1}=H_{\mathrm{SE}}-\sum_{\gamma v}h_{\gamma v}|\psi^\gamma\rangle\langle\psi^\gamma| \otimes |\omega^v\rangle\langle\omega^v|$, that is, $g_1^{\gamma v,\gamma'v'}=\langle\psi^\gamma\omega^v|H'_{tot1}|\psi^\gamma\omega^v\rangle$.

It must be emphasized that the operators in the above definitions and expressions are defined in the ISESR (that has been diagoalized in the degenerate subspaces if the degeneracy cases exist). However, $A_{\mathrm{I}k}(\pm t)$ including $\widetilde{E}$ have to be calculated using H'_1 that is the perturbing Hamiltonian via the redivision skill. Hence, it is important to distinguish $H_{tot0}=H_{\mathrm{S}}+H_{\mathrm{E}}$, $H_{tot1}=H_{\mathrm{SE}}$ and their redivision H'_{tot0}, H'_{tot1} in spite of them in the same ISESR. In addition, we assume all degeneracies are completely removed by the diagonalization procedure of degenerate subspaces and/or hamiltonian redivision for simplicity and determination. If the remained degeneracies are allowed, it requires that the off-diagonal elements of the perturbing Hamiltonian matrix between any two degenerate levels are always vanishing, in order to let our improved scheme of perturbation theory work well.

In the above derivation of our master equation, we do not use Born-Markov approximation, but only standard cut-off approximation. From our point of view, it is more reasonable in physics theory and its precision and reliability should be better in practical applications.

VI.   RE-DEDUCTION OF REDFIELD MASTER EQUATION

In order to compare our master equation with the known master equations and illustrate the validness of our master equation, we will deduce the Redfield master equation from our master equation without using the Born-Markov approximation in this section. In addition, we point out what differences between our master equation and the existed one, and provide the comments on well-known approximations using in open system dynamics.

Firstly, we assume a thermal equilibrium for the environment, that is,

$$\rho_{\mathrm{E}}(0) = \frac{e^{-\beta_{\mathrm{B}} H_{\mathrm{E}}}}{\mathrm{Tr} e^{-\beta_{\mathrm{B}} H_{\mathrm{E}}}} = \frac{1}{Z} \sum_v e^{-\beta_{\mathrm{B}} \varepsilon_{\mathrm{E}v}} |\omega^v\rangle\langle\omega^v|, \tag{79}$$

where $\beta_{\mathrm{B}} = 1/k_{\mathrm{B}} T$ with T the bath equilibrium. This is justified when the environment is "very large". Thus, it is easy to get

$$
\begin{aligned}
L^{(1)}(t) &= -R^{(1)}(t) = F^{(1)}(t) \\
&= \sum_{\beta,\gamma,u} \sum_m \frac{e^{-i(E_{\mathrm{S}\beta} - E_{\mathrm{S}\gamma})t} - 1}{E_{\mathrm{S}\beta} - E_{\mathrm{S}\gamma}} S_m^{\beta\gamma} B^{uu} \rho_{\mathrm{E}}^u |\psi^\beta\rangle\langle\psi^\gamma| \tag{80} \\
&= -i e^{-i h_{\mathrm{S}} t} \int_0^t d\tau \mathrm{Tr}_{\mathrm{E}} \left[\widehat{H}_{\mathrm{SE}}(\tau) \left(I_{\mathrm{S}} \otimes \rho_E(0) \right) \right] e^{i h_{\mathrm{S}} t} \tag{81} \\
&= -i \int_0^t d\tau \mathrm{Tr}_{\mathrm{E}} \left[\overline{H}_{\mathrm{SE}}(\tau) \left(I_{\mathrm{S}} \otimes \rho_E(0) \right) \right], \tag{82}
\end{aligned}
$$

where

$$\overline{H}_{\mathrm{SE}}(\tau) = e^{-i H_0 \tau} H_{\mathrm{SE}} e^{i H_0 \tau}. \tag{83}$$

Likewise, we have

$$
\begin{aligned}
C_{Lm}^{(1)}(t) &= -i e^{-i h_{\mathrm{S}} t} \int_0^t d\tau \mathrm{Tr}_{\mathrm{E}} \left[\left(I_{\mathrm{S}} \otimes \widehat{B}_m(t) \right) \widehat{H}_{\mathrm{SE}}(\tau) \left(I_{\mathrm{S}} \otimes \rho_E(0) \right) \right] e^{i h_{\mathrm{S}} t} \tag{84} \\
&= -i \int_0^t d\tau \mathrm{Tr}_{\mathrm{E}} \left[\left(I_{\mathrm{S}} \otimes B_m \right) \overline{H}_{\mathrm{SE}}(\tau) \left(I_{\mathrm{S}} \otimes \rho_E(0) \right) \right], \tag{85}
\end{aligned}
$$

$$
\begin{aligned}
C_{Rm}^{(1)}(t) &= i e^{-i h_{\mathrm{S}} t} \int_0^t d\tau \mathrm{Tr}_{\mathrm{E}} \left[\left(I_{\mathrm{S}} \otimes \rho_E(0) \right) \widehat{H}_{\mathrm{SE}}(\tau) \left(I_{\mathrm{S}} \otimes \widehat{B}_m(t) \right) \right] e^{i h_{\mathrm{S}} t} \tag{86} \\
&= i \int_0^t d\tau \mathrm{Tr}_{\mathrm{E}} \left[\left(I_{\mathrm{S}} \otimes B_m \right) \overline{H}_{\mathrm{SE}}(\tau) \left(I_{\mathrm{S}} \otimes \rho_E(0) \right) \right]. \tag{87}
\end{aligned}
$$

Therefore, our master equation (64) up to the second order approximation can be rewritten as

$$
\begin{aligned}
\dot{\rho}_{\mathrm{S}}(t) &= -i \left[h_{\mathrm{S}}, \rho_{\mathrm{S}}(t) \right] - i [J(t), \rho_{\mathrm{S}}(t)] - \int_0^t d\tau \mathrm{Tr}_{\mathrm{E}} \left\{ \left[H_{\mathrm{SE}}, \left[\overline{H}_{\mathrm{SE}}(\tau), \rho_{\mathrm{S}}(t) \otimes \rho_{\mathrm{E}}(0) \right] \right] \right\} \\
&\quad + \left[J(t), \int_0^t d\tau \mathrm{Tr}_{\mathrm{E}} \left\{ \left[\overline{H}_{\mathrm{SE}}(\tau), \rho_{\mathrm{S}}(t) \otimes \rho_{\mathrm{E}}(0) \right] \right\} \right]. \tag{88}
\end{aligned}
$$

If we introduce the interaction picture, that is, an operator $\widehat{O}$ in this picture is defined by a corresponding operator in the Schrödinger picture

$$\widehat{O}(t) = e^{i H_{tot0} t} O e^{-i H_{tot0} t}. \tag{89}$$

It is clear that for an operator F_{S} in the open system Hilbert space and an operator F_{E} in the environment Hilbert space, we have

$$
\begin{aligned}
\widehat{F}_{\mathrm{S}}(t) &= e^{i h_{\mathrm{S}} t} F_{\mathrm{S}} e^{-i h_{\mathrm{S}} t}, \tag{90} \\
\widehat{F}_{\mathrm{E}}(t) &= e^{i h_{\mathrm{E}} t} F_{\mathrm{E}} e^{-i h_{\mathrm{E}} t}. \tag{91}
\end{aligned}
$$

