Microscopic statistical description of incompressible Navier-Stokes granular fluids

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Based on the recently-established Master kinetic equation and related Master constant H-theorem which describe the statistical behavior of the Boltzmann-Sinai classical dynamical system for smooth and hard spherical particles, the problem is posed of determining a microscopic statistical description holding for an incompressible Navier-Stokes fluid. The goal is reached by introducing a suitable mean-field interaction in the Master kinetic equation. The resulting Modified Master Kinetic Equation (MMKE) is proved to warrant at the same time the condition of mass-density incompressibility and the validity of the Navier-Stokes fluid equation. In addition, it is shown that the conservation of the Boltzmann-Shannon entropy can similarly be warranted. Applications to the plane Couette and Poiseuille flows are considered showing that they can be regarded as final decaying states for suitable non-stationary flows. As a result, it is shown that an arbitrary initial stochastic $1-$body PDF evolving in time by means of MMKE necessarily exhibits the phenomenon of Decay to Kinetic Equilibrium (DKE), whereby the $1-$body PDF asymptotically relaxes to a stationary and spatially-uniform Maxwellian PDF.

1 - INTRODUCTION

Statistical approaches to the Incompressible Navier-Stokes Equations (INSE) usually adopt one of the following routes:

1) *Asymptotic approach adopting high-Knudsen number and suitable slow-velocity asymptotic approximations of the Boltzmann kinetic equation* (see for example Refs. [1–4]);

2) *Asymptotic approach based on so-called Lattice-Boltzmann Methods*, *i.e.*, adopting suitable discrete-velocities approximation schemes for the Boltzmann kinetic equation (see for instance Refs. [5–8]);

3) *Non-asymptotic approach based on the so-called Inverse Kinetic Theory* (IKT, see Refs. [9–16]). This is based on the adoption of a mean-field interaction acting on a set of collisionless point particles described by means of the Vlasov kinetic equation.

In all cases indicated above the underlying dynamical system is usually considered deterministic and time-reversible, despite the fact that the theory may exhibit in some sense a property of macroscopic irreversibility. In the case of the Boltzmann equation this is due to the celebrated namesake H-theorem, while for INSE this arises because of the irreversible behavior produced by a finite viscosity acting on the Navier-Stokes equation. Nevertheless, the difference between these two approaches is notable. In fact, in contrast to the IKT-approach, it is well

known that Boltzmann H-theorem is usually interpreted as being due to the phenomenon of decay to kinetic equilibrium, namely to the occurrence of an irreversible behavior for the kinetic probability density function (PDF) itself. Departing from these views, in accordance to the GENERIC statistical model proposed by Grmela and Oettinger (see Refs. [17, 18]), the underlying dynamical system should be, instead, time-irreversible too (*microscopic irreversibility*).

The issue arises, however, whether it is possible to reconcile the two approaches 1) and 3) indicated above, namely to formulate a statistical description for INSE which has the following features:

- first, it is non-asymptotic;

- second, the underlying classical dynamical system remains reversible across arbitrary instantaneous collision events while exhibiting at the same time a possible irreversible behavior due to the mean-field force acting on the individual particles;

- third, it may exhibit the phenomenon of decay to kinetic equilibrium, *i.e.*, the irreversible time-evolution of the related kinetic PDF.

As a part of a systematic investigation on the statistical description of granular fluids [19–25], identified here either with finite or large ensembles of finite-size extended particles (namely in which the number of particles N is

respectively considered of order $O(1)$ or $\gg 1$) , the aim of the paper is to look for a possible realization of a microscopic statistical description for INSE. In particular, the problem is formulated in the case of the so-called Boltzmann-Sinai classical dynamical system (CDS), or S_N-CDS [19]), which advances in time an ensemble of N finite-size hard-spheres undergoing instantaneous unary, binary or multiple elastic collisions and is also subject to the action of a suitably-prescribed external mean-field interaction.

Motivations and open problems

A key issue in fluid dynamics is the identification of the appropriate continuous fluid equations, if they exist at all, for granular systems, namely discrete ensembles of finite-size particles which are subject to mutual binary and/or multiple collisions as well as external interactions. Examples are ubiquitous including: *Example #1: Environmental* and *material-science granular fluids* (ambient atmosphere, sea-water and ocean dynamics, etc.); *Example #2: Biological granular fluids* (bacterial motion in fluids, cell-blood dynamics in the human body, blood-vessels, capillaries, etc.); *Example #3: Industrial granular fluids* (grain or pellet dynamics in metallurgical and chemical processes, air and water pollution dynamics, etc.); *Example #4: Geological fluids* (slow dynamics of highly viscous granular fluids, inner Earth-core dynamics, etc.). In most of the cases indicated above it is well known that a consistent statistical description, and in particular a fluid one, is still missing or remains largely unsatisfactory to date.

Crucial aspects involve actually the following key aspects: *A) first requirement:* the proper prescription of the dynamics of granular particles, to be intended as classical particles subject to suitable unary, binary and multiple interactions; *B) the second requirement:* their socalled microscopic statistical description based on classical statistical mechanics, which involves the prescription of a suitable Liouville equation which determines the time evolution of the corresponding phase-space PDF; *C) the third requirement* is to seek a possible finite set of continuous fluid fields $\{Z(\mathbf{r},t)\} \equiv \{Z_i(\mathbf{r},t),\ i=1,..,k\}$ which uniquely identify the "macroscopic" state to be associated with the same granular system and satisfy identically a closed set of PDE's, *i.e.*, a finite system of differential equations which are referred to as *fluid equations,*

$$F_j(Z(\mathbf{r},t),\mathbf{r},t) = 0, \tag{1}$$

with F_j for $j=1,..n$, suitably-smooth real functions. In particular, $Z_i(\mathbf{r},t)$, for $i=1,..,k$ represent continuous real tensor fields defined on the set $\Omega \times I$, with $\Omega \subset \mathbb{R}^3$ the configuration space (*fluid domain*), *i.e.*, either a bounded or unbounded open and connected subset of the real Euclidean space and $I \equiv \mathbb{R}$ the real time axis.

The possibility of actually achieving a closed set of PDE's of this type depends, however, critically on the realization of the requirements indicated above. In this paper we intend to show that such a goal can reached by suitably prescribing a mean-field interaction $\mathbf{F}$ acting on the individual granular particles while still taking into account the mutual interactions occurring among different particles. In particular, we intend to show that in this way a fluid description of granular fluids can be achieved in terms of the so-called incompressible Navier-Stokes equations (INSE). More, precisely this requires, first, identifying $\{Z(\mathbf{r}_1,t)\}$ with the so-called Navier-Stokes fluid fields $\{Z(\mathbf{r}_1,t)\} \equiv \{\rho, \mathbf{V}, p\}$, with $\rho(\mathbf{r}_1,t) \geq 0, \mathbf{V}(\mathbf{r}_1,t)$ and $p(\mathbf{r}_1,t)$ representing respectively the mass density, the fluid velocity and the fluid pressure. Notice, in particular, that here the fluid pressure is allowed for greater generality to take also negative values. These fluid fields are assumed to be defined and of class C^2 in the open set $\Omega \times I$, with $\mathbf{r}_1$ spanning the configuration space Ω, to be identified with a connected open subset of the Euclidean space $\mathbb{R}^3$, and t belonging to the oriented real time axis $I \equiv \mathbb{R}^+$. In particular in the same open set $\Omega \times I$ both $\rho(\mathbf{r}_1,t)$ and $p(\mathbf{r}_1,t)$ are assumed strictly positive. Second, the same fluid fields are required to satisfy in same domain a suitable initial and boundary-value problem associated with the set of fluid equations denoted as *incompressible Navier-Stokes equations* (INSE), namely respectively

$$\nabla_1 \cdot \mathbf{V} = 0, \tag{2}$$

$$\rho_o \frac{\partial}{\partial t}\mathbf{V} + \rho_o \mathbf{V} \cdot \nabla_1 \mathbf{V} + \nabla_1 p - \mathbf{f} - \mu \nabla_1^2 \mathbf{V} = 0, \tag{3}$$

where in particular in the open set Ω the initial conditions are of the form

$$\begin{cases} \rho(\mathbf{r}_1,t_o) = \rho_o, \\ \mathbf{V}(\mathbf{r}_1,t_o) = \mathbf{V}_o(\mathbf{r}_1), \\ p(\mathbf{r}_1,t_o) = p_o(\mathbf{r}_1), \end{cases} \tag{4}$$

with $\rho_o > 0$ a constant mass density, $p_o(\mathbf{r}_1)$ the initial fluid pressure and $\mathbf{V}_o(\mathbf{r}_1)$ the corresponding initial fluid velocity which by assumption must fulfill the isochoricity equation (2). Then by construction it follows that in the domain $\Omega \times I$ the fluid pressure takes the form

$$p(\mathbf{r}_1,t) = \int_\Omega d^3\mathbf{r}_1' \frac{S(\mathbf{r}_1',t)}{|\mathbf{r}_1-\mathbf{r}_1'|} + p_f, \tag{5}$$

with p_f here assumed to be an in principle arbitrary real constant. Moreover, requiring that the force density field $\mathbf{f}$ to be divergence-free and such that $\mathbf{f}$, $S(\mathbf{r}_1',t)$ is the source term

$$S(\mathbf{r}_1,t) = -\rho_o \nabla_1 \mathbf{V} : \nabla_1 \mathbf{V}. \tag{6}$$

