# Spin models of Calogero–Sutherland type and associated spin chains

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A Pilar. A mis padres y a mi hermana.

Per correr miglior acque alza le vele omai la navicella del mio ingegno, che lascia dietro a sé mar sì crudele; e canterò di quel secondo regno dove l'umano spirito si purga e di salire al ciel diventa degno.

Dante Alighieri, Divina Commedia

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## Preface

This dissertation intends to present a unified view of my research in the area of integrable systems, which has been carried out over the last four years under the insightful supervision of Profs. F. Finkel, A. González-López and M. A. Rodríguez. Particular emphasis has been laid on the connection between (partially) solvable spin models of Calogero–Sutherland (CS) type and Haldane–Shastry (HS) chains, which can be rightly regarded as the leitmotif of this thesis.

The dissertation is divided into five chapters. In the first one we summarize some aspects of the theory of CS and HS models that are needed in later chapters and introduce some notation. Related original results are introduced when appropriate. In Chapter 2 we rigorously analyze the CS/HS connection, which enables us to compute the partition function of the  $BC_N$  HS chain in closed form. In Chapter 3 we turn our attention to quasi-exactly solvable (QES) generalizations of the standard CS models presenting only near-neighbors interactions. A systematic explanation of their quasi-exact solvability is given in terms of known invariant flags. In Chapter 4 we make use of the results in the two preceding chapters to analyze the algebraic spectrum of a novel spin chain with nearest-neighbors interactions which is naturally related to a QES Schrödinger operator of CS type. This work ends with a short summary and a concluding note on future projects.

The most remarkable contributions presented in this thesis are the following:

- (i) We complete the first mathematically rigorous treatment of the CS/HS connection. Our approach hinges on several convergence results that make Polychronakos's "freezing trick" precise.
- (ii) We provide the first systematic analysis of spin QES models of CS type presenting near-neighbors interactions, thereby obtaining several families of totally explicit eigenfunctions.
- (iii) We define an HS chain with nearest-neighbors interactions that presents

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some of the features of a QES system. The study of its algebraic states relies on a nontrivial refinement of the freezing trick and on the explicit results obtained in Chapter 3.

(iv) We introduce a natural Hamiltonian system admitting the maximum number of independent first integrals, just as the original Calogero and hyperbolic Sutherland models. This is the first example of a maximally superintegrable system in an N-dimensional space of nonconstant curvature.

A significant part of the material presented in this memoir has been adapted from Refs. [15,63–70]. The presentation of the main results in Chapters 2 and 4, however, greatly differs from that of the original articles. In particular, most of the rigorous proofs provided in these chapters appear in printed form here for the first time.

## Chapter 1

## Calogero-Sutherland models

#### 1.1 Introduction

The discovery of Calogero–Sutherland (CS) models [35, 186, 187] is a land-mark in the theory of integrable systems which has exerted a far-reaching influence on many different areas of Mathematics and Physics. The intrinsic mathematical interest of these models and their classical analogs [153, 157] lies in their manifold connections with such disparate topics as representation theory [22, 72, 80, 177], the theory of special functions and orthogonal polynomials [14, 60, 190, 191], harmonic analysis [40, 108, 174], and complex and algebraic geometry [71, 114, 119, 154, 202]. On the other hand, CS models have found a wide variety of applications to diverse areas of Physics, including quantum field and string theory [8, 54, 85, 95, 149, 151], the quantum Hall effect [11, 127, 169], soliton theory [1, 33, 128, 133], fractionary statistics [37, 99, 163] and random matrix theory [38, 181].

In this chapter we summarize the basic properties of CS models that shall be needed in the sequel. We prefer not to follow a historical approach to this subject, but rather to develop a modern bird's-eye presentation of the essentials of the theory. Thus the most advanced machinery becomes available from the very beginning, while the interest of the reader is not suffocated by unnecessary details. Our survey of CS models does not intend to be exhaustive. In particular, we have omitted several aspects of the theory (e.g., spin CM models [121,140,142,204], Ruijsenaars—Schneider systems and q-Dunkl operators [9, 13, 43, 141, 176, 198], supersymmetry [48, 51, 93] and connections with random matrix theory [38,44,181]), which could fit perfectly within the mainstream of this dissertation and will indeed have some bearing on the content of Chapter 5.

The core of this section is the brief survey of spin CS models and Dunkl operators developed in Sections 1.6 and 1.7. While we introduce a wealth of relevant concepts, we do not include a single rigorous proof or statement: our presentation of these topics should be regarded as a large dry bone that will be fleshed in Chapters 2 and 3.

We occasionally present some original results as we go along; these are mainly taken from Refs. [15, 69, 70].

#### 1.2 Scalar CS models

Let  $R \subset \mathbb{R}^N$  be a (crystallographic, possibly nonreduced) root system, which we assume to be irreducible without loss of generality, and let  $R_+ \subset R$  a subsystem of positive roots [118]. Given  $\alpha \in R$ , we represent by

$$\sigma_{\alpha}(\mathbf{x}) = \mathbf{x} - 2 \frac{\boldsymbol{\alpha} \cdot \mathbf{x}}{\|\boldsymbol{\alpha}\|^2} \boldsymbol{\alpha}$$

the reflection across the hyperplane orthogonal to  $\alpha$ . The Weyl group corresponding to R is denoted by  $\mathfrak{W}(R) = \langle \sigma_{\alpha} : \alpha \in R_{+} \rangle$ .

Consider a function  $v \in C^{\infty}(\mathbb{R} \setminus \{0\})$  of one of the following forms:

$$v(x) = \begin{cases} x^{-2} & \text{(rational)} \\ \sinh^{-2} x & \text{(hyperbolic)} \\ \sin^{-2} x & \text{(trigonometric)}. \end{cases}$$
 (1.1)

The (scalar) N-body CS model of type R is defined by the Hamiltonian

$$H_{\rm sc} = -\Delta + \sum_{\alpha \in R_+} g_{\alpha}^2 v(\alpha \cdot \mathbf{x}), \qquad (1.2)$$

where

$$\Delta = \sum_{i=1}^{N} \partial_{x_i}^2$$

and the coupling constants  $g_{\alpha}^2$  are assumed to take the same value for roots of the same length. In the rational case we shall frequently include a confining harmonic term, i.e.,

$$H_{\rm sc} = -\Delta + \sum_{\boldsymbol{\alpha} \in R_+} \frac{g_{\boldsymbol{\alpha}}^2}{(\boldsymbol{\alpha} \cdot \mathbf{x})^2} + \omega^2 r^2,$$

with  $r = ||\mathbf{x}||$ . If we define a unitary action K of  $\mathfrak{W}(R)$  on  $L^2(\mathbb{R}^N)$  by

$$K(\sigma_{\alpha}) \equiv K_{\alpha} : f \mapsto f \circ \sigma_{\alpha} \,, \tag{1.3}$$

it is clear that  $\mathfrak{W}(R)$  is a symmetry group of the Hamiltonian (1.2).

Remark 1.1. More generally [158], the function v in Eq. (1.1) can be taken to be a Weierstrass  $\wp$  function with suitable periods. Despite the relevance of this case and the abundant related literature (cf. e.g. [135,136] and references therein), we have preferred to omit elliptic potentials in our presentation of CS models. On the one hand, they do not appear anywhere in the sequel; on the other, their solvability and integrability properties are rather different from those of their limiting cases listed in Eq. (1.1) (with  $\omega$  set to zero). The conditions on the root system can be also relaxed somewhat [29, 158].

The double pole that v possesses at 0 imposes that any function  $\psi$  in the domain of  $H_{\rm sc}$  must vanish at the hyperplanes  $\mathbf{x} \cdot \boldsymbol{\alpha} = 0$  with  $\boldsymbol{\alpha} \in R_+$ , which means that the particles cannot overtake one another. Hence one can assume that the configuration space for  $H_{\rm sc}$  is the Weyl chamber

$$C = \left\{ \mathbf{x} \in \mathbb{R}^N : \boldsymbol{\alpha} \cdot \mathbf{x} > 0, \ \forall \boldsymbol{\alpha} \in R_+ \right\}$$
 (1.4)

in the rational and hyperbolic cases, and the Weyl alcove

$$C = \left\{ \mathbf{x} \in \mathbb{R}^N : \boldsymbol{\alpha} \cdot \mathbf{x} > 0, \ \forall \boldsymbol{\alpha} \in R_+; \ \boldsymbol{\alpha}_{\text{max}} \cdot \mathbf{x} < \pi \right\}$$
 (1.5)

in the trigonometric one. Here we denote by  $\alpha_{\max}$  the highest root of the system R.

Remark 1.2. The symmetry of  $H_{sc}$  under  $\mathfrak{W}(R)$  guarantees the equivalence of the eigenvalue problems for  $H_{sc}$  in  $\mathbb{R}^N$  and in C. Namely, for any eigenfunction  $\varphi \in L^2(\mathbb{R}^N)$  of  $H_{sc}$  there exists an eigenfunction  $\psi \in L^2(C)$  of  $H_{sc}$  with the same eigenvalue and a one-dimensional representation  $\chi : \mathfrak{W}(R) \to \{\pm 1\}$  of the Weyl group such that for any  $w \in \mathfrak{W}(R)$  satisfying  $w \mathbf{x} \in C$  one has

$$\varphi(\mathbf{x}) = \chi(w) \psi(w\mathbf{x})$$
.

Clearly, the inclusion map defines a one-to-one correspondence between the eigenfunctions of  $H_{sc}$  on  $L^2(\mathbb{C})$  and its eigenfunctions on  $L^2(\mathbb{R}^N)$  with a given symmetry.

We shall be mainly interested in the root systems  $A_{N-1}$  and  $BC_N$ , which are respectively given by

$$R = \{\mathbf{e}_i - \mathbf{e}_j : 1 \leqslant i \neq j \leqslant N\}$$

and

$$R = \{ \pm \mathbf{e}_i, \pm 2\mathbf{e}_i, \pm \mathbf{e}_i \pm \mathbf{e}_j : 1 \leqslant i < j \leqslant N \},$$

 $\{\mathbf{e}_i: 1 \leqslant i \leqslant N\}$  being the canonical basis of  $\mathbb{R}^N$ . An appropriate choice of the coupling constants of the CS models associated with these systems suffices to exhaust all the possibilities arising from the non-exceptional Lie algebras. Physically, these systems are important because they are the only ones which allow for an arbitrary number N of particles, given by the rank of the algebra. For the sake of completeness and future reference, let us mention that set of positive roots  $R_+$  and the highest root  $\alpha_{\max}$  for the  $A_{N-1}$  and  $BC_N$  systems can be taken as

$$R_{+} = \{ \mathbf{e}_{i} - \mathbf{e}_{j} : 1 \leqslant i < j \leqslant N \}, \qquad \boldsymbol{\alpha}_{\text{max}} = \mathbf{e}_{1} - \mathbf{e}_{N},$$

and

$$R_{+} = \left\{ \mathbf{e}_{i} \pm \mathbf{e}_{j}, \mathbf{e}_{i}, 2\mathbf{e}_{i} : 1 \leqslant i < j \leqslant N \right\}, \qquad \boldsymbol{\alpha}_{\text{max}} = 2\mathbf{e}_{1}.$$

The  $A_{N-1}$  and  $BC_N$  scalar CS models (1.2) with rational and trigonometric potentials can therefore be written as

$$H_{\rm sc} = -\Delta + V_{\rm sc} \,, \tag{1.6}$$

where the potential is in each case given by

$$V_{\rm sc} = \sum_{i \neq j} \frac{a(a-1)}{(x_i - x_j)^2} + \omega^2 r^2, \qquad (1.7a)$$

$$V_{\rm sc} = \sum_{i \neq j} \frac{a(a-1)}{\sin^2(x_i - x_j)},$$
(1.7b)

$$V_{\rm sc} = a(a-1)\sum_{i\neq j} \left[ \frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right] + \sum_i \frac{b(b-1)}{x_i^2} + \omega^2 r^2, \quad (1.7c)$$

$$V_{\text{sc}} = a(a-1) \sum_{i \neq j} \left[ \frac{1}{\sin^2(x_i - x_j)} + \frac{1}{\sin^2(x_i + x_j)} \right] + \sum_{i} \frac{b(b-1)}{\sin^2 x_i} + \sum_{i} \frac{b'(b'-1)}{\cos^2 x_i} .$$
 (1.7d)

Here and in what follows the indices in all sums and products range from 1 to N unless otherwise stated.

#### 1.3 Integrability and solvability

CS models are the paradigm of quantum integrable N-body problems. The first satisfactory explanation of their remarkable algebraic properties was given by Olshanetsky and Perelomov [158], who constructed N commuting first integrals for the Hamiltonian (1.2) associated with any non-exceptional root system. A new proof of the integrability of these systems (which applies even to the case of non-crystallographic root systems) has been recently developed by Bordner, Manton and Sasaki [31] using universal quantum Lax pairs, and previously by Heckman and Opdam [106, 107] (in the context of degenerate Hecke algebras) and by Oshima and Sekiguchi [162]. For simplicity, we shall henceforth assume that R is a non-exceptional root system, typically  $A_{N-1}$  or  $BC_N$ .

For these root systems, the models (1.2) are **exactly solvable** in the sense that all their eigenvalues and eigenfunctions can be computed in closed form. (The solvability properties of CS models associated with exceptional Lie algebras are still uncertain, even though there have been some recent developments in this area [32].) The eigenfunctions  $\psi_{\mathbf{n}}$  are labeled by integer multiindices  $\mathbf{n} = (n_1, \dots, n_N)$  and factorize as

$$\psi_{\mathbf{n}}(\mathbf{x}) = \mu(\mathbf{x}) P_{\mathbf{n}}(\mathbf{z}), \qquad (1.8)$$

where  $\mu \equiv \psi_0$  is the ground state function,  $\mathbf{z} = (z_1, \dots, z_N)$  with

$$z_i = \begin{cases} x_i & \text{(rational)} \\ e^{2x_i} & \text{(hyperbolic)} \\ e^{2ix_i} & \text{(trigonometric)}, \end{cases}$$

and  $P_{\mathbf{n}}$  is a polynomial of degree

$$|\mathbf{n}| = \sum_{i} |n_i|. \tag{1.9}$$

Physically, the multiindex  $\mathbf{n}$  can be thought of as the quasimomentum of the particles [168,171], and the energy of the corresponding eigenfunction can be expressed neatly in terms of this quantity and the parameters appearing in the Hamiltonian.

**Example 1.3.** The original *Calogero model* (1.7a) is obtained by setting  $R = A_{N-1}$  and choosing the rational potential with a harmonic term. Its eigenfunctions are labeled by a multiindex  $\mathbf{n} \in \mathbb{N}_0^N$  and can be written as  $\psi_{\mathbf{n}} = \mu P_{\mathbf{n}}$ , where

$$\mu(\mathbf{x}) = e^{-\frac{\omega}{2}r^2} \prod_{i < j} |x_i - x_j|^a$$

is the ground state function and  $P_{\mathbf{n}}$  is a homogeneous polynomial of degree  $|\mathbf{n}|$ ). These polynomials are known as generalized Hermite polynomials [137, 200], since when N=1 they reduce to Hermite polynomials. Explicit formulas can be consulted, e.g., in Refs. [34, 103, 125]. The energy of the eigenfunction  $\psi_{\mathbf{n}}$  is given by

$$E_{\mathbf{n}} = 2\omega |\mathbf{n}| + E_0,$$

with 
$$E_0 = \omega N(a(N-1) + 1)$$
.

**Example 1.4.** If we keep the  $A_{N-1}$  root system and choose the trigonometric potential, we obtain the original **Sutherland model**. Its eigenfunctions can be written as  $\psi_{\mathbf{n}}(\mathbf{x}) = \mu(\mathbf{x})P_{\mathbf{n}}(\mathbf{z})$ , where

$$\mu(\mathbf{x}) = \prod_{i < j} |\sin(x_i - x_j)|^a,$$

 $\mathbf{n} \in \mathbb{Z}^N$ , and  $P_{\mathbf{n}}$  is a polynomial of degree  $|\mathbf{n}|$  in  $\mathbf{z} = (\mathrm{e}^{2\mathrm{i}x_1}, \ldots, \mathrm{e}^{2\mathrm{i}x_N})$ . These polynomials are called Jack polynomials [123], and play a fundamental role in Heckman and Opdam's theory of hypergeometric functions for root systems [104, 109, 159, 160].

A hyperbolic version of the trigonometric Sutherland system can be readily obtained. Its spectrum, however, is not discrete, and in fact this model only has a finite number of eigenvalues.

### 1.4 Orthogonal polynomials

Whereas the theory of orthogonal polynomials in one variable is a classical branch of Mathematics, significant results on multivariate polynomials were not obtained until the second half of the XX century [61]. Since then, orthogonal polynomials of several variables have become a major field of research in modern analysis and combinatorics [19,43,146,185,199]. As can be seen from Eq. (1.8), multivariate orthogonal polynomials appear as eigenfunctions of CS models, and the role that these models play in the latter theory is analogous to that of the Laguerre, Hermite or Jacobi equations in the case of classical orthogonal polynomials in one variable.

As a matter of fact, the analogy between the eigenfunctions of CS models and classical orthogonal polynomials is even closer. We showed in Example 1.3 that the eigenfunctions of the rational  $A_{N-1}$  CS model (the Hi-Jack polynomials) reduce to the Hermite polynomials in the case N=1 and indeed coincide with their multivariate generalization [137]. It is also well known [14]

that the multivariate generalizations of the Laguerre [139] and Jacobi [138] polynomials are similarly recovered from the  $BC_N$  rational and trigonometric CS models, respectively. Generally, CS models give rise to generalized hypergeometric functions associated with root systems, as extensively studied by Heckman and Opdam [104, 109, 159, 160].

A recent development is some kind of converse statement to the latter comment. Hallnäs and Langmann have constructed an algorithm that allows one to go from a family of orthogonal polynomials in one variable to an N-body Hamiltonian whose eigenfunctions are a symmetric generalization of this family to N variables. The N-body models associated with the classical families of orthogonal polynomials are in fact previously known CS Hamiltonians.

The fruitful connection between CS models and the theory of orthogonal polynomials has lead to a plethora of closed formulas for the eigenfunctions of these Hamiltonians (cf. [61,103,146] and references therein). Furthermore, an exciting new ingredient of this theory are supersymmetric (spin) polynomials [52,53], related directly to the spin CS models which will be the main subject of the remaining chapters of this memoir.

#### 1.5 A classical detour

It can be argued that the conceptually simplest approach to the exceptional solvability properties of CS models is through their classical counterparts, the Calogero–Moser (CM) models. Hence, we deem it important to present a brief account of the theory of CM models, which, apart from its academic interest, is essential in order to understand some recent developments on classical Haldane–Shastry spin chains [122].

CM models [157] are the Hamiltonian systems on  $\mathbb{R}^N$  whose Hamiltonian function is obtained from Eq. (1.2) by substitution of the quantum kinetic energy (the Laplacian  $-\Delta$ ) for the classical one (the squared momentum  $\|\mathbf{p}\|^2$ ). The mathematical intricacies that lurk behind quantum integrability [45,69,134,201] are absent in the classical setting, the Liouville integrability of CM models being related to the existence of an invariant foliation by Lagrangian cylinders in phase space [2]. The Liouville integrability of CM models for any (generalized) root system stems from Bordner, Corrigan and Sasaki's construction [30] of a universal Lax representation for these systems. Nonetheless, their integrability was also explicitly obtained [160] as a corollary in a series of papers by Heckman and Opdam ten years earlier by means of a detailed study of the corresponding CS models and a suitable

theorem on the classical limit of their quantum first integrals. An advantage of the first authors' approach is that the equations of motion of the system can be explicitly integrated for any root system (resp. for root systems associated with classical Lie algebras) when the potential is rational (resp. trigonometric or hyperbolic.)

Most remarkably, the  $A_N$  CM Hamiltonians with rational and hyperbolic potentials are also  $maximally \ superintegrable$  for any value of N, i.e., they admit the maximum number 2N-1 of functionally independent global first integrals. This property is quite exceptional: apart from these two models, the only known (natural) Hamiltonian systems which are maximally superintegrable in arbitrary dimension are the geodesic flow on the simply connected space forms, the (generalized) Kepler problem [73], the Smorodinsky-Winternitz system [74, 90], their generalizations to the simply connected spaces of constant sectional curvature (cf. [16] and references therein), the nonisotropic oscillator with rational frequencies, the nonperiodic  $A_N$  Toda lattice [5,194], and a very recent example living on a manifold of nonconstant curvature [15], namely

$$H(\mathbf{x}, \mathbf{p}) = \frac{\|\mathbf{p}\|^2 + \omega^2 r^2 + \sum_j b_j x_j^{-2}}{1 + r^2}.$$

This system can be understood as the intrinsic Smorodinsky–Winternitz model on an N-dimensional generalization of the Darboux space of type III [131].

The exceptional properties of the CM models can be understood by means of (singular) Hamiltonian reduction [161]. This technique, independently developed by Olshanetsky and Perelomov [155,156] and by Kazhdan, Konstant and Sternberg [130], was recently clarified and placed in a broader context by Fehér and Pusztai [76–78]. The latter works provide some kind of symplectic analog of Harish-Chandra's theory of c-functions [72,112].

We sketch the gist of the method for the simple case of the  $A_{N-1}$  trigonometric Sutherland model<sup>1</sup> (cf. Example 1.4), as developed by Kazhdan, Konstant and Sternberg [130]. Let us consider the geodesic motion on K = SU(N), which is given by the free Hamiltonian

$$H_0(k,\lambda) = ||\lambda||^2$$

on  $T^*K$ , where we trivialize  $T^*K \cong K \times \mathfrak{k}^*$  (with  $\mathfrak{k} = \text{Lie}(K)$ ) by right translations and identify  $\mathfrak{k}^* \cong \mathfrak{k}$  by means of the inner product. Here the

<sup>&</sup>lt;sup>1</sup>I am indebted to Prof. L. Fehér for valuable discussions on the state of the art of the projection method in CM models.

invariant symplectic form on  $T^*K$  is  $\omega = d\langle \lambda, (dk)k^{-1}\rangle$ . The Hamiltonian action

$$(k,\lambda) \mapsto (hkh^{-1}, h\lambda h^{-1}), \qquad h \in K$$
 (1.10)

is a symmetry of  $H_0$  and gives rise to a momentum map  $\psi: T^*K \to \mathfrak{k}$  defined by

$$\psi(k,\lambda) = \lambda - k^{-1}\lambda k.$$

Now let us look at the (singular) Marsden-Weinstein reduced phase spaces

$$M_{\mu} = \psi^{-1}(\mu)/K_{\mu}$$
,

where  $K_{\mu}$  denotes the isotropy group of  $\mu \in \mathfrak{k}$ . (Generally speaking, these symplectic spaces are not manifolds, but stratified sets whose strata are differentiable manifolds.) The easiest nontrivial choice for  $\mu$  is

$$\mu = \mathrm{i}(uu^{\dagger} - N^{-1}g^2) \,,$$

 $u \in \mathbb{C}^N$  being an arbitrary vector of norm g. In this case, the reduced phase space actually turns out to be  $M_{\mu} = T^*C \cong C \times \operatorname{Cartan}(K) \cong C \times \mathbb{R}^N$ , C being a Weyl alcove, and the reduced Hamiltonian is given by the trigonometric Sutherland model, with  $(\mathbf{x}, \mathbf{p}) \in C \times \mathbb{R}^N$ . The integrability and solvability properties of the latter system now follow from those of the geodesic flow on K. As a matter of fact [77], these ideas can be also applied to obtain a Lax representation. These ideas apply to any compact, connected simple Lie group K, but actually it can be proved that a scalar CM model is only obtained in the case that we have reviewed, whereas for a general K a spin CM model arises.

The above construction can be reinterpreted as follows. Let  $G = K \times K$  and consider its involution  $\Theta(k_1, k_2) = (k_2, k_1)$ . If  $G_+ = \{(k, k) : k \in K\}$  and  $G_+^R$  (resp.  $G_+^L$ ) denotes its right (resp. left) action on G, it is clear that geodesic motion on G enjoys a  $G_+^R \times G_+^L$  symmetry. If we reduce  $T^*G$  using the  $G_+^R$  symmetry at the zero value of the momentum map, we arrive precisely at the symplectic system  $(T^*K, \omega, H)$  analyzed in the preceding paragraph. Moreover, the  $G_+^L$  symmetry still survives, and its action on  $K \simeq G/G_+^R$  in fact coincides with the conjugation action (1.10) of K on itself. In this framework, hyperbolic CS models (possibly with spin) are obtained when considering the complexification G of K given by  $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ ,  $\Theta$  being a Cartan involution of G. The hyperbolic Sutherland model is obtained when  $K = \mathrm{SU}(N) \simeq G_+$ , so that  $G = \mathrm{SL}(N, \mathbb{C})$ ,  $G/G_+^R \simeq \exp(i\mathfrak{k})$  and  $M_\mu = T^*C$  for the above prescription for  $\mu$ , C now being a Weyl chamber.

The above results also hold true, mutatis mutandis, if we take G to be any noncompact simple Lie group with finite center,  $\Theta$  a Cartan involution, and

 $G_+ = \{g \in G : \Theta(g) = g\}$ , so that it is a maximal compact subgroup. Any element  $g \in G$  can be uniquely decomposed as  $g = g_+g_-$ , where  $g_{\pm} \in G_{\pm}$  and  $G_- = \{g \in G : \Theta(g) = g^{-1}\}$  is diffeomorphic to  $\mathfrak{g}_-$  via the exponential map. In this framework, one usually obtains spin CM models on  $T^*C$ , and further generalizations have been also considered. Moreover, the quantum analogs of these results are being actively pursued by the same authors, and connections with flat symmetric spaces have also been studied recently [6, 115].

The related topic of the reduction of CS models by discrete symmetries has also been considered in the Physics literature [167,170], both in the spin and spinless cases. A wide panoply of CS models can be obtained from the original Calogero model through this procedure, partially explaining why this system (with or without the harmonic confining potential) is sometimes regarded as the mother of all CM systems [171].

Interestingly, some purely quantum mechanical techniques in CS models were originally suggested by a careful analysis of their classical analogs. For instance, Polychronakos's rediscovery of Dunkl operators [164] was motivated by a careful derivation of the scattering Calogero model by reduction of free motion. Moreover, a short calculation allows one to prove that the only asymptotic effect of the interaction with the potential (1.7a) is an effective reshuffling of the particles. This property carries over to the quantum case, where it acquires an even greater relevance through the concept of identical particles. It is therefore amusing that some "physical" effects of the CS models were first observed at a classical level but only become really fundamental in the quantum realm. For instance, the ideas that we have just sketched lie at the heart of the applications of CS models to generalized statistics in one dimension.

Another fruitful connection between CM and CS models, which actually holds for a wide variety of one-dimensional quantum systems, appears when considering the exact **bosonization** of one such model. This technique essentially consists [27, 195, 203] in describing a fermionic system by means of a bosonic field theory. More precisely, we require that the algebra of bosonic operators be irreducible on the fermionic Hilbert space and that the bosonic field theory reproduce the exact eigenvalues of the system up to non-perturbative effects. It has been recently discovered [70] that an exact one-dimensional bosonization can be constructed via Weyl quantization of the classical picture, thus obtaining a new realization of the  $W_{\infty}$  algebra. This result is expected to shed new light on higher dimensional bosonization, but the difficulties encountered in this case are considerable.

### 1.6 Dunkl operators

The first attempt to provide a conceptual explanation of the exact solvability of CS models was made by Olshanetsky and Perelomov [158]. In the rational (with  $\omega=0$ ) and trigonometric cases, they showed that there exist some particular values of the coupling constants for which the Hamiltonian (1.2) is equivalent to the radial part of the Laplacian on a symmetric space associated with the root system R. The radial parts of the Laplacians on these spaces have been thoroughly studied [112], and quite explicit formulas are available. The general rational ( $\omega \neq 0$ ) and hyperbolic cases could also be dealt with using appropriate modifications. The main drawback of Olshanetsky and Perelomov's approach was that for the  $BC_N$  case and arbitrary coupling constants one did not obtain the radial Laplacian. This, however, has been recently amended by Fehér and Pusztai (although so far only the classical case has been carried out in detail [78]).

Concerning the connection between CS models and the theory of orthogonal polynomials, a major breakthrough was the introduction of *Dunkl* (or exchange) operators [59]

$$J_i^{\text{rat}} = \partial_{x_i} - \sum_{\alpha \in R_+} a_{\alpha} \frac{\alpha \cdot \mathbf{e}_i}{\alpha \cdot \mathbf{x}} (1 - K_{\alpha}), \qquad (1.11)$$

to study spherical harmonics associated with measures invariant under the Weyl group  $\mathfrak{W}(R)$ . Dunkl operators have become an essential tool in harmonic analysis, combinatorics and CS models [107, 175]. The differential-difference operators (1.11) have two key properties: they are homogeneous of degree -1, i.e., their action on a polynomial lowers its degree by one, and are mutually commutative. In this sense, one can think of the Dunkl operators as deformations of the usual partial derivatives that preserve most of their analytic properties. In fact, one can construct a Dunkl transform [46, 47] using the spectral resolution of the anti-self-adjoint operator  $\mathbf{J}^{\text{rat}}$ , and the resulting properties are fairly similar to those of the usual Fourier transform.

In the Physics community, Dunkl operators were rediscovered by Polychronakos [164] within the context of CS models. To explain the essentials of the connection between these objects, let us restrict ourselves to the case  $v(x) = x^{-2}$  and define

$$\mu(\mathbf{x}) = \prod_{\boldsymbol{\alpha} \in R_+} |\boldsymbol{\alpha} \cdot \mathbf{x}|^{a_{\boldsymbol{\alpha}}}.$$

A simple link can now be established through the equation

$$\mu H_{\rm sc} \mu^{-1} = -\sum_{i} (J_i^{\rm rat})^2 + E_0,$$
 (1.12)

where  $E_0$  is a constant and the constants  $g_{\alpha}$  are given by

$$g_{\alpha}^2 = \|\alpha\|^2 a_{\alpha} (a_{\alpha} + 2a_{2\alpha} - 1).$$

This equation is known to hold true [92] on the space  $L^2(\mathbb{R}^N)^{\mathfrak{W}(R)}$  of  $\mathfrak{W}(R)$ -invariant square-integrable functions on  $\mathbb{R}^N$ , which is canonically isometric to our (reduced) Hilbert space  $L^2(C)$ . Hence Dunkl operators appear in the rational ( $\omega = 0$ ) case (the scattering Calogero model) as first integrals of the system, and have the physical interpretation of (conserved) asymptotic momenta. Furthermore, Eq. (1.12) partially accounts for the relevance of CS models in harmonic analysis. External harmonic potentials can be introduced in this picture using either deformed destruction operators  $a_i = -\mathrm{i}(J_i^{\mathrm{rat}} + \omega x_i)$  [164] or auxiliary differential operators [81].

In the trigonometric case one can define an analogous set of commuting differential-difference operators by

$$J_i^{\text{trig}} = \partial_{x_i} + \sum_{\alpha \in R_+} a_{\alpha} \frac{\alpha \cdot \mathbf{e}_i}{1 - e^{-2i\alpha \cdot \mathbf{x}}} (1 - K_{\alpha}).$$
 (1.13)

The relation between the Sutherland system of type R and the sum of squares of these Dunkl operators goes along the lines of the rational case (1.12), with gauge factor

$$\mu(\mathbf{x}) = \prod_{\alpha \in R_{+}} \left| \sin(\alpha \cdot \mathbf{x}) \right|^{a_{\alpha}}. \tag{1.14}$$

It can be readily verified that the operators (1.13) leave invariant the subspace of polynomials in  $\mathbf{z} = (e^{2ix_1}, \dots, e^{2ix_N})$  of degree not higher than k, for all  $k \in \mathbb{N}_0$ .

As a matter of fact, the trigonometric Dunkl operators (1.13) were introduced by Cherednik [41, 42], while a closely related set of differential-difference operators has been previously considered by Heckman [105]. Several analogous families of operators are currently known, both in the rational and trigonometric cases. For simplicity, we shall understand by (generalized) **Dunkl operators** of type R [81, 107, 175] any set of first order differential-difference operators  $J_i$  (i = 1, ..., N) in N variables  $\mathbf{z}$  satisfying:

- (i) The operators  $J_i$  map the space of polynomials  $\mathbb{C}[\mathbf{z}]$  into itself and leave invariant finite-dimensional subspaces  $\mathcal{M}^n \subset \mathbb{C}[\mathbf{z}]$ .
- (ii) The operators  $\{J_i, K_{\alpha} : 1 \leq i \leq N, \alpha \in R_+\}$  span a degenerate Hecke algebra.

Degenerate (or graded) Hecke algebras were defined independently by Drinfeld [58] and Lusztig [144]. For the sake of completeness, let us recall that the degenerate Hecke algebra ( $\mathbb{H}, *$ ) associated with the system of positive roots  $R_+ \subset V$  and the parameters  $(a_{\alpha})_{\alpha \in R_+}$  is the linear space  $\mathbb{H} = \mathcal{S}(V^{\mathbb{C}}) \otimes \mathbb{C}[\mathfrak{W}(R)]$  endowed with the unique graded algebra structure satisfying

- (i)  $S(V^{\mathbb{C}}) \otimes 1$  and  $1 \otimes \mathbb{C}[\mathfrak{W}(R)]$  are subalgebras, so that  $S(V^{\mathbb{C}})$  and  $\mathbb{C}[\mathfrak{W}(R)]$  can be identified with their images in  $\mathbb{H}$ .
- (ii) If  $\alpha$  is a simple root, then  $\sigma_{\alpha} * \mathbf{v} (\sigma_{\alpha} \mathbf{v}) * \sigma_{\alpha} = -a_{\alpha} \alpha \cdot \mathbf{v}$  for all  $\mathbf{v} \in V^{\mathbb{C}}$ .

Here  $R_+$  is assumed to span V,  $\mathcal{S}(V^{\mathbb{C}})$  denotes the algebra of symmetric tensors in the complex linear space  $V^{\mathbb{C}}$ , and the grading in  $\mathbb{H}$  is in fact inherited from that of  $\mathcal{S}(V^{\mathbb{C}})$ . Clearly one can identify  $T*w=T\otimes w$ , where  $T\in\mathcal{S}(V^{\mathbb{C}})$  and  $w\in\mathbb{C}[\mathfrak{W}(R)]$ .

As in Eq. (1.12), Dunkl operators happen to give rise to (generalized) CS models through the addition of auxiliary operators, a similarity transformation and a global change of variables. Full details can be consulted in Refs. [81,82] and in Chapter 3.

### 1.7 Spin CS models

We shall comment now on another benefit granted by the Dunkl operator formalism that is essential for all the forthcoming sections: the definition of spin CS models. Since the study of (generalizations of) such models could be rightly considered to be the core of this Ph. D. dissertation, we shall have many opportunities in the following chapters to elaborate on this topic. Our goal in this section is to define the basic operators and present the fundamentals of the theory without providing any details on the computational difficulties involved. This and other aspects shall be covered in detail in Chapter 2.

For the sake of simplicity, we shall restrict ourselves to the non-exceptional root systems  $A_N$  and  $BC_N$ , which are the only ones that have been considered in the literature up to now. (In fact, we shall make use of the root system  $A_{N-1}$  embedded in  $\mathbb{R}^N$ , not of  $A_N$ . Nevertheless, we shall hereafter drop the subscript -1 and regard the subscript N as an abstract index.)

Let  $\Sigma \approx \mathbb{C}^{(2M+1)^N}$  be the Hilbert space of the internal degrees of freedom of N particles of spin  $M \in \frac{1}{2}\mathbb{N}$ , and let

$$\mathcal{B}_{\Sigma} = \left\{ |\mathbf{s}\rangle \equiv |s_1, \dots, s_N\rangle : s_i \in \{-M, -M+1, \dots, M\} \right\}$$
 (1.15)

be the canonical basis of  $\Sigma$ . Let us define the spin permutation and reversal operators by

$$S_{ij}|s_1,\ldots,s_i,\ldots,s_j,\ldots s_N\rangle = |s_1,\ldots,s_j,\ldots,s_i,\ldots s_N\rangle,$$
 (1.16a)

$$S_i|s_1,\ldots,s_i,\ldots s_N\rangle = |s_1,\ldots,-s_i,\ldots s_N\rangle,$$
 (1.16b)

$$\tilde{S}_{ij}|\mathbf{s}\rangle = S_i S_j S_{ij}|\mathbf{s}\rangle.$$
 (1.16c)

One can easily verify that they satisfy the algebraic relations

$$S_{ij}^2 = 1,$$
  $S_{ij}S_{jk} = S_{ik}S_{ij} = S_{jk}S_{ik},$   $S_{ij}S_{kl} = S_{kl}S_{ij},$   $S_i^2 = 1,$   $S_iS_j = S_jS_i,$   $S_{ij}S_k = S_kS_{ij},$   $S_{ij}S_j = S_iS_{ij},$ 

where the indices i, j, k, l take distinct values in the range  $1, \ldots, N$ .

The one-dimensional representations  $\chi:\mathfrak{W}(R)\to\{\pm 1\}$  of the Weyl group  $\mathfrak{W}(R)$  are given by [118]

$$\sigma_{\mathbf{e}_i - \mathbf{e}_j} \stackrel{\chi_{\epsilon}}{\mapsto} \epsilon, \qquad \epsilon = \pm 1$$

in the  $A_N$  case and by

$$\sigma_{\mathbf{e}_i - \mathbf{e}_j} \stackrel{\chi_{\epsilon \epsilon'}}{\mapsto} \epsilon, \qquad \sigma_{\mathbf{e}_i} \stackrel{\chi_{\epsilon \epsilon'}}{\mapsto} \epsilon', \qquad \epsilon, \epsilon' = \pm 1$$

in the  $BC_N$  case. For each of these one-dimensional representations  $\chi$ , one can define the **spin** CS **model** with scalar counterpart (1.6) as

$$H_{\chi} = -\Delta + V_{\chi} \,, \tag{1.17}$$

where the spin potentials  $V_{\chi}$  are given by

$$V_{\epsilon} = \sum_{i \neq j} \frac{a(a - \epsilon S_{ij})}{(x_i - x_j)^2} + \omega^2 r^2,$$
 (1.18a)

$$V_{\epsilon} = \sum_{i \neq j} \frac{a(a - \epsilon S_{ij})}{\sin^2(x_i - x_j)},$$
(1.18b)

$$V_{\epsilon\epsilon'} = \sum_{i \neq j} \left[ \frac{a(a - \epsilon S_{ij})}{(x_i - x_j)^2} + \frac{a(a - \epsilon \tilde{S}_{ij})}{(x_i + x_j)^2} \right] + \sum_i \frac{b(b - \epsilon' S_i)}{x_i^2} + \omega^2 r^2, \quad (1.18c)$$

$$V_{\epsilon\epsilon'} = \sum_{i \neq j} \left[ \frac{a(a - \epsilon S_{ij})}{\sin^2(x_i - x_j)} + \frac{a(a - \epsilon S_{ij})}{\sin^2(x_i + x_j)} \right] + \sum_{i} \frac{b(b - \epsilon' S_i)}{\sin^2 x_i} + \sum_{i} \frac{b'(b' - \epsilon' S_i)}{\cos^2 x_i}.$$
(1.18d)

We shall frequently omit the subscript if there is no risk of confusion. The extension to the hyperbolic case is straightforward.

