An Inverse Boundary Value Problem for the Magnetic Schrödinger Operator on a Half Space

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Abstract: This licentiate thesis is concerned with an inverse boundary value problem for the magnetic Schrödinger equation in a half space, for potentials $A \in W^{1,\infty}_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{R}^3)$ and $q \in L^{\infty}_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{C})$. We prove that q and the curl of A are uniquely determined by the knowledge of the Dirichlet-to-Neumann map on parts of the boundary of the half space. The existence and uniqueness of the corresponding direct problem are also considered.

Contents

1 Introduction

The main purpose of this thesis is to investigate an inverse problem for the magnetic Schrödinger operator in the half space geometry. The magnetic Schrödinger operator $L_{A,q}$ is defined by

$$L_{A,q} := \sum_{j=1}^{3} (-i\partial_j + A_j)^2 + q(x). \tag{1.1}$$

We fix the half space by considering $\mathbb{R}^3_- := \{x \in \mathbb{R}^3 \mid x_3 < 0\}$. We shall moreover assume that

$$A \in W^{1,\infty}_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{R}^3), \text{ and } q \in L^\infty_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{C}), \quad \operatorname{Im} q \leq 0. \tag{1.2}$$

Here

$$W^{1,\infty}_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{R}^3) := \{A|_{\overline{\mathbb{R}^3_-}} \mid A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3), \operatorname{supp}(A) \subset \mathbb{R}^3 \text{ compact}\}$$

and similarly, we define

$$L^\infty_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{C}) := \{q \in L^\infty(\overline{\mathbb{R}^3_-}, \mathbb{C}) \mid \operatorname{supp}(q) \subset \overline{\mathbb{R}^3_-} \text{ compact}\}.$$

The direct problem, from which the inverse problem stems, is the Dirichlet problem

$$\begin{aligned} (L_{A,q} - k^2)u &= 0 \quad \text{in } \mathbb{R}^3_-, \\ u|_{\partial \mathbb{R}^3_-} &= f, \end{aligned} \tag{1.3}$$

where $k > 0$ is fixed and $f \in H^{3/2}_{comp}(\partial \mathbb{R}^3_-)$. Furthermore, we will also require that the solution u should satisfy a boundary condition at infinity, which will be the *Sommerfeld radiation condition*

$$\lim_{|x| \to \infty} |x| \left(\frac{\partial u(x)}{\partial |x|} - iku(x) \right) = 0. \tag{1.4}$$

Solutions satisfying this condition are called *outgoing* or *radiating* solutions. We will also occasionally use the term *incoming* solution. This refers to a solution of (1.3) that satisfies (1.4), when the factor $-ik$ is replaced by ik.

The following result, which will be established in the first part of this work, gives the solvability of the direct problem (1.3), (1.4).

Theorem 1.1. *Let* $A \in W^{1,\infty}_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{R}^3)$ *and* $q \in L^\infty_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{C})$ *be such that* $\operatorname{Im} q \leq 0$. *Then for any* $f \in H^{3/2}_{comp}(\partial \mathbb{R}^3_-)$, *the Dirichlet problem* (1.3), (1.4) *has a unique solution* $u \in H^2_{loc}(\overline{\mathbb{R}^3_-})$.

Here

$$H^2_{loc}(\overline{\mathbb{R}^3_-}) := \{u|_{\overline{\mathbb{R}^3_-}} \mid u \in H^2_{loc}(\mathbb{R}^3)\}.$$

Theorem 1.1 permits us to define the so called *Dirichlet to Neumann map* $\Lambda_{A,q}$, (DN-map for short), $\Lambda_{A,q} : H^{3/2}_{comp}(\partial \mathbb{R}^3_-) \to H^{1/2}_{loc}(\partial \mathbb{R}^3_-)$ as

$$f \mapsto (\partial_n + iA \cdot n)u|_{\partial \mathbb{R}^3_-},$$

where u is the solution of the Dirichlet problem (1.3), (1.4) and f is the boundary condition. Here $n = (0, 0, 1)$ is the unit outer normal to the boundary $\partial \mathbb{R}^3_-$.

The inverse problem is then to determine if the DN-map uniquely determines the potentials A and q in $\mathbb{R}^3_-$. It turns out that the DN-map does not in general uniquely determine A. This is due to the gauge invariance of the DN-map, which was first noticed by [34]. It follows from the identities

$$e^{-i\psi} L_{A,q} e^{i\psi} = L_{A+\nabla\psi,q}, \quad e^{-i\psi} \Lambda_{A,q} e^{i\psi} = \Lambda_{A+\nabla\psi,q}, \tag{1.5}$$

that $\Lambda_{A,q} = \Lambda_{A+\nabla\psi,q}$ when $\psi \in C^{1,1}(\overline{\mathbb{R}^3_-}, \mathbb{R})$ compactly supported is such that $\psi|_{\partial \mathbb{R}^3_-} = 0$ (see Lemma 3.1). This shows that $\Lambda_{A,q}$ cannot uniquely determine A. The DN-map does however carry enough information to determine $\nabla \times A$, which is the magnetic field in the context of electrodynamics.

We shall use the notation $A_j = (A_{j,1}, A_{j,2}, A_{j,3})$ for the component functions, when considering a pair of magnetic potentials A_j, $j = 1, 2$. We now state the main result of this work, which generalizes the corresponding results of [21], obtained in the case of the Schrödinger operator without a magnetic potential.

Theorem 1.2. *Let $A_j \in W^{1,\infty}_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{R}^3)$ and $q_j \in L^{\infty}_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{C})$ be such that* $\operatorname{Im} q_j \leq 0$, *$j = 1, 2$. Denote by B an open ball in $\mathbb{R}^3$, containing the supports of A_j, and q_j, $j = 1, 2$. Let $\Gamma_1, \Gamma_2 \subset \partial \mathbb{R}^3_-$ be open sets such that*

$$(\partial \mathbb{R}^3_- \setminus \overline{B}) \cap \Gamma_j \neq \emptyset, \quad j = 1, 2. \tag{1.6}$$

Then if

$$\Lambda_{A_1,q_1}(f)|_{\Gamma_1} = \Lambda_{A_2,q_2}(f)|_{\Gamma_1}, \tag{1.7}$$

for all $f \in H^{3/2}_{comp}(\partial \mathbb{R}^3_-)$, $\operatorname{supp}(f) \subset \Gamma_2$, *then*

$$\nabla \times A_1 = \nabla \times A_2 \quad \textit{and} \quad q_1 = q_2 \quad \textit{in} \quad \mathbb{R}^3_-.$$

We would like to emphasize that in Theorem 1.2, the set Γ_1, where measurements are performed, and the set Γ_2, where the data is supported, can be taken arbitrarily small, provided that (1.6) holds. The result of Theorem 1.2 pertains therefore to inverse problems with partial data. Such problems are important from the point of view of applications, since in practice, performing measurements on the entire boundary could be impossible, due to limitations in resources or obstructions from obstacles.

The first uniqueness result, in the context of inverse boundary value problems for the magnetic Schrödinger operator on a bounded domain, was obtained by Sun in [34], under a smallness condition on A. Nakamura, Sun and Uhlmann proved the uniqueness without any smallness condition in [27], assuming that $A \in C^2$. Tolmasky extended this result to C^1 magnetic potentials, in [35] and Panchenko to some less regular but small magnetic potentials in [29]. Salo proved uniqueness for Dini continuous magnetic potentials in [31]. The most recent result is given by Krupchyk and Uhlmann in [20], where uniqueness is proved for L^∞ magnetic potentials. In all of these works, the inverse boundary value problem with full data was considered.

In [10], Eskin and Ralston obtained a uniqueness result for the closely related inverse scattering problem, assuming the exponential decay of the potentials. The partial data problem in the magnetic case was considered by Dos Santos Ferreira, Kenig, Sjöstrand and Uhlmann in [9] and by Chung in [6].

The inverse problem for the half space geometry, without a magnetic potential was examined by Cheney and Isaacson in [4]. The uniqueness for this problem in the case of compactly supported electric potentials was proved by Lassas, Cheney and Uhlmann in [21], assuming that the supports do not come close to the boundary of the half space. The result of Theorem 1.2 is therefore already a generalization of the work [21], even in the absence of magnetic potentials. Li and Uhlmann proved uniqueness for the closely related infinite slab geometry with $A = 0$, in [26]. Krupchyk, Lassas and Uhlmann did this for the magnetic case in [19]. In both of these works, the reflection argument of Isakov [16] played an important role. The uniqueness problem for the magnetic potentials in the slab and half space geometries has also been studied in a recent paper by Li [25]. The half space results in [25] differ from the ones given in this work, by concerning the more general matrix valued Schrödinger equation and by assuming C^6 regularity on the magnetic potential.

The half space is perhaps the simplest example of an unbounded region with an unbounded boundary. It is of special interest in many applications, such as geophysics, ocean acoustics, and optical tomography, since it provides a simple model for semi infinite geometries. We would like to mention that the magnetic Schrödinger equation is closely related to the diffusion approximation of the photon transport equation, used in optical tomography

[3]. The half space geometry is also of interest in optical tomography, since in practice, the source-detector pairs are often located on the same interface [23].

The thesis is divided into two main parts. Section 2 gives a detailed account of the solvability of the direct problem and provides a proof of Theorem 1.1. Subsection 2.1 develops some basic tools from scattering theory. In Subsection 2.2 we prove the existence and uniqueness for the direct scattering problem in all of $\mathbb{R}^3$, using the Lax–Phillips method. In Subsection 2.3 we extend this discussion to the half space case, using a reflection argument.

Section 3 deals with the inverse problem and contains the proof of Theorem 1.2. In Subsection 3.1 a central integral identity is derived. Subsection 3.2 contains the construction of complex geometric optics solutions for magnetic Schrödinger operators with Lipschitz continuous potentials. The proof of Theorem 1.2 is concluded in Subsections 3.3 and 3.4.

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2 The direct problem

The purpose of this section is to investigate the well-posedness of the boundary value problem (1.3), (1.4) for the magnetic Schrödinger operator in the half space, and to establish Theorem 1.1.

We prove existence and uniqueness results for this problem, but we will not concern ourselves with showing that solutions depend continuously on boundary data. This is enough to guarantee that the questions of the uniqueness of the inverse problem are sensible.

2.1 The free space outgoing Green function

The aim of this subsection is to introduce the outgoing Green function for the Helmholtz equation and develop some basic notions used in scattering theory that we need to attack the direct problem. For a more in depth exposition, see [7]. Having constructed the outgoing Green function, we use it to investigate the corresponding resolvent operator and the asymptotics of solutions.

A function G is generally speaking a Green function for the Helmholtz equation if it solves the problem

$$(-\Delta_x - k^2)G(x,y) = \delta(x-y), \quad y \in \Omega, \tag{2.1}$$

in some region Ω and satisfies some specific boundary condition on $\partial\Omega$. We are interested in the case $\Omega = \mathbb{R}^3$ with the boundary condition being the Sommerfeld radiation condition (1.4).

In the next proposition we construct a specific Green function called the *outgoing free space Green function* denoted by G_0, which satisfies (2.1) and the Sommerfeld condition.

Proposition 2.1. *Let $k > 0$. The function*

$$G_0(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{2.2}$$

is a free space outgoing Green function.

Proof. We start by assuming $y = 0$ in (2.1) and thinking of G_0 as a function of x only. Instead of considering the operator $-\Delta - k^2$ directly, we first look for outgoing Green's functions G_ϵ for the operators $-\Delta - k^2 - i\epsilon$, where $\epsilon > 0$. By inserting $i\epsilon$ into (2.1) and taking the Fourier transform we get that

$$\widehat{G}_\epsilon = \frac{1}{\xi^2 - k^2 - i\epsilon}.$$

Since this is a locally integrable function, decaying at infinity, we may take the inverse Fourier transform. Let us therefore proceed by calculating the inverse Fourier transform,

$$
\begin{aligned}
\mathcal{F}^{-1}\left(\frac{1}{\xi^2 - k^2 - i\epsilon}\right)(x) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{e^{i\xi\cdot x}}{\xi^2 - k^2 - i\epsilon} d\xi \qquad (2.3)\\
&= \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{e^{i\xi(r,\theta,\phi)\cdot x}}{r^2 - k^2 - i\epsilon} r^2 \sin\theta d\phi d\theta dr.
\end{aligned}
$$

The Fourier transform of a spherically symmetric function is spherically symmetric. We need therefore to calculate this integral only for e.g. $x = (0,0,R)$, $R = |x|$. By doing this and abbreviating $\alpha = k^2 + i\epsilon$ we get

$$
\begin{aligned}
\int_0^{\pi}\int_0^{\infty} \frac{e^{irR\cos\theta}}{r^2 - \alpha} r^2 \sin\theta d\theta dr &= \frac{-1}{iR}\int_0^{\infty}\left[\frac{e^{irRcos\theta}}{r^2-\alpha}\right]_0^{\pi} rdr\\
&= \frac{-1}{iR}\int_0^{\infty} \frac{e^{-irR} - e^{irR}}{r^2 - \alpha} rdr\\
&= \frac{1}{iR}\int_{-\infty}^{\infty} \frac{e^{irR}}{r^2 - \alpha} rdr.
\end{aligned}
$$

We can evaluate this last integral with the method of residues, if we temporarily replace r with a complex variable z. We choose as a contour, an origin centered half circle C_s that is in the upper half plane and goes along the real axis. The parameter s stands for radius of the circle. By choosing the branch of the complex square root in the upper half plane and a large enough s, we achieve that the pole $\sqrt{\alpha}$ will lie inside the contour. A residue integration gives us then

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{irR}}{r^2 - \alpha} rdr &= \lim_{s\to\infty} \oint_{C_s} \frac{e^{izR}}{(z - \sqrt{\alpha})(z + \sqrt{\alpha})} zdz\\
&= 2\pi i \operatorname{Res}\left(\frac{e^{izR} z}{(z - \sqrt{\alpha})(z + \sqrt{\alpha})}, \sqrt{\alpha}\right)\\
&= \pi i e^{iR\sqrt{\alpha}}.
\end{aligned}
$$

By going back to (2.3) we see that

$$
\mathcal{F}^{-1}\left(\frac{1}{\xi^2 - k^2 - i\epsilon}\right)(x) = \frac{e^{i\sqrt{k^2+i\epsilon}R}}{4\pi R}, \quad R = |x|.
$$

In other words, $G_\epsilon(x) = e^{i\sqrt{k^2+i\epsilon}|x|}/(4\pi|x|)$.

We will need to use the theory of distributions to extend the above argument to the case $\epsilon = 0$. We define the tempered distribution $(\xi^2 - k^2 - i0)^{-1}$, by taking the limit

$$\big\langle (\xi^2 - k^2 - i0)^{-1}, \varphi \big\rangle = \lim_{\epsilon \to 0+} \big\langle (\xi^2 - k^2 - i\epsilon)^{-1}, \varphi \big\rangle \tag{2.4}$$

for $\varphi \in \mathcal{S}(\mathbb{R}^3)$, where the distribution pairing on the right is given by an integral. One can check that this definition makes sense, using similar arguments as for the principal value distribution. We define

$$G_0 := \mathcal{F}^{-1}\big((\xi^2 - k^2 - i0)^{-1}\big).$$

The first part of the proof shows that

$$\begin{aligned}
\big\langle \mathcal{F}^{-1}\big((\xi^2 - k^2 - i0)^{-1}\big), \varphi \big\rangle &= \lim_{\epsilon \to 0+} \langle \mathcal{F}^{-1} \widehat{G}_\epsilon, \varphi \rangle \\
&= \lim_{\epsilon \to 0+} \int \frac{e^{i\sqrt{k^2 + i\epsilon}|x|}}{4\pi |x|} \varphi(x) dx \\
&= \int \frac{e^{ik}|x|}{4\pi |x|} \varphi(x) dx,
\end{aligned}$$

so that $G_0(x) = e^{ik|x|}/(4\pi|x|)$.

To check condition (2.1), we first note that G_ϵ is a Green function, and hence

$$\lim_{\epsilon \to 0+} \big\langle (-\Delta - k^2 - i\epsilon) G_\epsilon, \varphi \big\rangle = \big\langle \delta, \varphi \big\rangle. \tag{2.5}$$

On the other hand we have

$$\begin{aligned}
\lim_{\epsilon \to 0+} \big\langle (-\Delta - k^2 - i\epsilon) G_\epsilon, \varphi \big\rangle &= \lim_{\epsilon \to 0+} \Big(\big\langle (-\Delta - k^2) G_\epsilon, \varphi \big\rangle + \big\langle -i\epsilon G_\epsilon, \varphi \big\rangle \Big) \\
&= \big\langle G_0, (-\Delta - k^2)\varphi \big\rangle + \lim_{\epsilon \to 0+} \big\langle -i\epsilon G_\epsilon, \varphi \big\rangle.
\end{aligned}$$

For the last term we have $-i\epsilon \langle G_\epsilon, \varphi \rangle \to 0$, as $\epsilon \to 0$. Combining the above with (2.5) gives

$$\big\langle (-\Delta - k^2) G_0, \varphi \big\rangle = \big\langle \delta, \varphi \big\rangle,$$

so that (2.1) holds.

A direct calculation shows that G_0 satisfies the Sommerfeld radiation condition (1.4). Finally let us set $G_0(x, y) = G_0(x - y)$. $\square$

Remark 2.2. *Notice that $G_0 \in L^2_{loc}(\mathbb{R}^3)$. Since G_0 satisfies* (2.1) *in the sense of distributions, we have for $\varphi \in L^2_{comp}(\mathbb{R}^3)$*

$$(-\Delta - k^2)(G_0 * \varphi) = \delta * \varphi = \varphi.$$

Remark 2.2 motivates the notation,

$$(-\Delta - k^2 - i0)^{-1}\varphi(x) := G_0 * \varphi,$$

where $-i0$ marks the fact that we took the positive sided limit in (2.4). This is important since the positive limit guarantees that G_0 and $G_0 * \varphi$ are outgoing, i.e. satisfy the Sommerfeld condition. We would have ended up with an *incoming solution*, had we taken the negative sided limit.

The above operator is in fact a limiting value of the resolvent of the Laplace operator. This operator is not continuous on $L^2(\mathbb{R}^3)$, when $\operatorname{Im} k = 0$. It is however continuous between certain weighted spaces (see e.g. [2]).

The next lemma gives a more modest continuity result, which will be used later.

Lemma 2.3. *The operator*

$$(-\Delta - k^2 - i0)^{-1} : L^2_{comp}(\mathbb{R}^3) \to L^2_{loc}(\mathbb{R}^3)$$

is continuous.