In addition, Dirichlet boundary conditions of the form

$$\begin{cases} \rho(\mathbf{r}_1,t)|_{\mathbf{r}_1 \in \partial\Omega} = \rho_o, \\ \mathbf{V}(\mathbf{r}_1,t)|_{\mathbf{r}_1 \equiv \mathbf{r}_w \in \partial\Omega} = \mathbf{V}_w(\mathbf{r}_w), \end{cases} \tag{7}$$

are considered with $\mathbf{V}_w(\mathbf{r}_w)$ a suitably smooth vector function, while $p_1(\mathbf{r}_1, t)|_{\mathbf{r}_1 \equiv \mathbf{r}_w \in \partial\Omega}$ is considered uniquely prescribed by Eq. (5). Here the notation is standard. Thus, ∇_1 is the gradient operator $\nabla_1 \equiv \frac{\partial}{\partial \mathbf{r}}$. The first and second equations are known respectively as the so-called *isochoricity* and *Navier-Stokes* equations, while $\mathbf{f}, \mu > 0$ and $\rho(\mathbf{r}_1, t)$ identify respectively a suitable volume force, the constant fluid viscosity and the mass density. The latter is assumed constant in the set $\overline{\Omega} \times I$, with $\overline{\Omega}$ being the closure of Ω, *i.e.*,

$$\rho(\mathbf{r}_1, t) = \rho_o > 0, \tag{8}$$

the same equation being referred to as *incompressibility condition*. In addition we shall introduce for $\mathbf{f}$ the decomposition

$$\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2, \tag{9}$$

where respectively $\mathbf{f}_1$ and $\mathbf{f}_2$ are assumed of the form $\mathbf{f}_1 = \mathbf{f}_1(\mathbf{r}_1, t) \equiv -\nabla_1 \phi_1$ and $\mathbf{f}_2 = \mathbf{f}_2(\mathbf{r}_1) \equiv -\nabla_1 \phi_2$, namely are respectively assumed non-stationary and stationary. The task posed in this paper involves representing the same fluid fields in terms of suitable statistical averages - *i.e.*, velocity moments - to be evaluated in terms of an appropriate phase-space PDF. More precisely, as shown below, the goal can be realized by means of a microscopic statistical description based on the Master kinetic equation recently developed (see Refs.[19–25]). This refers, in particular, to the phase-space dynamics of smooth hard-spheres whose time-evolution is generated by the so-called Boltzmann-Sinai classical dynamical systems (briefly denoted as S_N-CDS). In addition, as indicated above, the same particles are assumed to be subject to the action of suitable mean-field interactions acting on the center of mass of each particle.

In this paper the problem is posed of formulating a novel statistical description for INSE for granular fluids *in conditions of arbitrary diluteness*. Although such an approach can in principle apply to a variety flows with different kinds of grains which are in turn also possibly embedded in another fluid (see *Examples #1-#4* above), in the following the attention will be focused on the kinetic treatment of a granular system formed by a single species of smooth hard spheres undergoing instantaneous and purely elastic collisions. Starting point is the axiomatic treatment recently developed for the statistical description of the Boltzmann-Sinai CDS [19–25]. The present study follows from two previous theoretical developments. The first one is related to the establishment of statistical descriptions based on so-called inverse kinetic theory (IKT). It relies on the axiomatic introduction of a Liouville equation describing the collisionless dynamics of particles subject to the action of suitable external mean-field forces (see Refs. [9–12, 14]).The second development is even more significant. It lays in the discovery of a new kinetic equation realized by Master equation [21], *i.e.*,

an exact, namely non-asymptotic, kinetic equation which advances in time an arbitrary ensemble of smooth hard-sphere particles which are advanced in time by means of the S_N-CDS indicated above. In particular, the Master equation is peculiar since it holds for a finite system S_N-CDS, *i.e.*, a finite number of smooth hard spheres which are in turn considered as finite-size, namely having a finite mass m and a finite diameter $\sigma > 0$. As discussed in Refs.[19–24] such features imply a radical change of viewpoint in kinetic theory. However, the introduction of mean-field (*i.e.*, external) forces into the Master kinetic equation, a task which is carried out in this paper, leads in principle to a novel form of the same equation which is referred to here as *modified Master kinetic equation* (MMKE). It is obvious that for such an equation the very validity of previous conclusions [25] is called into question. Therefore a number of basic related issues arise which concern in particular:

Problem #1: The first one is whether the incompressibility condition can achieved or not. This is realized by the requirement

$$n(\mathbf{r}_1, t) = n_o, \tag{10}$$

with $n(\mathbf{r}_1, t)$ denoting the Navier-Stokes configuration-space probability density and $n_o > 0$ a constant value to hold in the interior fluid domain Ω.

Problem #2: The second issue concerns the validity itself of the Navier-Stokes equation, *i.e.*, whether it holds identically in $\Omega \times I$. This means, in particular, that in the absence of external volume forces, except for gravity, its intrinsic time-irreversible character should be warranted (*macroscopic irreversibility*).

Problem #3: The third one concerns the possible validity of a constant H-theorem holding also for the modified Master kinetic equation, in terms of the Boltzmann-Shannon statistical entropy associated with the kinetic PDF.

Problem #4: The final issue is whether an intrinsic macroscopic irreversibility phenomenon can occur for the modified Master kinetic equation. More precisely, this is related to the possible occurrence of the decay to kinetic equilibrium (DKE) for the $1-$body PDF in the limit $t \to +\infty$, i.e., in which the same PDF coincides with a stationary solution of the modified Master kinetic equation.

Goals of the investigation

Given the premises indicated above, the goals to be pursued in the paper are as follows:

GOAL #1 (Section 2): Review of the Master kinetic equation and its basic properties.

GOAL #2 (Section 3): The introduction of the modified Master kinetic equation (MMKE). This concerns

also the establishment of the related moment equations and the related evolution equation for the Boltzmann-Shannon entropy.

GOAL #4 (Section 3, Subsection 3.1): The problem of incompressibility for MMKE. The goal here is to prescribe the a suitable form of the mean-field interaction in such a way to fulfill identically in $\Omega \times I$ the incompressibility condition.

GOAL #5 (Section 3, Subsection 3.2): The problem of Navier-Stokes equation for MMKE. We intend to investigate the possible realization of the Navier-Stokes equation based on MMKE.

GOAL #6 (Section 3, Subsection 3.3): The problem of the entropy conservation for MMKE. In this subsection the constant H-theorem is established.

GOAL #7 (Section 4): Example applications of MMKE: the plane Couette and the Poiseuille flows. The goal here is to test the validity of the modified Master kinetic equation to describe particular stationary solutions of INSE.

GOAL #8 (Section 5): Physical implications of the theory. These concern the possible occurrence of the phenomenon of DKE for MMKE.

2 - THE MASTER KINETIC EQUATION AND THE MASTER H-THEOREM

Although Boltzmann's namesake equation and Boltzmann H-Theorem [26–29] still represent nowadays a historical breakthrough, certain aspects of their derivation and in particular their extension to the treatment of granular systems, *i.e.*, formed by finite-size particles, have remained for a long time unsatisfactory [21, 30]. A critical issue in this connection is the physical basis of the involved microscopic statistical description [19, 32]. This refers in particular to the unique identification of the correct physical prescriptions - at arbitrary collision events - for the time-evolution laws of the $s-$body (for $s = 2, .., N$) probability density functions (PDF) associated with the Boltzmann-Sinai S_N-CDS namely a system formed by a finite number N of finite-size smooth hard spheres of diameter σ and mass m undergoing unary, binary as well in principle arbitrary multiple instantaneous elastic collisions [25]. More precisely, the issue is to determine the relationship between the $s-$body PDF's occurring immediately before and after an instantaneous collision event, *i.e.,*, the so-called incoming and outgoing PDF's $\rho_s^{(N)(-)}$ and $\rho_s^{(N)(+)}$ (with $s = 1, .., N$), referred to as *collision boundary condition*.(CBC) [19] for the $s-$body reduced PDF $\rho_s^{(N)}$ which is assumed function of the corresponding $s-$body state $\mathbf{x}^{(s)} \equiv \{\mathbf{x}_1, ..., \mathbf{x}_s\}$ and time t. Here for definiteness, $\mathbf{x}_i \equiv \{\mathbf{r}_i, \mathbf{v}_i\}$ for $i = 1, N$ denotes the $i-$the particle state of the S_N-CDS belonging to the corresponding $1-$body phase space $\Gamma_1 \equiv \Omega \times U_1$. In addition, $\mathbf{r}_i$ and $\mathbf{v}_i$ label its center of mass position

and velocity. The latter are assumed to span the configuration and velocity spaces, here identified respectively with the fluid domain Ω and $U_1 \equiv \mathbb{R}^3$. In this connection the following physical prescriptions are mandatory for the determination of the appropriate CBC:

Physical prescription #1: causality principle requires representing $\rho_s^{(N)(+)}$ as a function of $\rho_s^{(N)(-)}$, thus predicting the future from the past (rather than the opposite). Such a causal choice according to Cercignani [33] determines the arrow of time, *i.e.*, the orientation of the time axis;

Physical prescription #2: the *axiom of probability conservation* at arbitrary collision events must be fulfilled for appropriate subsets of the $N-$body phase-space (see related discussion in Refs. [19–21]);

