Mean–Field Approximation to the Master Equation for Sympathetic Cooling of Trapped Bosons

S. J. Wang¹, M. C. Nemes², A. N. Salgueiro³, and H. A. Weidenmüller³

¹ Department of Modern Physics, Lanzhou University, 730000 Lanzhou, P.R. China
²Department of Physics, University of Belo Horizonte, Belo Horizonte, Minas Gerais, Brasil
³Max-Planck-Institut für Kernphysik, D-69029 Heidelberg, Germany

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We use the mean-field approximation to simplify the master equation for sympathetic cooling of Bosons. For the mean single-particle occupation numbers, this approach yields the same equations as the factorization assumption introduced in an erlier paper. The stationary or equilibrium solution of the resulting master equation for the one-body density matrix shows that the mean-field approximation breaks down whenever the fraction of condensate Bosons exceeds ten percent or so of the total. Using group-theoretical methods, we also solve the time-dependent master equation for the one-body density matrix. Given the time dependence of the mean single-particle occupation numbers, this solution is obtained by quadratures. It tends asymptotically towards the equilibrium solution.

I. INTRODUCTION

Sympathetic cooling of Bosons or Fermions is an important tool in Atom Optics. It is used in cases where the interaction between atoms is too weak for evaporative cooling to work. This fact calls for the development of a theory describing the process. A master equation for sympathetic cooling was formulated in Ref. [1]. This equation turned out to be too unwieldy for practical purposes. A significant simplification was introduced in Refs. [2]. Two independent approximation schemes – microcanonical averaging, and a factorization

assumption – both led to equations that could be solved numerically. The results were in good agreement with each other [2].

In this paper, we introduce and analyze another approximation to the master equation of Ref. [1]. This is the mean–field approximation often used in many–body theory. We show that this approach leads to the same rate equation for the mean occupation numbers as the factorization assumption introduced in Refs. [2]. In this sense, the mean–field approach contains and generalizes the factorization assumption. It allows us to study not only the time evolution of the mean occupation numbers but that of the full one–body density matrix. We do so by using a group–theoretical technique in von–Neumann space. We are able to construct analytically both the equilibrium and the non–equilibrium solutions of the master equation for the one–body density matrix. This leads to a complete theoretical understanding of equilibrium and non–equilibrium properties of the solutions of the master equation and, at the same time, yields insight into the limitations of the mean–field approach itself. We focus attention on the case of Bosons which is theoretically more complex and, therefore, more interesting.

In Section II we formulate the master equation and use decoherence to reduce it to diagonal form. The mean–field approximation is introduced in Section III. Our algebraic technique is introduced in Section IV. We then investigate the equilibrium solution (Section V). While this investigation could have been done without using the algebra developed in Section IV, our method shows its full power when we diagonalize the rate operator in Section VI and when we construct the time–dependent solutions of the master equation in Section VII. Section VIII contains a brief summary.

II. THE MASTER EQUATION FOR THE DENSITY MATRIX AND DECOHERENCE

We follow the notation of Refs. [1,2]. System A is subject to sympathetic cooling and consists of N_A Bosons. The master equation for the time-dependence of the reduced density

matrix $\rho_A(t)$ has the form

$$\frac{\mathrm{d}\rho_A(t)}{\mathrm{d}t} = -\frac{i}{\hbar} \Big[H_A + H'_{A-A}, \rho(t) \Big] + \mathcal{L}\rho_A \ . \tag{1}$$

Here, H_A is the sum of the single–particle Hamiltonians for the atoms in system A containing the kinetic energy and the harmonic–oscillator potential of the trap, while H'_{A-A} contains the interaction between the atoms in system A. The action of the Liouvillean \mathcal{L} on the reduced density matrix $\rho_A(t)$ is given by

$$\mathcal{L}\rho_{A} = \sum_{\vec{n},\vec{n}',\vec{m},\vec{m}'} \Gamma_{\vec{n},\vec{n}'}^{\vec{m},\vec{m}'} \left(2a_{\vec{m}}^{\dagger} a_{\vec{m}'} \rho_{A}(t) a_{\vec{n}}^{\dagger} a_{\vec{n}'} - a_{\vec{n}}^{\dagger} a_{\vec{n}'} a_{\vec{m}}^{\dagger} a_{\vec{m}'} \rho_{A}(t) - \rho_{A}(t) a_{\vec{n}}^{\dagger} a_{\vec{n}'} a_{\vec{m}}^{\dagger} a_{\vec{m}'} \right). \tag{2}$$

The rate coefficients $\Gamma_{\vec{n},\vec{n}'}^{\vec{m},\vec{m}'}$ are given in Ref. [1]; we do not repeat the definition here. Suffice it to say that they account for the interaction between particles in system A and those in the cooling system B. The latter is assumed to be at thermal equilibrium at all times. The labels \vec{m} and \vec{n} refer to the three quantum numbers (m_x, m_y, m_z) defining the single–particle eigenstates of an isotropic harmonic oscillator in three dimensions representing the trap potential, with associated creation and annihilation operators $a_{\vec{m}}^{\dagger}$ and $a_{\vec{m}}$, respectively. Eqs. (1) and (2) are derived from the von Neumann equation for the density matrix of the combined system A plus B under the following assumptions: The process is Markovian, the correlation time for the interaction between systems A and B is much shorter than the cooling time, and a rotating—wave approximation applies. Equations of the form of Eqs. (1,2) are often used to discuss Boson condensation phenomena [3,4].