Let us introduce the coordinate permutation and sign reversal operators

$$(K_{ij}f)(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N) = f(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_N),$$
 (1.19a)

$$(K_i f)(x_1, \dots, x_i, \dots, x_N) = f(x_1, \dots, -x_i, \dots, x_N),$$
 (1.19b)

$$\tilde{K}_{ij}f = K_i K_j K_{ij} f \,, \tag{1.19c}$$

which are the reflection operators associated with the root systems  $A_N$  and  $BC_N$ . One obviously has the group isomorphisms  $\langle K_{ij} \rangle \approx \langle S_{ij} \rangle \approx \mathfrak{W}(A_{N-1})$  and  $\langle K_i, K_{ij} \rangle \approx \langle S_i, S_{ij} \rangle \approx \mathfrak{W}(BC_N)$ . We shall denote by  $\mathfrak{K}$ ,  $\mathfrak{S}$  and  $\mathfrak{W}$  respectively the realizations of  $\mathfrak{W}(R)$  generated by the operators (1.19), (1.16), and

$$\Pi_{ij} = K_{ij}S_{ij}, \qquad \Pi_i = K_iS_i. \tag{1.20}$$

When notationally convenient, we shall regard these realizations as unitary representations

$$K: w \mapsto K_w \in \text{End}(L^2)$$
, (1.21a)

$$S: w \mapsto S_w \in \text{End}(\Sigma),$$
 (1.21b)

$$\Pi: w \mapsto \Pi_w \in \operatorname{End}(L^2 \otimes \Sigma),$$
 (1.21c)

with  $w \in \mathfrak{W}(R)$ . Moreover, we shall not distinguish between a character of  $\mathfrak{W}(R)$  and its isomorphic extension to  $\mathfrak{W}, \mathfrak{S}$  or  $\mathfrak{K}$ .

The key property of the spin models (1.18) is their connection with Dunkl operators. This connection is implemented via the **star mapping**  $\mathfrak{U}(\mathfrak{K}) \to \mathfrak{U}(\mathfrak{S})$  associated with the one-dimensional representation  $\chi$ , which is the antihomomorphism defined by

$$K_{ij} \mapsto (K_{ij})_{\epsilon\epsilon'}^* = \epsilon S_{ij} ,$$
 (1.22a)

$$K_i \mapsto (K_i)_{\epsilon\epsilon'}^* = \epsilon' S_i$$
. (1.22b)

Here  $\mathfrak{U}(\mathfrak{g})$  stands for the (real) universal enveloping algebra of  $\mathfrak{g}$ , and the second equation (1.22b) is to be omitted in the  $A_N$  case. We shall also consider the extension of (1.22) to  $\mathfrak{D} \otimes \mathfrak{K} \to \mathfrak{D} \otimes \mathfrak{G}$  given by

$$(D \otimes K)^*_{\epsilon \epsilon'} = D \otimes (K)^*_{\epsilon \epsilon'}.$$

Here  $\mathfrak D$  is the space of (scalar) smooth differential operators, and D and K belong to  $\mathfrak D$  and  $\mathfrak K$  respectively. It is easy to show that the spin CS models (1.17) are obtained by applying the star mapping associated with  $\chi$  to the **exchange Hamiltonian** 

$$\overline{H} = -\Delta + \overline{V}, \qquad (1.23)$$

with

$$\overline{V} = \sum_{i \neq j} \frac{a(a - K_{ij})}{(x_i - x_j)^2} + \omega^2 r^2, \qquad (1.24a)$$

$$\overline{V} = \sum_{i \neq j} \frac{a(a - K_{ij})}{\sin^2(x_i - x_j)},$$
(1.24b)

$$\overline{V} = \sum_{i \neq j} \left[ \frac{a(a - K_{ij})}{(x_i - x_j)^2} + \frac{a(a - \tilde{K}_{ij})}{(x_i + x_j)^2} \right] + \sum_i \frac{b(b - K_i)}{x_i^2} + \omega^2 r^2, \quad (1.24c)$$

$$\overline{V} = \sum_{i \neq j} \left[ \frac{a(a - K_{ij})}{\sin^2(x_i - x_j)} + \frac{a(a - \tilde{K}_{ij})}{\sin^2(x_i + x_j)} \right] + \sum_{i} \frac{b(b - K_i)}{\sin^2 x_i} + \sum_{i} \frac{b'(b' - K_i)}{\cos^2 x_i} .$$
 (1.24d)

Since these are essentially the operators obtained from the sum of the squares of the Dunkl operators (1.11) and (1.13) through a gauge transformation and a change of variables, the eigenvalue problem for the spin Hamiltonian (1.17) can be effectively reduced to the analogous problem for a quadratic polynomial in the Dunkl operators plus, perhaps, some simple auxiliary operators.

The connection between CS models and Dunkl operators can also be used to define generalizations of the above solvable Hamiltonians (with or without spin). Such an operator H is called **of** CS type. Its spectrum cannot be generally computed by algebraic methods, but by construction [81, 82] H remains partially solvable in the sense that it leaves invariant some known finite-dimensional subspace of  $L^2(\mathbb{R}^N) \otimes \Sigma$ ; in this case, H is said to be  $quasi-exactly \ solvable$  (QES), and the eigenvalues that one can obtain from the latter subspaces are called algebraic.

Remark 1.5. A particularly interesting situation [196, 197] arises when one can find explicitly an infinite flag (i.e., a sequence  $\mathcal{M}_1 \subsetneq \mathcal{M}_2 \subsetneq \cdots \subset L^2(\mathbb{R}^N) \otimes \Sigma$ , where  $\mathcal{M}_n$  is finite-dimensional) invariant under H. In this case, the corresponding Hamiltonian H is sometimes said to be exactly solvable in the sense of Turbiner [196]. This expression is certainly misleading, the latter condition does not imply by any means that algebraic eigenvalues cover the whole point spectrum of H. In this dissertation we shall be interested in this stronger type of quasi-exact solvability, and in fact all the QES Hamiltonians that we shall encounter in Chapter 3 will indeed preserve an infinite-dimensional invariant flag.

#### 1.8 Solvable spin chains

Solvable spin chains have enjoyed a growing popularity in the last few years, due in part to their novel applications to SUSY Yang–Mills and string theories [20, 21, 89, 94, 150, 173]. The role of CS spin models in the study of solvable spin chains was unveiled by Polychronakos [164, 166] in a couple of insightful papers from the mid 90s which elucidated the connection between the original Sutherland model and the recently discovered Haldane–Shastry (HS) chain [100, 178]. We shall conclude this chapter with an overview of the essentials of the CS/HS connection and the definition of some celebrated spin chains that shall be frequently referred to in forthcoming chapters.

The **Heisenberg chain** [110], which was born as an attempt to model ferromagnetic materials, describes N particles on a lattice with isotropic nearneighbor interactions independent of the site. For particles of spin  $M \in \frac{1}{2}\mathbb{N}$ , the Hamiltonian of the system is customarily written as

$$\mathsf{H}_{\mathrm{He}} = \sum_{i} \mathbf{S}_{i} \cdot \mathbf{S}_{i+1} \,, \tag{1.25}$$

where  $\mathbf{S}_i$  is the (vector) spin operator of the particle  $i, \, \mathbf{S}_{N+1} \equiv \mathbf{S}_1$  and

$$\mathbf{S}_i \cdot \mathbf{S}_j = \sum_{a=1}^{4M(M+1)} S_i^a S_j^a.$$

For each fixed particle label i, we denote by  $S_i^a$   $(a=1,\ldots,4M(M+1))$  a basis of the fundamental representation of  $\mathfrak{su}(2M+1)$ , which we assume to be orthonormal with respect to the bilinear form  $\langle S,S'\rangle=2\operatorname{tr}(SS')$ . It is straightforward to write the Hamiltonian (1.25) in terms of the spin exchange operators (1.16a) using that

$$S_{ij} = 2\mathbf{S}_i \cdot \mathbf{S}_j + \frac{1}{2M+1}. \tag{1.26}$$

Note that for spin 1/2 particles we have  $\mathbf{S}_i = \frac{1}{2}(\sigma_i^1, \sigma_i^2, \sigma_i^3)$ , where  $\sigma_i^j$  is the j-th Pauli matrix acting on the spin space of the i-th particle.

It is well known [25, 50, 117] that the spin 1/2 Heisenberg chain can be solved exactly using the Bethe ansatz. Several (partially) solvable generalizations of the Heisenberg chain (1.25) with short-range interactions (at most between next-to-nearest neighbors) have been subsequently proposed in the literature. These include, in particular, the family of chains with arbitrary spin and nearest-neighbors interactions polynomial in  $\mathbf{S}_i \cdot \mathbf{S}_{i+1}$  of

Refs. [12, 192], as well as several models whose ground state can be written in terms of "valence bonds" [3, 147, 148].

The  $Haldane-Shastry\ model\ [100,178]$  is the prime example of solvable spin chain with long-range interactions. It describes an arrangement of N equally spaced spins on a circle which interact pairwise with a strength proportional to the inverse square of their chord distance. The Hamiltonian of the HS chain can be written as

$$\mathsf{H}_{\mathrm{HS}} = \sum_{i < j} \sin^{-2}(\xi_i - \xi_j)(1 - S_{ij}), \qquad (1.27)$$

where the positions of the sites are given by

$$\xi_i = \frac{i\pi}{N} \,.$$

This system quickly attracted considerable attention [86,101,120,129,188] due to its remarkable algebraic properties and its connection with conformal field theory [55] and the symplectic Dyson model [62]. The original interest of this model, however, lies in the fact that Gutzwiller's variational wave function for the Hubbard model [91,97,98], which also coincides with the one-dimensional version of the resonating valence bond state introduced by Anderson [7] and with the one-dimensional version of the Kalmeyer–Laughlin state [126], becomes an exact eigenfunction of the HS chain as the strength of the on-site interaction tends to infinity.

In the early nineties the integrability of the Hamiltonian (1.27) was rigorously proved and its Yangian symmetry [87, 102] was established. Not surprisingly, the Yangian algebra [57] also appears as the symmetry algebra [9, 23, 113] of the Sutherland model (1.7b). Although the obvious relationship between this model and the chain (1.27) was already remarked by Shastry in his original paper, their connection was first made explicit by Polychronakos through the so-called **freezing trick** [165]. This technique formally allows one to recover the HS Hamiltonian from the Sutherland spin model (1.18b) (with  $\epsilon = 1$ ) in the large coupling constant limit, which is clearly tantamount to the classical limit  $\hbar \downarrow 0$ . A rigorous proof of the classical limit and some extensions and improvements will be presented in Chapters 2 and 4.

A further application of the freezing trick enabled Polychronakos to obtain the integrable spin chain with long-range interactions given by the Hamiltonian

$$\mathsf{H}_{PF} = \sum_{i} (\xi_i - \xi_j)^{-2} (1 - S_{ij}), \qquad (1.28)$$

where the sites  $\xi_1 < \cdots < \xi_N$  satisfy the equation

$$\xi_i = \sum_{j \neq i} \frac{1}{\xi_i - \xi_j} \,. \tag{1.29}$$

This system is usually referred to as the *Polychronakos–Frahm (PF)* chain, and relates to the spin Calogero model (1.18a) as the HS chain does to the Sutherland model (1.18b).

As remarked by Frahm [88], the sites of the PF chain are no longer equally spaced, but Eq. (1.29) shows that they are given by the zeros of the N-th Hermite polynomial [189]. In particular [36], this implies that the density of sites for large N is given by

$$\rho_N^{\rm PF}(x) = \frac{1}{\pi N} \sqrt{2N - x^2} \,. \tag{1.30}$$

Remark 1.6. The CS/HS correspondence casts the original model of Haldane and Shastry into a much wider framework. Consequently, we shall say that a spin chain is **of HS type** if it can be derived from a spin Schrödinger operator of CS type via the freezing limit. In particular, both the HS and the PF chains are of HS type.

Interestingly, we shall see in Chapters 2 and 4 (cf. also [84, 166]) that the chains of HS type, featuring long-range position-dependent interactions, seem to enjoy stronger solvability properties than those of Heisenberg type, characterized by the short range and position independence of the interactions.

## Chapter 2

## HS spin chains of $BC_N$ type

#### 2.1 Introduction

In the previous chapter we saw that the connection between  $A_N$  CS models and spin chains of HS type has been thoroughly studied. In contrast,  $BC_N$  spin models [39, 60, 82, 122, 205] had received comparatively little attention. Two main reasons were responsible for this fact. On one hand,  $BC_N$  HS chains depend nontrivially on free parameters (one in the rational case and two in the trigonometric one). On the other hand, the sites of the  $BC_N$  chain are not equally spaced and no explicit description of their positions is available.

Let us give a very brief review of the existing results on these models. The integrability of the rational  $BC_N$  HS chain was established by Yamamoto and Tsuchiya [206] using Dunkl operators, but its spectrum has not been computed so far. The HS trigonometric spin chain of  $BC_N$  type was discussed by Bernard, Pasquier and Serban [24], but only for spin 1/2 and with the assumption that the sites are equally spaced, which restricts the space of free parameters in the model to just three particular values. Finkel et al. [83] discussed the integrability of the hyperbolic HS chain of  $BC_N$  type, but did not examine its spectrum either.

The main result of this chapter is the exact computation of the spectrum of the  $BC_N$  HS chain. In fact, four chains associated with the  $BC_N$  Sutherland model are considered, two of which are essentially different. This computation is based on a rigorous study of Polychronakos's "freezing trick", which is in fact promoted to a "freezing lemma" in Section 2.5.

The organization of this chapter is as follows. In Sections 2.2 to 2.4 we present a detailed study of the  $BC_N$  Sutherland spin model and prove its

exact solvability using Dunkl operators. In Section 2.5 we introduce the  $BC_N$  HS chains and compute their spectrum, whose statistical properties we analyze carefully in the following section. Finally, in Section 2.7 we consider the addition of a constant magnetic field to this picture.

The material presented in this chapter is based on Refs. [63–65].

### 2.2 The spin $BC_N$ Sutherland model

In this section we shall study in some detail the spin  $BC_N$  CS models associated with the one-dimensional representations  $\chi_{\epsilon\epsilon'}$  of  $\mathfrak{W}(BC_N)$ . Physically, the parameter  $\epsilon$  determines the ferromagnetic ( $\epsilon = 1$ ) or antiferromagnetic ( $\epsilon = -1$ ) behavior of the associated spin chains.

Let us recall that the potential  $V_{\epsilon\epsilon'}$  of the spin  $BC_N$  Sutherland model was given in Eq. (1.18d). For reasons that will be apparent by the end of this chapter, it is convenient to define the  $BC_N$  Sutherland Hamiltonian by the action of the differential operator

$$H_{\epsilon\epsilon'} = -\Delta + V_{\epsilon\epsilon'} \tag{2.1}$$

on the "reduced" Hilbert space

$$\mathcal{H} = L^2(C) \otimes \Sigma. \tag{2.2}$$

Here the Weyl alcove (1.5) reads

$$C = \{ \mathbf{x} : 0 < x_1 < \dots < x_N < \pi/2 \}.$$

For the sake of completeness, we shall prove rigorously that this choice of domain is perfectly admissible from a mathematical point of view. To this end, let us define the  $C^2$  function  $\mu$  as in Eq. (1.14), so that

$$\mu(\mathbf{x}) = \prod_{i < j} \left| \sin(x_i - x_j) \sin(x_i + x_j) \right|^a \prod_i \left| \sin x_i \right|^b \prod_i \left| \cos x_i \right|^{b'}. \tag{2.3}$$

Here we are assuming that a, b, b' > 1/2. It can be readily verified that

$$H_{\rm sc}\mu = E_0\mu$$
.

where the **scalar Hamiltonian**  $H_{\rm sc}$  is defined on  $L^2(C)$  by Eqs. (1.6)-(1.7d) and

$$E_0 = N\left[\frac{2}{3}(N-1)(2N-1)a^2 + 2(N-1)a(b+b') + (b+b')^2\right].$$
 (2.4)

Since  $\mu$  does not vanish on C,  $\mu$  is the ground state function of  $H_{\rm sc}$  [143].

**Proposition 2.1.** If a, b, b' > 2,  $H_{\epsilon\epsilon'}$  is essentially self-adjoint on  $\mathcal{D} \otimes \Sigma$ , where

$$\mathcal{D} = \mu \mathbb{C}[e^{\pm 2ix_1}, \dots, e^{\pm 2ix_N}].$$

*Proof.* A theorem of Komori and Takemura [132] shows that  $\mathcal{D}$  is a core for the operator  $H_{\text{sc}}$  (1.7d). Since  $H_{\epsilon\epsilon'} - H_{\text{sc}}$  is  $H_{\text{sc}}$ -bounded with relative bound

$$\max\{(a-1)^{-1},(b-1)^{-1},(b'-1)^{-1}\}\,,$$

the Kato–Rellich theorem [172] shows that  $H_{\epsilon\epsilon'}$  is essentially self-adjoint on  $\mathcal{D} \otimes \Sigma$ .

Remark 2.2. Although its proof is simple (in fact, a direct proof could have been provided without much additional effort, and the conditions on a, b, b' could have be easily relaxed), the inverse-square singularity prevents us from obtaining Proposition 2.1 from the usual theorems on self-adjointness [184]. Interestingly, the proof of this result is essentially algebraic, and leans on our precise knowledge of the eigenfunctions of the  $BC_N$  scalar Sutherland model.

Although it has some applications to CS models [17,18,79], hereafter we shall not touch upon this topic, and by "self-adjoint" we shall always mean "formally self-adjoint".

Some words on the choice of Hilbert space are in order. Consider the orthogonal projectors

$$\Lambda_{\epsilon\epsilon'} = \frac{1}{|\mathfrak{M}|} \sum_{\Pi \in \mathfrak{M}} \chi_{\epsilon\epsilon'}(\Pi) \Pi, \qquad (2.5a)$$

$$\Lambda_{\epsilon} = \frac{1}{N!} \sum_{\Pi \in \langle \Pi_{ij} \rangle} \chi_{\epsilon}(\Pi) \Pi, \qquad (2.5b)$$

which we shall call **symmetrizers**, and let

$$\mathbb{T}^N = \{ \mathbf{x} : -\pi/2 < x_i < \pi/2 \}$$

be the flat N-torus. Observe that the angle coordinates of the torus are given by  $2\mathbf{x}$ , so that every function in  $C^0(\mathbb{T}^N)$  will be  $\pi$ -periodic in  $x_i$ . Define the "physical" Hilbert spaces of N distinguishable particles as

$$\mathcal{H}_0 = L^2(\mathbb{T}^N) \otimes \Sigma$$
.

It is a standard result in representation theory [183] that  $\mathcal{H}_0$  can be decomposed into the direct sum

$$\mathcal{H}_0 = \bigoplus_{\epsilon, \epsilon' = \pm 1} \mathcal{H}_{\epsilon \epsilon'} \tag{2.6}$$

of the auxiliary spaces

$$\mathcal{H}_{\epsilon\epsilon'} = \Lambda_{\epsilon\epsilon'} \mathcal{H}_0$$
,

which are given by the image of the symmetrizers (2.5). We shall also be interested in the subspaces

$$\mathcal{H}_{\epsilon} = \mathcal{H}_{\epsilon,+} \oplus \mathcal{H}_{\epsilon,-} \,, \tag{2.7}$$

i.e., the fermionic ( $\epsilon = -1$ ) and bosonic ( $\epsilon = +1$ ) sectors of  $\mathcal{H}_0$ .

It is technically convenient to define other two self-adjoint operators defined by the differential operator (2.1). We shall denote by  $H^{\epsilon\epsilon'}$  (resp.  $T_{\epsilon\epsilon'}$ ) the Hamiltonian defined by the action of (2.1) on  $\mathcal{H}_0$  (resp.  $\mathcal{H}_{\epsilon\epsilon'}$ ). In the next proposition we summarize some elementary results relating the "physical" Hamiltonians to the "reduced" ones.

#### **Proposition 2.3.** The following statements hold:

- (i)  $\mathfrak{W}$  maps  $\mathcal{H}_{\epsilon\epsilon'}$  into itself.
- (ii)  $\mathfrak{W}$  is a symmetry group of the physical Hamiltonian  $H^{\epsilon\epsilon'}$ .
- (iii) The decompositions (2.6) and (2.7) are invariant under the physical Hamiltonians.
- (iv)  $H^{\epsilon\epsilon'}|_{\mathcal{H}_{\eta\eta'}}$  and  $H_{\epsilon\epsilon'}$  are isospectral, and the corresponding eigenfunctions  $\Psi \in \mathcal{H}$  and  $\Phi \in \mathcal{H}_{\eta\eta'}$  are related by

$$\Phi(\mathbf{x}) = \chi_{mn'}(w) S_w \Psi(w\mathbf{x}), \qquad (2.8)$$

where  $w \in \mathfrak{W}(BC_N)$  is such that  $w\mathbf{x} \in C$ .

*Proof.* The first part of the lemma follows most easily from Lemma 2.5 in the next section. To prove the remaining points it is convenient to keep the abstract root formulation, with  $g_{\alpha}^2 = \|\boldsymbol{\alpha}\|^2 a_{\alpha}(a_{\alpha} - 1)$ , and use the representation (1.21b). Then one can write

$$\Pi_{w}H^{\epsilon\epsilon'} = \Pi_{w} \left[ -\Delta + \sum_{\boldsymbol{\alpha} \in (BC_{N})_{+}} \|\boldsymbol{\alpha}\|^{2} a_{\boldsymbol{\alpha}} (a_{\boldsymbol{\alpha}} - \epsilon_{\boldsymbol{\alpha}} S_{\boldsymbol{\alpha}}) v(\boldsymbol{\alpha} \cdot \mathbf{x}) \right] 
= \left[ -\Delta + \sum_{\boldsymbol{\alpha} \in (BC_{N})_{+}} \|\boldsymbol{\alpha}\|^{2} a_{\boldsymbol{\alpha}} (a_{\boldsymbol{\alpha}} - \epsilon_{\boldsymbol{\alpha}} S_{w\boldsymbol{\alpha}}) v(w\boldsymbol{\alpha} \cdot \mathbf{x}) \right] \Pi_{w} 
= \left[ -\Delta + \sum_{\boldsymbol{\alpha} \in (BC_{N})_{+}} \|\boldsymbol{\alpha}\|^{2} a_{\boldsymbol{\alpha}} (a_{\boldsymbol{\alpha}} - \epsilon_{\boldsymbol{\alpha}} S_{\boldsymbol{\alpha}}) v(\boldsymbol{\alpha} \cdot \mathbf{x}) \right] \Pi_{w}$$

$$= H^{\epsilon\epsilon'} \Pi_{w}, \qquad (2.9)$$

so that  $[H^{\epsilon\epsilon'}, \Pi_w] = 0$ . Since the decompositions (2.6) and (2.7) are associated with a symmetry group of  $H^{\epsilon\epsilon'}$ , they are invariant under this operator.

Each eigenfunction  $\Phi$  of  $H^{\epsilon\epsilon'}$  with energy E belongs to some subspace  $\mathcal{H}_{\eta\eta'}$  by the  $\mathfrak{W}$  symmetry. The potential diverging quadratically at  $\partial C \subset \mathbb{T}^N$ ,  $\Phi$  must vanish at  $\partial C$ , and therefore  $\Psi = \Phi|_C$  must be an  $L^2$  solution to the PDE

$$(H_{\epsilon\epsilon'} - E)\Psi = 0$$
 in  $C$ ,  
 $\Psi|_{\partial C} = 0$ .

Hence  $\Psi$  is an eigenfunction of the self-adjoint operator  $H_{\epsilon\epsilon'}$  and  $\Phi$  can be recovered from  $\Psi$  through Eq. (2.8).

Corollary 2.4.  $\operatorname{spec}(T_{\epsilon\epsilon'}) = \operatorname{spec}(H_{\epsilon\epsilon'}).$ 

#### 2.3 $BC_N$ Dunkl operators

In this section we shall make use of the Dunkl operator formalism to compute the spectrum of the spin model  $H_{\epsilon\epsilon'}$  in closed form. Following Refs. [83, 205] and Eq. (1.13), let us consider the following set of commutative **Dunkl** operators

$$J_{i} = i \partial_{x_{i}} + a \sum_{j \neq i} \left[ (1 - i \cot(x_{i} - x_{j})) K_{ij} + (1 - i \cot(x_{i} + x_{j})) \tilde{K}_{ij} \right]$$

$$+ \left[ b (1 - i \cot x_{i}) + b' (1 + i \tan x_{i}) \right] K_{i} - 2a \sum_{j < i} K_{ij}.$$
 (2.11)

The following commutation relations with the exchange operators  $K_i$ ,  $K_{ij}$  can be readily checked:

$$[K_{ij}, J_k] = \begin{cases} 2a(K_{ik} - K_{jk})K_{ij}, & \text{if } i < k < j \\ 0, & \text{otherwise} \end{cases}$$

$$(2.12a)$$

$$K_{ij}J_i - J_jK_{ij} = 2a\left(1 + \sum_{i < l < j} K_{ij}K_{il}\right),$$
 (2.12b)

$$K_{ij}J_j - J_iK_{ij} = -2a\left(1 + \sum_{i < l < j} K_{ij}K_{jl}\right),$$
 (2.12c)

$$[K_i, J_j] = 2aK_{ij}(K_i - K_j), [K_j, J_i] = 0,$$
 (2.12d)

$$\{K_i, J_i\} = 2(b+b') + 2a \sum_{l>i} K_{il}(K_i + K_l),$$
 (2.12e)

Here we assume that i < j and k are distinct indices in the range  $1, \ldots, N$ .

As an application of the Dunkl operator formalism, let us show the complete integrability of the spin Hamiltonian  $H_{\epsilon\epsilon'}$ . The proof is based on the following lemma, which has been essentially taken from Ref. [83]. The definitions of the spaces  $\mathfrak{D}, \mathfrak{K}$  and  $\mathfrak{S}$  and of the star mapping were given in Section 1.7.

**Lemma 2.5.** Let  $A, B \in \mathfrak{D} \otimes \mathfrak{K}$  and  $T \in \mathfrak{D} \otimes \mathfrak{S}$ . Then the following statements hold:

- (i)  $A\Lambda_{\epsilon\epsilon'} = A^*_{\epsilon\epsilon'}\Lambda_{\epsilon\epsilon'}$ .
- (ii) If  $T\Lambda_{\epsilon\epsilon'} = 0$ , then T = 0.
- (iii) If B commutes with  $\Lambda_{\epsilon\epsilon'}$ , then  $(AB)^*_{\epsilon\epsilon'} = A^*_{\epsilon\epsilon'}B^*_{\epsilon\epsilon'}$ .
- (iv) If A, B commute with  $\Lambda_{\epsilon\epsilon'}$ , then  $[A, B]^*_{\epsilon\epsilon'} = [A^*_{\epsilon\epsilon'}, B^*_{\epsilon\epsilon'}]$ .
- (v) If A commutes with  $K_w$  for some  $w \in \mathfrak{W}(BC_N)$ , then  $A_{\epsilon\epsilon'}^*$  commutes with  $\Pi_w$ .

*Proof.* (i) Let us write

$$A = \sum_{\sigma} D_{\sigma} K_{\sigma} \,, \tag{2.13}$$

where  $\sigma$  takes values in  $\mathfrak{W}(BC_N)$ . By definition,  $K_{\sigma}\Lambda_{\epsilon\epsilon'} = \chi_{\epsilon\epsilon'}(\sigma)S_{\sigma}\Lambda_{\epsilon\epsilon'}$ , so

$$A\Lambda_{\epsilon\epsilon'} = \sum_{\sigma} \chi_{\epsilon\epsilon'}(\sigma) D_{\sigma} S_{\sigma} \Lambda_{\epsilon\epsilon'} = A_{\epsilon\epsilon'}^* \Lambda_{\epsilon\epsilon'}.$$

- (ii) Write  $T = \sum_{|\mathbf{n}| < k, \sigma} B_{\sigma, \mathbf{n}}(\mathbf{x}) \partial^{\mathbf{n}} S_{\sigma}$ , where  $\mathbf{n} \in \mathbb{N}_0^N$  and  $\partial^{\mathbf{n}} = \partial_{x_1}^{n_1} \cdots \partial_{x_N}^{n_N}$ . The statement follows by choosing a finite set of functions  $\{\varphi_{\mathbf{n}, \sigma}\} \subset \mathcal{H}_{\epsilon \epsilon'}$  such that the matrix  $(S_{\sigma} \partial^{\mathbf{n}} \varphi_{\mathbf{n}', \sigma'}(\mathbf{x}))$  is nonsingular.
- (iii) It follows from (i) and (ii) using that

$$AB\Lambda_{\epsilon\epsilon'} = A\Lambda_{\epsilon\epsilon'}B = A^*_{\epsilon\epsilon'}\Lambda_{\epsilon\epsilon'}B = A^*_{\epsilon\epsilon'}B^*_{\epsilon\epsilon'}\Lambda_{\epsilon\epsilon'}$$

- (iv) It is a direct consequence of (iii).
- (v) Let us denote by  $Ad_w$  the automorphism of  $\mathfrak{W}(BC_N)$  given by  $\sigma \mapsto w\sigma w$ . By hypothesis, and using the notation (2.13),

$$A = K_w A K_w = \sum_{\sigma} K_w(D_{\sigma}) K_{\mathrm{Ad}_w \sigma},$$

so  $K_w(D_\sigma) = D_{\operatorname{Ad}_w \sigma}$  for each  $\sigma$ . Hence

$$\Pi_w A_{\epsilon \epsilon'}^* \Pi_w = \sum_{\sigma} \chi_{\epsilon \epsilon'}(\sigma) K_w(D_{\sigma}) S_{\mathrm{Ad}_w \sigma}$$
$$= \sum_{\sigma} \chi_{\epsilon \epsilon'}(\mathrm{Ad}_w \sigma) D_{\mathrm{Ad}_w \sigma} S_{\mathrm{Ad}_w \sigma} = A_{\epsilon \epsilon'}^*,$$

as we wanted to prove.

Let us define the operators

$$I_p = \sum_i J_i^{2p} \tag{2.14}$$

acting on  $L^2(C)$ . It can be readily verified that  $I_1$  coincides with the exchange Hamiltonian (1.24d), so that the spin Hamiltonian (1.18d) can be recovered as a differential operator from the relation  $H_{\epsilon\epsilon'} = (I_1)^*_{\epsilon\epsilon'}$ . Since the Dunkl operators (2.11) are self-adjoint and commute among them, so do the positive operators  $I_p$ .

**Proposition 2.6.** The operators (2.14) are  $\Re$ -invariant.

*Proof.* Let us prove that the elementary permutation  $K_{i,i+1}$  commutes with  $I_p$ . Eq. (2.12a) shows that  $[K_{i,i+1}, J_j] = 0$  for  $j \neq i, i+1$ . Since

$$K_{i,i+1}J_i^{2p} = J_{i+1}^{2p}K_{i,i+1} + 2a\sum_{r=0}^{2p-1}J_i^{2p-r-1}J_{i+1}^r,$$
  
$$K_{i,i+1}J_{i+1}^{2p} = J_i^{2p}K_{i,i+1} - 2a\sum_{r=0}^{2p-1}J_i^{2p-r-1}J_{i+1}^r,$$

it stems that  $[K_{i,i+1}, \sum J_i^{2p}] = 0$ .  $K_N$  also commutes with  $I_p$  by virtue of the relations

$$K_N J_i = J_i K_N, \quad \text{if } i < N,$$
 
$$K_N J_N^2 = -J_N K_N J_N + 2(b+b') J_N = J_N^2 K_N.$$

Here we have used that  $\{K_N, J_N\} = 2(b + b')$ . Now it immediately follows that

$$[K_i,I_p]=K_{iN}[K_{iN}K_i,I_p]=K_{iN}[K_NK_{iN},I_p]=0$$
 for all  $i=1,\ldots,N.$ 

**Lemma 2.7.** The operator  $(I_p)_{\epsilon\epsilon'}^*$  is self-adjoint.

*Proof.* Proposition 2.6 shows that  $I_p$  commutes with  $\Lambda_{\epsilon\epsilon'}$ . Using Lemma 2.5 and the self-adjointness of  $I_p$  and  $\Lambda_{\epsilon\epsilon'}$ , one can see that

$$(I_p)_{\epsilon\epsilon'}^* \Lambda_{\epsilon\epsilon'} = I_p \Lambda_{\epsilon\epsilon'} = (I_p \Lambda_{\epsilon\epsilon'})^{\dagger} = [(I_p)_{\epsilon\epsilon'}^* \Lambda_{\epsilon\epsilon'}]^{\dagger} = [\Lambda_{\epsilon\epsilon'} (I_p)_{\epsilon\epsilon'}^*]^{\dagger} = [(I_p)_{\epsilon\epsilon'}^*]^{\dagger} \Lambda_{\epsilon\epsilon'},$$
so  $[(I_p)_{\epsilon\epsilon'}^*]^{\dagger} = (I_p)_{\epsilon\epsilon'}^*$  as a consequence of Lemma 2.5.

Thus we have proved the following

**Theorem 2.8.** The self-adjoint operators  $(I_p)_{\epsilon\epsilon'}^*$   $(p \in \mathbb{N})$  form a commuting family which includes the Hamiltonian  $H_{\epsilon\epsilon'} = (I_1)_{\epsilon\epsilon'}^*$ .

### 2.4 Spectrum of the dynamical model

There are two steps in the computation of the spectrum of  $H_{\epsilon\epsilon'}$ . First, a key observation is that the gauge Hamiltonian is given by

$$\overline{H} = I_1 = \sum_i J_i^2 \tag{2.15}$$

and that  $J_i$  leaves invariant an infinite flag

$$\overline{\mathcal{H}}^0 \subset \overline{\mathcal{H}}^1 \subset \cdots$$

of finite-dimensional subspaces. The action of  $\overline{H}$  is triangular in a suitably chosen basis and the eigenvalues can be read off the diagonal elements. Second, a careful analysis allows us to extend these results to the spin Hamiltonian  $H_{\epsilon\epsilon'}$ .

Let us begin with some definitions. Let us consider the flag

$$\overline{\mathcal{H}}^0 \subset \overline{\mathcal{H}}^1 \subset \cdots$$

where

$$\overline{\mathcal{H}}^k = \langle f_{\mathbf{n}} : |\mathbf{n}| \leqslant k \rangle. \tag{2.16}$$

Here  $\mathbf{n} \in \mathbb{Z}^N$ ,  $|\mathbf{n}|$  is defined as in Eq. (1.9) and

$$f_{\mathbf{n}}(\mathbf{x}) = \mu(\mathbf{x}) e^{2i\mathbf{n} \cdot \mathbf{x}},$$
 (2.17)

with the gauge factor  $\mu$  given by Eq. (2.3) and

$$\mathbf{n} \cdot \mathbf{x} = \sum_{i} n_i x_i \,.$$

Using Fourier analysis it is easy to see that the countable union  $\bigcup_{k=1}^{\infty} \overline{\mathcal{H}}^k$  is dense in  $L^2(\mathbb{T}^N)$ , and that one can obtain a Hilbert basis  $\overline{\mathcal{B}}$  of this space by applying the Gram–Schmidt procedure to the functions (2.17).

Given a multiindex  $\mathbf{n} \in \mathbb{Z}^N$ , we shall define the non-negative, nonincreasing multiindex

$$[\mathbf{n}] = (|n_{\pi(1)}|, \dots, |n_{\pi(N)}|),$$

where the permutation  $\pi \in S_N$  is chosen so that

$$|n_{\pi(1)}| \geqslant \cdots \geqslant |n_{\pi(N)}|$$
.

The set of non-negative, nonincreasing multiindices will be denoted by  $[\mathbb{Z}^N]$ . We shall order  $\mathbb{Z}^N$  by defining  $\mathbf{n} \prec \mathbf{m}$  if there exists a number  $i \geqslant 1$  such that  $[\mathbf{n}]_j = [\mathbf{m}]_j$  for all j < i and  $[\mathbf{n}]_i < [\mathbf{m}]_i$ , where  $[\mathbf{n}]_j$  denotes the jth component of  $[\mathbf{n}]$ . This partial order can be naturally extended to  $\{f_{\mathbf{n}}\}$  and hence to  $\overline{\mathcal{B}}$ . We shall assume that the ordering of  $\overline{\mathcal{B}}$  is compatible with this partial order.

In the following lemma we show that the order relations do in fact hold on orbits of  $\mathfrak{K}$ .

**Lemma 2.9.** For each  $w \in \mathfrak{W}(BC_N)$ ,  $K_w f_{\mathbf{n}} = f_{w\mathbf{n}}$  and  $[w\mathbf{n}] = [\mathbf{n}]$ . In particular, there exists as element  $K \in \mathfrak{K}$  such that  $K f_{\mathbf{n}} = f_{[\mathbf{n}]}$ .

*Proof.* The first part is trivial from the fact that w is a reflection, and thus  $\mathbf{n} \cdot w\mathbf{x} = w\mathbf{n} \cdot \mathbf{x}$ . The second statement holds because  $\mathfrak{W}(BC_N)$  is generated by  $K_{ij}$  and  $K_i$ , i.e., by permutations and sign reversals.

Now let us introduce some more notation. Given  $\mathbf{n} \in [\mathbb{Z}^N]$  and  $k \in \mathbb{N}_0$ , we define

$$\#(k) \equiv \#(k, \mathbf{n}) = \operatorname{card}\{i : n_i = k\},$$
  
$$\ell(k) \equiv \ell(k, \mathbf{n}) = \min (\{i : n_i = k\} \cup \{+\infty\}).$$

Now one can characterize the action of the Dunkl operators (2.11) on the functions (2.17) as follows.