Physical prescription #3: the *local prescription* of the appropriate collision boundary conditions must be adopted. More precisely, this means that CBC should be realized by means of a *local relationship* between $\rho_s^{(N)(+)}$ and $\rho_s^{(N)(-)}$ when both are evaluated *at the same state* belonging to the $s-$body phase-space Γ_s. In other words, this requires prescribing the functional form of the outgoing PDF in terms of the same outgoing state, *i.e.*, after collision. The latter one can be equivalently expressed in Lagrangian or Eulerian states, namely either in terms of $\mathbf{x}^{(s)(+)}(t_k)$ or $\mathbf{x}^{(s)(+)}$ (with $\mathbf{x}^{(s)(+)}(t_k)$ belonging to a prescribed Lagrangian trajectory $\left\{\mathbf{x}^{(s)(+)}(t), t \in I\right\}$). Consider for definiteness the case of a two-body collision.. Then, upon identifying for $i = 1, 2$, $\mathbf{x}_i^{(+)}(t_k) \equiv \left\{\mathbf{r}_i(t_k), \mathbf{v}_i^{(+)}(t_k)\right\}$ and $\mathbf{x}_i^{(-)}(t_k) \equiv \left\{\mathbf{r}_i(t_k), \mathbf{v}_i^{(-)}(t_k)\right\}$ with the corresponding outgoing and incoming states and letting $\mathbf{r}_2(t_k) \equiv \mathbf{r}_1(t_k) + \sigma \mathbf{n}_{21}(t_k)$, this means in particular that

$$\begin{cases} \mathbf{v}_1^{(+)}(t_k) = \mathbf{v}_1^{(-)}(t_k) - \mathbf{n}_{12}(t_k)\mathbf{n}_{12}(t_k) \cdot \mathbf{v}_{12}^{(-)}(t_k), \\ \mathbf{v}_2^{(+)}(t_k) = \mathbf{v}_2^{(-)}(t_k) - \mathbf{n}_{21}(t_k)\mathbf{n}_{21}(t_k) \cdot \mathbf{v}_{21}^{(-)}(t_k). \end{cases}$$
(11)

Here the notation is standard [19]. Thus, $\mathbf{r}_{12}(t_k) \equiv \mathbf{r}_1(t_k) - \mathbf{r}_2(t_k)$, $\mathbf{v}_{12}^{(-)}(t_k) \equiv \mathbf{v}_1(t_k) - \mathbf{v}_2(t_k)$ and $\mathbf{n}_{12}(t_k) = \mathbf{r}_{12}(t_k)/|\mathbf{r}_{12}(t_k)|$ denote respectively the relative position and velocity vectors and the corresponding relative position unit vector. As shown in Ref. [20] the correct causal prescription of CBC, denoted as *modified collision boundary conditions* (MCBC), is found to be provided respectively by the Lagrangian and Eulerian equations

$$\rho^{(+)(N)}(\mathbf{x}^{(+)}(t_k), t_k) = \rho^{(-)(N)}(\mathbf{x}^{(+)}(t_k), t_k), \quad (12)$$

and

$$\rho^{(+)(N)}(\mathbf{x}^{(+)}, t) = \rho^{(-)(N)}(\mathbf{x}^{(+)}, t). \quad (13)$$

These prescriptions emerge clearly when the special case is considered of the $N-$body deterministic PDF. This is realized by the $N-$body Dirac delta (or so-called *certainty function* [34]), namely the distribution $\rho_H^{(N)}(\mathbf{x}, t) =$

$\delta(\mathbf{x} - \mathbf{x}(t)) \equiv \prod_{i=1,N} \delta(\mathbf{x}_i - \mathbf{x}_i(t))$, with $\delta(\mathbf{x}_i - \mathbf{x}_i(t))$ being the $1-$body Dirac delta, $\mathbf{x}_i$ and $\{\mathbf{x}_i(t), t \in I\}$ denoting respectively the $i-$th particle state which spans the corresponding $1-$body phase space and the phase-space trajectory of the same particle. In fact, exclusively based on physical grounds, *i.e.*, thanks to the axioms of classical statistical mechanics [19, 21], the $N-$body Dirac delta must necessarily provide a particular possible realization for the $N-$body PDF associated the same CDS. On the other hand, each single-particle Dirac delta can be regarded as the limit function of an arbitrary, *i.e.*, intrinsically non-unique, sequence of smooth strictly positive real functions. Therefore, it is obvious that the same collision boundary conditions should manifestly be satisfied also by the sequence-functions themselves.

As shown in Ref. [20] the modified collision boundary conditions indicated above (see Eqs. (12) and (13)) differ from the traditional prescription earlier adopted in the literature and originally first introduced by Boltzmann (see related discussion in Refs. [33]) which is realized by the so-called PDF-conserving CBC, namely the Lagrangian requirement

$$\rho^{(+)(N)}(\mathbf{x}^{(+)}(t_k), t_k) = \rho^{(-)(N)}(\mathbf{x}^{(-)}(t_k), t_k), \qquad (14)$$

of the equivalent Eulerian one

$$\rho^{(+)(N)}(\mathbf{x}^{(+)}, t) = \rho^{(-)(N)}(\mathbf{x}^{(-)}, t), \qquad (15)$$

with $\mathbf{x}^{(-)}(t_k), \mathbf{x}^{(+)}(t_k)$ and $\mathbf{x}^{(-)}, \mathbf{x}^{(+)}$ denoting again the respectively the couples of Lagrangian and Eulerian incoming and outgoing particle states. Actually Eqs. (14) and (15) are peculiar since they assertively relate two PDF's evaluated ad different phase-space states. Thus, manifestly violating the *Physical prescription #3* indicated above, they should be regarded as un-physical ones. Nonetheless, the same CBC preserve by construction the customary Boltzmann collisional invariants (see Eq.(21) below; [23]). In addition, when then so-called Boltzmann-Grad limit is performed, *i.e.*, point-like particles are considered in validity of the dilute-gas asymptotic ordering, one can show [21, 24, 25] that the two choices indicated above (represented respectively by MCBC and the PDF-conserving CBC) actually lead to the same realization of the collision operator in the BBGKY hierarchy [22] and in the Boltzmann equation [19–21].

The adoption of MCBC as well the prescription of an extended functional setting for microscopic statistical description of the Boltzmann-Sinai CDS, lay at the basis of the new "*ab initio*" treatment of classical statistical mechanics (CSM), recently developed in Refs. [19–25], enabling the treatment of granular systems formed by a finite number of particles. As a consequence, it is found that the $N-$body PDF now can include among its physically-admissible realizations both stochastic (*i.e.*, smoothly differentiable ordinary functions), partially de-

terministic and deterministic (*i.e.*, in both cases distributions) probability density functions [19, 21]. In addition, the same Γ_N-phase-space Liouville equation necessarily admits among its physically-admissible solutions also a permutation-symmetric, particular realization of the $N-$body PDF which is factorized in terms of the corresponding $1-$body PDF's $\rho_1^{(N)}(\mathbf{x}_i, t)$ for all $i = 1, N$.

Such an approach has opened up a host of exciting new developments in the kinetic theory of granular systems.

In particular, as pointed out in Refs. [21] and [24], the really remarkable implication which follows is that the same $1-$body PDF $\rho_1^{(N)}(t) \equiv \rho_1^{(N)}(\mathbf{x}_1, t)$ satisfies an equivalent exact, *i.e.*, non-asymptotic kinetic equation holding for finite values of (N, σ, m), denoted as *Master kinetic equation* (see Eq.(16) below), while in addition the corresponding Boltzmann-Shannon $1-$body entropy $S(\rho_1^{(N)}(t))$ (see Eq. (23) below) is identically conserved for all $t \in I \equiv \mathbb{R}$ (*Master H-theorem equation*; see Eq.(27) below).

For the sake of greater clarity the two equations are briefly recalled below together with their Boltzmann-Grad limits.

2.1 - Master kinetic equation

In the case of a stochastic factorized $N-$body PDF the Master kinetic equation for the corresponding stochastic reduced $1-$body PDF can be represented in terms of the integro-differential equation

$$L_{1(1)} \rho_1^{(N)}(\mathbf{x}_1, t) = \mathcal{C}_1\left(\rho_1^{(N)} | \rho_1^{(N)}\right), \qquad (16)$$

where the operators $L_{1(1)}$ and $C_1\left(\rho_1^{(N)} | \rho_1^{(N)}\right)$ identify respectively the free-streaming operators and the Master collision operator which is consistent both with the causality principle and MCBC. These are given respectively by

$$L_{1(1)} = L_{1(1)} \equiv \frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{r}_1}. \qquad (17)$$

$$\mathcal{C}_1\left(\rho_1^{(N)} | \rho_1^{(N)}\right) \equiv (N-1)\sigma^2 \int_{U_{1(2)}} d\mathbf{v}_2 \int^{(-)} d\Sigma_{21}$$

$$\left[\widehat{\rho}_2^{(N)}(\mathbf{x}^{(2)(+)}, t) - \widehat{\rho}_2^{(N)}(\mathbf{x}^{(2)}, t)\right] |\mathbf{v}_{21} \cdot \mathbf{n}_{21}| \overline{\Theta}^*. \qquad (18)$$

Here the notation are standard [21]. Thus $U_{1(k)} \equiv \mathbb{R}^3$ is the $1-$body velocity space for the $k-$th particle, the symbol $\int^{(-)} d\Sigma_{21}$ denotes integration on the subset of the solid angle of incoming particles namely for which $\mathbf{v}_{12} \cdot \mathbf{n}_{12} < 0$. Furthermore

$$\widehat{\rho}_2^{(N)}(\mathbf{x}^{(2)}, t) \equiv \frac{k_2^{(N)}(\mathbf{r}_1, \mathbf{r}_2, t)}{k_1^{(N)}(\mathbf{r}_1, t) k_1^{(N)}(\mathbf{r}_2, t)}$$
$$\rho_1^{(N)}(\mathbf{r}_1, \mathbf{v}_1, t) \rho_1^{(N)}(\mathbf{r}_2, \mathbf{v}_2, t) \qquad (19)$$

identifies the incoming $2-$body PDF. Similarly $\widehat{\rho}_2^{(N)}(\mathbf{x}^{(2)(+)}, t)$ is the corresponding outgoing $2-$body PDF's as determined from MCBC, namely

$$\widehat{\rho}_2^{(N)}(\mathbf{x}^{(2)(+)}, t) \equiv \frac{k_2^{(N)}(\mathbf{r}_1, \mathbf{r}_2, t)}{k_1^{(N)}(\mathbf{r}_1, t) k_1^{(N)}(\mathbf{r}_2, t)}$$
$$\rho_1^{(N)}(\mathbf{r}_1, \mathbf{v}_1^{(+)}, t) \rho_1^{(N)}(\mathbf{r}_2, \mathbf{v}_2^{(+)}, t). \quad (20)$$