It was shown in Ref. [2] that if the number N_B of particles in the bath is very large, decoherence acts very quickly and reduces the density matrix to diagonal form. Then, Eq. (2) takes the form

$$\mathcal{L}\rho_{A} = \sum_{\vec{m}\neq\vec{n}} \Gamma_{\vec{n},\vec{m}}^{\vec{m},\vec{n}} \left(2a_{\vec{m}}^{\dagger} a_{\vec{n}} \rho_{A}(t) a_{\vec{n}}^{\dagger} a_{\vec{m}} - a_{\vec{n}}^{\dagger} a_{\vec{m}} a_{\vec{m}}^{\dagger} a_{\vec{n}} \rho_{A}(t) - \rho_{A}(t) a_{\vec{n}}^{\dagger} a_{\vec{m}} a_{\vec{m}}^{\dagger} a_{\vec{n}} \right) . \tag{3}$$

III. MEAN-FIELD APPROXIMATION

Eq. (3) serves as the starting point for our mean-field approximation. We apply this approximation in standard fashion by replacing on the right-hand side of Eq. (3) one pair of creation and annihilation operators referring to the same single-particle state \vec{m} or \vec{n} by its expectation value. We use the notation

$$\langle a_{\vec{m}}^{\dagger} a_{\vec{m}} \rangle = \operatorname{tr}(a_{\vec{m}}^{\dagger} a_{\vec{m}} \rho_A) = N_{\vec{m}} . \tag{4}$$

The quantities $N_{\vec{m}}$ are the mean occupation numbers of the orbital \vec{m} . For the term $\mathcal{L}\rho_A$, this procedure yields

$$\mathcal{L}\rho_{A} = \sum_{\vec{m}\neq\vec{n}} \left(\Gamma_{\vec{n},\vec{m}}^{\vec{m},\vec{n}} N_{\vec{n}} \left(2a_{\vec{m}}^{\dagger} \rho_{A}(t) a_{\vec{m}} - a_{\vec{m}}^{\dagger} a_{\vec{m}} \rho_{A}(t) - \rho_{A}(t) a_{\vec{m}} a_{\vec{m}}^{\dagger} - \rho_{A} \right) + \Gamma_{\vec{m}\vec{n}}^{\vec{n}\vec{m}} (N_{\vec{n}} + 1) \left(2a_{\vec{m}} \rho_{A}(t) a_{\vec{m}}^{\dagger} - a_{\vec{m}}^{\dagger} a_{\vec{m}} \rho_{A}(t) - \rho_{A}(t) a_{\vec{m}} a_{\vec{m}}^{\dagger} + \rho_{A} \right) \right). \tag{5}$$

Eqs.(1,4) and (5) imply

$$\frac{\mathrm{dtr}\rho_A(t)}{\mathrm{d}t} = 0. \tag{6}$$

In what follows, we explore the algebraic structure of Eq. (5). We show that this equation is linear in the generators of the group SU(1,1). In the algebraic theory of non–autonomous quantum systems [5,6], it is shown that any such linear dynamical system is integrable and, thus, can be solved analytically.

IV. ALGEBRAIC STRUCTURE OF THE MEAN-FIELD EQUATION

In the first two Subsections, we introduce the mathematical tools which are used in Subsection IVC to explore the structure of Eq. (5).

A. Right and Left Algebras

Let $|N\rangle_{\vec{m}}$ stand for the N-fold occupied single-particle state \vec{m} . In a Hilbert-space basis defined in terms of such states, the creation and annihilation operators $a_{\vec{m}}^{\dagger}$ and $a_{\vec{m}}$ appearing

in Eq. (5) may operate from the left on the ket state $|N\rangle_{\vec{m}}$ or from the right on the bra state $|m\rangle_{\vec{m}}$. In the latter case, we attach an upper index r to the operators and write them to the left of the density matrix ρ_A . We do so with the understanding that for a product of m operators $O_1 \times O_2 \times \ldots \times O_m$, we have

$$O_1^r \times O_2^r \times \ldots \times O_m^r \rho_A = \rho_A O_m \times \ldots \times O_2 \times O_1 . \tag{7}$$

This convention (the reversal of order as we bring the factors from one side of ρ_A to the other) is the cause for the differences in signs between the first and the second set of Eqs. (10) below. As a result, the pair of operators $a_{\vec{m}}^{\dagger}$ and $a_{\vec{m}}$ generates both the left and and the right representations of the group hw(4). We define accordingly

$$a_{\vec{m}}^{l\dagger} = a_{\vec{m}}^{\dagger} , a_{\vec{m}}^{l} = a_{\vec{m}} , n_{\vec{m}}^{l} = a_{\vec{m}}^{\dagger} a_{\vec{m}} ,$$
 (8)

if the operators act from the left on the ket state $|N\rangle_{\vec{m}}$, and

$$a_{\vec{m}}^{r\dagger} = a_{\vec{m}}^{\dagger} , a_{\vec{m}}^{r} = a_{\vec{m}} , n_{\vec{m}}^{r} = a_{\vec{m}}^{\dagger} a_{\vec{m}} ,$$
 (9)

if the operators act from the right on the bra state $_{\vec{m}}\langle N|$. Together with the unit operator, the three operators appearing on the left-hand sides of Eqs. (8) (of Eqs. (9)) constitute the left representation $\text{hw}(4)_l$ (the right representation $\text{hw}(4)_r$ of hw(4), respectively). These operators obey the following commutation relations.