Proof. Let us write $T := (-\Delta - k^2 - i0)^{-1}$. Let B_r be a ball centered at the origin of radius $r > 0$ and χ_{B_r} stand for the characteristic function of the ball B_r. It suffices to show that the operator $\chi_{B_r} T \chi_{B_r}$ is continuous on $L^2(\mathbb{R}^3)$. We have, for $x \in B_r$,

$$T\chi_{B_r}\varphi(x) = \int_{B_r} G_0(x-y)\varphi(y)dy, \quad \varphi \in L^2(\mathbb{R}^3).$$

Since $G_0 \in L^2(B_r \times B_r)$ we get by applying the Cauchy–Schwarz inequality, that

$$\begin{aligned}\int_{B_r} |T\chi_{B_r}\varphi(x)|^2 dx &\leq \int_{B_r} \left(\int_{B_r} G_0(x-y)\varphi(y)dy \right)^2 dx \\ &\leq \int_{B_r} \int_{B_r} |G_0(x-y)|^2 dy \, \|\varphi\|_2^2 dx \\ &\leq C\|\varphi\|_2^2.\end{aligned}$$

Hence $\|\chi_{B_r} T \chi_{B_r}\varphi\|_2 \leq C\|\varphi\|_2$. ∎

Next we use the closed graph theorem and elliptic regularity to extend the above result to case where the range is the Sobolev space $H^2_{loc}(\mathbb{R}^3)$.

Corollary 2.4. *The operator* $(-\Delta - k^2 - i0)^{-1} : L^2_{comp}(\mathbb{R}^3) \to H^2_{loc}(\mathbb{R}^3)$ *is continuous.*

Proof. Let us write $T := (-\Delta - k^2 - i0)^{-1}$. By elliptic regularity, we have

$$TL^2_{comp}(\mathbb{R}^3) \subset H^2_{loc}(\mathbb{R}^3).$$

The claim follows from the closed graph theorem, once we show that the operator $T : L^2(B) \to H^2(B)$ is closed. Here B is an open ball in $\mathbb{R}^3$. To that end, let $f_n \in L^2(B)$ be such that $f_n \to f$ in $L^2(B)$ and $Tf_n \to g$ in $H^2(B)$. By Lemma 2.3, we know that $Tf_n \to Tf$ in $L^2(B)$. Hence, $g = Tf$. This completes the proof. $\square$

We now begin investigating the asymptotics of radiating solutions to the Helmholtz equation. First we look at the asymptotics of G_0.

Lemma 2.5. *The outgoing Green function has the following asymptotics,*

$$G_0(x, y) = \frac{e^{ik(|x|-(x\cdot y)/|x|)}}{4\pi|x|} + O\left(\frac{1}{|x|^2}\right), \tag{2.6}$$

as $|x| \to \infty$, *uniform for* y *in a bounded set. It can be differentiated with respect to* y.

Proof. We will derive the asymptotic expression estimating the nominator and the reciprocal of the denominator of G_0 separately.

We start by writing

$$|x - y| = |x|\sqrt{1 - \frac{2x \cdot y}{|x|^2} + \frac{|y|^2}{|x|^2}}.$$

By looking at the Taylor series we see that $\sqrt{1 + x} = 1 + x/2 + O(x^2)$, as $x \to 0$. Applying this to the above gives

$$|x - y| = |x| - \frac{x \cdot y}{|x|} + O\left(\frac{1}{|x|}\right),$$

for $|x| \to \infty$. It follows that

$$\frac{1}{|x - y|} = \frac{1}{|x|} + O\left(\frac{1}{|x|^2}\right), \tag{2.7}$$

as $|x| \to \infty$.

For the nominator of G_0, let $f(x) = O(1/|x|)$ be the function, for which $ik|x-y| = ik|x| - ikx \cdot y/|x| + f(x)$. Then

$$\begin{aligned}\exp(ik|x-y|) &= \exp\left(ik|x| - ik\frac{x \cdot y}{|x|}\right)\exp(f(x)) \\ &= \exp\left(ik|x| - ik\frac{x \cdot y}{|x|}\right)\left(1 + f(x) + \frac{f(x)^2}{2!} + \dots\right),\end{aligned}$$

which gives the expression

$$\exp(ik|x-y|) = \exp\left(ik|x| - ik\frac{x \cdot y}{|x|}\right) + O\left(\frac{1}{|x|}\right),$$

as $|x| \to \infty$. Multiplying this with the asymptotic expression (2.7) gives the asymptotic expression for G_0. $\square$

The next small lemma shows that the L^2 norm over a sphere of an outgoing solution to the Helmholtz equation stays bounded as the radius of the sphere grows. Note that this applies also to incoming solutions. This will be used later. The lemma is mainly needed, since we use a weak form of the Sommerfeld radiation condition (1.4).

Lemma 2.6. *Let $u \in H^2_{loc}(\mathbb{R}^3)$ be an outgoing or incoming solution to the Helmholtz equation $(-\Delta - k^2)u = 0$ in $\mathbb{R}^3 \setminus \overline{B_r}$, where $B_r = \{x \mid |x| < r\}$. Then*

$$\|u\|_{L^2(\partial B_s)} = O(1), \quad \|\partial_n u\|_{L^2(\partial B_s)} = O(1), \quad s \to \infty. \tag{2.8}$$

Proof. Pick a ball $B_s \supset \overline{B_r}$. Assume that u is outgoing, i.e. it satisfies (1.4). Multiplying this condition with its complex conjugate and integrating over B_s gives

$$\int_{|x|=s} \big(k^2|u|^2 + |\partial_n u|^2 + 2k\,\mathrm{Im}(u\partial_n\overline{u})\big)dS = \int_{|x|=s} |\,\partial_n\, u - iku|^2 dS \to 0, \tag{2.9}$$

as $s \to \infty$. Here n is the unit outer normal to $\partial\, B_s$. Using Green's formulas we have that

$$\int_{\partial B_s \cup \partial B_r} (u\partial_n\overline{u} - \overline{u}\partial_n u)dS = \int_{B_s \setminus B_r} (u\Delta\overline{u} - \overline{u}\Delta u)dx = 0,$$

and therefore,

$$\int_{\partial B_r} (u\partial_n\overline{u} - \overline{u}\partial_n u)dS = -\int_{\partial B_s} (u\partial_n\overline{u} - \overline{u}\partial_n u)dS$$
$$= -2i\int_{\partial B_s} \mathrm{Im}(u\partial_n\overline{u})dS.$$

It follows that the last expression does not depend on s. This shows that the two first terms in (2.9), which are both positive, are bounded, and particularly we see that (2.8) holds.

The incoming case can be reduced to the outgoing case as follows. Firstly if u is an incoming solution of the Helmholtz equation, then $\overline{u}$ is an outgoing solution. Moreover $\|u\|_{L^2} = \|\overline{u}\|_{L^2}$. The incoming case follows therefore from the outgoing case. □

The following lemma gives a boundary integral representation for radiating solutions of the Helmholtz equation in exterior regions.

Lemma 2.7. *Let $u \in H^2_{loc}(\mathbb{R}^3)$ be an outgoing solution to the Helmholtz equation $(-\Delta - k^2)u = 0$ in $\mathbb{R}^3 \setminus \overline{B_r}$, where $B_r = \{x \in \mathbb{R}^3 \,\big|\, |x| < r\}$. Then*

$$u(x) = \int_{\partial B_{r_1}} (\partial_{n_y} G_0(x,y)u(y) - G_0(x,y)\partial_{n_y} u(y))dS(y),$$

for $x \in \mathbb{R}^3 \setminus \overline{B}_{r_1}$ and $r_1 > r$.

Proof. Let $r < r_1 < r_2$ and $x_0 \in \mathbb{R}^3$ be an arbitrary point in $\Omega := B_{r_2} \setminus \overline{B_{r_1}}$. Applying the Green formula to $\Omega_\epsilon := B_{r_2} \setminus (\overline{B_{r_1}} \cup \overline{B_\epsilon(x_0)})$, $\epsilon > 0$ sufficiently small, we get

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} (-\Delta - k^2)G_0(x,x_0)u(x) - G_0(x,x_0)(-\Delta - k^2)u(x)dx \\ &= \int_{\partial\Omega_\epsilon} (-\partial_n G_0(x,x_0)u(x) + G_0(x,x_0)\,\partial_n u(x))dS(x). \end{aligned} \tag{2.10}$$

We have

$$\left|\int_{\partial B_\epsilon(x_0)} G_0(x,x_0)\,\partial_n u(x)dS(x)\right| \le \int_{\partial B_\epsilon(x_0)} \frac{|\,\partial_n u(x)|}{4\pi\epsilon} dS(x) \le O(\epsilon).$$

Consider next

$$\int_{\partial B_\epsilon(x_0)} \partial_n G_0(x,x_0)u(x)dS(x) = -\int_{\partial B_\epsilon(x_0)} \frac{ike^{ik\epsilon}}{4\pi\epsilon}u(x)dS(x)$$
$$+ \int_{\partial B_\epsilon(x_0)} \frac{e^{ik\epsilon}}{4\pi\epsilon^2}u(x)dS(x).$$

It follows that
$$\Big|\int_{\partial B_\epsilon(x_0)} \frac{ike^{ik\epsilon}}{4\pi\epsilon} u(x)dS(x)\Big| = O(\epsilon),$$
as $\epsilon \to 0$. Using the fact that $u(x) = u(x_0) + O(\epsilon)$, we get
$$\int_{\partial B_\epsilon(x_0)} \frac{e^{ik\epsilon}}{4\pi\epsilon^2} u(x)dS(x) = e^{ik\epsilon}u(x_0) + O(\epsilon).$$
Letting $\epsilon \to 0$ in (2.10), we obtain that
$$u(x_0) = \int_{\partial\Omega} (-\,\partial_n\, G_0(x,x_0)u(x) + G_0(x,x_0)\,\partial_n\, u(x))dS(x).$$
The next step is to show that
$$I := \int_{\partial B_{r_2}} (-\,\partial_n\, G_0(x,x_0)u(x) + G_0(x,x_0)\,\partial_n\, u(x))dS(x) \to 0,$$
as $r_2 \to \infty$. By adding and subtracting ikG_0u, I can be written as
$$\begin{aligned} I = &-\int_{\partial B_{r_2}} (\partial_n\, G_0(x,x_0) - ikG_0(x,x_0))u(x)dS(x) \\ &+ \int_{\partial B_{r_2}} G_0(x,x_0)(\partial_n\, u(x) - iku(x))dS(x). \end{aligned}$$
We show that the first term goes to zero as $r_2 \to \infty$. The second term can be estimated in the same way. Because of (2.8) we can use the Cauchy-Schwarz inequality and write
$$\Big|\int_{\partial B_{r_2}} u(\partial_n G_0 - ikG_0)dS\Big|^2 \le \int_{\partial B_{r_2}} |u|^2 dS \int_{\partial B_{r_2}} |\partial_n G_0 - ikG_0|^2 dS \to 0,$$
as $r_2 \to \infty$. Here we have used that $|\partial_n G_0 - ikG_0|^2 = o(r_2^{-2})$, valid because of (1.4).

$\square$

The preceding lemmas allow us to prove the main result on the asymptotics of outgoing solutions to the Helmholtz equation.

Proposition 2.8. *Let $u \in H^2_{loc}(\mathbb{R}^3)$ be an outgoing solution to the Helmholtz equation $(-\Delta - k^2)u = 0$ in $\mathbb{R}^3 \setminus \overline{B_r}$, where $B_r = \{x \in \mathbb{R}^3 \,\big|\, |x| < r\}$. Then there exists $a \in L^2(S^2)$ such that*
$$u(x) = \frac{e^{ik|x|}}{|x|} a(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \quad \hat{x} := x/|x| \in S^2, \tag{2.11}$$
uniformly in all directions as $x \to \infty$.

Proof. An application of Lemma 2.5 allows us to write

$$G_0(x,y) = \frac{e^{ik|x|}}{|x|} a_1(\hat{x}, y) + f_1(x,y),$$

where $f_1 = O(1/|x|^2)$ uniformly in all directions as $|x| \to \infty$. Next, a straightforward computation using Lemma 2.5 shows that

$$\partial_{n_y} G_0(x,y) = \frac{e^{ik|x|}}{|x|} a_2(\hat{x}, y) + f_2(x,y),$$

where $f_2 = O(1/|x|^2)$ uniformly in all directions as $|x| \to \infty$. By the representation of Lemma 2.7 we have

$$\begin{aligned} u(x) = &\int_{\partial B_{r_1}} \left(\frac{e^{ik|x|}}{|x|} a_2(\hat{x}, y) + f_2(x,y) \right) u(y) dS(y) \\ &- \int_{\partial B_{r_1}} \left(\frac{e^{ik|x|}}{|x|} a_1(\hat{x}, y) + f_1(x,y) \right) \partial_n u(y) dS(y), \end{aligned}$$

with $r_1 > r$. Let us split this into four separate integrations corresponding to the individual terms. The terms involving a_1 and a_2 are clearly of the form of the first term on the right side of the equation (2.11). The two remaining terms give the contribution,

$$\int_{\partial B_{r_1}} (f_2(x,y)u(y) - f_1(x,y)\partial_n u(y)) dS(y).$$

Since $f_1, f_2 = O(1/|x|^2)$ uniformly in all directions as $|x| \to \infty$, we conclude that the integral above is $O(1/|x|^2)$, which proves the claim. $\square$

The function a in (2.11) is called *the far field pattern* or *scattering amplitude* and is of central interest in scattering theory.

2.2 The magnetic Schrödinger equation in $\mathbb{R}^3$

In this subsection we begin proving existence and uniqueness for problem (1.3), (1.4) by first considering the partial differential equation in the whole of $\mathbb{R}^3$. The main tool will be the Lax-Phillips method from scattering theory, see [22] and [15]. We will assume that the potentials and sources are compactly supported in $\mathbb{R}^3$. More precisely, we assume that

$$A \in W^{1,\infty}_{comp}(\mathbb{R}^3, \mathbb{R}^3), \quad q \in L^\infty_{comp}(\mathbb{R}^3, \mathbb{C}), \quad \operatorname{Im} q \leq 0, \text{ and } f \in L^2_{comp}(\mathbb{R}^3). \tag{2.12}$$

Our aim is to find an outgoing solution u to

$$(L_{A,q} - k^2)u \quad = f, \quad \text{in } \mathbb{R}^3 \tag{2.13}$$

and to show that it is unique.

We begin by recalling what is meant by a weak solution. We call $u \in H^1_{loc}(\mathbb{R}^3)$ a *weak solution* to (2.13) if for every $v \in C_0^\infty(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} (\nabla u \cdot \nabla \overline{v} - 2iA \cdot (\nabla u)\overline{v} + P_{k,q,A} u\overline{v})dx = \int_{\mathbb{R}^3} f\overline{v}dx, \tag{2.14}$$

where $P_{k,q,A} := -i\nabla \cdot A + A^2 + q - k^2$. We state explicitly the following regularity result for weak solutions.

Lemma 2.9. *Let u be a weak solution to* (2.13). *Then $u \in H^2_{loc}(\mathbb{R}^3)$.*

Proof. Since $u \in H^1_{loc}(\mathbb{R}^3)$ and $f \in L^2(\mathbb{R}^3)$, we have

$$-\Delta u = 2iA \cdot \nabla u + P_{k,q,A} u + f \in L^2_{loc}(\mathbb{R}^3).$$

By elliptic regularity, see [12], we conclude that $u \in H^2_{loc}(\mathbb{R}^3)$. $\square$

The asymptotics given by Proposition 2.8, combined with Rellich's lemma and the unique continuation principle, see the appendix, give the uniqueness for outgoing solutions to (2.13). This is the content of the following theorem.

Theorem 2.10. *Assume that A and q satisfy* (2.12). *If $u \in H^1_{loc}(\mathbb{R}^3)$ satisfies the Sommerfeld condition* (1.4) *and solves*

$$(L_{A,q} - k^2)u = 0 \quad \textit{in} \quad \mathbb{R}^3,$$

where $k > 0$, then $u \equiv 0$.

Proof. Let $B_r := B(0,r)$. Denote the $L^2(B_r)$ inner product by $(\cdot,\cdot)$. It follows from the Green's formula of Lemma 4.1, in the appendix that

$$\begin{aligned}&((L_{A,q} - k^2)u, u) - (u, (L_{A,0} - k^2)u) - (u, \overline{q}u)\\ &= (u, (\partial_n + iA \cdot n)u)_{L^2(\partial B_r)} - ((\partial_n + iA \cdot n)u, u)_{L^2(\partial B_r)}.\end{aligned}$$

The first term on the left side vanishes and the second term is (u, qu). Notice also that the vector field A vanishes along ∂B_r. The above equation reduces thus to

$$\operatorname{Im} \int_{\partial B_r} \overline{u}\partial_n u dS = \operatorname{Im} \int_{B_r} q|u|^2 dx \leq 0.$$

Using the asymptotic expansions in Proposition 2.8, we get

$$\begin{aligned}
&\operatorname{Im}\int_{\partial B_r}\left(\overline{a}\frac{e^{-ik|x|}}{|x|}+O\Big(\frac{1}{|x|^2}\Big)\right)\left(a\frac{ike^{ik|x|}}{|x|}+O\Big(\frac{1}{|x|^2}\Big)\right)dS\\
=\;&\operatorname{Im}\int_{\partial B_r}\left(|a|^2\frac{ik}{|x|^2}+O\Big(\frac{1}{|x|^3}\Big)\right)dS\\
=\;&\operatorname{Im}\int_{|x|=1}\left(|a(\theta,\varphi)|^2\frac{ik}{r^2}+O\Big(\frac{1}{r^3}\Big)\right)r^2 sin\theta d\theta d\varphi.
\end{aligned}$$

By taking the limit as $r\to\infty$, we obtain that

$$\int_{|x|=1} k|a(\theta,\varphi)|^2 sin\theta d\theta d\varphi \le 0,$$

and hence the far-field pattern a vanishes identically.

An application of Rellich's lemma, see Proposition 4.8 in the appendix, allows us to conclude that $u\equiv 0$ outside B_r. The unique continuation principle, see Theorem 4.6 in the appendix, implies that $u\equiv 0$ in $\mathbb{R}^3$. □

We now proceed to proving the existence of outgoing solutions to (2.13). In doing so, we shall first establish that the Dirichlet realization of $L_{A,q}$ on a ball has a discrete spectrum. Showing this is complicated by the presence of the imaginary part of q, which makes $L_{A,q}$ non-self-adjoint. We will use the notion of relative compactness for operators to resolve this issue.