In both equations above (19) and (20) the position vector $\mathbf{r}_2$ is identified with $\mathbf{r}_2 = \mathbf{r}_1 + \sigma \mathbf{n}_{21}$, while $k_1^{(N)}(\mathbf{r}_1, t), k_1^{(N)}(\mathbf{r}_2, t)$ and $k_2^{(N)}(\mathbf{r}_1, \mathbf{r}_2, t)$ identify suitably-prescribed $1-$ and $2-$body occupation coefficients (see the corresponding definitions reported in Ref. [21]). Finally $\overline{\Theta}^*$ denotes $\overline{\Theta}^* \equiv \overline{\Theta}\left(\left|\mathbf{r}_2 - \frac{\sigma}{2}\mathbf{n}_2\right| - \frac{\sigma}{2}\right)$, with $\overline{\Theta}(x)$ being the strong theta function.

Then it is immediate to show that the Master collision operator admits the standard Boltzmann collisional invariants which warrants that identically the equations

$$\int\limits_{U_1} d\mathbf{v}_1 G_i(\mathbf{x}_1, t) \mathcal{C}_1\left(\rho_1^{(N)} | \rho_1^{(N)}\right) = 0 \quad (21)$$

must hold for $G_i(\mathbf{x}_1, t) = 1, \mathbf{v}_1, v_1^2$.

2.2 - Master H-theorem equation

Let us require that $\rho_1^{(N)}(t) \equiv \rho_1^{(N)}(\mathbf{x}_1, t)$ is a stochastic $1-$body PDF solution of the Master kinetic equation (16) such that for the corresponding initial condition $\rho_1^{(N)}(t_o) \equiv \rho_1^{(N)}(\mathbf{x}_1, t_o)$ the Boltzmann-Shannon entropy associated with $\rho_1^{(N)}(t_o)$ is defined, namely the functional [26, 31]

$$S(\rho_1^{(N)}(t_o)) = -\int\limits_{\Gamma_1} d\mathbf{v}_1 \rho_1^{(N)}(\mathbf{x}_1, t_o) \ln \rho_1^{(N)}(\mathbf{x}_1, t_o) \quad (22)$$

exists. Then it follows that:

1. For all $t \in I \equiv \mathbb{R}$, the Boltzmann-Shannon entropy associated with a stochastic solution of the Master kinetic equation (16) $\rho_1^{(N)}(t)$, namely

$$S(\rho_1^{(N)}(t)) \equiv -\int\limits_{\Gamma_1} d\mathbf{v}_1 \rho_1^{(N)}(\mathbf{x}_1, t) \ln \rho_1^{(N)}(\mathbf{x}_1, t), \quad (23)$$

exists globally in time (see Ref. [25]);

2. For all $t \in I \equiv \mathbb{R}$, $S(\rho_1^{(N)}(t))$ is such that denoting $K_1(\rho_1^{(N)}(t))$ the weighted phase-space integral of the Master collision operator

$$K_1(\rho_1^{(N)}(t)) \equiv -\int\limits_{\Gamma_1} d\mathbf{v}_1 \ln \rho_1^{(N)}(\mathbf{x}_1, t) \mathcal{C}_1\left(\rho_1^{(N)} | \rho_1^{(N)}\right), \quad (24)$$

the entropy density $G(\mathbf{x}_1, t) \equiv \ln \rho_1^{(N)}(\mathbf{x}_1, t)$ is a generalized collisional invariant for the Master collision operators (see Ref.[23]) so that the identity

$$K_1(\rho_1^{(N)}(t)) \equiv 0 \quad (25)$$

necessarily holds.

3. As a consequence, since by construction

$$\frac{\partial}{\partial t} S(\rho_1^{(N)}(t)) \equiv K_1(\rho_1^{(N)}(t)), \quad (26)$$

i.e., $K_1(\rho_1^{(N)}(t))$ is the entropy production rate associated with the Master collision operator, the Boltzmann-Shannon entropy is constant in time so that necessarily the constant Master H-theorem equation

$$S(\rho_1^{(N)}(t)) = S(\rho_1^{(N)}(t_o)) \quad (27)$$

is identically fulfilled (see Ref. [24]).

2.3 - Boltzmann-Grad limit

For completeness it is convenient to recall here the relationship with Boltzmann kinetic theory, namely the asymptotic approximation by which the Master kinetic equation (16) recovers the customary form of the Boltzmann equation (see also Ref.[22]). This involves, besides suitable smoothness conditions on the $1-$ body PDF, invoking:

A) first, the so-called *dilute-gas asymptotic ordering* for N and σ, obtained by means of the asymptotic conditions

$$\begin{cases} N \equiv \frac{1}{\varepsilon} \gg 1 \\ 0 < \sigma \sim O(\varepsilon^{1/2}), \end{cases} \quad (28)$$

requiring the Knudsen number $K_n \equiv N\sigma^2$ to become of $O(\varepsilon^0)$ (see also Ref. [22]);

B) second, *the continuum limit*, or Boltzmann-Grad limit, obtained letting

$$\begin{cases} \varepsilon \equiv \frac{1}{N} \to 0^+, \\ K_n \equiv N\sigma^2 \to K_n^{(o)}, \end{cases} \quad (29)$$

with K_n denoting the Knudsen number and $K_n^{(o)} \sim O(\varepsilon^0)$ its limit value.

As shown in Refs. [21] and [24] these assumptions imply that the limit function

$$\rho_1(\mathbf{x}_1, t) = \lim_{\varepsilon \to 0^+} \rho_1^{(N)}(\mathbf{x}_1, t) \quad (30)$$

obtained consistently with Eqs.(28) and (29) by construction satisfies the Boltzmann kinetic equation and

the corresponding Boltzmann H-theorem. However, requirements A) and B) imply treating the hard-spheres as point-like and taking the continuum limit in such a way that the Knudsen number remains finite. Therefore this means that, unlike the case of a granular system, *i.e.*, formed by finite-size particles, the ensemble of particles corresponds to a rarefied gas, *i.e.*, an entirely different physical system from the one considered here.

3 - MODIFIED FORM OF THE MASTER KINETIC EQUATION

The task posed in this section is to determine a suitable *mean-field interaction*, represented by a smoothly-differentiable real vector field $\mathbf{F}$, occurring in the Master kinetic equation and representing a unary (*i.e.*, so-called *mean-field*) interaction depending only on the state of each individual particle. The latter is assumed to act individually on each particle belonging to a granular system, *i.e.*, an ensemble of finite-size hard spheres described by means of the Boltzmann-Sinai CDS. The goal is to show that the vector field $\mathbf{F}$ can be prescribed in such way to achieve GOALS #2-#6 stated in the introduction, *i.e.*, to satisfy identically INSE and to warrant at the same time conservation of the statistical Boltzmann-Shannon entropy associated with the relevant $1-$body PDF. The resulting statistical equation, obtained in terms of the Master kinetic equation recalled above (16) and to be referred to as *modified Master kinetic equation* (MMKE), takes the form

$$L_1(\mathbf{F})\rho_1^{(N)}(\mathbf{x}_1, t) = \mathcal{C}_1\left(\rho_1^{(N)}|\rho_1^{(N)}\right), \qquad (31)$$

with $L_1(\mathbf{F})$ denoting the modified streaming operator

$$L_1(\mathbf{F}) = \frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \nabla_1 + \frac{\partial}{\partial \mathbf{v}_1} \cdot (\mathbf{F}). \qquad (32)$$

The same vector field $\mathbf{F}$ will be assumed to be a smoothly-differentiable real vector field of the type $\mathbf{F} = \mathbf{F}(\mathbf{x}_1, t; \rho_1^{(N)})$, which depends functionally also on the $1-$body PDF $\rho_1^{(N)}(t) \equiv \rho_1^{(N)}(\mathbf{x}_1, t)$. Here we intend to prescribe $\mathbf{F}$ in such a way to satisfy the following constraint conditions: a) the first one is realized by the incompressibility condition prescribed by Eq.(8); b) the second constraint is provided by the validity of the Navier-Stokes equation 3); c) the third constraint condition is realized by the requirement that the Boltzmann Shannon entropy associated with the MMKE should be conserved, namely

$$\frac{\partial S(\rho_1^{(N)}(t))}{\partial t} = 0, \qquad (33)$$

where $S(\rho_1^{(N)}(t))$ denotes again the functional (23).