$$hw(4)_{l}: [n_{\vec{m}}^{l}, a_{\vec{m}}^{l,\dagger}] = +a_{\vec{m}}^{l,\dagger} ,$$

$$[n_{\vec{m}}^{l}, a_{\vec{m}}^{l}] = -a_{\vec{m}}^{l} ,$$

$$[a_{\vec{m}}^{l}, a_{\vec{m}}^{l,\dagger}] = +1 .$$

$$hw(4)_{r}: [n_{\vec{m}}^{r}, a_{\vec{m}}^{r,\dagger}] = -a_{\vec{m}}^{r,\dagger} ,$$

$$[n_{\vec{m}}^{r}, a_{\vec{m}}^{r}] = +a_{\vec{m}}^{r} ,$$

$$[a_{\vec{m}}^{r}, a_{\vec{m}}^{r,\dagger}] = -1 .$$

$$(10)$$

It is evident that $hw(4)_l$ ($hw(4)_r$) is isomorphic (antiisomorphic, respectively) to hw(4). Since $hw(4)_l$ and $hw(4)_r$ act in different spaces, the operators belonging to these two different groups commute with each other. We write this somewhat symbolically as

$$[hw(4)_l, hw(4)_r] = 0. (11)$$

B. Composite Algebra

In order to explore the algebraic structure of Eq. (3), it is not sufficient to list the basic algebras $hw(4)_l$ and $hw(4)_r$. It is neccessary, in addition, to display the structure of the composite algebra C obtained by joining these two algebras. The composite algebra C consists of the elements

$$C: \{n_{\vec{m}}^l, n_{\vec{m}}^r, a_{\vec{m}}^{r\dagger} a_{\vec{m}}^l, a_{\vec{m}}^{l\dagger} a_{\vec{m}}^r\} . \tag{12}$$

The elements of C obey the commutation relations

$$\begin{split} &[n_{\vec{m}}^{r}, a_{\vec{m}}^{r\dagger} a_{\vec{m}}^{l}] = -a_{\vec{m}}^{r\dagger} a_{\vec{m}}^{l} \;, \\ &[n_{\vec{m}}^{r}, a_{\vec{m}}^{l\dagger} a_{\vec{m}}^{r}] = +a_{\vec{m}}^{r} a_{\vec{m}}^{l\dagger} \;, \\ &[n_{\vec{m}}^{l}, a_{\vec{m}}^{r\dagger} a_{\vec{m}}^{l}] = -a_{\vec{m}}^{r\dagger} a_{\vec{m}}^{l} \;, \\ &[n_{\vec{m}}^{l}, a_{\vec{m}}^{l\dagger} a_{\vec{m}}^{r}] = +a_{\vec{m}}^{r} a_{\vec{m}}^{l\dagger} \;, \\ &[n_{\vec{m}}^{l\dagger}, a_{\vec{m}}^{r\dagger} a_{\vec{m}}^{r}] = +a_{\vec{m}}^{r} a_{\vec{m}}^{l\dagger} \;, \\ &[a_{\vec{m}}^{l\dagger} a_{\vec{m}}^{r}, a_{\vec{m}}^{r\dagger} a_{\vec{m}}^{l}] = -n_{\vec{m}}^{r} - n_{\vec{m}}^{l} \;, \\ &[n_{\vec{m}}^{r}, n_{\vec{m}}^{l}] = 0 \;. \end{split}$$

These relations follow from Eqs. (10) and (11). The algebra C is the direct sum $U(1) \oplus SU(1,1)$ of two commuting parts, a radical U(1) and a simple Lie algebra SU(1,1). Here

$$U(1): \{n_{\vec{m}} = n_{\vec{m}}^l - n_{\vec{m}}^r\}$$
(14)

and

$$SU(1,1): \{K_{\vec{m}}^{0} = \frac{1}{2}(n_{\vec{m}}^{r} + n_{\vec{m}}^{l}), K_{\vec{m}}^{+} = a_{\vec{m}}^{l\dagger} a_{\vec{m}}^{r}, K_{\vec{m}}^{-} = a_{\vec{m}}^{r\dagger} a_{\vec{m}}^{l}\} .$$
 (15)

The elements of SU(1,1) obey the relations

$$[K_{\vec{m}}^{0}, K_{\vec{m}}^{\pm}] = \pm K_{\vec{m}}^{\pm} ,$$

$$[K_{\vec{m}}^{-}, K_{\vec{m}}^{+}] = 2K_{\vec{m}}^{0} .$$
(16)

We list the action of the four elements of C on the projectors $\Pi_{\vec{m}}^N = |N\rangle_{\vec{m}\vec{m}}\langle N|$.

$$n_{\vec{m}}\Pi_{\vec{m}}^{N} = (-)\Pi_{\vec{m}}^{N} ,$$

$$K_{\vec{m}}^{0}\Pi_{\vec{m}}^{N} = \frac{1}{2}(2N+1)\Pi_{\vec{m}}^{N} ,$$

$$K_{\vec{m}}^{+}\Pi_{\vec{m}}^{N} = (N+1)\Pi_{\vec{m}}^{N+1} ,$$

$$K_{\vec{m}}^{-}\Pi_{\vec{m}}^{N} = N\Pi_{\vec{m}}^{N-1} .$$
(17)

Since $\rho(t)$ is diagonal in energy representation, we are concerned exclusively with the projectors $\Pi_{\vec{m}}^N$, i.e., with the eigenvalue (-1) of the operator $n_{\vec{m}}$. Our algebraic method is not confined to this case, however. This is shown in the Appendix.

