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Cohomologie BRST locale des théories de p-formes

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L'utilisation des méthodes de simulation des grandes échelles et de la modélisation des structures les plus petites permet de caractériser la turbulence magnéto-hydrodynamique.

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Chapter 1

Introduction

At present, the most promising candidates of models unifying the four known fundamental interactions consist of string theories [1, 2]. As their name indicates, the fundamental objects of these theories are strings and it is believed that all the particles found in nature can be viewed as their different vibration modes. At the lowest end of their energy spectrum, string theories contain states which can be described by antisymmetric tensor fields. These fields are therefore essential ingredients of various supergravity models which represent low energy approximations of string theories. The study of these fields is thus an important question of theoretical and mathematical physics.

Antisymmetric tensor fields, denoted $B_{\mu_1...\mu_p}$, generalize the vector potential A_μ used to describe Maxwell's theory of electromagnetism. They are antisymmetric with respect to the permutation of any two of their indices and are thus naturally viewed as the components of a p-form,

$$B = \frac{1}{p!} B_{\mu_1...\mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}. \tag{1.1}$$

As in the case of electromagnetism, the action of massless p-form gauge theories is invariant under gauge transformations. For Maxwell's theory, the gauge transformations read, $A_\mu \to A_\mu + \partial_\mu \epsilon$ where ϵ is an arbitrary spacetime function. In form notation, this symmetry is written as $A = A_\mu dx^\mu \to A + d\epsilon$ where $d = dx^\mu \partial_\mu$ is the spacetime exterior derivative. The gauge transformations of p-form gauge fields are straightforward generalization of the above transformations and are given by, $B \to B + d\eta$ where η is now an arbitrary spacetime dependent $p - 1$-form. Their is however a significant difference. The gauge transformations of p-forms $(p > 1)$ are reducible, i.e, they are not all independent. Indeed the gauge parameters η and $\eta' = \eta + d\rho$ where ρ is a $p - 2$-form produce the same transformation of B by virtue of $d^2 = 0$. As we shall see later, this is one of the particularities of p-form gauge

theories which makes their study a rich problem.

The aim of this thesis is to study some of the properties of p-form gauge theories by using the BRST field-antifield formalism. The essential ingredient of this formalism is the BRST differential s which was first discovered in the context of the quantization of Yang-Mills theory. As for any gauge theory, it is necessary when quantizing Yang-Mills theory to introduce a gauge fixing term in the action in order to obtain well-defined propagators and Green functions. Furthermore one must ensure that the measurable quantities calculated from the gauge fixed action do not depend on the choice of the gauge fixation and that the theory is unitary in the physical sector. An elegant way of achieving this program for Yang-Mills theory was devised by Faddeev and Popov [3]. Their method requires the introduction of new fields, called ghosts which only appear as intermediate states in transition amplitudes.

Becchi, Rouet, Stora [4, 5] and Tyutin [6] then discovered that the gauge-fixed Yang-Mills action incorporating the ghosts was invariant under a nilpotent symmetry, the BRST symmetry, which was then extended to theories with more general gauge structures [7, 8, 9, 10, 11] and in particular to reducible gauge theories. The most important property of the BRST symmetry is that for any gauge theory it allows one to construct an "extended action" which guarantees at the quantum level all the conditions recalled in the preceding paragraph.

Another benefit of the BRST symmetry is that one can study various problems of classical and quantum field theory by calculating the local cohomology $H(s|d)$ of its nilpotent generator s; this amounts to the classification of the "non-trivial" solutions of the *Wess-Zumino consistency condition* [12].

In quantum field theory, the local BRST cohomology is especially useful in renormalization theory. Indeed, one can show that in "ghost number" 1 (see Section **2.1**), $H(s|d)$ contains all the candidate gauge anomalies which can arise upon quantization of a gauge theory. The existence of these BRST cocycles does not imply that the theory is anomalous. However, a calculation of their coefficients indicates if the theory is consistent or not. Other important cohomological groups in renormalization theory are found in ghost number 0 and contain the counterterms needed for renormalization. The calculation of these BRST cocycles is therefore useful to examine the stability of the action and to determine if the divergences can be absorbed in a redefinition of the fields, masses, coupling constants or other parameters present in the classical action.

The calculation of the BRST cohomology is also relevant to address some questions of classical field theory. Indeed, the cohomological groups in ghost number 0 which restrict the form of the counterterms also determine the consistent interaction terms that can be added to a lagrangian. By consistent

it is meant vertices which preserve gauge invariance and therefore the number of degrees of freedom. The BRST cohomology is also very powerful to study the gauge invariant nature of the conserved currents of a theory. Indeed, as explained in Section **2.3**, this problem is related to the calculation of the cohomological groups in ghost number -1.

To obtain the BRST cohomology we will closely follow the approach developed for Yang-Mills theory[1]. This is possible mainly because the BRST differential of p-form gauge theories splits as in the Yang-Mills case as the sum of the Koszul-Tate differential δ and the longitudinal exterior derivative γ: $s = \delta + \gamma$ (for Chapline-Manton models (Chapter **7**), this form of s is obtained after a change of variables). The rôles of the Koszul-Tate differential and the longitudinal exterior derivative are respectively to implement the equations of motion and to take into account the gauge invariance of the theory.

Our analysis will be performed in the presence of antifields; in the terminology of path integral quantization, they are the sources of the BRST variations of the fields and ghosts. They are useful in renormalization theory to control how gauge invariance survives renormalization and they also provide a convenient way to take into account the dynamics of the model when one addresses the problems of classical field theory raised above.

As in the Yang-Mills case, our study of the BRST cohomology is composed of two parts. In the first one, we calculate the solutions of Wess-Zumino consistency condition which do not depend on the antifields. Our analysis relies on the resolution of the so-called "descent equations" from which one obtains the cohomology $H(\gamma|d)$ by relating it to the cohomology $H(\gamma)$.

In the second part of our study, we calculate the antifield dependent BRST cocycles. Their existence is closely tied to the presence of non-vanishing classes in the "characteristic cohomology" $H(\delta|d)$. This cohomology is defined as the set of q-forms w (see Section **2.5**) which satisfy the conditions:

$$dw \approx 0; \qquad w \not\approx d\psi. \tag{1.2}$$

In (1.2) d is the spacetime exterior derivative and the symbol $\approx$ means "equal when the equations of motion hold". The calculation of the characteristic cohomology is an interesting problem on its own but is also related to important questions of field theory (see Chapter **5**).

Our analysis will show that the BRST cocycles naturally fall into two categories. In the first category, one has the cocycles a which are strictly

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[1]A complete account of the analysis of the local BRST cohomology in Yang-Mills theory and a list of the original works on the subject can be found in [13, 14].

annihilated by s. They consist of the gauge invariant functions of the theory and are easily constructed. In second category, one has the cocycles which are s-closed only modulo the spacetime exterior derivative d, i.e., which satisfy $sa + db = 0$. A priori, any cocycle can depend on the components of the antisymmetric tensors, the corresponding ghosts and antifields as well as their derivatives. However, a remarkable property of the solutions of the Wess-Zumino consistency condition belonging to the second category is that they only depend on the exterior forms build up from the antisymmetric tensors, the ghosts and the antifields and not on the individual components of these forms. This property explains a posteriori why the calculation of the BRST cohomology in the algebra of forms and their exterior derivatives does not miss any of the cocycles of the second category[2].

From the analysis of the BRST cohomology, we will obtain for the models considered all the candidate anomalies, counterterms, consistent interactions and we will analyze the gauge structure of the conserved currents. Our main result will be the classification of all the first-order vertices which can be added to the action of a system of free p-forms ($p > 1$). Their construction has attracted a lot of interest in the past [15, 16, 17, 18, 19, 20, 21] but systematic results were only obtained recently [22]. We will show that besides the strictly gauge invariant ones, the first-order vertices are necessarily of the Noether form "conserved antisymmetric tensor" times "p-form potential" and exist only in particular spacetime dimensions. The higher-order vertices will also be studied for a system containing forms of two different degrees. This analysis will unveil only one type of consistent interacting theory which is not of the forms described in [15, 16, 18, 19].

Our thesis is organized as follows. In Chapter 2, we expose briefly the BRST field-antifield construction and describe how it addresses the problems of classical and quantum theory we have outlined. We will also define the mathematical framework in which the calculations will be done.

In Chapter 3, we perform the BRST construction for an arbitrary system of free p-forms, i.e, we introduce all the necessary antifields, ghosts and define their transformation under the BRST differential in terms of the Koszul-Tate differential and the longitudinal exterior derivative. We then give results concerning the cohomologies of the Koszul-Tate differential and the spacetime exterior derivative (Poincaré Lemma). By making use of Young diagrams we also calculate the cohomology $H(\gamma)$ of the longitudinal exterior derivative which represents an essential ingredient in the analysis of the antifield independent solutions of the Wess-Zumino consistency condition.

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[2]This statement is true except for the antifield dependent BRST cocycles related to the conserved currents of the theory (see Theorem **26**).

In Chapter 4 we begin our study of the BRST cohomology for the system of p-forms described in Chapter 3. Chapter 4 is divided into three sections and begins with an outline of the methods used to obtain the antifield (in)dependent BRST cocycles. In Section **4.2** we analyze the Wess-Zumino consistency condition in the absence of antifields. We begin by reviewing the technic of the "descent equations": we illustrate it with simple examples and then describe the general theory. We then show that for free p-forms, one can investigate the Wess-Zumino consistency condition in the so-called "small algebra". After that we introduce the "Universal algebra" and we formulate the *Generalized transgression lemma* from which one is able to obtain all the non-trivial cocycles. The results of the analysis are presented in Section **4.2.6** and we summarize them in Section **4.2.8** by listing the various (antifield independent) counterterms and anomalies. In Section **4.3** we turn our attention to the antifield dependent solutions of the Wess-Zumino consistency condition and we prove some preliminary results. In particular we show that the existence of antifield dependent BRST cocycles is related to the presence of non-vanishing classes in the "characteristic cohomology".

This cohomology is studied in Chapter **5** which begins by a general description of the subject and a summary of our results.

In Chapter 6, we continue the analysis of the Wess-Zumino consistency condition in the presence of antifields. Using the results of Chapter **5** we obtain in Section **6.1** all the antifield dependent BRST cocycles from which we infer in Section **6.2** and **6.3** the corresponding counterterms, first-order vertices, anomalies and the gauge structure of the conserved currents of the theory. The last Section of Chapter 6 is dedicated to the construction of interacting p-form theories which are consistent to all orders in the coupling constant.

In Chapter 7 we study the BRST cohomology for Chapline-Manton models. The analysis is similar to the one of the free theory since after a redefinition of the antifields the BRST differential is brought to the standard form $s = \delta + \gamma$. Thus we begin by studying the antifield independent cocycles, then the characteristic cohomology and finally the antifield dependent cocycles. The important lesson of the analysis of Chapline-Manton models is that the results gathered for the free theory are useful in the analysis of interacting theories since they allow to obtain the cohomological groups by "perturbative arguments".

Finally, in Chapter 8 we conclude this dissertation with a few comments.

Chapter 2

BRST formalism

One of the benefits of the BRST field-antifield formalism is that it allows one to formulate in a purely algebraic manner certain questions concerning the (classical or quantum) theory of fields. In this section we briefly describe how this formalism is introduced and how it deals with the following questions:

1. determination of the consistent deformations of a classical theory,

2. analysis of the gauge invariant nature of conserved currents of a classical theory.

3. classification of possible counterterms occurring in the renormalization of quantum theory,

4. determination of candidate anomalies which can occur as a result of the quantization of a classical theory.

We will also describe the mathematical framework in which the calculations will be done.

2.1 BRST action and master equation

Let us start by introducing rapidly the various ingredients of the BRST formalism which are relevant for our discussion of the questions raised above.

The starting point of the analysis is a local classical action I,

$$I = \int d^n x \mathcal{L}(\phi^i, \partial_\mu \phi^i, \dots). \tag{2.1}$$

By *local* we mean that $\mathcal{L}$ depends on the fields and their derivatives up to a finite order k.

The first step in the BRST construction is to associate with each field ϕ^i present in (2.1) another field called an *antifield*. These antifields, which we label ϕ_i^*, are assigned a degree called the antighost number which is equal to 1.

Besides being invariant under some global symmetries acting on the fields, the action I can be invariant under some *gauge transformations*. These have the property that their parameters, ϵ^α, can be set independently at each point in spacetime, i.e., $\epsilon^\alpha = \epsilon^\alpha(x)$. For each of these *gauge parameters*, one introduces another antifield ϕ_α^* of antighost number 2, as well as a field called a ghost, which we label C^α. This ghost is assigned a degree called the pureghost number which is equal to 1. The pureghost number of the antifields and the antighost number of the ghosts are set equal to zero. One also introduces a third degree called the *ghost number* which is equal to the difference between the pureghost number and the antighost number.

A gauge theory is said to be *reducible* when the gauge parameters are not all independent. In that case, the gauge transformations are themselves invariant under certain variations of the gauge parameters ϵ^α. For each parameter involved in the transformations of ϵ^α, one again introduces a new antifield ϕ_Γ^* of antighost number 3 and a ghost C^Γ of pureghost number 2.

This construction continues if the transformations of the gauge parameters are themselves reducible and so on. We will suppose here that the construction stops after a finite number of steps so that the number of antifields and ghosts introduced is finite. The last ghosts introduced are called the *ghosts of ghosts*.

Once all the necessary antifields and ghosts have been defined, one can introduce the BRST differential s and define its action on all the variables. The general expression of s is of the form,

$$s = \delta + \gamma + s_1 + s_2 + \dots . \tag{2.2}$$

In (2.2) the first part of s is the *Koszul-Tate differential* δ. Its rôle is to implement the equations of motion in cohomology, i.e., its cohomology is given by the on-shell functions and in particular, any function which vanishes on-shell is δ-exact. The second operator present in (2.2) is the *longitudinal exterior derivative* γ. Its rôle is to take into account the gauge invariance of the theory; γ is thus constructed in such a way that any *observable* is annihilated by γ on-shell. The general expressions for δ and γ may be found in [23] and will be given later for the various models considered in this thesis. Let us only note here that: δ lowers by one unit the antighost number while γ leaves it unchanged; δ is nilpotent and γ is nilpotent modulo δ, i.e, γ is a differential in the cohomology $H(\delta) \equiv \frac{\text{Ker } \delta}{\text{Im } \delta}$; δ and γ anticommute ($\delta\gamma + \gamma\delta = 0$).

The extra terms s_j present in s are of antighost number j (they raise the antighost number by j units). Their action on the variables is obtained by requiring that $s^2 = 0$.

For all the models we consider in this thesis, we will see that s reduces to the sum of δ and γ: $s = \delta + \gamma$.[1] In the rest of the text we will assume that s has this particular form.

Once the action of s on all the fields, ghosts and antifields has been obtained, one can construct a new action S of ghost number 0 called the BRST action which generates the BRST symmetry s. It is defined by[2],

$$S = \int d^n x (\mathcal{L} + (-)^{\epsilon_A} \phi_A^* s\phi^A), \qquad (2.3)$$

where ϵ_A is the Grassmann parity of ϕ^A. We have denoted all the antifields by $\phi_A^* \equiv (\phi_i^*, \phi_\alpha^*, \phi_\Gamma^*, \dots)$ and the fields and ghosts by $\phi^A \equiv (\phi^i, C^\alpha, C^\Gamma, \dots)$. By construction the BRST action is such that,

$$s\phi^A = (S, \phi^A) = -\frac{\delta^R S}{\delta \phi_A^*(x)}, \qquad (2.4)$$

$$s\phi_A^* = (S, \phi_A^*) = \frac{\delta^R S}{\delta \phi^A(x)}. \qquad (2.5)$$

Furthermore, S satisfies the *master equation*,

$$(S, S) = 0, \qquad (2.6)$$

where the *antibracket* $(.,.)$ of two local functionals A and B is defined by,

$$(A, B) = \int d^n x \left(\frac{\delta^R A}{\delta \phi^A(x)} \frac{\delta^L B}{\delta \phi_A^*(x)} - \frac{\delta^R A}{\delta \phi_A^*(x)} \frac{\delta^L B}{\delta \phi^A(x)} \right). \qquad (2.7)$$

The master equation (2.6) is a direct consequence of $s^2 = 0$.

An important feature is that by construction, the BRST action contains all the information about the action and its gauge structure. For instance one can directly read off from its expression the gauge transformations, their algebra, the reducibility identities etc.

In the following sections we quickly review how the BRST formalism addresses the questions raised at the beginning of this section.

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[1]For Chapline-Manton models this form of s is obtained by a redefinition of the antifields.

[2]This form of the BRST differential is not the most general one; indeed there are some theories for which S contains terms which are not linear in the antifields. However, the models examined in this thesis all admit a BRST action of the form (2.3).

2.2 Consistent deformations

Let us illustrate the problem of consistent deformations for an irreducible gauge theory (for a recent and more complete review see [24]).

Let I_0 be a "free"[3] action invariant under some gauge transformations $\delta_\epsilon \phi^i = R^i_{0\alpha} \epsilon^\alpha$, i.e.,

$$\int d^n x \frac{\delta I_0}{\delta \phi^i} R^i_{0\alpha} \epsilon^\alpha = 0. \tag{2.8}$$

A deformation of the free action consists in adding to I_0 interaction vertices,

$$I_0 \to I = I_0 + g I_1 + g^2 I_2 + \dots , \tag{2.9}$$

where g is a coupling constant. In order to be *consistent*, we require the deformation to preserve gauge invariance. Therefore, the action I must remain gauge invariant, possibly under modified gauge transformations:

$$\int d^n x \frac{\delta I}{\delta \phi^i} R^i{}_\alpha \epsilon^\alpha = 0. \tag{2.10}$$

with,

$$R^i{}_\alpha = R^i_{0\alpha} + g R^i_{1\alpha} + g^2 R^i_{2\alpha} + \dots . \tag{2.11}$$

Eq. (2.10) has to be satisfied at each order in the coupling constant.

The deformations of an action can be of two types. In the first one, gauge invariant terms are added to the original lagrangian and therefore no modification of the gauge transformations is required. In Maxwell theory, Chern-Simons terms produce such deformations. The second type of deformations are obtained by adding vertices which are not invariant under the original gauge transformations and which therefore require a modification in them. In such a case, the algebra of the new gauge transformations can also be altered. A famous example is the Yang-Mills theory in which the abelian $U(1)^N$ symmetry is replaced by the non-abelian $SU(N)$ symmetry.

For reducible theories, one also imposes the gauge transformations (2.11) to remain reducible and all higher order reducibility identities to hold (possibly in a deformed way). These consistency requirements guarantee that both the deformed theory and the free theory contain the same number of degrees of freedom.

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[3]Although the action I_0 can already contain interaction vertices, we call it *free* as it is the action we want to deform.

The problem of consistent interactions among fields can be elegantly formulated within the BRST field-antifield formalism [25]. Indeed, let S_0 be the solution of the master equation for a free theory,

$$(S_0, S_0) = 0. \tag{2.12}$$

If one deforms this free theory, the lagrangian, its gauge transformations, reducibility identities etc. are modified. Therefore, the BRST action S_0 which encodes all this information it is also modified and becomes,

$$S = S_0 + gS_1 + g^2 S_2 + \dots , \tag{2.13}$$

where all the terms are of ghost number 0. All our consistency requirements will be met if the new BRST action continues to satisfy the master equation (2.6). Indeed, $(S, S) = 0$ guarantees in particular that condition (2.10) is fulfilled but this is also true for all the higher order reducibility identities.

As we already stressed, the BRST action encodes all the information about the gauge structure of the theory. In its expansion (2.3) according to the antighost number, one encounters three terms of special interest in the present discussion:

1. the antifield independent term which corresponds to the full interacting lagrangian;

2. a term of the form $\phi_i^* R^i{}_\alpha C^\alpha$ (where ϕ_i^* are the antifields of antighost number 1 and the C^α are the ghosts of pureghost number 1). From this term one deduces the modified gauge transformations of the fields $\delta_\epsilon \phi^i = R^i{}_\alpha \epsilon^\alpha$. Note that if $S_1, S_2, \dots$ do not depend on the antifields then the gauge transformations leaving the deformed action invariant are identical to those of the free theory.

3. a term of the form $C^\alpha_{\beta\gamma} \phi_\alpha^* C^\beta C^\gamma$ (where ϕ_α^* are the antifields of antighost number 2). This term is present when the gauge transformations are not abelian even on-shell; the $C^\alpha_{\beta\gamma}$ are the structure "constants" of the algebra of the gauge transformations.

The advantage of using the BRST formalism to address the problem of consistent deformations is that one can make use of the cohomological techniques. Indeed, as Eq. $(S, S) = 0$ must be fulfilled at each order in the coupling constant, one gets a tower of equations which reads,

$$(S_0, S_0) = 0, \tag{2.14}$$

$$2(S_0, S_1) = 0, \tag{2.15}$$

$$2(S_0, S_2) + (S_1, S_1) = 0, \tag{2.16}$$

$$\vdots$$

The first equation is satisfied by hypothesis. The second one implies that S_1 is a cocycle of the BRST differential s of the *free* theory. This condition, known as the *Wess-Zumino consistency condition*, imposes very strong restrictions on S_1 and thus provides a convenient way to determine all the first order vertices.

The interesting S_1 are in fact elements of the cohomology $H^0(s)$. To see this, suppose that S_1 is of the form $S_1 = sT_1$ where T_1 is of ghost number -1. Since $s^2 = 0$, S_1 is automatically a BRST cocycle. However, such solutions correspond to "trivial" deformations because they amount to field redefinitions in the original action [25, 24]. Therefore, the non-trivial deformations of the BRST action are represented by the cohomological classes of $H^0(s) \equiv \frac{\text{Ker } s}{\text{Im } s}$.

At order g^2, condition (2.16) indicates that the antibracket of S_1 with itself must be s-exact. This is a new constraint for S_1 but it also defines S_2 up to an s-exact term. If locality (as defined above) is not imposed to the functionals, one can show that condition (2.16) is always satisfied because the antibracket of two closed functionals is always exact. This is also true for all the other equations beneath (2.16). It is thus only when one requires the functionals to be local that one can meet an obstruction in the construction of the higher order vertices corresponding to S_1.

To properly take into account locality one reformulates all the equations in terms of their integrands. For instance, Eq. (2.15) can be written,

$$sS_1 = 0 \Leftrightarrow s\left(\int \mathcal{S}_1\right) = 0 \Leftrightarrow \int (s\mathcal{S}_1) = 0. \tag{2.17}$$

In terms of the integrand $\mathcal{S}_1$ the last equation reads,

$$s\mathcal{S}_1 + d\mathcal{M}_1 = 0, \tag{2.18}$$

where $\mathcal{M}_1$ is a local form of degree $n - 1$ and ghost number 1 and d is the spacetime exterior derivative. Again one can show that BRST-exact terms modulo d are trivial solutions of (2.18) and correspond to trivial deformations. In order to implement locality, the proper cohomology to evaluate is thus $H_0^n(s|d)$ where the superscript and subscript respectively denote the form degree and ghost number.

The local equivalent of (2.16) is,

$$s\mathcal{S}_2 + (\mathcal{S}_1, \mathcal{S}_1) + d\mathcal{M}_2 = 0, \tag{2.19}$$

where we have set $S_2 = \int \mathcal{S}_2$ while $(\mathcal{S}_1, \mathcal{S}_1)$ is defined by $(S_1, S_1) = \int (\mathcal{S}_1, \mathcal{S}_1)$ up to an irrelevant d-exact term. Contrary to the previous situation in which

one did not impose locality, Eq. (2.19) is not automatically fulfilled if one requires $\mathcal{S}_2$ and $\mathcal{M}_2$ to be local functions. It indicates that $(\mathcal{S}_1, \mathcal{S}_1)$, which is of form degree n and ghost number 1, must be BRST-exact modulo d. Therefore, the construction of S can be obstructed at this stage if $H_1^n(s|d)$ does not vanish.

The consistency conditions at higher order in the coupling constant are also non-trivial for local functionals and it is immediate to check that further obstructions to the construction of S can arise only if $H_1^n(s|d)$ is not zero.

To summarize, the problem of consistent deformations is elegantly captured by the BRST-field antifield formalism. The non-trivial first order vertices are representatives of $H_0^n(s|d)$ and the construction of the higher order vertices can be obstructed if $H_1^n(s|d)$ is non-vanishing. These observations motivate the study of these two cohomological groups.

2.3 Gauge invariant conserved currents and global symmetries

In this section we discuss the rôle of the BRST cohomology in the study of the gauge structure of global symmetries and conserved currents [26].

A *global symmetry* of a field theory is a transformation of the fields depending on constant parameters,

$$\Delta \phi^i = a^i(\phi^j, \partial_\mu \phi^j, \dots), \tag{2.20}$$

which leaves the action invariant or in other words, which leaves the lagrangian invariant up to a total derivative, i.e.,

$$\frac{\delta \mathcal{L}}{\delta \phi^i} \Delta \phi^i + \partial_\mu j^\mu = 0. \tag{2.21}$$

In (2.21), j^μ is the conserved current associated with the symmetry $\Delta \phi^i$.

If we set $a_1 = \phi_i^* a^i d^n x$, (2.21) can be rewritten as,

$$\delta a_1 + dj = 0, \tag{2.22}$$

where j is the $n-1$ form dual to j^μ. We have used the fact that by *definition*, $\delta \phi_i^* = -\frac{\delta \mathcal{L}}{\delta \phi^i}$. The n-form a_1 of antighost number 1 therefore defines an element of the cohomological group $H_1^n(\delta|d)$.

A global symmetry is said to be trivial if the corresponding a_1 defines a trivial element in $H_1^n(\delta|d)$, i.e.,

$$a_1 = \delta r_2 + dc_1. \tag{2.23}$$

This is the case if and only if the global symmetry reduces on-shell to a gauge transformation [23, 27]. Two global symmetries are said to be equivalent if they differ by a trivial symmetry. Therefore, the equivalence classes of non-trivial symmetries correspond to the classes of $H_1^n(\delta|d)$.

From Eq. (2.21), we see that there is a well defined map between the cohomological classes of $H_1^n(\delta|d)$ and the classes of conserved currents, where two currents are said to be equivalent if they differ by a term of the form: $\delta e_1 + dm$ where m is a $n-2$ form. Such a term reduces on-shell to an identically conserved current and will be called a *trivial current*. This does not mean that trivial currents are devoid of physical interest but rather that their expression is easily obtained.

The question we want to answer is: can a_1 and j be redefined ("improved") by the addition of trivial terms in order to make them gauge invariant. In other words, does one have $\gamma a_1' = \gamma j' = 0$ with $a_1' = a_1 + \delta r_2 + dc_1$ and $j' = j + \delta e_1 + dm$?

Using the following isomorphism theorem, one can relate this question to the calculation of the local BRST cohomology $H(s|d)$:

Theorem 1. *(valid in particular for the p-form gauge theories considered in this thesis)*

$$H_k(s|d) \simeq \begin{cases} H_k(\gamma|d, H_0(\delta)) & k \geq 0 \\ H_{-k}(\delta|d) & k < 0 \end{cases} \qquad (2.24)$$

Proof: The proof of this result can be found in [28]. □

Using the case $k < 0$ of Theorem **1** we see that to any class of $H_1^n(\delta|d)$ representing a global symmetry we can associate a class of $H_{-1}^n(s|d)$ of ghost number -1. The map $q : H_{-1}^n(s|d) \to H_1^n(\delta|d)$ is realized in the following manner. Let a be a representative of a class of $H_{-1}^n(s|d)$. The expansion of a according to the antighost number is of the form:

$$a = \overline{a}_1 + \overline{a}_2 + \ldots + \overline{a}_k. \qquad (2.25)$$

Since a satisfies the Wess-Zumino consistency condition $sa+db = 0$, the term $\overline{a}_1$ satisfies $\delta\overline{a}_1 + db_0 = 0$ and therefore defines an element of $H_1^n(\delta|d)$. It is easy to see that the map $q : H_{-1}^n(s|d) \to H_1^n(\delta|d) : [a] \to [\overline{a}_1]$ is well defined in cohomology since a change of representative in $H_{-1}^n(s|d)$ $(a \to a + sr + dc)$ does not change the class of $\overline{a}_1$ in $H_1^n(\delta|d)$ $(\overline{a}_1 \to \overline{a}_1 + \delta r_2 + dc_1)$.

The BRST cocycles (2.25) can be classified according to the value of k at which their expansion (according to the antighost number) stops. This

expansion genuinely stops at order k if it is not possible to remove the term $\overline{a}_k$ by the addition of trivial terms.

Because a is a BRST cocycle, the last term $\overline{a}_k$ satisfies the equation $\gamma\overline{a}_k + db_k = 0$. In Section **4.3** however we will show that in each class of $H(s|d)$ there is a representative which satisfies the simpler equation $\gamma\overline{a}_k = 0$. This means that the only global symmetries which cannot be made gauge invariant are those for which the term $\overline{a}_k$ in the corresponding BRST cocycle is genuinely of order k. Indeed, if the terms in a of antighost numbers $j > 1$ can be removed by the addition of trivial terms then the corresponding class of $H_1^n(\delta|d)$ has a representative which obeys $\gamma a_1' = 0$ and which is therefore gauge invariant.

Our analysis of Section **6.3** will show that when one excludes an explicit dependence on the spacetime coordinates, the only BRST cocycles in ghost number -1 which cannot be assumed to stop at order 1 are those which correspond to the following global symmetries:

$$\delta B^a_{\mu_1\ldots\mu_p} = k^a{}_b B^b_{\mu_1\ldots\mu_p}, \quad \text{with } k^a{}_b \text{ an antisymmetric matrix.} \qquad (2.26)$$

Since the $k^a{}_b$ are antisymmetric matrices these global symmetries correspond to rotations of the p-forms among themselves. Only these are not equivalent to a gauge invariant global symmetry.

Once this result has been obtained, one may investigate which conserved currents can or cannot be made gauge invariant. Our analysis establishes that the only currents that cannot be made gauge invariant are those associated to the global symmetries (2.26).

The crucial part of our study of the gauge invariant nature of the global symmetries and the conserved currents is the classification of the solution of the Wess-Zumino consistency condition in ghost number -1.

2.4 Counterterms and Anomalies

In this section we analyze the rôle of the BRST field-antifield formalism in the theory of renormalization. Our presentation is based on [13].

In order to quantize a gauge theory by the method of path integral it is necessary to fix the gauge in order to obtain properly defined Green functions. A convenient way of achieving this is to introduce *auxiliary fields* and to define a gauge-fixing fermion.

The auxiliary fields consist of ghosts and antifields and will respectively be denoted by C^i, b^i and C_i^*, b_i^*. They are associated in pairs by the BRST

differential,

$$sC^i = b^i, \quad sb^i = 0, \tag{2.27}$$

$$sb_i^* = C_i^*, \quad sC_i^* = 0, \tag{2.28}$$

and their ghost numbers and Grassmann parities are related by,

$$ghost \ C^i = ghost \ b_i^* = ghost \ b^i - 1 = ghost \ C_i^* - 1, \tag{2.29}$$

$$\epsilon(C^i) = \epsilon(b_i^*) = \epsilon(b^i) - 1 = \epsilon(C_i^*) - 1. \tag{2.30}$$

The number of auxiliary fields required depends on the theory considered and in particular on its order of reducibility.

By virtue of (2.27) and (2.28), the BRST action generating the differential s in presence of auxiliary fields is still given by,

$$S = \int d^n x (\mathcal{L} + (-)^{\epsilon_A} \phi_A^* s \phi^A), \tag{2.31}$$

where the ϕ^A and ϕ_A^* now include the auxiliary fields.

The extended BRST action S_{ext} is defined as,

$$S_{ext}[\phi^A, \tilde{\phi}_A^*] = S[\phi^A, \tilde{\phi}_A^* + \frac{\delta^R \psi}{\delta \phi^A}]. \tag{2.32}$$

In (2.32), ψ is the gauge-fixing fermion. It is of ghost number -1 and depends only on the ϕ^A. It is chosen in such a way that the Feynman rules obtained from $S_{ext}[\phi^A, \tilde{\phi}_A^* = 0]$ are well defined. It is easy to see that the extended BRST action satisfies the master equation $(S_{ext}, S_{ext}) = 0$ where the antibracket is now expressed in terms of the variables ϕ^A and $\tilde{\phi}_A^*$. From (2.32) one can *formally* calculate all the Green functions of the theory.

The effective action Γ is defined as the generating functional for the connected one particle irreducible, amputated vertex functions,

$$\Gamma[\tilde{\varphi}] = \sum_{m=2}^{\infty} \frac{1}{m!} \int d^n x_1 ... d^n x_m \tilde{\varphi}(x_1) ... \tilde{\varphi}(x_m) < 0|T\varphi(x_1)...\varphi(x_m)|0 >^{1PI,amp}. \tag{2.33}$$

In (2.33) all the fields, ghosts and antifields have been collectively denoted φ. The $\tilde{\varphi}$ are the corresponding smooth, fast decreasing test functions.

As is well known, when one tries to calculate the effective action by using the Feynman rules, one encounters difficulties arising from ill-defined (infinite) quantities. Therefore, in order for the theory to make sense one needs a regularization scheme which eliminates those infinities.

To be more precise, Γ admits an expansion in powers of $\hbar$,

$$\Gamma = \Gamma^{(0)} + \hbar\Gamma^{(1)} + \hbar^2\Gamma^{(2)} + \dots . \qquad (2.34)$$

In (2.34) one can show that the first term is a tree diagram which actually coincides with the extended BRST action S_{ext}. The other terms are loop diagrams, the number of loops being given by the power in $\hbar$. The problems usually begin when one tries to calculate $\Gamma^{(1)}$. In order to eliminate the infinite quantity present in $\Gamma^{(1)}$ one needs to regularize the theory by giving a meaning to divergent integrals. This can be done for example but introducing a cut-off Λ. Eq. (2.34) then becomes,

$$\Gamma^{reg}(\tilde{\phi}^A, \tilde{\phi}_A^*) = S_{ext} + \hbar\Gamma^{(1)}_{div}(\Lambda) + \hbar\Gamma^{(1)}_f(\Lambda) + O(\hbar^2), \qquad (2.35)$$

where $\Gamma^{(1)}_{div}(\Lambda)$ is the sum of the one loop contributions to Γ which diverge when $\Lambda \to \infty$.

Problems can occur when the regularization introduced fails to preserve at the quantum level the local symmetries of the classical action. Indeed, the *quantum action principle* [29] states that the following identity holds,

$$\frac{1}{2}(\Gamma^{reg}, \Gamma^{reg})(\tilde{\phi}) = \hbar\Delta(\tilde{\phi}) + O(\hbar^2), \qquad (2.36)$$

where the antibracket is taken with respect to the sources $\tilde{\phi}^A$ and $\tilde{\phi}_A^*$. The term $\Delta(\tilde{\phi})$ is an integrated polynomial of ghost number 1. It represents at order $\hbar$ the obstruction for Γ^{reg} to satisfy the equation,

$$(\Gamma^{reg}, \Gamma^{reg}) = 0. \qquad (2.37)$$

If a term $\Delta(\tilde{\phi})$ is present and cannot be removed by adding local counterterms to the classical action, the theory is said to be anomalous. Indeed, Eq. (2.37) reflects gauge invariance and has to be satisfied in order to guarantee the consistency of the quantum theory and in particular its unitarity.

By taking the antibracket of Eq. (2.36) with Γ^{reg} and using the Jacobi identity $(\Gamma^{reg}, (\Gamma^{reg}, \Gamma^{reg})) \equiv 0$ one obtains a consistency requirement for the anomaly $\Delta(\tilde{\phi})$ [12]:

$$(S_{ext}, \Delta) = 0. \qquad (2.38)$$

This implies that the anomalies have to satisfy the Wess-Zumino consistency condition for the differential $s_{ext} = (S_{ext}, .)$. The problem of finding the possible anomalies of the theory is in fact of cohomological nature since trivial solutions of (2.38) can be easily eliminated by modifying the original

action S_{ext}. Indeed if $\Delta = (S_{ext}, \Sigma)$, the redefinition $S_{ext} \to S_{ext} - \hbar\Sigma$ yields at order $\hbar$, $(\Gamma^{reg}, \Gamma^{reg}) = 0$. Therefore, the non-trivial anomalies are elements of $H_1^n(s_{ext})$.

If the theory can be shown to be free from anomalies, then the BRST cohomology is also useful to determine the counterterms required to renormalize the theory. Indeed, in that case Eq. (2.36) implies at order $\hbar$ for the divergent contributions to Γ,

$$(S_{ext}, \Gamma_{div}^{(1)}) = 0. \tag{2.39}$$

The counterterms are thus solutions of the Wess-Zumino consistency condition in ghost number 0. As in the case of anomalies, there are some trivial solutions to (2.39) given by $\Gamma^{(1)} = (S_{ext}, \Omega)$ where Ω is of ghost number -1. They are called trivial because they amount to field redefinitions in the action S_{ext}.

The above discussion shows that two important questions concerning renormalization can be examined by the calculation of the BRST cohomology of the differential s_{ext}. One can in fact show that this cohomology is isomorphic to $H(s)$ where s is the BRST differential of the *free theory*. The isomorphism is implemented by making the change of variables $\tilde{\phi}_A^* \to \tilde{\phi}_A^* + \frac{\delta^R \psi}{\delta \phi^A}$ in the representatives of $H(s)$.

This concludes our survey of the use of the BRST cohomology in classical and quantum field theory. In the next section we will formulate precisely the mathematical framework in which the Wess-Zumino consistency condition will be studied.

2.5 General definitions

In the previous sections we have emphasized the rôle of the BRST cohomology in field theory. However, in order to focus the attention on the physical ideas we deliberately postponed the precise definition of the space in which the calculations have to be done.

The standard way to impose locality in the BRST formalism is to work in the so-called "jet spaces". To that end, we define a local function f as a function which depends on the spacetime coordinates x^μ, the fields (including the ghosts and the antifields) and their derivatives up to a finite order k,

$$f = f(x^\mu, \Phi, \partial_\mu \Phi, \dots, \partial_{\mu_1 \dots \mu_k} \Phi). \tag{2.40}$$

A local function is thus a function over a *finite dimensional space* V^k called a jet space which is parameterized by the spacetime coordinates x^μ, the fields

Φ and their subsequent derivatives $\partial_{\mu_1...\mu_k}\Phi$ up to a finite order k. In the sequel we will always assume that spacetime has the topology of $\mathbb{R}^n$.

A local functional F is then defined as the integral of local function:

$$F = \int d^n x\, f(x^\mu, \Phi, \partial_\mu\Phi, \dots, \partial_{\mu_1...\mu_k}\Phi). \qquad (2.41)$$

By definition, the local BRST cohomology $H(s)$ is the set of local functionals which satisfy the equation,

$$sF = s\int d^n x\, f = 0, \qquad (2.42)$$

(for all allowed field configurations) and which are not of the form,

$$F = sG = s\int d^n x\, g, \qquad (2.43)$$

with g a local function.

The next step in the analysis is to remove the integral sign. This is done by recalling that Eq. (2.42) is satisfied if and only if there exist a local $n-1$ form m such that,

$$sf + dm = 0, \qquad (2.44)$$

with $\oint m = 0$; a local q-form a is defined as a spacetime form with coefficients belonging to the algebra of local functions:

$$a = a_{\mu_1...\mu_q}(x^\mu, \Phi, \partial_\mu\Phi, \dots, \partial_{\mu_1...\mu_k}\Phi)dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q}. \qquad (2.45)$$

In (2.44) d is the algebraic spacetime exterior derivative defined on local forms as,

$$dm = dx^\mu \partial_\mu m, \text{ and} \qquad (2.46)$$

$$\partial_\mu m = \frac{\partial m}{\partial x^\mu} + \partial_\mu\Phi \frac{\partial^L m}{\partial\Phi} + \partial_{\mu\nu}\Phi \frac{\partial^L m}{\partial(\partial_\nu\Phi)} + \dots + \partial_{\mu\nu_1...\nu_s}\Phi \frac{\partial^L m}{\partial(\partial_{\nu_1...\nu_s}\Phi)}. \qquad (2.47)$$

Similarly, the coboundary condition (2.43) is equivalent to,

$$f = sg + dl, \qquad (2.48)$$

with l a local $n-1$ form such that $\oint l = 0$.

Accordingly, the cohomology $H(s)$ in the space of local functionals is isomorphic to the cohomology $H(s|d)$ in the space of local functions.

The conditions $\oint m = \oint l = 0$ are rather awkward to take into account (in particular they depend on the precise conditions imposed on the fields at the boundaries). Therefore, one usually investigates $H(s|d)$ without restrictions on the $(n-1)$-forms at the boundary. One thus allows elements of $H(s|d)$ that *do not define s*-closed local functionals because of non-vanishing surface terms.

Because we are interested in the physical problems described in Section **2.2-2.4** we will further restrict the algebra of local functions. Indeed, the counterterms and anomalies are in fact local integrated polynomials. We will also impose this condition on the consistent deformations (at each order in the coupling constant) and on the conserved currents. The local BRST cohomology $H(s|d)$ will therefore be investigated in the space of local q-forms with coefficients that are polynomials in fields and their derivatives. This algebra will be denoted by $\mathcal{P}$. Thus, a belongs to $\mathcal{P}$ if and only if

$$a = \alpha_{\nu_1 \ldots \nu_q}(x^\mu, [\Phi]) dx^{\nu_1} \ldots dx^{\nu_q}, \tag{2.49}$$

with α a polynomial. The notation $f([y])$ means that f depends on y and its successive derivatives. Note that in (2.49) we have dropped the $\wedge$ symbol between the dx^μ.

We will also perform many of the calculations in the algebra of local q-forms which do not depend explicitly on the spacetime coordinates x^μ. This algebra will be denoted $\mathcal{P}_-$.

Chapter 3

Free theory: BRST construction

3.1 Lagrangian, equations of motion and gauge invariance

The free action of a system of p-forms $B^a_{\mu_1 \ldots \mu_{p_a}}$ is given by,

$$I = \int d^n x \mathcal{L} \tag{3.1}$$

$$= \int d^n x \sum_a \left(\frac{-1}{2(p_a + 1)!} H^a_{\mu_1 \ldots \mu_{p_a+1}} H^{a\mu_1 \ldots \mu_{p_a+1}} \right), \tag{3.2}$$

where the field strengths $H^{a\mu_1 \ldots \mu_{p_a+1}}$ are defined by,

$$H^a = \frac{1}{(p_a + 1)!} H^a_{\mu_1 \ldots \mu_{p_a+1}} dx^{\mu_1} \ldots dx^{\mu_{p_a+1}} = dB^a, \tag{3.3}$$

$$B^a = \frac{1}{p_a!} B^a_{\mu_1 \ldots \mu_{p_a}} dx^{\mu_1} \ldots dx^{\mu_{p_a}}. \tag{3.4}$$

Otherwise stated, we will always suppose in the sequel that $p_a \geq 1$ and $n > p_a + 1$ for all values of a. The second condition is necessary in order for the p-forms to have local degrees of freedom.

From (3.2) one easily obtains the equations of motion by varying the fields $B^a_{\mu_1 \ldots \mu_{p_a}}$. They can be written equivalently as

$$\partial_\rho H^{a\rho\mu_1 \ldots \mu_{p_a}} = 0 \Leftrightarrow d\overline{H}^a = 0, \tag{3.5}$$

where $\overline{H}^a$ is the $n - p_a - 1$ form dual to $H^{a\rho\mu_1 \ldots \mu_{p_a}}$.