3.1 - The problem of incompressibility for MMKE

For definiteness let us first introduce appropriate velocity moments of the $1-$body PDF $\rho_1^{(N)}(t)$ which are associated with an arbitrary solution of MMKE and the corresponding velocity moments of MMKE. In particular, introducing the relative velocity $\mathbf{u} = \mathbf{v}_1 - \mathbf{V}_1(\mathbf{r}_1, t)$, let us consider the integrals $M_i\left[\rho_1^{(N)}\right] = \int_{U_1} d^3\mathbf{v}_1 X_i \rho_1^{(N)}(\mathbf{x}_1, t)$ which correspond to the weight-functions

$$X_i = m, m\mathbf{v}_1, \frac{m}{3}u^2 \qquad (34)$$

(with the third one identifying the scalar pressure density), namely

$$n_1(\mathbf{r}_1, t) = \int_{U_1} d^3\mathbf{v}_1 \rho_1^{(N)}(\mathbf{x}_1, t), \qquad (35)$$

$$\rho_1(\mathbf{r}_1, t) = mn_1(\mathbf{r}_1, t), \qquad (36)$$

$$\mathbf{V}_1(\mathbf{r}_1, t) \equiv \frac{1}{\rho_1(\mathbf{r}_1, t)} \int_{U_1} d^3\mathbf{v}_1 \mathbf{v}_1 \rho_1^{(N)}(\mathbf{x}_1, t), \qquad (37)$$

$$p_1(\mathbf{r}_1, t) = \int_{U_1} d^3\mathbf{v}_1 \frac{m}{3}u^2 \rho_1^{(N)}(\mathbf{x}_1, t). \qquad (38)$$

Here $\{\rho_1(\mathbf{r}_1, t) \geq 0, \mathbf{V}_1(\mathbf{r}_1, t), p_1(\mathbf{r}_1, t)\}$ identify respectively the configuration-space kinetic mass density, velocity and pressure and in particular it follows that by construction the kinetic pressure is necessarily strictly positive, *i.e.*, $p_1(\mathbf{r}_1, t) > 0$.

Following Refs. [19, 20], one can show that the vector field $\mathbf{F}$ can be (non-uniquely) determined in such a way to satisfy identically both the isochoricity equation (2) and incompressibility condition (8). The proof of the statement follows by suitable prescription of appropriate velocity moments of MMKE. For this purpose, one first notices that independent of the choice of the same vector field $\mathbf{F}$, the first velocity moment of MMKE corresponding to the weight functions $X_1 = m$ yields simply the mass-continuity equation, namely

$$\frac{\partial}{\partial t}\rho_1 + \nabla_1 \cdot (\rho_1 \mathbf{V}_1) = 0. \qquad (39)$$

Instead, straightforward algebra yields that the velocity moment associated with $X_3 = \frac{m}{3}u^2$ takes the form:

$$\frac{\partial p_1}{\partial t} + \nabla_1 \cdot (\mathbf{V}_1 p_1) + \frac{2}{3}\underline{\underline{\Pi}} : \nabla_1\mathbf{V} +$$

$$\nabla_1 \cdot \mathbf{Q} - \int_{U_1} d\mathbf{v}_1 \frac{2}{3}m\mathbf{u} \cdot \mathbf{F}\rho_1^{(N)}(\mathbf{x}_1, t) = 0, \qquad (40)$$

where $\underline{\underline{\Pi}}$ and $\mathbf{Q}$ are respectively the kinetic tensor pressure

$$\underline{\underline{\Pi}} = \int_{U_1} d\mathbf{v}_1 m\mathbf{u}\mathbf{u}\rho_1^{(N)}(\mathbf{x}_1, t) \qquad (41)$$

and the relative kinetic energy flux

$$\mathbf{Q} = \int_{U_1} d\mathbf{v}_1 \mathbf{u} \frac{m}{3} u^2 \rho_1^{(N)}(\mathbf{x}_1, t). \qquad (42)$$

Therefore, by suitable prescription of $\mathbf{F}$, one can always require that

$$\int_{U_1} d\mathbf{v}_1 \frac{2}{3} m\mathbf{u} \cdot \mathbf{F} \rho_1^{(N)}(\mathbf{x}_1, t) =$$
$$\left[\frac{2}{3} \underline{\underline{\mathbf{\Pi}}} : \nabla_1 \mathbf{V} + \nabla_1 \cdot \mathbf{Q} \right] + \frac{Dp}{Dt}. \qquad (43)$$

As a result, the velocity moment associated with $X_3 = \frac{m}{3} u^2$ takes the form

$$\frac{\partial p_1}{\partial t} + \nabla_1 \cdot (\mathbf{V}_1 p_1) - \frac{Dp_1}{Dt} = 0, \qquad (44)$$

with $\frac{D}{Dt}$ denoting Lagrangian derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}_1 \cdot \nabla_1$. Therefore, provided $p_1 > 0$ in the open set $\Omega \times I$, the last equation implies identically the validity of the isochoricity condition (analogous to Eq.(2))

$$\nabla_1 \cdot \mathbf{V}_1 = 0. \qquad (45)$$

Then, upon imposing the initial condition

$$\rho_1(\mathbf{r}_1, t_o) = \rho_o > 0, \qquad (46)$$

with ρ_o a non-vanishing constant in the closure $\overline{\Omega}$ of the set Ω, the simultaneous validity of Eqs.(39) and (44) requires identically that

$$\rho_1(\mathbf{r}_1, t) = \rho_o > 0 \qquad (47)$$

must hold in whole the set $\overline{\Omega} \times I$, thus implying the validity of the incompressibility condition (8).

As shown in Refs. NOI1,NOI2, in order to satisfy the constraint equation (44), this requires to select a vector field $\mathbf{F}$ expressed as a polynomial function of the kinetic relative velocity $\mathbf{u} = \mathbf{v}_1 - \mathbf{V}_1(\mathbf{r}_1, t)$. In particular to satisfy the constraint equation indicated above (see Eq.(45)) it is sufficient to require that $\mathbf{F}$ is represented by a 3rd degree polynomial for the type:

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4, \qquad (48)$$

where in particular $\mathbf{F}_0$ is independent of $\mathbf{u}$, $\mathbf{F}_1 = \mathbf{u} F_1$, is linearly-dependent on $\mathbf{u}$, while the remaining contributions do not contribute by assumption to the moment 43. Then, in order to satisfy Eq.(44) it is manifestly sufficient to require that

$$F_1 = \frac{1}{2p_1(\mathbf{r}_1, t)} \left[\frac{2}{3} \underline{\underline{\mathbf{\Pi}}} : \nabla_1 \mathbf{V}_1 \right.$$
$$\left. + \nabla_1 \cdot \mathbf{Q} \right] + \frac{1}{2} \frac{D \ln p_1}{Dt}. \qquad (49)$$

In fact this implies

$$\int_{U_1} d\mathbf{v}_1 \frac{2}{3} m u^2 F_1 \rho_1^{(N)}(\mathbf{x}_1, t) =$$
$$\left[\frac{2}{3} \underline{\underline{\mathbf{\Pi}}} : \nabla_1 \mathbf{V} + \nabla_1 \cdot \mathbf{Q} \right] + \frac{Dp}{Dt}, \qquad (50)$$

which fulfills identically (43).

3.2 - The problem of Navier-Stokes equation for MMKE

Let us now pose the problem of requiring validity of the Navier-Stokes equation (3). For this purpose let us first evaluate the second velocity moment of MMKE corresponding to the weight function $m\mathbf{v}_1$. Taking into account Eqs.(45) and (47) elementary algebra yields

$$\rho_o \frac{\partial \mathbf{V}_1}{\partial t} + \rho_o \mathbf{V}_1 \cdot \nabla_1 \mathbf{V}_1 +$$
$$\nabla_1 \cdot \underline{\underline{\mathbf{\Pi}}} - \int_{U_1} d\mathbf{v}_1 \mathbf{F} \rho_1^{(N)}(\mathbf{x}_1, t) = 0 \qquad (51)$$

Finally let us require, in particular, that $\mathbf{F}$ is prescribed so that

$$\int_{U_1} d\mathbf{v}_1 \mathbf{F} \rho_1^{(N)}(\mathbf{x}_1, t) = \nabla_1 \cdot \underline{\underline{\mathbf{\Pi}}} -$$
$$\nabla_1 p_1 + \mathbf{f}_1 + \mu \nabla_1^2 \mathbf{V}_1. \qquad (52)$$

As a consequence it follows that Eq.(51) takes a form analogous to the Navier-Stokes equation (3), namely

$$\rho_o \frac{\partial \mathbf{V}_1}{\partial t} + \rho_o \mathbf{V}_1 \cdot \nabla_1 \mathbf{V}_1 +$$
$$\nabla_1 p_1 - \mathbf{f}_1 - \mu \nabla_1^2 \mathbf{V}_1 = 0. \qquad (53)$$

To recover, however in a proper sense also the proper values of the Navier-Stokes fluid fields $\{\rho, \mathbf{V}, p\}$ a suitable mapping must be introduced to relate them to the kinetic moments $\{\rho_1, \mathbf{V}_1, p_1\}$ indicated above. More precisely, such a relationship is realized by the *kinetic correspondence principle*:

$$\begin{cases} \rho(\mathbf{r}_1, t) = \rho_1(\mathbf{r}_1, t) = \rho_o, \\ \mathbf{V}(\mathbf{r}_1, t) = \mathbf{V}_1(\mathbf{r}_1, t) + \mathbf{V}_2(\mathbf{r}_1), \\ p(\mathbf{r}_1, t) = p_1(\mathbf{r}_1, t) + p_2(\mathbf{r}_1), \end{cases} \qquad (54)$$

in which the fields $\{\rho_o, \mathbf{V}_2(\mathbf{r}_1), p_2(\mathbf{r}_1)\}$ identify here an - in principle arbitrary - particular stationary solution of INSE, namely such that identically in $\Omega \times I$:

$$\begin{cases} \nabla_1 \cdot \mathbf{V}_2(\mathbf{r}_1) = 0, \\ \rho_o \mathbf{V}_2(\mathbf{r}_1) \cdot \nabla_1 \mathbf{V}_2(\mathbf{r}_1) + \nabla_1 p_2 - \mathbf{f}_2 - \mu \nabla_1^2 \mathbf{V}_2 = 0. \end{cases}$$
$$(55)$$