It immediately follows the master equation in the interaction picture:

$$\frac{\mathrm{d}\widetilde{\rho}_\mathrm{S}(t)}{\mathrm{d}t} = -\mathrm{i}\left[\widetilde{J}(t), \widetilde{\rho}_\mathrm{S}(t)\right] - \int_0^t \mathrm{d}\tau \mathrm{Tr}_\mathrm{E}\left\{\left[\widetilde{H}_\mathrm{SE}, \left[\widetilde{H}_\mathrm{SE}(\tau), \widetilde{\rho}_\mathrm{S}(t)\otimes\rho_\mathrm{E}(0)\right]\right]\right\}$$
$$+ \left[\widetilde{J}(t), \int_0^t \mathrm{d}\tau \mathrm{Tr}_\mathrm{E}\left\{\left[\widetilde{H}_\mathrm{SE}(\tau), \widetilde{\rho}_\mathrm{S}(t)\otimes\rho_\mathrm{E}(0)\right]\right\}\right]. \tag{92}$$

It must be emphasized that $\widetilde{\rho}_\mathrm{S}(t)$ is equal to $\mathrm{e}^{\mathrm{i}h_\mathrm{S}t}\rho_\mathrm{S}(t)\mathrm{e}^{-\mathrm{i}h_\mathrm{S}t}$, but not $\mathrm{e}^{\mathrm{i}h_\mathrm{S}t}\rho_\mathrm{S}(0)\mathrm{e}^{-\mathrm{i}h_\mathrm{S}t}$.

In special,when we introduce the assumption

$$\mathrm{Tr}_\mathrm{E}\left\{\left[\widetilde{H}_\mathrm{SE}, \rho_{tot}(0)\right]\right\} = 0. \tag{93}$$

we have

$$\sum_m \sum_{\beta\beta',\gamma\gamma'} C_{\beta\beta',\gamma\gamma'}^{m,00} [S_m, \mathcal{P}_\mathrm{S}(\beta,\beta')\varrho_\mathrm{S}(t)\mathcal{P}(\gamma,\gamma')] = \mathrm{e}^{-\mathrm{i}h_\mathrm{S}t}\mathrm{Tr}_\mathrm{E}\left\{\left[\widetilde{H}_\mathrm{SE}, \rho_{tot}(0)\right]\right\}\mathrm{e}^{\mathrm{i}h_\mathrm{S}t} = 0. \tag{94}$$

Thus, Eq. (53) becomes

$$\dot{\rho}_\mathrm{S}(t) = -\mathrm{i}[h_\mathrm{S}, \rho_\mathrm{S}(t)] - \mathrm{i}\sum_m \sum_{\substack{k,l=0 \\ k+l>0}}^\infty \sum_{\beta,\beta',\gamma,\gamma'} C_{\beta\beta',\gamma\gamma'}^{m,kl}(t) [S_m, \mathcal{P}_\mathrm{S}(\beta,\beta')\varrho_\mathrm{S}(t)\mathcal{P}(\gamma,\gamma')], \tag{95}$$

that is, the perturbed form of master equation up to the second order approximation reads

$$\dot{\rho}_\mathrm{S}(t) = -\mathrm{i}[h_\mathrm{S}, \rho_\mathrm{S}] - \mathrm{i}\sum_m [S_m, \rho_\mathrm{S}(t)C_{Rm}^{(1)}(t) + C_{Lm}^{(1)}(t)\rho_\mathrm{S}(t)]. \tag{96}$$

This means that the approximation (93) leads to the following terms

$$\mathrm{i}[J(t), \rho_\mathrm{S}(t)] + \left[J(t), \int_0^t \mathrm{d}\tau \mathrm{Tr}_\mathrm{E}\left\{\left[\overline{H}_\mathrm{SE}(\tau), \rho_\mathrm{S}(t)\otimes\rho_\mathrm{E}(0)\right]\right\}\right], \tag{97}$$

or, equivalently, in the interaction picture

$$-\mathrm{i}\left[\widetilde{J}(t), \widetilde{\rho}_\mathrm{S}(t)\right] + \left[\widetilde{J}(t), \int_0^t \mathrm{d}\tau \mathrm{Tr}_\mathrm{E}\left\{\left[\widetilde{H}_\mathrm{SE}(\tau), \widetilde{\rho}_\mathrm{S}(t)\otimes\rho_\mathrm{E}(0)\right]\right\}\right] \tag{98}$$

are dropped by comparing Eq. (96) with Eq. (88) or (92).

Usually, the approximation is thought of as a unimportant restriction since one can absorb the the dropped terms into the system Hamiltonian H_S. However, based on the above result, we think that the approximation (93) is a real assumption because the second term in (97) or (98) is not nontrivial and it can not be absorbed into H_S in general. In other words, the the second order contribution to the master equation from the second term in (97) or (98) should be considered and the approximation (93) should be rechecked for the concrete open systems except for the cases when $J(t) = 0$. Actually, we think, if $J(t)$ is not equal to zero, the last term appears in our master equation (88) or (92) is obviously different from the existed master equations.

It is very interesting, when the approximation (93) can be used to some given open systems, we immediately from the equation (96) obtain

$$\frac{\mathrm{d}\widetilde{\rho}_\mathrm{S}(t)}{\mathrm{d}t} = -\int_0^t \mathrm{d}\tau \mathrm{Tr}_\mathrm{E}\left\{\left[\widetilde{H}_\mathrm{SE}, \left[\widetilde{H}_\mathrm{SE}(\tau), \widetilde{\rho}_\mathrm{S}(t)\otimes\rho_\mathrm{E}(0)\right]\right]\right\}. \tag{99}$$

This is just the well-known Redfield master equation. This conclusion implies that the Redfield master equation is still valid without introducing Born-Markov approximation. Therefore, we think that Born-Markov approximation is unnecessary for the master equation with the second order perturbed approximations. From our point of view, this is a real physical reason why ones should use jointly Born- and Markov approximations and why ones can obtain useful conclusions in the cases without Born-Markov approximation. In fact, those terms that are dropped by Born approximation are compensated by Markov approximation. In other words, Born approximation plus Markov approximation back to no approximation based on our results.

VII. MILBURN DYNAMICS FOR OPEN SYSTEMS

Historically, a useful dynamical model of open system is the Milburn model [11]. It provides a way to describe so-called "intrinsic" decoherence. However, in our point of view, perhaps, it can be called external-external environment decoherence. That is, Milburn dynamics might be alternatively explained as the effect of environment of the composite system or the large environment of the proper system. This explanation is, in fact, a conclusion from that we believe the von Neumann equation is uniquely correct for a closed system. One argues what mechanism results in that the external influence is reflected by the extra term in Milburn dynamics. We can not answer it at present, but we would like to ask what condition changes the dynamics from the von Neumann's to the Milburn's. If the answer is the Milburn dynamics is a nature of closed quantum systems, then it is very difficult how to understand the free parameter θ_0.