The main feature of theories involving p-form gauge fields is that their gauge symmetries are *reducible*. More precisely, in the present case, the Lagrangian (3.2) is invariant under the gauge transformations,

$$B^a \to B^{'a} = B^a + d\Lambda^a \tag{3.6}$$

where Λ^a are arbitrary $(p_a - 1)$-forms. Now, if $\Lambda^a = d\epsilon^a$, then, the variation of B^a vanishes identically. Thus, the gauge parameters Λ^a do not all provide independent gauge symmetries: the gauge transformations (3.6) are reducible. In the same way, if ϵ^a is equal to $d\mu^a$, then, it yields a vanishing Λ^a. There is "reducibility of reducibility" unless ϵ^a is a zero form. If ϵ^a is not a zero form, the process keeps going until one reaches 0-forms. For the theory with Lagrangian (3.2), there are thus $p_M - 1$ stages of reducibility of the gauge transformations (Λ^a is a $(p_a - 1)$-form), where p_M is the degree of the form of highest degree occurring in (3.2) [30, 31, 32, 20]. One says that the theory is a reducible gauge theory of reducibility order $p_M - 1$.

3.2   BRST differential

We define the action of the BRST differential s along the lines recalled in Section **2.1**. Therefore we need to introduce besides the fields, some antifields and some ghosts. Because the theory is reducible (of order $p_a - 1$ for each a), the following set of antifields [23, 10, 11] is required:

$$B^{*a\mu_1...\mu_{p_a}}, B^{*a\mu_1...\mu_{p_a-1}}, \ldots, B^{*a\mu_1}, B^{*a}. \tag{3.7}$$

The Grassmann parity and the *antighost* number of the antifields $B^{*a\mu_1...\mu_{p_a}}$ associated with the fields $B^a_{\mu_1...\mu_{p_a}}$ are equal to 1. The Grassmann parity and the *antighost* number of the other antifields is determined according to the following rule. As one moves from one term to the next one to its right in (3.7), the Grassmann parity changes and the antighost number increases by one unit. Therefore the parity and the antighost number of a given antifield $B^{*a\mu_1...\mu_{p_a-j}}$ are respectively $j + 1$ modulo 2 and $j + 1$.

Reducibility also imposes the following set of ghosts [23, 10, 11]:

$$C^a_{\mu_1...\mu_{p_a-1}}, \ldots, C^a_{\mu_1...\mu_{p_a-j}}, \ldots, C^a. \tag{3.8}$$

These ghosts carry a degree called the pure ghost number. The pure ghost number of $C^a_{\mu_1...\mu_{p_a-1}}$ and its Grassmann parity are equal to 1. As one moves from one term to the next one to its right in (3.8), the Grassmann parity changes and the ghost number increases by one unit up to p_a.

For each field, the *ghost number* is defined as the difference between the pureghost number and the antighost number.

The following table summarizes for each field the various values of the gradings we have introduced:

Fields	parity mod 2	pureghost	antighost	ghost
$B_{a\mu_1...\mu_{p_a}}$	0	0	0	0
$B^{*a\mu_1...\mu_{p_a}}$	1	0	1	-1
$B^{*a\mu_1...\mu_{p_a-1}}$	0	0	2	-2
...	...	...	...	...
$B^{*a\mu_1...\mu_{p_a-j}}$	$j+1$	0	$j+1$	$-j-1$
...	...	...	...	...
B^{*a}	p_a+1	0	p_a+1	$-p_a-1$
$C^a_{\mu_1...\mu_{p_a-1}}$	1	1	0	1
$C^a_{\mu_1...\mu_{p_a-2}}$	0	2	0	2
...	...	...	...	...
$C^a_{\mu_1...\mu_{p_a-j}}$	j	j	0	j
...	...	...	...	...
C^a	p_a	p_a	0	p_a
x^μ	0	0	0	0
dx^μ	1	0	0	0

As explained in Section **2.5**, we denote by $\mathcal{P}$ the algebra of spacetime forms depending explicitly on the spacetime coordinates x^μ with coefficients that are polynomials in the fields, antifields, ghosts and their derivatives. The corresponding algebra without explicit dependence on the spacetime coordinates is denoted $\mathcal{P}_-$.

The action of s in $\mathcal{P}$ is the sum of two parts, namely, the 'Koszul-Tate differential δ' and the 'longitudinal exterior derivative γ':

$$s = \delta + \gamma, \tag{3.9}$$

and by definition we have,

$$\delta B^a_{\mu_1 \ldots \mu_{p_a}} = 0, \tag{3.10}$$
$$\delta C^a_{\mu_1 \ldots \mu_{p_a - j}} = 0, \tag{3.11}$$
$$\delta \overline{B}^{*a}_1 + d\overline{H}^a = 0,$$
$$\delta \overline{B}^{*a}_2 + d\overline{B}^{*a}_1 = 0,$$
$$\vdots \tag{3.12}$$
$$\delta \overline{B}^{*a}_{p_a + 1} + d\overline{B}^{*a}_{p_a} = 0,$$

and,

$$\gamma B^{*a \mu_1 \ldots \mu_{p_a + 1 - j}} = 0, \tag{3.13}$$
$$\gamma B^a + dC^a_1 = 0, \tag{3.14}$$
$$\gamma C^a_1 + dC^a_2 = 0, \tag{3.15}$$
$$\vdots$$
$$\gamma C^a_{p_a - 1} + dC^a_{p_a} = 0, \tag{3.16}$$
$$\gamma C^a_{p_a} = 0. \tag{3.17}$$

In the above equations, C^a_j is the $p_a - j$ form whose components are $C^a_{\mu_1 \ldots \mu_{p_a - j}}$. Furthermore, we have denoted the duals by an overline (to avoid confusion with the *-notation of the antifields); for instance, $\overline{B}^{*a}_1$ is the dual of the antifield of antighost number 1, $B^{*a \mu_1 \ldots \mu_{p_a}}$.

The actions of δ and γ are extended to the rest of the generators of $\mathcal{P}$ and $\mathcal{P}_-$ by using the rules,

$$\delta \partial_\mu = \partial_\mu \delta, \quad \gamma \partial_\mu = \partial_\mu \gamma, \tag{3.18}$$

and

$$\delta(x^\mu) = \gamma(x^\mu) = 0. \tag{3.19}$$

By definition, we choose both δ and γ to act like left-antiderivations, e.g.,

$$\delta(ab) = (\delta a)b + (-)^{\epsilon_a} a(\delta b), \tag{3.20}$$

where ϵ_a is the Grassmann parity of a.

With these conventions we have:

Fields	parity mod 2	form-degree	pureghost	antighost	ghost
$\overline{H}^a$	$n - p_a - 1$	$n - p_a - 1$	0	0	0
$\overline{B}_1^{*a}$	$n - p_a - 1$	$n - p_a$	0	1	-1
$\overline{B}_2^{*a}$	$n - p_a - 1$	$n - p_a + 1$	0	2	-2
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\overline{B}_j^{*a}$	$n - p_a - 1$	$n - p_a - 1 + j$	0	j	$-j$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\overline{B}_{p_a+1}^{*a}$	$n - p_a - 1$	n	0	$p_a + 1$	$-p_a - 1$
C_1^a	p_a	$p_a - 1$	1	0	1
C_2^a	p_a	$p_a - 2$	2	0	2
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
C_j^a	p_a	$p_a - j$	j	0	j
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
C^a	p_a	0	p_a	0	p_a
x^μ	0	0	0	0	0
dx^μ	1	1	0	0	0

The rôle of the Koszul-Tate differential is to implement the equations of motion in cohomology. This statement is contained in the following theorem:

Theorem 2. *(Valid in $\mathcal{P}$ and $\mathcal{P}_-$) Let i be the antighost number. Then, $H_i(\delta) = 0$ for $i > 0$, i.e., the cohomology of δ vanishes in antighost number strictly greater than zero. Furthermore, the cohomology of δ in antighost number zero is given by the algebra of "on-shell spacetime forms".*

Proof: See [33, 34, 23, 35] and also [27] for the explicit proof in the case of 1-forms and 2-forms. $\square$

While δ controls the dynamics of the theory, the rôle of the exterior longitudinal derivative is to take care of the gauge invariance. Therefore, the only combinations of the fields $B^a_{\mu_1...\mu_{p_a}}$ and their derivatives appearing in the cohomology of γ should be the field strengths and their derivatives. Indeed, those are the only gauge invariant objects that can be formed out of the fields and their derivatives. The following theorem shows that the definition of γ is consistent with this requirement.

Theorem 3. *(Valid in $\mathcal{P}$ and $\mathcal{P}_-$) The cohomology of γ is given by,*

$$H(\gamma) = \mathcal{I} \otimes \mathcal{C}, \qquad (3.21)$$

where $\mathcal{C}$ is the algebra generated by the undifferentiated *ghosts $C^a_{p_a}$, and $\mathcal{I}$ is the algebra of spacetime forms with coefficients that are polynomials in the fields strengths, their derivatives, the antifields and their derivatives. In particular, in antighost and pure ghost numbers equal to zero, one can take as representatives of the cohomological classes the gauge invariant forms.*
<u>rem</u>: From now on, the variables generating $\mathcal{I}$ will be denoted by χ.
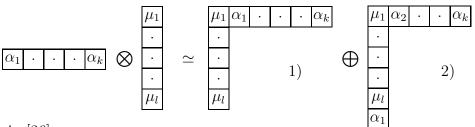
Proof: One follows the standard method which consists in separating the variables into three sets obeying respectively $\gamma x^i = 0$, $\gamma y^\alpha = z^\alpha$, $\gamma z^\alpha = 0$. The variables y^α and z^α form "contractible pairs" and the cohomology is then generated by the (independent) variables x^i. In our case, the x^i are given by x^μ, dx^μ, the fields strength components, the antifields and their derivatives as well as the last (undifferentiated) ghosts of ghosts (in $\mathcal{P}_-$ the x^μ are absent). Of course, for the proof to work, any identity verified by the y^α should also be verified by the z^α, so that the independent z^α are all killed in cohomology.

Let us first note that x^μ, dx^μ, the antifields and their derivatives are automatically part of the x^i since they are γ-closed and do not appear in the γ-variations.

To show that indeed the rest of the generators of $\mathcal{P}$, which are

$$C^a_{\mu_1...\mu_l}, \partial_{\alpha_1...\alpha_k} C^a_{\mu_1...\mu_l}, \ldots, C^a, \partial_{\alpha_1...\alpha_k} C^a, \ldots, \qquad (3.22)$$

(with the convention $B^a_{\mu_1...\mu_p} = C^a_{\mu_1...\mu_p}$) split as indicated, we decompose the $\partial_{\alpha_1...\alpha_k} C^a_{\mu_1...\mu_l}$ into irreducible tensors under the full linear group $GL(n)$. Since the $\partial_{\alpha_1...\alpha_k} C^a_{\mu_1...\mu_l}$ are completely symmetric in $\alpha_1 \ldots \alpha_k$ and completely antisymmetric in $\mu_1 \ldots \mu_l$ they transform under $GL(n)$ as the variables of the tensor product representation symbolically denoted by

$$\boxed{\alpha_1}\ \cdot\ \cdot\ \cdot\ \boxed{\alpha_k}\ \otimes\ \begin{array}{c}\boxed{\mu_1}\\ \cdot\\ \cdot\\ \cdot\\ \boxed{\mu_l}\end{array}\ \simeq\ \begin{array}{c}\boxed{\mu_1}\boxed{\alpha_1}\ \cdot\ \cdot\ \cdot\ \boxed{\alpha_k}\\ \cdot\\ \cdot\\ \boxed{\mu_l}\end{array}\ 1)\ \oplus\ \begin{array}{c}\boxed{\mu_1}\boxed{\alpha_2}\ \cdot\ \cdot\ \boxed{\alpha_k}\\ \cdot\\ \cdot\\ \boxed{\mu_l}\\ \boxed{\alpha_1}\end{array}\ 2)$$

in [36].

Convenient generators for the irreducible spaces corresponding to diagram 1) and 2) are respectively,

$$\partial_{(\alpha_1...\alpha_k}C^a_{[\mu_1)_1...\mu_l]_2} \quad \text{and} \quad \partial_{\alpha_2...\alpha_k}H^a_{\alpha_1\mu_1...\mu_l}, \tag{3.23}$$

with $H^a_{\mu_1...\mu_l} = \partial_{[\mu_1}C^a_{\mu_2...\mu_l]}$. [ ] and ( ) mean respectively antisymmetrization and symmetrization; the subscript indicates the order in which the operations are done.

A direct calculation shows that,

$$\gamma C^a_{\mu_1...\mu_l} = H^a_{\mu_1...\mu_l} \text{ for } 2 \le l \le p, \tag{3.24}$$
$$\gamma C^a_{\mu_1} = \partial_{\mu_1}C^a, \tag{3.25}$$
$$\gamma\partial_{(\alpha_1...\alpha_k}C^a_{[\mu_1)_1...\mu_l]_2} = c\partial_{\alpha_1...\alpha_k}H^a_{\mu_1...\mu_l} \text{ for } 2 \le l \le p, \tag{3.26}$$
$$\gamma\partial_{(\alpha_1...\alpha_k}C^a_{\mu_1)} = \partial_{\alpha_1...\alpha_k\mu_1}C^a, \tag{3.27}$$
$$\gamma H^a_{\mu_1...\mu_{p+1}} = 0, \tag{3.28}$$
$$\gamma\partial_{\alpha_1...\alpha_k}H^a_{\mu_1...\mu_{p+1}} = 0, \tag{3.29}$$
$$\gamma C^a = 0, \tag{3.30}$$

with c in (3.26) given by $c = \frac{k+l}{l(k+1)}$. All the generators are are now split according to the rule recalled at the beginning of the subsection. The cohomology is therefore generated by

$$C^a, \ H^a_{\mu_1...\mu_{p+1}} \quad \text{and} \quad \partial_{\alpha_1...\alpha_k}H^a_{\mu_1...\mu_{p+1}}. \tag{3.31}$$

This ends the proof of the theorem. Note that the generators are not independent but restricted by the Bianchi identity $dH^a = 0$. □

The last two theorems remain valid if we restrict ourselves to $\mathcal{P}_-$ except that the spacetime forms are now independent of the coordinates x^μ.

3.3   Poincaré Lemma

In the sequel, we will also need the following result on the cohomology of d:

Theorem 4. *The cohomology of d in the algebra of local forms $\mathcal{P}$ is given by,*

$$H^0(d) \simeq R, \qquad (3.32)$$

$$H^k(d) = 0 \text{ for } k \neq 0, k \neq n, \qquad (3.33)$$

$$H^n(d) \simeq \text{ space of equivalence classes of local forms,} \qquad (3.34)$$

where k is the form degree and n the spacetime dimension. In (3.34), two local forms are said to be equivalent if and only if they have identical Euler-Lagrange derivatives with respect to all the fields and the antifields.

Proof: This theorem is known as the algebraic Poincaré Lemma. It differs from the usual Poincaré lemma because here we only work with *local* forms in the various fields and their derivatives. For various proofs, see [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49]. $\square$

It should be mentioned that the theorem holds as such only in $\mathcal{P}$. In $\mathcal{P}_-$, (3.33) would have to be amended as

$$H^k(d) \simeq \{\text{constant forms}\} \text{ for } k \neq 0, k \neq n, \qquad (3.35)$$

where the constant forms are by definition the local exterior forms with constant coefficients. Indeed, the explicit x-dependence enables one to remove the constant k-forms ($k > 0$) from the cohomology, since these are exact, $c_{i_1 i_2 \ldots i_k} dx^{i_1} dx^{i_2} \ldots dx^{i_k} = d(c_{i_1 i_2 \ldots i_k} x^{i_1} \, dx^{i_2} \ldots dx^{i_k})$. Note that the constant exterior forms can be alternatively eliminated without introducing an explicit x-dependence, but by imposing Lorentz invariance (there is no Lorentz-invariant constant k-form for $0 < k < n$).

We shall denote in the sequel the algebra of constant forms by Λ^* and the subspace of constants forms of degree k by Λ^k.

The following formulation of the Poincaré lemma is also useful.

Theorem 5. *Let a be a local, closed k-form ($k < n$) that vanishes when the fields, the ghosts and the antifields are set equal to zero. Then, a is d-exact.*

Proof: The condition that a vanishes when the fields and the antifields are set equal to zero eliminates the constants. $\square$

This form of the Poincaré lemma holds in both the algebras of x-dependent and x-independent local exterior forms.

3.4   Mixed forms

In our analysis of the BRST cohomology, in turns out that two combinations of the fields and antifields play a central rôle. The first one groups the field strengths and the duals of the antifields and is denoted $\tilde{H}^a$,

$$\tilde{H}^a = \overline{H}^a + \sum_{j=1}^{p_a+1} \overline{B}_j^{*a}. \tag{3.36}$$

The second one combines the p_a-forms and their associated ghosts and is denoted $\tilde{B}^a$,

$$\tilde{B}^a = B^a + C_1^a + \ldots + C_{p_a}^a. \tag{3.37}$$

The regrouping of physical fields with ghost-like variables is quite standard in BRST theory [50]. Expressions similar (but not identical) to (3.36) have appeared in the analysis of the Freedman-Townsend model and of string field theory [51, 52], as well as in the context of topological models [53, 54]. Note that for a one-form, expression (3.36) reduces to relation (9.8) of [28].

It is easy to see that both $\tilde{H}^a$ and $\tilde{B}^a$ have a definite Grassmann parity respectively given by $n-p_a+1$ and p_a modulo 2. On the other hand, products of $\tilde{H}^a$ or $\tilde{B}^a$ are not homogeneous in form degree and ghost number. To isolate a component of a given form degree k and ghost number g we enclose the product in brackets $[\ldots]_g^k$.

Since products of $\tilde{B}^a$ very frequently appear in the rest of the text, we introduce the following notations,

$$\mathcal{Q}^{a_1\ldots a_m} = \tilde{B}^{a_1}\ldots\tilde{B}^{a_m} \quad \text{and} \quad \mathcal{Q}_{k,g}^{a_1\ldots a_m} = [\tilde{B}^{a_1}\ldots\tilde{B}^{a_m}]_g^k. \tag{3.38}$$

For example, using these conventions, we write the most general representative of $H(\gamma)$ as, $a = \sum_m \alpha_{a_1\ldots a_m}(\chi)[\tilde{B}^{a_1}\ldots\tilde{B}^{a_m}]_l^0 = \sum_m \alpha_{a_1\ldots a_m}(\chi)\mathcal{Q}_{0,l}^{a_1\ldots a_m}$, with $l = \sum_m p_{a_m}$.

We also define the three "mixed operators": $\Delta = \delta + d$, $\tilde{\gamma} = \gamma + d$ and $\tilde{s} = s + d$.

Using those definitions we have the following relations:

$$\Delta\tilde{H}^a = 0, \quad \Delta\tilde{B}^a = 0, \tag{3.39}$$

$$\tilde{\gamma}\tilde{H}^a = 0, \quad \tilde{\gamma}\tilde{B}^a = H^a, \tag{3.40}$$

$$\tilde{s}\tilde{H}^a = 0, \quad \tilde{s}\tilde{B}^a = H^a. \tag{3.41}$$

Eq. $\tilde{\gamma}\tilde{B}^a = H^a$ is known in the literature as the "horizontality condition" [20].

Chapter 4

Free theory: BRST cohomology (Part I)

4.1 Antifield dependence of solutions

In the previous chapters, we have introduced the various ingredients of the BRST field-antifield for the system of free p-form gauge fields (3.2). We have defined all the necessary fields, antifields, ghosts and their transformation law under the BRST differential which in this case reduces to the sum of the Koszul-Tate differential and the longitudinal exterior derivative. We now start the analysis of the BRST cohomology.

The equation we want to solve is,

$$sa_g^k + da_{g+1}^{k-1} = 0, \tag{4.1}$$

where a_g^k is a polynomial in $\mathcal{P}_{(-)}$ of ghost number g and form degree k.

Historically, solutions of (4.1) were first sought in the absence of antifields. This approach is incomplete since it does only take into account the gauge invariance of the theory but not its dynamics. For example, by not investigating the BRST cohomology in the presence of antifields, one would miss the Yang-Mills vertex when computing the various consistent interactions of Maxwell's theory. We will nevertheless start by determining the antifield independent solution of (4.1) for the following reasons. First, they are of course interesting on their own. But also, their knowledge is required in order to apply our method of analysis of the antifield dependent solutions.

We will analyze the antifield independent solutions of the Wess-Zumino consistency condition by studying the so-called descent equations [55, 50, 56]. They were first studied in the Yang-Mills theory context. They are however useful for a large class of theories for which the longitudinal exterior derivative

is nilpotent. This is the case for the system (3.2) but also for the "Chapline-Manton" theories as we shall see later. All the relevant details of these descent equations will be recalled in the next section.

The descent equations for the antifield dependent solutions also exist. However, they cannot be used straightforwardly to obtain the general solution of (4.1). The main difficulty lies in the fact that the antifields are of negative ghost number. Therefore, one cannot easily formulate a *Generalized "transgression" lemma* when they are present (see Theorem **9**).

In order to analyze the antifield dependent solutions we therefore follow another approach. First, we decompose the solutions of (4.1) according to the antighost number, $a_g^k = a_{g,0}^k + a_{g,1}^k + \ldots + a_{g,q}^k$. When performing such a decomposition, we suppose that the term of highest antighost number $(a_{g,q}^n)$ cannot be eliminated by adding trivial terms to a_g^n, i.e., $a_{g,q}^k \neq se_{g-1}^k + dc_g^{k-1}$; otherwise, the expansion would stop at $q-1$. If $q \neq 0$ then a_g^k is said to depend non-trivially on the antifields. To proceed, we will first determine the most general form for the term $a_{g,q}^n$. Then, substituing this form in (4.1) we will try to compute recursively the components $a_{q,q-1}^n, a_{g,q-2}^n$ etc. of lower antighost numbers in a_g^n. As we shall see, this construction is not obstructed only for a small number of $a_{q,q}^n$. This way of analyzing the antifield dependent solutions of the Wess-Zumino consistency condition has some resemblance with the method of the descent equations and we shall later comment on this.

4.2 Antifield independent solutions

As stated, we start by calculating the antifield independent solution of the Wess-Zumino consistency condition by solving the so-called "descent equation". We first recall the general procedure of the analysis and then apply it to free p-forms.

4.2.1 Descent equations

Because we work here in the absence of antifields, Eq. (4.1) becomes,

$$\gamma a_g^k + da_{q+1}^{k-1} = 0, \tag{4.2}$$

(where g now denotes the pureghost number). This is permissible since both γ and d are of antighost number 0. To a given solution a_g^k of (4.2), one can associate another solution of the Wess-Zumino consistency condition, namely, a_{g+1}^{k-1}. Indeed, the triviality of d (Theorem **5**) implies,

$$\gamma a_{g+1}^{k-1} + da_{g+2}^{k-2} = 0, \tag{4.3}$$

for some a_{g+2}^{k-2}. There are ambiguities in the choice of a_{g+1}^{k-1} given the class $[a_g^k]$ of a_g^k in $H^{(k,g)}(\gamma|d)$, but it is easy to verify that the map $\partial : H^{(k,g)}(\gamma|d) \to H^{(k-1,g+1)}(\gamma|d)$ is well defined.

The map ∂ is in general not injective. There are non trivial classes of $H^{*,*}(\gamma|d)$ that are mapped on zero through the descent. For instance, if one iterates ∂, one gets from a_g^k a chain of cocycles in $H^{*,*}(\gamma|d)$, $[a_g^k] \mapsto \partial[a_g^k] \in H^{(k-1,g+1)}(\gamma|d) \mapsto \partial^2[a_g^k] \in H^{(k-2,g+2)} \mapsto \cdots \mapsto \partial^l[a_g^k] \in H^{(k-l,g+l)} \mapsto 0$ which must eventually end on zero since there are no forms of negative form degree.

The equations defining the successive images of $[a_g^k]$ are

$$\gamma a_g^k + da_{g+1}^{k-1} = 0, \tag{4.4}$$
$$\gamma a_{g+1}^{k-1} + da_{g+2}^{k-2} = 0, \tag{4.5}$$

$$\vdots$$

$$\gamma a_{g+l}^{k-l} + da_{g+l+1}^{k-l-1} = 0, \tag{4.6}$$

and are known as the "descent equations" [55, 50, 56].

If a_{g+l+1}^{k-l-1} is trivial in $H^{(k-l-1,g+l+1)}(\gamma|d)$, i.e., $a_{g+l+1}^{k-l-1} = \gamma b_{g+l}^{k-l-1} + db_{g+l+1}^{k-l-2}$, one may redefine $a_{g+l}^{k-l} \to a_{g+l}^{k-l} - db^{k-l-1,g+l} = a'^{k-l,g+l)}$ in such a way that we have $\gamma a'^{k-l}_{g+l} = 0$, i.e., $a'^{k-l-1}_{g+l+1} = 0$. Conversely, if a_{g+l}^{k-l} is annihilated by γ, then $\partial[a_{g+l}^{k-l}] = 0$. Thus, the last non-trivial element a_{g+l}^{k-l}, or "bottom", of the descent is a γ-cocycle that is not exact in $H^{*,*}(\gamma|d)$. The non-injectivity of ∂ follows precisely from the existence of such cocycles.

The length of the descent associated with $[a_g^k]$ is the integer l for which $\partial^l[a_g^k]$ is the last non-trivial cocycle occurring in the chain. One says that a descent is non trivial if it has length ≥ 1. The idea of [57, 58] is to classify the elements of $H^{*,*}(\gamma|d)$ according to the length of the associated descent.

In order to achieve this, one must determine the possible bottoms, i.e., the elements of $H(\gamma)$ which are not trivial in $H(\gamma|d)$ and which can be lifted l times.

4.2.2 Lifts of elements of $H(\gamma)$ - An example

The difficulty in the analysis of the lift is that contrary to the descent which carries no ambiguity in cohomology, the lift is ambiguous because $H(\gamma)$ is not trivial. Furthermore, for the same reason, the lift can be obstructed, i.e., given $a \in H(\gamma)$, there may be no descent (i) which has a as bottom; and (ii) which starts with a solution b of the Wess-Zumino consistency condition such that $db = 0$ (while any descent ends always with an a such that $\gamma a = 0$). The "first" b may be such that $db \neq 0$ or even $db \neq \gamma(\text{something})$. In that case,

there is no element c above b such that $\gamma c + db = 0$ (while there is always an element e below a such that $\gamma a + de = 0$, namely $e = 0$: the descent effectively stops at a but is not obstructed at a).

In this subsection, we shall illustrate these features on a specific example: that of a free 1-form A and a free 2-form B, with BRST algebra

$$\gamma A + dA_1 = 0, \ \gamma A_1 = 0, \tag{4.7}$$
$$\gamma B + dB_1 = 0, \ \gamma B_1 + dB_2 = 0, \ \gamma B_2 = 0. \tag{4.8}$$

The curvatures are $F = dA$ and $H = dB$, with $\gamma F = \gamma H = 0$.

Consider the γ-cocycle $A_1 B_2$. It has form-degree zero and ghost number three. The descent that ends on this bottom has length one, and not the maximum length three. Indeed, the γ-cocycle $A_1 B_2$ can be lifted once, since there exists $a \in \mathcal{P}$ such that $\gamma a + d(A_1 B_2) = 0$. One may take $a = AB_2 + A_1 B_1$. Of course, a has form-degree one and ghost number two. If one tries to lift the given γ-cocycle $A_1 B_2$ once more, one meets an obstruction. Namely, there is no b such that $\gamma b + da = 0$. This is because da is in the same γ-class as FB_2, which is non-trivial, i.e., which cannot be written as a γ-variation. It is easy to verify that one cannot remove the obstruction by adding to a a γ-cocycle (which would not change γa). This provides an example of a γ-cocycle for which the lift is obstructed after one step.

Consider now the γ-cocycle $\frac{1}{2} F(B_2)^2$ with ghost number four and form-degree two. This cocycle can be lifted a first time, for instance $FB_1 B_2$ is above it,

$$\gamma[FB_1 B_2] + d[\frac{1}{2} F(B_2)^2] = 0. \tag{4.9}$$

It can be lifted a second time to $\frac{1}{2}(B_1)^2 + FBB_2$. However, if one tries to lift it once more, one meets apparently the obstruction FHB_2, since the exterior derivative of $\frac{1}{2}(B_1)^2 + FBB_2$ differs from the γ-cocycle FHB_2 by a γ-exact term. It is true that FHB_2 is a non-trivial γ-cocycle. However, the obstruction to lifting three times $\frac{1}{2} F(B_2)^2$ is really absent. What happens is that we made a "wrong" choice for the term above the γ-cocycle $\frac{1}{2} F(B_2)^2$ and should have rather taken a term that differs from $FB_1 B_2$ by an appropriate γ-cocycle. This is because FHB_2 is in fact the true obstruction to lifting twice the γ-cocycle $A_1 HB_2$. Thus if one replaces (4.9) by

$$\gamma[FB_1 B_2 - A_1 HB_2] + d[\frac{1}{2} F(B_2)^2] = 0, \tag{4.10}$$

which is permissible since $\gamma(A_1 HB_2) = 0$, one removes the obstruction to further lifting $\frac{1}{2} F(B_2)^2$. This shows that the obstructions to lifting k times

a γ-cocycle are not given by elements of $H(\gamma)$, but rather, by elements of $H(\gamma)$ that are not themselves obstructions of shorter lifts. The ambiguity in the choice of the lifts plays accordingly a crucial rôle in the analysis of the obstructions.

In fact, the given γ-cocycle $\frac{1}{2}F(B_2)^2$ is actually trivial in $H(\gamma|d)$

$$
\begin{aligned}
\frac{1}{2}F(B_2)^2 =&d[\frac{1}{2}A(B_2)^2 + A_1 B_1 B_2] \\
&+ \gamma[AB_1 B_2 + \frac{1}{2}A_1(B_1)^2 + A_1 B B_2]
\end{aligned}
\tag{4.11}
$$

and therefore, its lift can certainly never be obstructed.

4.2.3 Lifts of elements of $H(\gamma)$ - The first two steps

In order to control these features, it is convenient to introduce new differential algebras [57, 58]. Let $E_0 \equiv H(\gamma)$. We define a map $d_0 : E_0 \to E_0$ as follows:

$$
d_0[a] = [da],
\tag{4.12}
$$

where $[\ ]$ is here the class in $H(\gamma)$. This map is well defined because $\gamma da = -d\gamma a = 0$ (so da is a γ-cocycle) and $d(\gamma m) = -\gamma(dm)$ (so d maps a γ-coboundary on a γ-coboundary). Now, d_0 is a derivation and $d_0^2 = 0$, so it is a differential. Cocycles of d_0 are elements of $H(\gamma)$ that can be lifted at least once since $d_0[a] = 0 \Leftrightarrow da + \gamma b = 0$ for some b, so b descends on a. By contrast, if $d_0[a] \neq 0$, then a cannot be lifted and, in particular, a is not exact in $H(\gamma|d)$ (if it were, $a = \gamma m + dn$, one would have $da = -\gamma dm$, i.e., $da = 0$ in $H(\gamma)$). Let F_0 be a subspace of E_0 supplementary to $\operatorname{Ker} d_0$. One has the isomorphism (as vector spaces),

$$
E_0 \simeq \operatorname{Ker} d_0 \oplus F_0.
\tag{4.13}
$$

The next step is to investigate cocycles that can be lifted at least twice. In order to be liftable at least once, these must be in $\operatorname{Ker} d_0$. Among the elements of $\operatorname{Ker} d_0$, those that are in $\operatorname{Im} d_0$ are not interesting, because they are elements of $H(\gamma)$ that are trivial in $H(\gamma|d)$ ($[a] = d_0[b] \Leftrightarrow a = db + \gamma m$). Thus the relevant space is $E_1 \equiv H(d_0, E_0)$. One has,

$$
\operatorname{Ker} d_0 \simeq \operatorname{Im} d_0 \oplus E_1.
\tag{4.14}
$$

One then defines the differential $d_1 : E_1 \to E_1$,

$$
d_1[[a]] = [[db]],
\tag{4.15}
$$

where b is defined through $da + \gamma b = 0$ – recall that $d_0[a] = 0$ – and where $[[a]]$ is the class of $[a]$ in E_1. It is easy to see that (4.15) provides a well-defined differential in $E_1{}^1$.

If $[[a]] \in E_1$ is a d_1-cocycle, then it can be lifted at least twice since $[[db]] = 0$ in E_1 means $db = du + \gamma(\text{something})$ with $\gamma u = 0$. Thus one has $da + \gamma b' = 0$ with $b' = b - u$ and $db' = \gamma(\text{something})$. If on the contrary, $d_1[[a]] \neq 0$, then the corresponding elements in $H(\gamma)$ cannot be lifted twice, $d_1[[a]]$ being the obstruction to the lift. More precisely, the inequality $d_1[[a]] \neq 0$ in E_1 means $[db] \neq d_0[c]$ in E_0. Thus, db cannot be written as a γ-variation, even up to the exterior derivative of a γ- closed term (ambiguity in the definition of b). It is straightforward to see that in that case both a and b are non-trivial in $H(\gamma|d)$.

Analogous to the decomposition (4.13) one has,

$$E_1 \simeq \operatorname{Ker} d_1 \oplus F_1, \qquad (4.16)$$

where F_1 is a subspace of E_1 supplementary to $\operatorname{Ker} d_1$. The elements in $\operatorname{Im} d_1$ are trivial in $H(\gamma|d)$ and thus of no interest from the point of view of the Wess-Zumino consistency condition.

To investigate the (non-trivial) γ-cocycle that can be lifted at least three times one defines,

$$E_2 = H(d_1, E_1), \qquad (4.17)$$

and the differential d_2 through,

$$d_2 : \ E_2 \to E_2, \ d_2[[[a]]] = [[[dc]]], \qquad (4.18)$$

where the triple brackets denote the classes in E_2 and where c is defined through the successive lifts $da + \gamma b = 0$, $db + \gamma c = 0$ (which exist since $d_1[[a]] = 0$). It is straightforward to verify that d_2 is well-defined in E_2, i.e., that the ambiguities in b and c play no rôle in E_2. Furthermore, a γ-cocycle a such that $d_0[a] = 0$ (so that $[[a]] \in E_1$ is well-defined) and $d_1[[a]] = 0$ (so that $[[[a]]] \in E_2$ is well-defined) can be lifted a third time if and only if $d_2[[[a]]] = 0$. Indeed, the relation $d_2[[[a]]] = 0$ is equivalent to $[[[dc]]] = 0$, i.e. $dc = \gamma u + dv + dw$, with $\gamma v = 0$ (this is the d_0-term) and $\gamma w + dt = 0$, $\gamma t = 0$ (this is the d_1-term). Thus, by redefining b as $b - t$ and c as $c - v - w$, one

---

[1] Proof: $d_0[a] = 0 \Rightarrow da + \gamma b = 0 \Rightarrow \gamma db = 0$. Hence, db is a γ-cocycle, which is clearly annihilated by d_0, $d_0[db] = [d^2 b] = 0$. Furthermore, the class of db in E_1 does not depend on the ambiguity in the definition of b, since if b is replaced by $b + dm + u$ with $\gamma u = 0$, then db is replaced by $db + du$ which is equivalent to db in E_1 (the class of du in E_0 is equal to $d_0[u]$ since $\gamma u = 0$, and this is zero in E_1). The derivation property is also easily verified, $d_1(ab) = (d_1 a)b + (-1)^{\epsilon_a} a d_1 b$.

gets, $dc_{Redefined} = \gamma u$. If a cannot be lifted a third time then it is non-trivial in $H(\gamma|d)$; this is also true for b and c.

Note that we have again the decomposition,

$$E_2 \simeq \mathrm{Ker} d_2 \oplus F_2, \tag{4.19}$$

where F_2 is a subspace of E_2 supplementary to $\mathrm{Ker} d_2$.

To summarize, the above discussion shows that: 1) the elements of F_0 are the non-trivial γ-cocycles in $H(\gamma|d)$ that cannot be lifted at all; 2) the elements of F_1 are the non-trivial γ-cocycles in $H(\gamma|d)$ that can be lifted once; and 3) the elements of F_2 are the non-trivial γ-cocycles in $H(\gamma|d)$ that can be lifted twice. Furthermore, the successive lifts of elements of F_1 and F_2 are non-trivial solution of the Wess-Zumino consistency condition.

4.2.4 Lifts of elements of $H(\gamma)$ - General theory

One can proceed in the same way for the next lifts. One finds in that manner a sequence of spaces E_r and differentials d_r with the properties:

1. $E_r = H(E_{r-1}, d_{r-1})$.

2. There exists an antiderivation $d_r : E_r \to E_r$ defined by $d_r[[\ldots[X]\ldots]] = [[\ldots[db]\ldots]]$ for $[[\ldots[X]\ldots]] \in E_r$ where $[[\ldots[db]\ldots]]$ is the class of the γ-cocycle db in E_r and where b is defined through $dX + \gamma c_1 = 0$, $dc_1 + \gamma c_2 = 0$, ..., $dc_{r-1} + \gamma b = 0$. Similarly, $[[\ldots[X]\ldots]]$ denotes the class of the γ-cocycle X in E_r (assumed to fulfill the successive conditions $d_0[X] = 0$, $d_1[[X]] = 0$ etc ... so as to define an element of E_r).

3. $d_r^2 = 0$.

4. A γ-cocycle X can be lifted r times if and only if $d_0[X] = 0$, $d_1[[X]] = 0$, $d_2[[[X]]] = 0$, ..., $d_{r-1}[[\ldots[X]\ldots]] = 0$. If $d_r[[\ldots[X]\ldots]] \neq 0$, the γ-cocycle X cannot be lifted $(r+1)$ times and it is, along with its successive lifts, not exact in $H(\gamma|d)$.

5. A necessary and sufficient condition for an element Y in $H(\gamma)$ to be exact in $H(\gamma|d)$ is that there exists a k such that $d_i[\ldots[Y]\ldots] = 0$, $(i = 1, 2, \ldots, k - 1)$ and $[\ldots[Y]\ldots] = d_k[\ldots[Z]\ldots]$. This implies in particular $d_j[\ldots[Y]\ldots] = 0$ for all j's.

6. Conversely, if a γ-cocycle Y fulfills $d_i[\ldots[Y]\ldots] = 0$ for $i = 0, 1, \ldots, k-1$ and $d_k[\ldots[Y\ldots] \neq 0$, then, it is not exact in $H(\gamma|d)$. The condition

is not necessary, however, because there are elements of $H(\gamma)$ that are non trivial in $H(\gamma|d)$ but which are annihilated by all d_i's. This is due to the fact that there are no exterior form of degree higher than the spacetime dimension. We shall come back to this point below.

The meaning of the integer k for which $Y = d_k Z$ in item 5 (with $Y \neq d_i$(something) for $i < k$) is as follows (we shall drop the multiple brackets when no confusion can arise). If the γ-cocycle a is in $\mathrm{Im}\, d_0$, then $a = db + \gamma c$, where b is also a γ-cocycle. If a is a non-zero element of E_1 in the image of d_1, then again $a = db + \gamma c$, but b is now *not* a cocycle of γ since a would then be in $\mathrm{Im}\, d_0$ and thus zero in E_1. Instead, one has $\gamma b + d\beta = 0$ where β *is* a cocycle of γ ($\gamma\beta = 0$) which is not trivial in $H(\gamma|d)$. More generally, k characterizes the length of the descent below b in $a = db + \gamma c$, $\gamma b + d\beta = 0$ etc.

The proof of items 1 to 4 proceeds recursively. Assume that the differential algebras (E_i, d_i) have been constructed up to order $r-1$, with the properties 2 through 4. Then, one defines the next space E_r as in 1. Let x be an element of E_r, and let X be one of the γ-cocycles such that the class $[[\ldots [X] \ldots]]$ in E_r is precisely x. Since X can be lifted r times, one has a sequence $dX + \gamma c_1 = 0$, ..., $dc_{r-1} + \gamma b = 0$. The ambiguity in X is $X \to X + \gamma a + du_0 + du_1 + \cdots + du_{r-1}$, where u_0 is a γ-cocycle (this is the d_0-exact term), u_1 is the first lift of a γ-cocycle (this is the d_1-exact term) etc. Setting $u = u_0 + u_1 + \cdots u_{r-1}$, one sees that the ambiguity in X is of the form $X \to X + \gamma a + du$. On the other hand, the ambiguity in the successive lifts takes the form $c_1 \to c_1 + m_1$, where m_1 is a γ-cocycle that can be lifted $r-1$ times, $c_2 \to c_2 + n_1 + m_2$, where n_1 descends on m_1 and m_2 is a γ-cocycle that can be lifted $r-2$ times, ..., and finally $b \to b + a_1 + a_2 + \cdots + a_{r-1} + a_r$, where a_1 descends $(r-1)$ times, on m_1, a_2 descends $(r-2)$ times, on m_2, etc, and a_r is a γ-cocycle.

The element $X_r \equiv db$ is clearly a cocycle of γ, which is annihilated by d_0 and the successive derivations d_k because $dX_r = 0$ exactly and not just up to γ-exact terms. The ambiguity in the successive lifts of X plays no rôle in the class of X_r in E_r, since it can (suggestively) be written $db \to db + d_{r-1}m_1 + d_{r-2}m_2 + \cdots + d_1 m_{r-1} + d_0 a_r$. Thus, the map d_r is well-defined as a map from E_r to E_r. It is clearly nilpotent since $dX_r = 0$. It is also a derivation, because one may rewrite the lift equations for X as $\tilde{\gamma}(X + c_1 + c_2 + \cdots + b) = d_r X$ where

$$\tilde{\gamma} = \gamma + d. \tag{4.20}$$

Let Y be another element of E_r and e_1, e_2, ...β its successive lifts. Then, $\tilde{\gamma}(Y + e_1 + e_2 + \cdots + \beta) = d_r Y$. Because $\tilde{\gamma}$ is a derivation, one has $\tilde{\gamma}[(X +$

$c_1 + \cdots + b)(Y + e_1 + \cdots + \beta)] = (d_r X)Y + (-1)^{\epsilon_X} X d_r Y +$ forms of higher form-degree, which implies $d_r(XY) = (d_r X)Y + (-1)^{\epsilon_X} X d_r Y$: d_r is also an odd derivation and thus a differential. This establishes properties 2 and 3.

To prove property 4, one observes that X can be lifted once more if and only if one may choose its successive lifts such that db is γ-exact. This is equivalent to stating that $d_r X$ is zero in E_r. Properties 5 and 6 are rather obvious: if a is a γ-cocycle which is exact in $H(\gamma|d)$, $a = db + \gamma c$, then $a = d_k m$ where k is the length of the descent associated with $\gamma b + dn = 0$, which has bottom m.

As shown in [57, 58], the above construction may be elegantly captured in an exact couple [59]. The detailed analysis of this exact couple and the proof of the above results using directly the powerful tools offered by this couple may be found in [58, 60, 61].

