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The Batalin-Vilkovisky
Lagrangian quantisation scheme
with applications to the study of
anomalies in gauge theories

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I was born not knowing,
and have only had a little time
to change that here and there.

Richard Feynman

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The Batalin-Vilkovisky Lagrangian quantisation scheme, with applications to the study of anomalies in gauge theories.

Abstract

Although gauge theories took the centre stage in theoretical physics only in this century, they are the rule, not the exception. The two standard examples of gauge theories are Maxwell's theory for electromagnetic phenomena and Einstein's theory of general relativity. Recent attempts to unify the four fundamental forces have provided new and more complex examples of gauge theories.

We study the Batalin-Vilkovisky (BV) Lagrangian quantisation scheme. Most of the gauge theories known today can be quantised using this scheme. We show how the BV scheme can be constructed by combining the gauge symmetry with the quantum equations of motion, the so-called Schwinger-Dyson equations. The different quantisation prescriptions that were known for the different types of gauge theories can be reformulated in one unified framework, using this guiding principle. One of the most remarkable properties of the BV scheme is that it possesses a symplectic structure that is invariant under canonical transformations, in analogy with classical mechanics. The possibilities offered by the canonical transformations are exploited in some examples, e.g. the construction of four-dimensional topological Yang-Mills theory.

In a second part, we show how the Lagrangian BV scheme can be derived from the Hamiltonian description of gauge theories.

When the quantum effects destroy the gauge symmetry that was present in the classical theory, the gauge symmetry is said to be anomalous. In the third part, we use Pauli-Villars regularisation to determine whether a theory is anomalous or not. We study how one can influence which symmetries will be anomalous and a new derivation is given of the one-loop regularised BV scheme.

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Introduction

The number of rational hypotheses that can explain any given phenomenon, is infinite[1]. Physicists –and other scientists alike– have always used two criteria to construct an hierarchy in this vast infinity of hypotheses. First of all, an hypothesis –or a model– that is able to explain more phenomena than other hypotheses, is considered to be superior. Secondly, physicists are easily seduced by the presence of *symmetry* in a model. It goes without saying, that both aspects are not independent, as two seemingly different phenomena might be related by a symmetry principle. In that case, symmetry allows to build one model that describes both phenomena. After having selected some preferred hypotheses in this way, the final justification for the promotion of an hypothesis to a *scientific fact* is of course an experimental verification.
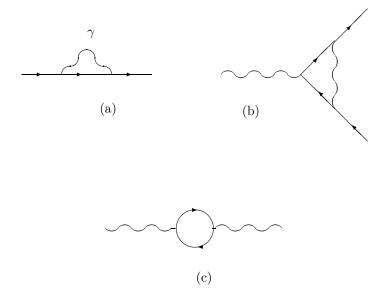
J.C. Maxwell was one of the first to unify two different forces in one model[2]. In 1864, he brought together the forces of electricity and magnetism in a set of coupled differential equations, the Maxwell equations. This Maxwell theory is also the first example of a gauge theory, that is, a theory with an invariance with space-time dependent transformation parameters. Even when trading the six degrees of freedom of the electric and magnetic vector field for the four degrees of freedom of the Lorentz vector potential field A_μ, not all configurations A_μ describe different physical systems. Indeed, the four field equations for the A_μ are not independent and the action is invariant under $A_\mu \to A_\mu + \partial_\mu \epsilon$.

Soon after a mathematical framework for quantum mechanics had been developed, people started studying the quantum theory of the interaction of electrons and photons (the particle excitations of the electromagnetic field), quantum electrodynamics. In the perturbative expansion, one immediately ran into trouble. The three one loop diagrams (see figures (a-c)) were found to lead to infinite corrections to the zeroth order (classical) result. This problem was finally solved by R.P. Feynman, S. Tomonaga and J. Schwinger[3] by what is now known as the *renormalisation procedure*. Schematically, this means that the infinities are removed by a redefinition of the free parameters of the theory, the charge e of the electron and its mass m. That the three divergences depicted in (a-c) can be removed using two parameters is the result of gauge invariance. The gauge invariance of the classical theory also manifests itself at the quantum level, and has as a consequence that the divergences (a) and (b) are not independent. Hence, gauge invariance plays a crucial role in the renormalisability of the theory.

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[1]R.M. Pirsig, *Zen, and the art of motorcycle maintenance*, Bantam book, New Age edition, 1981, p100.

[2]Phil. Trans. R. Soc. **155** (1864) 459.

[3]A collection of the early articles can be found in *Quantum Electrodynamics*, ed. J. Schwinger, Dover, New York, 1958.

The three one loop divergences of QED.
The lines with an arrow are fermions,
the wiggly lines photons.

Inspired by the example of quantum electrodynamics, C.N. Yang and R. Mills[4] proposed a model for the strong interaction between protons and neutrons, based on the $SU(2)$ Lie algebra. With every generator of this algebra, a vector boson is associated. These vector bosons are the force carriers, in analogy with the photon. This principle was generalised to arbitrary Lie algebras, and nowadays a whole industry of modelbuilding based on this method exists. However, more important for our discussion is that all these models have local symmetries, gauge symmetries. Moreover, it was recognised that also the theory of general relativity allows for a formulation as a gauge theory, where the Christoffel symbols play the role of gauge fields and where arbitrary coordinate transformations are the local gauge transformations.

It is one thing to construct a classical gauge theory, but it is a different story to quantise the theory and to prove that it is renormalisable. The attempts to demonstrate that Yang-Mills theories are renormalisable have stimulated the research on the quantisation itself of gauge theories[5]. What was needed was a means to quantise (Yang-Mills) gauge theories in different gauges, as the renormalisability is more transparant in other gauges than those gauges where the physical content of the theory is more clear. In solving this problem, the quantisation using path integrals proved to be superior to operator quantisation. In 1967, L.D. Faddeev and V.N. Popov[6] showed how the integration over configurations that are related by a gauge transformation can be factored out from the path integral. In their recipe, extra fields are introduced as a technical device to rewrite a Jacobian as resulting from fictitious particle interactions. These extra fields are called *ghost fields*, and were already foreshadowed in the work of R.P. Feynman and B. DeWitt a few years earlier. Ever since, ghosts wander

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[4]Phys. Rev. **96** (1954) 191.

[5]For a lively account, see M.J.G. Veltman, *The path to renormalisability*, Third International Symposium on the History of Particle Physics, SLAC, june 1992.

[6]Phys. Lett. **25B** (1967) 29.

over the battle fields of theoretical particle physics.

Gauge theories really took the centre stage after G. 't Hooft and M. Veltman proved in 1971 that gauge theories of the Yang-Mills type are renormalisable[7]. Let us stress that all phenomenological models in particle physics, namely the gauge theory of electroweak interactions and QCD for the strong interactions, are of that type. The theoretical framework of the early seventies is still sufficient to account for present day (1993) experimental particle physics. All subsequent developments in theoretical particle physics are of speculative nature and largely motivated by the attempts to unify *all* four forces, including gravity, in one (quantum) theory.

A nice reformulation of the results on quantisation and renormalisation of gauge theories, and that later proved to be the one amenable to generalisation, was presented by C. Becchi, A. Rouet and R. Stora[8]. They discovered that in the quantisation process the local gauge invariance is replaced by a global invariance, which now goes under the name BRST invariance. This transformation is encoded in a nilpotent operator δ ($\delta^2 = 0$). The nilpotency of this operator has led to the introduction of cohomological methods in the study of gauge theories.

In the second half of the seventies, it was found that in some supergravity models a term quartic in the ghosts is needed[9] for unitarity. Such terms can not be generated with the usual Faddeev-Popov procedure, which only gives quadratic ghost actions. The way out was to use BRST invariance of the quantum theory as a guiding principle. However, the nilpotency of the BRST operator then only holds upon using the field equations of the gauge fixed action. Theories where this is the case, are said to have an open algebra.

Another complication that was discovered, is that for some theories the action for the ghosts (obtained à la Faddeev-Popov) has itself a gauge symmetry that needs gauge fixing. This happens when the original gauge symmetries are not independent, and the theory is said to be a reducible gauge theory. What is needed in such cases is yet another enlargement of the field spectrum, with so-called ghosts for ghosts.

Although all these types of theories can be quantised using ad hoc rules, there is a quantisation scheme that encompasses all the previous developments in one unified formalism, the Batalin-Vilkovisky (BV) scheme[10]. It is the focus of our attention in this work. The scheme uses the Lagrangian formulation of path integral quantisation. Next to providing us with a powerful tool for the quantisation of gauge theories of any kind known today, the BV recipe also gives a natural environment for the study of the occurrence of anomalies in gauge theories[11]. A gauge theory is anomalous, when quantum effects destroy the BRST invariance of the theory. Here too, recent developments like string theory have led to new examples to study.

At this moment, we should give a long list of the assets of the BV scheme, to arouse the appetite of the reader and to motivate our effort to investigate both the BV scheme itself and its applications. But since it is difficult to describe

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[7] Nucl. Phys. **B50** (1972) 318.

[8] Phys. Lett. **52B** (1974) 344.

[9] R.E. Kallosh, Nucl. Phys. **B141** (1978) 141.
G. Sterman, P.K. Townsend and P. van Nieuwenhuizen, Phys. Rev. **D17** (1978) 1501.

[10] Phys. Lett. **102B** (1981) 27.

[11] W. Troost, P. van Nieuwenhuizen, A. Van Proeyen, Nucl. Phys. **B333** (1990) 727.

the beauty of the Alps to a Dutchman who has never left Holland, we postpone this motivation to our conclusions.

This thesis is divided in three main parts. In the first part, we study the Batalin-Vilkovisky Lagrangian quantisation scheme itself. In particular, we show how it naturally follows from and encompasses the Faddeev-Popov and BRST quantisation methods. The Lagrangian BV scheme is derived from the Hamiltonian quantisation scheme for gauge theories of Batalin-Fradkin-Vilkovisky in the second part of this dissertation. In the third part, we use the BV quantisation scheme to study some aspects of the gauge anomalies that can appear in the quantisation process. The appendices contain, apart from some technical rules, a discussion of the algebraic steps that are common for all the regularised calculations of anomalies in part III.

We start by introducing *gauge theories*, the subject of this dissertation, in chapter one. Different types of gauge theories exist, depending on the structure functions of the gauge algebra. Whatever their type, gauge theories lead to divergent functional integrals upon naive quantisation, since infinitely many elements of the configuration space describe the same physical system. The Faddeev-Popov quantisation procedure to remedy this problem, is presented in chapter two. Following Faddeev and Popov, one can construct a gauge fixed action that has no local gauge invariances anymore and that can be used as a starting point for perturbative calculations. However, this gauge fixed action has a global symmetry, the *BRST symmetry*. It is argued in chapter three that it is expedient to formulate the complete quantisation procedure from the point of view that the local gauge invariance of the classical theory is to be traded for BRST invariance of the quantum theory. The BRST symmetry of the quantum theory leads to the so-called *Ward identities*, which are relations between correlation functions that express the consequences of the gauge symmetry for the full quantum theory. When quantum effects destroy the BRST symmetry of the quantum theory, the gauge symmetry is said to be *anomalous(ly broken)*.

In the fourth chapter, the equations of motion of the quantum theory, the Schwinger-Dyson (SD) equations, are derived as Ward identities from a global BRST symmetry. This is done by introducing a gauge symmetry, called the *SD shift symmetry*, using a collective field formalism. By demanding that the BRST symmetry algebra of any gauge theory be enlarged such that it includes the SD shift symmetry, we reconstruct the BV scheme in chapter five. One of the most remarkable properties of that scheme is that for every field ϕ^A an *antifield* ϕ_A^* is introduced. Fields and antifields are canonically conjugated with respect to the *antibracket*, in much the same way as coordinates and momenta are conjugated in classical Hamiltonian mechanics. Most of the features of the BRST quantisation recipe are translated in the BV scheme using the antibracket: the construction of the BRST invariant gauge fixed action, the Ward identities, the condition that operators have to satisfy in order to have a gauge invariant expectation value etc. As examples, we[12] discuss the construction of topological Yang-Mills theory and we derive a general prescription for the construction of a BRST invariant energy-momentum tensor in the BV scheme.

If the commutator of two infinitesimal gauge transformations is only a linear combination of infinitesimal gauge transformations when acting on the classical configurations that satisfy the classical equations of motion, the algebra is said to be *open*. In chapter six, we describe the BRST quantisation of theories with

an open gauge algebra. Here too, we[13] enlarge the BRST symmetry such that it includes the SD shift symmetry, and derive the BV scheme for open algebras. We[14] construct an antifield scheme that is invariant under both BRST and anti-BRST symmetry in chapter seven (anti-BRST symmetry is introduced in chapter three).

Chapter eight, the final chapter of part one, contains a reformulation of the quantisation in the BV scheme using *canonical transformations*. Canonical transformations are transformations of fields and antifields that leave the antibracket invariant, in analogy with canonical transformations in classical mechanics that leave the Poisson bracket invariant. The examples are the continuation of the examples of chapter five[15].

All the developments in part I are based on the Lagrangian, Lorentz covariant description of field theory, in casu gauge theory. In part II, we derive the Lagrangian BV formalism from the Hamiltonian path integral quantisation procedure for gauge theories. In chapter nine, an overview is given of the description of gauge symmetries in the Hamiltonian canonical formalism and of the standard quantisation procedure using an Hamiltonian formulation of BRST symmetry. We[16] then apply this recipe to derive the Schwinger-Dyson equations as Ward identities of the Hamiltonian formalism in chapter ten. We also show that if we enlarge the BRST symmetry of the Hamiltonian system to include the SD shift symmetry, we naturally obtain the Lagrangian BV formalism after integration over the momenta.

Part III contains a detailed, one-loop regularised study of anomalies in gauge theories. Gauge anomalies are defined and some of their properties, in particular the Wess-Zumino consistency condition, are discussed in chapter eleven. All these results are also reformulated in the BV formalism. In order to calculate an explicit expression for the anomaly in a specific model, a regularisation scheme is required. As is explained in chapter twelve, the functional integral can be regularised up to one loop using Pauli-Villars regularisation. This regularisation scheme leads to an expression for the anomaly that is of the same type as proposed by K. Fujikawa. An important role is played by the mass term of the PV fields. The invariances of this mass term determine which symmetries will be anomaly free. We[17] show how the freedom in the choice of mass term can be exploited to calculate actions for gauge fields that are induced by matter fields (induced gravity, Wess-Zumino-Witten model). Chapter thirteen contains a new derivation of the regularised, one-loop master equation of the BV scheme. In the final chapter of the third part and of this work, we[18] demonstrate that one can keep preferred gauge symmetries anomaly free by the introduction of extra (scalar) degrees of freedom.

The models in the examples are always presented without giving a raison d' être for these models. They only serve as an illustration of the points raised in the general developments. In particular, the examples of the third part always involve two-dimensional models. The general recipe has been applied to four-

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[13]F. De Jonghe; CERN-TH-6858/93, KUL-TF-93/13, accepted for publication in J.Math.Phys.
[14]P.H. Damgaard and F. De Jonghe; Phys. Lett. **B305** (1993) 59.
F. De Jonghe; KUL-TF-93/37.
[15]F. De Jonghe and S. Vandoren, Op. Cit.
[16]F. De Jonghe; Phys. Lett. **B316** (1993) 503.
[17]F. De Jonghe, R. Siebelink and W. Troost; Phys. Lett. **B288** (1992) 47.
[18]F. De Jonghe, R. Siebelink and W. Troost; Phys. Lett. **B306** (1993) 295.

dimensional models[19] as well, although the amount of algebraic work increases drastically with the space-time dimension.

We do not discuss the quantisation of reducible gauge theories in this dissertation. Basically, the problem there is the construction of the correct particle spectrum, of the BRST transformation rules and of a suitable gauge fermion. There too, the BRST operator on the complete set of fields is either nilpotent or on-shell nilpotent. The BV scheme can then be developed for reducible gauge theories as well, following the steps of the chapters 5 and 6.

[19]F. De Jonghe, R. Siebelink, W. Troost, S. Vandoren, P. van Nieuwenhuizen and A. Van Proeyen; Phys. Lett. **B289** (1992) 354.

Part I

The Batalin-Vilkovisky Lagrangian quantisation scheme for gauge theories

Chapter 1

The definition and basic properties of gauge theories

We start by introducing in section one the study subject of this dissertation: *gauge theories*. It is shown why gauge theories require special quantisation procedures. The second section contains a definition of the *gauge algebra* and a classification of the different types of algebras that are known to exist.

1.1 What is a gauge theory and why does it need a special quantisation procedure ?

Before we start, some technical remarks. We will use the so-called DeWitt notation. This means that we denote the (field)degrees of freedom by ϕ^i. Here, the index i runs over the *internal* degrees of freedom (e.g. the Lorentz index of the vector potential in electromagnetism) as well as over the space-time variable. Repeated indices are summed over, except when explicitly indicated otherwise. The convention implies that whenever a summation over i occurs, an integral over space-time is understood.

Here and below we will loosely use notions like (in)dependence, completeness et cetera, hoping that the context makes clear what is meant. In the literature [1, 2, 3, 4, 5], these concepts are defined by using the rank of matrices. In the proofs of various properties, the results of finite dimensional analysis are then used. When treating the DeWitt indices as if they only run over a finite number of values, the difference between global and local symmetries is somewhat obscured. Moreover, the locality of the function(al)s of the fields is not always guaranteed.

In the following, when we use the term *quantisation*, we have in mind the quantisation method based on path integrals. This method consists of the following steps. One starts from a configuration space with degrees of freedom labelled by ϕ^i, which for the argument we take to be bosonic. On this configuration space, an action functional $S[\phi]$ is defined, which associates with every configuration a real number, and which specifies the dynamics. The partition

function is then defined as

$$\mathcal{Z} = \int [d\phi] e^{\frac{i}{\hbar} S[\phi]} \; ; \tag{1.1}$$

that is, $\mathcal{Z}$ is a summation over all configurations, where every configuration contributes a complex number of unit norm and with a phase determined by the value of the action functional for that configuration.

The action functional (which we will henceforth call the *action*) is such that the classical theory is described by the configurations ϕ_0^i that extremise it. These configurations are the solutions of the *field equations* (also known as *equations of motion* or *Euler-Lagrange equations*). Denoting

$$y_i(\phi^l) = \frac{\overleftarrow{\delta} S}{\delta \phi^i}, \tag{1.2}$$

the field equations are $y_i(\phi^l) = 0$. The subspace of configuration space consisting of all solutions of these field equations is called the *stationary surface*. We will always assume that at least one classical solution exists [5], as this is a basic requirement to set up the perturbation theory by considering quantum fluctuations around a classical solution [8, 9].

Suppose now that a set of operators $R_\alpha^i[\phi]$ exists, such that

$$y_i(\phi) R_\alpha^i[\phi] \epsilon^\alpha = 0, \tag{1.3}$$

for arbitrary values of the parameters ϵ^α. If the index α does *not* include a space-time index, we speak of a *global* or *rigid symmetry*. If the parameters are space-time dependent, the symmetry is a *local* or *gauge symmetry*. The latter are the main subject of this work. In that case, we will sometimes refer to the ϕ^i as *gauge fields*[1]. If no such operators R_α^i exist, and hence all field equations y_i are independent, one has a theory without global or local symmetries, and the stationary surface is really a stationary point, provided appropriate boundary conditions in space and time are specified.

One of the consequences of (1.3) is that the Hessian, defined by

$$H_{ij} = \frac{\overrightarrow{\delta}}{\delta \phi^j} \frac{\overleftarrow{\delta} S[\phi]}{\delta \phi^i}, \tag{1.4}$$

has zeromodes when evaluated in a point of the stationary surface[2]. This is easily proved by differentiating (1.3) with respect to ϕ^j:

$$\frac{\overrightarrow{\delta}}{\delta \phi^j} \frac{\overleftarrow{\delta} S[\phi_0^l]}{\delta \phi^i} \cdot R_\alpha^i[\phi_0^l] = 0 \; . \tag{1.5}$$

ϕ_0^l denotes a field configuration that is a solution of the field equations, and hence $y_i(\phi_0^l) = 0$ identically.

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[1] This terminology is somewhat different from the usual one. When considering models like, for instance, QCD one distinguishes between matter fields (the fermion fields for the quarks) that have well-defined propagators, and the vector bosons, the gauge fields, that have ill-defined propagators. We do not make this distinction and denote *all* fields present in a classical action that has gauge symmetries by *gauge fields*.

[2] When an expression is evaluated on the stationary surface, we will sometimes say that *it is taken on shell*.

The zeromodes[3] $R^i_\alpha[\phi^l_0]$ determine an infinitesimal transformation which maps a classical solution to another classical solution. Indeed, for infinitesimal parameters ϵ^α such that a Taylor series expansion to linear order makes sense, we have that

$$y_i(\phi^l_0 + R^l_\alpha[\phi^k_0]\epsilon^\alpha) = 0. \qquad (1.6)$$

It may be necessary to impose boundary conditions on the ϵ^α in order for $\phi^l_0 + R^l_\alpha[\phi^k_0]\epsilon^\alpha$ to satisfy the original boundary conditions. This result is valid for global as well as local symmetries. The upshot is that indeed the stationary point becomes a stationary surface. In the case of a global symmetry, it becomes a finite dimensional space, coordinatised by a finite set of parameters ϵ^α. For local symmetries, the stationary surface becomes infinite dimensional: the index α contains a space-time point, so we are free to choose a discrete set of parameters ϵ^i at every space-time point x ($\alpha = (i, x)$).

Hence, already at the level of classical field theory, the presence of zeromodes manifests itself. Also the difference between global and local symmetries shows up. Whereas global symmetries just relate a set of solutions of the field equations parametrised by a finite set of arbitrary parameters, the local symmetries really mean that not all field equations are independent, and hence not all field degrees of freedom ϕ^i are fixed by the classical field equations. Arbitrary fields appear in the classical solutions. In order to eliminate this arbitrariness, one needs in such cases to impose other constraints on the fields, the *gauge fixing conditions*. The example everybody is familiar with, is of course classical electromagnetism. The Maxwell equations are, even when expressed in terms of the Lorentz vector potential A_μ, redundant and one imposes a gauge condition before solving them.

If one would try to quantise theories with a gauge symmetry by exponentiating the action and summing (integrating) over all field configurations, one would run into problems when setting up the usual perturbation theory. Suppose that one picks a classical solution ϕ^l_0 and one makes the saddle point approximation [8, 9]. The quantum fluctuations around the classical solution are composed of two contributions: fluctuations along the stationary surface, which can be parametrised ϵ^α, and fluctuations which take the field away from the stationary surface, which we denote by $\delta_\perp\phi^i$. We have:

$$\phi^i = \phi^i_0 + R^i_\alpha\epsilon^\alpha + \delta_\perp\phi^i. \qquad (1.7)$$

Notice that only in the case of a local gauge symmetry the ϵ^α are field degrees of freedom. Expanding the action up to terms quadratic in the fluctuations (which is sufficient to study the one-loop structure of the theory, or, in other words, the first quantum corrections of order $\hbar$) we get:

$$S[\phi^i] = S[\phi^i_0] + \left(R^j_\alpha\epsilon^\alpha + \delta_\perp\phi^j\right)\left[\frac{\overleftarrow{\delta}}{\delta\phi^i}\frac{\overrightarrow{\delta}}{\delta\phi^j}S[\phi^l_0]\right] \cdot \left(R^i_\beta\epsilon^\beta + \delta_\perp\phi^i\right). \qquad (1.8)$$

Because of (1.5), all terms which depend on the fields (in the case of a local symmetry) ϵ^α drop out, and the previous expression reduces to

$$S[\phi^i] = S[\phi^i_0] + \delta_\perp\phi^j\frac{\overleftarrow{\delta}}{\delta\phi^i}\frac{\overrightarrow{\delta}}{\delta\phi^j}S[\phi^i_0]\delta_\perp\phi^i. \qquad (1.9)$$

Owing to the decomposition (1.7), the measure of the path integral $[d\phi]$ also splits up. We symbolically write $[d\epsilon][d\delta_\perp\phi]$. As the integrand is independent

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[3]We will use the name *gauge generators* below for the R^i_α.

of ϵ^α, the integration over these degrees of freedom factorises from the path integral. This is not really a problem for global symmetries. Even if the integral over the parameter space of the symmetries is divergent in that case, it can be cured by imposing boundary conditions on the integration domain of the path integral. For gauge symmetries however, we have field degrees of freedom $\epsilon^\beta(x)$ which have no quadratic part in the action, and hence have no propagator. The classical solution around which one expands is often taken to be $\phi_0^i = 0$. The Hessian is then just the quadratic part of the original action. This leads to the common criterion that gauge symmetries manifest themselves in the fact that the quadratic part of the action is not invertible.

Let us finally make a small comment on the notion of a *regular theory* [5]. A theory is called regular if the non-invertibility of the Hessian is only due to the symmetries generated by R_α^i. Two categories of irregular theories have been identified. First, it may occur that even when the stationary surface reduces to a stationary point, the Hessian has zeromodes in that stationary point. A pathological example is provided by $S = \phi^3$. The classical solution is $\phi = 0$, and the second derivative evaluated for this solution is also zero. From the previously mentioned semiclassical expansion it is clear that even when there are no symmetries, the perturbation series can not be set up for this kind of theories. A second type of irregular theory is found when the notion of non-degeneracy of the Hessian itself becomes ill-defined.

We will only consider (local) function(al)s of the fields such that if they vanish on the stationary surface, they are proportional to field equations everywhere in configuration space. That is,

$$X(\phi_0^i) = 0 \Rightarrow X(\phi^i) = y_j Y^j(\phi^i) , \qquad (1.10)$$

for some functions Y^j. Again, the standard proof of this property starting from the definition of regularity uses a discrete set of degrees freedom (see e.g.[6]).

Gauge theories are characterised by the fact that their field equations are not independent. We have demonstrated how this leads to divergences when setting up the perturbation theory in a naive path integral quantisation procedure.

1.2 The gauge algebra

Let us now further study the properties of the R_α^i. We consider a set of degrees of freedom ϕ^i, with Grassmann parity[4] ϵ_i. We start from a set of gauge generators R_α^i with the following properties:

$$y_i R_\alpha^i = 0$$
$$\epsilon_{R_\alpha^i} = \epsilon_i + \epsilon_\alpha. \qquad (1.11)$$

Again, we denote $y_i = \frac{\overleftarrow{\delta} S}{\delta \phi^i}$. From now on, we always have the case in mind of local symmetries, i.e. α contains a space-time point. Suppose that the set R_α^i is *complete*, in the sense that

$$\forall X_{\hat\beta}^i : y_i X_{\hat\beta}^i(\phi) = 0 \Rightarrow X_{\hat\beta}^i(\phi) = R_\alpha^i \epsilon_{\hat\beta}^\alpha(\phi) + y_j M_{\hat\beta}^{ij}(\phi), \qquad (1.12)$$

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[4]$\epsilon_i = 0$ for a bosonic field or monomial of fields and $\epsilon_i = 1$ for a fermionic field or monomial of fields. For some basic rules to keep in mind when working with quantities of different Grassmann parity, we refer to the appendices.

where $\bar{\beta}$ is an arbitrary set of indices, and where the $M_{\bar{\beta}}^{ij}$ have the graded antisymmetry property:

$$M_{\bar{\beta}}^{ij}(\phi) = (-1)^{\epsilon_i \epsilon_j + 1} M_{\bar{\beta}}^{ji}(\phi). \tag{1.13}$$

Owing to this graded antisymmetry, $y_i y_j M_{\bar{\beta}}^{ij}$ vanishes identically.

Consider now

$$\frac{\overleftarrow{\delta}}{\delta \phi^j} \left[y_i R_\alpha^i \right] . R_\beta^j - (-1)^{\epsilon_\alpha \epsilon_\beta} \frac{\overleftarrow{\delta}}{\delta \phi^j} \left[y_i R_\beta^i \right] . R_\alpha^j = 0, \tag{1.14}$$

where both terms vanish identically owing to (1.11). This leads to

$$y_i \left[\frac{\overleftarrow{\delta} R_\alpha^i}{\delta \phi^j} R_\beta^j - (-1)^{\epsilon_\alpha \epsilon_\beta} \frac{\overleftarrow{\delta} R_\beta^i}{\delta \phi^j} R_\alpha^j \right] = 0, \tag{1.15}$$

since the two terms with a second derivative of the action S cancel each other. The most general form of the *gauge algebra* now follows trivially from (1.12), which states that $T_{\beta\gamma}^\alpha[\phi]$ and $E_{\alpha\beta}^{ij}[\phi]$ exist, such that

$$\frac{\overleftarrow{\delta} R_\alpha^i}{\delta \phi^j} R_\beta^j - (-1)^{\epsilon_\alpha \epsilon_\beta} \frac{\overleftarrow{\delta} R_\beta^i}{\delta \phi^j} R_\alpha^j = 2 R_\gamma^i T_{\alpha\beta}^\gamma (-1)^{\epsilon_\alpha} - 4 y_j E_{\alpha\beta}^{ji} (-1)^{\epsilon_i} (-1)^{\epsilon_\alpha}. \tag{1.16}$$

We introduced some numerical and sign factors for later convenience. The different types of gauge algebras are classified as follows. When $E_{\alpha\beta}^{ji} = 0$, one says that the algebra is *closed*. Closed algebras can be divided in several categories, depending on the type of structure functions $T_{\alpha\beta}^\gamma$. If the structure functions are just arbitrary functions of the fields, the algebra is called *soft*. When they reduce to field independent constants, the *structure constants*, the algebra becomes an ordinary *Lie algebra*, or their infinite dimensional generalisations. When the structure constants all vanish, i.e. when $T_{\alpha\beta}^\gamma = 0$, the algebra is *abelian*. If the $E_{\alpha\beta}^{ji} \neq 0$, they are called the *non-closure functions*, and the algebra is an *open algebra*. Theories with this type of gauge algebra will require special attention. Examples of all types will be given below when discussing their quantisation.

Two more remarks are in order. In [4], I.A. Batalin and G.A. Vilkovisky showed that it is always possible to choose a different set of generators for the gauge algebra, such that it is closed. Once this is achieved, they even proved that any closed algebra is a disguised form of an abelian one. However, in rewriting an (open) algebra in its abelian form, other required features of the theory might get lost, like a Lorentz covariant formulation or locality in some preferred set of variables. Hence, the necessity remains to have a prescription for quantising theories with an arbitrary algebra.

Furthermore, the following properties of the structure functions T and E are useful for later developments. We find the Grassmann parities:

$$\begin{aligned} \epsilon_{T_{\alpha\beta}^\gamma} &= \epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma \\ \epsilon_{E_{\alpha\beta}^{ji}} &= \epsilon_\alpha + \epsilon_\beta + \epsilon_i + \epsilon_j. \end{aligned} \tag{1.17}$$

Under the exchange of α and β, we see that

$$T_{\beta\alpha}^\gamma = (-1)^{(\epsilon_\alpha + 1)(\epsilon_\beta + 1)} T_{\alpha\beta}^\gamma, \tag{1.18}$$

and an analogous property for $E^{ji}_{\beta\alpha}$. The structure functions also satisfy the *Jacobi identity.* This can be seen as follows. Define first the structure functions $t^{\gamma}_{\alpha\beta} = 2T^{\gamma}_{\alpha\beta}(-1)^{\epsilon_\alpha}$, for which under the exchange of α and β the property $t^{\gamma}_{\beta\alpha} = (-1)^{\epsilon_\alpha\epsilon_\beta+1}t^{\gamma}_{\alpha\beta}$ holds. Define now

$$
\begin{aligned}
Y^i_{\alpha\beta\gamma} &= \frac{\overleftarrow{\delta}}{\delta\phi^k}\left[\frac{\overleftarrow{\delta}R^i_\alpha}{\delta\phi^j}R^j_\beta - (-1)^{\epsilon_\alpha\epsilon_\beta}\frac{\overleftarrow{\delta}R^i_\beta}{\delta\phi^j}R^j_\alpha\right]R^k_\gamma \\
&\quad -(-1)^{(\epsilon_\alpha+\epsilon_\beta)\epsilon_\gamma}\frac{\overleftarrow{\delta}R^i_\gamma}{\delta\phi^j}\cdot\left[\frac{\overleftarrow{\delta}R^j_\alpha}{\delta\phi^k}R^k_\beta - (-1)^{\epsilon_\alpha\epsilon_\beta}\frac{\overleftarrow{\delta}R^j_\beta}{\delta\phi^k}R^k_\alpha\right] \ , \quad (1.19)
\end{aligned}
$$

which is just the generalisation of $[A,[B,C]]$ to the case of graded commutators. The Jacobi identity then follows from the fact that

$$
Y^i_{\alpha\beta\gamma} + (-1)^{\epsilon_\alpha\cdot(\epsilon_\beta+\epsilon_\gamma)}Y^i_{\beta\gamma\alpha}(-1)^{\epsilon_\gamma\cdot(\epsilon_\beta+\epsilon_\alpha)}Y^i_{\gamma\alpha\beta} = 0 \qquad (1.20)
$$

For a closed algebra, the Jacobi identity becomes

$$
\begin{aligned}
0 &= (-1)^{\epsilon_\alpha\epsilon_\gamma}t^\delta_{\alpha\beta,k}R^k_\gamma + (-1)^{\epsilon_\alpha\epsilon_\beta}t^\delta_{\beta\gamma,k}R^k_\alpha + (-1)^{\epsilon_\beta\epsilon_\gamma}t^\delta_{\gamma\alpha,k}R^k_\beta \qquad (1.21) \\
&\quad + (-1)^{\epsilon_\gamma(\epsilon_\beta+\epsilon_\mu)}t^\delta_{\mu\gamma}t^\mu_{\alpha\beta} + (-1)^{\epsilon_\alpha(\epsilon_\gamma+\epsilon_\mu)}t^\delta_{\mu\alpha}t^\mu_{\beta\gamma} + (-1)^{\epsilon_\beta(\epsilon_\alpha+\epsilon_\mu)}t^\delta_{\mu\beta}t^\mu_{\gamma\alpha} \ .
\end{aligned}
$$

Notice the terms with right derivatives of the structure functions with respect to the fields ϕ^k, denoted by $_{,k}$.

Now that we have classified the different types of gauge theories depending on the properties of their gauge algebra, one final concept has to be introduced, i.e. *reducible gauge theories.* As we have shown above, gauge theories are characterised by the fact that the Hessian has zeromodes on-shell, that is

$$
H_{ij}[\phi^l_0].R^i_\alpha[\phi^l_0] = 0. \qquad (1.22)
$$

Although we assumed completeness of the set of gauge generators, we did not consider the fact that they might be dependent with respect to the index α. Indeed, it can happen that operators $Z^\alpha_{\alpha_1}[\phi]$ exist, such that

$$
R^i_\alpha[\phi^l_0]Z^\alpha_{\alpha_1}[\phi^l_0] = 0. \qquad (1.23)
$$

Notice that this relation only has to hold on the stationary surface. Again, a distinction could be made depending on whether α_1 contains a space time index or not, but we will only consider the case where it does. If the $Z^\alpha_{\alpha_1}[\phi^l_0]$ themselves are an independent set labelled by α_1, one speaks of a *first order reducible gauge theory.* It is clear that (1.23) expresses the fact that not all zeromodes of the Hessian are independent. If no such $Z^\alpha_{\alpha_1}$ exist, and hence all gauge generators R^i_α are independent, one speaks of an *irreducible gauge theory.*

Using the regularity of the theory, (1.23) can be generalised to a relation between the gauge generators which is valid everywhere in configuration space:

$$
R^i_\alpha[\phi]Z^\alpha_{\alpha_1}[\phi] = 2y_j B^{ji}_{\alpha_1}(-1)^{\epsilon_i}. \qquad (1.24)
$$

Some factors were again introduced for later convenience. Like for the gauge generators themselves, we demand that the reducibility relations are complete. That is,

$$
\forall X^\alpha_{\bar\beta}[\phi] : R^i_\alpha[\phi^l_0]X^\alpha_{\bar\beta}[\phi^l_0] = 0 \implies X^\alpha_{\bar\beta}[\phi^l_0] = Z^\alpha_{\alpha_1}[\phi^l_0]N^{\alpha_1}_{\bar\beta}[\phi^l_0], \qquad (1.25)
$$

or, in an off-shell notation,

$$X_{\bar{\beta}}^{\alpha}[\phi^l] = Z_{\alpha_1}^{\alpha}[\phi^l]N_{\bar{\beta}}^{\alpha_1}[\phi^l] + y_i D_{\bar{\beta}}^{\alpha i}[\phi^l]. \tag{1.26}$$

Of course, the $Z_{\alpha_1}^{\alpha}[\phi_0^l]$ may not all be independent. Some relations may exist among them

$$Z_{\alpha_1}^{\alpha}[\phi_0^l]Z_{\alpha_2}^{\alpha_1}[\phi_0^l] = 0, \tag{1.27}$$

again on the stationary surface. If the $Z_{\alpha_2}^{\alpha_1}$ are independent with respect to α_2, we have a *second order reducible gauge theory*. As the reader has understood by now, this can go on to lead to reducible theories of an arbitrary order. Even theories of infinite order reducibility have been encountered (see e.g.[10]).

Owing to the regularity condition, one could again start eliminating the reducibility relations and construct a set of independent gauge generators. However, the same objections as raised against the abelianisation procedure also apply here: Lorentz covariance and space-time locality may be lost when rewriting the theory this way. We will not discuss the quantisation of reducible gauge theories.

This finishes our overview of the different types of gauge algebras and their basic properties. In the following chapters, we first quantise the simplest cases, gradually attacking the more difficult ones. We show how all the procedures we will develop, can be incorporated into one, the Batalin-Vilkovisky scheme for Lagrangian BRST quantisation. The latter is at present the most general quantisation prescription at our disposal.

Chapter 2

Finite gauge transformations and the Faddeev-Popov quantisation procedure

In this chapter, we present a procedure which allows the quantisation of theories with a closed, irreducible gauge algebra. In studying this problem, the path integral formulation of quantum field theory proved to be of major importance. The quantisation recipe described below, was first developed by L.D. Faddeev and V.N. Popov [11] using path integrals. Only much later, an operator quantisation prescription was given by T. Kugo and I. Ojima [12], based on BRST symmetry. We discuss BRST symmetry in extenso in the next chapter. Whatever the point of view, one is led to enlarging the set of field degrees of freedom by the introduction of *ghost fields*. This latter fact was already foreshadowed in [13, 14].

The basic rationale behind the definition of the path integral for gauge theories, is that one should not integrate over the whole configuration space, but only over the space of so-called *gauge orbits*. These are subsets of the configuration space consisting of configurations that are related by (finite) gauge transformations, and therefore have the same action. Every gauge orbit contributes one term to the path integral summation. In order to select one configuration on every gauge orbit, *gauge fixing conditions* are introduced. The action of such a selected configuration determines the contribution to the path integral of the gauge orbit it belongs to. Of course, the whole procedure should be independent of the gauge fixing conditions.

In the exposition, we will restrict ourselves again to bosonic degrees of freedom and bosonic gauge symmetries, in order not to obscure the structure of the derivation. In the first section, we describe the gauge orbits in a bit more detail. There we already see one important difference between theories with a closed or an open algebra. The Faddeev-Popov quantisation prescription for closed algebras is developed in the second section. The final section of this chapter contains an example: Yang-Mills theory. We will treat this type of theory using the Faddeev-Popov procedure.

2.1 Gauge orbits

Let us first define the gauge orbits more carefully. Every gauge orbit is co-ordinatised by a set of parameters θ^α, where the index α runs over all gauge symmetries determined by the generators R_α^i. Hence, loosely speaking, one can say that the dimension of the gauge orbits is equal to the number of gauge generators. The basic idea that we want to implement is that two configurations on a gauge orbit that have infinitesimally differing coordinates θ^α, are connected by an infinitesimal gauge transformation. Following [15], we introduce a function $\phi^i(\theta)$, which satisfies the *Lie equation* [1] :

$$\frac{\delta \phi^i(\theta)}{\delta \theta^\beta} = R_\alpha^i[\phi^l(\theta)]\lambda_\beta^\alpha(\theta) \ . \tag{2.1}$$

The unspecified functions λ_β^α express the fact that we can choose different sets of gauge generators R_α^i. On the next page, we show that doing a coordinate transformation on the gauge orbits leads to other functions λ_β^α. We can choose any configuration Φ_0^i as a boundary condition for solving the Lie equation, $\phi^i(\theta = 0) = \Phi_0^i$. All configurations one gets by taking all values for θ^α for a fixed solution of the Lie equation are said to form a gauge orbit. It is clear that all configurations on a gauge orbit have indeed the same action:

$$\frac{\delta S[\phi^i(\theta)]}{\delta \theta^\alpha} = y_i R_\alpha^i \lambda_\beta^\alpha = 0. \tag{2.2}$$

The functions λ_β^α are not completely arbitrary. They have to satisfy the analogue of the *Maurer-Cartan* equation, which follows from the requirement that (2.1) be integrable. This means that

$$\frac{\delta^2 \phi^i(\theta)}{\delta \theta^\gamma \delta \theta^\beta} - \frac{\delta^2 \phi^i(\theta)}{\delta \theta^\beta \delta \theta^\gamma} = 0, \tag{2.3}$$

which leads to

$$\frac{\delta \lambda_\beta^\alpha}{\delta \theta^\gamma} - \frac{\delta \lambda_\gamma^\alpha}{\delta \theta^\beta} + t_{\mu\nu}^\alpha \lambda_\beta^\mu \lambda_\gamma^\nu = 0. \tag{2.4}$$

To arrive at this result one has to use (2.1) and the commutation relation for a closed algebra ($E_{\alpha\beta}^{ij} = 0$ in (1.16)). For the Maurer-Cartan equation, one chooses the boundary condition $\lambda_\beta^\alpha(\theta = 0) = \delta_\beta^\alpha$. The equation (2.4) itself is integrable for closed algebras due to the Jacobi identity (1.21). This only works for closed algebras, because for open algebras an extra term, proportional to the field equations, appears in (2.4). The final result is that the Lie equation (2.1) as it stands above is not integrable for open algebras [4], except of course on the stationary surface where both the Maurer-Cartan equation and the Jacobi identity have the same form for open as for closed algebras. This does however not mean that one can not define finite gauge transformations for open algebras off the stationary surface. We will point out the appropriate starting point below (2.8).

The gauge orbits can be reparametrised by an invertible coordinate transformation, specified by some functions $\theta^\alpha(\xi^\beta)$. It is then easy to see that

---

[1]We import terminology from the theory of Lie groups and Lie algebras. If the structure functions of the closed algebra are structure constants, then our considerations reduce to that case. A point on a gauge orbit is then coordinatised by specifying a point on the group manifold for every space-time point.

$\tilde{\phi}^i(\xi) = \phi^i(\theta(\xi))$ satisfies the Lie equation

$$\frac{\delta \tilde{\phi}^i(\xi)}{\delta \xi^\alpha} = R^i_\gamma(\tilde{\phi})\tilde{\lambda}^\gamma_\alpha(\xi), \tag{2.5}$$

with $\tilde{\lambda}^\gamma_\alpha(\xi) = \lambda^\gamma_\beta(\theta(\xi))\frac{\delta \theta^\beta}{\delta \xi^\alpha}$. It is as straightforward to show that if the Maurer-Cartan equation is satisfied in one set of coordinates, it also is satisfied in another. Hence, it is clear that different choices of functions λ^α_β when writing down the Lie equation, give rise to different coordinatisations of the gauge orbits. When describing the Faddeev-Popov procedure, we will at every step keep invariance under coordinate transformations of the gauge orbits. In section 4 of chapter 8, changes of the set of generators are discussed from the point of view of canonical transformations of the BV scheme.

Let us now restrict the freedom in the choice of the functions λ^α_β by imposing the extra condition

$$\lambda^\alpha_\beta \theta^\beta = \theta^\alpha. \tag{2.6}$$

This restriction allows the derivation of a different differential equation that defines the gauge orbits and finite gauge transformations. Define the functions

$$\begin{aligned}
\varphi^i(x, \theta) &= \phi^i(x\theta), \\
\Lambda^\alpha_\beta(x, \theta) &= x\lambda^\alpha_\beta(x\theta).
\end{aligned} \tag{2.7}$$

The functions determining the gauge orbits are $\varphi^i(1, \theta)$. Using (2.6), the Lie equation (2.1) and the Maurer-Cartan equation (2.4) lead to the following two equations for φ^i and Λ^α_β:

$$\begin{aligned}
\frac{d\varphi^i(x)}{dx} &= R^i_\alpha[\varphi^l]\theta^\alpha, \\
\frac{d\Lambda^\alpha_\beta(x)}{dx} &= \delta^\alpha_\beta + t^\alpha_{\mu\nu}\theta^\mu \Lambda^\nu_\beta,
\end{aligned} \tag{2.8}$$

with the boundary conditions $\varphi^i(x = 0, \theta) = \Phi^i_0$ and $\Lambda^\alpha_\beta(x = 0, \theta) = 0$.

The latter equations (2.8) can be taken as the starting point for the definition of finite gauge transformations for closed as well as for open algebras [15, 4]. For closed algebras, the Lie equation, the Maurer-Cartan equation and (2.6) can be rederived. For open algebras however, the Lie equation that one obtains, contains an extra term proportional to field equations, the hallmark of open algebras:

$$\frac{\delta \phi^i}{\delta \theta^\beta} = R^i_\alpha \lambda^\alpha_\beta + y_j M^{ji}_\beta. \tag{2.9}$$

Instead of just two equations, one for ϕ^i and one for the functions λ^α_β, one gets an infinite sequence of differential equations. Every equation in this sequence is obtained by imposing integrability on the previous equation. For more details we refer again to [15, 4].

2.2   The Faddeev-Popov procedure

Having been through all this trouble to define gauge orbits, the derivation of the Faddeev-Popov procedure [11] becomes quite straightforward. The measure

of the path integral can be decomposed into two pieces, an integration over the different gauge orbits and an integration over all configurations on a fixed orbit:

$$[d\phi] = [d\Phi_0^i].\prod_l [d\theta_l^\alpha].\det\lambda. \qquad (2.10)$$

Symbolically, $[d\Phi_0^i]$ denotes the integration over the different gauge orbits. We introduced an index l, labelling the different gauge orbits. The coordinates of the l-th orbit are θ_l^α, so $[d\theta_l^\alpha]$ denotes the integration over the configurations that lie on the l-th orbit. We coordinatise all orbits in the same way, using the same functions λ_β^α, although this is not necessary. The $\det\lambda$ is introduced in order to define the integration over the gauge orbits in a coordinate invariant way.

The culprit for the infinities of the naive path integral is of course the integration over the coordinates of the gauge orbits, since the classical action, and hence also the contribution to the naive path integral, does not change under variation of θ^α (2.2). An obvious way to cure this problem, is to introduce δ-functions that select a specific set of values for the θ^α, i.e. that select one configuration on every orbit. Again maintaining coordinate invariance, we *replace* $[d\phi]$ by

$$[d\Phi_0^i]\prod_l [d\theta_l]\det\lambda.\frac{\delta\left(\theta_l^\alpha - \Theta_l^\alpha\right)}{\det\lambda} = [d\phi].\prod_l \frac{\delta\left(\theta_l^\alpha - \Theta_l^\alpha\right)}{\det\lambda}. \qquad (2.11)$$

The Θ_l^α are the coordinates of the configuration that is selected on gauge orbit l. Notice that we do not need to fix the same values for the coordinates on the different orbits.

This procedure is however not a very practical way of selecting one configuration on every orbit. A more useful, but less explicit, way is by choosing a set of *gauge fixing functions* $F^\alpha(\phi^i)$. There are as many gauge fixing functions as there are coordinates for the gauge orbit. Instead of introducing δ-functions that select specific values for the coordinates, we will introduce $\delta(F^\alpha(\phi)-f^\alpha(x))$ in the measure, for some space time dependent functions f^α. We will not dwell here on the questions whether $F^\alpha(\phi)-f^\alpha = 0$ has one solution on every gauge orbit or possibly more than one. If on every orbit there is not exactly one solution, there is omission or double counting of certain gauge orbits in the path integral. These possible problems go under the name *Gribov ambiguities*. Some details can be found in [9, 16, 17].

For every gauge orbit, we can relate the δ-function for the coordinates to the δ-function of the implicit gauge fixing conditions in the usual way:

$$\delta\left(F^\alpha(\phi(\theta_l)) - f^\alpha\right) = \frac{1}{\det M}\delta\left(\theta_l^\alpha - \Theta_l^\alpha\right). \qquad (2.12)$$

The Θ_l^α are such that $F^\alpha(\phi(\Theta_l^\alpha)) - f^\alpha = 0$. The matrix M is defined by

$$M_\beta^\alpha = \frac{\delta F^\alpha(\phi(\theta))}{\delta\theta^\beta}. \qquad (2.13)$$

Using the Lie equation for *closed* algebras (2.1), this can be rewritten as

$$M_\beta^\alpha = \frac{\delta F^\alpha(\phi(\theta))}{\delta\phi^i}R_\gamma^i\lambda_\beta^\gamma. \qquad (2.14)$$

Hence, the measure of the path integral is taken to be

$$[d\phi] \to [d\phi]\frac{1}{\det\lambda}.\det M.\delta(F^\alpha - f^\alpha), \qquad (2.15)$$

and the complete path integral becomes

$$\tilde{\mathcal{Z}} = \int [d\phi] \frac{1}{\det \lambda} . \det M . \delta(F^\alpha - f^\alpha) . e^{\frac{i}{\hbar}S} . \qquad (2.16)$$

Of course, $\tilde{\mathcal{Z}}$ is not a good starting point for diagrammatic calculations. Fortunately, both factors which appeared in the measure owing to the definition of the path integral, can be rewritten using extra fields. Enlarge the set of field degrees of freedom with one pair of fields (b_α, c^α) of odd Grassmann parity for every gauge generator R^i_α. Then we can write

$$\det M = \int [db][dc] \exp \left[\frac{i}{\hbar} b_\alpha M^\alpha_\beta c^\beta \right] = \int [db][dc] e^{\frac{i}{\hbar}S_{ghost}}. \qquad (2.17)$$

These extra fields are called *ghost fields* or *ghosts*. The c^β are the *ghosts*, the b_α the *antighosts*. In the literature, the antighosts are often denoted by $\bar{c}_\alpha$ for historical reasons. However, in order to prevent the misleading interpretations this might cause (see e.g. [12] in this context), we will not follow this tradition.

The dependence of S_{ghost} on the coordinatisation of the gauge orbit, via M, can be removed, by redefining c^γ as $\lambda^\gamma_\beta c^\beta$, leading to

$$S_{ghost} = b_\alpha \frac{\delta F^\alpha}{\delta \phi^i} R^i_\beta c^\beta. \qquad (2.18)$$

The Jacobian of this redefinition, $\det \lambda$, cancels with its inverse in the path integral.

As could already be seen from (2.16), the gauge fixing conditions F^α are only *admissible* if $\det M \neq 0$. As M^{-1} is the propagator for the ghost, we see that this condition is equivalent to saying that the ghosts have a well-defined propagator.