Actually, only one can do something within the near environment in order to control decoherence, for example, the self-interaction of environment, and it is possible that one only knows how to appropriately describe the dynamics of near environment and the interaction between the system and near environment but are short of the knowledge about the remote environment. Therefore, in this section, we intend to use the Milburn model to consider the dynamics of the composite system made up of the system and its near environment. The conclusions obtained here imply the our solution and methods are also applicable to more general open systems such as the Milburn model.

Dynamics in the Milburn model replaces the usual von-Neumann equation of the density matrix by

$$\dot{\rho}_{tot}(t) = -\mathrm{i}[H_{tot}, \rho_{tot}(t)] - \frac{\theta_0}{2}[H_{tot}, [H_{tot}, \rho_{tot}(t)]], \tag{100}$$

where θ_0 is a constant meaning that there is some minimum unitary-phase transformation. This implies that coherence is destroyed as the physical properties of the system approach a macroscopic level. Hence, seemingly, the "intrinsic" decoherence explanation looks like to be reasonable. However, the minimum unitary-phase transformation is not clear. The parameter θ_0 in the Milburn model is still "free". In other words, θ_0 is not been given by the theory. If we think the extra term is resulted in by the remote environment, θ_0 should be able be known by the experiment.

Now, we directly extend Milburn dynamics to a Milburn-type closed quantum system consisting of the concerned system and its near environment. The Hamiltonian in eq.(100) still reads $H_{tot} = H_{\mathrm{S}} + H_{\mathrm{E}} + H_{\mathrm{SE}}$. Here, a Milburn-type closed quantum system is not really closed system from the view that a really closed system must obey the von Neumann equation. Actually, an alternative explanation is that a Milburn-type closed quantum system is still affected by the remote (larger environment), and this influence is represented by an extra term with θ_0 multiplier because one cannot know the Hamiltonian of its remote environment and the interaction form between the interesting system and its remote environment. Obviously, when $\theta_0 = 0$, Milburn dynamics back to von Neumann dynamics. This implies that the (very) remote environment can be ignored.

The formal solution of Milburn dynamics for the composite system can be written as [14]

$$\rho_{tot}(t) = \exp\left\{-\mathrm{i}H_{tot}t - \theta_0 H_{tot}^2 t/2\right\}\left[\mathrm{e}^{\mathfrak{M}t}\rho_{tot}(0)\right]\exp\left\{\mathrm{i}H_{tot}t - \theta_0 H_{tot}^2 t/2\right\} = \sum_{k}^{\infty} M_k(t)\rho_{tot}(0)M_k^{\dagger}(t), \tag{101}$$

where $\mathfrak{M}$ is a superoperator, i.e, $\mathfrak{M}\rho_{tot} = \theta_0 H_{tot}\rho_{tot}H_{tot}$, and the Kraus operators $M_k(t)$ is in the form

$$M_k(t) = \sqrt{\frac{(\theta_0 t)^k}{k!}} H_{tot}^k \exp\left\{-\mathrm{i}H_{tot}t - \theta_0 H_{tot}^2 t/2\right\}. \tag{102}$$

Without loss of generality, using of the denotation $(A + B)^K = A^K + f^K(A, B)$ and $f^0(A, B) = 0$, we can write

$$M_k(t) = \sqrt{\frac{(\theta_0 t)^k}{k!}}\left[H_{tot0}^k \exp\left\{-\mathrm{i}H_{tot0}t - \theta_0 H_{tot0}^2 t/2\right\} + \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-\mathrm{i}t)^n}{n!}\left(\frac{\theta_0}{2\mathrm{i}}\right)^m C_n^m f^{n+m+k}(H_{tot0}, H_{tot1})\right]. \tag{103}$$

Just as we find the exact solution of open system in von Neumann dynamics, we need a system-environment separated representation (SESR), which has been given in Sec. II. Thus, in this SESR, based on the our expansion formula of operator binomials power [12] we have

$$f^K(H_{tot0}, H_{tot1}) = \sum_{l=1}^{K}\sum_{\gamma_1,\cdots,\gamma_{l+1}}\sum_{v_1,\cdots,v_{l+1}} C_l^K(E[\gamma v, l])\left[\prod_{j=1}^{l} H_{tot1}^{\gamma_j v_j, \gamma_{j+1}, v_{j+1}}\right]|\psi^{\gamma_1}\rangle\langle\psi^{\gamma_{l+1}}| \otimes |\omega^{v_1}\rangle\langle\omega^{v_{l+1}}|, \tag{104}$$

$$C_l^K(E[\gamma v, l]) = \sum_{i=1}^{l+1} (-1)^{i-1} \frac{E_{\gamma v}^K}{d_i(E[\gamma v, l])}, \tag{105}$$

where $d_i(E[\gamma v, l])$ is defined in Sec. II. Therefore, the expression of $M_k(t)$ is changed to a summation according to the order (or power) of the H_{tot1} as follows

$$M_k^{\gamma v, \gamma' v'}(t) = \sqrt{\frac{(\theta_0 t)^k}{k!}} E_{\gamma v}^k g(E_{\gamma v}, t) \delta_{\gamma \gamma'} \delta_{vv'} + \sqrt{\frac{(\theta_0 t)^k}{k!}} \sum_{l=1}^{\infty} \sum_{\gamma_1, \cdots, \gamma_{l+1}} \sum_{v_1, \cdots, v_{l+1}} \left[\sum_{i=1}^{l+1} (-1)^{i-1} \frac{E_{\gamma_i v_i}^k g(E_{\gamma_i v_i}; t)}{d_i(E[\gamma v, l])} \right]$$

$$\times \prod_{j=1}^{l} H_{tot1}^{\gamma_j v_j, \gamma_{j'} v_j'} \delta_{\gamma_1 \gamma} \delta_{\gamma_{l+1} \gamma'} \delta_{v_1 v} \delta_{v_{l+1} v'}, \tag{106}$$

where the time evolution function $g(x; t)$ with the exponential form is defined by

$$g(x; t) = \exp\left\{ -\mathrm{i}xt - \theta_0 x^2 t/2 \right\}. \tag{107}$$

Obviously, $M_k^{\dagger \gamma v, \gamma' v'}(t)$ can be given via replacing $g(x; t)$ by $g^*(x; t)$. Furthermore, we obtain the expression of time evolution of reduced density matrix of open systems, that is a general and explicit solution of open systems in Milburn dynamics:

$$\rho_{\mathcal{S}}(t) = \sum_{\beta, u, \beta', u'} \sum_{\gamma, v, \gamma', v'} M_k^{\beta u, \beta u'}(t) \rho^{\beta' u', \gamma' v'}(0) M_k^{\dagger \gamma' v', \gamma v} \delta_{uv} |\psi^\beta\rangle \langle \psi^\gamma|$$

$$= \sum_{\beta, \gamma, v} \rho^{\beta v, \gamma v}(0) g(E_{\beta v} - E_{\gamma v}; t) |\psi^\beta\rangle \langle \psi^\gamma| + \sum_{\beta, u} \sum_{\gamma', v', \gamma} \rho^{\beta u, \gamma' v'}(0) \sum_{l=1}^{\infty} \sum_{\gamma_1, \cdots, \gamma_{l+1}} \left[\sum_{i=1}^{l+1} (-1)^{i-1} \frac{g(E_{\beta u} - E_{\gamma_i v_i}; t)}{d_k(E[\gamma v, l])} \right]$$

