A Derivation of Moment Evolution Equations for Linear Open Quantum Systems*

Shan Ma

School of Information Science and Engineering
Central South University
Changsha 410083, China
School of Engineering and Information Technology
UNSW Canberra
Canberra ACT 2600, Australia
shanma.adfa@gmail.com

Matthew J. Woolley
School of Engineering and IT
UNSW Canberra
Canberra ACT 2600, Australia
m.woolley@adfa.edu.au

Ian R. Petersen
Research School of Engineering
The Australian National University
Canberra ACT 2601 Australia
i.r.petersen@gmail.com

Abstract—Given a linear open quantum system which is described by a Lindblad master equation, we detail the calculation of the moment evolution equations from this master equation. We stress that the moment evolution equations are well-known, but their explicit derivation from the master equation cannot be found in the literature to the best of our knowledge, and so we provide this derivation for the interested reader.

Index Terms—Linear quantum system, open quantum system, Lindblad master equation, moment, evolution, mean vector, covariance matrix, Gaussian state, drift matrix, diffusion matrix.

I. INTRODUCTION

Quantum systems unavoidably interact with their surrounding environments, and such quantum systems are referred to as open quantum systems. The dynamics of open quantum systems can be classified into two categories, Markovian and non-Markovian regimes, depending on whether the system is weakly coupled to a memoryless environment [1], [2]. For open quantum systems coupled weakly to a memoryless environment, a well-established treatment known as a Markovian Lindblad master equation can be used to approximate the time evolution of such systems [1]–[4]. Typical examples of such systems include quantum optical systems [5].

Let us consider a continuous-variable open quantum system with N degrees of freedom, the time evolution of which is described by the following Markovian Lindblad master equation (we set $\hbar=1$)

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \,\hat{\rho}] + \sum_{j=1}^{K} \mathfrak{D}[\hat{c}_j]\hat{\rho},\tag{1}$$

where $\hat{H} = \hat{H}^*$ is the system Hamiltonian, $\hat{c} = \begin{bmatrix} \hat{c}_1 & \hat{c}_2 & \cdots & \hat{c}_K \end{bmatrix}^\top$ is the vector of Lindblad operators, K is the number of decoherence channels, and $\mathfrak{D}[\hat{c}_j]\hat{\rho} \triangleq \hat{c}_j\hat{\rho}\hat{c}_j^* - \frac{1}{2}\left(\hat{c}_j^*\hat{c}_j\hat{\rho} + \hat{\rho}\hat{c}_j^*\hat{c}_j\right)$.

*The material presented here has been partially presented in thesis form [16]. This work was supported by the 111 Project (B17048), the Australian Research Council (ARC) under grant FL110100020, the Air Force Office of Scientific Research (AFOSR), under agreement number FA2386-16-1-4065, and the Australian Academy of Science.

Let (\hat{q}_j, \hat{p}_j) , $j = 1, \dots, N$, be the position and momentum operators of this quantum system. They satisfy the canonical commutation relation

$$\begin{bmatrix} \hat{x}, \hat{x}^{\top} \end{bmatrix} = \hat{x}\hat{x}^{\top} - \begin{pmatrix} \hat{x}\hat{x}^{\top} \end{pmatrix}^{\top} = i\Sigma, \quad \Sigma \triangleq \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}, \quad (2)$$

where $\hat{x} \triangleq \begin{bmatrix} \hat{q}_1 & \cdots & \hat{q}_N & \hat{p}_1 & \cdots & \hat{p}_N \end{bmatrix}^{\top}$. Suppose that the system Hamiltonian \hat{H} is quadratic in the quadrature operators; i.e., $\hat{H} = \frac{1}{2}\hat{x}^{\top}M\hat{x}$, with $M = M^{\top} \in \mathbb{R}^{2N \times 2N}$. Also, suppose that the vector of Lindblad operators \hat{c} is linear in the quadrature operators; i.e., $\hat{c} = C\hat{x}$, with $C \in \mathbb{C}^{K \times 2N}$. An open quantum system with such \hat{H} and \hat{c} is said to be a linear open quantum system. For a linear open quantum system, given Gaussian initial conditions, the Lindblad master equation (1) will always have a Gaussian state as its solution.

The mean value of \hat{x} is given by $\langle \hat{x} \rangle = \left[\operatorname{tr}(\hat{x}_1 \hat{\rho}) \operatorname{tr}(\hat{x}_2 \hat{\rho}) \cdots \operatorname{tr}(\hat{x}_{2N} \hat{\rho}) \right]^{\top}$ and the covariance matrix of \hat{x} is given by $V = \frac{1}{2} \langle \triangle \hat{x} \triangle \hat{x}^{\top} + (\triangle \hat{x} \triangle \hat{x}^{\top})^{\top} \rangle$, where $\triangle \hat{x} = \hat{x} - \langle \hat{x} \rangle$. The moment evolution equations for $\langle \hat{x} \rangle$ and V are given by

$$\begin{cases}
\frac{d\langle \hat{x} \rangle}{dt} = \mathscr{A} \langle \hat{x} \rangle, \\
\frac{dV}{dt} = \mathscr{A} V + V \mathscr{A}^{\top} + \mathscr{D},
\end{cases} \tag{3}$$

where $\mathscr{A} = \Sigma \left(M + \operatorname{Im}(C^{\dagger}C) \right)$ is the *drift matrix* and $\mathscr{D} = \Sigma \operatorname{Re}(C^{\dagger}C)\Sigma^{\top}$ is the *diffusion matrix*.