Definition 2.11. *Let $\mathcal{B}$ be a Banach space and let T be a closed densely defined linear operator on $\mathcal{B}$ such that* $\operatorname{Spec}(T)\neq\mathbb{C}$. *Assume that A is a linear operator on $\mathcal{B}$ such that $D(T)\subset D(A)$. We say that A is relatively compact with respect to T if for any sequence $\{u_n\}\subset D(T)$, such that both $\{u_n\}$ and $\{Tu_n\}$ are bounded, $\{Au_n\}$ has a convergent subsequence.*

Lemma 2.12. *Let $\mathcal{B}$ be a Banach space, let T be a closed densely defined linear operator on $\mathcal{B}$, and assume that $\lambda\notin$* $\operatorname{Spec}(T)$. *Then A is relatively compact with respect to T if and only if $A(\lambda-T)^{-1}$ is compact.*

Proof. Suppose that we have a sequence $\{u_n\}\subset D(T)$ such that $\{u_n\}$ and $\{Tu_n\}$ are bounded. Then there is a constant M such that

$$\|(\lambda-T)u_n\|\le|\lambda|\|u_n\|+\|Tu_n\|<M<\infty,$$

for all n, so that $\{(\lambda-T)u_n\}$ is bounded. Since $A(\lambda-T)^{-1}$ is compact, the sequence

$$\{A(\lambda-T)^{-1}(\lambda-T)u_n\}=\{Au_n\},$$

has a convergent subsequence.

A similar deduction shows the opposite direction of the lemma. □

We will now show that the Dirichlet realization of $L_{A,q}$ on a ball has a discrete spectrum.

Lemma 2.13. *Assume that* A *and* q *satisfy* (2.12) *and that*

$$\operatorname{supp}(q), \operatorname{supp}(A) \subset B_r = \{x \in \mathbb{R}^3 \,\big|\, |x| < r\}.$$

The operator $L_{A,q} : L^2(B_r) \to L^2(B_r)$*, equipped with the domain* $H^2(B_r) \cap H_0^1(B_r)$*, is closed and has discrete spectrum.*

Proof. We let L_0 be the operator $-\Delta$ on $L^2(B_r)$, equipped with the domain $H^2(B_r) \cap H_0^1(B_r) =: D(L_0)$. We know that L_0 is self-adjoint with discrete spectrum, see [12].

We will show that $L_{A,q}$ is a relatively compact perturbation of L_0. According to Theorem 11.2.6 in [8], the essential spectrum is preserved under relatively compact perturbations, as is closedness. Thus $L_{A,q}$ will have no essential spectrum, since L_0 has none. By writing

$$L_{A,q} = -\Delta - 2iA \cdot \nabla + p, \quad p = -i\nabla \cdot A + A^2 + q \in L^\infty(B_r),$$

we see that our task is reduced to showing that $-2iA \cdot \nabla + p$ is relatively compact with respect to L_0.

Assume that $\lambda \notin \operatorname{Spec}(L_0)$. By the criterion of Lemma 2.12, the operator $-2iA \cdot \nabla + p$ is relatively compact with respect to L_0 if and only if

$$(-2iA \cdot \nabla + p)(\lambda I - L_0)^{-1}$$

is compact on $L^2(B_r)$. We split this as

$$-2iA \cdot \nabla(\lambda I - L_0)^{-1} + p(\lambda I - L_0)^{-1},$$

and show that both of the resulting operators are compact on $L^2(B_r)$. The resolvent operator $(\lambda I - L_0)^{-1}$ is continuous: $L^2(B_r) \to H^2(B_r) \cap H_0^1(B_r)$. The latter space is however compactly imbedded into $L^2(B_r)$, by the Rellich-Kondrachov theorem. If we view p as a multiplication operator, then it is continuous: $L^2(B_r) \to L^2(B_r)$. This shows that the second operator is compact.

The operator $2iA \cdot \nabla$ is continuous from $H^2(B_r) \to H^1(B_r)$, since A is Lipschitz. The latter space is however compactly imbedded into $L^2(B_r)$. The first operator is therefore also compact.

□

We are now ready to prove the existence of outgoing solutions to (2.13), using the Lax-Phillips method.

Theorem 2.14. *Assume that A and q satisfy* (2.12). *Let $k > 0$. Then for any $f \in L^2_{comp}(\mathbb{R}^3)$, there exists $u \in H^2_{loc}(\mathbb{R}^3)$, satisfying the Sommerfeld radiation condition, that solves*

$$(L_{A,q} - k^2)u = f \quad \text{in} \quad \mathbb{R}^3.$$

Proof. Let $B_r = \{x \in \mathbb{R}^3 \,\big|\, |x| < r\}$ be such that $\mathrm{supp}(q), \mathrm{supp}(A) \subset B_r$. Let $s > r$ be such that $\mathrm{supp}(f) \subset B_s$. We pick a function $\varphi \in C_0^\infty(\mathbb{R}^3, [0,1])$ such that $\varphi = 1$ on B_r and $\mathrm{supp}(\varphi) \subset B_s$.

Let $\lambda \in \mathbb{C}$, $\mathrm{Im}\,\lambda \neq 0$, be such that λ avoids the spectrum of the Dirichlet realization of $L_{A,q}$ on the ball B_s. The existence of such λ is guaranteed by Lemma 2.13.

We begin by looking for a solution u of the form

$$u = \varphi w + (1 - \varphi)v. \tag{2.15}$$

Here $w \in H^2(B_s) \cap H_0^1(B_s)$ is the unique solution to the Dirichlet problem

$$\begin{aligned} (L_{A,q} - \lambda)w &= g \quad \text{in } B_s, \\ w|_{\partial B_s} &= 0, \end{aligned}$$

where $g \in L^2(\mathbb{R}^3)$, with $\mathrm{supp}(g) \subset B_s$. And v is the unique outgoing solution of the equation,

$$(-\Delta - k^2)v = g \quad \text{in } \mathbb{R}^3. \tag{2.16}$$

Remark 2.2 gives an explicit formula for v, and according to Corollary 2.3, we know that $v \in H^2_{loc}(\mathbb{R}^3)$.

Inserting u into the original equation will result in an operator equation for the unknown function g. Abbreviating $L := (L_{A,q} - k^2)$, we have[1]

$$\begin{aligned} Lu &= L(\varphi w) + L((1-\varphi)v) \\ &= [L,\varphi]w + \varphi L w + [L,(1-\varphi)]v + (1-\varphi)Lv \\ &= [L,\varphi]w + \varphi(L_{A,q} - \lambda)w + \varphi(\lambda - k^2)w + [L,(1-\varphi)]v + (1-\varphi)g \\ &= [L,\varphi]w + \varphi g + \varphi(\lambda - k^2)w + [L,(1-\varphi)]v + (1-\varphi)g \end{aligned}$$

Noting that the commutator of k^2 and φ is zero, we get from the above,

$$\begin{aligned} Lu &= [L_{A,q},\varphi]w + \varphi(\lambda - k^2)w + [L_{A,q},(1-\varphi)]v + g \\ &= [L_{A,q},\varphi]w + \varphi(\lambda - k^2)w - [L_{A,q},\varphi]v + g. \end{aligned}$$

[1] Here we use the following bracket notation for the commutator operator $[A,B] := AB - BA$.

By setting $Tg := \varphi(\lambda - k^2)w + [L_{A,q}, \varphi](w - v)$, we see that g is to satisfy the operator equation

$$(I + T)g = f. \tag{2.17}$$

Our problem of finding a solution of the special form (2.15) is thus reduced to showing that the equation (2.17) has a solution.

Our aim is to use the Fredholm theory. In order to do this we need to show that $T : L^2(B_s) \to L^2(B_s)$ is compact. Notice that the first term of T is $\varphi(\lambda - k^2)(L_{A,q} - \lambda)^{-1}g$. But since the resolvent $(L_{A,q} - \lambda)^{-1} : L^2(B_s) \to H^2(B_s)$ is bounded and since the inclusion map: $H^2(B_s) \to L^2(B_s)$ is compact, we see that the first term of T is compact on $L^2(B_s)$.

For the second term of T, we first note that $(-\Delta - k^2 - i0)^{-1} : L^2(B_s) \to H^2(B_s)$ is bounded by Corollary 2.4. Next note that $\mathrm{supp}([L_{A,q}, \varphi]v) \subset B_s$, since $\mathrm{supp}(\varphi) \subset B_s$. The first order operator $[L_{A,q}, \varphi]$ is explicitly given by

$$[L_{A,q}, \varphi] = -\Delta\varphi - 2\nabla\varphi \cdot \nabla - 2iA \cdot \nabla\varphi.$$

It also has $W^{1,\infty}$ coefficients, so that it maps $H^2(B_s)$ to $H^1(B_s)$ continuously. Now $H^1(B_s)$ is compactly embedded in $L^2(B_s)$ and hence we see that T is compact on $L^2(B_s)$.

According to the Fredholm theory, we need only to show that $I + T$ is injective, to have the surjectivity and thus a solution to (2.17), for a given f.

Assume that $(I + T)g = 0$. Theorem 2.10 gives that

$$u = \varphi w + (1 - \varphi)v = 0 \quad \text{in } \mathbb{R}^3. \tag{2.18}$$

We want to show that $g \equiv 0$. This follows if $w \equiv 0$, which we will prove next. First note that (2.18) gives $\varphi w = (\varphi - 1)v$, so that

$$w = 0 \quad \text{when} \quad \varphi = 1. \tag{2.19}$$

In particular, $w = 0$ on B_r. This gives that

$$g = (L_{A,q} - \lambda)w = (-\Delta - \lambda)w$$

in B_s, because $\mathrm{supp}(A), \mathrm{supp}(q) \subset B_r$. We also have that

$$(-\Delta - k^2)v \;=\; g,$$

in B_s. Subtracting the last two equations and using that $v = \varphi(v - w)$, we get

$$\begin{aligned}
(-\Delta - k^2)v + (-\Delta - \lambda)(-w) &= (-\Delta - \lambda)(v - w) - k^2 v + \lambda v \\
&= (-\Delta - \lambda)(v - w) - (k^2 - \lambda)\varphi(v - w) \\
&= 0.
\end{aligned}$$

Set $r := v - w$. Now $(-\Delta - \lambda)r - (k^2 - \lambda)\varphi r = 0$ in B_s. Multiplying this by $\overline{r}$, using integration by parts and noting that $r|_{\partial B_s} = -w|_{\partial B_s} = 0$, because of (2.18) we get

$$\int_{B_s} |\nabla r|^2 - \lambda |r|^2 dx = \int_{B_s} (k^2 - \lambda)\varphi |r|^2 dx.$$

Taking the imaginary part gives

$$\operatorname{Im} \lambda \int_{B_s} (1 - \varphi)|r|^2 dx = 0.$$

From this we see that $r = 0$ in the region where $\varphi \neq 1$. It follows that $v = \varphi r = 0$ when $\varphi \neq 1$. Hence,

$$w = v - r = 0 \quad \text{when} \quad \varphi \neq 1. \tag{2.20}$$

Combining (2.19) and (2.20), we see that $w = 0$ in B_s.

$\square$

In summary, we see that Theorem 2.10 and Theorem 2.14 show the existence and uniqueness of outgoing solutions for the problem (2.13).

2.3 The magnetic Schrödinger equation in a half space

The main objective of this subsection is to extend the existence and uniqueness results of the previous subsection to the half space case.

Let $k > 0$ and let

$$A \in W^{1,\infty}_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{R}^3), \quad q \in L^\infty_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{C}), \quad \operatorname{Im} q \leq 0. \tag{2.21}$$

Given $f \in L^2_{comp}(\overline{\mathbb{R}^3_-}, \mathbb{C}) := \{f \in L^2(\overline{\mathbb{R}^3_-}, \mathbb{C}) \bigm| \operatorname{supp}(f) \subset \overline{\mathbb{R}^3_-} \text{ compact}\}$. we first prove the existence and uniqueness of an outgoing solution to the following problem,

$$\begin{aligned} &(L_{A,q} - k^2)u = f \quad \text{in} \quad \mathbb{R}^3_-, \\ &u|_{\partial \mathbb{R}^3_-} = 0. \end{aligned} \tag{2.22}$$

We will reduce this problem to the case of $\mathbb{R}^3$, by using an extension argument and then use Theorems 2.10 and 2.14. The extension argument relies on the following Lemma, which will also be of importance later (see also Theorem 1.3.3 in [14]).

Lemma 2.15. Let $v \in C^{0,1}(\mathbb{R}^2)$, with compact support. Then there is a $\psi \in C^{1,1}(\mathbb{R}^3)$, with compact support for which

$$\psi(x,0) = 0 \quad \text{and} \quad \partial_3\psi(x,0) = v(x), \tag{2.23}$$

for $x \in \mathbb{R}^2$.

Proof. Let φ be the usual mollifier function in $\mathbb{R}^2$, with $\varphi \in C_0^\infty(\mathbb{R}^2)$, $\varphi \geq 0$ and $\int \varphi = 1$. Set $\varphi_t(x) = 1/t^2\varphi(x/t)$. We define $u(x,t) := (v * \varphi_t)(x)$, for $t \neq 0$ and $u(x,0) := v(x) = (v * \delta)(x)$. More explicitly

$$u(x,t) = \frac{1}{t^2}\int_{\mathbb{R}^2} v(y)\varphi\Big(\frac{x-y}{t}\Big)dy = \int_{\mathbb{R}^2} v(x-ty)\varphi(y)dy, \tag{2.24}$$

for $t \neq 0$. From the right hand side we see that u is Lipschitz in (x,t), because v is Lipschitz. We define Ψ as

$$\Psi(x,t) := tu(x,t). \tag{2.25}$$

Notice that Ψ satisfies the first condition in (2.23).

Next we show that the partial derivatives of Ψ are Lipschitz. When $t \neq 0$ we have, using (2.24) that

$$\begin{aligned}\partial_{x_i}(tu) &= t\int_{\mathbb{R}^2} v(y)(\partial_i\,\varphi)\Big(\frac{x-y}{t}\Big)\frac{1}{t^3}dy\\ &= \int_{\mathbb{R}^2} v(y)(\partial_i\,\varphi)\Big(\frac{x-y}{t}\Big)\frac{1}{t^2}dy\\ &= \int_{\mathbb{R}^2} v(x-ty)\,\partial_i\,\varphi(y)dy.\end{aligned}$$

It follows that this identity also holds when $t = 0$. The right hand side of this identity is easily seen to be Lipschitz in both x and t, since v is Lipschitz. It follows that $\partial_{x_i}(tu)$ is Lipschitz in $\mathbb{R}^3$.

The next step is to show that $\partial_t\,\Psi$ is Lipschitz. To see that $\partial_t\,\Psi = \partial_t(tu)$ is continuous, we compute the derivative at $t = 0$

$$\partial_t(tu)|_{t=0} = \lim_{h\to 0}\frac{hu(x,h) - 0\,u(x,0)}{h} = u(x,0) = v(x),$$

Notice that this shows that Ψ also satisfies the second condition in (2.23). And further that $\partial_t(tu)$ is Lipschitz in x when $t = 0$.

For $t \neq 0$, observe firstly that $\partial_t(tu) = u + t\,\partial_t\,u$. Thus we need only to check that the later term is Lipschitz. We write using (2.24)

$$\partial_t\,u = -\int_{\mathbb{R}^2} v(x-ty)\Big(\nabla\varphi(y)\cdot\frac{y}{t} + \varphi(y)\frac{2}{t}\Big)dy. \tag{2.26}$$

This gives that

$$t\,\partial_t\, u = -\int_{\mathbb{R}^2} v(x-ty)\big(\nabla\varphi(y)\cdot y + 2\varphi(y)\big)dy,$$

which is easily seen to be Lipschitz in both x and t, because v is Lipschitz.

To obtain ψ we pick a $\chi \in \mathbb{C}_0^\infty(\mathbb{R}^3)$, s.t. $\chi|\operatorname{supp}(v) = 1$. Then $\psi := \chi\Psi \in C^{1,1}(\mathbb{R}^3)$, $\chi\Psi|_{t=0} = 0$ and $\partial_t(\chi\Psi)|_{t=0} = (\chi\,\partial_t\,\Psi)|_{t=0} = v$.

$\square$

We use the above Lemma to show that, it is sufficient to consider existence and uniqueness in problem (2.22), for potentials A for which $\partial_3\, A|_{x_3=0} = 0$. To see this notice that Lemma 2.15 guarantees the existence of a $\psi \in C^{1,1}(\mathbb{R}^3,\mathbb{R})$ with compact support, for which $\psi|_{x_3=0} = 0$ and $\nabla\psi|_{x_3=0} = (0,0,-A_3)|_{x_3=0}$. A straight forward computation shows that

$$e^{-i\psi}L_{A,q}e^{i\psi} = L_{A+\nabla\psi,q},$$

Using this we see that u is an outgoing solution to the problem (2.22) if and only if $\tilde{u} = e^{-i\psi}u$ is an outgoing solution to the problem,

$$\begin{aligned}&(L_{A+\nabla\psi,q} - k^2)\tilde{u} = e^{-i\psi}f \quad \text{in} \quad \mathbb{R}^3_-,\\ &\tilde{u}|_{\partial\,\mathbb{R}^3_-} = 0.\end{aligned}$$

We can thus, without loss of generality, assume that $A_3 = 0$ along $\partial\,\mathbb{R}^3_-$, when showing that the solution of problem of (2.22) exists and is unique.

We have the following result.

Theorem 2.16. *Let A and q satisfy* (2.21) *Then for any $f \in L^2_{comp}(\overline{\mathbb{R}^3_-})$, there exists a unique outgoing solution $u \in H^2_{loc}(\overline{\mathbb{R}^3_-})$ to the problem* (2.22).

Proof. We shall first prove the existence. In doing so, we shall reduce the problem (2.22) to all of $\mathbb{R}^3$ by making use of appropriate even and odd extensions of the coefficients of the operator $L_{A,q}$. The discussion preceding the Lemma 2.15 shows that we may without loss of generality, assume that $A_3 = 0$ along $\partial\,\mathbb{R}^3_-$.

We extend the potentials $A = (A_1, A_2, A_3)$ and q, and the source term f to the whole of $\mathbb{R}^3$. Let $\tilde{x} := (x_1, x_2, -x_3)$. For A_1, A_2, and q, we do even extensions in x_3, i.e.,

$$\tilde{A}_j(x) = \begin{cases} A_j(x), & x_3 < 0,\\ A_j(\tilde{x}), & x_3 > 0,\end{cases} \qquad j = 1, 2,$$

$$\tilde{q}(x) = \begin{cases} q(x), & x_3 < 0,\\ q(\tilde{x}), & x_3 > 0.\end{cases}$$

For A_3 and f, we do odd extensions in x_3,

$$\tilde{A}_3(x) = \begin{cases} A_3(x), & x_3 < 0, \\ -A_3(\tilde{x}), & x_3 > 0, \end{cases}$$

$$\tilde{f}(x) = \begin{cases} f(x), & x_3 < 0, \\ -f(\tilde{x}), & x_3 > 0. \end{cases}$$

Since $A_3 = 0$ when $x_3 = 0$, we see that $\tilde{A} \in W^{1,\infty}_{comp}(\mathbb{R}^3, \mathbb{R})$. Furthermore, $\tilde{q} \in L^\infty_{comp}(\mathbb{R}^3)$ and $\tilde{f} \in L^2_{comp}(\mathbb{R}^3)$.