$$\times \left[\prod_{j=1}^{l} H_{tot1}^{\gamma_j \gamma_{j+1}} \right] \delta_{\gamma' \gamma_1} \delta_{v' v_1} \delta_{\gamma_{l+1} \gamma} \delta_{v_{l+1} u} |\psi^\beta\rangle \langle \psi^\gamma| + \sum_{\beta, \beta', u'} \sum_{\gamma, v} \rho^{\beta' u', \gamma v}(0)$$

$$\times \sum_{l=1}^{\infty} \sum_{\beta_1, \cdots, \beta_{l+1}} \left[\sum_{i=1}^{l+1} (-1)^{i-1} \frac{g(E_{\beta_i v_i} - E_{\gamma v}; t)}{d_k(E[\beta u, l])} \right] \left[\prod_{j=1}^{l} H_{tot1}^{\beta_j \beta_{j+1}} \right] \delta_{\beta \beta_1} \delta_{v, u_1} \delta_{\beta_{l+1} \beta'} \delta_{u_{l+1} u'} |\psi^\beta\rangle \langle \psi^\gamma|$$

$$+ \sum_{\beta, u, \beta', u'} \sum_{\gamma, v, \gamma', v'} \rho^{\beta' u', \gamma' v'}(0) \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{\beta_1, \cdots, \beta_{k+1}} \sum_{\gamma_1, \cdots, \gamma_{l+1}} \left[\sum_{i=1}^{k+1} \sum_{j=1}^{l+1} (-1)^{i+j} \frac{g(E_{\beta_i u_i} - E_{\gamma_j v_j}; t)}{d_i(E[\beta u]) d_j(E[\gamma v, l])} \right]$$

$$\times \left[\prod_{i=1}^{k} H_{tot1}^{\beta_i v_i, \beta_{i+1} v_{i+1}} \right] \left[\prod_{j=1}^{l} H_{tot1}^{\gamma_j v_j, \gamma_{j+1} v_{j+1}} \right] \delta_{\beta \beta_1} \delta_{\beta_{k+1} \beta'} \delta_{u u_1} \delta_{u_{l+1} u'} \delta_{\gamma' \gamma_1} \delta_{\gamma_{l+1} \gamma} \delta_{v' v_1} \delta_{v_{l+1} v} \delta_{uv} |\psi^\beta\rangle \langle \psi^\gamma|. \tag{108}$$

It is clear that if $\theta_0 = 0$, this solution is just the form of solution of van Neumann dynamics that is obtained in Sec. II. Usually, the finite (even often low) order approximation about H_{tot1} can be taken, thus this expression will be cut off to the finite terms. Similar to the methods used in Secs. III, IV and V, we can study the perturbed solution and motion equation of open systems in Milburn dynamics. It is not difficult, so we omit them in order to save space.

VIII. EXAMPLE AND APPLICATION

In order to concretely illustrate our general and explicit solution of open system dynamics, we recall an exactly solvable two-state open system for decoherence that was first introduced by Zurek [5, 15]. In this Zurek model, the "free" (unperturbed) Hamiltonian H_{S0} and the self-interaction (perturbing) H_{S1} of concerning two-state system and the 'free' (unperturbed) Hamiltonian H_{E0} and the self-interaction (perturbing) H_{E1} of the environment are taken as to be equal to zero. The total Hamiltonian of the composite system made of the interesting system plus the environment only has their interaction term, that is

$$H_{\text{Zurek}} = H_{\text{SE}} = \sigma_S^z \otimes B_{z\text{E}}, \tag{109}$$

where the environment operator $B_{z\mathrm{E}}$ is defined by

$$B_{z\mathrm{E}} = \sum_{k=1}^{N_\mathrm{E}} \left(\bigotimes_{i=1}^{k-1} I_{\mathrm{E}i}\right) \otimes (Z_k \sigma^{z_k}) \otimes \left(\bigotimes_{j=k+1}^{N_\mathrm{E}} I_{\mathrm{E}j}\right), \tag{110}$$

and N_E is the degree of freedom of the environment, which is very large even infinite.

It is clear that this Zurek model can be exactly solved out. Its eigenvectors are so-called natural bases

$$|n_\mathrm{S} n_\mathrm{E}\rangle = |n_\mathrm{S}; n_1, n_2 \cdots\rangle = |n_\mathrm{S}\rangle \otimes \bigotimes_{k=1}^{N_\mathrm{E}} |n_k\rangle, \tag{111}$$

where $n_\mathrm{S}, n_1, n_2, \cdots = 0, 1$ and

$$|0\rangle_\mathrm{S} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle_\mathrm{S} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{112}$$

$$|0\rangle_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle_k = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{113}$$

The corresponding eigenvalues are

$$E_{n_\mathrm{S} n_\mathrm{E}} = E_{n_\mathrm{S} n_1 n_2 \cdots} = (-1)^{n_\mathrm{S}} \sum_{k=1}^{N_\mathrm{E}} (-1)^{n_k} Z_k. \tag{114}$$

Note that we use a simple notation n_E to denote $n_1, n_2, \cdots$ here and after.

Now, let we solve this Zurek model by using our exact solution or the improved form of the perturbed solution. From our point of view, the assumption that H_S and H_E are taken as zero is a theoretical simplification. In fact, we can think that H_S and H_E are constants so that we can absorb them into energy eigenvalues or, equivalently, directly omit them since these constants do not affect physics. Therefore, the base of the SESR can be taken as the natural bases (111).

Since H_SE is completely diagonal in this SESR, that is

$$\langle m_\mathrm{S} m_\mathrm{E} | H_\mathrm{SE} | n_\mathrm{S} n_\mathrm{E}\rangle = (-1)^{n_\mathrm{S}} \sum_{k=1}^{N_\mathrm{E}} (-1)^{n_k} Z_k \delta_{m_\mathrm{S} n_\mathrm{S}} \prod_{i=1}^{N_\mathrm{E}} \delta_{m_i n_i}. \tag{115}$$

We should use the Hamiltonian redivision skill, and then

$$H'_{tot0} = H_{Zurek}. \tag{116}$$

It is easy to get

$$\widetilde{E}_{n_\mathrm{S} n_\mathrm{E}} = \widetilde{E}_{n_\mathrm{S} n_1 n2 \cdots} = (-1)^{n_\mathrm{S}} \sum_{k=1}^{N_\mathrm{E}} (-1)^{n_k} Z_k. \tag{117}$$

$$A_{\mathrm{I}l}(t) = 0, \quad (l > 0). \tag{118}$$

where we have used the fact $g_1^{m_\mathrm{S} m_\mathrm{E}, n_\mathrm{S} n_\mathrm{E}} = 0$ based on $H'_{tot1} = 0$. This means that the perturbed solution part of higher than the zeroth order approximation is vanishing. Therefore, our exact solution or the improved form of the perturbed solution including only non-vanishing zeroth order part becomes

$$\rho_{Zurek}(t) = \sum_{m_\mathrm{S}, n_\mathrm{S}=0}^{1} \sum_{m_\mathrm{E}, n_\mathrm{E}} e^{-i\widetilde{E}_{m_\mathrm{S} m_\mathrm{E}} t} \rho^{m_\mathrm{S} m_\mathrm{E}, n_\mathrm{S} n_\mathrm{E}}(0) e^{i\widetilde{E}_{n_\mathrm{S} n_\mathrm{E}} t} |m_\mathrm{S} m_\mathrm{E}\rangle \langle n_\mathrm{S} n_\mathrm{E}| = e^{-iH_{Zurek} t} \rho(0) e^{iH_{Zurek} t}. \tag{119}$$