One has, for each r, the vector space isomorphisms,

$$E_r \simeq \operatorname{Ker} d_r \oplus F_r \simeq \operatorname{Im} d_r \oplus E_{r+1} \oplus F_r, \qquad (4.21)$$

where F_r is a subspace supplementary to $\operatorname{Ker} d_r$ in E_r. Thus,

$$E_0 \simeq \oplus_{k=0}^{k=r-1} F_k \oplus_{k=0}^{k=r-1} \operatorname{Im} d_k \oplus E_r. \qquad (4.22)$$

Because there is no form of degree higher than the spacetime dimension, $d_n = 0$ ($d_n a$ has form-degree equal to $FormDeg(a) + n + 1$). Therefore, $E_n = E_{n+1} = E_{n+2} = \ldots$. This implies

$$E_0 \simeq \oplus_{k=0}^{k=n-1} F_k \oplus_{k=0}^{k=n-1} \operatorname{Im} d_k \oplus E_n. \qquad (4.23)$$

The elements in any of the F_k's are non trivial bottoms of the descent which can be lifted exactly k times. All the elements above them in the descent are also non trivial solutions of the Wess-Zumino consistency condition. The elements in $\operatorname{Im} d_k$ are bottoms which are trivial in $H(\gamma|d)$ and which therefore define trivial solutions of the Wess-Zumino consistency condition. Finally, the elements in E_n are bottoms that can be lifted all the way up to form degree n. These are non trivial in $H(\gamma|d)$, since they are not equal to $d_i m$ for some i and m. The difference between the elements in $\oplus F_k$ and those in E_n is that the former cannot be lifted all the way up to form-degree n: one meets an obstruction before, which is $d_k a$ (if $a \in F_k$). By contrast, the elements in E_n can be lifted all the way up to form degree n. This somewhat unpleasant distinction between γ-cocycles that are non-trivial in $H(\gamma|d)$ will be removed below, where we shall assign an obstruction to the elements of E_n in some appropriate higher dimensional space.

In order to solve the Wess-Zumino consistency condition, our task is now to determine explicitly the spaces E_r and F_r.

4.2.5 Invariant Poincaré lemma – Small algebra – Universal Algebra

To that end, we first work out the cohomology of d_0 in $E_0 \equiv H(\gamma)$. Let u be a γ-cocycle. Without loss of generality, we may assume that u takes the form

$$u = \sum P_I \omega^I \tag{4.24}$$

where the ω^I are polynomials in the last undifferentiated ghosts of ghosts $C_{p_a}^a$ and where the P_I are polynomials in the field strength components and their derivatives, with coefficients that involve dx^μ and x_μ. The P_I are called "gauge-invariant polynomials". A direct calculation using the fact that the d-variation of the last ghosts is γ-exact yields $du = \sum (dP_I)\omega^I + \gamma v'$. The r.h.s of the previous equation is γ-exact if and only if $dP_I = 0$.

The first consequence is that F_0 is isomorphic to the space of polynomials of the form,

$$F_0 \simeq \{u = \sum P_I \omega^I \text{with } dP_I \neq 0 \text{ and } P_I \sim P_I + dQ_I\}. \tag{4.25}$$

F_0 exhausts all the solutions of the Wess-Zumino consistency condition in form degree $< n$ which descend trivially.

Now, if $P_I = dR_I$ where R_I is also a gauge invariant polynomial, then u is d-exact modulo γ, $u = da + \gamma b$, with $\gamma a = 0$. Conversely, if $u = da + \gamma b$ with $\gamma a = 0$, then P_I is d-exact in the space of invariant polynomials. Thus, the class of u (in E_0) is a non trivial cocycle of d_0 if and only if P_I is a non trivial cocycle of the *invariant* cohomology of d which we now calculate.

Since we are interested in lifts of γ-cocycles from form-degree k to form-degree $k + 1$, we shall investigate the d-invariant cohomology only in form-degree strictly smaller than the spacetime dimension n. This will be assumed throughout the remainder of this section. [In form-degree n, there is clearly further invariant cohomology since any invariant n-form is annihilated by d, even when it cannot be written as the d of an invariant form].

Invariant Poincaré Lemma

In the literature, the result covering the invariant cohomology of d is known as the invariant Poincaré Lemma. It is contained in the following theorem which we formulate in the presence of antifields because we will also apply it further on the antifield dependent solutions of the Wess-Zumino consistency condition.

Theorem 6. *Let $\mathcal{H}^k$ be the subspace of form degree k of the finite dimensional algebra $\mathcal{H}$ of polynomials in the curvature $(p_a + 1)$-forms H^a, $\mathcal{H} = \oplus_k \mathcal{H}^k$. One has*

$$H_j^{k,inv}(d) = 0, \ k < n, \ j > 0 \tag{4.26}$$

and

$$H_0^{k,inv}(d) = \mathcal{H}^k, \ k < n. \tag{4.27}$$

Thus, in particular, if $a = a(\chi)$ with $da = 0$, antighost $a > 0$ and deg $a < n$ then $a = db$ with $b = b(\chi)$. And if a has antighost number zero, then $a = P(H^a) + db$, where $P(H^a)$ is a polynomial in the curvature forms and $b = b([H])$.

This generalizes the result established for 1-forms in [62, 63, 49].

Proof: Let us first show that in antighost number > 0 the invariant cohomology of d is trivial. Let α be a solution of $d\alpha = 0$ with $\alpha = \alpha(\chi)$. We decompose α according to the number of derivatives of the antifields:

$$\alpha = \alpha_0 + \ldots + \alpha_k. \tag{4.28}$$

With d written as $d = \overline{d} + \overline{\overline{d}}$ where $\overline{d}$ acts only on the antifields and $\overline{\overline{d}}$ on all the others variables, equation $d\alpha = 0$ then implies $\overline{d}\alpha_k = 0$. According to Theorem **5** we have, $\alpha_k = \overline{d}\beta_{k-1}(\chi)$ (the fact that all the antifields and their derivatives appear in the χ is crucial here) and therefore by redefining in α the terms of order less than k, one can get rid of α_k, so that $\alpha = \alpha_0 + \ldots + \alpha_{k-1} + d\beta_{k-1}$. In the same way, one shows that all the α_i up to α_1 can be removed from α by adding the d-exact term $d\beta_{i-1}$. Finally, α_0 has to vanish because the condition $\overline{d}\alpha_0 = 0$ implies $\alpha_0 = \alpha_0(x^\mu, [H])$ and we are in antighost > 0.

We now prove that in antighost number 0 the invariant cohomology of d is exhausted by the polynomials in the curvatures. We first establish the result for one p-form and then extend the analysis to an arbitrary system of p-forms.

So let us start with one p-form. It can be either of odd or even degree. Let us begin with the odd-case. Because we have $dP_J = 0$ and

$\gamma P_J = 0$ we can build a descent equation as follows:

$$dP_J^k = 0 \Rightarrow P_J^k = da_0^{k-1} + K \tag{4.29}$$

$$0 = \gamma a_0^{k-1} + da_1^{k-2} \tag{4.30}$$

$$\vdots \tag{4.31}$$

$$0 = \gamma a_{j-1}^{k-j} + da_j^{k-j-1} \tag{4.32}$$

$$0 = \gamma a_j^{k-j-1}, \tag{4.33}$$

where K is a constant. In the case of one p-form of odd degree, the last equation of the descent and Theorem **3** tell us that $a_j^{k-j-1} = a_p^{k-p-1} = M_J C_p$ where M_J is a polynomial in the field strength components and their derivatives. If we substitute this in (4.32) we obtain $dM_J C_P + \gamma(a_{j-1}^{k-j} - M_J C_{p-1}) = 0$. This implies $dM_J = 0$ (Theorem **3** again). Because the form degree of M_J is strictly less than k (the form degree of P_J), we make the recurrence hypothesis that the theorem holds for M_J, i.e, $M_J = M_J(H)$. a_p^{k-p-1} then lifts with no ambiguity (except for γ and d exact terms which are irrelevant) up to $a_1^{k-2} = M_J(H)C_1$. Equation (4.30) then implies $a_0^{k-1} = M_J(H)B + R_J$ where R_J is a polynomial in the field strength components and their derivatives. Therefore we have $P_J^k = M_J(H)H + dR_J$ which proves the theorem for one p-form of odd degree.

In the case where the p-form is of even degree the proof proceed as follows. We first construct the same descent as previously with a_j^{k-j-1} this time of the form $a_j^{k-j-1} = a_{pl}^{k-pl-1} = M_J C_p^l$. Just as in the the odd case, (4.32) implies $dM_J = 0$. We thus make the recurrence hypothesis $M_J = M_J(H) = \alpha H + \beta$ where α and β are constants. Therefore, $a_{pl}^{k-pl-1} = (\alpha H + \beta)C_p^l$. We then note that $\alpha H C_p^l$ is γ-exact modulo d. One can see this by using the horizontality condition (3.40). Indeed, $\tilde{\gamma}(\alpha \tilde{B}^{l+1}) = (l+1)\alpha H \tilde{B}^l$ which implies $(l+1)\alpha H C_p^l = \gamma[\alpha \tilde{B}^{l+1}]_{pl-1}^{p+1} + d[\alpha \tilde{B}^{l+1}]_{pl}^p$. Thus we may suppose that $a_{pl}^{k-pl-1} = a_{pl}^0 = \beta C_p^l$. Let us now show that if $l > 1$ then $\beta = 0$. In that case, the bottom a_{pl}^0 can be lifted without any ambiguity up to $a_{p(l-1)+1}^{p-1} = [\beta \tilde{B}^l]_{p(l-1)+1}^{p-1}$ (using Eq. (3.40) again). The next equation of the descent then yields: $a_{p(l-1)}^p = [\beta \tilde{B}^l]_{p(l-1)}^p + R_J C_p^{l-1}$ where R_J is a polynomial in the field strength components and their derivatives. Substituting this into $\gamma a_{p(l-1)-1}^{p+1} + da_{p(l-1)}^p = 0$ we get $\beta l H C_p^{l-1} + dR_J C_p^{l-1} + \gamma(a_{p(l-1)-1}^{\prime p+1}) = 0 \Rightarrow \beta l H + dR_J = 0$. Thus, using our recurrence hypothesis, we see that for $l > 1$ we necessarily have $\beta = 0$. If $l = 1$, the above obstruction is not present. The bottom βC_p then yields $P_J^k = k + \beta H + dR_J$ which proves the theorem for the even-case.

Let us finally prove the theorem for an arbitrary system of p-forms. We label one of the p-form with A and the rest with B and decompose P_J according to the number of derivatives of the field strengths of the p-form labeled by A: $P_J = P_0 + \ldots + P_k$. Because $dP_J = 0$ we have $d_A P_k = 0$, where d_A only acts on the fields of the sector labeled by A. Using the theorem in the case of a single p-form we get $P_k = d_A R_{k-1} + V$ where V is only present for $k = 0$ because it only depends on the sector A through H_A and on the sector B through the field strengths and their derivatives. Thus, except when $k = 0$, P_k can be removed from P_J by subtracting the coboundary $dR_{k-1} \Rightarrow P_J = V$ up to trivial terms. We now expand V in powers of H_A: $V = \sum (H_A)^k v_k$. The condition $dV = 0$ then implies $dv_k = 0$. By induction on the number of p-forms occurring in v_k we obtain the desired result. $\square$

If the local forms are not taken to be explicitly x-dependent, Equation (4.27) must be replaced by

$$H_0^{k,inv}(d) = (\Lambda \otimes \mathcal{H})^k. \tag{4.34}$$

Small Algebra

From now on, we restrict our attention to the algebra of x-dependent space-time forms $\mathcal{P}$. In Section **4.2.7** we comment on how the results are affected when the analysis is pursued in $\mathcal{P}_-$.

Theorem **6** implies, according to the general analysis of the descent equation given above, that the only bottoms u ($\gamma u = 0$) that can be lifted at least once can be expressed in terms of exterior products of the curvature forms H^a and the last ghosts of ghosts (up to trivial redefinitions). Out of the infinitely many generators of $H(\gamma)$, only H^a and $C_{p_a}^a$ survive in E_1.

Because the objects that survive the first step in the lift can be expressed in terms of forms, it is convenient to introduce the so-called "small algebra" $\mathcal{A}$ generated in the exterior product by the exterior forms, B^a, H^a, $C_{p_a-k}^a$ and $dC_{p_a-k}^a$ ($k = 0, ..., p_a - 1$). This algebra is stable under γ and d. Denoting by E_0^{small} the cohomology of γ in the small algebra, one finds

$$E_0^{small} \equiv H(\gamma, \mathcal{A}) \simeq \mathcal{B} \tag{4.35}$$

where $\mathcal{B}$ is the subalgebra of $\mathcal{A}$ generated by the curvatures H^a and the last ghosts of ghosts $C_{p_a}^a$.

One also defines E_1^{small} as $H(d_0^{small}, E_0^{small})$, where d_0^{small} is the restriction of d_0 to E_0^{small}. Because $dH^a = 0$ and $dC_{p_a}^a = \gamma(\text{something})$, the restriction

d_0^{small} identically vanishes. Thus

$$E_1^{small} \simeq E_0^{small} \simeq \mathcal{B}. \tag{4.36}$$

What is the relationship between E_1^{small} and E_1? These two spaces are in fact isomorphic,

$$E_1 \simeq E_1^{small}. \tag{4.37}$$

Indeed, let q be the map from E_1^{small} to E_1 that assigns to a cohomological class in E_1^{small} its cohomological class in E_1 ($a \in E_1^{small} \simeq \mathcal{B}$ fulfills $\gamma a = 0$ and $d_0 a = 0$ and so defines of course an element of E_1). It follows from the above theorem that the map q is surjective since any class in E_1 possesses a representative in the small algebra. The map q is also injective because there is no non trivial class in E_1^{small} that becomes trivial in E_1. If the small-algebra γ-cocycle $r = \sum P_I \omega^I$ with $P_I, \omega^I \in \mathcal{B}$ can be written as $r = du + \gamma t$ with u and v in the big algebra and u a γ-cocycle, then r is actually zero. Indeed, if $\gamma u = 0$ we have, $u = Q_I \omega^I + \gamma m$. This implies, $P_I \omega^I = dQ_I \omega^I + \gamma t'$ and thus $P_I = dQ_I$ which is impossible according to Theorem **6**.

In fact, it is easy to see that we also have $E_k \simeq E_k^{small}$ for each $k > 1$. Indeed, suppose the result holds for E_{k-1}^{small} and E_{k-1} and let q be the bijective map between these two spaces. If $a \in E_{k-1}^{small}$ than there exists $c_1, \ldots, c_{k-1} \in \mathcal{A}$ such that $da + \gamma c_1 = 0, \ldots, dc_{k-2} + dc_{k-1} = 0$ and we have by definition,

$$d_{k-1}^{small}[a]_{E_{k-1}^{small}} = [dc_{k-1}]_{E_{k-1}^{small}}. \tag{4.38}$$

Furthermore, because $E_{k-1}^{small} \simeq E_{k-1}$, a also represents a non trivial class of E_{k-1} and we may choose as its successive $k-1$ lifts the previous $c_1, \ldots, c_{k-1}$. So by definition,

$$d_{k-1}[a]_{E_k} = [dc_{k-1}]_{E_k}. \tag{4.39}$$

Equation (4.38) and (4.39) imply that the differentials d_{k-1}^{small} and d_{k-1} are mapped on each other through the isomorphism, $q d_{k-1}^{small} = d_{k-1} q$. Therefore $E_k^{small} = H(d_{k-1}^{small}, E_{k-1}^{small}) \simeq H(d_{k-1}, E_{k-1}) = E_{k-1}$ which proves the result.

By virtue of this result, one can equivalently compute the spaces E_k^{small} instead of the spaces E_k in order to obtain the elements of $H(\gamma)$ which can be lifted k times and which are not d_k-exact.

What about the relationship between F_k^{small} and F_k for $k > 0$? Suppose $a \in F_k^{small}$. This means that even when taking into account the ambiguities in the definitions of $c_1, \ldots, c_k$ in the *small algebra* one does not have $dc_k' + \gamma c_{k+1} = 0$. One may ask whether or not the obstruction to the lift of c_k can

be removed when the ambiguities are not restricted to the small algebra? The answer is negative for the following reason. The ambiguities in any of the c_i ($i < k$) have to be lifted at least once; so up to trivial terms they can be supposed to be in the small algebra as well as their successive lifts. This implies that the ambiguity in c_k is $c_k \rightarrow c_k + m + u_0$ with m in $\mathcal{A}$ and u_0 a γ-closed term. c_k can be lifted if it is possible to adjust m and u_0 so that $d(c_k + m) + du_0 = \gamma r$. However, the same argument used in the proof of the injectivity of the map q from E_1^{small} to E_1 shows that this is impossible.

We can summarize the above discussion in the following theorem:

Theorem 7. *There is no loss of generality in investigating in the small algebra the solutions of the Wess-Zumino consistency condition that descend non trivially.*

Universal Algebra

The small algebra $\mathcal{A}$ involves only exterior forms, exterior products and exterior derivatives. It does "remember" the spacetime dimension since its generators are not free: any product of generators with form-degree exceeding the spacetime dimension vanishes.

It is useful to drop this relation and to work in the algebra freely generated by the potentials, the ghosts and their exterior derivatives with the sole condition that these commute or anti-commute (graded commutative algebra) but without imposing any restriction on the maximally allowed form degree [58, 64]. This algebra is called the universal algebra and denoted by $\mathcal{U}$. In this algebra, the cohomology of d is trivial in all form-degrees and the previous theorems on the invariant cohomology of d are also valid in form-degree $\geq n$. Furthermore, one can sharpen the condition for a cocycle in $H(\gamma)$ to be non trivial in $H(\gamma|d)$.

Theorem 8. *A necessary and sufficient condition for $X \in H(\gamma)$ to be non-trivial in $H(\gamma|d)$ is that there exists r such that $d_r X \neq 0$. That is, the lift of X must be obstructed at some stage. (For the equation $d_r X \neq 0$ to make sense, $d_i X$ must vanish for $i < r$. Here also we denote by the same letter $X \in E_0$ and its representative in E_r).*

Proof: The decomposition of E_n is now non-trivial since da does not necessarily vanish even when a is a n-form. Thus, d_n is not necessarily zero and the procedure of lifting can be pursued above form-degree n. Suppose that one encounters no obstruction in the lifting of X. That is, one can go all the way up to ghost number zero, the last two

equations being $dc_k + \gamma b = 0$ (with b of ghost number zero) and $db = 0$ (so b lifts to zero). Then, one can write $b = dm$ since the cohomology of d is trivial in any form-degree in the universal algebra $\mathcal{U}$ (except for the constants, which cannot arise here since b involves the fields). The triviality of the top-form b implies the triviality in $H(\gamma|d)$ of all the elements below it. Thus, a necessary condition for the bottom to be non trivial in $H(\gamma|d)$ is that one meets an obstruction in the lift at some stage. The condition is also clearly sufficient. $\square$

One can summarize our results as follows:

Theorem 9. *(Generalized "transgression" lemma) Let $X \in E_0$ be a non-trivial element of $H(\gamma|d)$. Then there exists an integer r such that $d_i X = 0$, $i < r$ and $d_r X = Y \neq 0$. The element Y is defined through the chain $dX + \gamma c_1 = 0$, ..., $dc_{r-1} + \gamma c_r = 0, dc_r + \gamma c_{r+1} = Y$, where the elements $c_i \in \mathcal{U}$ ($i = 1, r + 1$) are chosen so as to go all the way up to c_{r+1}. One has $\gamma Y = 0$ and Y should properly viewed as an element of E_r (reflecting the ambiguities in the lift). One calls the obstruction Y to a further lift of X the (generalized) "transgression" of X. The element X and its transgression have opposite statistics.*

This is the direct generalization of the analysis of [58] to the case of p-forms. "Primitive elements" of E_0 are those that have form-degree zero and for which the transgression has ghost number zero, i.e., they are the elements that can be lifted all the way up to ghost number zero ("that can be transgressed"). We refer to [58, 65, 66] for more background information applicable to the Yang-Mills case.

Because the space E_n and the next ones can be further decomposed in the universal algebra,

$$E_n \simeq \operatorname{Im} d_n \oplus E_{n+1} \oplus F_n, \ E_{n+1} \simeq \text{etc} \tag{4.40}$$

where the decomposition for a given γ-cocycle ultimately ends at form-degree equal to the ghost number, one has

$$E_0 \simeq \oplus_{k=0}^{\infty} F_k \oplus_{k=0}^{\infty} \operatorname{Im} d_k. \tag{4.41}$$

4.2.6   Results

We now compute the spaces E_k for the system of free p-forms.

Let $0 < p_1 < p_2 < \cdots < p_M$ be the form degrees of the gauge potentials B^a. We denote by $B_1^{a_1}$ the forms of degree p_1, $B_2^{a_2}$ the forms of degree p_2 etc.

The first non-vanishing differential (in E_0^{small}) is d_{p_1} so that $E_0^{small} = E_1 = E_2 = ... = E_{p_1}$. Any bottom in E_0^{small} can be lifted at least p_1 times. In E_1, the differential d_{p_1} acts as follows,

$$d_{p_1} C_{p_1}^{a_1} = H_1^{a_1}, \; d_{p_1} H_1^{a_1} = 0, \tag{4.42}$$

in the sector of the forms of degree p_1 and,

$$d_{p_1} C_{p_k}^{a_k} = 0, \; d_{p_1} H_k^{a_k} = 0, \; k > 1 \tag{4.43}$$

in the other sectors. The form of the differential d_{p_1} makes explicit the contractible part of (E_{p_1}, d_{p_1}). The variables $C_{p_1}^{a_1}$ and $H_1^{a_1}$ are removed from the cohomology, so that E_{p_1+1} is isomorphic to the algebra generated by the curvatures $H_k^{a_k}$ of form-degree $> p_1 + 1$ and the last ghosts of ghosts of ghost number $> p_1$.

A subspace F_{p_1} complementary to $\operatorname{Ker} d_{p_1}$ is easily constructed. In fact, a monomial in $C_{p_1}^{a_1}$ and $H_1^{a_1}$ is defined by a tensor $f_{a_1...a_k b_1...b_m}$ which is symmetric (respectively antisymmetric) in $a_1, \ldots, a_k$ and antisymmetric (respectively symmetric) in $b_1, \ldots, b_m$ if the last ghosts of ghosts are commuting (respectively anticommuting). For definiteness, suppose that the $H_1^{a_1}$ are anticommuting and the $C_{p_1}^{a_1}$ commuting. The irreducible components of $f_{a_1...a_k b_1...b_m}$ are then of the two following Young-symmetry types,

so that the polynomial in $C_{p_1}^{a_1}$ and $H_1^{a_1}$ can be written as,

$$f_{a_1...a_k b_1...b_m} C_{p_1}^{a_1} \ldots C_{p_1}^{a_k} H_1^{b_1} \ldots H_1^{b_m} = f^{(1)}_{a_1...a_k b_1...b_m} C_{p_1}^{a_1} \ldots C_{p_1}^{a_k} H_1^{b_1} \ldots H_1^{b_m} \tag{4.44}$$

$$+ f^{(2)}_{a_1...a_k b_1...b_m} C_{p_1}^{a_1} \ldots C_{p_1}^{a_k} H_1^{b_1} \ldots H_1^{b_m}.$$

The first term on the r.h.s (4.44) is annihilated by d_{p_1} while the second term is not and therefore defines an element of F_{p_1}. The space F_{p_1} can be taken to be the space generated by the monomials of this symmetry type (not annihilated by d_{p_1}), tensored by the algebra generated by the curvatures and last ghosts of ghosts of higher degree. Together with their successive lifts, the elements in F_{p_1} provide all the non-trivial solutions of the Wess-Zumino

consistency condition which are involved in descents whose bottoms can be lifted exactly p_1 times.

In the case where the $H_1^{a_1}$ are commuting and the $C_{p_1}^{a_1}$ anticommuting one simply exchanges the rôles of a_i and b_j in the previous discussion.

Similarly, one finds that the next non-vanishing differential is d_{p_2}. The generators $C_{p_2}^{a_2}$ and $H_2^{a_2}$ drop from the cohomology of d_{p_2} while those of higher degree remain. A space F_{p_2} can be constructed along exactly the same lines as the space F_{p_1} above and characterizes the solutions of the Wess-Zumino consistency condition involved in descents whose bottoms can be lifted exactly p_2 times.

More generally, the non-vanishing differentials are d_{p_k}. They are defined (in E_{p_k}, which is isomorphic to the algebra generated by the curvatures of form-degree $> p_{k-1} + 1$ and the last ghosts of ghosts of ghost number $> p_{k-1}$) through

$$d_{p_k} C_{p_k}^{a_k} = H_k^{a_k}, \ d_{p_k} H_k^{a_k} = 0 \qquad (4.45)$$

and

$$d_{p_k} C_{p_j}^{a_j} = 0, \ d_{p_k} H_j^{a_j} = 0, \ j > k. \qquad (4.46)$$

The generators $C_{p_k}^{a_k}$ and $H_k^{a_k}$ disappear in cohomology. The subspace F_{p_k} is again easily constructed along the previous lines. Together with their successive lifts, the elements in F_{p_k} provide all the non-trivial solutions of the Wess-Zumino consistency condition which are involved in descents whose bottoms can be lifted exactly p_k times.

Along with F_0 (see (4.25)), the F_{p_i}'s constructed in this section provide in form degree $< n$ all the non-trivial γ-cocycle remain non-trivial as elements of $H(\gamma|d)$. Together with their successive lifts they form a complete set of representatives of $H(\gamma|d)$ involved in non-trivial descents.

Example: Let us illustrate this discussion in the case of the simple model with one free 1-form A and one free 2-form B considered in Section **4.2.2**. The space E_0^{small} is isomorphic to the space of polynomials in the curvature-forms F, H and the last ghosts of ghosts A_1, B_2. The differential d_0^{small} vanishes so $E_1 \simeq E_0^{small}$. One next finds $d_1 A_1 = F$, $d_1 F = 0$, $d_1 B_2 = 0$ and $d_1 H = 0$. The space E_2 is isomorphic to the space of polynomials in B_2 and H. One may take for F_1 the space of polynomials linear in A_1. These can be lifted exactly once, their lifts being linear in A and A_1,

$$a \in F_1 \Leftrightarrow a = A_1 \sum (B_2)^l F^k H^m \ \ (m = 0 \text{ or } 1). \qquad (4.47)$$

Then, one gets

$$da + \gamma b = 0, \tag{4.48}$$

with

$$b = \sum \left(A(B_2)^l F^k H^m + l A_1 B_1 (B_2)^{l-1} F^k H^m \right). \tag{4.49}$$

They cannot be further lifted since the obstruction $d_1 a = \sum (B_2)^l \; F^{k+1} H^m$ does not vanish. The above a's and b's are the most general solutions of the Wess-Zumino consistency condition involved in descents of length 1.

The differential d_2 in E_2 is given by $d_2 B_2 = H$, $d_2 H = 0$. Because $H^2 = 0$, one may take for F_2 the space of polynomials in B_2 only. For those, the descent reads,

$$\alpha = (B_2)^l \quad , \quad \gamma \alpha = 0,$$
$$\beta = l B_1 (B_2)^{l-1} \quad , \quad d\alpha + \gamma \beta = 0,$$
$$\lambda = l B (B_2)^{l-1} + \frac{l(l-1)}{2} (B_2)^{l-2} (B_1)^2 \quad , \quad d\beta + \gamma \lambda = 0. \tag{4.50}$$

The elements of the form α, β or λ are the most general solutions of the Wess-Zumino consistency condition involved in descents of length 2. With the solutions involved in descents of length 1 and those that do not descend (i.e., which are strictly annihilated by γ), they exhaust all the (antifield-independent) solutions of the Wess-Zumino consistency condition.

A straightforward consequence of our discussion is the following theorem, which will prove useful in the analysis of antifield dependent solutions.

Theorem 10. *Let ω be a γ-cocycle of the form*

$$\omega = \alpha(H_s^{a_s} \; C_{p_s}^{a_s}) \beta(C_{p_k}^{a_k}, H_k^{a_k}), \; k > s \tag{4.51}$$

where α vanishes if $H_s^{a_s}$ and $C_{p_s}^{a_s}$ are set equal to zero (no constant term) and which fulfills,

$$d_{p_s} \alpha = 0. \tag{4.52}$$

(i.e. the first possible obstruction in the lift of ω is absent). Then, ω is trivial in $H(\gamma|d)$.

Proof: The proof is direct: one has $\alpha = d_{p_s} \mu$ for some $\mu(H_s^{a_s}, C_{p_s}^{a_s})$ since d_{p_s} is acyclic in the space of the $\alpha(H_s^{a_s} C_{p_s}^{a_s})$ with no constant term. Thus ω is d_{p_s}-exact, $\omega = d_{p_s}(\mu\beta)$; so ω is the first obstruction to the further lift of $\mu\beta$ and as such, is trivial. $\square$

The theorem applies in particular when α is an arbitrary polynomial of strictly positive degree in the curvatures $H_s^{a_s}$.

4.2.7 Results in $\mathcal{P}_-$

In the algebra of x-independent spacetime forms $\mathcal{P}_-$, the analysis proceeds similarly as in $\mathcal{P}$.

Because of (4.34), we now define the small algebra $\mathcal{A}_-$ as the algebra generated by the exterior products of the forms H^a, B^a, $C^a_{p_a-k}$, $dC^a_{p_a-k}$ ($k = 0, \ldots p_a - 1$) but also dx^μ since we now need the constant forms. By exactly the same arguments used in $\mathcal{P}$, one shows that the isomorphisms $E^{small}_k \simeq E_k$, $k \geq 1$ still hold in $\mathcal{P}_-$ and that one cannot remove obstructions to lifts by going to the "big algebra". Therefore, Theorem **7** remains valid in $\mathcal{P}_-$.

The universal algebra is then defined as the algebra freely generated by the exterior products of the forms H^a, B^a, $C^a_{p_a-k}$, $dC^a_{p_a-k}$ ($k = 0, \ldots p_a - 1$) and dx^μ without any restriction on the maximally allowed form degree. In that algebra the cohomology of d and the invariant cohomology of d are respectively given by the constant forms and (4.34) (in all form-degrees including form-degrees $\geq n$). Theorems **8** and **9** are then proved as they are in $\mathcal{P}$.

Therefore, the calculation of the spaces E^{small}_k in $\mathcal{P}_-$ can proceed as in Section **4.2.6**. The only difference is that the $f_{a_1\ldots a_k b_1 \ldots b_m}$ in (4.44) are now constant forms instead of just constants.

Note that even in $\mathcal{P}_-$, the constant forms can be eliminated by requiring Lorentz invariance.

4.2.8 Counterterms and anomalies

In this section we summarize the above results by giving explicitly the anti-field-independent counterterms and anomalies: $H^{(n,0)}(\gamma|d)$ and $H^{(n,1)}(\gamma|d)$. These can be of two types: (i) those that descend trivially ("type A") and can be assumed to be strictly annihilated by γ; they are described by $H(\gamma)$ up to trivial terms; and (ii) those that lead to a non-trivial descent ("type B") and can be assumed to be in the small algebra modulo solutions of the previous type. For small ghost number, it turns out to be more convenient to determine the solutions of "type B" directly from the obstructions sitting above them rather than from the bottom. That this procedure, which works in the universal algebra, yields all the solutions, is guaranteed by the analysis of Section **4.2.5**.

The following results apply equally to $\mathcal{P}$ and $\mathcal{P}_-$ since for the counterterms and anomalies we impose Lorentz-invariance which eliminates the constant forms.

Counterterms and anomalies of type A

The counterterms that lead to a trivial descent involve in general the individual components of the gauge-invariant field strengths and their derivatives and cannot generically be expressed as exterior products of the forms F or H. They are the gauge-invariant polynomials introduced above and read explicitly,

$$a = a([H^a])d^n x. \tag{4.53}$$

In order to be non-trivial in $H(\gamma|d)$ the above cocycles must satisfy the condition $a \neq db$ which is equivalent to the condition that the variational derivatives of a with respect to the fields do not identically vanish. We have assumed that the spacetime forms dx^μ occur only through the product $dx^0 dx^1 \cdots dx^{n-1} \equiv d^n x$ as this is required by Lorentz-invariance.

The anomalies that lead to a trivial descent are sums of terms of the form $a = P C d^n x$ where P is a gauge-invariant polynomial and C is a last ghost of ghost with ghost number one. These anomalies exist only for a theory with 1-forms. One has explicitly

$$a = P_A([H^a])C_1^A \tag{4.54}$$

where A runs over the 1-forms. a will be trivial if and only if $P = dR$ where R is an invariant polynomial or if $P_A = P_A(H^a)$ with $P_A H^A = 0$. Indeed, suppose that a is trivial, i.e., $P_A C^A = \gamma c + de$. By acting with γ on this equation we see that e satisfies $\gamma e + dm = 0$. We can thus decompose e as the sum of an element of $H(\gamma|d)$ which descends trivially and a term v in the small algebra which is the lift of a γ-cocycle: $e = R_A([H^a])C_1^A + v$. This implies $de = dR_A C_1^A + Q_A(H^a)C_1^A + \gamma u$ and thus $P_A([H^a]) = dR_A([H^a]) + Q_A(H^a)$ where $Q_A H^A = 0$ (see Eq. (4.62) in the section on Anomalies of type B).

The existence of such anomalies - which cannot generically be expressed as exterior products of curvatures and ghosts - was pointed out in [67] for Yang-Mills gauge models with $U(1)$ factors.

Counterterms of type B

The solutions that lead to a non trivial descent can be assumed to live in the small algebra, i.e., can be expressed in terms of exterior products of the fields, the ghosts (which are all exterior forms) and their exterior derivatives (modulo solutions of type A). If a is a non-trivial solution of the Wess-Zumino consistency condition with ghost number zero, then $da \neq 0$ (in the universal algebra). Since a has ghost number zero, it is the top of the descent and da

is the obstruction to a further lift. Because da is a γ-cocycle, it is a gauge-invariant polynomial. It must, in addition, be d-closed but not d-exact in the space of gauge-invariant polynomials since otherwise, a could be redefined to be of type A. Therefore, da is an element of the invariant cohomology of d and it will be easier to determine a directly from the obstruction da rather than from the bottom of the descent because one knows the invariant cohomology of d.

Thus we may assume that,

$$da = P(H) = dQ(H, B) \tag{4.55}$$

where Q is linear in the forms B^a, and so up to trivial terms,

$$a = R_a(H^b)B^a. \tag{4.56}$$

Note that up to a d-exact term, one may in fact assume that a involves only the potentials B^a of the curvatures of smaller form-degree present in P. These are the familiar Chern-Simons terms, which exist provided one can match the spacetime dimension n with a polynomial in the curvatures H^a and the forms B^a, linear in B^a.

The whole descent associated with a is generated through the "Russian formula" [55, 50, 56, 20]

$$
\begin{align}
P &= \tilde{\gamma}a(H, \tilde{B}) \tag{4.57} \\
\tilde{\gamma} &= d + \gamma \tag{4.58} \\
\tilde{B}^a &= B^a + C_1^a + \cdots C_{p_a}^a, \tag{4.59}
\end{align}
$$

which follows from the "horizontability condition" (Eq. (3.40)),

$$\tilde{\gamma}\tilde{B}^a = H^a. \tag{4.60}$$

By expanding (4.57) according to the ghost number, one gets the whole tower of descent equations. The bottom takes the form $R_a(H^b)C_{p_a}^a$ and is linear in the last ghosts of ghosts associated with the forms of smaller form degree involved in P. That the bottoms should take this form might have been anticipated since these are the only bottoms with the right degrees that can be lifted all the way to form-degree n. The non-triviality of the bottom implies also the non-triviality of the whole tower.

It is rather obvious that the Chern-Simons terms are solutions of the Wess-Zumino consistency condition. The main result here is that these are the only solutions that descend non trivially (up to solutions of type A).

Anomalies of type B

The anomalies a of type B can themselves be of two types. They can arise from an obstruction that lives one dimension higher or from an obstruction that lives two dimensions higher. In the first case, the obstruction da has form degree $n + 1$ and ghost number 1. This occurs only for models with 1-forms since other models have no γ-cohomology in ghost number one. In the other case, the anomaly can be lifted once, $da + \gamma b = 0$. The obstruction db to a further lift is then a $(n + 2)$-form of ghost number 0.

In the first case, the obstruction da reads

$$da + \gamma(\text{something}) = P_A(H)C_1^A \tag{4.61}$$

where A runs over the 1-forms. The right-hand side of (4.61) is necessarily the d_1 of something. Indeed, it cannot be the d_k $(k > 1)$ of something, say m, since this would make m trivial: the first obstruction to the lift of m would have to vanish and m involves explicitly the variables of the 1-form sector (see Theorem **10** above). This implies

$$P_A(H)C_1^A = C_{AB}(H)H^A C_1^B, \quad C_{AB}(H) = -C_{BA}(H) \tag{4.62}$$

so that $P_A(H)C_1^A = d_1(\frac{1}{2}C_{AB}(H)C_1^A C_1^A$. One thus needs at least two 1-forms to construct such solutions. If $C_{AB}(H)$ involves the curvatures H^A of the 1-forms, it must be such that (4.62) is not zero. The anomaly following from (4.61) is

$$a = C_{AB}(H)B^A C_1^A \tag{4.63}$$

and the associated descent is generated through

$$C_{AB}(H)H^A C_1^A = \tilde{\gamma}(\frac{1}{2}C_{AB}(H)\tilde{B}^A \tilde{B}^B) \tag{4.64}$$

In the second case, the obstruction $P \in H^{inv}(d)$ is a polynomial in H^a of form-degree $n + 2$, which can be written $P = dQ$ with Q linear in the potentials associated to the curvatures of lowest degree present in P. The solution a and the descent are obtained from the Russian formula (4.57), exactly as for the counterterms,

$$a = R_a(H^b)C_1^a. \tag{4.65}$$

They are linear in the ghosts and exist only for models with forms of degree > 1 which are the only ones that can occur in P since otherwise a is either trivial or of type A. Indeed, if variables from the 1-form sector are present in P, then $P = d_1 a$ (if P is non trivial) and the descent has only two steps. But this really means that a is the bottom of the descent and is actually of type A.

4.2.9 Conclusions

In this section, we have derived the general solution of the antifield-independent Wess-Zumino consistency condition for models involving p-forms. We have justified in particular why one can assume that the solutions can be expressed in terms of exterior products of the fields, the ghosts (which are all exterior forms) and their exterior derivatives, *when these solutions appear in non trivial descents.* This is not obvious to begin with since there are solutions that are not expressible in terms of forms (those that descend trivially) and justify the usual calculations made for determining the anomalies. Once one knows that the solutions involved in non trivial descents can be expressed in terms of forms (up to solutions that descend trivially), one can straightforwardly determine their explicit form in ghost numbers zero and one. This has been done in the last section where all counterterms and anomalies have been classified. The counterterms are either strictly gauge invariant and given by (4.53) or of the Chern-Simons type (when available) and given by (4.56).

The anomalies are also either strictly annihilated by γ, or lead to a non-trivial descent. The first type generalizes the anomalies of Dixon and Ramon Medrano [67] and are given by (4.54). The more familiar anomalies with a non trivial descent are listed in Eqs. (4.63) and (4.65).

The method applies also to other values of the ghost number, which are relevant in the analysis of the antifield-dependent cohomology.

This result is also valid for more general lagrangians than (3.2). Indeed, if one adds to the lagrangian of free p-forms interactions which are gauge invariant ((4.53) or (4.56)), then the gauge transformations of the resulting theory are identical to those of the free theory. Therefore, the definition of the longitudinal exterior derivative γ is unchanged and the results are not modified.

As we will show in the next section, the natural appearance of exterior forms holds also for the characteristic cohomology: all higher order conservation laws are naturally expressed in terms of exterior products of field strengths and duals to the field strengths. It is this property that makes the gauge symmetry-deforming consistent interactions for p-form gauge fields expressible also in terms of exterior forms and exterior products.

4.3   Antifield dependent solutions

We now turn our attention to the antifield dependent solutions of the Wess-Zumino consistency condition.

4.3.1   Preliminary results

Any antifield dependent solution of equation (4.1) may be decomposed according to the antighost number $a_g^n = a_{g,0}^n + a_{g,1}^n + \ldots + a_{g,q}^n$ with $q \neq 0$.[2] When this is done, the Wess-Zumino consistency condition,

$$sa_g^k + db_{g+1}^{k-1} = 0, \tag{4.66}$$

splits as,

$$\gamma a_0 + \delta a_1 + db_0 = 0, \tag{4.67}$$
$$\gamma a_1 + \delta a_2 + db_1 = 0, \tag{4.68}$$
$$\vdots$$
$$\gamma a_{q-1} + \delta a_q + db_{q-1} = 0, \tag{4.69}$$
$$\gamma a_q + db_q = 0, \tag{4.70}$$

where we have dropped the indices labeling the ghost numbers and form degrees (which are fixed) and we have performed the decomposition $b_{g+1}^{k-1} = b_{g+1,0}^{k-1} + \ldots + b_{g+1,q}^{k-1}$ (because of the Algebraic Poincaré Lemma, we may assume that up to a d-exact term, the expansion of b_{g+1}^{k-1} according to the antighost number stops at order q).

These equations resemble the descent equations used in the analysis of antifield independent BRST cocycles. The bottom equation now defines an element of $H(\gamma|d)$ which we denote $\mathcal{E}_0$. In order to 'lift' this cocycle, one needs to analyze whether or not its δ-variation is trivial in $H(\gamma|d)$. To address this question we define the map $\delta_0 : \mathcal{E}_0 \to \mathcal{E}_0$ as follows,

$$\delta_0[a_q] = [\delta a_q], \tag{4.71}$$

where $[\ ]$ denotes the class in $H(\gamma|d)$. This map is well-defined because: 1) $\gamma \delta a_q = -\delta \gamma a_q = \delta db_q = -d\delta b_q$ (so δa_q is a γ mod d cocycle; 2) $\delta(\gamma r_q + dc_q) = -\gamma(\delta r_q) - d(\delta c_q)$ (so δ maps a coboundary on a coboundary). Furthermore

---

[2]In this section we limit our attention to solutions of the Wess-Zumino consistency condition in form degree n. In particular, this applies to consistent interactions, counterterms, anomalies and solutions related to the analysis of the gauge invariance of conserved currents. Other values of the form-degree are treated along the same lines.

$\delta_0^2 = 0$ so δ_0 is a differential and one can define its cohomology. Cocycles of δ_0 are elements of $H(\gamma|d)$ which can be lifted at least once. By contrast, if $\delta_0[a_q] \neq 0$ then one cannot lift a_q to construct the component of antighost number $q - 1$ of a_g^n. Furthermore, if $[a_q] = \delta_0[c_{q+1}]$ then one can eliminate a_q from a_g^n by the addition of trivial terms and the redefinition of the terms of lower antighost numbers. The interesting a_q involved in the construction of BRST-cocycles are therefore the representatives of $H(\mathcal{E}_0, \delta_0)$.

To analyze the next lifts, one can define a sequence of spaces $\mathcal{E}_r$ and differentials δ_r which satisfy similar properties as those encountered in the standard descent equations:

1. $\mathcal{E}_r \equiv H(\mathcal{E}_{r-1}, \delta_{r-1})$.

2. There exist a map $\delta_r : \mathcal{E}_r \to \mathcal{E}_r$ which is defined by $\delta_r[[\ldots [a_q] \ldots]] = [[\ldots [\delta a_{q-r}] \ldots]]$, for $[[\ldots [a_q] \ldots]] \in \mathcal{E}_r$ where $[[\ldots [\delta a_{q-r}] \ldots]]$ is the class of the γ mod d cocycle δa_{q-r} and where a_{q-r} is defined by the equations (4.67)-(4.70).