Therefore, from the initial conditions holding for the Navier-Stokes fluid fields (4), it follows that the kinetic moments $\{\rho_1, \mathbf{V}_1, p_1\}$ must fulfill the initial conditions

$$\begin{cases} \rho_1(\mathbf{r}_1, t_o) = \rho_o, \\ \mathbf{V}_1(\mathbf{r}_1, t_o) = \mathbf{V}_o(\mathbf{r}_1) - \mathbf{V}_2(\mathbf{r}_1), \\ p_1(\mathbf{r}_1, t_o) = p_o(\mathbf{r}_1) - p_2(\mathbf{r}_1). \end{cases} \quad (56)$$

Notice that in Eqs.(55) $p_2(\mathbf{r}_1)$ is actually defined up to an arbitrary constant. Therefore one can always require the initial kinetic pressure $p_1(\mathbf{r}_1, t_o)$ to be strictly positive, $i.e.$,

$$p_1(\mathbf{r}_1, t_o) = p_o(\mathbf{r}_1) - p_2(\mathbf{r}_1) > 0. \quad (57)$$

Based on the representation (48), for the fulfillment of the constraint equation (52) it is manifestly sufficient to require

$$\mathbf{F}_0 = \frac{1}{\rho_o} \left[\nabla_1 \cdot \underline{\underline{\mathbf{\Pi}}} - \nabla_1 p + \mu \nabla_1^2 \mathbf{V}_1 \right], \quad (58)$$

while also requiring that

$$\int_{U_1} d\mathbf{v}_1 \mathbf{F} \rho_1^{(N)}(\mathbf{x}_1, t) = \int_{U_1} d\mathbf{v}_1 \mathbf{F}_0 \rho_1^{(N)}(\mathbf{x}_1, t), \quad (59)$$

$i.e.$, the remaining terms ($\mathbf{F}_2, \mathbf{F}_3$ and $\mathbf{F}_4$) in the representation (48) do not contribute, so that identically

$$\int_{U_1} d\mathbf{v}_1 \mathbf{F}_0 \rho_1^{(N)}(\mathbf{x}_1, t) = \nabla_1 \cdot \underline{\underline{\mathbf{\Pi}}} - \nabla_1 p - \mu \nabla_1^2 \mathbf{V}.$$

3.3 - The problem of entropy conservation for MMKE

Let us now pose the problem of determining a possible non-unique realization of the remaining terms in the polynomial expansion (48) in such a way to fulfill the constant entropy theorem (33). For definiteness we consider here only the additional contributions arising from $\mathbf{F}_2, \mathbf{F}_3$ and $\mathbf{F}_4$, more precisely requiring that they are of the form see

$$\begin{aligned} \mathbf{F}_2 &= \mathbf{F}_2(\mathbf{r}_1, t), \\ \mathbf{F}_3 &= \frac{mu^2}{3p_1} F_3(\mathbf{r}_1, t), \\ \mathbf{F}_4 &= \frac{m\mathbf{u}}{3p_1} F_4(\mathbf{r}_1, t). \end{aligned} \quad (60)$$

Here the real vector/scalar fields $\mathbf{F}_2(\mathbf{r}_1, t)$, $F_3(\mathbf{r}_1, t)$ and $F_4(\mathbf{r}_1, t)$ still remain in principle completely arbitrary. Starting point is provided by the theoretical results established Refs. [21, 22]. These warrant the conservation laws of the Master collision operator, and in particular also the validity of a constant H−theorem analogous to Eq. (33) for the Master kinetic equation ($i.e.$, when

the mean-field interaction $\mathbf{F}$ vanishes identically). However, the same H−theorem is generally now warranted in case of the MMKE. This follows from direct evaluation of the phase-space moment of MMKE determined in terms of the weight-function $X = \ln \rho_1^{(N)}(\mathbf{x}_1, t)$, yielding the Boltzmann-Shannon entropy (23). In fact, elementary algebra yields in the case of the MMKE:

$$\frac{\partial S(\rho_1^{(N)}(t))}{\partial t} = K_1(\rho_1^{(N)}(t)) + K_F(\rho_1^{(N)}(t)), \quad (61)$$

where the entropy production rate $K_1(\rho_1^{(N)}(t))$ is defined by Eq.(24) and hence vanishes identically in the case of the Master collision operator [21, 22] (see Eq.(25) in subsection 2.2); furthermore $K_F(\rho_1^{(N)}(t))$ is given by

$$K_F(\rho_1^{(N)}(t)) = - \int_{\Gamma_1} d\mathbf{x}_1 \rho_1^{(N)}(\mathbf{x}_1, t) \frac{\partial}{\partial \mathbf{v}_1} \cdot \mathbf{F} \quad (62)$$

and hence, unless a further specific constraint is placed on the choice of the vector field $\mathbf{F}$, it is generally non-vanishing. Therefore the question is whether the remaining contributions $\mathbf{F}_2, \mathbf{F}_3$ and $\mathbf{F}_4$ in the polynomial representation (48) of the mean-field interaction $\mathbf{F}$ can be prescribed in such a way to fulfill the further constraint condition

$$\int_{\Gamma_1} d\mathbf{x}_1 \rho_1^{(N)}(\mathbf{x}_1, t) \frac{\partial}{\partial \mathbf{v}_1} \cdot \mathbf{F} = 0 \quad (63)$$

($entropic constraint$). However, one should add for consistency also the requirement that the same terms ($\mathbf{F}_2, \mathbf{F}_3$ and $\mathbf{F}_4$) should leave unaffected the two previous constraint conditions (43) and (52). In view of the representation (48) and Eqs. (60) it follows in particular:

$$\begin{aligned} \frac{\partial}{\partial \mathbf{v}_1} \cdot \mathbf{F} &= \frac{3}{2p_1} \left[\nabla_1 \cdot \mathbf{Q} + \frac{2}{3} \nabla_1 \mathbf{V} : \underline{\underline{\mathbf{\Pi}}} \right] + \\ & 3 \frac{D \ln p_1}{Dt} + \frac{m}{p} F_4 + \left[\frac{mu^2}{p_1} + \frac{2mu^2}{3p_1} \right] F_3. \end{aligned} \quad (64)$$

This means that the vector and scalar fields $\mathbf{F}_2, F_4$ and F_3 should be prescribed in such a way to fulfill identically in $\Omega \times I$ the following velocity-moment constraint equations

$$\int_{U_1} d\mathbf{v}_1 \left[\mathbf{F}_2 + \mathbf{u} \frac{mu^2}{3p_1} F_3 \right] \rho_1^{(N)}(\mathbf{x}_1, t) = \mathbf{0}, (65)$$

$$\int_{U_1} d\mathbf{v}_1 \mathbf{u} \cdot \left[\frac{m\mathbf{u}}{3p_1} F_4 + \mathbf{u} \frac{mu^2}{3p_1} F_3 \right] \rho_1^{(N)}(\mathbf{x}_1, t) = 0. (66)$$

Then a sufficient condition in order to satisfy the entropic constraint indicated above is manifestly to require validity in the whole set $\Omega \times I$ of the following integral identity

$$\int_{U_1} d\mathbf{v}_1 \rho_1^{(N)}(\mathbf{x}_1, t) \frac{\partial}{\partial \mathbf{v}_1} \cdot \mathbf{F} = 0. \quad (67)$$

Evaluation of the velocity integrals indicated above yields therefore the following constraint equations for the still undetermined vector and scalar fields $\mathbf{F}_2, F_3$ and F_4:

$$n_1(\mathbf{r}_1, t)\mathbf{F}_2 + \frac{1}{p_1}\mathbf{Q}F_3 = 0, \tag{68}$$

$$F_4 + \frac{1}{p_1}WF_3 = 0, \tag{69}$$

$$\frac{3n_1(\mathbf{r}_1, t)}{2p_1}\left[\nabla_1 \cdot \mathbf{Q} + \frac{2}{3}\nabla_1\mathbf{V} :\underline{\underline{\mathbf{\Pi}}}\right] + $$
$$3n_1(\mathbf{r}_1, t)\frac{D\ln p_1}{Dt} + \frac{mn_1(\mathbf{r}_1, t)}{p}F_4 + 5F_3 = 0, \tag{70}$$

where

$$W(\mathbf{r}_1, t) = \int_{U_1} d\mathbf{v}_1 \frac{mu^4}{3}\rho_1^{(N)}(\mathbf{x}_1, t). \tag{71}$$

The solution of the first equation gives for $\mathbf{F}_2$ a unique prescription in terms of the scalar field F_3, namely

$$\mathbf{F}_2 = -\frac{\mathbf{Q}}{n_1(\mathbf{r}_1, t)}F_3. \tag{72}$$

Then, provided

$$\Delta \equiv 5 - \frac{mn_1(\mathbf{r}_1, t)}{p_1^2}W > 0, \tag{73}$$

the functions F_3 and F_4 are given respectively by

$$F_4 = \frac{1}{p_1}WF_3, \tag{74}$$

$$F_3 = \frac{S_3(\mathbf{r}_1, t)}{5 - \frac{mn_1(\mathbf{r}_1, t)}{p_1^2}W}, \tag{75}$$

with

$$S_3(\mathbf{r}_1, t) \equiv \frac{3n_1(\mathbf{r}_1, t)}{2p_1}\left[\nabla_1 \cdot \mathbf{Q} + \frac{2}{3}\nabla_1\mathbf{V} :\underline{\underline{\mathbf{\Pi}}}\right] + $$
$$3n_1(\mathbf{r}_1, t)\frac{D\ln p_1}{Dt}. \tag{76}$$

Regarding the validity of the requirement (73) one notices that, thanks to Schwartz's inequality, necessarily

$$\frac{mn_1(\mathbf{r}_1, t)}{p_1^2}W \leq 3 \tag{77}$$

must hold, thus implying in turn the validity of the inequality (73). Therefore, Eqs.(72),(74) and (75) realize a unique solution to the constraint condition (63) which warrants the validity of the constant H-theorem (33).