Obviously, it is equal to the exact solution of the Zurek model (109) via directly solving it. Of course, the solutions $\rho_\mathrm{S}(t)$ of this open system obtained by our exact solution or improved form of perturbed solution formula or directly solving method are consistent. Therefore, we can say our improved form of perturbed solution indeed absorbs the contributions from all order approximations of the perturbing Hamiltonian H_SE since it is diagonal. In addition, we would like to point out that although there are the degeneracies in $\widetilde{E}_{n_\mathrm{S} n_\mathrm{E}}$ when $m_\mathrm{S} + m_k = n_\mathrm{S} + n_k$, our improved form of perturbed solution can work well since $H'_{tot1} = 0$.

In order to reveal the advantages of our exact solution and perturbed solution, we add two transverse fields, respectively, in the system and the environment, that is

$$H_{tot} \; = \; \mu\sigma_{\mathrm{S}}^x \otimes I_{\mathrm{E}} + H_{\mathrm{Zurek}} + \sigma_{\mathrm{S}}^z \otimes B_{x\mathrm{E}} = \mu\sigma_{\mathrm{S}}^x \otimes I_{\mathrm{E}} + \sigma_{\mathrm{S}}^z \otimes (B_{x\mathrm{E}} + B_{z\mathrm{E}}), \tag{120}$$

where

$$B_{x\mathrm{E}} = \left[\sum_{k}^{N_{\mathrm{E}}} \left(\bigotimes_{i=1}^{k-1} I_{\mathrm{E}i} \right) \otimes (X_k \sigma_{\mathrm{E}}^{x_k}) \otimes \left(\bigotimes_{j=k+1}^{N_{\mathrm{E}}} I_{\mathrm{E}j} \right) \right]. \tag{121}$$

The problem only with the system transverse field was studied in Ref. [16]. The model (120) is not exactly solvable unless $\mu = 0$. Obviously, there are four kinds of the SESRs.

Case one: The Hamiltonian split is

$$H_{tot0} = H_{\mathrm{S}} + H_{\mathrm{E}} = \mu\sigma_{\mathrm{S}}^x \otimes I_{\mathrm{E}}, \quad H_{tot1} = \sigma_{\mathrm{S}}^z \otimes (B_{x\mathrm{E}} + B_{z\mathrm{E}}). \tag{122}$$

The bases of unperturbed SESR are

$$|\psi_{\mathrm{S}}^{n_{\mathrm{S}}} \chi^{n_{\mathrm{E}}}\rangle = |\psi_{\mathrm{S}}^{n_{\mathrm{S}}}\rangle \otimes \bigotimes_{k=1}^{N_{\mathrm{E}}} |\chi^{n_k}\rangle, \tag{123}$$

where

$$|\psi^{n_{\mathrm{S}}}\rangle \; = \; \frac{1}{\sqrt{2}} \left[|0\rangle_{\mathrm{S}} + (-1)^{n_{\mathrm{S}}} |1\rangle_{\mathrm{S}} \right] \tag{124}$$

$$|\chi^{n_k}\rangle \; = \; \frac{1}{\sqrt{X_k^2 + (Z_k + (-1)^{n_k} Y_k)^2}} \left[(Z_k + (-1)^{n_k} Y_k) |0\rangle_k + X_k |1\rangle_k \right]. \tag{125}$$

where $Y_k = \sqrt{X_k^2 + Z_k^2}$. Here, $|\chi^{n_k}\rangle$ are the eigenvectors of the environment operator $B_{x\mathrm{E}} + B_{z\mathrm{E}} = (X_k\sigma^x + Z_k\sigma^z)$, and corresponding eigenvalues are $(-1)^{n_k} Y_k$. Thus, the eigenvalues of H_{tot0} acting on $|\psi_{\mathrm{S}}^{n_{\mathrm{S}}} \chi^{n_{\mathrm{E}}}\rangle$ are

$$E_{n_{\mathrm{S}} n_{\mathrm{E}}} = E_{n_{\mathrm{S}}, n_1 n_2 \cdots} = \mu(-1)^{n_{\mathrm{S}}}. \tag{126}$$

Case two: The Hamiltonian split is the same as (122), and the corresponding eigenvalues of H_{tot0} are then the same as (126). But the bases of unperturbed SESR can be taken as

$$|\psi^{n_{\mathrm{S}}} n_{\mathrm{E}}\rangle = |\psi^{n_{\mathrm{S}}}\rangle \otimes \bigotimes_{k=1}^{N_{\mathrm{E}}} |n_k\rangle. \tag{127}$$

Case three: The Hamiltonian split is

$$H_{tot0} = 0 \text{ or constant}, \quad H_{tot1} = \mu\sigma_{\mathrm{S}}^x \otimes I_{\mathrm{E}} + \sigma_{\mathrm{S}}^z \otimes (B_{x\mathrm{E}} + B_{z\mathrm{E}}). \tag{128}$$

The bases of a selected SESR are just the natural bases $|n_{\mathrm{S}} n_{\mathrm{E}}\rangle$ defined in (111). Then, we use our Hamiltonian redivision skill to obtain

$$H'_{tot0} = H_{\mathrm{Zurek}}, \quad H'_{tot1} = \mu\sigma_{\mathrm{S}}^x \otimes I_{\mathrm{E}} + \sigma_{\mathrm{S}}^z \otimes B_{x\mathrm{E}}. \tag{129}$$

The corresponding eigenvalues of H'_{tot0} is given by (114).

Case four: The Hamiltonian split is the same as (128). But the bases of the unperturbed SESR can be chosen as

$$|n_{\mathrm{S}} \chi_{n_{\mathrm{E}}}\rangle = |n_{\mathrm{S}}\rangle \otimes \bigotimes_{k=1}^{N_{\mathrm{E}}} |\chi^{n_k}\rangle. \tag{130}$$

Then, we use our Hamiltonian redivision skill to obtain

$$H'_{tot0} = H_{\mathrm{Zurek}} + \sigma_{\mathrm{S}}^z \otimes B_{x\mathrm{E}}, \quad H'_{tot1} = \mu\sigma_{\mathrm{S}}^x \otimes I_{\mathrm{E}}. \tag{131}$$

The corresponding eigenvalue is

$$E'_{n_{\mathrm{S}} n_{\mathrm{E}}} = E'_{n_{\mathrm{S}} n_1 n_2 \cdots} = (-1)^{n_{\mathrm{S}}} \sum_{k=1}^{N_{\mathrm{E}}} (-1)^{n_k} Y_k. \tag{132}$$

It must be emphasized that the four kinds of choices on the SESRS aim at the different preconditions if the cut-off approximation of perturbation is necessary. Cases one and two are used to the preconditions that $\mu \gg Z_k$ and/or $\mu \gg X_k$, that is, the transverse field μ is strong. Case three is chosen when $Z_k \gg \mu$ and $Z_k \gg X_k$. In other words, two transverse fields are weak. Case four is suitable to solve the problem under $Z_k \gg \mu$ and/or $X_k \gg \mu$. This means that the transverse field μ is weak.