Finally, we can also rewrite the δ-function in a more tractable way. By construction, $\tilde{\mathcal{Z}}$ does not depend on the specific choice of f^α in the gauge fixing. This implies that we are allowed to integrate over f^α with a suitable weight factor $W[f]$ such that

$$\int [df] W[f] = 1. \qquad (2.19)$$

A very popular choice for $W[f]$ is a Gaussian damping factor, $W[f] = N e^{\frac{i}{2\hbar}f^2}$. We then define

$$\begin{aligned} \widehat{\mathcal{Z}} &= \int [df] \, \tilde{\mathcal{Z}} \, W[f] \\ &= \int [d\phi][db][dc] \, e^{\frac{i}{\hbar}S_{com}}. \end{aligned} \qquad (2.20)$$

Here, the complete action is given by

$$\begin{aligned} S_{com} &= S + S_{ghost} + S_{gf} \\ &= S + b_\alpha \frac{\delta F^\alpha}{\delta \phi^i} R^i_\beta c^\beta + \frac{1}{2} F^2. \end{aligned} \qquad (2.21)$$

This way of gauge fixing is called *Gaussian gauge fixing*. The path integral (2.20) can be used as a starting point for perturbative, diagrammatic calculations. Before applying the Faddeev-Popov recipe to an example, let us finish this section with some comments.

- First of all, the generalisation to the case where the fields and the gauge symmetries can have either Grassmann parities is straightforward. The general rule is that for bosonic gauge symmetries fermionic ghost fields are added to the configuration space and vice versa.

- In our derivation, we used the Lie equation for closed algebras, so we should not apply this formalism to open algebras. However, it also fails for reducible gauge theories. This can be seen as follows. In the semiclassical approximation, the propagator for the quantum fluctuations of the ghosts is the inverse of

$$\frac{\delta F^\alpha(\phi_0^i)}{\delta \phi^i} R_\beta^i[\phi_0^i] . \tag{2.22}$$

But for reducible gauge theories, this matrix has a zeromode owing to the on-shell reducibility relations (1.23). Notice that although the symptoms are the same as for non admissible gauge choices, i.e. $\det M = 0$, the rationale is of course different. For reducible gauge theories the problem exists for whatever gauge fixing function one chooses, while as far as admissibility is concerned, the problem can be solved by choosing the gauge fixing functions judiciously.

- A tacit assumption in the derivation is also that the final action S_{com} describes a local field theory. This restricts the choice of gauge fixing functions F^α, which are normally polynomials in the fields and a finite order of their derivatives.

- Another weighing procedure that is often used is

$$\int [d\lambda][df] \ e^{\frac{i}{\hbar}\lambda_\alpha f^\alpha} = 1, \tag{2.23}$$

leading to the gauge fixed action $S_{com} = S + S_{ghost} + \lambda_\alpha F^\alpha$. The *auxiliary field* λ_α is sometimes called the *Nakanishi-Lautrup* field. This way of gauge fixing is called *delta function gauge fixing*.

This concludes the description of the first quantisation procedure that was developed for gauge theories. To make the abstract construction more concrete, we now turn to the example of Yang-Mills gauge theory.

2.3   Example : non-abelian Yang-Mills theory

Historically, the first gauge theories that were encountered are electromagnetism and the theory of general relativity. The former is the abelian case of the more general type of gauge theories which go under the name *Yang-Mills theories*, as C.N. Yang and R.L.Mills were the first to consider them [18]. All the experimentally confirmed models fall in this class. The Faddeev-Popov quantisation is hence sufficient to construct a quantum theory for any phenomenological model in particle physics.

These models are based on a Lie algebra, in the true mathematical sense, defined by a set of structure constants f_{ab}^c:

$$[T_a, T_b] = T_c f_{ab}^c. \tag{2.24}$$

Here, the T_a form a set of matrices that satisfy this commutation rule[2]. Introduce the *covariant derivative*

$$\mathbf{D}_\mu \mathbf{B} = \partial_\mu \mathbf{B} + [\mathbf{A}_\mu, \mathbf{B}]. \tag{2.25}$$

The curvature tensor of the gauge field is then defined by

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu]. \tag{2.26}$$

Notice that the curvature tensor is antisymmetric in its two Lorentz indices, $\mathbf{F}_{\mu\nu} = -\mathbf{F}_{\nu\mu}$.

The action for pure Yang-Mills theory is

$$S^{YM} = -\frac{1}{4} \int d^4x \, \mathrm{tr}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}, \tag{2.27}$$

We work in the 4 space-time dimensions of experimental physics and the trace of two generators of the algebra is normalised to $\mathrm{tr}T_a T_b = \delta_{ab}$ in this representation. S^{YM} is not an action of free fields, as cubic and quartic terms in the fields A_μ^a occur, except when $f_{bc}^a = 0$, in which case everything reduces to electromagnetism. By varying the action S^{YM} with respect to the field variable A_μ^a, we get

$$\delta S^{YM} \sim \int d^4x \, \mathrm{tr}[\mathbf{D}_\mu \mathbf{F}^{\mu\nu} T_a] \delta A_\mu^a, \tag{2.28}$$

which gives the field equations

$$\mathrm{tr}[\mathbf{D}_\mu \mathbf{F}^{\mu\nu}(x) T_a] = 0. \tag{2.29}$$

Comparing with the general notation for field equations, we see that $i = (\nu, a, x)$. The index α of the gauge generators runs over a space-time point y (local symmetry) and an index of the algebra c. The gauge generators are then given by

$$R_{y,c}^{\nu,a,x} = \left[\partial_\nu^x \delta_c^a + A_\nu^b f_{bc}^a\right] \delta(x-y). \tag{2.30}$$

∂^x denotes a derivative with respect to x. The R_α^i do indeed satisfy

$$
\begin{aligned}
y_i R_\alpha^i &= \int d^4x \, \mathrm{tr}[\mathbf{D}_\mu \mathbf{F}^{\mu\nu}(x) T_a].\left[\partial_\nu \delta_c^a + A_\nu^b f_{bc}^a\right] \delta(x-y) \\
&= -\mathrm{tr}[\mathbf{D}_\nu \mathbf{D}_\mu \mathbf{F}^{\mu\nu}(y) T_c] \\
&= +\frac{1}{2}\mathrm{tr}[[\mathbf{F}_{\mu\nu}, \mathbf{F}^{\mu\nu}] T_c] \\
&= 0, \tag{2.31}
\end{aligned}
$$

owing to the antisymmetry of $\mathbf{F}^{\mu\nu}$. It is easy to verify that owing to the Jacobi identity, the gauge generators satisfy the algebra commutation relations (1.16) with the same structure constants as the Lie algebra we started from and with $E_{\alpha\beta}^{ij} = 0$.

A popular, and Lorentz covariant, gauge fixing condition for Yang-Mills theories is given by the functions F^α:

$$F^a(x) = \partial^\mu A_\mu^a(x). \tag{2.32}$$

---

[2]For a space-time dependent field which takes its values in the algebra, we use the boldface style. For example, $\mathbf{A}_\mu(x) = A_\mu^a(x) T_a$, $\mathbf{B}(x) = B^a(x) T_a$.

The ghost fields are $b_a(x)$ and $c^a(x)$ in this case. Following the general prescription (2.18), we get the ghost action

$$S_{ghost} = \int d^4x \, b_a \partial^\mu \left[\partial_\mu \delta_c^a + A_\mu^b f_{bc}^a \right] c^c. \tag{2.33}$$

We rewrite this using the trace in any representation of the algebra as

$$
\begin{aligned}
S_{ghost} &= \int d^4x \, \mathrm{tr} \left[\mathbf{b} \partial^\mu (\partial_\mu \mathbf{c} + [\mathbf{A}_\mu, \mathbf{c}]) \right] \\
&= \int d^4x \, \mathrm{tr} \left[\mathbf{b} \partial^\mu \mathbf{D}_\mu \mathbf{c} \right].
\end{aligned}
\tag{2.34}
$$

If the algebra is abelian, the case of electromagnetism, then the ghosts and the antighosts are not coupled to the gauge fields A_μ and are free fields, since the covariant derivative in the ghostaction reduces to an ordinary one. This is why quantum electrodynamics could be quantised without the need for (anti)ghosts. If the structure constants are not zero, a three point vertex is present in the theory, where a gauge field couples to both the ghost fields. The complete gauge fixed action becomes

$$S_{com} = -\frac{1}{4} \int d^4x \, \mathrm{tr} \mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu} + \int d^4x \, \mathrm{tr} \left[\mathbf{b} \partial^\mu \mathbf{D}_\mu \mathbf{c} \right] + \frac{1}{2} \int d^4x \, \mathrm{tr}(\partial_\mu \mathbf{A}^\mu)^2 , \tag{2.35}$$

if we use Gaussian gauge fixing.

Chapter 3

BRST quantisation

The gauge fixed action we have obtained (2.21) using the Faddeev-Popov construction, has no local gauge invariances anymore for admissible gauge choices. This is of course the goal we wanted to achieve. However, the gauge invariance of the classical theory is expected to manifest itself in the quantum theory as well. This is indeed the case. The classical, local gauge invariance is traded for a global invariance, the so-called *BRST-invariance*, which is present during the complete quantisation process. This acronym stands for C. Becchi, A. Rouet, R. Stora and I.V. Tyutin who were the first to notice and use the fact that the gauge fixed action (2.21) has a global invariance [19].

We start by constructing the BRST transformation rules and by pointing out how the gauge fixed action S_{com} can be obtained from them. This way, it will be immediately clear that S_{com} is indeed invariant under the transformation. This fact will then be used to derive the most general Ward identity, which is at the heart of the proofs of perturbative renormalisability and unitarity of gauge theories. We point out the possible need for quantum corrections in order to guarantee that the naive Ward identities are valid. We also introduce the concept of BRST-cohomology. In a final subsection, BRST–anti-BRST symmetry is briefly discussed.

3.1 The BRST operator

Starting from a classical action $S_0[\phi^i]$ with gauge invariances determined by $R^i_\alpha[\phi]$, the BRST transformation rules can be constructed as follows. First of all, the BRST operator, denoted by δ, is a fermionic, linear differential operator, acting from the right. This means that

$$\delta(X.Y) = X.\delta Y + (-1)^{\epsilon_Y}\delta X.Y, \tag{3.1}$$

and that

$$\epsilon_{\delta X} = \epsilon_X + 1. \tag{3.2}$$

δ is completely known if we specify its action on the fields, as the Leibnitz rule (3.1) allows to work out how it acts on any polynomial. With every gauge generator R^i_α we associate a ghost field c^α with statistics *opposite* to α, i.e.

$\epsilon_{c^\alpha} = \epsilon_\alpha + 1$. The BRST transformations of the classical fields ϕ^i are defined to be

$$\delta\phi^i = R^i_\alpha[\phi]c^\alpha. \tag{3.3}$$

Notice that for function(al)s that only depend on these classical fields, gauge invariance is equivalent to BRST invariance. Especially, the classical action is BRST invariant : $\delta S_0 = y_i R^i_\alpha c^\alpha = 0$.

A crucial property that we impose on δ, is that it be a *nilpotent* operator, i.e. that $\delta^2 = 0$. One easily verifies that if δ is nilpotent on a set of fields $-\delta^2 A^l = 0-$, it is also nilpotent when acting on any function(al) of these fields, $\delta^2 F(A^l) = 0$. For *closed* algebras, the nilpotency of the BRST operator can be guaranteed easily by choosing the BRST transformation of the ghost field. Imposing nilpotency on the classical fields ϕ^i, we get

$$
\begin{aligned}
0 &= \delta^2\phi^i \\
&= R^i_\alpha\delta c^\alpha + \frac{\overleftarrow{\delta} R^i_\alpha c^\alpha}{\delta\phi^j}R^j_\beta c^\beta.
\end{aligned} \tag{3.4}
$$

If we choose

$$\delta c^\gamma = T^\gamma_{\alpha\beta}[\phi]c^\beta c^\alpha , \tag{3.5}$$

the condition (3.4) is satisfied. Indeed, consider the closed gauge algebra ($E^{ji}_{\alpha\beta} = 0$ in (1.16)) and multiply both sides from the right with $(-1)^{\epsilon_\alpha\epsilon_\beta+\epsilon_\beta}c^\alpha c^\beta$. We obtain

$$\frac{\overleftarrow{\delta} R^i_\alpha c^\alpha}{\delta\phi^j}R^j_\beta c^\beta + R^i_\gamma T^\gamma_{\alpha\beta}c^\beta c^\alpha = 0. \tag{3.6}$$

The numerical and signfactors were precisely introduced in (1.16) to have this simple form here. Hence, we have that δ is nilpotent when acting on ϕ^i.

It remains to verify that with the two definitions above, δ is also nilpotent when acting on the ghost c^γ, i.e. that $\delta^2 c^\gamma = 0$. To see that for closed algebras this is indeed the case, it suffices to use the Jacobi identity (1.21). We multiply all terms of that identity from the right with $(-1)^{\epsilon_\alpha\epsilon_\gamma+\epsilon_\beta}c^\gamma c^\beta c^\alpha$, which gives

$$\frac{\overleftarrow{\delta} T^\gamma_{\alpha\beta}[\phi]c^\beta c^\alpha}{\delta\phi^i}R^i_\mu c^\mu + 2T^\gamma_{\alpha\beta}c^\beta T^\alpha_{\mu\nu}c^\nu c^\mu = 0. \tag{3.7}$$

This result straightforwardly implies $\delta^2 c^\gamma = 0$.

So far, the BRST operator acts on fields and ghosts. One can however always enlarge the set of fields by pairs of fields A^l and B^l, for an arbitrary set of indices l, with the following set of BRST transformation rules:

$$
\begin{aligned}
\delta A^l &= B^l \\
\delta B^l &= 0 .
\end{aligned} \tag{3.8}
$$

The interpretation of this goes as follows. The classical action we started from, does not depend on the fields A^l. Therefore, shifting the A^l over an arbitrary amount is a symmetry of that action. It is a local symmetry and the B^l are the ghosts associated with this shift symmetry. The nilpotency of the BRST operator is clearly maintained by this way of extending the set of fields. Such pairs of fields form together a *trivial system*. Although this is at first sight a rather trivial construction, hence the name, it has many applications. For

instance, the antighosts which were used in the Faddeev-Popov procedure are exactly introduced in this way:

$$\begin{aligned} \delta b_\alpha &= \lambda_\alpha \\ \delta \lambda_\alpha &= 0 \end{aligned} \tag{3.9}$$

In the configuration space, one can define *gradings*. Every field can be assigned a sort of *charge*, and, as is typical for the $U(1)$ charge that is known from electromagnetism, the *charge* of a monomial in the fields is the sum of the *charges* of the fields of which it is the product. We already have been using an example of such a grading, the Grassmann parity of every field. In that case, the *charge*, i.e. the parity, takes the values 0 or 1. A new grading that we will use very often is called *ghost number*, and the two basic assignments are:

$$\begin{aligned} \mathrm{gh}\left(\phi^i\right) &= 0 \\ \mathrm{gh}\left(c^\alpha\right) &= 1. \end{aligned} \tag{3.10}$$

The classical action satisfies $\mathrm{gh}\left(S_0\right) = 0$, and we impose the same requirement on the gauge fixed action (2.21) which we obtained using the Faddeev-Popov prescription. As a result, we have that $\mathrm{gh}\left(b_\alpha\right) = -1$ and $\mathrm{gh}\left(\lambda_\alpha\right) = 0$.

The BRST operator also carries a ghostnumber, $\mathrm{gh}\left(\delta\right) = 1$. This means that for any functional F with a specific ghostnumber,

$$\mathrm{gh}\left(\delta F\right) = \mathrm{gh}\left(F\right) + 1. \tag{3.11}$$

All the examples above, were the F are just fields, satisfy this rule. The ghost-number serves many times as a good bookkeeping device.

We come to the main purpose of the construction of the nilpotent BRST operator: it can be used to construct the gauge fixed Faddeev-Popov action of the previous chapter (2.21). The claim is that

$$S_{com} = S_0 + \delta \Psi. \tag{3.12}$$

Ψ is called the *gauge fermion*, as it obviously has to have odd Grassmann parity. Since we want the gauge fixed action to have ghostnumber zero, and since the BRST operator raises the ghostnumber with one unit, $\mathrm{gh}\left(\Psi\right) = -1$. The ϕ^i and c^α have respectively ghostnumber zero and one, which does not allow the construction of a suitable Ψ. Therefore, we introduce a trivial pair $(b_\alpha, \lambda^\alpha)$ with $\mathrm{gh}\left(b_\alpha\right) = -1$. Given any set of admissible gauge fixing functions F^α, we can consider $\Psi = b_\alpha(F^\alpha - \lambda^\alpha a)$, which leads to the terms

$$\begin{aligned} \delta \Psi &= \delta\left[b_\alpha(F^\alpha - \lambda^\alpha a)\right] \\ &= b_\alpha \frac{\overleftarrow{\delta} F^\alpha(\phi)}{\delta \phi^i} R^i_\beta c^\beta + F^\alpha \lambda_\alpha - a\lambda^2(-1)^{\epsilon_\alpha}. \end{aligned} \tag{3.13}$$

These are precisely the terms we got in the previous chapter (2.21) by rewriting the determinant and the gauge fixing δ-function. A commuting parameter a was introduced. Notice that the a-dependent term is only present for bosonic gauge symmetries, because for fermionic gauge symmetries λ is fermionic and hence its square is zero. For bosonic gauge symmetries, $a = 1$ gives the Gaussian gauge fixing after integrating over λ, while $a = 0$ leads to the δ-function gauge.

Now that we have established that the gauge fixed action that one obtains using the Faddeev-Popov procedure is of the form $S_{com} = S_0 + \delta \Psi$, it is clear that

$\delta S_{com} = 0$, since the classical action is BRST invariant. Our main conclusion is then that *after the gauge fixing, the original gauge invariance manifests itself as the global BRST invariance of the gauge fixed action.* In section 3, we argue that also for operators other than the action, gauge invariance is to be replaced by BRST invariance (3.26).

In the BRST quantisation scheme we are not obliged to take for the gauge fermion the one taken above (3.13) in order to reproduce the Faddeev-Popov expression for the gauge fixed action. More general choices are allowed, leading e.g. to four ghost interactions. In the next section we prove –formally– that as long as Ψ leads to path integrals that are well-defined, meaning that they do not have gauge invariances, the partition function is indeed independent of the gauge fixing fermion.

The quantisation based on BRST symmetry also needs to be modified in order to handle gauge theories with an open algebra. With the two basic definitions of $\delta\phi^i$ and δc^γ of above, the BRST operator is not nilpotent when the algebra is open. Indeed, $\delta^2\phi^i$ becomes proportional to field equations, as such a term appears in (3.6) for open algebras. As the nilpotency of δ is crucial for having a BRST invariant action after gauge fixing, we see that this latter invariance is not present for open algebras, when quantised as described above. In chapter 6, we show how the BRST transformations and the gauge fixing procedure have to be modified in the case of an open algebra, in order to end up with a BRST invariant gauge fixed action.

3.2   Ward identities

In the previous section, we have seen that the classical gauge symmetry gives rise to the BRST invariance of the gauge fixed action. In this section, we derive the Ward identity $\langle\delta X\rangle = 0$.

Given an action[1] $S[\phi^A]$ that is invariant under the BRST transformation rules $\delta\phi^A$, we have that for any $X(\phi)$

$$\langle\delta X(\phi)\rangle = \int [d\phi].\delta X.e^{\frac{i}{\hbar}S[\phi]} = 0 . \qquad (3.14)$$

The simple proof goes as follows. Consider the expectation value of an operator $X(\phi)$:

$$\chi = \int [d\phi].X(\phi).e^{\frac{i}{\hbar}S[\phi]} . \qquad (3.15)$$

We are always allowed to redefine the integration variables [2] $\phi \to \phi + \delta\phi.\mu$. If we assume for the moment that this redefinition does not give a Jacobian, we have

$$\chi = \int [d\phi].[X(\phi) + \delta X.\mu].e^{\frac{i}{\hbar}S[\phi+\delta\phi.\mu]} . \qquad (3.16)$$

Since $S[\phi]$ is BRST invariant, we can subtract (3.15) from (3.16) to find

$$\int [d\phi].\delta X.e^{\frac{i}{\hbar}S[\phi]} = 0 , \qquad (3.17)$$

---

[1]The ϕ^A denote all the fields present in the gauge fixed action:ϕ^i,c^α,trivial pairs.

[2]We use a global parameter μ of Grassmann parity one (a fermionic parameter) to construct $\tilde{\delta}A = \delta A.\mu$. Then $\epsilon_{\tilde{\delta}A} = \epsilon_A$.

which is the desired result. We will come back to the Jacobian that can appear in the redefinition at the end of this section. Notice that once a BRST invariant action is given, nowhere in this proof the nilpotency of δ is required.

From this simple property, all Ward-Takahashi-Slavnov-Taylor identities[3] [20] can be derived by choosing X. The use of these Ward identities is manifold. In the first place, they allow us to prove that the gauge fixed actions of the form $S_{com} = S_0 + \delta\Psi$ do indeed lead to partition functions which are independent of the gauge fixing, that is, of Ψ. Consider two partition functions, one calculated with the gauge fermion Ψ and one with an infinitesimally different gauge fermion $\Psi + d\Psi$. In an obvious notation we have that

$$
\begin{aligned}
\mathcal{Z}_{\Psi+d\Psi} - \mathcal{Z}_\Psi &= \int [d\phi] \exp\left[\frac{i}{\hbar}(S_0 + \delta[\Psi + d\Psi])\right] - \int [d\phi] \exp\left[\frac{i}{\hbar}(S_0 + \delta\Psi)\right] \\
&= \frac{i}{\hbar}\langle \delta d\Psi \rangle_\Psi \\
&= 0.
\end{aligned} \tag{3.18}
$$

This shows that the quantisation based on BRST symmetry is internally consistent. No reference is needed to the results obtained with the Faddeev-Popov procedure.

Perturvative proofs of unitarity and renormalisability are also heavily based on the Ward identities. For instance, the different divergences that can occur when doing loop calculations are related as a consequence of these Ward identities. This retricts the number of independent divergences that have to be absorbed in the parameters of the theory and thus aids in proving renormalisability. For examples in practical calculations, see e.g. chapter 7 of [21]. A treatment which strongly stresses the perturbative, diagrammatic point of view can be found in [22]. More formal discussions, e.g. on the use of BRST invariance to constrain the renormalised effective action, can be found in [8, 23].

Let us now correct for our carelessness in the proof of the Ward identity and investigate what effect a possible Jacobian has on the derivation of the Ward identity. The Jacobian is given by[4]

$$
\text{sdet}\left[\delta_B^A + \frac{\overrightarrow{\delta}\left(\delta\phi^A\right)}{\delta\phi^B}\mu\right] = \exp\left[\text{str}\frac{\overrightarrow{\delta}\left(\delta\phi^A\right)}{\delta\phi^B}\mu\right] \overset{not.}{\equiv} e^{\mathcal{A}\mu}. \tag{3.21}
$$

Instead of (3.16), we get

$$
\chi = \int [d\phi].[X(\phi) + \delta X.\mu].e^{\frac{i}{\hbar}S[\phi]}.[1 + \mathcal{A}.\mu] . \tag{3.22}
$$

---

[3]Below we will use the name Ward identities.

[4]We follow the conventions of [24]. When doing a redefinition of integration variables $y^i = y^i(x)$, the measure changes by

$$
dy = \text{sdet}\left[\frac{\overrightarrow{\delta}\, y^i(x)}{\delta x^j}\right] dx . \tag{3.19}
$$

The supertrace and the superdeterminant are related by $\delta \ln \text{sdet}M = \text{str}(M^{-1}\delta M)$. The supertrace is defined here by

$$
\text{str}\left[\frac{\overrightarrow{\delta}\, y^i(x)}{\delta x^j}\right] = (-1)^{\epsilon_i}\frac{\overrightarrow{\delta}\, y^i(x)}{\delta x^i} . \tag{3.20}
$$

If we subtract (3.15) from (3.22) we find

$$\langle \delta X(\phi) \rangle + \langle X\mathcal{A} \rangle = 0. \qquad (3.23)$$

So, this Jacobian really makes a difference, it gives a correction to the Ward identity. Suppose now that a functional $M(\phi)$ –a *local* function of the fields– exists such that $\delta M = i\mathcal{A}$. Then, by considering as weight of the path integral $S + \hbar M$ instead of S, one can still derive the result that

$$\int [d\phi] e^{\frac{i}{\hbar}(S+\hbar M)} . \delta X = 0, \qquad (3.24)$$

since the possible non-invariance of the measure is then cancelled by the non-invariance of this M when repeating the derivation above. Hence, we see that possible Jacobians are quite harmless if they are the BRST variation of something, because adding this something as so-called *counterterm* enables us to derive the so desired Ward identity. When such a local M can not be found, and consequently we can not get our naive result, we speak of a *genuine anomaly*. The Ward identities are then modified in the quantum theory, and one says that the gauge symmetry is *anomalously broken*. Notice also that M is imaginary as $\mathcal{A}$ is real, at least when we work in Minkowski space.

Let us stress once again that the manipulations above are at a formal level. As soon as one tries to evaluate such a Jacobian, one runs into the usual infinities of quantum field theory. The Jacobians are often neglected on formal grounds (see e.g. the argumentation in section 4 of [23], or in the introduction of [25]). K. Fujikawa was the first to point out that genuine anomalies can be understood in the functional integral framework for quantisation as coming from Jacobians [26]. He also proposed a regularisation method to obtain finite expressions for these Jacobians. The third part of this work is devoted to a one-loop regularised treatment of these counterterms and of genuine anomalies.

3.3   BRST cohomology

Using the Ward identity, we can also see that the classical gauge invariance is replaced by BRST invariance for the full quantum theory. In classical electrodynamics, for instance, the quantities that one can measure experimentally, the electric and magnetic vector fields $\vec{E}$ and $\vec{B}$, are independent of the chosen gauge. Analogously, we can look for operators that have a quantum expectation value that is invariant under (infinitesimal) changes of the gauge fixing fermion. Consider any local operator $\Omega[\phi^A]$, i.e. a function(al) of the fields ϕ^A and a finite number of their derivatives. Adopting the same notation as above (3.18), we express this gauge invariance condition as

$$\begin{aligned}
0 &= \langle \Omega[\phi] \rangle_{\Psi+d\Psi} - \langle \Omega[\phi] \rangle_{\Psi} \\
&= \frac{i}{\hbar} \int [d\phi] \ \delta d\Psi . \Omega . e^{\frac{i}{\hbar}(S_0+\delta\Psi+\hbar M)} \qquad (3.25) \\
&= \frac{i}{\hbar} \int [d\phi] \ \delta\Omega . d\Psi . e^{\frac{i}{\hbar}(S_0+\delta\Psi+\hbar M)}.
\end{aligned}$$

The last step is obtained by using the Ward identity, which allows to *integrate by parts* the BRST operator. As this should be zero for any infinitesimal deformation $d\Psi$ of the gauge fermion, we see that *gauge invariant operators* are

characterised by the condition

$$\delta\Omega[\phi] = 0, \tag{3.26}$$

that is, they *have to be BRST invariant*. Hence, the classical gauge invariance, which is a property of functions of the original fields ϕ^i, is replaced by BRST invariance, which is a property of functions depending on all variables ϕ^A, including the ghosts. This generalises the conclusion of the previous section that gauge invariance of the classical action is traded for BRST invariance of the gauge fixed action.

Owing to the Ward identity, one sees that two operators that differ by the BRST variation of something, i.e. $\Omega_2 = \Omega_1 + \delta\Theta$ have the same expectation value. Indeed,

$$\langle\Omega_2\rangle_\Psi = \langle\Omega_1\rangle_\Psi + \langle\delta\Theta\rangle_\Psi = \langle\Omega_1\rangle_\Psi. \tag{3.27}$$

Since the BRST operator is nilpotent for closed algebras, we can combine the latter two results in the following **theorem**. *The gauge invariant operators are the non-trivial cohomology classes of the BRST operator δ at ghostnumber zero.*

What do we mean by this? Whenever one has a nilpotent operator acting on some space, one can define equivalence classes by

$$X \sim Y \Leftrightarrow \exists Z : X = Y + \delta Z. \tag{3.28}$$

Two elements of the same equivalence class lead to the same result when δ acts on them because of the nilpotency, $X \sim Y \Rightarrow \delta X = \delta Y$. Because of the condition of BRST invariance, we are only interested in those equivalence classes which are in the kernel of δ. We adopt terminology familiar from differential geometry, where the exterior derivative d which acts on forms is also nilpotent. A quantity Z is *BRST exact* if it is the BRST variation of something, that is if $Z = \delta X$. A quantity Y is *BRST closed*, if it has a vanishing BRST variation, $\delta Y = 0$. The equivalence classes are called *cohomology classes*. Of course, we can construct operators of any ghostnumber. Hence the addition *at ghostnumber zero* for the physical observables, which means that we only consider the cohomology classes of operators with ghostnumber zero. The cohomology classes at ghostnumber one are important for the nonperturbative study of anomalies, as will be pointed out below (see the discussion following (11.10)).

One final comment. Above we based the equivalence of operators on whether they had the same quantum expectation value or not. We can replace in the above discussion *expectation value* by *correlation functions with BRST invariant operators*. However, two operators which differ by a BRST exact term need not have the same correlation functions with non BRST invariant operators.

3.4 BRST–anti-BRST symmetry

In this section, we give a short discussion of an extension of the BRST construction of above, which goes under the name *extended BRST symmetry* or *BRST–anti-BRST symmetry*[5]. In the literature also the name $Sp(2)$ *invariant*

---

[5]We will use the name BRST–anti-BRST symmetry when we refer to the construction presented in this section. In the rest of this work, we will sometimes use *extended BRST transformation rules* which will *not* refer to this section.

quantisation is sometimes used. This anti-BRST symmetry was first discovered as *yet another symmetry* of gauge fixed actions, where instead of the ghost field, the antighost field plays a more important role [27]. Although this BRST–anti-BRST symmetry pops up from time to time, it does not seem to add anything substantial to the BRST quantisation procedure. However, a superfield formulation of Yang-Mills theory has been constructed using it [23, 28].

Instead of introducing one BRST operator, two operators are defined, δ_1 and δ_2, also sometimes denoted δ and $\bar{\delta}$. Both are fermionic, linear differential operators acting from the right. δ_1 is called BRST-operator, δ_2 anti-BRST operator. We construct them such that the BRST operator raises the ghostnumber by one, while the anti-BRST operators lowers it with the same amount. The configuration space is extended by the introduction of two ghost fields, c_1^α and c_2^α, with $\mathrm{gh}\,(c_a^\alpha) = -(-1)^a$. So, using the terminology of above, c_1^α is a ghost and c_2^α an antighost, as far as ghostnumber is concerned. The basic transformation rules are given by:

$$
\begin{aligned}
\delta_a \phi^i &= R_\alpha^i c_a^\alpha \\
\delta_a c_a^\alpha &= T_{\beta\gamma}^\alpha c_a^\gamma c_a^\beta,
\end{aligned}
\tag{3.29}
$$

where in the last line there is *no* summation over a. Further,

$$
\delta_1 c_2^\alpha + \delta_2 c_1^\alpha = 2 T_{\beta\gamma}^\alpha c_1^\gamma c_2^\beta.
\tag{3.30}
$$

This requirement does not yet fix $\delta_1 c_2^\alpha$ and $\delta_2 c_1^\alpha$. This is done by introducing an extra field b^α with $\mathrm{gh}\,(b) = 0$, and with transformation properties

$$
\begin{aligned}
\delta_1 c_2^\alpha &= b^\alpha \\
\delta_1 b^\alpha &= 0.
\end{aligned}
\tag{3.31}
$$

The transformation of the ghost under the anti-BRST operator is then

$$
\delta_2 c_1^\alpha = -b^\alpha + 2 T_{\beta\gamma}^\alpha c_1^\gamma c_2^\beta,
\tag{3.32}
$$

as a consequence of (3.30).

Finally, $\delta_2 b^\alpha$ is determined by imposing nilpotency of the anti-BRST operator on c_1^α. We find:

$$
\delta_2 b^\alpha = -2 T_{\gamma\beta}^\alpha c_2^\beta b^\gamma - \frac{\overleftarrow{\delta}\, T_{\mu\nu}^\alpha c_1^\nu c_2^\mu}{\delta \phi^k} R_\gamma^k c_2^\gamma.
\tag{3.33}
$$

We used the equality that one obtains by multiplying the Jacobi identity with $(-1)^{\epsilon_\alpha \epsilon_\gamma + \epsilon_\beta} c_2^\gamma c_1^\beta c_2^\alpha$, in analogy with (3.7). With these transformation rules, both the BRST and the anti-BRST operator are nilpotent. Moreover,

$$
\delta_1 \delta_2 + \delta_2 \delta_1 = 0.
\tag{3.34}
$$

This can again be checked with the by now familiar tricks based on the closed gauge algebra and the Jacobi identity.

Gauge fixing in a BRST–anti-BRST invariant way is done by adding

$$
S_{gf} = \frac{1}{2} \epsilon^{ab} \delta_a \delta_b \Psi
\tag{3.35}
$$

to the classical action [6]. Since the classical action is invariant under both BRST and anti-BRST transformations and since $\delta_1 + \delta_2$ is nilpotent, this leads indeed

---

[6] $\epsilon^{ab} = -\epsilon^{ba}$ is the antisymmetric tensor, which is typical for $Sp(2)$. Our convention is such that $\epsilon^{12} = -1$.

to gauge fixed actions which are invariant under BRST–anti-BRST symmetry. In contrast to ordinary BRST quantisation, $\text{gh}(\Psi) = 0$.

As the gauge fixed action is now invariant under both the BRST and the anti-BRST symmetry, there are two general Ward identities. We have

$$
\begin{aligned}
\langle \delta_1 X \rangle &= 0 \\
\langle \delta_2 X \rangle &= 0.
\end{aligned}
\tag{3.36}
$$

Again, these Ward identities guarantee that the partition functions are gauge independent.

We will come back to BRST–anti-BRST symmetry only once below. We construct an antifield scheme for BRST–anti-BRST invariant quantisation in chapter 7.

3.5   Overview

In this chapter, we have shown that in the process of constructing a gauge fixed action, the local gauge invariance is replaced by a global invariance, the BRST invariance. The BRST invariance of the quantum theory implies Ward identities $\langle \delta X \rangle = 0$ if the theory is anomaly free. These Ward identities are relations between correlation functions, encoding the consequences of gauge symmetry for the full quantum theory.

We have argued that gauge invariant operators are the cohomology classes of the BRST operator at ghostnumber zero. Finally, we have discussed a generalisation of the BRST symmetry, that treats ghosts and antighosts on an equal footing. In the next chapter, we show how Schwinger-Dyson equations can be obtained as Ward identities of a BRST symmetry.

Chapter 4

The Schwinger-Dyson BRST-symmetry

The Schwinger-Dyson equations [29] are equations satisfied by the Green's functions of any theory, with or without gauge symmetries. In principle, they determine the quantum theory completely. The Schwinger-Dyson equations (SD equations) are the quantum equations of motion. In the standard textbook arguments [21, 8], they are derived as a consequence of the generalisation to path integrals of the invariance of an integral under a redefinition of the integration variable from x to $x+a$. They are used in several domains of quantum field theory, like the study of bound states and the study of theories with spontaneously or dynamically broken symmetries, to name only two. In this brief chapter, we show how Schwinger-Dyson equations can be obtained as Ward identities of an extra symmetry following [30, 31]. This serves both as an illustration of the previous section in a simple setting and as one of the cornerstones of the subsequent developments.

Start from any action $S_0[\phi^i]$ which leads to a well defined path integral. This may be an action without gauge symmetries, or an action obtained after gauge fixing. In the latter case the ϕ^i contain the gauge fields, the ghosts and antighosts and possible auxiliary fields. Double now the configuration space, by copying every field ϕ^i with a *collective field* φ^i, and consider the action $S_0[\phi^i - \varphi^i]$. This action now has a gauge symmetry, it is invariant under $\delta\phi^i = \epsilon^i$, $\delta\varphi^i = \epsilon^i$. Introducing a ghost field for this *shift symmetry*, we have the following BRST transformation rules:

$$
\begin{aligned}
\delta\phi^i &= c^i \\
\delta\varphi^i &= c^i \\
\delta c^i &= 0,
\end{aligned}
\tag{4.1}
$$

which is obviously nilpotent. We want to remove the gauge symmetry by gauge fixing the collective field to zero. For that purpose, we introduce a trivial pair consisting of an antighost ϕ_i^* and an auxiliary field b_i. They have the BRST transformations $\delta\phi_i^* = b_i$ and $\delta b_i = 0$. The gauge fixing functions are taken to be $F^i = \varphi^i$, i.e. we fix the collective field φ^i to zero. The gauge fixing is done by adding to the classical action S_0 the term $\delta[\phi_k^*\varphi^k]$. We obtain

$$
S_{com} = S_0[\phi - \varphi] + \delta[\phi_k^*\varphi^k]
$$

$$= S_0[\phi - \varphi] + \phi_k^* c^k + (-1)^{\epsilon_k} b_k \varphi^k. \tag{4.2}$$

The second term is the ghost action (2.18), the third the delta function gauge fixing[1]. The ghostnumber assignments here are

$$
\begin{aligned}
\mathrm{gh}\left(\phi^i\right) &= \mathrm{gh}\left(\varphi^i\right) = x^i, \\
\mathrm{gh}\left(c^i\right) &= x^i + 1, \\
\mathrm{gh}\left(\phi_i^*\right) &= -x^i - 1 = -\mathrm{gh}\left(\phi^i\right) - 1.
\end{aligned}
\tag{4.3}
$$

Notice especially the last line, which we will encounter again in the next chapter.

Owing to the BRST invariance of this gauge fixed action, we have the Ward identity (3.14):

$$
0 = \langle \delta\left[\phi_i^* X(\phi)\right]\rangle = \int [d\phi][d\varphi][dc][d\phi^*][db]\left[\phi_i^* \frac{\overleftarrow{\delta X}}{\delta \phi^j} c^j + (-1)^{\epsilon_X} b_i X\right] e^{\frac{i}{\hbar} S_{gf}}.
\tag{4.4}
$$

Notice that, at least at the formal level, no counterterms are needed to maintain the BRST invariance, as the measure of the functional integral is assumed to be translational invariant. We will now integrate over the Nakanishi-Lautrup field b_i, the collective field φ^i and over the ghost and antighost field. As a result, we see that this Ward identity is nothing but the Schwinger-Dyson equation satisfied by $X(\phi)$.

In the first term the integral over b and φ can be done trivially, fixing the collective field φ to zero. This leads to

$$
T_1 = \int [d\phi][dc][d\phi^*]\left(\phi_i^* \frac{\overleftarrow{\delta X}}{\delta \phi^j} c^j\right) e^{\frac{i}{\hbar}\left(S_0[\phi] + \phi_k^* c^k\right)}.
\tag{4.5}
$$

Integrating over the ghosts is a bit more subtle, as they also occur outside the exponentiated action. We rewrite

$$
T_1 = \int [d\phi][dc][d\phi^*] \cdot \phi_i^* \frac{\overleftarrow{\delta X}}{\delta \phi^j} e^{\frac{i}{\hbar} S_0[\phi]} \cdot \frac{\hbar}{i} \frac{\overrightarrow{\delta}}{\delta \phi_j^*} e^{\frac{i}{\hbar} \phi_k^* c^k}.
\tag{4.6}
$$

Integrating by parts, and integrating over the ghostfields, we get

$$
T_1 = -(-1)^{(\epsilon_X + 1)(\epsilon_i + 1)} \int [d\phi] \frac{\overleftarrow{\delta X}}{\delta \phi^i} \cdot \frac{\hbar}{i} e^{\frac{i}{\hbar} S_0}.
\tag{4.7}
$$

For the second term, we do an analogous manipulation to integrate out the collective and auxiliary field:

$$
T_2 = \int [d\phi][d\varphi][dc][d\phi^*][db](-1)^{\epsilon_X} \frac{\hbar}{i} \frac{\overrightarrow{\delta}}{\delta \varphi^i}\left[e^{\frac{i}{\hbar} \varphi^k b_k}\right] \cdot X(\phi) \cdot e^{\frac{i}{\hbar}(S_0[\phi - \varphi] + \phi_k^* c^k)}.
\tag{4.8}
$$

Again integrating by parts, using that $\frac{df(x-y)}{dx} = -\frac{df(x-y)}{dy}$, and integrating out all the fields of the BRST shift symmetry leads to

$$
T_2 = -\int [d\phi](-1)^{(\epsilon_X + 1)(\epsilon_i + 1)} X(\phi) \frac{\overleftarrow{\delta S_0[\phi]}}{\delta \phi^i} e^{\frac{i}{\hbar} S_0}.
\tag{4.9}
$$

---

[1]Notice the notation. The ϕ_i^* is now the antighost (b_α in (2.18)) while b_i is now the Nakanishi-Lautrup field (λ^α of (3.13)).

Adding up both terms, we find that

$$\langle X(\phi)\frac{\overleftarrow{\delta} S_0}{\delta\phi^i} + \frac{\hbar}{i}\frac{\overleftarrow{\delta} X(\phi)}{\delta\phi^i}\rangle = \int [d\phi] \left[X(\phi)\frac{\overleftarrow{\delta} S_0}{\delta\phi^i} + \frac{\hbar}{i}\frac{\overleftarrow{\delta} X(\phi)}{\delta\phi^i} \right] e^{\frac{i}{\hbar}S_0} = 0. \quad (4.10)$$

This is the general form of the Schwinger-Dyson equations. We derived it here using the *collective field formalism* of J. Alfaro and P.H. Damgaard. The BRST symmetry which implies these Schwinger-Dyson equations as Ward identities, will below be referred to as *Schwinger-Dyson BRST symmetry*.

In the next chapter we will study the interplay between this Schwinger-Dyson BRST symmetry and gauge symmetries which may originally be present in the theory in more detail. This will lead to the Batalin-Vilkovisky scheme for the quantisation of gauge theories. We will show that the Batalin-Vilkovisky scheme combines BRST symmetry and the quantum equations of motion (the Schwinger-Dyson equations) in one formalism [32].

Chapter 5

The BV antifield formalism for closed, irreducible gauge algebras

The Batalin-Vilkovisky scheme [1, 2, 3, 4, 5] combines, loosely speaking, the BRST symmetry associated with gauge symmetries and the Schwinger-Dyson shift symmetry. This was first noticed by J. Alfaro and P.H. Damgaard recently [32]. In this and the following chapter, we present the BV scheme from this point of view. This way, the key features of the BV scheme (classical and quantum master equation, quantum BRST operator) are linked with their counterparts in BRST quantisation.

In the first two sections of this chapter, we develop the BV scheme following the idea of [32]. The BRST symmetry of the theory is enlarged such that the Schwinger-Dyson equations are Ward identities of the theory. We can do this by using the collective field formalism of the previous chapter. In contrast to [32], we argue that the collective field formalism is more than a technical means to implement the Schwinger-Dyson BRST symmetry. This becomes clear in section 3, where a slight generalisation of [32] is discussed[1]. Section 4 contains some examples, mainly from [33].

5.1 Classical BV

Whatever the gauge structure of the theory that we want to quantise, the basic requirement will be that the BRST symmetry of the theory is extended in such a way that the Schwinger-Dyson equations are included in the BRST Ward identities of the theory [32]. It is clear that the collective field formalism of the previous chapter is the appropriate tool to implement this.

We start from a classical action $S_0[\phi^i]$, depending on a set of fields ϕ^i. Suppose that this classical action has gauge invariances which are irreducible

---

[1]For further arguments that emphasize the importance of the collective field, see the next chapter on open algebras and chapter 7 where an antifield formalism for BRST–anti-BRST invariant quantisation is developed.

and form a closed algebra. Then one can construct a nilpotent BRST operator, acting on an extended set of fields ϕ^A as is described in section 1 of chapter 3. The ϕ^A include the original gauge fields ϕ^i, the ghosts c^α and the pairs of trivial systems used for the construction of the gauge fermion and for the gauge fixing. We summarise all their BRST transformation rules by

$$\delta\phi^A = \mathcal{R}^A[\phi^B]. \tag{5.1}$$

The nilpotency of δ,

$$\delta^2\phi^A = \frac{\overleftarrow{\delta}\mathcal{R}^A[\phi]}{\delta\phi^B}\mathcal{R}^B[\phi] = 0, \tag{5.2}$$

contains both the commutation relations of the algebra (for $\phi^A = \phi^i$) and the Jacobi identity (for $\phi^A = c^\alpha$). With this BRST operator, we can in principle construct the gauge fixed action.

Instead, we first enlarge the set of fields by replacing the field ϕ^A wherever it occurs, by $\phi^A - \varphi^A$. Again, φ^A is called the *collective field*. Like in the previous chapter, this leads to a new symmetry, the shift symmetry, for which we introduce a ghost field c^A, and a trivial pair consisting of an antighost field ϕ_A^* and an auxiliary field B_A. The BRST transformation rules are taken to be:

$$\begin{aligned}
\delta\phi^A &= c^A \\
\delta\varphi^A &= c^A - \mathcal{R}^A[\phi - \varphi] \\
\delta c^A &= 0 \\
\delta\phi_A^* &= B_A \\
\delta B_A &= 0.
\end{aligned} \tag{5.3}$$

These rules are constructed such that $\delta[\phi^A - \varphi^A] = \mathcal{R}^A[\phi - \varphi]$, as the classical action $S_0[\phi - \varphi]$ should be BRST invariant. This leaves of course the arbitrariness to take

$$\begin{aligned}
\delta\phi^A &= c^A + \alpha\mathcal{R}^A[\phi - \varphi] \\
\delta\varphi^A &= c^A - (1 - \alpha)\mathcal{R}^A[\phi - \varphi].
\end{aligned} \tag{5.4}$$

We have chosen the parameter α to be zero, as it is this choice which leads to what is known as the Batalin-Vilkovisky quantisation scheme. In the next chapter, when describing the quantisation of open algebras, we will give a more compelling reason for this choice (6.5). Possible other choices of α lead also to well-defined path integrals, but will not be considered. Notice that the BRST operator (5.3) is still nilpotent.

Instead of one, we now have two gauge symmetries to fix, the original symmetries and the shift symmetry. Like in the previous chapter, we will impose as gauge fixing condition $\varphi^A = 0$. So, we subtract $\delta[\phi_A^*\varphi^A]$ from the classical action, which gives the terms

$$-\delta[\phi_A^*\varphi^A] = -\phi_A^*\left(c^A - \mathcal{R}^A[\phi - \varphi]\right) - (-1)^{\epsilon_A}B_A\varphi^A. \tag{5.5}$$

The original gauge symmetry is fixed by constructing a gauge fermion Ψ that we take to be a function of the fields ϕ^A only. Remember that the ϕ^A include the trivial systems which are introduced in the usual BRST quantisation precisely for the purpose of gauge fixing. We add

$$\delta\Psi[\phi] = \frac{\overleftarrow{\delta}\Psi}{\delta\phi^A}c^A \tag{5.6}$$

to $S_0[\phi - \varphi]$. The complete gauge fixed action that we obtain is then given by

$$
\begin{aligned}
S_{com} &= S_0[\phi^i - \varphi^i] - \delta[\phi_A^* \varphi^A] + \delta\Psi[\phi^A] \\
&= S_0[\phi^i - \varphi^i] + \phi_A^* \mathcal{R}^A[\phi - \varphi] - \phi_A^* c^A + \frac{\overleftarrow{\delta}\Psi}{\delta\phi^A} c^A - \varphi^A B_A. \quad (5.7)
\end{aligned}
$$

Define now $S_{BV}(\phi^A, \phi_A^*) = S_0[\phi^i] + \phi_A^* \mathcal{R}^A[\phi]$, which allows us to rewrite the gauge fixed action as

$$
S_{com} = S_{BV}(\phi - \varphi, \phi^*) - \left(\phi_A^* - \frac{\overleftarrow{\delta}\Psi[\phi]}{\delta\phi^A} \right) c^A - \varphi^A B_A. \quad (5.8)
$$

Below we will study the properties of S_{BV}, the so-called *extended action* of the Batalin-Vilkovisky scheme, in more detail.

The partition function constructed with the gauge fixed action becomes

$$
\mathcal{Z}_\Psi = \int [d\phi^A][d\varphi^A][d\phi_A^*] \; e^{\frac{i}{\hbar} S_{BV}(\phi-\varphi,\phi^*)} \; \delta\left(\phi_A^* - \frac{\overleftarrow{\delta}\Psi[\phi]}{\delta\phi^A} \right) \delta(\varphi^A). \quad (5.9)
$$

The two delta-functions appear because we have integrated over the ghost of the shift-symmetry c^A and over the auxiliary field B_A. Integrating out the collective field –it is to put to zero owing to the δ-function– brings the partition function in the form which is typical for the BV scheme:

$$
\mathcal{Z}_\Psi = \int [d\phi^A][d\phi_A^*] e^{\frac{i}{\hbar} S_{BV}(\phi,\phi^*)} \delta\left(\phi_A^* - \frac{\overleftarrow{\delta}\Psi[\phi]}{\delta\phi^A} \right). \quad (5.10)
$$

Gauge fixed path integrals are obtained by exponentiating the extended action, which is a function of fields and antifields, and afterwards replacing the antifields ϕ_A^* by a derivative of an admissible gauge fermion with respect to the corresponding field.

From section 1 of chapter 3 on BRST quantisation, we know that for $\phi^A = \phi^i$ one has

$$
\mathcal{R}^A[\phi^B] = R_\alpha^i[\phi^j] c^\alpha, \quad (5.11)
$$

and that for $\phi^A = c^\gamma$,

$$
\mathcal{R}^A[\phi^B] = T_{\alpha\beta}^\gamma[\phi^i] c^\beta c^\alpha. \quad (5.12)
$$

This gives the following terms in the extended action

$$
\phi_A^* \mathcal{R}^A[\phi] = \phi_i^* R_\alpha^i[\phi] c^\alpha + c_\gamma^* T_{\alpha\beta}^\gamma[\phi] c^\beta c^\alpha. \quad (5.13)
$$

Analogously, the trivial system $\delta b_\alpha = \lambda_\alpha$, $\delta\lambda_\alpha = 0$ leads to the extra term $b^{*\alpha}\lambda_\alpha$. This is the form trivial systems take in the BV scheme. The complete extended action for a theory with a closed, irreducible gauge algebra plus some trivial systems, is

$$
S_{BV}(\phi^A, \phi_A^*) = S_0[\phi^i] + \phi_i^* R_\alpha^i[\phi] c^\alpha + c_\gamma^* T_{\alpha\beta}^\gamma[\phi] c^\beta c^\alpha + b^{*\alpha}\lambda_\alpha. \quad (5.14)
$$

If the gauge fermion is of the simple form

$$
\Psi[\phi^A] = b_\alpha F^\alpha(\phi^i), \quad (5.15)
$$

we have that

$$\frac{\overleftarrow{\delta}\,\Psi}{\delta\phi^i} = b_\alpha \frac{\overleftarrow{\delta}\,F^\alpha}{\delta\phi^i}$$

$$\frac{\overleftarrow{\delta}\,\Psi}{\delta c^\gamma} = 0 \qquad\qquad (5.16)$$

$$\frac{\overleftarrow{\delta}\,\Psi}{\delta b_\alpha} = F^\alpha(\phi^i),$$

which leads to the by now familiar expression for the gauge fixed action (cfr. $a = 0$ in (3.13)):

$$S_{BV}\left(\phi^A, \phi_A^* = \frac{\overleftarrow{\delta}\,\Psi[\phi]}{\delta\phi^A}\right) = S_0[\phi^i] + b_\alpha \frac{\overleftarrow{\delta}\,F^\alpha}{\delta\phi^i}R^i_\beta c^\beta + F^\alpha \lambda_\alpha. \qquad (5.17)$$

So, the Faddeev-Popov procedure is still contained in the BV scheme. Notice that the choice of the gauge fermion may be more complicated than the one used above to rederive the old expressions, as was already the case for BRST quantisation.