3. $\delta_r^2 = 0$.

4. A γ mod d cocycle a_q can be 'lifted' q times if and only if $\delta_0[a_q] = 0$, $\delta_1[[a_q]] = 0, \ldots, \delta_{q-1}[[\ldots [a_q] \ldots]] = 0$. Such a γ mod d cocycle and its successive lifts constitute the components of a BRST cocycle. If $\delta_r[[\ldots [a_q] \ldots]] \neq 0$ $(r < q)$, the γ mod d cocycle cannot be lifted $r + 1$ times and it is not an s mod d coboundary up to terms of lower antighost number.

5. One can eliminate a_q from a_g^n by the addition of trivial terms and the redefinition of terms of lower antighost numbers if and only if there exists a k such $\delta_i[[\ldots [a_q] \ldots]] = 0, (i = 0, \ldots, k-1)$ and $[[\ldots [a_q] \ldots]] = \delta_k[[\ldots [a_{q+k+1}] \ldots]]$.

The proof of these properties proceed as in the case of the standard descent equations.

According to these properties, the γ mod d cocycles which yield nontrivial BRST cocycles belong to $\mathcal{E}_r \equiv H(\mathcal{E}_{r-1}, \delta_r)$, $\forall r$. This condition seems awkward to work with since it implies the calculation of an infinite number of cohomologies $\mathcal{E}_r$. The reason for this is that the antighost number is not bounded. However, for a large class of theories such as linear theories and normal theories [27] the cohomologies $\mathcal{E}_r$ coincide for $r > k$ where k is a finite number. This result is a consequence of the fact that their characteristic cohomology (to be defined below) vanishes above a certain value of the antighost number.

There is thus a strong analogy between the analysis of antifield independent and antifield dependent solutions of the Wess-Zumino consistency condition. However, compared to the d_r, the operators δ_r are not antiderivations. Indeed they are maps in vector spaces $\mathcal{E}_r$ and not in algebras. For instance, if a_q and b_q are representatives of $H(\gamma|d) \equiv \mathcal{E}_0$ their product is not necessarily in $\mathcal{E}_0$. The product of two elements of $\mathcal{E}_r$ is therefore not an internal operation and one cannot even speak of derivations. Because of this particularity, we cannot calculate the spaces $\mathcal{E}_k$ as we did for the E_k. Instead we will have to lift the possible γ mod d cocycles a_q "by hand" and find those which can be lifted all the way up to produce BRST cocycles.

Before we do so, we begin by characterizing the "bottoms" a_q as much as possible. To this end, we first prove that without changing the class of a_g^k in $H(s|d)$ we can replace (4.70) by the simpler Eq. $\gamma a_q = 0$. To prove this, we use the following theorem:

Theorem 11. *Let a^k be a γ-closed solution of $da^k = \gamma b^{k+1}$ of form degree $k < n$ and antighost > 0. Then, $a^k = dc^{k-1} + \gamma b^k$, with c^{k-1} in $H(\gamma)$. In other words, the cohomology in antighost > 0 and form degree $< n$ of d in $H(\gamma)$ vanishes.*

Proof: Since a^k is γ-closed we have $a^k = P_J(\chi)\omega^J + \gamma m^k$. Equation $da^k = \gamma b^{k+1}$ then implies $dP_J\omega^J = \gamma(b^{k+1}+dm^k+P_J\hat{\omega}^J)$, where $d\omega^J+\gamma\hat{\omega}^J = 0$. Therefore we have $dP_J = 0$ and using Theorem **6**, $P_J = dQ_J$. Thus, $a^k = dQ_J\omega^J + \gamma m^k = d(Q_J\omega^J) + \gamma(Q_J\hat{\omega}^J + m^k)$ which proves the theorem. $\square$

Theorem **11** now allows us assume $\gamma a_q = 0$. Indeed, if we act on (4.70) with γ we can build a descent equation. However, the bottom of this descent satisfies the condition of Theorem **11** which implies that it is trivial in $H(\gamma|d)$ and thus that a_q cannot descend non-trivially. Therefore we have $a_q = P_J(\chi)\omega^J + \gamma m_q + dr_q = P_J(\chi)\omega^J + sm_q + dr_q - \delta m_q$. Because δm_q is of antighost number $q - 1$ we see that in each class of $H(s|d)$ there is a representative satisfying $a_q = P_J(\chi)\omega^J$.

The next equation we have to examine is (4.69). If we substitute a_q in this equation we obtain,

$$\delta P_J\omega^J + \gamma a_{q-1} + db_{q-1} = 0. \qquad (4.72)$$

By acting on this equation with γ we can once more build a descent equation for b_{q-1}. If $q - 1 > 0$ then again the bottom satisfies the conditions of

Theorem **11** and therefore $b_{q-1} = N_J\omega^J + \gamma h_{q-1} + dl_{q-1}$. If we substitute this form in (4.72) we obtain the condition,

$$\delta P_J + dN_J = 0. \tag{4.73}$$

If $q = 1$ then we have for the descent,

$$\delta P_J\omega^J + \gamma a_0 + db_0 = 0, \tag{4.74}$$

$$\gamma b_0 + dr_0 = 0, \tag{4.75}$$

$$\vdots \tag{4.76}$$

$$\gamma m_0 = 0. \tag{4.77}$$

According to our analysis of $H(\gamma|d)$, we know that b_0 can be written as $b_0 = N_J\omega^J + \overline{b}_0$, where $\overline{b}_0$ is an element of $H(\gamma|d)$ which does not descend trivially and which is in the small algebra. Thus we have $db_0 = dN_J\omega^J - \gamma(N_J\hat{\omega}^J) + R_J(H^a)\omega^J + \gamma c_0$, where $R_J(H^a)\omega^J + \gamma c_0 = d\overline{b}_0$. The condition (4.72) therefore becomes,

$$\delta P_J + dN_J + R_J(H^a) = 0. \tag{4.78}$$

In the algebra of x-independent forms $\mathcal{P}_-$, one easily obtains $R_J = 0$ by counting the number of derivatives of $H_{\mu\nu\rho}$. The above equation therefore reduces to (4.73). In the algebra of x-dependent forms $\mathcal{P}$, this is no longer true and we shall comment later on the consequences of this.

A solution of (4.73) is called trivial when it is of the form, $P_J = \delta M_J + dR_J$ where M_J and R_J are polynomials in the χ. Such solutions are irrelevant in the study of BRST cocycles because the corresponding a_q can be eliminated from a_g^n. Indeed we have, $a_q = (\delta M_J + dR_J)\omega^J = \delta(M_J\omega^J) + d(R_J\omega^J) + \gamma(R_J\hat{\omega}^J) = s(M_J\omega^J + R_J\hat{\omega}^J) + d(R_J\omega^J) - \delta R_J\hat{\omega}^J$ so we may assume that the expansion of a_g^n stops at order $q - 1$ in the antighost number.

We therefore need to solve equation (4.73) for invariant polynomials P_J and identify two solutions which differ by trivial terms of the form $\delta M_J + dR_J$.

These equivalence classes define the invariant δ mod d cohomology denoted $H^{inv}(\delta|d)$. This cohomology is related to the so-called characteristic cohomology and both will be calculated in the next section.

To summarize, we have shown in this section that the antifield dependent solutions of the Wess-Zumino consistency condition can be chosen of the form $a_q = P_J\omega^J$ where P_J has to be a non-trivial element of $H^{inv}(\delta|d)$ (for $q = 1$ this is only valid in the algebra of x-independent forms).

In the next Chapter we study the characteristic cohomology in detail and return to the calculation of $H(s|d)$ in Chapter **6**.

Chapter 5

Characteristic Cohomology

5.1 Introduction

The characteristic cohomology [68] plays a central role in the analysis of
any local field theory. The easiest way to define this cohomology, which is
contained in the so-called Vinogradov C-spectral sequence [38, 39, 69, 70, 43,
71], is to start with the more familiar notion of conserved current. Consider
a dynamical theory with field variables ϕ^i $(i = 1, \ldots, M)$ and Lagrangian
$\mathcal{L}(\phi^i, \partial_\mu \phi^i, \ldots, \partial_{\mu_1 \ldots \mu_k} \phi^i)$. The field equations read

$$\mathcal{L}_i = 0, \tag{5.1}$$

with

$$\mathcal{L}_i = \frac{\delta \mathcal{L}}{\delta \phi^i} = \frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \Big) + \cdots + (-1)^k \partial_{\mu_1 \ldots \mu_k} \Big(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1 \ldots \mu_k} \phi^i)} \Big). \tag{5.2}$$

A (local) conserved current j^μ is a vector-density which involves the fields
and their derivatives up to some finite order and which is conserved modulo
the field equations, i.e., which fulfills

$$\partial_\mu j^\mu \approx 0. \tag{5.3}$$

Here and in the sequel, $\approx$ means "equal when the equations of motion hold"
or, as one also says equal "on-shell". Thus, (5.3) is equivalent to

$$\partial_\mu j^\mu = \lambda^i \mathcal{L}_i + \lambda^{i\mu} \partial_\mu \mathcal{L}_i + \cdots + \lambda^{i\mu_1 \ldots \mu_s} \partial_{\mu_1 \ldots \mu_s} \mathcal{L}_i \tag{5.4}$$

for some $\lambda^{i\mu_1 \ldots \mu_j}$, $j = 0, \ldots, s$. A conserved current is said to be trivial if it
can be written as

$$j^\mu \approx \partial_\nu S^{\mu\nu} \tag{5.5}$$

for some local antisymmetric tensor density $S^{\mu\nu} = -S^{\nu\mu}$. The terminology does not mean that trivial currents are devoid of physical interest, but rather, that they are easy to construct and that they are trivially conserved. Two conserved currents are said to be equivalent if they differ by a trivial one. The characteristic cohomology in degree $n - 1$ is defined to be the quotient space of equivalence classes of conserved currents. One assigns the degree $n - 1$ because the equations (5.3) and (5.5) can be rewritten as $d\omega \approx 0$ and $\omega \approx d\psi$ in terms of the $(n-1)$-form ω and $(n-2)$-form ψ respectively dual to j^{μ} and $S^{\mu\nu}$.

One defines the characteristic cohomology in degree k $(k < n)$ along exactly the same lines, by simply considering other values of the form degree. So, one says that a local k-form ω is a cocycle of the characteristic cohomology in degree k if it is weakly closed,

$$d\omega \approx 0; \quad \text{"cocycle condition"} \tag{5.6}$$

and that it is a coboundary if it is weakly exact,

$$\omega \approx d\psi, \quad \text{"coboundary condition"} \tag{5.7}$$

just as it is done for $k = n - 1$. For instance, the characteristic cohomology in form degree $n - 2$ is defined, in dual notations, as the quotient space of equivalence classes of weakly conserved antisymmetric tensors,

$$\partial_{\nu} S^{\mu\nu} \approx 0, \ S^{\mu\nu} = S^{[\mu\nu]}, \tag{5.8}$$

where two such tensors are regarded as equivalent iff

$$S^{\mu\nu} - S'^{\mu\nu} \approx \partial_{\rho} R^{\rho\mu\nu}, \ R^{\rho\mu\nu} = R^{[\rho\mu\nu]}. \tag{5.9}$$

We shall denote the characteristic cohomological groups by $H^{k}_{char}(d)$.

Higher order conservation laws involving antisymmetric tensors of degree 2 or higher are quite interesting in their own right. In particular, conservation laws of the form (5.8), involving an antisymmetric tensor $S^{\mu\nu}$ have attracted a great deal of interest in the past [72] as well as recently [73, 74] in the context of the "charge without charge" mechanism developed by Wheeler [75]. Higher order conservation laws also enter the analysis of the BRST field-antifield formalism extension incorporating global symmetries [76, 77].

But as we have seen in section (4.3) the characteristic cohomology is also important because it appears as an important step in the calculation of the local BRST cohomology.

In this section we will carry out the calculation of the characteristic cohomology for a system of free p-form gauge fields. We give complete results

in degree $< n - 1$; that is, we determine all the solutions to the equation $\partial_\mu S^{\mu\nu_1\dots\nu_s} \approx 0$ with $s > 0$. Although we do not solve the characteristic cohomology in degree $n - 1$, we comment at the end of the section on the gauge invariance properties of the conserved currents and provide an infinite number of them, generalizing earlier results of the Maxwell case [78, 79, 80, 81]. The determination of all the conserved currents is of course also an interesting question, but it is not systematically pursued here for two reasons. First, for the free theories considered here, the characteristic cohomology $H^{n-1}_{char}(d)$ is infinite-dimensional and does not appear to be completely known even in the Maxwell case in an arbitrary number of dimensions. By contrast, the cohomological groups $H^k_{char}(d)$, $k < n - 1$, are all finite-dimensional and can explicitly be computed. Second, the group $H^{n-1}_{char}(d)$ plays no role either in the analysis of the consistent interactions of antisymmetric tensor fields of degree > 1, nor in the analysis of candidate anomalies if the antisymmetric tensor fields all have degree > 2.

An essential feature of theories involving p-form gauge fields is that their gauge symmetries are *reducible* (see (3.6)). General vanishing theorems have been established in [68, 38, 39, 27] showing that the characteristic cohomology of reducible theories of reducibility order $p-1$ vanishes in form degree strictly smaller than $n - p - 1$. Accordingly, in the case of p-form gauge theories, there can be a priori non-vanishing characteristic cohomology only in form degree $n - p_M - 1$, $n - p_M$, etc, up to form degree $n - 1$ (conserved currents). In the 1-form case, these are the best vanishing theorems one can prove, since a set of free gauge fields A^a_μ has characteristic cohomology both in form degree $n - 1$ and $n - 2$ [27]. Representatives of the cohomology classes in form degree $n - 2$ are given by the duals to the field strengths, which are indeed closed on-shell due to Maxwell equations.

Our main result is that the general vanishing theorems of [68, 38, 39, 69, 70, 27] can be considerably strengthened when $p > 1$. For instance, if there is a single p-form gauge field and if $n - p - 1$ is odd, there is only one non-vanishing group of the characteristic cohomology in degree $< n - 1$. This is $H^{n-p-1}_{char}(d)$, which is one-dimensional. All the other groups $H^k_{char}(d)$ with $n - p - 1 < k < n - 1$ happen to be zero, even though the general theorems of [68, 38, 39, 69, 70, 27] left open the possibility that they might not vanish. The presence of these additional zeros give p-form gauge fields and their gauge transformations a strong rigidity.

Besides the standard characteristic cohomology, one may consider the invariant characteristic cohomology, in which the local forms ω and ψ occurring in (5.6) and (5.7) are required to be invariant under the gauge transformations (3.6). This is the relevant cohomology for the resolution of the Wess-Zumino consistency condition. We also completely determine in this

section the invariant characteristic cohomology in form degree $< n - 1$.

Our method for computing the characteristic cohomology is based on the reformulation performed in [27] of the characteristic cohomology in form degree k in terms of the cohomology $H^n_{n-k}(\delta|d)$ of the Koszul-Tate differential δ modulo the spacetime exterior derivative d. Here, n is the form degree and $n - k$ is the antighost number.

This section is organized as follows. First, since the calculation of the characteristic cohomology is rather long and intricate, we begin by formulating precisely our main results, which state (i) that the characteristic cohomology $H^k_{char}(d)$ with $k < n - 1$ is generated (in the exterior product) by the exterior forms $\overline{H}^a$ dual to the field strengths H^a; these are forms of degree $n - p_a - 1$; and (ii) that the invariant characteristic cohomology $H^{k,inv}_{char}(d)$ with $k < n - 1$ is generated (again in the exterior product) by the exterior forms H^a and $\overline{H}^a$. Then, we recall the relation between the characteristic cohomology and the Koszul-Tate complex and show how they relate to the cohomology of the differential $\Delta = \delta + d$. After that we analyze the gauge invariance properties of δ-boundaries modulo d. We then determine the characteristic cohomology for a single p-form gauge field and afterwards extend the results to an arbitrary system of p-forms. Next, we calculate the invariant cohomology and use the results to obtain the cohomological groups $H^{*,inv}(\delta|d)$. Finally, we show that the existence of representatives expressible in terms of the $\overline{H}^a$'s does not hold for the characteristic cohomology in form degree $n - 1$, by exhibiting an infinite number of (inequivalent) conserved currents which are not of that form. We will also comment on how the results on the free characteristic cohomology in degree $< n - 1$ generalize straightforwardly if one adds to the free Lagrangian (3.2) gauge invariant interaction terms that involve the fields $B^a_{\mu_1...\mu_{p_a}}$ and their derivatives only through the gauge invariant field strength components and their derivatives.

5.2   Results

5.2.1   Characteristic cohomology

Remember that the equations of motion (3.5) can be written as,

$$d\overline{H}^a \approx 0, \tag{5.10}$$

in terms of the $(n - p_a - 1)$-forms $\overline{H}^a$ dual to the field strengths. It follows that any polynomial in the $\overline{H}^a$'s is closed on-shell and thus defines a cocycle of the characteristic cohomology.

The remarkable feature is that these polynomials are not only inequivalent in cohomology, but also that they *completely exhaust the characteristic cohomology in form degree strictly smaller than $n-1$*. Indeed, one has:

Theorem 12. *Let $\overline{\mathcal{H}}$ be the algebra generated by the $\overline{H}^a$'s and let $\mathcal{V}$ be the subspace containing the polynomials in the $\overline{H}^a$'s with no term of form degree exceeding $n-2$. The subspace $\mathcal{V}$ is isomorphic to the characteristic cohomology in form degree $< n-1$.*

We stress again that this theorem does not hold in degree $n-1$ because there exist conserved currents not expressible in terms of the $\overline{H}^a$'s.

Since the form degree is limited by the spacetime dimension n, and since $\overline{H}^a$ has strictly positive form degree $n-p_a-1$ (as explained previously, we assume $n-p_a-1 > 0$ for each a), the algebra $\overline{\mathcal{H}}$ is finite-dimensional. In this algebra, the $\overline{H}^a$ with $n-p_a-1$ even commute with all the other generators, while the $\overline{H}^a$ with $n-p_a-1$ odd are anticommuting objects.

5.2.2 Invariant characteristic cohomology

While the cocycles of Theorem **12** are all gauge invariant, there exists coboundaries of the characteristic cohomology that are gauge invariant, i.e., that involve only the field strength components and their derivatives, but which cannot, nevertheless, be written as coboundaries of gauge invariant local forms, even weakly. Examples are given by the field strengths $H^a = dB^a$ themselves. For this reason, the invariant characteristic cohomology and the characteristic cohomology do not coincide. We shall denote by $\mathcal{H}$ the finite-dimensional algebra generated by the (p_a+1)-forms H^a, and by $\mathcal{J}$ the finite-dimensional algebra generated by the field strengths H^a and their duals $\overline{H}^a$. One has,

Theorem 13. *Let $\mathcal{W}$ be the subspace of $\mathcal{J}$ containing the polynomials in the H^a's and the $\overline{H}^a$'s with no term of form degree exceeding $n-2$. The subspace $\mathcal{W}$ is isomorphic to the invariant characteristic cohomology in form degree $< n-1$.*

This chapter is devoted to proving these theorems.

5.2.3 Cohomologies in the algebra of x-independent forms

The previous theorems hold as such in the algebra of local forms that are allowed to have an explicit x-dependence. If one restricts one's attention to

the algebra of local forms with no explicit dependence on the spacetime co-ordinates, then, one must replace in the above theorems the polynomials in the curvatures and their duals with coefficients that are *numbers* by the polynomials in the curvatures and their duals with coefficients that are *constant exterior forms*.

5.3 Characteristic Cohomology and Koszul-Tate Complex

Our analysis of the characteristic cohomology relies upon the isomorphism established in [27] between $H^*_{char}(d)$ and the cohomology $H^*_*(\delta|d)$ of δ modulo d. The cohomology $H^k_i(\delta|d)$ in form degree k and antighost number i is obtained by solving in the algebra $\mathcal{P}$ of local exterior forms the equation,

$$\delta a^k_i + db^{k-1}_{i-1} = 0, \tag{5.11}$$

and by identifying solutions which differ by δ-exact and d-exact terms, i.e,

$$a^k_i \sim a'^k_i = a^k_i + \delta n^k_{i+1} + dm^{k-1}_i. \tag{5.12}$$

One has,

Theorem 14.

$$H^k_{char}(d) \simeq H^n_{n-k}(\delta|d),\ 0 < k < n \tag{5.13}$$

$$\frac{H^0_{char}(d)}{R} \simeq H^n_n(\delta|d). \tag{5.14}$$

$$0 \simeq H^n_{n+k}(\delta|d),\ k > 0 \tag{5.15}$$

Proof: Although the proof is standard and can be found in [49, 27], we shall repeat it explicitly here because it involves ingredients which will be needed below. Let α be a class of $H^k_{char}(d)$ $(k < n)$ and let a^k_0 be a representative of α, $\alpha = [a^k_0]$. One has,

$$\delta a^{k+1}_1 + da^k_0 = 0, \tag{5.16}$$

for some a^{k+1}_1 since any antifield-independent form that is zero on-shell can be written as the δ of something. By acting with d on this equation, one finds that da^{k+1}_1 is δ-closed and thus, by Theorem **2**,

that it is δ-exact, $\delta a_2^{k+2} + d a_1^{k+1} = 0$ for some a_2^{k+2}. One can repeat the procedure until one reaches degree n, the last term a_{n-k}^n fulfilling

$$\delta a_{n-k}^n + d a_{n-1-k}^{n-1} = 0, \tag{5.17}$$

and, of course, $d a_{n-k}^n = 0$ (it is a n-form). For future reference we collect all the terms appearing in this tower of equations as

$$a^k = a_{n-k}^n + a_{n-1-k}^{n-1} + \cdots + a_1^{k+1} + a_0^k. \tag{5.18}$$

Eq. (5.17) shows that a_{n-k}^n is a cocycle of the cohomology of δ modulo d, in form-degree n and antighost number $n-k$. Now, given the cohomological class α of $H_{char}^k(d)$, it is easy to see, using again Theorem **2**, that the corresponding element a_{n-k}^n is well-defined in $H_{n-k}^n(\delta|d)$. Consequently, the above procedure defines an non-ambiguous map m from $H_{char}^k(d)$ to $H_{n-k}^n(\delta|d)$.

This map is surjective. Indeed, let a_{n-k}^n be a cocycle of $H_{n-k}^n(\delta|d)$. By acting with d on Eq. (5.17) and using the second form of the Poincaré lemma (Theorem **5**), one finds that a_{n-1-k}^{n-1} is also δ-closed modulo d. Repeating the procedure all the way down to antighost number zero, one sees that there exists a cocycle a_0^k of the characteristic cohomology such that $m([a_0^k]) = [a_{n-k}^n]$.

The map m is not quite injective, however, because of the constants. Assume that a_0^k is mapped on zero. This means that the corresponding a_{n-k}^n is trivial in $H_{n-k}^n(\delta|d)$, i.e., $a_{n-k}^n = \delta b_{n-k+1}^n + d b_{n-k}^{n-1}$. Using the Poincaré lemma (in the second form) one then successively finds that all a_{n-k-1}^{n-1} ... up to a_1^{k+1} are trivial. The last term a_0^k fulfills $d a_0^k + \delta d b_1^k = 0$ and thus, by the Poincaré lemma (Theorem **4**), $a_0^k = \delta b_1^k + d b_0^{k-1} + c^k$. In the algebra of x-dependent local forms, the constant k-form c^k is present only if $k = 0$. This establishes (5.13) and (5.14).

That $H_m^n(\delta|d)$ vanishes for $m > n$ is proved as follows. If a_m^n is a solution of $\delta a_m^n + d a_{m-1}^{n-1}$ then one can build a descent by acting with δ on this equation. The bottom of this descent satisfies, $\delta a_{m-j}^{n-j} = 0$. Since $j \leq n$ and $m > n$, Theorem **2** implies that the bottom of the descent and all the cocycles above him are trivial. $\square$

The proof of the theorem also shows that (5.13) holds as such because one allows for an explicit x-dependence of the local forms. Otherwise, one must take into account the constant forms c^k which appear in the analysis of injectivity and which are no longer exact even when $k > 0$, so that (5.13)

becomes,

$$\frac{H_{char}^k(d)}{\Lambda^k} \simeq H_{n-k}^n(\delta|d), \tag{5.19}$$

while (5.14) and (5.15) remain unchanged.

5.4 Characteristic Cohomology and Cohomology of $\Delta = \delta + d$

We have seen in Section **3.4** that in form notation, the action on the antifields of the Koszul-Tate differential can be written in the compact form,

$$\Delta \tilde{H}^a = 0, \tag{5.20}$$

with,

$$\Delta = \delta + d, \tag{5.21}$$

and

$$\tilde{H}^a = \overline{H}^a + \sum_{j=1}^{p_a+1} \overline{B}_j^{*a}. \tag{5.22}$$

The parity of the mixed form $\tilde{H}^a$ is equal to $n - p_a - 1$. Quite generally, it should be noted that the dual $\overline{H}^a$ to the field strength H^a is the term of lowest form degree in $\tilde{H}^a$. It is also the term of lowest antighost number, namely, zero. At the other end, the term of highest form degree in $\tilde{H}^a$ is $\overline{B}_{p_a+1}^{*a}$, which has form degree n and antighost number $p_a + 1$. If we call the difference between the form degree and the antighost number the "Δ-degree", all the terms present in the expansion of $\tilde{H}^a$ have same Δ-degree, namely $n - p_a - 1$.

The differential $\Delta = \delta + d$ enables one to reformulate the characteristic cohomology as the cohomology of Δ. Indeed one has,

Theorem 15. *The cohomology of Δ is isomorphic to the characteristic cohomology,*

$$H^k(\Delta) \simeq H_{char}^k(d), \ 0 \le k \le n \tag{5.23}$$

where k in $H^k(\Delta)$ means the Δ-degree, and in $H_{char}^k(d)$ k is the form degree.

Proof: Let a_0^k $(k < n)$ be a cocycle of the characteristic cohomology. Construct a^k as in the proof of Theorem **14**, formula (5.18). The form a^k is easily seen to be a cocycle of Δ, $\Delta a^k = 0$, and furthermore, to be uniquely defined in cohomology given the class of a_0^k. It is also immediate to check that the map so defined is both injective and surjective. This proves the theorem for $k < n$. For $k = n$, the isomorphism of $H^n(\Delta)$ and $H_{char}^n(d)$ is even more direct ($da_0^n = 0$ is equivalent to $\Delta a_0^n = 0$ and $a_0^n = db_0^{n-1} + \delta b_1^n$ is equivalent to $a_0^n = \Delta(b_0^{n-1} + b_1^n)$). $\square$

Our discussion has also established the following useful rule: the term of lowest form degree in a Δ-cocycle a is a cocycle of the characteristic cohomology. Its form degree is equal to the Δ-degree k of a. For $a = \tilde{H}^a$, this reproduce the rule discussed above Theorem **15**. Similarly, the term of highest form degree in a has always form degree equal to n if a is not a Δ-coboundary (up to a constant), and defines an element of $H_{n-k}^n(\delta|d)$.

Because Δ is a derivation, its cocycles form an algebra. Therefore, any polynomial in the $\tilde{H}^a$ is also a Δ-cocycle. Since the form degree is limited by the spacetime dimension n, and since the term $\overline{H}^a$ with minimum form degree in $\tilde{H}^a$ has strictly positive form degree $n - p_a - 1$, the algebra generated by the $\tilde{H}^a$ is finite-dimensional. We shall show below that these Δ-cocycles are not exact and that any cocycle of form degree $< n - 1$ is a polynomial in the $\tilde{H}^a$ modulo trivial terms. According to the isomorphism expressed by Theorem **15**, this is equivalent to proving Theorem **12**.

Remarks: (i) The Δ-cocycle associated with a conserved current contains only two terms,

$$a = a_1^n + a_0^{n-1}, \tag{5.24}$$

where a_0^{n-1} is the dual of the conserved current. The product of such a Δ-cocycle with a Δ-cocycle of Δ-degree k has Δ-degree $n - 1 + k$ and therefore vanishes unless $k = 0$ or 1.

(ii) It will be useful below to introduce another degree N as follows. One assigns N-degree 0 to the undifferentiated fields and N-degree 1 to all the antifields irrespective of their antighost number. One then extends the N-degree to the differentiated variables according to the rule $N(\partial_\mu \Phi) = N(\Phi) + 1$. Thus, N counts the number of derivatives and of antifields. Explicitly,

$$N = \sum_a N_a \tag{5.25}$$

with

$$N_a = \sum_J \left[(|J| \sum_i \partial_J B_i^a \frac{\partial}{\partial_J B_i^a} + (|J| + 1) \sum_\alpha \partial_J \phi_\alpha^{*a} \frac{\partial}{\partial_J \phi_\alpha^{*a}} \right]. \tag{5.26}$$

where (i) the sum over J is a sum over all possible derivatives including the zeroth order one; (ii) $|J|$ is the differential order of the derivative ∂_J (i.e., $|J| = k$ for $\partial_{\mu_1...\mu_k}$); (iii) the sum over i stands for the sum over the independent components of B^a; and (iv) the sum over α is a sum over the independent components of all the antifields appearing in the tower associated with B^a (but there is *no* sum over the p-form species a in (5.26)). The differential δ increases N by one unit. The differentials d and Δ have in addition an inhomogeneous piece not changing the N-degree, namely $dx^\mu(\partial^{explicit}/\partial x^\mu)$, where $\partial^{explicit}/\partial x^\mu$ sees only the explicit x^μ-dependence. The forms $\tilde{H}^a$ have N-degree equal to one.

5.5   Acyclicity and Gauge Invariance

5.5.1   Preliminary results

In the sequel, we shall denote by $\mathcal{I}_{Small}$ the algebra of local exterior forms with coefficients $\omega_{\mu_1...\mu_J}$ that depend only on the field strength components and their derivatives (and possibly x^μ); these are the only invariant objects that can be formed out of the "potentials" $B^a_{\mu_1...\mu_{p_a}}$ and their derivatives (c.f. Theorem **3**). The algebras $\mathcal{H}$, $\overline{\mathcal{H}}$ and $\mathcal{J}$ respectively generated by the (p_a+1)-forms H^a, $(n - p_a - 1)$-forms $\overline{H}^a$ and $(H^a, \overline{H}^a)$ are subalgebras of $\mathcal{I}_{Small}$. Remember that we have also defined the algebra $\mathcal{I}$ as the algebra of invariant polynomials in the field strength components, the antifield components and their derivatives and that in pureghost number 0 the cohomology $H(\gamma)$ is given by $\mathcal{I}$.

Since the field equations are gauge invariant and since d maps $\mathcal{I}_{Small}$ on $\mathcal{I}_{Small}$, one can consider the cohomological problem (5.6), (5.7) in the algebra $\mathcal{I}_{Small}$. This defines the invariant characteristic cohomology $H^{*,inv}_{char}(d)$. Furthermore, the differentials δ, d and Δ map the algebra $\mathcal{I}$ on itself. Clearly, $\mathcal{I}_{Small} \subset \mathcal{I}$. The invariant cohomologies $H^{*,inv}(\Delta)$ and $H^{n,inv}_j(\delta|d)$ are defined by considering only local exterior forms that belong to $\mathcal{I}$.

In order to analyze the invariant characteristic cohomology and to prove the non triviality of the cocycles listed in Theorem **12**, we shall need some preliminary results on the invariant cohomology of the Koszul-Tate differential δ and on the gauge invariant δ-boundaries modulo d.

We define $\mathcal{A}$ and $\mathcal{A}_-$ respectively as the restrictions of $\mathcal{P}$ and $\mathcal{P}_-$ in pureghost number 0.

The variables generating $\mathcal{A}$ are, together with x^μ and dx^μ,

$$B_{a\mu_1...\mu_{p_a}}, \partial_\rho B_{a\mu_1...\mu_{p_a}}, \ldots, B^{*a\mu_1...\mu_{p_a-m}}, \partial_\rho B^{*a\mu_1...\mu_{p_a-m}}, \ldots, B^{*a}, \partial_\rho B^{*a}, \ldots .$$

These variables can be conveniently split into two subsets. The first subset consists of the variables χ defined below Theorem **3** (field strengths, antifields and their derivatives). Note that the field strengths and their derivatives which are present in χ are not independent, since they are constrained by the identity $dH^a = 0$ and its differential consequences, but this is not a difficulty for the considerations of this section. The χ's are invariant under the gauge transformations and they generate the algebra $\mathcal{I}$ of invariant polynomials. In order to generate the full algebra $\mathcal{A}$ we need to add to the χ's some extra variables that will be collectively denoted Ψ. The Ψ's contain the field components $B^{a\mu_1\cdots\mu_{p_a}}$ and their appropriate derivatives not already present in the χ's. The explicit form of the Ψ's may be found in the proof of Theorem **3** and have the property that they are algebraically independent from the χ's and that, together with the χ's, they generate $\mathcal{A}$.

Theorem 16. *Let a be a polynomial in the χ: $a = a(\chi)$. If $a = \delta b$ then we can choose b such that $b = b(\chi)$. In particular,*

$$H_j^{inv}(\delta) \simeq 0 \; for \; j > 0. \tag{5.27}$$

Proof: We can decompose b into two parts: $b = \bar{b} + \bar{\bar{b}}$, with $\bar{b} = \bar{b}(\chi) = b(\Psi = 0)$ and $\bar{\bar{b}} = \sum_m R_m(\chi)S_m(\Psi)$, where $S_m(\Psi)$ contains at least one Ψ. Because $\delta\Psi = 0$, we have, $\delta(\bar{b}+\bar{\bar{b}}) = \delta\bar{b}(\chi)+\sum_m \delta R_m(\chi)S_m(\Psi)$. Furthermore if $M = M(\chi)$ then $\delta M(\chi) = (\delta M)(\chi)$. We thus get,

$$a(\chi) = (\delta\bar{b})(\chi) + \sum_m (\delta R_m)(\chi)S_m(\Psi).$$

The above equation has to be satisfied for all the values of the Ψ's and in particular for $\Psi = 0$. This means that $a(\chi) = (\delta\bar{b})(\chi) = \delta\bar{b}(\chi)$. $\square$

5.5.2 Gauge invariant δ-boundary modulo d

For the following theorem we assume that the antisymmetric tensors $B^a_{\mu_1\mu_2\ldots\mu_p}$ all have the same degree p. This covers, in particular, the case of a single p-form.

Theorem 17. *(Valid when the $B^a_{\mu_1\mu_2\ldots\mu_p}$'s have all the same form degree p). Let $a_q^k = a_q^n(\chi) \in \mathcal{I}$ be an invariant local k-form of antighost number $q > 0$. If a_q^k is δ-exact modulo d, $a_q^k = \delta\mu_{q+1}^k+d\mu_q^{k-1}$, then one can assume that μ_{q+1}^k and μ_q^{k-1} only depend on the χ's, i.e., are invariant (μ_{q+1}^k and $\mu_q^{k-1} \in \mathcal{I}$).*

Proof: The proof goes along exactly the same lines as the proof of a similar
statement made in [28] (theorem 6.1) for 1-form gauge fields.

By acting with d and δ on the equation $a_q^k = \delta\mu_{q+1}^k + d\mu_q^{k-1}$ one
can construct a ladder of equations:

$$a_{q+n-k}^n = \delta\mu_{q+n-k+1}^n + d\mu_{q+n-k}^{n-1} \tag{5.28}$$

$$\vdots$$

$$a_q^k = \delta\mu_{q+1}^k + d\mu_q^{k-1} \tag{5.29}$$

$$\vdots$$

$$\begin{cases} a_1^{k-q+1} = \delta\mu_2^{k-q+1} + d\mu_1^{k-q} \\ \text{or} \\ a_{q-k}^0 = \delta\mu_{q-k+1}^0, \end{cases} \tag{5.30}$$

where the a_j^i only depend on the variables χ. To go up the ladder one
acts with d on one of the equations and uses Theorem **16** and the fact
that d maps a χ on a χ. To go down the ladder one acts with δ on one
of the equations and uses Theorem **6** and the fact that δ also maps a
χ on a χ.

Using the same theorems, it is easy to see that if one of the μ_j^i
may be assumed to depend only on the χ, then one can suppose that
it is also true for the other μ_j^i. Therefore, the theorem will hold if
we can prove it for the equation at the top of the ladder and thus for
$a_q^n = \delta\mu_{q+1}^n + d\mu_q^{n-1}$ with $q > 0$.

Let us first treat the case where $q > n$ in which the bottom of the
ladder is $a_{q-n}^0 = \delta\mu_{q-n+1}^0$. A direct application of Theorem **16** tells us
that we may assume that μ_{q-n+1}^0 only depends on the χ and therefore
that the property also holds for the other μ_j^i. This ends the proof for
$q > n$.

Let us now examine the more difficult case $q \leq n$. We shall work
in dual notations where the equation reads,

$$a_q = \delta b_{q+1} + \partial_\mu j_q^\mu. \tag{5.31}$$

All we need to prove is that one can assume that b_{q+1} only depends
on the χ. To this end, let us take the Euler-Lagrange derivatives of
(5.31) with respect to all the fields and the antifields. Using Theorem
2 and the fact that the variational derivative of a divergence vanishes,

we obtain:

$$\frac{\delta^L a_q}{\delta B^{*a}} = \delta Y_a, \tag{5.32}$$

$$\vdots$$

$$\frac{\delta^L a_q}{\delta B^{*a\mu_1\dots\mu_{p-m}}} = \delta Y_{a\mu_1\dots\mu_{p-m}} + (-1)^{m+1}\partial_{[\mu_1} Y_{a\mu_2\dots\mu_{p-m}]}, \tag{5.33}$$

$$\vdots$$

$$\frac{\delta^L a_q}{\delta B^{*a\mu_1\dots\mu_p}} = \delta Y_{a\mu_1\dots\mu_p} - \partial_{[\mu_1} Y_{a\mu_2\dots\mu_p]}, \tag{5.34}$$

$$\vdots$$

$$\frac{\delta^L a_q}{\delta B^{a\mu_1\dots\mu_p}} = \delta X_{a\mu_1\dots\mu_p} - (p+1)\partial^\rho\partial_{[\rho} Y_{a\mu_1\dots\mu_p]}, \tag{5.35}$$

where [ ] denotes complete antisymmetrization (factorial included), $Y_{a\mu_1\dots\mu_{p-m}} = (-1)^{m+1}\frac{\delta^L b_{q+1}}{\delta B^{a*\mu_1\dots\mu_{p-m}}}$ and $X = \frac{\delta^L b_{q+1}}{\delta B^{q\mu_1\dots\mu_p}}$. Using Theorem **16** successively in all the above equations we conclude that all the $Y_{a\mu_1\dots\mu_{p-m}}$ may be assumed to depend only on the χ. Furthermore, since a_q depends on $B^{a\mu_1\dots\mu_p}$ only through $H^{a\mu_1\dots\mu_{p+1}}$, its Euler-Lagrange derivatives with respect to $B^{a\mu_1\dots\mu_p}$ are of the form $\partial^\rho Z_{a\rho\mu_1\dots\mu_p}$ where $Z_{a\rho\mu_1\dots\mu_p}$ only depends on $H^{a\mu_1\dots\mu_{p+1}}$ and is completely antisymmetric. This implies that we may also assume that $X_{a\mu_1\dots\mu_p}$ only depends on the χ.

Eq. (5.35) means that $X_{a\mu_1\dots\mu_p}$ is dual to an element of $H^{n-p}_{q+1}(\delta|d)$ $\simeq H^n_{q+p+1}(\delta|d) \equiv 0$ according to Theorem **18** (see below). We thus have $X_{a\mu_1\dots\mu_p} = \partial^\rho S_{a\rho\mu_1\dots\mu_p} + \delta N_{a\mu_1\dots\mu_p}$. Let us make the recurrence hypothesis that the theorem holds for $q' = q+p+1$, so that $S_{a\rho\mu_1\dots\mu_p}$ and $N_{a\mu_1\dots\mu_p}$ can be chosen in such a way that they only depend on the χ. Setting $a_q(t) = a_q(tB^{\mu_1\dots\mu_p}, t\partial_\rho B^{\mu_1\dots\mu_p}, \dots, tB^{*\mu_1\dots\mu_p}, t\partial_\rho B^{*\mu_1\dots\mu_p}, \dots)$ we can reconstruct a_q from its Euler-Lagrange derivatives as follows:

$$a_q = \int_0^1 dt \frac{d}{dt} a_q(t) \tag{5.36}$$

$$= \int_0^1 dt (B^{a\mu_1\dots\mu_p} \frac{\delta^L a}{\delta B^{a\mu_1\dots\mu_p}}(t) + B^{*a\mu_1\dots\mu_p} \frac{\delta^L a}{\delta B^{*a\mu_1\dots\mu_p}}(t) + \dots$$

$$+ B^{*a\mu_1} \frac{\delta^L a}{\delta B^{*a\mu_1}}(t) + B^{*a} \frac{\delta^L a}{\delta B^{a*}}(t)) + \partial_\mu j^{'\mu} \tag{5.37}$$

$$= \delta (B^{a\mu_1\dots\mu_p} \int_0^1 dt \partial^\rho S_{a\rho\mu_1\dots\mu_p}(t) - B^{a*\mu_1\dots\mu_p} \int_0^1 dt Y_{a\mu_1\dots\mu_p}(t)$$

$$+ (-1)^{p+1} B^{a*} \int_0^1 dt Y_a(t)) + \partial_\mu j^{''\mu} \tag{5.38}$$

$$= \delta (\frac{1}{p+1} H^{a\rho\mu_1\dots\mu_p} \int_0^1 dt S_{a\rho\mu_1\dots\mu_p}(t) - B^{a*\mu_1\dots\mu_p} \int_0^1 dt Y_{a\mu_1\dots\mu_p}(t)$$

$$+ \dots + (-1)^{p+1} B^{a*} \int_0^1 dt Y_a(t)) + \partial_\mu j^{''\mu} = \delta m_{p+1} + \partial_\mu j^{'''\mu}, \tag{5.39}$$

where m_{p+1} manifestly only depends on the χ.

The theorem will thus be proven if the recurrence hypothesis is correct. This is the case since we have proved that the theorem holds for $q > n$ and therefore for $q'' = q + r(p+1)$ for a sufficiently great r. $\square$

Remark : The theorem does not hold if the forms have various form degrees (see Theorem **25** below).

5.6   Characteristic Cohomology for a Single p-Form Gauge Field

Our strategy for computing the characteristic cohomology is as follows. First, we compute $H_*^n(\delta|d)$ (cocycle condition, coboundary condition) for a single p-form. We then use the isomorphism theorems to infer $H_{char}^*(d)$. Finally, we solve the case of a system involving an arbitrary (but finite) number of p-forms of various form degrees.

5.6.1   General theorems

Before we compute $H_*^n(\delta|d)$ for a single abelian p-form gauge field $B_{\mu_1\dots\mu_p}$, we will recall some general results which will be needed in the sequel. These

theorems hold for an arbitrary linear theory of reducibility order $p-1$.