**Proposition 2.10.** For each  $\mathbf{n} \in [\mathbb{Z}^N]$ , the action of  $J_i$  on the function  $f_{\mathbf{n}}$  is given by

$$J_i f_{\mathbf{n}} = \lambda_{i,\mathbf{n}} f_{\mathbf{n}} + \sum_{\mathbb{Z}^N \ni \mathbf{m} \prec \mathbf{n}} c_{i,\mathbf{n}}^{\mathbf{m}} f_{\mathbf{m}},$$

where

$$\lambda_{i,\mathbf{n}} = \begin{cases} 2n_i + b + b' + 2a(N + i + 1 - \#(n_i) - 2\ell(n_i)), & \text{if } n_i > 0, \\ -b - b' + 2a(i - N), & \text{if } n_i = 0, \end{cases}$$

and  $c_{i,\mathbf{n}}^{\mathbf{m}} \in \mathbb{R}$ .

*Proof.* Let  $z_i = e^{2ix_i}$ ,  $y_{ij} = z_i z_j^{-1}$  y  $w_{ij} = z_i z_j$ . A tedious but straightforward computation shows that

$$J_{i}f_{\mathbf{n}} = 2f_{\mathbf{n}} \left[ n_{i} + a(N-1) + \frac{1}{2} (b+b') + a \sum_{j < i} \left( \frac{1 - y_{ij}^{n_{j} - n_{i}}}{y_{ij} - 1} + \frac{1 - w_{ij}^{1 - n_{i} - n_{j}}}{w_{ij} - 1} \right) + a \sum_{j > i} \left( \frac{1 - y_{ij}^{1 + n_{j} - n_{i}}}{y_{ij} - 1} + \frac{1 - w_{ij}^{1 - n_{i} - n_{j}}}{w_{ij} - 1} \right) + b \frac{1 - z_{i}^{1 - 2n_{i}}}{z_{i} - 1} - b' \frac{1 + z_{i}^{1 - 2n_{i}}}{z_{i} + 1} \right].$$

It can be readily verified that all the multiindices appearing in this formula are smaller than or equal to  $\mathbf{n}$ . In particular, the value of  $\lambda_{i,\mathbf{n}}$  can be computed from the constant terms in the above formula.

Corollary 2.11. For all  $n \in \mathbb{Z}^N$ ,

$$J_i f_{\mathbf{n}} = \sum_{\mathbb{Z}^N \ni \mathbf{m} \prec \mathbf{n}} \gamma_{i,\mathbf{n}}^{\mathbf{m}} f_{\mathbf{m}}$$

for some real constants  $\gamma_{i,\mathbf{n}}^{\mathbf{m}}$ .

Proof. Let  $K \in \mathfrak{K}$  be defined as in Lemma 2.9. Then  $J_iKf_{\mathbf{n}} = J_if_{[\mathbf{n}]}$  is given by Proposition 2.10, and the result follows from the commutation relations (2.12a)–(2.12e) and the invariance of the partial order under the Weyl group (cf. Lemma 2.9).

**Proposition 2.12.** For all  $\mathbf{n} \in \mathbb{Z}^N$  there exist some real constants  $c_{\mathbf{n}}^{\mathbf{m}}$  such that

$$\overline{H}f_{\mathbf{n}} = E_{\mathbf{n}}f_{\mathbf{n}} + \sum_{\mathbb{Z}^{N} \ni \mathbf{m} \prec \mathbf{n}} c_{\mathbf{n}}^{\mathbf{m}} f_{\mathbf{m}} ,$$

where

$$E_{\mathbf{n}} = \sum_{i} (2[\mathbf{n}]_{i} + b + b' + 2a(N - i))^{2}$$
(2.18)

only depends on [n].

*Proof.* By Proposition 2.6,  $\overline{H} = I_1$  is  $\mathfrak{K}$ -invariant, so  $\overline{H}f_{\mathbf{n}} = K\overline{H}Kf_{\mathbf{n}}$ . By Lemma 2.9, one can assume that  $Kf_{\mathbf{n}} = f_{[\mathbf{n}]}$ . By Proposition 2.10,

$$\begin{split} \overline{H}f_{\mathbf{n}} &= K \overline{H}f_{[\mathbf{n}]} = K \sum_{i} J_{i}^{2} f_{[\mathbf{n}]} \\ &= \sum_{i} \lambda_{i,[\mathbf{n}]}^{2} f_{\mathbf{n}} + \sum_{i,\mathbf{m} \prec \mathbf{n}} \lambda_{i,[\mathbf{n}]} c_{i,[\mathbf{n}]}^{\mathbf{m}} K f_{\mathbf{m}} + \sum_{i,\mathbf{m} \prec \mathbf{n}} c_{i,[\mathbf{n}]}^{\mathbf{m}} K J_{i} f_{\mathbf{m}} \,. \end{split}$$

By Lemma 2.9 and Corollary 2.11, it suffices to show that  $\sum_{i} \lambda_{i,[\mathbf{n}]}^{2}$  is given by Eq. (2.18).

There is no loss of generality in assuming that  $\mathbf{n} \in [\mathbb{Z}^N]$ . To compute the sum, let us gather those components of the multiindex that have the same value. Thus, take those components of the non-negative, nonincreasing multiindex  $\mathbf{n}$  such that  $n_{k-1} \neq n_k = \cdots = n_{k+p} \neq n_{k+p+1}$ . Then  $\ell(n_{k+j}) = k$  and  $\#(n_{k+j}) = p+1$  for all  $0 \leq j \leq p$ . If  $n_k \neq 0$ ,

$$\lambda_{k+j,\mathbf{n}} = 2n_{k+j} + b + b' + 2a(N-k-p+j) = 2n_{k+p-j} + b + b' + 2a(N-(k+p-j)),$$

so that

$$\sum_{i=k}^{k+p} \lambda_{i,\mathbf{n}}^2 = \sum_{i=k}^{k+p} \left( 2n_i + b + b' + 2a(N-i) \right)^2.$$
 (2.19)

If  $n_k = 0$ , Proposition 2.10 shows directly that Eq. (2.19) also holds in this case.

Corollary 2.13. The spectrum of  $\overline{H}$  as a self-adjoint operator on  $L^2(\mathbb{T}^N)$  is given by  $\{E_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^N\}$ . Thus the degeneracy of each eigenvalue is at least  $|\mathfrak{W}| = 2^N N!$ .

*Proof.* It follows from the density of the flag (2.16) and the fact that  $\overline{H}|_{\overline{\mathcal{H}}^k}$  is a triangular matrix in the basis  $\overline{\mathcal{B}}$  with diagonal elements given by  $E_{\mathbf{n}}$ .  $\square$ 

Remark 2.14. Here we loosely (but safely) regard the spectra of the models  $H_{\epsilon\epsilon'}$  as "sets with multiplicities". We shall use this convention whenever appropriate.

Our goal now is to extend Corollary 2.13 to the spin and spinless Hamiltonians  $H_{\epsilon\epsilon'}$  and  $H_{\rm sc}$ . For simplicity, let us start with the latter, considered as an operator on  $L^2(C)$ .

Theorem 2.15. 
$$\operatorname{spec}(H_{\operatorname{sc}}) = \{E_{\mathbf{n}} : \mathbf{n} \in [\mathbb{Z}^N]\}.$$

Proof. By definition of the Weyl alcove,  $|\mathfrak{M}|^{-1}$  times the inclusion  $C \subset \mathbb{T}^N$  yields a unitary transformation U from  $L^2(C)$  to the space of  $\mathfrak{K}$ -invariant functions  $L^2(\mathbb{T}^N)^{\mathfrak{K}}$ . Its inverse is given by  $|\mathfrak{M}|$  times the symmetric extension  $\psi(w\mathbf{x}) = \psi(\mathbf{x})$  (for all  $w \in \mathfrak{M}(BC_N)$ ). Since the action of  $H_{\mathrm{sc}}$  and  $\overline{H}$  agree on  $\mathfrak{K}$ -invariant functions,  $\overline{H}|_{L^2(\mathbb{T}^N)^{\mathfrak{K}}}$  and  $H_{\mathrm{sc}}$  are isospectral.

For each  $\mathbf{n} \in \mathbb{Z}^N$ , the orbit  $\mathfrak{W}(BC_N) \cdot \mathbf{n}$  intersects  $[\mathbb{Z}^N]$  exactly once, at  $[\mathbf{n}]$ . By Lemma 2.9, this implies that

$$\{f_{\mathbf{n}}: |\mathbf{n}| \leqslant k, \ \mathbf{n} \in [\mathbb{Z}^N]\}$$

is a basis of  $(\overline{\mathcal{H}}^k)^{\mathfrak{K}}$ , yielding a Hilbert basis of  $L^2(\mathbb{T}^N)^{\mathfrak{K}}$  via the Gram-Schmidt procedure. By Proposition 2.12, the matrix of  $\overline{H}|_{(\overline{\mathcal{H}}^k)^{\mathfrak{K}}}$  is triangular in an ordered basis and its eigenvalues are given by  $E_{\mathbf{n}}$ , where now  $\mathbf{n} \in [\mathbb{Z}^N]$ .  $\square$ 

The computation of the spectrum of  $H_{\epsilon\epsilon'}$  is similar, although the characterization of the bases is considerably more involved.

**Lemma 2.16.** A basis of  $\Lambda_{\epsilon\epsilon'}(\overline{\mathcal{H}}^k \otimes \Sigma)$  is given by

$$\mathcal{B}_{\epsilon\epsilon'}^{k} = \left\{ \Lambda_{\epsilon\epsilon'}(f_{\mathbf{n}}|\mathbf{s}) : \mathbf{n} \in [\mathbb{Z}^{N}], |\mathbf{s}\rangle \in \mathcal{B}_{\Sigma}^{\mathbf{n},\epsilon,\epsilon'} \right\}, \tag{2.20}$$

where

$$\mathcal{B}_{\Sigma}^{\mathbf{n},\epsilon,\epsilon'} = \left\{ |\mathbf{s}\rangle \in \mathcal{B}_{\Sigma} \mid \begin{array}{cc} s_i - s_j \geqslant \delta_{-1,\epsilon} & \text{when } n_i = n_j \text{ and } i < j \\ s_i \geqslant \frac{1}{2}\delta_{-1,\epsilon'} & \text{when } n_i = 0 \end{array} \right\}.$$

Proof. Since

$$\Pi_w \Lambda_{\epsilon \epsilon'} \Psi = \chi_{\epsilon \epsilon'}(w) \Lambda_{\epsilon \epsilon'} \Psi , \qquad (2.21)$$

it is clear that

$$\left\{ \Lambda_{\epsilon\epsilon'}(f_{\mathbf{n}}|\mathbf{s}\rangle) : \mathbf{n} \in [\mathbb{Z}^N], \ |\mathbf{s}\rangle \in \mathcal{B}_{\Sigma} \right\}$$

must span the whole  $\Lambda_{\epsilon\epsilon'}(\overline{\mathcal{H}}^k\otimes\Sigma)$ . Moreover, a state of the form  $\Lambda_{\epsilon\epsilon'}(f_{\mathbf{n}}|\mathbf{s}\rangle)$  ( $|\mathbf{s}\rangle\in\mathcal{B}_{\Sigma}$ ) vanishes if and only if some of the following conditions hold:

- (i)  $n_i = n_i$ ,  $s_i = s_i$  and  $\epsilon = -1$ .
- (ii)  $n_i = n_i = 0$ ,  $s_i = -s_i$  and  $\epsilon = \epsilon' = -1$ .
- (iii)  $n_i = 0, s_i = 0 \text{ and } \epsilon' = -1.$

It is easy to check that the above states do not vanish and are in fact linearly independent modulo  $\ker \Lambda_{\epsilon\epsilon'}$ . Furthermore, it is immediate to see that for any element  $\Lambda_{\epsilon\epsilon'}(f_{\mathbf{n}}|\mathbf{s}\rangle)$  not satisfying any of the above conditions one can use permutations and sign reversals to construct an element  $w \in \mathfrak{W}(BC_N)$  in the stabilizer of  $\mathbf{n}$  such that  $\Pi_w[\Lambda_{\epsilon\epsilon'}(f_{\mathbf{n}}|\mathbf{s}\rangle)]$  belongs to the basis (2.20). By Eq. (2.21), the claim follows.

Corollary 2.17. For each element  $\Lambda_{\epsilon\epsilon'}(f_{\mathbf{n}}|\mathbf{s}\rangle)$  of the basis  $\mathcal{B}_{\epsilon\epsilon'}^k$ ,

$$\#(n_i) \leqslant \begin{cases} 2M+1 & \text{if } n_i > 0\\ M_{\epsilon'} & \text{if } n_i = 0 \end{cases},$$

where

$$M_{+} = |M| + 1, \qquad M_{-} = \lceil M \rceil,$$
 (2.22)

and  $\lfloor x \rfloor$  and  $\lceil x \rceil$  respectively denote the integer part of x and the smallest integer greater than or equal to x.

Theorem 2.18. spec
$$(H_{\epsilon\epsilon'}) = \{E_{\mathbf{n}} : \mathbf{n} \in [\mathbb{Z}^N], | \mathbf{s} \rangle \in \mathcal{B}_{\Sigma}^{\mathbf{n}, \epsilon, \epsilon'} \}.$$

*Proof.* By Proposition 2.3,  $H_{\epsilon\epsilon'}$  and  $H^{\epsilon\epsilon'}$  are isospectral. We can assume that the basis (2.20) of  $\Lambda_{\epsilon\epsilon'}(\overline{\mathcal{H}}^k\otimes\Sigma)$  is ordered according to the partial order  $\prec$ . By Lemma 2.5,

$$H^{\epsilon \epsilon'} \Lambda_{\epsilon \epsilon'}(\psi | \mathbf{s} \rangle) = \Lambda_{\epsilon \epsilon'} ((\overline{H} \psi) | \mathbf{s} \rangle)$$

for each  $\psi \in L^2(\mathbb{T}^N)$ . Now the same arguments used in the proofs of Corollary 2.13 and Theorem 2.15 show that the action of  $H_{\epsilon\epsilon'}$  in the basis given by Lemma 2.16 is triangular with diagonal elements  $E_{\mathbf{n}}$ .

## 2.5 The spin chain

We define the  $BC_N$  spin chain Hamiltonian as

$$8 H_{\epsilon \epsilon'} = \sum_{i \neq j} \left[ \sin^{-2}(\xi_i - \xi_j) (1 - \epsilon S_{ij}) + \sin^{-2}(\xi_i + \xi_j) (1 - \epsilon \tilde{S}_{ij}) \right] + \sum_{i} \left( \beta \sin^{-2} \xi_i + \beta' \cos^{-2} \xi_i \right) (1 - \epsilon' S_i), \quad (2.23)$$

where the spin sites  $0 < \xi_1 < \cdots < \xi_N < \pi/2$  satisfy

$$\beta' \tan \xi_i - \beta \cot \xi_i = \sum_{j \neq i} \left[ \cot(\xi_i - \xi_j) + \cot(\xi_i + \xi_j) \right]. \tag{2.24}$$

Remark 2.19. A cursory glance at the spin chain (1.27) reveals that the Hamiltonian (2.23) is in fact the appropriate  $BC_N$  version of the celebrated HS chain.

In this section we shall study the Hamiltonian (2.23) and compute its spectrum by taking the large coupling constant limit of the model (1.18d). To this end, let us rescale the parameters of the  $BC_N$  Sutherland Hamiltonian as

$$b = a\beta, \qquad b' = a\beta'. \tag{2.25}$$

The ground state function (2.3) can thus be written as

$$\mu = e^{a\lambda},$$

where the smooth function

$$\lambda(\mathbf{x}) = \sum_{i < j} \left[ \log \sin(x_i - x_j) + \log \sin(x_i + x_j) \right] + \sum_i \left[ \beta \log \sin x_i + \beta' \log \cos x_i \right]$$
(2.26)

is independent of the coupling parameter a. It is immediate to show that Eq. (2.24) merely states that  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in C$  is a critical point of  $\lambda$ . In the following proposition we shall prove an existence and uniqueness result for the solutions of this equation.

**Proposition 2.20.** The function  $\lambda \in L^2(C) \cap C^{\infty}(C)$  has a unique critical point  $\boldsymbol{\xi}$  in C, which is a hyperbolic maximum.

Proof. Since C is bounded and  $\lambda$  tends to  $-\infty$  at  $\partial C$ ,  $\lambda$  must attain its maximum in C. Uniqueness shall follow easily from the fact that the Hessian  $D^2\lambda = (\partial^2\lambda/\partial x_i\partial x_j)$  is negative definite in the convex domain C. To show the definiteness of  $D^2\lambda$ , let us compute

$$\frac{\partial^2 \lambda}{\partial x_i^2} = -\sum_{j \neq i} \left[ \sin^{-2}(x_i - x_j) + \sin^{-2}(x_i + x_j) \right] - \beta \sin^{-2} x_i - \beta' \cos^{-2} x_i < 0,$$

$$\frac{\partial^2 \lambda}{\partial x_i \partial x_j} = -\sin^{-2}(x_i - x_j) + \sin^{-2}(x_i + x_j).$$

By Gerschgorin's theorem [96, Theorem 15.814], the eigenvalues of  $D^2\lambda(\mathbf{x})$  lie in

$$\bigcup_{i} \left[ \frac{\partial^{2} \lambda}{\partial x_{i}^{2}} - \gamma_{i}, \frac{\partial^{2} \lambda}{\partial x_{i}^{2}} + \gamma_{i} \right],$$

with

$$\gamma_i = \sum_{j \neq i} \left| \frac{\partial^2 \lambda}{\partial x_i x_j} \right|.$$

As  $\partial_{x_i}^2 \lambda + \gamma_i \leq -\beta \sin^{-2} x_i - \beta' \cos^{-2} x_i < 0$ ,  $D^2 \lambda(\mathbf{x})$  is negative definite. Hence, any critical point of  $\lambda$  must be a hyperbolic maximum.

To complete the proof of uniqueness, let us suppose that  $\boldsymbol{\xi}_1$  and  $\boldsymbol{\xi}_2$  are two critical points of  $\lambda$ . Consider the segment  $\boldsymbol{\gamma}: t \in [0,1] \mapsto t\boldsymbol{\xi}_2 + (1-t)\boldsymbol{\xi}_1 \in C$  and set  $f = \lambda \circ \boldsymbol{\gamma}$ . Clearly f'(0) = f'(1) = 0, so there must exist  $t \in (0,1)$  such that

$$f''(t) = \langle \gamma'(t), D^2 \lambda(\gamma(t)) \gamma'(t) \rangle = 0$$

contradicting the fact that  $D^2\lambda$  is negative definite in C.

Remark 2.21. It can be proved [116, 182] that  $\xi$  is also a (global) minimum of the part of the potential  $V_{\rm sc}$  quadratic in a, namely

$$U(\mathbf{x}) = \sum_{i \neq j} \sin^{-2}(x_i - x_j) + \sin^{-2}(x_i + x_j) + \beta^2 \sum_{i} \sin^{-2} x_i + \beta'^2 \sum_{i} \cos^{-2} x_i.$$

Let us now consider the positive function

$$\varphi_0 = \frac{\mu^2}{\|\mu\|^2} = \frac{e^{2a\lambda}}{\int_C e^{2a\lambda}} \in L^1(C) \cap C^{\infty}(C).$$
(2.27)

It is intuitively obvious that  $\varphi_0$  concentrates near its maximum  $\boldsymbol{\xi}$  as  $a \to \infty$ . More precise, one can prove the following

**Lemma 2.22.** Let  $\boldsymbol{\xi}$  be the unique maximum of  $\lambda$ . Then  $\varphi_0$  converges to  $\delta_{\boldsymbol{\xi}}$  in the sense of distributions and falls off exponentially fast away from  $\boldsymbol{\xi}$ .

*Proof.* We shall show that

$$\lim_{a \to \infty} \int_C \varphi_0 \phi = \phi(\boldsymbol{\xi})$$

for any  $\phi \in C_0^{\infty}(C)$ . By continuity, for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\lambda(\boldsymbol{\xi}) - \lambda(\mathbf{x}) < \varepsilon, \qquad |\phi(\boldsymbol{\xi}) - \phi(\mathbf{x})| < \varepsilon,$$

for all  $\mathbf{x}$  in the ball B centered at  $\boldsymbol{\xi}$  and of radius  $\delta$ . Furthermore, one can assume that  $\lambda|_{C\setminus B} \leqslant -\frac{c}{2}\delta^2$ , with  $c \geqslant c_0 > 0$ . Since the ground state function  $\mu$  is only defined up to a multiplicative constant and the claim obviously applies to constant functions, we can assume that  $\lambda(\boldsymbol{\xi}) = \phi(\boldsymbol{\xi}) = 0$  and  $\max_{C} |\phi| \leqslant 1$ . The method of steepest descent ensures that

$$\|\mu\|^2 = \int_C e^{2a\lambda} = \left(\frac{\pi}{a}\right)^{N/2} \frac{1 + o(1)}{\det[D^2\lambda(\xi)]}$$

as  $a \to \infty$ , so that

$$\int_{C \setminus B} \varphi_0 = \|\mu\|^{-2} \int_{C \setminus B} e^{2a\lambda} \leqslant C_1 a^{N/2} e^{-ac\delta^2} = \mathcal{O}(a^{N/2} e^{-ac\delta^2}). \tag{2.28}$$

Hence

$$\lim_{a \to \infty} \left| \int_{C} \varphi_{0} \phi \right| \leqslant \varepsilon \lim_{a \to \infty} \int_{B} \varphi_{0} + \lim_{a \to \infty} \int_{C \setminus B} \varphi_{0} |\phi|$$
$$\leqslant \varepsilon + \lim_{a \to \infty} \mathcal{O}(a^{N/2} e^{-ac\delta^{2}})$$
$$= \varepsilon.$$

for any  $\varepsilon > 0$ .

**Proposition 2.23.** Let  $\psi$  be a normalized eigenfunction of  $H_{\rm sc}$ . Then  $\varphi = |\psi|^2 \to \delta_{\xi}$  as  $a \to \infty$  and  $\varphi$  falls off exponentially fast away from  $\xi$ .

*Proof.* It is well known [104,109,159,160] that  $g = \mu^{-1}\psi$  is a hypergeometric polynomial in  $\mathbf{z} = (e^{2ix_1}, \dots, e^{2ix_N})$  associated with the  $BC_N$  root system. Therefore,  $||g||^{-2}|g|^2$  is polynomially bounded in a and

$$\int_{C \setminus B} \varphi \to 0$$

exponentially fast as  $a \to \infty$  on account of Lemma 2.22. Since  $\varphi$  is obviously positive and  $\int_C \varphi = 1$ , the previous argument shows that

$$\lim_{a \to \infty} \int_C \varphi \phi = \phi(\boldsymbol{\xi})$$

for all  $\phi \in C_0^{\infty}(C)$ , as claimed.

Remark 2.24. Finer control of the decay could have been obtained if needed [4, 111].

We shall use the above lemma to relate the spectrum of the  $BC_N$  chain with those of the  $BC_N$  scalar and spin Sutherland models. The precise relationship can be introduced by means of the partition function of these systems, i.e., the trace of the exponential of minus their Hamiltonian operators. In the large coupling constant limit, we obtain an exact formula for the eigenvalues of the  $BC_N$  chain Hamiltonian.

To this end, let us write the spin differential operator (1.18d) as

$$H_{\epsilon\epsilon'} = H_{\rm sc} + 8a\,\mathsf{h}_{\epsilon\epsilon'}\,,\tag{2.29}$$

where

$$h_{\epsilon\epsilon'}(\mathbf{x}) = \sum_{i \neq j} \left[ \sin^{-2}(x_i - x_j) (1 - \epsilon S_{ij}) + \sin^{-2}(x_i + x_j) (1 - \epsilon \tilde{S}_{ij}) \right] + \sum_{i} \left( \beta \sin^{-2} x_i + \beta' \cos^{-2} x_i \right) (1 - \epsilon' S_i) \quad (2.30)$$

is an  $\operatorname{End}(\Sigma)$ -valued multiplication operator and

$$\mathsf{H}_{\epsilon\epsilon'} = \mathsf{h}_{\epsilon\epsilon'}(\boldsymbol{\xi}) \,. \tag{2.31}$$

Gloss 2.25. The essence of Polychronakos's freezing trick is that the eigenfunctions of  $H_{\epsilon\epsilon'}$  become sharply peaked at  $\xi$  as  $a \to \infty$ , so that

$$H_{\epsilon\epsilon'}\Psi(\mathbf{x}) \simeq H_{\mathrm{sc}}\Psi(\mathbf{x}) + 8a\,\mathsf{h}_{\epsilon\epsilon'}(\boldsymbol{\xi})\Psi(\mathbf{x}) = (H_{\mathrm{sc}} + 8a\,\mathsf{H}_{\epsilon\epsilon'})\Psi(\mathbf{x}),$$

in view of Eqs. (2.29) and (2.31). This idea suggests that the eigenvalue equation

$$(H_{\epsilon\epsilon'} - E_{\epsilon\epsilon'})\Psi = 0$$

should yield approximate eigenfunctions of  $H_{sc}$  and  $H_{\epsilon\epsilon'}$ , i.e.,

$$(H_{\rm sc} - E_{\rm sc})\Psi \simeq 0$$
,  $(H_{\epsilon\epsilon'} - E_{\epsilon\epsilon'})\Psi \simeq 0$ 

for some eigenvalues  $E_{sc}$ ,  $\mathsf{E}_{\epsilon\epsilon'}$ . This leads to Polychronakos's heuristic formula [166]

$$E_{\epsilon\epsilon'} \simeq E_{\rm sc} + 8a \mathsf{E}_{\epsilon\epsilon'}$$

relating the spectra of  $H_{\epsilon\epsilon'}$ ,  $H_{\rm sc}$  and  $H_{\epsilon\epsilon'}$ . A major drawback of this relation is that it does not determine a priori which eigenvalues of the spin and scalar models should be combined to obtain an eigenvalue of the spin chain in the large coupling constant limit. This difficulty can be overcome [65, 84, 166] by summing over the spectra of these operators and promoting the above relation to the more precise formula (2.34) relating the partition functions of  $H_{\epsilon\epsilon'}$ ,  $H_{\rm sc}$  and  $H_{\epsilon\epsilon'}$  respectively. In what follows, we provide a full proof of this technique and apply it to compute the spectrum of the  $BC_N$  chain.

The **partition function** of a self-adjoint operator H acting on a Hilbert space  $\mathcal{H}$  is the map  $Z_H : \mathbb{R}^+ \to \mathbb{R}^+$  defined by

$$Z_H(T) = \operatorname{tr}_{\mathcal{H}_1} e^{-H/T} = \sum_{\psi \in \mathcal{B}} (\psi, e^{-H/T} \psi)$$
 (2.32)

provided that the above sum converges (i.e., that  $e^{-H/T}$  is trace class for all T > 0.) Here  $\mathcal{B}$  is an orthonormal basis of  $\mathcal{H}$ . Clearly the above definition amounts to setting

$$Z_H(T) = \sum_{E \in \operatorname{spec}(H)} e^{-E/T}$$

when H has a pure point spectrum. We shall use the shorthand notation  $Z_{\epsilon\epsilon'} \equiv Z_{H_{\epsilon\epsilon'}}$ ,  $Z_{\rm sc} \equiv Z_{H_{\rm sc}}$  and  $Z_{\epsilon\epsilon'} \equiv Z_{H_{\epsilon\epsilon'}}$  and define the equivalence relation

$$A \sim B \iff \lim_{n \to \infty} \frac{A}{B} = 1$$
. (2.33)

Moreover, in the rest of this chapter we shall set  $E_0 = 0$  by adding a constant term to the Hamiltonians  $H_{\epsilon\epsilon'}$  and  $H_{\rm sc}$ . The relation (2.29) is not modified by this rescaling.

**Lemma 2.26** (Freezing trick). For all T > 0,

$$\mathsf{Z}_{\epsilon\epsilon'}(T) = \frac{\lim_{a \to \infty} Z_{\epsilon\epsilon'}(8aT)}{\lim_{a \to \infty} Z_{\mathrm{sc}}(8aT)}.$$
 (2.34)

Proof. Let  $\mathcal{B}_{sc} = \{ \psi \equiv \psi(\mathbf{x}; a) \} \subset L^2(C)$  and  $\mathcal{B}_{spin} = \{ |\mathbf{s}\rangle \} \subset \Sigma$  be orthonormal eigenbases of  $H_{sc}$  and  $H_{\epsilon\epsilon'}$  respectively, and consider the orthonormal basis of  $\mathcal{H}$  given by the tensor product of these bases, i.e.,  $\mathcal{B} = \{ \Psi \equiv \psi | \mathbf{s} \rangle \}$ . As we have set  $E_0 = 0$ ,

$$Z_{\epsilon\epsilon'}(8aT) = \sum_{E_{\epsilon\epsilon'} \in \operatorname{spec}(H_{\epsilon\epsilon'})} e^{-E_{\epsilon\epsilon'}/8aT} = Z_{H_{\epsilon\epsilon'}/8a}(T).$$

By Theorem 2.18, the eigenvalues of  $H_{\epsilon\epsilon'}$  with  $E_0 = 0$  depend linearly on a, and hence it is not difficult to check that the sum defining  $Z_{H_{\epsilon\epsilon'}/8a}$  converges uniformly on compact sets. In particular, the limit

$$\lim_{a\to\infty} Z_{\epsilon\epsilon'}(8aT)$$

exists and depends continuously on  $T \in \mathbb{R}^+$ . The same result holds for the scalar Hamiltonian  $H_{sc}$ .

Let us estimate the value of  $(\Psi, e^{-H_{\epsilon\epsilon'}/8aT}\Psi)$ . By continuity and the fact that  $\boldsymbol{\xi}$  is the unique maximum of  $\lambda$ , for any  $\varepsilon > 0$  there exist  $\varepsilon_1 > 0$  and an open neighborhood B of  $\boldsymbol{\xi}$  such that

$$-\varepsilon < \mathsf{h}_{\epsilon\epsilon'} - \mathsf{H}_{\epsilon\epsilon'} < \varepsilon \tag{2.35}$$

in B and  $\lambda(\boldsymbol{\xi}) - \lambda > \varepsilon_1$  in  $C \backslash B$ . As  $H_{\epsilon \epsilon'} \geqslant 0$ , it follows from Proposition 2.23 that

$$\int_{C \setminus B} \langle \Psi, e^{-H_{\epsilon \epsilon'}/8aT} \Psi \rangle \leqslant \int_{C \setminus B} |\Psi|^2 \to 0$$

as  $a \to \infty$ . Therefore,

$$\lim_{a \to \infty} (\Psi, e^{-H_{\epsilon\epsilon'}/8aT} \Psi) = \lim_{a \to \infty} \int_{B} \langle \Psi, e^{-H_{\epsilon\epsilon'}/8aT} \Psi \rangle, \qquad (2.36)$$

and similarly for  $H_{\text{sc}} + ca$  ( $c \in \mathbb{R}$ ) since it only differs from  $H_{\epsilon\epsilon'}$  by a times a relatively bounded operator. Moreover, Eq. (2.35) implies that

$$e^{-\frac{\varepsilon}{T}} \int_{B} \langle \Psi, e^{-\frac{H_{sc}}{8aT}} e^{-\frac{H_{e\epsilon'}}{T}} \Psi \rangle < \int_{B} \langle \Psi, e^{-\frac{H_{e\epsilon'}}{8aT}} \Psi \rangle < e^{\frac{\varepsilon}{T}} \int_{B} \langle \Psi, e^{-\frac{H_{sc}}{8aT}} e^{-\frac{H_{e\epsilon'}}{T}} \Psi \rangle,$$

and hence

$$\lim_{a \to \infty} (\Psi, e^{-H_{\epsilon\epsilon'}/8aT} \Psi) = \langle \mathbf{s} | e^{-\mathsf{H}_{\epsilon\epsilon'}/T} | \mathbf{s} \rangle \lim_{a \to \infty} (\psi, e^{-H_{\mathrm{sc}}/8aT} \psi)$$
 (2.37)

for each  $\Psi \in \mathcal{B}$ , on account of Eq. (2.36). As we have uniform convergence in compact sets we can exchange limits, so that

$$\lim_{a \to \infty} Z_{\epsilon \epsilon'}(8aT) = \lim_{a \to \infty} \sum_{\Psi \in \mathcal{B}} (\Psi, e^{-H_{\epsilon \epsilon'}/8aT} \Psi)$$

$$= \sum_{\Psi \in \mathcal{B}} \lim_{a \to \infty} (\Psi, e^{-H_{\epsilon \epsilon'}/8aT} \Psi)$$

$$= \sum_{|\mathbf{s}| \in \mathcal{B}_{\text{spin}}} \langle \mathbf{s} | e^{-H_{\epsilon \epsilon'}/T} | \mathbf{s} \rangle \sum_{\psi \in \mathcal{B}_{\text{sc}}} \lim_{a \to \infty} (\psi, e^{-H_{\text{sc}}/8aT} \psi)$$

$$= Z_{\epsilon \epsilon'}(T) \lim_{a \to \infty} Z_{\text{sc}}(8aT),$$

as claimed.  $\Box$ 

We shall now compute the partition functions of  $H_{\epsilon\epsilon'}$  and  $H_{sc}$  and obtain the spectrum of the spin chain (2.23) using the freezing trick. A first observation is that the eigenvalues (2.18) can be written in a more compact form as

$$E_{\mathbf{n}} = a^2 E_0 + 8a \sum_{i} n_i (\overline{\beta} + N - i) + \mathcal{O}(1),$$
 (2.38)

where  $\mathbf{n} \in [\mathbb{Z}^N]$ ,  $\overline{\beta} = \frac{1}{2}(\beta + \beta')$  and  $E_0$  was defined in Eq. (2.4).

**Proposition 2.27.** Set  $q = e^{-1/T}$ . Then

$$\lim_{a \to \infty} Z_{\rm sc}(8aT) = \prod_{i} \left[ 1 - q^{i[\overline{\beta} + N - \frac{1}{2}(i+1)]} \right]^{-1}.$$

*Proof.* By Theorem 2.15 and Eq. (2.38),

$$Z_{\rm sc}(8aT) = \sum_{\mathbf{n} \in [\mathbb{Z}^N]} e^{-E_{\mathbf{n}}/8aT} \sim \sum_{\mathbf{n} \in [\mathbb{Z}^N]} \prod_i q^{n_i(\overline{\beta} + N - i)}.$$

Defining  $p_i = n_i - n_{i+1}$  for  $1 \le i \le N-1$  and  $p_N = n_N$ , we have

$$\prod_i q^{n_i(\overline{\beta}+N-i)} = \prod_{i \leqslant j} q^{p_j(\overline{\beta}+N-i)} = \prod_j q^{p_j \sum_{i=1}^j (\overline{\beta}+N-i)} = \prod_j q^{jp_j[\overline{\beta}+N-\frac{1}{2}(j+1)]},$$

and hence

$$Z_{\rm sc}(aT) \sim \sum_{\mathbf{p} \in \mathbb{N}_0^N} \prod_i q^{ip_i[\overline{\beta} + N - \frac{1}{2}(i+1)]} = \prod_i \sum_{p_i \geqslant 0} q^{ip_i[\overline{\beta} + N - \frac{1}{2}(i+1)]}$$
$$= \prod_i \left[ 1 - q^{i[\overline{\beta} + N - \frac{1}{2}(i+1)]} \right]^{-1}. \tag{2.39}$$

The computation of the partition function of the spin model is considerably more involved. First, let us define the set of partitions of a natural number n as

$$\mathfrak{P}_n = \bigcup_{r=1}^n \left\{ \mathbf{k} \in \mathbb{N}^r : |\mathbf{k}| = n \right\},$$

where  $|\mathbf{k}| = k_1 + \cdots + k_r$ . We find it convenient to represent each positive nonincreasing multiindex  $\mathbf{n} \in [\mathbb{Z}^N]$  as

$$\mathbf{n} = (\overbrace{m_1, \dots, m_1}^{k_1}, \overbrace{m_2, \dots, m_2}^{k_2}, \dots, \overbrace{m_r, \dots, m_r}^{k_r}), \tag{2.40}$$

where  $m_1 > m_2 > \cdots > m_r \geqslant 0$  and  $k_i = \#(m_i)$ . In particular,  $\mathbf{k} \in \mathfrak{P}_N \cap \mathbb{N}^r$  and condition (2.17) is satisfied if  $\epsilon = -1$ .

**Example 2.28.**  $\mathfrak{P}_3 = \{(3), (2,1), (1,2), (1,1,1)\}$ . For  $\mathbf{n} = (6,3,3,3,2,1,1) \in [\mathbb{Z}^7]$  one has  $\mathbf{m} = (6,3,2,1) \in [\mathbb{Z}^4]$  and  $\mathbf{k} = (1,3,1,2) \in \mathfrak{P}_7 \cap \mathbb{N}^4$ .