So far, we have verified that upon integration over all the fields that were introduced in the collective field formalism, i.e. φ^A, c^A, ϕ_A^* and B_A, the previously derived expressions for well-defined path integrals are found back. Although this is necessary for consistency, we have of course not gone through all the trouble of introducing the extra shift symmetry and extra fields just to integrate them out again immediately.

The BV formalism corresponds to the stage where we have integrated over all extra fields, except over the antighosts of the shift symmetry, ϕ_A^*. In the context of BV, one calls ϕ_A^* the *antifield* of ϕ^A. Like before (4.3), we have the following important relations:

$$\epsilon_{\phi_A^*} = \epsilon_{\phi^A} + 1$$

$$\mathrm{gh}\,(\phi_A^*) = -\mathrm{gh}\,(\phi^A) - 1. \qquad (5.18)$$

Notice that the antifields have an indexstructure opposite to that of their associated field. For example, with a covariant vectorfield A_μ a contravariant antifield vector $A^{*\mu}$ is to be associated. The reason for keeping precisely the antifield in the scheme, is that it leads to an elegant formulation of BRST invariance using the *antibracket*. This we discuss below (5.25).

Before turning to the Ward identities, let us derive the most important property of the extended action. $S_{BV}(\phi, \phi^*)$ satisfies the so-called *classical master equation*. It is precisely this equation which generalises to the cases of open and reducible gauge algebras. It follows from the fact that S_{com} (5.7) is BRST invariant under the transformation rules (5.3). We have that

$$0 = \delta S_{com}$$

$$= \delta S_{BV}(\phi - \varphi, \phi^*) - \delta\left[\left(\phi_A^* - \frac{\overleftarrow{\delta}\,\Psi[\phi]}{\delta\phi^A}\right)c^A\right] - \delta\left[\varphi^A B_A\right] \quad (5.19)$$

$$= \frac{\overleftarrow{\delta}\,S_{BV}(\phi - \varphi, \phi^*)}{\delta\phi^A}\mathcal{R}^A[\phi - \varphi].$$

Using the explicit expression for S_{BV}, which allows to rewrite $\delta[\phi^A - \varphi^A] = \frac{\overrightarrow{\delta} S_{BV}(\phi - \varphi, \phi^*)}{\delta \phi_A^*}$, this can be cast in the form

$$\frac{\overleftarrow{\delta} S_{BV}(\phi, \phi^*)}{\delta \phi^A} \frac{\overrightarrow{\delta} S_{BV}(\phi, \phi^*)}{\delta \phi_A^*} = 0. \qquad (5.20)$$

This is the *classical master equation* of the BV formalism. It is a consequence of the BRST invariance of the gauge fixed action S_{gf} and hence of the nilpotency of the BRST operator (5.3). In the next section we will rederive the classical master equation as the $\hbar = 0$ stage of an infinite tower of equations, which are all gathered in the *quantum master equation*.

Let us finish this section with a brief summary of the BV recipe for the quantisation of gauge theories as it has emerged so far. One first has to find a solution of the classical master equation (5.20). For closed, irreducible gauge algebras, the standard solution is of the form (5.14). In fact, this is not the only solution. Also the classical action $S_0[\phi]$ itself, for instance, satisfies the classical master equation, as it does not depend on antifields. Hence, to find the solution of the desired form when solving the master equation, one has to impose the extra condition that it be of the form $S_{BV} = S_0[\phi^i] + \phi_i^* R_\alpha^i c^\alpha + \ldots$ Here, the R_α^i have to be a complete set of gauge generators (1.12). If the R_α^i form a complete set, we say that S_{BV} is *proper*[2]. The solution obtained in this way is called the *minimal proper solution*. Before one is able to construct the gauge fermion, which has ghost number -1, we have to enlarge the set of fields with trivial systems. This we can do by taking the non minimal solution of the master equation

$$S_{n.m.} = S_{BV} + b^{*X} \lambda_X, \qquad (5.21)$$

for an arbitrary set of indices X. The non minimal solution satisfies the classical master equation if the minimal one does. Owing to the interpretation of the added term as a set of fields b_X with their arbitrary shift symmetries with ghosts λ_X, we still have a proper solution[3]. Once this is done, the gauge fixed action is obtained by replacing the antifields by the derivative of an admissible gauge fermion with respect to the associated field.

5.2 Ward identities, the quantum master equation and the quantum cohomology

5.2.1 Ward identities

As was argued above (section 2 of chapter 3), the Ward identities are crucial properties of any gauge theory. For instance, they guarantee that the partition function is independent of the gauge fixing. We also showed that it may be

---

[2]Basically, this condition expresses that we have to introduce a gauge fixing and ghost action for *all* gauge symmetries of the classical action. If we do not do that, we can not start the perturbative expansion (cfr. chapter 1).

[3]A different way to construct a non-minimal proper solution is by adding trivial systems of the form [34]

$$S_{n.m.} = S_{min} + b^{*X} b^{*Y} M_{XY} \qquad (5.22)$$

for an invertible matrix M_{XY}. The fields b_X and b^{*X} form a trivial pair for the classical antibracket cohomology (5.43).

necessary to add quantum corrections, or counterterms, to the action in order for the identities to be valid. Here, we will study how all this is translated in the BV scheme. This will lead us to the introduction of the antibracket and a second order differential operator. With these two ingredients, we can define the quantum BRST operator.

From the expressions for the BRST transformations (5.3), we see that the possible variation of the measure under a BRST transformation gives a Jacobian that is a function of $\phi^A - \varphi^A$. As $\delta(\phi^A - \varphi^A) = \mathcal{R}^A(\phi - \varphi)$, we are led to add a counterterm $\hbar M(\phi^A - \varphi^A)$. However, we will also allow dependence on the antifields. We consider the Ward identity

$$
\begin{aligned}
0 &= \langle \delta X(\phi^A, \phi_A^*) \rangle \\
&= \int [d\phi^A][d\varphi^A][d\phi_A^*][dc^A][dB_A] \left[\frac{\overleftarrow{\delta} X}{\delta \phi^B} c^B + \frac{\overleftarrow{\delta} X}{\delta \phi_B^*} B_B \right] \\
&\quad \times \exp \left[\frac{i}{\hbar} \left(W[\phi^i - \varphi^i, \phi^*] - \phi_A^* c^A + \frac{\overleftarrow{\delta} \Psi}{\delta \phi^A} c^A - \varphi^A B_A \right) \right].
\end{aligned}
$$
(5.23)

The *quantum action* is defined by $W(\phi, \phi^*) = S_{BV}(\phi, \phi^*) + \hbar M(\phi, \phi^*)$. We restricted ourselves here to quantities $X(\phi, \phi^*)$, as all the other fields are integrated out to get the BV formalism. We could have considered quantities $X(\phi - \varphi, \phi^*)$, but this would not change our final result, as the collective field φ is fixed to zero anyway. The integration over B_A, φ^A and c^A can be done following the same steps that are described in detail in the previous chapter on the Schwinger-Dyson equations. We find that the Ward identity becomes

$$
\begin{aligned}
0 &= \int [d\phi][d\phi^*] \left[(X, W) - i\hbar \Delta X \right] e^{\frac{i}{\hbar} W(\phi, \phi^*)} \delta \left(\phi_A^* - \frac{\overleftarrow{\delta} \Psi[\phi]}{\delta \phi^A} \right) \\
&\stackrel{not.}{=} \langle \sigma X \rangle_\Psi.
\end{aligned}
$$
(5.24)

Let us first explain the multitude of new notations introduced here. The *antibracket* between two quantities F and G of arbitrary Grassmann parity is defined by

$$
(F, G) = \frac{\overleftarrow{\delta} F}{\delta \phi^A} \frac{\overrightarrow{\delta} G}{\delta \phi_A^*} - \frac{\overleftarrow{\delta} F}{\delta \phi_A^*} \frac{\overrightarrow{\delta} G}{\delta \phi^A}.
$$
(5.25)

It plays a crucial part in the whole BV formalism. Many of its properties are listed in the appendices. The antibracket ressembles a lot the Poisson bracket used in classical Hamiltonian mechanics. Fields and antifields are *canonically conjugated* with respect to the antibracket, $(\phi^A, \phi_B^*) = \delta_B^A$, in much the same way as coordinates and momenta are conjugated in classical mechanics with respect to the Poisson bracket. Below, in chapter 8, we present the BV formalism from a more algebraic point of view, based on the properties of *canonical transformations*, which are transformations of the fields and antifields that leave the antibracket invariant. This is in analogy with the canonical transformations which are used in Hamiltonian mechanics and which leave the Poisson bracket invariant.

Furthermore, there is the fermionic second order differential operator which we will often refer to with the name *delta operator* or *box*. It is defined by

$$
\Delta X = (-1)^{\epsilon_A + 1} \frac{\overleftarrow{\delta}}{\delta \phi_A^*} \frac{\overleftarrow{\delta}}{\delta \phi^A} X = (-1)^{\epsilon_X} (-1)^{\epsilon_A} \frac{\overrightarrow{\delta}}{\delta \phi_A^*} \frac{\overrightarrow{\delta}}{\delta \phi^A} X .
$$
(5.26)

This operator Δ has two nasty properties. Firstly, it is a non-linear differential operator, which means that $\Delta(XY) \neq X\Delta Y + (-1)^{\epsilon_Y}\Delta X.Y$. An extra term, (X, Y) is present. The correct formula, and a few more, can again be found in the appendices. Secondly, when acting on local functionals of fields and antifields, Δ leads to expressions proportional to $\delta(0)$. The third part of this work is devoted completely to a one-loop regularisation prescription to handle this difficulty. For the moment, we will neglect this problem and all the manipulations with path integrals are understood to be *formal*. Finally, the operator σX is defined to be

$$\sigma X = (X, W) - i\hbar\Delta X. \tag{5.27}$$

Below we will argue that σ is the *quantum BRST operator*.

Let us present two applications of this Ward identity. Consider first the partition function constructed with the gauge fermion $\Psi[\phi]$:

$$\mathcal{Z}_\Psi = \int [d\phi][d\phi^*]e^{\frac{i}{\hbar}W(\phi,\phi^*)}\delta(\phi_A^* - \Psi_A). \tag{5.28}$$

The new notation is $\Psi_A = \frac{\overleftarrow{\delta}\Psi}{\delta\phi^A}$. An infinitesimal change of the gauge fermion from $\Psi[\phi]$ to $\Psi[\phi] + d\Psi[\phi]$ gives the partition function

$$\mathcal{Z}_{\Psi+d\Psi} = \int [d\phi][d\phi^*]e^{\frac{i}{\hbar}W(\phi,\phi^*)}\delta(\phi_A^* - \Psi_A - d\Psi_A), \tag{5.29}$$

in an obvious notation. Redefine now the integration variable $\phi_A^{*'} = \phi_A^* - d\Psi_A$, which formally gives Jacobian 1. Using the fact that $d\Psi$ is infinitesimal, we can expand to linear order to get (dropping the primes)

$$\mathcal{Z}_{\Psi+d\Psi} = \int [d\phi][d\phi^*]e^{\frac{i}{\hbar}W(\phi,\phi^*)}\left(1 + \frac{\overleftarrow{\delta}W}{\delta\phi_B^*}\frac{\overrightarrow{\delta}\,d\Psi}{\delta\phi^B}\right)\delta(\phi_A^* - \Psi_A). \tag{5.30}$$

Subtracting (5.28) from (5.30) and using that $d\Psi$ does not depend on antifields, we get

$$\mathcal{Z}_{\Psi+d\Psi} - \mathcal{Z}_\Psi = \langle\sigma d\Psi\rangle_\Psi = 0. \tag{5.31}$$

Hence, the Ward identity still implies gauge independence.

As a second application, consider $X(\phi, \phi^*) = \phi_A^* F(\phi, \phi^*)$. Using the properties of the antibracket and of the box operator as listed in the appendix, we easily see that

$$\sigma[\phi_A^* F] = \phi_A^* \sigma F + (-1)^{(\epsilon_A+1)(\epsilon_F+1)}\left[F\frac{\overleftarrow{\delta}W}{\delta\phi^A} + \frac{\hbar}{i}\frac{\overleftarrow{\delta}F}{\delta\phi^A}\right]. \tag{5.32}$$

If $\sigma F = 0$, we see that we find back the Schwinger-Dyson equation as a Ward identity. This was already pointed out in [35].

5.2.2 Quantum master equation

Of course, turning the argument around, the fact that the Ward identity is valid for all $X(\phi, \phi^*)$ implies that $W(\phi, \phi^*)$ has to satisfy certain conditions.

We now derive the *quantum master equation* by removing, by means of partial integrations, all derivatives acting on X. We start from

$$
\begin{aligned}
0 &= \int [d\phi][d\phi^*] \left[\frac{\overleftarrow{\delta}X}{\delta\phi^A}\frac{\overrightarrow{\delta}W}{\delta\phi_A^*} - \frac{\overleftarrow{\delta}X}{\delta\phi_A^*}\frac{\overrightarrow{\delta}W}{\delta\phi^A} - i\hbar(-1)^{\epsilon_A+1}\frac{\overleftarrow{\delta}}{\delta\phi_A^*}\frac{\overleftarrow{\delta}}{\delta\phi^A}X \right] \\
&\quad \times e^{\frac{i}{\hbar}W(\phi,\phi^*)}\ \delta\left(\phi_A^* - \frac{\overleftarrow{\delta}\Psi[\phi]}{\delta\phi^A} \right).
\end{aligned}
\tag{5.33}
$$

We will do a field redefinition of the variables in the path integral: $\phi_A^* = \phi_A^{*'} + \overleftarrow{\delta}\Psi[\phi]/\delta\phi^A$ and the fields themselves are not altered. The Jacobian of this transformation is, at least formally, 1. We use the notation Ψ_A as defined on the previous page. We will make use of the fact that

$$
\frac{\overleftarrow{\delta}Y(\phi,\phi^*)}{\delta\phi^A}\Big|_{\phi_A^* \to \phi_A^* + \Psi_A} = \frac{\overleftarrow{\delta}Y(\phi,\phi_A^*+\Psi_A)}{\delta\phi^A} - \frac{\overleftarrow{\delta}Y(\phi,\phi_A^*+\Psi_A)}{\delta\phi_B^*}\cdot\frac{\overleftarrow{\delta}}{\delta\phi^A}\frac{\overleftarrow{\delta}}{\delta\phi^B}\Psi.
\tag{5.34}
$$

This allows us to rewrite the first contribution to the Ward identity as

$$
i\hbar \int [d\phi][d\phi^*]\ X(\phi,\phi_A^*+\Psi_A).\delta(\phi^*).\Delta \exp\left[\frac{i}{\hbar}W(\phi,\phi_A^*+\Psi_A) \right] \tag{5.35}
$$

$$
- \int [d\phi][d\phi^*]\ \frac{\overleftarrow{\delta}X(\phi,\phi_A^*+\Psi_A)}{\delta\phi_B^*}\cdot\frac{\overleftarrow{\delta}}{\delta\phi^A}\frac{\overleftarrow{\delta}}{\delta\phi^B}\Psi.\frac{\hbar}{i}\frac{\overrightarrow{\delta}e^{\frac{i}{\hbar}W(\phi,\phi_A^*+\Psi_A)}}{\delta\phi_A^*}\delta(\phi^*).
$$

Before doing the field redefinition in the second and third term of the Ward identity, we combine them by isolating the derivative with respect to ϕ^A. Then doing the field redefinition, applying the shift trick (5.34) and dropping a total divergence, leads to

$$
\begin{aligned}
-\frac{\hbar}{i}\int[d\phi][d\phi^*]\ (-1)^{\epsilon_A+1}\frac{\overleftarrow{\delta}}{\delta\phi_B^*}&\left[e^{\frac{i}{\hbar}W(\phi,\phi_A^*+\Psi_A)}\frac{\overleftarrow{\delta}X(\phi,\phi_A^*+\Psi_A)}{\delta\phi_A^*} \right] \\
&\times \left(\frac{\overleftarrow{\delta}}{\delta\phi^A}\frac{\overleftarrow{\delta}}{\delta\phi^B}\Psi \right).\delta(\phi^*).
\end{aligned}
\tag{5.36}
$$

Working out the derivative acting on the square brackets gives two terms, one which vanishes identically and a second one which cancels the second term of (5.36). Thus the first term of (5.36) is zero for all possible choices of X, which gives the *quantum master equation*

$$
0 = \Delta \exp\left[\frac{i}{\hbar}W(\phi^A,\phi_A^*+\Psi_A) \right].
\tag{5.37}
$$

This master equation can be rewritten in the more tractable form

$$
(W,W) - 2i\hbar\Delta W = 0\ ,
\tag{5.38}
$$

by using the explicit expression for Δ. The natural Ansatz for solving this equation, is by expanding W in a powerseries in $\hbar$:

$$
W(\phi,\phi_A^*+\Psi_A) = S_{BV}(\phi,\phi_A^*+\Psi_A) + \sum_{n=1}\hbar^n M_n.
\tag{5.39}
$$

Grouping the terms order by order in $\hbar$, we get the set of equations

$$
\begin{array}{ll}
\hbar^0 & (S_{BV}, S_{BV}) = 0 \\
\hbar^1 & (S_{BV}, M_1) = i\Delta S_{BV} \\
\hbar^2 & (S_{BV}, M_2) + \frac{1}{2}(M_1, M_1) = i\Delta M_1 \\
& \cdots
\end{array}
\tag{5.40}
$$

The $\mathcal{O}(\hbar^0)$ is the only one where Δ does not appear, and hence it is the only one that can be studied without having to introduce a regularisation scheme. Although the set of equations that one has to solve is possibly infinite, one very rarely has to worry about more than the first two equations. In fact, the regularisation prescription that we will discuss in the third part of this work, is only capable to give a regularised expression for $\mathcal{O}(\hbar)$ equation. No regularised computations have been done in the BV scheme for the $\mathcal{O}(\hbar^2)$ equation[4]. The equation at $\mathcal{O}(\hbar^0)$ is just the classical master equation we encountered before. The extra term generated by the fact that we here (5.39) have the gauge fixed action rather than just $S_{BV}(\phi, \phi^*)$, is identically zero, as an explicit calculation shows. In chapter 8, we will see that this is due to the fact that gauge fixing is a canonical transformation.

Instead of expanding in integer powers of $\hbar$, one can also take an expansion in halfinteger powers as Ansatz: $W = S_{BV} + \sqrt{\hbar}M_{1/2} + \hbar M_1 + \ldots$ This leads to the set of equations

$$
\begin{array}{ll}
\hbar^0 & (S_{BV}, S_{BV}) = 0 \\
\hbar^{1/2} & (S_{BV}, M_{1/2}) = 0 \\
\hbar^1 & (S_{BV}, M_1) + \frac{1}{2}(M_{1/2}, M_{1/2}) = i\Delta S_{BV} \\
& \cdots
\end{array}
\tag{5.41}
$$

It has been found ([37], see also [38]) that the method of introducing *background charges* (in conformal field theory) to cancel the anomalies, can be incorporated in BV in this way. We come back to this issue in the last chapter of this dissertation.

5.2.3 Classical and quantum cohomology

The form of the Ward identity, $\langle \sigma X \rangle = 0$, suggests that we should consider

$$
\sigma X = (X, W) - i\hbar \Delta X
\tag{5.42}
$$

as the BRST operator in the BV formalism. Moreover, it is easy to show that σ is a nilpotent operator if the quantum action W satisfies the quantum master equation. Owing to the non-linearity of the delta operator and the fact that a regularisation scheme is needed to calculate σX, the *quantum BRST operator* and its cohomology have not been studied in detail yet. In the examples in section 4, we give a formal derivation of an operator satisfying $\sigma X = 0$.

The classical part of σ, $\mathcal{S}X = (X, S_{BV})$ is easier to handle. As S_{BV} satisfies the classical master equation, it is easy to see that this is also a nilpotent

---

[4]It is known that W_3 gravity (the model is presented as an example in the next chapter) has a two loop anomaly [36]. It would hence be interesting to study this model in a two loop regularised BV formalism.

operator: $\mathcal{S}^2 X = 0$. In contrast to the full quantum BRST operator, it is a linear differential operator acting from the right:

$$\mathcal{S}X(\phi, \phi^*) = \frac{\overleftarrow{\delta} X}{\delta \phi^A} \mathcal{S}\phi^A + \frac{\overleftarrow{\delta} X}{\delta \phi^*_A} \mathcal{S}\phi^*_A. \tag{5.43}$$

This differential operator acts on the space of (local) function(al)s of the fields and antifields. For an extensive study of the antibracket cohomology, see [6, 7, 38].

A third BRST operator can be defined that only acts on function(al)s of the fields, not of the antifields. It is defined by

$$QF(\phi) = (F(\phi), S_{BV})|_{\phi^*=0}. \tag{5.44}$$

If $S_{gf} = S_{BV}(\phi, \Psi_A)$ denotes the gauge fixed action, it follows from evaluating the master equation for $\phi^* = 0$ that $QS_{gf} = 0$. For closed algebras, one has that $Q^2 F(\phi) = 0$. For open algebras, which we discuss in the next chapter, it is easy to show that the nilpotency of Q holds only using the field equations.

5.3 Room for generalisation

We close the theoretical developments of this chapter with two comments, both concerning the choice of the gauge fermion Ψ (5.7). They will provide the link with chapter 8 on canonical transformations. First off all, consider a specific configuration for the antifields ϕ^*_A, say θ^*_A, and take as gauge fermion:

$$\Psi[\phi] = \theta^*_A \phi^A + \Psi_0[\phi]. \tag{5.45}$$

This leads to the partition function

$$
\begin{aligned}
\mathcal{Z}(\theta^*) &= \int [d\phi][d\phi^*] e^{\frac{i}{\hbar} W(\phi, \phi^*)} \delta(\phi^*_A - \theta^*_A - \Psi_{0A}) \\
&= \int [d\phi][d\phi^*] e^{\frac{i}{\hbar} W(\phi, \phi^*_A + \Psi_{0A})} \delta(\phi^*_A - \theta^*_A).
\end{aligned} \tag{5.46}
$$

Hence, we do not have to fix the antifields to zero, but we can keep them as arbitrary, external sources for the BRST transformations.

Secondly, we can lift the restriction that the gauge fermion only depends on the fields ϕ^A. We can consider $\Psi[\phi, \phi^*]$, and still we have that $\delta^2 \Psi = 0$[5]. Gauge fixing the original symmetries with such a gauge fermion, leads instead of (5.8) to

$$S_{com} = S_{BV}(\phi - \varphi, \phi^*) - \left(\phi^*_A - \frac{\overleftarrow{\delta} \Psi[\phi, \phi^*]}{\delta \phi^A}\right) c^A - \left(\varphi^A - \frac{\overleftarrow{\delta} \Psi[\phi, \phi^*]}{\delta \phi^*_A}\right) B_A. \tag{5.47}$$

For such a gauge fermion, the collective field is not fixed to zero. We obtain the partition function

$$\mathcal{Z} = \int [d\phi][d\phi^*] \exp\left[\frac{i}{\hbar} S_{BV}\left(\phi - \frac{\delta\Psi[\phi, \phi^*]}{\delta \phi^*}, \phi^*\right)\right] \delta(\phi^*_A - \frac{\delta\Psi[\phi, \phi^*]}{\delta \phi^A}), \tag{5.48}$$

---

[5]The property that $\delta^2 \Psi = 0$ is of crucial importance in the quantisation of open algebras, as we will see in the next chapter. There too, this property is valid for gauge fermions depending on both fields and antifields.

If we make a specific choice for the gauge fermion of the form $\Psi[\phi, \phi^*] = \theta_A^* \phi^A + \Psi_0[\phi, \phi^*]$ for an infinitesimal Ψ_0, we can solve the condition in the delta-function for ϕ^*, up to terms quadratic in Ψ_0. The antifields can then be integrated out. Below we will see that this expression can be reinterpreted elegantly using canonical transformations. The derivation of the Ward indentity and hence of the quantum master equation as sketched above becomes more difficult. We discuss these generalisations from the more algebraic point of view of canonical transformations in section 2 of chapter 8.

5.4   Some hopefully clarifying examples

5.4.1   Extended action of W_2-gravity

Let us consider a first example: W_2-gravity [39]. The classical action is given by

$$S_0 = \frac{1}{2\pi} \int d^2x \left[\partial \phi \bar{\partial} \phi - h(\partial \phi)^2 \right].$$

(5.49)

The fields in the configuration space, $\phi(z, \bar{z})$ and $h(z, \bar{z})$, are both of even Grassmann parity and we used the notation $\partial = \frac{d}{dz}$ and $\bar{\partial} = \frac{d}{d\bar{z}}$. The action is invariant under the transformations

$$
\begin{aligned}
\delta_\epsilon \phi &= \epsilon \partial \phi \\
\delta_\epsilon h &= \bar{\partial} \epsilon - h \partial \epsilon + \partial h . \epsilon.
\end{aligned}
$$

(5.50)

From this we can derive the R_α^i. They are given by

$$
\begin{aligned}
R_y^{\phi(x)} &= \partial_x \phi(x) . \delta(x - y) \\
R_y^{h(x)} &= \partial_x h(x) . \delta(x - y) - \bar{\partial}_y \delta(x - y) + \partial_y [h(x) \delta(x - y)]
\end{aligned}
$$

(5.51)

(5.52)

Here, x and y are complex coordinates. We can calculate the structure functions $T_{\beta\gamma}^\alpha = T_{y\tilde{y}}^z$ explicitly by evaluating the commutator of the gauge generators (1.16) for any choice of i, by using $i = \phi(x)$ or $i = h(x)$. We find

$$2T_{y\tilde{y}}^z = \delta(z - y)\partial_z \delta(z - \tilde{y}) - \delta(z - \tilde{y})\partial_z \delta(z - y).$$

(5.53)

Notice that in order to verify that the algebra is closed, one has to calculate the commutator for every value of i. Using all this, we find the extended action

$$
\begin{aligned}
S &= S_0 + \phi_i^* R_\alpha^i c^\alpha + c_\alpha^* T_{\beta\gamma}^\alpha c^\gamma c^\beta \\
&= \frac{1}{2\pi} \left[\partial \phi \bar{\partial} \phi - h(z, \bar{z})(\partial \phi)^2 \right] \\
&\quad + \phi^* c \partial \phi + h^* (\bar{\partial} c - h \partial c + \partial h . c) + c^* (\partial c) c.
\end{aligned}
$$

(5.54)

In practice, we do not evaluate the algebra and its possible non-closure functions. One takes $S = S_0 + \phi_i^* R_\alpha^i c^\alpha + \ldots$ and one tries to find the dots such that $(S, S) = 0$.

To construct a gauge fixed action, we first introduce a trivial system, as up to now, all fields have positive ghostnumber. Take the non-minimal solution of the classical master equation (5.21)

$$S_{n.m.} = S + b^* \lambda.$$

(5.55)

Hence, we have introduced an antighost b and an auxiliary field λ. The gauge field h can for instance be fixed to a background field $\hat{h}$ by taking as gauge fermion $\Psi = b(h - \hat{h})$. This gives

$$S_{n.m.}(\phi, \phi_A^* + \Psi_A) = S + (b^* + h - \hat{h})\lambda + b(\bar{\partial}c - h\partial c + \partial h.c). \qquad (5.56)$$

Of course, the first extra term is the gauge fixing while the second extra term is the ghost action. Notice that upon integration over the auxiliary field λ, we are left with the delta-function $\delta(b^* + h - \hat{h})$. If we integrate over h, imposing this gauge fixing, h is replaced everywhere by $\hat{h} - b^*$. Normally, the gauge fixing of h on this background field is done in order to prevent the accidental vanishing of the anomaly. Now we see that $\hat{h}$ occurs always together with b^*. From this we can conclude that if we do not put b^* to zero, it may play the part of the background field. This is a first reason to keep the antifield dependence after gauge fixing [25].

5.4.2 Construction of topological Yang-Mills theory

In this second example, we [33] construct the extended action for topological Yang-Mills (YM) theory [40, 41, 42]. This model is presented as it exemplifies a non-standard gauge fixing procedure where the antighosts that are introduced, have a different (tensor) structure than the ghosts.

We start from a compact, four-dimensional manifold, endowed with a metric $g_{\alpha\beta}$ which may be of Euclidean or Minkowski signature. On this manifold we define the Yang-Mills fields $A_\mu = A_\mu^a T_a$. The T_a are the generators of a Lie algebra. The classical action is the topological invariant known as the *Pontryagin index* or *winding number*. So we have

$$S_0 = \int_{\mathcal{M}} d^4x \sqrt{|g|} F_{\mu\nu} \tilde{F}^{\mu\nu} . \qquad (5.57)$$

The *dual* of an antisymmetric tensor $G_{\mu\nu}$ is defined by

$$\tilde{G}_{\mu\nu} = \frac{1}{2}[\epsilon]_{\mu\nu\sigma\tau} G_{\alpha\beta} g^{\alpha\sigma} g^{\beta\tau} . \qquad (5.58)$$

The Levi-Civita tensor tensor is defined by $[\epsilon]_{\mu\nu\sigma\tau} = \sqrt{g}\epsilon_{\mu\nu\sigma\tau}$, where $\epsilon_{\mu\nu\sigma\tau}$ is the permutation symbol and $g = \det g_{\alpha\beta}$. Remark that it is complex for a Minkowski metric. We normalise our representation for the algebra such that $Tr[T_a T_b] = \delta_{ab}$, and a trace over the Yang-Mills indices is always understood. The classical action is invariant under continuous deformations of the gauge fields that do not change the winding number:

$$\delta A_\mu = \epsilon_\mu . \qquad (5.59)$$

We will not specify the conditions to be imposed on ϵ_μ. We associate the ghosts ψ_μ with the shift parameters ϵ_μ. Then we immediately obtain the BV extended action

$$S = S_0 + A^{*\mu}\psi_\mu . \qquad (5.60)$$

In the literature [40, 41, 42, 43], one works with the reducible set of gauge symmetries consisting of the shift symmetries $\delta A_\mu = \epsilon_\mu$ and the usual Yang-Mills symmetry $\delta A_\mu = D_\mu\epsilon$. We will not do this, as this is merely a complicated disguise of the construction that we will describe here. We can always reintroduce this reducible set of symmetries by adding a trivial system and doing two

canonical transformations. We refer to section 4 of chapter 8, where we discuss this in detail as an example of the use of canonical transformations to enlarge the set of fields.

Let us now gauge-fix the shift symmetry (5.59) in order to obtain the topological field theory that is related to the moduli space of self dual YM instantons [40]. We take the gauge fixing conditions

$$
\begin{aligned}
F^+_{\mu\nu} &= 0 \\
\partial_\mu A^\mu &= 0 \ ,
\end{aligned}
\tag{5.61}
$$

where $G^\pm_{\mu\nu} = \frac{1}{2}(G_{\mu\nu} \pm \tilde{G}_{\mu\nu})$. Fields $G_{\mu\nu}$ that satisfy $G_{\mu\nu} = G^+_{\mu\nu}$ are *selfdual*, while $G_{\mu\nu} = G^-_{\mu\nu}$ is the *anti-selfduality* condition. These projectors are orthogonal to eachother, so that we have for general X and Y that $X^+_{\alpha\beta}Y^{-\alpha\beta} = 0$. The above gauge choice does not fix all the gauge freedom because there may not be a unique solution of (5.61) for a given winding number. To use the picture of the gauge orbits, the gauge fixing may not select one configuration on every orbit. If that is the case, then the moduli space would consist out of one single point for every winding number. However, this gauge choice is admissible in the sense that the gauge fixed action will have well defined propagators. Moreover, the degrees of freedom that are left (the space of solutions of (5.61)) form exactly the moduli space of the instantons that one wants to explore. We now introduce auxiliary fields in order to construct a gauge fermion. Obviously we should add

$$
S_{nm} = S + \chi^*_{0\alpha\beta}\lambda_0^{\alpha\beta} + b^*\lambda \ ,
\tag{5.62}
$$

and consider the gauge fermion

$$
\Psi_1 = \chi_0^{\alpha\beta}(F^+_{\alpha\beta} - x\lambda_{0\alpha\beta}) + b(\partial_\mu A^\mu - y\lambda) \ ,
\tag{5.63}
$$

where x and y are some arbitrary gauge parameters. We introduced here an antisymmetric field $\chi_0^{\alpha\beta}$[6]. This field has six components, which are used to impose three gauge conditions. After the gauge fixing, the action has the gauge symmetry $\chi_0^{\alpha\beta} \to \chi_0^{\alpha\beta} + \epsilon_0^{-\alpha\beta}$. So we fix this symmetry by imposing the condition $\chi_0^- = 0$. This can be done by adding an extra trivial system $(\chi_{1\alpha\beta}, \lambda_{1\alpha\beta})$ and with the extra gauge fermion $F = \chi_{1\alpha\beta}\chi_0^{-\alpha\beta}$. But then we have again introduced too much fields, and this leads to a new symmetry $\chi_{1\alpha\beta} \to \chi_{1\alpha\beta} + \epsilon^+_{1\alpha\beta}$ which we have to gauge fix. One easily sees that this procedure repeats itself ad infinitum. We could, in principle, also solve this problem by only introducing $\chi_0^{+\alpha\beta}$ as a field. Then we have to integrate over the space of self dual fields. To construct the measure on this space, we have to solve the constraint $\chi = \chi^+$. Since this in general can be complicated (as e.g. in the topological σ–model) we will keep the $\chi_{\alpha\beta}$ as the fundamental fields. The path integral is with the measure $[d\chi_0^{\alpha\beta}]$ and we do not split this into the measures in the spaces of self and anti–selfdual fields. The price we have to pay is an infinite tower of auxiliary fields. These we denote by $(\chi_n^{\alpha\beta}, \lambda_n^{\alpha\beta})$[7] with Grassmann parities $\epsilon_{\lambda_n} = n, \epsilon_{\chi_n} = n + 1$ (modulo 2) and ghostnumbers $\mathrm{gh}(\lambda_n)$ equal to zero for n even and one for n odd. Similarly, $\mathrm{gh}(\chi_n)$ equals -1 for n even and zero for

---

[6]In the literature [40, 41, 42, 43], one usually introduces a selfdual two-tensor as antighost. However, in taking the variation of the action, to obtain the field equations or to calculate the energy-momentum tensor, ad hoc rules are then needed to maintain this selfduality. This becomes particularly cumbersome when the projection operators on the selfdual or anti-selfdual pieces are dependent of other fields in the theory, as is the case for topological σ-models.

[7]One remark has to be made here concerning the place of the indices. We choose the indices of χ_n and λ_n to be upper resp. lower indices when n is even resp. odd. Their antifields have the opposite property, as usual.

n odd. We then add to the action (5.62) the term $\sum_{n=1}^{\infty} \chi_{n,\alpha\beta}^* \lambda_n^{\alpha\beta}$ and take as gauge fixing fermion

$$\Psi_2 = \sum_{n=1}^{\infty} \chi_n^{\alpha\beta} \chi_{n-1,\alpha\beta}^{(-)^n} + \Psi_1, \tag{5.64}$$

where $G_{\alpha\beta}^{(-)^n}$ is the selfdual part of $G_{\alpha\beta}$ if n is even and the anti-selfdual part if n is odd. After doing the gauge fixing we end up with the following non–minimal solution of the classical master equation [8] :

$$
\begin{aligned}
S_{nm} &= S_0 + A^{*\mu}\psi_\mu + (\partial_\mu A^\mu + b^*)\lambda + (F_{\alpha\beta}^+ + \chi_{1\alpha\beta}^- + \chi_{0\alpha\beta}^*)\lambda_0^{\alpha\beta} \\
&\quad -y\lambda^2 - x\lambda_0^{\alpha\beta}\lambda_{0\alpha\beta} + \chi_0^{+\alpha\beta} D_{[\alpha}\psi_{\beta]} + b\partial_\mu\psi^\mu \\
&\quad + \sum_{n=1}^{\infty}(\chi_{n\alpha\beta}^* + \chi_{n+1,\alpha\beta}^{(-)^{(n+1)}} + \chi_{n-1,\alpha\beta}^{(-)^n})\lambda_n^{\alpha\beta} .
\end{aligned}
\tag{5.65}
$$

Performing the $\lambda_n, n \geq 1$ integrals would give the gauge fixing delta functions $\delta(\chi_{n+1}^{(-)^{n+1}} + \chi_{n-1}^{(-)^n} + \chi_n^*)$. Doing only the Gaussian λ_0 and λ integral, we arrive at

$$
\begin{aligned}
S &= S_0 + \frac{1}{4x}(\partial_\mu A^\mu + b^*)^2 + \frac{1}{4y}(F^+ + \chi_1^- + \chi_0^*)^2 \\
&\quad + b\partial_\mu\psi^\mu + \chi_0^{+\alpha\beta} D_{[\alpha}\psi_{\beta]} + A_\mu^*\psi^\mu \\
&\quad + \sum_{n=1}^{\infty}(\chi_{n\alpha\beta}^* + \chi_{n+1,\alpha\beta}^{(-)^{(n+1)}} + \chi_{n-1,\alpha\beta}^{(-)^n})\lambda_n^{\alpha\beta} .
\end{aligned}
\tag{5.66}
$$

Notice that we now have terms quadratic in the antifields. This means that the BRST operator defined by $Q\phi^A = (\phi^A, S)|_{\phi^*=0}$ is only nilpotent using field equations. Indeed, $Q^2 b = \frac{1}{2x}\partial_\mu\psi^\mu \approx 0$, using the field equation of the field b. The fact that we have to use the field equations to prove the nilpotency of the BRST operator is the hallmark of an open algebra. In the next chapter, we will show that open algebras manifest themselves in the BV scheme by non-linear terms in the antifields in the extended action.

5.4.3 The energy-momentum tensor as BRST invariant operator in BV

We [33] construct formally the energy-momentum tensor $T_{\alpha\beta}^q$ that satisfies the condition $\sigma T_{\alpha\beta}^q = (T_{\alpha\beta}^q, W) - i\hbar\Delta T_{\alpha\beta}^q = 0$ or, classically only, $(T_{\alpha\beta}, S) = 0$, provided $W(S)$ satisfies the quantum (classical) master equation. In a first subsection, we derive expressions for the derivation of the antibracket and Δ-operator with respect to the metric. We then define an energy-momentum tensor that is classical or quantum BRST invariant. In the chapter on canonical transformations, we show that the energy momentum tensor as we define it here is canonically invariant (8.45).

5.4.3.1 Metric dependence of the antibracket and Δ

We have to be precise on the occurences of the metric in all our expressions, and specify a consistent set of conventions. All integrations are with the volume

---

[8]Note that from $(\chi, \chi^*) = 1$, it follows that $(\chi^\pm, \chi^{*\pm}) = P^\pm$ and $(\chi^+, \chi^{*-}) = 0$, where $P^\pm$ are the projectors onto the (anti)-selfdual sectors.

element $dx\sqrt{|g|}$. The functional derivative is then defined as

$$\frac{\delta\phi^A}{\delta\phi^B} = \frac{1}{\sqrt{|g|}_B}\frac{d\phi^A}{d\phi^B} = \frac{1}{\sqrt{|g|}_B}\delta_{AB} \ , \tag{5.67}$$

and the same for the antifields. The notation is that A and B contain both the discrete and space-time indices, such that δ_{AB} contains both space-time δ-functions (without $\sqrt{|g|}$) and Kronecker deltas (1 or zero) for the discrete indices . g is $\det g_{\alpha\beta}$, and its subscript B denotes that we evaluate it in the space-time index contained in B. Using this, the antibracket and box operator are defined by[9]

$$(A,B) = \sum_i \int dx\sqrt{|g|}_X \left(\frac{\overleftarrow{\delta}A}{\delta\phi^X}\frac{\overrightarrow{\delta}B}{\delta\phi^*_X} - \frac{\overleftarrow{\delta}A}{\delta\phi^*_X}\frac{\overrightarrow{\delta}B}{\delta\phi^X}\right)$$

$$\Delta A = \sum_i \int dx\sqrt{|g|}_X (-1)^{\epsilon_X+1}\frac{\overleftarrow{\delta}}{\delta\phi^X}\frac{\overleftarrow{\delta}}{\delta\phi^*_X}A \ . \tag{5.68}$$

For once, we made the summation that is hidden in the DeWitt summation more explicit. X contains the discrete indices i and the space-time index x. These definitions guarantee that the antibracket of two functionals is again a functional. Using the notation introduced above, we have that

$$(A,B) = \sum_i \int dx\frac{1}{\sqrt{|g|}_X} \left(\frac{\overleftarrow{d}A}{d\phi^X}\frac{\overrightarrow{d}B}{d\phi^*_X} - \frac{\overleftarrow{d}A}{d\phi^*_X}\frac{\overrightarrow{d}B}{d\phi^X}\right)$$

$$\Delta A = \sum_i \int dx\frac{1}{\sqrt{|g|}_X} (-1)^{\epsilon_X+1}\frac{\overleftarrow{d}}{d\phi^X}\frac{\overleftarrow{d}}{d\phi^*_X}A \ . \tag{5.69}$$

It is now simple to differentiate with respect to the metric. We use the following rule :

$$\frac{\delta g^{\alpha\beta}(x)}{\delta g^{\rho\gamma}(y)} = \frac{1}{2}(\delta^\alpha_\rho\delta^\beta_\gamma + \delta^\alpha_\gamma\delta^\beta_\rho)\delta(x-y) \ , \tag{5.70}$$

where the δ–function does *not* contain any metric, i.e. $\int dx\delta(x-y)f(x) = f(y)$. This we do in order to agree with the familiar recipe to calculate the energy-momentum tensor. Then we find that

$$\frac{\delta(A,B)}{\delta g^{\alpha\beta}(y)} = \left(\frac{\delta A}{\delta g^{\alpha\beta}(y)},B\right) + \left(A,\frac{\delta B}{\delta g^{\alpha\beta}(y)}\right) + \frac{1}{2}g_{\alpha\beta}(y)\sqrt{|g|}(y)[A,B](y) \ , \tag{5.71}$$

and

$$\frac{\delta\Delta A}{\delta g^{\alpha\beta}(y)} = \Delta\frac{\delta A}{\delta g^{\alpha\beta}} + \frac{1}{2}g_{\alpha\beta}(y)\sqrt{|g|}(y)[\Delta A](y) \ , \tag{5.72}$$

[9]We then have that $(\phi,\phi^*) = \frac{1}{\sqrt{|g|}}$. In this convention the extended action takes the form $S = \int dx\sqrt{|g|}[\mathcal{L}_0+\phi^*_i\delta\phi^i+\phi^*\phi^*...]$. Demanding that the total Lagrangian is a scalar amounts to taking the antifield of a scalar to be a scalar, the antifield of a covariant vector to be a contravariant vector, etc. One could also use the following set of conventions. We integrate with the volume element dx without metric, and define the functional derivative (5.67) without $\sqrt{|g|}$. Also the antibracket is defined having no metric in the integration. Therefore, $(\phi,\phi^*)' = 1$. With this bracket the extended action takes the form $S' = \int dx[\sqrt{|g|}\mathcal{L}_0 + \phi^*_i\delta\phi^i + \frac{1}{\sqrt{|g|}}\phi^*\phi^*...]$. The relation between the two sets of conventions is a transformation that scales the antifields with the metric, i.e. $\phi^* \to \sqrt{|g|}\phi^*$. In these variables, general covariance is not explicit and requires a good book–keeping of the $\sqrt{|g|}$'s in the extended action and in other computations. Therefore, we will not use this convention.

with the notation that $(A, B) = \int dx \sqrt{|g|}[A, B]$ and $\Delta A = \int dx \sqrt{|g|}[\Delta A]$. Notice that in $[A, B]$ and $[\Delta A]$ a summation over the discrete indices is understood, but no integration over space-time. Before applying this to define the energy momentum tensor in the BV scheme, consider the following properties. For any two operators A and B, we have that

$$\sum_i \phi_X^* \frac{\vec{\delta}}{\delta \phi_X^*}(A, B) = (\sum_i \phi_X^* \frac{\vec{\delta} A}{\delta \phi_X^*}, B) + (A, \sum_i \phi_X^* \frac{\vec{\delta} B}{\delta \phi_X^*}) - [A, B](x) , \quad (5.73)$$

and

$$\sum_i \phi_X^* \frac{\vec{\delta} \Delta A}{\delta \phi_X^*} = \Delta \left(\sum_i \phi_X^* \frac{\vec{\delta} A}{\delta \phi_X^*} \right) - [\Delta A](x) . \quad (5.74)$$

In both expressions, $X = (i, x)$ with discrete indices i and continuous indices x. There is no integration over x understood, only a summation over i, which is explicitised.

Let us define the differential operator

$$D_{\alpha\beta} = \frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\alpha\beta}} + g_{\alpha\beta} \sum_i \phi_X^* \frac{\vec{\delta}}{\delta \phi_X^*} . \quad (5.75)$$

Then it follows from (5.71) and (5.73) that this operator satisfies

$$D_{\alpha\beta}(A, B) = (D_{\alpha\beta} A, B) + (A, D_{\alpha\beta} B) . \quad (5.76)$$

Owing to (5.72) and (5.74), $D_{\alpha\beta}$ is seen to commute with the Δ-operator:

$$D_{\alpha\beta} \Delta A = \Delta D_{\alpha\beta} A . \quad (5.77)$$

5.4.3.2 Definition of the energy-momentum tensor

Let us now apply all these results to define an expression which can be interpreted as being the BRST invariant energy-momentum tensor and that is invariant under the BRST transformations in the antibracket sense. Define

$$\theta_{\alpha\beta} = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\alpha\beta}} . \quad (5.78)$$

By differentiating the classical master equation $(S, S) = 0$ with respect to the metric $g^{\alpha\beta}(y)$, and by multiplying with $2/\sqrt{|g|}$, we find from (5.71) that

$$0 = 2(\theta_{\alpha\beta}(y), S) + 2g_{\alpha\beta}(y) \sum_i \frac{\overleftarrow{\delta} S}{\delta \phi^X} \frac{\vec{\delta} S}{\delta \phi_X^*} . \quad (5.79)$$

In the second term, $X = (y, i)$ and there is only a summation over i. Hence, we see that $\theta_{\alpha\beta}$ is not BRST invariant in the antibracket sense.

However, if we define the energy-momentum tensor by

$$T_{\alpha\beta} = D_{\alpha\beta} S, \quad (5.80)$$

then it follows immediately that

$$D_{\alpha\beta}(S, S) = 0 \iff (T_{\alpha\beta}, S) = 0. \quad (5.81)$$

It is then clear that $T_{\alpha\beta}$ is a BRST invariant energy-momentum tensor[10]. Moreover, $\theta_{\alpha\beta}|_{\phi^*=0} = T_{\alpha\beta}|_{\phi^*=0}$. Whether this is a trivial element of the cohomology, i.e. equivalent to zero, can of course not be derived on general grounds. By adding to this expression for $T_{\alpha\beta}$ terms of the form $(X_{\alpha\beta}, S)$, one can obtain cohomologically equivalent expressions. For example, by subtracting the term $(\frac{1}{2}g_{\alpha\beta}\sum_i \phi_X^* \phi^X, S)$, the terms that have to be added to $\theta_{\alpha\beta}$ to obtain $T_{\alpha\beta}$ take a form that is symmetric in fields and antifields.

We can generalise this result and define an energy-momentum tensor that is quantum BRST invariant. Consider the quantum extended action W that satisfies the quantum master equation $(W, W) - 2i\hbar\Delta W = 0$. Define the quantum analogue of $\theta_{\alpha\beta}$, i.e.

$$\theta^q_{\alpha\beta} = \frac{2}{\sqrt{|g|}}\frac{\delta W}{\delta g^{\alpha\beta}} \ . \tag{5.82}$$

Again, one easily sees that this is not a quantum BRST invariant quantity. Define however

$$T^q_{\alpha\beta} = D_{\alpha\beta}W, \tag{5.83}$$

then it follows by letting $D_{\alpha\beta}$ act on the quantum master equation that

$$\begin{aligned} D_{\alpha\beta}[(W, W) - 2i\hbar\Delta W] = 0 \\ \Updownarrow \\ \sigma T^q_{\alpha\beta} = (T^q_{\alpha\beta}, W) - i\hbar\Delta T^q_{\alpha\beta} = 0 \ . \end{aligned} \tag{5.84}$$

5.5   Overview

Since this is a rather long chapter where we have introduced many new concepts, a short recapitulation is in place. Imposing that the quantum equations of motion (the Schwinger-Dyson equations) are included in the Ward identities of any theory, we have constructed the BV antifield scheme for closed, irreducible algebras. One can do this by enlarging the symmetry algebra to include the shift symmetries. The BRST invariance of the gauge fixed action under the enlarged symmetry implies that the extended action of the BV scheme satisfies the classical master equation. We have defined the quantum BRST operator σ, and have derived the quantum master equation as the condition that guarantees that the Ward identities $\langle \sigma X \rangle = 0$ hold. After a short discussion of different cohomologies, we have given some examples. Let us stress that most of the developments above generalise to open algebras, as we show in the next chapter.

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[10] Notice that this quantity is the energy momentum tensor that one immediately obtains when using the variables mentioned in the previous footnote, i.e. after scaling the antifields. One can then check that $T_{\alpha\beta} = \frac{2}{\sqrt{|g|}}\frac{\delta S'}{\delta g^{\alpha\beta}}$. In this sense the modification of $\theta_{\alpha\beta}$ is an artifact of the used conventions.

Chapter 6

Open gauge algebras

In this chapter, we give a quantisation recipe for theories with an open gauge algebra. First, we describe in detail an inductive approach [44], combining the recipe of B. de Wit and J.W. van Holten [45] for the BRST quantisation of gauge theories with an open algebra with the requirement of J. Alfaro and P.H. Damgaard that the BRST Ward identities must include the Schwinger-Dyson equations. This way, we show that also for open algebras the central object is the extended action that is a solution of the BV classical master equation. Secondly, we turn the argument around, and point out that the collective field formalism leads in a straightforward fashion to a quantisation recipe for open algebras.

6.1 From BRST to BV quantisation for open algebras

Whatever the quantisation procedure for closed algebras that we have discussed, be it the Faddeev-Popov recipe, the quantisation based on BRST symmetry or the reformulation of the latter in the BV scheme, an essential step in the quantisation process is the choice of gauge. Let us assume that the gauge choice is encoded in the gauge conditions $F^\alpha = 0$. Any quantisation scheme should at least satisfy the following two requirements. First of all, the functional integrals for the partition function and for the correlation functions can be made well-defined. This means that all fields can be given a propagator by a careful choice of the functions F^α. This has previously been called the *admissability of the gauge choice*. No general prescription can be given for this, *gauge fixing is an art*. Moreover, although the partition function is defined using a specific choice for the F^α, it should nevertheless be invariant under (infinitesimal) deformations of these functions. This means that the partition function should be gauge independent.