The result described in (3) and (4) and the explicit formulas for \mathscr{A} and \mathscr{D} can be found in several references; e.g., [6]–[10]. Recent results on the preparation of pure Gaussian states are all built upon the moment evolution equations (3) and (4) [11]–[20]. To the best of the authors' knowledge, the explicit derivation for (3) and (4) from the master equation (1) cannot be found in the literature. So we provide a detailed derivation here for the interested reader.

II. DERIVATION OF MOMENT EVOLUTION EQUATIONS

For simplicity, we consider a linear open quantum system with a single decoherence channel (K = 1). The extension

of the derivation to multiple decoherence channels is straightforward; see Remark 1. We will first calculate the evolution equation for the mean vector $\langle \hat{x} \rangle$. Then, we will calculate the evolution equation for the covariance matrix V.

Part 1: Calculation of the Evolution Equation for the Mean Vector $\langle \hat{x} \rangle$

Let \hat{x}_{ℓ} be the ℓ th entry of the column vector \hat{x} . Then the commutation relations (2) can be written as

$$[\hat{x}_{\ell},\,\hat{x}_m]=\hat{x}_{\ell}\hat{x}_m-\hat{x}_m\hat{x}_{\ell}=i\Sigma_{\ell m},$$

where $\Sigma_{\ell m}$ is the (ℓ, m) entry of the matrix Σ . The equation of motion for $\langle \hat{x}_{\ell} \rangle$ is

$$\frac{d\langle \hat{x}_{\ell} \rangle}{dt} = \operatorname{tr}\left(\hat{x}_{\ell} \frac{d\hat{\rho}}{dt}\right) = \operatorname{tr}\left(\hat{x}_{\ell} \left(-i[\hat{H}, \hat{\rho}] + \mathfrak{D}[\hat{c}]\hat{\rho}\right)\right)
= \operatorname{tr}\left(\hat{x}_{\ell} \left(-i[\hat{H}, \hat{\rho}]\right)\right) + \operatorname{tr}\left(\hat{x}_{\ell} \left(\mathfrak{D}[\hat{c}]\hat{\rho}\right)\right).$$
(5)

Let us calculate $\operatorname{tr}\left(\hat{x}_{\ell}\left(-i[\hat{H},\,\hat{\rho}]\right)\right)$ and $\operatorname{tr}\left(\hat{x}_{\ell}\left(\mathfrak{D}[\hat{c}]\hat{\rho}\right)\right)$ sep- $=\sum_{j=1}^{2N}\sum_{k=1}^{2N}\sum_{k=1}^{2N}\sum_{k=1}^{2N}\left(C_{j}^{\dagger}C_{k}\right)\langle\hat{x}_{k}\rangle\right)$ arately. First, we have

$$\operatorname{tr}\left(\hat{x}_{\ell}\left(-i[\hat{H},\hat{\rho}]\right)\right) = -\frac{i}{2}\operatorname{tr}\left(\hat{x}_{\ell}\hat{x}^{\top}M\hat{x}\hat{\rho} - \hat{x}_{\ell}\hat{\rho}\hat{x}^{\top}M\hat{x}\right)$$

$$= -\frac{i}{2}\operatorname{tr}\left(\sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\hat{x}_{\ell}\hat{x}_{j}\hat{x}_{k}\hat{\rho} - \sum_{j=1}^{2N}\sum_{k=1}^{2N}\hat{x}_{\ell}\hat{\rho}M_{jk}\hat{x}_{j}\hat{x}_{k}\right)$$

$$= -\frac{i}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\operatorname{tr}\left(\hat{x}_{\ell}\hat{x}_{j}\hat{x}_{k}\hat{\rho} - \hat{x}_{j}\hat{x}_{k}\hat{x}_{\ell}\hat{\rho}\right)$$

$$= -\frac{i}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\operatorname{tr}\left((\hat{x}_{j}\hat{x}_{\ell} + i\Sigma_{\ell j})\hat{x}_{k}\hat{\rho} - \hat{x}_{j}\hat{x}_{k}\hat{x}_{\ell}\hat{\rho}\right)$$

$$= -\frac{i}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\operatorname{tr}\left(i\Sigma_{\ell j}\hat{x}_{k}\hat{\rho} + \hat{x}_{j}\hat{x}_{\ell}\hat{x}_{k}\hat{\rho} - \hat{x}_{j}\hat{x}_{k}\hat{x}_{\ell}\hat{\rho}\right)$$

$$= -\frac{i}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\operatorname{tr}\left(i\Sigma_{\ell j}\hat{x}_{k}\hat{\rho} + i\Sigma_{\ell k}\hat{x}_{j}\hat{\rho}\right)$$

$$= -\frac{i}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\operatorname{tr}\left(i\Sigma_{\ell j}\hat{x}_{k}\hat{\rho} + i\Sigma_{\ell k}\hat{x}_{j}\hat{\rho}\right)$$

$$= \frac{1}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}\sum_{k=1}^{2N}M_{jk}\langle\hat{x}_{k}\rangle + \frac{1}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}\sum_{k=1}^{2N}\Sigma_{\ell k}M_{jk}\langle\hat{x}_{j}\rangle$$

$$= \frac{1}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}\Sigma_{\ell j}M_{jk}\langle\hat{x}_{k}\rangle + \frac{1}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}\Sigma_{\ell k}M_{kj}\langle\hat{x}_{j}\rangle$$

$$= \sum_{j=1}^{2N}\sum_{k=1}^{2N}\Sigma_{\ell j}M_{jk}\langle\hat{x}_{k}\rangle$$

$$= \sum_{j=1}^{2N}\sum_{k=1}^{2N}\Sigma_{\ell j}M_{jk}\langle\hat{x}_{k}\rangle$$