By Theorem 2.10 and Theorem 2.14, the problem

$$(L_{\tilde{A},\tilde{q}} - k^2)\tilde{u} = \tilde{f} \quad \text{in} \quad \mathbb{R}^3$$

has a unique outgoing solution $\tilde{u} \in H^2_{loc}(\mathbb{R}^3)$.

Next we want to show that $\tilde{u}$ is odd in x_3. To that end it is convenient to write,

$$L_{\tilde{A},\tilde{q}} = -\Delta - 2i\tilde{A} \cdot \nabla + \tilde{p}, \quad \tilde{p} = -i\nabla \cdot \tilde{A} + \tilde{A}^2 + \tilde{q}. \tag{2.27}$$

Here one sees easily that the operators Δ, $\tilde{A}_3\partial_3$ and $\tilde{p}$ all preserve the parity in x_3 of a function that they operate on. Hence, the operator $L_{\tilde{A},\tilde{q}} - k^2$ preserves the parity in x_3.

Decompose $\tilde{u}$ into an even and odd part with respect to x_3, i.e.

$$\tilde{u} = \tilde{u}_e + \tilde{u}_o,$$

where

$$\tilde{u}_e(x) = \frac{1}{2}\big(\tilde{u}(x) + \tilde{u}(\tilde{x})\big), \quad \tilde{u}_o(x) = \frac{1}{2}\big(\tilde{u}(x) - \tilde{u}(\tilde{x})\big).$$

Then

$$\tilde{f} = (L_{\tilde{A},\tilde{q}} - k^2)\tilde{u}_e + (L_{\tilde{A},\tilde{q}} - k^2)\tilde{u}_0,$$

and using that $\tilde{f}$ is odd with respect to x_3, we conclude that

$$(L_{\tilde{A},\tilde{q}} - k^2)\tilde{u}_e = 0 \quad \text{in} \quad \mathbb{R}^3.$$

Now a direct computation shows that the function $x \mapsto \tilde{u}(\tilde{x})$ is outgoing on $\mathbb{R}^3$, since $\tilde{u}$ has this property. Thus, $\tilde{u}_e$ is outgoing, and by Theorem 2.10, $\tilde{u}_e = 0$. Hence, $\tilde{u}$ is odd in x_3.

The Sobolev embedding theorem shows that $\tilde{u}$ is continuous in $\mathbb{R}^3$, since $\tilde{u} \in H^2_{loc}(\mathbb{R}^3)$. Hence, $\tilde{u}|_{\partial\mathbb{R}^3_-} = 0$, so that $\tilde{u}|_{\mathbb{R}^3_-}$ is a solution to the half space Dirichlet problem (2.22).

In order to prove uniqueness, we assume that $u \in H^2_{loc}(\overline{\mathbb{R}^3_-})$ is an outgoing solution to the problem (2.22) with $f = 0$. We need to show that $u \equiv 0$. To that end, let us consider the odd extension of u with respect to x_3, i.e.

$$\tilde{u}(x) = \begin{cases} u(x), & x_3 < 0, \\ -u(\tilde{x}), & x_3 > 0. \end{cases} \tag{2.28}$$

Notice that since $u = 0$ along $x_3 = 0$, the function $\tilde{u}$ is continuous across $x_3 = 0$.

Let us now show that $\tilde{u}$ satisfies the equation,

$$(L_{\tilde{A},\tilde{q}} - k^2)\tilde{u} = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{2.29}$$

with $\tilde{A}$ and $\tilde{q}$ as in the first part of the proof. Indeed, computing the first order partial derivatives of $\tilde{u}$, given by (2.28), in the sense of distributions on $\mathbb{R}^3$, we obtain that

$$\begin{aligned} \partial_j \tilde{u}(x) &= \begin{cases} (\partial_j u)(x), & x_3 < 0, \\ -(\partial_j u)(\tilde{x}), & x_3 > 0, \end{cases} \quad j = 1, 2, \\ \partial_3 \tilde{u}(x) &= \begin{cases} (\partial_3 u)(x), & x_3 < 0, \\ (\partial_3 u)(\tilde{x}), & x_3 > 0. \end{cases} \end{aligned} \tag{2.30}$$

Hence, we see that $\tilde{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$.

One has to be more careful when computing the second order partial derivatives of $\tilde{u}$. For this reason, we shall give the details of the computation below. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$. Then denoting by $\langle \cdot, \cdot \rangle$ the duality between distributions and test functions, $x' = (x_1, x_2)$, and $\partial^2_{x'} = \partial_1^2 + \partial_2^2$, we have

$$\begin{aligned} \langle -\Delta\tilde{u}, \varphi \rangle = -\int_{\mathbb{R}^3} \tilde{u}\Delta\varphi dx = -\int_{-\infty}^0 \int_{\mathbb{R}^2} u(x', x_3)(\partial^2_{x'} + \partial_3^2)\varphi dx' dx_3 \\ + \int_0^{+\infty} \int_{\mathbb{R}^2} u(x', -x_3)(\partial^2_{x'} + \partial_3^2)\varphi dx' dx_3 \\ = -\int_{-\infty}^0 \int_{\mathbb{R}^2} (\partial^2_{x'} u)(x', x_3)\varphi dx' dx_3 + \int_0^{+\infty} \int_{\mathbb{R}^2} (\partial^2_{x'} u)(x', -x_3)\varphi dx' dx_3 \\ + \int_{\mathbb{R}^2} I(x') dx', \end{aligned}$$

where

$$
\begin{aligned}
I(x') &= -\int_{-\infty}^{0} u(x',x_3)\,\partial_3^2\,\varphi dx_3 + \int_{0}^{+\infty} u(x',-x_3)\,\partial_3^2\,\varphi dx_3 \\
&= \int_{-\infty}^{0} \partial_3\, u(x',x_3)\,\partial_3\,\varphi dx_3 - u(x',x_3)\,\partial_3\,\varphi|_{-\infty}^{0} \\
&\quad + \int_{0}^{+\infty} (\partial_3\, u)(x',-x_3)\,\partial_3\,\varphi dx_3 + u(x',-x_3)\,\partial_3\,\varphi|_{0}^{+\infty} \\
&= -\int_{-\infty}^{0} \partial_3^2\, u(x',x_3)\varphi dx_3 + \partial_3\, u(x',x_3)\varphi|_{-\infty}^{0} \\
&+ \int_{0}^{+\infty} (\partial_3^2\, u)(x',-x_3)\varphi dx_3 + (\partial_3\, u)(x',-x_3)\varphi|_{0}^{+\infty} \\
&= -\int_{-\infty}^{0} \partial_3^2\, u(x',x_3)\varphi dx_3 + \int_{0}^{+\infty} (\partial_3^2\, u)(x',-x_3)\varphi dx_3.
\end{aligned}
$$

Hence, we have

$$
-(\Delta\tilde{u})(x) = \begin{cases} -(\Delta u)(x), & x_3 < 0, \\ (\Delta u)(\tilde{x}), & x_3 > 0, \end{cases} \tag{2.31}
$$

in the sense of distributions. Using (2.27), (2.30) and (2.31), we obtain that

$$
(L_{\tilde{A},\tilde{q}} - k^2)\tilde{u} = \begin{cases} -\Delta u(x) - 2iA(x)\cdot\nabla u(x) + (\tilde{p}(x) - k^2)u(x), & x_3 < 0, \\ (\Delta u)(\tilde{x}) + 2iA(\tilde{x})\cdot\nabla u(\tilde{x}) + (-\tilde{p}(\tilde{x}) + k^2)u(\tilde{x}), & x_3 > 0. \end{cases}
$$

We have therefore verified that $\tilde{u}$ solves (2.29).

Taking into account the fact that $\tilde{u}$ is outgoing and applying Theorem 2.10, we finally get $u \equiv 0$ in $\mathbb{R}^3_-$.

□

It is now straightforward to establish the main result of this section, Theorem 1.1.

Proof of Theorem 1.1. Uniqueness follows from Theorem 2.16, since it implies that $v_1 - v_2 \equiv 0$ for any two solutions v_1 and v_2 of problem (1.3) and (1.4).

It remains to check the existence of a solution. To that end, given $f \in H^{3/2}_{comp}(\partial\,\mathbb{R}^3_-)$, let $F \in H^2_{comp}(\mathbb{R}^3)$ be such that $F|_{\partial\mathbb{R}^3_-} = f$. Theorem 2.16 gives the existence of an outgoing solution v to

$$
\begin{aligned}
(L_{A,q} - k^2)v &= -(L_{A,q} - k^2)F, \\
v|_{\partial\mathbb{R}^3_-} &= 0.
\end{aligned}
$$

Then $u = v + F$ solves the problem (1.3) and satisfies (1.4). The proof of Theorem 1.1 is complete.

$\square$

3 The inverse problem: Proof of Theorem 1.2

The main task of this section is to prove Theorem 1.2. It will be convenient to set

$$B_{-} := \mathbb{R}^3_{-} \cap B, \quad l := \partial \mathbb{R}^3_{-} \cap B,$$

where $B \subset \mathbb{R}^3$ is the open ball of Theorem 1.2 containing the supports of A_j and q_j, $j = 1, 2$. Recall that we assume that

$$(\partial \mathbb{R}^3_{-} \setminus \overline{B}) \cap \Gamma_j \neq \emptyset, \quad j = 1, 2.$$

We can thus choose $\tilde{\Gamma}_j$, such that

$$\tilde{\Gamma}_j \subset \Gamma_j, \quad \tilde{\Gamma}_j \subset\subset \partial \mathbb{R}^3_{-} \setminus \overline{B}, \quad j = 1, 2.$$

Then it follows from (1.7) that

$$\Lambda_{A_1,q_1}(f)|_{\tilde{\Gamma}_1} = \Lambda_{A_2,q_2}(f)|_{\tilde{\Gamma}_1}, \tag{3.1}$$

for any $f \in H^{3/2}(\partial \mathbb{R}^3_{-})$, $\operatorname{supp}(f) \subset \tilde{\Gamma}_2$. In order to prove Theorem 1.2 we shall only use the data (3.1), which turns out to be enough to determine the magnetic field and the electric potential.

The gauge invariance of the DN-map plays an important role in the sequel. We state therefore the following results.

Lemma 3.1. *Let $A \in W^{1,\infty}(\overline{\mathbb{R}^3_{-}}, \mathbb{R}^3)$ and $q \in L^\infty(\overline{\mathbb{R}^3_{-}})$. Then*

(i) For all $\psi \in C^{1,1}(\overline{\mathbb{R}^3_{-}}, \mathbb{R})$ we have

$$e^{-i\psi} \Lambda_{A,q} e^{i\psi} = \Lambda_{A+\nabla\psi,q}.$$

(ii) There exists $\psi \in C^{1,1}(\overline{\mathbb{R}^3_{-}}, \mathbb{R})$ with $\psi|_{\{x_3=0\}} = 0$, for which

$$\Lambda_{A,q} = \Lambda_{A+\nabla\psi,q}$$

and $(A + \nabla\psi)|_{\{x_3=0\}} = (A_1, A_2, 0)$.

Proof. (i). A straight forward computation shows that

$$e^{-i\psi} L_{A,q} e^{i\psi} = L_{A+\nabla\psi,q}. \tag{3.2}$$

So that a function u solves $L_{A+\nabla\psi,q} u = 0$ if and only if $v = e^{i\psi} u$ solves $L_{A,q} v = 0$. Moreover we have

$$\begin{aligned} (e^{-i\psi} \Lambda_{A,q} e^{i\psi}) f &= e^{-i\psi} (\partial_n + in \cdot A)(e^{i\psi} u) \\ &= i\, \partial_n \psi u + \partial_n u + in \cdot Au \\ &= \Lambda_{A+\nabla\psi,q} f. \end{aligned}$$

(ii). By Lemma 2.15 there exists a $\psi \in C^{1,1}(\overline{\mathbb{R}^3_-}, \mathbb{R})$ with $\psi|_{\{x_3=0\}} = 0$ and $\nabla\psi|_{\{x_3=0\}} = (0,0,-A_3)$. By part *(i)* we have

$$\begin{aligned}\Lambda_{A+\nabla\psi,q} f &= e^{-i\psi}\Lambda_{A,q}(e^{i\psi})f \\ &= \Lambda_{A,q} f.\end{aligned}$$

□

Remark 3.2. *Notice that part (ii) of the Proposition says in other words that we can change a potential A to $\tilde{A} = (A_1, A_2, A_3 - \partial_3\,\psi) = (A_1, A_2, 0)$, while still retaining that $\Lambda_{A,q} = \Lambda_{\tilde{A},q}$. It follows that we can always assume that $n \cdot A|_{\{x_3=0\}} = 0$, where n is unit normal to the plane $\{x_3 = 0\}$, without altering the DN-map.*

Notice also that this is the reason why we can take the DN-map as having the value $(\partial_n + in \cdot A)u|_{\{x_3=0\}}$, instead of just $\partial_n\, u|_{\{x_3=0\}}$, even though we do not "know" A on the boundary.

3.1 An integral identity

One central step in the ideas that are used in proving uniqueness results for Calderon's problem, is to derive an integral equation that expresses L^2 orthogonality between the product of two solutions u_1 and u_2, and the difference of two potentials q_1 and q_2, see [36]. One shows that

$$\int (q_1 - q_2)u_1 u_2 = 0,$$

provided that the DN-maps for q_1 and q_2 are equal.

A similar thing will be done in this subsection, for the magnetic case. The integral equation, is however more involved in the case of a magnetic potential and cannot by itself be interpreted as an orthogonality relation. We will be considering the integral equation in conjunction with solutions that depend on a small positive parameter h. In the later sections we will see that in the limit $h \to 0$, we obtain a criterion for the curl being zero.

We now begin deriving the integral identity. We assume that A_j, q_j and Γ_j are as in Theorem 1.2 and that

$$\Lambda_{A_1,q_1}(f)|_{\Gamma_1} = \Lambda_{A_2,q_2}(f)|_{\Gamma_1},$$

for any $f \in H^{3/2}_{comp}(\partial\,\mathbb{R}^3_-)$, $\mathrm{supp}(f) \subset \Gamma_2$, so that (3.1) also applies.

Let $u_1 \in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_-})$ be the radiating solution to

$$\begin{aligned}(L_{A_1,q_1} - k^2)u_1 = 0, \text{ in } \mathbb{R}^3_-,\\ u_1|_{\partial\mathbb{R}^3_-} = f,\end{aligned}$$

with $f \in H^{3/2}(\partial\,\mathbb{R}^3_-)$, $\mathrm{supp}(f) \subset \tilde{\Gamma}_2$. Let $v \in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_-})$ be the radiating solution to

$$(L_{A_2,q_2} - k^2)v = 0, \text{ in } \mathbb{R}^3_-,$$
$$v|_{\partial\mathbb{R}^3_-} = f.$$

Define $w := v - u_1$. Then

$$(L_{A_2,q_2} - k^2)w = 2i(A_2 - A_1)\cdot\nabla u_1 + i\nabla\cdot(A_2 - A_1)u_1 + (A_1^2 - A_2^2)u_1 + (q_1 - q_2)u_1. \tag{3.3}$$

It follows from (3.1)that

$$(\partial_n + iA_1\cdot n)u_1|_{\tilde{\Gamma}_1} = (\partial_n + iA_2\cdot n)v|_{\tilde{\Gamma}_1}. \tag{3.4}$$

By Remark 3.2 we may assume that $A_1\cdot n = A_2\cdot n = 0$ on $\partial\,\mathbb{R}^3_-$, so that $\partial_n\, w = 0$ on $\tilde{\Gamma}_1$. We also conclude from (3.3) that w satisfies the equation

$$(-\Delta - k^2)w = 0 \quad \text{in} \quad \mathbb{R}^3_- \setminus \overline{B_-}.$$

As $w|_{\tilde{\Gamma}_1} = \partial_n\, w|_{\tilde{\Gamma}_1} = 0$, by unique continuation, we get that $w = 0$ in $\mathbb{R}^3_- \setminus \overline{B_-}$. See Theorem 4.6 and Corollary 4.7 in the appendix. Since $w \in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_-})$, we have

$$w = \partial_n\, w = 0 \quad \text{on} \quad \partial\, B_- \cap \mathbb{R}^3_-.$$

Let $u_2 \in H^2(B_-)$ be a solution to $(L_{A_2,\overline{q_2}} - k^2)u_2 = 0$ in B_-. Then by Green's formula, we get

$$\begin{aligned}((L_{A_2,q_2} - k^2)w, u_2)_{L^2(B_-)} &= (w, (L_{A_2,\overline{q_2}} - k^2)u_2)_{L^2(B_-)} \\ &\quad - ((\partial_n + iA_2\cdot n)w, u_2)_{L^2(\partial B_-)} \\ &\quad + (w, (\partial_n + iA_2\cdot n)u_2)_{L^2(\partial B_-)} \\ &= -(\partial_n w, u_2)_{L^2(l)}.\end{aligned}$$

Assuming that

$$u_2 = 0 \quad \text{on} \quad l,$$

we conclude that

$$((L_{A_2,q_2} - k^2)w, u_2)_{L^2(B_-)} = 0.$$

Using equation (3.3) we may write this as follows,

$$\int_{B_-} (2i(A_2 - A_1)\cdot(\nabla u_1)\overline{u_2} + i\nabla\cdot(A_2 - A_1)u_1\overline{u_2})\, dx + \int_{B_-} (A_1^2 - A_2^2 + q_1 - q_2)u_1\overline{u_2}\, dx = 0.$$

Using the fact that $(A_2 - A_1) \cdot n = 0$ on ∂B_- and an integration by parts, we get

$$i \int_{B_-} \nabla \cdot (A_2 - A_1) u_1 \overline{u_2} \, dx = -i \int_{B_-} (A_2 - A_1) \cdot (\nabla u_1 \overline{u_2} + u_1 \nabla \overline{u_2}) dx.$$

Thus, we obtain that

$$\begin{aligned} \int_{B_-} & i(A_2 - A_1) \cdot (\nabla u_1 \overline{u_2} - u_1 \nabla \overline{u_2}) \, dx \\ & + \int_{B_-} (A_1^2 - A_2^2 + q_1 - q_2) u_1 \overline{u_2} \, dx = 0, \end{aligned} \tag{3.5}$$

where $u_1 \in W_1(\mathbb{R}^3_-)$ and $u_2 \in W_2^*(B_-)$. Here

$$\begin{aligned} W_1(\mathbb{R}^3_-) := \{ u \in H^2_{\text{loc}}(\overline{\mathbb{R}^3_-}) \mid (L_{A_1,q_1} - k^2)u = 0 \text{ in } \mathbb{R}^3_-, \\ \text{supp}(u_1|_{\partial \mathbb{R}^3_-}) \subset \tilde{\Gamma}_2, u \text{ radiating} \}, \end{aligned}$$

and

$$W_2^*(B_-) := \{ u \in H^2(B_-) \mid (L_{A_2,\overline{q_2}} - k^2)u = 0 \text{ in } B_-, u|_l = 0 \}.$$

We shall next extend the integral identity (3.5) to a richer class of solutions to the magnetic Schrödinger operators. To that end, let us introduce the following space of solutions,

$$W_1(B_-) := \{ u \in H^2(B_-) \mid (L_{A_1,q_1} - k^2)u = 0 \text{ in } B_-, u|_l = 0 \}.$$

The following Runge type approximation result is similar to those found in [16], [26] and [19].