Let us recapitulate how the BRST quantisation for closed algebras that was described in chapter 3, satisfies these requirements. Having a nilpotent BRST operator δ, the gauge fixed action is constructed by adding the BRST variation of a gauge fermion –which encodes the gauge functions F^α– to the classical action: $S_{com} = S_0 + \delta\Psi$. Owing to the nilpotency of δ, we have that $\delta S_{com} = 0$,

which allows the derivation of the Ward identities $\langle \delta X \rangle = 0$ (3.14). These Ward identities then imply invariance of the partition under infinitesimal deformation of the gauge fermion (3.18). One may have to add quantum counterterms to the action to cancel the BRST non-invariance of the measure in the derivation of the Ward identity (section 2 of chapter 3).

This recipe fails for open algebras. Remember that by *open algebra*, we mean that a term proportional to the classical field equations appears in the commutation relations of the algebra[1] (1.16):

$$\frac{\overleftarrow{\delta} R^i_\alpha}{\delta \phi^j} R^j_\beta - (-1)^{\epsilon_\alpha \epsilon_\beta} \frac{\overleftarrow{\delta} R^i_\beta}{\delta \phi^j} R^j_\alpha = 2 R^i_\gamma T^\gamma_{\alpha\beta} (-1)^{\epsilon_\alpha} - 4 y_j E^{ji}_{\alpha\beta} (-1)^{\epsilon_i} (-1)^{\epsilon_\alpha}. \quad (6.1)$$

Multiplying both sides with $(-1)^{(\epsilon_\alpha + 1)\epsilon_\beta} c^\alpha c^\beta$, we get

$$\frac{\overleftarrow{\delta} R^i_\alpha c^\alpha}{\delta \phi^j} R^j_\beta c^\beta + R^i_\gamma T^\gamma_{\alpha\beta} c^\beta c^\alpha = 2(-1)^{\epsilon_i} y_j E^{ji}_{\alpha\beta} c^\beta c^\alpha. \quad (6.2)$$

If we take the naive BRST transformation rules of section 1 of chapter 3, i.e. $\delta \phi^i = R^i_\alpha c^\alpha$ and $\delta c^\gamma = T^\gamma_{\alpha\beta} c^\beta c^\alpha$, we see that (6.2) implies that the BRST operator δ is not nilpotent when acting on ϕ^i, instead we have that

$$\delta^2 \phi^i = 2(-1)^{\epsilon_i} y_j E^{ji}_{\alpha\beta} c^\beta c^\alpha. \quad (6.3)$$

Hence, the BRST operator is only nilpotent on the stationary surface . Similar terms, proportional to field equations, may appear when calculating $\delta^2 c^\alpha$. If we again adopt the condensed notation ϕ^A for ϕ^i, c^α and trivial systems and $\delta \phi^A = \mathcal{R}^A[\phi]$, we can sum up the situation by

$$\delta^2 \phi^A = \frac{\overleftarrow{\delta} \mathcal{R}^A}{\delta \phi^B} \mathcal{R}^B = y_j M^{jA}. \quad (6.4)$$

It is then clear that if we would construct the gauge fixed action like for closed algebras, it would no longer be BRST invariant off the stationary surface. Consequently, we can not prove gauge independence in the way we have described in section 2 of chapter 3, using Ward identities.

Gauge theories with an open algebra were first encountered in the study of supergravity theories in the second half of the seventies. It was found that terms quartic in the ghost fields are needed, which can of course *not* be obtained from the Faddeev-Popov procedure like we presented it in chapter 2. The action with the quartic ghost term is still invariant under modified BRST transformations [46]. Inspired by these results, B. de Wit and J.W. van Holten gave a general recipe for BRST quantisation of theories with an open algebra [45]. The basic observation is that one can drop the nilpotency requirement of the BRST transformation and just demand that one constructs a gauge fixed action that is invariant under a set of BRST transformation rules. If the theory is anomaly free, a BRST invariant action allows the derivation of Ward identities, as we described in section 2 of chapter 3. Furthermore, the gauge fixed action should be such that changing the gauge (fermion) infinitesimally, amounts to adding the BRST variation of an infinitesimal quantity. The Ward identities then guarantee gauge independence of the partition function.

---

[1] This term proportional to field equations is not arbitrary. $E^{ji}_{\alpha\beta}(-1)^{\epsilon_i}$ is graded antisymmetric in i and j (1.12).

The recipe of [45] is to consider the gauge fermion $F = b_\alpha F^\alpha(\phi^i)$ and to define $F_i = \overleftarrow{\delta} F/\delta\phi^i$. The gauge fixed action S_{com} is obtained by adding an expansion in powers of these F_i to the classical action, where the linear term is taken to be the Faddeev-Popov quadratic ghost action. The modified BRST transformation rules for the gauge fields and the ghosts are also obtained by adding an expansion in these F_i to the BRST transformation rules for closed algebras. All field dependent coefficients in these expansions are fixed by requiring $\delta S_{com} = 0$.

In accordance with our collective field approach to enlarge the BRST algebra in such a way that the Ward identities include the Schwinger-Dyson equations, we introduce again collective fields φ^A for ϕ^A, shift ghosts c^A and the trivial pair (ϕ_A^*, B_A). The naive BRST transformation rules are taken to be

$$
\begin{aligned}
\delta_N \phi^A &= c^A \\
\delta_N \varphi^A &= c^A - \mathcal{R}^A[\phi - \varphi] \\
\delta_N c^A &= 0 \\
\delta_N \phi_A^* &= B_A \\
\delta_N B_A &= 0.
\end{aligned}
\tag{6.5}
$$

Notice that these transformation rules are still only nilpotent using the field equations, as can be seen from $\delta^2(\phi^A - \varphi^A)$. More importantly however, the BRST transformation as we have constructed it, is now nilpotent when acting on $\phi^A : \delta_N^2 \phi^A = 0$. This implies that the original gauge symmetry can be gauge fixed in a BRST invariant way by adding $\delta_N \Psi[\phi]$. This is the real reason for using the freedom to shift the $\mathcal{R}^A[\phi - \varphi]$ between the BRST transformation of the field and the collective field the way we do. We shift the off-shell non-nilpotency in the transformation rules of the collective field.

If we impose the same gauge fixing conditions as in chapter 5 (5.7), the complete gauge fermion F is given by

$$
F = -\phi_A^* \varphi^A + \Psi[\phi].
\tag{6.6}
$$

The prescription of de Wit and van Holten then implies that we should add expansions in

$$
\frac{\overleftarrow{\delta} F}{\delta \phi^A} = \frac{\overleftarrow{\delta} \Psi}{\delta \phi^A} \overset{not.}{=} \Psi_A,
\tag{6.7}
$$

and

$$
\frac{\overleftarrow{\delta} F}{\delta \varphi^A} = -\phi_A^*
\tag{6.8}
$$

to the classical action and to the naive BRST transformation rules. Owing to the nilpotency of δ_N when acting on ϕ^A, we only get a linear term in Ψ_A and an expansion in the antighost (or antifield) ϕ_A^* remains. We then look for quantities $M_n^{A_1 \ldots A_n}(\phi)$, with the Grassmann parities

$$
\epsilon(M_n^{A_1 \ldots A_n}) = \sum_{i=1}^{n} (\epsilon_{A_i} + 1),
\tag{6.9}
$$

and the antisymmetry property

$$
M_n^{A_1 \ldots A_i A_{i+1} \ldots A_n} = (-1)^{(\epsilon_{A_i}+1)(\epsilon_{A_{i+1}}+1)} M_n^{A_1 \ldots A_{i+1} A_i \ldots A_n},
\tag{6.10}
$$

such that we can construct the gauge fixed action

$$
\begin{aligned}
S_{com} &= S_0[\phi - \varphi] - \varphi^A B_A - (\phi_A^* - \Psi_A)c^A \\
&\quad + \phi_A^* \mathcal{R}^A[\phi - \varphi] - \sum_{n \geq 2} \frac{1}{n} \phi_{A_1}^* \dots \phi_{A_n}^* M_n^{A_1 \dots A_n}(\phi - \varphi) \quad (6.11)
\end{aligned}
$$

to be invariant under the BRST transformation rules

$$
\begin{aligned}
\delta \phi^A &= c^A \\
\delta \varphi^A &= c^A - \mathcal{R}^A[\phi - \varphi] + \sum_{n \geq 2} \phi_{A_2}^* \dots \phi_{A_n}^* M_n^{A A_2 \dots A_n}(\phi - \varphi) \\
\delta c^A &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.12) \\
\delta \phi_A^* &= B_A \\
\delta B_A &= 0.
\end{aligned}
$$

We introduced a factor $\frac{1}{n}$ in the last term of the gauge fixed action. Therefore, we have

$$
\begin{aligned}
\delta \left[\frac{1}{n} \phi_{A_1}^* \dots \phi_{A_n}^* M_n^{A_1 \dots A_n} \right] &= \frac{1}{n} \phi_{A_1}^* \dots \phi_{A_n}^* \delta M_n^{A_1 \dots A_n} \qquad (6.13) \\
&\quad + (-1)^{\epsilon_{A_1}+1} B_{A_1} \phi_{A_2}^* \dots \phi_{A_n}^* M_n^{A_1 \dots A_n}.
\end{aligned}
$$

This makes all B-dependent terms cancel in the expression for δS_{com}.

Calculating δS_{com} and equating it order by order in ϕ_A^* to zero, we get a set of equations. At the **order** $(\phi^*)^0$ we have

$$
\frac{\overleftarrow{\delta} S_0(\phi - \varphi)}{\delta \phi^A} \mathcal{R}^A(\phi - \varphi) = 0 \,. \qquad (6.14)
$$

The first condition, the term in δS_{com} that is independent of antifields, is just that the classical action has invariances. From the **order** $(\phi^*)^1$ in $\delta S_{com} = 0$ we get the condition

$$
\frac{\overleftarrow{\delta} \mathcal{R}^A[\phi - \varphi]}{\delta \phi^B} \mathcal{R}^B(\phi - \varphi) - (-1)^{(\epsilon_A+1)\epsilon_B} \frac{\overleftarrow{\delta} S_0(\phi - \varphi)}{\delta \phi^B} M_2^{BA}(\phi - \varphi) = 0 \quad (6.15)
$$

If we take for ϕ^A the original gauge fields ϕ^i, we have that

$$
\frac{\overleftarrow{\delta} R_\alpha^i c^\alpha}{\delta \phi^j} R_\beta^j c^\beta + R_\gamma^i T_{\alpha\beta}^\gamma c^\beta c^\alpha - (-1)^{(\epsilon_i+1)\epsilon_j} y_j M_2^{ji} = 0. \qquad (6.16)
$$

Comparing this with (6.2), we find that we can take

$$
M_2^{ij} = -2E_{\alpha\beta}^{ji} c^\beta c^\alpha, \qquad (6.17)
$$

leading to a contribution $\phi_i^* \phi_j^* E_{\alpha\beta}^{ji} c^\beta c^\alpha$ in S_{com}. Analogously, for $\phi^A = c^\alpha$, we find

$$
\frac{\overleftarrow{\delta} T_{\beta\gamma}^\alpha c^\gamma c^\beta}{\delta \phi^j} R_\mu^j c^\mu + 2T_{\beta\gamma}^\alpha c^\gamma T_{\mu\nu}^\beta c^\nu c^\mu - (-1)^{\epsilon_\alpha \epsilon_j} y_j M_2^{j\alpha} = 0. \qquad (6.18)
$$

Provided that the first two terms equal an expression proportional to a field equation, we can conclude that we can take

$$
M_2^{i\alpha} = -D_{\mu\nu\sigma}^{i\alpha} c^\sigma c^\nu c^\mu, \qquad (6.19)
$$

for some function $D^{i\alpha}_{\mu\nu\sigma}$. This gives an extra term in S_{com} of the form $\frac{1}{2}\phi^*_i c^*_\alpha D^{i\alpha}_{\mu\nu\sigma} c^\sigma c^\nu c^\mu$. No further terms come from this condition. All the other fields that are included in ϕ^A belong to trivial systems and hence have that

$$\frac{\overleftarrow{\delta}\mathcal{R}^A}{\delta\phi^B}\mathcal{R}^B = 0. \tag{6.20}$$

Thus, we see that for all these fields $M^{jA}_2 = 0$.

In principle, the set of equations coming from the condition $\delta S_{com} = 0$ may be infinite. It is clear that every M_n is to be determined from an equation of the form

$$-\frac{\overleftarrow{\delta}S_0}{\delta\phi^A}\phi^*_{A_2}\cdots\phi^*_{A_n}M^{AA_2\ldots A_n}_n + G(\mathcal{R}^A, M_2, \ldots, M_{n-1}) = 0. \tag{6.21}$$

Although knowing $M_2,\ldots,M_{n-1}$ allows in principle to determine M_n, it is not a priori guaranteed that $G(\mathcal{R}^A, M_2, \ldots, M_{n-1})$ is indeed of the required form proportional to a field equation. It can be proven however that this is indeed the case, i.e. that a BRST invariant action of the form (6.11) exists. Moreover, the solution is not unique. It has however been shown that different solution are related by a canonical transformation of fields and antifields[2]. For more details on the existence proof, we refer to [45, 6, 7, 38, 47, 48].

Like for closed algebras (5.8), we decompose

$$S_{com} = S_{BV}(\phi - \varphi, \phi^*) - (\phi^*_A - \Psi_A)c^A - \varphi^A B_A. \tag{6.22}$$

From all the above, we then conclude that

$$S_{BV}(\phi, \phi^*) = S_0[\phi] + \phi^*_A\mathcal{R}^A[\phi] + \phi^*_i\phi^*_j E^{ji}_{\alpha\beta}c^\beta c^\alpha + \frac{1}{2}\phi^*_i c^*_\alpha D^{i\alpha}_{\mu\nu\sigma}c^\sigma c^\nu c^\mu + \ldots, \tag{6.23}$$

where the dots stand for terms cubic and higher order in the antifields. Notice that in the terms that are non-linear in the antifields, at least one antifield ϕ^*_i of the original gauge fields ϕ^i is present, as S_0 only depends on these.

As was the case for closed algebras (5.20), we have that

$$\delta(\phi^A - \varphi^A) = \frac{\overrightarrow{\delta}S_{BV}(\phi - \varphi, \phi^*)}{\delta\phi^*_A}. \tag{6.24}$$

Therefore, BRST invariance of S_{com} implies that also for open algebras, the extended action S_{BV} satisfies the classical master equation

$$\frac{\overleftarrow{\delta}S_{BV}}{\delta\phi^A}\frac{\overrightarrow{\delta}S_{BV}}{\delta\phi^*_A} = 0. \tag{6.25}$$

We now have a unifying principle for quantising theories with an algebra that may be closed or open. In both cases we have to solve the classical master equation $(S_{BV}, S_{BV}) = 0$, with the boundary condition that $S_{BV} = S_0 + \phi^*_i R^i_\alpha c^\alpha + \ldots$ As for the rest of the recipe to obtain a well-defined partition function, everything goes through as for closed algebras (5.21). Especially, for a gauge fermion

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[2]Canonical transformations are discussed in chapter 8.

of the form $\Psi = b_\alpha F^\alpha(\phi)$, we see that the replacement in S_{BV} (6.23) of anti-fields by derivatives with respect to the fields, gives the four ghost interaction mentioned above:

$$\phi_i^* \phi_j^* E_{\alpha\beta}^{ji} c^\beta c^\alpha \rightarrow b_\alpha \frac{\overleftarrow{\delta} F^\alpha}{\delta \phi^i} b_\beta \frac{\overleftarrow{\delta} F^\beta}{\delta \phi^j} E_{\mu\nu}^{ji} c^\nu c^\mu \ . \tag{6.26}$$

Hence, we see that for theories with an open gauge algebra, the combination of the BRST quantisation recipe with the SD BRST symmetry, gives rise to the same (BV) scheme as was derived for closed algebras in the previous chapter.

6.2 The role of the collective field

A posteriori, we can see that precisely by introducing the collective fields, we can gauge fix the original gauge symmetries in the same way for open algebras as for closed algebras and prove that the functional integral is formally gauge independent. We discuss this point of view here in some detail, as it will be our starting point for the derivation of an antifield scheme for BRST–anti-BRST invariant quantisation of theories with an open gauge algebras in the next chapter.

The introduction of collective fields allows us to construct the BRST transformation rules such that $\delta^2 \phi^A = 0$, since we can shift the off-shell nilpotency problem of open algebras to the transformation rules of the collective field by defining $\delta \phi^A = c^A$. We can then fix the originally present gauge symmetry in a manifest BRST invariant way by adding $\delta \Psi[\phi]$. As (ϕ_A^*, B_A) form a trivial system, here too we can generalise the choice of gauge fermion to $\Psi[\phi, \phi^*]$ without spoiling the fact that $\delta^2 \Psi = 0$.

The gauge fixed action can be decomposed as $S_{com} = S_{inv} + \delta \Psi$, and we consider the gauge fixed partition function

$$\mathcal{Z}_\Psi = \int [d\phi][d\phi^*][d\varphi][dc][dB] \ e^{\frac{i}{\hbar} S_{com}} \ . \tag{6.27}$$

It now remains to make sure that the Ward identities are valid. Then the partition function $\mathcal{Z}_\Psi$ is invariant under infinitesimal deformations of the gauge fermion Ψ. For that purpose, we construct S_{inv} to be BRST invariant. If we make the decomposition,

$$S_{inv} = S_{BV}(\phi - \varphi, \phi^*) - \phi_A^* c^A - \varphi^A B_A, \tag{6.28}$$

we know that $\delta S_{inv} = 0$, under the BRST transformation rules

$$\begin{aligned}
\delta \phi^A &= c^A \\
\delta \varphi^A &= c^A - \frac{\overrightarrow{\delta} S_{BV}(\phi - \varphi, \phi^*)}{\delta \phi_A^*} \\
\delta c^A &= 0 \\
\delta \phi_A^* &= B_A \\
\delta B_A &= 0,
\end{aligned} \tag{6.29}$$

if S_{BV} satisfies the classical master equation of the BV scheme; $(S_{BV}, S_{BV}) = 0$. The question whether open algebras can be quantised then amounts to proving

that the classical master equation of the BV scheme can be solved for open algebras [47, 38]. The reason for the decomposition (6.28) is that the auxiliary collective field φ^A is fixed to zero to remove it from the final functional integrals. The $\phi_A^* c^A$ is the Faddeev-Popov ghost-antighost action associated with this gauge fixing. Moreover, we know that this way the SD equations with the gauge fixed action are included in the Ward identities of the theory.

Things get only slightly more complicated when quantum counterterms have to be used to cancel the BRST variation of the measure in the derivation of the Ward identity. In that case, we decompose $S_{com} = S_q + \delta\Psi$ after the introduction of the collective field. Now S_q is *not* BRST invariant. Its BRST variation has to cancel the Jacobian of the measure under BRST transformations. We decompose

$$S_q = W(\phi - \varphi, \phi^*) - \phi_A^* c^A - \varphi^A B_A . \tag{6.30}$$

With the BRST transformation rules of (6.29), except for the collective field for which we take

$$\delta\varphi^A = c^A - \frac{\overrightarrow{\delta} W(\phi - \varphi, \phi^*)}{\delta\phi_A^*}, \tag{6.31}$$

we find that

$$\delta S_q.\mu = \frac{1}{2}(W, W).\mu , \tag{6.32}$$

where the expression of the antibracket is evaluated in $(\phi - \varphi, \phi^*)$. With these BRST transformation rules (6.29,6.31), the Jacobian from the measure in the derivation of the Ward identity (3.21) is

$$J = \exp[\frac{i}{\hbar}(-i\hbar\Delta W).\mu]. \tag{6.33}$$

Then we see that the product of the measure and $e^{\frac{i}{\hbar}S_q}$ is BRST invariant if W satisfies the quantum master equation

$$(W, W) - 2i\hbar\Delta W = 0. \tag{6.34}$$

We are now back at the result derived in previous chapter for closed algebras (5.38). If the quantum master equation is satisfied, we can reverse steps of section 2 of chapter 5 to prove that the Ward identity $\langle\sigma X\rangle = 0$. This then implies gauge independence of the partition function $\mathcal{Z}_\Psi$.

In this chapter, we have shown how the quantisation of gauge theories with an open algebra leads to the same classical and quantum master equation of the BV scheme. We have also argued that the design of the collective field formalism itself plays a crucial role in the construction of a quantisation scheme for open algebras. We close this chapter by giving an example of a theory with an open algebra.

6.3   Example: W_3 gravity

We give here W_3 gravity [49] (again a two dimensional model) as an example of a theory with an open algebra and give its extended action [38]. Consider the classical action

$$S_0 = \int d^2x \left[\frac{1}{2}\partial\phi\bar{\partial}\phi - \frac{1}{2}h(\partial\phi)^2 - \frac{1}{3}B(\partial\phi)^3\right]. \tag{6.35}$$

The fields ϕ, h and B all have even Grassmann parity. Notice that, up to a scale factor $\frac{1}{\pi}$, the first two terms are the classical action of W_2 gravity (5.49). The invariances of this action are

$$
\begin{aligned}
\delta\phi &= \epsilon\partial\phi + \lambda(\partial\phi)^2 \\
\delta h &= (\bar{\partial}\epsilon - h\partial\epsilon + \partial h.\epsilon) + (\lambda.\partial B - B.\partial\lambda)(\partial\phi)^2 \\
\delta B &= (\epsilon.\partial B - 2B.\partial\epsilon) + (\bar{\partial}\lambda - h.\partial\lambda + 2\lambda.\partial h).
\end{aligned}
\tag{6.36}
$$

The transformation parameters are ϵ and λ, for which we introduce respectively the ghosts c and l.

A long but straightforward calculation allows to verify that

$$
\begin{aligned}
S_{BV} &= \frac{1}{2}\partial\phi\bar{\partial}\phi - \frac{1}{2}h(\partial\phi)^2 - \frac{1}{3}B(\partial\phi)^3 \\
&\quad +\phi^* \left[c.\partial\phi + l(\partial\phi)^2 \right] \\
&\quad +h^* \left[(\bar{\partial}c - h.\partial c + \partial h.c) + (l.\partial B - B.\partial l)(\partial\phi)^2 \right] \\
&\quad +B^* \left[(c.\partial B - 2B.\partial c) + (\bar{\partial}l - h.\partial l + 2l.\partial h) \right] \\
&\quad +c^*.\partial c.c + c^*.\partial l.l.(\partial\phi)^2 + 2l^*.\partial c.l - l^*.c.\partial l \\
&\quad +2\phi^* h^*.\partial l.l.\partial\phi \, ,
\end{aligned}
\tag{6.37}
$$

satisfies the classical masterequation. Moreover, it clearly is a proper solution. An integration over two dimensional space time is understood. As is discussed in [38], this particular solution involves a choice: what are $T^\alpha_{\beta\gamma}$ and $E^{ji}_{\alpha\beta}$. In particular, the term $c^*.\partial l.l.(\partial\phi)^2$ is due to taking a non-vanishing structure function $T^\epsilon_{\lambda\lambda}$. However, $(\partial\phi)^2 \sim y_h$, and one can also take $T^\epsilon_{\lambda\lambda} = 0$ and modify the non-closure functions. The price to pay is that then also terms proportional h^*h^* and h^*B^* are needed to construct a solution of the master equation. All these different solutions are related by canonical transformations. For some examples of the change of the gauge generators and the structure functions under canonical transformations, we refer to section 4 of chapter 8.

Chapter 7

An antifield scheme for BRST–anti-BRST invariant quantisation

In this chapter, we derive an antifield scheme for quantisation in a BRST–anti-BRST invariant way. Instead of only one antifield for every field, our construction will lead to three antifields: one acting as a source term for BRST transformations, one as a source term for anti-BRST transformations and one as a source term for mixed transformations. The whole structure of BV is doubled. There are two master equations, one of ghostnumber one and one of ghostnumber minus one. They correspond to two BRST operators, one that raises the ghostnumber by one and one that lowers the ghostnumber by the same amount. For closed algebras, the scheme was derived using the usual collective field formalism in [50]. However, in order to obtain a better agreement, especially for the gauge fixing, with the earlier algebraic derivation of [51], an improved collective field formalism was set up in [52]. There, we introduced *two* collective fields for every field. We first describe how the Schwinger-Dyson equations can be derived using this formalism with two collective fields. Then we derive the antifield scheme. For some alternative formulations of BRST–anti-BRST symmetry using antifields, we refer to [53].

7.1 Schwinger-Dyson Equations from two collective fields

In this section, we present the collective field formalism with *two* collective fields. We derive the SD equation as a Ward identity using this formalism and postpone the complication of possible gauge symmetries of the classical action to the next section.

We start from an action $S_0[\phi]$, depending on bosonic degrees of freedom ϕ^i and that has no gauge symmetries. The index i is suppressed in this section. We introduce *two* copies of the original field, the two so-called *collective fields*, φ_1 and φ_2 and consider the action $S_0[\phi - \varphi_1 - \varphi_2]$. There now are two gauge

symmetries for which we introduce two ghostfields π_1 and ϕ_2^* and two antighost fields ϕ_1^* and π_2. The BRST–anti-BRST transformation rules are taken to be (we follow the construction of section 4 of chapter 3)

$$
\begin{array}{ll}
\delta_1\phi = \pi_1 & \delta_2\phi = \pi_2 \\
\delta_1\varphi_1 = \pi_1 - \phi_2^* & \delta_2\varphi_1 = -\phi_1^* \\
\delta_1\varphi_2 = \phi_2^* & \delta_2\varphi_2 = \pi_2 + \phi_1^* \\
\delta_1\pi_1 = 0 & \delta_2\pi_2 = 0 \\
\delta_1\phi_2^* = 0 & \delta_2\phi_1^* = 0.
\end{array}
\tag{7.1}
$$

Imposing $(\delta_2\delta_1 + \delta_1\delta_2)\phi = 0$ gives the extra condition $\delta_2\pi_1 + \delta_1\pi_2 = 0$, while analogously $(\delta_2\delta_1 + \delta_1\delta_2)\varphi_1 = 0$ gives $\delta_1\phi_1^* + \delta_2\phi_2^* = \delta_2\pi_1$, and $(\delta_2\delta_1 + \delta_1\delta_2)\varphi_2 = 0$ leads to no new condition. We introduce two extra bosonic fields B and λ and the BRST transformation rules:

$$
\begin{array}{ll}
\delta_1\pi_2 = B & \delta_2\pi_1 = -B \\
\delta_1 B = 0 & \delta_2 B = 0 \\
\delta_1\phi_1^* = \lambda - \frac{B}{2} & \delta_2\phi_2^* = -\lambda - \frac{B}{2} \\
\delta_1\lambda = 0 & \delta_2\lambda = 0.
\end{array}
\tag{7.2}
$$

All these rules together guarantee that $\delta_1^2 = \delta_2^2 = \delta_1\delta_2 + \delta_2\delta_1 = 0$ on the complete set of fields.

With all these BRST transformation rules at hand, we can construct a gauge fixed action that is invariant under BRST–anti-BRST symmetry. We will fix both the collective fields to be zero. To that end, we add

$$
\begin{aligned}
S_{col} &= \frac{1}{2}\delta_1\delta_2[\varphi_1^2 - \varphi_2^2] \\
&= -(\varphi_1 + \varphi_2)\lambda + \frac{B}{2}(\varphi_1 - \varphi_2) + (-1)^a\phi_a^*\pi_a.
\end{aligned}
\tag{7.3}
$$

In the last term, there is a summation over $a = 1, 2$. Denoting $\varphi_\pm = \varphi_1 \pm \varphi_2$, we have the gauge fixed action

$$
S_{com} = S_0[\phi - \varphi_+] - \varphi_+\lambda + \frac{B}{2}\varphi_- + (-1)^a\phi_a^*\pi_a.
\tag{7.4}
$$

S_{com} has both BRST and anti-BRST symmetry.

The Schwinger-Dyson equations can be derived as Ward identities in the following way.

$$
\begin{aligned}
0 &= \langle \delta_1[\phi_1^* F(\phi)]\rangle \\
&= \int d\mu \left[\phi_1^* \frac{\overleftarrow{\delta} F}{\delta\phi}\pi_1 + (\lambda - \frac{B}{2})F(\phi)\right] e^{\frac{i}{\hbar}S_{com}}.
\end{aligned}
\tag{7.5}
$$

$d\mu$ denotes the integration measure over all fields. The term $\langle BF(\phi)\rangle$ is zero. This can be seen by noticing that $B = \delta_1\delta_2\varphi_+$. The Ward identities themselves allow to *integrate by parts* to get

$$
\langle BF(\phi)\rangle = -\langle \varphi_+\delta_2\delta_1 F(\phi)\rangle,
\tag{7.6}
$$

which drops out as φ_+ is fixed to zero.

The SD equation then results as in chapter 4 or in [30, 31], by integrating out $\pi_a, \phi_a^*, \lambda, B, \varphi_+$ and φ_-. Of course, the SD equations can also be derived as Ward identities of the anti-BRST transformation δ_2.

7.2 Closed algebras

Given any classical action $S_0[\phi^i]$ with a closed and irreducible gauge algebra, the configuration space is enlarged by introducing the necessary ghosts, antighosts and auxiliary fields, needed for the construction of BRST–anti-BRST transformation rules as is described in section 4 of chapter 3. The complete set of fields is denoted by ϕ_A and their BRST–anti-BRST transformation rules are all summarised by $\delta_a\phi_A = \mathcal{R}_{Aa}(\phi)$. For $a = 1$, we have the BRST transformation rules, for $a = 2$ the anti-BRST transformation. Since the algebra is closed, we have that $(\delta_a^2\phi_A = 0)$

$$\frac{\overleftarrow{\delta}\,\mathcal{R}_{Aa}(\phi)}{\delta\phi_B}\mathcal{R}_{Ba}(\phi) = 0 \qquad (7.7)$$

and that $((\delta_1\delta_2 + \delta_2\delta_1)\phi_A = 0)$

$$\frac{\overleftarrow{\delta}\,\mathcal{R}_{A1}(\phi)}{\delta\phi_B}\mathcal{R}_{B2}(\phi) + \frac{\overleftarrow{\delta}\,\mathcal{R}_{A2}(\phi)}{\delta\phi_B}\mathcal{R}_{B1}(\phi) = 0. \qquad (7.8)$$

In the first formula, there is no summation over a.

Instead of constructing a gauge fixed action that is invariant under the BRST–anti-BRST symmetry, we introduce collective fields and associated extra shift symmetries. We introduce *two* collective fields φ_{A1} and φ_{A2}, collectively denoted by φ_{Aa}, and replace everywhere ϕ_A by $\phi_A - \varphi_{A1} - \varphi_{A2}$. There now are two shift symmetries for which we introduce the ghosts π_{A1} and ϕ_A^{*2} with ghostnumber $\text{gh}(\pi_{A1}) = \text{gh}(\phi_A^{*2}) = \text{gh}(\phi_A) + 1$ and the antighosts ϕ_A^{*1} and π_{A2} with ghostnumber $\text{gh}(\pi_{A2}) = \text{gh}(\phi_A^{*1}) = \text{gh}(\phi_A) - 1$. Again, we will use π_{Aa} and ϕ_A^{*a} as compact notation. Of course, one has to keep in mind that for $a = 1$, π_{Aa} is a ghost, while for $a = 2$, π_{Aa} is an antighost and vice versa for ϕ_A^{*a}.

We construct the BRST–anti-BRST transformations as follows:

$$\delta_a\phi_A = \pi_{Aa} \qquad (7.9)$$
$$\delta_a\varphi_{Ab} = \delta_{ab}\left[\pi_{Aa} - \epsilon_{ac}\phi_A^{*c} - \mathcal{R}_{Aa}(\phi - \varphi_1 - \varphi_2)\right] + (1 - \delta_{ab})\epsilon_{ac}\phi_A^{*c},$$

with no summation over a in the second line[1]. These rules are constructed such that

$$\delta_a(\phi_A - \varphi_{A1} - \varphi_{A2}) = \mathcal{R}_{Aa}(\phi - \varphi_1 - \varphi_2). \qquad (7.10)$$

The two collective fields lead to even more freedom to shift the $\mathcal{R}_{Aa}$ in the transformation rules, than the one in the collective field formalism for BV (5.3). The choice above incorporates the antifield formalism for BRST–anti-BRST symmetry [51]. Furthermore, the discussion of open algebras in the previous chapter (6.5) also indicates that it is useful to construct the rules such that $\delta_a^2\phi_A = 0$ and $(\delta_1\delta_2 + \delta_2\delta_1)\phi_A = 0$, independently of the closure of the algebra. We can make sure that $\delta_a^2 = 0$ ($a = 1, 2$) and that $\delta_1\delta_2 + \delta_2\delta_1 = 0$ when acting on any field, by the introduction of two extra fields B_A and λ_A and the new transformationrules:

$$\delta_a\pi_{Ab} = \epsilon_{ab}B_A$$
$$\delta_a B_A = 0 \qquad (7.11)$$
$$\delta_a\phi_A^{*b} = -\delta_a^b\left[(-1)^a\lambda_A + \frac{1}{2}\left(B_A + \frac{\overleftarrow{\delta}\,\mathcal{R}_{A1}(\phi - \varphi_1 - \varphi_2)}{\delta\phi_B}\mathcal{R}_{B2}(\phi - \varphi_1 - \varphi_2)\right)\right]$$
$$\delta_a\lambda_A = 0.$$

[1] Our convention: $\epsilon_{12} = 1, \epsilon^{12} = -1$.

We gauge fix both the collective fields to zero in a BRST–anti-BRST invariant way. For that purpose, we need a matrix M^{AB}, with constant c-number entries and which is invertible. Moreover, it has to have the symmetry property $M^{AB} = (-1)^{\epsilon_A \epsilon_B} M^{BA}$ and all the entries of M between Grassmann odd and Grassmann even sectors have to vanish. It has to be such that $\phi_A M^{AB} \phi_B$ has over all ghostnumber zero and has even Grassmann parity. Except for the constraints above, the precise form of M is of no concern. It will drop out completely in the end. The collective fields are then gauge fixed to zero in a BRST–anti-BRST invariant way by adding the term

$$
\begin{aligned}
S_{col} &= -\frac{1}{4}\epsilon^{ab}\delta_a\delta_b \left[\varphi_{A1} M^{AB}\varphi_{B1} - \varphi_{A2}M^{AB}\varphi_{B2} \right] \\
&= -(\varphi_{A1} + \varphi_{A2})M^{AB}\lambda_B + \frac{1}{2}(\varphi_{A1} - \varphi_{A2})M^{AB}B_B \qquad (7.12)\\
&+ (-1)^{\epsilon_B+1}\phi_A^{*1}M^{AB}\pi_{B1} + (-1)^{\epsilon_B}\phi_A^{*2}M^{AB}\pi_{B2} \\
&+ (-1)^{\epsilon_B}\phi_A^{*1}M^{AB}R_{B1}(\phi - \varphi_1 - \varphi_2) + (-1)^{\epsilon_B+1}\phi_A^{*2}M^{AB}R_{B2}(\phi - \varphi_1 - \varphi_2) \\
&+ \frac{1}{2}(\varphi_{A1} - \varphi_{A2})M^{AB}\frac{\overleftarrow{\delta}R_{B1}(\phi - \varphi_1 - \varphi_2)}{\delta\phi_C}R_{C2}(\phi - \varphi_1 - \varphi_2).
\end{aligned}
$$

The relative sign between the two contributions of the gauge fixing is needed to make two terms containing the product $\phi_A^{*1}M^{AB}\phi_B^{*2}$, cancel. Redefine now $\varphi_{A\pm} = \varphi_{A1} \pm \varphi_{A2}$, which allows us to rewrite the gauge fixing terms in a more compact and suggestive form:

$$
\begin{aligned}
S_{col} &= -\varphi_{A+}M^{AB}\lambda_B + \frac{1}{2}\varphi_{A-}M^{AB}B_B + (-1)^a(-1)^{\epsilon_B}\phi_A^{*a}M^{AB}\pi_{Ba} \\
&+ \frac{1}{2}\varphi_{A-}M^{AB}\frac{\overleftarrow{\delta}R_{B1}(\phi - \varphi_+)}{\delta\phi_C}R_{C2}(\phi - \varphi_+) \qquad (7.13)\\
&+ (-1)^{a+1}(-1)^{\epsilon_B}\phi_A^{*a}M^{AB}R_{Ba}(\phi - \varphi_+).
\end{aligned}
$$

Notice that this time a summation over a is understood in the third and fifth term. The ϕ^{*a} have indeed become source terms for the BRST and anti-BRST transformation rules of the fields $\phi - \varphi_+$, while the difference of the two collective fields φ_- acts as a source for mixed transformations. The sum of the two collective fields is fixed to zero.

The original gauge symmetry can be fixed in a BRST–anti-BRST invariant way by adding the variation of a gauge *boson* $\Psi(\phi)$, of ghostnumber zero. We take it to be only a function of the original fields ϕ_A. This gives the extra terms

$$
\begin{aligned}
S_\Psi &= \frac{1}{2}\epsilon^{ab}\delta_a\delta_b\Psi(\phi) \\
&= -\frac{\overleftarrow{\delta}\Psi}{\delta\phi_A}B_A + \frac{1}{2}\epsilon^{ab}(-1)^{\epsilon_B+1}\left[\frac{\overleftarrow{\delta}}{\delta\phi_A}\frac{\overleftarrow{\delta}}{\delta\phi_B}\Psi\right].\pi_{Aa}\pi_{Bb}. \qquad (7.14)
\end{aligned}
$$

In order to make contact with the antifield formalism that was derived on algebraic grounds in [51], we first have to make the following (re)definitions. We incorporate the matrix M^{AB} introduced above in the antifields:

$$
\phi^{*Aa'} = (-1)^{\epsilon_A}\phi_B^{*\ a}M^{BA}(-1)^{a+1} \qquad a = 1,2
$$
$$
(7.15)
$$
$$
\bar{\phi}^A = \frac{1}{2}\varphi_{B-}M^{BA}.
$$

Owing to the properties of the matrix M^{AB} above, the ghostnumber assignments *after the redefinition* are given by

$$\begin{aligned}
\text{gh}\left(\phi^{*Aa'}\right) &= (-1)^a - \text{gh}\left(\phi_A\right) \\
\text{gh}\left(\bar{\phi}^A\right) &= -\text{gh}\left(\phi_A\right),
\end{aligned} \tag{7.16}$$

while the Grassmann parities are of course

$$\varepsilon_{\phi^{*Aa'}} = \varepsilon_{\phi_A} + 1 \;\; ; \;\; \varepsilon_{\bar{\phi}^A} = \varepsilon_{\phi_A}. \tag{7.17}$$

In [51], I.A. Batalin, P.M. Lavrov and I.V. Tyutin introduced the so-called *extended action*, which we denote by S_{BLT}. Using the new variables and dropping the primes, it is defined by

$$S_{BLT}(\phi_A, \phi^{*Aa}, \bar{\phi}^A) = S_0[\phi_A] + \phi^{*Aa}R_{Aa}(\phi) + \bar{\phi}^A \frac{\overleftarrow{\delta}R_{A1}(\phi)}{\delta\phi_B}R_{B2}(\phi). \tag{7.18}$$

S_{BLT} is the sum of the classical action, plus the last two terms of (7.13), up to a substitution of ϕ by $\phi - \varphi_+$. The remaining terms of S_{col}, are denoted by S_δ, hence

$$S_\delta = -\varphi_{A+}M^{AB}\lambda_B + \bar{\phi}^A B_A - \phi^{*Aa}\pi_{Aa}. \tag{7.19}$$

Integrating over π_{Aa}, B_A and λ_B, S_δ leads to a set of δ-functions removing all the fields of the collective field formalism. The situation is then analogous to the BV scheme. Before the gauge fixing term S_Ψ is added, all antifields are fixed to zero.

With all these definitions at hand, we have that

$$\begin{aligned}
S_{com} &= S_0[\phi - \varphi_+] + S_{col} + S_\Psi \\
&= S_{BLT}[\phi - \varphi_+, \phi^{*a}, \bar{\phi}] + S_\delta + S_\Psi,
\end{aligned} \tag{7.20}$$

which gives the gauge fixed partition function

$$\mathcal{Z} = \int [d\phi][d\phi^{*a}][d\bar{\phi}][d\pi_a][dB]e^{\frac{i}{\hbar}S_{BLT}[\phi,\phi^{*a},\bar{\phi}]}e^{\frac{i}{\hbar}S_\Psi}e^{\frac{i}{\hbar}\tilde{S}_\delta}. \tag{7.21}$$

We already integrated out λ and φ_+, and $\tilde{S}_\delta$ is S_δ with the term $-\varphi_{A+}M^{AB}\lambda_B$ omitted. The gauge fixing term $\exp(\frac{i}{\hbar}S_\Psi)$ can be obtained by acting with an operator $\hat{V}$ on $\exp(\frac{i}{\hbar}\tilde{S}_\delta)$, i.e.

$$e^{\frac{i}{\hbar}S_\Psi}e^{\frac{i}{\hbar}\tilde{S}_\delta} = \hat{V}e^{\frac{i}{\hbar}\tilde{S}_\delta}. \tag{7.22}$$

From the explicit form of $\tilde{S}_\delta$ and S_Ψ, and using that $e^{a(y)\frac{\delta}{\delta x}}f(x) = f(x + a(y))$, we see that $\hat{V}(\Psi) = e^{-T_1(\Psi)-T_2(\Psi)}$ with

$$\begin{aligned}
T_1(\Psi) &= \frac{\overleftarrow{\delta}\Psi(\phi)}{\delta\phi_A} \cdot \frac{\overrightarrow{\delta}}{\delta\bar{\phi}^A} \\
T_2(\Psi) &= \frac{i\hbar}{2}\varepsilon^{ab}\frac{\overrightarrow{\delta}}{\delta\phi^{*Bb}}\left[\frac{\overleftarrow{\delta}}{\delta\phi_A}\frac{\overleftarrow{\delta}}{\delta\phi_B}\Psi\right]\frac{\overrightarrow{\delta}}{\delta\phi^{*Aa}}.
\end{aligned} \tag{7.23}$$

The convention is that the derivatives with respect to the antifields ϕ^* and $\bar{\phi}$ act on everything standing to the right of them. The operator $\hat{V}$ can be integrated by parts, such that

$$\mathcal{Z} = \int [d\phi][d\phi^{*a}][d\bar{\phi}]\delta(\phi^{*A1})\delta(\phi^{*A2})\delta(\bar{\phi}^A)\left[\hat{U}(\Psi)e^{\frac{i}{\hbar}S_{BLT}}\right], \tag{7.24}$$

with the operator $\hat{U} = e^{+T_1 - T_2}$. This form of the path integral agrees with [51].

Let us finally derive the *classical master equations* which are satisfied by S_{BLT}. They follow from the fact that S_{com} (7.20) is invariant under both the BRST and the anti-BRST transformation. Furthermore, one has to use the fact that the matrix M^{AB} only has non-zero entries for $\varepsilon_A = \varepsilon_B$, and hence that $M^{AB} = (-1)^{\varepsilon_A} M^{BA} = (-1)^{\varepsilon_B} M^{BA}$. Also, in the collective field BRST–anti-BRST transformation rules, we may replace $R_{Aa}(\phi - \varphi_+)$ by $\overrightarrow{\delta} S_{BLT}/\delta\phi^{*Aa'}$. Since $\delta_a S_\Psi = 0$, we have that

$$
\begin{aligned}
0 &= \delta_a S_{com} \\
&= \delta_a S_{BLT} + \delta_a S_\delta \\
&= \frac{\overleftarrow{\delta} S_{BLT}}{\delta\phi_A} \cdot \frac{\overrightarrow{\delta} S_{BLT}}{\delta\phi^{*Aa'}} + \varepsilon_{ab}\phi^{*Ab} \frac{\overrightarrow{\delta} S_{BLT}}{\delta\bar{\phi}^A}
\end{aligned}
\tag{7.25}
$$

We introduce two antibrackets, one for every ϕ^{*Aa}, defined by

$$
(F, G)_a = \frac{\overleftarrow{\delta} F}{\delta\phi_A} \cdot \frac{\overrightarrow{\delta} G}{\delta\phi^{*Aa}} - \frac{\overleftarrow{\delta} F}{\delta\phi^{*Aa}} \cdot \frac{\overrightarrow{\delta} G}{\delta\phi_A} .
\tag{7.26}
$$

Of course, they have the same properties as the antibrackets from the usual BV scheme, so that we finally can write the classical master equations as

$$
\frac{1}{2}(S_{BLT}, S_{BLT})_a + \varepsilon_{ab}\phi^{*Ab} \frac{\overrightarrow{\delta} S_{BLT}}{\delta\bar{\phi}^A} = 0 .
\tag{7.27}
$$

For closed, irreducible algebras, we know that the proper solution is of the form (7.18), if a complete set of gauge generators R^i_α is used.

The quantisation prescription is then to construct S_{BLT}, function of fields and antifields, by solving the classical master equations. The gauge fixing is done by acting with the operator $\hat{U}(\Psi)$. Then the antifields ϕ^{*Aa} and $\bar{\phi}^A$ are removed by the δ-functions which fix them to zero. Notice however that instead of acting with $\hat{U}$ on $e^{\frac{i}{\hbar} S_{BLT}}$, it is a lot easier to take as realisation of the gauge fixing $S_\Psi + \tilde{S}_\delta$, especially when S_{BLT} becomes non-linear in the antifields.

7.3 Ward identities and quantum master equations

In this section, we first derive the Ward identities for the BRST–anti-BRST symmetry and then we take these identities as a starting point to derive the quantum master equation. This is in analogy with section 2 of chapter 5.

7.3.1 Ward identities

Since the gauge fixed action we constructed (7.20) is invariant under both the BRST and anti-BRST transformation rules, the standard procedure of section 2 of chapter 3 allows the derivation of 2 types of Ward identities. For any X, we have that

$$
\begin{aligned}
\langle \delta_1 X \rangle &= 0 \\
\langle \delta_2 X \rangle &= 0 ,
\end{aligned}
\tag{7.28}
$$

where $\langle \mathcal{O} \rangle$ denotes the quantum expectation value using the gauge fixed action (7.20) of an operator $\mathcal{O}$. As we are only interested in the theory after having integrated out φ_+, we will restrict ourselves to quantities $X(\phi_A, \phi^{*Aa}, \bar{\phi})$. For a closed algebra, the Jacobian of the measure of the functional integral under either a BRST or an anti-BRST transformation is a function of $\phi - \varphi_+$ (7.9). Since $\delta_a(\phi_A - \varphi_{A+}) = \mathcal{R}_{Aa}(\phi - \varphi_+)$, we consider quantum counterterms of the form $M(\phi - \varphi_+)$. The quantum extended action is defined by

$$W_{BLT}(\phi, \phi^{*Aa}, \bar{\phi}) = S_{BLT}(\phi, \phi^{*Aa}, \bar{\phi}) + \hbar M(\phi). \tag{7.29}$$

The Ward identities become

$$
\begin{aligned}
0 &= \langle \delta_a X \rangle \\
&= \int [d\phi][d\phi^{*a}][d\bar{\phi}][d\varphi_+][d\pi_a][dB][d\lambda] \ \delta_a X \cdot e^{\frac{i}{\hbar} W_{BLT}(\phi - \varphi_+, \phi^{*Aa}, \bar{\phi})} \\
&\quad \cdot e^{\frac{i}{\hbar} S_\Psi} e^{\frac{i}{\hbar} S_\delta} .
\end{aligned}
\tag{7.30}
$$

Let us take $a = 1$. Then

$$
\begin{aligned}
\delta_1 X &= \frac{\overleftarrow{\delta} X}{\delta \phi_A} \cdot \pi_{A1} + \frac{\overleftarrow{\delta} X}{\delta \phi^{*A1'}} (-1)^{\epsilon_A} M^{BA} \left[\lambda_B - \frac{1}{2} \left(B_B + \frac{\overleftarrow{\delta} R_{B1}}{\delta \phi_C} R_{C2} \right) \right] \\
&\quad + \frac{\overleftarrow{\delta} X}{\delta \bar{\phi}^A} \cdot \frac{1}{2} M^{BA}[-2\phi_B^{*2} + \pi_{B1} - R_{B1}(\phi - \varphi_+)] .
\end{aligned}
\tag{7.31}
$$

We reintroduced the primes for the $\phi^{*Aa'}$ in order to distinguish the antifields before and after the redefinition (7.15). In the second term of (7.31), we can replace

$$B_B + \frac{\overleftarrow{\delta} R_{B1}(\phi - \varphi_+)}{\delta \phi_C} R_{C2}(\phi - \varphi_+) , \tag{7.32}$$

by

$$\frac{\overrightarrow{\delta}}{\delta \bar{\phi}^B} (S_\delta + W_{BLT}(\phi - \varphi_+, \phi^{*Aa}, \bar{\phi})) . \tag{7.33}$$

In the third term of (7.31), $\pi_{B1} - R_{B1}(\phi - \varphi_+)$ equals

$$- \frac{\overrightarrow{\delta}}{\delta \phi^{*B1'}} (S_\delta + W_{BLT}(\phi - \varphi_+, \phi^{*Aa}, \bar{\phi})) . \tag{7.34}$$

Under the path integral, $(\overrightarrow{\delta} S_\delta / \delta Q) . e^{\frac{i}{\hbar} S_\delta}$ can be replaced by $(\hbar/i) \overrightarrow{\delta} e^{\frac{i}{\hbar} S_\delta} / \delta Q$ ($Q = \bar{\phi}^B$ or $Q = \phi^{*B1'}$), and analogously for the derivatives on W_{BLT}. By partial integrations, one sees that these two contributions cancel.

$d\mu$ denotes the complete measure of the path integral. The remaining Ward identity is

$$
\begin{aligned}
0 &= \int d\mu \left[\frac{\overleftarrow{\delta} X}{\delta \phi_A} \pi_{A1} + \frac{\overleftarrow{\delta} X}{\delta \phi^{*A1'}} (-1)^{\epsilon_A} M^{BA} \lambda_B + \phi^{*A2'} (-1)^{\epsilon_X} \frac{\overrightarrow{\delta} X}{\delta \bar{\phi}^A} \right] \\
&\quad \cdot e^{\frac{i}{\hbar} W_{BLT}(\phi - \varphi_+, \phi^{*Aa}, \bar{\phi})} \cdot \left[\hat{V} e^{\frac{i}{\hbar} \tilde{S}_\delta} \right] . e^{-\frac{i}{\hbar} \varphi_{A+} M^{AB} \lambda_B} .
\end{aligned}
\tag{7.35}
$$

In the first term, the φ_{A+} can trivially be integrated out. Then, considering the expressions for $\hat{V}$ and $\tilde{S}_\delta$, we see that π_{A1} can be replaced by $-\frac{\hbar}{i} \overrightarrow{\delta} (\exp \frac{i}{\hbar} \tilde{S}_\delta) / \delta \phi^{*A1'}$.