Proposition 2.29. Given  $k \in \mathfrak{P}_N$ , set

$$N_j \equiv N_j(\mathbf{k}) = \left(\sum_{i=1}^{j} k_i\right) \left(\overline{\beta} + N - \frac{1}{2} - \frac{1}{2} \sum_{i=1}^{j} k_i\right),$$
 (2.41)

where  $1 \leq j \leq r$  and  $r \equiv r(\mathbf{k})$  is the length of  $\mathbf{k}$ . Then one can write the large a limit of the partition function of the Hamiltonian (1.18d) as

$$\lim_{a \to \infty} Z_{\epsilon \epsilon'}(8aT) = \sum_{\mathbf{k} \in \mathfrak{P}_N} \left\{ \prod_{j=1}^{r-1} \left[ \binom{2M+1+\delta_{1,\epsilon}(k_j-1)}{k_j} \frac{q^{N_j}}{1-q^{N_j}} \right] \times \left[ \binom{M_{\epsilon'}+\delta_{1,\epsilon}(k_r-1)}{k_r} + \binom{2M+1+\delta_{1,\epsilon}(k_r-1)}{k_r} \frac{q^{N_r}}{1-q^{N_r}} \right] \right\}. \quad (2.42)$$

*Proof.* It follows from Eq. (2.40) that the eigenvalues (2.38) can be written as

$$E_{\mathbf{n}} \sim 8a \sum_{i=1}^{r} m_{i} \sum_{j=k_{1}+\dots+k_{i-1}+1}^{k_{1}+\dots+k_{i-1}+k_{i}} (\overline{\beta} + N - j)$$

$$= 8a \sum_{i=1}^{r} m_{i} k_{i} (\overline{\beta} + N - \frac{1}{2} - \frac{k_{i}}{2} - \sum_{j=1}^{i-1} k_{j})$$

$$\equiv 8a \sum_{i=1}^{r} m_{i} \nu_{i},$$

since we have set  $E_0 = 0$ . The degeneracy  $d_{\epsilon\epsilon'}(\mathbf{n}) \equiv d_{\epsilon\epsilon'}(\mathbf{k}, \mathbf{m})$  of each eigenvalue  $E_{\mathbf{n}}$  is given by the cardinal of the set  $\mathcal{B}^{\mathbf{n},\epsilon,\epsilon'}_{\Sigma}$  introduced in Lemma 2.16. It is not difficult to check that  $d_{\epsilon\epsilon'}(\mathbf{k}, \mathbf{m})$  solely depends on whether  $m_r = 0$  or not, and is in fact given by

$$d_{\epsilon\epsilon'}(\mathbf{k}, \mathbf{m}) = \begin{cases} \prod_{j=1}^{r} {2M+1+\delta_{1,\epsilon}(k_j-1) \choose k_j} \equiv d_{\epsilon\epsilon'}^+(\mathbf{k}), & \text{if } m_r > 0, \\ {M_{\epsilon'}+\delta_{1,\epsilon}(k_r-1) \choose k_r} \prod_{j=1}^{r-1} {2M+1+\delta_{1,\epsilon}(k_j-1) \choose k_j} \equiv d_{\epsilon\epsilon'}^0(\mathbf{k}) & \text{if } m_r = 0. \end{cases}$$

The partition function can thus be written as

$$Z_{\epsilon\epsilon'}(8aT) = \sum_{\mathbf{n} \in [\mathbb{Z}^N]} d_{\epsilon\epsilon'}(\mathbf{n}) e^{-E_{\mathbf{n}}/8aT}$$

$$\sim \sum_{\mathbf{k} \in \mathfrak{P}_N} \sum_{m_1 > \dots > m_r \geqslant 0} d_{\epsilon\epsilon'}(\mathbf{k}, \mathbf{m}) \prod_{i=1}^r q^{m_i \nu_i}$$

$$= \sum_{\mathbf{k} \in \mathfrak{P}_N} \left[ \sum_{m_1 > \dots > m_r > 0} d_{\epsilon\epsilon'}^+(\mathbf{k}) \prod_{i=1}^r q^{m_i \nu_i} + \sum_{m_1 > \dots > m_{r-1} > 0} d_{\epsilon\epsilon'}^0(\mathbf{k}) \prod_{i=1}^{r-1} q^{m_i \nu_i} \right].$$

Since

$$\sum_{m_1 > \dots > m_s > 0} \prod_{i=1}^s q^{m_i \nu_i} = \sum_{\mathbf{p} \in \mathbb{N}^s} \prod_{i=1}^s q^{\nu_i \sum_{j=i}^s p_j} = \sum_{\mathbf{p} \in \mathbb{N}^s} \prod_{i=1}^s \prod_{j=i}^s q^{p_j \nu_i}$$

$$= \sum_{\mathbf{p} \in \mathbb{N}^s} \prod_{j=1}^s q^{p_j \sum_{i=1}^j \nu_i} = \prod_{j=1}^s \sum_{p_j > 0} q^{p_j \sum_{i=1}^j \nu_i}$$

$$= \prod_{j=1}^s \frac{q^{N_j}}{1 - q^{N_j}}, \qquad (2.43)$$

we easily obtain Eq. (2.42).

From Propositions 2.27 and 2.29 we immediately derive the following theorem, which is the main result of this chapter.

**Theorem 2.30.** The partition function of the spin chain (2.23) is given by

$$Z_{\epsilon\epsilon'}(T) = \prod_{i=1}^{N} \left[ 1 - q^{i[\overline{\beta} + N - \frac{1}{2}(i+1)]} \right]$$

$$\sum_{\mathbf{k} \in \mathfrak{P}_{N}} \left\{ \left[ \binom{M_{\epsilon'} + \delta_{1,\epsilon}(k_{r} - 1)}{k_{r}} + \binom{2M + 1 + \delta_{1,\epsilon}(k_{r} - 1)}{k_{r}} \frac{q^{N_{r}}}{1 - q^{N_{r}}} \right] \right.$$

$$\left. \prod_{j=1}^{r-1} \left[ \binom{2M + 1 + \delta_{1,\epsilon}(k_{j} - 1)}{k_{j}} \frac{q^{N_{j}}}{1 - q^{N_{j}}} \right] \right\}. \quad (2.44)$$

Remark 2.31. The eigenvalues E of the spin chain Hamiltonian and their degeneracies  $d(E) \equiv d_{\epsilon\epsilon'}(E)$  can be recovered by identifying

$$\mathsf{Z}_{\epsilon\epsilon'}(T) = \sum_{\mathsf{E}\in\operatorname{spec}(\mathsf{H}_{\epsilon\epsilon'})} d(\mathsf{E}) q^{\mathsf{E}}, \qquad (2.45)$$

where the sum runs over distinct eigenvalues.

### 2.6 Statistical properties of the spectrum

Several remarkable properties of the spectrum of the spin chain (2.23) can be inferred from Theorem 2.30. First of all, for half-integer spin the partition function (2.44) does not depend on  $\epsilon'$ , since in this case  $M_{\pm} = M + \frac{1}{2}$ . Hence the spectrum of the spin chain is independent of  $\epsilon'$  when M is a half-integer, a property that is not apparent from the expression of the Hamiltonian (2.23). Secondly, all the denominators  $1 - q^{N_j}$ ,  $1 \leq j \leq r$ , appearing in Eq. (2.44) are included as factors in the product in the first line. Hence the partition function (2.44) can be rewritten as

$$\mathsf{Z}_{\epsilon\epsilon'}(T) = \sum_{\boldsymbol{\delta} \in \{0,1\}^N} d_{\epsilon\epsilon'}(M, \boldsymbol{\delta}) \, q^{\mathsf{E}_{\boldsymbol{\delta}}} \,, \tag{2.46}$$

where  $\mathsf{E}_{\boldsymbol{\delta}}$  is given by

$$\mathsf{E}_{\delta} = \sum_{i=1}^{N} i \delta_i (\overline{\beta} + N - \frac{1}{2} (i+1)), \qquad (2.47)$$

and the degeneracy factor  $d_{\epsilon\epsilon'}(M, \delta)$  is a polynomial of degree N in M. Therefore,

**Proposition 2.32.** For all values of  $\overline{\beta}$ ,  $\epsilon$ ,  $\epsilon'$  and M, the following statements hold:

- (i) For half-integer spin,  $H_{\epsilon\epsilon'}$  and  $H_{\epsilon,-\epsilon'}$  are isospectral.
- (ii) spec( $\mathsf{H}_{\epsilon\epsilon'}$ )  $\subset \{\mathsf{E}_{\delta} : \delta \in \{0,1\}^N\}$ , and it exactly coincides with this set (of cardinal at most  $2^N$ ) for sufficiently large M.

(iii) Set 
$$\mathsf{E}_{\max} = \frac{1}{6} N(N+1)(2N+3\overline{\beta}-2)$$
. Then 
$$\mathsf{H}_{\epsilon\epsilon'} = \mathsf{E}_{\max} - \mathsf{H}_{-\epsilon,-\epsilon'} \tag{2.48}$$

*Proof.* The first two statements follow from the above discussion. Let us prove the last claim. First, observe that the above equation holds with the constant

$$4\mathsf{E}_{\max} = \sum_{i \neq j} \left[ \sin^{-2}(\xi_i - \xi_j) + \sin^{-2}(\xi_i + \xi_j) \right] + \sum_i \left( \beta \sin^{-2} \xi_i + \beta' \cos^{-2} \xi_i \right)$$
(2.49)

by the definition of the spin chains. Since  $H_{\epsilon\epsilon'} \geq 0$  and  $\ker(H_{\epsilon\epsilon'}) = \Lambda_{\epsilon\epsilon'}(\Sigma)$  is nontrivial for sufficiently high M,  $E_{\max}$  must coincide with the highest possible eigenvalue of  $H_{-\epsilon,-\epsilon'}$ , which can be easily obtained using Eq. (2.47):

$$\mathsf{E}_{\max} = \sum_{i=1}^{N} i(\overline{\beta} + N - \frac{1}{2}(i+1)) = \frac{1}{6}N(N+1)(2N+3\overline{\beta}-2).$$

Note, however, that in the antisymmetric case some of the possible energies may not be attained if M is kept fixed and N increases. For instance, it is obvious that the kernel  $\ker(\mathsf{H}_{-,\epsilon}) = \Lambda_{-,\epsilon}(\Sigma)$  becomes trivial in this case.  $\square$ 

Remark 2.33. Eq. (2.48) allows us to restrict our attention to the ferromagnetic chains  $\mathsf{H}_{-,\pm}$  without loss of generality. Moreover,  $\mathsf{E}_{\max} = \max \operatorname{spec}(\mathsf{H}_{\epsilon\epsilon'})$ .

We shall now present several concrete examples where we analyze the spectrum of the chains by means of Theorem 2.30.

**Example 2.34.** The structure of Eq. (2.44) makes it straightforward to compute the spectrum of the spin chains for any fixed number of particles as a function of the spin. For instance, for N=3 sites and integer M the eigenvalues of the spin chain  $\mathsf{H}_{--}$  are  $0, \overline{\beta}+2, 2\overline{\beta}+3, 3\overline{\beta}+3, 3\overline{\beta}+5, 4\overline{\beta}+5, 5\overline{\beta}+6, 6\overline{\beta}+8$ , with respective degeneracies

$$\frac{1}{6}M(M-1)(M-2), \quad \frac{5}{6}M(M^2-1), \quad \frac{1}{6}M(M+1)(11M-2), \\
\frac{1}{6}M(M+1)(7M-4), \quad \frac{1}{6}M(M+1)(7M+11), \quad \frac{1}{6}M(M+1)(11M+13), \\
\frac{5}{6}M(M+1)(M+2), \quad \frac{1}{6}(M+1)(M+2)(M+3).$$

Note that in this case spec $(H_{--}) = \{ \mathsf{E}_{\delta} : \delta \in \{\pm 1\} \}$  for  $M \geqslant 3$ , in agreement Proposition 2.32. The above results have been numerically checked for M = 1 and a few values of  $\beta$  and  $\beta'$  by representing the operators  $S_{ij}$  and  $S_i$  as  $27 \times 27$  matrices. The obtained results are in complete agreement with those listed above.

For a fixed value of the spin M, it is not apparent how to derive an explicit formula expressing the eigenvalues and their multiplicities in terms of the number of particles N. We shall next present two concrete examples for the cases M = 1/2 and M = 1.

**Example 2.35** (Spin 1/2). By Proposition 2.32,  $H_{-+}$  and  $H_{--}$  are isospectral. We have computed the partition function  $Z_{-,\pm}$  for up to 20 particles. For instance, for N=6 the antiferromagnetic spin chain eigenvalues and their corresponding multiplicities (denoted by subindices) are

$$(9\overline{\beta} + 32)_2$$
,  $(10\overline{\beta} + 36)_2$ ,  $(11\overline{\beta} + 38)_2$ ,  $(11\overline{\beta} + 41)_4$ ,  $(12\overline{\beta} + 38)_1$ ,  $(12\overline{\beta} + 43)_6$ ,  $(13\overline{\beta} + 43)_3$ ,  $(13\overline{\beta} + 46)_6$ ,  $(14\overline{\beta} + 46)_4$ ,  $(14\overline{\beta} + 50)_4$ ,  $(15\overline{\beta} + 47)_2$ ,  $(15\overline{\beta} + 50)_3$ ,  $(15\overline{\beta} + 55)_6$ ,  $(16\overline{\beta} + 51)_2$ ,  $(16\overline{\beta} + 55)_5$ ,  $(17\overline{\beta} + 53)_1$ ,  $(17\overline{\beta} + 56)_4$ ,  $(18\overline{\beta} + 58)_3$ ,  $(19\overline{\beta} + 61)_2$ ,  $(20\overline{\beta} + 65)_1$ ,  $(21\overline{\beta} + 70)_1$ .

The number of energy levels increases rapidly with the number of particles N. For example, if N=10 the number of distinct eigenvalues (for generic values of  $\overline{\beta}$ ) is 136, while for N=20 this number becomes 7756. It is therefore convenient to plot the eigenvalues E and their corresponding degeneracies d(E), as is done in Fig. 2.1 for N=10 particles. Note that Eq. (2.47) implies that when  $\overline{\beta} \gg N$  the levels cluster around integer multiples of  $\overline{\beta}$ . In fact, for all N up to 20 we have observed that these integers take all values in a certain range  $j_0, j_0 + 1, \ldots, N(N+1)/2$ . For example, in the case N=6 presented above,  $j_0=9$ .

Example 2.36 (Spin 1). We have computed the partition functions  $Z_{-,\pm}$  of the spin chains  $H_{-,\pm}$  with spin M=1 for up to 15 particles. As remarked in the previous section, for integer M the partition functions  $Z_{-,\pm}$  are expected to be essentially different. This is immediately apparent from Fig. 2.2, where the energy spectra of the even and odd spin chains  $Z_{-,\pm}$  with  $\overline{\beta}=\sqrt{2}$  for N=10 particles are graphically compared. However, the standard deviation of the energy is exactly the same for both chains. This rather unexpected result will be relevant in the ensuing discussion of the level density (see Conjecture 2.39 below). We also note that, just as for spin 1/2, for N up to (at least) 15 and  $\overline{\beta} \gg N$  the energy levels cluster around an equally spaced set of nonnegative integer multiples of  $\overline{\beta}$ .

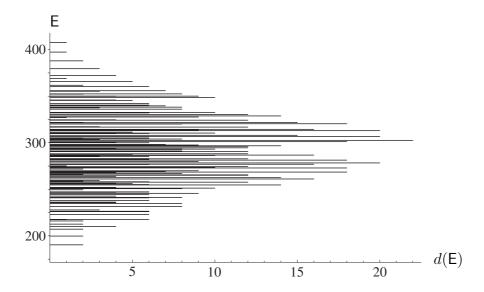


Figure 2.1: Eigenvalues E and degeneracies d(E) of the spin 1/2 chain  $H_{-,\pm}$  for N=10 particles and  $\overline{\beta}=\sqrt{2}$ .

The numerical explorations that we have carried off naturally give rise to several conjectures that we shall now discuss in detail.

Conjecture 2.37. For  $\overline{\beta} \gg N$ , the eigenvalues cluster around an equally spaced set of levels of the form  $j\overline{\beta}$ , with  $j = j_0, j_0 + 1, \dots, N(N+1)/2$ 

Remark 2.38. For sufficiently large values of the spin M this conjecture (with  $j_0 = 0$ ) follows directly from Eq. (2.47). Numerical calculations for a wide range of values of N and M fully corroborate the above conjecture.

Conjecture 2.39. For  $N \gg 1$ , the level density follows a Gaussian distribution

More precisely, we claim that the number of eigenvalues (counting their degeneracies) in an interval  $I \subset \mathbb{R}$  is approximately given by

$$(2M+1)^N \int_I \mathcal{N}(\mathsf{E}; \mu, \sigma) d\mathsf{E}, \qquad (2.50)$$

where

$$\mathcal{N}(\mathsf{E}; \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\mathsf{E}-\mu)^2}{2\sigma^2}}$$
 (2.51)

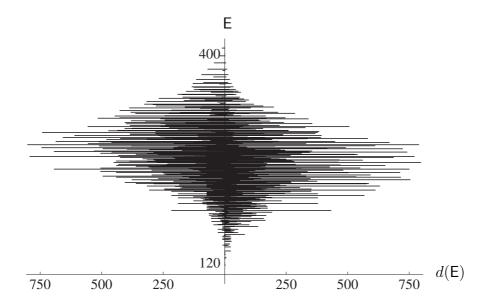


Figure 2.2: Comparison of the energy levels E and degeneracies d(E) of the spin 1 chains  $H_{--}$  (left) and  $H_{-+}$  (right) for N=10 particles and  $\overline{\beta} = \sqrt{2}$ .

is the normal (Gaussian) distribution with parameters  $\mu$  and  $\sigma$  respectively equal to the mean and standard deviation of the spectrum of the spin chain. Although the shape of the plots in Figs. 2.1 and 2.2 make this conjecture quite plausible, for its precise numerical verification it is preferable to compare the distribution function

$$F_{\mathcal{N}}(\mathsf{E}) = \int_{-\infty}^{\mathsf{E}} \mathcal{N}(\mathsf{e}; \mu, \sigma) d\mathsf{e}$$
 (2.52)

of the Gaussian probability density with its discrete analog

$$F(\mathsf{E}) = (2M+1)^{-N} \sum_{\mathsf{E} \geqslant \mathsf{e} \in \mathrm{spec}(H_{\epsilon\epsilon'})} d(\mathsf{e}) , \qquad (2.53)$$

where  $d(\mathbf{e}) \equiv d_{\epsilon\epsilon'}(\mathbf{e})$  denotes the degeneracy of the eigenvalue  $\mathbf{e}$ . Indeed, our computations for a wide range of values of M and  $N \gtrsim 10$  are in total agreement with the latter conjecture for all four chains (2.23). This is apparent, for instance, in the case  $\overline{\beta} = \sqrt{2}$ , M = 1/2, and N = 10 presented in Fig. 2.3. The agreement between the distribution functions (2.52) and (2.53) improves dramatically as N increases. In fact, their plots are virtually indistinguishable for  $N \gtrsim 15$ .

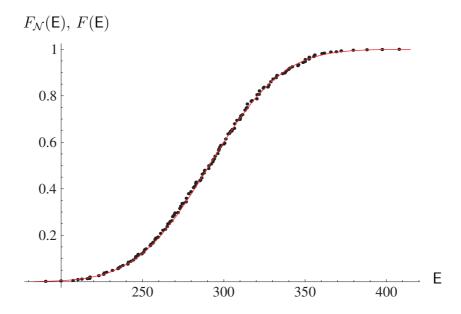


Figure 2.3: Distribution functions  $F_{\mathcal{N}}(\mathsf{E})$  (continuous line) and  $F(\mathsf{E})$  (at its discontinuity points) for  $\overline{\beta} = \sqrt{2}$ , M = 1/2, and N = 10.

It is well known in this respect that a Gaussian level density is a characteristic feature of the embedded Gaussian ensemble (EGOE) in Random Matrix Theory [152]. It should be noted, however, that the EGOE applies to a system of N particles with up to n-body interactions (n < N) in the high dilution regime  $N \to \infty$ ,  $\kappa \to \infty$  and  $N/\kappa \to 0$ , where  $\kappa$  is the number of one-particle states. Since in our case  $\kappa = 2M + 1$  is fixed, the fact that the level density is Gaussian does not follow from the above general result.

If Conjecture 2.39 is true, the spectrum for large N is completely characterized by the parameters  $\mu$  and  $\sigma$  through the Gaussian law (2.51). It is therefore of great interest to compute these parameters in closed form as functions of N and M. To this end, let us write

$$\mathsf{H}_{-,\pm} = \sum_{i \neq j} \left[ h_{ij} (1 + S_{ij}) + \tilde{h}_{ij} (1 + \tilde{S}_{ij}) \right] + \sum_{i} h_{i} (1 \mp S_{i}), \qquad (2.54)$$

where the constants  $h_{ij}$ ,  $\tilde{h}_{ij}$  and  $h_i$  can be easily read off from Eq. (2.23). We shall denote by  $\mu_{\pm}$  the average energy of  $H_{-,\pm}$ .

Operator	Trace (integer $M$ )	Trace (half-integer $M$ )
$S_i$	$(2M+1)^{N-1}$	0
$S_{ij}, \ \tilde{S}_{ij}$	$(2M+1)^{N-1}$	$(2M+1)^{N-1}$
$S_iS_j$	$(2M+1)^{N-2+2\delta_{ij}}$	$(2M+1)^N \delta_{ij}$
$S_{ij}S_k, \ \tilde{S}_{ij}S_k$	$(2M+1)^{N-2}$	0
$S_{ij}\tilde{S}_{kl}$	$(2M+1)^{N-2}$	$(2M+1)^{N-2}(1-\delta_{ik}\delta_{jl})(1-\delta_{il}\delta_{jk})$
$S_{ij}S_{kl}, \ \tilde{S}_{ij}\tilde{S}_{kl}$	$(2M+1)^{N-2+2\delta_{ik}\delta_{jl}+2\delta_{il}\delta_{jk}}$	$(2M+1)^{N-2+2\delta_{ik}\delta_{jl}+2\delta_{il}\delta_{jk}}$

Table 2.1: Traces of products of the spin operators.

**Proposition 2.40.** If the spin M is an integer,

$$\mu_{-} = \frac{M+1}{6(2M+1)} N(N+1)(2N+3\overline{\beta}-2), \qquad (2.55)$$

$$\mu_{+} = \frac{1}{2M+1} \left[ (M+1) \mathsf{E}_{\max} - 2\Sigma_{1} \right], \tag{2.56}$$

with  $\Sigma_1 = \sum_i h_i$ . If it is a half-integer,

$$\mu_{\pm} = \frac{1}{2M+1} \left[ (M+1) \mathsf{E}_{\max} - \Sigma_1 \right]. \tag{2.57}$$

*Proof.* We shall begin with the case of integer spin. Using the formulas for the traces of the spin operators given in Table 2.1 we immediately obtain

$$\mu_{-} = (2M+1)^{-N} \operatorname{tr} \mathsf{H}_{--}$$

$$= \frac{2(M+1)}{2M+1} \left[ \sum_{i \neq j} (h_{ij} + \tilde{h}_{ij}) + \sum_{i} h_{i} \right] = \frac{M+1}{2M+1} \mathsf{E}_{\max}$$

$$= \frac{M+1}{6(2M+1)} N(N+1)(2N+3\overline{\beta}-2).$$

Here we have used Proposition 2.32 and Eq. (2.49) to compute the sum. On the other hand, the average energy  $\mu_{+}$  of the chain  $H_{-+}$  is given by

$$\begin{split} \mu_{+} &= (2M+1)^{-N} \operatorname{tr} \mathsf{H}_{-+} \\ &= \frac{2(M+1)}{2M+1} \sum_{i \neq j} (h_{ij} + \tilde{h}_{ij}) + \frac{2M}{2M+1} \sum_{i} h_{i} \\ &= \frac{1}{2M+1} \left[ (M+1) \mathsf{E}_{\max} - 2\Sigma_{1} \right], \qquad M \text{ integer,} \end{split}$$

where  $\Sigma_1 = \sum_i h_i$ .

When M is a half-integer,  $\mu_{+} = \mu_{-}$  due to Proposition 2.32. A similar calculation shows that the formulas for the traces of the spin operators in Table 2.1 yield the expression (2.57) for the mean energy.

Let us turn now to the (squared) standard deviation of the spectrum of  $H_{-,\pm}$ , which is given by

$$\sigma_{\pm}^2 = \frac{\operatorname{tr}(\mathsf{H}_{-,\pm}^2)}{(2M+1)^N} - \frac{(\operatorname{tr}\mathsf{H}_{-,\pm})^2}{(2M+1)^{2N}}.$$

A long but straightforward calculation using the formulas in Table 2.1 yields the following

**Proposition 2.41.** For integer spin, the standard deviation is given by

$$\sigma_{\pm}^2 = \frac{4M(M+1)}{(2M+1)^2} \Sigma_2, \qquad (2.58)$$

with  $\Sigma_2 = 2\sum_{i\neq j}(h_{ij}^2 + \tilde{h}_{ij}^2) + \sum_i h_i^2$ . If M is a half-integer, then

$$\sigma_{\pm}^{2} = \frac{4M(M+1)}{(2M+1)^{2}} \left[ \Sigma_{2} + \frac{\Sigma_{3}}{M(M+1)} \right], \tag{2.59}$$

where 
$$\Sigma_3 = \frac{1}{4} \sum_i h_i^2 - \sum_{i \neq j} h_{ij} \tilde{h}_{ij}$$
.

Since  $\Sigma_1$ ,  $\Sigma_2$  and  $\Sigma_3$  do not depend on M, the above propositions completely determine the dependence of  $\mu_{\pm}$  and  $\sigma_{\pm}$  on the spin. A nontrivial consequence of Proposition 2.41 is the equality of the standard deviation of the energy for the even and odd antiferromagnetic chains (for half-integer spin, this trivially follows from the fact that the even and odd chains have the same spectrum). This result is quite surprising, since for integer spin the energy spectra of the chains  $H_{-,\pm}$  are essentially different, as shown in Fig. 2.2.

To analyze the dependence on the number of particles of the spectrum we still need to evaluate  $\Sigma_1$ ,  $\Sigma_2$  and  $\Sigma_3$  as functions of N. Although it is not at all apparent how to compute these quantities in closed form, in view of Eqs. (2.55)–(2.57) one is lead to formulate a natural

Conjecture 2.42. The average energy  $\mu_{\pm}$  and its squared standard deviation  $\sigma_{\pm}^2$  depend polynomially on N.

In fact, since  $\mathsf{E}_{\max}$  is a polynomial of degree 3 in N by Proposition 2.32, it follows that the degrees in N of  $\mu_{\pm}$  and  $\sigma_{\pm}^2$  cannot exceed 3 and 6, respectively. The latter conjecture and this fact allow us to determine the quantities  $\Sigma_i(N)$  by evaluating  $\mu_{\pm}$  and  $\sigma_{\pm}^2$  for  $N=2,\ldots,8$  and M=1/2,1 using Theorem 2.30 (cf. Eqs. (2.56), (2.58) and (2.59)). Hence one immediately proves the following

Proposition 2.43. If Conjecture 2.42 is true, then

$$\Sigma_{1} = 8\Sigma_{3} = \frac{N}{4} (2\overline{\beta} + N - 1),$$

$$\Sigma_{2} = \frac{N}{144} \left[ 2(2N^{2} + 3N + 13)\overline{\beta}^{2} + (N - 1)(5N^{2} + 7N + 20)\overline{\beta} + \frac{1}{5} (N - 1)(8N^{3} + 3N^{2} + 13N - 12) \right]. \quad (2.60)$$

These expressions, together with Eqs. (2.55)–(2.59), completely determine  $\mu_{\pm}$  and  $\sigma_{\pm}$  for all values of M and N. It has been numerically verified that the resulting formulas yield the exact values of  $\mu_{\pm}$  and  $\sigma_{\pm}$  computed from the partition function (2.44) for a wide range of values of M and N. This provides a very solid confirmation of Conjecture 2.42. Let us mention, in closing, that formulas analogous to (2.55)–(2.59) expressing the mean and standard deviations of the energy for the ferromagnetic chains  $H_{+,\pm}$  can be immediately deduced from the previous expressions and Eq. (2.48).

## 2.7 Models with constant magnetic fields

In this section we shall briefly show how a constant magnetic field can be added into the picture that we have presented in the previous sections. For the sake of simplicity, we restrict ourselves to the case M=1/2,  $\epsilon=-1$  ("one-dimensional electrons"), and define the **magnetic BC**<sub>N</sub> **Hamiltonian** as

$$H_{\epsilon}^{\text{mag}} = H_{-,\epsilon} - 8aB \sum_{i} S_{i}. \qquad (2.61)$$

Gloss 2.44. The motion of a (three-dimensional) electron in the presence of static magnetic and electric fields  $\mathbf{B} = \nabla \wedge \mathbf{A}$  and  $\mathbf{E} = -\nabla V$  in  $\mathbb{R}^3$  is governed by the Pauli operator

$$H = (i\nabla + e\mathbf{A})^2 - 2e\mathbf{B} \cdot \mathbf{S} + eV,$$

which is the nonrelativistic limit (in the resolvent norm topology) of the Dirac operator [193]. Here  $\mathbf{x} = (x, y, z)$ , e is the electric charge of the electron,

 $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$  is the spin operator of the particle and  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  are the Pauli matrices. If  $\mathbf{B}$  is constant, one can take  $\mathbf{A} = \frac{1}{2}\mathbf{x} \wedge \mathbf{B}$ , so that

$$H = -\Delta + ie\mathbf{B} \cdot \mathbf{x} \wedge \mathbf{\nabla} + \frac{e^2}{4} ||\mathbf{x} \wedge \mathbf{B}||^2 - 2e\mathbf{B} \cdot \mathbf{S} + eV.$$

Since we are interested in a one-dimensional scenario, we drop the second and third summands in order to preserve the symmetries  $\partial_y$  and  $\partial_z$ . This is a priori a good approximation to the problem when the angular momentum and the charge is small and the electric potential is strongly binding.

Hence one is led to define the magnetic  $BC_N$  Hamiltonian for spin 1/2 particles as

$$H_{-,\epsilon}^{\text{mag}} = H_{\epsilon\epsilon'} - 2e \sum_{i} \mathbf{B} \cdot \mathbf{S}_{i}.$$

As the exchange operator  $S_{ij}$  is written in terms of the SU(2)-invariant product as in Eq. (1.26), in this model of one-dimensional particles there is only one privileged direction  $\mathbf{e}$ , namely, the one which implements the Weyl reflections in spin space via

$$S_i = 2\mathbf{e} \cdot \mathbf{S}_i$$
.

Therefore it is natural to choose B parallel to this direction, so that

$$2e\sum_{i} \mathbf{B} \cdot \mathbf{S}_{i} = e \|\mathbf{B}\| \sum_{i} S_{i}.$$

If we now rescale  $e \| \mathbf{B} \|$  to 8aB as in Eqs. (2.25) we arrive at Eq. (2.61).

Since  $H_{\epsilon\epsilon'}$  and  $\sum_i S_i$  commute, they possess a common basis of eigenfunctions. Let  $\mathbf{n} \in [\mathbb{Z}^N]$  and  $|\mathbf{s}\rangle \in \mathcal{B}^{\mathbf{n},-,\epsilon}_{\Sigma}$  (cf. Eq. Lemma 2.16), and define

$$\rho(\mathbf{k}) = \text{card} \left\{ i : k_i = 1 \right\},$$

$$\rho_+(\mathbf{n}, \mathbf{s}) = \text{card} \left\{ i : s_i = \frac{1}{2}, \ n_i \neq 0, \ \#(n_i) = 1 \right\},$$

where  $\mathbf{k}$  is defined as in Eq. (2.40). Then one can easily prove the following

Theorem 2.45. Let  $\mathcal{B} \equiv \mathcal{B}_{\Sigma}^{-,\epsilon,\mathbf{n}}$ . Then

$$\operatorname{spec}(H_{\epsilon}^{\operatorname{mag}}) = \left\{ E_{\mathbf{n}} + 8aB \left[ \rho(\mathbf{k}) - 2\rho_{+}(\mathbf{n}, \mathbf{s}) - \delta_{n_{N}, 0}(1 + \epsilon) \right] : \mathbf{n} \in [\mathbb{Z}^{N}], \ |\mathbf{s}\rangle \in \mathcal{B} \right\}.$$

*Proof.* Let  $\Sigma_1 = \mathbb{C}|\frac{1}{2}\rangle \oplus \mathbb{C}|-\frac{1}{2}\rangle$  be the one-particle spin space, so that  $\Sigma = \Sigma_1^{\otimes N}$ , and define the basis of one-particle states

$$|\pm\rangle = |\frac{1}{2}\rangle \pm |-\frac{1}{2}\rangle \in \Sigma_1$$
.

Clearly  $S_1|\pm\rangle=\pm|\pm\rangle$ . Now define the one- and two-particle operators

$$\begin{split} & \Lambda_{\epsilon}^{(1)} : |\pm \frac{1}{2}\rangle \mapsto \pm |\epsilon\rangle \,, \\ & \Lambda_{\epsilon}^{(2)} : |\pm \frac{1}{2}, \mp \frac{1}{2}\rangle \mapsto |\epsilon, -\epsilon\rangle \,, \end{split}$$

with  $\epsilon = \pm$ . By Propositions 2.3 one can consider the action of the differential operator (2.61) on  $\Lambda_{-,\epsilon}(L^2(\mathbb{T}^N)\otimes\Sigma)$ . Theorem 2.18 states that  $H_{-,\epsilon}$  is diagonal in the basis (2.20). Expressing the nonnegative, nonincreasing multiindex  $\mathbf{n}$  in terms of  $\mathbf{m}$  and  $\mathbf{k}$  as in Eq. (2.40), one can use Conditions (i)–(iii) in Lemma 2.16 to readily verify that

$$\langle \Lambda_{-,\epsilon}(f_{\mathbf{n}}|\mathbf{s}\rangle) : |\mathbf{s}\rangle \in \mathcal{B}\rangle = \langle \Psi_{\mathbf{n},\mathbf{s}} : |\mathbf{s}\rangle \in \mathcal{B}\rangle.$$

Here we have set

$$\Psi_{\mathbf{n},\mathbf{s}} = \Lambda_{-,\epsilon} \left( f_{\mathbf{n}} \bigotimes_{j=1}^{r} \mathbf{v}_{\epsilon}(s_{\kappa_{j}+1}, k_{j}, m_{j}) \right),$$

with

$$\kappa_j \equiv \kappa_j(\mathbf{k}) = \sum_{i=1}^{j-1} k_i$$

and

$$\mathbf{v}_{\epsilon}(\pm \frac{1}{2}, 1, m) = \begin{cases} |\pm\rangle & \text{if } m \neq 0, \\ |\epsilon\rangle & \text{if } m = 0, \end{cases}$$
$$\mathbf{v}_{\epsilon}(\pm \frac{1}{2}, 2, m) = |+\rangle \otimes |-\rangle,$$

so that for each j = 1, 2 one has

$$\sum_{i=1}^{j} S_i \mathbf{v}_{\epsilon}(\pm \frac{1}{2}, j, m) = \begin{cases} \pm \delta_{j,1} & \text{if } m \neq 0, \\ \epsilon & \text{if } m = 0. \end{cases}$$

Since  $\sum_{i} S_{i}$  commutes with the symmetrizer and  $m_{j} = n_{\kappa_{j}+1}$ , one immediately derives from these equations that

$$\sum_{i} S_{i} \Psi_{\mathbf{n},\mathbf{s}} = \Psi_{\mathbf{n},\mathbf{s}} \left[ \sum_{\eta = \pm 1} \eta \operatorname{card} \left\{ i : s_{i} = \frac{\eta}{2}, \ n_{i} \neq 0, \ \#(n_{\kappa_{i}+1}) = 1 \right\} + \epsilon \delta_{n_{N},0} \right]$$
$$= \left[ 2\rho_{+}(\mathbf{n},\mathbf{s}) - \rho(\mathbf{k}) + \delta_{n_{N},0}(1+\epsilon) \right] \Psi_{\mathbf{n},\mathbf{s}},$$

i.e.,  $\sum_i S_i$  is diagonal in this basis. As the action of  $H_{-,\epsilon}$  and  $\overline{H}$  agree on  $\Psi_{\mathbf{n},\mathbf{s}}$  and, on account of Proposition 2.12, the action of the latter operator on  $\{f_{\mathbf{n}}\}$  is triangular with diagonal elements  $E_{\mathbf{n}}$ , the result follows.

The spin chain associated with the magnetic Hamiltonian (2.61) is

$$\mathsf{H}_{\epsilon}^{\mathrm{mag}} = \mathsf{H}_{-,\epsilon} - B \sum_{i} S_{i} \,. \tag{2.62}$$

It is straightforward to show that Lemma 2.26 also applies, mutatis mutandis, to the magnetic Hamiltonian (2.61). Hence we shall compute its spectrum by means of the freezing trick. To this end, let us denote by  $Z_{\epsilon}^{\text{mag}}$  and  $Z_{\epsilon}^{\text{mag}}$  the partition functions of (2.61) and (2.62), and again we shall set the ground state energy  $E_0$  equal to 0. We shall freely use the notation of Section 2.5.

### Proposition 2.46. Let

$$\mathfrak{B}_N = \bigcup_{j=1}^N \{ \mathbf{k} \in \{1, 2\}^j : |\mathbf{k}| = N \} = \mathfrak{P}_N \cap \bigcup_{j=1}^N \{1, 2\}^j$$

the set of partitions of N by  $\{1,2\}$ . Then

$$\lim_{a \to \infty} Z_{\epsilon}^{\text{mag}}(8aT) = \sum_{\mathbf{k} \in \mathfrak{B}_{N}} \left( q^{B} + q^{-B} \right)^{\rho(\mathbf{k})} \prod_{j=1}^{r-1} \frac{q^{N_{j}}}{1 - q^{N_{j}}} \left( \frac{q^{N_{r}}}{1 - q^{N_{r}}} + \delta_{k_{r}, 1} \frac{q^{\epsilon B}}{q^{B} - q^{-B}} \right). \tag{2.63}$$

*Proof.* From Proposition 2.29 one immediately obtains

$$Z_{\epsilon}^{\text{mag}}(8aT) \sim \sum_{\mathbf{k} \in \mathfrak{B}_N} \sum_{m_1 > \dots > m_r \geqslant 0} q^{\sum_{i=1}^r m_i \nu_i} \sum_{|\mathbf{s}| \in \mathcal{B}} q^{B[2\rho_+(\mathbf{n}, \mathbf{s}) - \rho(\mathbf{k})]},$$

so that the sum over  $|\mathbf{s}\rangle$  depends on whether  $m_r > 0$  or not. If this is the case,

$$\operatorname{card}\left\{|\mathbf{s}\rangle \in \mathcal{B} : \rho_{+}(\mathbf{n}, \mathbf{s}) = \varrho\right\} = \begin{pmatrix} \rho(\mathbf{k}) \\ \varrho \end{pmatrix}$$
 (2.64)

for any  $0 \le \varrho \le \rho(\mathbf{k})$ , so

$$\sum_{|\mathbf{s}| \in \mathcal{B}} q^{B[2\rho_{+}(\mathbf{n},\mathbf{s}) - \rho(\mathbf{k})]} = q^{-B\rho(\mathbf{k})} \sum_{\rho=0}^{\rho(\mathbf{k})} \binom{\rho(\mathbf{k})}{\varrho} q^{2B\varrho} = (q^{B} + q^{-B})^{\rho(\mathbf{k})}.$$

If, on the contrary,  $m_r = 0$ , then  $k_r = 1$  and it is not difficult to see that the formula (2.64) becomes

card 
$$\{|\mathbf{s}\rangle \in \mathcal{B} : \rho_+(\mathbf{n}, \mathbf{s}) = \varrho\} = \begin{pmatrix} \rho(\mathbf{k}) - 1 \\ \varrho \end{pmatrix},$$

where  $0 \le \varrho \le \rho(\mathbf{k}) - 1$ . Hence in this case

$$\sum_{|\mathbf{s}| \in \mathcal{B}} q^{B[2\rho_{+}(\mathbf{n},\mathbf{s})-\rho(\mathbf{k})+1]} = q^{-B[\rho(\mathbf{k})-1]} \sum_{\varrho=0}^{\rho(\mathbf{k})-1} \binom{\rho(\mathbf{k})-1}{\varrho} q^{2B\varrho} = \left(q^B + q^{-B}\right)^{\rho(\mathbf{k})-1},$$

and therefore

$$Z_{\epsilon}^{\text{mag}}(8aT) \sim \sum_{\mathbf{k} \in \mathfrak{B}_{N}} \left( q^{B} + q^{-B} \right)^{\rho(\mathbf{k})} \sum_{m_{1} > \dots > m_{r-1} > 0} q^{\sum_{i=1}^{r-1} m_{i} \nu_{i}} \left( \frac{q^{\sum_{i=1}^{r} m_{i} \nu_{i}}}{1 - q^{\sum_{i=1}^{r} m_{i} \nu_{i}}} + \frac{\delta_{k_{r}, 1} q^{\epsilon B}}{q^{B} - q^{-B}} \right).$$

Taking Eqs. (2.41) and (2.43) into account we obtain the desired formula.  $\square$ 

The freezing trick formula (2.34) and the scalar partition function (cf. Proposition 2.27) then yield the following

**Theorem 2.47.** When M = 1/2, the partition function of the spin chain (2.62) is given by

$$Z_{\epsilon}^{\text{mag}}(T) = \prod_{i} \left[ 1 - q^{i[\overline{\beta} + N - \frac{1}{2}(i+1)]} \right]$$

$$\times \sum_{\mathbf{k} \in \mathfrak{R}_{N}} \left( q^{B} + q^{-B} \right)^{\rho(\mathbf{k})} \prod_{i=1}^{r-1} \frac{q^{N_{j}}}{1 - q^{N_{j}}} \left( \frac{q^{N_{r}}}{1 - q^{N_{r}}} + \delta_{k_{r}, 1} \frac{q^{\epsilon B}}{q^{B} - q^{-B}} \right).$$

# Chapter 3

# Spin models with NN interactions

### 3.1 Introduction

In this chapter we analyze in detail three QES N-body spin models which can be regarded as cyclic versions of (generalized) CS models, as constructed by Finkel et al. [81]. As this kind of systems feature only near-neighbors interactions, we shall generically call them NN models. There are two reasons that make this kind of systems very promising from a physical point of view. First, some of them are related to the short-range Dyson model in random matrix theory [28]. Second, the HS chains associated with these models occupy an interesting intermediate position between the Heisenberg chain (short-range, position-independent interactions) and the usual HS chains (long-range, position-dependent interactions). We shall consider this latter aspect in detail in the next chapter.