3.4 - IMPLICATIONS

Let us briefly summarize the implications of subsections 3.1-3.3. Based on the kinetic correspondence principle (54) and the moment equations of MMKE determined for the weight-functions (34) it follows that the isochoricity and incompressibility conditions are fulfilled respectively by the kinetic velocity moment $\mathbf{V}_1(\mathbf{r}_1, t)$ (37) and the corresponding kinetic/fluid mass density moment ρ_o (36) given respectively by Eqs. (45) and (8) (subsection 3.1). As a consequence, the Navier-Stokes equation (53) is satisfied by the same kinetic velocity moment $\mathbf{V}_1(\mathbf{r}_1, t)$ and the kinetic scalar pressure p_1 (38), so that the kinetic moments $\{\rho_1, \mathbf{V}_1, p_1\}$ provide a particular solution of INSE (see Eqs.(2) and (3)). However, in order to satisfy the initial conditions (4, and in particular the corresponding ones holding for the kinetic moments one must generally identify the Navier-Stokes fluid fields $\{\rho, \mathbf{V}, p\}$ in terms of the kinetic moments $\{\rho_1, \mathbf{V}_1, p_1\}$ determined above only via the kinetic correspondence principle (54). Therefore, Eqs. (54) yield the general solution of INSE (2) and (3) holding globally in $\Omega \times I$ and subject to the initial and boundary conditions provided by Eqs. (56) and (7). Finally, by suitably prescribing the mean-field interaction $\mathbf{F}$ (see Eqs. (48) and (60)), the entropic constraint (63) is identically fulfilled thanks to the local condition (67). This means that by construction the Boltzmann-Shannon entropy associated to an arbitrary solution of MMKE is conserved.

4 - EXAMPLE APPLICATIONS OF MMKE: THE PLANE COUETTE AND THE POISEUILLE FLOWS

Let us now test the validity of the statistical description of the INSE problem based on MMKE. Indeed, an interesting issue is whether one can recover in this way well-known particular possible realizations of incompressible Navier-Stokes fluids. Here two examples are considered which concern respectively:

1. *the plane Couette flow*, represented by the solution

$$\begin{cases} \rho(\mathbf{r}_1, t) = \rho_o, \\ p_2(\mathbf{r}_1) = p_o, \\ \mathbf{V}_2(\mathbf{r}_1) = V_o y\widehat{\mathbf{x}}. \end{cases} \tag{78}$$

with $\mathbf{r}_1 \equiv (x, y, z)$ and $p_o > 0$ a constant. The corresponding INSE are realized by the equations

$$\begin{cases} V_o\nabla_1 \cdot y\widehat{\mathbf{x}} = 0, \\ \rho_o V_o^2 y\widehat{\mathbf{x}} \cdot \nabla_1 y\widehat{\mathbf{x}} - \mu V_o\nabla_1^2 y\widehat{\mathbf{x}} = 0. \end{cases} \tag{79}$$

2. *the plane Poiseuille flow*, represented by the solu-

tion

$$\begin{cases} \rho(\mathbf{r}_1,t) = \rho_o, \\ p_2(\mathbf{r_1}) = p_o + \beta x, \\ \mathbf{V}_2(\mathbf{r_1}) = V_o\left(1 - y^2\right)\widehat{\mathbf{x}} \end{cases} \quad (80)$$

with $\beta > 0$ a constant. Notice, however, that the pressure indicated in Eq.(80) is equivalent to the one occurring due to a constant (*i.e.*, equilibrium) gravitational force acting along the $x-$direction. Therefore the corresponding set of INSE can be identified with

$$\begin{cases} V_o\nabla_1 \cdot y\widehat{\mathbf{x}} = 0, \\ \rho_o V_o^2 y\left(1 - y^2\right)\widehat{\mathbf{x}} \cdot \nabla_1\left(1 - y^2\right)\widehat{\mathbf{x}} - \\ \mu V_o\nabla_1^2\left(1 - y^2\right)\widehat{\mathbf{x}} = 0. \end{cases} \quad (81)$$

Let us examine separately the two cases.

The first one is of immediate realization. In fact the fluid fields (78) correspond manifestly to a particular stationary solution $\{\rho_o, \mathbf{V}_2(\mathbf{r}_1), p_2(\mathbf{r}_1)\}$ of INSE (see Eqs. (55)) in which the fluid pressure is constant and positive while the fluid domain is unbounded and two dimensional. The second one is analogous. The only difference arises because now the pressure can become negative and is unbounded too. These features, however - namely the occurrence of an unbounded fluid domain Ω and possibly also the appearance of an unbounded stationary fluid pressure - do not pose any restriction on the validity of the kinetic theory here developed. Therefore, in both cases the kinetic correspondence principle (54) provides a representation of the general solution of INSE which admits either Eqs.(78) or (55) as particular stationary solutions.

We remark that in both cases the fluid domain Ω is realized in principle by a $2-$dimensional infinite strip between two parallel co-oriented straight lines lying at distance H, y being the Cartesian coordinates orthogonal to both ones and x the Cartesian coordinate along the same lines. Notice, however, that Eqs. (79) and (81) can be equivalently realized by replacing Ω with a bounded domain Ω_p identified with a rectangle of finite width L along the $x-$direction and introducing for the fluid velocity $\mathbf{V}(\mathbf{r_1},t)$ periodic boundary conditions at the two boundaries located respectively at $x = 0$ and $x = L$, namely letting

$$\mathbf{V}(x = 0, y, t) = \mathbf{V}(x = L, y, t). \quad (82)$$

Under these conditions, therefore, the fluid domain $\Omega \equiv \Omega_p$ is bounded so that integration of the Navier-Stokes equation (see (3)) delivers in a straightforward way the time-evolution of the kinetic energy of the fluid, namely

$$\frac{\partial}{\partial t}\int_{\Omega_p} d^3\mathbf{r}_1\rho_o^2 V^2(\mathbf{r_1},t) =$$

$$-\mu\int_{\Omega_p} d^3\mathbf{r}_1\nabla_1\mathbf{V}(\mathbf{r_1},t){:}\nabla_1\mathbf{V}(\mathbf{r_1},t) < 0. \quad (83)$$

This implies necessarily that $\mathbf{V}(\mathbf{r_1},t)$ must decay asymptotically to a stationary solution, namely

$$\lim_{t\to+\infty}\mathbf{V}(\mathbf{r_1},t) = \mathbf{V}_2(\mathbf{r_1}), \quad (84)$$

with $\mathbf{V}_2(\mathbf{r_1})$ corresponding either to Eqs. (78) or (80), In a similar way, thanks to Eq.(5), one expects also the decay of the related fluid pressure $p(\mathbf{r_1},t)$, namely

$$\lim_{t\to+\infty}p(\mathbf{r_1},t) = p_2(\mathbf{r_1}), \quad (85)$$

to occur.

The two example cases indicated above have, therefore, an important physical implication. This is precisely realized by the *fluid decay conditions* (84)-(85), *i.e.*, from the fact that both the stationary planar Couette and Poiseuille flows actually coincide with the final equilibrium states of the fluid. In other words in both cases they are the result of the decay of a suitable non-stationary Navier-Stokes fluids $\{\mathbf{V}(\mathbf{r_1},t), p(\mathbf{r_1},t)\}$ which, in turn, both correspond in principle to arbitrary initial conditions (4).

5 - DECAY TO KINETIC EQUILIBRIUM FOR MMKE

The interesting issue which arises is whether the fluid decay phenomenon indicated above may correspond or not to the occurrence of a *global decay to kinetic equilibrium* (DKE) for the relevant $1-$body kinetic PDF, *i.e.*, occurring globally in the phase space Γ_1. More precisely, this means that time-dependent $1-$body kinetic PDF solution of MMKE should decay uniformly to a stationary solution of MMKE, in the sense that provided $\rho_1^{(N)}(\mathbf{x}_1,t)$ is suitably smooth in the extended phase-space $\Gamma_1 \times I$ it should occur uniformly in Γ_1 that:

$$\lim_{t\to+\infty}\rho_1^{(N)}(\mathbf{x}_1,t) = \rho_M^{(N)}(\mathbf{v}_1), \quad (86)$$

i.e., in other words $\rho_1^{(N)}(\mathbf{x}_1,t)$ should decay asymptotically to a prescribed stationary $1-$body kinetic PDF. As shown in the Appendix $\rho_M^{(N)}(\mathbf{v}_1)$ this means that it should necessarily coincide with a *spatially uniform kinetic equilibrium,* which is realized by *a spatially-uniform local Maxwellian PDF*

$$\rho_M^{(N)}(\mathbf{v}_1) = n_o\frac{1}{\left(\pi v_{th}\right)^{3/2}}\exp\left\{-\frac{\mathbf{v}_1^2}{v_{th}^2}\right\}. \quad (87)$$

Here the notation is standard. Thus

$$v_{th}^2 = 2p_{1o}/mn_o \quad (88)$$

is the thermal velocity, n_o the constant configuration-space probability density, m the mass of a hard sphere

and $p_{1o} > 0$ an arbitrary non-vanishing constant kinetic pressure.

As shown in the Appendix, the validity of the kinetic decay limit (86) for MMKE arises because, in validity of the fluid decay conditions (84)-(85), the only admissible kinetic equilibrium is provided by Eq. (87).