Lemma 3.3. *The space* $V_1 := W_1(\mathbb{R}^3_-)|_{B_-}$ *is dense in* $W_1(B_-)$ *in the* $L^2(B_-)$*–topology.*

Proof. Suppose that V_1 is not dense in $W_1(B_-)$. First notice that $\text{span}(V_1) = V_1$ so that $\overline{V_1}$ is a linear subspace of $L^2(B_-)$. Since V_1 is not dense in $W_1(B_-)$, we have a vector $u_0 \in W_1(B_-)$ such that $u_0 \notin \overline{V_1}$. We can decompose u_0 as $u_0 = a + b$, where $a \in \overline{V_1}$, $b \in \overline{V_1}^\perp$ and $b \neq 0$. Let T be the linear functional on $L^2(B_-)$, defined by $T(x) := \text{proj}_{\overline{V_1}^\perp}(x)/\|b\|_{L^2}$, where $\text{proj}_{\overline{V_1}^\perp}$ is the orthogonal projection to $\overline{V_1}^\perp$. Now clearly $\|T(u_0)\|_{L^2} = 1$ and $T|_{V_1} = 0$.

By the Riesz representation theorem, there is $g_T \in L^2(B_-)$ that corresponds to T. Extend g_T by zero to the complement of B_- in $\mathbb{R}^3_-$. Let $U \in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_-})$ be the incoming solution to

$$\begin{aligned}(L_{A_1,\overline{q_1}} - k^2)U &= g_T \quad \text{in} \quad \mathbb{R}^3_-,\\ U|_{\partial\mathbb{R}^3_-} &= 0.\end{aligned}$$

The existence of such a solution follows from Theorem 1.1, since we can find a $\tilde{U}$ outgoing with $(L_{-A_1,q_1} - k^2)\tilde{U} = \overline{g_T}$ and $\tilde{U}|_{\partial\mathbb{R}^3_-} = 0$. It then suffices to take $U = \overline{\tilde{U}}$.

Now let $u \in W_1(\mathbb{R}^3_-)$. Then because $T|_{V_1} = 0$ and $\mathrm{supp}(g_T) \subset B_-$, we get by the Green's formula of Lemma 4.2 that

$$\begin{aligned}0 = (u, g_T)_{L^2(\mathbb{R}^3_-)} &= (u, (L_{A_1,\overline{q_1}} - k^2)U)_{L^2(\mathbb{R}^3_-)}\\ &= ((L_{A_1,q_1} - k^2)u, U)_{L^2(\mathbb{R}^3_-)}\\ &\quad - (u, (\partial_n + iA_1 \cdot n)U)_{L^2(\partial\mathbb{R}^3_-)}\\ &\quad + ((\partial_n + iA_1 \cdot n)u, U)_{L^2(\partial\mathbb{R}^3_-)}\\ &= -(u, \partial_n U)_{L^2(\tilde{\Gamma}_2)}.\end{aligned}$$

Since the boundary condition $u|_{\tilde{\Gamma}_2}$ can be chosen arbitrarily from $C_0^\infty(\tilde{\Gamma}_2)$, we get that $\partial_n U|_{\tilde{\Gamma}_2} = 0$. Since $U|_{\tilde{\Gamma}_2} = 0$, we apply the unique continuation principle to conclude that $U|_{\mathbb{R}^3_- \setminus \overline{B_-}} = 0$. As $U \in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_-})$, we have

$$U|_{\partial B_- \cap \mathbb{R}^3_-} = \partial_n U|_{\partial B_- \cap \mathbb{R}^3_-} = 0.$$

Now applying Green's formula and doing the same calculation as above for u_0 and B_- instead of u yields

$$\begin{aligned}(u_0, g_T)_{L^2(B_-)} &= (u_0, (L_{A_1,\overline{q_1}} - k^2)U)_{L^2(B_-)}\\ &= ((L_{A_1,q_1} - k^2)u_0, U)_{L^2(B_-)}\\ &\quad - (u_0, (\partial_n + iA_1 \cdot n)U)_{L^2(\partial B_-)}\\ &\quad + ((\partial_n + iA_1 \cdot n)u_0, U)_{L^2(\partial B_-)}\\ &= -(u_0, \partial_n U)_{L^2(l)} = 0.\end{aligned}$$

Here we have used that $u_0|_l = 0$. It follows that $T(u_0) = 0$. This contradiction completes the proof. $\square$

Since $(A_2 - A_1) \cdot n = 0$ on ∂B_-, we can rewrite (3.5) in the following form,

$$
\begin{aligned}
-\int_{B_-} u_1 i \nabla \cdot ((A_2 - A_1)\overline{u_2})\, dx - \int_{B_-} i(A_2 - A_1) \cdot (u_1 \nabla \overline{u_2})\, dx \\
+ \int_{B_-} (A_1^2 - A_2^2 + q_1 - q_2) u_1 \overline{u_2}\, dx = 0.
\end{aligned}
$$

Hence, an application of Lemma 3.3 implies that the integral identity (3.5) is valid for any $u_1 \in W_1(B_-)$ and any $u_2 \in W_2^*(B_-)$.

We summarize the discussion in this subsection in the following result.

Proposition 3.4. *Assume that A_j, q_j and Γ_j, $j = 1, 2$ are as in Theorem 1.2 and that the DN-maps satisfy*

$$
\Lambda_{A_1,q_1}(f)|_{\Gamma_1} = \Lambda_{A_2,q_2}(f)|_{\Gamma_1}, \tag{3.6}
$$

for any $f \in H^{3/2}_{\mathrm{comp}}(\partial \mathbb{R}^3_-)$, $\mathrm{supp}(f) \subset \Gamma_2$. Then

$$
\begin{aligned}
\int_{B_-} i(A_2 - A_1) \cdot (\nabla u_1 \overline{u_2} - u_1 \nabla \overline{u_2})\, dx \\
+ \int_{B_-} (A_1^2 - A_2^2 + q_1 - q_2) u_1 \overline{u_2}\, dx = 0,
\end{aligned} \tag{3.7}
$$

for $u_1 \in W_1(B_-)$ and any $u_2 \in W_2^*(B_-)$.

□

Remark 3.5. *Notice that the proof of Proposition 3.4 only uses the assumption* (3.1)*, which follows from* (3.6)*. Proposition 3.4 holds therefore also under the weaker assumption.*

The next step in proving Theorem 1.2 is to use the integral identity (3.7) with u_j, $j = 1, 2$, taken to be special solutions, which are called complex geometric optics solutions.

3.2 Complex geometric optics solutions

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary, and let $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$, $q \in L^\infty(\Omega, \mathbb{C})$. The task of this subsection is to review the construction of complex geometric optics solutions for the magnetic Schrödinger equation,

$$
L_{A,q} u = 0 \quad \text{in} \quad \Omega. \tag{3.8}
$$

A complex geometric optics solution to (3.8) is a solution of the form

$$u(x, \zeta; h) = e^{x \cdot \zeta / h}(a(x, \zeta; h) + r(x, \zeta; h)), \tag{3.9}$$

where $\zeta \in \mathbb{C}^3$, $\zeta \cdot \zeta = 0$, a is a smooth amplitude, r is a remainder, and $h > 0$ is a small parameter.

In the case when $A \in C^2(\overline{\Omega})$ and $q \in L^\infty(\Omega)$, such solutions were constructed in [9] using the method of Carleman estimates, and the construction was extended to the case of less regular potentials in [17], see also [19].

Let $\varphi(x) = \alpha \cdot x$, $\alpha \in \mathbb{R}^3$, $|\alpha| = 1$. The fundamental role in the construction of complex geometric optics solutions is played by the following Carleman estimate,

$$h\|u\|_{H^1_{\mathrm{scl}}(\Omega)} \leq C\|e^{\varphi/h} h^2 L_{A,q} e^{-\varphi/h} u\|_{L^2(\Omega)}, \tag{3.10}$$

valid for all $u \in C_0^\infty(\Omega)$ and $0 < h \leq h_0$, which was proved in [9] and [17]. Here $\|u\|_{H^1_{\mathrm{scl}}(\Omega)} = \|u\|_{L^2(\Omega)} + \|h\nabla u\|_{L^2(\Omega)}$. For the convenience of the reader, we shall present a derivation of (3.10) in the appendix.

Based on the estimate (3.10), the following solvability result was established in [17, Proposition 4.3]. See also the discussion in the appendix.

Proposition 3.6. *Let $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$, $q \in L^\infty(\Omega, \mathbb{C})$, $\alpha \in \mathbb{R}^3$, $|\alpha| = 1$ and $\varphi(x) = \alpha \cdot x$. Then there is $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0]$, and any $f \in L^2(\Omega)$, the equation*

$$e^{\varphi/h} h^2 L_{A,q} e^{-\varphi/h} u = f \quad \text{in} \quad \Omega,$$

has a solution $u \in H^1(\Omega)$ with

$$\|u\|_{H^1_{\mathrm{scl}}(\Omega)} \leq \frac{C}{h}\|f\|_{L^2(\Omega)}.$$

Proof. See appendix. □

Our basic strategy in constructing solutions of the form (3.9) is to write (3.8), as

$$L_\zeta r = -L_\zeta a, \tag{3.11}$$

where $L_\zeta := e^{-x \cdot \zeta / h} h^2 L_{A,q} e^{x \cdot \zeta / h}$. Then we first search for a suitable a, after which we will get r by Proposition 3.6. We must however take some care in choosing a and the way it depends on h, since we need later that $\|r\|_{H^1_{scl}(\Omega)} \to 0$, sufficiently fast as $h \to 0$. We need a also to be smooth enough. This will be handled as in [17].

We extend $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ to a Lipschitz vector field, compactly supported in $\tilde{\Omega}$, where $\tilde{\Omega} \subset \mathbb{R}^3$ is an open bounded set such that $\Omega \subset\subset \tilde{\Omega}$. We consider the mollification $A^\sharp := A * \psi_\epsilon \in C_0^\infty(\tilde{\Omega}, \mathbb{R}^3)$. Here $\epsilon > 0$ is small and $\psi_\epsilon(x) = \epsilon^{-3}\psi(x/\epsilon)$ is the usual mollifier with $\psi \in C_0^\infty(\mathbb{R}^3)$, $0 \le \psi \le 1$, and $\int \psi dx = 1$. We write $A^\flat = A - A^\sharp$. Notice that we have the following estimates for $A^\flat$,

$$\|A^\flat\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon), \tag{3.12}$$
$$\| \partial^\alpha A^\sharp\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-|\alpha|}) \quad \text{for all} \quad \alpha,$$

as $\epsilon \to 0$.

We shall work with a complex $\zeta = \zeta_0 + \zeta_1$ depending slightly on h, for which

$$\begin{aligned} &\zeta \cdot \zeta = 0,\ \zeta_0 := \alpha + i\beta,\ \alpha, \beta \in S^2,\ \alpha \cdot \beta = 0, \\ &\zeta_0 \ \text{independent of } h \text{ and } \ \zeta_1 = \mathcal{O}(h),\ \text{as } h \to 0. \end{aligned} \tag{3.13}$$

By expanding the conjugated operator we write the right hand side of (3.11) as

$$\begin{aligned} L_\zeta a =&(-h^2\Delta - 2i(-i\zeta_0 + hA) \cdot h\nabla - 2\zeta_1 \cdot h\nabla + h^2A^2 \\ &- 2ih\zeta_0 \cdot (A^\sharp + A^\flat) - 2ih\zeta_1 \cdot A - ih^2(\nabla \cdot A) + h^2q)a. \end{aligned} \tag{3.14}$$

Now we want a to be such that this expression decays more rapidly than $\mathcal{O}(h)$, as $h \to 0$.

Consider the operator in (3.14), ignoring for the time being a and its possible dependence on h. We would like to eliminate from this operator the terms that are first order in h. Notice first that $\zeta_1 \in \mathcal{O}(h)$ and that we can control $\|A^\flat\|_{L^\infty(\Omega)}$ with h, if we choose ϵ to be dependent on h. Then in an attempt to eliminate first order terms in h, it is natural to search for an a for which

$$\zeta_0 \cdot \nabla a = -i\zeta_0 \cdot A^\sharp a, \quad \text{in } \Omega. \tag{3.15}$$

We will look for a solution of the form $a = e^\Phi$. The above equation becomes then

$$\zeta_0 \cdot \nabla\Phi = -i\zeta_0 \cdot A^\sharp, \quad \text{in } \Omega. \tag{3.16}$$

Pick a $\gamma \in S^2$, such that $\gamma \perp \{\alpha, \beta\}$.

Next we consider the above equation in coordinates y, associated with the basis $\{\alpha, \beta, \gamma\}$. Let T be the coordinate transform $y = Tx := (x\cdot\alpha, x\cdot\beta, x\cdot\gamma)$. Using the chain rule and the fact that $T^{-1} = T^*$, one gets that[2]

$$\nabla(\Phi \circ T^{-1})(Tx) = T[\nabla\Phi(x)]^*.$$

[2]Here T^* is the transpose of T.

We therefore have that

$$\begin{aligned}(1,i,0)\cdot\nabla(\Phi\circ T^{-1})(Tx) &= (1,i,0)\cdot T[\nabla\Phi(x)]^* \\ &= (\alpha\cdot\nabla + i\beta\cdot\nabla)\Phi(x) \\ &= \zeta_0\cdot\nabla\Phi(x).\end{aligned}$$

Equation (3.16) gives hence the $\bar{\partial}$-equation

$$2\,\partial_{\bar{z}}\cdot(\Phi\circ T^{-1})(y) = -i\zeta_0\cdot(A^\sharp\circ T^{-1})(y), \tag{3.17}$$

where $\partial_{\bar{z}} = (\partial_{y_1} + i\,\partial_{y_2})/2$. We will solve this using the Cauchy operator

$$N^{-1}f(x) := \frac{1}{\pi}\int_{\mathbb{R}^2}\frac{1}{s_1+is_2}f(x-(s_1,s_2,0))ds_1ds_2,$$

which is an inverse for the $\bar{\partial}$-operator, $N := (\partial_{y_1} + i\partial_{y_2})/2$ (see e.g. [13] Theorem 1.2.2). We will need the following straight forward continuity result for the Cauchy operator.

Lemma 3.7. Let $r > 0$ and $f \in W^{k,\infty}(\mathbb{R}^3)$, $k \geq 0$ and assume that $\operatorname{supp}(f) \subset B(0,r)$. Then

$$\|N^{-1}f\|_{W^{k,\infty}(\mathbb{R}^3)} \leq C_k\|f\|_{W^{k,\infty}(\mathbb{R}^3)}$$

for some constant $C_k > 0$. And if $f \in C_0(\mathbb{R}^3)$, then $N^{-1}f \in C(\mathbb{R}^3)$.

Proof. See e.g. [32]. □

Returning to (3.17) we get that $\Phi = \frac{1}{2}N^{-1}(-i\zeta_0\cdot(A^\sharp\circ T^{-1}))\circ T$. Or more explicitly that

$$\Phi(x,\zeta_0;\epsilon) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{-i\zeta_0\cdot A^\sharp(x-T^{-1}(s_1,s_2,0))}{s_1+is_2}ds_1ds_2, \tag{3.18}$$

here $T^{-1}(s_1,s_2,0) = s_1\alpha + s_2\beta$. We have thus found a solution $a = e^\Phi$ to equation (3.15). We will choose ϵ so that it depends on h, which implies that a will depend on h. In order to determine how the norm of r will depend on h and also for later estimates, we will need to see how $\|\,\partial^\alpha a\|_{L_\infty}$ depends on h.

Lemma 3.8. *Equation* (3.15) *has a solution* $a \in C^\infty(\overline{\Omega})$ *satisfying the estimates*

$$\|\,\partial^\alpha a\|_{L^\infty(\Omega)} \leq C_\alpha\epsilon^{-|\alpha|} \quad \textit{for all} \quad \alpha. \tag{3.19}$$

Proof. Existence of a solution is a consequence of the considerations above. Therefore we need only to prove the norm estimate.

For $\alpha = 0$, Lemma 3.7 gives that $\|\Phi\|_{L^\infty(\Omega)} \leq C$. From this it follows that $\|e^\Phi\|_{L^\infty(\Omega)} \leq C'$. For $|\alpha| > 1$ argue similarly using the estimates (3.12). $\square$

We can now write the $L^\infty(\Omega)$ norm of (3.14) as

$$\|L_\zeta a\|_{L^\infty(\Omega)} = \| - h^2 L_{A,q} a + 2ih\zeta_0 \cdot A^\flat a + 2\zeta_1 \cdot h\nabla a + 2ih\zeta_1 \cdot Aa\|_{L^\infty(\Omega)}.$$

Using (3.12), (3.19) and the fact that $\zeta_1 = \mathcal{O}(h)$ we have that

$$\|L_\zeta a\|_{L^\infty(\Omega)} = \mathcal{O}(h^2\epsilon^{-2} + h\epsilon).$$

Choosing $\epsilon = h^{1/3}$, gives finally $\|L_\zeta a\|_{L^\infty(\Omega)} = \mathcal{O}(h^{4/3})$, as $h \to 0$.

Finally to solve (3.11) for r, we rewrite it as

$$e^{-x\cdot \mathrm{Re}\,\zeta/h} h^2 L_{A,q} e^{x\cdot \mathrm{Re}\,\zeta/h}(e^{ix\cdot \mathrm{Im}\,\zeta/h} r) = -e^{ix\cdot \mathrm{Im}\,\zeta/h} L_\zeta a. \tag{3.20}$$

If we replace $e^{ix\cdot \mathrm{Im}\,\zeta/h} r$ by $\tilde{r}$, then the solvability result 3.6, shows that we can find a solution $\tilde{r}$, so that a solution r to (3.20) is given by $r = e^{-ix\cdot \mathrm{Im}\,\zeta/h}\tilde{r}$.