C. The Structure of Eq. (5)

We define

$$\Gamma_{\vec{m}} = \sum_{\vec{n} \neq \vec{m}} \Gamma_{\vec{n}, \vec{m}}^{\vec{m}, \vec{n}} N_{\vec{n}} ,$$

$$\Gamma^{\vec{m}} = \sum_{\vec{n} \neq \vec{m}} \Gamma_{\vec{m}, \vec{n}}^{\vec{n}, \vec{m}} (N_{\vec{n}} + 1)$$

$$\tag{18}$$

and use Eqs. (14,15) and (17). The master equation takes the form of a rate equation,

$$\frac{\mathrm{d}\rho_A(t)}{\mathrm{d}t} = \sum_{\vec{m}} \{ 2\Gamma_{\vec{m}} K_{\vec{m}}^+ + 2\Gamma^{\vec{m}} K_{\vec{m}}^- - 2(\Gamma_{\vec{m}} + \Gamma^{\vec{m}}) K_{\vec{m}}^0 - (\Gamma_{\vec{m}} - \Gamma^{\vec{m}}) \} \rho_A(t) \ . \tag{19}$$

Eq. (19) shows that aside from the mean values $N_{\vec{m}}$ contained in the rate coefficients $\Gamma^{\vec{m}}$ and $\Gamma_{\vec{m}}$, the time-dependence of $\rho_A(t)$ is given by a sum of terms each referring to one of the orbital subspaces only. Moreover, each such term is linear in the generators of SU(1,1) and is, thus, integrable. This makes it possible to solve Eq. (19) algebraically.

We reduce the rate equation Eq. (19) by taking a partial trace. We use a Hilbert–space representation in terms of products of states $|N_{\vec{m}}\rangle$. We take the trace of $\rho_A(t)$ over all states \vec{n} with $\vec{n} \neq \vec{m}$ and denote the result by $\rho_{\vec{m}}(t)$. From Eq. (19), we find

$$\frac{\mathrm{d}\rho_{\vec{m}}(t)}{\mathrm{d}t} = \left[2\Gamma_{\vec{m}}K_{\vec{m}}^{+} + 2\Gamma^{\vec{m}}K_{\vec{m}}^{-} - 2(\Gamma_{\vec{m}} + \Gamma^{\vec{m}})K_{\vec{m}}^{0} - (\Gamma_{\vec{m}} - \Gamma^{\vec{m}})\right]\rho_{\vec{m}}(t) \ . \tag{20}$$

The omission of the interaction term H_{A-A} from Eq. (20) was justified in Refs. [2]. We recall that $\rho_A(t)$ is diagonal in energy representation. It follows that $\rho_{\vec{m}}$ can be written as a sum of the projectors $\Pi_{\vec{m}}^N$,

$$\rho_{\vec{m}}(t) = \sum_{N=0}^{N_A} P_{\vec{m}}^N \Pi_{\vec{m}}^N . \tag{21}$$

We use this form in Eq. (20) and find for $N < N_A$

$$\frac{\mathrm{d}P_{\vec{m}}^{N}(t)}{\mathrm{d}t} = 2\Gamma_{\vec{m}}NP_{\vec{m}}^{N-1} + 2\Gamma^{\vec{m}}(N+1)P_{\vec{m}}^{N+1} - 2(\Gamma_{\vec{m}} + \Gamma^{\vec{m}})NP_{\vec{m}}^{N} - 2\Gamma_{\vec{m}}P_{\vec{m}}^{N} \ . \tag{22}$$

Eq. (22) is similar to the time–dependent Hartree–Fock equation used in Nuclear Physics [7]. From Eqs. (20,21) it follows that the mean particle number $N_{\vec{m}}(t)$ in orbit \vec{m} obeys the equation

$$\frac{\mathrm{d}N_{\vec{m}}(t)}{\mathrm{d}t} = 2\Gamma_{\vec{m}}(N_{\vec{m}} + 1) - 2\Gamma^{\vec{m}}N_{\vec{m}} \ . \tag{23}$$

Using the definitions for the Γ 's in Eqs. (18) we see that Eq. (23) coincides with Eq. (39) of the second of Refs. [2]. In particular, we have $d\sum_{\vec{m}} N_{\vec{m}}/dt = 0$ so that particle number $N_A = \sum_{\vec{m}} N_{\vec{m}}$ is conserved. This shows that the factorization assumption used in Refs. [2] is equivalent to the mean–field approximation.