Theorem 18. *For a linear gauge theory of reducibility order p-1, one has,*

$$H_j^n(\delta|d) = 0, \qquad j > p+1. \tag{5.40}$$

Proof: The technic of the proof is the same as the one used in the previous theorem. Although it is valid for any linear theory [27] we only prove it for free p-forms. Suppose there exists some a with antighost$(a) > p+1$, satisfying,

$$\delta a + \partial_\rho b^\rho = 0. \tag{5.41}$$

If we take the Euler-Lagrange derivatives of (5.41) with respect to all the fields and antifields we get:

$$\delta\left(\frac{\delta^L a}{\delta B^*}\right) = 0, \tag{5.42}$$

$$\vdots$$

$$\delta\left(\frac{\delta^L a}{\delta B^{*\mu_1\ldots\mu_{p-m}}}\right) = (-1)^{m+1}\partial_{[\mu_1}\frac{\delta^L a}{\delta B^{*\mu_2\ldots\mu_{p-m}]}}, \tag{5.43}$$

$$\vdots$$

$$\delta\left(\frac{\delta^L a}{\delta B^{\mu_1\ldots\mu_p}}\right) = -(p+1)\partial^\rho\partial_{[\rho}\frac{\delta^L a}{\delta B^{*\mu_1\ldots\mu_p]}}. \tag{5.44}$$

Because antighost$(a) > p+1$ we have antighost$\left(\frac{\delta^L a}{\delta B^*}\right) > 1$; by using Theorem **2** successively in all the above equations we obtain:

$$\frac{\delta^L a}{\delta B^*} = \delta f, \tag{5.45}$$

$$\vdots$$

$$\frac{\delta^L a}{\delta B^{*\mu_1\ldots\mu_{p-m}}} = \delta f_{\mu_1\ldots\mu_{p-m}} + (-1)^{m+1}\partial_{[\mu_1}f_{\mu_2\ldots\mu_{p-m}]}, \tag{5.46}$$

$$\vdots$$

$$\frac{\delta^L a}{\delta B^{\mu_1\ldots\mu_p}} = \delta g_{\mu_1\ldots\mu_p} - (p+1)\partial^\rho\partial_{[\rho}f_{\mu_1\ldots\mu_p]}. \tag{5.47}$$

Using those relations we can reconstruct a as in Theorem **17**:

$$a = \int_0^1 dt \frac{d}{dt} a(t) \tag{5.48}$$

$$= \delta(B^{\mu_1 \ldots \mu_p} \int_0^1 dt g_{\mu_1 \ldots \mu_p}(t) - B^{*\mu_1 \ldots \mu_p} \int_0^1 dt f_{\mu_1 \ldots \mu_p}(t) + \ldots$$

$$+ (-1)^{p+1} B^* \int_0^1 dt f(t)) + \partial_\mu j^{'\mu}. \tag{5.49}$$

Equation (5.49) shows explicitly that a is trivial. $\square$

Theorem **18** is particularly useful because it limits the number of possible non-vanishing cohomologies. The calculation of the characteristic cohomology is further simplified by the following theorem:

Theorem 19. *Any solution of $\delta a + \partial_\rho b^\rho = 0$ that is at least bilinear in the antifields is necessarily trivial.*

Proof: the proof proceed along the same lines of Theorem **18**. One takes the Euler-Lagrange derivatives of the equation $\delta a + \partial_\rho b^\rho = 0$ with respect to all the fields and antifields keeping in mind that since a is at least bilinear in the antifields it cannot depend on B^*. One then uses the reconstruction formula (5.49) but with $f = 0$. $\square$

Both theorems hold whether the local forms are assumed to have an explicit x-dependence or not.

5.6.2   Cocycles of $H^n_{p+1}(\delta|d)$

We have just seen that the first possible non-vanishing cohomological group is $H^n_{p+1}(\delta|d)$. We show in this section that this group is one-dimensional and provide explicit representatives. We systematically use the dual notation involving divergences of antisymmetric tensor densities.

Theorem 20. $H^n_{p+1}(\delta|d)$ *is one-dimensional. One can take as representatives of the cohomological classes $a = kB^*$ where B^* is the last antifield, of antighost number $p + 1$ and where k is a number.*

Proof: Any polynomial of antighost number $p + 1$ can be written $a = fB^* + f^\rho \partial_\rho B^* + \ldots + \mu$, where f does not involve the antifields and where μ is at least bilinear in the antifields. By adding a divergence

to a, one can remove the derivatives of B^*, i.e., one can assume $f^\rho = f^{\rho\sigma} = \cdots = 0$. The cocycle condition $\delta a + \partial_\rho b^\rho = 0$ then implies $-\partial_\rho f B^{*\rho} + \delta\mu + \partial_\rho(b^\rho + f B^{*\rho}) = 0$. By taking the Euler-Lagrange derivative of this equation with respect to $B^{*\rho}$, one gets

$$-\partial_\rho f + \delta((-1)^p \frac{\delta^L \mu}{\delta B^{*\rho}}) = 0. \tag{5.50}$$

This shows that f is a cocycle of the characteristic cohomology in degree zero since $\delta(\text{anything of antighost number one}) \approx 0$. Furthermore, if f is trivial in $H^0_{char}(d)$, then a can be redefined so as to be at least bilinear in the antifields and thus is also trivial in the cohomology of δ modulo d. Now, the isomorphism of $H^0_{char}(d)/R$ with $H^n_n(\delta|d)$ implies $f = k + \delta g$ with k a constant ($H^n_n(\delta|d) = 0$ because $n > p + 1$). As we already pointed out, the second term can be removed by adding a trivial term, so we may assume $f = k$. Writing $a = kB^* + \mu$, we see that μ has to be a solution of $\delta\mu + \partial_\rho b'^\rho = 0$ by itself and is therefore trivial by Theorem **19**. So $H^n_{p+1}(\delta|d)$ can indeed be represented by $a = kB^*$. In form notations, this is just the n-form $k\overline{B}^*_{p+1}$. Note that the calculations are true both in the x-dependent and x-independent cases.

To complete the proof of the theorem, it remains to show that the cocycles $a = kB^*$, which belong to the invariant algebra $\mathcal{I}$ and which contain the undifferentiated antifields, are non trivial. If they were trivial, one would have according to Theorem **17**, that $\overline{B}^*_{p+1} = \delta u + dv$ for some u, v also in $\mathcal{I}$. But this is impossible, because both δ and d bring in one derivative of the invariant generators χ while $\overline{B}^*_{p+1}$ does not contain derivatives of χ. [This derivative counting argument is direct if u and v do not involve explicitly the spacetime coordinates x^μ. If they do, one must expand u, v and the equation $\overline{B}^*_{p+1} = \delta u + dv$ according to the number of derivatives of the fields in order to reach the conclusion. Explicitly, one sets $u = u_0 + \cdots + u_k$, $v = v_0 + \cdots + v_k$, where k counts the number of derivatives of the $H_{\mu_1 \ldots \mu_{p+1}}$ and of the antifields. The condition $\overline{B}^*_{p+1} = \delta u + dv$ implies in degree $k + 1$ in the derivatives that $\delta u_k + d'v_k = 0$, where d' does not differentiate with respect to the explicit dependence on x^μ. This relation implies in turn that u_k is δ-trivial modulo d' since there is no cohomology in antighost number $p + 2$. Thus, one can remove u_k by adding trivial terms. Repeating the argument for u_{k-1}, and then for u_{k-2} etc, leads to the desired conclusion]. $\square$

5.6.3 Cocycles of $H_i^n(\delta|d)$ with $i \leq p$

We now solve the cocycle condition for the remaining degrees. First we prove

Theorem 21. *Let K be the greatest integer such that $n - K(n-p-1) > 1$. The cohomological groups $H_j^n(\delta|d)$ ($j > 1$) vanish unless $j = n - k(n-p-1)$, $k = 1, 2, \ldots, K$. Furthermore, for those values of j, $H_j^n(\delta|d)$ is at most one-dimensional.*

Proof: We already know that $H_j^n(\delta|d)$ is zero for $j > p + 1$ and that $H_{p+1}^n(\delta|d)$ is one-dimensional. Assume thus that the theorem has been proved for all j's strictly greater than $J < p + 1$ and let us extend it to J. In a manner analogous to what we did in the proof of Theorem **20**, we can assume that the cocycles of $H_J^n(\delta|d)$ take the form,

$$f_{\mu_1 \ldots \mu_{p+1-J}} B^{*\mu_1 \ldots \mu_{p+1-J}} + \mu, \tag{5.51}$$

where $f_{\mu_1 \ldots \mu_{p+1-J}}$ does not involve the antifields and defines an element of $H_{char}^{p+1-J}(d)$. Furthermore, if $f_{\mu_1 \ldots \mu_{p+1-J}}$ is trivial, then the cocycle (5.51) is also trivial. Now, using the isomorphism $H_{char}^{p+1-J}(d) \simeq H_{n-p-1+J}^n(\delta|d)$ ($p + 1 - J > 0$), we see that f is trivial unless $j' = n - p - 1 + J$, which is strictly greater than J and is of the form $j' = n - k(n - p - 1)$. In this case, $H_{j'}^n$ is at most one-dimensional. Since $J = j' - (n - p - 1) = n - (k+1)(n-p-1)$ is of the required form, the property extends to J. This proves the theorem. $\square$

Because we explicitly used the isomorphism $H_{char}^{p+1-J}(d) \simeq H_{n-p-1+J}^n(\delta|d)$, which holds only if the local forms are allowed to involve explicitly the coordinates x^μ, the theorem must be amended for x-independent local forms. This will be done in section (5.6.5).

Theorem **21** goes beyond the vanishing theorems of [68, 38, 39, 27] since it sets further cohomological groups equal to zero, in antighost number smaller than $p + 1$. This is done by viewing the cohomological group $H_i^n(\delta|d)$ as a subset of $H_{n-p-1+i}^n(\delta|d)$ at a *higher* value of the antighost number, through the form (5.51) of the cocycle and the isomorphism between $H_{char}^{p+1-i}(d)$ and $H_{n-p-1+i}^n(\delta|d)$. In that manner, the known zeros at values of the ghost number greater than $p + 1$ are "propagated" down to values of the ghost number smaller than $p + 1$.

To proceed with the analysis, we have to consider two cases:
(i) Case I: $n - p - 1$ is even.
(ii) Case II: $n - p - 1$ is odd.

We start with the simplest case, namely, case I. Here, $\tilde{H}$ is a commuting object and we can consider its various powers $(\tilde{H})^k$, $k = 1, 2, \ldots, K$ with K as in Theorem **21**. These powers have Δ-degree $k(n - p - 1)$. By Theorem **15**, the term of form degree n in $(\tilde{H})^k$ defines a cocycle of $H^n_{n-k(n-p-1)}(\delta|d)$, which is non trivial as is indicated by the same invariance argument used in the previous subsection. Thus, $H^n_{n-k(n-p-1)}(\delta|d)$, which we know is at most one-dimensional, is actually exactly one-dimensional and one may take as representative the term of form degree n in $(\tilde{H})^k$. This settles the case when $n - p - 1$ is even.

In the case when $n - p - 1$ is odd, $\tilde{H}$ is an anticommuting object and its powers $(\tilde{H})^k$, $k > 0$ all vanish unless $k = 1$. We want to show that $H^n_{n-k(n-p-1)}(\delta|d)$ similarly vanishes unless $k = 1$. To that end, it is enough to prove that $H^n_{n-2(n-p-1)}(\delta|d) = H^n_{2p+2-n}(\delta|d) = 0$ as the proof of Theorem **21** indicates (we assume, as before, that $2p + 2 - n > 1$ since we only investigate here the cohomological groups $H^n_i(\delta|d)$ with $i > 1$). Now, as we have seen, the most general cocycle in $H^n_{2p+2-n}(\delta|d)$ may be assumed to take the form $a = f_{\mu_{p+2}\ldots\mu_n}B^{*\mu_{p+2}\ldots\mu_n} + \mu$, where μ is at least quadratic in the antifields and where $f_{\mu_{p+2}\ldots\mu_n}$ does not involve the antifields and defines an element of $H^{n-p-1}_{char}(d)$. But $H^{n-p-1}_{char}(d) \simeq H^n_{p+1}(\delta|d)$ is one-dimensional and one may take as representative of $H^{n-p-1}_{char}(d)$ the dual $k\epsilon_{\mu_1\ldots\mu_n}H^{\mu_1\ldots\mu_{p+1}}$ of the field strength. This means that a is necessarily of the form,

$$a = k\epsilon_{\mu_1\ldots\mu_n}H^{\mu_1\ldots\mu_{p+1}}B^{*\mu_{p+2}\ldots\mu_n} + \mu, \tag{5.52}$$

so the question to be answered is: for which value of k can one adjust μ in (5.52) so that,

$$k\epsilon_{\mu_1\ldots\mu_n}H^{\mu_1\ldots\mu_{p+1}}\delta B^{*\mu_{p+2}\ldots\mu_n} + \delta\mu + \partial_\rho b^\rho = 0 \ ? \tag{5.53}$$

In (5.53), μ does not contain $B^{*\mu_{p+2}\ldots\mu_n}$ and is at least quadratic in the antifields. Without loss of generality, we can assume that it is exactly quadratic in the antifields and that it does not contain derivatives, since δ and ∂ are both linear and bring in one derivative. [That μ can be assumed to be quadratic is obvious. That it can also be assumed not to contain the derivatives of the antifields is a little bit less obvious since we allow for explicit x-dependence, but can be easily checked by expanding μ and b^ρ according to the number of derivatives of the variables and using the triviality of the cohomology of δ modulo d in the space of local forms that are at least quadratic in the fields and the antifields]. Thus we can write,

$$\mu = \sigma_{\mu_1\ldots\mu_n}B^{*\mu_1\ldots\mu_p}_{(1)}B^{*\mu_{p+1}\ldots\mu_n}_{(2p+1-n)} + \mu',$$

where μ' neither involves $B^{*\mu_{p+2}...\mu_n}_{(2p+2-n)}$ nor $B^{*\mu_{p+1}...\mu_n}_{(2p+1-n)}$. We have explicitly indicated the antighost number in parentheses in order to keep track of it. Inserting this form of μ in (5.53) one finds by taking the Euler-Lagrange derivatives of the result by $B^{*\mu_{p+1}...\mu_n}_{(2p+1-n)}$ that $\sigma_{\mu_1...\mu_n}$ is equal to $k\epsilon_{\mu_1...\mu_n}$ if $2p+1-n > 1$ (if $2p+1-n = 1$, see below). One can then successively eliminate $B^{*\mu_p...\mu_n}_{(2p-n)}$, $B^{*\mu_{p-1}...\mu_n}_{(2p-n-1)}$ etc from μ.

So the question ultimately boils down to: is

$$k\epsilon_{\mu_1...\mu_{2j}} B^{*\mu_1...\mu_j}_{(p+1-j)} \delta B^{*\mu_{j+1}...\mu_{2j}}_{(p+1-j)} \quad \text{for } n \text{ even} = 2j$$

or

$$k\epsilon_{\mu_1...\mu_{2j+1}} B^{*\mu_1...\mu_{j+1}}_{(p-j)} \delta B^{*\mu_{j+2}...\mu_{2j+1}}_{(p+1-j)} \quad \text{for } n \text{ odd} = 2j+1$$

δ-exact modulo d, i.e., of the form $\delta\nu + \partial_\rho c^\rho$, where ν does not involve the antifields B^*_s for $s > p+1-j$ (n even) or $s > p-j$ (n odd)?

That the answer to this question is negative unless $k = 0$ and accordingly that a is trivial, is easily seen by trying to construct explicitly ν. We treat for definiteness the case n even ($n = 2j$).

One has

$$\nu = \lambda_{\mu_1...\mu_{2j}} B^{*\mu_1...\mu_j}_{(p+1-j)} B^{*\mu_{j+1}...\mu_{2j}}_{(p+1-j)}$$

where $\lambda_{\mu_1...\mu_{2j}}$ is antisymmetric (respectively symmetric) for the exchange of $(\mu_1...\mu_j)$ with $(\mu_{j+1}...\mu_{2j})$ if j is even (respectively odd) (the j-form $\overline{B}^*_{(p+1-j)}$ is odd by assumption and this can happen only if the components $B^{*\mu_1...\mu_j}_{(p+1-j)}$ are odd for j even, or even for j odd). From the equation

$$k\epsilon_{\mu_1...\mu_{2j}} B^{*\mu_1...\mu_j}_{(p+1-j)} \delta B^{*\mu_{j+1}...\mu_{2j}}_{(p+1-j)} = \delta\nu + \partial_\rho c^\rho \tag{5.54}$$

one gets

$$k\epsilon_{\mu_1...\mu_{2j}} B^{*\mu_1...\mu_j}_{(p+1-j)} \partial_\rho B^{*\rho\mu_{j+1}...\mu_{2j}}_{(p-j)} = \pm 2\lambda_{\mu_1...\mu_{2j}} B^{*\mu_1...\mu_j}_{(p+1-j)} \partial_\rho B^{*\rho\mu_{j+1}...\mu_{2j}}_{(p-j)} + \partial_\rho c^\rho \tag{5.55}$$

Taking the Euler-Lagrange derivative of this equation with respect to $B^{*\mu_1...\mu_j}_{(p+1-j)}$ yields next

$$\left(k\epsilon_{\mu_1...\mu_{2j}} \mp 2\lambda_{\mu_1...\mu_{2j}}\right) \partial_\rho B^{*\rho\mu_{j+1}...\mu_{2j}}_{(p-j)} = 0,$$

which implies $k\epsilon_{\mu_1...\mu_{2j}} = \pm 2\lambda_{\mu_1...\mu_{2j}}$. This contradicts the symmetry properties of $\lambda_{\mu_1...\mu_{2j}}$, unless $k = 0$, as we wanted to prove.

5.6.4   Characteristic Cohomology

By means of the isomorphism Theorems **14** and **15**, our results on $H_*^n(\delta|d)$ can be translated in terms of the characteristic cohomology as follows.

(i) If $n - p - 1$ is odd, the only non-vanishing group of the characteristic cohomology in form degree $< n - 1$ is $H_{char}^{n-p-1}(d)$, which is one-dimensional. All the other groups vanish. One may take as representatives for $H_{char}^{n-p-1}(d)$ the cocycles $k\overline{H}$. Similarly, the only non-vanishing group $H^j(\Delta)$ with $j < n-1$ is $H^{n-p-1}(\Delta)$ with representatives $k\tilde{H}$ and the only non-vanishing group $H_i^n(\delta|d)$ with $i > 1$ is $H_{p+1}^n(\delta|d)$ with representatives $k\overline{B}_{p+1}^*$.

(ii) If $n - p - 1$ is even, there is further cohomology. The degrees in which there is non trivial cohomology are multiples of $n - p - 1$ (considering again values of the form degree strictly smaller than $n - 1$). Thus, there is characteristic cohomology only in degrees $n - p - 1$, $2(n - p - 1)$, $3(n - p - 1)$ etc. The corresponding groups are one-dimensional and one may take as representatives $k\overline{H}$, $k(\overline{H})^2$, $k(\overline{H})^3$ etc. There is also non-vanishing Δ-cohomology for the same values of the Δ-degree, with representative cocycles given by $k\tilde{H}$, $k(\tilde{H})^2$, $k(\tilde{H})^3$, etc. By expanding these cocycles according to the form degree and keeping the terms of form degree n, one gets representatives for the only non-vanishing groups $H_i^n(\delta|d)$ (with $i > 1$), which are respectively $H_{p+1}^n(\delta|d)$, $H_{p+1-(n-p-1)}^n$, $H_{p+1-2(n-p-1)}^n$ etc.

   An immediate consequence of our analysis is the following useful theorem:

Theorem 22. *If the polynomial $P^k(H)$ of form degree $k < n$ in the curvature $(p + 1)$-form H is δ-exact modulo d in the invariant algebra $\mathcal{I}$, then $P^k(H) = 0$.*

Proof: The theorem is straightforward in the algebra of x-independent local forms; it follows from a direct derivative counting argument. To prove it when an explicit x-dependence is allowed, one proceeds as follows. If $P^k(H) = \delta a_1^k + da_0^{k-1}$ where a_1^k and $a_0^{k-1} \in \mathcal{I}$, then $da_1^k + \delta a_2^{k+1} = 0$ for some invariant a_2^{k+1}. Using the results on the cohomology of δ modulo d that we have just established, we know that a_1^k differs from the component of form degree k and antighost number 1 of a polynomial $Q(\tilde{H})$ by a term of the form $\delta\rho + d\sigma$, where ρ and σ are both invariant. But then, δa_1^k has the form $d\big([Q(\tilde{H})]_0^{k-1} + \delta\sigma\big)$, which implies $P^k(H) = d\big(-[Q(\tilde{H})]_0^{k-1} - \delta\sigma + a_0^{k-1}\big)$, i.e., $P^k(H) = d(\text{invariant})$. According to the theorem on the invariant cohomology of d, this can only occur if $P^k(H) = 0$. $\square$

5.6.5   Characteristic cohomology in the algebra of x-independent local forms

Let us denote $(\tilde{H})^m$ by P_m $(m = 0, \ldots, K)$. We have just shown (i) that the most general cocycles of the Δ-cohomology are given, up to trivial terms, by the linear combinations $\lambda_m P_m$ with λ_m real or complex numbers; and (ii) that if $\lambda_m P_m$ is Δ-exact, then the λ_m are all zero. In establishing these results, we allowed for an explicit x-dependence of the local forms (see comments after the proof of Theorem **21**). How are our results affected if we work exclusively with local forms with no explicit x-dependence?

In the above analysis, it is in calculating the cocycles that arise in anti-ghost number $< p + 1$ that we used the x-dependence of the local forms, through the isomorphism $H_{char}^{p+1-J}(d) \simeq H_{n-p-1+J}^n(\delta|d)$. If the local exterior forms are not allowed to depend explicitly on x, one must take into account the constant k-forms $(k > 0)$. The derivation goes otherwise unchanged and one finds that the cohomology of Δ in the space of x-independent local forms is given by the polynomials in the P_m with coefficients λ_m that are constant forms, $\lambda_m = \lambda_m(dx)$. In addition, if $\lambda_m P_m$ is Δ-exact, then, $\lambda_m P_m = 0$ for each m. One cannot infer from this equation that λ_m vanishes, because it is an exterior form. One can simply assert that the components of λ_m of form degree $n - m(n - p - 1)$ or lower are zero (when multiplied by P_m, the other components of λ_m yields forms of degree $> n$ that identically vanish, no matter what these other components are).

It will be also useful in the sequel to know the cohomology of Δ', where Δ' is the part of Δ that acts only on the fields and antifields, and not on the explicit x-dependence. One has $\Delta = \Delta' + d_x$, where $d_x \equiv \partial^{explicit}/\partial x^\mu$ sees only the explicit x-dependence. By the above result, the cohomology of Δ' is clearly given by the polynomials in the P_m with coefficients λ_m that are now arbitrary spacetime forms, $\lambda_m = \lambda_m(x, dx)$.

5.7   Characteristic Cohomology in the General Case

To compute the cohomology $H_i^n(\delta|d)$ for an arbitrary set of p-forms, one proceeds along the lines of the Kunneth theorem. Let us illustrate explicitly the procedure for two fields $B^1_{\mu_1 \ldots \mu_{p_1}}$ and $B^2_{\mu_1 \ldots \mu_{p_2}}$. One may split the differential Δ as a sum of terms with definite N_a-degrees,

$$\Delta = \Delta_1 + \Delta_2 + d_x \qquad (5.56)$$

(see (5.26)). In (5.56), d_x leaves both N_1 and N_2 unchanged. By contrast, Δ_1 increases N_1 by one unit without changing N_2, while Δ_2 increases N_2 by one unit without changing N_1. The differential Δ_1 acts only on the fields B^1 and its associated antifields ("fields and antifields of the first set"), whereas the differential Δ_2 acts only on the fields B^2 and its associated antifields ("fields and antifields of the second set"). Note that $\Delta_1 + \Delta_2 = \Delta'$.

Let a be a cocycle of Δ with Δ-degree $< n - 1$. Expand a according to the N_1-degree,

$$a = a_0 + a_1 + a_2 + \cdots + a_m, \ N_1(a_j) = j. \tag{5.57}$$

The equation $\Delta a = 0$ implies $\Delta_1 a_m = 0$ for the term a_m of highest N_1-degree. Our analysis of the Δ'-cohomology for a single p-form then yields $a_m = c_m(\tilde{H}^1)^k + \Delta_1(\text{something})$, where c_m involves only the fields and antifields of the second set, as well as dx^μ and possibly x^μ. There can be no conserved current in a_m since we assume the Δ-degree of a - and thus of each a_j - to be strictly smaller than $n - 1$. Now, the exact term in a_m can be absorbed by adding to a_m a Δ-exact term, through a redefinition of a_{m-1}. Once this is done, one finds that the next equation for a_m and a_{m-1} following from $\Delta a = 0$ reads

$$[(\Delta_2 + d_x)c_m](\tilde{H}^1)^k + \Delta_1 a_{m-1} = 0. \tag{5.58}$$

But we have seen that $\lambda_m(\tilde{H}^1)^k$ cannot be exact unless it is zero, and thus this last equation implies both

$$[(\Delta_2 + d_x)c_m](\tilde{H}^1)^k = 0 \tag{5.59}$$

and

$$\Delta_1 a_{m-1} = 0. \tag{5.60}$$

Since $(\tilde{H}^1)^k$ has independent form components in degrees $k(n-p-1)$, $k(n-p-1)+1$ up to degree n, we infer from (5.59) that the form components of $(\Delta_2 + d_x)c_m$ of degrees 0 up to degree $n - k(n-p-1)$ are zero. If we expand c_m itself according to the form degree, $c_m = \sum c_m^i$, we obtain the equations

$$\delta c_m^i + d c_m^{i-1} = 0, \ i = 1, \ldots, n - k(n-p-1), \tag{5.61}$$

and

$$\delta c_m^0 = 0. \tag{5.62}$$

Our analysis of the relationship between the Δ-cohomology and the cohomology of δ modulo d indicates then that one can redefine the terms of form degree $> n - k(n - p - 1)$ of c_m in such a way that $\Delta c_m = 0$. This does not affect the product $c_m(\tilde{H}^1)^k$. We shall assume that the (irrelevant) higher order terms in c_m have been chosen in that manner. With that choice, c_m is given, up to trivial terms that can be reabsorbed, by $\lambda_m(\tilde{H}^2)^l$, with λ_m a number (or a constant form in the case of $\mathcal{P}_-$), so that $a_m = \lambda_m(\tilde{H}^2)^l(\tilde{H}^1)^k$ is a Δ-cocycle by itself. One next successively repeats the analysis for a_{m-1}, a_{m-2} until one reaches the desired conclusion that a may indeed be assumed to be a polynomial in the $\tilde{H}^a$'s, as claimed above.

The non-triviality of the polynomials in the $\tilde{H}^a$'s is also easy to prove. If $P(\tilde{H}) = \Delta\rho$, with $\rho = \rho_0 + \rho_1 + \cdots + \rho_m$, $N_1(\rho_k) = k$, then one gets at N_1-degree $m+1$ the condition $(P(\tilde{H}))_{m+1} = \Delta_1\rho_m$, which implies $(P(\tilde{H}))_{m+1} = 0$ and $\Delta_1\rho_m = 0$, since no polynomial in $\tilde{H}^1$ is Δ_1-trivial, except zero. It follows that $\rho_m = u(\tilde{H}^1)^m$ up to trivial terms that play no role, where u is a function of the variables of the second set as well as of x^μ and dx^μ. The equation of order m implies then $(P(\tilde{H}))_m = \big((\Delta_2 + d_x)u\big)(\tilde{H}^1)^m + \Delta_1\rho_{m-1}$. The non-triviality of the polynomials in $\tilde{H}^1$ in Δ_1-cohomology yields next $\Delta_1\rho_{m-1} = 0$ and $(P(\tilde{H}))_m = \big((\Delta_2 + d_x)u\big)(\tilde{H}^1)^m$. Since the coefficient of $(\tilde{H}^1)^m$ in $(P(\tilde{H}))_m$ is a polynomial in $\tilde{H}^2$, which cannot be $(\Delta_2 + d_x)$-exact, one gets in fact $(P(\tilde{H}))_m = 0$ and $(\Delta_2 + d_x)u = 0$. It follows that ρ_m fulfills $\Delta\rho_m = 0$ and can be dropped. The analysis goes on in the same way at the lower values of the Δ_1-degree, until one reaches the desired conclusion that the exact polynomial $P(\tilde{H})$ indeed vanishes.

In view of the isomorphism between the characteristic cohomology and $H^*(\Delta)$, this completes the proof of Theorem **12** in the case of two p-forms. The case of more p-forms is treated similarly.

5.8   Invariant Characteristic Cohomology

5.8.1   Isomorphism theorems for the invariant cohomologies

To compute the invariant characteristic cohomology, we proceed as follows. First, we establish isomorphism theorems between $H_{char}^{k,inv}(d)$, $H_{n-k}^{n,inv}(\delta|d)$ and $H^{k,inv}(\Delta)$. Then, we compute $H^{k,inv}(\Delta)$ for a single p-form. Finally, we extend the calculation to an arbitrary systems of p-forms.

Theorem 23.

$$\frac{H_{char}^{k,inv}(d)}{\mathcal{H}^k} \quad \simeq \quad H_{n-k}^{n,inv}(\delta|d),\ 0 \le k < n \qquad (5.63)$$

$$0 \quad \simeq \quad H_{n+k}^{n,inv}(\delta|d),\ k > 0 \qquad (5.64)$$

Theorem 24. *The invariant cohomology of Δ is isomorphic to the invariant characteristic cohomology,*

$$H^{k,inv}(\Delta) \simeq H_{char}^{k,inv}(d),\ 0 \le k \le n. \qquad (5.65)$$

Proof: First we prove (5.63). To that end we observe that the map m introduced in the demonstration of Theorem **14** associates $H_{char}^{k,inv}(d)$ and $H_{n-k}^{n,inv}(\delta|d)$. Indeed, in the expansion (5.18) for a, all the terms can be assumed to be invariant on account of Theorem **16**. The surjectivity of m is also direct, provided that the polynomials in the curvature $P(H)$ are not trivial in $H^*(\delta|d)$, which is certainly the case if there is a single p-form (Theorem **22**). We shall thus use Theorem **23** first only in the case of a single p-form. We shall then prove that Theorem **22** extends to an arbitrary system of forms of various form degrees, so that the proof of Theorem **23** will be completed.

To compute the kernel of m, consider an element $a_0^k \in \mathcal{I}$ such that the corresponding a_{n-k}^n is trivial in $H_{n-k}^{n,inv}(\delta|d)$. Then, again as in the proof of Theorem **14**, one finds that all the terms in the expansion (5.18) are trivial, except perhaps a_0^k, which fulfills $da_0^k + \delta db_1^k = 0$, where $b_1^k \in \mathcal{I}$ is the k-form appearing in the equation expressing the triviality of a_1^{k+1}, $a_1^{k+1} = db_1^k + \delta b_2^{k+1}$. This implies $d(a_0^k - \delta b_1^k) = 0$, and thus, by Theorem **6**, $a_0^k = P + db_0^{k-1} + \delta b_1^k$ with $P \in \mathcal{H}^k$ and $b_0^{k-1} \in \mathcal{I}$. This proves (5.63), since P is not trivial in $H^*(\delta|d)$ (Theorem **22**). [Again, we are entitled to use this fact only for a single p-form until we have proved the non-triviality of P in the general case].

The proof of (5.64) is a direct consequence of Theorem **16** and parallels step by step the proof of a similar statement demonstrated for 1-forms in [28] (lemma 6.1). It will not be repeated here. Finally, the proof of Theorem **24** amounts to observing that the map m' that sends $[a_0^k]$ on $[a]$ (Equation (5.18)) is indeed well defined in cohomology, and is injective as well as surjective (independently of whether $P(H)$ is trivial in the invariant cohomology of δ modulo d). $\square$

Note that if the forms do not depend explicitly on x, on must replace (5.63) by

$$\frac{H_{char}^{k,inv}(d)}{(\Lambda \otimes \mathcal{H})^k} \simeq H_{n-k}^{n,inv}(\delta|d). \tag{5.66}$$

5.8.2 Case of a single p-form gauge field

Theorem **17** enables one to compute also the invariant characteristic cohomology for a single p-form gauge field. Indeed, this theorem implies that $H_{n-k}^{n,inv}(\delta|d)$ and $H_{n-k}^n(\delta|d)$ actually coincide since the cocycles of $H_{n-k}^n(\delta|d)$ are invariant and the coboundary conditions are equivalent. The isomorphism of Theorem **23** shows then that the invariant characteristic cohomology for a single p-form gauge field in form degree $< n-1$ is isomorphic to the subspace of form degree $< n-1$ of the direct sum $\mathcal{H} \oplus \overline{\mathcal{H}}$. Since the product $H \wedge \overline{H}$ has form degree n, which exceeds $n-1$, this is the same as the subspace $\mathcal{W}$ of Theorem **13**. The invariant characteristic cohomology in form degree $k < n-1$ is thus given by $(\mathcal{H} \otimes \overline{\mathcal{H}})^k$, i.e., by the invariant polynomials in the curvature H and its dual $\overline{H}$ with form degree $< n-1$. Similarly, by the isomorphism of Theorem **24**, the invariant cohomology $H^{k,inv}(\Delta)$ of Δ is given by the polynomials in $\tilde{H}$ and H with Δ-degree smaller than $n-1$.

5.8.3 Invariant cohomology of Δ in the general case

The invariant Δ-cohomology for an arbitrary system of p-form gauge fields follows again from a straightforward application of the Kunneth formula and is thus given by the polynomials in the $\tilde{H}^a$'s and H^a's with Δ-degree smaller than $n-1$. The explicit proof of this statement works as in the non-invariant case (for that matter, it is actually more convenient to use as degrees not N_1 and N_2, but rather, degrees counting the number of derivatives of the invariant variables χ's. These degrees have the advantage that the cohomology is entirely in degree zero). In particular, none of the polynomials in the $\tilde{H}^a$'s and H^a's is trivial.

The isomorphism of Theorem **24** implies next that the invariant characteristic cohomology $H_{char}^{k,inv}(d)$ $(k < n-1)$ is given by the polynomials in the curvatures H^a and their duals $\overline{H}^a$, restricted to form degree smaller than $n-1$. Among these, those that involve the curvatures H^a are weakly exact, but not invariantly so. The property of Theorem **22** thus extends as announced to an arbitrary system of dynamical gauge forms of various form degrees.

Because the forms have now different form degrees, one may have elements in $H_{char}^{k,inv}(d)$ $(k < n-1)$ that involve both the curvatures and their duals.

For instance, if B^1 is a 2-form and B^2 is a 4-form, the cocycle $H^1 \wedge \overline{H}^2$ is a $(n-2)$-form. It is trivial in $H^k_{char}(d)$, but not in $H^{k,inv}_{char}(d)$.

5.9   Invariant cohomology of δ mod d

The easiest way to work out explicitly $H^{n,inv}_{n-k}(\delta|d)$ in the general case is to use the above isomorphism theorems, which we are now entitled to do. Thus, one starts from $H^{k,inv}(\Delta)$ and one works out the component of form degree n in the associated cocycles.

Because one has elements in $H^{k,inv}(\Delta)$ that involve simultaneously both the curvatures and their Δ-invariant duals $\tilde{H}^a$, the property that $H^{n,inv}_{n-k}(\delta|d)$ and $H^n_{n-k}(\delta|d)$ coincide can no longer hold. In the previous example, one would find that $H^{(1)}_{\lambda\mu\nu} B^{*(2)\lambda\mu\nu}$, which has antighost number two, is a δ-cocycle modulo d, but it cannot be written invariantly so. An important case where the isomorphism $H^{n,inv}_{n-k}(\delta|d) \simeq H^n_{n-k}(\delta|d)$ $(k > 1)$ does hold, however, is when the forms have all the same degrees.

To write down the generalization of Theorem **17** in the case of p-forms of different degrees, let $P(H^a, \tilde{H}^a)$ be a polynomial in the curvatures $(p_a + 1)$-forms H^a and their Δ-invariant duals $\tilde{H}^a$. One has $\Delta P = 0$. We shall be interested in polynomials of Δ-degree $< n$ that are of degree > 0 in both H^a and $\tilde{H}^a$. The condition that P be of degree > 0 in H^a implies that it is trivial (but not invariantly so), while the condition that it be of degree > 0 in $\tilde{H}^a$ guarantees that when expanded according to the antighost number, P has non-vanishing components of antighost number > 0,

$$P = \sum_{j=k}^{n} [P]^j_{j-k}. \tag{5.67}$$

From $\Delta P = 0$, one has $\delta[P]^n_{n-k} + d[P]^{n-1}_{n-k-1} = 0$.

There is no polynomial in H^a and $\tilde{H}^a$ with the required properties if all the antisymmetric tensors $B^a_{\mu_1 \ldots \mu_{p_a}}$ have the same form degree ($p_a = p$ for all a's) since the product $H^a \tilde{H}^b$ has necessarily Δ-degree n. When there are tensors of different form degrees, one can construct, however, polynomials P with the given features.

The analysis of the previous subsection implies straightforwardly:

Theorem 25. *Let $a^n_q = a^n_q(\chi) \in \mathcal{I}$ be an invariant local n-form of antighost number $q > 0$. If a^n_q is δ-exact modulo d, $a^n_q = \delta\mu^n_{q+1} + d\mu^{n-1}_q$, then one has*

$$a^n_q = [P]^n_q + \delta\mu'^n_{q+1} + d\mu'^{n-1}_q \tag{5.68}$$

for some polynomial $P(H^a, \tilde{H}^a)$ of degree at least one in H^a and at least one in $\tilde{H}^a$, and where μ'^n_{q+1} and μ'^{n-1}_q can be assumed to depend only on the χ's, i.e., to be invariant. In particular, if all the p-form gauge fields have the same form degree, $[P]^n_q$ is absent and one has

$$a^n_q = \delta\mu'^n_{q+1} + d\mu'^{n-1}_q \tag{5.69}$$

where one can assume that μ'^n_{q+1} and μ'^{n-1}_q are invariant (μ'^n_{q+1} and $\mu'^{n-1}_q \in \mathcal{I}$).

5.10 Remarks on Conserved Currents

That the characteristic cohomology is finite-dimensional and that it is entirely generated by the duals $\overline{H}^a$'s to the field strengths holds only in form degree $k < n - 1$. This property is not true in form degree equal to $n - 1$, where there are conserved currents that cannot be expressed in terms of the forms $\overline{H}^a$, even up to trivial terms.

An infinite number of conserved currents that cannot be expressible in terms of the forms $\overline{H}^a$ are given by,

$$T_{\mu\nu\alpha_1...\alpha_s\beta_1...\beta_r} = -\frac{1}{2}(\frac{1}{p!}H_{\mu\rho_1...\rho_p,\alpha_1...\alpha_s}H_\nu{}^{\rho_1...\rho_p}{}_{,\beta_1...\beta_r}$$

$$-\frac{1}{(n-p-2)!}H^*_{\mu\rho_2...\rho_{n-p-1},\alpha_1...\alpha_s}H^{*\rho_2...\rho_{n-p-1}}_\nu{}_{,\beta_1...\beta_r}). \tag{5.70}$$

These quantities are easily checked to be conserved,

$$T^\mu_{\nu\alpha_1...\alpha_s\beta_1...\beta_r,\mu} \equiv 0, \tag{5.71}$$

and generalize the conserved currents given in [78, 79, 80, 81] for free electromagnetism. They are symmetric for the exchange of μ and ν and are duality invariant in the critical dimension $n = 2p + 2$ where the field strength and its dual have same form degree $p + 1$. In this critical dimension, there are further conserved currents which generalize the "zilches",

$$Z^{\mu\nu\alpha_1...\alpha_r\beta_1...\beta_s} = H^{\mu\sigma_1...\sigma_p,\alpha_1...\alpha_r}H^{*\nu}{}_{\sigma_1...\sigma_p}{}^{,\beta_1...\beta_s}$$

$$-H^{*\mu\sigma_1...\sigma_p,\alpha_1...\alpha_r}H^\nu{}_{\sigma_1...\sigma_p}{}^{,\beta_1...\beta_s}. \tag{5.72}$$

Let us prove that the conserved currents (5.70) which contain an even total number of derivatives are not trivial in the space of x-independent local

forms. To avoid cumbersome notations we will only look at the currents with no β indices. One may reexpress (5.70) in terms of the field strengths as

$$T^{\mu\nu\alpha_1...\alpha_m} = -\frac{1}{2p!}(H^{\mu\sigma_1...\sigma_p,\alpha_1...\alpha_m}H^\nu{}_{\sigma_1...\sigma_p} + H^{\mu\sigma_1...\sigma_p}H^\nu{}_{\sigma_1...\sigma_p}{}^{,\alpha_1...\alpha_m})$$

$$+\eta^{\mu\nu}\frac{1}{2(p+1)!}H_{\sigma_1...\sigma_{p+1}}H^{\sigma_1...\sigma_{p+1},\alpha_1...\alpha_m}. \tag{5.73}$$

If one takes the divergence of this expression one gets,

$$T^{\mu\nu\alpha_1...\alpha_m}{}_{,\mu} = \delta K^{\nu\alpha_1...\alpha_m} \tag{5.74}$$

where $K^{\nu\alpha_1...\alpha_m}$ differs from $kH^\nu{}_{\sigma_1...\sigma_p}{}^{,\alpha_1...\alpha_m}B^{*\sigma_1...\sigma_p}$ by a divergence. It is easy to see that $T^{\mu\nu\alpha_1...\alpha_m}$ is trivial if and only if $H^\nu{}_{\sigma_1...\sigma_p}{}^{,\alpha_1...\alpha_m}B^{*\sigma_1...\sigma_p}$ is trivial. So the question is: can we write,

$$H^\nu{}_{\sigma_1...\sigma_p}{}^{,\alpha_1...\alpha_m}B^{*\sigma_1...\sigma_p} = \delta M^{\nu\alpha_1...\alpha_m} + \partial_\rho N^{\rho\nu\alpha_1...\alpha_m}, \tag{5.75}$$

for some $M^{\nu\alpha_1...\alpha_m}$ and $N^{\rho\nu\alpha_1...\alpha_m}$? Without loss of generality, one can assume that M and N have the Lorentz transformation properties indicated by their indices type (the parts of M and N transforming in other representations would cancel by themselves). We can also decompose both sides of Eq. (5.75) according to tensors of given symmetry type (under the permutations of $\nu\alpha_1...\alpha_m$); in particular (5.75) implies:

$$H^{(\nu}{}_{\sigma_1...\sigma_p}{}^{,\alpha_1...\alpha_m)}B^{*\sigma_1...\sigma_p} = \delta M^{(\nu\alpha_1...\alpha_m)} + \partial_\rho N^{\rho(\nu\alpha_1...\alpha_m)}. \tag{5.76}$$

Moreover, according to Theorem **17**, one can also assume that M and N are gauge invariant, i.e., belong to $\mathcal{I}$. If one takes into account all the symmetries of the left-hand side and use the identity $dH = 0$, Eq. (5.76) reduces to,

$$H^{(\nu}{}_{\sigma_1...\sigma_p}{}^{,\alpha_1...\alpha_m)}B^{*\sigma_1...\sigma_p} = \partial_\rho N^{\rho(\nu\alpha_1...\alpha_m)} + \text{terms that vanish on-shell.} \tag{5.77}$$

This is a consequence of the fact that there is no suitable polynomial in $(\partial)H_{\mu_1...\mu_r}$ and $(\partial)B^*_{\mu_1...\mu_s}$ which can be present in $M^{(\nu\alpha_1...\alpha_m)}$.

If one takes the Euler-Lagrange derivative of this equation with respect to $B^{*\sigma_1...\sigma_p}$ one gets,

$$H_{(\nu|\sigma_1...\sigma_p|,\alpha_1...\alpha_m)} \approx 0, \tag{5.78}$$

which is not the case.