It is easy to see that in cases one and two there are two degenerate subspaces with N_E dimensions, which cannot be completely removed via the usual diagonalization procedure of the degenerate subspaces and our Hamiltonian redivision. However, the conditions that degeneracies happen are $\delta_{m_S n_S}$. For case one

$$g_1^{m_S m_E, n_S n_E} = \delta_{m_S(1-n_S)} \left[\sum_{k=1}^{N_E} (-1)^{m_k} Y_k \right] \prod_{l=1}^{N_E} \delta_{m_l n_l}, \tag{133}$$

while for case two

$$g_1^{m_S m_E, n_S n_E} = \delta_{m_S(1-n_S)} \left[\sum_{k=1}^{N_E} (-1)^{m_k} Z_k \left(\prod_{l=1}^{N_E} \delta_{m_l n_l} \right) + \sum_{k=1}^{N_E} \left(\prod_{i=1}^{k-1} \delta_{m_i n_i} \right) X_k \delta_{m_k(1-n_k)} \left(\prod_{j=k+1}^{N_E} \delta_{m_j n_j} \right) \right]. \tag{134}$$

This implies such a fact that in both case one and case two $g_1^{m_S m_E, n_S n_E}$ are vanishing between any two degenerate levels, that is $g_1^{m_S m_E, n_S n_E} \delta_{m_S n_S} = 0$. Therefore, our improved scheme of perturbation theory can work well. However, note that the preconditions that $\mu \gg Z_k$ and/or $\mu \gg X_k$ in cases one and two are the same, we prefer to use the choice of case one because its calculation is easier than case two in our improved scheme of perturbation theory.

As to case three, we also can not completely remove the degeneracies via the usual diagonalization procedure of the degenerate subspaces and our Hamiltonian redivision. The conditions that degeneracies happen are solutions of the following equation

$$\sum_{k=1}^{N_E} Z_k \left[(-1)^{m_S+m_k} - (-1)^{n_S+n_k} \right] = 0, \tag{135}$$

while the off-diagonal elements of the perturbing Hamiltonian matrix are

$$g_1^{m_S m_E, n_S n_E} = \mu \delta_{m_S(1-n_S)} \prod_{k=1}^{N_E} \delta_{m_k, n_k} + \delta_{m_S n_S} \sum_{k=1}^{N_E} \left(\prod_{i=1}^{k-1} \delta_{m_i n_i} \right) X_k \delta_{m_k(1-n_k)} \left(\prod_{j=k+1}^{N_E} \delta_{m_j n_j} \right). \tag{136}$$

It is clear that we can not guarantee, in general, that $g_1^{m_S m_E, n_S n_E}$ are vanishing between any two degenerate levels. If the result is indeed so. This SESR is not a good choice because the remained degeneracies will result in the difficulty to use the usual perturbation theory and complication in our improved ones. If we still intend to use the cut-off approximation of perturbation, the results from case three will not be satisfied enough if the evolution time is long enough, because $A_l(t)$ or $A_{Il}(t)$ has the extra terms proportional to the evolution time that can not be simply absorbed to the exponential power for $l \geq 2$ in our views.

Fortunately, that case four can be covered the precondition that $Z_k \gg \mu$ and $Z_k \gg X_k$ in case three. Hence, we give up the choice of case three and only use the SESR in case four. Actually, the above problems originally motivate us to consider how to choose the appropriate SESR for open systems, which has been seen in Sec. II.

It is easy to get that the conditions that degeneracies happen in case four are solutions of the following equation

$$\sum_{k=1}^{N_E} Y_k \left[(-1)^{m_S+m_k} - (-1)^{n_S+n_k} \right] = 0, \tag{137}$$

while the off-diagonal elements of the perturbing Hamiltonian matrix are

$$g_1^{m_S m_E, n_S n_E} = \mu \delta_{m_S(1-n_S)} \prod_{k=1}^{N_E} \delta_{m_k, n_k}. \tag{138}$$

Our interesting task is to seek for the conditions that degeneracies happen when $g_1^{m_S m_E, n_S n_E} \neq 0$. Hence, we substitute $m_k = n_k$ for any k, into Eq. (137) and rewrite it as

$$\left[\sum_{k=1}^{N_E} Y_k (-1)^{m_k} \right] \left[(-1)^{m_S} - (-1)^{n_S} \right] = 0. \tag{139}$$

Its solution is $m_{\mathrm{S}} = n_{\mathrm{S}}$ unless the exception $\sum_{k=1}^{N_{\mathrm{E}}} Y_k (-1)^{m_k} = 0$. However, this exception is not valid limiting our problem to open systems because it means that $H'_{tot0} = 0$ from Eq. (132), or equivalently, the total H_{tot} becomes $\mu \sigma_{\mathrm{S}}^x \otimes I_{\mathrm{E}}$. Again jointly considering it and the expression (138) of $g_1^{m_{\mathrm{S}} m_{\mathrm{E}}, n_{\mathrm{S}} n_{\mathrm{E}}}$ in case four, we obtain the conclusion that $g_1^{m_{\mathrm{S}} m_{\mathrm{E}}, n_{\mathrm{S}} n_{\mathrm{E}}}$ are indeed vanishing between any two degenerate levels.

In the following discussion, we only focus on case one with the strong transverse field μ and case four with weak transverse field μ in order to illustrate our exact solution and improved form of perturbed solution more simply and better.

Let us define

$$\delta_{m_{\mathrm{E}} n_{\mathrm{E}}} = \prod_{i=1}^{N_{\mathrm{E}}} \delta_{m_i n_i}. \tag{140}$$

$$f_{n_{\mathrm{E}}} = \sum_{n_1, n_2, \cdots = 0}^{1} Y_k (-1)^{n_k} = \sum_{n_{\mathrm{E}} = 0}^{1} Y_k (-1)^{n_k}. \tag{141}$$

Thus, for case one, we can rewrite the off-diagonal elements of the perturbing Hamiltonian matrix as

$$g_1^{m_{\mathrm{S}} m_{\mathrm{E}}, n_{\mathrm{S}} n_{\mathrm{E}}} = \delta_{m_{\mathrm{S}}(1 - n_{\mathrm{S}})} f_{m_{\mathrm{E}}} \delta_{m_{\mathrm{E}} n_{\mathrm{E}}}. \tag{142}$$

Substituting it into the definition of $G_{n_{\mathrm{S}} n_{\mathrm{E}}}^{(a)}$, we obtain

$$G_{n_{\mathrm{S}} n_{\mathrm{E}}}^{(2)} = \frac{(-1)^{n_{\mathrm{S}}} f_{n_{\mathrm{E}}}^2}{2\mu}, \quad G_{n_{\mathrm{S}} n_{\mathrm{E}}}^{(3)} = 0, \quad G_{n_{\mathrm{S}} n_{\mathrm{E}}}^{(4)} = -\frac{(-1)^{n_{\mathrm{S}}} f_{n_{\mathrm{E}}}^4}{8\mu^3}. \tag{143}$$