The study of these systems was initiated a few years ago, when Jain and Khare presented a novel class of partially solvable models in which each particle interacted only with its nearest and next-to-nearest neighbors [124]. In a subsequent paper [10], Auberson, Jain and Khare discussed a generalization of these models to the  $BC_N$  root system and to higher dimensions. The latter papers, however, left open some important issues, such as the exact or quasi-exact solvability of these models, the derivation of general explicit formulas for their eigenfunctions, or even the existence of similar models for particles with spin. The last question was first addressed by Deguchi and Ghosh [48], who introduced spin 1/2 extensions of the scalar models of Jain and Khare and used the supersymmetric approach to obtain some eigenfunc-

tions. All these authors solely managed to construct a few exact solutions, with trivial spin dependence. Moreover, their approach was by no means systematic.

Our construction of NN models is based on a nontrivial modification of the Dunkl operator technique and yields fully explicit formulas for a wide variety of spin eigenfunctions. The presentation of these results is organized as follows. In Section 3.2 we define the Hamiltonians of the spin N-body models which are the subject of this chapter (cf. Eqs. (3.1)-(3.2)), and show that they can be expressed in terms of suitable differential operators with near-neighbors exchange terms. Section 3.3 is entirely devoted to the characterization of certain finite-dimensional spaces of polynomial spin functions invariant under the latter operators. In Section 3.4 we show that the eigenvalue problems for the Hamiltonians of the models (3.2) restricted to their invariant spaces reduce to finding the polynomial solutions of a corresponding system of partial differential equations. By completely solving the latter problem, we obtain several (infinite) families of eigenfunctions of the models (3.2) in closed form.

The material presented in this chapter is taken from Refs. [66–68].

### 3.2 The NN models

In this chapter we shall study in detail the N-body NN spin models defined by the Schrödinger operators

$$H_{\epsilon} = -\Delta + V_{\epsilon} \,, \tag{3.1}$$

where  $\epsilon = 0, 1, 2$  and

$$V_0 = \omega^2 r^2 + \sum_i \frac{2a^2}{(x_i - x_{i-1})(x_i - x_{i+1})} + \sum_i \frac{2a(a - S_{i,i+1})}{(x_i - x_{i+1})^2},$$
 (3.2a)

$$V_1 = \omega^2 r^2 + \sum_i \frac{b(b-1)}{x_i^2} + \sum_i \frac{8a^2 x_i^2}{(x_i^2 - x_{i-1}^2)(x_i^2 - x_{i+1}^2)}$$

$$+4a\sum_{i}\frac{x_{i}^{2}+x_{i+1}^{2}}{(x_{i}^{2}-x_{i+1}^{2})^{2}}(a-S_{i,i+1}),$$
 (3.2b)

$$V_2 = 2a^2 \sum_{i} \cot(x_i - x_{i-1}) \cot(x_i - x_{i+1}) + \sum_{i} \frac{2a(a - S_{i,i+1})}{\sin^2(x_i - x_{i+1})}, \quad (3.2c)$$

where we identify  $x_0 \equiv x_N$  and  $x_{N+1} \equiv x_1$ . As we did in the study of CS models, we shall assume  $a, b > \frac{1}{2}$  and take into account the inverse-square

singularities to consider the above Hamiltonians as self-adjoint operators on

$$\mathcal{H}_{\epsilon} = L^2(C_{\epsilon}) \otimes \Sigma$$
,

where the "reduced" configuration spaces are given by

$$C_0 = \{ \mathbf{x} : x_1 < \dots < x_N \},$$
 (3.3a)

$$C_1 = \{ \mathbf{x} : 0 < x_1 < \dots < x_N \},$$
 (3.3b)

$$C_2 = \{ \mathbf{x} : 0 < x_{i+1} - x_i < \pi, \ \forall i < N \}.$$
 (3.3c)

Remark 3.1. Unlike CS models, the Hamiltonians defined by (3.2) on  $\mathcal{H}_{\epsilon}$  and on  $L^2(\mathbb{R}^N) \otimes \Sigma$  (or  $L^2(\mathbb{T}^N) \otimes \Sigma$  for the third potential) are a priori different. This is due to the fact that the Weyl symmetry present in CS models has been replaced in the NN case by the cyclic group

$$\mathfrak{Z} = \left\langle \Pi_{12} \cdots \Pi_{N1} \right\rangle$$

$$\approx \left\langle \sigma_{\mathbf{e}_1 - \mathbf{e}_2} \cdots \sigma_{\mathbf{e}_N - \mathbf{e}_1} \right\rangle,$$
(3.4)

and this latter group does not tessellate  $\mathbb{R}^N$  (or  $\mathbb{T}^N$ ). Nonetheless, all the eigenfunctions presented in Theorems 3.17, 3.19 and 3.23 can be clearly extended to eigenfunctions of  $H_{\epsilon}$  on the "unreduced" Hilbert space since all our developments are essentially algebraic.

We shall also make use of the **scalar reductions** of these models, which are the self-adjoint operators on  $L^2(C_{\epsilon})$  given by

$$H_{\epsilon}^{\rm sc} = H_{\epsilon}|_{S_{i,i+1} \to 1} \,. \tag{3.5}$$

The identity

$$H_{\epsilon}(\psi|\mathbf{s}\rangle) = (H_{\epsilon}^{\mathrm{sc}}\psi)|\mathbf{s}\rangle,$$

with  $|\mathbf{s}\rangle$  a symmetric spin vector, is obvious.

The models (3.1) share a common property that is ultimately responsible for their partial solvability, namely that each Hamiltonian  $H_{\epsilon}$  is related to a scalar differential-difference operator involving near-neighbors exchange operators. In fact, consider the operators

$$T_{\epsilon} = \sum_{i} z_{i}^{\epsilon} \partial_{i}^{2} + 2a \sum_{i} \frac{1}{z_{i} - z_{i+1}} \left( z_{i}^{\epsilon} \partial_{i} - z_{i+1}^{\epsilon} \partial_{i+1} \right) - 2a \sum_{i} \frac{\vartheta_{\epsilon}(z_{i}, z_{i+1})}{(z_{i} - z_{i+1})^{2}} \left( 1 - K_{i,i+1} \right), \tag{3.6}$$

where  $\partial_i \equiv \partial_{z_i}$ ,  $z_{N+1} \equiv z_1$ , and

$$\vartheta_0(x,y) = 1$$
,  $\vartheta_1(x,y) = \frac{1}{2}(x+y)$ ,  $\vartheta_2(x,y) = xy$ .

	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 2$
c	-1	-4	4
<i>c</i> _	0	-2(2b+1)	0
$c_0$	$2\omega$	$4\omega$	4(1-2a)
$E_0$	$N\omega(2a+1)$	$N\omega(4a+2b+1)$	$2Na^2$
$\mu(\mathbf{x})$	$e^{-\frac{\omega}{2}r^2} \prod_i  x_i - x_{i+1} ^a$	$e^{-\frac{\omega}{2}r^2} \prod_i  x_i^2 - x_{i+1}^2 ^a x_i^b$	$\prod_{i} \sin^a  x_i - x_{i+1} $
$\zeta(x)$	x	$x^2$	$e^{\pm 2ix}$

Table 3.1: Parameters, gauge factor and change of variable.

Each Hamiltonian  $H_{\epsilon}$  is related to a linear combination

$$\overline{H_{\epsilon}} = c T_{\epsilon} + c_{-} J^{-} + c_{0} J^{0} + E_{0} \tag{3.7}$$

of its corresponding operator  $T_{\epsilon}$  and the first-order differential operators

$$J^{-} = \sum_{i} \partial_{i}, \qquad J^{0} = \sum_{i} z_{i} \partial_{i}$$
 (3.8)

through the star mapping (1.22), a change of variables and a gauge transformation. More precisely,

$$H_{\epsilon} = \mu \overline{H_{\epsilon}}^* \big|_{z_i = \zeta(x_i)} \mu^{-1}, \qquad \epsilon = 0, 1, 2,$$
 (3.9)

where the constants c,  $c_-$ ,  $c_0$ ,  $E_0$ , the gauge factor  $\mu$ , and the change of variables  $\zeta$  for each model are listed in Table 3.1. Here and in what follows we write the star mapping and the symmetrizer as  $* \equiv *_+$  and  $\Lambda \equiv \Lambda_+$ , since for the sake of concreteness we shall restrict our attention to the symmetric case.

From Lemma 2.5 and Eq. (3.9) it follows that if  $\Phi(\mathbf{z}) \in \Lambda(C^{\infty} \otimes \Sigma)$  is a symmetric (formal) eigenfunction of  $\overline{H}_{\epsilon}$ , then

$$\Psi(\mathbf{x}) = \mu(\mathbf{x})\Phi(\mathbf{z})|_{z_i = \zeta(x_i)}$$
(3.10)

is a (formal) eigenfunction of  $H_{\epsilon}$  with the same eigenvalue. In the next section we shall construct a flag  $\mathcal{H}^0_{\epsilon} \subset \mathcal{H}^1_{\epsilon} \subset \cdots$  of finite-dimensional subspaces of  $\Lambda(\mathbb{C}[\mathbf{z}] \otimes \Sigma)$  invariant under each  $\overline{H}_{\epsilon}$ . We will show that the problem of diagonalizing  $\overline{H}_{\epsilon}$  in each subspace  $\mathcal{H}^n_{\epsilon}$  is equivalent to the computation of

the polynomial solutions of a system of linear differential equations. We shall completely solve this problem, thereby obtaining several infinite families of eigenfunctions of  $H_{\epsilon}$  for each  $\epsilon$ . From the expressions for the change of variable and the gauge factor in Table 3.1, and the fact that the functions  $\Phi$  in Eq. (3.10) are in all cases polynomials, it immediately follows that the eigenfunctions thus obtained are in fact normalizable.

Gloss 3.2. The above procedure parallels the construction of (generalized) CS models of  $A_N$  type [81], which also applies to the  $BC_N$  case [82]. Each operator  $T_{\epsilon}$  is the NN analog of the sum of squares of one of the three known families of  $A_N$  Dunkl operators. E.g., the operator  $T_0$  is to be compared with the sum of the squares of the Dunkl operators (1.11) associated with the  $A_N$  system, i.e.,

$$T_0^{\text{CS}} = \sum_{i} (J_i^{\text{rat}})^2 = \sum_{i} \partial_i^2 + 2a \sum_{i < j} \frac{1}{z_i - z_j} (\partial_i - \partial_j) - 2a \sum_{i < j} \frac{1}{(z_i - z_j)^2} (1 - K_{ij}).$$

Hence, one can regard  $H_0$  as the proper NN analog of the Calogero model from the point of view of quasi-exact solvability. In contrast with the usual CS models, however, the physical Hamiltonians  $H_{\epsilon}$  associated with the cyclic differential-difference operators  $T_{\epsilon}$  include a three-body term which turns out to be crucial in the analysis of the solvability properties of these models.

Two natural questions arise from the above discussion. First, it should be ascertained whether the full Dunkl operator approach can be extended to the NN setting or one is actually obliged to do without it. Secondly, one could wonder whether one would be able to use the above construction to prove the quasi-exact solvability of the models (3.1) as in the case of CS models. The answer to both questions is negative. Since 3 is not a Coxeter group, there are no known procedures to construct a set of Dunkl operators associated with this problem. Moreover, a couple of technical details prevents the usual approach to CS systems to work successfully in the NN context. Let us elaborate on this.

The proof of the quasi-exact solvability of the Hamiltonian obtained from  $T_0^{\text{CS}}$  (or a linear combination similar to (3.9)) through a gauge transformation is based on two facts:

(i)  $T_0^{\text{CS}}$  leaves invariant the space of polynomials

$$\mathcal{P}^n = \left\{ f \in \mathbb{C}[\mathbf{z}] : \deg f \leqslant n \right\}$$

for any  $n \in \mathbb{N}_0$ , where here and in what follows we denote by deg f the degree in  $\mathbf{z}$  of the polynomial f.

(ii)  $T_0^{\text{CS}}$  commutes with the symmetrizer  $\Lambda$ , which satisfies  $K_{ij}\Lambda = S_{ij}\Lambda$ .

These conditions immediately imply that

$$(T_0^{\text{CS}})^* \Lambda(\mathcal{P}^n | \mathbf{s} \rangle) = \Lambda[(T_0^{\text{CS}} \mathcal{P}^n) | \mathbf{s} \rangle] \subset \Lambda(\mathcal{P}^n | \mathbf{s} \rangle)$$

for any  $n \in \mathbb{N}_0$ ,  $|\mathbf{s}\rangle \in \Sigma$ . Whereas  $T_0$  certainly satisfies the first condition, it does not commute with  $\Lambda$ , as it is not invariant under arbitrary permutations of the particles. It is also clear that any attempt of replacing  $\Lambda$  by the symmetrizer under the cyclic group (3.4) is doomed to fail, as this symmetry is not enough to exchange  $K_{i,i+1}$  by  $S_{i,i+1}$ , and thus  $\overline{H}_{\epsilon}$  by  $H_{\epsilon}$ .

Hence, the extension of the CS methods to the context of NN models is not at all trivial, and in fact  $H_{\epsilon}$  is not a priori guaranteed to admit finite-dimensional invariant subspaces of  $\mu\Lambda(\mathbb{C}[\mathbf{z}]\otimes\Sigma)$ . We shall next show that this is the case nonetheless.

### 3.3 The invariant spaces

In this section we shall prove that each operator  $T_{\epsilon}$  leaves invariant a flag  $\mathcal{T}_{\epsilon}^{0} \subset \mathcal{T}_{\epsilon}^{1} \subset \cdots$ , where  $\mathcal{T}_{\epsilon}^{n}$  is a finite-dimensional subspace of  $\Lambda(\mathcal{P}^{n} \otimes \Sigma)$ . This result will then be used to construct a corresponding invariant flag  $\mathcal{H}_{\epsilon}^{0} \subset \mathcal{H}_{\epsilon}^{1} \subset \cdots$  for the operator  $\overline{H}_{\epsilon}$ , where  $\mathcal{H}_{\epsilon}^{n} \subset \mathcal{T}_{\epsilon}^{n}$  for all n.

Let us first introduce the following two sets of elementary symmetric polynomials:

$$\sigma_k = \sum_i z_i^k$$
,  $\tau_k = \sum_{i_1 < \dots < i_k} z_{i_1} \cdots z_{i_k}$ ;  $k = 1, \dots, N$ .

It is well known [145] that any symmetric polynomial in  $\mathbf{z}$  can be expressed as a polynomial in either  $\boldsymbol{\sigma} \equiv (\sigma_1, \dots, \sigma_N)$  or  $\boldsymbol{\tau} \equiv (\tau_1, \dots, \tau_N)$ .

We shall denote by  $2aX_{\epsilon}$  the terms of  $T_{\epsilon}$  linear in derivatives, that is

$$X_{\epsilon} = \sum_{i} \frac{1}{z_i - z_{i+1}} \left( z_i^{\epsilon} \partial_i - z_{i+1}^{\epsilon} \partial_{i+1} \right).$$

In the next lemma we show that each vector field  $X_{\epsilon}$  leaves invariant a corresponding flag  $\mathcal{X}_{\epsilon}^{0} \subset \mathcal{X}_{\epsilon}^{1} \subset \cdots$  of finite-dimensional subspaces of the space  $\Lambda(\mathbb{C}[\mathbf{z}]) = \mathbb{C}[\boldsymbol{\sigma}] = \mathbb{C}[\boldsymbol{\tau}]$  of symmetric polynomials in  $\mathbf{z}$ . If f is a function of the symmetric variables  $\sigma_{1}, \sigma_{2}, \sigma_{3}, \tau_{N-1}, \tau_{N}$ , we shall use from now on the convenient notation

$$f_k = \begin{cases} \partial_{\sigma_k} f, & k = 1, 2, 3, \\ \partial_{\tau_k} f, & k = N - 1, N. \end{cases}$$

**Lemma 3.3.** For each n = 0, 1, ..., the operator  $X_{\epsilon}$  leaves invariant the linear space  $\mathcal{X}_{\epsilon}^{n}$ , where

$$\mathcal{X}_0^n = \mathbb{C}[\sigma_1, \sigma_2, \sigma_3] \cap \mathcal{P}^n, \quad \mathcal{X}_1^n = \mathbb{C}[\sigma_1, \sigma_2, \tau_N] \cap \mathcal{P}^n, \quad \mathcal{X}_2^n = \mathbb{C}[\sigma_1, \tau_{N-1}, \tau_N] \cap \mathcal{P}^n.$$

*Proof.* Let us first consider the vector field  $X_0$ . Since

$$X_0 \sigma_k = k \sum_i z_i^{k-1} X_0 z_i = k \left( \sum_i \frac{z_i^{k-1}}{z_i - z_{i+1}} - \sum_i \frac{z_i^{k-1}}{z_{i-1} - z_i} \right) = \begin{cases} 0, & k = 1, \\ 2N, & k = 2, \\ 6\sigma_1, & k = 3, \end{cases}$$

if  $f \in \mathcal{X}_0^n$  we have

$$X_0 f = 2(N f_2 + 3\sigma_1 f_3) \in \mathcal{X}_0^n$$
. (3.11a)

The proof for the remaining two cases follows from the analogous formulas

$$X_1 f = N f_1 + 4\sigma_1 f_2,$$
  $f \in \mathcal{X}_1^n;$  (3.11b)

$$X_2 f = 2\sigma_1 f_1 + N(\tau_{N-1} f_{N-1} + \tau_N f_N), \qquad f \in \mathcal{X}_2^n.$$
 (3.11c)

Remark 3.4. It should be noted that these flags cannot be trivially enlarged, since, e.g.,

$$\frac{1}{4}X_0\sigma_4 = 2\sigma_2 + \sum_i z_i z_{i+1},$$

$$\frac{1}{3}X_1\sigma_3 = 2\sigma_2 + \sum_i z_i z_{i+1}, \qquad X_1\tau_{N-1} = \tau_N \sum_i (z_i z_{i+1})^{-1},$$

$$\frac{1}{2}X_2\sigma_2 = 2\sigma_2 + \sum_i z_i z_{i+1}, \qquad X_2\tau_{N-2} = N\tau_{N-2} - \tau_N \sum_i (z_i z_{i+1})^{-1}$$

are not symmetric polynomials.

We note that the restriction of  $T_{\epsilon}$  to  $\mathcal{X}_{\epsilon}^{n} \subset \Lambda(\mathbb{C}[\mathbf{z}])$  obviously satisfies

$$T_{\epsilon}|_{\mathcal{X}_{\epsilon}^{n}} = \sum_{i} z_{i}^{\epsilon} \partial_{i}^{2} + 2aX_{\epsilon}. \tag{3.12}$$

The second-order terms of the operator (3.12), however, do not preserve the corresponding space  $\mathcal{X}_{\epsilon}^{n}$ , unless one imposes the additional restrictions specified in the following proposition:

**Proposition 3.5.** For each  $n \in \mathbb{N}_0$ , the operator  $T_{\epsilon}$  leaves invariant the linear space  $\mathcal{S}_{\epsilon}^n$ , where

$$S_0^n = \{ f \in \mathcal{X}_0^n : f_{33} = 0 \},$$
  

$$S_1^n = \{ f \in \mathcal{X}_1^n : f_{22} = f_{NN} = 0 \},$$
  

$$S_2^n = \{ f \in \mathcal{X}_2^n : f_{11} = f_{N-1,N-1} = 0 \}.$$

*Proof.* Let us begin with the operator  $T_0$ . If  $f \in \mathcal{X}_0^n$ , an elementary computation shows that

$$\partial_i f = f_1 + 2z_i f_2 + 3z_i^2 f_3 \tag{3.13}$$

and therefore

$$\sum_{i} \partial_{i}^{2} f = N(f_{11} + 2f_{2}) + 2(2f_{12} + 3f_{3})\sigma_{1} + 2(3f_{13} + 2f_{22})\sigma_{2} + 12f_{23}\sigma_{3} + 9f_{33}\sigma_{4}. \quad (3.14)$$

From the previous formula and Eq. (3.11a) it follows that  $T_0 f \in \mathcal{S}_0^n$  whenever  $f \in \mathcal{S}_0^n$ . Similarly, if  $f \in \mathcal{X}_1^n$  we have

$$\partial_i f = f_1 + 2z_i f_2 + z_i^{-1} \tau_N f_N , \qquad (3.15)$$

so that

$$\sum_{i} z_{i} \partial_{i}^{2} f = (f_{11} + 2f_{2})\sigma_{1} + 4f_{12}\sigma_{2} + 4f_{22}\sigma_{3} + 2Nf_{1N}\tau_{N} + 4f_{2N}\sigma_{1}\tau_{N} + f_{NN}\tau_{N-1}\tau_{N}, \quad (3.16)$$

which together with Eq. (3.11b) implies that  $T_1 f \in \mathcal{S}_1^n$  for all  $f \in \mathcal{S}_1^n$ . Finally, if  $f \in \mathcal{X}_2^n$  then

$$\partial_i f = f_1 + \left( z_i^{-1} \tau_{N-1} - z_i^{-2} \tau_N \right) f_{N-1} + z_i^{-1} \tau_N f_N \tag{3.17}$$

and hence

$$\sum_{i} z_{i}^{2} \partial_{i}^{2} f = f_{11} \sigma_{2} + 2f_{1,N-1} (\sigma_{1} \tau_{N-1} - N \tau_{N})$$

$$+ 2f_{1N} \sigma_{1} \tau_{N} + f_{N-1,N-1} [(N-1)\tau_{N-1}^{2} - 2\tau_{N-2} \tau_{N}]$$

$$+ 2(N-1)f_{N-1,N} \tau_{N-1} \tau_{N} + Nf_{NN} \tau_{N}^{2}. \quad (3.18)$$

The statement follows again from the previous equation and Eq. (3.11c).  $\Box$ 

The last proposition implies that each operator  $T_{\epsilon}$  preserves "trivial" symmetric spaces  $\mathcal{S}^n_{\epsilon} \otimes \Lambda(\Sigma)$  spanned by factorized states. The main theorem of this section shows that in fact the latter operator leaves invariant a flag of nontrivial finite-dimensional subspaces of  $\Lambda(\mathcal{P}^n \otimes \Sigma)$ . Before stating this theorem we need to make a few preliminary definitions. Given a spin state  $|\mathbf{s}\rangle \in \Sigma$ , we set

$$|\mathbf{s}_{i}\rangle = \frac{1}{N!} \sum_{\substack{\pi \in S_{N} \\ \pi(1)=i}} \pi |\mathbf{s}\rangle, \qquad |\mathbf{s}_{ij}^{\pm}\rangle = \frac{1}{N!} \sum_{\substack{\pi \in S_{N} \\ \pi(1)=i,\pi(2)=j}} \pi (1 \pm S_{12}) |\mathbf{s}\rangle, \qquad (3.19)$$

where  $S_N$  is the symmetric group on N elements and we identify an abstract permutation  $\pi$  with its realization as a permutation of the particles' spins. From Eq. (3.19) we have

$$\Lambda(f(z_1)|\mathbf{s}\rangle) = \sum_{i} f(z_i)|\mathbf{s}_i\rangle, \qquad \Lambda(g_{\pm}(z_1, z_2)|\mathbf{s}\rangle) = \sum_{i < j} g_{\pm}(z_i, z_j)|\mathbf{s}_{ij}^{\pm}\rangle,$$
(3.20)

where the last identity holds if  $g_{\pm}(x,y) = \pm g_{\pm}(y,x)$ . We also define the subspace

$$\Sigma' = \left\{ |\mathbf{s}\rangle \in \Sigma : \sum_{i} |\mathbf{s}_{i,i+1}^{+}\rangle \in \Lambda(\Sigma) \right\} \subset \Sigma.$$
 (3.21)

#### Theorem 3.6. Let

$$\mathcal{T}_{0}^{n} = \left\langle f(\sigma_{1}, \sigma_{2}, \sigma_{3}) \Lambda | \mathbf{s} \right\rangle, g(\sigma_{1}, \sigma_{2}, \sigma_{3}) \Lambda(z_{1} | \mathbf{s} \rangle), h(\sigma_{1}, \sigma_{2}) \Lambda(z_{1}^{2} | \mathbf{s} \rangle), 
\tilde{h}(\sigma_{1}, \sigma_{2}) \Lambda(z_{1} z_{2} | \mathbf{s}' \rangle), w(\sigma_{1}, \sigma_{2}) \Lambda(z_{1} z_{2} (z_{1} - z_{2}) | \mathbf{s} \rangle) : f_{33} = g_{33} = 0 \right\rangle, 
\mathcal{T}_{1}^{n} = \left\langle f(\sigma_{1}, \sigma_{2}, \tau_{N}) \Lambda | \mathbf{s} \right\rangle, g(\sigma_{1}, \tau_{N}) \Lambda(z_{1} | \mathbf{s} \rangle) : f_{22} = f_{NN} = g_{NN} = 0 \right\rangle, 
\mathcal{T}_{2}^{n} = \left\langle f(\sigma_{1}, \tau_{N-1}, \tau_{N}) \Lambda | \mathbf{s} \right\rangle, g(\tau_{N-1}, \tau_{N}) \Lambda(z_{1} | \mathbf{s} \rangle), 
\tau_{N} q(\sigma_{1}, \tau_{N}) \Lambda(z_{1}^{-1} | \mathbf{s} \rangle) : f_{11} = f_{N-1, N-1} = g_{N-1, N-1} = q_{11} = 0 \right\rangle,$$

where  $|\mathbf{s}\rangle \in \Sigma$ ,  $|\mathbf{s}'\rangle \in \Sigma'$ ,  $\deg f \leqslant n$ ,  $\deg g \leqslant n-1$ ,  $\deg h \leqslant n-2$ ,  $\deg \tilde{h} \leqslant n-2$ ,  $\deg w \leqslant n-3$ ,  $\deg q \leqslant n-N+1$ . Then  $\mathcal{T}_{\epsilon}^n$  is invariant under  $T_{\epsilon}$  for all  $n \in \mathbb{N}_0$ .

*Proof.* By Proposition 3.5 it suffices to show that  $T_{\epsilon}$  maps  $\mathcal{T}_{\epsilon}^{n}/(\mathcal{S}_{\epsilon}^{n} \otimes \Lambda(\Sigma))$  into  $\mathcal{T}_{\epsilon}^{n}$ . We shall first deal with the operator  $T_{0}$ . Consider the states of the form  $g\Lambda(z_{1}|\mathbf{s}\rangle)$ , with  $g \in \mathcal{S}_{0}^{n-1}$ . Since

$$(\partial_l - \partial_{l+1})z_i = \frac{1}{z_l - z_{l+1}} (1 - K_{l,l+1})z_i, \quad \forall i, l,$$

we have

$$T_0(gz_i) = (T_0g)z_i + 2\partial_i g.$$

Calling

$$\Phi^{(k)} \equiv \Phi^{(k)}(|\mathbf{s}\rangle) = \Lambda(z_1^k |\mathbf{s}\rangle), \qquad k \in \mathbb{Z}, \tag{3.22}$$

from Eqs. (3.13) and (3.20) we obtain

$$T_0(g\Phi^{(1)}) = \sum_i T_0(gz_i)|\mathbf{s}_i\rangle = (T_0g)\Phi^{(1)} + 2\sum_{k=1}^3 kg_k\Phi^{(k-1)} \in \mathcal{T}_0^{n-2}. \quad (3.23)$$

Similarly, if  $h(\sigma_1, \sigma_2) \in \mathcal{S}_0^{n-2}$ , the identity

$$(\partial_l - \partial_{l+1})z_i^2 = \frac{1}{z_l - z_{l+1}} (1 - K_{l,l+1})z_i^2 + (z_l - z_{l+1})(\delta_{li} + \delta_{l,i-1}), \quad \forall i, l$$

implies that

$$T_0(hz_i^2) = (T_0h)z_i^2 + 4z_i\partial_i h + 2(2a+1)h$$

and therefore

$$T_0(h\Phi^{(2)}) = \sum_i T_0(hz_i^2)|\mathbf{s}_i\rangle$$
  
=  $(T_0h + 8h_2)\Phi^{(2)} + 4h_1\Phi^{(1)} + 2(2a+1)h\Phi^{(0)}$  (3.24)

belongs to  $\mathcal{T}_0^{n-2}$  on account of Eqs. (3.11a) and (3.14). On the other hand, from the equality

$$(\partial_l - \partial_{l+1})z_i z_j = \frac{1}{z_l - z_{l+1}} (1 - K_{l,l+1}) z_i z_j - (z_l - z_{l+1}) \delta_{j,i+1} \delta_{l,i}, \quad \forall i < j, \forall l$$

it follows that

$$T_0(\tilde{h}z_iz_j) = (T_0\tilde{h})z_iz_j + 2(z_i\partial_j\tilde{h} + z_j\partial_i\tilde{h}) - 2a\tilde{h}\delta_{j,i+1}.$$

Setting

$$\widetilde{\Phi}^{(2)} \equiv \widetilde{\Phi}^{(2)}(|\mathbf{s}\rangle) = \Lambda(z_1 z_2 |\mathbf{s}\rangle) \tag{3.25}$$

and using again Eqs. (3.13) and (3.20) we then have

$$T_{0}(\tilde{h}\widetilde{\Phi}^{(2)}) = \sum_{i < j} T_{0}(\tilde{h}z_{i}z_{j})|\mathbf{s}_{ij}^{+}\rangle$$

$$= (T_{0}\tilde{h} + 8\tilde{h}_{2})\widetilde{\Phi}^{(2)} + 2\tilde{h}_{1}\Lambda[(z_{1} + z_{2})|\mathbf{s}\rangle] - 2a\tilde{h}\sum_{i} |\mathbf{s}_{i,i+1}^{+}\rangle. \quad (3.26)$$

Since  $\Lambda[(z_1+z_2)|\mathbf{s}\rangle] = \Lambda[z_1(1+S_{12})|\mathbf{s}\rangle]$ , the RHS of Eq. (3.26) belongs to  $\mathcal{T}_0^{n-2}$  if and only if  $|\mathbf{s}\rangle \in \Sigma'$ . The last type of states generating the module  $\mathcal{T}_0^n$  are of the form  $w(\sigma_1, \sigma_2)\widehat{\Phi}^{(3)}$ , where

$$\widehat{\Phi}^{(3)} \equiv \widehat{\Phi}^{(3)}(|\mathbf{s}\rangle) = \Lambda(z_1 z_2 (z_1 - z_2)|\mathbf{s}\rangle). \tag{3.27}$$

From the equality

$$\frac{1}{z_{l}-z_{l+1}} \left(\partial_{l}-\partial_{l+1}\right) \left[z_{i}z_{j}(z_{i}-z_{j})\right] = \frac{1}{(z_{l}-z_{l+1})^{2}} \left(1-K_{l,l+1}\right) \left[z_{i}z_{j}(z_{i}-z_{j})\right] + \left(\delta_{l,i-1}+\delta_{li}\right) z_{j} - \left(\delta_{l,j-1}+\delta_{lj}\right) z_{i}, \quad \forall i < j, \forall l$$

it follows that

$$T_0[wz_iz_j(z_i-z_j)] = (T_0w)z_iz_j(z_i-z_j) + 2z_j(2z_i-z_j)\partial_i w - 2z_i(2z_j-z_i)\partial_j w - 2(2a+1)(z_i-z_j)w.$$
(3.28)

Using again Eqs. (3.13) and (3.20) we obtain

$$T_{0}(w\widehat{\Phi}^{(3)}) = \sum_{i < j} T_{0}(wz_{i}z_{j}(z_{i} - z_{j}))|\mathbf{s}_{ij}^{-}\rangle = (T_{0}w + 12w_{2})\widehat{\Phi}^{(3)} + 2w_{1}\Lambda[(z_{1}^{2} - z_{2}^{2})|\mathbf{s}\rangle] - 2(2a + 1)w\Lambda[(z_{1} - z_{2})|\mathbf{s}\rangle]. \quad (3.29)$$

Since  $\Lambda[(z_1^k - z_2^k)|\mathbf{s}\rangle] = \Lambda[z_1^k(1 - S_{12})|\mathbf{s}\rangle]$ , the RHS of the latter equation clearly belongs to  $\mathcal{T}_0^{n-2}$ . This shows that  $T_0(\mathcal{T}_0^n) \subset \mathcal{T}_0^{n-2} \subset \mathcal{T}_0^n$ .

Consider next the action of the operator  $T_1$  on a state of the form  $g(\sigma_1, \tau_N)\Phi^{(1)}$ , with  $g \in \mathcal{S}_1^{n-1}$ . From the identity

$$(z_l \partial_l - z_{l+1} \partial_{l+1}) z_i = \frac{1}{2} \frac{z_l + z_{l+1}}{z_l - z_{l+1}} (1 - K_{l,l+1}) z_i + \frac{1}{2} (z_l - z_{l+1}) (\delta_{l,i} + \delta_{l,i-1}), \quad \forall i, l$$

we easily obtain

$$T_1(gz_i) = (T_1g)z_i + 2z_i\partial_i g + 2ag,$$

and therefore, by Eqs. (3.11b), (3.15) and (3.16),

$$T_1(g\Phi^{(1)}) = \sum_i T_1(gz_i)|\mathbf{s}_i\rangle = (T_1g + 2g_1)\Phi^{(1)} + 2(ag + \tau_N g_N)\Phi^{(0)} \in \mathcal{T}_1^{n-1}.$$
(3.30)

Thus  $T_1(\mathcal{T}_1^n) \subset \mathcal{T}_1^{n-1} \subset \mathcal{T}_1^n$ , as claimed.

Consider, finally, the operator  $T_2$ . If  $g(\tau_{N-1}, \tau_N) \in \mathcal{S}_2^{n-1}$ , the identity

$$(z_l^2 \partial_l - z_{l+1}^2 \partial_{l+1}) z_i = \frac{z_l z_{l+1}}{z_l - z_{l+1}} (1 - K_{l,l+1}) z_i + z_i (z_l - z_{l+1}) (\delta_{l,i} + \delta_{l,i-1}), \quad \forall i, l$$

vields

$$T_2(gz_i) = (T_2g)z_i + 2z_i^2\partial_i g + 4az_i g,$$

and hence, by Eq. (3.17),

$$T_2(g\Phi^{(1)}) = \sum_i T_2(gz_i)|\mathbf{s}_i\rangle$$

$$= \left[T_2g + 2(\tau_{N-1}g_{N-1} + \tau_Ng_N + 2ag)\right]\Phi^{(1)} - 2\tau_Ng_{N-1}\Phi^{(0)} \quad (3.31)$$

clearly belongs to  $\mathcal{T}_2^n$  on account of Eqs. (3.11c) and (3.18). The last type of spin states we need to study are of the form  $\hat{q}\Phi^{(-1)}$ , where  $\hat{q} \equiv \tau_N q(\sigma_1, \tau_N)$  with  $q_{11} = 0$ . Since

$$(z_l^2 \partial_l - z_{l+1}^2 \partial_{l+1}) z_i^{-1} = \frac{z_l z_{l+1}}{z_l - z_{l+1}} (1 - K_{l,l+1}) z_i^{-1}, \qquad \forall i, l,$$

we obtain

$$T_2(\hat{q}z_i^{-1}) = (T_2\hat{q})z_i^{-1} - 2\partial_i\hat{q} + 2\hat{q}z_i^{-1},$$

and thus, by Eqs. (3.17) and (3.20),

$$T_2(\hat{q}\Phi^{(-1)}) = \sum_i T_2(\hat{q}z_i^{-1})|\mathbf{s}_i\rangle = (T_2\hat{q} - 2\tau_N\hat{q}_N + 2\hat{q})\Phi^{(-1)} - 2\hat{q}_1\Phi^{(0)}. \quad (3.32)$$

From Eqs. (3.11c) and (3.18), it follows that the RHS of the previous equation belongs to  $\mathcal{T}_2^n$ . Hence  $T_2(\mathcal{T}_2^n) \subset \mathcal{T}_2^n$ , which concludes the proof.