$$= \sum_{\ell \in M}\langle\hat{x}_{\ell}\rangle, \tag{6}$$

where Σ_{ℓ} : denotes the ℓ th row of Σ . Second, we have

$$\begin{split} &\operatorname{tr}\left(\hat{x}_{\ell}\left(\mathfrak{D}[\hat{c}]\hat{\rho}\right)\right) = \operatorname{tr}\left(\hat{x}_{\ell}\left(\hat{c}\hat{\rho}\hat{c}^{*} - \frac{1}{2}\hat{c}^{*}\hat{c}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{c}^{*}\hat{c}\right)\right) \\ &= \operatorname{tr}\left(\hat{c}^{*}\hat{x}_{\ell}\hat{c}\hat{\rho} - \frac{1}{2}\hat{x}_{\ell}\hat{c}^{*}\hat{c}\hat{\rho} - \frac{1}{2}\hat{c}^{*}\hat{c}\hat{x}_{\ell}\hat{\rho}\right) \\ &= \operatorname{tr}\left(\left(\sum_{i=1}^{2N} C_{j}^{*}\hat{x}_{j}\right)\hat{x}_{\ell}\left(\sum_{k=1}^{2N} C_{k}\hat{x}_{k}\right)\hat{\rho} - \frac{1}{2}\hat{x}_{\ell}\left(\sum_{i=1}^{2N} C_{j}^{*}\hat{x}_{j}\right)\left(\sum_{k=1}^{2N} C_{k}\hat{x}_{k}\right)\hat{\rho} \end{split}$$

Here $C_j \in \mathbb{C}$ denotes the *j*th entry of the row vector *C*. Substituting (6) and (7) into (5), we obtain

$$\frac{d\langle \hat{x}_{\ell} \rangle}{dt} = \Sigma_{\ell} : \left(M + \operatorname{Im} \left(C^{\dagger} C \right) \right) \langle \hat{x} \rangle. \tag{8}$$

The evolution equation (3) follows immediately from (8).

Part 2: Calculation of the Evolution Equation for the Covariance Matrix V

Suppose $V_{\ell m}$ is the (ℓ, m) entry of the covariance matrix V. Then we have

$$\begin{split} &V_{\ell m} \\ &= \frac{1}{2} \langle \triangle \hat{x}_{\ell} \triangle \hat{x}_{m} + \triangle \hat{x}_{m} \triangle \hat{x}_{\ell} \rangle \\ &= \frac{1}{2} \langle (\hat{x}_{\ell} - \langle \hat{x}_{\ell} \rangle) (\hat{x}_{m} - \langle \hat{x}_{m} \rangle) + (\hat{x}_{m} - \langle \hat{x}_{m} \rangle) (\hat{x}_{\ell} - \langle \hat{x}_{\ell} \rangle) \rangle \\ &= \frac{1}{2} \left(\langle \hat{x}_{\ell} \hat{x}_{m} + \hat{x}_{m} \hat{x}_{\ell} \rangle - \langle \hat{x}_{\ell} \rangle \langle \hat{x}_{m} \rangle - \langle \hat{x}_{m} \rangle \langle \hat{x}_{\ell} \rangle \right) \\ &= \frac{1}{2} \langle \hat{x}_{\ell} \hat{x}_{m} + \hat{x}_{m} \hat{x}_{\ell} \rangle - \langle \hat{x}_{\ell} \rangle \langle \hat{x}_{m} \rangle. \end{split}$$

Therefore, the equation of motion for $V_{\ell m}$ is given by

$$\frac{dV_{\ell m}}{dt} = \frac{1}{2} \frac{d\langle \hat{x}_{\ell} \hat{x}_{m} + \hat{x}_{m} \hat{x}_{\ell} \rangle}{dt} - \frac{d\langle \hat{x}_{\ell} \rangle \langle \hat{x}_{m} \rangle}{dt} \\
= \frac{1}{2} \operatorname{tr} \left((\hat{x}_{\ell} \hat{x}_{m} + \hat{x}_{m} \hat{x}_{\ell}) \frac{d\hat{\rho}}{dt} \right) - \frac{d\langle \hat{x}_{\ell} \rangle \langle \hat{x}_{m} \rangle}{dt} \\
= \frac{1}{2} \operatorname{tr} \left((\hat{x}_{\ell} \hat{x}_{m} + \hat{x}_{m} \hat{x}_{\ell}) (-i[\hat{H}, \hat{\rho}]) \right)$$

$$\begin{split} &+\frac{1}{2}\operatorname{tr}\left(\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\left(\Sigma[\hat{c}]\hat{\rho}\right)\right)-\frac{d\left(\left(\hat{x}_{\ell}\hat{b}_{m}\hat{b}_{m}\right)\right)}{dt}, \qquad =2\sum_{j=1}^{2N}\sum_{k=1}^{N}M_{jk}\Sigma_{mk}\operatorname{tr}\left(\hat{x}_{\ell}\hat{x}_{j}\hat{\rho}\right)-2\sum_{j=1}^{2N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\operatorname{tr}\left(\hat{x}_{\ell}\hat{x}_{k}\hat{a}_{k}+\hat{x}_{m}\hat{x}_{\ell}\right)\left(-i[\hat{H},\hat{\rho}]\right)\right), \\ &\operatorname{Ir}\left(\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\left(\Sigma[\hat{c}]\hat{\rho}\right)\right) \quad \operatorname{and} \quad \frac{2i(ij_{\ell}|\hat{x}_{m}|)}{dt} \quad \operatorname{separately.} \quad \operatorname{First}, \\ &-\sum_{j=1}^{N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\operatorname{tr}\left(\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}+i\Sigma_{\ell j}\right)\hat{\rho}\right)\right) \\ &\operatorname{tr}\left(\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\left(-i[\hat{H},\hat{\rho}]\right)\right) \quad \operatorname{and} \quad \frac{2i(ij_{\ell}|\hat{x}_{m}|)}{dt} \quad \operatorname{separately.} \quad \operatorname{First}, \\ &-\sum_{j=1}^{N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\operatorname{tr}\left(\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}+i\Sigma_{\ell j}\right)\hat{\rho}\right)\right) \\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}+i\Sigma_{\ell j}\right)\hat{\rho}\right) \\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}+i\Sigma_{\ell j}\right)\hat{\rho}\right) \\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}+i\Sigma_{\ell j}\right)\hat{\rho}\right) \\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}\right)\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}\right)\hat{\rho} \\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}\right)\hat{\rho} \\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{\rho} \\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{\rho} \\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{\rho} \\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}M_{jk}\Sigma_{jk}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{\rho} \\ &=\sum_{j=1}^{N}$$