To get a norm estimate for r, notice that for the left hand side of (3.20) we have

$$\|e^{ix\cdot \mathrm{Im}\,\zeta/h} L_\zeta a\|_{L^\infty(\Omega)} = \mathcal{O}(h^{4/3}),$$

as $h \to 0$. The solvability result 3.6 gives then that

$$\|\tilde{r}\|_{H^1_{scl}(\Omega)} = \mathcal{O}(h^{1/3}),$$

as $h \to 0$, which implies that $\|r\|_{H^1_{scl}(\Omega)} = \mathcal{O}(h^{1/3})$, as $h \to 0$.

Thus we have obtained the following existence result for complex geometric optics solutions.

Proposition 3.9. *Let $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ and $q \in L^\infty(\Omega, \mathbb{C})$. Then for $h > 0$ small enough, there exist solutions $u \in H^1(\Omega)$, of the equation*

$$L_{A,q} u = 0 \quad in \quad \Omega,$$

that are of the form

$$u(x, \zeta; h) = e^{x\cdot\zeta/h}(a(x, \zeta; h) + r(x, \zeta; h)),$$

where $\zeta \in \mathbb{C}^3$, is of the form given by (3.13), *$a \in C^\infty(\overline{\Omega})$ solves the equation* (3.15), *and where a and r satisfy the estimates*

$$\|\,\partial^\alpha\, a\|_{L^\infty(\Omega)} \leq C_\alpha h^{-|\alpha|/3} \quad and \quad \|r\|_{H^1_{\mathrm{scl}}(\Omega)} = \mathcal{O}(h^{1/3}).$$

$\square$

Remark 3.10. *In the sequel, we need complex geometric optics solutions belonging to* $H^2(\Omega)$*. To obtain such solutions, let* $\Omega' \supset\supset \Omega$ *be a bounded domain with smooth boundary, and let us extend* $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ *and* $q \in L^\infty(\Omega)$ *to* $W^{1,\infty}(\Omega', \mathbb{R}^3)$ *and* $L^\infty(\Omega')$*-functions, respectively. By elliptic regularity, the complex geometric optics solutions, constructed on* Ω'*, according to Proposition* 3.9*, belong to* $H^2(\Omega)$*.*

Remark 3.11. *Recall that* $\Phi = \frac{1}{2}N^{-1}(-i(\alpha + i\beta)\cdot(A^\sharp \circ T^{-1}))\circ T$*. Lemma 3.7 implies that* $N^{-1} : C_0(\Omega) \to C(\Omega)$ *is continuous. The estimates* (3.12) *show that* $A^\sharp \to A$ *uniformly on* Ω*. It follows that there is an* Φ^0*, s.t.*

$$\|\Phi(x, \zeta_0; h^{1/3}) - \Phi^0\|_{L^\infty(\Omega)} \to 0, \quad h \to 0,$$

where $\Phi^0 = \frac{1}{2}N^{-1}(-i(\alpha + i\beta)\cdot(A \circ T^{-1}))\circ T$ *solves the equation*

$$\zeta_0 \cdot \nabla\Phi^0 = -i\zeta_0 \cdot A \quad in \quad \Omega, \tag{3.21}$$

as $h \to 0$*.*

Remark 3.12. *We shall later use a slightly more general form for the amplitude* a *in the CGO solutions. Namely we suppose that* $a = ge^\Phi$*, where* $g \in C^\infty(\overline{\Omega})$*, with*

$$\zeta_0 \cdot \nabla g = 0. \tag{3.22}$$

This means that g *is holomorphic in a plane spanned by* α *and* β*. Notice also that by picking* $a = ge^\Phi$*, we get by* (3.15) *that*

$$\zeta_0 \cdot g\nabla\Phi = -i\zeta_0 \cdot gA^\sharp,$$

in place of (3.16)*. But the* Φ *solving* (3.16) *also solves the above. Hence we can use the same argument to obtain the* Φ *for the above equation, as earlier.*

We thus obtain CGO solutions of the form

$$u = e^{x\cdot\zeta/h}(ge^\Phi + r_g),$$

where Φ *is the same as when* a *is of the earlier of form with no* g*.*

Notice also that setting $a = ge^\Phi$ *does not affect the norm estimates on* a *in Proposition 3.9, since* g *does not depend on* h*.*

3.3 Recovering the magnetic field

The aim of this section is to prove the first part of Theorem 1.2, by showing that the curl of the magnetic potential is determined by the DN-map. We use again similar notations as in Subsection 3.1, i.e.

$$B_- := \mathbb{R}^3_- \cap B, \quad B_+ := \mathbb{R}^3_+ \cap B, \quad l := \partial \mathbb{R}^3_- \cap B,$$

where B is an open ball in $\mathbb{R}^3$, containing the supports of the potentials A_j and q_j, $j = 1, 2$. The first step in the argument will be to construct complex geometric optics solutions u_1 and u_2, belonging to the spaces $W_1(B_-)$ and $W_2^*(B_-)$ (defined in Section 3.1) and then to examine the limit of (3.7) as $h \to 0$.

For $u_1 \in W_1(B_-)$ and $u_2 \in W_2^*(B_-)$, we have that $u_j|_l = 0$, $j = 1, 2$. To obtain solutions that satisfy this condition, we will first choose solutions defined on the bigger set $B = B_+ \cup l \cup B_-$.

The parameters ζ for the complex geometric optics solutions will be picked as follows. We will assume that

$$\xi, \gamma_1, \gamma_2 \in \mathbb{R}^3, \ |\gamma_1| = |\gamma_2| = 1 \ \text{ and that } \{\gamma_1, \gamma_2, \xi\} \text{ is orthogonal.} \tag{3.23}$$

Similarly to [34], we set

$$\begin{aligned} \zeta_1 &= \frac{ih\xi}{2} + \gamma_1 + i\sqrt{1 - h^2\frac{|\xi|^2}{4}}\gamma_2, \\ \zeta_2 &= -\frac{ih\xi}{2} - \gamma_1 + i\sqrt{1 - h^2\frac{|\xi|^2}{4}}\gamma_2, \end{aligned} \tag{3.24}$$

so that $\zeta_j \cdot \zeta_j = 0$, $j = 1, 2$, and $(\zeta_1 + \overline{\zeta_2})/h = i\xi$. Here $h > 0$ is a small semiclassical parameter.

We need to extend the potentials A_j and q_j, $j = 1, 2$, to B_+. For the component functions $A_{j,1}$, $A_{j,2}$, and q_j, we do an even extension, and for $A_{j,3}$, we do an odd extension, i.e., for $j = 1, 2$ we set,

$$\tilde{A}_{j,k}(x) = \begin{cases} A_{j,k}(x), & x_3 < 0, \\ A_{j,k}(\tilde{x}), & x_3 > 0, \end{cases} \quad k = 1, 2,$$

$$\tilde{A}_{j,3}(x) = \begin{cases} A_{j,3}(x), & x_3 < 0, \\ -A_{j,3}(\tilde{x}), & x_3 > 0, \end{cases}$$

$$\tilde{q}_j(x) = \begin{cases} q_j(x), & x_3 < 0, \\ q_j(\tilde{x}), & x_3 > 0, \end{cases}$$

where $\tilde{x} := (x_1, x_2, -x_3)$. By Remark 3.2 we can take $A_{j,3}|_{x_3=0} = 0$, from which it follows that $\tilde{A}_j \in W^{1,\infty}(B)$ and $\tilde{q}_j \in L^\infty(B)$, $j = 1, 2$.

We can now by Proposition 3.9 and Remark 3.10 pick complex geometric optics solutions $\tilde{u}_1$ in $H^2(B)$,

$$\tilde{u}_1(x, \zeta_1; h) = e^{x\cdot\zeta_1/h}(e^{\Phi_1(x,\gamma_1+i\gamma_2;h)} + r_1(x, \zeta_1; h))$$

of the equation $(L_{\tilde{A}_1,\tilde{q}_1} - k^2)\tilde{u}_1 = 0$ in B, where $\Phi_1 \in C^\infty(\overline{B})$. By Remark 3.11, $\Phi_1 \to \Phi_1^0$ in the L^∞-norm as $h \to 0$, where Φ_1^0 solves the equation

$$(\gamma_1 + i\gamma_2) \cdot \nabla\Phi_1^0 = -i(\gamma_1 + i\gamma_2) \cdot \tilde{A}_1 \quad \text{in} \quad B. \tag{3.25}$$

To obtain a function that is zero on the plane $x_3 = 0$, we set

$$u_1(x) := \tilde{u}_1(x) - \tilde{u}_1(\tilde{x}), \quad x \in B_- \cup l. \tag{3.26}$$

Then it is easy to check that $u_1 \in W_1(B_-)$.

We can similarly pick by Proposition 3.9 and Remark 3.10, complex geometric optics solutions $\tilde{u}_2$ in $H^2(B)$,

$$\tilde{u}_2(x, \zeta_2; h) = e^{x\cdot\zeta_2/h}(e^{\Phi_2(x,-\gamma_1+i\gamma_2;h)} + r_2(x, \zeta_2; h))$$

of the equation $(L_{\tilde{A}_2,\overline{\tilde{q}_2}} - k^2)\tilde{u}_2 = 0$ in B, where $\Phi_2 \in C^\infty(\overline{B})$. By Remark 3.11, $\Phi_2 \to \Phi_2^0$ in the L^∞-norm as $h \to 0$, where Φ_2^0 solves the equation

$$(-\gamma_1 + i\gamma_2) \cdot \nabla\Phi_2^0 = -i(-\gamma_1 + i\gamma_2) \cdot \tilde{A}_1 \quad \text{in} \quad B. \tag{3.27}$$

To obtain a function that is zero on the plane $x_3 = 0$, we set

$$u_2(x) := \tilde{u}_2(x) - \tilde{u}_2(\tilde{x}), \quad x \in B_- \cup l. \tag{3.28}$$

Then it is easy to check that $u_1 \in W_2^*(B_-)$.

The next step is to substitute the complex geometric optics solutions u_1 and u_2, given by (3.26) and (3.28), respectively, into the integral identity (3.7). This will be done in the Lemma bellow. We will use the abbreviations $P_1(x) := e^{\Phi_1(x)} + r_1(x)$ and $P_2(x) := e^{\Phi_2(x)} + r_2(x)$, so that

$$\begin{aligned} u_1(x) &= e^{x\cdot\zeta_1/h}P_1(x) - e^{\tilde{x}\cdot\zeta_1/h}P_1(\tilde{x}), \\ u_2(x) &= e^{x\cdot\zeta_2/h}P_2(x) - e^{\tilde{x}\cdot\zeta_2/h}P_2(\tilde{x}). \end{aligned}$$

For future references, it will be convenient to compute the product of the phases that occur in the terms $u_1\overline{u}_2$, $\nabla u_1\overline{u_2}$ and $u_1\nabla\overline{u_2}$

$$\begin{aligned} &e^{x\cdot\zeta_1/h}e^{x\cdot\overline{\zeta_2}/h} = e^{ix\cdot\xi}, \quad e^{\tilde{x}\cdot\zeta_1/h}e^{\tilde{x}\cdot\overline{\zeta_2}/h} = e^{i\tilde{x}\cdot\xi}, \\ &e^{\tilde{x}\cdot\zeta_1/h}e^{x\cdot\overline{\zeta_2}/h} = e^{ix\cdot\xi}e^{i(0,0,-2x_3)\cdot\zeta_1/h} = e^{ix\cdot\xi_- - 2\gamma_{1,3}x_3/h}, \\ &e^{x\cdot\zeta_1/h}e^{\tilde{x}\cdot\overline{\zeta_2}/h} = e^{i\tilde{x}\cdot\xi}e^{i(0,0,2x_3)\cdot\zeta_1/h} = e^{ix\cdot\xi_+ + 2\gamma_{1,3}x_3/h}, \end{aligned} \tag{3.29}$$

where $\gamma_j = (\gamma_{j,1}, \gamma_{j,2}, \gamma_{j,3})$, $j = 1, 2$ and

$$\xi_{\pm} = \left(\xi_1, \xi_2, \pm\frac{2}{h}\sqrt{1 - \frac{h^2|\xi|^2}{4}}\gamma_{2,3}\right).$$

We restrict the choices of γ_1, by assuming that

$$\gamma_{1,3} = 0 \quad \text{and} \quad \gamma_{2,3} \neq 0. \tag{3.30}$$

We need these conditions for the proof of the next Lemma. The first condition makes the above phases purely imaginary, which avoids exponential growth of the terms, as $h \to 0$. The second condition implies that $|\xi_{\pm}| \to \infty$ as $h \to 0$. This will be needed since, we will use the Riemann-Lebesgue Lemma to eliminate unwanted imaginary exponentials.

Finally it will also be convenient to explicitly state the following norm estimates, which follow from Proposition 3.9

$$\begin{aligned} &\|e^{\Phi_j}\|_{L^\infty} = \mathcal{O}(1), \quad \|\nabla e^{\Phi_j}\|_{L^\infty} = \mathcal{O}(h^{-1/3}), \\ &\|r_j\|_{L^2} = \mathcal{O}(h^{1/3}), \quad \|\nabla r_j\|_{L^2} = \mathcal{O}(h^{-2/3}), \quad j = 1, 2, \end{aligned} \tag{3.31}$$

as $h \to 0$.

Lemma 3.13. *If the assumptions of Proposition 3.4 hold, then*

$$(\gamma_1 + i\gamma_2) \cdot \int_B (\tilde{A}_2 - \tilde{A}_1) e^{ix\cdot\xi} e^{\Phi_1^0 + \overline{\Phi_2^0}} dx = 0, \tag{3.32}$$

where γ_1, γ_2 *and* ξ *satisfy* (3.23) *and* (3.30).

Proof. We will prove the statement by multiplying the integral equation of Proposition 3.4 by h, when u_1 and u_2 are given by (3.26) and (3.28), and then take the limit as $h \to 0$.

We first show that for the second term in (3.7) we have

$$h \int_{B_-} (A_1^2 - A_2^2 + q_1 - q_2) u_1 \overline{u}_2 \to 0, \tag{3.33}$$

as $h \to 0$. Using the phase computations (3.29) we get that

$$\begin{aligned} u_1 \overline{u}_2 = \; & e^{i\xi\cdot x} P_1(x) \overline{P}_2(x) - e^{ix\cdot\xi_+} P_1(x) \overline{P}_2(\tilde{x}) \\ & - e^{ix\cdot\xi_-} P_1(\tilde{x}) \overline{P}_2(x) + e^{i\xi\cdot\tilde{x}} P_1(\tilde{x}) \overline{P}_2(\tilde{x}). \end{aligned}$$

This is multiplied by an L^∞ function in (3.33). Since we restricted the choice of γ_1 to make the exponents purely imaginary, we see easily using the estimates (3.31) that (3.33) holds.

Equation (3.7) multiplied by h, is thus reduced in the limit to

$$\lim_{h\to 0}\left(h\int_{B_-} i(A_2-A_1)\cdot\nabla u_1\overline{u}_2 - h\int_{B_-} i(A_2-A_1)\cdot u_1\nabla\overline{u}_2\right)=0. \tag{3.34}$$

We will proceed by examining the first term. Using (3.29) we write $\nabla u_1\overline{u}_2$ as

$$\begin{aligned}\nabla u_1\overline{u}_2 =&\frac{\zeta_1}{h}\big(e^{ix\cdot\xi}P_1(x)\overline{P_2(x)} - e^{ix\cdot\xi_+}P_1(x)\overline{P_2(\tilde{x})}\big)\\ &+ e^{ix\cdot\xi}\nabla P_1(x)\overline{P_2(x)} - e^{ix\cdot\xi_+}\nabla P_1(x)\overline{P_2(\tilde{x})}\\ &-\frac{\tilde{\zeta}_1}{h}\big(e^{ix\cdot\xi_-}P_1(\tilde{x})\overline{P_2(x)} - e^{i\tilde{x}\cdot\xi}P_1(\tilde{x})\overline{P_2(\tilde{x})}\big)\\ &- e^{ix\cdot\xi_-}\nabla P_1(\tilde{x})\overline{P_2(x)} + e^{i\tilde{x}\cdot\xi}\nabla P_1(x)\overline{P_2(\tilde{x})},\end{aligned}$$

where $\tilde{\zeta}_j := \zeta_j\cdot(0,0,-1)$, $j=1,2$. The terms of the product that do not contain the factor $1/h$, result in integrals similar to the one in (3.33). And one sees similarly using estimates (3.31) that they are zero in the limit of (3.34). The first term inside the limit in (3.34) is therefore reduced to

$$\begin{aligned}\lim_{h\to 0}\int_{B_-} i(A_2-A_1)\cdot\big(&\zeta_1 e^{ix\cdot\xi}P_1(x)\overline{P_2(x)} - \tilde{\zeta}_1 e^{ix\cdot\xi_-}P_1(\tilde{x})\overline{P_2(x)}\\ &-\zeta_1 e^{ix\cdot\xi_+}P_1(x)\overline{P_2(\tilde{x})} - \tilde{\zeta}_1 e^{i\tilde{x}\cdot\xi}P_1(\tilde{x})\overline{P_2(\tilde{x})}\big).\end{aligned}$$

Now we use the Riemann-Lebesgue Lemma to conclude that the terms with exponents containing ξ_+ and ξ_- are zero in the limit. To see this, notice that by Remark 3.11, we see that $\|\Phi_i\|_{L^\infty(B_-)} < C$, for some $C>0$, when h is small enough. Estimates (3.31) show that $\|r_i\|_{L^1(B_-)} = \mathcal{O}(h^{1/3})$. Hence $\|P_i\|_{L_1(B_-)} < C$, for some $C>0$ when h is small enough. Finally we have $\xi_\pm \to \infty$, as $h\to 0$, because of the restrictions (3.30).

The first term in (3.34) is therefore

$$\lim_{h\to 0}\int_{B_-} i(A_2-A_1)\cdot\big(\zeta_1 e^{ix\cdot\xi}P_1(x)\overline{P_2(x)} + \tilde{\zeta}_1 e^{i\tilde{x}\cdot\xi}P_1(\tilde{x})\overline{P_2(\tilde{x})}\big)$$

as $h\to 0$. The terms containing r_i in the products of P_1 and P_2 are, because of (3.31), zero in the limit. The above limit is thus equal to

$$\lim_{h\to 0}\int_{B_-} i(A_2-A_1)\cdot\big(\zeta_1 e^{ix\cdot\xi}e^{\Phi_1(x)+\overline{\Phi_2(x)}} + \tilde{\zeta}_1 e^{i\tilde{x}\cdot\xi}e^{\Phi_1(\tilde{x})+\overline{\Phi_2(\tilde{x})}}\big).$$

Finally we split the integral and do a change of variable in the second term and arrive at the expression

$$\lim_{h \to 0} \int_B i(\tilde{A}_2 - \tilde{A}_1) \cdot \zeta_1 e^{ix\cdot\xi} e^{\Phi_1(x)+\overline{\Phi_2(x)}}, \tag{3.35}$$

for the first term of (3.34).