Integrating by parts over $\phi^{*A1'}$, gives

$$\frac{\hbar}{i}(-1)^{\varepsilon_X(\varepsilon_A+1)}\frac{\overrightarrow{\delta}}{\delta\phi^{*A1'}}\left[\frac{\overleftarrow{\delta}X}{\delta\phi_A}e^{\frac{i}{\hbar}W_{BLT}}\right]\cdot\left[\hat{V}e^{\frac{i}{\hbar}\tilde{S}_\delta}\right] \qquad (7.36)$$

$$=\left[-i\hbar\Delta_1 X+\frac{\overleftarrow{\delta}X}{\delta\phi_A}\cdot\frac{\overrightarrow{\delta}W_{BLT}}{\delta\phi^{*A1'}}\right]e^{\frac{i}{\hbar}W_{BLT}}\cdot\left[\hat{V}e^{\frac{i}{\hbar}\tilde{S}_\delta}\right]$$

in the path integral. Here, we generalised that other operator well-known from BV (5.26) :

$$\Delta_a X=(-1)^{\varepsilon_A+1}\frac{\overleftarrow{\delta}}{\delta\phi^{*Aa'}}\frac{\overleftarrow{\delta}}{\delta\phi_A}X \,. \qquad (7.37)$$

For the second term we can proceed analogously by replacing $M^{AB}\lambda_B e^{-\frac{i}{\hbar}\phi_A+M^{AB}\lambda_B}$ by $\left(-\frac{\hbar}{i}\right)\frac{\overrightarrow{\delta}}{\delta\varphi_{A+}}e^{-\frac{i}{\hbar}\varphi_A+M^{AB}\lambda_B}$. Integrating by parts over φ_{A+}, we see that the derivative can only act on $W_{BLT}(\phi-\varphi_+,\phi^{*a},\bar{\phi})$, and we get under the path integral

$$\frac{\overleftarrow{\delta}X}{\delta\phi^{*A1'}}\frac{\hbar}{i}\frac{\overrightarrow{\delta}}{\delta\varphi_{A+}}e^{\frac{i}{\hbar}W_{BLT}(\phi-\varphi_+,\phi^{*a},\bar{\phi})}\cdot\left[\hat{V}e^{\frac{i}{\hbar}\tilde{S}_\delta}\right]\cdot\delta(\varphi_+)\,. \qquad (7.38)$$

The derivative with respect to φ_+ can be replaced by a derivative with respect to ϕ. This leads finally to

$$-\int d\mu \ \frac{\overleftarrow{\delta}X}{\delta\phi^{*A1'}}\frac{\overrightarrow{\delta}W_{BLT}}{\delta\phi_A}\cdot\left[\hat{V}e^{\frac{i}{\hbar}\tilde{S}_\delta}\right]\,. \qquad (7.39)$$

The complete Ward identity hence becomes, dropping the primes again,

$$0 \ = \ \left\langle(X,W_{BLT})_1-i\hbar\Delta_1 X+(-1)^{\varepsilon_X}\phi^{*A2}\frac{\overrightarrow{\delta}X}{\delta\bar{\phi}^A}\right\rangle \qquad (7.40)$$

$$= \ \int[d\phi][d\phi^{*a}][d\bar{\phi}]\left[(X,W_{BLT})_1-i\hbar\Delta_1 X+(-1)^{\varepsilon_X}\phi^{*A2}\frac{\overrightarrow{\delta}X}{\delta\bar{\phi}^A}\right]$$

$$e^{\frac{i}{\hbar}W_{BLT}}\cdot\left[\hat{V}e^{\frac{i}{\hbar}\tilde{S}_\delta}\right]\,.$$

An analogous property is obtained by going through the same steps for the Ward identities $\langle\delta_2 X\rangle=0$.

7.3.2 Quantum Master Equation

Analogous to the case of the BV formalism (section 2 of chapter 5), the fact that these Ward identities are valid for all $X(\phi,\phi^{*a},\bar{\phi})$, leads to two conditions on W_{BLT}, the so-called *quantum master equations*. Starting from the most general Ward identity, the purpose is to remove all derivative operators acting on X by partial integrations. Again, $d\mu$ denotes the measure of the path integral. We start from

$$0 \ = \ \int d\mu\left[\frac{\overleftarrow{\delta}X}{\delta\phi_A}\frac{\overrightarrow{\delta}W_{BLT}}{\delta\phi^{*Aa}}-\frac{\overleftarrow{\delta}X}{\delta\phi^{*Aa}}\frac{\overrightarrow{\delta}W_{BLT}}{\delta\phi_A}-i\hbar(-1)^{\varepsilon_A+1}\frac{\overleftarrow{\delta}}{\delta\phi^{*Aa}}\frac{\overleftarrow{\delta}}{\delta\phi_A}X\right.$$

$$\left.+(-1)^{\varepsilon_X}\epsilon_{ab}\phi^{*Ab}\frac{\overrightarrow{\delta}X}{\delta\bar{\phi}^A}\right]e^{\frac{i}{\hbar}(W_{BLT}+S_\Psi+\tilde{S}_\delta)}\,. \qquad (7.41)$$

Notice that the operator $\hat{V}$ was explicitised again as $e^{\frac{i}{\hbar}S_\Psi}$.

By integrating by parts over ϕ_A in the first term, we get the following two terms:

$$\int d\mu \;\; i\hbar.X.\Delta_a e^{\frac{i}{\hbar}W_{BLT}}.e^{\frac{i}{\hbar}(S_\Psi+\tilde{S}_\delta)}$$

$$+ \int d\mu \;\; i\hbar.X.(-1)^{\epsilon_A+1}\frac{\overleftarrow{\delta}\,e^{\frac{i}{\hbar}W_{BLT}}}{\delta\phi^{*Aa}}.\frac{\overleftarrow{\delta}}{\delta\phi_A}\left[e^{\frac{i}{\hbar}(S_\Psi+\tilde{S}_\delta)}\right]. \qquad (7.42)$$

The second and third contribution to the Ward identity (7.41) can be combined to give

$$\int d\mu \;\; (-i\hbar)(-1)^{\epsilon_A+1}\frac{\overleftarrow{\delta}}{\delta\phi_A}\left[\frac{\overleftarrow{\delta}X}{\delta\phi^{*Aa}}e^{\frac{i}{\hbar}W_{BLT}}\right].e^{\frac{i}{\hbar}(S_\Psi+\tilde{S}_\delta)}. \qquad (7.43)$$

Integrating by parts twice, first over ϕ_A, then over ϕ^{*Aa} gives us the terms:

$$\int d\mu \;\; i\hbar(-1)^{\epsilon_A}.X.e^{\frac{i}{\hbar}W_{BLT}}.\frac{\overleftarrow{\delta}}{\delta\phi^{*Aa}}\frac{\overleftarrow{\delta}}{\delta\phi_A}\left[e^{\frac{i}{\hbar}(S_\Psi+\tilde{S}_\delta)}\right]$$

$$+ \int d\mu \;\; i\hbar(-1)^{\epsilon_A}.X.\frac{\overleftarrow{\delta}\,e^{\frac{i}{\hbar}W_{BLT}}}{\delta\phi^{*Aa}}.\frac{\overleftarrow{\delta}}{\delta\phi_A}\left[e^{\frac{i}{\hbar}(S_\Psi+\tilde{S}_\delta)}\right]. \qquad (7.44)$$

Notice that the second term of (7.42) cancels the second term of (7.44).

Also in the fourth term of (7.41), we have to integrate by parts, over $\bar{\phi}^A$. This gives us again two terms:

$$- \int d\mu \;\; X.\epsilon_{ab}\phi^{*Ab}\frac{\overrightarrow{\delta}\,e^{\frac{i}{\hbar}W_{BLT}}}{\delta\bar{\phi}^A}e^{\frac{i}{\hbar}(S_\Psi+\tilde{S}_\delta)}$$

$$- \int d\mu \;\; X.\epsilon_{ab}\phi^{*Ab}e^{\frac{i}{\hbar}W_{BLT}}\frac{\overrightarrow{\delta}}{\delta\bar{\phi}^A}\left[e^{\frac{i}{\hbar}(S_\Psi+\tilde{S}_\delta)}\right]. \qquad (7.45)$$

We now show that the first term in (7.44) and the second term in (7.45) cancel. Working out the two derivatives and using the explicit form of $\tilde{S}_\delta$, we rewrite the first term of (7.44) as

$$\int d\mu \;\; (i\hbar)(\frac{i}{\hbar})^2.X.e^{\frac{i}{\hbar}W_{BLT}}e^{\frac{i}{\hbar}(S_\Psi+\tilde{S}_\delta)}\frac{\overleftarrow{\delta}S_\Psi}{\delta\phi_A}\pi_{Aa}. \qquad (7.46)$$

Now, we know that $\delta_a S_\Psi = 0$, which allows us to replace $\frac{\overleftarrow{\delta}S_\Psi}{\delta\phi_A}\pi_{Aa}$ by $-\frac{\overleftarrow{\delta}S_\Psi}{\delta\pi_{Ab}}\epsilon_{ab}B_A$. Using the explicit form of $\tilde{S}_\delta$ again, this is

$$- \int d\mu \;\; (i\hbar).X.e^{\frac{i}{\hbar}W_{BLT}}.\frac{\overleftarrow{\delta}\,e^{\frac{i}{\hbar}S_\Psi}}{\delta\pi_{Ab}}.\frac{\overrightarrow{\delta}\,e^{\frac{i}{\hbar}\tilde{S}_\delta}}{\delta\bar{\phi}^A}\epsilon_{ab}. \qquad (7.47)$$

One more partial integration, over π_{Ab}, is needed to see that the terms do cancel as mentioned above.

Summing up (7.42,7.44,7.45), we see that the Ward identities (7.41) are equivalent to

$$0 = \int d\mu \;\; X\left[\hat{V}e^{\frac{i}{\hbar}\tilde{S}_\delta}\right]\left[i\hbar\Delta_a - \epsilon_{ab}\phi^{*Ab}\frac{\overrightarrow{\delta}}{\delta\bar{\phi}^A}\right]e^{\frac{i}{\hbar}W_{BLT}}. \qquad (7.48)$$

As this is valid for all possible choices for $X(\phi, \phi^{*a}, \bar{\phi})$, we see that W_{BLT} has to satisfy the *quantum master equation*

$$\left[i\hbar\Delta_a - \epsilon_{ab}\phi^{*Ab} \frac{\vec{\delta}}{\delta\bar{\phi}^A} \right] e^{\frac{i}{\hbar}W_{BLT}} = 0. \tag{7.49}$$

This is equivalent to

$$\frac{1}{2}(W_{BLT}, W_{BLT})_a + \epsilon_{ab}\phi^{*Ab} \frac{\vec{\delta} W_{BLT}}{\delta\bar{\phi}^A} = i\hbar\Delta_a W_{BLT}. \tag{7.50}$$

Remember that these are two equations, $a = 1, 2$. By doing the usual expansion $W_{BLT} = S_{BLT} + \hbar M_1 + \hbar^2 M_2 + \ldots$, we find the classical master equation (7.27) for S_{BLT} back.

7.4 Open Algebras

In the previous chapter, we pointed out how combining the collective field approach and the recipe of [45], one is naturally led to the construction of an extended action that contains terms of quadratic and higher order in the antifields for BRST invariant quantisation. As we do not have a principle analogous to the one described in [45] and chapter 6, for constructing a gauge fixed action that is invariant under BRST–anti-BRST symmetry for the case of an open algebra, we will have to take the other point of view advocated in section 2 of chapter 6.

The collective field method is *a method sometimes employed in French cuisine : a piece of pheasant meat is cooked between two slices of veal, which are then discarded* [54]. Nevertheless, like in the case of ordinary BRST collective field quantisation (see chapter 5 and 6), the introduction of the collective fields allows to shift the problem of the off-shell non-nilpotency to the (anti-)BRST transformations of the collective fields. Indeed, $\delta_a\phi_A = \pi_{Aa}$, $\delta_a\pi_{Ab} = \epsilon_{ab}B_A$ and $\delta_a B_A = 0$ guarantee that $\delta_a^2\phi_A = 0$ and that $(\delta_1\delta_2 + \delta_2\delta_1)\phi_A = 0$. Therefore, the originally present gauge symmery can be fixed in a BRST–anti-BRST invariant way like for closed algebras, i.e. by adding $S_\Psi = \frac{1}{2}\epsilon^{ab}\delta_a\delta_b\Psi$ to a BRST–anti-BRST invariant action, S_{inv}. This way, the BRST and anti-BRST Ward identities guarantee that whatever way we choose to construct S_{inv}, the partition function will be independent of the gauge choice if S_{inv} is BRST–anti-BRST invariant.

We decompose again

$$\begin{aligned}
S_{inv} &= S_{BLT}(\phi - \varphi_+, \phi^{*a'}, \bar{\phi}) \tag{7.51}\\
&\quad -\varphi_{A+}M^{AB}\lambda_B + \bar{\phi}^A B_A - \phi^{*Aa'}\pi_{Aa}.
\end{aligned}$$

It is useful to keep the redefinitions (7.15) in mind in the following. S_{inv} will be BRST–anti-BRST invariant, that is $\delta_a S_{inv} = 0$, under the transformation rules (7.9,7.11), except for the generalisations

$$\delta_a\varphi_{Ab} = \delta_{ab}\left[\pi_{Aa} - \epsilon_{ac}\phi_A^{*c} - \frac{\vec{\delta} S_{BLT}(\phi - \varphi_+)}{\delta\phi^{*Aa'}} \right] + (1 - \delta_{ab})\epsilon_{ac}\phi_A^{*c}$$

$$\delta_a\phi_A^{*b} = -\delta_a^b\left[(-1)^a\lambda_A + \frac{1}{2}(B_A + \frac{\vec{\delta} S_{BLT}(\phi - \varphi_+)}{\delta\bar{\phi}^A}) \right], \tag{7.52}$$

if S_{BLT} satisfies the two classical master equations of the antifield scheme. Hence, we see that the question whether open algebras can be quantised in a BRST–anti-BRST invariant way, reduces to the fact whether a solution to (7.27) can be found for open algebras with the extra condition that $S_{BLT} = S_0 + \phi^{*Aa}\mathcal{R}_{Aa} + \ldots$ It has been proved that such solutions exist [51, 55, 56].

When quantum counterterms are needed to derive the Ward identities, we obtain the two quantum master equations, following the same steps as in section 2 of chapter 6, as the conditions that guarantee the validity of the Ward identities. These identities then imply gauge independence of the partition function.

Chapter 8

Canonical transformations

After the inductive approach of the previous chapters, where we have shown how the antifield scheme for BRST invariant quantisation can be constructed from the BRST quantisation recipes, we present here a more algebraic approach. As was already pointed out above (5.25), the antibracket with fields and antifields ressembles the Poisson bracket of classical mechanics in its Hamiltonian formulation. Inspired by this analogy, we will look for *canonical transformations* of the fields and the antifields that leave the antibracket of two function(al)s invariant.

A large class of canonical transformations, although not all, are those that are obtained from a generating function, which has to be fermionic here. We will first show that such transformations leave both the classical and quantum cohomology invariant. For the former, this follows trivially from the definition of canonical transformations itself. For the latter however, we have to study carefully how Δ transforms, which is related to the transformation of the measure of the path integral.

The gauge fixing procedure as defined above (5.10), is a canonical transformation generated by $F = \mathbb{1} + \Psi$ [57], where $\mathbb{1}$ is a symbolic notation for the identity tranformation. We replace the condition of gauge invariance of the expectation value of an arbitrary operator X by invariance under arbitrary canonical transformations. This way we rederive the quantum master equation and the condition that the operator $X(\phi, \phi^*)$ has to satisfy in order to have the same expectation value in two different sets of canonical coordinates.

Besides gauge fixing, the use of canonical transformations is manifold. First of all, they can be used to construct other realisations as a field theory of the same physical degrees of freedom, i.e. to construct cohomologically equivalent theories with a different field content. This will be demonstrated in the examples below and used in chapter 14 on the hiding of anomalies. Conversely, auxiliary fields can be removed in a consistent way by doing canonical transformations that bring them under the form of a trivial system, which can then be discarded [58, 59]. Taking a different set of gauge generators R^i_α can also be seen as a canonical transformation [4].

Before studying the canonical transformations in more detail, let us make another comment. The analogy with the Poisson bracket has also served as a starting point for recent investigations on the geometrical structure of BV

[60]. Like for Poisson brackets, antibrackets have been defined using a general Grassmann odd symplectic 2-form which has to be closed (Jacobi identity). The form of the antibracket that we always use in this work corresponds to working with the Darboux coordinates. The same 2-form is then used to define a second order differential operator Δ, that has all the properties listed in the appendix. It is believed that these constructions will serve in the attempts at a construction of string field theories [61, 62].

8.1 Canonical transformations and the cohomologies

Two sets of canonical variables (=fields and antifields) are respectively denoted by $\{\phi^A, \phi_A^*\}$ and $\{\phi^{A'}, \phi_A^{*'}\}$. A transformation from the unprimed to the primed indices is then said to be *canonical* if for any two function(al)s $A(\phi, \phi^*)$ and $B(\phi, \phi^*)$ calculating the antibracket in the unprimed variables and transforming the result gives the same expression as first transforming A and B to the primed variables and then calculating the antibracket with respect to the primed variables.

A large class of canonical transformations consists of those transformations for which

$$\frac{\overleftarrow{\delta}\, \phi^B(\phi', \phi^{*'})}{\delta \phi'^A}\Big|_{\phi^{*'}} \tag{8.1}$$

is invertible. It is possible to show [63, 59] that they can be obtained from a fermionic generating function $F(\phi, \phi^{*'})$ of ghostnumber -1. The transformation rules are then given by

$$\begin{aligned}
\phi^{A'} &= \frac{\delta F(\phi, \phi^{*'})}{\delta \phi_A^{*'}} \\
\phi_A^* &= \frac{\delta F(\phi, \phi^{*'})}{\delta \phi^A}.
\end{aligned} \tag{8.2}$$

The other way around, if $F(\phi, \phi^{*'})$ is such that

$$\frac{\overrightarrow{\delta}\, \delta}{\delta \phi_A^{*'} \delta \phi^B} F(\phi, \phi^{*'}) \tag{8.3}$$

is invertible, then the transformation given by (8.2) is canonical.

We study here infinitesimal canonical transformations generated by

$$F(\phi, \phi^{*'}) = \mathbb{1} + f(\phi, \phi^{*'}) = \phi^A \phi_A^{*'} + f(\phi, \phi^{*'}), \tag{8.4}$$

with f small. Below we will use the name *generating fermion* for f. The transformation rules become

$$\begin{aligned}
\phi^{A'} &= \phi^A + \frac{\delta f(\phi, \phi^*)}{\delta \phi_A^*} \\
\phi_A^{*'} &= \phi_A^* - \frac{\delta f(\phi, \phi^*)}{\delta \phi^A}.
\end{aligned} \tag{8.5}$$

We replaced $\phi^{*'}$ on the RHS by ϕ^* since we are making an infinitesimal transformation.

The expression in the primed coordinates for any function(al) given in the unprimed coordinates can be obtained by direct substitution of the transformation rules. Owing to the infinitesimal nature of the transformation, we can expand in a Taylor series to linear order in f and we find

$$
\begin{aligned}
X'(\phi', \phi^{*'}) &= X\left(\phi^{A'} - \frac{\delta f(\phi', \phi^{*'})}{\delta \phi_A^{*'}}, \phi_A^{*'} + \frac{\delta f(\phi', \phi^{*'})}{\delta \phi^{A'}}\right) \\
&= X(\phi', \phi^{*'}) - (X, f)(\phi', \phi^{*'}).
\end{aligned} \tag{8.6}
$$

By $(\phi', \phi^{*'})$ we do of course not mean the antibracket of a field and an antifield but only that the preceding expression is a function of fields and antifields. We can drop the primes of the arguments and denote the transformed functional by

$$
X'(\phi, \phi^*) = X(\phi, \phi^*) - (X, f)(\phi, \phi^*). \tag{8.7}
$$

With this result, it becomes easy to show that (infinitesimal) transformations of the form (8.2) are indeed canonical. Consider

$$
\begin{aligned}
(A', B') &= (A - (A, f), B - (B, f)) \\
&= (A, B) - ((A, B), f) + \mathcal{O}(f^2) \\
&= (A, B)',
\end{aligned} \tag{8.8}
$$

where in the second step the Jacobi identity for the antibracket is used. Remember that $\mathcal{O}(f^2)$ is neglected. As a corollary of this, it is straightforward to see that the classical cohomology is invariant under (infinitesimal) canonical transformations. Also, a solution of the classical master equation is transformed in a solution of the classical master equation. We discuss two examples in section 4 of canonical transformations that are merely redefinitions of the gauge generators R_α^i. The structure functions and non-closure functions of the algebra (1.16) of the new gauge generators can then be determined from the transformed extended action.

Let us now turn to the box operator. Using again the expressions of the appendix, we have that

$$
\begin{aligned}
\Delta X' &= \Delta X - \Delta(X, f) \\
&= (\Delta X)' - (X, \Delta f).
\end{aligned} \tag{8.9}
$$

Hence, we see that acting with Δ is not a canonical invariant operation. An extra term $-(X, \Delta f)$ appears. We show in the next section that $\Delta f = \ln J$, with J the Jacobian of the change of integration variables in the path integral.

We want to define the quantum BRST operator in the transformed coordinates, $\sigma' Y = (Y, \tilde{W}) - i\hbar \Delta Y$. In order for this operator to be nilpotent, we know that $\tilde{W}$ has to satisfy the quantum master equation. In contrast with the classical master equation, the transformation of a solution W of the quantum master equation does *not* give a solution in the new variables. Indeed, we have that

$$
\begin{aligned}
(W', W') - 2i\hbar \Delta W' &= [(W, W) - 2i\hbar \Delta W]' + 2i\hbar(W, \Delta f) \\
&= 2i\hbar(W, \Delta f),
\end{aligned} \tag{8.10}
$$

if W satisfies the quantum master equation (5.38) in the original variables. Instead,

$$
\tilde{W} = W' - i\hbar \Delta f = W + \sigma f \tag{8.11}
$$

does satisfy the quantum master equation in the transformed coordinates, as follows from the nilpotency of Δ. Notice that (8.11) generalises (8.7) in the sense that both the classical and quantum extended action transform under an infinitesimal canonical transformation by the addition of respectively the classical and quantum BRST transformation of the generating fermion f. Then we have that

$$
\begin{aligned}
\sigma' X' &= (X', \tilde{W}) - i\hbar \Delta X' \\
&= (\sigma X)'.
\end{aligned}
\tag{8.12}
$$

Hence, we see that also the quantum cohomology is invariant under infinitesimal transformations. If $\sigma X = 0$, then $\sigma' X' = 0$ and if $Y = \sigma X$ then $Y' = \sigma' X'$.

Although we have only shown that the antibracket and both the classical and quantum cohomology are invariant under *infinitesimal* canonical transformations, these results also hold for finite transformations. For proofs of this statement, see [59].

8.2 From gauge invariance to invariance under canonical transformations

As was pointed out in section 3 of chapter 5, we can choose more general gauge fermions, depending on both the fields and the antifields, leading to a gauge fixed action (5.48) $S(\phi - \frac{\delta \Psi}{\delta \phi^*}, \phi^* + \frac{\delta \Psi}{\delta \phi})$. This gauge fixed action is obtained from $S(\phi, \phi^*)$ by doing a canonical transformation. This observation will be our starting point here. In chapter 13 we copy the derivation of the master equation and of the quantum BRST operator that we give in this section, using one-loop regularised path integrals.

We construct the expression for the expectation value of an arbitrary operator $X(\phi, \phi^*)$ in two sets of coordinates, related by an infinitesimal canonical transformation generated by a fermion f and we calculate their difference in function of the fermion f. So we have

$$
\chi(\phi^*) = \int [d\phi] X(\phi, \phi^*) e^{\frac{i}{\hbar} W(\phi, \phi^*)}
\tag{8.13}
$$

and

$$
\chi'(\phi^*) = \int [d\phi] \left[X - (X, f) \right] e^{\frac{i}{\hbar}(W - (W, f) - i\hbar \Delta f)}.
\tag{8.14}
$$

Expanding in $\chi'(\phi^*)$ to linear order in f, and using the notation $\mathcal{W} = \frac{i}{\hbar} W$, we find that

$$
\begin{aligned}
\delta \chi &= \chi'(\phi^*) - \chi(\phi^*) \\
&= \int [d\phi] \left[-X(\mathcal{W}, f) + X \Delta f - (X, f) \right] e^{\mathcal{W}}
\tag{8.15} \\
&= \int [d\phi] \left[-X \frac{\overleftarrow{\delta} e^{\mathcal{W}}}{\delta \phi^A} \frac{\overrightarrow{\delta} f}{\delta \phi_A^*} + X \frac{\overleftarrow{\delta} e^{\mathcal{W}}}{\delta \phi_A^*} \frac{\overrightarrow{\delta} f}{\delta \phi^A} + X.\Delta f.e^{\mathcal{W}} \right. \\
&\qquad \left. - \frac{\overleftarrow{\delta} X}{\delta \phi^A} \frac{\overrightarrow{\delta} f}{\delta \phi_A^*} e^{\mathcal{W}} + \frac{\overleftarrow{\delta} X}{\delta \phi_A^*} \frac{\overrightarrow{\delta} f}{\delta \phi^A} e^{\mathcal{W}} \right].
\tag{8.16}
\end{aligned}
$$

The first, third and fourth term can be combined to a total derivative with respect to ϕ^A, and can hence be discarded. In the two remaining terms, we integrate by parts over ϕ^A, which leads to:

$$\delta\chi = \int [d\phi] \left[X.\Delta e^{\mathcal{W}}.f + [(X,\mathcal{W}) + \Delta X].e^{\mathcal{W}}.f \right]. \tag{8.17}$$

Imposing that $\delta\chi = 0$ for all f, i.e. that the expectation value of the operator X is invariant under infinitesimal canonical transformations (=infinitesimal deformations of the gauge choice), we have the sufficient conditions[1]

$$\begin{aligned} \Delta e^{\mathcal{W}} &= 0 \\ (X,\mathcal{W}) + \Delta X &= 0. \end{aligned} \tag{8.18}$$

These are easily seen to be equivalent to the quantum master equation

$$(W,W) - 2i\hbar\Delta W = 0, \tag{8.19}$$

and the condition that a gauge invariant operator has a vanishing quantum BRST variation

$$\sigma X = (X,W) - i\hbar\Delta X = 0. \tag{8.20}$$

We can also impose gauge independence [2] on correlation functions. Consider for $X = Ye^{J(\phi,\phi^*)}$, where the $J(\phi,\phi^*)$ can be interpreted as source terms. From the expression for $\sigma[AB]$ in the appendix, we conclude:

$$\begin{aligned} \sigma e^J &= 0 \\ (Y,e^J) &= 0 \\ \sigma Y &= 0. \end{aligned} \tag{8.21}$$

The first line is the gauge independence condition for the sources J. If the sources J do not satisfy this requirement, the correlation functions become gauge dependent with a dependence given by (8.17).

It is straightforward to show, using again partial integrations, that for all functions $X(\phi,\phi^*)$,

$$\int [d\phi]\, \sigma X(\phi,\phi^*).e^{\frac{i}{\hbar}W(\phi,\phi^*)} = 0, \tag{8.22}$$

provided W satisfies the quantum master equation. This is the form of the Ward identities in the BV scheme (see section 2 of chapter 5). Here too, by taking $X = Ye^J$, Ward identities for the correlation functions of σY are obtained.

Notice that demanding that the partition function is the same in both sets of coordinates is a non-trivial condition. Of course, the fact that we can re-define $\phi^A = \phi^{A'} - \delta f/\delta\phi_A^{*'}$ is a mere consequence of the freedom to redefine the variables in a (path) integral. It is the invariance under redefinition of the antifields that leads to conditions on the integrand. The extra term $-i\hbar\Delta f$ in the transformation the quantum extended action W is exactly the Jacobian that one expects from the change of the integration variables:

$$\begin{aligned} J &= \mathrm{sdet}\left[\delta_B^A - \frac{\overrightarrow{\delta}}{\delta\phi^{B'}} \frac{\overleftarrow{\delta} f}{\delta\phi_A^{*'}} \right] \\ &= e^{\frac{i}{\hbar}(-i\hbar\Delta f)}, \end{aligned} \tag{8.23}$$

---

[1]In fact, it seems that from $\delta\chi = 0$ for all f it only follows that $\Delta[Xe^{\mathcal{W}}] = 0$. However, as we want that for $X = 1$ the partition function is independent of f, we separate this one condition in two.

[2]We will use equivalently *independent of the set of canonical coordinates* and *gauge independent*.

again neglecting $\mathcal{O}(f^2)$ corrections.

8.3 The Zinn-Justin equation

Let us here mention a nice result for the sake of completeness, but which we will not explicitly use below [35, 59]. After introducing a (non gauge invariant) sourceterm for all fields, the generating functional $\mathcal{W}_c$ for connected diagrams is defined as:

$$e^{-\frac{i}{\hbar}\mathcal{W}_c(J,\phi^*)} = \int [d\phi] e^{\frac{i}{\hbar}W(\phi,\phi^*)+\frac{i}{\hbar}J_A\phi^A}. \tag{8.24}$$

As usual, $\mathcal{W}_c$ depends on the sources of the fields, but it now also depends on the sources ϕ^* of the BRST transformations. Using a Legendre transform, we can pass from $\mathcal{W}_c$ to the *effective action* Γ. This goes as follows. The *classical field* is defined by

$$\phi_{cl}^A = -\frac{\overrightarrow{\delta}\mathcal{W}_c}{\delta J_A}. \tag{8.25}$$

We assume that this relation is invertible to give $J(\phi_{cl}, \phi^*)$. The effective action is then defined by

$$\Gamma(\phi_{cl}, \phi^*) = \mathcal{W}_c(J(\phi_{cl}, \phi^*), \phi^*) + J_B(\phi_{cl}, \phi^*)\phi_{cl}^B. \tag{8.26}$$

If we further define the antibracket

$$(\Gamma, \Gamma) = \frac{\overleftarrow{\delta}\Gamma}{\delta\phi_{cl}^A}\frac{\overrightarrow{\delta}\Gamma}{\delta\phi_A^*} - \frac{\overleftarrow{\delta}\Gamma}{\delta\phi_A^*}\frac{\overrightarrow{\delta}\Gamma}{\delta\phi_{cl}^A}, \tag{8.27}$$

it is easy to show that

$$\frac{1}{2}(\Gamma, \Gamma)e^{-\frac{i}{\hbar}\mathcal{W}_c(J(\phi_{cl}, \phi^*), \phi^*)}$$
$$= \left[\int[d\phi][\frac{1}{2}(W, W) - i\hbar\Delta W]e^{\frac{i}{\hbar}W(\phi,\phi^*)+\frac{i}{\hbar}J_A\phi^A}\right]_{J(\phi_{cl}, \phi^*)}. \tag{8.28}$$

If the quantum extended action W satisfies the quantum master equation (5.38), we clearly have that

$$(\Gamma, \Gamma) = 0. \tag{8.29}$$

This equation goes under the name *Zinn-Justin equation*. It is the generalisation to all types of algebras of the result discussed in chapter 19 of [8]. *The effective action satisfies the classical master equation if the theory is anomaly free.*

8.4 Examples

We give here some examples of the use of canonical transformations. First we show how a change of basis of gauge generators can be realised by a canonical transformation. Then we give two more examples that are continuations [33] of the examples given in chapter 5, the topological Yang-Mills theory and the BRST invariant energy-momentum tensor.

8.4.1 Changing gauge generators using canonical transformations

The form of the minimal proper extended action for a specific choice of a complete set gauge generators R^i_α is given by (see chapters 5 and 6):

$$S(\phi^A, \phi^*_A) = S_0[\phi^i] + \phi^*_i R^i_\alpha[\phi]c^\alpha + c^*_\gamma T^\gamma_{\alpha\beta}[\phi]c^\beta c^\alpha + \phi^*_i \phi^*_j E^{ji}_{\alpha\beta}c^\beta c^\alpha + \dots \quad (8.30)$$

$T^\gamma_{\alpha\beta}$ and $E^{ji}_{\alpha\beta}$ are determined by (1.16).

Consider now the canonical transformation generated by

$$F_1 = \phi^{*'}_i \phi^i + c^{*'}_\alpha M^\alpha_\beta[\phi]c^\beta, \quad (8.31)$$

where $M^\alpha_\beta[\phi]$ is invertible. From (8.2), we have that

$$
\begin{aligned}
\phi^{i'} &= \phi^i \\
c^{\alpha'} &= M^\alpha_\beta c^\beta \\
\phi^*_i &= \phi^{*'}_i + c^{*'}_\alpha \frac{\overleftarrow{\delta} M^\alpha_\beta[\phi]c^\beta}{\delta\phi^i} \\
c^*_\beta &= c^{*'}_\alpha M^\alpha_\beta[\phi] \, .
\end{aligned}
\quad (8.32)
$$

If we only consider the terms linear in the antifields, we see that the gauge generators R^i_α and the structure functions $T^\gamma_{\alpha\beta}$ of the algebra change. In particular,

$$R^{'i}_\alpha = \left[\frac{\overrightarrow{\delta}}{\delta\phi^{*'}_i} \frac{\overleftarrow{\delta}}{\delta c^{\alpha'}} S' \right]_{\phi^*_A = 0} = R^i_\beta[\phi] M^{-1\beta}_\alpha[\phi] \quad (8.33)$$

We see that this canonical transformation transforms a complete set of generators (1.12) R^i_α in a different complete set $R^{'i}_\alpha$ if $M^\alpha_\beta[\phi]$ is invertible.

On the other hand, the fermion

$$F_2 = \mathbb{1} + \frac{1}{2}\phi^{*'}_i \phi^{*'}_j M^{ji}_\alpha c^\alpha \, , \quad (8.34)$$

where $M^{ij}_\alpha = (-1)^{(\epsilon_i+1)(\epsilon_j+1)} M^{ji}_\alpha$, and where M^{ij}_α is field independent, generates the transformation rules

$$
\begin{aligned}
\phi^{i'} &= \phi^i + \phi^{*'}_j M^{ji}_\alpha c^\alpha \\
c^{\alpha'} &= c^\alpha \\
\phi^*_i &= \phi^{*'}_i \\
c^*_\alpha &= c^{*'}_\alpha + \frac{1}{2}\phi^{*'}_i \phi^{*'}_j M^{ji}_\alpha \, .
\end{aligned}
\quad (8.35)
$$

The transformed gauge generators are

$$R^{'i}_\alpha = \left[\frac{\overrightarrow{\delta}}{\delta\phi^*_i} \frac{\overleftarrow{\delta}}{\delta c^\alpha} S' \right]_{\phi^*_A = 0} = R^i_\alpha + (-1)^{\epsilon_i+1} \frac{\overleftarrow{\delta} S_0}{\delta\phi^j} M^{ji}_\alpha \, . \quad (8.36)$$

The non-closure functions are also altered by this transformation, owing to the transformation of c^*_α. Clearly, using this type of canonical transformation we can open a closed algebra.

8.4.2 Enlarging the set of fields

Canonical transformations combined with the introduction of trivial systems, can be used to enlarge the set of fields and the set of gauge symmetries without changing the cohomology of the theory. In other words, they allow us to construct other field realisations of the same physics.

Take an extended action $S(\phi^A, \phi^*_A)$ that satisfies the classical master equation. Suppose that we want to enlarge the set of fields by a set α_k. As S does not depend on α_k, the action is invariant under arbitrary shifts of these fields. The properness condition implies that we have to introduce ghosts β_k for that symmetry and consider the new extended action $S' = S + \alpha^{*k}\beta_k$. It is trivial to see that these extra fields do not change the (classical) cohomology. We are now allowed to disguise these extra fields and extra symmetries by doing whatever canonical transformation we want.

A nice application of this procedure is for instance smooth bosonisation [64]. There, one starts from fermions (in 2d) coupled to an external source. An extra scalar field is introduced via a trivial system, and the transformation that is done to disguise this trivial system is a chiral rotation, where the extra scalar field gives the space-time dependent rotation angle. The Jacobian of the transformation plays an important part, it provides the coupling of this extra scalar field to the sources. Afterwards, the fermionic degrees of freedom one started from, can be decoupled from the source using a gauge fixing, and one is left with the bosonised theory. An analogous scenario has been used to extract mesonic degrees of freedom from the QCD field theory [65]. We describe the first step of this bosonisation procedure using a one loop regularised BV scheme as an example in the third chapter of the third part. Another application, which we will discuss in extenso later on, is the hiding of anomalies [37]. We will now briefly discuss how in the model of topological Yang-Mills a realisation with a reducible set of gauge symmetries can be obtained along these lines.

We had (5.59) that $\delta A_\mu = \epsilon_\mu$, but usually [41, 42, 43] the Yang–Mills gauge symmetry $\delta A_\mu = D_\mu \epsilon$ is included in the R^i_α and one starts from $\delta A_\mu = \epsilon_\mu + D_\mu \epsilon$. This is clearly a reducible set of gauge generators as for $\epsilon_\mu = D_\mu \eta$ and $\epsilon = -\eta$, we have $\delta A_\mu = 0$. We can go over to this reducible set following the general lines sketched above. First, we enlarge the configuration space by introducing a fermionic ghost field c. As it does not appear in the extended action so far, the extended action is invariant under arbitrary shifts of c, for which we introduce a *ghost for ghost* ϕ. The new extended action then becomes

$$S = S_0 + A^{*\mu}\psi_\mu + c^*\phi \; . \tag{8.37}$$

Now we do a canonical transformation, generated by the fermion

$$F = \mathbb{1} - \psi'^{*\mu} D_\mu c \; . \tag{8.38}$$

This gives the transformation rules

$$\begin{aligned}
\psi_\mu &= \psi'_\mu + D_\mu c \\
c^* &= c'^* + \partial_\mu \psi'^{*\mu} - \psi'^{*\mu}[A_\mu, \cdot] \\
A^{*\mu} &= A'^{*\mu} - \psi'^{*\mu}[\cdot, c] \; .
\end{aligned} \tag{8.39}$$

The transformed extended action is then (dropping the primes) :

$$S = S_0 + c^*\phi - \psi^{*\mu} D_\mu(\phi - cc) + \psi^{*\mu}(\psi_\mu c + c\psi_\mu) + A^{*\mu}(\psi_\mu + D_\mu c) \; . \tag{8.40}$$

Notice that the antifields of the ghosts c and ψ_μ now act as sources for the reducibility transformations: $c^*\phi$ and $-\psi^{\mu*}D_\mu\phi$. Also, A_μ transforms under the shifts as well as under the Yang–Mills symmetry. These are two typical properties of the solution of the classical master equation in the case of reducible gauge symmetries. In order to make the connection to the description with reducible symmetries complete, we do yet another canonical transformation that makes the familiar c^*cc term of the Yang-Mills symmetry appear. This transformation is generated by

$$G = \mathbf{1} - \phi'^* cc \ . \tag{8.41}$$

This gives $\phi' = \phi - cc$ and $c^* = c'^* - \phi'^*[c, \cdot]$. After doing these two canonical transformations, we have that

$$
\begin{aligned}
S &= S_0 + A^{*\mu}(\psi_\mu + D_\mu c) + \psi^{*\mu}(\psi_\mu c + c\psi_\mu - D_\mu\phi) \\
&\quad + c^*(\phi + cc) - \phi^*[c, \phi] \ .
\end{aligned}
\tag{8.42}
$$

Of course, this extra symmetry with ghost ϕ, has to be gauge fixed too. This is done by introducing a Lagrange multiplier and antighost (sometimes called η and $\bar\phi$).

8.4.3 Canonical invariance of the energy-momentum tensor

We now show that the definition of the energy-momentum tensor that we have given in section 4 of chapter 5, is invariant under (infinitesimal) canonical transformations, up to a BRST exact term. Under an infinitesimal canonical transformation generated by the fermion $F = \mathbf{1} + f$, the classical action and the energy-momentum tensor transform as follows:

$$
\begin{aligned}
S' &= S - (S, f) \\
T'_{\alpha\beta} &= T_{\alpha\beta} - (T_{\alpha\beta}, f).
\end{aligned}
\tag{8.43}
$$

Here, $T_{\alpha\beta}$ is the energy-momentum tensor that is obtained following the recipe given in chapter 5 starting from the extended action S [3]. Analogously, we can apply the recipe to the transformed action S', which leads to an energy-momentum tensor $\tilde T_{\alpha\beta}$. Using (5.71) and (5.73), it is easy to show that

$$
\begin{aligned}
\tilde T_{\alpha\beta} &= \frac{2}{\sqrt{|g|}}\frac{\delta S'}{\delta g^{\alpha\beta}} + g_{\alpha\beta}\sum_i \phi_X^* \frac{\overrightarrow{\delta}S'}{\delta\phi_X^*} \\
&= T'_{\alpha\beta} - (S, \frac{2}{\sqrt{|g|}}\frac{\delta f}{\delta g^{\alpha\beta}} + g_{\alpha\beta}\sum_i \phi_X^*\frac{\overrightarrow{\delta}f}{\delta\phi_X^*}) \\
&= T'_{\alpha\beta} + (D_{\alpha\beta}f, S'),
\end{aligned}
\tag{8.45}
$$

as for infinitesimal transformations terms of order f^2 can be neglected.

---

[3]Let us briefly recapitulate this recipe. Define the operator

$$D_{\alpha\beta} = \frac{2}{\sqrt{|g|}}\frac{\delta}{\delta g^{\alpha\beta}} + g_{\alpha\beta}\sum_i \phi_X^*\frac{\overrightarrow{\delta}}{\delta\phi_X^*} \ . \tag{8.44}$$

If S satisfies the classical master equation, then $T_{\alpha\beta} = D_{\alpha\beta}S$ is a classical gauge invariant operator: $(T_{\alpha\beta}, S) = 0$. If W satisfies the quantum master equation, then $T^q_{\alpha\beta} = D_{\alpha\beta}W$ is quantum BRST invariant (at least formally): $\sigma T^q_{\alpha\beta} = 0$.

We finally verify that also $T^q_{\alpha\beta}$ is canonically invariant. Under an infinitesimal canonical transformation, we have the following transformation properties:

$$
\begin{aligned}
\tilde{W} &= W + \sigma f = W + (f, W) - i\hbar \Delta f \\
T^{q'}_{\alpha\beta} &= T^q_{\alpha\beta} - (T^q_{\alpha\beta}, f),
\end{aligned}
\tag{8.46}
$$

with the same definition of f as above. Let $\tilde{T}^q_{\alpha\beta}$ denote the energy-momentum tensor that we obtain by applying the recipe to the transformed action $\tilde{W}$. We then easily see that

$$
\begin{aligned}
\tilde{T}^q_{\alpha\beta} &= \frac{2}{\sqrt{|g|}} \frac{\delta \tilde{W}}{\delta g^{\alpha\beta}} + g_{\alpha\beta} \sum_i \phi^*_X \frac{\overrightarrow{\delta}\, \tilde{W}}{\delta \phi^*_X} \\
&= T^{q'}_{\alpha\beta} + \sigma\, [D_{\alpha\beta} f].
\end{aligned}
\tag{8.47}
$$

Here too, rewriting the last term using σ', the quantum BRST operator in the transformed basis, only involves f^2 corrections.

Part II

The equivalence of the Hamiltonian Batalin-Fradkin-Vilkovisky and the Lagrangian Batalin-Vilkovisky quantisation schemes

Chapter 9

Hamiltonian quantisation of gauge theories

In this chapter, the Hamiltonian approach to the quantisation of gauge theories is outlined, without any justification or proof. Since we believe that most of the introduced concepts and procedures are evident from the preceding discussion of the Lagrangian quantisation procedure, this chapter is somewhat less self-contained. In the next chapter, we apply the recipe given here, to describe the Schwinger-Dyson BRST symmetry in the Hamiltonian formalism. This will allow us to prove the equivalence of the Hamiltonian scheme given in this chapter and the Lagrangian approach on which the rest of this work is focussed. We again restrict ourselves to irreducible gauge symmetries.

In the Hamiltonian approach, two main parts can be distinguished. First, one has to perform an analysis of the classical system, using what is often denoted by *Dirac's constraints analysis* [66]. With the results of this analysis at hand, one formulates the classical system in terms of the global BRST symmetry, if gauge symmetries are present. We discuss the method of I.A. Batalin, E.S. Fradkin and G.A. Vilkovisky [67]. This prepares the stage for the quantisation of the model, using either operator or path integral methods. In the spirit of this work, we choose the latter.

A detailed account of both steps can be found in [17, 47]. A discussion of the second step, the BFV Hamiltonian formalism, is given in [68].

9.1 Dirac's constraints analysis

Consider a classical dynamical system, described by a Lagrangian function $L(q_n(t), \dot{q}_n(t))$. We use the notation of a system with a discrete set of degrees of freedom, although n may represent a possibly continuous set of indices (=classical field theory). The $q_n(t)$ may be of either Grassmann parity and as usual, $\dot{q}_n$ denotes the time derivative of q_n. The Hamiltonian formalism is obtained by trading the velocities $\dot{q}_n$ for momenta p_n via a Legendre transformation. The momenta are defined by

$$p_n(q, \dot{q}) = \frac{\vec{\delta} L}{\delta \dot{q}_n}. \tag{9.1}$$

If the matrix $\overset{\rightarrow}{\delta} p_n / \delta \dot{q}_m$ is of maximal rank, then the relation (9.1) between the momenta and velocities can be inverted to $\dot{q}_n(p, q)$ and the Hamiltonian is defined by

$$H(p, q) = \dot{q}_n(p, q) p_n - L(q, \dot{q}(p, q)).\qquad(9.2)$$

Theories for which this procedure can be followed, are not of interest as gauge theories.

Much more exciting is the case where the equations (9.1) can not be solved for the velocities. A carefull analysis [66] leads to the following results. There exists a set of *constraints*, that is, a set of functions $\phi_j(p, q)$ such that for the classical system $\phi_j(p, q) = 0$. Functions $F(p, q)$ that vanish on the subspace of phase space that is defined by $\phi_j = 0$, are said to be *weakly vanishing*, and we write $F(p, q) \approx 0$. Especially, $\phi_j \approx 0$. We restrict the functions $F(p, q)$ to these functions such that $F \approx 0 \Leftrightarrow F = \sum_j c^j(p, q) \phi_j(p, q)$ for some phase space functions $c^j(p, q)$ (regularity condition).

Functions F defined on the phase space can now be divided into two categories, *first class* and *second class* functions. A function F is said to be first class, if it has a weakly vanishing Poisson bracket[1] with *all* constraints. That is, F is first class if

$$\forall j : [F, \phi_j] \approx 0 \quad \Leftrightarrow \quad \forall j : [F, \phi_j] = \sum_k C_j^k(p, q) \phi_k .\qquad(9.3)$$

All other functions defined on phase space are called second class.

Of course, this divides also the set of constraints themselves into two classes, first class constraints and second class constraints. A constraint is first class if it has a weakly vanishing Poisson bracket with all other constraints. A constraint is second class if there exists at least one other constraint with which it does not have a weakly vanishing Poisson bracket. Second class constraints can be accounted for by replacing the Poisson brackets by *Dirac brackets*. These Dirac brackets have the same properties as Poisson brackets (distributivity, Jacobi-identities), but they are constructed using a symplectic two form that is determined from the Poisson bracket of the second class constraints. We will in this and the following chapter assume that no second class constraints are present. If they were, we only have to put in the associated symplectic form at the right places.

After the elimination of the second class constraints, we are left with a set of first class constraints, ϕ_α; i.e.

$$[\phi_\alpha, \phi_\beta] \approx 0 \quad \Leftrightarrow \quad [\phi_\alpha, \phi_\beta] = C_{\alpha\beta}^\gamma(p, q) \phi_\gamma .\qquad(9.4)$$

The phase space functions $C_{\alpha\beta}^\gamma$ are called *(first order) structure functions*. It was conjectured, by P.A.M. Dirac, that all first class constraints generate gauge symmetries.

Besides constraints, the second ingredient of the Hamiltonian formalism is a first class Hamiltonian H_0:

$$[H_0, \phi_\alpha] \approx 0 \Leftrightarrow [H_0, \phi_\alpha] = V_\alpha^\beta(p, q) \phi_\beta.\qquad(9.5)$$

---

[1]For the definition and a list of properties of Poisson brackets, see the appendix at the end of *this chapter*.

With this Hamiltonian and this set of constraints a first order action (linear in the velocities) can be constructed:

$$S = \int dt \left[\dot{q}_n p_n - H_0 - \sum_\alpha \lambda^\alpha(t)\phi_\alpha \right].$$ (9.6)

Here, $\lambda^\alpha(t)$ is a Lagrange multiplier, imposing the constraints. This action is invariant under gauge transformations. The transformation of an arbitrary phase space function F is generated by the constraints:

$$\delta_\epsilon F(p,q) = [F, \epsilon^\alpha(t)\phi_\alpha].$$ (9.7)

The ϵ^α are the parameters of the transformation. The Lagrange multiplier λ^α has to be given the transformation rule

$$\delta_\epsilon \lambda^\alpha(t) = \dot{\epsilon}^\alpha - \epsilon^\beta V_\beta^\alpha + \lambda^\gamma \epsilon^\beta C_{\beta\gamma}^\alpha,$$ (9.8)

in order for S (9.6) to be invariant. These are the main results of Dirac's constraints analysis, the first step of the Hamiltonian quantisation of (gauge) theories.

9.2 The BFV formalism

We now turn to the Batalin-Fradkin-Vilkovisky (BFV) scheme [67]. A first step is to promote the Lagrange multiplier λ^α in (9.6) to a full dynamical variable. When one considers (9.6) as a Lagrangian action and one starts the constraints analysis, an extra set of constraints appears. The momenta that are canonically conjugated to the λ^α have to vanish: $\pi_{\lambda^\alpha} = 0$. The set of constraints $(\pi_{\lambda^\alpha}, \phi_\alpha) = G_a$ is the set we will use. Notice that we could now introduce Lagrange multipliers for the constraints G_a, and repeat the procedure. This goes on ad infinitum. A discussion of this possibly infinitely nested structure and of the reason to treat the Lagrange multipliers as dynamical degrees of freedom can be found in respectively chapter 2 and 3 of [17]. With the extra pair of conjugate variables λ^α and π_{λ^α} and the extended set of constraints G_a, we still have

$$\begin{array}{rcl} [G_a, G_b] & = & C_{ab}^c G_c \\ [H_0, G_a] & = & V_a^b G_b. \end{array}$$ (9.9)

The second step in the BFV procedure is the construction of the *extended phase space*. For every constraint G_a a ghost field η^a is introduced. Its Grassmann parity is opposite to the Grassmann parity of the constraint it is associated with: $\epsilon_{\eta^a} = \epsilon_{G_a} + 1 = \epsilon_a + 1$. The momenta $\mathcal{P}_a$ canonically conjugated to η^a (i.e. $[\mathcal{P}_a, \eta^b] = -\delta_a^b$) have the same Grassmann parity as η^a. Furthermore, gh $(\eta) = 1$ and gh $(\mathcal{P}) = -1$.

In this extended phase space, one constructs the *generator of the BRST transformations* Ω, which acts on functions of the phase space variables. This Ω is also called the *BRST charge*. Ω is a functional of odd Grassmann parity and has ghostnumber 1. It has to satisfy

$$[\Omega, \Omega] = 0,$$ (9.10)

with the extra condition that

$$\Omega = \eta^a G_a + \dots$$ (9.11)

Depending on the symmetry algebra, more terms of higher order in the ghosts and their momenta may appear where the dots are. For any function $F(p, q)$, we have $\delta F = [F, \Omega] = [F, \eta^a G_a] + \ldots$ In analogy with the extended action of the Lagrangian BV formalism, Ω is only specified by these two conditions up to canonical transformations in the extended phase space [68].