$$+\frac{1}{2}\operatorname{tr}\left(\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\left(\mathfrak{D}\left[\hat{c}\right]\hat{\rho}\right)\right)-\frac{d\left(\left\langle\hat{x}_{\ell}\right\rangle\langle\hat{x}_{m}\right\rangle)}{dt}. \quad (9) = 2\sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\Sigma_{mk}\operatorname{tr}\left(\hat{x}_{\ell}\hat{x}_{j}\hat{\rho}\right)-2\sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\Sigma_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}\hat{\rho}\right)$$

$$\text{us calculate }\operatorname{tr}\left(\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\left(-i[\hat{H},\hat{\rho}]\right)\right), = \sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\Sigma_{mk}\operatorname{tr}\left(\left(\hat{x}_{\ell}\hat{x}_{j}+\hat{x}_{j}\hat{x}_{\ell}+i\Sigma_{\ell j}\right)\hat{\rho}\right)$$

$$\frac{1}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\Sigma_{j\ell}\operatorname{tr}\left(\left(\hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}+i\Sigma_{km}\right)\hat{\rho}\right)$$

$$= \sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\Sigma_{j\ell}\operatorname{tr}\left(\left(\hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}+i\Sigma_{km}\right)\hat{\rho}\right)$$

$$= \sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\Sigma_{j\ell}\operatorname{tr}\left(\left(\hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}+i\Sigma_{km}\right)\hat{\rho}\right)$$

$$= \sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\Sigma_{mk}\langle\hat{x}_{\ell}\hat{x}_{j}+\hat{x}_{j}\hat{x}_{\ell}\rangle - \sum_{j=1}^{2N}\sum_{k=1}^{2N}M_{jk}\Sigma_{j\ell}\langle\hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}\rangle$$

$$= \sum_{j=1}^{2N}\sum_{k=1}^{2N}\Sigma_{\ell}M_{jk}\langle\hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}\rangle - \sum_{j=1}^{2N}\sum_{k=1}^{2N}\langle\hat{x}_{\ell}\hat{x}_{j}+\hat{x}_{j}\hat{x}_{\ell}\rangle M_{jk}\Sigma_{km}.$$

$$\sum_{k=1}^{2N}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{\rho}M_{jk}\hat{x}_{j}\hat{x}_{k}$$

$$= \sum_{j=1}^{2N}\sum_{k=1}^{2N}\Sigma_{\ell}M_{jk}\langle\hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}\rangle - \sum_{j=1}^{2N}\sum_{k=1}^{2N}\langle\hat{x}_{\ell}\hat{x}_{j}+\hat{x}_{j}\hat{x}_{\ell}\rangle M_{jk}\Sigma_{km}.$$