V. THERMODYNAMIC EQUILIBRIUM: STATIONARY SOLUTION

In thermodynamic equilibrium, the reduced density matrix becomes independent of time, and the stationary solution of the rate equation Eq. (22) obeys $d\rho_A(t)/dt = 0$ or

$$\sum_{\vec{m}} \{ 2\Gamma^{\vec{m}} K_{\vec{m}}^{+} + 2\Gamma^{\vec{m}} K_{\vec{m}}^{-} - 2(\Gamma_{\vec{m}} + \Gamma^{\vec{m}}) K_{\vec{m}}^{0} - (\Gamma_{\vec{m}} - \Gamma^{\vec{m}}) \} \rho_{A} = 0 .$$
 (24)

Inserting the expansion Eq. (21), we find the following set of equations.

$$2\Gamma_{\vec{m}}NP_{\vec{m}}^{N-1} + 2\Gamma^{\vec{m}}(N+1)P_{\vec{m}}^{N+1} - 2(\Gamma_{\vec{m}} + \Gamma^{\vec{m}})NP_{\vec{m}}^{N} - 2\Gamma_{\vec{m}}P_{\vec{m}}^{N} = 0 \text{ if } N < N_{A};$$

$$2\Gamma_{\vec{m}}N_{A}P_{\vec{m}}^{N_{A}-1} - 2(\Gamma_{\vec{m}} + \Gamma^{\vec{m}})N_{A}P_{\vec{m}}^{N_{A}} - 2\Gamma_{\vec{m}}P_{\vec{m}}^{N_{A}} = 0;$$

$$2\Gamma_{\vec{m}}(N_{A} + 1)P_{\vec{m}}^{N_{A}} = 0.$$
(25)

It is easily seen that these equations only have the trivial solution $P_{\vec{m}}^N = 0$ for all $N \leq N_A$. This fact shows that a consistent non-trivial solution of the mean-field equations does not exist if we impose on ρ_A the constraint expressed by Eq. (21). Therefore, we modify this equation by writing

$$\rho_{\vec{m}}(t) = \sum_{N=0}^{\infty} P_{\vec{m}}^{N} \Pi_{\vec{m}}^{N} . \tag{26}$$

All we now require is that the sum $\sum_{N=N_A}^{\infty} P_{\vec{m}}^N$ be negligibly small. Then the first of Eqs. (25) applies for all N. The solution is

$$P_{\vec{m}}^N = P_{\vec{m}}^0 \chi^N \text{ where } \chi = \frac{\Gamma_{\vec{m}}}{\Gamma^{\vec{m}}}.$$
 (27)

From Eq. (23) we see that in the stationary case we always have $\chi < 1$. The normalization condition yields $P_{\vec{m}}^0 = (1 - \chi)$. The mean value $N_{\vec{m}}$ is given by $N_{\vec{m}} = \chi/(1 - \chi)$. Conversely, we may replace χ everywhere by $N_{\vec{m}}/(N_{\vec{m}}+1)$. To discuss the validity of the mean-field solution, we impose the constraint that $P_{\vec{m}}^{N_A}/P_{\vec{m}}^0 < \exp(-a)$. For $N_A \gg 1$, this yields $N_{\vec{m}}/N_A < 1/a$. This shows that for the condensate (the only case where we expect $N_{\vec{m}}$ to take sizeable values), the mean-field approximation (as defined in the framework of this paper) begins to fail whenever the ratio $N_{\vec{m}}/N_A$ grows beyond ten percent or so.

The $P_{\vec{m}}^N \sim \chi^N$ form a sequence which decreases monotonically with increasing N. This behavior is in marked contrast to that of the equilibrium solution found first by Scully [3] and reproduced, in the present context, in Ref. [2]. Scully's solution is not based upon the mean–field approximation. We believe that it describes correctly the equilibrium behavior of the fully developed condensate.

VI. DIAGONALIZATION OF THE RATE OPERATOR

We turn to the time dependence of the solutions of the rate equation. As a preparatory step, we determine in this Section the eigenvalues and eigenvectors of the rate operator Γ , defined as the operator which appears on the right-hand side of Eq. (20),

$$\Gamma = \left[2\Gamma_{\vec{m}}K_{\vec{m}}^{+} + 2\Gamma^{\vec{m}}K_{\vec{m}}^{-} - 2(\Gamma_{\vec{m}} + \Gamma^{\vec{m}})K_{\vec{m}}^{0} - (\Gamma_{\vec{m}} - \Gamma^{\vec{m}})\right]. \tag{28}$$

For simplicity, we omit reference to the single-particle state \vec{m} , define

$$a = 2\Gamma_{\vec{m}}, \ b = 2\Gamma^{\vec{m}} \tag{29}$$

and have

$$\Gamma = aK_{\vec{m}}^{+} + bK_{\vec{m}}^{-} - (a+b)K_{\vec{m}}^{0} - \frac{1}{2}(a-b) . \tag{30}$$

We note that Γ is not self-adjoint, and that

$$\Gamma^{\dagger} = aK_{\vec{m}}^{-} + bK_{\vec{m}}^{+} - (a+b)K_{\vec{m}}^{0} - \frac{1}{2}(a-b) . \tag{31}$$

The mean-field equation (20) takes the form

$$\frac{\mathrm{d}\rho_{\vec{m}}(t)}{\mathrm{d}t} = \Gamma\rho_{\vec{m}} \ . \tag{32}$$

The equilibrium solution determined in section V obviously corresponds to the eigenvalue zero of the rate operator. The eigenfunction is given by Eqs. (26,27). We now construct the complete set of eigenvalues and eigenvectors of the rate operator. A similar procedure has also been used in Quantum Optics [8].