The last important ingredient of the BFV formalism is the Hamiltonian, H. We impose that H is BRST invariant, i.e. we look for a solution of

$$[H, \Omega] = 0, \tag{9.12}$$

with the extra condition that $H = H_0 + \ldots$ and with $\mathrm{gh}\,(H) = 0$. Again, terms that are a function of the ghosts and their momenta may appear where the dots are.

If the structure functions C_{bc}^a and V_a^b are independent of the phase space variables, we have that:

$$\begin{aligned} \Omega &= \eta^a G_a - \frac{1}{2}(-1)^{\epsilon_b}\eta^b\eta^c C_{cb}^a \mathcal{P}_a \\ H &= H_0 + \eta^a V_a^b \mathcal{P}_b. \end{aligned} \tag{9.13}$$

Owing to (9.10), H is only determined up to a term $-[\Psi, \Omega]$. We define

$$H_{eff} = H - [\Psi, \Omega], \tag{9.14}$$

and Ψ is called the *gauge fermion*.

So far, the classical gauge system has been reformulated using BRST technology. The quantisation using functional integrals goes as follows. Consider the action

$$S_{eff} = \int dt \left[\dot{q}_n p_n + \dot{\lambda}^\alpha \pi_{\lambda\alpha} + \dot{\eta}^a \mathcal{P}_a - H_{eff} \right]. \tag{9.15}$$

The gauge fermion that is contained in H_{eff} is to be chosen such that all fields in this functional integral have propagators. The first three terms are determined by the symplectic form of the extended phase space. A general theorem, the Fradkin-Vilkovisky theorem, states that the path integral

$$\mathcal{Z}_\Psi = \int [d\mu] e^{\frac{i}{\hbar} S_{eff}} \tag{9.16}$$

is independent of the specific choice of the gauge fermion Ψ. The measure is the product over time of the Liouville measure that is defined on the phase space. As a corollary, it is easy to derive the form of the Ward identities in the Hamiltonian formalism. By considering $\Psi \to \Psi + \epsilon X$ for an infinitesimal parameter ϵ, we find that

$$\int [d\mu][X, \Omega] e^{\frac{i}{\hbar} S_{eff}} = 0. \tag{9.17}$$

This expression is the Hamiltonian version of the Ward identities (3.14).

Two final remarks are in order before applying the recipe in the next chapter. Although we introduced the η^a as ghost fields and the $\mathcal{P}_a$ as their momenta, the following reinterpretation will be made:

G_a	$\pi_{\lambda\alpha}$	ϕ_α
η^a	$-i^{\epsilon_\alpha+1}\mathcal{P}^\alpha$	c^α
$\mathcal{P}_a$	$i^{\epsilon_\alpha+1}b_\alpha$	$\bar{\mathcal{P}}_\alpha$

$$\tag{9.18}$$

Now the fields are taken to be the ghost c^α and the antighost b_α. The $\bar{\mathcal{P}}_\alpha$ and $\mathcal{P}^\alpha$ are considered to be momenta. The canonical Poisson brackets are $[\bar{\mathcal{P}}_\alpha, c^\beta] = [\mathcal{P}^\beta, b_\alpha] = -\delta_\alpha^\beta$.

One often chooses a gauge fermion Ψ of the form

$$\Psi = i^{\epsilon_\alpha+1} b_\alpha X^\alpha + \bar{\mathcal{P}}_\alpha \lambda^\alpha. \tag{9.19}$$

Here, the X^α are gauge fixing functions that do not depend on the ghosts, antighosts nor on the momenta of both.

This concludes our outline of the Hamiltonian treatment of gauge theories. In the next chapter, we first demonstrate this recipe by applying it to a Lagrangian with a shift symmetry. This serves as a preparation for the incorporation of the shift symmetry in a general Hamiltonian system with first class constraints. Thus we will prove the equivalence of the Hamiltonian BFV and Lagrangian BV formalism.

Appendix: Poisson brackets

Suppose that we can divide the coordinates q_n and momenta p_n in bosonic degrees of freedom (q^i, p_i) and fermionic degrees of freedom $(\theta^\alpha, \pi_\alpha)$. Here, q_i and θ_α are the coordinates, p_i and π_α the momenta. The canonical Poisson bracket of two phase space function F and G is then defined by:

$$
\begin{aligned}
[F, G] &= \frac{\overleftarrow{\delta} F}{\delta q^i} \frac{\overrightarrow{\delta} G}{\delta p_i} - \frac{\overleftarrow{\delta} F}{\delta p_i} \frac{\overrightarrow{\delta} G}{\delta q^i} \\
&\quad - \frac{\overleftarrow{\delta} F}{\delta \theta^\alpha} \frac{\overrightarrow{\delta} G}{\delta \pi_\alpha} - \frac{\overleftarrow{\delta} F}{\delta \pi_\alpha} \frac{\overrightarrow{\delta} G}{\delta \theta^\alpha}.
\end{aligned} \tag{9.20}
$$

The Poisson bracket is a bosonic bracket, $\epsilon_{[F,G]} = \epsilon_F + \epsilon_G$, and $\mathrm{gh}([F,G]) = \mathrm{gh}(F) + \mathrm{gh}(G)$. Furthermore, we have the following list of properties:

1. $\qquad [F, G] = (-1)^{\epsilon_F \epsilon_G + 1}[G, F]$
2. $\qquad [F, GH] = [F, G]H + (-1)^{\epsilon_F \epsilon_G} G[F, H]$
3. $[[F, G], H] + (-1)^{\epsilon_F(\epsilon_G + \epsilon_H)}[[G, H], F] + (-1)^{\epsilon_H(\epsilon_F + \epsilon_G)}[[H, F], G] = 0.$

$$\tag{9.21}$$

As a consequence of the definition of the Poisson brackets, the following derivative rules hold. Suppose that $x^k(p_n, q_n)$ are some phase space functions and suppose $F(x^k)$. Then

$$
\begin{aligned}
[F, G] &= \frac{\overleftarrow{\delta} F}{\delta x^k}[x^k, G] \\
[G, F] &= [G, x^k]\frac{\overrightarrow{\delta} F}{\delta x^k}.
\end{aligned} \tag{9.22}
$$

Chapter 10

The equivalence of Hamiltonian BFV and Lagrangian BV

We now show [69] how the Lagrangian antifield formalism can be derived from the Hamiltonian BFV formalism that is described in the previous chapter. The transition from the Hamiltonian to the Lagrangian description of systems with gauge symmetries comprises two aspects. The first, and this is not restricted to gauge theories, concerns the question how the Hamiltonian description, with its special treatment of the time coordinate, leads to the Lorentz covariant Lagrangian formalism. We will not address this question. Let us only mention that the Lagrange multipliers of the Hamiltonian formalism (9.6) are considered dynamical fields for exactly the purpose of Lorentz covariance in the Lagrangian formulation. For instance, in Maxwell's theory for electromagnetism, the fourth component A_0 of the Lorentz vector is a Lagrange multiplier that imposes the constraint $\partial_i F^{0i} = 0$ (Gauss' law). A detailed discussion of this example can be found in [17]. The second aspect is the relation between the Hamiltonian BFV and the Lagrangian BV formalisms themselves. Our approach will be to enlarge the BRST symmetry of the Hamiltonian system with the Schwinger-Dyson shift symmetry. This way, we introduce the antifields (antighosts of the shift symmetry) in the Hamiltonian path integral. Integrating out the momenta of the Hamiltonian formalism, the gauge fixed action of the BV formalism is obtained. The Fradkin-Vilkovisky theorem, which guarantees that the Hamiltonian path integral is independent of the gauge fermion, is shown to imply the BV quantum master equation.

Given the importance of this equivalence, a large effort has already been devoted to its study. In [70, 71], an approach different from ours, is followed. There, the starting point is the first order action (9.6) of the previous chapter, which is treated as any other Lagrangian using the BV antifield scheme. The ϕ^i of the BV formalism are the phase space variables of the Hamiltonian scheme, i.e. the fields *and* their momenta. The basic observation is that an extended action that satisfies the classical master equation can be obtained by taking as antifield dependent terms $\phi_i^*[\phi^i, \Omega]$ if $[\Omega, \Omega] = 0$. By gauge fixing, i.e. by replacing the antifields of the fields and the momenta of the Hamiltonian formalism by derivatives with respect to a gauge fermion, the action (9.15) as prescribed by

the BFV formalism is reobtained [71]. An analogous, although less general, result is described in [72].

The approach we follow is closer to that of [73], where the transition from the Hamiltonian to Lagrangian formalism was made directly at the path integral level. However, in [73], the antifields are introduced in a rather ad hoc way as sources for the BRST transformations and the Lagrangian quantum master equation requires a seperate proof, whereas one would expect it to follow from the Fradkin-Vilkovisky theorem. Our presentation below clarifies exactly these two points.

As a preparation to the equivalence proof, we first derive the Schwinger-Dyson equation as Ward identity in the Hamiltonian formalism. In the second section of this chapter we present our proof of the equivalence of the Hamiltonian and Lagrangian formalism.

10.1 Schwinger-Dyson equation in the BFV formalism

Consider a Lagrangian depending on fields and their time-derivatives and which describes a system without gauge symmetries: $L(t) = L(\phi^a, \dot{\phi}^a)$. Introducing collective fields amounts here to considering $L(\phi^a - \varphi^a, \dot{\phi}^a - \dot{\varphi}^a)$. The momenta conjugate to ϕ^a are denoted by π_a and to φ^a by ϖ_a. The first class constraints are

$$\chi_a = \pi_a + \varpi_a = 0. \tag{10.1}$$

The structure constants C_{bc}^a of the constraints algebra vanish, $[\chi_b, \chi_c] = 0$, since $[\pi_a, \phi^b] = [\varpi_a, \varphi^b] = -\delta_a^b$ are the only non-vanishing Poisson brackets. The constraints are clearly first class. These constraints also have a vanishing Poisson bracket with the Hamiltonian, as the latter only depends on the difference $\phi^a - \varphi^a$. This is equivalent to saying that the structure constants V_a^b vanish. With every constraint χ_a, we associate one Lagrange multiplier λ^a and its canonical momentum $\pi_{\lambda a}$ ($[\pi_{\lambda a}, \lambda^b] = -\delta_a^b$). The complete set of constraints $G_a = (\pi_{\lambda a}, \chi_a)$ still has vanishing structure constants. We construct the extended phase space, following the prescription of the previous chapter. The ghost and antighost fields are introduced as in (9.18):

$$\begin{array}{c|c|c|c} G_a & \pi_{\lambda^a} & \chi_a & \text{ghnr.} \\ \hline \eta^a & -i^{\epsilon_a+1}\mathcal{P}^a & c^a & 1 \\ P_a & i^{\epsilon_a+1}\phi_a^* & \bar{\mathcal{P}}_a & -1 \end{array} \tag{10.2}$$

with the only non-vanishing brackets $[\bar{\mathcal{P}}_a, c^b] = [\mathcal{P}^b, \phi_a^*] = -\delta_a^b$. The Grassmann parities are as follows: $\epsilon_{\phi^a} = \epsilon_{\varphi^a} = \epsilon_{\pi_a} = \epsilon_{\varpi_a} = \epsilon_{\lambda^a} = \epsilon_{\pi_{\lambda a}} = a$ and $\epsilon_{\mathcal{P}^a} = \epsilon_{\bar{\mathcal{P}}_a} = \epsilon_{c^a} = \epsilon_{\phi_a^*} = a + 1$. Notice that ϕ_a^* denotes again the antighost. In the extended phase space, one can straightforwardly construct the BRST generator of the shift symmetries. All structure constants vanish, so we have $\Omega_s = \eta^a G_a$, which gives

$$\Omega_s = -i^{\epsilon_a+1}\mathcal{P}^a \pi_{\lambda a} + c^a(\pi_a + \varpi_a). \tag{10.3}$$

Indeed, $[\phi^a, \Omega_s] = [\varphi^a, \Omega_s] = (-1)^{\epsilon_a} c^a$. Since $[H_0, \Omega_s] = 0$, we have that $H = H_0$. Gauge fixing the collective field to zero can be done by taking as gauge fermion

$$\Psi_s = \bar{\mathcal{P}}_a \lambda^a + \frac{i^{\epsilon_a+1}}{\beta} \phi_a^* \varphi^a, \tag{10.4}$$

where β is an arbitrary parameter. We will take the $\beta \to 0$ limit later. This is a standard procedure in the Hamiltonian formalism. See e.g. [68], see also [17] for a critical examination of this procedure. The Poisson bracket of this gauge fermion with the BRST-charge gives

$$[\Psi_s, \Omega_s] = i^{\epsilon_a+1} \mathcal{P}^a \bar{\mathcal{P}}_a - \frac{1}{\beta} \varphi^a \pi_{\lambda a} - \lambda^a (\pi_a + \varpi_a) + \frac{i^{\epsilon_a+1}(-1)^{\epsilon_a}}{\beta} \phi_a^* c^a. \quad (10.5)$$

The action to be used in the path integral, is then given by (9.15)

$$S = \dot{\phi}^a \pi_a + \dot{\varphi}^a \varpi_a + \dot{\lambda}^a \pi_{\lambda a} + \dot{\eta}^a P_a - H(\pi, \varpi, \phi - \varphi) + [\Psi, \Omega_s]. \quad (10.6)$$

An integration over time is understood. We want to construct a Ward identity like (9.17), with $X = \frac{i^{\epsilon_a+1} \phi_a^*}{\beta} F(\phi)$. We calculate

$$[X, \Omega_s] = \frac{i^{\epsilon_a+1} \phi_a^*}{\beta} \frac{\overleftarrow{\delta} F}{\delta \phi^b} (-1)^{\epsilon_b} c^b - \frac{1}{\beta} (-1)^{\epsilon_F} \pi_{\lambda a} F. \quad (10.7)$$

We now redefine[1]

$$\frac{1}{\beta} \pi_{\lambda a} \quad \to \quad \pi_{\lambda a}$$

$$\frac{i^{\epsilon_a+1}}{\beta} \phi_a^* \quad \to \quad \phi_a^* \quad\quad (10.8)$$

$$(-1)^{\epsilon_a} c^a \quad \to \quad c^a.$$

After this rescaling, we take the limit $\beta \to 0$. Thereafter, the momenta of the ghosts, $\mathcal{P}^a$ and $\bar{\mathcal{P}}_a$, can be integrated out trivially, leading to

$$S = \dot{\phi}^a \pi_a + \dot{\varphi}^a \varpi_a - H(\pi, \varpi, \phi - \varphi) - \varphi^a \pi_{\lambda a} - \lambda^a (\pi_a + \varpi_a) + \phi_a^* c^a. \quad (10.9)$$

The first three terms are grouped in S_1, the other three in S_2. S_2 clearly removes all the extra fields of the collective field formalism. Indeed, integration over $\pi_{\lambda a}$ gives a delta-function fixing the collective field to zero, integration over the Lagrange multiplier λ^a leads to a delta-function imposing the constraint χ_a and integration over c^a gives a delta-function fixing the antifield to zero. However, in the path integral for the expectation value of $[X, \Omega_s]$, these integrations can not be done immediately. After the rescaling (10.8), we have

$$[X, \Omega_s] = \phi_a^* \frac{\overleftarrow{\delta} F}{\delta \phi^b} c^b - (-1)^{\epsilon_F} \pi_{\lambda a} F. \quad (10.10)$$

The Ward identity hence becomes

$$0 = \int [d\phi][d\varphi][d\pi][d\varpi][dc][d\phi^*][d\lambda][d\pi_\lambda] e^{\frac{i}{\hbar} S_1}$$

$$\times \frac{\hbar}{i} \left[\phi_a^* \frac{\overleftarrow{\delta} F}{\delta \phi^b} \frac{\overrightarrow{\delta} e^{\frac{i}{\hbar} S_2}}{\delta \phi_b^*} + (-1)^{\epsilon_F} \frac{\overrightarrow{\delta} e^{\frac{i}{\hbar} S_2}}{\delta \varphi^a} F \right]. \quad (10.11)$$

In the first term, we integrate by parts over ϕ^*. Thereafter, the integrations contained in S_2 can be done. In the second term, we integrate by parts over

[1] Formally, this redefinition leads to a β independent Jacobian in the path integral measure as the two fields that are rescaled by the inverse of β have opposite Grassmann parity. However, these formal manipulations may require a more careful treatment on a case by case basis. See [74].

φ, which again allows to do the integrations leading to $\delta(\varphi)\delta(\pi + \varpi)\delta(\phi^*)$. We integrate out the momentum ϖ of the collective field, after which we can replace the derivative with respect to φ by minus a derivative over ϕ. If we then integrate out the momenta π_a of the original fields and define

$$\exp[\frac{i}{\hbar}\mathcal{S}] = \int [d\pi_a]\exp\left[\frac{i}{\hbar}\left(\int dt\,\dot{\phi}^a\pi_a - H(\pi, -\pi, \phi)\right)\right], \qquad (10.12)$$

we find back the Schwinger-Dyson equation

$$\int [d\phi]\ e^{\frac{i}{\hbar}\mathcal{S}}\left[F\frac{\overleftarrow{\delta}\mathcal{S}}{\delta\phi^a} + \frac{\hbar}{i}\frac{\overleftarrow{\delta}F}{\delta\phi^a}\right] = 0. \qquad (10.13)$$

This parallels the result of chapter 4.

10.2 From Hamiltonian BFV to Lagrangian BV via Schwinger-Dyson symmetry

Now we turn to the case of an Hamiltonian system with gauge symmetries. We start from an extended phase space, on which a nilpotent BRST charge $\Omega_0(\Pi_A, \phi^A)$ and a BRST invariant Hamiltonian $H_0(\Pi_A, \phi^A)$ are defined. More specifically, the extended phase space has coordinates $(\phi^i, c^\alpha, b_\alpha, \lambda^\alpha$ for the case of first class irreducible theories) which we collectively denote by ϕ^A, and conjugate momenta $(\pi_i, \bar{\mathcal{P}}_\alpha, \mathcal{P}^\alpha, \pi_{\lambda\alpha})$ denoted by Π_A. The fundamental Poisson bracket is $[\Pi_A, \phi^B] = -\delta_A^B$. We use the specific field content of ϕ^A and Π_A only in the end. The BRST-charge and the the BRST-invariant Hamiltonian satisfy the standard conditions $[\Omega_0, \Omega_0] = [H_0, \Omega_0] = 0$.

We double the phase space by introducing for every field ϕ^A a collective field φ^A which has the conjugate momentum Υ_A. The fundamental Poisson brackets are also copied. We take $[\Upsilon_A, \varphi^B] = -\delta_A^B$. With every function $F(\Pi_A, \phi^A)$ defined on the original extended phase space, we associate a function $\widetilde{F} = F(-\Upsilon_A, \phi^A - \varphi^A)$, defined on the doubled extended phase space. The following two properties are then easily seen to hold:

$$[\widetilde{F}, \widetilde{G}] = [\widetilde{F, G}], \qquad (10.14)$$

and

$$[\widetilde{F}, \chi_A] = 0. \qquad (10.15)$$

Again, $\chi_A = \Pi_A + \Upsilon_A$ is the constraint that generates the shift symmetry. In the case that the original extended phase space has an arbitrary symplectic form $[\Pi_A, \phi^B] = \omega_A^B(\Pi, \phi)$ that depends on the phase space variables, the brackets in the doubled extended phase space have to be defined as $[\Pi_A, \phi^B] = \tilde{\omega}_A^B$ and $[\Upsilon_A, \varphi^B] = \tilde{\omega}_A^B$. This way, the two crucial properties (10.14,10.15) on which all following developments are based, can be generalised. Notice that the bracket on the RHS of (10.14) is the one defined in the original extended phase space.

The next step is to see how the BRST charge and the Hamiltonian have to be modified to take the new symmetry (the shift symmetry) into account. We look for

$$\Omega = \widetilde{\Omega_0} + \Omega_s \qquad (10.16)$$

$$H = \widetilde{H_0} + \Delta H \qquad (10.17)$$

and demand that $[\Omega, \Omega] = [H, \Omega] = 0$, with the extra condition that Ω generates *all* BRST symmetries, also the BRST shift symmetry. For that purpose, we construct the extended phase space[2] as in the case of no internal symmetries discussed above:

G_A	π_{λ^A}	χ_A	ghnr.
η^A	$-i^{\epsilon_A+1}\mathcal{P}^A$	c^A	1
P_A	$i^{\epsilon_A+1}\phi_A^*$	$\bar{\mathcal{P}}_A$	-1

$$(10.18)$$

This table summarises the following, by now familiar, steps (9.18). For every shift constraint χ_A we have introduced a Lagrange multiplier λ^A and its conjugate momentum π_{λ^A}. The latter is constrained to zero. For the set of constraints $G_A = (\pi_{\lambda^A}, \chi_A)$ we have added ghosts, antighosts and their momenta. As usual, the fundamental Poisson brackets are defined $[\mathcal{P}^B, \phi_A^*] = [\bar{\mathcal{P}}_A, c^B] = -\delta_A^B$.

The total BRST-charge is obtained by taking for Ω_s the expression of the case without gauge symmetries (10.3):

$$\Omega = \widetilde{\Omega_0} - i^{\epsilon_A+1}\mathcal{P}^A\pi_{\lambda^A} + c^A(\Pi_A + \Upsilon_A). \qquad (10.19)$$

This quantity satisfies $[\Omega, \Omega] = 0$ owing to (10.14) and (10.15). It is also clear that $\Delta H = 0$, i.e. $H = \widetilde{H_0}$. In fact, these two results simply reflect the vanishing of the structure constants associated with the Poisson brackets of the original constraints and of the original Hamiltonian with the constraints of the shift symmetry, when the former are evaluated in $(-\Upsilon_A, \phi^A - \varphi^A)$.

Let us calculate the BRST transformations of the fields ϕ^A and of the collective fields φ^A. This clarifies the meaning of the abstract construction above. Moreover, it *explains* why the $\tilde{F}$ operation also involves a substitution of the momenta by minus the momenta of the collective fields. The fields only transform under the shift symmetry

$$[\phi^A, \Omega] = (-1)^{\epsilon_A}c^A, \qquad (10.20)$$

while the original gauge transformations have shifted to the collective fields

$$[\varphi^A, \Omega] = (-1)^{\epsilon_A}c^A + [\varphi^A, \widetilde{\Omega_0}]. \qquad (10.21)$$

It is precisely by our momentum substitution rule that the BRST transformations of the originally present gauge symmetries end up entirely in the collective field transformation. This is analogous to what we did in the construction of the BV scheme from BRST quantisation (5.3). Here, we have no a priori reason for doing this, contrary to our discussion in the chapters 5,6 and 7. There the BRST (–anti-BRST) transformation rules were organised this way in order to be able to gauge fix theories with an open gauge algebra in the same way as theories with a closed algebra. Making the same choice here, we will obtain terms where the antifields act as sources for the BRST transformation rules of their associated field.

To gauge fix the collective field to zero, we use again (10.4)

$$\Psi_s = \bar{\mathcal{P}}_A\lambda^A + \frac{i^{\epsilon_A+1}}{\beta}\phi_A^*\varphi^A , \qquad (10.22)$$

---

[2]In fact we mean the extended phase space associated with the shift symmetry in the doubled extended phase space, but for obvious linguistic reasons we speak of *the extended phase space.*

leading to

$$
\begin{aligned}
{[\Psi_s, \Omega]} &= i^{\epsilon_A+1}\mathcal{P}^A\bar{\mathcal{P}}_A - \frac{1}{\beta}\varphi^A\pi_{\lambda^A} - \lambda^A(\Pi_A + \Upsilon_A) \\
&\quad + \frac{i^{\epsilon_A+1}(-1)^{\epsilon_A}}{\beta}\phi_A^* c^A + \frac{i^{\epsilon_A+1}}{\beta}\phi_A^*[\varphi^A, \widetilde{\Omega_0}].
\end{aligned}
\tag{10.23}
$$

We can rewrite the last term as

$$
[\varphi^A, \widetilde{\Omega_0}] = -[\widetilde{\phi^A}, \widetilde{\Omega_0}] = -[\widetilde{\phi^A, \Omega_0}],
\tag{10.24}
$$

since $\widetilde{\Omega_0}$ does not depend on the momenta conjugate to ϕ^A.

It only remains to gauge fix the original symmetries. This can be done by taking a fermion $\Psi(\Pi_A, \phi^A)$ and adding

$$
[\widetilde{\Psi}, \Omega] = [\widetilde{\Psi, \Omega_0}].
\tag{10.25}
$$

Before taking the $\beta \to 0$ limit, we scale like in (10.8), with a replacement of the indices a by A. This leads to the standard BFV action (9.15)

$$
\begin{aligned}
S &= \dot{\phi}^A\Pi_A + \dot{\varphi}^A\Upsilon_A + \beta\dot{\lambda}^A\pi_{\lambda^A} - \beta i^{\epsilon_A+1}\dot{\mathcal{P}}^A\phi_A^* + (-1)^A\dot{c}^A\bar{\mathcal{P}}_A \\
&\quad - \widetilde{H_0} + i^{\epsilon_A+1}\mathcal{P}^A\bar{\mathcal{P}}_A - \varphi^A\pi_{\lambda^A} - \lambda^A(\Pi_A + \Upsilon_A) \\
&\quad + \phi_A^* c^A - \phi_A^*[\widetilde{\phi^A, \Omega_0}] + [\widetilde{\Psi, \Omega_0}].
\end{aligned}
\tag{10.26}
$$

Taking the limit $\beta \to 0$, the integrations over the ghost momenta $\mathcal{P}$ and $\bar{\mathcal{P}}$ become trivial. Integrating over the momenta π_{λ^A} of the Lagrange multipliers, we obtain a delta-function $\delta(\varphi)$ while integrating out the Lagrange multipliers λ^A themselves leads to $\delta(\Pi_A + \Upsilon_A)$, imposing the constraint. It is trivial to see that these two delta-function constraints allow us to drop the tildes upon integration over the collective field and its momentum. We obtain in this way

$$
S = \dot{\phi}^A\Pi_A - H_0 - \phi_A^*[\phi^A, \Omega_0] + \phi_A^* c^A + [\Psi, \Omega_0].
\tag{10.27}
$$

Again, the antighosts (antifields) start acting as a source term for the BRST transformations of ϕ^A.

We will now integrate out the momenta of the Hamiltonian formalism to obtain an action that satisfies the Lagrangian quantum master equation. In order to do that, we rewrite (10.27) in a more useful form. We make the most popular choice for the gauge fermion[3] (9.19)

$$
\Psi = \Psi_0(\phi^A, \pi_{\lambda\alpha}) + \bar{\mathcal{P}}_\alpha\lambda^\alpha.
\tag{10.28}
$$

Ω_0 is generally of the form

$$
\Omega_0 = -i^{\epsilon_\alpha+1}\mathcal{P}^\alpha\pi_{\lambda\alpha} + \Omega_{min},
\tag{10.29}
$$

where Ω_{min} does not depend on the Lagrange multiplier λ^α nor on its momentum $\pi_{\lambda\alpha}$ [68]. Taking these two facts into account, the terms for the gauge fixing of the original gauge symmetries are of the form:

$$
[\Psi, \Omega_0] = \frac{\overleftarrow{\delta}\Psi_0}{\delta\phi^A}[\phi^A, \Omega_0] + i^{\epsilon_\alpha+1}\mathcal{P}^\alpha\bar{\mathcal{P}}_\alpha + [\bar{\mathcal{P}}_\alpha\lambda^\alpha, \Omega_{min}].
\tag{10.30}
$$

---

[3]The Lagrange multipliers, ghosts and antighosts that appear below, must not be confused with the analogous fields for the shift symmetries, which have been integrated out above. The former have indices α, while the latter had A. In order to proceed further we have for the first time to specify in detail what fields are contained in ϕ^A and Π_A.

The important term here is the first one. It is now convenient to define $\hat{S}$ by

$$S = \widehat{S}(\phi^A, \phi^*_A, \pi_{\lambda\alpha}, \mathcal{P}^\alpha, \bar{\mathcal{P}}_\alpha, \pi_i) + \phi^*_A c^A - \frac{\overleftarrow{\delta}\Psi_0}{\delta\phi^A}\frac{\overrightarrow{\delta}\widehat{S}}{\delta\phi^*_A}. \tag{10.31}$$

Let us stress that $\widehat{S}$ is linear in the antifields (10.27), which allows us to write the last term the way we do. The partition function, obtained by Hamiltonian methods is then seen to be given by

$$\mathcal{Z} = \int [d\phi^A][d\pi_{\lambda\alpha}][d\phi^*_A][dc^A][d\mathcal{P}^\alpha][d\bar{\mathcal{P}}_\alpha][d\pi_i] \exp\left[\frac{i}{\hbar}\phi^*_A c^A\right] \hat{U}\left(\exp\left[\frac{i}{\hbar}\widehat{S}\right]\right). \tag{10.32}$$

The differential operator $\hat{U}$ produces the gauge fixing term and is defined by

$$\hat{U} = \exp\left[-\frac{\overleftarrow{\delta}\Psi_0}{\delta\phi^A}\frac{\overrightarrow{\delta}}{\delta\phi^*_A}\right]. \tag{10.33}$$

Define the quantum extended action W by

$$e^{\frac{i}{\hbar}W(\phi,\phi^*)} = \int [d\mathcal{P}^\alpha][d\bar{\mathcal{P}}_\alpha][d\pi_i] \exp[\frac{i}{\hbar}\widehat{S}]. \tag{10.34}$$

The operator $\hat{U}$ commutes with the integrations over the momenta that define W. Integrating out the momenta, the partition function becomes

$$\mathcal{Z} = \int [d\phi^A][d\pi_{\lambda\alpha}][d\phi^*_A][dc^A] \exp\left[\frac{i}{\hbar}\left(\phi^*_A c^A + W(\phi^A, \phi^*_A - \frac{\overleftarrow{\delta}\Psi_0}{\delta\phi^A})\right)\right]. \tag{10.35}$$

The shift in the antifields is obtained by using the well-known relation $\exp\left[a(y)\frac{\delta}{\delta x}\right] f(x) = f(x + a(y))$. We have now recovered the form of the BV path integral (5.10), as the integration over the ghost c^A leads to a $\delta(\phi^*_A)$, removing the antifields. In contrast with [73], we do not integrate over the Lagrange multipliers, but include them together with their momenta in the set of degrees of freedom of the obtained Lagrangian system. Notice that W need not be linear in the antifields. This suggests that the non-linear terms in the antifields that are typical for open algebras in the BV scheme, appear when integrating out the momenta.

It is now trivial to show that W satisfies the BV quantum master equation. The Fradkin-Vilkovisky theorem states that changing the gauge fermion Ψ_0 to $\Psi_0 + d\Psi$ leaves the partition function $\mathcal{Z}$ invariant. For an infinitesimal change in the gauge fermion we have

$$\int [d\phi^A][d\phi^*_A][dc^A][d\pi_{\lambda\alpha}] \; e^{\frac{i}{\hbar}\phi^*_A c^A} \cdot \frac{\overleftarrow{\delta}\,d\Psi}{\delta\phi^A} \cdot \frac{\overrightarrow{\delta}}{\delta\phi^*_A} \exp[\frac{i}{\hbar}\hat{W}] = 0, \tag{10.36}$$

for any choice of $d\Psi$. We denoted $W(\phi^A, \phi^*_A - \frac{\overleftarrow{\delta}\Psi_0}{\delta\phi^A}) = \hat{W}$. As $d\Psi$ is completely arbitrary, a partial integration gives us the by now well-known quantum master equation:

$$\Delta e^{\frac{i}{\hbar}\hat{W}} = 0. \tag{10.37}$$

Let us finally try to get a better understanding of W defined in (10.34). We consider the familiar expansion in $\hbar$ (5.39)

$$\hat{W} = S_0 + \sum_{i=1}^{+\infty} \hbar^i M_i. \tag{10.38}$$

We will only discuss S_0, which satisfies the classical master equation $(S_0, S_0) = 0$. It can be calculated by applying the saddle-point approximation to the momentum integrals in (10.34). Solving the field equations (the equations that determine the extrema of the integrand)

$$\frac{\delta \hat{S}}{\delta \mathcal{P}^\alpha} = \frac{\delta \hat{S}}{\delta \bar{\mathcal{P}}_\alpha} = \frac{\delta \hat{S}}{\delta \pi_i} = 0 \qquad (10.39)$$

leads to functions $\mathcal{P}^\alpha(\phi^A, \phi^*_A, \pi_{\lambda\alpha})$, $\bar{\mathcal{P}}_\alpha(\phi^A, \phi^*_A, \pi_{\lambda\alpha})$ and $\pi_i(\phi^A, \phi^*_A, \pi_{\lambda\alpha})$. When we plug in these solutions, we denote this by $|_\Sigma$. Clearly,

$$S_0(\phi, \phi^*, \pi_{\lambda\alpha}) = \hat{S}|_\Sigma. \qquad (10.40)$$

Finally, we have that the BRST transformations in the Lagrangian formalism (5.3) are

$$\mathcal{R}^A[\phi] = \left[\frac{\overrightarrow{\delta} S_0}{\delta \phi^*_A}\right]_{\phi^*=0} = -[\phi^A, \Omega_0]_{\Sigma, \phi^*=0} + \ldots \qquad (10.41)$$

This finishes our proof of the equivalence of the Hamiltonian BFV and the Lagrangian BV formalism. Our guiding principle was that the Schwinger-Dyson shift symmetry allows for a natural introduction of antifields. Using the prescriptions of the BFV scheme to implement the Schwinger-Dyson BRST symmetry, we see that the presence of the antifields need not be restricted to Lagrangian BV. However, integrating out the momenta leads straightforwardly to an interpretation like that of the Lagrangian scheme of BV. The Lagrangian action we have in the end, satisfies the BV quantum master equation, as a result of the Fradkin-Vilkovisky theorem. We thus have linked the two principles which assure that the Hamiltonian and Lagrangian method can be used for quantising gauge theories at all, namely, that the partition functions constructed following their prescription, are independent of the chosen gauge fixing.

Part III

A regularised study of anomalies in BV

Chapter 11

Gauge anomalies and BV

The third part of this dissertation is devoted entirely to one-loop aspects of the quantisation of gauge theories. We will especially focus on the occurrence of gauge anomalies (for a first definition, see the discussion following (3.23)) and on their description in the BV scheme. In this chapter, we first highlight some aspects of gauge anomalies in the case of a closed algebra. In a second part of this chapter, we rephrase gauge anomalies in the BV terminology, which generalises the preceding results to all types of algebras. In this chapter, we only work on the formal level, that is, without using a regularisation scheme. This will be remedied in extenso in the following chapters. Regularisation should be one of the cornerstones of any discussion on anomalies. In the next chapter, we introduce a one-loop regularisation scheme for the path integral, which is then used to derive a regularised version of the BV quantum master equation. In the final chapter, the previously discussed regularisation techniques are applied to study how extra fields can be introduced to keep some preferred symmetries anomaly free.

11.1 Gauge anomalies in quantum field theory

In general, when a symmetry of the classical action, be it a local or a global one, is not a symmetry of the effective action owing to quantum corrections, then this symmetry is said to be *anomalous* or *anomalously broken*. Ideally, one would like to turn this procedure around and define a quantum theory with a certain set of symmetries. This set may then enlarge upon taking the classical limit. This way, the negative consequences of gauge anomalies, which we will point out below, maybe are to be seen as an artifact of our as yet incomplete understanding of quantisation of gauge theories [75].

We study *perturbative gauge anomalies*[1], i.e. anomalies in local symmetries. Usually, the classical gauge symmetry leads to BRST symmetry of the gauge fixed action and of the quantum theory (see chapter 3). When the product of the measure of the path integral and the exponential of the (quantum) action can not be made BRST invariant, we have a gauge anomaly, see (3.23). A crucial ingredient is the Jacobian of the path integral measure under BRST

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[1]*Perturbative* here is in contrast with non-perturbative anomalies, so-called *global anomalies* [76].

transformations. When the Jacobian is different from 1, we can distinguish two cases. Either the Jacobian can be cancelled by the addition of a *local* quantum counterterm to the action, or it can not be cancelled this way. We will loosely speak of *anomaly* when the Jacobian is different from 1 and denominate a Jacobian that can not be countered by adding a *local* quantum counterterm, a *genuine anomaly*. Genuine anomalies lead to a quantum correction to the Ward indentity. The consequences are that the partition function becomes gauge dependent and that the customary proofs of renormalisability and unitarity of gauge theories are jeopardized. The other way around, imposing the absence of anomalies has served as a criterion in the selection of *healthy* theories. Here, the famous example is of course the structure of the matter families in the standard model of electroweak interactions. Within one generation of matter fields, the contributions of the different particles to the anomaly cancel. This requires a careful choice of the representations of the gauge group under which both chiralities of the fermions transform and gives evidence for the existence of quarks in three colours [77]. Also, the interest in string theory was triggered by the observation that the anomalies in these models can be cancelled by working in specific space-time dimensions [78].

The most important consequence of a gauge anomaly, which manifests itself time and again, is the fact that degrees of freedom that can classically be fixed to zero by a choice of gauge, start propagating in the quantum theory. This can be seen as follows. Suppose that we start from a classical action $S_0[\phi^i]$, with gauge generators R^i_α that form a closed algebra. We take both the ϕ^i and the gauge generators to be bosonic for the sake of the argument. Enlarge the configuration space, as usual, with ghosts c^α and the trivial systems consisting of the antighosts b_α and the Lagrange multipliers λ^α. On this enlarged configuration space, the nilpotent BRST operator δ is defined (see the chapters 2 and 3). In order to gauge fix, one chooses as many gauge fixing functions, denoted e.g. by F^α, as there are gauge symmetries. The choice of the F^α can be interpreted as the selection of degrees of freedom of the system that will be fixed to zero. By taking as gauge fermion $\Psi = b_\alpha F^\alpha$, the gauge fixed action is of the form (3.12), with $a = 0$ in (3.13). Instead of fixing these degrees of freedom F^α to zero, we could fix them on any configuration θ^α with the gauge fermion

$$\Psi_\theta = b_\alpha(F^\alpha - \theta^\alpha). \tag{11.1}$$

The gauge fixed path integral then becomes

$$\mathcal{Z}_\theta = \int [d\phi^A] e^{\frac{i}{\hbar}(S_0 + \delta\Psi_\theta)}. \tag{11.2}$$

By taking all possible configurations for θ^α, we cover a range of gauge fixings. When there is no anomaly, one has that

$$\frac{\delta \mathcal{Z}_\theta}{\delta \theta^\alpha} = 0. \tag{11.3}$$

However, in the case of an anomaly, it follows from (3.18,3.23), that

$$\frac{\delta \mathcal{Z}_\theta}{\delta \theta^\alpha} = \frac{i}{\hbar} \langle \mathcal{A} b_\alpha \rangle. \tag{11.4}$$

The different gauge choices, i.e. the different choices for the configurations θ^α give now different partition functions. It is then natural to consider the θ^α as new degrees of freedom and to include an integration over them in the functional integral. We will come back to these extra degrees of freedom in the last chapter of this dissertation.

Let us now derive the *Wess-Zumino consistency condition* for anomalies [79]. Consider a theory where the fields can be divided in matter fields ϕ^i and external gauge fields A^a, and where the classical action $S[\phi, A]$ has a gauge invariance. This classical gauge invariance is traded for BRST invariance. δ denotes the BRST transformation. Define

$$\exp[\frac{i}{\hbar}W[A]] = \int [d\phi] e^{\frac{i}{\hbar}S[\phi,A]+iM}, \tag{11.5}$$

where we have included a possible *local* counterterm M. We suppose that the matter fields have well-defined propagators and therefore, we do not need to gauge fix. Despite all these restrictions, in many interesting examples these requirements are satisfied (see next chapter for an example). Consider now

$$
\begin{aligned}
\delta e^{\frac{i}{\hbar}W[A]}.\mu &= \int [d\phi] \exp\left[\frac{i}{\hbar}S[\phi, A + \delta A.\mu] + iM[\phi, A + \delta A.\mu]\right] \\
&\quad - \int [d\phi] \exp\left[\frac{i}{\hbar}S[\phi, A] + iM\right],
\end{aligned}
\tag{11.6}
$$

where μ is again a space-time independent, Grassmann odd parameter. As in the derivation of the Ward identities (section 2 of chapter 3), we can now redefine the integration variables in the first integral, and use that S is BRST invariant to obtain

$$e^{\frac{i}{\hbar}W[A]}.\frac{i}{\hbar}\delta W[A].\mu = \int [d\phi] e^{\frac{i}{\hbar}S+iM}(\mathcal{A} + i\delta M)\mu, \tag{11.7}$$

where $\mathcal{A}$ is defined as in (3.21). If neither $\mathcal{A}$ nor M depend on the matter fields ϕ^i, we are led to

$$\frac{i}{\hbar}\delta W[A] = \mathcal{A} + i\delta M. \tag{11.8}$$

Again, if we can find a *local* M such that the RHS of (11.8) is zero, the naively expected result $\delta W[A] = 0$ is obtained. Whether one can or can not find such an M, we always have that

$$\delta \mathcal{A} = 0, \tag{11.9}$$

owing to the nilpotency of the BRST operator for closed algebras. This condition is the *Wess-Zumino consistency condition*. The logarithm of the Jacobian of the measure of the path integral under a BRST transformation is BRST invariant. $\mathcal{A}$ is called a *consistent anomaly*. A different form of the anomaly has been introduced and used in [80], the so-called *covariant anomaly*. We restrict our attention to consistent anomalies and we will introduce a regularisation scheme in next chapter that gives consistent anomalies.

Notice that $\mathrm{gh}(\mathcal{A}) = 1$, since it is the BRST variation of $W[A]$ and M, both of ghostnumber zero. In a theory with an irreducible gauge algebra, the ghosts c^α are the only fields with ghostnumber 1. In such cases, the general form of $\mathcal{A}$ is given by

$$\mathcal{A} = c^\alpha \mathcal{A}_\alpha, \tag{11.10}$$

with $\mathrm{gh}(\mathcal{A}_\alpha) = 0$.

As a result of all the above, it is sometimes possible to determine whether a theory is anomaly free or is possibly anomalous without doing perturbative calculations. Indeed, if all BRST invariant functions of ghostnumber 1 are BRST exact, then there can be no consistent genuine anomaly as any possible Jacobian can then be neutralised by choosing a counterterm. On the other hand, if there

exist BRST invariant functions of ghostnumber 1 that are not BRST exact, one only knows that the theory is possibly anomalous. The mathematical structure to investigate is clearly the cohomology of the BRST operator at ghostnumber 1. Notice however, that the actual expression for the local counterterm or the anomaly in a specific regularised calculation, can not be obtained from cohomological arguments. As we show on a few examples in the next chapters, these actual expressions depend on the regularisation scheme that one uses.

The consistency condition has been used as a starting point for geometric approaches to anomalies [80, 81]. This has led to an input of results from mathematics, centered around the Atiyah-Singer index theorems. The interested reader is referred to [82] for a physicist's introduction.

In the next section, we first indicate how genuine anomalies manifest themselves in the context of BV [63]. The Wess-Zumino condition for a consistent anomaly naturally appears there.

11.2 Gauge anomalies in BV

In this section, we translate the results of the previous section in the language of BV [63]. This generalises them at the same time to all types of gauge algebras.

As was pointed out in the previous section, if the theory has a genuine gauge anomaly, the partition function becomes gauge dependent. One has that (3.23),

$$\mathcal{Z}_{\Psi+d\Psi} - \mathcal{Z}_\Psi = \frac{i}{\hbar}\langle \mathcal{A}d\Psi\rangle. \tag{11.11}$$

This is easily compared with (8.17) for $X = 1$, where the change of the partition function under infinitesimal canonical transformations is considered. In analogy, we define the genuine anomaly in the BV scheme as

$$(W, W) - 2i\hbar\Delta W = -2i\hbar\mathcal{A}. \tag{11.12}$$

Hence, we see that the genuine anomaly expresses the failure to construct a quantum extended action W that satisfies the quantum master equation. Notice that a genuine anomaly also implies that the Zinn-Justin equation changes (8.28). The effective action $\Gamma(\phi_{cl}, \phi^*)$ does not satisfy the classical master equation in that case.

If we take for W the usual expansion in $\hbar$: $W = S + \hbar M_1 + \ldots$ and for the anomaly $\mathcal{A} = \mathcal{A}_0 + \hbar\mathcal{A}_1 + \hbar^2\mathcal{A}_2 + \ldots$, we get from (11.12), order in order in $\hbar$,

$$\begin{align}
(S, S) &= 0 \tag{11.13}\\
(S, M_1) - i\Delta S &= -i\mathcal{A}_0 \tag{11.14}
\end{align}$$

$$\ldots$$

The dots denote the infinite tower of equations, one for every order in $\hbar$. Since we will use a regularisation prescription that is only capable to handle one loop in the next chapters, we will not discuss them.

It was already pointed out in chapter 6 that the classical master equation can always be solved [6, 7, 38, 47], starting from a given classical action S_0 and a given complete set of gauge generators R^i_α. With the extended action

S at hand, one then has to calculate ΔS. Remember that in the antifield scheme, the BRST transformation in the antibracket sense is defined by (5.43) $\mathcal{S}\phi^A = (\phi^A, S) = \overrightarrow{\delta} S/\delta\phi_A^*$. It is then not difficult to see that formally, ΔS is indeed proportional to the logarithm of the Jacobian of the path integral measure $[d\phi^A]$ under BRST transformation[2]. Since S is a local functional of fields and antifields, one has that $\Delta S \sim \delta(0)$. To remedy this situation, a regularisation scheme is required. We postpone a discussion of this to the next chapter.

If $\Delta S = 0$, then that is all there is to it. The M_i are then only determined by the renormalisation process. However, if $\Delta S \neq 0$, we have what we called an *anomaly* in the previous section. In that case, one has to look for a *local* M_1, function of fields and antifields, such that $(S, M_1) = i\Delta S$. For a specific ΔS, the *counterterm* M_1 is not uniquely defined. Indeed, we can always consider $\tilde{M}_1 = M_1 + (S, X)$, because $(S, \tilde{M}_1) = (S, M_1)$, owing to the fact that $(S, S) = 0$. This X may contain divergent terms of order $\hbar$ that are needed in the renormalisation process etc. We will not study this in detail and refer to [83, 84]. If no M_1, local in a preferred set of variables, can be found such that $\mathcal{A}_0 = 0$, then there is a *genuine anomaly*. A specific expression for $\mathcal{A}_0$ is obtained by choosing a counterterm M_1. As we showed above (11.10), $\mathcal{A}_0 = c^\alpha \mathcal{A}_{0\alpha}$. By different choices of the counterterm M_1, it may be possible to make $\mathcal{A}_{0\alpha} = 0$ for certain values of α. This way, one can specify which gauge symmetries are kept anomaly free by a carefull choice of M_1.

We can also see how $\mathcal{A}_0$ of (11.14) transforms under infinitesimal canonical transformations generated by $F = \mathbb{1} + f$. We find,

$$\tilde{\mathcal{A}}_0 = i(S', M_1') + \Delta S' = \mathcal{A}_0' - (S, \Delta f). \tag{11.15}$$

Here we used the notation (8.7). The extra term comes from the transformation of ΔS, given in (8.9). Hence, we see that in a different set of coordinates, an extra counterterm $i\Delta f$ appears. In a formal reasoning, which needs justification in a regularised treatment, we can even go further. For a closed algebra, S is linear in the antifields. Therefore, it is expected that $\mathcal{A}_0$ does not depend on antifields (for $M_1 = 0$). If one then only considers $f(\phi)$, independent of the antifields, one has that $\mathcal{A}_0' = \mathcal{A}_0$ and one sees that changing gauge only results in a change of counterterm. Consequently, in different gauges, different gauge symmetries may be anomalous.

Let us finally reformulate the Wess-Zumino consistency condition in the BV scheme [63, 85]. From the definition of the genuine anomaly (11.12), and from the properties of the antibracket and the box operator as listed in the appendix, we easily arrive at

$$\sigma\mathcal{A} = (\mathcal{A}, W) - i\hbar\Delta\mathcal{A} = 0. \tag{11.16}$$

The full anomaly is quantum BRST invariant. The usual expansion in $\hbar$ gives

$$(\mathcal{A}_0, S) = 0$$
$$(\mathcal{A}_0, M_1) + (\mathcal{A}_1, S) - i\Delta\mathcal{A}_0 = 0 \tag{11.17}$$
$$\cdots$$

The first condition at $\mathcal{O}(\hbar^0)$ is that the one loop anomaly is classical BRST invariant. This is the condition that we will meet below in the examples and

---

[2]Notice the slight change of notation with respect to the previous section and section 2 of chapter 3. What we there denoted by $\mathcal{A}$ corresponds to ΔS here. The notation $\mathcal{A}$ is now reserved for Jacobians that can not be countered by an M_1 (11.12).

we will denote it by *Wess-Zumino consistency condition*. In fact, since S satisfies the classical master equation, the $\mathcal{O}(\hbar^0)$ consistency condition becomes $(\Delta S, S) = 0$.

This finishes our overview of the definition and the major properties of the description of gauge anomalies in BV. It is important to remember that we have so far two related ways of changing the explicit expression of the genuine anomaly, viz the choice of the *local* counterterm M_1 and the choice of gauge. This way, we obtain other representants of the same cohomology class that determines the anomaly. In the second chapter of this part, we will discuss the regularisation of the formal expressions of section 1. Only in the following chapter we will return to the regularised treatment of anomalies in BV.

Chapter 12

Pauli-Villars regularisation for consistent anomalies

In this and the following chapters, a one-loop regularised study of anomalies is presented. In the first section, we sketch the core idea of the procedure proposed by K. Fujikawa [26] to obtain a regularised expression for the Jacobian associated with a transformation of the fields. Typically, the explicit regularised expression that one gets for such a Jacobian is determined by the transformation itself *and* by the way one chooses to regularise the determinant by means of what we will call below a *regulator*. Not all choices for the regulator that are allowed in the Fujikawa scheme, give consistent anomalies. In the second section, we follow A. Diaz, M. Hatsuda, W. Troost, P. van Nieuwenhuizen and A. Van Proeyen [86, 87, 63] and use Pauli-Villars regularisation [88] to obtain regulators that give consistent anomalies. In the final section, the freedom one has in choosing a mass term for the Pauli-Villars fields, is exploited [89] to calculate the induced action for W_2 gravity.