$$(10)$$

Second, we have

$$\begin{split} &\text{tr}\left(\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\left(\mathfrak{D}\left[\hat{c}\right]\hat{\rho}\right)\right) \\ &=\text{tr}\left(\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\left(\hat{c}\hat{\rho}\hat{c}^{*}-\frac{1}{2}\hat{c}^{*}\hat{c}\hat{\rho}-\frac{1}{2}\hat{\rho}\hat{c}^{*}\hat{c}\right)\right) \\ &=\text{tr}\left(\hat{c}^{*}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{c}\hat{\rho}-\frac{1}{2}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{c}^{*}\hat{c}\hat{\rho}\right) \\ &=\text{tr}\left(\hat{c}^{*}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{\rho}\right) \\ &=\text{tr}\left(\left(\sum_{j=1}^{2N}C_{j}^{*}\hat{x}_{j}\right)\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{\rho}\right) \\ &=\text{tr}\left(\left(\sum_{j=1}^{2N}C_{j}^{*}\hat{x}_{j}\right)\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{\rho}\right) \\ &-\frac{1}{2}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\left(\sum_{j=1}^{2N}C_{j}^{*}\hat{x}_{j}\right)\left(\sum_{k=1}^{2N}C_{k}\hat{x}_{k}\right)\hat{\rho} \\ &-\frac{1}{2}\left(\sum_{j=1}^{2N}C_{j}^{*}\hat{x}_{j}\right)\left(\sum_{k=1}^{2N}C_{k}\hat{x}_{k}\right)\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{\rho}\right) \\ &=\sum_{j=1}^{2N}\sum_{k=1}^{2N}C_{j}^{*}C_{k}\,\text{tr}\left(\left(\hat{x}_{j}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{x}_{k}\right) \\ &-\frac{1}{2}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\hat{x}_{j}\hat{x}_{k}-\frac{1}{2}\hat{x}_{j}\hat{x}_{k}\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\right)\hat{\rho}\right) \\ &=\sum_{j=1}^{2N}\sum_{k=1}^{2N}C_{j}^{*}C_{k}\,\text{tr}\left(\left(\hat{x}_{j}\left(2\hat{x}_{\ell}\hat{x}_{m}-i\Sigma_{\ell m}\right)\hat{x}_{k}\right) \\ &-\frac{1}{2}\left(2\hat{x}_{\ell}\hat{x}_{m}-i\Sigma_{\ell m}\right)\hat{x}_{j}\hat{x}_{k}-\frac{1}{2}\hat{x}_{j}\hat{x}_{k}\left(2\hat{x}_{\ell}\hat{x}_{m}-i\Sigma_{\ell m}\right)\hat{\rho}\right) \\ &=\sum_{j=1}^{2N}\sum_{k=1}^{2N}C_{j}^{*}C_{k}\,\text{tr}\left(\left(2\hat{x}_{j}\hat{x}_{\ell}\hat{x}_{m}\hat{x}_{k}-\hat{x}_{\ell}\hat{x}_{m}\hat{x}_{j}\hat{x}_{k}-\hat{x}_{j}\hat{x}_{k}\hat{x}_{\ell}\hat{x}_{m}\right)\hat{\rho}\right) \\ &=\sum_{j=1}^{2N}\sum_{k=1}^{2N}C_{j}^{*}C_{k}\,\text{tr}\left(\left(2\left(\hat{x}_{\ell}\hat{x}_{j}+i\Sigma_{j\ell}\right)\left(\hat{x}_{k}\hat{x}_{m}+i\Sigma_{mk}\right) \\ &-\hat{x}_{\ell}\left(\hat{x}_{j}\hat{x}_{m}+i\Sigma_{mj}\right)\hat{x}_{k}-\hat{x}_{j}\left(\hat{x}_{\ell}\hat{x}_{k}+i\Sigma_{k\ell}\right)\hat{x}_{m}\right)\hat{\rho}\right) \\ &=\sum_{j=1}^{2N}\sum_{k=1}^{2N}C_{j}^{*}C_{k}\,\text{tr}\left(\left(2\hat{x}_{\ell}\hat{x}_{j}\hat{x}_{k}\hat{x}_{m}+2i\Sigma_{j\ell}\hat{x}_{k}\hat{x}_{m}+2i\Sigma_{mk}\hat{x}_{\ell}\hat{x}_{j}\right) \\ &=\sum_{j=1}^{2N}\sum_{k=1}^{2N}C_{j}^{*}C_{k}\,\text{tr}\left(\left(2\hat{x}_{\ell}\hat{x}_{j}\hat{x}_{k}\hat{x}_{m}+2i\Sigma_{j\ell}\hat{x}_{k}\hat{x}_{m}+2i\Sigma_{mk}\hat{x}_{\ell}\hat{x}_{j}\right) \\ &=\sum_{j=1}^{2N}\sum_{k=1}^{2N}C_{j}^{*}C_{k}\,\text{tr}\left(\left(2\hat{x}_{\ell}\hat{x}_{j}\hat{x}_{k}\hat{x}_{m}+2i\Sigma_{j\ell}\hat{x}_{k}\hat{x}_{m}+2i\Sigma_{mk}\hat{x}_{\ell}\hat{x}_{j}\right) \\ &=\sum_{j=$$

$$\begin{split} &-2\sum_{j=1}^{N}\sum_{k=1}^{N}C_{j}^{k}\mathcal{L}_{j}\hat{x}_{m}\hat{x}_{k}-i\Sigma_{mj}\hat{x}_{l}\hat{x}_{k}-\hat{x}_{j}\hat{x}_{l}\hat{x}_{k}\hat{x}_{m}-i\Sigma_{k\ell}\hat{x}_{j}\hat{x}_{m})\hat{\rho}\Big)\\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}C_{j}^{*}C_{k}\operatorname{tr}\left(\left(i\Sigma_{km}\hat{x}_{k}\hat{x}_{j}+i\Sigma_{\ell j}\hat{x}_{k}\hat{x}_{m}+2i\Sigma_{j\ell}\hat{x}_{k}\hat{x}_{m}\right)\\ &+2i\Sigma_{mk}\hat{x}_{\ell}\hat{x}_{j}^{*}-2\Sigma_{j\ell}\Sigma_{mk}-i\Sigma_{mj}\hat{x}_{\ell}\hat{x}_{k}-i\Sigma_{k\ell}\hat{x}_{j}\hat{x}_{m}\Big)\hat{\rho}\Big)\\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}C_{j}^{*}C_{k}\operatorname{tr}\left(\left(i\Sigma_{j\ell}\hat{x}_{k}\hat{x}_{m}+i\Sigma_{mk}\hat{x}_{\ell}\hat{x}_{j}\right)\\ &-2\Sigma_{j\ell}\Sigma_{mk}-i\Sigma_{mj}\hat{x}_{\ell}\hat{x}_{k}-i\Sigma_{k\ell}\hat{x}_{j}\hat{x}_{m}\Big)\hat{\rho}\Big)\\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}iC_{j}^{*}C_{k}\Sigma_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}\hat{\rho}\right)+\sum_{j=1}^{N}\sum_{k=1}^{N}iC_{j}^{*}C_{k}\Sigma_{mk}\operatorname{tr}\left(\hat{x}_{\ell}\hat{x}_{j}\hat{\rho}\right)\\ &-2\sum_{j=1}^{N}\sum_{k=1}^{N}iC_{j}^{*}C_{k}\Sigma_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}\hat{\rho}\right)+\sum_{j=1}^{N}\sum_{k=1}^{N}iC_{j}^{*}C_{k}\Sigma_{mk}\operatorname{tr}\left(\hat{x}_{\ell}\hat{x}_{j}\hat{\rho}\right)\\ &-\sum_{j=1}^{N}\sum_{k=1}^{N}iC_{j}^{*}C_{k}\Sigma_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}\hat{\rho}\right)+\sum_{j=1}^{N}\sum_{k=1}^{N}iC_{j}^{*}C_{k}\Sigma_{mk}\operatorname{tr}\left(\hat{x}_{\ell}\hat{x}_{j}\hat{\rho}\right)\\ &-\sum_{j=1}^{N}\sum_{k=1}^{N}iC_{j}^{*}C_{k}\Sigma_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}\hat{\rho}\right)+\sum_{j=1}^{N}\sum_{k=1}^{N}iC_{j}^{*}C_{k}\Sigma_{mk}\operatorname{tr}\left(\hat{x}_{\ell}\hat{x}_{j}\hat{\rho}\right)\\ &-\sum_{j=1}^{N}\sum_{k=1}^{N}iC_{j}^{*}C_{k}\sum_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}\hat{\rho}\right)\\ &+\sum_{j=1}^{N}\sum_{k=1}^{N}i\left(C_{j}^{*}C_{k}-C_{k}^{*}C_{j}\right)\Sigma_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}\hat{\rho}\right)\\ &+\sum_{j=1}^{N}\sum_{k=1}^{N}2\operatorname{Im}\left(C_{j}C_{k}^{*}\right)\Sigma_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}\hat{\rho}\right)\\ &+\sum_{j=1}^{N}\sum_{k=1}^{N}2\operatorname{Im}\left(C_{j}C_{k}^{*}\right)\Sigma_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}\hat{\rho}\right)\\ &+\sum_{j=1}^{N}\sum_{k=1}^{N}2\operatorname{Im}\left(C_{j}C_{k}^{*}\right)\Sigma_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}\hat{\rho}\right)\\ &+\sum_{j=1}^{N}\sum_{k=1}^{N}2\operatorname{Im}\left(C_{j}C_{k}^{*}\right)\Sigma_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}+i\Sigma_{m}\hat{\rho}\right)\\ &+\sum_{j=1}^{N}\sum_{k=1}^{N}2\operatorname{Im}\left(C_{j}C_{k}^{*}\right)\Sigma_{j\ell}\operatorname{tr}\left(\frac{\hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}+i\Sigma_{m}\hat{\rho}}{2}\right)\\ &-2\sum_{j=1}^{N}\sum_{k=1}^{N}2\operatorname{Im}\left(C_{j}C_{k}^{*}\right)\Sigma_{j\ell}\operatorname{tr}\left(\hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}\right)\end{aligned}$$