This shows that $T^{\mu\nu\alpha_1\cdots\alpha_m}$ (with m even) is not trivial in the algebra of x-independent local forms. It then follows, by a mere counting of derivative argument, that the $T^{\mu\nu\alpha_1\cdots\alpha_m}$ define independent cohomological classes and cannot be expressed as polynomials in the undifferentiated dual to the field strengths $\overline{H}$ with coefficients that are constant forms.

The fact that the conserved currents are not always expressible in terms of the forms $\overline{H}^a$ makes the validity of this property for higher order conservation laws more striking. In that respect, it should be indicated that the computation of the characteristic cohomology in the algebra generated by the $\overline{H}^a$ is clearly a trivial question. The non trivial issue is to demonstrate that this computation does not miss other cohomological classes in degree $k < n - 1$.

Finally, we point out that the conserved currents can all be redefined so as to be strictly gauge-invariant, apart from a few of them whose complete list can be systematically determined for each given system of p-forms. This point will be fully established in Section **6.3**; it extends to higher degree antisymmetric tensors a property established in [26] for one-forms (see also [82] in this context).

5.11 Introduction of Gauge Invariant Interactions

The analysis of the characteristic cohomology proceeds in the same fashion if one adds to the Lagrangian (3.2) interactions that involve gauge invariant terms of higher dimensionality. These interactions may increase the derivative order of the field equations. The resulting theories should be regarded as effective theories and can be handled through a systematic perturbation expansion [83].

The new equations of motion read,

$$\partial_\mu \mathcal{L}^{a\mu\mu_1\mu_2\cdots\mu_{p_a}} = 0, \tag{5.79}$$

where $\mathcal{L}^{a\mu\mu_1\mu_2\cdots\mu_{p_a}}$ are the Euler-Lagrange derivatives of the Lagrangian with respect to the field strengths (by gauge invariance, $\mathcal{L}$ involves only the field strength components and their derivatives). These equations can be rewritten as,

$$d\overline{\mathcal{L}}^a \approx 0, \tag{5.80}$$

where $\overline{\mathcal{L}}^a$ is the $(n - p_a - 1)$-form dual to the Euler-Lagrange derivatives.

The Euler-Lagrange equations obey the same Noether identities as in the free case, so that the Koszul-Tate differential takes the same form, with $\overline{H}^a$ everywhere replaced by $\overline{\mathcal{L}}^a$. It then follows that

$$\tilde{\mathcal{L}}^a = \overline{\mathcal{L}}^a + \sum_{j=1}^{p+1} \overline{B}_j^{*a} \tag{5.81}$$

fulfills

$$\Delta \tilde{\mathcal{L}}^a = 0. \tag{5.82}$$

This implies, in turn, that any polynomial in the $\tilde{\mathcal{L}}^a$ is Δ-closed. It is also clear that any polynomial in the $\overline{\mathcal{L}}^a$ is weakly d-closed. By making the regularity assumptions on the higher order terms in the Lagrangian explained in [27], one easily verifies that these are the only cocycles in form degree $< n - 1$, and that they are non-trivial. The characteristic cohomology of the free theory possesses therefore some amount of "robustness" since it survives deformations. By contrast, the infinite number of non-trivial conserved currents is not expected to survive interactions (even gauge-invariant ones).

[In certain dimensions, one may add Chern-Simons terms to the Lagrangian. These interactions are not strictly gauge invariant, but only gauge-invariant up to a surface term. The equations of motion still take the form $d(\text{something}) \approx 0$, but now, that "something" is not gauge invariant. Accordingly, with such interactions, some of the cocycles of the characteristic cohomology are no longer gauge invariant. These cocycles are removed from the invariant cohomology].

5.12   Summary of Results and Conclusions

In this section, we have completely worked out the characteristic cohomology $H_{char}^k(d)$ in form degree $k < n - 1$ for an arbitrary collection of free, antisymmetric tensor theories. We have shown in particular that the cohomological groups $H_{char}^k(d)$ are finite-dimensional and take a simple form, in sharp contrast with $H_{char}^{n-1}(d)$, which is infinite-dimensional and appears to be quite complex. Thus, even in free field theories with an infinite number of conserved local currents, the existence of higher degree local conservation laws is quite constrained. For instance, in ten dimensions, there is one and only one (non trivial) higher degree conservation law for a single 2-, 3-, 4-, 6-, or 8-form gauge field, in respective form degrees 7, 6, 5, 3 and 1. It is $d\overline{H} \approx 0$. For a 5-form, there are two higher degree conservation laws, namely $d\overline{H} \approx 0$ and $d(\overline{H})^2 \approx 0$, in form degrees 4 and 8. For a 7-form, there are four

higher degree conservation laws, namely $d\overline{H} \approx 0$, $d(\overline{H})^2 \approx 0$, $d(\overline{H})^3 \approx 0$ and $d(\overline{H})^4 \approx 0$, in form degrees 2, 4, 6 and 8.

Our results provide at the same time the complete list of the isomorphic groups $H^k(\Delta)$, as well as of $H^n_{n-k}(\delta|d)$. We have also worked out the invariant characteristic cohomology, which is central in the investigation of the BRST cohomology since it controls the antifield dependence of BRST cohomological classes.

An interesting feature of the characteristic cohomology in form degree $< n - 1$ is its "robustness" to the introduction of strictly gauge invariant interactions, in contrast to the conserved currents.

Chapter 6

Free theory: BRST cohomology (Part II)

6.1 Main theorems

Having studied the characteristic cohomology, we can return to our analysis of the antifield dependent solutions of the Wess-Zumino consistency condition. In Section **4.3.1** we have shown that the most general form for the component of highest antighost number of a BRST cocycle is, $a_{g,q}^n = P_J \omega^J$ where P_J is an element of the invariant characteristic cohomology. According to the results of the previous section and in particular Theorem **25**, we are able to specify further $a_{g,q}^n$. These terms fall into two categories:

1. If $q = 1$ we have,

$$a_{g,q=1}^n = k_{\Delta a_1 \ldots a_r} a^\Delta \mathcal{Q}_{0,g+1}^{a_1 \ldots a_r}, \tag{6.1}$$

   where the a^Δ form a complete set of non-trivial gauge invariant global symmetries [27] of the action (3.2); they satisfy, $\delta a_\Delta + \partial_\mu j_\Delta^\mu = 0$, where the j_Δ^μ form a complete set of non-trivial conserved currents [27]. These belong to an infinite dimensional space and are not all known explicitly. In the previous section we have exhibited an infinite number of them.

2. If $q \geq 2$ we have,

$$a_{g,q}^n = [P_J]_{-q}^n \omega^J \tag{6.2}$$
$$= [P_{a_1 \ldots a_m}]_{-q}^n \mathcal{Q}_{0,g+q}^{a_1 \ldots a_m}, \tag{6.3}$$

   where $P_{a_1 \ldots a_m}$ is a polynomial in the variables H^a and $\tilde{H}^a$ and thus $[P_{a_1 \ldots a_m}]_{-q}^n$ is a representative of the cohomology $H^{inv}(\delta|d)$.

These results clearly exhibits the fact that the existence of antifield dependent solutions is closely tied to the presence of a non-vanishing invariant characteristic cohomology. Furthermore, the component of form degree n of $P_{a_1...a_m}$ is necessarily of antighost number $q \leq p_M + 1$. This means that the expansion of a_g^n according to the antighost number cannot stop after degree $p_M + 1$.

To find elements of the BRST cohomology we must try to complete the possible $a_{g,q}^n$ with components of lower antighost number in order to have $sa_g^n + da_{g+1}^{n-1} = 0$.

When $q = 1$, this construction is immediate and we have,

Theorem 26. *The term $a_{g,q=1}^n = k_{\Delta a_1...a_r} a^\Delta \mathcal{Q}_{0,g+1}^{a_1...a_r}$ can be completed in a solution of the Wess-Zumino consistency condition a_g^n given by,*

$$a_g^n = k_{\Delta a_1...a_r}(j^\Delta \mathcal{Q}_{1,g}^{a_1...a_r} + a^\Delta \mathcal{Q}_{0,g+1}^{a_1...a_r}), \tag{6.4}$$

where $\delta a_\Delta + dj_\Delta = 0$.

When $q \geq 2$ the situation is more complicated. The next theorem classifies a first set of solutions:

Theorem 27. *If $a_{g,q}^n = [P_J]_{-q}^n \omega^J$ only involves the ghosts of ghosts corresponding to p-forms of degree $\geq q$, then it can be completed in a solution of the Wess-Zumino consistency condition a_g^n given by,*

$$a_g^n = [P_{a_1...a_m} \mathcal{Q}^{a_1...a_m}]_g^n. \tag{6.5}$$

Proof: The Δ-degree of P_J is $n - q$. This implies that it is a sum of terms of form degrees $\geq n - q$. Furthermore, since $\mathcal{Q}^{a_1...a_m}$ only involves the ghosts of ghosts corresponding to p-forms of degrees $\geq q$, the term of lowest degree occurring in $\tilde{s}\mathcal{Q}^{a_1...a_m}$ is at least of form degree $\geq q + 1$. Therefore $\tilde{s}P_{a_1...a_m} \mathcal{Q}^{a_1...a_m} = 0$ and $a_g^n = [P_{a_1...a_m} \mathcal{Q}^{a_1...a_m}]_g^n$ is a solution of the Wess-Zumino consistency condition. $\square$

We now investigate what happens when $a_{g,q}^n$ involves ghosts of ghosts corresponding to p-forms of degrees $< q$.

Theorem 28. *If $a_{g,q}^n = [P_J]_{-q}^n \omega^J$ involves ghosts of ghosts corresponding to p-forms of degree $p < q$, then it can only be completed in a solution of the Wess-Zumino consistency condition a_g^n if it is of the form $a_{g,q}^n = k_{A_1 A_2...A_r b_1...b_s} \overline{B}_{p+1}^{*A_1} \mathcal{Q}_{0,g+p+1}^{A_2...A_r b_1...b_s}$, where the labels A_i (resp. b_j) correspond*

to the variables of forms of degree p (resp. $r > p$) and $k_{A_1 A_2 ... A_r b_1 ... b_s} = -k_{A_2 A_1 ... A_r b_1 ... b_s}$.

The corresponding solution of the Wess-Zumino consistency condition is

$$a_g^n = [k_{A_1 A_2 ... A_r b_1 ... b_s} \tilde{H}_{p+1}^{A_1} \mathcal{Q}^{A_2 ... A_r b_1 ... b_s}]_g^n, \qquad (6.6)$$

$$\text{with} \quad k_{A_1 A_2 ... A_r b_1 ... b_s} = -k_{A_2 A_1 ... A_r b_1 ... b_s}. \qquad (6.7)$$

Proof: Let p be the lowest form degree appearing in $\mathcal{Q}_{0,g+q}^{a_1 ... a_m}$. We can then write $a_{g,q}^n$ as,

$$a_{g,q}^n = [P_{a_1 ... a_r b_1 ... b_s}]_{-q}^n [\tilde{B}_1^{a_1} ... \tilde{B}_1^{a_r} \tilde{B}_2^{b_1} ... \tilde{B}_2^{b_s}]_{g+q}^0, \qquad (6.8)$$

where the $\tilde{B}_1^{a_i}$ correspond to forms of degree p and $\tilde{B}_2^{b_j}$ correspond to forms of any higher degree. A direct calculation then shows that we have,

$$a_{g,q-j}^n = [P_{a_1 ... a_r b_1 ... b_s}]_{-q+j}^{n-j} [\tilde{B}_1^{a_1} ... \tilde{B}_1^{a_r} \tilde{B}_2^{b_1} ... \tilde{B}_2^{b_s}]_{g+q-j}^j, \qquad (6.9)$$
$$\text{for } 0 \leq j \leq p.$$

The ambiguity in $a_{g,q-j}^n$ is $a_{g,q-j}^n \to a_{g,q-j}^n + m_0 + m_1 + ... + m_{j-1}$ where m_0 satisfies $\gamma m_0 = 0$, m_1 satisfies $\gamma m_1 + \delta n_1 + db_1 = 0$, $\gamma n_1 = 0$, m_2 satisfies $\gamma m_2 + \delta n_2 + db_2 = 0$, $\gamma n_2 + dl_2 + dc_2 = 0$, $\gamma l_2 = 0$, etc. Using the vocabulary of Section **4.2**, we will say that the ambiguity in $a_{g,q-j}^n$ is the sum of a γ-cocycle (m_0), the first "lift" (m_1) of a γ-cocycle, the second "lift" (m_2) of a γ-cocycle, etc.

However, none of these ambiguities except m_0 in a_{q-p} can play a role in the construction of a non-trivial solution. To see this, we note that δ, γ and d conserve the polynomial degree of the variables of any given sector. We can therefore work at fixed polynomial degree in the variables of all the different p-forms. Since n_1, l_2, etc. are γ-closed terms which have to be lifted at least once, they have the generic form $R[H, \tilde{H}]\mathcal{Q}$ where $\mathcal{Q}$ has to contain a ghost of ghost of degree $p_A < p$. Because we work at fixed polynomial degree, the presence of such terms imply that $P_{a_1 ... a_r b_1 ... b_s}$ depends on either H^A or on $\tilde{H}^A$. However, if P depends on $\tilde{H}^A$ then its component of degree n is of antighost $q \leq p_A + 1 < p + 1$ which is in contradiction with our assumption $q > p$. This means that P has to depend on H^A so that,

$$[P_{a_1 ... b_s}]_{-q}^n \mathcal{Q}_{0,g+q}^{a_1 ... b_s} = [M_{A_1 a_1 ... b_s}]_{-q}^{n-p_A-1} H^{A_1} \mathcal{Q}_{0,g+q}^{a_1 ... b_s}$$
$$= s((-)^{\epsilon_M} [M_{A_1 a_1 ... b_s} \mathcal{Q}^{A_1 a_1 ... b_s}]_{g-1}^n)$$
$$+ d((-)^{\epsilon_M} [M_{A_1 a_1 ... b_s} \mathcal{Q}^{A_1 a_1 ... b_s}]_g^{n-1})$$
$$+ \text{ terms of lower antighost number.}$$

Thus, if $P_{a_1 \ldots a_m}$ depends on H^A, one can eliminate a_q from a by the addition of trivial terms and the redefinition of the terms of antighost number $< q$. Therefore we may now assume that a_q does not contain H^A and that the only ambiguity in the definitions of the a_{q-j} is m_0 in a_{q-p}.

Since $p < q$, we have to substitute $a^n_{g,q-p}$ in the equation $\gamma a^n_{g,q-p-1} + \delta a^n_{g,q-p} + db^{n-1}_{g+1,q-p-1} = 0$. We then get,

$$\gamma a^n_{g,q-p-1} + \delta [P_{a_1 \ldots a_r b_1 \ldots b_s}]^{n-p}_{-q+p} [\tilde{B}_1^{a_1} \ldots \tilde{B}_1^{a_r} \tilde{B}_2^{b_1} \ldots \tilde{B}_2^{b_s}]^p_{g+q-p}$$
$$+ \delta m_0 + db^{n-1}_{g+1,q-p-1} = 0, \quad (6.10)$$

which can be written as,

$$\gamma a'^n_{g,q-p-1} + db'^{n-1}_{g+1,q-p-1} + \delta m_0$$
$$+ (-)^{\epsilon_P} r [P_{a_1 \ldots a_r b_1 \ldots b_s}]^{n-p-1}_{-q+p+1} H_1^{a_1} \mathcal{Q}^{a_2 \ldots a_r b_1 \ldots b_s}_{0,g+q-p} = 0. \quad (6.11)$$

If we act with γ on the above equation we obtain $d\gamma b'^{n-1}_{g+1,q-p-1} = 0 \Rightarrow \gamma b'^{n-1}_{g+1,q-p-1} + db'^{n-2}_{g+2,q-p-1} = 0$ which means that $\gamma b'^{n-1}_{g+1,q-p-1}$ is a γ mod d cocycle. There are two possibilities according to whether $q - p - 1 > 0$ or $q - p - 1 = 0$. In the first case we may assume that $b'^{n-1}_{g+1,q-p-1}$ is strictly annihilated by γ so that $db'^{n-1}_{g+1,q-p-1} = [d\beta_{a_2 \ldots a_r b_1 \ldots b_s}(\chi)] \mathcal{Q}^{a_2 \ldots a_r b_1 \ldots b_s}_{0,g+q-p} + \gamma l^n_{g,q-p-1}$. Equation (6.11) then reads,

$$(-)^{\epsilon_P} r [P_{a_1 \ldots a_r b_1 \ldots b_s}]^{n-p-1}_{-q+p+1} H_1^{a_1}$$
$$+ \delta \alpha_{a_2 \ldots a_r b_1 \ldots b_s}(\chi) + d\beta_{a_2 \ldots a_r b_1 \ldots b_s}(\chi) = 0,$$
$$(6.12)$$

where we have set $m_0 = \alpha_{a_2 \ldots a_r b_1 \ldots b_s}(\chi) \mathcal{Q}^{a_2 \ldots a_r b_1 \ldots b_s}_{0,g+q-p}$. If we restrict ourselves to the algebra of x-independent forms, Eq. (6.12) implies,

$$[P_{a_1 \ldots a_r b_1 \ldots b_s}]^{n-p-1}_{-q+p+1} H_1^{a_1} = 0, \quad (6.13)$$

since δ and d both increase the number of derivatives of the χ. The situation in the algebra of x-dependent forms is more complicated and we shall discuss it at the end of the section. Let us first note that $P_{a_1 \ldots a_r b_1 \ldots b_s}$ cannot depend on $\tilde{H}_1^c$ because in that case we would have $q - p - 1 \leq 0$ which contradicts our assumption. This means that $P_{a_1 \ldots a_r b_1 \ldots b_s}$ will satisfy (6.13) only if it is of the form, $P_{a_1 \ldots a_r b_1 \ldots b_s} = R_{ca_1 \ldots a_r b_1 \ldots b_s} H_1^c$ with $R_{ca_1 \ldots a_r b_1 \ldots b_s}$ symmetric in $c \leftrightarrow a_1$ (resp. antisymmetric) if H_1 is anticommuting (resp. commuting). However the same

calculation as in (6.10) shows that $a_{g,q}^n$ can then be absorbed by the addition of trivial terms and a redefinition of the components of lower antighost number of a_g^n.

We now turn to the case $q - p - 1 = 0$. According to our analysis of Section **4.2** the obstruction to writing $db_{g+1,q-p-1}^{\prime n-1}$ as a γ-exact term is of the form, $[d\beta_{a_2...a_r b_1...b_s}(\chi) + V_{a_2...a_r b_1...b_s}(H^a)]\mathcal{Q}_{0,g+q-p}^{a_2...a_r b_1...b_s}$. Equation (6.11) then reads,

$$(-)^{\epsilon_P} r[P_{a_1...a_r b_1...b_s}]_0^{n-p-1} H_1^{a_1} + V_{a_2...a_r b_1...b_s}(H^a)$$
$$+ \delta\alpha_{a_2...a_r b_1...b_s}(\chi) + d\beta_{a_2...a_r b_1...b_s}(\chi) = 0,$$
(6.14)

which becomes,

$$(-)^{\epsilon_P} r[P_{a_1...a_r b_1...b_s}]_0^{n-p-1} H_1^{a_1} + V_{a_2...a_r b_1...b_s}(H^a) = 0, \qquad (6.15)$$

in the algebra of x-independent local forms. The fact that $V_{a_2...a_r b_1...b_s}$ is d-exact implies that the variational derivatives with respect to all the fields of $[P_{a_1...a_r b_1...b_s}]_0^{n-p-1} H_1^{a_1}$ must vanish. If $P_{a_1...a_r b_1...b_s}$ depends on $\tilde{H}_1^c$ then the condition $q = p + 1$ implies that

$$P_{a_1...a_r b_1...b_s} = k_{ca_1...a_r b_1...b_s} \tilde{H}_1^c, \qquad (6.16)$$

where the $k_{ca_1...a_r b_1...b_s}$ are constants. If we take the Euler-Lagrange derivative of (6.15) with respect to $B_{\mu_1...\mu_p}^b$ we obtain,

$$k_{ca_1...a_r b_1...b_s} = -k_{a_1 c...a_r b_1...b_s}. \qquad (6.17)$$

In that case,

$$a_g^n = [k_{ca_1...a_r b_1...b_s} \tilde{H}_1^c \tilde{B}_1^{a_1} \ldots \tilde{B}_1^{a_r} \tilde{B}_2^{b_1} \ldots \tilde{B}_2^{b_s}]_g^n \qquad (6.18)$$

is a solution of the Wess-Zumino consistency condition.

If $P_{a_1...a_r b_1...b_s}$ does not depend on $\tilde{H}_1^c$ it is of the form,

$$P_{a_1...a_r b_1...b_s} \sim (H_{p_a+1}^a)^k \ldots (H_{p_b+1}^b)^i (\tilde{H}_{n-p_c-1}^c)^u \ldots (\tilde{H}_{n-p_d-1}^d)^v, \quad (6.19)$$

with $p \leq p_a < \ldots < p_b < p_c \ldots < p_d$. If we insert this expression in (6.15) and take the Euler-Lagrange derivative with respect to $B_{\mu_1...\mu_p}^b$ we obtain identically 0 only if,

$$[P_{a_1...a_r b_1...b_s}]_{-q+p+1}^{n-p-1} H_1^{a_1} = 0, \qquad (6.20)$$

as in (6.13). By repeating exactly the discussion following (6.13) we reach the conclusion that (6.20) implies that $a_{g,q}^n$ can be absorbed by the addition of trivial terms and a redefinition of the components of lower antighost number of a_g^n. All the results stated in the theorem have now been proved. $\square$

Coboundary condition for antifield dependent solutions

In this section we analyze the coboundary condition for the antifield dependent solutions of the Wess-Zumino consistency condition.

Let $a_g^n = a_{g,0}^n + \ldots + a_{g,q}^n$ be a BRST cocycle. From our general analysis we know that $a_{g,q}^n$ is of the form (for $q > 1$), $a_{q,q}^n = [P_{a_1 \ldots a_m}(H^a, \tilde{H}^a)]_{-q}^n \mathcal{Q}_{0,g+q}^{a_1 \ldots a_m}$.

If a_g^n is trivial then there exist $c_{g-1}^n = c_{g-1,0}^n + \ldots + c_{g-1,l}^n$ and $e_g^{n-1} = e_{g,0}^{n-1} + \ldots + e_{g,l}^{n-1}$ such that $a_g^n = sc_{g-1}^n + de_g^{n-1}$. Decomposing this equation according to the antighost number we get:

$$a_0 = \delta c_1 + \gamma c_0 + de_0, \tag{6.21}$$

$$a_1 = \delta c_2 + \gamma c_1 + de_1, \tag{6.22}$$

$$\vdots$$

$$a_q = \delta c_{q+1} + \gamma c_q + de_q, \tag{6.23}$$

$$0 = \delta c_{q+2} + \gamma c_{q+1} + de_{q+1}, \tag{6.24}$$

$$\vdots$$

$$0 = \delta c_l + \gamma c_{l-1} + de_{l-1}, \tag{6.25}$$

$$0 = \gamma c_l + de_l. \tag{6.26}$$

We have dropped the indices labeling the ghost numbers and form degrees which are fixed. In the above equations, $a_{g,q}^n$ appears as the obstruction to lifting $l - q$ times the term c_l.

From our analysis of Section **4.3.1** we know that (6.26) implies $c_l = Q_J(\chi)\omega^J$. If $l = q + 1$ we then have,

$$a_{g,q}^n = \delta Q_J(\chi)\omega^J + \gamma c_q + de_q. \tag{6.27}$$

Because $q > 1$ we may assume that $de_q = dS_J(\chi)\omega^J + \gamma v_q$ and therefore,

$$[P_{a_1 \ldots a_m}(H^a, \tilde{H}^a)]_{-q}^n = \delta Q_J(\chi) + dS_J(\chi), \tag{6.28}$$

which is not the case since $[P]_{-q}^n$ defines a non-trivial class of $H^{inv}(\delta|d)$.

Thus we now assume that $l > q + 1$ and in that case c_l has to be lifted at least once. According to the analysis of Section **4.3.1**, the most general expression for c_l is then

$$c_l = [R_{a_1 \ldots a_r}(H^a, \tilde{H}^a)]_{-l}^n \mathcal{Q}_{0,g+l}^{a_1 \ldots a_r}. \tag{6.29}$$

By examining the calculations performed in the proof of Theorem **28** we conclude that a_g^n is trivial if and only if $a_{g,q}^n$ is given by,

$$[M_{a_1 \ldots a_r b_1 \ldots b_s}(H^a, \tilde{H}^a)]_{-q+p+1}^{n-p-1} H^{a_1} \mathcal{Q}_{0,g+q-p}^{a_2 \ldots a_r b_1 \ldots b_s}, \tag{6.30}$$

since this is the obstruction arising when one tries to lift a term of the form (6.29). In (6.30) the a_i label p-forms of the same degree while the b_j label forms of higher degree; furthermore $M_{a_1...a_r b_1...b_s}$ is symmetric (resp. antisymmetric) in $(a_1 ... a_r)$ if H^{a_j} is anticommuting (resp. commuting).

In particular, the BRST cocycles described in Theorem **28** are not trivial.

Results in the algebra of x-dependent forms

The proofs of the previous theorems hold because we have limited our attention to the algebra of x-independent local forms in which the exterior derivative d maps a polynomial containing i derivatives of the fields onto a polynomial containing $i+1$ derivatives of the fields. This observation allowed us in particular to obtain Eq. (6.13) from Eq. (6.12) and Eq. (6.15) from Eq. (6.14). The complete calculations in the algebra of x-dependent forms of the BRST cocycles will not be done here. Instead we give an example to illustrate the problem.

Let us examine a system of 1-forms for which we want to construct in ghost number -1 the solutions corresponding to $a^n_{-1,2} = f_{ab}\overline{A}^{*a}_2 C^b$. By examining the BRST cocycle condition at antighost number 1 one easily gets $a_1 = f_{ab}\overline{A}^{*a}_1 A^b + m_1$ where m_1 can be assumed of the form $m_1 = \alpha(\chi)$. At antighost number 0 we then have the condition, $f_{ab}\overline{F}^a F^b + V(F^a) + \delta\alpha(\chi) + d\beta(\chi) = 0$ which is just Eq. (6.14) for our particular example. In the algebra of x-independent variables we have seen that this condition implies that the symmetric part of f_{ab} vanishes. In $\mathcal{P}$ this is no longer true. Indeed for the symmetric part of f_{ab} one finds the BRST cocycles [28],

$$a = a_1 + a_2 = f_{(ab)} \left(\frac{n-4}{2} C^{a*} C^b + A^{*a\mu} \left[x^\nu F^b_{\nu\mu} + \frac{n-4}{2} A^b_\mu \right] \right). \quad (6.31)$$

These are available in all dimensions except $n = 4$ where they are actually described by Theorem **26**.

6.2   Counterterms, first order vertices and a-nomalies

Using the results of the previous section, we obtain the counterterms, first-order vertices and the anomalies which depend non-trivially on the antifields.

6.2.1   Counterterms and first order vertices

In ghost number 0, Theorems **26**, **27** and **28** imply the existence of the following BRST cocycles:

1. $a_0^n = k_{\Delta a}(j^\Delta B^a + a^\Delta C^a)$ (Theorem **26**).

2. $a_0^n = [P_c(H^a, \tilde{H}^b)Q^c]_0^n$,   where P_c has Δ-degree $n - p_c$. (Theorem **27**).

3. $a_0^n = [f_{abc}\tilde{H}^a\tilde{B}^b\tilde{B}^c]_0^n$, with f_{abc} completely antisymmetric.
                                                            (Theorem **28**).

The cocycles of the first and third type are only available for 1-*forms.*

To obtain the first order vertices and counterterms one isolates from the above cocycles the component of antighost number 0. In particular, we see that a vertex of the Yang-Mills type is only available for 1-forms. In the absence of 1-forms we have:

Theorem 29. *For a system of free p-form gauge fields with $p \geq 2$, the counterterms and the first order vertices are given by,*

$$\mathcal{V} = P_c(H^a, \overline{H}^b)B^c. \tag{6.32}$$

where in (6.32) *the form degrees of the various forms present must add up to* n.

For such systems, the first order vertex $P_c(H^a, \overline{H}^b)B^c$ is as announced a generalized Noether coupling since $P_c(H^a, \overline{H}^b)$ is a higher order conserved current. All these vertices have the remarkable property to be linear in the gauge potentials B^a; we stress again that this is in sharp contrast to the situation where 1-forms are present since in those cases couplings of the Yang-Mills type exist.

To first order in the coupling constants, the classical action corresponding to (6.32) reads,

$$I_{int} = \int d^n x \Big\{ \sum_a \Big(\frac{-1}{2(p_a + 1)!} H^a_{\mu_1 \ldots \mu_{p_a+1}} H^{a\mu_1 \ldots \mu_{p_a+1}} + g_a S^{a\mu_1 \ldots \mu_{p_a}} B^a_{\mu_1 \ldots \mu_{p_a}} \Big) \Big\},$$
$$\tag{6.33}$$

where $S^{c\mu_1 \ldots \mu_{p_a}}$ are the components of the form dual to $P^c(H^a, \overline{H}^b)$.

This action is no longer gauge invariant under the original gauge transformations. Since $S^{a\mu_1 \ldots \mu_{p_a}}$ is gauge invariant and has an on-shell vanishing

divergence, we have, $\partial_\mu S^{a\mu_1\ldots\mu_{p_a}} = k^{a\mu_2\ldots\mu_{p_a}}_{b\nu_1\ldots\nu_k} \partial_\rho H^{b\rho\nu_1\ldots\nu_k}$, where the $k^{a\mu_2\ldots\mu_{p_a}}_{b\nu_1\ldots\nu_k}$ are gauge invariant. It is then easy to check that up to terms of order g^2, I_{int} is invariant under,

$$\delta^{New} B^a_{\mu_1\ldots\mu_{p_a}} = \partial_{[\mu_1}\Lambda^a_{\mu_2\ldots\mu_p]} + g_b p_a! p_b k^{a\nu_2\ldots\nu_{p_b}}_{b\mu_1\ldots\mu_{p_a}} \Lambda^b_{\nu_2\ldots\nu_{p_b}}. \tag{6.34}$$

Since the $k^{a\nu_2\ldots\nu_{p_b}}_{b\mu_1\ldots\mu_{p_a}}$ are gauge invariant, the new gauge algebra remains abelian up to order g^2. Another way to see this is to remember from the general theory that in order to deform the gauge algebra to order g one needs a term in a_0^n which is quadratic in the ghosts.

6.2.2 Anomalies

Using our analysis we can compute the "anomalies" of the theory. Taking into account Theorems **26**, **27** and **28** we obtain in ghost number 1 the following BRST cocycles:

1. $a_1^n = k_{\Delta ab}(2j^\Delta B^a C^b + a^\Delta C^a C^b)$, for 1-forms (Theorem **26**).

   $a_1^n = k_{\Delta a}(j^\Delta B^a + a^\Delta C^a)$, for 2-forms (Theorem **26**).

2. $a_1^n = [P_c(H^a, \tilde{H}^b)Q^c]_1^n$, where P_c has Δ-degree $n - p_c - 1$.
   (Theorem **27**).

3. $a_1^n = [f_{ABc}\tilde{H}^A\tilde{B}^B\tilde{B}^c]_1^n$, with $f_{ABc} = -f_{ABc}$ and where B^A is a 1-form and B^c a 2-form. (Theorem **28**).

   $a_1^n = [f_{ABCD}\tilde{H}^A\tilde{B}^B\tilde{B}^C\tilde{B}^D]_1^n$, for 1-forms and where f_{ABCD} is completely antisymmetric (Theorem **28**).

   Notice that anomalies of type 1 and 2 only exist in the presence of 1-forms or 2-forms. Therefore,

Theorem 30. *In the absence of* 1-*forms and* 2-*forms, the antifield dependent candidate anomalies are given by,*

$$a_1^n = [P_c(H^a, \tilde{H}^b)Q^c]_1^n. \tag{6.35}$$

6.3 Gauge invariance of conserved currents

In this section we list all the conserved currents which can not be covariantized. As we recalled in Section **2.3** these are related to the representatives of $H^n_{-1}(s|d)$ for which $a = a_1 + \ldots + a_q$ with $q > 1$.

The results of Section **6.1** imply that these BRST cocycles are necessarily of the form,

$$a_{-1}^n = [k_{ab} \tilde{H}_{p+1}^a \tilde{B}^b]_{-1}^n \quad \text{with} \quad k_{ab} = -k_{ba}, \tag{6.36}$$

where the $\tilde{H}^a$ and $\tilde{B}^b$ are mixed forms associated to exterior forms of the same degree.

The global symmetries associated with the component $a_1 = k_{ab} \overline{B}_1^{*a} B^b$ of the BRST cocycles (6.36) are,

$$\delta B_{\mu_1 \dots \mu_p}^a = k^a{}_b B_{\mu_1 \dots \mu_p}^b, \tag{6.37}$$

and correspond to "rotations" of the forms B^a among themselves since the generators of these symmetries are antisymmetric matrices k_{ab}.

Let us now prove that a conserved current j associated to a *gauge invariant* global symmetry through the relation $\delta a_1 + dj = 0$ can be assumed to be gauge invariant as well.

Proof: If a_1 is gauge invariant, then this is also true of δa_1. By assumption we have $d\delta a_1 = 0$ since $\delta a_1 + dj = 0$. Therefore, according to Theorem **6** we have $\delta a_1 = dR$, where R is a gauge invariant polynomial (there can be no polynomial in H^a present in δa_1 because δ and d bring one derivatives of the χ). Thus we conclude that up to a d-exact term, $j = R$. $\square$

Conversely, using Theorem **16** it is immediate to see that any global symmetry associated to a gauge invariant conserved current may be assumed gauge invariant. Therefore, we have:

Theorem 31. *The only conserved currents which cannot be assumed to be gauge invariant are associated to the global symmetries $\delta B_{\mu_1 \dots \mu_p}^a = k^a{}_b B_{\mu_1 \dots \mu_p}^b$ and given by,*

$$j = k_{ab} \overline{H}^a B^b \ \text{with} \ k_{ab} = -k_{ba}. \tag{6.38}$$

6.4   Conclusions

In this section we have computed all the solutions of the Wess-Zumino consistency condition which depend on the antifields. Apart from those described by Theorem **26** which related to the conserved currents of the theory, they are all explicitly known and given by Theorems **27** and **28**. The latter have the property to be expressible in terms of the forms $B^a, H^a, \overline{H}^a, \overline{B}_j^{*a}$ and C_j^a.

This extends to antifield dependent solutions the similar property we established in Section **4.2** concerning the antifield independent BRST cocycles belonging to non-trivial descents.

From the BRST cocycles in ghost number 0, we obtained all the first-order vertices, counterterms and "anomalies" of the theory. We also determined all the Noether conserved currents which are not equivalent to gauge invariant currents.

In the absence of 1-forms, all the first order vertices were shown to be of the Noether type (conserved current x potential) and exist only in particular spacetime dimensions. As a consequence there is in that case no vertex of the Yang-Mills type and accordingly no interaction which deforms the algebra of the gauge transformations at first order. If 1-forms are present in the system, one finds additional interactions which are also of the Noether form $j^\mu A_\mu$. However, the conserved current j^μ which couples to the 1-form need not be gauge-invariant. There is actually only one non gauge-invariant current that is available and it leads to the Yang-Mills cubic vertex, which deforms the gauge algebra to order g. All other currents j^μ may be assumed to be gauge-invariant and thus do not lead to algebra-deforming interactions. There is in particular no vertex of the form $\overline{H}BA$ where A are 1-forms and B are p-forms ($p > 1$) with curvature H, which excludes charged p-forms (i.e. p-forms transforming in some representation of a Lie algebra minimally coupled to a Yang-Mills potential).

The "anomalies" have also been computed and we have shown that for systems of p-forms of degree ≥ 3, they are all linear in the ghosts variables. When 1-forms and 2-forms are included in the system, one finds additional anomalies related to the conserved currents of the theory. Among these, there are only two for which the corresponding conserved currents cannot be made gauge invariant.

Some of the above conclusions are based on our analysis of the gauge invariant nature of the global symmetries and conserved currents. Our results are: 1) the only global symmetries which are not gauge invariant up to trivial terms are the rotation of forms among themselves; 2) the only conserved currents which cannot be improved to become gauge invariant are those related to these global symmetries.

6.5   Higher order vertices

Once all the first order vertices are known, one can pursue the analysis of the consistent deformations. This is done by requiring that the equation $(S, S) = 0$ should be satisfied to all orders in the coupling constant. One

then obtains a succession of equations which read,

$$(S_0, S_0) = 0, \tag{6.39}$$

$$(S_0, S_1) = 0, \tag{6.40}$$

$$2(S_0, S_2) + (S_1, S_1) = 0, \tag{6.41}$$

$$(S_0, S_3) + (S_1, S_2) = 0, \tag{6.42}$$

$$\vdots$$

The construction of the full interacting action $S = S_0 + gS_1 + g^2S_2 + \ldots$, consistent to all orders, can be obstructed if one equation of the tower fails to be satisfied. In this section we investigate this problem for a system of exterior forms with degrees limited to two values p and q such that $2 \leq p < q \leq n - 2$ and we construct lagrangians which are consistent to all orders in the coupling constant.

The p-forms are denoted A^a ($a = 1, \ldots m$) and the q-forms B^A ($A = 1, \ldots, M$); their curvatures $F^a = dA^a$ and $H^A = dB^A$ are respectively $(p+1)$- and $q + 1$-forms. Their duals,

$$\overline{F}^a = \frac{1}{(n - p - 1)!} \epsilon_{\mu_1 \ldots \mu_n} F^{a\mu_1 \ldots \mu_{p+1}} dx^{\mu_{p+2}} \ldots dx^{\mu_n}, \tag{6.43}$$

$$\overline{H}^A = \frac{1}{(n - q - 1)!} \epsilon_{\mu_1 \ldots \mu_n} H^{A\mu_1 \ldots \mu_{q+1}} dx^{\mu_{q+2}} \ldots dx^{\mu_n}, \tag{6.44}$$

are respectively $(n - p - 1)$- and $(n - q - 1)$-forms. In form notation, the free lagrangian can be written as,

$$\mathcal{L} = -\frac{1}{2(p+1)!} F^a \overline{F}_a - \frac{1}{2(q+1)!} H^A \overline{H}_A. \tag{6.45}$$

From the previous section we know that the first order vertices are exterior products of form degree n of one of the forms, the curvatures and their duals and are thus given by,

$$(H^A)^k (F^a)^l (\overline{H}^A)^m (\overline{F}^a)^r A^a, \tag{6.46}$$

or

$$(H^A)^k (F^a)^l (\overline{H}^A)^m (\overline{F}^a)^r B^A, \tag{6.47}$$

where the form degrees must add up to n. This imposes the condition $k(q + 1) + l(p + 1) + m(n - q - 1) + r(n - p - 1) + p = n$ in the first case and $k(q + 1) + l(p + 1) + m(n - q - 1) + r(n - p - 1) + q = n$ in the second

case. Furthermore, in order for these vertices to truly deform the gauge transformations they must contain at least one dual so that $m + r > 1$. Using those conditions in addition to $n > q + 1 > p + 1$ one finds only three types of first order couplings which truly deform the gauge transformations:

(i) Chapline-Manton couplings, which are linear in the duals [19, 84, 85, 86, 18],

$$V_1 = \int f_A^a \overline{F}_a B^A, \quad (q = p + 1), \tag{6.48}$$

$$V_2 = \int f_{a_1 \ldots a_{k+1}}^A \overline{H}_A F^{a_1} \ldots F^{a_k} A^{a_{k+1}},$$
$$(k(p + 1) + p = q + 1). \tag{6.49}$$

Here, f_A^a and $f_{a_1 \ldots a_{k+1}}^A$ are arbitrary constants. The $f_{Aa_1 \ldots a_{k+1}}$ may be assumed to be completely symmetric (antisymmetric) in the a's if p is odd (even). The Chapline-Manton coupling (6.48) only exists for $q = p + 1$ while the Chapline-Manton coupling (6.49) only exists if there is some integer k such that $k(p + 1) + p = q + 1$.

(ii) Freedman-Townsend couplings, which are quadratic in the duals [16],

$$V_3 = \int f_{BC}^A \overline{H}^B \overline{H}^C B_A, \tag{6.50}$$

$$V_4 = \int t_{Ab}^a \overline{H}^A \overline{F}_a A^b. \tag{6.51}$$

Here, f_{BC}^A and t_{Ab}^a are constants arbitrary at first order but restricted at second order. The Freedman-Townsend vertices (6.50) and (6.51) only exist for $q = n - 2$.

(iii) Generalized couplings, which are at least quadratic in the duals $\overline{H}^A$,

$$V_5 = \int k_{a_1 \ldots a_{k+1}}^{A_1 \ldots A_l} \overline{H}_{A_1} \ldots \overline{H}_{A_l} F^{a_1} \ldots F^{a_k} A^{a_{k+1}} \tag{6.52}$$

where $k_{a_1 \ldots a_{k+1}}^{A_1 \ldots A_l}$ are arbitrary constants with the obvious symmetries. These interactions exist only if there are integers k, l (with $l \geq 2$) such that $l(n - q - 1) + k(p + 1) + p = n$.

We next show how the above first order vertices can be extended to higher orders in order to obtain a theory which is consistent to all orders.

6.5.1   Chapline-Manton couplings

Instead of using the master equation $(S, S) = 0$ to get the higher order vertices, it is sometimes easier to try to guess the full interacting action and then show that without imposing any conditions on the arbitrary parameters of the first order vertex, one has a consistent deformation in the sense recalled in the introduction. The antibracket analysis is then facultative. This is how we proceed for the Chapline-Manton couplings (6.48), (6.49) and the generalized couplings (6.52).

For V_1, the complete lagrangian which reduces to $\mathcal{L} + V_1$ at order 1 in the coupling constant is:

$$\mathcal{L}_{I,1} = -\frac{1}{(p+1)!} F_I^a \overline{F}_{Ia} - \frac{1}{(q+1)!} H^A \overline{H}_A, \tag{6.53}$$

where $F_I^a = F^a + g' f_A^a B^A$ and $g' = -\frac{(p+1)!}{2}(-)^{q(n-q)} g$. The "improved" field strengths are invariant under the gauge transformations:

$$A^a \rightarrow A^a + d\epsilon^a - g' f_A^a \eta^A, \quad B^A \rightarrow B^A + d\eta^A, \tag{6.54}$$

from which it follows that $\mathcal{L}_{I,1}$ is gauge invariant under (6.54) to all orders.

For V_2, the role of the p-form and the q-form are in a certain sense exchanged. The complete lagrangian is:

$$\mathcal{L}_{I,2} = -\frac{1}{(p+1)!} F^a \overline{F}_a - \frac{1}{(q+1)!} H_I^A \overline{H}_{IA}, \tag{6.55}$$

with $H_I^A = H^a + g' f_{a_1 \ldots a_{k+1}}^A F^{a_1} \ldots F^{a_k} A^{a_{k+1}}$ and $g' = -\frac{(q+1)!}{2}(-)^{(q+1)(n-q-1)} g$. The "improved" field strengths are invariant under the gauge transformations:

$$A^a \rightarrow A^a + d\epsilon^a, \quad B^A \rightarrow B^A + d\eta^A - (-)^{k(q+1)} g' f_{a_1 \ldots a_{k+1}}^A F^{a_1} \ldots F^{a_k} \epsilon^{a_{k+1}}, \tag{6.56}$$

from which it follows that $\mathcal{L}_{I,2}$ is gauge invariant under (6.56) to all orders.