12.1 Fujikawa's proposal for regularised Jacobians

Following [26], we give here –schematically– a procedure to calculate a regularised expression for anomalies. Although a few steps are rather ad hoc, this procedure provides a first contact with the type of regularised expressions that we will meet below. Typically, one starts from a path integral (cfr. (11.5))

$$e^{-W[A]} = \int [d\phi] \exp[-\phi^\dagger D[A]\phi] \ . \tag{12.1}$$

We have put $\hbar = 1$, as in the rest of this chapter. We work now in Euclidean space, and consider bosonic fields ϕ. A space-time integration is understood in the exponent on the RHS. $D[A]$ is a first or second order differential operator, depending on an external field A. Internal indices are understood, i.e. $D[A]$ may actually be a matrix and $\phi^\dagger$ and ϕ respectively a row and a column. We assume that $D[A]$ has a complete set of orthonormal eigenfunctions, denoted by ϕ_n:

$$D[A]\phi_n \quad = \quad \lambda_n \phi_n$$

$$\int dx\, \phi_n^\dagger \phi_m \;=\; \delta_{nm}. \tag{12.2}$$

Every function ϕ can then be expanded in this complete basis $\phi(x) = \sum_n a_n \phi_n(x)$. Therefore, we define the measure of the functional integral by

$$[d\phi] = \prod_n da_n \;. \tag{12.3}$$

We do an infinitesimal transformation that may or may not leave the action $S = \phi^\dagger D[A]\phi$ invariant. We only focus on a possible Jacobian. Suppose that the transformation is given by

$$\phi' = \phi + \epsilon.M.\phi \;, \tag{12.4}$$

with an infinitesimal parameter ϵ. M is a field independent matrix that determines the transformation. In principle, it can be a differential operator too, but for the sake of the argument we restrict ourselves to matrices. This case includes for instance the important example of chiral rotations of fermions, where $M = \gamma_5$. We can expand $\phi' = \sum_n a'_n \phi_n$, with

$$a'_n = a_n + \epsilon \sum_m M_{nm} a_m. \tag{12.5}$$

The matrix $\mathcal{M}$ is defined by its elements

$$M_{nm} = \int dx\, \phi_n^\dagger M \phi_m. \tag{12.6}$$

It is then clear that

$$\prod_n da'_n \;=\; \det[\delta_{nm} + \epsilon M_{nm}] \prod_n da_n$$
$$\approx\; e^{\epsilon \mathrm{tr}\mathcal{M}} \prod_n da_n. \tag{12.7}$$

So far, we have only given a specification of the integration measure and have determined what form the Jacobian then takes.

$\mathrm{tr}\mathcal{M}$ is often ill-defined as an infinite sum. K. Fujikawa proposed to replace it with the regularised expression

$$\mathrm{tr}_\alpha \mathcal{M} = \sum_n \int dx\, \phi_n^\dagger M f(\lambda_n, \alpha)\phi_n \;, \tag{12.8}$$

where $f(\lambda_n, \alpha)$ is such that it suppresses the contributions to the trace of the eigenfunctions associated with large eigenvalues λ_n. $f(\lambda_n, \alpha = 0) = 1$, so that the original, divergent expression is reobtained in the limit $\alpha \to 0$. We make the typical choice $f(\lambda_n, \alpha) = \exp(-\lambda_n^\beta \alpha)$. Since the ϕ_n are eigenfunctions of $D[A]$ with precisely the eigenvalue λ_n, we find

$$\mathrm{tr}\mathcal{M} = \lim_{\alpha \to 0} \sum_n \int dx\, \phi_n^\dagger M \exp(-\alpha D[A]^\beta)\phi_n. \tag{12.9}$$

β is chosen such that in the exponent there is a term of second order in derivatives, leading to a Gaussian damping in momentum representation. $D[A]^\beta$ is called the *regulator*. Using the methods of appendix C, a regularised expression for the Jacobian can be obtained.

This method has however some shortcomings. First of all, there is the arbitrariness in the choice $f(\lambda_n, \alpha)$. Certain choices give an expression for the Jacobian that satisfies the consistency condition (11.9) while others do not [86]. Furthermore, there seems to be no a priori reason for considering a complete orthonormal set of eigenfunctions of the operator $D[A]$ that determines the action of the theory. Clearly, the relation between the regulator and action of the model under consideration is to be put in by hand. This situation will be clarified in the next section.

Despite this criticism, we will always have a form like (12.9) for Jacobians, i.e. a trace over functional space of a matrix determining the transformation times the exponent of a regulator that dampens the contributions to this trace of eigenfunctions with a large eigenvalue. There are other methods to regularise Jacobians, see e.g. ζ-regularisation [9], but in the next section we will take the different point of view that one does not need regularise the determinant but the complete path integral. This then implies regularised expressions for the Jacobians [86, 63].

12.2   Pauli-Villars regularisation

In this section, we describe a method [86, 87, 63] to calculate regularised expressions for Jacobians. The method starts by giving a one-loop regularisation prescription for the complete functional integral. This is in contrast with the method of Fujikawa which only regularises the Jacobian determinant itself. Then we consider the BRST transformation of this regularised functional integral. Remembering the way the consistency condition was derived in the previous chapter, we see that we are thus guaranteed to obtain a consistent anomaly (11.9).

We introduce the regularisation method by means of a simple example that nevertheless keeps many of the characteristic features. We take as classical action the action for W_2 gravity (5.49):

$$S_0[\phi, h] = \frac{1}{2\pi} \int d^2x \left[\partial\phi\bar{\partial}\phi - h(\partial\phi)^2 \right].$$ (12.10)

The BRST transformation rules are given by

$$
\begin{aligned}
\delta\phi &= c\partial\phi \\
\delta h &= \bar{\partial}c - h.\partial c + \partial h.c \\
\delta c &= \partial c.c \ ,
\end{aligned}
$$ (12.11)

and $\delta^2 = 0$ (the extended action (5.54) is linear in the antifields). We define the *induced action* $\Gamma[h]$ by

$$e^{-\Gamma[h]} = \int [d\phi] e^{-S_0[\phi, h]}.$$ (12.12)

Notice that this is an example that satisfies the requirements that were imposed to derive the consistency condition (11.5,11.9). By a partial integration in both terms of the classical action, we can bring S_0 in the form

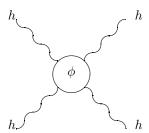$$S_0[\phi, h] = -\frac{1}{2\pi} \int d^2x \, \phi\partial\nabla\phi \ ,$$ (12.13)

Figure 12.1: An example of a one loop diagram contributing to the induced action.

where $\nabla = \bar{\partial} - h\partial$. As the integration over ϕ is Gaussian, we can formally write

$$e^{-\Gamma[h]} = [\det \partial\nabla]^{-\frac{1}{2}}. \tag{12.14}$$

The $\det A$ is formally the product of the eigenvalues of the operator A.

$\Gamma[h]$ is the generating function of the connected correlation functions of an arbitrary number n of operators $T = \frac{1}{2\pi}(\partial\phi)^2$. To every such correlation function contributes only one one-loop diagram. For example, for $n = 4$, we have the diagram Fig. 12.1.

At the diagrammatic level, these diagrams can be regularised à la Pauli-Villars [88] as follows. From every diagram[1] we *subtract* an identical one, with the only difference that instead of a massless, bosonic particle ϕ in the loop, we use a massive, bosonic particle χ, with mass M. These regulating diagrams disappear formally again upon taking the limit $M \to \infty$. Clearly, such diagrams can be obtained from the action $S_{PV} + S_M$, with

$$
\begin{aligned}
S_{PV} &= S_0[\chi, h] \\
&= -\frac{1}{2\pi} \int d^2x\, \chi\partial\nabla\chi \\
S_M &= -\frac{1}{4\pi} \int d^2x\, M^2\chi^2. \tag{12.15}
\end{aligned}
$$

The fact that the diagrams generated by this action have to be subtracted, is implemented by the rule that every closed loop gets a minus sign. Owing to this extra minus sign, we formally have that

$$e^{-\Gamma_{PV}[h]} = \int [d\chi]e^{-S_{PV}-S_M} = [\det \partial\nabla + \frac{M^2}{2}]^{+\frac{1}{2}}. \tag{12.16}$$

Notice that the power is now $+\frac{1}{2}$ in contrast to the integration of the original ϕ-fields (12.14). $\Gamma_{PV}[h]$ generates all the regularising diagrams. The regularised

---

[1]We use *diagram* here as a metonymy for the *mathematical expression associated with the diagram*.

expression for the induced action is then by definition (we omit sometimes space-time integrations and consider them understood):

$$
\begin{aligned}
e^{-\Gamma_M[h]} &= \int [d\phi][d\chi] e^{-S_0[\phi,h] - S_0[\chi,h] + \frac{1}{4\pi} M^2 \chi^2} \\
&= \left[\frac{\det \partial\nabla + M^2/2}{\det \partial\nabla} \right]^{\frac{1}{2}}.
\end{aligned}
\tag{12.17}
$$

In a formal sense, the latter expression already shows that Pauli-Villars regularisation removes the contributions to the induced action of the large eigenvalues λ_n of the operator $\partial\nabla$ and favours the small eigenvalues. Indeed,

$$
e^{-\Gamma_M[h]} \sim \prod_n \left(\frac{\lambda_n + M^2/2}{\lambda_n} \right)^{\frac{1}{2}},
\tag{12.18}
$$

where the RHS in fact defines the ratio of the two determinants in (12.17) in this regularisation scheme. For a fixed M, we see that eigenvalues $\lambda_n \gg M^2$ give a contribution 1 to the product, while for $\lambda_n \ll M^2$, the contribution is proportional to $\frac{1}{\sqrt{\lambda_n}}$.

Instead of implementing the minus sign by hand with a Pauli-Villars (PV) field of the same (even) Grassmann parity as the original field, we could have used a PV field ψ of odd Grassmann parity. As with one such field we would not be able to construct, for instance, a mass term ($\psi^2 = 0$), we would have to introduce two, ψ_1 and ψ_2. But then the PV path integral with the action $S_{PV} + S_M \sim \psi_1 (\partial\nabla + M^2/2)\psi_2$ would be proportional to $\det(\partial\nabla + M^2/2)$, i.e. half a power too much. Therefore, a third, again bosonic, PV field would be needed to generate an extra $\det(\partial\nabla + M_*^2/2)^{-1/2}$. Instead of using this correct but cumbersome procedure, we will always use the sleight-of-hand above, knowing that the procedure with three times as many PV fields provides us with an ultimate justification if needed.

It is also important to notice that $S_{PV}[\chi,h]$ is BRST invariant if we copy the BRST transformation of ϕ for the PV field

$$
\delta\chi = c\partial\chi.
\tag{12.19}
$$

The mass term S_M is then the only term of the complete action $S_R + S_M = S_0[\phi,h] + S_0[\chi,h] + S_M[\chi]$ that may not be BRST invariant. As a matter of fact, this mass term is the only possible source of BRST non-invariance of (12.17). This is a consequence of the fact that the complete measure of the fields ϕ and the PV fields χ is (BRST) invariant, as the Jacobian of the ϕ measure is compensated by the Jacobian of the χ measure[2] [26, 86], owing to the extra minus sign for PV loops.

Taking this into account, we find that the BRST variation of the regularised induced action is

$$
\begin{aligned}
\delta e^{-\Gamma_M[h]} &= -\int [d\phi][d\chi] e^{-S_R + \frac{1}{4\pi} M^2 \chi^2} . \delta S_M \\
&= -\int [d\phi][d\chi] e^{-S_R + \frac{1}{4\pi} M^2 \chi^2} . \frac{M^2}{4\pi} . \partial c.\chi^2.
\end{aligned}
\tag{12.20}
$$

[2]See the appendix of *this* chapter for more details.

We can now use (C.4) of the appendix to integrate out the PV fields

$$\delta\Gamma_M[h] = -\frac{M^2}{4}.\partial c.\text{tr}[\frac{1}{-\partial\nabla - M^2/2}], \qquad (12.21)$$

where the factor I on the RHS of (C.4) allowed us to divide out a factor $e^{-\Gamma_M[h]}$ on both sides. Notice that ∂c can be brought out of the trace over the function space, as it does not contain derivatives that act further nor matrix indices. As is also explained in the appendix (C.8), we can now make contact with the expressions of Fujikawa (12.9), using [86, 63]

$$\int_0^\infty d\lambda e^{-\lambda} e^{\lambda\mathcal{R}} = -\frac{1}{\mathcal{R} - \mathbb{1}}. \qquad (12.22)$$

$\mathcal{R}$ is called the *regulator*. In our case, this gives,

$$\delta\Gamma_M[h] = \frac{1}{2}\int d^2x \ \partial c. \int_0^\infty d\lambda e^{-\lambda}.\text{tr}\left[\exp\left(-\frac{\lambda}{M^2}2\partial\nabla\right)\right]. \qquad (12.23)$$

Using (C.34), we can calculate the trace of the exponential of the regulator in the limit $M \to \infty$. We find

$$\delta\Gamma_M[h] = \frac{1}{8\pi}\int d^2x \ \partial c. \int_0^\infty d\lambda e^{-\lambda}\left[\frac{M^2}{\lambda} - \frac{1}{3}\partial^2 h + \mathcal{O}(\frac{1}{M^2})\right]. \qquad (12.24)$$

But now we see that we are in trouble. Even for a finite value of M, the term of (12.24) that is proportional to M^2 diverges owing to the integration over λ: $\int_0^\infty d\lambda \ e^{-\lambda}\lambda^{-1} = \Gamma(0)$ (this is of course the gamma-function, not the induced action !). The remedy is well-known (see e.g. chapter 7 of [21]) and goes as follows. Instead of only one PV field, one introduces several copies χ_i with a mass M_i. The precise number may be determined from requirements to be specified below, but is irrelevant. We take

$$S_{PV}^i + S_M^i = -\frac{1}{2\pi}\int d^2x \ \chi_i\partial\nabla\chi_i - \frac{1}{4\pi}\int d^2x \ M_i^2\chi_i^2, \qquad (12.25)$$

and $\delta\chi_i = c\partial\chi_i$. The formal integration is defined by

$$\int [d\chi_i]e^{-S_{PV}^i - S_M^i} = [\det\partial\nabla + M_i^2/2]^{x_i/2}. \qquad (12.26)$$

All M_i are taken to infinity in the end. The precise values of the x_i follow from some relations which we will now specify. We should certainly have that

$$\sum_i x_i = 1 , \qquad (12.27)$$

in order to keep the complete measure of the original fields and all PV fields BRST invariant (12.57). When taking the BRST variation of the regularised action (12.20), we now have $\sum_i \delta S_M^i$ instead of δS_M. From (C.4), we see that upon integration over the PV fields, we get

$$\delta\Gamma_M[h] = \frac{1}{8\pi}\int d^2x \ \partial c. \int_0^\infty d\lambda e^{-\lambda}\left[\frac{\sum_i x_i M_i^2}{\lambda} - \sum_i x_i\frac{1}{3}\partial^2 h + \sum_i x_i\mathcal{O}(\frac{1}{M_i^2})\right]. \qquad (12.28)$$

We now impose the extra condition

$$\sum_i x_i M_i^2 = 0, \qquad (12.29)$$

to get rid of the diverging λ integration. As is clear from the formulas in the appendix (C.13), when working in 4 or more space-time dimensions, the expansion of the exponent of the regulator starts with terms proportional to M_i^4 or more. Even more conditions like (12.29), with higher powers of M_i, are imposed then. After having obtained the full set of such conditions, one can specify a minimal number of PV fields, their x_i and the relations between their masses.

We finally find that

$$\mathcal{A} \equiv \lim_{M \to \infty} \delta\Gamma_M[h] = \frac{1}{24\pi} \int d^2x \, c\partial^3 h. \tag{12.30}$$

It is easy to verify that $\delta\mathcal{A} = 0$, with the BRST transformations (12.11). It can be shown [39] that this expression for $\delta\Gamma[h]$ is satisfied for

$$\Gamma[h] = -\frac{1}{48\pi} \int d^2x \, \partial^2 h \frac{1}{\partial\nabla} \partial^2 h, \tag{12.31}$$

which is non-local. In the next section, we will derive this expression for $\Gamma[h]$. No expression for $\Gamma[h]$, local in h, with $\delta\Gamma[h] = \mathcal{A}$ has been found.

Before turning to a further study of the use of the mass term in the PV regularisation, let us briefly repeat the important steps of the above procedure. We started by regularising the complete functional integral by the introduction of a Pauli-Villars field χ for every field ϕ. χ has the same Grassmann parity as ϕ, but with a closed χ-loop we associate an extra minus sign. The BRST transformation of the PV fields and their action were obtained by direct substitution of χ for ϕ in the original BRST transformation rules and action. As a result of this construction, the measure of the regularised path integral is BRST invariant. To complete the PV regularisation, one has to choose a mass term S_M for the PV fields. This mass term is the only possibly BRST non-invariant factor in the regularised partition function. In the PV scheme, anomalies come from the BRST variation of the mass term. At first sight, one might think that one has no freedom in choosing the mass term. However, the contrary is true. The possibilities this freedom offers are explored in the next section. The regularised functional integral gives rise to a regularised expression for Jacobians that is of the same type as the regularised expressions proposed by Fujikawa (12.9). The heat kernel methods of the appendix C can then be used to obtain an expression for the anomaly.

12.3 Mass term dependence of the anomaly

We now show how the freedom that one has in choosing a mass term for the PV fields, can be exploited [89]. As it is this mass term that determines which gauge symmetries are anomalous (12.20), one can try to keep preferred symmetries anomaly free by a judicious choice of the mass term. If the mass term is invariant under a certain symmetry, that symmetry will not become anomalous. In other words, we have yet another factor (the mass term) that determines the actual expression of the anomaly. If the ϕ^A denote the original fields, the most general choice of mass term that we can make is $S_M = -\frac{1}{2}\chi^A T_{AB}(\phi)\chi^B M^2$. Here, T_{AB} is an invertible, field dependent matrix that satisfies $T_{AB} = (-1)^{\epsilon_A + \epsilon_B + \epsilon_A \epsilon_B} T_{BA}$. Of course, in general such mass

terms are not fit for diagrammatic calculations, but they pose no problem for the procedure sketched in the previous section.

Let us consider the W_2 example. A.M. Polyakov [39] found that a redefinition of the field –from h to f– makes the non-local induced action (12.31) local in this new variable f. The field f is defined by

$$h = \frac{\bar{\partial} f}{\partial f} \ . \tag{12.32}$$

The BRST transformation of f is $\delta f = c\partial f$. This reproduces the BRST transformation of h in (12.11). Suppose now that we take as mass term for the PV field

$$\tilde{S}_M = -\frac{1}{4\pi} \int d^2x \, M^2 \chi^2 \partial f \ . \tag{12.33}$$

Although this is a local expression in f, it is non-local in the variable h. Hence, the term *non-local regularisation* was coined for this procedure. This mass term (12.33) is clearly BRST invariant. Therefore, we find that (cfr. (12.20))

$$e^{-\Gamma_M}.\delta\Gamma_M = 0, \tag{12.34}$$

and we can take $\Gamma[h] = 0$ in this regularisation scheme.

So far, we have in this chapter neglected the possibility to modify the expression for the anomaly using a local counterterm M_1. More specifically, the question arises whether one can compensate for the change in the expression for the anomaly owing to a different choice of local mass term by the addition of a *local* quantum counterterm. It turns out that we can indeed do that. A different mass term only leads to a different representant of the same cohomology class for the anomaly.

Consider two induced actions, obtained by a regularised calculation with two different mass terms. We allow for the presence of a counterterm M, which only depends on the external gauge field h. We have

$$e^{-\Gamma^{(0)}[h]} = \int [d\phi][d\chi] e^{-S_0[\phi,h]-S_0[\chi,h]-S_M^{(0)}-M^{(0)}[h]}, \tag{12.35}$$

and

$$e^{-\Gamma^{(1)}[h]} = \int [d\phi][d\chi] e^{-S_0[\phi,h]-S_0[\chi,h]-S_M^{(1)}-M^{(1)}[h]}. \tag{12.36}$$

We want to find the relation between the two counterterms, $X[h] = M^{(0)} - M^{(1)}$, in order to have that the two induced actions are equal: $\Gamma^{(0)}[h] = \Gamma^{(1)}[h]$. We suppose that an *interpolating mass term* $S_M^{(\alpha)}$ exists, i.e. a mass term depending on a parameter α, such that $S_M^{(\alpha=1)} = S_M^{(1)}$ and $S_M^{(\alpha=0)} = S_M^{(0)}$. With this mass term, we define

$$\mathcal{Z}(\alpha) = \int [d\phi][d\chi] e^{-S_0[\phi,h]-S_0[\chi,h]-S_M^{(\alpha)}+\alpha.X[h]}. \tag{12.37}$$

From the definition of X, and from (12.35),(12.36) it follows that if we want $\Gamma^{(0)} = \Gamma^{(1)}$, we have to take X such that

$$\begin{aligned} 0 &= \ln \mathcal{Z}(1) - \ln \mathcal{Z}(0) \\ &= \int_0^1 d\alpha \, \frac{1}{\mathcal{Z}(\alpha)} \frac{d\mathcal{Z}(\alpha)}{d\alpha} \ . \end{aligned} \tag{12.38}$$

Let us focus for a moment on $\frac{d\mathcal{Z}(\alpha)}{d\alpha}$. We write

$$S_M^{(\alpha)} = -\frac{1}{2}\chi^A T_{AB}(\alpha)\chi^B, \tag{12.39}$$

where $T_{AB}(\alpha)$ may depend on the original fields ϕ. Notice that we have also absorbed the mass parameter M in T_{AB}. Analogously, we define $\mathcal{O}_{AB}$ by

$$S_0[\chi, h] = \frac{1}{2}\chi^A \mathcal{O}_{AB}\chi^B . \tag{12.40}$$

Then we have that

$$
\begin{aligned}
\frac{d\mathcal{Z}(\alpha)}{d\alpha} &= \int [d\phi][d\chi]e^{-S_0[\phi,h]-S_0[\chi,h]-S_M^{(\alpha)}+\alpha.X[h]} \\
&\times \left[\frac{1}{2}\chi^A \frac{dT_{AB}(\alpha)}{d\alpha}\chi^B + X[h]\right].
\end{aligned} \tag{12.41}
$$

Integrating out the PV fields (C.4), gives

$$\frac{d\mathcal{Z}(\alpha)}{d\alpha} = \mathcal{Z}(\alpha)\left[-\frac{1}{2}\mathrm{tr}\left(T^{-1}(\alpha)\frac{dT(\alpha)}{d\alpha}.\frac{1}{T^{-1}(\alpha)\mathcal{O}-1}\right) + X[h]\right]. \tag{12.42}$$

Plugging this back in (12.38) we arrive at the most important result of this section

$$M^{(0)} - M^{(1)} = \frac{1}{2}\int_0^1 d\alpha\,\mathrm{tr}\left(T^{-1}(\alpha)\frac{dT(\alpha)}{d\alpha}.\frac{1}{T^{-1}(\alpha)\mathcal{O}-1}\right) . \tag{12.43}$$

Of course, the limiting procedure $M \to \infty$ and the introduction of copies of the PV field are understood. This relation (12.43) was foreshadowed in [90], conjectured in [63] and proven in [37].

Let us now apply this result to our example [89] and find the counterterm $M^{(0)}$, needed to obtain $\Gamma^{(0)}[h] = 0$ with the PV mass term $S_M^{(0)} = -\frac{1}{4\pi}M^2\chi^2$. When calculated with the BRST invariant mass term $S_M^{(1)} = -\frac{1}{4\pi}M^2\chi^2\partial f$ and with $M^{(1)} = 0$, we have the induced action $\Gamma^{(1)}[h] = 0$. Clearly we have

$$
\begin{aligned}
T(\alpha) &= \frac{1}{2\pi}(\partial f)^\alpha M^2 \\
\mathcal{O} &= -\frac{1}{\pi}\partial\nabla \\
T^{-1}(\alpha)\mathcal{O} &= \frac{\mathcal{R}(\alpha)}{M^2} = -2(\partial f)^{-\alpha}\partial\nabla\frac{1}{M^2} \\
T^{-1}\frac{dT}{d\alpha} &= \ln(\partial f) .
\end{aligned} \tag{12.44}
$$

Using (12.22), we have then

$$M^{(0)} = \lim_{M\to\infty} -\frac{1}{2}\int_0^1 d\alpha\int_0^\infty d\lambda e^{-\lambda}\,\mathrm{tr}\left(\ln\partial f.\exp[\frac{\lambda}{M^2}\mathcal{R}(\alpha)]\right) . \tag{12.45}$$

$\mathcal{R}(\alpha)$ can again be rewritten as a covariant Laplacian $\mathcal{R} = \frac{1}{\sqrt{|g|}}\partial_\mu\sqrt{|g|}g^{\mu\nu}\partial_\nu$ with

$$g^{\mu\nu} = (\partial f)^{-\alpha}\begin{pmatrix} 2h & -1 \\ -1 & 0 \end{pmatrix} . \tag{12.46}$$

With the results listed in the appendix (C.15) (we need E_2 as we work in two dimensions), we find that

$$M^{(0)} = \frac{1}{48\pi} \int_0^1 d\alpha \int d^2x \, \ln \partial f . \sqrt{|g|} R[g^{\mu\nu}] \ . \qquad (12.47)$$

The factor $\sqrt{|g|}$ comes from $dvol(x)$ and $R[g^{\mu\nu}]$ is the Riemann scalar of the metric $g^{\mu\nu}$. We have taken the limit $M \to \infty$, and extra PV fields were added so that we could integrate out λ without running into divergences. Observe that

$$\sqrt{|g|} R[g^{\mu\nu}] = R[\tilde{g}^{\mu\nu}] + \frac{1}{2} \partial_\mu \{ \sqrt{|g|} g^{\mu\nu} \partial_\nu \ln |g| \} \qquad (12.48)$$

for $\tilde{g}^{\mu\nu} = \sqrt{|g|} g^{\mu\nu} = \begin{pmatrix} 2h & -1 \\ -1 & 0 \end{pmatrix}$. This gives us finally

$$\begin{aligned} M^{(0)} &= \frac{1}{48\pi} \int_0^1 d\alpha \int d^2x \, \ln \partial f . 2(1-\alpha) \partial^2 h \\ &= \frac{1}{48\pi} \int d^2x \, \partial^2 h \ln \partial f \ . \end{aligned} \qquad (12.49)$$

It is clear that when written in terms of f, this is a local expression. We can also reexpress it as a functional of h, using that $\partial \nabla \ln \partial f = \partial^2 h$, which gives

$$M^{(0)} = \frac{1}{48\pi} \int d^2x \, \partial^2 h \frac{1}{\partial \nabla} \partial^2 h. \qquad (12.50)$$

We find back the Polyakov-action (12.31). With this $M^{(0)}$, we have that

$$1 = \int [d\phi][d\chi] e^{-S_0[\phi,h] - S_0[\chi,h] - S_M^{(0)} - M^{(0)}[h]}, \qquad (12.51)$$

such that the induced action $\Gamma[h]$ defined in the previous section (12.17) is actually $-M^{(0)}$. It is easy to verify that indeed

$$-\delta M^{(0)} = \frac{1}{24\pi} \int d^2x \, c\partial^3 h. \qquad (12.52)$$

Our method has hence become a way to calculate the induced action. Moreover, it provides us with insight on why the induced action can be written in a local way when the variable f is used instead of h. Indeed, we have obtained the Polyakov-action expressed in f, using a procedure that is completely local in f.

We [89] have also applied the same method to calculate the action for gauge fields, induced by chiral fermions in two dimensions. There, the reparametrisation corresponding to (12.32) is $A = \bar{\partial} g.g^{-1}$ [91], i.e. the algebra element A is written as a function of a group element g. With this g, we can again construct an invariant mass term for the PV regularisation fields. The induced action is in that case, as is well-known, the Wess-Zumino-Witten model [79, 92]. The α integral can not always be done explicitly, it becomes the third dimension that is present in the topological term of that model.

In this section, we have pointed out a third factor that influences the actual expression for the anomaly, namely the mass term of the PV regularisation. We have shown that the expressions for the anomaly obtained with two different

local mass terms are in general related by the BRST variation of a local counterterm that can be determined from (12.43). Hence, whatever the chosen mass term, we obtain an anomaly of the same cohomology class. As an application, we have seen how we can calculate the induced action of W_2 gravity from a counterterm, since an invariant mass term can be found. In our example, a redefinition of the field variables was done from h to f. The complete procedure, in particular the two different mass terms and the counterterm, is local in the Polyakov variable f. However, when everything is expressed in function of the original variable h, non-local expressions appear.

Appendix

The measure of the functional integral over the PV fields is defined implicitly by the definition of the Gaussian integrals over the PV fields (12.16,12.26). Hence, we can take the definition of these integrals as a starting point to derive a consistent prescription for the Jacobian of the PV measure under transformations. Denote by z^i a (finite) set of (bosonic) variables, and define

$$\int [dz] \exp[-\frac{1}{2}z^i D_{ij} z^j] = [\det D]^{x/2}. \tag{12.53}$$

Here, x is a constant. In section 2 of this chapter, we have $x = 1$ for the PV fields and $x = -1$ for the original fields. Suppose that we want to change integration variables to y^i:

$$z^i = L^i{}_j y^j, \tag{12.54}$$

with

$$\int [dy] \exp[-\frac{1}{2}y^i D_{ij} y^j] = [\det D]^{x/2}. \tag{12.55}$$

If we assume that $[dz] = [dy].J$, we have that

$$[\det D]^{x/2} = J[\det(L^t D L)]^{x/2}. \tag{12.56}$$

From this we find that we should take for consistency

$$J = [\det L]^{-x}. \tag{12.57}$$

This result generalises to superintegrations and superdeterminants. Notice that $L^i{}_j$ may and will often depend on the fields ϕ if the z^i are the PV fields χ.

Chapter 13

PV regularisation for BV

We now use Pauli-Villars regularisation to construct a one-loop regularised version of the BV scheme. In particular, we copy the derivation of the quantum master equation based on canonical transformations (8.17) using one-loop regularised path integrals. This will lead us to a regularised $\mathcal{O}(\hbar)$ master equation and a regularised expression for σX, also to $\mathcal{O}(\hbar)$. The basic features of the previous chapter will reappear. Specifically, the Δ-operators do not appear in the regularised treatment, but are replaced by expressions of the Fujikawa type (12.9). For instance, ΔS is replaced by an expression determined by a regulator and by the antibracket of the regularised extended action with the PV mass term. This statement is of course the BV version of the result of the previous chapter (12.20).

This chapter is rather technical. Its purpose is to translate the insights of the previous chapter in the BV language. In section one[1], we discuss the construction of a one-loop regularised path integral for the expectation value of an operator $X(\phi, \phi^*)$ in the BV scheme. It is shown in section two what this one-loop regularised path integral looks like in a second set of canonical coordinates, related to the first one by an infinitesimal canonical transformation. By imposing that the two expectation values are equal, i.e. by imposing gauge independence, we derive a one-loop regularised expression for the one-loop master equation and for the quantum BRST operator of the BV scheme. After a discussion of the results in section three, we give an example in section four, where we use our regularised expression for σX to calculate the Jacobian of the measure under an infinitesimal canonical transformation.

13.1   Setting up the PV regularisation

In this section, we construct a one-loop regularised expression for the expectation value of an operator $X(\phi, \phi^*)$ in the PV regularisation scheme. We first describe the set-up of the PV regularisation scheme and give some justification for this construction at the end of this section.

The ϕ^A denote, as usual, the complete set of fields and the ϕ^*_A their antifields.

---

[1]Some of the results in section one were developed in discussions with R. Siebelink and W. Troost.

By the new notation ϕ^α, we denominate both the ϕ^A *and* the ϕ^*_A. In the same vein, we write the PV fields as $\chi^\alpha = \{\chi^A, \chi^*_A\}$. With every functional depending on fields and antifields $A(\phi, \phi^*)$ we associate

$$A_{PV} = \frac{1}{2}\chi^\alpha \left[\frac{\overrightarrow{\delta}}{\delta\phi^\alpha} \frac{\overleftarrow{\delta}}{\delta\phi^\beta} A \right] \chi^\beta . \tag{13.1}$$

The regularised version of A is then taken to be $A_R = A + A_{PV}$. One can straightforwardly prove that

$$\begin{aligned} (A, B)_R &= (A, B) + (A, B)_{PV} \\ &= (A + A_{PV}, B + B_{PV})_{\phi,\chi} + \mathcal{O}(\chi^4) \\ &= (A_R, B_R)_{\phi,\chi} + \mathcal{O}(\chi^4). \end{aligned} \tag{13.2}$$

By the notation $(A, B)_{\phi,\chi}$, we mean that the antibracket is to be calculated with respect to the fields *and* with respect to the PV fields. If no such underscore is added to the antibracket, only derivatives with respect to ϕ and ϕ^* are meant. We will consistently drop all the terms of quartic or higher order in the PV fields. When integrating over the PV fields, they lead to contributions of higher orders in $\hbar$, or they disappear when putting the PV antifields χ^*_A to zero after all antibrackets etc. have been evaluated.

As a first application of (13.2), consider the extended action $S(\phi, \phi^*)$, that satisfies the classical master equation $(S, S) = 0$. If we consider $S_R = S + S_{PV}$, we have that

$$(S_R, S_R)_{\phi,\chi} = (S, S)_R + \mathcal{O}(\chi^4) \approx 0 . \tag{13.3}$$

This corresponds to our observation in the previous chapter that S_{PV} of (12.15) is BRST invariant if we take the BRST transformation of the PV field as (12.19). Here we see this as follows. S contains a term $\phi^* c\partial\phi$ (5.54), and hence a term $\chi^* c\partial\chi$ is present in S_{PV}. The antibracket $(\chi, S_{PV})_\chi$ then gives $\delta\chi = c\partial\chi$.

To complete the regularisation scheme, we have to introduce a mass term for the PV fields. We take as most general form for this mass term

$$S_M = -\frac{1}{2}\chi^A T_{AB}(\phi, \phi^*)\chi^B, \tag{13.4}$$

where T_{AB} also depends on the mass M. T_{AB} is invertible and satisfies $T_{AB} = (-1)^{\epsilon_A + \epsilon_B + \epsilon_A \epsilon_B} T_{BA}$. Notice that this mass term is independent of the PV antifields, but may depend on both the original fields ϕ and their antifields ϕ^*. The *regularised extended action* is then

$$\mathcal{S} = S + S_{PV} + S_M = S_R + S_M. \tag{13.5}$$

If S satisfies the classical master equation, we have that
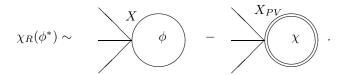
$$\begin{aligned} (\mathcal{S}, \mathcal{S})_{\phi,\chi} &= 2(S_R, S_M)_{\phi,\chi} + \mathcal{O}(\chi^4) \\ &= 2(S, S_M)_\phi + 2(S_{PV}, S_M)_\chi + \mathcal{O}(\chi^4). \end{aligned} \tag{13.6}$$

The regularised extended action does not necessarily satisfy the classical master equation owing to the presence of the mass term. Notice however that the terms on the RHS of (13.6) are at least quadratic in the PV fields, and hence effectively of order $\hbar$.

With this regularised action, we can construct a regularised expression for the expectation value of any operator $X(\phi, \phi^*)$:

$$\chi_R(\phi^*) = \int [d\phi][d\chi] \left[X_R e^{\frac{i}{\hbar}(\mathcal{S} + \hbar M_1)} \right]_{\chi^* = 0}. \tag{13.7}$$

That this is a one loop regularised expression follows from the general principles of PV regularisation. In particular, from every diagram with a closed loop that contributes to the order $\hbar$ in perturbation theory to the expectation value of X, a diagram is *subtracted* (remember the extra minus sign for PV loops) corresponding to the expectation value of the terms quadratic in the PV fields χ^A of X_{PV}. The terms in X_{PV} that contain one PV field and one PV antifield have expectation value zero. The terms of X_{PV} that are quadratic in the PV antifields χ_A^* might cause new divergences owing to ϕ^A-loops. To prevent this, we construct the regularised path integral with $\chi_A^* = 0$. At the diagrammatic level, we have for instance

From the one-loop contribution (the ϕ loop is denoted by a single circle) of the expectation value for the operator X a one-loop diagram is subtracted of the expectation value of X_{PV} (the χ loop is denoted by a double circle).

In the next section, we construct the regularised expectation value of $X(\phi, \phi^*)$ in a different set of coordinates, related to the first one by an infinitesimal canonical transformation generated by $F = \mathbb{1} + f$. By demanding that the two expectation values are equal, we will obtain a regularised version of the master equation and of σX. We copy the steps of section 2 of chapter 8 for one-loop regularised path integrals.

13.2 Regularised derivation of the master equation

Under an infinitesimal canonical transformation generated by $F = \mathbb{1} + f$, a functional $A(\phi, \phi^*)$ transforms to $A'(\phi, \phi^*) = A - (A, f)$, as was shown in chapter 8 (8.7). We can apply the recipe of the previous section to construct a regularised operator A'_R :

$$\begin{aligned} A'_R &= A - (A, f) + A_{PV} - (A, f)_{PV} \\ &= A_R - (A_R, f_R)_{\phi, \chi} + \mathcal{O}(\chi^4). \end{aligned} \tag{13.8}$$

We used again (13.2). This result shows that we have in fact two options to obtain the regularised functional in the transformed coordinates. We can first transform the unregularised operator and apply then the regularisation prescription, or we can first regularise the operator and then transform this regularised expression A_R. The transformation rules are then to be derived

from f_R. The PV fields need to be transformed too, and the transformation rules of the original fields ϕ^α may get corrections, quadratic in the PV fields [93, 59].

This result holds of course especially for S_R, which transforms to $S_R' = S_R - (S_R, f_R)_{\phi,\chi}$. Having S_R' is not sufficient to construct a regularised path integral in the transformed coordinates. We have to choose a mass term. We take $S_M' = S_M - (S_M, f_R)_{\phi,\chi}$. This choice is made for two reasons. First of all, this choice makes a derivation of the master equation copying the steps leading to (8.17) possible. Secondly, we have seen in the previous chapter that taking a different mass term only results in the addition of an extra counterterm M_1. Hence, we have that the regularised extended action transforms as

$$\mathcal{S}' = \mathcal{S} - (\mathcal{S}, f_R)_{\phi,\chi} \ . \tag{13.9}$$

The possible counterterm $\hbar M_1$ transforms as before, $M_1' = M_1 - (M_1, f)$, but we can take there too $M_1' = M_1 - (M_1, f_R)_\phi$. The extra terms are of order $\hbar \mathcal{O}(\chi^2)$, and hence negligible.

When doing the canonical transformation on the fields in the unregularised path integral, we had to take a Jacobian (8.23) into account. Like in the previous chapter (section 2 and appendix), we see that in the regularised functional integrals the Jacobian from the fields ϕ is cancelled by the Jacobian of the PV fields χ. Indeed, we would now have $\ln J \sim \Delta_\phi f_R - \Delta_\chi f_R$ instead of $\ln J \sim \Delta_\phi f$. The relative minus sign is again a result of the implicit definition of the path integral measure of the PV fields (12.57). It is easy to see that $\Delta_\chi f_{PV} = \Delta_\phi f$, such that we now have $\ln J \sim \Delta_\phi f_{PV}$, which we can drop as this would contribute a term of order $\hbar \mathcal{O}(\chi^2)$ to the transformed quantum extended action.

Finally, we have in the two sets of canonical coordinates the following two regularised functional integrals for the expectation value of $X(\phi, \phi^*)$:

$$\chi_R(\phi^*) = \int [d\phi][d\chi] \left[X_R e^{\frac{i}{\hbar}(\mathcal{S} + \hbar M_1)} \right]_{\chi^*=0} \tag{13.10}$$

and

$$\begin{aligned} \chi_R'(\phi^*) &= \int [d\phi][d\chi] \left[X_R - (X_R, f_R)_{\phi,\chi} \right]_{\chi^*=0} \\ &\quad \times \exp \left[\frac{i}{\hbar} \left(\mathcal{S} + \hbar M_1 - (\mathcal{S} + \hbar M_1, f_R)_{\phi,\chi} \right) \right]_{\chi^*=0} \ . \end{aligned} \tag{13.11}$$

Subtracting the first from the second, we can again expand to linear order in the infinitesimal fermion f_R and find

$$\begin{aligned} \delta\chi_R(\phi^*) &= \chi_R'(\phi^*) - \chi_R(\phi^*) \tag{13.12} \\ &= \int [d\phi][d\chi] \left[-X_R(\mathcal{W}, f_R)_{\phi,\chi} e^{\mathcal{W}} - (X_R, f_R)_{\phi,\chi} e^{\mathcal{W}} \right]_{\chi^*=0} \ . \end{aligned}$$

We denoted $\mathcal{W} = \frac{i}{\hbar}(\mathcal{S} + \hbar M_1)$. By writing out the antibrackets, we can bring this in the form

$$\delta\chi_R(\phi^*) = \int [d\phi][d\chi] \left[-\frac{\overleftarrow{\delta} X_R e^{\mathcal{W}}}{\delta \phi^A} \frac{\overrightarrow{\delta} f_R}{\delta \phi_A^*} - \frac{\overleftarrow{\delta} X_R e^{\mathcal{W}}}{\delta \chi^A} \frac{\overrightarrow{\delta} f_R}{\delta \chi_A^*} \right.$$

$$+ X_R \frac{\overleftarrow{\delta} \mathcal{W}}{\delta \phi_A^*} . e^{\mathcal{W}} . \frac{\overrightarrow{\delta} f_R}{\delta \phi^A} + X_R \frac{\overleftarrow{\delta} \mathcal{W}}{\delta \chi_A^*} . e^{\mathcal{W}} . \frac{\overrightarrow{\delta} f_R}{\delta \chi^A}$$

$$\left. + \frac{\overleftarrow{\delta} X_R}{\delta \phi_A^*} \frac{\overrightarrow{\delta} f_R}{\delta \phi^A} e^{\mathcal{W}} + \frac{\overleftarrow{\delta} X_R}{\delta \chi_A^*} \frac{\overrightarrow{\delta} f_R}{\delta \chi^A} e^{\mathcal{W}} \right]_{\chi^* = 0} . \tag{13.13}$$

In analogy with the unregularised derivation of the quantum master equation, we would now like to do partial integrations to arrive at the regularised version of (8.17). Owing to the implicit definition of the measure of the PV fields, it is unclear how the PV fields are to be integrated by parts. Instead, we will use the following property:

$$\int [d\phi][d\chi] \left[\frac{\overleftarrow{\delta} A_R}{\delta \phi_A^*} \frac{\overrightarrow{\delta} B(\phi, \chi)}{\delta \phi^A} + \frac{\overleftarrow{\delta} A_R}{\delta \chi_A^*} \frac{\overrightarrow{\delta} B(\phi, \chi)}{\delta \chi^A} \right] = 0 , \tag{13.14}$$

for any $A(\phi, \phi^*)$ and $B(\phi, \phi^*, \chi, \chi^*)$. The proof of this property, which is based on the freedom to redefine integration variables, is given in the appendix of *this* chapter. As a first consequence of this lemma, the first line of (13.13) is seen to vanish identically ($A = f$). Also, the third line of (13.13) is equal to

$$\int [d\phi][d\chi] \left[- \frac{\overleftarrow{\delta} X_R}{\delta \phi_A^*} \frac{\overrightarrow{\delta} \mathcal{W}}{\delta \phi^A} . e^{\mathcal{W}} . f_R - \frac{\overleftarrow{\delta} X_R}{\delta \chi_A^*} \frac{\overrightarrow{\delta} \mathcal{W}}{\delta \chi^A} . e^{\mathcal{W}} . f_R \right]_{\chi^* = 0} , \tag{13.15}$$

by taking $A_R = X_R$ and $B = f_R e^{\mathcal{W}}$. As for the second line of (13.13), instead of A_R we now have $S_R + S_M + \hbar M_1$. The extra contribution to the logarithm of the Jacobian is proportional to $\Delta_\phi (S_M + \hbar M_1)$ and would lead to $\mathcal{O}(\hbar^2)$ corrections to the quantum extended action, which we drop. We find that the second line of (13.13) equals

$$\int [d\phi][d\chi] \quad \left[- X_R \frac{\overleftarrow{\delta} \mathcal{W}}{\delta \phi_A^*} \frac{\overrightarrow{\delta} \mathcal{W}}{\delta \phi^A} . e^{\mathcal{W}} . f_R - X_R \frac{\overleftarrow{\delta} \mathcal{W}}{\delta \chi_A^*} \frac{\overrightarrow{\delta} \mathcal{W}}{\delta \chi^A} . e^{\mathcal{W}} . f_R \right.$$

$$\left. + \frac{\overleftarrow{\delta} X_R}{\delta \phi^A} \frac{\overrightarrow{\delta} \mathcal{W}}{\delta \phi_A^*} . e^{\mathcal{W}} . f_R + \frac{\overleftarrow{\delta} X_R}{\delta \chi^A} \frac{\overrightarrow{\delta} \mathcal{W}}{\delta \chi_A^*} . e^{\mathcal{W}} . f_R \right]_{\chi^* = 0} , \tag{13.16}$$

if $A_R = \mathcal{W}$ and $B = X_R . e^{\mathcal{W}} . f_R$.

If we combine (13.15) and (13.16), we find the one loop regularised version of (8.17):

$$\delta \chi_R(\phi^*) = \int [d\phi][d\chi] \left[X_R \frac{1}{2} (\mathcal{W}, \mathcal{W})_{\phi, \chi} . e^{\mathcal{W}} . f_R \right.$$

$$\left. + (X_R, \mathcal{W})_{\phi, \chi} . e^{\mathcal{W}} . f_R \right]_{\chi^* = 0} . \tag{13.17}$$

In contrast to (8.17), we see that no Δ operators have appeared, as the Δ_ϕ were always cancelled by Δ_χ terms. Notice that the first term of this very important result should be interpreted as the regularised master equation, while the second term gives a regularised expression for σX. We discuss both expressions in some detail in the next section.

13.3   Discussion

We first consider the (one loop) **regularised master equation**. It is contained in the first term of (13.17):

$$\int [d\phi][d\chi] \left[X_R \frac{(-1)}{2\hbar^2}(\mathcal{S} + \hbar M_1, \mathcal{S} + \hbar M_1)_{\phi,\chi} \cdot e^{\frac{i}{\hbar}(\mathcal{S}+\hbar M_1)} \cdot f_R \right]_{\chi^*=0} = 0. \quad (13.18)$$

As we have already pointed out (13.6), $(\mathcal{S},\mathcal{S})_{\phi,\chi} = 2(S, S_M)_\phi + 2(S_{PV}, S_M)_\chi + \mathcal{O}(\chi^4)$. The only other contribution in (13.18) that is not of second or higher order in $\hbar$ with respect to $(S,S)_\phi$, is $2(S, M_1)$. Taking these two remarks into account, we are led to the *regularised one loop master* equation

$$\hbar(S, M_1)I(PV) + \int [d\chi][(S, S_M)_\phi + (S_{PV}, S_M)_\chi] e^{\frac{i}{\hbar}(S_{PV}+S_M)}|_{\chi^*=0} = 0. \quad (13.19)$$

We denoted $I(PV) = \int [d\chi] \exp[\frac{i}{\hbar}(S_{PV} + S_M)]|_{\chi^*=0}$. At this point, the PV fields can be integrated out (C.4) in the second term, and all steps following (12.20) can be copied. In particular, the limit $M \to \infty$ is understood, as is the introduction of copies of the PV fields, if needed (cfr. (12.25)). By comparing (13.19) with the formal, unregularised one loop master equation, $(S, M_1) - i\Delta S = 0$, we see that the second term of the RHS is to be interpreted as a regularised expression for ΔS [63, 59]:

$$(\Delta S)_R = \frac{i}{I(PV)\hbar} \int [d\chi][(S_{PV}, S_M)_\chi + (S, S_M)_\phi] e^{\frac{i}{\hbar}(S_{PV}+S_M)}|_{\chi^*=0}. \quad (13.20)$$

We denominate this regularised expression by $(\Delta S)_R$.

An important property of the formal, unregularised ΔS is that $(S, \Delta S) = 0$, if S satisfies the classical master equation. This is the translation in the BV language of the Wess-Zumino consistency condition (see section 2 of chapter 11). As we have duplicated the complete structure of BV for the PV fields, we expect this property to be valid for the regularised expression too. In the appendix of [38], it is explicitly proven that $((\Delta S)_R, S) = 0$.

The second term of (13.17) is a one loop **regularised expression for σX**, as follows from a comparison with (8.17). We have

$$(\sigma X)_R I(PV) = \int [d\chi] (X_R, S_R + S_M + \hbar M_1)_{\phi,\chi} e^{\frac{i}{\hbar}(S_{PV}+S_M)}|_{\chi^*=0}. \quad (13.21)$$

Let us consider the integrand in some more detail:

$$(X_R, S_R + S_M + \hbar M_1)_{\phi,\chi} = (X, S)_R + \hbar(X, M_1)_\phi + (X_R, S_M)_{\phi,\chi} + \mathcal{O}(\chi^4). \quad (13.22)$$

The first two terms are the (regularised) order $\hbar$ contribution of the antibracket part of σX, $(X, S + \hbar M_1)$. The remaining term is then interpreted as the regularised version of $-i\hbar\Delta X$:

$$(\Delta X)_R = \frac{i}{I(PV)\hbar} \int [d\chi][(X_{PV}, S_M)_\chi + (X, S_M)_\phi] e^{\frac{i}{\hbar}(S_{PV}+S_M)}|_{\chi^*=0}. \quad (13.23)$$

For $X = S$, we find back the expression for $(\Delta S)_R$. Possible applications of this result are 1) infinitesimal canonical transformations, under which the quantum extended action transforms by the addition of σf (8.11) and 2) the study of the quantum cohomology at one loop.

Again, the properties derived for the formal, unregularised σX like nilpotency etc. have to be verified for the regularised expression. This remains to be done for the general result. In the next section, we will give an example of a regularised calculation of σf for an infinitesimal canonical transformation. In this example, it is easy to see that $(\sigma(\sigma f)_R)_R = 0$.

13.4   Example

We consider an example in two dimensions. We sketch the first step of the bosonisation procedure that is presented in [64], from the BV point of view (see section 4 of chapter 8). We start from an action[2] containing massless fermions, with an external source A_μ coupled to their axial current:

$$S_0 = i\bar\psi\not\partial\psi + A_\mu\bar\psi\gamma^\mu\gamma^5\psi \; . \tag{13.24}$$

An integration over two dimensional space-time is understood. Including also a source term for the vector current, would only make the algebra slightly more complicated.

We enlarge the set of fields with an extra scalar degree of freedom $\alpha(x)$. As this field is not present in the classical action, S_0 is invariant under arbitrary shifts of this scalar field. We introduce a ghost c for this shift symmetry and consider the extended action

$$S = i\bar\psi\not\partial\psi + A_\mu\bar\psi\gamma^\mu\gamma^5\psi + \alpha^* c \; . \tag{13.25}$$

We now have a gauge symmetry that has to be gauge fixed. Therefore, we consider the non-minimal solution of the master equation

$$S_{n.m.} = S + b^*\lambda \; , \tag{13.26}$$

and do the canonical transformation generated by $F = \mathbb{1} - b\alpha$. We get the gauge fixed action

$$S_{com} = i\bar\psi\not\partial\psi + A_\mu\bar\psi\gamma^\mu\gamma^5\psi - bc - \alpha\lambda + \alpha^* c + b^*\lambda \; . \tag{13.27}$$

Neither the ghost action bc nor the term fixing α to zero lead to propagating degrees of freedom, so that we only have to introduce PV fields for the fermion fields ψ and $\bar\psi$. We denote the PV fields by χ and $\bar\chi$.

Let us now disguise the shift symmetry by doing an infinitesimal canonical transformation generated by

$$F = \mathbb{1} - i\psi^{*'}\epsilon\alpha\gamma^5\psi - i\bar\psi\epsilon\alpha\gamma^5\bar\psi^{*'} \; . \tag{13.28}$$

Here, ϵ is a global, infinitesimal parameter. This canonical transformation is an infinitesimal chiral rotation, as can be seen from the transformation rules it generates:

$$\begin{aligned} \psi &= (1 + i\epsilon\alpha\gamma^5)\psi' \\ \bar\psi &= \bar\psi'(1 + i\epsilon\alpha\gamma^5) \; . \end{aligned} \tag{13.29}$$

---

[2]We take the following conventions. The two Dirac γ^0 and γ^1 matrices satisfy the anticommutation relation $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$. We define $\gamma^5 = \gamma^0\gamma^1$ and $\gamma^\mu\gamma^5 = \epsilon^\mu{}_\nu\gamma^\nu$ with $\epsilon^\mu{}_\nu g^{\nu\sigma} = \epsilon^{\mu\sigma}$ the antisymmetric 2x2 tensor with $\epsilon^{01} = -1$. We also have that $\{\gamma^\mu,\gamma^5\} = 0$.