To this end, it is useful to first look at a simpler problem. Given the three components J_1, J_2, J_3 of spin which obey the cyclic commutation relations $[J_1, J_2] = iJ_3$, we ask for the eigenvalues and eigenvectors of the operator

$$\mathcal{O} = aJ_2 + bJ_3 \ . \tag{33}$$

The constants a, b are real. Writing $a = \sqrt{a^2 + b^2} \sin \theta$ and $b = \sqrt{a^2 + b^2} \cos \theta$, we have

$$\mathcal{O} = \sqrt{a^2 + b^2} \left(\sin \theta J_2 + \cos \theta J_3 \right) \,, \tag{34}$$

and we recognize immediately that \mathcal{O} can be diagonalized by a rotation in the (2,3) plane by the angle $-\theta$, i.e., by the unitary transformation $U = \exp(-i\theta J_1)$,

$$\overline{\mathcal{O}} = U\mathcal{O}U^{\dagger} = \sqrt{a^2 + b^2} J_3 . \tag{35}$$

This result can be verified algebraically. The eigenvalues of \mathcal{O} are proportional to those of the operator J_3 , the eigenvectors are obtained by applying the unitary transformation U^{\dagger} to the eigenvectors of J_3 . The diagonalizing matrix U is not unique: Replacing θ by $\theta \pm \pi$ also yields a diagonal form, with J_3 replaced by $(-J_3)$.

The form of the rate operator in Eq. (30) is somewhat more complicated than that of the operator \mathcal{O} . Moreover, the underlying group is not SU(2) but SU(1,1). Therefore, it takes a similarity rather than a unitary transformation to diagonalize Γ . Geometrically speaking, the vector Γ is not confined to the (2,3) plane. This makes it necessary to define a pair of similarity transformations,

$$T_{-} = \exp(-\alpha_{-}K^{-}), T_{+} = \exp(-\alpha_{+}K^{+}).$$
 (36)

Eqs. (16) imply the identities

$$\exp(-\alpha_{-}K^{-})K^{+}\exp(\alpha_{-}K^{-}) = K^{+} - 2\alpha_{-}K^{0} - \alpha_{-}^{2}K^{-};$$

$$\exp(-\alpha_{-}K^{-})K^{0}\exp(\alpha_{-}K^{-}) = K^{0} - \alpha_{-}K^{-};$$

$$\exp(-\alpha_{+}K^{+})K^{-}\exp(\alpha_{+}K^{+}) = K^{-} + 2\alpha_{+}K^{0} + \alpha_{+}^{2}K^{+};$$

$$\exp(-\alpha_{+}K^{+})K^{0}\exp(\alpha_{+}K^{+}) = K^{0} + \alpha_{+}K^{+}.$$
(37)

It is easy to see that the product

$$T = T_{-}T_{+} \tag{38}$$

diagonalizes Γ (in the sense that $T\Gamma T^{-1}$ is diagonal) provided the coefficients α_{-} and α_{+} obey the equations

$$a + b \alpha_{+}^{2} - (a + b) \alpha_{+} = 0 ; b (1 - 2\alpha_{+}\alpha_{-}) + (a + b) \alpha_{-} = 0 .$$
 (39)

The transformed rate operator reads

$$T\Gamma T^{-1} = [2b\alpha_{+} - (a+b)] K^{0} - \frac{1}{2} (a-b) .$$
 (40)

The eigenvectors of $T\Gamma T^{-1}$ belonging to the eigenvalue -1 of the operator $n_{\vec{m}}$, are those of K^0 and have the form Π^n , with $n=0,1,2,\ldots$ The right-hand eigenvectors of the rate operator Γ are correspondingly given by $c_n T^{-1} \Pi^n$ with c_n a normalization constant.

Eqs. (39) have two solutions, yielding two sets of eigenvalues γ_n for the rate operator. This fact is due to the same type of ambiguity as found for the operator \mathcal{O} in the last but one paragraph. With $n = 0, 1, 2, \ldots$, the solutions are

(a)
$$\alpha_{+} = \chi$$
, $\alpha_{-} = \frac{1}{\chi - 1}$, $\gamma_{n} = n(a - b)$;
(b) $\alpha_{+} = 1$, $\alpha_{-} = \frac{1}{1 - \chi}$, $\gamma_{n} = (n + 1)(b - a)$. (41)

For the right-hand eigenvectors of Γ , this yields

$$\rho_n = c_n T^{-1} \Pi^n = c_n \exp(\alpha_+ K^+) \exp(\alpha_- K_-) \Pi^n , \qquad (42)$$

where c_n is a normalization constant. Eq. (42) shows that the solutions (b) of Eq. (41) are not admissible. Indeed, we have $\exp(\alpha_+ K^+)$ $\Pi^0 = \sum_{N=0}^{\infty} \alpha_+^N \Pi^N$, and a corresponding result when the operator $\exp(\alpha_+ K^+)$ acts on Π^N with $N \geq 1$. For solution (a) of Eq. (41), where $\alpha_+ = \chi < 1$, the resulting series behaves as required while for solution (b) where $\alpha_+ = 1$, the coefficients multiplying Π^N do not decrease with decreasing N and, thus, violate the condition formulated in Section V for the viability of the mean–field solution. Solution (a) does contain the zero eigenvalue discussed in Section V. Moreover, all non–zero eigenvalues γ_n of solution (a) are negative, as is to be expected on physical grounds.