$$+ \sum_{j=1}^{2N} \sum_{k=1}^{2N} \operatorname{Im}(C_{j}C_{k}^{*}) (i\Sigma_{j\ell}\Sigma_{km})$$

$$+ \sum_{j=1}^{2N} \sum_{k=1}^{2N} \operatorname{Im}(C_{j}C_{k}^{*}) \Sigma_{mk} \langle \hat{x}_{\ell}\hat{x}_{j} + \hat{x}_{j}\hat{x}_{\ell} \rangle$$

$$+ \sum_{j=1}^{2N} \sum_{k=1}^{2N} \operatorname{Im}(C_{j}C_{k}^{*}) (i\Sigma_{mk}\Sigma_{\ell j})$$

$$- 2 \sum_{j=1}^{2N} \sum_{k=1}^{2N} C_{j}^{*}C_{k}\Sigma_{j\ell}\Sigma_{mk}$$

$$= \sum_{j=1}^{2N} \sum_{k=1}^{2N} \sum_{k=1}^{2N} \operatorname{Im}(C_{j}^{*}C_{k}) \langle \hat{x}_{k}\hat{x}_{m} + \hat{x}_{m}\hat{x}_{k} \rangle$$

$$+ 2i \sum_{j=1}^{2N} \sum_{k=1}^{2N} \sum_{k=1}^{2N} \sum_{k=1}^{2N} \operatorname{Im}(C_{j}^{*}C_{k}) \Sigma_{km}$$

$$+ \sum_{j=1}^{2N} \sum_{k=1}^{2N} \langle \hat{x}_{\ell}\hat{x}_{j} + \hat{x}_{j}\hat{x}_{\ell} \rangle \operatorname{Im}(C_{j}^{*}C_{k}) \Sigma_{km}$$

$$- 2 \sum_{j=1}^{2N} \sum_{k=1}^{2N} \sum_{k=1}^{2N} \sum_{k=1}^{2N} \sum_{k=1}^{2N} \operatorname{Im}(C_{j}^{*}C_{k}) \langle \hat{x}_{k}\hat{x}_{m} + \hat{x}_{m}\hat{x}_{k} \rangle$$

$$+ \sum_{j=1}^{2N} \sum_{k=1}^{2N} \sum_{k=1}^{2N} \sum_{k=1}^{2N} \operatorname{Im}(C_{j}^{*}C_{k}) \langle \hat{x}_{k}\hat{x}_{m} + \hat{x}_{m}\hat{x}_{k} \rangle$$

$$+ \sum_{j=1}^{2N} \sum_{k=1}^{2N} \sum_{k=1}^{2N} \langle \hat{x}_{\ell}\hat{x}_{j} + \hat{x}_{j}\hat{x}_{\ell} \rangle \operatorname{Im}(C_{j}^{*}C_{k}) \Sigma_{km}$$

$$- 2 \sum_{j=1}^{2N} \sum_{k=1}^{2N} \sum$$

Third, using (8), we have

$$\frac{d\langle\hat{x}_{\ell}\rangle\langle\hat{x}_{m}\rangle}{dt} = \frac{d\langle\hat{x}_{\ell}\rangle}{dt}\langle\hat{x}_{m}\rangle + \langle\hat{x}_{\ell}\rangle\frac{d\langle\hat{x}_{m}\rangle}{dt}$$

$$= \left(\sum_{j=1}^{2N}\sum_{k=1}^{2N}\sum_{k=1}^{2N}\sum_{k=j}^{2N}\sum_{k=1}^{2N}\sum_{k=1}^{2N}\sum_{k=1}^{2N}\sum_{k=j}^{2N}$$