Remark 3.7. We have chosen to allow a certain overlap between the different types of states spanning the spaces  $\mathcal{T}_{\epsilon}^{n}$ . For instance, if  $|s\rangle$  is symmetric a state  $g(\sigma_{1}, \sigma_{2}, \sigma_{3})\Lambda(z_{1}|s\rangle) \in \mathcal{T}_{0}^{n}$  is also of the form  $f(\sigma_{1}, \sigma_{2}, \sigma_{3})\Lambda|s\rangle$ . Less trivially, in the case of spin 1/2 the identity

$$\widehat{\Phi}^{(3)}(|\mathbf{s}\rangle) = \frac{2}{N} \left( \sigma_1 \Phi^{(2)}(|\mathbf{s}\rangle) - \sigma_2 \Phi^{(1)}(|\mathbf{s}\rangle) \right), \qquad S_{12}|\mathbf{s}\rangle = -|\mathbf{s}\rangle,$$

where  $\Phi^{(k)}$  and  $\widehat{\Phi}^{(3)}$  are respectively defined in Eqs. (3.22) and (3.27), implies that the states of the form  $w(\sigma_1, \sigma_2)\widehat{\Phi}^{(3)}$  in the space  $\mathcal{T}_0^n$  can be expressed in terms of the other generators of this space.

Corollary 3.8. For each  $\epsilon = 0, 1, 2$ , the gauge Hamiltonian  $\overline{H}_{\epsilon}$  leaves invariant the space  $\mathcal{H}_{\epsilon}^n$  defined by

$$\mathcal{H}_0^n = \mathcal{T}_0^n$$
,  $\mathcal{H}_1^n = \mathcal{T}_1^n \big|_{f_N = q_N = 0}$ ,  $\mathcal{H}_2^n = \mathcal{T}_2^n$ .

*Proof.* We shall begin by showing that each space  $\mathcal{T}_{\epsilon}^{n}$  is invariant under the operator  $J^{0}$ . Note first that

$$J^0 \Phi^{(j)} = j \Phi^{(j)}, \qquad J^0 \widetilde{\Phi}^{(2)} = 2 \widetilde{\Phi}^{(2)}, \qquad J^0 \widehat{\Phi}^{(3)} = 3 \widehat{\Phi}^{(3)}; \qquad j \in \mathbb{Z}, (3.33)$$

where the states  $\Phi^{(j)}$ ,  $\widetilde{\Phi}^{(2)}$  and  $\widehat{\Phi}^{(3)}$  are defined in Eqs. (3.22), (3.25) and (3.27), respectively. Using Eqs. (3.13), (3.15) and (3.17) one can immediately establish the identities

$$J^{0}f = \sigma_{1}f_{1} + 2\sigma_{2}f_{2} + 3\sigma_{3}f_{3}, \qquad \forall f(\sigma_{1}, \sigma_{2}, \sigma_{3}), \qquad (3.34a)$$

$$J^{0}f = \sigma_{1}f_{1} + 2\sigma_{2}f_{2} + N\tau_{N}f_{N}, \qquad \forall f(\sigma_{1}, \sigma_{2}, \tau_{N}), \qquad (3.34b)$$

$$J^{0}f = \sigma_{1}f_{1} + (N-1)\tau_{N-1}f_{N-1} + N\tau_{N}f_{N}, \qquad \forall f(\sigma_{1}, \tau_{N-1}, \tau_{N}). \quad (3.34c)$$

From Eqs. (3.33)-(3.34) and the fact that  $J^0$  is a derivation it follows that  $J^0$  leaves invariant the spaces  $\mathcal{T}^n_{\epsilon}$  for all  $\epsilon = 0, 1, 2$ . This implies that  $\overline{H}_{\epsilon}$  preserves  $\mathcal{T}^n_{\epsilon}$  for  $\epsilon = 0, 2$ , since the coefficient  $c_-$  vanishes in these cases (cf. Table 3.1). On the other hand, for  $\epsilon = 1$  the coefficient  $c_-$  is nonzero, and thus we have to consider the action of the operator  $J^-$  on the space  $\mathcal{T}^n_1$ . We now have

$$J^{-}\Phi^{(j)} = j\Phi^{(j-1)}, \qquad j \in \mathbb{Z},$$
 (3.35)

and, from Eq. (3.15),

$$J^{-}f = Nf_1 + 2\sigma_1 f_2 + \tau_{N-1} f_N, \qquad \forall f(\sigma_1, \sigma_2, \tau_N).$$
 (3.36)

Hence  $J^-$  leaves invariant the subspace  $\mathcal{H}_1^n$  of  $\mathcal{T}_1^n$  defined by the restrictions  $f_N = g_N = 0$ . From the obvious identity  $T_1(f\Phi^{(0)}) = (T_1f)\Phi^{(0)}$  and Eq. (3.30), together with (3.11b), (3.12) and (3.16), it follows that the operator  $T_1$  also preserves  $\mathcal{H}_1^n$ . Likewise, Eqs. (3.33) and (3.34b) imply that  $\mathcal{H}_1^n$  is invariant under  $J^0$ , and hence under the gauge Hamiltonian  $\overline{H}_1$ .

Theorem 3.6 characterizes the invariant space  $\mathcal{T}_0^n$  in terms of the subspace  $\Sigma' \subset \Sigma$  in Eq. (3.21) that we shall now study in detail. In fact, from the definition of the invariant space  $\mathcal{T}_0^n$  it follows that we can consider without loss of generality the quotient space  $\Sigma'/\sim$ , where  $|\mathbf{s}\rangle \sim |\tilde{\mathbf{s}}\rangle$  if  $\Lambda(z_1z_2|\mathbf{s}\rangle) = \Lambda(z_1z_2|\tilde{\mathbf{s}}\rangle)$ . For instance, from Eq. (3.19) it immediately follows that if  $|\mathbf{s}\rangle \in \Sigma'$  and  $\pi \in S_N$  is a permutation such that  $\pi(i) \in \{1,2\}$  for i=1,2, then  $\pi|\mathbf{s}\rangle$  belongs to  $\Sigma'$  and is equivalent to  $|\mathbf{s}\rangle$ .

In the rest of this section, we shall denote  $|\mathbf{s}_{ij}^{+}\rangle$  simply as  $|\mathbf{s}_{ij}\rangle$  for the sake of conciseness. From Eq. (3.19) it easily follows that any symmetric state belongs to  $\Sigma'$ , since

$$\sum_{i} |\mathbf{s}_{i,i+1}\rangle = \frac{2}{N-1} |\mathbf{s}\rangle, \quad \text{for all } |\mathbf{s}\rangle \in \Lambda(\Sigma).$$
 (3.37)

On the other hand, if  $|\mathbf{s}\rangle \in \Lambda(\Sigma)$  the corresponding state  $h(\sigma_1, \sigma_2)\Lambda(z_1z_2|\mathbf{s}\rangle)$  is a trivial (factorized) state. We shall next show that the reciprocal of this statement is also true, up to equivalence.

**Lemma 3.9.** For every  $|\mathbf{s}\rangle \in \Sigma$ ,  $\Lambda(z_1z_2|\mathbf{s}\rangle)$  is a factorized state if and only if  $|\mathbf{s}\rangle \sim \Lambda|\mathbf{s}\rangle$ .

Proof. Suppose that

$$\Lambda(z_1 z_2 | \mathbf{s} \rangle) = |\hat{\mathbf{s}}\rangle \sum_{i < j} c_{ij} z_i z_j$$

is a factorized state. Since the LHS of the previous formula is symmetric,  $c_{ij} = c$  for all i, j and  $|\hat{\mathbf{s}}\rangle \in \Lambda(\Sigma)$ . By absorbing the constant c into  $|\hat{\mathbf{s}}\rangle$  we can take c = 1 without loss of generality, and therefore

$$\Lambda(z_1 z_2 | \mathbf{s} \rangle) = \sum_{i < j} z_i z_j | \mathbf{s}_{ij} \rangle = \tau_2 | \hat{\mathbf{s}} \rangle \implies | \mathbf{s}_{ij} \rangle = | \hat{\mathbf{s}} \rangle, \qquad i, j = 1, \dots, N.$$

From Eq. (3.20) with  $f(z_1, z_2) = 1$  it then follows that

$$\Lambda |\mathbf{s}\rangle = \sum_{i \leq j} |\mathbf{s}_{ij}\rangle = \frac{1}{2} N(N-1) |\hat{\mathbf{s}}\rangle.$$

Setting  $|\mathbf{s}_0\rangle = |\mathbf{s}\rangle - \Lambda |\mathbf{s}\rangle$  and using the previous identity we obtain

$$\Lambda(z_1 z_2 | \mathbf{s}_0 \rangle) = \Lambda(z_1 z_2 | \mathbf{s} \rangle) - \frac{2\tau_2}{N(N-1)} \Lambda | \mathbf{s} \rangle = |\hat{\mathbf{s}} \rangle \tau_2 - \frac{2\tau_2}{N(N-1)} \Lambda | \mathbf{s} \rangle = 0.$$

Hence  $|\mathbf{s}\rangle \sim \Lambda |\mathbf{s}\rangle$ , as claimed.

By the previous lemma, it suffices to characterize the nonsymmetric states in  $\Sigma'$ . To this end, let us introduce the linear operator  $A: \Sigma \to \Sigma$  by

$$A|\mathbf{s}\rangle = \sum_{i} |\mathbf{s}_{i,i+1}\rangle.$$
 (3.38)

Given an element  $|\mathbf{s}\rangle \equiv |s_1...s_N\rangle$  of the canonical basis of  $\Sigma$ , we shall also denote by  $\{s^1,...,s^n\}$  the set of distinct components of  $\mathbf{s} \equiv (s_1,...,s_N)$ , and by  $\nu_i$  the number of times that  $s^i$  appears among the components of  $\mathbf{s}$ . For instance, if  $|\mathbf{s}\rangle = |-2,0,1,-2,1\rangle$ , then we can take  $s^1 = -2$ ,  $s^2 = 0$ ,  $s^3 = 1$ , so that  $\nu_1 = \nu_3 = 2$ ,  $\nu_2 = 1$ . Consider the spin states  $|\chi_i(\mathbf{s})\rangle \equiv |\chi_i\rangle$ ,

 $i = 1, \ldots, n$ , given by

$$|\chi_i\rangle = \nu_i(\nu_i - 1)|s^i s^i \dots\rangle - \sum_{\substack{1 \le j,k \le n \\ j,k \ne i}} \nu_j(\nu_k - \delta_{jk})|s^j s^k \dots\rangle, \qquad \nu_i > 1,$$
(3.39a)

$$|\chi_i\rangle = \sum_{\substack{1 \le j \le n \\ j \ne i}} \nu_j \left( |s^i s^j \dots\rangle + |s^j s^i \dots\rangle \right), \qquad \qquad \nu_i = 1.$$
(3.39b)

Here we have adopted the following convention: an ellipsis inside a ket stands for an arbitrary ordering of the components in  $\mathbf{s}$  not indicated explicitly. Note that the states (3.39) are defined only up to equivalence, and that  $|\chi_i(\mathbf{s})\rangle = |\chi_i(\pi \mathbf{s})\rangle$  for any permutation  $\pi \in S_N$ .

**Proposition 3.10.** Given a basic spin state  $|\mathbf{s}\rangle$ , the associated spin states  $|\chi_i(\mathbf{s})\rangle$  are all in  $\Sigma'/\sim$ .

*Proof.* Consider first a state  $|\chi_i\rangle$  of the type (3.39a). Using the definition of the operator A in Eq. (3.38) we obtain

$$N!A|\chi_{i}\rangle = 2\nu_{i}(\nu_{i} - 1) \sum_{l} \sum_{\substack{\pi \in S_{N-2} \\ 1 \leq j,k \leq n \\ j,k \neq i}} \pi|\dots s^{i} s^{i} \dots\rangle$$

$$-2 \sum_{l} \sum_{\substack{1 \leq j,k \leq n \\ j,k \neq i}} \sum_{\pi \in S_{N-2}} \nu_{j}(\nu_{k} - \delta_{jk})\pi|\dots s^{j} s^{k} \dots\rangle, \quad (3.40)$$

where the permutations  $\pi$  act only on the N-2 spin components specified by an ellipsis. On the other hand, we have

$$N \cdot N! \Lambda |\mathbf{s}\rangle = \sum_{l} \sum_{\substack{\pi \in S_{N-2} \\ 1 \le j \le n}} \nu_{i} (\nu_{i} - 1) \pi |\dots s^{i} s^{i} \dots\rangle$$

$$+ 2 \sum_{l} \sum_{\substack{1 \le j \le n \\ j \ne i}} \sum_{\pi \in S_{N-1}} \nu_{j} \pi |\dots s^{j} \dots\rangle$$

$$- \sum_{l} \sum_{\substack{1 \le j, k \le n \\ j, k \ne i}} \sum_{\pi \in S_{N-2}} \nu_{j} (\nu_{k} - \delta_{jk}) \pi |\dots s^{j} s^{k} \dots\rangle. \quad (3.41)$$

Comparing Eqs. (3.40) and (3.41) we obtain

$$A|\chi_{i}\rangle = 2\left(N\Lambda|\mathbf{s}\rangle - \frac{2}{N!} \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \nu_{j} \sum_{\pi \in S_{N-1}} \pi|\dots s^{j} \dots\rangle\right)$$
$$= 2\left(N - 2 \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \nu_{j}\right) \Lambda|\mathbf{s}\rangle = 2(2\nu_{i} - N)\Lambda|\mathbf{s}\rangle. \tag{3.42}$$

This shows that any state of the form (3.39a) belongs to  $\Sigma'/\sim$ . Suppose next that  $\nu_i = 1$ , so that  $|\chi_i\rangle$  is given by Eq. (3.39b). Since

$$A|\chi_{i}\rangle = \frac{2}{N!} \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \sum_{\pi \in S_{N-2}} \nu_{j} \pi \left( | \dots \underset{i}{s^{i}} s^{j} \dots \rangle + | \dots \underset{i}{s^{j}} s^{i} \dots \rangle \right) = 4\Lambda |\mathbf{s}\rangle,$$

$$(3.43)$$

it follows that in this case  $|\chi_i\rangle$  is also in  $\Sigma'/\sim$ .

Remark 3.11. Just as symmetric spin states, cf. Eq. (3.37), the states  $|\chi_i\rangle$  satisfy the relation

$$A|\chi_i\rangle = \frac{2}{N-1}\Lambda|\chi_i\rangle. \tag{3.44}$$

Indeed, if  $\nu_i > 1$ , from Eqs. (3.39a) and (3.42) we have

$$\Lambda |\chi_{i}\rangle = \left[\nu_{i}(\nu_{i} - 1) + \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \nu_{j} - \sum_{\substack{1 \leq j, k \leq n \\ j, k \neq i}} \nu_{j}\nu_{k}\right] \Lambda |\mathbf{s}\rangle$$

$$= \left[\nu_{i}(\nu_{i} - 1) + N - \nu_{i} - (N - \nu_{i})^{2}\right] \Lambda |\mathbf{s}\rangle$$

$$= (N - 1)(2\nu_{i} - N)\Lambda |\mathbf{s}\rangle = \frac{N - 1}{2} A|\chi_{i}\rangle.$$

On the other hand, if  $\nu_i = 1$  Eqs. (3.39b) and (3.43) imply that

$$\Lambda |\chi_i\rangle = 2 \left( \sum_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} \nu_j \right) \Lambda |\mathbf{s}\rangle = 2(N-1)\Lambda |\mathbf{s}\rangle = \frac{N-1}{2} A |\chi_i\rangle.$$

**Example 3.12.** We shall now present all the states of the form (3.39) for spin 1/2. In this case, up to a permutation the basic state  $|\mathbf{s}\rangle$  is given by

$$|\mathbf{s}\rangle = |\overbrace{+\cdots + -\cdots -}^{\nu}\rangle.$$
 (3.45)

If  $\nu$  is either 0 or N, then n=1 and thus  $|\chi_1\rangle$  is of the type (3.39a) and proportional to  $|\mathbf{s}\rangle$ . If  $\nu=1$ , then n=2 and we can take (dropping inessential factors)

$$|\chi_1\rangle = \frac{1}{2} (|+-\cdots\rangle + |-+\cdots\rangle) \sim |+-\cdots\rangle, \qquad |\chi_2\rangle = |--\cdots\rangle.$$

Although the states  $|\chi_1\rangle$  and  $|\chi_2\rangle$  are linearly independent, the combination  $2|\chi_1\rangle + (N-2)|\chi_2\rangle$  is equivalent to a symmetric state. In the case  $\nu = N-1$  the states  $|\chi_i\rangle$  are obtained from the previous ones by flipping the spins. Finally, if  $2 \leq \nu \leq N-2$  then n=2 and the states  $|\chi_i\rangle$  are now given by

$$|\chi_1\rangle = -|\chi_2\rangle = \nu(\nu - 1)|++\cdots\rangle - (N - \nu)(N - \nu - 1)|--\cdots\rangle. \quad (3.46)$$

According to the previous example, for spin 1/2 there are exactly n-1 independent states of the form (3.39) associated with each basic state  $|\mathbf{s}\rangle$ , up to symmetric states. We shall see next that this fact actually holds for arbitrary spin:

**Proposition 3.13.** Let  $|\mathbf{s}\rangle$  be a basic spin state. If n is the number of distinct components of  $\mathbf{s}$ , there are exactly n-1 independent states of the form (3.39) modulo symmetric states.

*Proof.* Let p be the number of distinct components  $s^i$  of s such that  $\nu_i > 1$ . A straightforward computation shows that the combination

$$\sum_{i=1}^{n} |\chi_i\rangle \sim (2-p) \sum_{i,j=1}^{n} \nu_i(\nu_j - \delta_{ij}) |s^i s^j \cdots\rangle \sim (2-p) N(N-1) \Lambda |\mathbf{s}\rangle \quad (3.47)$$

is equivalent to a symmetric state. Suppose first that  $p \neq 2$ . It is immediate to check that in this case the set  $\{|\chi_i\rangle: i=1,\ldots,n\}$  is linearly independent. If a linear combination  $\sum_{i=1}^n c_i|\chi_i\rangle$  is equivalent to a symmetric state  $|\hat{\mathbf{s}}\rangle$ , then  $|\hat{\mathbf{s}}\rangle$  must be proportional to  $\Lambda|\mathbf{s}\rangle$ , so that we can write

$$\sum_{i=1}^{n} c_i |\chi_i\rangle \sim \lambda(2-p)N(N-1)\Lambda |\mathbf{s}\rangle.$$

Hence  $\sum_{i=1}^{n} (c_i - \lambda) |\chi_i\rangle \sim 0$ , and the linear independence of the states  $|\chi_i\rangle$  implies that  $c_i = \lambda$  for all i. On the other hand, if p = 2 the set  $\{|\chi_i\rangle: i = 1, \ldots, n\}$  is linearly dependent on account of Eq. (3.47), but removing one of the two states with  $\nu_i > 1$  clearly yields a linearly independent set. It is also obvious from the coefficients of the states  $|s^i s^i \ldots\rangle$  that no linear combination  $\sum_{i=1}^n c_i |\chi_i\rangle$  can be equivalent to a nonzero symmetric state.  $\square$ 

The next natural question to be addressed is whether the states of the form (3.39) span the space  $\Sigma'/\sim$  up to symmetric states:

**Proposition 3.14.** The space  $(\Sigma'/\Lambda(\Sigma))/\sim$  is spanned by states of the form (3.39).

Proof. For conciseness, we present the proof of this result only for the case M=1/2. Let  $\Sigma_{\nu}$  denote the subspace of  $\Sigma$  spanned by basic spin states with  $\nu$  "+" spins, and set  $\Sigma'_{\nu}=\Sigma'\cap\Sigma_{\nu}$ . Since the operators A and  $\Lambda$  involved in the definition (3.21) of  $\Sigma'$  clearly preserve  $\Sigma_{\nu}$ , it suffices to show that the states  $|\chi_i(\mathbf{s})\rangle$  with  $\mathbf{s}$  given by (3.45) span the space  $\Sigma'_{\nu}/\sim$  up to symmetric states. Note first that the statement is trivial for  $\nu=0,1,N-1,N,$  since in this case the states of the form (3.39) obviously generate the whole space  $\Sigma_{\nu}/\sim$ . Suppose, therefore, that  $2 \leq \nu \leq N-2$ , so that

$$\Sigma_{\nu}/\sim = \langle |++\cdots\rangle, |+-\cdots\rangle, |--\cdots\rangle\rangle.$$

Since the state (3.46) and the symmetric state (up to equivalence)

$$\nu(\nu-1)|++\cdots\rangle+2\nu(N-\nu)|+-\cdots\rangle+(N-\nu)(N-\nu-1)|--\cdots\rangle$$

both belong to  $\Sigma'_{\nu}/\sim$ , we need only show that (for instance)  $|+-\cdots\rangle$  is not in  $\Sigma'_{\nu}/\sim$ , i.e., that  $A|+-\cdots\rangle$  is not symmetric. But this is certainly the case, since a state of the form

$$|\overbrace{+-\cdots+-}^{2k}\overbrace{-\cdots-}^{N-\nu-k}\overbrace{+\cdots+}^{\nu-k}\rangle, \qquad k=1,2,\ldots,\min(\nu,N-\nu),$$

appears in  $A|+-\cdots\rangle$  with coefficient  $2k(\nu-1)!(N-\nu-1)!$  depending on k.

## 3.4 The algebraic eigenfunctions

In the previous section we have provided a detailed description of the spaces  $\mathcal{H}^n_{\epsilon} \subset \Lambda(\mathbb{C}[\mathbf{z}] \otimes \Sigma)$  invariant under the corresponding gauge Hamiltonian  $\overline{H}_{\epsilon}$ . In this section we shall explicitly compute all the eigenfunctions of the restrictions of the operators  $\overline{H}_{\epsilon}$  to their invariant spaces  $\mathcal{H}^n_{\epsilon}$ . This yields several infinite<sup>1</sup> families of eigenfunctions for each of the models (3.2), which is the main result of this paper. We shall use the term *algebraic* to refer

<sup>&</sup>lt;sup>1</sup>For the model (3.2c) we shall see below that the number of eigenfunctions with a given total momentum is finite.

to these eigenfunctions and their corresponding energies. It is important to observe that the eigenfunctions of the gauge Hamiltonian  $\overline{H}_{\epsilon}$  that can be constructed in this way are necessarily invariant under the whole symmetric group, in spite of the fact that  $\overline{H}_{\epsilon}$  is symmetric only under cyclic permutations. In fact, the explicit solutions of all known CS models with near-neighbors interactions (both in the scalar and spin cases) can be factorized as the product of a simple gauge factor analogous to  $\mu$  times a completely symmetric function [10, 48, 75, 124]. This, however, does not rule out the existence of other eigenfunctions of the gauge Hamiltonian  $\overline{H}_{\epsilon}$  invariant only under the subgroup of cyclic permutations, which is indeed an interesting open problem.

### Case a

We shall begin with the model (3.2a), which is probably the most interesting one due to the rich structure of its associated invariant flag. In order to find the algebraic energies of the model, note first that one can clearly construct a basis  $\mathcal{B}_0^n$  of  $\mathcal{H}_0^n$  whose elements are homogeneous polynomials in **z** with coefficients in  $\Sigma$ . If  $f \in \mathcal{B}_0^n$  has degree k, then  $J^0 f = kf$  and  $T_0 f$  has degree at most k-2. If  $\mathcal{B}_0^n$  is ordered according to the degree, the operator  $\overline{H}_0$ is represented in this basis by a triangular matrix with diagonal elements  $E_0 + kc_0$ , where  $k = 0, \ldots, n$  is the degree. Thus the algebraic eigenfunctions are the numbers

$$E_k = E_0 + 2k\omega, \qquad k = 0, 1, \dots$$

We shall next show that the algebraic eigenfunctions of  $\overline{H}_0$  can be expressed in closed form in terms of generalized Laguerre and Jacobi polynomials. The computation is basically a two-step procedure. In the first place, one encodes the eigenvalue problem in the invariant space  $\mathcal{H}_0^n$  as a system of linear partial differential equations. The second step then consists in finding the polynomial solutions of this system.

Regarding the first step, we shall need the following preliminary lemma:

**Lemma 3.15.** The operator  $\overline{H}_0$  preserves the following subspaces of  $\mathcal{H}_0^n$ :

$$\overline{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n} = \langle f\Phi^{(0)}, g\Phi^{(1)}, h\Phi^{(2)}\rangle, \qquad |\mathbf{s}\rangle \in \Sigma, \tag{3.48}$$

$$\overline{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n} = \langle f\Phi^{(0)}, g\Phi^{(1)}, h\Phi^{(2)}\rangle, \qquad |\mathbf{s}\rangle \in \Sigma, \qquad (3.48)$$

$$\widetilde{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n} = \overline{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n} + \langle \widetilde{h}\widetilde{\Phi}^{(2)}\rangle, \qquad |\mathbf{s}\rangle \in \Sigma', \quad S_{12}|\mathbf{s}\rangle = |\mathbf{s}\rangle, \qquad (3.49)$$

$$\widehat{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n} = \overline{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n} + \langle w\widehat{\Phi}^{(3)}\rangle, \qquad |\mathbf{s}\rangle \in \Sigma, \quad S_{12}|\mathbf{s}\rangle = -|\mathbf{s}\rangle, \qquad (3.50)$$

$$\widehat{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n} = \overline{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n} + \langle w\widehat{\Phi}^{(3)}\rangle, \qquad |\mathbf{s}\rangle \in \Sigma, \quad S_{12}|\mathbf{s}\rangle = -|\mathbf{s}\rangle, \qquad (3.50)$$

where f, g, h,  $\tilde{h}$ , w are as in the definition of  $\mathcal{T}_0^n$  in Theorem 3.6, and  $\Phi^{(k)}$ ,  $\widehat{\Phi}^{(2)}$ ,  $\widehat{\Phi}^{(3)}$  are respectively given by (3.22), (3.25) and (3.27).

Proof. The identity  $T_0(f\Phi^{(0)}) = (T_0f)\Phi^{(0)}$  and Eqs. (3.23), (3.24) and (3.33) clearly imply that the subspace  $\overline{\mathcal{H}}_{0,|\mathbf{s}\rangle}^n$  is invariant under  $\overline{H}_0$ . Consider next the action of  $\overline{H}_0$  on a function of the form  $\tilde{h}\widetilde{\Phi}^{(2)}$ . Since  $|\mathbf{s}\rangle$  is symmetric under  $S_{12}$ , we can replace  $\Lambda[(z_1 + z_2)|\mathbf{s}\rangle]$  by  $2\Phi^{(1)}$  in Eq. (3.26). Secondly, any state  $|\mathbf{s}\rangle \in \Sigma'$  satisfies the identity

$$\sum_{i} |\mathbf{s}_{i,i+1}^{+}\rangle = \frac{2}{N-1} \Lambda |\mathbf{s}\rangle.$$
 (3.51)

Indeed, we already know that the previous identity holds for symmetric states (cf. Eq. (3.37)) and for states of the form (3.39) (cf. Eqs. (3.38) and (3.44)). On the other hand, by Proposition 3.14 every state in  $\Sigma'$  is a linear combination of a symmetric state, states of the form (3.39), and a state  $|\mathbf{s}\rangle$  such that  $\Lambda(z_1z_2|\mathbf{s}\rangle) = 0$ . But for the latter "null" state  $|\mathbf{s}_{ij}\rangle = 0$  for all i < j, and hence  $A|\mathbf{s}\rangle = \Lambda|\mathbf{s}\rangle = 0$ . Therefore, Eq. (3.26) can be written as

$$T_0(\tilde{h}\widetilde{\Phi}^{(2)}) = (T_0\tilde{h} + 8\tilde{h}_2)\widetilde{\Phi}^{(2)} + 4\tilde{h}_1\Phi^{(1)} - \frac{4a}{N-1}\tilde{h}\Phi^{(0)}. \tag{3.52}$$

From the previous equation and Eq. (3.33) it follows that  $\overline{H}_0(\tilde{h}\widetilde{\Phi}^{(2)}) \in \widetilde{\mathcal{H}}_{0,|\mathbf{s}\rangle}^n$ . Finally, if  $S_{12}|\mathbf{s}\rangle = -|\mathbf{s}\rangle$ , Eq. (3.29) reduces to

$$T_0(w\widehat{\Phi}^{(3)}) = (T_0w + 12w_2)\widehat{\Phi}^{(3)} + 4w_1\Phi^{(2)} - 4(2a+1)w\Phi^{(1)}, \qquad (3.53)$$

which, together with Eq. (3.33), implies that 
$$\overline{H}_0(w\widehat{\Phi}^{(3)}) \in \widehat{\mathcal{H}}_{0,|\mathbf{s}\rangle}^n$$
.

Remark 3.16. The requirement that  $|\mathbf{s}\rangle$  be symmetric (respectively antisymmetric) under  $S_{12}$  in the definition of the space  $\widetilde{\mathcal{H}}^n_{0,|\mathbf{s}\rangle}$  (respectively  $\widehat{\mathcal{H}}^n_{0,|\mathbf{s}\rangle}$ ) is no real restriction, since the antisymmetric (respectively symmetric) part of  $|\mathbf{s}\rangle$  does not contribute to the state  $\widetilde{\Phi}^{(2)}$  (respectively  $\widehat{\Phi}^{(3)}$ ).

By the previous lemma, we can consider without loss of generality eigenfunctions of  $\overline{H}_0$  of the form

$$\Phi = f\Phi^{(0)} + g\Phi^{(1)} + h\Phi^{(2)} + \tilde{h}\widetilde{\Phi}^{(2)} + w\widehat{\Phi}^{(3)}, \qquad \deg \Phi = k, \qquad (3.54)$$

where the spin functions  $\Phi^{(k)}$ ,  $\widetilde{\Phi}^{(2)}$  and  $\widehat{\Phi}^{(3)}$  are all built from the same spin state  $|\mathbf{s}\rangle$ . Note that we can assume that  $\widetilde{h}w = 0$ , and that the spin state  $|\mathbf{s}\rangle$  is symmetric under  $S_{12}$  and belongs to  $\Sigma'$  if  $\widetilde{h} \neq 0$ , whereas it is antisymmetric under  $S_{12}$  if  $w \neq 0$ .

Using Eqs. (3.23), (3.24), (3.33), (3.52) and (3.53), it is straightforward to show that the eigenvalue equation  $\overline{H}_0\Phi = (E_0 + 2k\omega)\Phi$  is equivalent to

the system

$$[-T_0 + 2\omega(J^0 + 3 - k)]w - 12w_2 = 0, (3.55a)$$

$$[-T_0 + 2\omega(J^0 + 2 - k)]\tilde{h} - 8\tilde{h}_2 = 0, \qquad (3.55b)$$

$$[-T_0 + 2\omega(J^0 + 2 - k)]h - 8h_2 = 6g_3 + 4w_1, \qquad (3.55c)$$

$$[-T_0 + 2\omega(J^0 + 1 - k)]g - 4g_2 = 4(h_1 + \tilde{h}_1) - 4(2a + 1)w, \qquad (3.55d)$$

$$[-T_0 + 2\omega(J^0 - k)]f = 2(g_1 + (2a+1)h - \frac{2a}{N-1}\tilde{h}).$$
(3.55e)

Since f and g are linear in  $\sigma_3$ , we can write

$$f = p + \sigma_3 q, \qquad g = u + \sigma_3 v, \tag{3.56}$$

where p, q, u and v are polynomials in  $\sigma_1$  and  $\sigma_2$ . Taking into account that the action of  $T_0$  on scalar symmetric functions is given by the RHS of Eq. (3.12) with  $\epsilon = 0$ , and using Eqs. (3.11a), (3.14) and (3.34a), we finally obtain the following linear system of PDEs:

$$[L_0 - 2\omega(k-3)]w - 12w_2 = 0, (3.57a)$$

$$[L_0 - 2\omega(k-2)]\tilde{h} - 8\tilde{h}_2 = 0, \qquad (3.57b)$$

$$[L_0 - 2\omega(k-2)]h - 8h_2 = 6v + 4w_1, \qquad (3.57c)$$

$$[L_0 - 2\omega(k-1)]u - 4u_2 = 4h_1 + 4\tilde{h}_1 + 6\sigma_2 v_1$$

$$+6(2a+1)\sigma_1 v - 4(2a+1)w$$
, (3.57d)

$$[L_0 - 2\omega(k-4)]v - 16v_2 = 0, (3.57e)$$

$$(L_0 - 2\omega k)p = 2u_1 + 2(2a+1)h - \frac{4a}{N-1}\tilde{h} + 6\sigma_2 q_1 + 6(2a+1)\sigma_1 q,$$
(3.57f)

$$[L_0 - 2\omega(k-3)]q - 12q_2 = 2v_1, \qquad (3.57g)$$

where

$$L_0 = -\left(N\partial_{\sigma_1}^2 + 4\sigma_1\partial_{\sigma_1}\partial_{\sigma_2} + 4\sigma_2\partial_{\sigma_2}^2 + 2(2a+1)N\partial_{\sigma_2}\right) + 2\omega(\sigma_1\partial_{\sigma_1} + 2\sigma_2\partial_{\sigma_2}). \quad (3.58)$$

As a consequence of the general discussion of the previous Section, the latter system is guaranteed to possess polynomials solutions. In fact, these polynomial solutions can be expressed in closed form in terms of generalized Laguerre polynomials  $L^{\lambda}_{\nu}$  and Jacobi polynomials  $P^{(\gamma,\delta)}_{\nu}$ .

Theorem 3.17. Let

$$\alpha = N\left(a + \frac{1}{2}\right) - \frac{3}{2}, \qquad \beta \equiv \beta(m) = 1 - m - N\left(a + \frac{1}{2}\right), \qquad t = \frac{2r^2}{N\bar{x}^2} - 1$$

where  $\bar{x} = \frac{1}{N} \sum_i x_i$  is the center of mass coordinate. The Hamiltonian  $H_0$  possesses the following families of spin eigenfunctions with eigenvalue  $E_{lm} = E_0 + 2\omega(2l+m)$ , with  $l \geqslant 0$  and m as indicated in each case:

Here

$$\Phi^{(k)} = \Lambda(x_1^k | \mathbf{s} \rangle), \qquad \widetilde{\Phi}^{(2)} = \Lambda(x_1 x_2 | \mathbf{s} \rangle), \qquad \widehat{\Phi}^{(3)} = \Lambda(x_1 x_2 (x_1 - x_2) | \mathbf{s} \rangle),$$

where the spin state  $|\mathbf{s}\rangle$  is symmetric under  $S_{12}$  and belongs to  $\Sigma'$  for the eigenfunction  $\widetilde{\Psi}_{lm}^{(2)}$ , and is antisymmetric under  $S_{12}$  for the eigenfunction  $\widehat{\Psi}_{lm}^{(3)}$ . The functions  $\varphi_m$ ,  $\varphi_m$  and  $\chi_m$  are polynomials given explicitly by

$$\begin{split} \varphi_m &= \frac{m+2\alpha+2}{m-1} P_{\frac{m}{2}}^{(\alpha+2,\beta-2)} - P_{\frac{m}{2}-1}^{(\alpha+3,\beta-1)} - \frac{4\alpha+7}{m-1} P_{\frac{m}{2}-1}^{(\alpha+2,\beta-1)} + \frac{1}{3} P_{\frac{m}{2}-2}^{(\alpha+3,\beta)} \,, \\ \phi_m &= P_{\frac{m}{2}-1}^{(\alpha+4,\beta-1)} - 2 P_{\frac{m}{2}-1}^{(\alpha+3,\beta-1)} - \frac{m+2\alpha+3}{(m-1)(m-3)} P_{\frac{m}{2}-1}^{(\alpha+2,\beta-1)} \\ &\qquad \qquad - \frac{1}{3} P_{\frac{m}{2}-2}^{(\alpha+4,\beta)} + \frac{m+2\alpha-1}{m-3} P_{\frac{m}{2}-2}^{(\alpha+3,\beta)} \,, \\ \chi_m &= \frac{3m+2\alpha}{(m-1)(m-3)} P_{\frac{m}{2}-1}^{(\alpha+2,\beta-1)} + \frac{2m-7}{m-3} P_{\frac{m}{2}-1}^{(\alpha+3,\beta-1)} - P_{\frac{m}{2}-1}^{(\alpha+4,\beta-1)} \\ &\qquad \qquad - \frac{m+2\alpha+2}{(m-1)(m-3)} P_{\frac{m}{2}-2}^{(\alpha+2,\beta)} - \frac{m+2\alpha}{m-3} P_{\frac{m}{2}-2}^{(\alpha+3,\beta)} + \frac{1}{3} P_{\frac{m}{2}-2}^{(\alpha+4,\beta)} \,, \end{split}$$

for even m, and

$$\begin{split} \varphi_m &= 2P_{\frac{m-1}{2}}^{(\alpha+2,\beta-1)} - P_{\frac{m-1}{2}}^{(\alpha+3,\beta-1)} + \frac{1}{3} P_{\frac{m-3}{2}}^{(\alpha+3,\beta)} + \frac{m+2\alpha+2}{m(m-2)} P_{\frac{m-3}{2}}^{(\alpha+1,\beta)} \\ &\quad - \frac{m+2\alpha+2}{m-2} P_{\frac{m-3}{2}}^{(\alpha+2,\beta)} \,, \\ \phi_m &= P_{\frac{m-3}{2}}^{(\alpha+4,\beta-1)} - \frac{2m-5}{m-2} P_{\frac{m-3}{2}}^{(\alpha+3,\beta)} - \frac{1}{3} P_{\frac{m-5}{2}}^{(\alpha+4,\beta)} + \frac{m+2\alpha-1}{m-2} P_{\frac{m-5}{2}}^{(\alpha+3,\beta)} \,, \\ \chi_m &= \frac{2m-3}{m(m-2)} P_{\frac{m-3}{2}}^{(\alpha+2,\beta-1)} + \frac{2(m-3)}{m-2} P_{\frac{m-3}{2}}^{(\alpha+3,\beta-1)} - P_{\frac{m-3}{2}}^{(\alpha+4,\beta-1)} \\ &\quad - \frac{m+2\alpha+1}{m(m-2)} P_{\frac{m-5}{2}}^{(\alpha+2,\beta)} - \frac{m+2\alpha}{m-2} P_{\frac{m-5}{2}}^{(\alpha+3,\beta)} + \frac{1}{3} P_{\frac{m-3}{2}}^{(\alpha+4,\beta)} \,, \end{split}$$

for odd m. For every n = 0, 1, ..., the above eigenfunctions with  $2l + m \le n$  span the whole  $H_0$ -invariant space  $\mu \mathcal{H}_0^n$ .