In both cases, the number of fields, gauge invariances and order of reducibility are the same as for the free theory. Let us check this explicitly for V_1. The fields are A^a and B^A; their gauge transformations (6.56) are not all independent since they vanish when ϵ^a and η^A are of the form,

$$\epsilon^a = d\mu^a + g f_A^a \Lambda^A, \quad \eta^A = d\Lambda^A. \tag{6.57}$$

From (6.57) we see that the number of parameters in the reducibility identities of order 1 are the same for the free theory and $\mathcal{L}_{I,1}$. In the same way,

one shows that this property also holds for the reducibility identities at all orders.

Furthermore, no restrictions on the coefficients f^a_A and $f^A_{a_1 \dots a_{k+1}}$ are imposed so the antibracket analysis is not required. Therefore, $\mathcal{L}_{I,1}$ and $\mathcal{L}_{I,2}$ are the most general consistent interactions corresponding to V_1 and V_2.

6.5.2 Generalized couplings

Before examining the Freedman-Townsend couplings, we study the "generalized couplings" because as in the Chapline-Manton case, no antibracket analysis is required.

First-order formulation A convenient way to analyze the Freedman-Townsend couplings and the generalized couplings is to reformulate those theories by introducing auxiliary fields. This has the advantage that the full interacting theories are then polynomial.

In the first-order formulation, the lagrangian (6.45) is replaced by,

$$\mathcal{L} = -\frac{a}{2}(2B^A d\beta_A + \overline{\beta}^A \beta_A) - \frac{b}{2}(2A^a d\alpha_a + \overline{\alpha}^a \alpha_a), \qquad (6.58)$$

with $a = -(n-q-1)!(-1)^{(q+1)(n-q-1)}$ and $b = -(n-p-1)!(-1)^{(p+1)(n-p-1)}$.

The equations of motion are,

$$d\beta^A = 0, \quad d\alpha^a = 0, \quad \beta^A = c\overline{H}^A, \quad \alpha^a = k\overline{F}^a, \qquad (6.59)$$

where $c = \frac{1}{(n-q-1)!(q+1)!}(-)^{(n-q-1)(q+1)+q}$ and $k = c(q \rightarrow p)$. The original lagrangian (6.45) is recovered by inserting (6.59) in (6.58).

The gauge transformations of the first-order lagrangian (6.58) are:

$$\delta_\Lambda B^A = d\Lambda^A, \quad \delta_\Lambda A^a = d\Lambda^a, \quad \delta_\Lambda \beta^A = 0, \quad \delta_\Lambda \alpha^a = 0. \qquad (6.60)$$

According to the rules of the BRST formalism, the differential s can then be written as $s = \delta + \gamma$ with:

$$
\begin{array}{cc}
\delta\overline{B}_1^{*A} + d\beta^A = 0, & \gamma B^A + dC_1^A = 0, \\
\delta\overline{B}_2^{*A} + d\overline{B}_1^{*A} = 0, & \gamma C_{q-1}^A + dC_q^A = 0, \\
\quad \vdots & \quad \vdots \\
\delta\overline{B}_{q+1}^{*A} + d\overline{B}_q^{*a} = 0, & \gamma C_q^A = 0, \\
\delta\beta^{*A} + \beta^A - c\overline{H}^A = 0, & \gamma\beta^A = 0.
\end{array}
$$

along with similar expressions for the p-form sector.

It has been shown in [27] that the cohomology $H(s|d)$ for the theory with auxiliary fields and for the original theory are isomorphic. In our case, the mapping between the BRST cohomologies of the two formulations is implemented through the replacement of $\overline{H}^A$ by β^A and $\overline{F}^a$ by α^a in the BRST cocycles of the theory without auxiliary fields. This is easily seen from the above definitions of δ and γ.

Generalized couplings   In the first-order formulation, the full interacting lagrangian corresponding to the generalized couplings can be written as,

$$
\begin{aligned}
\mathcal{L}_{I,5} &= -\frac{a}{2}(2B^A d\beta_A + \overline{\beta}^A \beta_A) - \frac{1}{2(p+1)!}F^a \overline{F}_a \\
&\quad + g' k^{A_1 \ldots A_l}_{a_1 \ldots a_{k+1}} \beta_{A_1} \ldots \beta_{A_l} F^{a_1} \ldots F^{a_k} A^{a_{k+1}},
\end{aligned}
\tag{6.61}
$$

with $g' = c^{-l}g$. To first order in g, the action $\int \mathcal{L}_{I,5}$ reduce to $I + g \int V_5$ upon elimination of the auxiliary fields (the auxiliary fields α^a have already been eliminated).

This lagrangian is invariant to all orders in the coupling constant under the following gauge transformations:

$$
\delta_\Lambda A^a = d\Lambda^a, \; \delta_\Lambda \beta_A = 0,
$$
$$
\delta_\Lambda B^{A_1} = d\Lambda^{A_1} - \alpha k^{A_1 \ldots A_l}_{a_1 \ldots a_{k+1}} \beta_{A_2} \ldots \beta_{A_l} F^{a_1} \ldots F^{a_k} \Lambda^{a_{k+1}}, \tag{6.62}
$$

where $\alpha = \frac{l}{a}g'(-)^{(n-q-1)[k(p-1)-l]-(n-q)(p-1)}$ and no restriction on the $k^{A_1 \ldots A_l}_{a_1 \ldots a_{k+1}}$ is needed to achieve gauge invariance. Furthermore, as in the case of V_1 it is easy to show that the number of fields, gauge parameters and the order of reducibility after elimination of β_A are the same as for the free theory (6.58).

6.5.3   Freedman-Townsend couplings

For the Freedman-Townsend couplings the situation is more complicated. However, it is easy to see that

$$
\begin{aligned}
\mathcal{L}_I = -\frac{a}{2}(2B^A d\beta_A + \overline{\beta}^A \beta_A) - \frac{b}{2}(2A^a d\alpha_a + \overline{\alpha}^a \alpha_a) \\
+ g f^A_{BC} \beta^B \beta^C B_A + g t^a_{Ab} \beta^A \alpha_a A^b
\end{aligned}
\tag{6.63}
$$

is to all orders a consistent interacting lagrangian invariant under the following gauge transformations,

$$\delta_\Lambda B_A = d\Lambda_A + \frac{2}{a} g f_{AC}^B (-)^q \Lambda_B \beta^C + \frac{1}{a} (-)^{p+1} g t_{Ab}^a \Lambda^b \alpha_a, \qquad (6.64)$$

$$\delta_\Lambda A^a = d\Lambda^a + (-)^{p+1} \frac{1}{b} g t_{Ab}^a \beta^A \Lambda^b, \quad \delta_\Lambda \beta^A = 0, \quad \delta_\Lambda \alpha_a = 0, \qquad (6.65)$$

provided we impose the conditions,

$$f_{AC}^B f_{DE}^A + f_{AD}^B f_{EC}^A + f_{AE}^B f_{CD}^A = 0, \qquad (6.66)$$

$$t_{Ab}^d t_{Bd}^a - t_{Bb}^d t_{Ad}^a = t_{Cb}^a f_{AB}^C. \qquad (6.67)$$

The first condition expresses that the f_{BC}^A are the structure constants of a Lie algebra while the second condition states that the t_{Bc}^a are the matrices of a representation of that Lie algebra.

Upon elimination of the auxiliary fields, (6.63) reduces to $\mathcal{L} + g V_3 + g V_4$ to first order in the coupling constant.

Notice that we are not sure at this stage that the conditions (6.66) and (6.67) are mandatory and that they cannot be dropped by adding to (6.63) higher order terms and by further modifying the gauge transformations. To answer this question we use the antibracket analysis recalled at the beginning of the section.

The solution of the Wess-Zumino consistency condition corresponding to the Freedman-Townsend vertex $g f_{BC}^A \beta^B \beta^C B_A + g t_{Ab}^a \beta^A \alpha_a A^b$ is,

$$S_1 = [f_{BC}^A \tilde{H}^B \tilde{H}^C \tilde{B}_A + t_{Ab}^a \tilde{H}^A \tilde{F}_a \tilde{A}^b]_0^n. \qquad (6.68)$$

The construction of the second order vertex will be possible only if the antibracket of S_1 with itself is s-exact. A direct calculation yields,

$$
\begin{aligned}
(S_1, S_1) &= \int [\{2 f_{AC}^B f_{DE}^A \tilde{H}^C \tilde{H}^D \tilde{H}^E \tilde{B}_B \\
&\quad + (t_{Cb}^a f_{AB}^C - t_{Ab}^d t_{Bd}^a) \tilde{H}^A \tilde{H}^B \tilde{F}_a \tilde{A}^b \}]_1^n. \qquad (6.69)
\end{aligned}
$$

Using our results on the antifield dependent anomalies, we see that the r.h.s. of (6.69) is the sum of two *non-trivial* BRST cocyles. Therefore (S_1, S_1) can only be s-exact if these two terms vanish which means that the conditions (6.66) and (6.67) are needed[1].

---

[1]In a recent paper [21], F. Brandt and N. Dragon have described an interaction between two 1-forms A^a, $a = 1, 2$ and one 2-form B^1. Their example is a particular case of (6.63) with $f_{BC}^A = 0$, $t_{11}^2 = 1$ and other components of $t_{1b}^a = 0$. It is clear that this choice of f_{BC}^A and t_{Bc}^a satisfies the requirements (6.66,6.67). One recovers the action they obtain upon elimination of the auxiliary fields. Their approach is complementary to ours and is based on the gauging of a global symmetry of the free lagrangian (3.2).

It is interesting to notice that (6.63) remains polynomial after the elimination of the auxiliary fields α^a. Indeed, by using their equation of motions one gets:

$$\mathcal{L}_I = -\frac{a}{2}(2B^A\Phi_A + \overline{\beta}^A\beta_A) - \frac{1}{2(p+1)!}F'^a\overline{F'}_a, \qquad (6.70)$$

where $\Phi^A = d\beta^A - \frac{1}{a}g_1 f^A_{BC}\beta^B\beta^C$ and $F'^a = dA^a + \frac{(-)^{(n-p)}}{b}g_2 t^a_{Ab}\beta^A A^b$. The gauge invariances for (6.70) read:

$$\delta_\Lambda B_A = d\Lambda_A + \frac{2}{a}g_1 f^B_{AC}(-)^q\Lambda_B\beta^C + \frac{1}{ab(p+1)!}(-)^{(n-p+q)}g_2 t^a_{Ab}\Lambda^b\overline{F}_a, \quad (6.71)$$

$$\delta_\Lambda A^a = d\Lambda^a + (-)^{(n-p)}\frac{1}{b}g_2 t^a_{Ab}\beta^A\Lambda^b, \quad \delta_\Lambda\beta^A = 0. \quad (6.72)$$

6.5.4   Remarks

1) For the three types of couplings described above, the algebra of the gauge transformations remains abelian on-shell to all orders in the coupling constant. This is a particularity of systems which contain exterior forms of degrees limited to two values. Indeed, with three (or more) degrees, one can modify the algebra at order g^2. For instance, if A, B and C are respectively 3-, 4- and 7-forms, the Lagrangian $\sim \overline{F}\wedge F + \overline{H}\wedge H + \overline{G}\wedge G$ is invariant under the gauge transformations $\delta A = d\epsilon + g\Lambda$, $\delta B = d\Lambda$ and $\delta C = d\mu + g\epsilon dB - g^2\Lambda B$, where ϵ, Λ and μ are respectively 2-, 3- and 6-forms. Here, $F = dA - gB$, $H = dB$ and $G = dC - gAdB + (1/2)g^2 B^2$. The commutator of two Λ-transformations is a μ-transformation with $\mu = g^2\Lambda_1\Lambda_2$. This model will be studied in detail in Chapter **7** where we discuss the BRST cohomology for Chapline-Manton models.

2) The interaction vertices of this section are still available in the presence of 1-forms. They just fail to exhaust all the possible vertices.

3) The basic interaction vertices described above can of course be combined, or can be combined with Chern-Simons terms. This leads, in general, to additional constraints on their coefficients (which may actually have no non trivial solutions in some cases). One example of a non-trivial combination is the description of massive vector fields worked out in [16, 31], which combines the Freedman-Townsend vertex with a Chern-Simons term [87]. Another example is given in [88], where both the Freedman-Townsend vertex and the Yang-Mills vertex are introduced simultaneously.

Chapter 7

Chapline-Manton models

7.1 Introduction

In the previous chapter we studied the BRST cohomology for a system of free p-forms. From this analysis we were able to get all the first order vertices that could be added to the free lagrangian. We have proved that except for those which are strictly gauge invariant, the interactions are quite constrained, i.e., for a fixed spacetime dimension, a given system of p-forms only allows for a finite number of consistent vertices (even to first order). This result complements the geometric analysis performed in [17] where it is shown that the non-abelian Yang-Mills construction cannot be generalized to p-forms viewed as connections for extended objects. [Topological field theory offers ways to bypass some of the difficulties [89], but will not be discussed here].

In this chapter we present the BRST cohomology for different Chapline-Manton couplings. These are particularly interesting because their gauge algebra remains closed off-shell and the reducibility identities hold strongly even after the interactions are switched on. Using those properties one can study their cohomologies essentially along the lines drawn for the system of free p-forms.

For the other couplings, the new gauge algebra generally closes only on-shell and the reducibility identities become on-shell relations. This occurs for the Freedman-Townsend interaction (6.63) and also for the generalized couplings (6.61). In those cases, $\gamma^2 \approx 0$ and it becomes meaningless to consider the cohomology of γ since it is no longer a differential. One cannot therefore study antifield independent BRST cocycles as we did in the free case; the antifields must be incorporated from the beginning.

In section **6.5** we have limited our study of Chapline-Manton couplings to a system of forms of only two different degrees. However, Chapline-Manton

type interactions can be constructed for more general systems of p-forms along the lines discussed in [20]. Rather than facing the general case, which would lead to non informative and uncluttered formulas, we shall illustrate the general construction through four particular example.

Chapline-Manton models are characterized by gauge-invariant curvatures H^a which differ from the free ones by terms proportional to the coupling constant g,

$$H^a = dB^a + g\mu^a + O(g^2).\qquad(7.1)$$

The gauge transformations are,

$$\delta_\epsilon B^a = d\epsilon^a + g\rho^a + O(g^2).\qquad(7.2)$$

Here, μ^a is a sum of exterior products of B's and dB's – which must match the form degree of dB^a – while ρ^a is a sum of exterior products of B's, dB's and ϵ's (linear in the ϵ's). The modified curvatures and gauge transformations must fulfill the consistency condition,

$$\delta_\epsilon H^a = 0,\qquad(7.3)$$

which means that the modified curvatures should be invariant under the modified gauge transformations. Furthermore, off-shell reducibility must be preserved, i.e., $\delta_\epsilon B^a$ should identically vanish for $\epsilon^a = d\lambda^a + \theta^a$ for some appropriate $\theta^a(\epsilon, B, dB, g)$. The Lagrangian being a function of the curvatures and their derivatives, $\mathcal{L} = \mathcal{L}([H^a_{\mu_1 \dots \mu_{p_a+1}}])$ is automatically gauge-invariant. To completely define the model, it is thus necessary to specify, besides the field spectrum, the modified curvatures and the gauge transformations leaving them invariant. In many cases, the curvatures are modified by the addition of Chern-Simons forms of same degree, but this is not the only possibility as Example 3 below indicates. In the sequel we set the coupling constant g equal to one.

7.2   The models

Model 1

The first example contains one p-form, denoted $A \equiv A_0^p$, and one $(p+1)$-form, denoted $B \equiv B_0^{p+1}$. The new field strengths are,

$$F = dA + B, \ H = dB,\qquad(7.4)$$

while the modified gauge transformations take the form,

$$\delta_{\epsilon,\eta}A = d\epsilon - \eta, \tag{7.5}$$
$$\delta_{\epsilon,\eta}B = d\eta, \tag{7.6}$$

where ϵ is a $(p-1)$-form and η a p-form. The gauge transformations are abelian and remain reducible off-shell since the particular gauge parameters $\epsilon = d\rho + \sigma$, $\eta = d\sigma$ clearly do not affect the fields. The BRST transformations of the undifferentiated fields and ghosts are,

$$sA_k^{p-k} + dA_{k+1}^{p-k-1} + B_{k+1}^{p-k} = 0, \tag{7.7}$$

for the A-variables, and

$$sB_k^{p+1-k} + dB_{k+1}^{p-k} = 0, \tag{7.8}$$
$$sB_{p+1}^0 = 0, \tag{7.9}$$

$(k = 0, \ldots, p)$ for the B-ones. One has,

$$sF = 0 = sH. \tag{7.10}$$

This model describes in fact a massive $(p+1)$-form. One can indeed use the gauge freedom of B to set $A = 0$. Once this is done, one is left with the Lagrangian of a massive $(p+1)$-form.

The BRST action of this model can be written as,

$$S = \int d^n x (\mathcal{L} + (-)^{\epsilon_A} \phi_A^* s\phi^A) \tag{7.11}$$

$$= \int d^n x (\mathcal{L} + \frac{1}{p!} B^{*\mu_1 \ldots \mu_{p+1}} \partial_{\mu_1} C_{\mu_2 \ldots \mu_{p+1}}^B + \ldots + B^{*\mu} \partial_\mu C^B$$

$$+ \frac{1}{p!} A^{*\mu_1 \ldots \mu_p} (p\partial_{\mu_1} C_{\mu_2 \ldots \mu_p}^A - C_{\mu_1 \ldots \mu_p}^B) + \ldots + A^{*\mu} (\partial_\mu C^A + (-)^p C_\mu^B)). \tag{7.12}$$

Using, $s\phi_A^* = \frac{\delta^R S}{\delta \phi^A}$, we extract from (7.12) the BRST transformations of the antifields:

$$sA^{*\mu_1 \ldots \mu_p} = \partial_\nu F^{\nu \mu_1 \ldots \mu_p}, \tag{7.13}$$
$$sA^{*\mu_1 \ldots \mu_{p-j}} = -\partial_\nu A^{*\nu \mu_1 \ldots \mu_{p-j}}, \tag{7.14}$$
$$sB^{*\mu_1 \ldots \mu_{p+1}} = \partial_\nu H^{\nu \mu_1 \ldots \mu_{p+1}} - F^{\mu_1 \ldots \mu_{p+1}}, \tag{7.15}$$
$$sB^{*\mu_1 \ldots \mu_{p+1-j}} = -\partial_\nu B^{*\nu \mu_1 \ldots \mu_{p+1-j}} + (-)^{j+1} A^{*\mu_1 \ldots \mu_{p+1-j}}. \tag{7.16}$$

The action of s on the fields, antifields and ghosts decomposes (as in the free theory) as the sum of the Koszul-Tate differential and the longitudinal exterior derivative: $s = \delta + \gamma$ with $s\phi_A^* = \delta\phi_A^*$ and $s\phi^A = \gamma\phi^A$.

Model 2

The second example contains an abelian 1-form $A \equiv A_0^1$ and a $2r$-form $B \equiv B_0^{2r}$ ($r > 0$). The field strengths are,

$$F = dA, \; H = dB + F^r A, \tag{7.17}$$

with $F^r \equiv FF \cdots F$ (r times). The gauge transformations read,

$$\delta_{\epsilon,\eta} A \;=\; d\epsilon, \tag{7.18}$$
$$\delta_{\epsilon,\eta} B \;=\; d\eta - F^r \epsilon, \tag{7.19}$$

and they clearly leave the curvatures invariant.

The BRST transformations of the fields and the ghosts are,

$$sA_0^1 + dA_1^0 \;=\; 0, \tag{7.20}$$
$$sA_1^0 \;=\; 0, \tag{7.21}$$
$$sB_0^{2r} + dB_1^{2r-1} + F^r A_1^0 \;=\; 0, \tag{7.22}$$
$$sB_k^{2r-k} + dB_{k+1}^{2r-k-1} \;=\; 0, \tag{7.23}$$
$$sB_{2r}^0 \;=\; 0, \tag{7.24}$$

($k = 1, ..., 2r - 1$).

The BRST action of this model can be written as,

$$S = \int d^n x (\mathcal{L} + (-)^{\epsilon_A} \phi_A^* s\phi^A) \tag{7.25}$$

$$= \int d^n x (\mathcal{L} + \frac{1}{(2r)!} B^{*\mu_1...\mu_{2r}} (2r \partial_{\mu_1} C_{\mu_2...\mu_{2r}}^B \tag{7.26}$$

$$- \frac{(2r)!}{2^r} F_{\mu_1\mu_2} \dots F_{\mu_{2r-1}\mu_{2r}} C^A) + \dots + B^{*\mu} \partial_\mu C^B + A^{*\mu} \partial_\mu C^A.$$

Therefore, the BRST transformations of the antifields are:

$$sA^{*\mu} = \partial_\nu F^{\nu\mu} - \frac{1}{2^r} H^{\nu_1...\nu_{2r}\mu} F_{\nu_1\nu_2} \dots F_{\nu_{2r-1}\nu_{2r}}$$

$$- \frac{2r}{2^r} \partial_\rho (H^{\mu\rho\nu_1...\nu_{2r-1}} F_{\nu_1\nu_2} \dots F_{\nu_{2r-3}\nu_{2r-2}} A_{\nu_{2r-1}})$$

$$- \frac{2r}{2^r} \partial_\rho (B^{*\mu\rho\nu_1...\nu_{2r-2}} F_{\nu_1\nu_2} \dots F_{\nu_{2r-3}\nu_{2r-2}} C^A), \tag{7.27}$$

$$sA^* = -\partial_\mu A^{*\mu} - \frac{1}{2^r} B^{*\mu_1...\mu_{2r}} F_{\mu_1\mu_2} \dots F_{\mu_{2r-1}\mu_{2r}}, \tag{7.28}$$

$$sB^{*\mu_1...\mu_{2r}} = \partial_\nu H^{\nu\mu_1...\mu_{2r}}, \tag{7.29}$$

$$sB^{*\mu_1...\mu_{2r-j}} = -\partial_\nu B^{*\nu\mu_1...\mu_{2r-j}}. \tag{7.30}$$

The action of s on the fields, antifields and ghosts decomposes as in the free theory as the sum of the Koszul-Tate differential and the longitudinal exterior derivative: $s = \delta + \gamma$. However, here there are two undesirable features. First, the variation of $A^{*\mu}$ involves contributions of antighost number 0 so that $\gamma A^{*\mu} \neq 0$. Furthermore, $sA^{*\mu}$ also contains the undifferentiated fields A_μ which are not invariant under the gauge transformations. This is in contrast with the free theory where all the antifields were in $H(\gamma)$ and their variations were invariant. Both defects can be corrected by making the following invertible transformations,

$$A^{*\mu} \to A^{*\mu} - \frac{2r}{2^r} F_{\nu_1\nu_2} \dots F_{\nu_{2r-3}\nu_{2r-2}} (B^{*\mu\nu_1\dots\nu_{2r-1}} A_{\nu_{2r-1}} - B^{*\mu\nu_1\dots\nu_{2r-2}} C^A),$$
(7.31)

$$A^* \to A^* - \frac{2r}{2^r} F_{\nu_1\nu_2} \dots F_{\nu_{2r-3}\nu_{2r-2}} (B^{*\nu_1\dots\nu_{2r-1}} A_{\nu_{2r-1}} - B^{*\nu_1\dots\nu_{2r-2}} C^A),$$
(7.32)

which cast s into the form,

$$sA^{*\mu} = \partial_\nu F^{\nu\mu} - \frac{r+1}{2^r} H^{\nu_1\dots\nu_{2r}\mu} F_{\nu_1\nu_2} \dots F_{\nu_{2r-1}\nu_{2r}},$$
(7.33)

$$sA^* = -\partial_\mu A^{*\mu} - \frac{r+1}{2^r} B^{*\mu_1\dots\mu_{2r}} F_{\mu_1\mu_2} \dots F_{\mu_{2r-1}\mu_{2r}}.$$
(7.34)

The s-variations of the new antifields are now gauge invariant and $\gamma \phi_A^* = 0$, $\forall \phi_A^*$.

Model 3

Let A, B and C be respectively 1-, 2- and 3-forms. The curvatures are defined through,

$$F = dA + B, \ H = dB, \ G = dC + AdB + (1/2)B^2.$$
(7.35)

The gauge transformations are

$$\delta_{\epsilon,\Lambda,\mu} A = d\epsilon - \Lambda,$$
(7.36)

$$\delta_{\epsilon,\Lambda,\mu} B = d\Lambda,$$
(7.37)

$$\delta_{\epsilon,\Lambda,\mu} C = d\mu - \epsilon dB - \Lambda B,$$
(7.38)

where ϵ, Λ and μ are respectively 0-, 1- and 2-forms. Their gauge algebra is non-abelian and the gauge transformations are off-shell reducible, as is easily verified.

The BRST differential on the fields and ghosts is defined by

$$sA_0^1 + dA_1^0 + B_1^1 = 0, \tag{7.39}$$
$$sA_1^0 + B_2^0 = 0, \tag{7.40}$$
$$sB_0^2 + dB_1^1 = 0, \tag{7.41}$$
$$sB_1^1 + dB_2^0 = 0, \tag{7.42}$$
$$sB_2^0 = 0, \tag{7.43}$$
$$sC_0^3 + dC_1^2 + A_1^0 H + B_1^1 B_0^2 = 0, \tag{7.44}$$
$$sC_1^2 + dC_2^1 + \frac{1}{2}B_1^1 B_1^1 + B_2^0 B_0^2 = 0, \tag{7.45}$$
$$sC_2^1 + dC_3^0 + B_1^1 B_2^0 = 0, \tag{7.46}$$
$$sC_3^0 + \frac{1}{2}B_2^0 B_2^0 = 0. \tag{7.47}$$

This example arises in some formulations of massive supergravity in 10 dimensions [90, 91].

The BRST action can again be written as,

$$S = \int d^n x (\mathcal{L} + (-)^{\epsilon_A} \phi_A^* s\phi^A) \tag{7.48}$$

$$\int d^n x (\mathcal{L} + A^{*\mu}(\partial_\mu C^A - C_\mu^B) + A^* C^B$$
$$+ B^{*\mu\nu}\partial_\mu C_\nu^B + B^{*\mu}\partial_\mu C^B$$
$$+ \frac{1}{3!}C^{*\mu\nu\rho}(3\partial_\mu C_{\nu\rho}^C - 3C^A\partial_\mu B_{\nu\rho} - 3C_\mu^B B_{\nu\rho})$$
$$- \frac{1}{2!}C^{*\mu\nu}(-2\partial_\mu C_\nu^C + C_\mu^B C_\nu^B - C^B B_{\mu\nu})$$
$$+ C^{*\mu}(\partial_\mu C^C - C_\mu^B C^B)$$
$$+ \frac{1}{2}C^*(C^B)^2). \tag{7.49}$$

Therefore, the variations of the antifields are,

$$sA^{*\mu} = \partial_\nu F^{\nu\mu} - \frac{1}{6}H_{\nu\rho\alpha}G^{\mu\nu\rho\alpha}, \tag{7.50}$$

$$sA^* = -\partial_\mu A^{*\mu} - \frac{1}{6}C^{*\mu\nu\rho}H_{\mu\nu\rho}, \tag{7.51}$$

$$sB^{*\mu\nu} = \partial_\rho H^{\rho\mu\nu} - F^{\mu\nu} - \frac{1}{2}G^{\mu\nu\rho\alpha}B_{\rho\alpha} + \partial_\alpha(G^{\rho\alpha\mu\nu}A_\rho)$$
$$\partial_\rho(C^{*\mu\nu\rho}C^A) - C^{*\mu\nu\rho}C_\rho^B - C^{*\mu\nu}C^B, \tag{7.52}$$

$$sB^{*\mu} = -\partial_\nu B^{*\nu\mu} - A^{*\mu} - \frac{1}{2}C^{*\mu\nu\rho}B_{\nu\rho} + C^{*\mu\nu}C_\nu^B - C^{*\mu}C^B, \tag{7.53}$$

$$sB^* = -\partial_\mu B^{*\mu} + A^* + \frac{1}{2}C^{*\mu\nu}B_{\mu\nu} - C^{*\mu}C_\mu^B + C^*C^B, \tag{7.54}$$

$$sC^{*\mu\nu\rho} = \partial_\alpha G^{\alpha\mu\nu\rho}, \tag{7.55}$$

$$sC^{*\mu\nu} = -\partial_\rho C^{*\rho\mu\nu}, \tag{7.56}$$

$$sC^{*\mu} = -\partial_\rho C^{*\rho\mu}, \tag{7.57}$$

$$sC^* = -\partial_\rho C^{*\rho}. \tag{7.58}$$

The definition BRST differential's action on the antifields of the B-sector suffers from defects similar to those encountered in Chapline-Manton model 2 with the original antifields. For example, $sB^{*\mu}$ involves components of antighost number 0 and 1 and the undifferentiated fields $B_{\mu\nu}$ and C_μ^B. This justifies the replacements,

$$B^{*\mu\nu} \to B^{*\mu\nu} + C^{*\rho\mu\nu}C_\rho^A + C^{*\mu\nu}C^A, \tag{7.59}$$

$$B^{*\mu} \to B^{*\mu} + C^{*\mu\rho}C_\rho^A + C^{*\mu}C^A, \tag{7.60}$$

$$B^* \to B^* + C^{*\rho}C_\rho^A + C^*C^A. \tag{7.61}$$

In terms of these modified antifields, the action of s reads,

$$sB^{*\mu\nu} = \partial_\rho H^{\rho\mu\nu} - F^{\mu\nu} - \frac{1}{2}G^{\mu\nu\rho\alpha}F_{\rho\alpha},$$

$$sB^{*\mu} = -\partial_\nu B^{*\nu\mu} - A^{*\mu} - \frac{1}{2}C^{*\mu\nu\rho}F_{\nu\rho}, \tag{7.62}$$

$$sB^* = -\partial_\mu B^{*\mu} + A^* + \frac{1}{2}C^{*\mu\nu}F_{\mu\nu}. \tag{7.63}$$

The BRST differential now decomposes as $s = \delta + \gamma$ and the new antifields are gauge invariant, $\gamma\phi_A^* = 0, \ \forall\phi_A^*$.

Model 4

Our last example is the original Chapline-Manton model, coupling a Yang-Mills connection A^a with a 2-form B. We will assume the gauge group to

be $SU(N)$ for definiteness although the analysis holds for any other compact group. The curvatures are,

$$F = dA + A^2, \tag{7.64}$$
$$H = dB + \omega_3, \tag{7.65}$$

where $\omega_3(A, dA)$ is the Chern-Simons 3-form,

$$\omega_3 = \frac{1}{2}[tr(AdA + \frac{2}{3}A^3)]. \tag{7.66}$$

The BRST differential's action on the fields and ghosts reads,

$$sA + DC = 0, \tag{7.67}$$
$$sC - C^2 = 0, \tag{7.68}$$
$$sB + \omega_2 + d\eta = 0, \tag{7.69}$$
$$s\eta + \omega_1 + d\rho = 0, \tag{7.70}$$
$$s\rho + \frac{1}{3}trC^3 = 0. \tag{7.71}$$

Here, the one-form ω_1 and the two-form ω_2 are related to the Chern-Simons form ω_3 through the descent,

$$s\omega_3 + d\omega_2 = 0, \quad \omega_2 = tr(CdA), \tag{7.72}$$
$$s\omega_2 + d\omega_1 = 0, \quad \omega_1 = tr(C^2A), \tag{7.73}$$
$$s\omega_1 + d(\frac{1}{3}trC^3) = 0. \tag{7.74}$$

The BRST action for this model is,

$$S = \int d^n x (\mathcal{L} + (-)^{\epsilon_A}\phi_A^* s\phi^A) \tag{7.75}$$

$$\int d^n x (\mathcal{L} + A_a^{*\mu}(\partial_\mu C^a - C_{bc}^a A_\mu^b C^c) - \frac{1}{2}C^* C_{bc}^a C^b C^c \tag{7.76}$$

$$+ \frac{1}{2}B^{*\mu\nu}(2C_a(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + (\partial_\nu \eta_\nu - \partial_\nu \partial_\mu)) \tag{7.77}$$

$$- \eta^{*\mu}(C_{abc}C^a C^b A_\mu^c - \partial_\mu \rho) + \frac{1}{3}\rho^* C_{abc}C^a C^b C^c). \tag{7.78}$$

Therefore, the variation of the antifields are,

$$sB^{*\mu\nu} = \partial_\rho H^{\rho\mu\nu}, \tag{7.79}$$

$$s\eta^{*\mu} = -\partial_\nu B^{*\nu\mu}, \tag{7.80}$$

$$s\rho^* = -\partial_\mu \eta^{*\mu}, \tag{7.81}$$

$$sA_a^{*\mu} = D_\nu F_a^{\nu\mu} + 2H^{\mu\nu\rho}F_{a\nu\rho} - 2\partial_\rho H^{\rho\mu\nu}A_{a\nu} \tag{7.82}$$

$$- 2\partial_\nu(B^{*\nu\mu}C_a) - \eta^{*\mu}C_{abc}C^bC^c + C_{abc}A^{*b\mu}C^c, \tag{7.83}$$

$$sC_a^* = -D_\mu A_a^{*\mu} + 2B^{*\mu\nu}\partial_\mu A_{a\nu} + 2C_{abc}\eta^{*\nu}C^bA_\mu^c \tag{7.84}$$

$$+ C_{abc}\rho^*C^bC^c - C_{abc}C^{*b}C^c. \tag{7.85}$$

In terms of the above variables, the BRST differential suffers once more from some defects. First it has a component of antighost number 1, e.g. $\eta^{*\mu}C_{abc}C^aC^b$ in (7.82), as a consequence of which the BRST differential does not split as the sum of the Koszul-Tate differential and the longitudinal exterior derivative. The second undesirable feature is that the BRST variations of the antifields of the Yang-Mills sector contain contributions which are not covariant under the gauge transformations, e.g. $\partial_\rho H^{\rho\mu\nu}A_{a\nu}$ in (7.82). One can remedy both problems by redefining the antifields of the Yang-Mills sector according to the following invertible transformations:

$$A_a^{*\mu} \rightarrow A_a^{*\mu} + 2B^{*\mu\nu}A_{a\nu} - 2\eta^{*\mu}C_a, \tag{7.86}$$

$$C_a^* \rightarrow C_a^* + 2\eta^{*\mu}A_{a\mu} - 2\rho^*C_a. \tag{7.87}$$

In terms of the new variables, the BRST differential now takes the familiar form, $s = \delta + \gamma$ with:

$$\delta B^{*\mu\nu} = \partial_\rho H^{\rho\mu\nu}; \quad \delta\eta^{*\mu} = -\partial_\nu B^{*\nu\mu}; \quad \delta\rho^* = -\partial_\mu \eta^{*\mu}; \tag{7.88}$$

$$\delta A_a^{*\mu} = D_\nu F_a^{\nu\mu} + 2\lambda H^{\mu\nu\rho}F_{a\nu\rho}; \quad \delta C_a^* = 2\lambda B^{*\mu\nu}F_{a\mu\nu} - D_\mu A_a^{*\mu}, \tag{7.89}$$

and

$$\gamma B^{*\mu\nu} = \gamma\eta^{*\mu} = \gamma\rho^* = 0; \quad \gamma A_a^{*\mu} = C_{abc}A^{*b\mu}C^c; \quad \gamma C_a^* = -C_{abc}C^{*b}C^c; \tag{7.90}$$

$$\gamma\,(fields) = s\,(fields). \tag{7.91}$$

The γ variations of the Yang-Mills variables are now identical to those of the uncoupled theory and A_μ^{*a} and C_a^* transform according to the adjoint representation.

7.3   Cohomology of γ

The cohomology $H^*(\gamma)$ of the Chapline-Manton model can be worked out as in the free case, by exhibiting explicitly the contractible part of the algebra. This contractible part typically gets larger with the coupling: some cocycles are removed from $H^*(\gamma)$. This happens for the models 1, 3 and 4.

Chapline-Manton model 1

In the absence of couplings, the γ-cohomology for the first model is given, according to Theorem **3**, by the polynomials in $(dA)_{\mu_1...\mu_{p+1}}$, $(dB)_{\mu_1...\mu_{p+2}}$, the antifields, their derivatives and the last ghosts A_p^0, B_{p+1}^0. When the coupling is turned on, however, some of these "x"-variables become contractible pairs and get canceled in cohomology. Specifically, it is the last ghosts of ghosts that disappear.

Indeed, as in the proof of Theorem **3** one can replace the original variables by,

$$\partial_{(\alpha_1...\alpha_k} A_{[\mu_1)_1...\mu_l]_2}, \partial_{\alpha_2...\alpha_k} F^0_{\alpha_1\mu_1...\mu_l}, \partial_{(\alpha_1...\alpha_k} B_{[\mu_1)_1...\mu_l]_2}, \text{ and } \partial_{\alpha_2...\alpha_k} H_{\alpha_1\mu_1...\mu_l},$$
$$(7.92)$$

with $F^0_{\mu_1...\mu_l} = \partial_{[\mu_1} A_{\mu_2...\mu_r]}$ $(2 \leq r \leq p)$ and $H_{\mu_1...\mu_l} = \partial_{[\mu_1} B_{\mu_2...\mu_l]}$ $(2 \leq l \leq p+1)$. One then makes a further change of coordinates by replacing $F^0_{\mu_1...\mu_l}$ with $F_{\mu_1...\mu_l} = \partial_{[\mu_1} A_{\mu_2...\mu_r]} - B_{\mu_1...\mu_r}$, which is obviously invertible. The variables can now be associated as in Theorem **3** except that a new contractible pair appears, i.e., $\gamma A_p^0 = B_{p+1}^0$; the x-variables are now made of the $H_{\mu_1...\mu_l}$, $F_{\mu_1...\mu_l}$ and their derivatives.

The important point which makes the argument correct is that $F^0_{\mu_1...\mu_l}$ and $F_{\mu_1...\mu_l}$ only differ by terms which are of lower order in the derivatives of the A-sector (here they even differ only by variables of the B-sector). As this property also holds for the other Chapline-Manton models, the same change of variables will also be possible.

Note that the Bianchi identities for the new field strengths read

$$dF = H, \;\; dH = 0. \tag{7.93}$$

They can be used to express the H-components and their derivatives in terms of the components $F_{\mu_1...\mu_{p+1}}$ and their derivatives, which thus completely generate the cohomology.

To summarize, we have proved:

Theorem 32. *For the Chapline-Manton model 1, the cohomology $H(\gamma)$ is given by the polynomials in the improved field strength components $F_{\mu_1 \ldots \mu_{p+1}}$, the antifields and their derivatives,*

$$\gamma\omega = 0 \Leftrightarrow \omega = \frac{1}{q!}\omega_{\nu_1 \ldots \nu_q}([F_{\mu_1 \ldots \mu_{p+1}}], [\phi_A^*])dx^{\nu_1} \ldots dx^{\nu_q}. \tag{7.94}$$

In particular, there is no *cohomology at non-vanishing pureghost number.*

The situation is very similar to the discussion of the gauged principal $U(1)$ sigma model [92] (see also [93] in this context).

Chapline-Manton model 2

In this case, the change of variables proceeds as for model 1 so that the γ-cohomology is unchanged compared with the free case except that the free curvatures are replaced by the improved, gauge invariant curvatures (7.17). The last ghosts remain in cohomology because $A^{(0,1)}$ is still γ-closed, so the mechanism of the previous subsection is not operative. We thus have,

Theorem 33. *The cohomology of γ for the Chapline-Manton model 2, is given by*

$$H(\gamma) = \tilde{\mathcal{I}} \otimes \mathcal{C} \tag{7.95}$$

where $\mathcal{C}$ is the algebra generated by the last, undifferentiated ghosts A_1^0 and B_{2r}^0, and where $\tilde{\mathcal{I}}$ is the algebra generated by the gauge invariant field strength components $F_{\mu\nu}$, $H_{\mu_1 \ldots \mu_{2r+1}}$, the antifields and their derivatives.

Note the new form of the Bianchi identities on the curvatures,

$$dF = 0, \quad dH = F^{r+1}. \tag{7.96}$$

Chapline-Manton model 3

The discussion of the third example proceeds to a large extent like that of the first one. The last ghosts of ghosts A_1^0 and B_2^0 form a contractible pair and disappear in cohomology; the improved last ghost of ghost

$$\tilde{C}_3^0 = C_3^0 - \frac{1}{2}A_1^0 B_2^0 \tag{7.97}$$

remains. Thus one has,

Theorem 34. *The cohomology of γ for the Chapline-Manton model 3 is given by,*

$$H(\gamma) = \tilde{\tilde{\mathcal{I}}} \otimes \tilde{\mathcal{C}}, \qquad (7.98)$$

where $\tilde{\tilde{\mathcal{I}}}$ is the algebra generated by the gauge invariant field strength components $F_{\mu\nu}$, $G_{\mu\nu\rho\sigma}$, the antifields and their derivatives, and where $\tilde{\mathcal{C}}$ is the algebra generated by the last, improved ghost of ghost $\tilde{C}_3^0 = C_3^0 - \frac{1}{2}A_1^0 B_2^0$.

Again, note the new form of the Bianchi identities,

$$dF = -H, \ dH = 0, \ dG = -FH, \qquad (7.99)$$

which express H in terms of the derivatives of F.

Chapline-Manton model 4

In the absence of coupling, the cohomology of γ is given by the tensor product of the pure Yang-Mills cohomology [94, 95, 96, 97, 62, 63, 48, 98] and of the free 2-form cohomology. We collectively denote by χ_0 (i) the Yang-Mills field strengths, their covariant derivatives $D_{\alpha_1} \ldots D_{\alpha_k} F_{\mu\nu}^a$, the antifields and their covariant derivatives $D_{\alpha_1} \ldots D_{\alpha_k} A_a^{*\mu}, D_{\alpha_1} \ldots D_{\alpha_k} C_a^*$; these transform according to the adjoint representation; and (ii) the free 2-form field strengths $H_{\mu\nu\rho}^0 = (dB)_{\mu\nu\rho}$, their derivatives, the antifields $B^{*\mu\nu}$, $\eta^{*\mu}$, ρ^*, their derivatives and the undifferentiated ghost of ghost ρ. Then the representatives of $H(\gamma)$ in the uncoupled case can be written as $a = \sum_J \alpha_J(\chi_0)\omega^J(C^a)$, where the $\alpha_J(\chi_0)$ are invariant polynomials (under $SU(N)$) in the χ_0 and where the $\omega^J(C^a)$ form a basis of the Lie algebra cohomology of the Lie algebra of the gauge group. The ω^J are polynomials in the so-called "primitive forms", i.e trC^3, trC^5 if $trC^5 \neq 0$, etc.