Eq. (40) shows that the operator $T\Gamma T^{-1}$ is self-adjoint and, thus, equal to $(T^{-1})^{\dagger}\Gamma^{\dagger}T^{\dagger}$. This implies that Γ^{\dagger} has the same eigenvalues as Γ and is diagonalized by the similarity transformation $(T^{-1})^{\dagger}$. The right-hand eigenvectors of Γ^{\dagger} are given by $c_n T^{\dagger} \Pi^n$. They coincide with the left-hand eigenvectors $\tilde{\rho}_n$ of Γ . It is easy to show that the right- and left-hand eigenvectors of Γ form a biorthogonal set [9],

$$Tr(\tilde{\rho}_n \rho_m) = \delta_{nm} . (43)$$

VII. TIME-DEPENDENT SOLUTIONS OF THE MEAN-FIELD EQUATIONS

In the fully time–dependent mean–field equations Eq. (32), the rate operator Γ depends upon time. This is because Γ is defined in terms of the coefficients a and b which in turn depend on time via the mean occupation numbers $N_{\vec{m}}$, see Eqs.(29) and (18). We now show that given the solutions $N_{\vec{m}}(t)$ of the rate equations Eq. (23) and, thus, the time dependence of the coefficients a(t) and b(t), the solutions to the full equations (32) are obtained by quadratures. We also show that these solutions tend asymptotically towards the equilibrium solution found in Section V. As in Section VI, we suppress the label for the single–particle state \vec{m} . We retain the symbol χ introduced in Section V to denote the asymptotic or equilibrium value of the ratio a/b.

We assume that at time t=0, we are given the initial density matrix

$$\rho(0) = \sum_{N=0}^{\infty} P^N \Pi^N \tag{44}$$

with the proviso that for $N > N_A$, the coefficients P^N are negligible. To solve Eq. (32), we define the time–dependent similarity transformation

$$\overline{\rho(t)} = T(t) \ \rho(t) = \exp[-\alpha_{-}(t) \ K^{-}] \ \exp[-\alpha_{+}(t) \ K^{+}] \ \rho(t) \ .$$
(45)

The time–dependent coefficients $\alpha_{\pm}(t)$ are solutions of the differential equations

$$\frac{d\alpha_{+}(t)}{dt} = a + b\alpha_{+}^{2} - (a+b) \alpha_{+}; \quad \frac{d\alpha_{-}(t)}{dt} = b (1 - 2\alpha_{+}\alpha_{-}) + (a+b) \alpha_{-}. \tag{46}$$

Comparison with Eqs. (38,39) shows that T(t) provides the time-dependent generalization of the similarity transformation T used in Section VI. As initial conditions for the differential equations (46), we choose $\alpha_{\pm}(0) = 0$ or $\overline{\rho(0)} = \rho(0)$. Using the identities Eq. (37), we see that the transformed density matrix $\overline{\rho(t)}$ obeys

$$\frac{\mathrm{d}\overline{\rho(t)}}{\mathrm{d}t} = \left[\gamma(t) \ K^0 - \frac{1}{2}(a-b)\right] \overline{\rho(t)} , \qquad (47)$$

where

$$\gamma(t) = 2b\alpha_+ - (a+b) . \tag{48}$$

Integrating this equation and using the relation inverse to Eq. (45), we obtain

$$\rho(t) = \exp[\alpha_{+}(t)K^{+}] \exp[\alpha_{-}(t)K^{-}] \exp\left(\int_{0}^{t} d\tau [\gamma(\tau)K^{0} - \frac{1}{2}(a(\tau) - b(\tau))]\right) \rho(0) . \tag{49}$$

We have accomplished our first goal and shown that given the time dependence of the coefficients a and b, the time dependence of the density matrix is obtained by quadratures.

We now show that $\rho(t)$ as given by Eq. (49) tends asymptotically $(t \to \infty)$ to the equilibrium solution established in Section V. To this end, we first investigate the asymptotic behavior of $\alpha_{\pm}(t)$. The first of Eqs. (46) reads $d\alpha_{+}(t)/dt = b(\alpha_{+} - a/b)(\alpha_{+} - 1)$. We recall that a > 0, b > 0 and distinguish two cases. (a) We have a/b < 1 for all times t. Then, $d\alpha_{+}(t)/dt > 0(<0)$ if $0 < \alpha_{+}(t) < a/b$ (if $a/b < \alpha_{+}(t) < 1$, respectively). With the initial condition $\alpha_{+} = 0$, we see that α_{+} approaches the value $a(\infty)/b(\infty) = \chi < 1$ asymptotically from below. This implies $\gamma(t) \to (a - b) < 0$ as $t \to \infty$. (b) We have a/b > 1 for some time interval, although $a/b \to \chi < 1$ must hold asymptotically. As long as a/b > 1, the initial condition $\alpha_{+} = 0$ implies $\alpha_{+}(t) < 1$. As soon as a/b intersects the value unity from above, $\alpha_{+}(t)$ has a negative (positive) slope if it is larger (smaller) than a/b. In either case, α_{+} again tends toward $\chi < 1$ asymptotically, and γ attains the asymptotic value $a(\infty) - b(\infty) < 0$ as in case (a). To study the asymptotic behavior of α_{-} , we define $f(t) = \alpha_{-} \exp[\int_{0}^{t} d\tau \gamma(\tau)]$. The time derivative of f(t) is given by $b \exp[\int_{0}^{t} d\tau \gamma(\tau)]$. Since b is bounded and γ negative for large t, this derivative tends to zero and, hence, f towards a constant f_{0} . This implies that α_{-} diverges asymptotically.