Proof. Recall, to begin with, that the algebraic eigenfunctions of  $H_0$  are of the form  $\Psi = \mu \Phi$ , with  $\mu$  given in Table 3.1 and  $\Phi$  an eigenfunction of  $\overline{H}_0$  of the form (3.54)-(3.56). In order to determine  $\Phi$ , we must find the most general polynomial solution of the inhomogeneous linear system (3.57). From the structure of this system it follows that there are seven types of independent solutions, characterized by the vanishing of certain subsets of the unknown functions  $p, q, u, v, h, \tilde{h}, w$ . These types are listed in Table 3.2, where in the last column we have indicated the eigenfunction of  $H_0$  obtained from each case. We shall present here in detail the solution of the system (3.57) for the case  $q = v = h = \tilde{h} = w = 0$  and  $u \neq 0$ , which yields the eigenfunctions of

Conditions	Eigenfunction
$q = u = v = h = \tilde{h} = w = 0,  p \neq 0$	$\Psi_{lm}^{(0)}$
$u = v = h = \tilde{h} = w = 0,  q \neq 0$	$\Psi_{lm}^{(3)}$
$q = v = h = \tilde{h} = w = 0,  u \neq 0$	$\Psi_{lm}^{(1)}$
$q = v = \tilde{h} = w = 0,  h \neq 0$	$\Psi_{lm}^{(2)}$
$q = v = h = w = 0,  \tilde{h} \neq 0$	$\widetilde{\Psi}_{lm}^{(2)}$
$q = v = \tilde{h} = 0,  w \neq 0$	$\widehat{\Psi}_{lm}^{(3)}$
$\tilde{h} = w = 0,  v \neq 0$	$\Psi_{lm}^{(4)}$

Table 3.2: The seven types of solutions of the system (3.57) and their corresponding eigenfunctions.

the form  $\Psi_{lm}^{(1)}$  (the procedure for the other cases is essentially the same). In this case the system (3.57) reduces to

$$[L_0 - 2\omega(k-1)]u - 4u_2 = 0, \qquad (L_0 - 2\omega k)p = 2u_1.$$
 (3.59)

Let us begin with the homogeneous equation for u. We shall look for polynomial solutions of this equation of the form  $u = Q(\sigma_1, \sigma_2)R(\sigma_2)$ , where Q is a homogeneous polynomial of degree m-1 in  $\mathbf{z}$  and R is a polynomial of degree l in  $\sigma_2$ , so that  $k = \deg \Phi = 2l + m$  by Eq. (3.54). From Eq. (3.58) and the homogeneity of Q we have

$$L_0(QR) = (L_0Q)R + Q(L_0R) - 4\sigma_1Q_1R_2 - 8\sigma_2Q_2R_2$$
  
=  $(L_0Q)R + Q(L_0 - 4(m-1)\partial_{\sigma_2})R$ .

Hence the equation for u can be written as

$$R(\widehat{L}_0 - 4\partial_{\sigma_2})Q = Q(-L_0 + 4m\partial_{\sigma_2} + 4l\omega)R,$$

where  $\widehat{L}_0 = L_0|_{\omega=0}$ . Since  $(\widehat{L}_0 - 4\partial_{\sigma_2})Q$  is a homogeneous polynomial of degree m-3 in **z**, both sides of the latter equation must vanish separately. We are thus led to the following decoupled equations for Q and R:

$$\left(\widehat{L}_0 - 4\partial_{\sigma_2}\right)Q = 0\,, (3.60)$$

$$(-L_0 + 4m\partial_{\sigma_2} + 4l\omega)R = 0.$$
(3.61)

In terms of the variable  $\rho = \omega \sigma_2$ , Eq. (3.61) can be written as

$$4\omega \mathcal{L}_l^{-\beta}(R) = 0\,,$$

where

$$\mathcal{L}^{\lambda}_{\nu} = \rho \partial_{\rho}^{2} + (\lambda + 1 - \rho) \partial_{\rho} + \nu \tag{3.62}$$

is the generalized Laguerre operator. Hence R is proportional to the generalized Laguerre polynomial  $L_l^{-\beta}(\omega\sigma_2)$ . On the other hand, we can write  $Q=\sigma_1^{m-1}P(t)$  where P is a polynomial in the homogeneous variable  $t=\frac{2N\sigma_2}{\sigma_1^2}-1$ . With this substitution, Eq. (3.60) becomes

$$4N\sigma_1^{m-3}\mathcal{J}_{\lfloor\frac{m-1}{2}\rfloor}^{(\alpha+1,\beta)}(P)=0\,,$$

where the Jacobi operator  $\mathcal{J}_{\nu}^{(\gamma,\delta)}$  is given by

$$\mathcal{J}_{\nu}^{(\gamma,\delta)} = (1-t^2)\partial_t^2 + \left[\delta - \gamma - (\gamma + \delta + 2)t\right]\partial_t + \nu(\nu + \gamma + \delta + 1).$$

Thus P(t) is proportional to the Jacobi polynomial  $P_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)}(t)$ , so that we can take

$$u = \sigma_1^{m-1} L_l^{-\beta}(\omega \sigma_2) P_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)}(t).$$
 (3.63)

We must next find a particular solution of the inhomogeneous equation for p in (3.59), since the general solution of the corresponding homogeneous equation yields an eigenfunction of the simpler type  $\Psi_{lm}^{(0)}$ . Since

$$u_1 = \sigma_1^{m-2} L_l^{-\beta}(\omega \sigma_2) \left[ (m-1) P_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)}(t) - 2(t+1) \dot{P}_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)}(t) \right]$$

(where the dot denotes derivative with respect to t), we make the ansatz  $p = \overline{Q}(\sigma_1, \sigma_2)\overline{R}(\sigma_2)$ , where  $\overline{Q}$  is a homogeneous polynomial of degree m in  $\mathbf{z}$  and  $\overline{R}$  is a polynomial of degree l in  $\sigma_2$ . Substituting this ansatz into the second equation in (3.59) and proceeding as before we immediately obtain

$$\overline{R}(\widehat{L}_0\overline{Q}) + \overline{Q}(L_0 - 4m\partial_{\sigma_2} - 4l\omega)\overline{R} = 2u_1.$$

If we set  $\overline{R} = L_l^{-\beta}(\omega \sigma_2)$  the second term of the LHS vanishes, and canceling the common factor  $L_l^{-\beta}(\omega \sigma_2)$  we are left with the following equation for  $\overline{Q}$ :

$$\widehat{L}_0 \overline{Q} = 2\sigma_1^{m-2} \left[ (m-1) P_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)}(t) - 2(t+1) \dot{P}_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)}(t) \right].$$

The form of the RHS of this equation suggests the ansatz  $\overline{Q} = \sigma_1^m \overline{P}(t)$ , with  $\overline{P}$  a polynomial in the variable t. The previous equation then yields

$$\mathcal{J}_{\lfloor \frac{m}{2} \rfloor}^{(\alpha,\beta)}(\overline{P}) = \frac{1}{2N} \left[ (m-1) P_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)}(t) - 2(t+1) \dot{P}_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)}(t) \right]. \tag{3.64}$$

From the definition of the Jacobi operator we easily obtain

$$\mathcal{J}_{\lfloor \frac{m}{2} \rfloor}^{(\alpha,\beta)} = \mathcal{J}_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)} + (1+t) \,\partial_t - \frac{1}{2} (m-1) \,,$$

which implies that  $\overline{P}=-\frac{1}{N}\,P_{\lfloor\frac{m-1}{2}\rfloor}^{(\alpha+1,\beta)}$  is a particular solution of Eq. (3.64). Hence

$$p = -\frac{1}{N} \sigma_1^m L_l^{-\beta}(\omega \sigma_2) P_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)}(t)$$
(3.65)

is a particular solution of the inhomogeneous equation in (3.59). We have thus shown that  $\Phi = p \Phi^{(0)} + u \Phi^{(1)}$ , with  $\underline{u}$  and p respectively given by Eqs. (3.63) and (3.65), is an eigenfunction of  $\overline{H}_0$  with eigenvalue  $E_0 + 2\omega(2l + m)$ . Multiplying  $\Phi$  by the gauge factor  $\mu$  we obtain the eigenfunction  $\Psi_{lm}^{(1)}$  of  $H_0$  in the statement.

It remains to show that the states  $\Psi_{lm}^{(k)}$   $(k=0,\ldots,4)$ ,  $\widetilde{\Psi}_{lm}^{(2)}$  and  $\widehat{\Psi}_{lm}^{(3)}$  with  $2l+m\leqslant n$  generate the spaces (3.48)–(3.50). Consider first the "monomials" of the form  $\mu\sigma_1^m\sigma_2^l\Phi^{(0)}$ , which belong to  $\mu\overline{\mathcal{H}}_{0,|\mathbf{s}\rangle}^n$  if  $2l+m\leqslant n$ . We can order such monomials as follows: we say that  $\mu\sigma_1^{m'}\sigma_2^l\Phi^{(0)}\prec\mu\sigma_1^m\sigma_2^l\Phi^{(0)}$  if 2l'+m'<2l+m, or 2l'+m'=2l+m and m'< m. From the expansion [96, Eq. 8.962.1]

$$P_{\nu}^{(\gamma,\delta)}(t) = \frac{1}{\nu!} \sum_{k=0}^{\nu} \frac{1}{2^k k!} (-\nu)_k (\gamma + \delta + \nu + 1)_k (\gamma + k + 1)_{\nu-k} (1 - t)^k,$$

where  $(x)_k$  is the Pochhammer symbol

$$(x)_k = x(x+1)\cdots(x+k-1),$$

it follows that  $P_{\nu}^{(\gamma,\delta)}(0) > 0$  provided that  $\gamma + 1 > 0$  and  $\gamma + \delta + 2\nu < 0$ . In particular,  $P_{\lfloor \frac{m}{2} \rfloor}^{(\alpha,\beta)}(0) > 0$  since

$$\alpha + 1 = N\left(a + \frac{1}{2}\right) - \frac{1}{2} > N - \frac{1}{2} > 0 , \quad \alpha + \beta + 2\left\lfloor \frac{m}{2} \right\rfloor = 2\left\lfloor \frac{m}{2} \right\rfloor - m - \frac{1}{2} \leqslant -\frac{1}{2} < 0 .$$

Hence we can write

$$\Psi_{lm}^{(0)} = \mu \Phi^{(0)} \left( c_{lm} \sigma_1^m \sigma_2^l + \text{l.o.t.} \right),$$

where  $c_{lm} \neq 0$ , so that

$$\langle \Psi_{lm}^{(0)} : 2l + m \leqslant n \rangle = \langle \mu \sigma_1^m \sigma_2^l \Phi^{(0)} : 2l + m \leqslant n \rangle.$$

Likewise, a similar argument shows that for  $m \ge 1$ 

$$\left\langle \mu \bar{x}^{m-1} L_l^{-\beta}(\omega r^2) P_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)}(t) \Phi^{(1)} : 2l + m \leqslant n \right\rangle = \left\langle \mu \sigma_1^{m-1} \sigma_2^l \Phi^{(1)} : 2l + m \leqslant n \right\rangle,$$

and therefore

$$\left< \Psi_{lm}^{(0)}, \, \Psi_{lm}^{(1)} : 2l + m \leqslant n \right> = \left< \mu \sigma_1^m \sigma_2^l \Phi^{(0)}, \, \mu \sigma_1^{m-1} \sigma_2^l \Phi^{(1)} : 2l + m \leqslant n \right>.$$

Proceeding in the same way with the remaining spin eigenfunctions we can finally show that

$$\langle \Psi_{lm}^{(k)} : k = 0, \dots, 4, 2l + m \leqslant n \rangle = \mu \overline{\mathcal{H}}_{0,|\mathbf{s}\rangle}^n,$$

and that

$$\mu \overline{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n} + \left\langle \widetilde{\Psi}_{lm}^{(2)} : 2l + m \leqslant n \right\rangle = \mu \widetilde{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n}, \qquad |\mathbf{s}\rangle \in \Sigma', \quad S_{12}|\mathbf{s}\rangle = |\mathbf{s}\rangle,$$
  
$$\mu \overline{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n} + \left\langle \widehat{\Psi}_{lm}^{(3)} : 2l + m \leqslant n \right\rangle = \mu \widehat{\mathcal{H}}_{0,|\mathbf{s}\rangle}^{n}, \qquad S_{12}|\mathbf{s}\rangle = -|\mathbf{s}\rangle,$$

as claimed.  $\Box$ 

#### Case b

Since  $c_0 = 4\omega$  in this case, reasoning as before we conclude that the algebraic energies are the numbers

$$E_k = E_0 + 4k\omega, \qquad k = 0, 1, \dots,$$

where k is the degree in  $\mathbf{z}$  of the corresponding eigenfunctions of  $\overline{H}_1$ . We shall see below that these eigenfunctions can be written in terms of generalized Laguerre polynomials. To this end, we begin by identifying certain subspaces of  $\mathcal{H}_1^n$  invariant under  $\overline{H}_1$ .

**Lemma 3.18.** For any given spin state  $|\mathbf{s}\rangle \in \Sigma$ , the operator  $\overline{H}_1$  preserves the subspace

$$\mathcal{H}_{1,|\mathbf{s}\rangle}^{n} = \langle f(\sigma_1, \sigma_2) \, \Phi^{(0)}, \, g(\sigma_1) \, \Phi^{(1)} : f_{22} = 0 \rangle \subset \mathcal{H}_1^{n},$$
 (3.66)

where f and g are polynomials of degrees at most n and n-1 in  $\mathbf{z}$ , respectively, and  $\Phi^{(k)}$  is given by (3.22).

*Proof.* The statement follows from the obvious identity  $T_1(f\Phi^{(0)}) = (T_1f)\Phi^{(0)}$  and Eqs. (3.11b), (3.12), (3.16), (3.30), (3.33) and (3.35).

By the previous lemma we can assume that the eigenfunctions of  $\overline{H}_1$  in  $\mathcal{H}^n_{1,|\mathbf{s}\rangle}$  are of the form

$$\Phi = f\Phi^{(0)} + g\Phi^{(1)}, \qquad \deg \Phi = k \leqslant n. \tag{3.67}$$

From Eqs. (3.30), (3.33) and (3.35) it easily follows that the eigenvalue equation  $\overline{H}_1\Phi = (E_0 + 4k\omega)\Phi$  can be cast into the system

$$\left[ -T_1 + \omega(J^0 + 1 - k) - \left(b + \frac{1}{2}\right)J^- \right]g - 2g_1 = 0,$$
 (3.68a)

$$\left[ -T_1 + \omega(J^0 - k) - \left(b + \frac{1}{2}\right)J^- \right] f = \left(2a + b + \frac{1}{2}\right)g. \tag{3.68b}$$

Since f is linear in  $\sigma_2$  (cf. Eq. (3.66)), we can write

$$f = p + \sigma_2 q, \tag{3.69}$$

where p and q are polynomials in  $\sigma_1$ . Using Eqs. (3.11b), (3.12), (3.16), (3.34b) and (3.36) we can easily rewrite the system (3.68) as follows:

$$[L_1 - \omega(k-1)]g - 2g_1 = 0, \qquad (3.70a)$$

$$[L_1 - \omega(k-2)]q - 4q_1 = 0,$$
 (3.70b)

$$(L_1 - \omega k)p = (2a + b + \frac{1}{2})g + 2(4a + b + \frac{3}{2})\sigma_1 q,$$
 (3.70c)

where

$$L_1 = -\sigma_1 \partial_{\sigma_1}^2 + \left[\omega \sigma_1 - \left(2a + b + \frac{1}{2}\right)N\right] \partial_{\sigma_1}. \tag{3.71}$$

The last step is to construct the polynomials solutions of the system (3.70), which can be expressed in terms of generalized Laguerre polynomials, according to the following theorem.

**Theorem 3.19.** The Hamiltonian  $H_1$  possesses the following families of spin eigenfunctions with eigenvalue  $E_k = E_0 + 4k\omega$ :

$$\begin{split} &\Psi_k^{(0)} = \mu L_k^{\alpha-1}(\omega r^2) \, \Phi^{(0)} \,, \qquad k \geqslant 0 \,, \\ &\Psi_k^{(1)} = \mu L_{k-1}^{\alpha+1}(\omega r^2) \big[ N \Phi^{(1)} - r^2 \Phi^{(0)} \big] \,, \qquad k \geqslant 1 \,, \\ &\Psi_k^{(2)} = \mu L_{k-2}^{\alpha+3}(\omega r^2) \Big[ N(\alpha+1) \sum_i x_i^4 - \beta r^4 \Big] \, \Phi^{(0)} \,, \qquad k \geqslant 2 \,, \end{split}$$

where  $\alpha = N(2a+b+\frac{1}{2})$ ,  $\beta = N(4a+b+\frac{3}{2})$ , and  $\Phi^{(j)} = \Lambda(x_1^{2j}|\mathbf{s}\rangle)$ , with j = 0, 1 and  $|\mathbf{s}\rangle \in \Sigma$ . For each  $|\mathbf{s}\rangle \in \Sigma$  and  $n = 0, 1, \ldots$ , the above eigenfunctions with  $k \leq n$  span the whole  $H_1$ -invariant space  $\mu \mathcal{H}_{1,|\mathbf{s}\rangle}^n$ .

*Proof.* As in the previous case, the algebraic eigenfunctions of  $H_1$  are of the form  $\Psi = \mu \Phi$ , where  $\mu$  is given in Table 3.1 and  $\Phi$  is an eigenfunction of  $\overline{H}_1$  of the form (3.67)-(3.69). The functions p, q and g are polynomials in  $\sigma_1$  determined by the system (3.70), which in terms of the variable  $t = \omega \sigma_1 \equiv \omega r^2$  can be written as

$$\mathcal{L}_{k-1}^{\alpha+1}g = \mathcal{L}_{k-2}^{\alpha+3}q = 0, \qquad \mathcal{L}_{k}^{\alpha-1}p = -\frac{\alpha}{N\omega}g - \frac{2\beta}{N\omega^2}tq,$$
 (3.72)

where  $\mathcal{L}^{\lambda}_{\nu}$  is the generalized Laguerre operator (cf. Eq. (3.62)). The general polynomial solutions of the first two equations in (3.72) are respectively given by

$$g = c_1 L_{k-1}^{\alpha+1}(t), \qquad q = c_2 L_{k-2}^{\alpha+3}(t).$$
 (3.73)

On the other hand, from the elementary identity

$$\mathcal{L}^{\lambda}_{\nu}\Big(t^lL^{\lambda+2l}_{\nu-l}(t)\Big) = l(l+\lambda)t^{l-1}L^{\lambda+2l}_{\nu-l}(t)\,,$$

it follows that the general polynomial solution of the third equation in (3.72) is given by

$$p = c_0 L_k^{\alpha - 1}(t) - \frac{c_1}{N\omega} t L_{k-1}^{\alpha + 1}(t) - \frac{c_2 \beta}{N\omega^2(\alpha + 1)} t^2 L_{k-2}^{\alpha + 3}(t).$$
 (3.74)

Equations (3.73) and (3.74) immediately yield the formulas of the eigenfunctions in this case. The last assertion in the statement of the theorem follows from the fact that the functions p, q and g in Eqs. (3.73) and (3.74) are the most general polynomial solution of the system (3.72).

Remark 3.20. It should be noted that for  $\omega = 0$  the potentials (3.2a) and (3.2b) scale as  $r^{-2}$  under dilations of the coordinates (as is the case for the original Calogero model). The standard argument used in the solution of the Calogero model shows that there is a basis of eigenfunctions of these models of the form  $\mu(\mathbf{x})L_{\nu}^{\lambda}(\omega r^2)F(\mathbf{x})$ , where F is a homogeneous spin-valued function. The eigenfunctions presented in Theorems 3.17 and 3.19 are indeed of this form.

#### Case c

In this case the number of independent algebraic eigenfunctions is essentially finite. We shall take, for definiteness, the plus sign in the change of variable listed in Table 3.1 (it will be apparent from the discussion that follows that the minus sign does not yield additional solutions).

Let us first note that the potential for this model is translationally invariant, so that the total momentum operator  $P = -i \sum_k \partial_{x_k}$  commutes with the Hamiltonian  $H_2$ . Hence the eigenfunctions of  $H_2$  can be assumed to have well-defined total momentum. Equivalently, since

$$\mu^{-1}P\mu\Big|_{x_k = -\frac{i}{2}\log z_k} = 2J^0,\tag{3.75}$$

the eigenfunctions of  $\overline{H}_2$  can be assumed to be homogeneous in  $\mathbf{z}$ . Let  $\Phi$  be a homogeneous eigenfunction of  $\overline{H}_2$  of degree k and eigenvalue E, so that  $\Psi = \mu \Phi$  is an eigenfunction of  $H_2$  with total momentum 2k (cf. Eq. (3.75)) and energy E. By Eq. (3.75), the function  $\tau_N^{\lambda}\Psi$  clearly has total momentum  $2(k+N\lambda)$ . In fact, the following lemma implies that  $\tau_N^{\lambda}\Psi$  is also an eigenfunction of  $H_2$  with a suitably boosted energy:

**Lemma 3.21.** Let  $\Phi$  be a homogeneous eigenfunction of  $\overline{H}_2$  of degree k and eigenvalue E. Then  $\tau_N^{\lambda}\Phi$  is an eigenfunction of  $\overline{H}_2$  with eigenvalue  $E + 8k\lambda + 4N\lambda^2$ .

*Proof.* From the identity

$$\sum_{i} \frac{1}{z_{i} - z_{i+1}} \left( z_{i}^{2} \partial_{i} - z_{i+1}^{2} \partial_{i+1} \right) = J^{0} + \frac{1}{2} \sum_{i} \frac{z_{i} + z_{i+1}}{z_{i} - z_{i+1}} \left( D_{i} - D_{i+1} \right),$$

where  $D_i = z_i \partial_i$ , we immediately obtain the following expression for the gauge Hamiltonian  $\overline{H}_2$ :

$$\frac{1}{4} (\overline{H}_2 - E_0) = \sum_{i} D_i^2 + a \sum_{i} \frac{z_i + z_{i+1}}{z_i - z_{i+1}} (D_i - D_{i+1}) - 2a \sum_{i} \frac{z_i z_{i+1}}{(z_i - z_{i+1})^2} (1 - K_{i,i+1}). \quad (3.76)$$

Since  $\tau_N^{-\lambda} D_i \tau_N^{\lambda} = D_i + \lambda$  for any real  $\lambda$ , it follows that

$$\tau_N^{-\lambda}\,\overline{H}_2\tau_N^\lambda=\,\overline{H}_2+8\lambda J^0+4N\lambda^2\,.$$

Taking into account that  $J^0\Phi = k\Phi$ , we conclude that

$$\overline{H}_2(\tau_N^{\lambda}\Phi) = (E + 8k\lambda + 4N\lambda^2)(\tau_N^{\lambda}\Phi),$$

as claimed.  $\Box$ 

By the previous discussion, in what follows any two eigenfunctions of  $\overline{H}_2$  that differ by a power of  $\tau_N$  shall be considered equivalent. From Theorem 3.6 and Corollary 3.8 it easily follows that in this case the number of independent algebraic eigenfunctions is finite, up to equivalence. More precisely:

**Lemma 3.22.** Up to equivalence, the algebraic eigenfunctions of  $\overline{H}_2$  can be assumed to belong to a space of the form

$$\mathcal{H}_{2,|\mathbf{s}\rangle} = \langle \sigma_1, \, \tau_{N-1}, \, \sigma_1 \tau_{N-1}, \, \tau_N \rangle \, \Phi^{(0)} + \langle 1, \, \tau_{N-1} \rangle \, \Phi^{(1)} + \langle 1, \, \sigma_1 \rangle \, \tau_N \Phi^{(-1)}$$
 (3.77)

for some spin state  $|\mathbf{s}\rangle$ , where  $\Phi^{(k)}$  is given by (3.22).

*Proof.* Given a spin state  $|\mathbf{s}\rangle$ , the obvious identity

$$T_2(f\Phi^{(0)}) = (T_2f)\Phi^{(0)}$$
 (3.78)

and Eqs. (3.7), (3.31), (3.32), and (3.33) imply that the gauge Hamiltonian  $\overline{H}_2$  preserves the space

$$\mathcal{H}_{2,|\mathbf{s}\rangle}^{n} = \left\langle f\Phi^{(0)}, g\Phi^{(1)}, \tau_N q\Phi^{(-1)} \right\rangle, \tag{3.79}$$

where f, g and q are as in the definition of  $\mathcal{T}_2^n$  in Theorem 3.6. Let  $\Phi \in \mathcal{H}_{2,|\mathbf{s}\rangle}^n$  be an eigenfunction of  $\overline{H}_2$ , which as explained above can be taken as a homogeneous function of  $\mathbf{z}$ . From the conditions satisfied by the functions f, g and q in (3.79) and the homogeneity of  $\Phi$ , it readily follows that  $\Phi \in \mathcal{T}_N^l \mathcal{H}_{2,|\mathbf{s}\rangle}$  for some l, as claimed.

**Theorem 3.23.** The Hamiltonian  $H_2$  possesses the following spin eigenfunctions with zero momentum

$$\Psi_0 = \mu \Phi^{(0)}, \qquad \Psi_{1,2} = \mu \sum_i \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (2(x_i - \bar{x})) |\mathbf{s}_i\rangle,$$

$$\Psi_3 = \mu \left[ \frac{2a}{2a+1} \Phi^{(0)} + \sum_{i \neq j} \cos(2(x_i - x_j)) |\mathbf{s}_j\rangle \right], \quad \Psi_4 = \mu \sum_{i \neq j} \sin(2(x_i - x_j)) |\mathbf{s}_j\rangle,$$

where  $\bar{x}$  is the center of mass coordinate. Their energies are respectively given by

$$E_0$$
,  $E_{1,2} = E_0 + 4\left(2a + 1 - \frac{1}{N}\right)$ ,  $E_{3,4} = E_0 + 8(2a + 1)$ .

Any algebraic eigenfunction with well-defined total momentum is equivalent to a linear combination of the above eigenfunctions.

*Proof.* By Lemma 3.22, in order to compute the algebraic eigenfunctions of  $\overline{H}_2$  it suffices to diagonalize  $\overline{H}_2$  in the spaces  $\mathcal{H}_{2,|\mathbf{s}\rangle}$  given by (3.77). Equations (3.31), (3.32), (3.33) and (3.78), and the fact that  $\overline{H}_2$  preserves the

degree of homogeneity, imply that the following subspaces of  $\mathcal{H}_{2,|\mathbf{s}\rangle}$  are invariant under  $\overline{H}_2$ :

$$\langle \sigma_1 \tau_{N-1}, \tau_N \rangle \Phi^{(0)},$$
 (3.80a)

$$\langle \sigma_1 \Phi^{(0)} \rangle$$
,  $\langle \sigma_1 \Phi^{(0)}, \Phi^{(1)} \rangle$ ,  $\langle \sigma_1 \tau_{N-1} \Phi^{(0)}, \tau_N \Phi^{(0)}, \tau_{N-1} \Phi^{(1)} \rangle$ , (3.80b)

$$\langle \tau_{N-1} \Phi^{(0)} \rangle$$
,  $\langle \tau_{N-1} \Phi^{(0)}, \tau_N \Phi^{(-1)} \rangle$ ,  $\langle \sigma_1 \tau_{N-1} \Phi^{(0)}, \tau_N \Phi^{(0)}, \sigma_1 \tau_N \Phi^{(-1)} \rangle$ . (3.80c)

From Eqs. (3.80) it follows that the alternative change of variables  $z_k = e^{-2ix_k}$  does not yield additional eigenfunctions of  $H_2$ . Indeed, the latter change corresponds to the mapping  $z_k \mapsto 1/z_k$ , which up to equivalence leaves the subspace (3.80a) invariant and exchanges each subspace in (3.80b) with the corresponding one in (3.80c). For this reason, we can safely ignore the subspaces (3.80c) in the computation that follows, provided that we add to the eigenfunctions of  $\overline{H}_2$  obtained from the subspaces (3.80b) their images under the mapping  $z_k \mapsto 1/z_k$ .

For the subspaces (3.80a)-(3.80b), using Eqs. (3.7), (3.12), (3.18), (3.31), (3.33), and (3.34c) we easily obtain the following eigenfunctions of  $\overline{H}_2$ :

$$\tau_N \Phi^{(0)}, \qquad E = E_0 + 4N, \qquad (3.81a)$$

$$\Phi^{(1)}, E = E_0 + 4(2a+1), (3.81b)$$

$$\tau_{N-1}\Phi^{(1)} - \frac{\tau_N}{2a+1}\Phi^{(0)}, \qquad E = E_0 + 4(N+4a+2).$$
(3.81c)

We have omitted the two additional eigenfunctions

$$\sigma_1 \Phi^{(0)}, \qquad \left(\sigma_1 \tau_{N-1} - \frac{N \tau_N}{2a+1}\right) \Phi^{(0)}$$

from the above list, since they are respectively obtained from (3.81b) and (3.81c) when the spin state  $|\mathbf{s}\rangle$  is symmetric. The eigenfunctions (3.81) are equivalent to the following "zero momentum" eigenfunctions:

$$\Phi^{(0)}, E = E_0, (3.82a)$$

$$\tau_N^{-1/N} \Phi^{(1)}, \qquad E = E_0 + 4\left(2a + 1 - \frac{1}{N}\right), \qquad (3.82b)$$

$$\frac{\tau_{N-1}}{\tau_N} \Phi^{(1)} - \frac{1}{2a+1} \Phi^{(0)}, \qquad E = E_0 + 8(2a+1), \tag{3.82c}$$

where the energies have been computed from those in Eqs. (3.81) using Lemma 3.21. The eigenfunctions of  $H_2$  listed in the statement are readily obtained from these spin functions together with the transforms of (3.82b) and (3.82c) under the mapping  $z_k \mapsto 1/z_k$ .

Remark 3.24. If the spin state  $|\mathbf{s}\rangle$  is symmetric, then  $|\mathbf{s}_i\rangle = \Phi^{(0)}/N$  for all i, and one easily obtains from Theorem 3.23 the following eigenfunctions of the scalar Hamiltonian  $H_2^{\text{sc}}$ :

$$\psi_0 = \mu, \ \psi_{1,2} = \mu \sum_i {\cos \brace \sin} (2(x_i - \bar{x})), \ \psi_3 = \mu \left[ \frac{aN}{2a+1} + \sum_{i < j} \cos(2(x_i - x_j)) \right].$$

# Chapter 4

# NN spin chains of HS type

## 4.1 Introduction

As we saw in Chapter 2, one can successfully solve spin chains of HS type by means of Polychronakos's freezing trick, which essentially consists in computing the large coupling constant limit of the quotient of the partition functions of a spin CS model and its scalar counterpart.

In the present chapter we shall extend this technique to the case of QES models of CS type with spin. This extension will prove crucial in the analysis of the spectrum of the novel N-site chain

$$H = \sum_{i} (\xi_i - \xi_{i+1})^{-2} (1 - S_{i,i+1}), \qquad (4.1)$$

with sites  $\boldsymbol{\xi} \in C_0 \subset \mathbb{R}^N$  defined by the algebraic system

$$\xi_i = \frac{1}{\xi_i - \xi_{i-1}} + \frac{1}{\xi_i - \xi_{i+1}}, \qquad 1 \leqslant i \leqslant N.$$
 (4.2)

Here and in what follows, we identify  $\xi_0 \equiv \xi_N$ ,  $\xi_{N+1} \equiv \xi_1$  (and similarly for  $x_i$  and  $S_{i,i+1}$ ), and keep denoting by  $C_0$  the domain (3.3a) and by M the particles' spin. It is not difficult to check that the spin chain (4.1), which we shall simply call the **NN** chain, is related to the NN Hamiltonian  $H_0$  in Eq. (3.2a) along the lines of the freezing trick.

We shall see in this chapter that this spin chain is in some sense intermediate between the Heisenberg chain (short-range, position-independent interactions) and the PF chain (long-range, position-dependent interactions), from which one can derive the Hamiltonian (4.1) by retaining only nearest-neighbors interactions. Moreover, H turns out to be QES in the sense that a

nontrivial proper subset of the spectrum can be explicitly computed for any value of N and M. While QES models have long played a relevant role in the theory of CS systems [179, 180], this is to our best knowledge the first QES phenomenon spotted in the (as has been made clear in this dissertation) closely related field of HS spin chains.

The results in this chapter can be easily modified to deal with the NN chains

$$H_1 = \sum_{i} \frac{\xi_i^2 + \xi_{i+1}^2}{(\xi_i^2 - \xi_{i+1}^2)^2} (1 - S_{i,i+1}),$$

$$H_2 = \sum_{i} \sin^{-2}(\xi_i - \xi_{i+1}) (1 - S_{i,i+1}),$$

where the chain sites are respectively defined by

$$\xi_i = \frac{1}{\xi_i} + \frac{\xi_i}{\xi_i^2 - \xi_{i+1}^2} + \frac{\xi_i}{\xi_i^2 - \xi_{i-1}^2},$$

$$0 = \cot(\xi_i - \xi_{i+1}) + \cot(\xi_i - \xi_{i-1}).$$

These spin chains are obtained from the NN models (3.2b) and (3.2c) respectively by a freezing trick argument. Note, however, that the sites of the second chain turn out to be equally spaced, so  $H_2$  is essentially the Heisenberg chain.

This chapter is organized as follows. In Section 4.2 we analyze the distribution of the chain sites using both exact results and numerical calculations. In Section 4.3 we show the QES nature of the chain by providing several families of eigenvectors, valid for any choice of N and M. The main technical tool is a nontrivial refinement of the freezing trick developed in Chapter 2 which also applies to QES systems. Finally, we discuss some qualitative properties of the spectrum and formulate an intriguing conjecture claiming that an eigenvalue of the chain is algebraic if and only if it is an integer.

The material presented in this chapter is taken from Ref. [63].

### 4.2 The chain sites

We shall begin with a detailed analysis of the distribution of the chain sites on the line in which we shall combine exact results and numerical computations to extract as much information as possible. An easy observation, which stems from the spin chain's connection with the dynamical model (3.2a) that will be developed in the next section, is that the sites are given by the coordinates of a critical point of the function

$$\lambda(\mathbf{x}) = \sum_{i} \log|x_i - x_{i+1}| - \frac{r^2}{2},$$
 (4.3)

which is the scaled logarithm of the ground state function of the NN Hamiltonian (3.2a).

The first questions to ascertain refer to the existence and uniqueness of the critical points of  $\lambda$ . In this direction we have the following result, which is roughly analogous to Proposition (2.20):

**Proposition 4.1.** The function  $\lambda$  has a unique critical point  $\xi$  in  $C_0$ , which is a hyperbolic maximum. Moreover,  $\xi$  satisfies

$$\xi_i = -\xi_{N-i+1}, \qquad i = 1, \dots, N.$$
 (4.4)

Proof. The existence of a maximum of  $\lambda$  in  $C_0$  is clear, since it is continuous in  $C_0$  and tends to  $-\infty$  both on its boundary and as  $r \to \infty$ . Uniqueness and hyperbolicity follow from the fact that the Hessian of  $\lambda$  is negative definite in  $C_0$ . Indeed, by Gerschgorin's theorem [96, 15.814], the eigenvalues of the Hessian of  $\lambda$  at  $\mathbf{x}$  lie in the union of the intervals

$$\left[\frac{\partial^2 \lambda}{\partial x_i^2} - \gamma_i, \frac{\partial^2 \lambda}{\partial x_i^2} + \gamma_i\right], \quad \text{where } \gamma_i = \sum_{j \neq i} \left|\frac{\partial^2 \lambda}{\partial x_i \partial x_j}\right|.$$

Since

$$\frac{\partial^2 \lambda}{\partial x_i^2} = -1 - (x_i - x_{i+1})^{-2} - (x_i - x_{i-1})^{-2},$$

$$\frac{\partial^2 \lambda}{\partial x_i \partial x_{i\pm 1}} = (x_i - x_{i\pm 1})^{-2}, \qquad \frac{\partial^2 \lambda}{\partial x_i \partial x_j} = 0 \text{ if } j \neq i, i \pm 1,$$

we have

$$\frac{\partial^2 \lambda}{\partial x_i^2} + \gamma_i = -1 \,,$$

and thus all the eigenvalues of the Hessian of  $\lambda$  are strictly negative.

Furthermore, the mapping

$$x_i \mapsto -x_{N-i+1}, \qquad i = 1, \dots, N,$$

is a symmetry of  $\lambda$  and maps the domain  $C_0$  into itself. Since  $\xi$  is the unique critical point of  $\lambda$  in  $C_0$ , it must be a fixed point of the above transformation, and thus Eqs. (4.4) follow.

From the previous proposition (and also directly from Eq. (4.2)), it immediately follows that the center of mass of the spins vanishes, i.e.,

$$\sum_{i} \xi_i = 0. \tag{4.5}$$

**Proposition 4.2.**  $\|\xi\|^2 = N$ .

*Proof.* Let us write  $\lambda$  in terms of the variables  $r = ||\mathbf{x}|| \in \mathbb{R}^+$  and  $\mathbf{y} = \mathbf{x}/r \in \mathbb{S}^{N-1}$ , i.e.,

$$\lambda = \sum_{i} \log |y_i - y_{i+1}| + N \log r - \frac{r^2}{2}.$$

Obviously

$$0 = \frac{\partial \lambda}{\partial r}(\boldsymbol{\xi}) = \frac{N}{\|\boldsymbol{\xi}\|} - \|\boldsymbol{\xi}\|,$$

as we wanted to show.

It is convenient to think of the spins as if located on a circle. In this way, the chain (4.1) truly presents only near-neighbors interactions in spite of the fact that the first spin interacts with the last one. Indeed, if the site coordinate  $\xi_i$  is understood as an arc length in a circle of radius

$$r_N = \frac{2\xi_N}{\pi} \,, \tag{4.6}$$

as shown in Fig. 4.1, the spins at the sites  $\xi_1$  and  $\xi_N$  become nearest-neighbors and, moreover, the strength of the interaction is inversely proportional to the squared distance between the spins, measured along the arc.