Substituting (10), (11) and (12) into (9), we obtain $= \frac{1}{2} \operatorname{tr} \left(\left(\hat{x}_{\ell} \hat{x}_{m} + \hat{x}_{m} \hat{x}_{\ell} \right) \left(-i [\hat{H}, \hat{\rho}] \right) \right)$ $+\frac{1}{2}\operatorname{tr}\left(\left(\hat{x}_{\ell}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{\ell}\right)\left(\mathfrak{D}\left[\hat{c}\right]\hat{\boldsymbol{\rho}}\right)\right)-\frac{d\left(\left\langle\hat{x}_{\ell}\right\rangle\left\langle\hat{x}_{m}\right\rangle\right)}{dt}$ $=\sum_{N=1}^{2N}\sum_{k=1}^{2N}\sum_{\ell}\sum_{\ell}M_{jk}\frac{\langle\hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}\rangle}{2}$ $-\sum_{k=1}^{2N}\sum_{j=1}^{2N}\frac{\langle\hat{x}_{\ell}\hat{x}_{j}+\hat{x}_{j}\hat{x}_{\ell}\rangle}{2}M_{jk}\Sigma_{km}$ $+\sum_{i=1}^{2N}\sum_{j=1}^{2N}\sum_{\ell j}\operatorname{Im}\left(C_{j}^{*}C_{k}\right)\frac{\left\langle \hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}\right\rangle }{2}$ $+\sum_{i=1}^{2N}\sum_{j=1}^{2N}\frac{\langle\hat{x}_{\ell}\hat{x}_{j}+\hat{x}_{j}\hat{x}_{\ell}\rangle}{2}\operatorname{Im}\left(C_{j}^{*}C_{k}\right)\Sigma_{km}$ $-\sum_{i=1}^{2N}\sum_{j=1}^{2N}\Sigma_{\ell j}\operatorname{Re}\left(C_{j}^{*}C_{k}\right)\Sigma_{km}$ $-\sum_{i=1}^{2N}\sum_{j=1}^{2N}\sum_{\ell j}\left(M_{jk}+\operatorname{Im}\left(C_{j}^{*}C_{k}\right)\right)\langle\hat{x}_{k}\rangle\langle\hat{x}_{m}\rangle$ $-\sum_{i=1}^{2N}\sum_{j=1}^{2N}\langle\hat{x}_{\ell}\rangle\langle\hat{x}_{j}
angleigg(-M_{jk}+\operatorname{Im}ig(C_{j}^{st}C_{k}ig)igg)\Sigma_{km}$ $=\sum_{i=1}^{2N}\sum_{j=1}^{2N}\sum_{\ell j}\left(M_{jk}+\operatorname{Im}\left(C_{j}^{*}C_{k}\right)\right)\left(\frac{\left\langle \hat{x}_{k}\hat{x}_{m}+\hat{x}_{m}\hat{x}_{k}\right\rangle }{2}-\left\langle \hat{x}_{k}\right\rangle \left\langle \hat{x}_{m}\right\rangle \right)$ $+\sum_{i=1}^{2N}\sum_{j=1}^{2N}\left(rac{\langle\hat{x}_{\ell}\hat{x}_{j}+\hat{x}_{j}\hat{x}_{\ell}
angle}{2}-\langle\hat{x}_{\ell}
angle\langle\hat{x}_{j}
angle
ight)\left(\operatorname{Im}\left(C_{j}^{st}C_{k}
ight)-M_{jk}
ight)\Sigma_{km}$ $-\sum_{i=1}^{2N}\sum_{j=1}^{2N}\sum_{\ell j}\operatorname{Re}\left(C_{j}^{*}C_{k}\right)\Sigma_{km}$ $=\sum_{j=1}^{2N}\sum_{k=1}^{2N}\Sigma_{\ell j}igg(M_{jk}+\operatorname{Im}ig(C_{j}^{st}C_{k}ig)igg)V_{km}$ $+\sum_{i=1}^{2N}\sum_{j=1}^{2N}V_{\ell j}\bigg(-M_{jk}+\operatorname{Im}\left(C_{j}^{*}C_{k}\right)\bigg)\Sigma_{km}$ $-\sum_{j=1}^{2N}\sum_{k=1}^{2N}\Sigma_{\ell j}\operatorname{Re}\left(C_{j}^{*}C_{k}\right)\Sigma_{km}$ $= \Sigma_{\ell:} \left(M + \operatorname{Im} \left(C^{\dagger} C \right) \right) V_{:m} + V_{\ell:} \left(-M + \operatorname{Im} \left(C^{\dagger} C \right) \right) \Sigma_{:m}$

where $V_{:m}$ denotes the *m*th column of V and V_{ℓ} : denotes the ℓ th row of V. It follows from (13) that

(13)

 $-\Sigma_{\ell}$ Re $(C^{\dagger}C)\Sigma_{m}$,

$$\begin{split} \frac{dV}{dt} &= \Sigma \bigg(M + \operatorname{Im} \left(C^{\dagger} C \right) \bigg) V + V \bigg(- M + \operatorname{Im} \left(C^{\dagger} C \right) \bigg) \Sigma \\ &- \Sigma \operatorname{Re} \left(C^{\dagger} C \right) \Sigma \\ &= \Sigma \bigg(M + \operatorname{Im} \left(C^{\dagger} C \right) \bigg) V + V \bigg(M + \operatorname{Im} \left(C^{\dagger} C \right) \bigg)^{\top} \Sigma^{\top} \\ &+ \Sigma \operatorname{Re} \left(C^{\dagger} C \right) \Sigma^{\top}. \end{split}$$

That is, Equation (4) holds. This completes the derivation.