However, also a transformation for α^* is generated:

$$\alpha^* = \alpha^{*'} - i\psi^{*'}\epsilon\gamma^5\psi' - i\bar{\psi}'\epsilon\gamma^5\bar{\psi}^{*'} \tag{13.30}$$

up to corrections of $\mathcal{O}(\epsilon^2)$. With these infinitesimal transformations, we find that

$$(f, S) = i\psi^*\epsilon\gamma^5 c\psi - i\bar{\psi}\epsilon\gamma^5 c\bar{\psi}^* - \epsilon\partial_\mu\alpha\bar{\psi}\gamma^\mu\gamma^5\psi , \tag{13.31}$$

with $f = -i\psi^*\epsilon\alpha\gamma^5\psi - i\bar{\psi}^{*a}\epsilon\alpha\gamma^5_{ba}\bar{\psi}^b$.

We also have to take into account the Jacobian of this transformation, i.e. we have to calculate $(\Delta f)_R$. As we already pointed out, we only have PV fields for the fermions, so we have

$$f_R = f - i\chi^*\epsilon\alpha\gamma^5\chi - i\bar{\chi}^{*a}\epsilon\alpha\gamma^5_{ba}\bar{\chi}^b . \tag{13.32}$$

The PV action for the fermions is given by

$$S_{PV} = i\bar{\chi}\slashed{\partial}\chi + A_\mu\bar{\chi}\gamma^\mu\gamma^5\chi , \tag{13.33}$$

and we take the mass term that is invariant under ordinary phase rotations $S_M = -M\bar{\chi}\chi$.

We can now calculate $(\Delta f)_R$. $(f, S_{PV})_\phi = 0$, but we have a contribution

$$(f_{PV}, S_M)_\chi = -2i\epsilon\alpha M\bar{\chi}\gamma^5\chi . \tag{13.34}$$

Hence, we find that

$$\begin{aligned}
(\Delta f)_R &= \frac{2\epsilon\alpha M}{\hbar I(PV)} \int [d\chi] \; \bar{\chi}\gamma^5\chi \; e^{\frac{i}{\hbar}(S_{PV}+S_M)} \\
&= 2i\epsilon\alpha M \, \mathrm{tr}\left[\gamma^5 \frac{1}{i\slashed{\partial} + \slashed{A}\gamma^5 - M}\right] ,
\end{aligned} \tag{13.35}$$

where in the last step we used again (C.4) ($x = -2$). At this stage, we can not yet use (12.22) to obtain a Gaussian damping regulator. If we symbolically write $\mathcal{O} = i\slashed{\partial} + \slashed{A}\gamma^5$, we have (using (C.10)):

$$M\mathrm{tr}[\gamma^5\frac{1}{\mathcal{O} - M}] = M\mathrm{tr}[\gamma^5(\mathcal{O} + M)\frac{1}{\mathcal{O}^2 - M^2}] . \tag{13.36}$$

As we consider the limit $M \to \infty$, only the second term (with M^2 in the numerator) survives and we get [86]

$$(\Delta f)_R = 2i\epsilon\alpha\mathrm{tr}\left[\gamma^5\frac{1}{\mathcal{O}^2/M^2 - 1}\right] \tag{13.37}$$

with $\mathcal{O}^2 = -\square + 2(iA_\nu\epsilon^{\nu\mu})\partial_\mu - A^2 + i\slashed{\partial}\slashed{A}\gamma^5$. We denoted $\slashed{\partial}\slashed{A} = \gamma^\mu\gamma^\nu\partial_\mu A_\nu$. We now go through the familiar steps to obtain a Fujikawa type expression (12.22) and use the results of the appendix (C.17). As we work in two dimensions, and provided additional PV fields are introduced to remove terms proportional to M^2, we only need E_2 (C.17). We find that $\mathcal{O}^2 = -(\partial_\mu + Y_\mu)(\partial^\mu + Y^\mu) - E$ for $Y^\mu = iA_\nu\epsilon^{\nu\mu}$ and $E = -\alpha^2 - \partial^\mu\alpha_\mu + A^2 - i\slashed{\partial}\slashed{A}\gamma^5$. Only the last term in E contributes as $\mathrm{tr}\gamma^5 = 0$. Therefore, we have that

$$\begin{aligned}
(\Delta f)_R &= -2i\epsilon\alpha\mathrm{tr}[\gamma^5 E_2] \\
&= \frac{\epsilon}{2\pi}\partial^\mu\alpha.A_\mu.
\end{aligned} \tag{13.38}$$

The quantum extended action after canonical transformation is then seen to be

$$
\begin{aligned}
\tilde{W} &= S_{com} + \sigma f \\
&= S_{com} + i\psi^* \epsilon \gamma^5 c\psi - i\bar{\psi}\epsilon\gamma^5 c\bar{\psi}^* \\
&\quad - \epsilon \partial_\mu \alpha \bar{\psi}\gamma^\mu \gamma^5 \psi - \frac{i\hbar}{2\pi}\epsilon\partial^\mu \alpha . A_\mu \ .
\end{aligned}
\tag{13.39}
$$

Let us mention two technical aspects of this result. The transformation of α^* has generated ψ^* and $\bar{\psi}^*$ terms. Canonical transformations produce the transformation rules in the transformed coordinates. Furthermore, it is not difficult to see that $(\sigma(\sigma f)_R)_R = 0$.

As for the physical point of view, we see that the Jacobian gives a term of the form $\hbar \partial^\mu \alpha . A_\mu$. We find back the typical bosonisation rule that the axial current of the fermions gets replaced by $\partial^\mu \alpha$. If we also include an external source V^μ for the vector current of the fermions, a term $\epsilon_{\mu\nu}V^\mu \partial^\nu \alpha$ appears. However, in $\tilde{W}$ a term $\alpha\lambda$ is present that fixes α to zero, so that all the α-dependent terms are effectively zero. The second step of the bosonisation procedure, as described in [64], consists of changing the gauge fixing. Instead of the gauge $\alpha = 0$, a gauge is chosen where some fermion degrees of freedom are fixed such that the fermion currents decouple from the sources. This way, it is seen that 2D bosonisation is a matter of gauge choice.

Appendix

Here we give a proof of the property (13.14) used in the main text. We start from an integral

$$
I = \int [d\phi][d\chi] B(\phi, \chi) \ ,
\tag{13.40}
$$

where the ϕ are ordinary fields and χ PV fields. The latter statement implies that $[d\chi]$ has the tranformation properties derived in the previous chapter (12.57). Consider any quantity $A_R = A + A_{PV}$, and let us redefine the integration variables as

$$
\begin{aligned}
\phi^A &\rightarrow \phi^A + \epsilon . \frac{\overleftarrow{\delta} A_R}{\delta \phi^*_A} \\
\chi^A &\rightarrow \chi^A + \epsilon . \frac{\overleftarrow{\delta} A_R}{\delta \chi^*_A} \ ,
\end{aligned}
\tag{13.41}
$$

where ϵ is an infinitesimal parameter. As was already dicussed in the main text, such redefinitions lead to a Jacobian 1 (up to higher orders in $\hbar$), owing to the definition of A_R and of the measure $[d\chi]$. We then have

$$
\int [d\phi][d\chi] B(\phi, \chi) = \int [d\phi][d\chi] B(\phi + \epsilon . \frac{\overleftarrow{\delta} A_R}{\delta \phi^*_A}, \chi + \epsilon . \frac{\overleftarrow{\delta} A_R}{\delta \chi^*_A}) \ ,
\tag{13.42}
$$

which immediately implies the result

$$
\int [d\phi][d\chi] \left[\frac{\overleftarrow{\delta} A_R}{\delta \phi^*_A} \frac{\overrightarrow{\delta} B(\phi, \chi)}{\delta \phi^A} + \frac{\overleftarrow{\delta} A_R}{\delta \chi^*_A} \frac{\overrightarrow{\delta} B(\phi, \chi)}{\delta \chi^A} \right] = 0.
\tag{13.43}
$$

Chapter 14

Anomalous theories

We have seen how we can detect genuine anomalies, at least to one-loop order. A theory has a genuine, one loop-anomaly if the one-loop master equation $(S, M_1) - i\Delta S = 0$ can not be solved for a local M_1. As was already stressed a few times, we can not derive the Ward identity $\langle \sigma X \rangle = 0$ in that case, such that the proofs of unitarity and renormalisability are jeopardized. However, in the last decade, a large effort has been devoted to attempts to proceed further with anomalous theories. These developments are particularly stimulated by the interest in non-critical string theories, i.e. string theories where one does *not* work in the specific dimensions (26 for the bosonic string) required for the vanishing of the anomaly. We will first briefly discuss the method of background charges in conformal field theory. In the second section of this chapter, we discuss in extenso the appearance of extra degrees of freedom in the case of an anomalous theory [37].

14.1   Background charges

Instead of expanding the quantum master equation in powers of $\hbar$ as Ansatz, we can also try $W = S + \sqrt{\hbar}M_{1/2} + \hbar M_1 + \ldots$ [37]. When this expansion is plugged in in the quantum master equation, we get the hierarchy of equations

$$
\begin{array}{lll}
\hbar^0 & (S, S) = 0 & \\
\hbar^{1/2} & (S, M_{1/2}) = 0 & \\
\hbar & (S, M_1) + \frac{1}{2}(M_{1/2}, M_{1/2}) = i\Delta S & \\
\ldots & &
\end{array}
\tag{14.1}
$$

Let us again consider the example of W_2 gravity, where we take the matter fields ϕ to be the only propagating quantum fields. The extended action is given by (5.54):

$$
\begin{aligned}
S &= -\frac{1}{2\pi}\phi\partial\nabla\phi + \phi^* c\partial\phi \\
&\quad + h^*(\bar{\partial}c - h\partial c + \partial h.c) + c^*(\partial c)c.
\end{aligned}
\tag{14.2}
$$

It is easy to see that for

$$
M_{1/2} = a(h\partial^2\phi + \pi\phi^*\partial c) ,
\tag{14.3}
$$

with a an arbitrary constant, we have that $(S, M_{1/2}) = 0$. Furthermore,

$$\frac{1}{2}(M_{1/2}, M_{1/2}) = -a^2\pi.c\partial^3 h . \tag{14.4}$$

As ΔS is also proportional to $c\partial^3 h$ for this model (12.30), we can choose the constant a such that

$$\frac{1}{2}(M_{1/2}, M_{1/2}) - i\Delta S = 0 . \tag{14.5}$$

Therefore, no (non-local) counterterm M_1 is needed for the $\mathcal{O}(\hbar)$ master equation to be satisfied. This is the translation in the BV language of the method of *background charges* familiar in conformal field theory (see e.g. [94])[1]. For a more elaborate example, where background charges were used to cancel the W_3 one loop-anomaly, we refer to [95] and to its translation in BV [38].

14.2   Hiding anomalies

As we already heuristically argued above (11.4), the most interesting feature of gauge theories with genuine anomalies is that some degrees of freedom that can be fixed to zero classically, start propagating at the quantum level. When faced with an anomalous theory, at least two strategies have been followed in the literature up to now. The first strategy (for examples, see e.g. [39, 96]) consists in making a judicious choice of the order in which the functional integrals are done. Consider for instance the W_2 example, with the classical action the antifield independent part of (14.2). Both the matter field ϕ and the field h are considered to be dynamical, i.e. are integrated over in the functional integral. If we stipulate that one first has to do the ϕ integral, we see that h gets a non-trivial action, the induced action $\Gamma[h]$ (12.31). Thereafter, the integral over h can be done. As is generally the case, the induced action that one obtains, is non-local. This need not worry us, precisely because we know this non-local action is induced by a local quantum field theory. Moreover, it has been shown that many of the (regularisation) procedures of local field theory can be transplanted to non-local theories without any problem [97].

The second approach started with the work of L.D. Faddeev [98]. He argued that anomalies make the first class constraints of the gauge symmetries in the Hamiltonian formalism second class and that they can be made first class again by the introduction of extra degrees of freedom. Later, it was recognised that in the functional integral approach to quantisation, these extra degrees of freedom arise naturally. Indeed, the integration over the volume of the gauge group does not factor out when following the Faddeev-Popov [11] procedure for gauge fields coupled to chiral fermions [99]. The integration measure for the fermions has an (anomalous) dependence on the gauge variables, effectively producing the Wess-Zumino action for the extra variables.

In [100], the idea of adding extra degrees of freedom has been implemented in the BV scheme. For every anomalous symmetry, an extra field is introduced. The transformation rules for these new fields under the original symmetry are chosen such that a local M_1 can be constructed to solve the quantum master

---

[1]We can interpret the terms of $\sqrt{\hbar}M_{1/2}$ as follows. The term $a\sqrt{\hbar}\partial^2\phi.h$ indicates that the energy-momentum tensor T that is coupled to h gets a correction $\sim \sqrt{\hbar}\partial^2\phi$. The transformation rule for ϕ generated by T also changes, it gets an extra term $\sim \sqrt{\hbar}\partial c$. This is expressed by the extra term $a\sqrt{\hbar}\pi\phi^*\partial c$ in $\sqrt{\hbar}M_{1/2}$.

equation to one loop. In other words, by adding extra fields, and by choosing their BRST transformation rules, ΔS is made BRST exact. This M_1 provides the dynamics for these extra fields. In this approach, the anomaly has apparently disappeared, where the price to pay is a minor change of the classical theory.

We [37] show here that these extra degrees of freedom can be introduced *without changing the classical theory* in the cohomological sense. The choice that one makes for the transformation rules of the extra fields under the original symmetries, is determined by the condition that one can construct a PV mass term, using these extra fields, that is invariant under these symmetries. However, together with the new fields, one has also introduced new symmetries (to keep the classical cohomology unchanged) and the PV mass term is in general not invariant under these new symmetries. As a consequence, the anomaly has not disappeared, but is shifted to these extra symmetries. Since these extra symmetries are often ignored in the literature, the anomaly is in fact hidden in this way.

14.2.1 Discussion of our method

Suppose that we start from an extended action $S[\phi^A]$, that is a proper solution of the classical master equation $(S, S) = 0$. After having set up the PV regularisation (see the two previous chapters), we can calculate the operator ΔS. At least for theories with an irreducible gauge algebra, the result is always of the form

$$(\Delta S)_{reg} = c^\alpha \mathcal{A}_\alpha, \tag{14.6}$$

at least up to antifield dependent terms. Here, the values that α takes in the sum, depend on the invariances of the PV mass term. In other words, the mass term determines which symmetries are anomalous. Let us assume that we have a genuine anomaly, i.e. that no local functional M_1 exists such that $(S, M_1) = i(\Delta S)_{reg}$.

Now we propose to add trivial systems to S, one for every anomalous gauge symmetry:

$$S \to S + \theta_\alpha^* d^\alpha . \tag{14.7}$$

θ^α has the Grassmann parity ϵ_α and d^α has the Grassmann parity $\epsilon_\alpha + 1$. Although these extra fields, and the entailing extra symmetries, are completely trivial at this point, regularisation in the quantum theory will interfere with this. The new degrees of freedom clearly do not change the classical theory, as they are cohomologically trivial for the (new) cohomology-operator $(S, \cdot)$.

At this point, we are still free to specify how these newly introduced fields transform under the original symmetries related to the $c-$ghosts. This choice can be encoded in the extended action by doing a canonical transformation. Remember that canonical transformations do not change the (classical) cohomology (8.7). Our approach is to choose this transformation such that a local PV-mass term can be constructed that *is* invariant under the c-symmetries.

Suppose that one would like to have the transformation rules

$$\delta_c \theta^\alpha = f^\alpha(\phi^A, \theta) \tag{14.8}$$

for the θ^α fields. This can be achieved simply by taking as generating fermion for the canonical transformation

$$F = \mathbb{1} - d_a^{*'} f^a(\phi^A, \theta) \ . \tag{14.9}$$

The fact that we used a canonical transformation guarantees that the transformed extended action is still a solution of the classical master equation, i.e. the new action is still BRST invariant with the modified transformation rules. It is also important to remark that the extended action after the transformation still contains terms with the d ghosts. These are necessary to ensure the properness of the action. It is clear that, if we use the extra θ fields to construct a PV mass term that is invariant under the c-symmetries, it can not also be invariant under the d-symmetries, which shift θ. As a result the anomalies will have been shifted to the new symmetries and one finds

$$(\Delta S)_{reg} = d^b \mathcal{B}_b \ . \tag{14.10}$$

It is only when one neglects the d-symmetries that one would conclude that there are no anomalies left.

To carry out calculations, one may still want to use the old (c non-invariant) mass-term, for technical or other reasons. The anomaly will still be left in the d-symmetries, if a counterterm is added that matches the interpolation between the two different regularisations, viz the two mass terms. This has extensively been discussed above (12.43). In practise, it is this counterterm that provides non-trivial dynamics for the variables that were introduced as a trivial system (Wess-Zumino term).

The next step is to integrate over these extra fields θ^α, as they acquired non-trivial dynamics. However, at this moment, it is not clear what measure one should take for these extra fields. This is related to the fact that there may be different ways to introduce extra fields (see the example in the next subsection). Clearly, a guiding principle can be that one tries to construct the complete theory to be invariant under the c-symmetries.

14.2.2   Example

We again use the W_2 gravity model as an example. Most of the technical manipulations are the same as those discussed in the sections 2 and 3 of chapter 12, and will therefore not be repeated in detail. However, there is an important difference. In chapter 12, we constructed a local invariant mass term for the PV fields using the Polyakov variable f, that is a reparametrisation of h. Here, we introduce an extra scalar field in the theory to construct a local invariant mass term.

We start from the extended action for W_2 gravity (14.2), and consider only the matter field ϕ as a quantum field. The PV field for ϕ is denoted by χ, and we have $S_{PV} = -1/(2\pi)\chi\partial\nabla\chi + \chi^* c\partial\chi$ (13.1). To complete the regularisation scheme, we try to construct a PV mass term that is invariant under the c-symmetry, where we allow ourselves to introduce an extra field θ with a transformation rule $\delta_c\theta$ that may be chosen. It is easy to see that

$$S_M = -\frac{1}{4\pi}.M^2.\chi^2.e^\theta \tag{14.11}$$

is invariant under the c-symmetry if we take $\delta_c\theta = c\partial\theta + \partial c$. We also could have taken

$$S_M = -\frac{1}{4\pi}.M^2.\chi^2.\xi \qquad (14.12)$$

with $\delta_c\xi = \partial(c\xi)$. It is clear that introducing a flat measure in the functional integral for ξ differs from the introduction of a flat measure for θ by a Jacobian of the redefinition by $\xi = e^\theta$.

Let us now calculate the anomaly, using the first invariant mass term. Define $S_M^{(\alpha)} = -1/(4\pi).M^2.\chi^2.e^{\alpha\theta}$. For $\alpha = 0$, we have the c non-invariant mass term while for $\alpha = 1$ we have the conformal c invariant mass term, if $\delta_c\theta = c\partial\theta + \partial c$. We add a trivial system and consider the extended action $\tilde{S} = S + \theta^*d$. The field d is the ghost field for the shift symmetry of the extra field θ. In order to have a term $\theta^*\delta_c\theta$ in the extended action, we do a canonical transformation generated by

$$F = \mathbb{1} - d^{*'}(\partial c + c.\partial\theta). \qquad (14.13)$$

After the canonical transformation we have

$$\tilde{S}' = S + \theta^*(d + \partial c + c\partial\theta) + d^*.\partial d.c . \qquad (14.14)$$

This action is the same as in [100], except for the terms containing the d-ghost. These terms are needed to assure that we have a proper solution of the classical master equation. They also guarantee that the classical cohomology of the original theory is the same as the classical cohomology of the theory with the extra field θ.

We now calculate the anomaly due to the matter fields[2]. We use (13.23), for $X = S$. The first ingredient is

$$(\tilde{S}'_R, S_M^{(\alpha)})_{\phi,\chi} = \frac{M^2}{4\pi}[\partial c(1-\alpha) - \alpha d].e^{\alpha\theta}.\chi^2 . \qquad (14.15)$$

Copying the steps of section 2 of chapter 12, we find

$$(\Delta S)_R(\alpha) = -\frac{1}{24\pi}\int d^2x\ [\partial c(1-\alpha) - \alpha d][\partial^2 h - \alpha\partial\nabla\theta] . \qquad (14.16)$$

For $\alpha = 0$, we find back (12.30), but for $\alpha = 1$, we have

$$(\Delta S)_R(\alpha = 1) = \frac{1}{24\pi}\int d^2x\ d.[\partial^2 h - \partial\nabla\theta] . \qquad (14.17)$$

The c-symmetry is seen to be anomaly free, the anomaly has been shifted to the extra symmetry that comes with the extra field.

Instead of using the c invariant mass terms ($\alpha = 1$) one might prefer to use the c non-invariant mass term ($\alpha = 0$) in practical calculations. The anomaly will still be proportional to the d-ghost, if we add a counterterm that compensates the change of mass term. The way to calculate such counterterms is described in section 3 of chapter 12. This counterterm provides non-trivial dynamics for the extra field θ. We find

$$M_1 = \frac{1}{24\pi}\int d^2x\ \left[-\frac{1}{2}\theta\partial\nabla\theta + h\partial^2\theta\right] . \qquad (14.18)$$

---

[2]We work again in Euclidean space and $\hbar = 1$.

Such counterterms are named *Wess-Zumino terms*. In the literature, one often introduces extra fields and adds a Wess-Zumino term to make the anomaly BRST exact. As we have shown, the anomaly has not disappeared, it has been shifted to extra symmetries that come with the extra fields and that are needed to keep the classical theories cohomologically equivalent. Moreover, the presence of the Wess-Zumino term involves a choice: one works with the c non-invariant mass term and the counterterm is added to shift the symmetry to the d symmetries.

Conclusions

After this extensive discussion of the **Batalin-Vilkovisky scheme**, we can list some of its **assets**, as was promised 14 chapters ago in the introduction.

- First of all, whatever the properties of the gauge algebra are, be it an open or a closed, a reducible or an irreducible algebra, in the BV scheme one has to solve the classical master equation $(S, S) = 0$ as the first step in the quantisation process (chapters 5 and 6). This way, the BV scheme provides a unified approach for all types of gauge theories.

- To formulate the classical master equation, we have already used the most remarkable feature of the BV scheme. The fields and antifields are canonically conjugated with respect to a Grassmann odd symplectic bracket, the antibraket (chapters 5 and 8). All manipulations that constitute the BRST quantisation formalism (chapter 3) can be rephrased using antibrackets *and* their canonical transformations (chapter 8): gauge fixing, taking a different set of gauge generators, the cohomology of gauge independent operators, Ward identities, ...

- The antifields serve many purposes in the BV scheme. One is that they act as sources for the BRST transformation of their associated field. This allows, for instance, for a natural derivation of the Zinn-Justin equation for the generating function of connected diagrams in the BV scheme (chapter 8). The equation holds not only for Yang-Mills theory, its original application area, but for all types of gauge theories, provided that the theory is anomaly free.

- Moreover, owing to the antifields, one does not have to choose a gauge. If one keeps track of all antifield dependence in practical calculations, one can afterwards always transform the result obtained in one gauge (one set of canonical coordinates) to other gauges by canonical transformations. For one thing, this prevents the accidental vanishing of anomalies that may be caused by gauge fixing symmetries that are anomalous. In the BV scheme, the anomaly will then be function of the antifields.

- Finally, both the classical and the quantum cohomology contain respectively the classical and quantum equations of motion.

Given all these assets of the BV scheme, we found it worthwhile to give a derivation of the main features of the scheme with a strong emphasis on their relation with the analogous constructions in the BRST quantisation prescriptions. This way it is clear that the BV scheme is an elegant, unified reformulation of previously known (BRST) quantisation recipes. Here, the BV scheme

is constructed by imposing that the BRST symmetry of a gauge theory be enlarged such that the Schwinger-Dyson equations for all fields can be obtained as Ward identities of this enlarged BRST symmetry. This requirement can be implemented using a collective field formalism. Owing to a doubling of the configuration space in that formalism, a new gauge symmetry –the Schwinger-Dyson (SD) shift symmetry– is present, and the antighosts introduced to gauge fix this symmetry become the antifields of the BV scheme. Starting from the BRST quantisation prescription, we have derived both for closed and open algebras the classical and quantum master equation of the BV scheme. Especially for open algebras, the collective fields are seen to play a crucial role in the construction of a BRST invariant, gauge fixed action.

Using a slightly modified collective field formalism, we have constructed an antifield scheme for BRST–anti-BRST invariant quantisation. Instead of one, one has three antifields for every field in that case.

We have also shown how the Lagrangian Batalin-Vilkovisky scheme follows from the Hamiltonian description of gauge theories. Here too, our guiding principle has been that the BRST symmetry of the theory has to be enlarged to include the SD shift symmetry. After integration over the momenta of the Hamiltonian formalism, a Lagrangian extended action is obtained that satisfies the quantum master equation of the BV scheme, owing to the gauge independence of the Hamiltonian path integral.

In a third part of this dissertation, some aspects of a one-loop regularised study of anomalies are presented. For that purpose, we have used a Pauli-Villars (PV) regularisation scheme. In particular, we have studied the effect of the mass term of the regulating PV fields on the actual expression of the gauge anomaly. We have shown how in some examples, preferred symmetries can be kept anomaly free by constructing a mass term that is invariant under these symmetries. This can be done by using either reparametrisations of fields that are already present in the field content of the theory or by introducing extra fields. We have also used PV regularisation to give a new derivation of the one-loop regularised quantum master equation and of the one-loop regularised quantum BRST operator of the BV scheme.

Some results where obtained in the study of concrete models as well, using the general methods that are described. We have reexamined the construction of four-dimensional topological Yang-Mills theory, taking advantage of the many uses of the canonical transformations of the BV scheme. Prompted by this example, we have derived a general recipe for a classical and quantum BRST invariant energy-momentum tensor in the BV scheme. We have studied induced W_2 gravity. Using the Polyakov variable, we can construct an invariant mass term for the regularisation of this model, which was exploited to calculate the induced action. This method provides more insight on why the Polyakov reparametrisation leads to a local expression for the induced action.

Part IV

Appendices

Appendix A

Grassmannology

With every field ϕ^A of the configuration space, we associate a Grassmann parity, which we denote by ϵ_A. ϵ_A is an element of $\mathbb{Z}_2$. By definition, $\epsilon_A = 0$ for a bosonic degree of freedom, and $\epsilon_A = 1$ for a fermionic one. The antifield of ϕ^A, the field ϕ^*_A, has Grassmann parity $\epsilon_A + 1$. Although our notation seems to allow for the possibility that the Grassmann parity of a field depends on the space-time point, this will never be the case.

The Grassmann parity of a product of two or more fields is the sum of their Grassman parities:

$$\epsilon_{\prod_i A_i} = \sum_i \epsilon_{A_i}. \tag{A.1}$$

The Grassmann parities are mainly introduced to keep track of the signs which appear when exchanging places between two fields (or monomials of fields):

$$X.Y = (-1)^{\epsilon_X \cdot \epsilon_Y} Y.X, \tag{A.2}$$

so only when both X and Y are fermionic, an extra minus sign has to be taken into account. The relation between the right derivative $\frac{\overleftarrow{\delta} X}{\delta \phi^A}$ and the left derivative $\frac{\overrightarrow{\delta} X}{\delta \phi^A}$ is

$$\frac{\overleftarrow{\delta} X}{\delta \phi^A} = (-1)^{(\epsilon_X + 1)\epsilon_A} \frac{\overrightarrow{\delta} X}{\delta \phi^A}. \tag{A.3}$$

In the Leibnitz rule for directional derivatives, the signs are as follows:

$$\frac{\overleftarrow{\delta}}{\delta \phi^A}(FG) = F.\frac{\overleftarrow{\delta} G}{\delta \phi^A} + (-1)^{\epsilon_G \epsilon_A}\frac{\overleftarrow{\delta} F}{\delta \phi^A}.G$$

$$\frac{\overrightarrow{\delta}}{\delta \phi^A}(FG) = \frac{\overrightarrow{\delta} F}{\delta \phi^A}.G + (-1)^{\epsilon_F \epsilon_A}F.\frac{\overrightarrow{\delta} G}{\delta \phi^A}. \tag{A.4}$$

Two directional derivatives both acting from the left or from the right do not commute. We have for instance,

$$\frac{\overrightarrow{\delta}}{\delta \phi^A}\frac{\overrightarrow{\delta}}{\delta \phi^B}X = (-1)^{\epsilon_A \epsilon_B}\frac{\overrightarrow{\delta}}{\delta \phi^B}\frac{\overrightarrow{\delta}}{\delta \phi^A}X. \tag{A.5}$$

However,

$$\frac{\overrightarrow{\delta}}{\delta \phi^A}\frac{\overleftarrow{\delta}}{\delta \phi^B}X = \frac{\overleftarrow{\delta}}{\delta \phi^B}\frac{\overrightarrow{\delta}}{\delta \phi^A}X. \tag{A.6}$$

Finally, $\epsilon_{\frac{\delta X}{\delta \phi^A}} = \epsilon_X + \epsilon_A$, whatever the derivative.

Appendix B

Properties of the antibracket and the $\triangle$-operator

As defined in the main text (5.25), the **antibracket** of two function(al)s F and G of arbitrary Grassmann parity is given by

$$(F, G) = \frac{\overleftarrow{\delta} F}{\delta \phi^A} \frac{\overrightarrow{\delta} G}{\delta \phi_A^*} - \frac{\overleftarrow{\delta} F}{\delta \phi_A^*} \frac{\overrightarrow{\delta} G}{\delta \phi^A}. \tag{B.1}$$

It is then clear that $\epsilon_{(F,G)} = \epsilon_F + \epsilon_G + 1$. Another obvious property of the antibracket is that $\mathrm{gh}\,((F, G)) = \mathrm{gh}\,(F) + \mathrm{gh}\,(G) + 1$, since the sum of the ghostnumber of a field with the ghostnumber of its antifield is -1.

From the properties of directional derivatives listed in the previous appendix, we can derive that

$$(G, F) = (-1)^{\epsilon_F \epsilon_G + \epsilon_F + \epsilon_G} (F, G). \tag{B.2}$$

It follows that for any function(al) F of odd Grassmann parity $(F, F) = 0$ trivially. In contrast to the Poisson brackets of classical mechanics, the antibracket of a bosonic B function(al) –like the extended action– with itself is not necessarily zero. Indeed,

$$(B, B) = 2 \frac{\overleftarrow{\delta} B}{\delta \phi^A} \frac{\overrightarrow{\delta} B}{\delta \phi_A^*}. \tag{B.3}$$

From the Leibnitz rule for directional derivatives, one easily arrives at

$$(F, GH) = (F, G)H + (-1)^{(\epsilon_F + 1)\epsilon_G} G(F, H). \tag{B.4}$$

The antibracket version of the Jacobi identity is

$$(F, (G, H)) = ((F, G), H) + (-1)^{(\epsilon_F + 1)(\epsilon_G + 1)} (G, (F, H)). \tag{B.5}$$

The **delta-operator** is defined by

$$\Delta X = (-1)^{\epsilon_A + 1} \frac{\overleftarrow{\delta}}{\delta \phi_A^*} \frac{\overleftarrow{\delta}}{\delta \phi^A} X = (-1)^{\epsilon_X} (-1)^{\epsilon_A} \frac{\overrightarrow{\delta}}{\delta \phi_A^*} \frac{\overrightarrow{\delta}}{\delta \phi^A} X, \tag{B.6}$$

and it emerged naturally by integrating out the fields in the collective field

formalism[1] (5.26). Δ is a fermionic operator, $\epsilon_{\Delta X} = \epsilon_X + 1$. The basic properties are

$$
\begin{array}{rllll}
1. & \Delta^2 & = & 0 & \\
2. & \Delta(FG) & = & F\Delta G + (-1)^{\epsilon_G}\Delta F.G + (-1)^{\epsilon_G}(F,G) & \text{(B.8)} \\
3. & \Delta(F,G) & = & (F,\Delta G) - (-1)^{\epsilon_G}(\Delta F, G). &
\end{array}
$$

Using all these results, we can derive the following property of the quantum BRST operator $\sigma X = (X, W) - i\hbar\Delta X$:

$$
\sigma[A.B] = A.\sigma B + (-1)^{\epsilon_B}\sigma A.B - i\hbar(-1)^{\epsilon_B}(A,B). \tag{B.9}
$$

---

[1] Notice that in [59] a slightly different definition of the delta-operator is used:

$$
\Delta X = (-1)^{\epsilon_A}\frac{\overrightarrow{\delta}}{\delta\phi_A^*}\frac{\overrightarrow{\delta}}{\delta\phi^A}X. \tag{B.7}
$$

It is clear that this only differs by a minus sign when the quantity X is fermionic.

Appendix C

Technical details of the anomaly calculation

This rather technical appendix contains a detailed description of the steps needed to calculate an expression for the regularised Jacobian, once the regulator is chosen. First, we show how the Pauli-Villars regularisation leads to Fujikawa-type expressions [86]. The mathematical results needed to evaluate these expressions are then presented. Thereafter, we treat a special case in great detail, which allows an understanding from a low-brow point of view of many of the characteristic features of the mathematical results.

C.1 From Pauli-Villars regularisation to a consistent Fujikawa regulator

Pauli-Villars regularisation typically leads to the evaluation of the following type of functional integral over PV fields (see e.g. (12.20)):

$$I[K] = \int [d\chi] \ \chi^C K_{CD} \chi^D \ e^{-\frac{1}{2}\chi^A D_{AB}\chi^B}. \tag{C.1}$$

Here, the supermatrices of the bosonic type K_{CD} and D_{AB}[1] are considered to be of the form

$$
\begin{aligned}
K_{CD} &= K_{kl}(x).\delta(x-y) \\
D_{AB} &= D_{ij}(x).\delta(x-y),
\end{aligned}
\tag{C.2}
$$

where $K_{kl}(x)$ and $D_{ij}(x)$ are differential operators in the variable x that act on the δ-functions. We have split up the capital indices $A,B,\dots$ in space-time indices $x,y,\dots$ and internal indices $i,j,\dots$ Remember that $\frac{1}{2}\chi^A D_{AB}\chi^B = S_{PV} + S_M$. Hence, D_{AB} is by construction supersymmetric, which means $D_{BA} = D_{AB}(-1)^{\epsilon_A + \epsilon_B + \epsilon_A \epsilon_B}$. On the other hand, K_{CD} has in general no such property.

In order to have a consistent evaluation of the integral $I[K]$, we can derive it from

$$I = \int [d\chi] e^{-\frac{1}{2}\chi^A D_{AB}\chi^B} = (\mathrm{sdet}D)^{x/2}, \tag{C.3}$$

which is defined in the regularisation procedure (12.26). Here, x is the *PV weight* of the field when several copies of PV fields are introduced. $I[K]$ is seen

---

[1]This means that $\chi^A K_{AB}\chi^B$ has even Grassmann parity, and hence $\epsilon_{K_{AB}} = \epsilon_A + \epsilon_B$

to be

$$I[K] = -2K_{CD}\frac{\overrightarrow{\delta}I}{\delta D_{CD}}$$

$$= -2K_{CD}\frac{\overrightarrow{\delta}\,(\text{sdet}D)^{x/2}}{\delta D_{CD}}$$

$$= -x.I.\text{str}\left(K\frac{1}{D}\right). \tag{C.4}$$

In the last step, we used that $\delta\text{sdet}D = \text{sdet}D.\text{str}\left(D^{-1}\delta D\right)$ [24]. Notice that the *PV weight* multiplies the whole expression. Using the explicit form (C.2) of the two operators in the supertrace, we can rewrite this as

$$I[K] = -x.I.\text{str}\int dz \int dz'\, K(z).\delta(z-z').\frac{1}{D(z)}.\delta(z-z'), \tag{C.5}$$

where the str now only runs over the internal indices.

In order to show how this procedure leads to Fujikawa regulators (12.9) [86], we take D_{lk} to be of the form (see e.g. (12.21)):

$$D_{lk}(z) = \mathcal{O}_{lk}(z) - T_{lk}(z)M^2. \tag{C.6}$$

The first term $\mathcal{O}$ contains all the differential operators, and is determined by S_{PV}. The second term T is the matrix determining the mass term of the PV fields. Using the cyclicity of the supertrace, we can rewrite

$$I[K] = -x.I.\text{str}\int dz \int dz'\, J(z).\delta(z-z').\frac{1}{\frac{\mathcal{R}}{M^2}-\mathbb{1}}.\delta(z-z'). \tag{C.7}$$

Here, $J = T^{-1}\mathcal{K}$ and $\mathcal{R} = T^{-1}\mathcal{O}$, where $\mathcal{K}$ is defined by $K = M^2\mathcal{K}$. The operator $\mathcal{R}$ is the regulator in the Fujikawa approach, as can be seen by using

$$\int_0^\infty d\lambda e^{-\lambda}e^{\lambda\mathcal{R}} = -\frac{1}{\mathcal{R}-\mathbb{1}}, \tag{C.8}$$

which leads to

$$I[K] = x.I.\text{str}\int dz \int dz' \int_0^\infty d\lambda e^{-\lambda}J(z).\delta(z-z')e^{-\lambda\frac{\mathcal{R}}{M^2}}.\delta(z-z'). \tag{C.9}$$

In the next section we discuss some mathematical results useful for the evaluation of this supertrace in the limit $M \to \infty$, for the case of a regulator $\mathcal{R}$ that is an elliptic second order differential operator. If this is not the case, see for instance the example in section 4 of chapter 13, one can use the equality

$$\text{str}\left(K\frac{1}{D}\right) = \text{str}\left(KA\frac{1}{(DA)}\right), \tag{C.10}$$

to get a second order differential operator in the denominator, by a judicious choice of the operator A.

C.2 Mathematical results for calculating the supertrace

We first present the main results of [101], which allow us to evaluate the supertrace (C.9). These results belong to the branch of differential geometry where

the properties of elliptic operators are studied. In the mathematical literature the names of Seeley and Gilkey are associated with this study. Physicists have time after time rederived these results from a more algebraic point of view, unaware of the complete results obtained by the mathematicians. One such derivation can be found in the third section of this appendix. In the physics literature, these results are associated with Schwinger, DeWitt, 't Hooft and Veltman, Avramidi ... and they are often denoted by the name *heat kernel expansion*[2].

The results sketched below are derived for bosonic matrices, but they can be generalised for supermatrices. We start from a second order differential operator

$$D = - \left(h^{ij} \frac{d^2}{dx_i dx_j} + a_i \frac{d}{dx_i} + b \right). \tag{C.11}$$

Here, a_i and b can be arbitrary square matrices of dimension r and $h^{ij} = g^{ij} \mathbb{1}_r$, with $\mathbb{1}_r$ the unit matrix of dimension r. With the operator e^{-tD}, where t is a positive parameter, we can associate an integral kernel, which describes how it acts on an arbitrary function $f(x)$:

$$e^{-tD} f(x) = \int dvol(y) \, K(t, x, y) f(y). \tag{C.12}$$

$dvol(y)$ is the Riemannian volume element. In the limit $t \overset{>}{\to} 0$, and for equal points $x = y$, this kernel has the following expansion:

$$K(t, x, x) = \sum_{n=0} E_n(x) t^{\frac{n-d}{2}}. \tag{C.13}$$

Here, d is the dimension of space-time. Notice that the expansion starts with terms of negative power in t, depending on the dimension d. The origin of these terms will become clear in the explicit calculation in the next section. In fact, we are not interested in terms with a strict positive power of t, as they disappear in the limit $t \to 0$, which corresponds to taking the mass M of the PV fields to infinity.

We now rewrite the operator by defining a metric and gauge connections. The quantities E_n can then be expressed using the invariant geometrical objects that can be constructed from these ingredients. This includes the Riemann- and Ricci tensor, the curvature of the gauge fields, covariant derivatives. Although results are available for operators where both a gauge and a metric connection is needed, we will restrict ourselves to the case where we have either a non-flat metric and no gauge connection, or a flat metric and a gauge connection, as these are the only cases encountered in the main text. The explicit example of the third section belongs to the first category.

First, suppose that no gauge connection has to be introduced to rewrite D. Then, we define g^{ij} and the (matrix) E by:

$$D = -\frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j \mathbb{1} - E. \tag{C.14}$$

The first term is the covariant Laplacian associated with the metric g^{ij}. All the derivatives are explicit, meaning that E does not contain free derivative operators. If we denote the Riemann curvature tensor associated with this metric by R_{ijkl}, then the first three non-zero contributions to (C.13) are determined by:

$$E_0 \quad = \quad \frac{1}{(4\pi)^{d/2}} I,$$

_____

[2]For a sketch of the field, we refer to [102].

$$E_2 = \frac{1}{(4\pi)^{d/2}} \left(E - \frac{1}{6} R_{ijij} \right), \tag{C.15}$$

$$E_4 = \frac{1}{(4\pi)^{d/2}} \left(-\frac{1}{30} R_{ijij;kk} + \frac{1}{72} R_{ijij} R_{kmkm} \right.$$
$$\left. -\frac{1}{180} R_{ijik} R_{njnk} + \frac{1}{180} R_{ijkn} R_{ijkn} - \frac{1}{6} R_{ijij} E + \frac{1}{2} E^2 + \frac{1}{6} E_{;kk} \right).$$

The covariant derivative of the metric connection is denoted by ; and repeated indices are summed over.

Secondly, suppose that we can rewrite D using a flat metric η^{ij}, a gauge connection A_i and a matrix E defined by:

$$D = -(\partial_i \mathbb{1} + A_i) \eta^{ij} (\partial_j \mathbb{1} + A_j) - E. \tag{C.16}$$

Again, all derivatives are explicit. E and A_i do not contain free derivative operators. Define now a covariant derivative $\nabla_i X = \partial_i X + [A_i, X]$, and the curvature of the gauge field by $W_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$. We then have:

$$E_0 = \frac{1}{(4\pi)^{d/2}} I,$$

$$E_2 = \frac{1}{(4\pi)^{d/2}} E,$$

$$E_4 = \frac{1}{(4\pi)^{d/2}} \left(\frac{1}{12} W_{ij} W^{ij} + \frac{1}{2} E^2 + \frac{1}{6} \nabla_i \eta^{ij} \nabla_j E \right). \tag{C.17}$$

We will now show how these results can be used to calculate the the supertrace derived in the first section of this appendix. First of all, identifying $t = \frac{\lambda}{M^2}$, we see that we indeed are studying the limit $t \to 0$ if $M \to \infty$. Introducing the integral kernel $K(t, x, y)$ for the operator $e^{-t\mathcal{R}}$, we can derive

$$\int dz \int dz' J(z).\delta(z - z') e^{-t\mathcal{R}}.\delta(z - z')$$

$$= \int dz \int dz' \int dvol(y) J(z).\delta(z - z') K(t, z, y) \delta(y - z')$$

$$= \int dz \int dvol(y) J(z).\delta(z - y) K(t, z, y). \tag{C.18}$$

If we now make the extra restriction that $J(z)$ does not contain differential operators acting on the δ-function, then the integral over z can also be done, leading to

$$I[K] = x.I.\text{str} \int_0^\infty d\lambda e^{-\lambda} \int dvol(y) J(y) K(\frac{\lambda}{M^2}, y, y). \tag{C.19}$$

Here it is clear that indeed the expansion (C.13) above can be used. Thereafter, calculating the supertrace over the internal degrees of freedom yields the final result. Notice that the terms with a strict negative power of t lead to ill-defined integrals for λ. These have to be removed by the regularisation or renormalisation, for which we refer to the main text.

We close this section with a comment on the restriction that $J(z)$ does not contain derivatives. If it does, one of the two following tricks may be useful. Using the fact that D is a supersymmetric matrix, and that the supertrace is invariant under supertransposition, the supertrace can be replaced by a supertrace over a symmetrised Jacobian. See [25], where this trick was used to remove linear derivatives from J. Another possibility, which works if J is up to second order in the derivatives, is to study $\text{str}(e^{-B+\alpha J})$ instead of $\text{str}(Je^{-B})$, as the latter can be obtained from the former by a derivation with respect to α.

C.3 An explicit example

In this section, we will calculate one explicit example. This will enable us to understand some of the features of the previous section from an algebraic point of view. In the mean time, it will become clear that a non-negligible amount of work is saved by using the results of [101] as they are listed in the previous section. As anounced, we will study the case where there are no gauge fields, only a metric. $J(z)$ will be taken to contain no derivatives, and $E = 0$. Thus we study

$$I = \int d^2 z \int d^2 z' \int_0^\infty d\lambda e^{-\lambda} J(z) \delta(z - z') e^{\lambda t \mathcal{R}(z)} . \delta(z - z'), \qquad (C.20)$$

with $\mathcal{R} = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu$ the covariant Laplacian, and with a two-dimensional space-time. The limit $t \to 0$ is of course understood. We work in two space-time dimensions.

The first thing to do is to write both the δ-functions as Fourier-integrals:

$$I = \int d^2 z \int d^2 z' \int_0^\infty d\lambda e^{-\lambda} \int \frac{d^2 p}{(2\pi)^2} \int \frac{d^2 q}{(2\pi)^2} J(z) e^{ip(z-z')} e^{a\mathcal{R}(z)} e^{iq(z-z')}.$$
$$(C.21)$$

We introduced $a = \lambda t$. Now pull the $e^{iq(z-z')}$ through the operator to the left. For $e^{-iqz'}$ this is trivial, as $\mathcal{R}$ acts on z, not on z'. Integrating over z' then leads to $(2\pi)^2 \delta(p + q)$. Some more effort is required to pull through the e^{iqz}. First observe that $\partial_\mu(e^{iqz} g(z)) = e^{iqz}[\partial_\mu + iq_\mu] g(z)$, so that

$$\mathcal{R}(z) \left[e^{iqz} g(z) \right] = e^{iqz} \frac{1}{\sqrt{g}} (\partial_\mu + iq_\mu) \sqrt{g} g^{\mu\nu} (\partial_\nu + iq_\nu) g(z) \stackrel{def.}{=} e^{iqz} \mathcal{R}_q(z) g(z).$$
$$(C.22)$$

This defines $\mathcal{R}_q(z)$. It straightforwardly follows that

$$I = \int d^2 z \int \frac{d^2 q}{(2\pi)^2} \int d\lambda e^{-\lambda} J(z) e^{a\mathcal{R}_q(z)} . 1, \qquad (C.23)$$

where we explicitly wrote the 1 on which the operator is acting.

We now regroup the terms in $\mathcal{R}_q(z)$ according to their power in the momentum q. $\mathcal{R}_q(z) = -A + B + C$, where

$$-A = -q_\mu q_\nu g^{\mu\nu}, \qquad (C.24)$$

$$B = iq_\nu \left[2g^{\mu\nu} \partial_\mu + \frac{1}{2} \partial_\mu \ln g . g^{\mu\nu} + \partial_\mu g^{\mu\nu} \right], \qquad (C.25)$$

$$C = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu. \qquad (C.26)$$

By doing the rescaling of the integrationvariable $q' = \sqrt{a} q = bq$, we get, dropping the primes:

$$I = \int d^2 z \int \frac{d^2 q}{(2\pi)^2} \int d\lambda e^{-\lambda} \frac{1}{b^2} e^{-A+bB+b^2C} . 1. \qquad (C.27)$$

Notice that a factor t^{-1} has appeared (in the b). For arbitrary dimension of space-time this generalises to $t^{-\frac{d}{2}}$, which is exactly the power of t of the first term in (C.13).

Let us now concentrate on the operator $\frac{1}{b^2} e^{-A+bB+b^2C} = \frac{1}{b^2} e^{-A} F$, which defines F. F is determined using the Baker-Campbell-Hausdorf formula. Defining $F(x) = e^{xA} e^{x(-A+bB+b^2C)}$, we can write

$$F = F(1) = \sum_{n=0} \frac{1}{n!} F^{(n)}(0). \qquad (C.28)$$

We have that

$$\frac{dF}{dx} = \left(bB + b^2C + xb[A,B] + xb^2[A,C] + \frac{x^2}{2}b^2[A,[A,C]]\right)F(x), \quad (C.29)$$

owing to the fact that A is a space-time dependent scalar, B a linear differential operator and C is a second order differential operator, which makes all possible higher order commutators vanish. As we take the limit $b \to 0$, we are only interested in the terms of the Taylorseries up to quadratic degree in b. Therefore, the $F^{(4)}(0)$ is the last term we need, and we find

$$\begin{aligned}
F &= 1 + bB + b^2C + \frac{1}{2}b[A,B] + \frac{1}{2}b^2[A,C] + \frac{1}{2}b^2B^2 \quad (C.30)\\
&\quad + \frac{1}{6}b^2[A,[A,C]] + \frac{1}{3}b^2[A,B]B + \frac{1}{6}b^2B[A,B] + \frac{1}{8}b^2[A,B]^2.
\end{aligned}$$

This operator acts on 1, so we can drop the term b^2C. Also, the two terms linear in the operator B, bB and $\frac{1}{2}b[A,B]$ disappear upon integration over q, as B is linear in q.

We now take our metric to be of the form discussed in the main text (12.24). That is (μ,ν run over $z,\bar{z}$):

$$g^{\mu\nu} = \begin{pmatrix} 2h(z,\bar{z}) & -1 \\ -1 & 0 \end{pmatrix}. \quad (C.31)$$

Constructing the operators A,B and C using this form for the metric and calculating the terms in $F.1$, one ends up with momentumintegrals of the form

$$\int d^2q f(q,\bar{q})e^{-A}, \quad (C.32)$$

where we denoted q_z by q and $q_{\bar{z}}$ by $\bar{q}$. $f(q,\bar{q})$ can be of the form $q\bar{q},q^2\bar{q}^2,q^2$, $q^4,q^3\bar{q},q^6,q^4\bar{q}^2$ and $q^5\bar{q}$. Only the first two give a non-zero contribution, because

$$\begin{aligned}
\int d^2q e^{-A} &= \pi\sqrt{g}, \\
\int d^2q q_\mu q_\nu e^{-A} &= \frac{1}{2}\pi\sqrt{g}g_{\mu\nu}, \\
\int d^2q q_\mu q_\nu q_\rho q_\sigma e^{-A} &= \frac{1}{4}\pi\sqrt{g}\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right), \quad (C.33)
\end{aligned}$$

and because $g_{zz} = 0$. The two remaining terms come from $B^2.1$, which contributes a $4q\bar{q}\partial^2h$, and from $B[A,B].1$, giving a contribution $8q^2\bar{q}^2\partial^2h$. Putting it all together, the final result is:

$$I = \frac{1}{4\pi}\int d^2z \int d\lambda e^{-\lambda}J(z)\frac{1}{b^2}\left(1 - \frac{1}{6}2\partial^2hb^2\right). \quad (C.34)$$

Notice that the scalar curvature associated with the chosen metric is indeed $2\partial^2h$, so that our result agrees with the E_0 and E_2 of (C.15).

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