Hence, we have

$$\widetilde{E}_{n_{\mathrm{S}} n_{\mathrm{E}}} = (-1)^{n_{\mathrm{S}}} \mu \left[1 + \frac{1}{2} \left(\frac{f_{n_{\mathrm{E}}}}{2\mu} \right)^2 - \frac{1}{2} \left(\frac{f_{n_{\mathrm{E}}}}{2\mu} \right)^4 \right]. \tag{144}$$

Similarly, for case four, from Eqs. (132) and (138) it follows that

$$G_{n_{\mathrm{S}} n_{\mathrm{E}}}^{(2)} = \frac{(-1)^{n_{\mathrm{S}}} \mu^2}{2 f_{n_{\mathrm{E}}}}, \quad G_{n_{\mathrm{S}} n_{\mathrm{E}}}^{(3)} = 0, \quad G_{n_{\mathrm{S}} n_{\mathrm{E}}}^{(4)} = -\frac{(-1)^{n_{\mathrm{S}}} \mu^4}{8 f_{n_{\mathrm{E}}}^3}. \tag{145}$$

Hence, we have

$$\widetilde{E}_{n_{\mathrm{S}} n_{\mathrm{E}}} = (-1)^{n_{\mathrm{S}}} f_{n_{\mathrm{E}}} \left[1 + \frac{1}{2} \left(\frac{\mu}{2 f_{n_{\mathrm{E}}}} \right)^2 - \frac{1}{2} \left(\frac{\mu}{2 f_{n_{\mathrm{E}}}} \right)^4 \right]. \tag{146}$$

It is clear that there is a corresponding relation between the case of the strong transverse field μ and the case of weak transverse field μ, that is, their perturbed solutions will be the same under the exchanging transformation $\mu \Leftrightarrow f_{n_{\mathrm{E}}}$. Hence, we only write down, for case one, the zeroth, first and second order parts of total system density matrix at time t, respectively

$$\rho_{tot}^{(0)}(t) = \sum_{m_{\mathrm{S}}, m_{\mathrm{E}} = 0}^{1} \sum_{n_{\mathrm{S}}, n_{\mathrm{E}} = 0}^{1} e^{-\mathrm{i}\left(\widetilde{E}_{m_{\mathrm{S}} m_{\mathrm{E}}} - \widetilde{E}_{n_{\mathrm{S}} n_{\mathrm{E}}}\right)t} \rho_{tot}^{m_{\mathrm{S}} m_{\mathrm{E}}, n_{\mathrm{S}} n_{\mathrm{E}}}(0) |\psi_{\mathrm{S}}^{m_{\mathrm{S}}}\rangle \langle \psi_{\mathrm{S}}^{n_{\mathrm{S}}}| \otimes |\chi^{m_{\mathrm{E}}}\rangle \langle \chi^{n_{\mathrm{E}}}|, \tag{147}$$

$$\begin{aligned}
\rho_{tot}^{(1)}(t) = & \sum_{m_{\mathrm{S}}, m_{\mathrm{E}} = 0}^{1} \sum_{n_{\mathrm{S}}, n_{\mathrm{E}} = 0}^{1} \left(e^{-\mathrm{i}\left(\widetilde{E}_{m_{\mathrm{S}} m_{\mathrm{E}}} - \widetilde{E}_{n_{\mathrm{S}} n_{\mathrm{E}}}\right)t} - e^{-\mathrm{i}\left(\widetilde{E}_{m_{\mathrm{S}} m_{\mathrm{E}}} - \widetilde{E}_{(1 - n_{\mathrm{S}}) n_{\mathrm{E}}}\right)t} \right) \frac{(-1)^{n_{\mathrm{S}}} f_{n_{\mathrm{E}}}}{2\mu} \\
& \times \rho_{tot}^{m_{\mathrm{S}} m_{\mathrm{E}}, n_{\mathrm{S}} n_{\mathrm{E}}}(0) |\psi_{\mathrm{S}}^{m_{\mathrm{S}}}\rangle \langle \psi_{\mathrm{S}}^{1 - n_{\mathrm{S}}}| \otimes |\chi^{m_{\mathrm{E}}}\rangle \langle \chi^{n_{\mathrm{E}}}| \\
& + \sum_{m_{\mathrm{S}}, m_{\mathrm{E}} = 0}^{1} \sum_{n_{\mathrm{S}}, n_{\mathrm{E}} = 0}^{1} \left(e^{-\mathrm{i}\left(\widetilde{E}_{m_{\mathrm{S}} m_{\mathrm{E}}} - \widetilde{E}_{n_{\mathrm{S}} n_{\mathrm{E}}}\right)t} - e^{-\mathrm{i}\left(\widetilde{E}_{(1 - m_{\mathrm{S}}) m_{\mathrm{E}}} - \widetilde{E}_{n_{\mathrm{S}} n_{\mathrm{E}}}\right)t} \right) \frac{(-1)^{m_{\mathrm{S}}} f_{m_{\mathrm{E}}}}{2\mu} \\
& \times \rho_{tot}^{m_{\mathrm{S}} m_{\mathrm{E}}, n_{\mathrm{S}} n_{\mathrm{E}}}(0) |\psi_{\mathrm{S}}^{1 - m_{\mathrm{S}}}\rangle \langle \psi_{\mathrm{S}}^{n_{\mathrm{S}}}| \otimes |\chi^{m_{\mathrm{E}}}\rangle \langle \chi^{n_{\mathrm{E}}}|,
\end{aligned} \tag{148}$$