Returning to the second term in (3.34), containing $u_1 \nabla \overline{u_2}$. This is of the same form as the first one. By doing the above derivation by simply exchanging the roles of u_1 and $\overline{u_2}$, we similarly see that the second term becomes

$$\lim_{h \to 0} \quad -\int_B i(\tilde{A}_2 - \tilde{A}_1) \cdot \overline{\zeta_2} e^{ix\cdot\xi} e^{\Phi_1(x)+\overline{\Phi_2(x)}}. \tag{3.36}$$

Now $\zeta_1 \to (\gamma_1 + i\gamma_2)$ and $\overline{\zeta_2} \to -(\gamma_1 + i\gamma_2)$, as $h \to 0$. Thus by using (3.35) with (3.36), we can rewrite (3.34) as

$$\begin{aligned}\lim_{h \to 0} \quad & \int_B i(\tilde{A}_2 - \tilde{A}_1) \cdot \left(\zeta_1 e^{ix\cdot\xi} e^{\Phi_1(x)+\overline{\Phi_2(x)}} - \overline{\zeta_2} e^{ix\cdot\xi} e^{\Phi_1(x)+\overline{\Phi_2(x)}}\right) \\ &= \int_B i(\tilde{A}_2 - \tilde{A}_1) \cdot (\gamma_1 + i\gamma_2) e^{ix\cdot\xi} e^{\Phi_1^0(x)+\overline{\Phi_2^0(x)}} = 0.\end{aligned}$$

□

The next Proposition shows that (3.32) holds even when the exponential function depending on Φ_i^0, $i = 1, 2$ is removed.

Proposition 3.14. *The equality* (3.32) *implies that*

$$(\gamma_1 + i\gamma_2) \cdot \int_B (\tilde{A}_2 - \tilde{A}_1) e^{ix\cdot\xi} dx = 0, \tag{3.37}$$

for γ_1, γ_2 *and* ξ *which satisfy* (3.23) *and* (3.30).

Proof. By (3.25) and (3.27) we have that

$$(\gamma_1 + i\gamma_2) \cdot \nabla(\Phi_1^0 + \overline{\Phi_2^0}) = -i(\gamma_1 + i\gamma_2) \cdot (\tilde{A}_1 - \tilde{A}_2) \quad \text{in} \quad B. \tag{3.38}$$

Remark 3.12 furthermore implies that the amplitude e^{Φ_1} in the definition of u_1 can be replaced by ge^{Φ_1}, if $g \in C^\infty(\overline{B})$ is a solution of

$$(\gamma_1 + i\gamma_2) \cdot \nabla g = 0 \quad \text{in} \quad B. \tag{3.39}$$

Let $\Psi(x) := \Phi_1^0(x) + \overline{\Phi_2^0}(x)$. Then instead of (3.32) we can write,

$$(\gamma_1 + i\gamma_2) \cdot \int_B (\tilde{A}_2 - \tilde{A}_1) g e^{ix\cdot\xi} e^{\Psi(x)} dx = 0.$$

We conclude from (3.38) that

$$(\gamma_1 + i\gamma_2) \cdot (\tilde{A}_2 - \tilde{A}_1) g e^{\Psi} = -i(\gamma_1 + i\gamma_2) \cdot (g \nabla e^{\Psi}),$$

and therefore, we get

$$\int_B g e^{ix\cdot\xi} (\gamma_1 + i\gamma_2) \cdot \nabla e^{\Psi} dx = 0, \tag{3.40}$$

for all g satisfying (3.39).

We pick a γ_3, with $|\gamma_3| = 1$, so that we obtain an orthonormal basis $\{\gamma_1, \gamma_2, \gamma_3\}$. Let T be the coordinate transform into this basis, i.e. $y = Tx = (x \cdot \gamma_1, x \cdot \gamma_2, x \cdot \gamma_3)$. Set $z = y_1 + iy_2$, so that $\partial_{\bar{z}} = (\partial_{y_1} + i\, \partial_{y_2})/2$ and

$$(\gamma_1 + i\gamma_2) \cdot \nabla = 2\, \partial_{\bar{z}}\, .$$

Rewriting (3.40) using this and a change of variable given by T we have

$$\int_{TB} g e^{iy\cdot\xi}\, \partial_{\overline{z}}\, e^{\Psi} dy = 0,$$

for all g satisfying (3.39).

Notice that $y \cdot \xi = y_3 \xi_3$, since ξ is in the y-coordinates of the form $(0, 0, \xi_3)$. The above integral is therefore a Fourier transform w.r.t. ξ_3. Let $g \in C^\infty(\overline{TB})$ satisfy $\partial_{\bar{z}}\, g = 0$ and be independent of y_3. Then taking the inverse Fourier transform we write

$$\begin{aligned} 0 &= \int_{T_{y_3}} g\, \partial_{\overline{z}}\, e^{\Psi} dy_1 dy_2 \\ &= \int_{T_{y_3}} \partial_{\overline{z}} (g e^{\Psi}) dy_1 dy_2, \end{aligned}$$

where $T_{y_3} := TB \cap \Pi_{y_3}$ and $\Pi_{y_3} = \{(y_1, y_2, y_3) : (y_1, y_2) \in \mathbb{R}^2\}$. Notice that the boundary of T_{y_3} is piecewise smooth. Multiplying the above by $2i$ and using Stokes' theorem we get that

$$\begin{aligned} 0 &= 2i \int_{T_{y_3}} \partial_{\overline{z}} (g e^{\Psi}) dy_1 dy_2 \\ &= \int_{T_{y_3}} \nabla \times (g e^{\Psi}, i g e^{\Psi}, 0) dy_1 dy_2 \\ &= \int_{\partial\, T_{y_3}} (g e^{\Psi}, i g e^{\Psi}, 0) \cdot dl \\ &= \int_{\partial\, T_{y_3}} g e^{\Psi} dz, \end{aligned} \tag{3.41}$$

for all holomorphic functions $g \in C^\infty(\overline{T_{y_3}})$.

Next we shall show that (3.41) implies that there exists a nowhere vanishing holomorphic function $F \in C(\overline{T_{y_3}})$ such that

$$F|_{\partial T_{y_3}} = e^{\Psi}|_{\partial T_{y_3}}. \tag{3.42}$$

To this end, we define F to be

$$F(z) = \frac{1}{2\pi i} \int_{\partial T_{y_3}} \frac{e^{\Psi(\zeta)}}{\zeta - z} d\zeta, \quad z \in \mathbb{C} \setminus \partial T_{y_3}.$$

The function F is holomorphic away from ∂T_{y_3}. As e^{Ψ} is Lipschitz, we know because of the Plemelj-Sokhotski-Privalov formula (see e.g. [18]), that

$$\lim_{z \to z_0, z \in T_{y_3}} F(z) - \lim_{z \to z_0, z \notin T_{y_3}} F(z) = e^{\Psi(z_0)}, \quad z_0 \in \partial T_{y_3}. \tag{3.43}$$

Now the function $\zeta \mapsto (\zeta - z)^{-1}$ is holomorphic on T_{y_3} when $z \notin T_{y_3}$. By choosing $g(z) = \zeta \mapsto (\zeta - z)^{-1}$ in (3.41), get therefore that $F(z) = 0$, when $z \notin T_{y_3}$. Hence, the second limit in (3.43) vanishes, and therefore, F is holomorphic function on T_{y_3}, such that (3.42) holds.

Next we show that F is non-vanishing in T_{y_3}. When doing so, let ∂T_{y_3} be parametrized by $z = \gamma(t)$, and N be the number of zeros of F in T_{y_3}. Then by the argument principle, we get

$$N = \frac{1}{2\pi i} \int_\gamma \frac{F'(z)}{F(z)} dz = \frac{1}{2\pi i} \int_{F \circ \gamma} \frac{1}{\zeta} d\zeta = \frac{1}{2\pi i} \int_{e^{\Psi \circ \gamma}} \frac{1}{\zeta} d\zeta = 0.$$

To see that the last integral is zero, notice that this the winding number of the path $e^{\Psi \circ \gamma}$. And that $e^{\Psi(\gamma(t))}$ is homotopic to the constant contour $\{1\}$, with the homotopy given by $e^{s\Psi(\gamma(t))}$, $s \in [0,1]$.

Next, since F is a non-vanishing holomorphic function on T_{y_3} and T_{y_3} is simply connected, it admits a holomorphic logarithm. Hence, (3.42) implies that

$$(\log F)|_{\partial T_{y_3}} = \Psi|_{\partial T_{y_3}}.$$

Because $\log F = \Psi$ is continuous on ∂T_{y_3}, we have by the Cauchy theorem,

$$\int_{\partial T_{y_3}} g \Psi dz = \int_{\partial T_{y_3}} g \log F dz = 0,$$

where $g \in C^\infty(\overline{T_{y_3}})$ is an arbitrary function such that $\partial_{\bar z}\, g = 0$. Using Stokes' formula as in (3.41) allows us to write this as

$$\int_{T_{y_3}} g\, \partial_{\bar z}\, \Psi dy_1 dy_2 = 0.$$

Taking the Fourier transform with respect to y_3, we get

$$\int_{T(B)} e^{iy\cdot\xi} g\, \partial_{\bar z}\, \Psi dy = 0,$$

for all $\xi = (0, 0, \xi_3)$, $\xi_3 \in \mathbb{R}$. Hence, returning back to the x variable, we obtain that

$$(\gamma_1 + i\gamma_2) \cdot \int_B e^{ix\cdot\xi} g(x) \nabla\Psi(x) dx = 0,$$

where $g \in C^\infty(\overline{B})$ is such that $(\gamma_1 + i\gamma_2) \cdot \nabla g = 0$ in B.

Using (3.38), we finally get

$$(\gamma_1 + i\gamma_2) \cdot \int_B (\tilde{A}_2 - \tilde{A}_1) g(x) e^{ix\cdot\xi} dx = 0. \tag{3.44}$$

Setting $g = 1$, we obtain (3.37).

□

By replacing the vector γ_2 by $-\gamma_2$ in (3.37), we see that

$$(\gamma_1 - i\gamma_2) \cdot \int_B (\tilde{A}_2 - \tilde{A}_1) e^{ix\cdot\xi} dx = 0. \tag{3.45}$$

Hence, (3.37) and (3.45) imply that

$$\gamma \cdot \int_B (\tilde{A}_2 - \tilde{A}_1) e^{ix\cdot\xi} dx = 0, \tag{3.46}$$

for all $\gamma \in \mathrm{span}\{\gamma_1, \gamma_2\}$ and all $\xi \in \mathbb{R}^3$ such that (3.23) and (3.30) hold.

In the proof of the next Proposition we see that (3.37) is actually a condition for having $\nabla \times (\tilde{A}_1 - \tilde{A}_2) = 0$. This is therefore the last step in proving that the DN-map determines the curl of the magnetic potential.

Proposition 3.15. *Assume that A_j, q_j and Γ_j, $j = 1, 2$ are as in Theorem 1.2 and that the DN-maps satisfy*

$$\Lambda_{A_1,q_1}(f)|_{\Gamma_1} = \Lambda_{A_2,q_2}(f)|_{\Gamma_1},$$

for any $f \in H^{3/2}_{\mathrm{comp}}(\partial\,\mathbb{R}^3_-)$, $\mathrm{supp}(f) \subset \Gamma_2$. Then

$$\nabla \times \tilde{A}_1 = \nabla \times \tilde{A}_2 \quad in \quad B. \tag{3.47}$$

Proof. Assume that $\xi \in \mathbb{R}^3$ is not on the line $L := (0,0,t)$, $t \in \mathbb{R}$. Then the vectors γ_1 and γ_2 given by

$$\begin{aligned} \tilde{\gamma}_1 &:= (-\xi_2, \xi_1, 0), \quad & \gamma_1 &:= \tilde{\gamma}_1/|\tilde{\gamma}_1|, \\ \tilde{\gamma}_2 &:= \xi \times \gamma_1, & \gamma_2 &:= \tilde{\gamma}_2/|\tilde{\gamma}_2|, \end{aligned} \tag{3.48}$$

where $\xi \times \gamma_1$ stands for the vector cross product, satisfy (3.30) and (3.23). Thus, for any vector $\xi \in \mathbb{R}^3 \setminus L$, (3.46) says that

$$\gamma \cdot v(\xi) = 0, \quad v(\xi) := \widehat{\tilde{A}_2\chi}(\xi) - \widehat{\tilde{A}_1\chi}(\xi), \tag{3.49}$$

for all $\gamma \in \operatorname{span}\{\gamma_1, \gamma_2\}$. Here χ is the characteristic function of the set B. For any vector $\xi \in \mathbb{R}^3$, we have the following decomposition,

$$v(\xi) = v_\xi(\xi) + v_\perp(\xi),$$

where $\operatorname{Re} v_\xi(\xi)$, $\operatorname{Im} v_\xi(\xi)$ are multiples of ξ, and $\operatorname{Re} v_\perp(\xi)$, $\operatorname{Im} v_\perp(\xi)$ are orthogonal to ξ. Now we have $\operatorname{Re} v_\perp(\xi), \operatorname{Im} v_\perp(\xi) \in \operatorname{span}\{\gamma_1, \gamma_2\}$, and therefore, it follows from (3.49) that $v_\perp(\xi) = 0$, for all $\xi \in \mathbb{R}^3 \setminus L$.

Hence, $v(\xi) = \alpha(\xi)\xi$, so that that

$$\xi \times v(\xi) = 0,$$

for all $\xi \in \mathbb{R}^3 \setminus L$, and thus, everywhere, by the analyticity of the Fourier transform. Taking the inverse Fourier transform, we obtain (3.47). □

3.4 Determining the electric potential

In order to complete the proof of Theorem 1.2, we need to show that the electric potential is also determined by the DN-map. Again we assume that A_j, q_j and Γ_j, $j = 1, 2$ are as in Theorem 1.2 and that the DN-maps satisfy (1.7), and hence (3.1).

Since B is simply connected, it follows from the Helmholtz decomposition of $\tilde{A}_1 - \tilde{A}_2$ and (3.47) that there exists $\psi \in C^{1,1}(\overline{B})$ with $\psi = 0$ near ∂B such that

$$\tilde{A}_1 = \tilde{A}_2 + \nabla\psi \quad \text{in} \quad B.$$

We extend ψ to a function of class $C^{1,1}$ on all of $\mathbb{R}^3$ such that $\psi = 0$ on $\mathbb{R}^3 \setminus \overline{B}$. Then

$$\tilde{A}_1 = \tilde{A}_2 + \nabla\psi \quad \text{in} \quad \mathbb{R}^3.$$

In particular, $\psi = 0$ on $\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$. It follows then from Lemma 3.1 part (i) and (3.1) that for all f with $\operatorname{supp}(f) \subset \tilde{\Gamma}_2$,

$$\Lambda_{A_1,q_1}(f)|_{\tilde{\Gamma}_1} = \Lambda_{A_2,q_2}(f)|_{\tilde{\Gamma}_1} = \Lambda_{A_2+\nabla\psi,q_2}(f)|_{\tilde{\Gamma}_1} = \Lambda_{A_1,q_2}(f)|_{\tilde{\Gamma}_1}.$$

We can now, by Remark 3.5 use this with Proposition 3.4. That is we consider equation (3.7), in the case $A_1 = A_2$. This gives

$$\int_{B_-} (q_1 - q_2) u_1 \overline{u_2} dx = 0, \tag{3.50}$$

for all $u_1 \in W_1(B_-)$ and $u_2 \in W_2^*(B_-)$.

Choosing in (3.50) u_1 and u_2 as the complex geometric optics solutions, given by (3.26) and (3.28), and letting $h \to 0$, we have

$$\int_B (\tilde{q}_1 - \tilde{q}_2) e^{ix\cdot\xi} e^{\Phi_1^0(x) + \overline{\Phi_2^0(x)}} dx = 0. \tag{3.51}$$

By Remark 3.12 e^{Φ_1} in the definition (3.26) of u_1 can be replaced by ge^{Φ_1} if $g \in C^\infty(\overline{B})$ is a solution of

$$(\gamma_1 + i\gamma_2) \cdot \nabla g = 0 \quad \text{in} \quad B.$$

Then (3.51) can be replaced by

$$\int_B (\tilde{q}_1 - \tilde{q}_2) g(x) e^{ix\cdot\xi} e^{\Phi_1^0(x) + \overline{\Phi_2^0(x)}} dx = 0.$$

Now (3.38) has the form,

$$(\gamma_1 + i\gamma_2) \cdot \nabla(\Phi_1^0 + \overline{\Phi_2^0}) = 0 \quad \text{in} \quad B,$$

since we consider that $\tilde{A}_1 = \tilde{A}_2$. Thus, we can take $g = e^{-(\Phi_1^0 + \overline{\Phi_2^0})}$ and obtain that

$$\int_B (\tilde{q}_1 - \tilde{q}_2) e^{ix\cdot\xi} dx = 0, \tag{3.52}$$

for all $\xi \in \mathbb{R}^3$ such that there exist $\gamma_1, \gamma_2 \in \mathbb{R}^3$, satisfying (3.23) and (3.30). Since for any $\xi \in \mathbb{R}^3$ not of the form $\xi = (0, 0, \xi_3)$, the vectors, given by (3.48), satisfy (3.23) and (3.30), we conclude that (3.52) holds for all $\xi \in \mathbb{R}^3$ except those of the form $\xi = (0, 0, \xi_3)$, and therefore, by analyticity of the Fourier transform, for all $\xi \in \mathbb{R}^3$. Hence, $q_1 = q_2$ in B_-. This completes the proof of Theorem 1.2.

4 Appendix

4.1 Magnetic Green's formulas

Let us first recall, following [9], the standard Green formula applied to the magnetic Schrödinger operator.

Lemma 4.1. *Suppose that $\Omega \subset \mathbb{R}^3$ is open and bounded, with piecewise C^1 boundary. Let $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ and $q \in L^\infty(\Omega)$. Then we have,*

$$\begin{aligned}&(L_{A,q}u, v)_{L^2(\Omega)} - (u, L_{A,\overline{q}}v)_{L^2(\Omega)}\\ &= (u, (\partial_n + iA \cdot n)v)_{L^2(\partial\Omega)} - ((\partial_n + iA \cdot n)u, v)_{L^2(\partial\Omega)},\end{aligned}$$

for all $u, v \in H^1(\Omega)$, with $\Delta u, \Delta v \in L^2(\Omega)$, where n is the exterior unit normal to $\partial\Omega$.