When the Chern-Simons coupling is turned on, the results are very similar but with two modifications: (i) one must replace in the above cocycles the free field strengths $H_{\mu\nu\rho}^0$ and their derivatives by the improved invariant field strengths $H_{\mu\nu\rho}$ and their derivatives (we shall denote the new set of improved variables defined in this manner by χ); (ii) the ghost of ghost ρ and the primitive form trC^3 now drop from the cohomology since these elements are related by $\gamma\rho = \frac{\lambda}{3}trC^3$, which indicates that trC^3 is exact, while ρ is no longer closed. This last feature underlies the Green-Schwarz anomaly cancellation mechanism. We thus have:

Theorem 35. *The cohomology of γ for the Chapline-Manton model 4 is given by,*

$$H(\gamma) = \mathcal{J} \otimes \mathcal{D}, \qquad (7.100)$$

where (i) $\mathcal{J}$ is the algebra of the invariant polynomials in the Yang-Mills curvature components, the antifields and their covariant derivatives, as well as in the components of the gauge invariant curvature H and their derivatives; and (ii) $\mathcal{D}$ is the algebra generated by the "primitive forms" trC^5, trC^7, ..., trC^{2N-1}.

We recall that the Lie algebra cohomology for $SU(N)$ is generated by the primitive forms trC^3, trC^5, ... up to trC^{2N-1} [65, 99]. The Bianchi identities read,

$$DF = 0, \quad dH = trF^2. \tag{7.101}$$

7.4 $H(s|d)$ - Antifield independent solutions

7.4.1 Covariant Poincaré Lemma

In the free case, we have shown that the various spaces $E_k, k > 0$ used in the lifts of elements of $H(\gamma)$ might be calculated in the so-called "small algebra". This result relies upon the invariant Poincaré lemma which states that each class of the invariant cohomology of d has a representative in $\mathcal{A}$. We now show that it is also the case for the Chapline-Manton models and therefore that $\mathcal{A}$ is again the relevant space in which to calculate the spaces $E_k, k > 0$. One has,

Theorem 36. *Let P be a gauge invariant polynomial. If P is closed, then P is the sum of a closed, gauge invariant polynomial belonging to the small algebra and of the exterior derivative of an invariant polynomial,*

$$dP = 0 \Leftrightarrow P = Q + dR, \; Q \in \mathcal{A}, \; dQ = 0, \tag{7.102}$$

(with P, Q and R all gauge-invariant). Furthermore, if Q is d-exact in the algebra of gauge-invariant polynomials, $Q = dS$ with S gauge-invariant, one may assume that S is in the small algebra (and gauge-invariant). Therefore, the invariant cohomology of d can be evaluated in $\mathcal{A}$ instead of the bigger algebra $\mathcal{P}$.

Note that while in the free case the conditions $Q \in \mathcal{A}$ and $Q = dS$ (with S gauge-invariant) imply $Q = 0$, this is no longer true here.

Proof: We shall prove the theorem for the specific case of the second model. The proof proceeds in the same way for the other models. We

introduce a grading N that counts the number of derivatives of the B field. According to this grading P and d split as,

$$P = P_k + P_{k-1} + \cdots + P_1 + P_0, \quad d = D_1 + D_0, \qquad (7.103)$$

with,

$$N(P_i) = i, \quad N(D_i) = i. \qquad (7.104)$$

The differential D_1 takes derivatives only of the B-field, the differential D_0 takes derivatives only of the A-field. Because P is gauge-invariant, the B-field enters P only through the components of dB and their derivatives. Furthermore, even though the P_i's with $i < k$ may involve the components A_μ's and their symmetrized derivatives, P_k depends on A only through the $F_{\mu\nu}$ and their derivatives.

The equation $dP = 0$ yields $D_1 P_k = 0$ at the highest value of the N-degree. According to the results for the free case, this implies $P_k = D_1 R_{k-1} + m_k$ where R_{k-1} is a polynomial in the components of dB and their derivatives, while m_k is a polynomial in the form dB, both with coefficients made of the components of F and their derivatives (which fulfill $D_1 F_{\mu\nu} = 0$). One then covariantize R_{k-1} and m_k by completing dB into H. This only introduces terms of lower N-degree. We denote the covariant objects by r and m, respectively. One has $P_k = [dr + m]_k$ and $P = dR_{k-1} + m_k + $ $more$, where "$more$" is an invariant polynomial of maximum N-degree strictly smaller than k. The invariant polynomial m - which exists only if $k = 1$ or 0 since $H^2 = 0$ - is of order k in the exterior form H. It must be closed by itself since there can be no compensation between $D_0 m$ and $D_1(more)$ which is necessarily of lower degree in the components of H and their derivatives. It follows from $D_0 m = 0$ that $m = \mu(F, H) + ds$, where μ is a polynomial in the forms F and H and where s is an invariant polynomial (using again the results for the free case and $H dp = -d(Hp) + more$). Thus one can get rid of P_k by adding to P terms of the form (7.102) of the theorem. By repeating the argument at the successive lower degrees, one reaches the desired conclusion.

To prove the second part of the theorem, one first observes that if $dQ = 0$, then $Q(F, H)$ does in fact not involve H, $Q = Q(F)$ (because $d(\alpha(F) + \beta(F)H) = 0 \Rightarrow \beta(F)F^{r+1} = 0 \Rightarrow \beta(F) = 0$). Assume then that $Q = dU$, where U is a gauge-invariant polynomial, $U = U([H], [F])$. By expanding U according to the N-degree, $U = U_0 + U_1 + ... + U_l$, one finds at higher order $D_1 U_l = 0$, which implies as above $U_l = D_1 R_{l-1} + m_l$ where m_l is a polynomial in the form dB.

One can remove $D_1 R_{l-1}$ from U_l by subtracting dR_{l-1} from U, which does not affect Q. Thus, only m_l, which is present for $l = 1$ or $l = 0$, is relevant. By repeating the argument, one finally arrives at,

$$U = Ha([F]) + b([F]). \tag{7.105}$$

The condition $Q = dU$ further implies $da = 0$ and thus $a = d\nu([F]) + \rho(F)$ where $\rho(F)$ is a polynomial in the form F. The term $Hd\nu([F])$ is irrelevant since it can be absorbed into $b([F])$ with a d-exact term. Thus, $U = H\rho(F) + b'([F])$. The condition $Q(F) = dU$ now reads $Q(F) = k(F) + db'([F])$ where $k(F)$ is a polynomial in F and implies $db' = 0$ (invariant Poincaré Lemma in the free case). But then, again, one can drop b' from U, which proves the second assertion. $\square$

It follows from this theorem that there is no restriction in investigating the invariant d-cohomology in the small algebra. Elements of $H(\gamma)$ that can be lifted at least once necessarily belong to $\mathcal{A}$ up to trivial terms. There is no restriction in the investigation of the next lifts either because again $E_1^{small} \simeq E_1$. If a γ-cocycle $a \in \mathcal{A}$ can be written as $a = du + \gamma v$ where u and v are in the big algebra and $\gamma u = 0$, then one may find u' and v' in $\mathcal{A}$ such that $a = du' + \gamma v'$ (with $\gamma u' = 0$). This follows from the second part of the theorem. Obstructions to lifts within $\mathcal{A}$ cannot be removed by going to the big algebra.

Chapline-Manton model 1

For the first Chapline-Manton model discussed above, the invariant cohomology of d is trivial. Indeed, in the algebra generated by F and H, the differential d takes the contractible form $dF = H$, $dH = 0$. Thus

$$E_1 \equiv H(d_0, E_0^{small}) = 0 \tag{7.106}$$

where E_0^{small} is the algebra generated by F and H.

Chapline-Manton model 2

In the algebra generated by the gauge-invariant curvatures, d takes the form

$$dF = 0, \; dH = F^{r+1}. \tag{7.107}$$

Since $H^2 = 0$, any element in this algebra is of the form

$$a = \alpha(F) + \beta(F)H, \tag{7.108}$$

where $\alpha(F)$ and $\beta(F)$ are polynomials in F. The condition that a is closed implies $\beta(F)F^{k+1} = 0$, which forces $\beta(F)$ to vanish. Furthermore $a \equiv \alpha(F)$ is exact if it is in the ideal generated by F^{r+1}. Thus, we have the theorem:

Theorem 37. *The invariant cohomology of d for the Chapline-Manton model 2 is the quotient of the algebra generated by the F's by the ideal generated by F^{r+1}.*

Chapline-Manton model 3

For the third model, d is given by (7.99). By redefining the curvature G as

$$G_M = G - \frac{F^2}{2}, \tag{7.109}$$

the algebra is brought to the form

$$dF = -H, \ dH = 0, \ dG_M = 0, \tag{7.110}$$

from which it follows that:

Theorem 38. *For the third model, the invariant cohomology of d is given by the polynomials in the variable $G_M = G - F^2/2$.*

Chapline-Manton model 4

The invariant polynomials in the small algebra are the polynomials in the gauge-invariant curvature H of the 2-form and in the "fundamental" invariants trF^2, trF^3, ... trF^N for $SU(N)$ (this is a basis for the $SU(N)$ symmetric polynomials). These polynomials are all closed, except H, which obeys $dH = trF^2$. Hence, H and trF^2 do not appear in the cohomology.

Theorem 39. *For the fourth model, the invariant cohomology of d is given by the polynomials in trF^3, trF^4, ... trF^N.*

7.4.2   Results

We can now compute the various E_k for the Chapline-Manton models.

Chapline-Manton model 1

The analysis is obvious in this case since there is no non trivial descent. All solutions of the Wess-Zumino consistency condition can be taken to be strictly annihilated by γ, i.e., can be taken to be in E_0 ($E_1 = 0$). They are thus completely described by Theorem **32** (from which one must remove the d-exact terms $d\alpha([F])$).

Chapline-Manton model 2

The second model is more interesting. Using Theorem **37** we know that $E_1 \simeq E_1^{small}$ is isomorphic to the algebra generated by F, A_1^0 and B_{2r}^0, with the relation $F^{r+1} = 0$. This is no longer a free algebra contrary to the situation encountered in the free case.

The differential d_1 is non trivial and given by,

$$d_1 A_1^0 = F, \; d_1 F = 0, \; d_1 B_{2r}^0 = 0, \tag{7.111}$$

when $r > 1$, which we shall assume at first. Because F is subject to the relation $F^{r+1} = 0$, the cohomological space $E_2 \equiv H(d_1, E_1)$ is isomorphic to the algebra generated by B_{2r}^0 and $\mu(A, F)$ with

$$\mu(A, F) = -A_1^0 F^r. \tag{7.112}$$

One can take for F_1 the space of polynomials of the form $(B_{2r}^0)^l Q_l(F) A_1^0$ where Q_l is a polynomial in F of degree strictly less than r. To obtain the lifts of these cocycles, one can use modified Russian formulas:

$$\tilde{\gamma} \tilde{B} = H - F\tilde{A}, \; \tilde{\gamma} \tilde{A} = F, \tag{7.113}$$

with $\tilde{B} = B_2^0 + B_1^1 + B_0^2$ and $\tilde{A} = A_1^0 + A_0^1$. The lifts are therefore, $l(B_{2r}^0)^{l-1} B_{2r-1}^1 Q_l(F) A_1^0 + (B_{2r}^0)^l Q_l(F) A_0^1$.

The next differentials d_2, d_3 ... vanish up to d_{2r-1}. So, $E_2 = E_3 = \cdots = E_{2r-1}$. One has

$$d_{2r-1} B_{2r}^0 = \mu(A, F), \; d_{2r-1} \mu(A, F) = 0. \tag{7.114}$$

Thus $E_{2r} = 0$.

One may take for F_{2r-1} the space of polynomials in B_{2r}^0 (with no constant piece). The k-th lift of the monomial $(B_{2r}^0)^l = [(\tilde{B})^l)]_{2rl}^0$ is $[(\tilde{B})^l)]_{2rl-k}^k$.

Note in particular that $\mu(A, F)$ does not appear in any of the spaces F_k, because it is now trivial. In the free case, $\mu(A, F)$ was an element of F_1 and the bottom of a non-trivial descent of length two. The coupling to the $2r$-form makes it disappear from the cohomology. At the same time, the cocycle F^{r+1}, which was in the invariant cohomology of d in the free case, is now d-exact in the space of invariant polynomials. Also, while B_{2r}^0 could be transgressed all the way up to H in the free case, its lift now stops at ghost number one with μ.

The situation for $r = 1$ is similar. The two steps corresponding to the differentials d_1 and d_{2r-1} are now combined in a single one so that the space E_2 vanishes. The easiest way to see this is to observe that $H(d_1, E_1)$ (with

$d_1 A^{(0,1)} = F$, $d_1 F = 0$ and $d_1 B^{(0,2)} = \mu(A, F)$ for $r = 1$) is isomorphic to $H(D, E_0)$ with $DA^{(0,1)} = F$, $DF = 0$, $DH = F^{r+1}$, $DB^{(0,2)} = \mu(A, F) + H$. Indeed, one may view the generator H as Koszul generator for the equation $F^{r+1} = 0$. The change of variable $H \to H' = H + \mu$ brings then D to the manifestly contractible form.

Chapline-Manton model 3

The third model is essentially a combination of the first model in the (A, B)-sector and of the free model for the improved 3-form $C_M = C - AB - \frac{1}{2}AdA$, with curvature $G_M = dC_M$ and improved last ghost of ghost $\tilde{C}^{(0,3)}$ (7.97). Using Theorem **38** we know that $E_1 \simeq E_1^{small}$ has generators G_M and $\tilde{C}_3^0$. The differentials d_1 and d_2 vanish so $E_1 = E_2 = E_3$. One next finds that the differential d_3 acts as,

$$d_3 \tilde{C}_3^0 = G_M, \; d_3 G_M = 0, \tag{7.115}$$

so that $E_4 = 0$.

For F_3, one can take as representatives polynomials of the form $P(G_M)\tilde{C}_3^0$. Their successive lifts are obtained by using the Russian formula,

$$\tilde{\gamma}\tilde{C}_M = G_M, \tag{7.116}$$

with $\tilde{C}_M = C_M + E_M + L_M + \tilde{C}_3^0$, $C_M = C - AB - \frac{1}{2}AdA$, $E_M = C_1^2 - \frac{1}{2}A_0^1 B_1^1 - \frac{1}{2}dAA_1^1 - BA_1^0$ and $L_M = C_2^1 - \frac{1}{2}AB_2^0 - \frac{1}{2}A_1^0 B_1^1$. The successive lifts of $P(G_M)\tilde{C}_3^0$ are therefore $P(G_M)L_M$, $P(G_M)E_M$ and $P(G_M)C_M$.

Chapline-Manton model 4

The cohomology $H(\gamma|d)$ for the 2-form has been studied in Section **4.2.6** while $H(\gamma|d)$ for Yang-Mills theory has been extensively studied in the literature [58, 60]. Here we only highlight the points which are relevant when the two models interact.

In the absence of coupling, the non trivial differentials are,

$$d_2 B_2^0 = H, \; d_2 H = 0, \tag{7.117}$$

$(B_2^0 \equiv \rho)$ and

$$d_3 tr C^3 = tr F^2, \; d_3 tr F^2 = 0, \tag{7.118}$$

$$d_5 tr C^5 = tr F^3, \; d_5 tr F^3 = 0, \tag{7.119}$$

$$\vdots \tag{7.120}$$

$$d_{2N-1} tr C^{2N-1} = tr F^N, \tag{7.121}$$

(see [58]). We have here only written down explicitly the actions of the non trivial d_k's on the contractible pairs. The last ghost of ghost B_2^0 is non trivial and can be lifted twice; trC^3 is non trivial and can be lifted three times; trC^5 is non trivial and can be lifted five times; more generally, trC^{2k+1} is non trivial and can be lifted $(2k+1)$ times.

When the coupling is turned on, the variables ρ and trC^3 disappear from the γ-cohomology. It follows that all the solutions of the Wess-Zumino consistency condition that previously were above a polynomial in ρ and trC^3 disappear or become trivial. This last feature is known as the Green-Schwarz anomaly cancellation mechanism [100]. At the same time, the differential d_0 becomes non trivial, as for the previous Chapline-Manton models. One has

$$d_0 H = trF^2, \ d_0 trF^2 = 0 \tag{7.122}$$

which explicitly shows that trF^2 disappears from the invariant cohomology. The other differentials (7.119) through (7.121) remain unchanged.

7.4.3 Counterterms and anomalies

As in the free case, we summarize the previous results by producing explicitly the antifield-independent counterterms and anomalies, i.e., $H_0^n(\gamma|d)$ and $H_1^n(\gamma|d)$.

Counterterms and anomalies of type A

The counterterms that lead to a trivial descent involve in general the individual components of the gauge-invariant field strengths and their derivatives and cannot generically be expressed as exterior products of the forms F or H. They are the gauge-invariant polynomials and read explicitly,

$$a = a([F])d^n x, \tag{7.123}$$

for the Chapline-Manton model 1,

$$a = a([F], [H])d^n x, \tag{7.124}$$

for the Chapline-Manton model 2,

$$a = a([F], [G])d^n x, \tag{7.125}$$

for the Chapline-Manton model 3 and,

$$a = P_I([F], [H])d^n x, \tag{7.126}$$

for the Chapline-Manton model 4, where P_I is an invariant function of $F_{\mu\nu}^a$ and their covariant derivatives, as well as of $[H]$.

In order for those counterterms to be non-trivial a should satisfy $a \neq db$ in all cases which is equivalent to the condition that its variational derivatives with respect to the fields do not identically vanish.

We have assumed that the spacetime forms dx^μ could occur only through the product $dx^0 dx^1 \cdots dx^{n-1} \equiv d^n x$ as is required by Lorentz-invariance.

The anomalies that lead to a trivial descent are sums of terms of the form $a = P C d^n x$ where P is a gauge-invariant polynomial and C is a last ghost of ghost of ghost number one, which must be non trivial in $H(\gamma)$. These anomalies exist only in the Chapline-Manton model 2 which has last ghost of ghosts with ghost number one. One has explicitly,

$$a = P([F], [H]) A_1^0 d^n x \quad \text{(second CM models).} \quad (7.127)$$

a will be trivial if and only if $P = dR([F], [H])$. Indeed, if a is trivial then it is of the form, $a = \gamma c + de$ with $\gamma e + dm = 0$, where e is of ghost number one and form degree $n-1$. Using the results of Section **7.4.2** we see that no element of an F_k can be lifted on a solution in ghost number one and form degree $n-1$. Therefore, up to irrelevant terms, e is of the form $e = R([F], [H]) A_1^0$ which implies $P = dR([F], [H])$.

7.4.4 Counterterms of type B

As in the free case (Section **4.2.8**), we determine the solutions a which descend non-trivially starting directly from the obstruction $P = da$ since the invariant cohomology of d is known.

Chapline-Manton model 1

There is in this case no non trivial solution of type B since there is no non trivial descent.

Chapline-Manton model 2

One may proceed as for the free theory. The polynomial $P = da$ must be taken in the invariant cohomology of d and so is a polynomial in the curvatures F with F^{r+1} identified with zero. This implies,

$$da = F^{m+1} = d(F^m A) \text{ with } m < r. \quad (7.128)$$

As in the free theory, this leads to the Chern-Simons terms,

$$a = F^m A, \quad (7.129)$$

except that $F^r A$ is now absent because it can be brought into class A by the addition of exact terms. However, due to the restriction $m < r$, these Chern-Simons terms are never of form degree n and therefore, they do not contribute to the counterterms.

Chapline-Manton model 3

In this case, the obstruction $P = da$ is a polynomial in the improved field strength G_M. Therefore, one has $P = dQ(G_M, C_M)$ and so up to trivial terms $a = Q(G_M, C_M) = R(G_M)C_M$ and a is linear in the improved potential C_M. The Chern-Simons solution Q exists only in spacetime dimension $4k - 1$.

Chapline-Manton model 4

Again, one finds as solutions the familiar higher order Yang-Mills Chern-Simons not involving $tr F^2$ or ω_3. These are only available in odd dimensions > 3.

7.4.5 Anomalies of type B

As in the free theory, the anomalies a of type B can be of two types. They can arise from an obstruction that lives one dimension higher or from an obstruction that lives two dimensions higher.

In the first case, the obstruction da has form degree $n + 1$ and ghost number 1. This case is only possible in the second Chapline-Manton model 2, since there is no γ-cohomology in ghost number one for the other models. The obstruction da reads,

$$da + \gamma(\text{something}) = P(F)A_1^0. \tag{7.130}$$

The right-hand side of (7.130) is necessarily the d_k of some element in F_{k-1} of ghost number > 1. According to the results of Section **7.4.2** we see that the only term which has as obstruction a polynomial of the form $P(F)A_1^0$ is B_{2r}^0. However, the corresponding lift a is B_1^{2r-1} which is not of form degree n. There is thus no anomaly in this case.

In the second case, the anomaly can be lifted once, $da + \gamma b = 0$. The obstruction db to a further lift is then a $(n + 2)$-form of ghost number 0.

There is no solution of this type for the Chapline-Manton model 1 because of the lack of a non-trivial descent.

For the Chapline-Manton model 2, there is again no anomaly that can be lifted once since the obstruction $db = kF^m \in H^{inv}(d)$ cannot be of form-degree $n + 2$ due to the restriction $m < r + 1$.

For the Chapline-Manton model 3, solutions descending from polynomials $P(G_M)$ in two dimensions higher exist only in spacetime dimensions equal to $4k - 2$. They are given by $a = Q(G_M)L_M$ with L_M defined in Section **7.4.2** above.

Finally, for the Chapline-Manton model 4, one has all the anomalies of the $SU(N)$ pure Yang-Mills theory, except those involving the cocycle trC^3 and its lifts which are now trivial.

7.5 $H(s|d)$ - Antifield dependent solutions

The calculation of the antifield dependent solutions of the Wess-Zumino consistency condition for the Chapline-Manton models proceeds in very much the same way as for the free theory.

To begin with, one repeats the analysis of Section **4.3.1**. In particular, Theorem **11** which states that there can be no non-trivial descents in $H(\gamma|d)$ involving the antifields remains valid. Using this result, it is again easy to prove that up to allowed redefinitions, the component of highest antighost number of a BRST cocycle can be written as,

$$a_{g,q}^n = P_J \omega^J. \tag{7.131}$$

In (7.131), P_J is in the invariant cohomology $H^{inv}(\delta|d)$ while the ω^J are a basis of the polynomials in the ghosts belonging to $H(\gamma)$.

To make use of this result we must calculate $H^{inv}(\delta|d)$ for the Chapline-Manton models. This is the subject of the next section. Afterwards, we will study which of the terms (7.131) can be completed by components of lower antighost numbers to produce solutions of the Wess-Zumino consistency condition.

7.5.1   Invariant characteristic cohomology

The calculation of the cohomology $H^{inv}(\delta|d)$ proceeds virtually identically for the four Chapline-Manton models considered. One decomposes the representatives of $H^{inv}(\delta|d)$ and the Koszul-Tate differential δ according to specific degrees in order to use the results on the invariant characteristic cohomology for the free theory. To avoid repetition, the method will only be explicited for the first Chapline-Manton model.

Chapline-Manton model 1

Theorem 40. *For the Chapline-Manton model 1, the invariant characteristic cohomology $H^{inv}(\delta|d)$ in antighost > 1 and form degree n vanishes.*

Proof: Let us first recall for this CM model the action of the Koszul-Tate differential on the antifields,

$$\delta A^{*\mu_1\ldots\mu_p} = \partial_\nu F^{\nu\mu_1\ldots\mu_p}, \tag{7.132}$$

$$\delta A^{*\mu_1\ldots\mu_{p-j}} = -\partial_\nu A^{*\nu\mu_1\ldots\mu_{p-j}}, \tag{7.133}$$

$$\delta B^{*\mu_1\ldots\mu_{p+1}} = \partial_\nu H^{\nu\mu_1\ldots\mu_{p+1}} - F^{\mu_1\ldots\mu_{p+1}}, \tag{7.134}$$

$$\delta B^{*\mu_1\ldots\mu_{p+1-j}} = -\partial_\nu B^{*\nu\mu_1\ldots\mu_{p+1-j}} + (-)^{j+1} A^{*\mu_1\ldots\mu_{p+1-j}}. \tag{7.135}$$

In the absence of coupling, Theorem **25** indicates that the invariant characteristic cohomology in antighost > 1 and form degree n is given by the linear combinations of the monomials $[\tilde{H}^m(\tilde{F}^0)^l]_q^n$ with $F^0 = dA$.

When the coupling is turned on, there are two modifications in the definition of the Koszul-Tate differential: in (7.132) the curvature F^0 is replaced by the improved curvature $F = dA + B$; in (7.134) and (7.135) there are some extra (invariant) terms in the variations of the antifields of the B-sector.

To obtain the elements a of $H^{inv}(\delta|d)$ in the interacting case, we first decompose a according to the number of derivatives of the invariant variables (field strengths, antifields): $a = a_0 + \ldots + a_k$. According to this degree δ splits as $\delta_1 + \delta_0$; δ_1 increases by one the number of derivatives of the invariants while δ_0 leaves it unchanged and has a non-vanishing action only on the antifields of the B-sector.

At highest degree in the derivatives, Eq. $\delta a + db = 0$ implies,

$$\delta_1 a_k + db_k = 0. \tag{7.136}$$

The differential δ_1 is identical to the Koszul-Tate differential of the free theory except for the substitution $F^0 \to F$ with the consequence that F is now subject to the Bianchi identity $dF = H$ instead of $dF^0 = 0$. To properly take this into account we decompose the solutions c of

$$\delta_1 c + dm = 0, \tag{7.137}$$

according to the polynomial degree of the A-sector. According to this degree we have $\delta_1 = \delta_f + \delta'$ where: 1) δ_f has the same action on the antifields as the Koszul-Tate differential of the free theory; 2) δ' decreases the polynomial degree in A and has a vanishing action on all the antifields except $A^{*\mu_1\ldots\mu_p}$ for which $\delta' A^{*\mu_1\ldots\mu_p} = \partial_\nu B^{\nu\mu_1\ldots\mu_p}$. If we set $c = c_0 + \ldots + c_l$, (7.137) implies $\delta_f c_l + dm_l = 0$ where c_l is now a polynomial in $F^0_{\mu_1\ldots\mu_{p+1}}$ and $H_{\mu_1\ldots\mu_{p+2}}$. Using the results on $H^{inv}(\delta|d)$

for the free case we then have, $c_l = \lambda_l [\tilde{H}^{r_l}(\tilde{F}^0)^l]_q^n + \delta_f \mu([F^0], [H]) + d\nu([F^0], [H])$ for $l > 0$ with λ_l a constant. By a redefinition of the terms of lower polynomial degree in the A-sector and the addition of trivial terms we conclude that $c = c_0 + \ldots c_{l-1} + \lambda_l [\tilde{H}^{r_l}\tilde{F}^l]_q^n$, where $c_0 + \ldots + c_{m-1}$ is of maximal order $m - 1$ in the variables of the A-sector and has to satisfy (7.137) on its own. By recurrence we thus have up to trivial terms $c = \lambda_0 [\tilde{H}^{r_0}]_q^n + \sum_l \lambda_l [\tilde{H}^{r_l}\tilde{F}^l]_q^n$.

Using this result, we deduce that unless $k = 0$ in (7.136) a_k can be removed from a so we necessarily have $a = a_0 = \lambda_0 [\tilde{H}^{r_0}]_q^n + \sum_l \lambda_l [\tilde{H}^{m_l}\tilde{F}^l]_q^n$.

Finally, the last condition which a has to satisfy is $\delta_0 a_0 = 0$ which immediately implies $k_0 = k_l = 0$ and therefore $a = \sum_l \lambda_l [\tilde{F}^l]_q^n$. However, because $(\delta + d)\tilde{H} + \tilde{F} = 0$, these cocycles are all trivial. $\square$

Chapline-Manton model 2

Theorem 41. *For the Chapline-Manton model 2, the invariant characteristic cohomology $H^{inv}(\delta|d)$ in antighost > 1 and form degree n is given by linear combinations of the monomials $[F^l \tilde{H}^k]_{q(l,k)}^n$, with l and k such that $q(k, l) = n - 2l - k(n - p - 1) > 1$.*

Chapline-Manton model 3

Theorem 42. *For the Chapline-Manton model 3, the invariant characteristic cohomology $H^{inv}(\delta|d)$ in antighost > 1 and form degree n is given by linear combinations of the monomials $[\tilde{G}^k]_{q(k)}^n$, with k such that $q(k) = n - k(n-4) > 1$.*

Chapline-Manton model 4

Theorem 43. *For the Chapline-Manton model 4, the invariant characteristic cohomology $H^{inv}(\delta|d)$ in antighost > 1 and form degree n is given by linear combinations of the monomials $[\tilde{H}^k]_{q(k)}^n$, with k such that $q(k) = n - k(n-3) > 1$.*

7.5.2   Results

Using the above four theorems we can continue our construction of the antifield dependent solutions of the Wess-Zumino consistency condition. Here, we will focus our attention on the counterterms and the anomalies. The other values of the ghost number are analyzed similarly.

Counterterms and anomalies of type I

As in the free case, the representatives of $H(s|d)$ for which the expansion according to the antighost number stops at order 1 are related to the gauge invariant conserved currents of the theory. These solutions exist a priori for the four CM models and are given by,

$$a_g^n = k_{\Delta a_1 \ldots a_r}(j^\Delta \mathcal{Q}_{1,g}^{a_1 \ldots a_r} + a^\Delta \mathcal{Q}_{0,g+1}^{a_1 \ldots a_r}), \tag{7.138}$$

where the $k_{\Delta a_1 \ldots a_r}$ are constants and the a^Δ form a complete set of non-trivial gauge invariant global symmetries of the model and satisfy $\delta a_\Delta + dj_\Delta = 0$.

Counterterms and anomalies of the form (7.138) exist only if the cohomology $H(\gamma)$ has non-trivial elements in pureghost number 1 or 2. This occurs only in the Chapline-Manton model 2. If $r = 1$, the counterterms and anomalies are given respectively by,

$$a_g^n = k_\Delta(j^\Delta A_0^1 + a^\Delta A_1^0) \quad \text{(counterterm)}, \tag{7.139}$$

$$a_g^n = k_\Delta(j^\Delta B_1^1 + a^\Delta B_2^0) \quad \text{(anomaly)}. \tag{7.140}$$

If $r > 1$ then only the counterterms (7.139) are present.

Counterterms and anomalies of type II

The counterterms and anomalies of this type correspond to solutions of the Wess-Zumino consistency condition for which the expansion according to the antighost number stops at order ≥ 2.

Chapline-Manton model 1 There is no counterterm or anomaly of this type because the cohomology $H(\gamma)$ vanishes at pureghost number > 0. Alternatively, one can view the absence of solutions of type II as a consequence of Theorem **40**.

Chapline-Manton model 2 For this model, the ghosts of ghosts available to construct the component of highest antighost number of a BRST cocycle are A_1^0 and B_{2r}^0. Combining this with Theorem **41**, we obtain for the counterterms $a_{g,q}^n = a_{0,2r+1}^n = k[\tilde{H}]_{2r+1}^n B_{2r}^0 A_1^0$ and for the anomalies $a_{g,q}^n = a_{1,2r-1}^n = k[F\tilde{H}]_{2r-1}^n B_{2r}^0$ or $a_{g,q}^n = a_{1,2r-1}^n = k[\tilde{H}^2]_{2r-1}^n B_{2r}^0$ (the last term is only available in spacetime dimension $n = 2r + 3$).

The $a_{g,q}^n$ corresponding to the anomalies are easily completed into solutions of the Wess-Zumino consistency conditions. They yield the following

representatives of $H(s|d)$,

$$a_1^n = k[F\tilde{H}\tilde{B}]_1^n \quad \text{(anomaly)}, \tag{7.141}$$

$$a_1^n = k[\tilde{H}^2\tilde{B}]_1^n \quad \text{(anomaly)} \text{ in spacetime dimension } n = 2r + 3. \tag{7.142}$$

Note that in (7.141) and (7.142) we suppose $r \geq 2$ otherwise the corresponding anomalies are of type I.

For the counterterms, the situation is more complicated. Indeed, $a_{g,q}^n = a_{0,2r+1}^n = k[\tilde{H}]_{2r+1}^n B_{2r}^0 A_1^0$ cannot be completed in a BRST cocycle. This implies that for the second CM model there are no counterterms of type II. The proof is the following:

We have,

$$\delta(k[\tilde{H}]_{2r+1}^n B_{2r}^0 A_1^0) = -d(k[\tilde{H}]_{2r}^{n-1} B_{2r}^0 A_1^0)$$
$$- \gamma(k[\tilde{H}]_{2r}^{n-1}(B_{2r-1}^1 A_1^0 + B_{2r}^0 A_0^1)), \quad (7.143)$$

and thus

$$a_{0,2r}^n = k[\tilde{H}]_{2r}^{n-1}(B_{2r-1}^1 A_1^0 + B_{2r}^0 A_0^1) + m_{2r} B_{2r}^0, \tag{7.144}$$

where m_{2r} is a polynomial in the invariants. At order $2r - 1$ in the antighost number we then have,

$$\gamma(a_{2r-1} - k[\tilde{H}]_{2r-1}^{n-2}(B_{2r-2}^2 A_1^0 + B_{2r-1}^1 A_0^1))$$
$$+ d(b_{2r-1} - k[\tilde{H}]_{2r-1}^{n-2}(B_{2r-1}^1 A_1^0 + B_{2r}^0 A_0^1)) + (\delta m_{2r})B_{2r}^0$$
$$+ (-)^{n-2r-1} k[\tilde{H}]_{2r-1}^{n-2} B_{2r}^0 F = 0. \quad (7.145)$$

Acting with γ on this equation we see that $b'_{2r-1} = b_{2r-1} - k[\tilde{H}]_{2r-1}^{n-2}(B_{2r-1}^1 A_1^0 + B_{2r}^0 A_0^1)$ is an element of $H(\gamma|d)$. Because $2r - 1 > 0$ we have up to irrelevant terms $b'_{2r-1} = u_{2r-1} B_{2r}^0$ where u_{2r-1} only depends on the invariants. Eq. (7.145) then implies,

$$(-)^{n-2r-1} k[\tilde{H}]_{2r-1}^{n-2} F + \delta m_{2r} + d u_{2r-1} = 0. \tag{7.146}$$

If $r > 1$, we must have $k = 0$ because according to Theorem **41**, $[\tilde{H}]_{2r-1}^{n-2} F$ defines a non-trivial class of $H^{inv}(\delta|d)$. Our statement is thus proved for $r > 1$.

If $r = 1$, Eq. (7.146) admits solutions. Indeed in form notation, the action of δ on the antifields of the 1-form (Eq. (7.33)-(7.34)) reads,

$$\delta\overline{A}_1^* + d\overline{F} + \alpha F^r \overline{H} = 0, \tag{7.147}$$

$$\delta\overline{A}_2^* + d\overline{A}_1^* + \alpha F^r \overline{B}_1^* = 0, \tag{7.148}$$

where $\alpha = \frac{2(r+1)}{(2r+1)!}$. Up to trivial terms, the solutions of (7.146) are therefore,

$$m_{2r} = \frac{1}{\alpha}(-)^{n-3}k\overline{A}_2^*; \quad u_1 = \frac{1}{\alpha}(-)^{n-3}k\overline{A}_1^*. \qquad (7.149)$$

Returning to Eq. (7.145) we thus have,

$$a_1 = k[\tilde{H}]_1^{n-2}(B_0^2 A_1^0 + B_1^1 A_0^1) + \frac{1}{\alpha}(-)^{n-3}k\overline{A}_1^* B_1^1 + m_1 A_1^0, \qquad (7.150)$$

where m_1 only depends on the invariants. It thus appears that for some Chapline-Manton models, it is possible to eliminate the first obstruction which is met in the construction of BRST cocycles. This is in contrast with the free case where those obstructions cannot be eliminated without imposing constraints on the arbitrary parameters present in $a_{g,q}^n$.

However, we now show that for the second CM Model, the construction of the counterterm is obstructed at the next step. Indeed, at order 0 in the antighost number we have,

$$\gamma a_0' + db_0'$$
$$+ k(-)^{n-3}\overline{H}HA_1^0 + \frac{1}{\alpha}(-)^{n-3}k\overline{F}FA_1^0 + (\delta m_1)A_1^0 = 0. \quad (7.151)$$

Acting with γ on this equation and using our results on $H(\gamma|d)$ we see that the obstruction for db_0' to be γ-exact is of the form $P(F) + du_0$ where u_0 only depends on the invariants. Therefore, (7.151) reduces to,

$$k(-)^{n-3}\overline{H}H + \frac{1}{\alpha}(-)^{n-3}k\overline{F}F + P(F) + \delta m_1 + du_0 = 0. \qquad (7.152)$$

Since m_1 is linear in the antifields of antighost number 1, it can be written as $m_1 = A^{*\mu}I_\mu d^n x + B^{*\mu_1\mu_2}G_{\mu_1\mu_2}d^n x$ where I_μ and $G_{\mu\nu}$ are functions of the fields strength and their derivatives (terms containing derivatives of the antifields are absorbed in a redefinition of u_0 in (7.152)). We thus have the condition,

$$k(-)^{n-3}\overline{H}H + \frac{1}{\alpha}(-)^{n-3}k\overline{F}F + P(F) + \partial_\rho H^{\rho\mu\nu}G_{\mu\nu}d^n x$$
$$+ (\partial_\nu F^{\nu\mu} - H^{\nu\alpha\mu}F_{\nu\alpha})I_\mu d^n x + du_0 = 0. \quad (7.153)$$

Because d is a linear operator which increases by one the number of derivatives, Eq. (7.153) reads at order 2 in the derivatives of the fields and polynomial degree 2 in the invariants,

$$k(-)^{n-3}\overline{H}H + \frac{1}{\alpha}(-)^{n-3}k\overline{F}F + fF^2 + H^{\nu\alpha\mu}F_{\nu\alpha}f_\mu d^n x = 0, \quad (7.154)$$

where f_μ and f are constants. To obtain this condition one use the fact that u_0, I_μ and $G_{\mu\nu}$ only depend on the field strengths and their derivatives.

If we now take the Euler-Lagrange derivative of (7.154) with respect to $B_{\mu\nu}$ we reach the conclusion that $k = 0$ and this proves our statement. $\square$

Chapline-Manton model 3   For this model, the ghost part of the cohomology $H(\gamma)$ is generated by the improved anticommuting ghost $\tilde{C}_3^0 - \frac{1}{2}A_1^0 B_2^0$.

In order to construct counterterms we thus need elements of $H^{inv}(\delta|d)$ in antighost number 3. However, using Theorem **42** we conclude that there are no such elements (the only candidates arise in spacetime dimension $n = 5$ and are of the form $[\tilde{G}^2]_3^6$ but vanish because $\tilde{G}$ is anticommuting for $n = 5$.)

To construct an anomaly we need elements of the $H^{inv}(\delta|d)$ in antighost number 2. Again using Theorem **42** we see that such terms exist only in spacetime dimension 6 and are of the form $[k\tilde{G}^2]_2^6$. The corresponding anomalies are given by,

$$a_1^n = k[\tilde{G}^2\tilde{C}_M]_1^6, \qquad (7.155)$$

where $\tilde{C}_M$ is defined below (7.116).

Chapline-Manton model 4   According to Theorem **35**, the ω^J are at least of pureghost number 5. In order to construct counterterms or anomalies we therefore need elements of $H^{inv}(\delta|d)$ in antighost number ≥ 4. However, Theorem **43** implies that $H^{inv}(\delta|d)$ vanishes in antighost number > 3. Therefore there are no counterterms or anomalies of type II for the fourth CM model.

Remarks

To summarize, we have shown that one can construct antifield dependent candidate anomalies for the Chapline-Manton models 2 and 3. However, for the four models considered, there are no antifield dependent BRST cocycles in ghost number 0. This shows that these models are quite rigid because it is impossible to construct consistent interactions which deform their gauge transformations.

7.6   Conclusions

In this section we have discussed the Wess-Zumino consistency condition for Chapline-Manton models by explicitly analyzing four examples.

The cohomology $H(s|d)$ was worked out but using the same procedure as for free p-forms. This is possible because the four models considered share the following properties:

1. The gauge algebra is closed on-shell and therefore the action of the longitudinal exterior derivative γ on the fields and ghosts is nilpotent: $\gamma^2 = 0$;

2. Although the action of the BRST differential s on the antifields contains components of antighost numbers > -1, it is possible to redefine the antifields to eliminate those components. Furthermore, the BRST variations of the new antifields only involve combinations of the invariant variables (denoted χ in the text).

The first property allows to calculate separately the BRST cocycles which do not depend on the antifields. The analysis explicitly shows that the "number" of such solutions is reduced compared to the free theory. This is due to the fact that the cohomologies $H(\gamma)$ and $H^{inv}(d)$ typically get smaller because of the emergence of new contractible pairs. In the case of $H(\gamma)$, those contractible pairs consist of ghosts of ghosts while for $H(d)$ they are made up of curvatures since these obey new Bianchi identities.

In the calculation of the antifield dependent BRST cocycles one observes the same reduction in the "number" of solutions. This is a consequence of the fact that $H(\gamma)$ contains less elements but also because the invariant characteristic cohomology $H^{inv}(\delta|d)$ vanishes for more values of the antighost number.

That the BRST cohomology contains less elements for interacting theories than for the free ones is easily understood by considering the following argument. Let us decompose the BRST action of an interacting theory in powers of the coupling constant, $S = S_0 + gS_1 + g^2 S_2 + \ldots$ and let $A = A_0 + gA_1 + g^2 A_2 + \ldots$ be a BRST cocycle: $(S, A) = 0$. This condition implies,

$$(S_0, A_0) = 0, \tag{7.156}$$

$$(S_0, A_1) + (S_1, A_0) = 0, \tag{7.157}$$

$$(S_0, A_2) + (S_1, A_1) + (S_2, A_0) = 0, \tag{7.158}$$

$$\vdots \tag{7.159}$$

The first equation tells us that A_0 is an elements of the cohomology $H(s|d)$ of the free theory. All the other equations beneath (7.156) are conditions which the free BRST cocycle must satisfy. It is therefore natural that for many of them the construction of the higher order terms $A_1, A_2, \ldots$ gets obstructed.

Chapter 8

Comments

In this thesis we have solved the Wess-Zumino consistency condition for an arbitrary system of free p-forms but also for models of the Chapline-Manton type. Using this analysis we have listed for each theory the first-order vertices, counterterms and anomalies; for the free system we have also discussed the gauge structure of the conserved currents.

We insist that our calculations were done in the algebra of forms depending on the components of the antisymmetric tensors, the ghost, the antifields and their derivatives up to an arbitrary high order. However we have shown that:

1. All the antifield independent BRST cocycles can be expressed in terms of exterior products of the fields and the ghost *when these solutions occur in non-trivial descents*; this justifies why previous calculations made in the so-called "small algebra" to obtain counterterms and anomalies are nearly exhaustive.

2. The natural appearance of exterior forms also holds for antifield dependent solutions of the Wess-Zumino consistency condition. Indeed, except for those related to the conserved currents of the theory, one may assume that the BRST cocycles only depend $B_{p_a}^a, H^a, C_1^a, \ldots, C_{p_a}^a, \overline{H}^a$, $\overline{B}_1^{*a}, \ldots, \overline{B}_{p_a+1}^{*a}$. This is a direct consequence of our analysis of the characteristic cohomology

A second feature which deserves to be highlighted is the fact that the calculation of the local BRST cohomology of interacting theories such as the Chapline-Manton models is greatly simplified when $H(s|d)$ is known in the free case. Indeed we have seen in Chapter 7 how we could obtain the BRST cocycles in the Chapline-Manton models from those of the free theory by "perturbative arguments". This encourages future works on the BRST

cohomology in the context of supersymmetric theories and supergravity theories where the p-forms are coupled to other gauge fields.

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