$$
\begin{aligned}
\rho_{tot}^{(2)}(t) = &-\sum_{m_{\mathrm{S}},m_{\mathrm{E}}=0}^{1}\sum_{n_{\mathrm{S}},n_{\mathrm{E}}=0}^{1}\left(\mathrm{e}^{-\mathrm{i}\left(\widetilde{E}_{m_{\mathrm{S}}m_{\mathrm{E}}}-\widetilde{E}_{n_{\mathrm{S}}n_{\mathrm{E}}}\right)t}-\mathrm{e}^{-\mathrm{i}\left(\widetilde{E}_{m_{\mathrm{S}}m_{\mathrm{E}}}-\widetilde{E}_{(1-n_{\mathrm{S}})n_{\mathrm{E}}}\right)t}\right)\left(\frac{f_{n_{\mathrm{E}}}}{2\mu}\right)^{2}\\
&\times\rho_{tot}^{m_{\mathrm{S}}m_{\mathrm{E}},n_{\mathrm{S}}n_{\mathrm{E}}}(0)|\psi_{\mathrm{S}}^{m_{\mathrm{S}}}\rangle\langle\psi_{\mathrm{S}}^{n_{\mathrm{S}}}|\otimes|\chi^{m_{\mathrm{E}}}\rangle\langle\chi^{n_{\mathrm{E}}}|\\
&-\sum_{m_{\mathrm{S}},m_{\mathrm{E}}=0}^{1}\sum_{n_{\mathrm{S}},n_{\mathrm{E}}=0}^{1}\left(\mathrm{e}^{-\mathrm{i}\left(\widetilde{E}_{m_{\mathrm{S}}m_{\mathrm{E}}}-\widetilde{E}_{n_{\mathrm{S}}n_{\mathrm{E}}}\right)t}-\mathrm{e}^{-\mathrm{i}\left(\widetilde{E}_{(1-m_{\mathrm{S}})m_{\mathrm{E}}}-\widetilde{E}_{n_{\mathrm{S}}n_{\mathrm{E}}}\right)t}\right)\left(\frac{f_{m_{\mathrm{E}}}}{2\mu}\right)^{2}\\
&\times\rho_{tot}^{m_{\mathrm{S}}m_{\mathrm{E}},n_{\mathrm{S}}n_{\mathrm{E}}}(0)|\psi_{\mathrm{S}}^{m_{\mathrm{S}}}\rangle\langle\psi_{\mathrm{S}}^{n_{\mathrm{S}}}|\otimes|\chi^{m_{\mathrm{E}}}\rangle\langle\chi^{n_{\mathrm{E}}}|\\
&+\sum_{m_{\mathrm{S}},m_{\mathrm{E}}=0}^{1}\sum_{n_{\mathrm{S}},n_{\mathrm{E}}=0}^{1}\left(\mathrm{e}^{-\mathrm{i}\widetilde{E}_{m_{\mathrm{S}}m_{\mathrm{E}}}t}-\mathrm{e}^{-\mathrm{i}\widetilde{E}_{(1-m_{\mathrm{S}})m_{\mathrm{E}}}t}\right)\left(\mathrm{e}^{-\mathrm{i}\widetilde{E}_{n_{\mathrm{S}}n_{\mathrm{E}}}t}-\mathrm{e}^{-\mathrm{i}\widetilde{E}_{(1-n_{\mathrm{S}})n_{\mathrm{E}}}t}\right)\\
&\times\frac{(-1)^{m_{\mathrm{S}}+n_{\mathrm{S}}}f_{m_{\mathrm{E}}}f_{n_{\mathrm{E}}}}{4\mu^{2}}\rho_{tot}^{m_{\mathrm{S}}m_{\mathrm{E}},n_{\mathrm{S}}n_{\mathrm{E}}}(0)\big|\psi_{\mathrm{S}}^{1-m_{\mathrm{S}}}\big\rangle\big\langle\psi_{\mathrm{S}}^{1-n_{\mathrm{S}}}\big|\otimes|\chi^{m_{\mathrm{E}}}\rangle\langle\chi^{n_{\mathrm{E}}}|.
\end{aligned}
\tag{149}
$$

It is easy to give the solution of the reduced density matrix of the open system up to the improved form of the second order approximation by tracing off the environment space, that is

$$
\rho_{\mathrm{S}}(t) = \mathrm{Tr}_{\mathrm{E}}\left[\rho_{tot}^{(0)}(t)+\rho_{tot}^{(1)}(t)+\rho_{tot}^{(2)}(t)\right].
\tag{150}
$$

For a given initial state, this trace is very easy to calculate and the explicit form of of the open system solution is obtained. Then we can discuss the decoherence and entanglement dynamics according to the methods in, for example, [16, 17], and they are arranged in our forthcoming manuscript (in preparing) [18].

IX. DISCUSSION AND CONCLUSION

This paper studies open systems dynamics, which is the third in our serial studies on quantum mechanics in general quantum systems. Its conclusions are obtained based on our previous two works [12, 13].

It must be emphasized that we study open system dynamics according to the "the first principle", that is, the Schrödinger equation and the von Neumann equation, and we do not consider the phenomenological methods and theories. For generality in theory, we obtain the exact solution of the open system without using any approximation. The deduction of our exact master equation only uses the factorizing initial condition. Particularly, we derive out our perturbed master equation and its improved form, but we give up all of approximations used in the traditional methods and formulism except for the factorizing initial condition. It is very interesting that we get the Redfield master equation without using the Born-Markov approximation. This implies the Born-Markov approximation is unnecessary based on our results.

A simple but key idea to obtain our exact solution of open systems is an appropriate choice of the SESR. In fact, it closely connects with the Hamiltonian redivision skill [13]. In Sec. VIII, we have clearly stated its reasons. Originally, the aim that we propose this idea is to break the accustomed choice of $H_{\mathrm{S}}+H_{\mathrm{E}}$, and build a picture to allow the interaction between the open system and its environment into our unperturbed representation. This makes the Hamiltonian redivision skill to look like more natural.

Our exact solution and master equation of open systems are general and explicit in form because all order approximations of the perturbing Hamiltonian not only are completely included but also are clearly expressed, although it is an infinite series. In special, they are in c-number function forms rather than operator forms. This means that they can inherit the same advantage as the Feynman path integral expression. Moreover, they are power series of the perturbing Hamiltonian like as the Dyson series in the interaction picture. This implies that the cut-off approximation of perturbation can be made for the needed precision of the problems.

Based on our improved scheme of perturbation theory, the improved forms of perturbed solution and perturbed master equation can absorb the partial contributions from the high order even all order approximations of perturbation. Therefore, we can say that our open system dynamics is actually calculable, operationally efficient, conclusively more accurate.

In order to extend our method, we also discuss Milburn model of open systems. In fact, from our point of view, Miburn model of dynamics should be applied to so-called Milburn-type closed quantum systems made up of the interesting open system and its near environment. A Milburn-type closed quantum system is not really closed system from the view that a really closed system must obey the von Neumann equation. If one cannot know the Hamiltonian of its remote environment and the interaction form between the interested system and its remote environment, Milburn model of dynamics might be a choice scheme to study this kind of open systems. In the above sense, the extra term

with θ_0 multiplier in the Milburn equation represents the influence from the remote environment. Obviously, when $\theta_0 = 0$, Milburn dynamics back to von Neumann dynamics. This implies that the (very) remote environment can be ignored. We obtain the exact solution that can provide a general tool to investigate those interesting and complicated open systems when the environment model is partially known. However, there a free parameter θ_0 in the Milburn model. It is still not been given by the theory, but it should be able be known by the experiment if we think the extra term in the Milburn dynamics is resulted in by the remote environment.

Note that our open system dynamics is derived from the first principle, our open system dynamics is not applicable to the cases that ones do not clearly know the Hamiltonians of the open system, its environment and the interaction between the system and the environment unless at this time Milburn model is suitable. How to relate with some phenomenological theory of open systems will be done in the near future.

As examples, Zurek model of two-state open system and its extension with two transverse fields are studied, respectively, in the strong and weak fields acting on the system. We specially display how to choose the appropriate SESR. They indicate that our open system dynamics is a powerful theory and tool. We are sure that our open system dynamics can be used to more open systems since its generality and clearness, and the calculations are simpler and more efficient, the results are more accurate and more reliable than the existed scheme.

In summary, our results can be thought of as theoretical developments of open system dynamics, and they are helpful for understanding the theory of quantum mechanics and providing some powerful tools for the calculation of decoherence, entanglement dynamics, quantum dissipation, quantum transport in general quantum systems and so on. Together with our exact solution and perturbation theory [12, 13], they can finally form the foundation of theoretical formulism of quantum mechanics in general quantum systems. Further study on quantum mechanics of general quantum systems is on progressing.

Acknowledgments

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