We shall also need a version of the above result where Ω is replaced by $\mathbb{R}^3_-$. We shall then need to put some restrictions on v and u, because $\mathbb{R}^3_-$ is unbounded. To this end we assume that u and v are solutions to the Helmholtz equation outside some compact set, that obey some form of radiation condition. To be precise, let $A \in W^{1,\infty}_{comp}(\mathbb{R}^3_-, \mathbb{R}^3)$, $q \in L^\infty_{comp}(\mathbb{R}^3_-)$, and let $u \in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_-})$ be such that

$$(L_{A,q} - k^2)u = 0 \quad \text{in} \quad \mathbb{R}^3_-,$$

$\mathrm{supp}(u|_{\partial\mathbb{R}^3_-})$ is compact, and u is outgoing. Assume also that $v \in H^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_-})$ satisfies

$$(L_{A,\overline{q}} - k^2)v \in L^2_{\mathrm{comp}}(\overline{\mathbb{R}^3_-}),$$

$\mathrm{supp}(v|_{\partial\mathbb{R}^3_-})$ is compact, and v is incoming.

Lemma 4.2. *With u and v as above, we have*

$$\begin{aligned}&((L_{A,q} - k^2)u, v)_{L^2(\mathbb{R}^3_-)} - (u, (L_{A,\overline{q}} - k^2)v)_{L^2(\mathbb{R}^3_-)}\\ &= (u, (\partial_n + iA \cdot n)v)_{L^2(\partial\mathbb{R}^3_-)} - ((\partial_n + iA \cdot n)u, v)_{L^2(\partial\mathbb{R}^3_-)}.\end{aligned} \tag{4.1}$$

Proof. Let $B_R := \{x \in \mathbb{R}^3 \,\big|\, |x| < R\}$ be an open ball in $\mathbb{R}^3$ of radius R, and choose $R > 0$ large enough so that

$$\mathrm{supp}(A), \mathrm{supp}(q) \subset B_R.$$

Set $\Omega = \mathbb{R}^3_- \cap B_R$. By Lemma 4.1, we know that

$$\begin{aligned}&((L_{A,q} - k^2)u, v)_{L^2(\Omega)} - (u, (L_{A,\overline{q}} - k^2)v)_{L^2(\Omega)}\\ &= (u, (\partial_n + iA \cdot n)v)_{L^2(\partial\Omega)} - ((\partial_n + iA \cdot n)u, v)_{L^2(\partial\Omega)}.\end{aligned}$$

Thus, to obtain (4.1) we need to show that

$$\int_{\partial B_R \cap \mathbb{R}^3_-} (u\overline{\partial_n v} - (\partial_n u)\overline{v}) dS_R \to 0, \quad R \to \infty. \tag{4.2}$$

Let us rewrite the left hand side of the above as follows,

$$\int_{\partial B_R \cap \mathbb{R}^3_-} (\partial_n \overline{v} - ik\overline{v}) u dS_R - \int_{\partial B_R \cap \mathbb{R}^3_-} (\partial_n u - iku)\overline{v} dS_R.$$

We show that first term goes to zero as $R \to \infty$. The second term can be handled in the same way. Applying Cauchy-Schwarz gives

$$\left| \int_{\partial B_R \cap \mathbb{R}^3_-} (\partial_n \overline{v} - ik\overline{v}) u dS_R \right|^2 \leq \int_{\partial B_R \cap \mathbb{R}^3_-} |\partial_n \overline{v} - ik\overline{v}|^2 dS_R \int_{\partial B_R \cap \mathbb{R}^3_-} |u|^2 dS_R.$$

Here the first integral goes to zero, since $\overline{\partial_n \overline{v} - ik\overline{v}} = \partial_n v + ikv$ and $|\partial_n v + ikv|^2$ is $o(1/r^2)$ as $r = |x| \to \infty$, since v is incoming. The second integral is bounded, because of Lemma 2.6. We conclude that (4.2) holds. $\square$

4.2 Carleman estimates and solvability

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with C^∞ boundary, and let $\varphi(x) = \alpha \cdot x$ with $\alpha \in \mathbb{R}^n$, $|\alpha| = 1$. Consider the conjugated operator

$$L_\varphi := e^{\varphi/h} h^2 L_{A,q} e^{-\varphi/h}.$$

Note that L_φ depends on A, q and h. In the beginning of this subsection we shall establish the following Carleman estimate, where we write

$$\|u\|^2_{H^1_{\mathrm{scl}}(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|h\nabla u\|^2_{L^2(\Omega)}.$$

Theorem 4.3. *Let $A \in W^{1,\infty}(\Omega, \mathbb{C}^n)$ and $q \in L^\infty(\Omega, \mathbb{C})$. Then there exist $C > 0$ and $h_0 > 0$ such that for all $u \in C_0^\infty(\Omega)$, we have*

$$h\|u\|_{H^1_{\mathrm{scl}}(\Omega)} \leq C\|L_\varphi u\|_{L^2(\Omega)}, \tag{4.3}$$

when $0 < h \leq h_0$.

Proof. In what follows, the $L^2(\Omega)$-norm is abbreviated to $\|\cdot\|$. Let $\epsilon > 0$. Define

$$\varphi_\epsilon(x) = \alpha \cdot x + \frac{\epsilon}{2}(\alpha \cdot x)^2.$$

Denote by $L_{0,\varphi_\epsilon} := -e^{\varphi_\epsilon/h}h^2\Delta e^{-\varphi_\epsilon/h}$. This can be decomposed as $L_{0,\varphi_\epsilon} = A_\epsilon + iB_\epsilon$, where

$$\begin{aligned}
A_\epsilon &:= -h^2\Delta - (1+\epsilon\alpha\cdot x)^2,\\
B_\epsilon &:= -2h(1+\epsilon\alpha\cdot x)\alpha\cdot i\nabla - i\epsilon h.
\end{aligned}$$

A direct calculation gives that

$$\|L_{0,\varphi_\epsilon}u\|^2 = \|A_\epsilon u\|^2 + \|B_\epsilon u\|^2 + i([A_\epsilon, B_\epsilon]u, u), \quad u \in C_0^\infty(\Omega). \tag{4.4}$$

Another straightforward calculation shows that the commutator can be written as

$$i[A_\epsilon, B_\epsilon] = -4\epsilon h^3 \sum_{j,k=1}^{3} \alpha_j\alpha_k\partial_j\partial_k + 4h\epsilon(1+\epsilon\alpha\cdot x)^2.$$

The first term is an operator with a positive semi-definite semiclassical symbol of the form $4\epsilon h(\alpha\cdot\xi)^2 \geq 0$. The inner product in (4.4), with this part is therefore non-negative. We can hence drop it and estimate (4.4) as follows,

$$\begin{aligned}
\|L_{0,\varphi_\epsilon}u\|^2 &\geq \|A_\epsilon u\|^2 + 4h\epsilon\|(1+\epsilon\alpha\cdot x)u\|^2\\
&\geq \|A_\epsilon u\|^2 + h\epsilon\|u\|^2,
\end{aligned} \tag{4.5}$$

for all $\epsilon > 0$ such that $\epsilon|\alpha\cdot x| \leq 1/2$, $x \in \Omega$.

The next step is to obtain a similar estimate for the first order perturbation $L_{\varphi_\epsilon} := e^{\varphi_\epsilon/h}h^2L_{A,q}e^{-\varphi_\epsilon/h}$. We decompose this as $L_{\varphi_\epsilon} = L_{0,\varphi_\epsilon} + Q_\epsilon$, where

$$Q_\epsilon := e^{\varphi_\epsilon/h}h^2(-2iA\cdot\nabla - i\nabla\cdot A + A^2 + q)e^{-\varphi_\epsilon/h}.$$

This can be estimated as

$$\begin{aligned}
\|Q_\epsilon u\|^2 &= h^4\| - 2iA\cdot\nabla u + 2iA\cdot\nabla\frac{\varphi_\epsilon}{h}u - i\nabla\cdot Au + A^2u + qu\|^2\\
&\leq C\big(h^4\|\nabla u\|^2 + h^2\|u\|^2\big),
\end{aligned}$$

when h is sufficiently small. For L_{φ_ϵ}, we get

$$\|L_{0,\varphi_\epsilon}u\|^2 \leq C\big(\|L_{\varphi_\epsilon}u\|^2 + h^4\|\nabla u\|^2 + h^2\|u\|^2\big), \tag{4.6}$$

when h is sufficiently small. Next we rewrite the gradient term and use the Cauchy inequality to obtain

$$\begin{aligned}
h^2\|\nabla u\|^2 &= (-h^2\Delta u, u)\\
&= (A_\epsilon u, u) + \|(1+\epsilon\alpha\cdot x)u\|^2\\
&\leq \frac{\delta}{2}\|A_\epsilon u\|^2 + \frac{2}{\delta}\|u\|^2 + 4\|u\|^2,
\end{aligned} \tag{4.7}$$

where $\delta > 0$. Combining this with (4.5) and (4.6), gives

$$\|A_\epsilon u\|^2 + h\epsilon\|u\|^2 \le C\bigg(\|L_{\varphi_\epsilon} u\|^2 + h^2\|u\|^2 \\ + \frac{\delta}{2}h^2\|A_\epsilon u\|^2 + (\frac{2}{\delta}+4)h^2\|u\|^2\bigg).$$

Choose $\delta = 2/C$. Rearranging the above gives

$$(1-h^2)\|A_\epsilon u\|^2 + h(\epsilon - (C^2+5C)h)\|u\|^2 \le C\|L_{\varphi_\epsilon} u\|^2.$$

We may assume that $h < 1/2$. The first term is then larger than $h^2\|A_\epsilon u\|^2$. Next we pick ϵ so that $(\epsilon - (C^2+5C)h) = 6h$, i.e. $\epsilon = Mh$, $M > 0$ is fixed. This gives

$$h^2\|A_\epsilon u\|^2 + 6h^2\|u\|^2 \le C\|L_{\varphi_\epsilon} u\|^2.$$

Using (4.7) with $\delta = 2$, gives then

$$h^4\|\nabla u\|^2 + h^2\|u\|^2 \le C\|L_{\varphi_\epsilon} u\|^2. \tag{4.8}$$

Written in another way we have

$$h^2\|u\|^2_{H^1_{\mathrm{scl}}} \le C\|L_{\varphi_\epsilon} u\|^2_{L^2},$$

which is almost the desired estimate.

To finish the proof we need to replace L_{φ_ϵ} in (4.8) by L_φ. To this end let $g := \epsilon(\alpha\cdot x)^2/2$ and let $u = e^{g/h}\tilde{u}$. Notice that $g/h = M(\alpha\cdot x)^2/2$ is independent of h. Estimate (4.8) gives now

$$h^4\|\nabla u\|^2 + h^2\|u\|^2 \le C\|e^{g/h}L_\varphi \tilde{u}\|^2 \le C'\|L_\varphi \tilde{u}\|^2,$$

for some constants $C, C' > 0$. To obtain (4.8) with L_φ, we need only to show that

$$h^4\|\nabla \tilde{u}\|^2 + h^2\|\tilde{u}\|^2 \le C(h^4\|\nabla u\|^2 + h^2\|u\|^2) \tag{4.9}$$

for some constant $C > 0$. Using the triangle and Cauchy inequalities we see that

$$\begin{aligned}\|e^{g/h}\nabla\tilde{u}\|^2 &\le 2\|\nabla(e^{g/h})\tilde{u} + e^{g/h}\nabla\tilde{u}\|^2 + 2\|\nabla(g/h)e^{g/h}\tilde{u}\|^2 \\ &\le 2\|\nabla(e^{g/h}\tilde{u})\|^2 + Ch^{-2}\|e^{g/h}\tilde{u}\|^2,\end{aligned}$$

for some some constant $C > 0$. Hence,

$$h^4\|\nabla\tilde{u}\|^2 \le C(h^4\|\nabla u\|^2 + h^2\|u\|^2),$$

for some constant $C > 0$, which shows that (4.9) holds. The proof is complete.

□

Let

$$\|u\|^2_{H^s_{\mathrm{scl}}(\mathbb{R}^n)} = (2\pi)^{-3}\int_{\mathbb{R}^n}(1+h^2|\xi|^2)^s|\hat{u}(\xi)|^2 d\xi, \quad s\in\mathbb{R}.$$

We have the following consequence of Theorem 4.3, see also [9].

Corollary 4.4. *Let $A \in W^{1,\infty}(\Omega,\mathbb{C}^n)$ and $q \in L^\infty(\Omega,\mathbb{C})$. Then there exist $C > 0$ and $h_0 > 0$ such that for all $u \in C_0^\infty(\Omega)$, we have*

$$h\|u\|_{L^2(\Omega)} \le C\|L_\varphi u\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}, \tag{4.10}$$

when $0 < h \le h_0$.

The formal adjoint of L_φ is given by

$$L^*_\varphi = e^{-\varphi/h}h^2 L_{\overline{A},\overline{q}}e^{\varphi/h},$$

and Corollary 4.4 still holds for L^*_φ.

Next we prove the following solvability result, see also [17, Proposition 4.3].

Proposition 4.5. *Let $A \in W^{1,\infty}(\Omega,\mathbb{C}^n)$, $q \in L^\infty(\Omega,\mathbb{C})$, $\alpha\in\mathbb{R}^n$, $|\alpha| = 1$ and $\varphi(x) = \alpha\cdot x$. Then there is $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0]$, and any $f \in L^2(\Omega)$, the equation*

$$L_\varphi u = f \quad in \quad \Omega,$$

has a solution $u \in H^1(\Omega)$ with

$$\|u\|_{H^1_{\mathrm{scl}}(\Omega)} \le \frac{C}{h}\|f\|_{L^2(\Omega)}.$$

Proof. The Carleman estimate of Corollary 4.4 applied to L^*_φ shows that L^*_φ is injective on $C^\infty_0(\Omega)$. This lets us define the functional $T : L^*_\varphi C^\infty_0(\Omega) \to \mathbb{C}$, given by

$$T(w) := \big((L^*_\varphi)^{-1}w, f\big)_{L^2(\mathbb{R}^n)}.$$

The set $\mathrm{Dom}(T) \subset L^\infty(\Omega) \subset H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)$ is a linear subspace. The linear functional T is moreover bounded, since by Corollary 4.4,

$$\begin{aligned} |T(w)| &= |\big((L^*_\varphi)^{-1}w, f\big)_{L^2(\Omega)}| \\ &\le \|(L^*_\varphi)^{-1}w\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} \\ &\le \frac{C}{h}\|f\|_{L^2(\Omega)}\|w\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}, \quad C > 0. \end{aligned} \tag{4.11}$$

The Hahn-Banach theorem allows us to extend T, without increasing its norm, to an operator $\tilde{T} : H^{-1}_{\mathrm{scl}}(\mathbb{R}^n) \to \mathbb{C}$. By the Riesz representation theorem there exists $r \in H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)$ such that

$$\tilde{T}(w) = (w, r)_{H^{-1}_{\mathrm{scl}}}, \quad \text{for } w \in H^{-1}_{\mathrm{scl}}(\mathbb{R}^n),$$

and

$$\|r\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)} = \|\tilde{T}\| \le \|T\| \le \frac{C}{h}\|f\|_{L^2(\Omega)}. \tag{4.12}$$

Here we have used (4.11).

Furthermore, we have

$$(w, r)_{H^{-1}_{\mathrm{scl}}} = (w, u)_{L^2},$$

with

$$u = \mathcal{F}^{-1}(1 + h^2\xi^2)^{-1}\mathcal{F}r \in H^1_{\mathrm{scl}}(\mathbb{R}^n).$$

Here $\mathcal{F}$ is the Fourier transformation on $\mathbb{R}^n$.

We now show that u solves $L_\varphi u = f$ in the weak sense in Ω. For every $\psi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}(L_\varphi u, \psi)_{L^2(\Omega)} &= (u, L^*_\varphi \psi)_{L^2(\mathbb{R}^n)} \\ &= \overline{T(L^*_\varphi \psi)} \\ &= \overline{\big((L^*_\varphi)^{-1} L^*_\varphi \psi, f\big)_{L^2(\mathbb{R}^n)}} \\ &= (f, \psi)_{L^2(\Omega)}.\end{aligned}$$

Hence u is a weak solution.

To obtain the norm estimate, we observe that $\|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} = \|r\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}$ and use (4.12). The proof is complete.

□

4.3 The unique continuation principle

In this work we make heavy use of the so called *unique continuation principle*. The unique continuation principle can be seen as an extension of the familiar property that an analytic function that is zero on some open set is identically zero.

Let $\Omega \subset \mathbb{R}^n$ be an open connected set, and let

$$Pu = \sum_{i,j=1}^{n} a_{ij}(x)\, \partial_i \partial_j u + \sum_i b_i(x)\, \partial_i u + c(x)u.$$

Here $a_{ij} \in C^1(\overline{\Omega})$ are real-valued, $a_{ij} = a_{ji}$, and there is $C > 0$ so that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq C|\xi|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n.$$

Furthermore, $b_i \in L^\infty(\Omega, \mathbb{C})$ and $c \in L^\infty(\Omega, \mathbb{C})$. We have the following result, see [5] and [24].

Theorem 4.6. *Let $u \in H^2_{\mathrm{loc}}(\Omega)$ be such that $Pu = 0$ in Ω. Let $\omega \subset \Omega$ be open non-empty. If $u = 0$ on ω, then u vanishes identically in Ω.*

Corollary 4.7. *Assume that $\partial\Omega$ is of class C^2. Let $\Gamma \subset \partial\Omega$ be open non-empty. Let $u \in H^2(\Omega)$ be such that $Pu = 0$ in Ω. Assume that*

$$u = \mathcal{B}_\nu u = 0 \quad on \quad \Gamma.$$

Here $\mathcal{B}_\nu u$ is the conormal derivative of u, given by

$$\mathcal{B}_\nu u = \sum_{i,j=1}^{n} \nu_i(a_{ij}\,\partial_j\, u)|_{\partial\Omega} \in H^{1/2}(\partial\Omega).$$

Then u vanishes identically in Ω.

4.4 Rellich's lemma

Rellich's lemma is a fundamental result in the scattering theory of the Helmholtz equation, see see e.g. [7].

Proposition 4.8. *Let $k > 0$ and let $u \in \mathcal{D}'(\mathbb{R}^3)$ satisfy the Helmholtz equation $(-\Delta - k^2)u = 0$ outside a ball B in $\mathbb{R}^3$. Assume that*

$$\lim_{R\to\infty} \int_{|x|=R} |u|^2 dS \to 0.$$

Then $u \equiv 0$ in $\mathbb{R}^3 \setminus \overline{B}$.

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