Remark 1. The above results can be easily extended to linear open quantum systems with multiple decoherence channels. By adding extra decoherence-induced terms (which are analogous to those obtained in (7) and (11)), we can obtain the moment evolution equations for linear open quantum systems with multiple decoherence channels. The results are given by (3) and (4).

REFERENCES

- H. P. Breuer and F. Petruccione, The Theory of Open Quantum Systems. Oxford University Press, 2002.
- [2] C. W. Gardiner and P. Zoller, Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics, 2nd ed. Springer, 2000.
- [3] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, "Completely positive dynamical semigroups of N-level systems," *Journal of Mathematical Physics*, vol. 17, no. 5, pp. 821–825, 1976.
- [4] G. Lindblad, "On the generators of quantum dynamical semigroups," Communications in Mathematical Physics, vol. 48, no. 2, pp. 119–130, 1976
- [5] D. F. Walls and G. J. Milburn, *Quantum Optics*, 2nd ed. Springer-Verlag Berlin Heidelberg, 2008.
- [6] H. M. Wiseman and A. C. Doherty, "Optimal unravellings for feedback control in linear quantum systems," *Physical Review Letters*, vol. 94, no. 7, p. 070405, 2005.
- [7] A. C. Doherty and H. M. Wiseman, "Feedback control of linear quantum systems," in *Proceedings of Quantum Electronics and Laser Science Conference*, May 2005, pp. 41–43.
- [8] H. M. Wiseman and G. J. Milburn, Quantum Measurement and Control. Cambridge University Press, 2010.
- [9] K. Jacobs, Quantum Measurement Theory and its Applications, 1st ed. Cambridge University Press, 2014.
- [10] M. G. Genoni, L. Lami, and A. Serafini, "Conditional and unconditional Gaussian quantum dynamics," *Contemporary Physics*, vol. 57, no. 3, pp. 331–349, 2016.
- [11] K. Koga and N. Yamamoto, "Dissipation-induced pure Gaussian state," Physical Review A, vol. 85, no. 2, p. 022103, 2012.
- [12] N. Yamamoto, "Pure Gaussian state generation via dissipation: a quantum stochastic differential equation approach," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 370, no. 1979, pp. 5324–5337, 2012.
- [13] Y. Ikeda and N. Yamamoto, "Deterministic generation of Gaussian pure states in a quasilocal dissipative system," *Physical Review A*, vol. 87, no. 3, p. 033802, 2013.
- [14] S. Ma, M. J. Woolley, I. R. Petersen, and N. Yamamoto, "Preparation of pure Gaussian states via cascaded quantum systems," in *Proceedings* of *IEEE Conference on Control Applications (CCA)*, October 2014, pp. 1970–1975.
- [15] F. Nicacio, M. Paternostro, and A. Ferraro, "Determining stationarystate quantum properties directly from system-environment interactions," *Physical Review A*, vol. 94, no. 5, p. 052129, 2016.
- [16] S. Ma, "Pure Gaussian States in Open Quantum Systems," Ph.D. thesis, University of New South Wales, Canberra, Australia, April 2017.
- [17] S. Ma, M. J. Woolley, I. R. Petersen, and N. Yamamoto, "Cascade and locally dissipative realizations of linear quantum systems for pure Gaussian state covariance assignment," *Automatica*, vol. 90, pp. 263– 270, 2018.
- [18] S. Ma, I. R. Petersen, and M. J. Woolley, "Linear quantum systems with diagonal passive Hamiltonian and a single dissipative channel," *Systems & Control Letters*, vol. 99, pp. 64–71, 2017.
- [19] S. Ma, M. J. Woolley, I. R. Petersen, and N. Yamamoto, "Pure Gaussian quantum states from passive Hamiltonians and an active local dissipative process," in *Proceedings of IEEE 55th Annual Conference* on *Decision and Control (CDC)*, December 2016, pp. 2519–2522. [Online]. Available: http://arxiv.org/abs/1608.02698

[20] S. Ma, M. J. Woolley, I. R. Petersen, and N. Yamamoto, "Pure Gaussian states from quantum harmonic oscillator chains with a single local dissipative process," *Journal of Physics A: Mathematical and Theoretical*, vol. 50, no. 13, p. 135301, 2017.

Appendix: Notation

$\mathbb{R},\ \mathbb{R}^{m imes n}$	Real numbers, real $m \times n$ matrices.
\mathbb{C} , $\mathbb{C}^{m \times n}$	Complex numbers, complex $m \times n$ matrices.
I_n	The $n \times n$ identity matrix.
a^* , \hat{a}^*	The complex conjugate of a complex number a , the adjoint of an operator \hat{a} .
$A^{ op}$	Transpose of A, i.e., $(A^{\top})_{jk} = A_{kj}$.
$A^{\#}$	Entrywise conjugate of A , i.e., $(A^{\#})_{jk} = A_{jk}^*$.
A^\dagger	Conjugate transpose of A , i.e., $(A^{\dagger})_{jk} = A_{kj}^*$.
Re(A), $Im(A)$	The real part, imaginary part of A , i.e., $\operatorname{Re}(A) = \left(A + A^{\#}\right)/2$ and $\operatorname{Im}(A) = \left(A - A^{\#}\right)/(2i)$.
tr(A)	Trace.