We saw in Section 1.8 that the site coordinates of the PF chain define the N-th Hermite polynomial. It is natural to wonder whether a similar result holds for the family of monic polynomials

$$p_N(t) = \prod_i (t - \xi_i), \qquad \boldsymbol{\xi} \in C_0 \subset \mathbb{R}^N, \tag{4.7}$$

defined by H, which can be regarded as the closest NN analog of the PF chain. In the following proposition we answer this question negatively:

**Proposition 4.3.** The polynomials (4.7) do not form an orthogonal family.

*Proof.* In order to show that the family  $\{p_N\}$  is not orthogonal, it suffices to prove that the polynomials (4.7) do not satisfy a three-term recursion relation of the form

$$p_{N+1}(t) = t p_N(t) + c_N p_{N-1}(t) ,$$

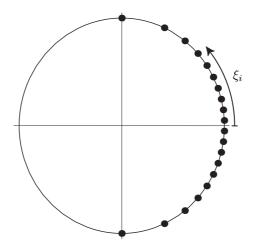


Figure 4.1: Sites of the chain H for N=20 spins.

with  $c_N$  constant. In fact, solving the system (4.2) for N=2,3,4 one immediately obtains

$$p_2(t) = t^2 - 1$$
,  $p_3(t) = t^3 - \frac{3t}{2}$ ,  $p_4(t) = t^4 - 2t^2 + \frac{1}{4}$ .

Thus the previous recursion relation already fails for N=3.

The sites are certainly not equally spaced; in fact, our numerical computations lead to the following

**Claim 4.4.** For  $N \gg 1$ , the site distribution follows the Gaussian law with zero mean and unit variance.

Note that, should the site distribution follow a Gaussian law, its mean would be zero on account of Corollary 4.5. We have solved numerically Eq. (4.2) for the positions of the chain sites for up to N=200 spins, and the results completely support the validity of the above conjecture. More precisely, we claim that the cumulative density of sites

$$\mathcal{F}(x) = N^{-1} \sum_{i} \theta(x - \xi_i), \qquad (4.8)$$

where  $\theta$  is Heaviside's step function, is asymptotically given by

$$F(x) = \frac{1}{2} \left[ 1 + \operatorname{erf}\left(x/\sqrt{2}\right) \right]. \tag{4.9}$$

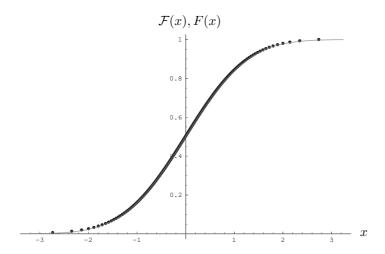


Figure 4.2: Cumulative distribution functions  $\mathcal{F}(x)$  (at its discontinuity points) and F(x) (continuous grey line) for N=150 spins.

The agreement between the functions  $\mathcal{F}$  and F is remarkably good for  $N \gtrsim 100$  (see Fig. 4.2 for the case N=150) and increases steadily with N. For example, the mean square error of the fit for 100, 150 and 200 spins is respectively  $2.6 \times 10^{-5}$ ,  $1.1 \times 10^{-5}$  and  $7.9 \times 10^{-6}$ .

Heuristic proof of Claim 4.4. The fact that for large N the cumulative density of sites is well approximated by the Gaussian law (4.9) can be justified by the following semi-rigorous argument. Let x(t,N) be a smooth function such that  $x(i,N) = \xi_i$  for i = 1, ..., N, and define the rescaled function  $y(s,\epsilon) = x(s/\epsilon,1/\epsilon)$ . By Eq. (4.2), the latter function must satisfy the relation

$$\frac{1}{y(s,\epsilon) - y(s-\epsilon,\epsilon)} + \frac{1}{y(s,\epsilon) - y(s+\epsilon,\epsilon)} = y(s,\epsilon)$$
 (4.10)

for  $\epsilon=1/N\ll 1$  and  $s=\frac{1}{N},\frac{2}{N},\ldots,1$ . Let us now assume that Eq. (4.10) holds for all  $s\in\mathbb{R}$  and all  $\epsilon\ll 1$ . Writing

$$y(s,\epsilon) = \sum_{k=0}^{\infty} y_k(s)\epsilon^k$$
,

and using the expansion

$$y(s,\epsilon) - y(s \pm \epsilon, \epsilon) = \mp y_0'(s)\epsilon - \left(\frac{y_0''(s)}{2} \pm y_1'(s)\right)\epsilon^2 + O(\epsilon^3)$$

the leading term in Eq. (4.10) yields the differential equation

$$y_0'' = y_0 y_0'^2.$$

The general solution of this equation is implicitly given by

$$s = c_0 + c_1 \operatorname{erf}(y_0(s)/\sqrt{2}).$$
 (4.11)

Hence, up to terms of order  $\epsilon = 1/N$ , the cumulative distribution function of the chain sites (normalized to unity) is approximated by the continuous function

$$F(x) = c_0 + c_1 \operatorname{erf}(x/\sqrt{2}).$$

The normalization conditions  $F(-\infty) = 0$  and  $F(\infty) = 1$  imply that  $c_0 = c_1 = 1/2$ , and thus the empiric law (4.9) is recovered.

From Eq. (4.11) (with  $c_0 = c_1 = 1/2$ ) it follows that the site  $\xi_k$  can be determined up to terms of order 1/N by the formula

$$\xi_k \simeq \sqrt{2} \operatorname{erf}^{-1} \left( \frac{2k - N}{N} \right).$$
 (4.12)

Note that, were the sites  $\xi_k$  exactly given by the previous formula, they would satisfy the identity

$$\operatorname{erf}(\xi_k/\sqrt{2}) + \operatorname{erf}(\xi_{N-k+1}/\sqrt{2}) = \frac{1}{N},$$

which is clearly inconsistent with the exact relation (4.4). However, the slightly modified formula

$$\xi_k \simeq \sqrt{2} \operatorname{erf}^{-1} \left( \frac{2k - N - 1}{N} \right) \tag{4.13}$$

differs from (4.12) by a term of order 1/N and is fully consistent with the relation (4.4). Although both (4.12) and (4.13) provide an excellent approximation to the chain sites for large N, the latter equation is always more accurate than the former, and can be used to estimate  $\xi_k$  with remarkable precision even for relatively low values of N, cf. Fig. 4.3.

In the rest of this section we shall analyze several relevant features of the site distribution using the above heuristic approach. It is certainly of interest to determine whether the position of the last spin tends to infinity as  $N \to \infty$ , since according to our interpretation of the chain's geometry the number  $2\xi_N/\pi$  is the radius of the circle on which the spins lie. According to

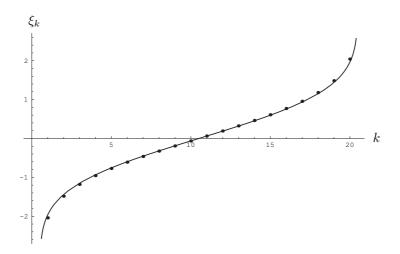


Figure 4.3: Sites coordinates  $\xi_k$  and their continuous approximation (4.13) for N = 20 spins.

Eq. (4.13), for large N the last spin's coordinate  $\xi_N$  is approximately given by

$$\xi_N \simeq \sqrt{2} \operatorname{erf}^{-1} \left( 1 - \frac{1}{N} \right), \tag{4.14}$$

so that  $\xi_N$  should diverge as  $N \to \infty$ . Of course, this assertion should be taken with some caution, since in Eq. (4.13) the argument of the inverse error function is correct only up to terms of order 1/N. In order to check the correctness of the approximation (4.14), we use the asymptotic expansion of  $\operatorname{erf}^{-1}(u)$  for  $u \to 1$  in Ref. [26] to replace (4.14) by the simpler formula

$$\xi_N \simeq \sqrt{2\eta - \log \eta} \,, \tag{4.15}$$

where

$$\eta = \log\left(\frac{N}{\sqrt{\pi}}\right).$$

As can be seen in Fig. 4.4, the approximate formula (4.15) qualitatively reproduces the growth of  $\xi_N$  when N ranges from 100 to 250. A greater accuracy can be achieved by introducing an adjustable parameter in Eq. (4.14) through the replacement  $1/N \to \alpha/N$ , so that in Eq. (4.15)  $\eta$  becomes

$$\eta = \log\left(\frac{N}{\alpha\sqrt{\pi}}\right). \tag{4.16}$$

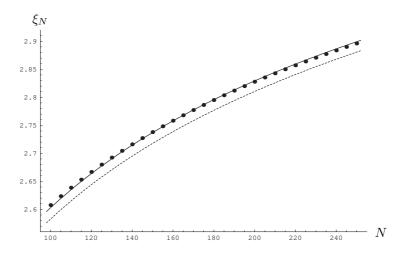


Figure 4.4: Position of the last spin  $\xi_N$  for  $N = 100, 105, \dots, 250$  and its continuous approximation (4.15)-(4.16) for  $\alpha = 0.94$  (solid line) and  $\alpha = 1$  (dashed line).

In Fig. 4.4, we have also plotted the law (4.15)-(4.16) with the optimal value  $\alpha = 0.94$ , which is in excellent agreement with the numerical values of  $\xi_N$  for  $N = 100, 105, \ldots, 250$ .

The last property of the spin chain (4.1) that we shall analyze in this section is the dependence of the coupling between neighboring spins on their mean coordinate. Calling

$$h_k = (\xi_k - \xi_{k+1})^{-2}, \qquad \overline{\xi}_k = \frac{\xi_k + \xi_{k+1}}{2},$$

we shall now see that when  $N\gtrsim 100$  the Gaussian law

$$h_k \simeq \frac{N^2}{2\pi} e^{-\bar{\xi}_k^2}$$
 (4.17)

holds with remarkable precision, cf. Fig. 4.5. Indeed, if x = x(k) denotes the RHS of Eq. (4.13) we have

$$2k = N\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + N + 1,$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}k} = \frac{\sqrt{2\pi}}{N} \,\mathrm{e}^{\frac{1}{2}x^2}$$

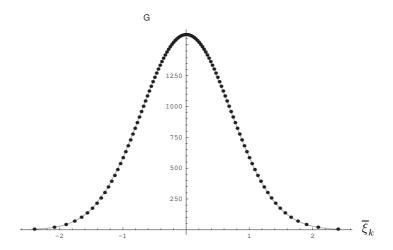


Figure 4.5: Plot of the coupling between the spins k and k+1 versus their mean position, fitted by the Gaussian (4.17), for N=100 spins.

is of order 1/N. Hence, up to terms of order 1/N we have

$$\left[x\left(k - \frac{1}{2}\right) - x\left(k + \frac{1}{2}\right)\right]^{-2} \simeq \left(\frac{\mathrm{d}x}{\mathrm{d}k}\right)^{-2} = \frac{N^2}{2\pi} e^{-x^2(k)}.$$
 (4.18)

Since (up to terms of order  $1/N^2$ )

$$x(k) \simeq \frac{1}{2} \left[ x \left( k - \frac{1}{2} \right) + x \left( k + \frac{1}{2} \right) \right],$$

Eq. (4.17) follows from (4.18) replacing k by  $k + \frac{1}{2}$ .

A brief comparison of the previous properties with those of the PF chain (1.28) is in order. For large N, the density of sites of the PF chain (normalized to unity), that is the density of zeros of the N-th Hermite polynomial, is asymptotically given by [36]

$$\rho_N(x) = \frac{1}{\pi N} \sqrt{2N - x^2}.$$
 (4.19)

The last site  $\zeta_N$  of the PF chain grows with N much faster than the corresponding site  $\xi_N$  of the chain (4.1), for the largest zero of the N-th Hermite polynomial behaves as  $\sqrt{2N} + \mathcal{O}(N^{-1/6})$ ; see, for instance, the recent paper [56].

## 4.3 Quasi-exact solvability of the NN chain

In this section we shall explore the solvability properties of the spin chain (4.1) by exploiting its connection with the QES model  $H_0$  defined in (3.2a). As in Section 2.5, we shall rescale the constant  $\omega$  in the latter Schrödinger operator and its scalar counterpart  $H^{sc}$  and remove the ground state energy, defining

$$H = (H_0 - E_0)|_{\omega = a}$$
,  $H^{\text{sc}} = (H_0^{\text{sc}} - E_0)|_{\omega = a}$ .

In terms of the matrix multiplication operator

$$h(\mathbf{x}) = \sum_{i} (x_i - x_{i+1})^{-2} (1 - S_{i,i+1}),$$

the connection between H and  $H^{sc}$  can be simply written as

$$H = H^{\rm sc} + 2a\mathsf{h}\,,\tag{4.20}$$

whereas the spin chain Hamiltonian (4.1) is recovered as

$$\mathsf{H} = \mathsf{h}(\boldsymbol{\xi}) \,. \tag{4.21}$$

Recall that the ground state function of the scalar Hamiltonian  $H^{\text{sc}}$  can be written as  $\mu(\mathbf{x}) = e^{a\lambda(\mathbf{x})}$ , cf. Table 3.1 and Eq. (4.3). We shall also denote by

$$\varphi_0 = \|\mu\|^{-2} \mu^2 \in L^1(C_0) \cap C^{\infty}(C_0)$$

its normalized square. The following result is easily established by repeating the proof of Lemma 2.22.

**Lemma 4.5.**  $\varphi_0$  converges to the delta distribution supported at  $\boldsymbol{\xi}$  and decays exponentially fast away from  $\boldsymbol{\xi}$ .

One would be tempted to believe that, mutatis mutandis, the discussion of Section (2.5) would provide an appropriate bridge to link the algebraic states of H and H. However, the fact that the dynamical models H and  $H^{\rm sc}$  are only quasi-exactly solvable, and thus only a proper subset of their spectra is known, makes it impossible to compute their partition functions from their algebraic energies. Hence Lemma 2.26, which constitutes the core of the application of CS models to the study of spin chains, cannot be invoked in this case.

In the rest of this chapter we shall develop a finer version of the freezing trick that will enable us to circumvent this limitation. We shall obtain a number of eigenstates of the chain (which we call algebraic), valid for any choice of N and M. As mentioned in the introduction, these algebraic states do not span the whole spin space, the chain Hamiltonian (4.1) thus inheriting the QES structure of the dynamical model.

In the following lemma we prove the key convergence result ("freezing trick") that shall enable us to compute the algebraic eigenvectors of H in closed form. This result is most easily stated in terms of the function  $\hat{\mu} = \|\mu\|^{-1}\mu$ , that is, the normalized ground state of  $H^{\text{sc}}$ . Moreover, given a scalar function  $\varphi \in L^2(C_0)$  and a spin state  $\Phi \in L^2(C_0) \otimes \Sigma$ , we shall use the notation

$$(\varphi,\Phi) = \int_{C_0} \overline{\varphi} \Phi$$

to denote their  $\Sigma$ -valued scalar product.

**Lemma 4.6.** Let  $\Psi$  be an algebraic eigenfunction of H with energy E (cf. Theorem 3.17). Assume that there exists an eigenfunction  $\psi$  of  $H^{\rm sc}$  and a  $\Sigma$ -valued polynomial  $F \in \mathbb{C}[\mathbf{x}, a^{-1}] \otimes \Sigma$  such that

$$\hat{\mu}^{-1}(\Psi - \psi F) = \mathcal{O}(a^{-1}), \tag{4.22}$$

and suppose, moreover, that these eigenfunctions can be normalized so that

$$\hat{\mu}^{-1}\Psi \in \mathbb{C}[\mathbf{x}, a^{-1}] \otimes \Sigma, \qquad \hat{\mu}^{-1}\psi \in \mathbb{C}[\mathbf{x}, a^{-1}].$$
 (4.23)

Then the limits

$$c = \lim_{a \to \infty} \frac{\psi(\xi)}{\hat{\mu}(\xi)}, \qquad \chi = \lim_{a \to \infty} F(\xi), \qquad \mathsf{E} = \lim_{a \to \infty} \frac{E}{2a}$$

exist, and

$$c(\mathsf{H} - \mathsf{E})\chi = 0. \tag{4.24}$$

*Proof.* The existence of the above limits trivially follows from the polynomial dependence of  $\hat{\mu}\psi$  and F on  $a^{-1}$  and the fact that E is linear in a. Furthermore, the self-adjointness of  $H^{\rm sc}$ , Eqs. (4.21)-(4.22) and Lemma 4.5 readily imply that

$$\begin{split} (\hat{\mu}, H^{\mathrm{sc}}\Psi) &= (H^{\mathrm{sc}}\hat{\mu}, \Psi) = 0 \,, \\ (\hat{\mu}, \mathsf{h}\Psi) &= \int_{C_0} \varphi_0 \left[ \hat{\mu}^{-1} \psi \, \mathsf{h} F + \mathcal{O}(a^{-1}) \right] = c \mathsf{H} \chi + \mathcal{O}(a^{-1}) \,. \end{split}$$

If we now use Eq. (4.20) and Lemma 4.5 to write

$$(\hat{\mu}, H^{\mathrm{sc}}\Psi) + 2a(\hat{\mu}, \mathsf{h}\Psi) = (\hat{\mu}, H\Psi) = Ec\chi + \mathcal{O}(1)\,,$$

substitute the former equations into this identity and take the limit  $a \to \infty$ , we readily derive Eq. (4.24).

**Corollary 4.7.** If  $c\chi \neq 0$ , then  $\chi$  is an eigenvector of H with energy E.

Remark 4.8. Since, by Lemma 4.5,

$$c = \lim_{a \to \infty} (\hat{\mu}, \psi)$$
,

the previous corollary is of no practical use unless  $\psi$  can be taken as the normalized ground state  $\hat{\mu}$ , in which case c = 1.

We are now ready to prove the main result of this chapter.

**Theorem 4.9.** If  $|\mathbf{s}\rangle \in \Sigma$  and  $|\mathbf{s}'\rangle \in \Sigma'$ , the states

$$\chi_0 \equiv \chi_0(|\mathbf{s}\rangle) = \Lambda|\mathbf{s}\rangle,$$
 (4.25a)

$$\chi_1 \equiv \chi_1(|\mathbf{s}\rangle) = \sum_i \xi_i |\mathbf{s}_i\rangle,$$
(4.25b)

$$\chi_2 \equiv \chi_2(|\mathbf{s}'\rangle) = \sum_i \xi_i^2 |\mathbf{s}'_i\rangle + (N-1) \sum_{i < i} \xi_i \xi_j |\mathbf{s}'_{ij}\rangle, \quad S_{12} |\mathbf{s}'\rangle = |\mathbf{s}'\rangle, \quad (4.25c)$$

$$\chi_3 \equiv \chi_3(|\mathbf{s}\rangle) = \sum_{i < j} \xi_i \xi_j (\xi_i - \xi_j) |\mathbf{s}_{ij}^-\rangle + 2 \sum_i \xi_i |\mathbf{s}_i\rangle, \quad S_{12} |\mathbf{s}\rangle = -|\mathbf{s}\rangle, \quad (4.25d)$$

satisfy the equations

$$(H-i)\chi_i = 0$$
,  $i = 0, 1, 2, 3$ .

*Proof.* The proof consists in the application of Lemma 4.6 to appropriate triples  $(\Psi, \psi, F)$ , where in fact  $\psi = \hat{\mu}$  (and hence c = 1) in view of the previous remark.

In the first place, it is easy to show that  $\chi_0 \in \ker \mathsf{H}$  directly. However, we prefer to prove it using the same freezing trick argument as in the other cases. To this end, consider the functions

$$\Psi = \frac{\Psi_{00}^{(0)}}{\|\hat{\mu}\|} = \hat{\mu}\Lambda |\mathbf{s}\rangle, \qquad F = \Lambda |\mathbf{s}\rangle,$$

which trivially satisfy the hypotheses in Lemma 4.6 with E=0 and  $\psi=\hat{\mu}$ . Since in this case

$$\mathsf{E} = 0$$
,  $\gamma = \Lambda | \mathbf{s} \rangle$ ,

from Corollary 4.7 it follows that  $\chi_0$ , if non-vanishing, is an eigenvector of H with eigenvalue 0.

The next case is slightly more complicated. Let us set

$$\Psi = \frac{\Psi_{01}^{(1)}}{\|\mu\|} = \hat{\mu}(\Phi^{(1)} - \bar{x}\Phi^{(0)}), \qquad F = \Phi^{(1)} - \bar{x}\Phi^{(0)}.$$

It is not difficult to show that  $(\Psi, \hat{\mu}, F)$  satisfies the hypotheses in Lemma 4.6 with E = 2a. Since  $\mathsf{E} = 1$  in this case, Corollary 4.7 shows that the state

$$\chi = \lim_{a \to \infty} F(\boldsymbol{\xi}) = \sum_{i} \xi_{i} |\mathbf{s}_{i}\rangle$$

is either zero or an eigenvector of H with energy 1. Here we have used the identity  $\sum_{i} \xi_{i} = 0$ , cf. Corollary 4.5.

The other two states are obtained in a similar manner starting with  $\psi = \hat{\mu}$  and the functions

$$\Psi = \frac{\Psi_{02}^{(2)} + (N-1)\widetilde{\Psi}_{02}^{(2)}}{\|\mu\|} = \hat{\mu} \left( \Phi^{(2)} + (N-1)\widetilde{\Phi}^{(2)} - 2N\bar{x}\Phi^{(1)} + (N+2)\bar{x}^2\Phi^{(0)} \right),$$

$$F = \frac{\Psi}{\hat{\mu}}, \qquad E = 4a, \qquad |\mathbf{s}'\rangle \in \Sigma', \qquad S_{12}|\mathbf{s}'\rangle = |\mathbf{s}'\rangle,$$

and

$$\Psi = \frac{\widehat{\Psi}_{03}^{(3)} - 4\Psi_{03}^{(1)} - \frac{8}{3}\Psi_{03}^{(0)}}{\|\mu\|} = \hat{\mu} \left[ \widehat{\Phi}^{(3)} - 2\bar{x}\Phi^{(2)} + \frac{2r^2}{N}\Phi^{(1)} + \bar{x} \left( \frac{r^2}{2N} - \frac{4\bar{x}^2}{3} \right) \Phi^{(0)} \right],$$

$$F = \frac{\Psi}{\hat{\mu}}, \qquad E = 6a, \qquad S_{12} |\mathbf{s}\rangle = -|\mathbf{s}\rangle$$

in each case. Indeed, with the former set of functions we immediately arrive at

$$\chi = \sum_{i} \xi_{i}^{2} |\mathbf{s}_{i}'\rangle + (N-1) \sum_{i < j} \xi_{i} \xi_{j} |\mathbf{s}_{ij}'^{+}\rangle,$$

while in the latter case we obtain

$$\chi = \sum_{i < j} \xi_i \xi_j (\xi_i - \xi_j) |\mathbf{s}_{ij}^-\rangle + \frac{2}{N} ||\boldsymbol{\xi}||^2 \sum_i \xi_i |\mathbf{s}_i\rangle.$$

Using Proposition 4.2, the result follows.

Remark 4.10. All the algebraic energies of H being integral multiples of a, it follows that any eigenvalue of the chain constructed from them via a freezing argument must be an integer.

Remark 4.11. The linear combinations of eigenfunctions with energy 2 and 3 used in Theorem 4.9 are not casual. Should one start, e.g., with

$$\Psi = \frac{\Psi_{02}^{(2)}}{a\|\mu\|}, \qquad \psi = \hat{\mu}, \qquad F = \hat{\mu}^{-1}\Psi,$$

for which  $F = \mathcal{O}(1)$  and c = 1, one would be lead to

$$\chi = -\frac{2N}{N-1} \left(\sum_{i} \xi_{i}\right)^{2} \Phi^{(0)} = 0.$$

The above linear combinations have been chosen so as to suppress the dominant parts in  $\widetilde{\Psi}_{02}^{(2)}$  and  $\widehat{\Psi}_{03}^{(3)}$ , which respectively are proportional to  $\bar{x}^2$  and  $\bar{x}^3$ .

It is also clear that the state  $\chi_2$  (resp.  $\chi_3$ ) is zero whenever the spin vector  $|\mathbf{s}\rangle$  (resp.  $|\mathbf{s}'\rangle$ ) is not symmetric (resp. antisymmetric) under  $S_{12}$ , which accounts for the condition on  $|\mathbf{s}\rangle$  and  $|\mathbf{s}'\rangle$  imposed in Eqs. (4.25b) and (4.25d).

It is natural to wonder whether additional eigenvectors of the chain can be computed using some different linear combinations of the algebraic eigenfunctions of H listed in Theorem 3.17. Nevertheless, since

$$c\chi = \lim_{a \to \infty} F(\boldsymbol{\xi}) = \lim_{a \to \infty} \frac{\Psi(\boldsymbol{\xi})}{\hat{\mu}(\boldsymbol{\xi})},$$

it is clear that an algebraic eigenfunction in the linear span of  $\Psi_{lm}^{(j)}$ ,  $\widetilde{\Psi}_{lm}^{(j)}$  and  $\widehat{\Psi}_{lm}^{(j)}$  can only give rise to a nonvanishing vector  $c\chi$  when l=0 and m=j. Indeed,  $\phi=a^{-l}\widehat{\mu}L_l^{\beta(0)}(ar^2)\in\widehat{\mu}\mathbb{C}[\mathbf{x},a^{-1}]$ , with  $\beta(0)=1-N(a+\frac{1}{2})$ , is an eigenfunction of  $H^{\mathrm{sc}}$  with energy 4al, which implies that

$$\lim_{a \to \infty} a^{-l} L_l^{-\beta - k}(aN^2) = 0 \qquad \forall l > 0$$

as  $\|\boldsymbol{\xi}\|^2 = N$ , proving the former statement. The latter condition follows directly from the factor  $\bar{x}^{m-j}$  present at the above eigenfunctions of H. This discussion strongly suggests (albeit does not rigorously prove) that one cannot obtain any additional eigenvectors of H using the technique developed in Lemma 4.6 and Theorem 4.9.

## 4.4 Qualitative properties of the algebraic spectrum

In this section we shall analyze which portion of the spectrum of (4.1) is algebraic and how the algebraic eigenvalues are embedded into the whole

spectrum. Regarding the first point, it is convenient to start with the following states  $\Phi_i \in L^2(C_0) \otimes \Sigma$ , whose evaluation at  $\boldsymbol{\xi}$  yields the chain eigenvectors (4.25):

$$\Phi_{0}(|\mathbf{s}\rangle) = \Phi^{(0)}(|\mathbf{s}\rangle), 
\Phi_{1}(|\mathbf{s}\rangle) = \Phi^{(1)}(|\mathbf{s}\rangle), 
\Phi_{2}(|\mathbf{s}\rangle) = \Phi^{(2)}(|\mathbf{s}\rangle) + (N-1)\widetilde{\Phi}^{(2)}(|\mathbf{s}\rangle), 
\Phi_{3}(|\mathbf{s}\rangle) = \widehat{\Phi}^{(3)}(|\mathbf{s}\rangle) - 2\bar{x}\Phi^{(2)}(\mathbf{s}) + \frac{2r^{2}}{N}\Phi^{(1)}(|\mathbf{s}\rangle).$$

It may be easily shown that if  $|\mathbf{s}\rangle$  is symmetric then  $\Phi_1(|\mathbf{s}\rangle)$  and  $\Phi_2(|\mathbf{s}\rangle)$  vanish at  $\mathbf{x} = \boldsymbol{\xi}$ . Likewise, if  $|\mathbf{s}\rangle \sim 0$  then  $\Phi_2(|\mathbf{s}\rangle)$  is also zero when  $\mathbf{x} = \boldsymbol{\xi}$ . For this reason, we shall consider the following spaces:

$$\mathcal{V}_{0} = \Lambda(\Sigma), \qquad \mathcal{V}_{1} = \left\langle \Phi_{1}(|\mathbf{s}\rangle) : |\mathbf{s}\rangle \in \Sigma/\Lambda(\Sigma) \right\rangle,$$

$$\mathcal{V}_{2} = \left\langle \Phi_{2}(|\mathbf{s}\rangle) : |\mathbf{s}\rangle \in \left(\Sigma'/\Lambda(\Sigma)\right)/\sim \right\rangle,$$

$$\mathcal{V}_{3} = \left\langle \Phi_{3}(|\mathbf{s}\rangle) : |\mathbf{s}\rangle \in \Sigma, \ S_{12}|\mathbf{s}\rangle = -|\mathbf{s}\rangle \right\rangle.$$

**Proposition 4.12.** The dimensions of the spaces  $V_i$  are given by

$$\dim \mathcal{V}_0 = \binom{N+2M}{N}, \tag{4.26}$$

$$\dim \mathcal{V}_1 = \dim \mathcal{V}_2 = (N-1) \binom{N+2M-1}{N},$$
 (4.27)

$$\dim \mathcal{V}_3 = \binom{N-1}{2} \binom{N+2M-2}{N}. \tag{4.28}$$

*Proof.* First of all, the dimension of  $\mathcal{V}_0$  is simply the number of permutations with repetitions of N elements from 2M + 1.

Consider next the space  $\mathcal{V}_1$ . If two basic states  $|\mathbf{s}\rangle$  and  $|\mathbf{s}'\rangle$  differ by a permutation of the last N-1 spins, then  $\Phi_1(|\mathbf{s}\rangle) = \Phi_1(|\mathbf{s}'\rangle)$ . Hence the dimension of  $\mathcal{V}_1$  is given by

$$(2M+1)\binom{N+2M-1}{2M}-\binom{N+2M}{2M},$$

where the first quantity is the number of possible choices for  $s_1$ , the second one is that of symmetric states of N-1 particles, and the last one, that of symmetric states of N particles.

The dimension of  $\mathcal{V}_2$  coincides with that of the space  $(\Sigma'/\Lambda(\Sigma))/\sim$ , a basis of which was computed in Propositions 3.10 and 3.13. Using the previous notation we can say that, for a given spin content  $\{s^1,\ldots,s^n\}\subset \{-M,\ldots,M\}$  with multiplicities  $(\nu_1,\ldots,\nu_n)\in\mathbb{N}^n$  (with  $\sum_{j=1}^n\nu_j=N$ ), there are n-1 independent states in  $(\Sigma'/\Lambda(\Sigma))/\sim$ . Hence

$$\dim \mathcal{V}_2 = \sum_{n=1}^{\min\{2M+1,N\}} (n-1) \operatorname{card}(\mathfrak{P}_N \cap \mathbb{N}^n) \binom{2M+1}{n},$$

where the binomial term accounts for the different choices of  $\{s^1, \ldots, s^n\}$ , and the cardinal for all the possible values of the multiplicities. Noting  $\operatorname{card}(\mathfrak{P}_N \cap \mathbb{N}^n) = \binom{N-1}{n-1}$  and using the identity

$$\sum_{k=0}^{\infty} \binom{n}{k} \binom{m}{j+k} = \binom{m+n}{m-j},$$

we arrive at the desired formula.

It only remains to prove Eq. (4.28). To this end, let us concentrate on the spin vectors  $\Sigma_0$  with a fixed spin content  $\{s^1, \ldots, s^n\}$ ,  $(\nu_1, \ldots, \nu_n)$ . It is clear that  $\mathcal{V}_3(\Sigma_0) = \langle \Phi_3(|\mathbf{s}\rangle) : |\mathbf{s}\rangle \in \Sigma_0 \rangle$  is spanned by the  $\binom{n}{2}$  states

$$\mathbf{v}(s^i, s^j) = \nu_i \nu_j \, \Phi_3(|s^i s^j \cdots\rangle - |s^j s^i \cdots\rangle) \,.$$

We shall shortly show that, for all i,

$$\sum_{j} \mathbf{v}(s^i, s^j) = 0. \tag{4.29}$$

By antisymmetry, the double sum  $\sum_{i,j} \mathbf{v}(s^i, s^j)$  vanishes, so there are at most n-1 independent relations of the above form. It can be verified that  $\{\mathbf{v}(s^i, s^j) : 1 \leq i < j < n\}$  is in fact a basis of  $\mathcal{V}_3(\Sigma_0)$ , so the dimension of  $\mathcal{V}_3$  is obtained from (ii) by substituting n-1 for  $\binom{n-1}{2}$ .

Finally, let us prove (4.29). A first observation is that

$$r^2\Phi^{(1)}(|s^is^j\cdots\rangle) - N\bar{x}\Phi^{(2)}(|s^is^j\cdots\rangle) = -\sum_k \nu_k \widehat{\Phi}^{(3)}(|s^is^k\cdots\rangle).$$

Hence

$$\frac{N}{2\nu_i} \sum_{i,j} \mathbf{v}(s^i, s^j) = N \sum_j \nu_j \widehat{\Phi}^{(3)}(|s^i s^j \cdots\rangle) 
- \sum_{j,k} \nu_j \nu_k \widehat{\Phi}^{(3)}(|s^i s^k \cdots\rangle) + \sum_{j,k} \nu_j \nu_k \widehat{\Phi}^{(3)}(|s^j s^k \cdots\rangle). \quad (4.30)$$

The first two terms of the RHS obviously cancel, and the last one vanishes by antisymmetry.  $\Box$ 

This proposition, which is interesting in its own right because of its connection with the dynamical model (3.2a), has the following consequences for the spin chain (4.1):

Corollary 4.13. For each i = 0, 1, 2, 3, the number of algebraic eigenvectors with energy i is bounded by dim  $V_i$ .

Corollary 4.14. The ratio

$$\frac{number\ of\ algebraic\ states}{total\ number\ of\ states}$$

tends to zero as  $N \to \infty$ . The whole kernel of H is composed of algebraic eigenvectors.

*Proof.* The first part of the statement follows by direct comparison with the total number of states  $(2M+1)^N$ . The second part is due to the fact that H is a linear combination with positive coefficients of the nonnegative operators  $1 - S_{i,i+1}$ , which implies that ker  $H = \Lambda(\Sigma)$ .

These observations are corroborated by our numerical simulations: the number of algebraic levels increase as M and N become larger, but nevertheless they span a lesser and lesser part of the spin space. The kernel of the chain, e.g., shows clearly this behavior. But most remarkable is the following conjecture regarding the embedding of the algebraic eigenvalues in the spectrum.

**Conjecture 4.15.** The algebraic eigenvectors (4.25a)–(4.25c) span the three lowest levels of the spin chain (4.1). When  $M \ge 1$  the fourth algebraic energy,  $\mathsf{E} = 3$ , is however the fifth lowest level of the chain, and its whole eigenspace is spanned by the states of the form (4.25d). The algebraic energies are singled out in the spectrum of  $\mathsf{H}$  as being the only integer ones, i.e.,

$$\operatorname{spec}(\mathsf{H}) \cap \mathbb{Z} = \{0, 1, 2, 3\}.$$

We have numerically diagonalized the matrix representing the Hamiltonian H for  $M \leq 2$  and up to 12 particles, and our results fully support the above conjecture. Another important observation is that the standard freezing trick relation between the energies of the spin chain H and of the dynamical models H and  $H^{\rm sc}$ , namely

$$\mathsf{E} = \lim_{a \to \infty} \frac{E - E^{\mathrm{sc}}}{2a} \,,$$

and the fact that H has noninteger eigenvalues imply that the spin dynamical model (3.2a) has noninteger energies, and hence that some of its eigenfunctions are not among those listed in Theorem 3.17.

## Chapter 5

## Conclusions and perspectives

In this thesis we have studied three QES models of CS type presenting short-range interactions and developed a rigorous treatment of the CS/HS correspondence, valid for both ES and QES Hamiltonians. In a nutshell, the main conclusions and results that one should extract from this dissertation are the following:

- (i) On the  $BC_N$  HS spin chain: We have computed the spectrum of the trigonometric  $BC_N$  Sutherland model by expressing its Hamiltonian as the sum of squares of an appropriate set of commuting Dunkl operators. The  $BC_N$  HS chain is obtained from this system by "freezing" the particles at their classical equilibrium positions. We have proved a rigorous version of the freezing trick which has enabled us to compute the partition function of the chain in terms of those of the corresponding scalar and spin models. We have resorted to numerical computations to explore the level distribution of the chain, finding out that the energies accurately follow the Gaussian law. A constant magnetic field was added to the picture in a rather straightforward way.
- (ii) On NN spin models: We defined three QES spin models with nearneighbors interactions that are related to NN versions of the sum of the squares of appropriate sets of  $A_N$  Dunkl operators. Furthermore, we have obtained infinite flags of finite-dimensional spaces invariant under these models, and computed several families of eigenfunctions in closed form by diagonalizing the restriction of the Hamiltonians to these spaces. All the algebraic solutions, whose energies are integers, can be expressed as the ground state times a symmetric function. We conjecture that some of their eigenfunctions are not of this form, and that their corresponding eigenvalues are not natural numbers.

(iii) On NN spin chains: We have defined a spin chain presenting nearest-neighbors interactions that is related to one of the previous NN models by a freezing trick argument. We have discussed its site distribution in detail, both analytically and numerically, and obtained a number of algebraic eigenvectors and energies using a refinement of the techniques developed in Chapter 2 that also applies to QES systems. We have raised the conjecture that the only integer energies of this chain are 0, 1, 2, and 3, and that the eigenstates corresponding to these energies are all algebraic.

The conjectures scattered throughout the text do not exhaust all the open questions that we find worth exploring in connection with our work. In what follows, we shall briefly mention some wide research lines that in our opinion would be of interest, and that we intend to follow in the near future:

- (i) The structure of the (non-algebraic) eigenfunctions of  $H_0$  (and other QES operators): From the previous discussion, it looks unlikely that all the eigenfunctions of  $H_0$  admit the ground state as a Jastrow factor. Nevertheless, this does happen in the theory of CS models, and the functions that are explicitly constructed in the usual QES models are generally of this form. A thorough study of this phenomenon is, to the best of our knowledge, still lacking, and would probably shed some light on the connections between Schrödinger operators and orthogonal polynomials. The particular case of  $H_0$  would be of special interest.
- (ii) HS/CS and symmetric spaces: A powerful procedure for constructing CS models with spin starting out with a symmetric space of nonpositive curvature has been recently developed by Fehér and Pusztai, as described in Section 1.5. It should not be difficult to extend the techniques described in this thesis to solve the associated spin chains.
- (iii) Statistical properties of the spectrum of CS models: The remarkable statistical properties of the  $BC_N$  HS chain occupied a significant portion of Section 2.6. It is difficult to obtain pen and paper results on these aspects, but the rigorous literature studying these topics is always increasing (see [38,49] and references therein). Further study of the topics mentioned in this section and extensions to the case of NN models would make a valuable addition to the existing literature.
- (iv) Integrability vs. solvability in spin chains of HS type: A satisfactory analysis of the integrability of the chains of HS type has only been accomplished in a few instances. A natural approach to this problem

would be to define a set of integrals of the chain using the freezing trick, but in general it is not easy to grant the linear independence of these conserved operators. If one managed to overcome these technical problems, this method (which has already been exploited in the study of some particular models) would provide a clean proof of the integrability of the chains associated with integrable spin models.

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