We turn to the asymptotic behavior of $\rho(t)$. We use Eqs. (44) and (17) to write

$$\rho(t) = \exp[\alpha_{+}(t)K^{+}] \exp[(1/2) \int_{0}^{t} d\tau (\gamma(\tau) - (a(\tau) - b(\tau))] \times \sum_{N=0}^{\infty} f(t)^{N} P^{N}(0) \sum_{p=0}^{N} {N \choose p} (\alpha_{-})^{-p} \Pi^{p} .$$
(50)

With the help of the asymptotic behavior of f(t), α_+ , and α_- established in the previous paragraph, this becomes asymptotically

$$\rho(t) \to \exp[\chi K^+] \exp[(1/2) \int_0^\infty d\tau (\gamma(\tau) - (a(\tau) - b(\tau))] (\sum_{N=0}^\infty f(0)^N P^N(0)) \Pi^0 . \tag{51}$$

As γ tends asymptotically towards $(a(\infty) - b(\infty))$, the integral in the exponent exists. Actually we ought to show that the difference $\gamma - (a - b)$ vanishes asymptotically at least as strongly as t^{-2} . This we have not done. We know, however, from Eq. (23) that both tr ρ and the mean value $N_{\vec{m}}$ of the number operator remain finite for $t \to \infty$. Therefore, the normalization constant of our solution cannot diverge asymptotically. Together with $(\sum_{N=0}^{\infty} f_0^N P^N(0))$, the exponential can be lumped into a new constant C. This yields

$$\rho(t) \to C \exp[\chi K^+] \Pi^0 . \tag{52}$$

This shows that asymptotically, ρ becomes proportional to the equilibrium solution found in Section V.

VIII. SUMMARY AND CONCLUSIONS

We have used the mean-field approximation to simplify the master equation for sympathetic cooling of Bosons. We have shown that the factorization assumption of Refs. [2] is equivalent to this approximation. Studying the stationary or equilibrium solution of the resulting master equation, we have shown that the mean-field approximation to sympathetic cooling is expected to break down whenever the fraction of condensate Bosons exceeds ten percent or so of the total Boson number in the cooled gas. This conclusion is supported by the observation that the equilibrium solution for the one-body density matrix differs markedly from the form derived [3,2] for the same quantity for the case of a fully developed condensate. Using group—theoretical methods, we have solved the time-dependent master equation for the one-body density matrix. Given the time-dependence of the mean single-particle occupation numbers, the solution is obtained by quadratures. It tends asymptotically $(t \to \infty)$ towards the equilibrium solution. For the time-dependence of the fully developed condensate, quantum fluctuations are important. These can probably be calculated along the lines described in Section V of the second of Refs. [2].

We expect that the mean-field approximation will play an important role in studies of sympathetic cooling. In this paper, we believe to have given a comprehensive theoretical treatment of this approximation.

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APPENDIX A: GENERAL APPROACH TO EQUILIBRIUM

We remove the restriction to the eigenvalues -1 of the operator $n_{\vec{m}}$ and define

$$\Pi_{\vec{m}}^{n,k} = |n\rangle_{\vec{m}\vec{m}}\langle k| \ . \tag{A1}$$

Eqs. (17) are replaced by

$$n_{\vec{m}}\Pi_{\vec{m}}^{n,k} = (n-k-1)\Pi_{\vec{m}}^{n,k} ,$$

$$K_{\vec{m}}^{0}\Pi_{\vec{m}}^{n,k} = \frac{1}{2}(n+k+1)\Pi_{\vec{m}}^{n,k} ,$$

$$K_{\vec{m}}^{+}\Pi_{\vec{m}}^{n,k} = \sqrt{(n+1)(k+1)} \Pi_{\vec{m}}^{n+1,k+1} ,$$

$$K_{\vec{m}}^{-}\Pi_{\vec{m}}^{n,k} = \sqrt{nk} \Pi_{\vec{m}}^{n-1,k-1} .$$
(A2)

The eigenvalues $\gamma_{n,k}$ of the transformed operator in Eq. (40) are given by

$$\gamma_{n,k} = \frac{1}{2}(a-b)(n+k+1) - \frac{1}{2}(a-b) , \qquad (A3)$$

where n and k run over non-negative integers and the eigenvectors are $c_{n,k}\Pi^{n,k}$. The right-hand eigenvectors of Γ are

$$\rho_{n,k} = c_{n,k} T^{-1} \Pi^{n,k} \ . \tag{A4}$$

The approach to equilibrium can also be generalized by considering

$$\rho(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P^{n,k}(0) \Pi^{n,k}(t)$$
(A5)

where $P^{n,k}$ is defined by the initial condition. In analogy with Eq. (50), Eq. (A5) can be rewriten as

$$\rho(t) = \exp[\alpha_{+}K^{+}] \exp[(\frac{1}{2}) \int_{0}^{t} d\tau (\gamma(\tau) - (a(\tau) - b(\tau)))] \exp[\frac{n}{2} \int_{0}^{t} dt' \gamma(t')]$$

$$\times \sum_{n,k=0}^{\infty} f^{k}(t) P^{n,k}(0) \sum_{p=0}^{k} \frac{(\alpha_{-})^{-p}}{(k-p)!} \sqrt{\frac{n!k!}{(n-k+p)!p!}} |n-k+p\rangle\langle p|.$$
 (A6)

We have assumed here k < n for simplicity. Given the asymptotic limit of the time-dependent function in the equation above one can immediately conclude that as $t \to \infty$, only the term corresponding to n = k = 0 will survive, yielding the equilibrium state.

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