This conclusion appears, in our view, interesting and in some respects also surprising.

The notable aspect lies in the physical mechanism at the basis of the global-DKE phenomenology. In fact it is manifest that, at least for case of the MMKE considered here, the phenomenon of DKE occurs specifically as a consequence of collisions occurring within the particles of the hard-sphere $N-$body system here described by means of the Master collision operator. This warrants that the entropy production rate associated with by the same collision operator vanishes identically (see subsection 2.2). Nevertheless as shown above the Boltzmann-Shannon entropy remains constant by construction also for MMKE, namely when the mean-field force $\mathbf{F}$ is introduced and a proper prescription is adopted for the same vector field.

This conclusion is potentially in conflict with the customary interpretation of Boltzmann kinetic theory. In fact, in accordance with Boltzmann H-theorem the physical origin of DKE is usually ascribed to the macroscopic irreversibility property of the Boltzmann-Shannon entropy functional together with the requirement that the same quantity should remain finite also in the time-asymptotic limit $t \to +\infty$.

Nevertheless, according to Boltzmann's own original interpretation, Boltzmann equation and Boltzmann H-theorem are only supposed to hold when both the dilute gas asymptotic ordering and the continuum Boltzmann-Grad limit apply, while for according to Boltzmann's own conjecture finite-size hard sphere the Boltzmann-Shannon entropy "*should tend to a constant*" . These viewpoints are included in Boltzmann's replies to Zermelo (1896-1897 [27–29]). In other words, the same equations cannot hold when the hard-spheres are considered finite-sized as in the present case. Indeed, in striking departure from Boltzmann kinetic theory, MMKE holds for an arbitrary finite Boltzmann-Sinai CDS, i.e., which both the number of particles N and their diameter σ are considered as finite. Nevertheless, the phenomenon of global DKE occurs also in the present case independent of the time-behavior of the Boltzmann-Shannon entropy.

The key differences arising between the two theories, *i.e.*, the one based on the Boltzmann equation and the other discussed here based on the modified Master kinetic equation, are of course related to the different and peculiar intrinsic properties of the Boltzmann and Master collision operators. In particular, as discussed elsewhere (see Refs.[21, 24, 25]), precisely because the Boltzmann equation is only an asymptotic approximation of the Master kinetic equation explains why a loss of information occurs in Boltzmann kinetic theory and consequently the related Boltzmann-Shannon entropy is not conserved.

Nevertheless, as recalled in Section 2, the Boltzmann-Shannon entropy for the Master kinetic equation is exactly conserved due to the symmetry properties of the Master collision operator (see subsection 2.2). Therefore its behavior is manifestly unrelated and independent from the occurrence of the phenomenon of DKE here pointed out. The present investigation shows, notwithstanding, that a macroscopic irreversibility property which is realized by DKE actually occurs. As indicated above this can be explained at a fundamental level, *i.e.*, based specifically on the collision processes described by the Master collision operator.

6 - CONCLUSIONS

In this paper the problem has been addressed of identifying a possible microscopic statistical description for the incompressible Navier-Stokes equations. The statistical approach has been based on the axiomatic statistical theory of the Boltzmann-Sinai classical dynamical system recently developed [19–25].

The theory presented here departs significantly, in several respects, from previous literature and notably from approaches based on the Boltzmann kinetic theory. The main differences actually arise because of the non-asymptotic character of the new theory, *i.e.*, the fact that it applies to arbitrary dense or rarefied systems for which the finite number and size of the constituent particles is accounted for [21]. In this paper basic consequences of the new theory have been investigated which concern the statistical treatment of an incompressible Navier-Stokes fluid based on the adoption of the Master kinetic equation and the introduction of a suitable mean-field interaction acting on a system of finite-sized smooth hard spheres. The phenomenon of decay to global kinetic equilibrium (DKE) has been pointed out. Remarkably, despite the fact that the related Boltzmann-Shannon entropy remains preserved, DKE occurs specifically because of the effect of collisions which are taken into account in the Master collision operator.

The present results are believed to be crucial both from the theoretical viewpoint and for applications of the "*ab initio*" statistical theory, *i.e.*, the Master kinetic equation. Indeed, regarding possible challenging future developments of the theory one should mention among others the following examples of possible (and mutually-related) directions worth to be explored:

- One is the possible ubiquitous occurrence of the DKE phenomenon for the Master kinetic equation, i.e., even in the absence of the mean-field interaction $\mathbf{F}$ introduced here.

- The second is related to the investigation of the time-asymptotic properties of the same kinetic equation, for which the present paper may represent a useful basis.

- The third goal refers to the possible extension of the theory to mixtures formed by hard spheres of different type, i.e., with different masses, diameters and undergoing either elastic or inelastic collisions.

- The fourth one concerns the investigation of hydrodynamic regimes for which a key prerequisite is provided by the DKE theory here established.

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APPENDIX: CONDITIONS OF VALIDITY OF KINETIC EQUILIBRIUM FOR MMKE

In this appendix we pose the problem of establishing necessary conditions for the existence of stationary solutions, or so-called *kinetic equilibrium solutions*, of MMKE, *i.e.*, such that

$$\frac{\partial}{\partial t}\rho_1^{(N)}(\mathbf{x}_1,t) \equiv 0 \qquad (89)$$

namely

$$\mathbf{v}_1 \cdot \nabla_1 \rho_1^{(N)}(\mathbf{x}_1,t) + \frac{\partial}{\partial \mathbf{v}_1} \cdot \left(\mathbf{F}\rho_1^{(N)}(\mathbf{x}_1,t) \right) = 0, \quad (90)$$

$$\mathcal{C}_1\left(\rho_1^{(N)}|\rho_1^{(N)}\right) = 0. \quad (91)$$

Let us introduce in addition, consistent with the fluid decay conditions (84)-(85), the requirements

$$\lim_{t\to+\infty} n_1(\mathbf{r}_1,t) = \int_{U_1} d^3\mathbf{v}_1 \lim_{t\to+\infty} \rho_1^{(N)}(\mathbf{x}_1,t) = n_o > 0 \quad (92)$$

$$\lim_{t\to+\infty} \mathbf{V}_1(\mathbf{r}_1,t) \equiv \frac{1}{n_o}\int_{U_1} d^3\mathbf{v}_1\mathbf{v}_1 \lim_{t\to+\infty}\rho_1^{(N)}(\mathbf{x}_1,t) \equiv 0 \quad (93)$$

Then it follows that necessarily the 1−body PDF $\rho_1^{(N)}$ must coincide with $\rho_1^{(N)} \equiv \rho_M^{(N)}(\mathbf{v}_1)$, namely the kinetic equilibrium local Maxwellian PDF (87). Conversely, let us assume that the fluid velocity moments associated with the MMKE are identified with the stationary equations given by Eqs. (55), namely are such that, besides Eqs.(92) and (93), they satisfy identically in $\Omega \times I$ also the constraint:

$$\lim_{t\to+\infty} p_1(\mathbf{r}_1,t) == p_{1o}. \qquad (94)$$

where $p_1(\mathbf{r}_1,t) = \int_{U_1} d^3\mathbf{v}_1 \frac{m}{3}\left(\mathbf{v}_1 - \mathbf{V}_1(\mathbf{r}_1,t)\right)^2 \rho_1^{(N)}(\mathbf{x}_1,t)$.
The issue if whether $\lim_{t\to+\infty}\rho_1^{(N)}(\mathbf{x}_1,t)$ must necessarily coincide or not with the Maxwellian PDF given above (87), or in other words whether the 1−body PDF $\rho_1^{(N)}$ can still be a non-stationary solution of the MMKE once the constraints Eqs. (92),(93) and (94) are placed on the same PDF. For definiteness let us consider the particular case in which for $t \to +\infty$

$$\rho_1^{(N)}(\mathbf{x}_1,t) = \rho_M^{(N)}(\mathbf{v}_1) + \delta\rho_1^{(N)}(\mathbf{x}_1,t), \qquad (95)$$

where $\delta\rho_1^{(N)}(\mathbf{x}_1,t)$ being an infinitesimal perturbation of order ε, ε denoting a prescribed real infinitesimal. Without loss of generality the latter can always be represented in terms of a polynomial expansion of the type

$$\delta\rho_1^{(N)}(\mathbf{x}_1,t) = \rho_M^{(N)}(\mathbf{v}_1)\, \mathbf{a}_1\cdot\mathbf{v}_1 + a_2\mathbf{v}_1^2 + \mathbf{a}_3\cdot\mathbf{v}_1\mathbf{v}_1^2 + a_4\mathbf{v}_1^4 + ..,] \qquad (96)$$

where the coefficients $a_i \equiv a_i(\mathbf{r}_1,t)$ for $i \in \mathbb{N}$ are all assumed of order $O(\varepsilon)$.Then validity of the same equations (92),(93) and (94) requires identically in $\Omega \times I$:

$$\int_{U_1} d^3\mathbf{v}_1\delta\rho_1^{(N)}(\mathbf{x}_1,t) = 0 \quad (97)$$

$$\int_{U_1} d^3\mathbf{v}_1\mathbf{v}_1\delta\rho_1^{(N)}(\mathbf{x}_1,t) = 0, \quad (98)$$

$$\int_{U_1} d^3\mathbf{v}_1\frac{m}{3}\left(\mathbf{v}_1 - \mathbf{V}_1(\mathbf{r}_1,t)\right)^2 \delta\rho_1^{(N)}(\mathbf{x}_1,t) = 0. \quad (99)$$

These equations yield linear constraint equations for the coefficients of the polynomial expansion (96) of the form:

$$\int_{U_1} d^3\mathbf{v}_1\rho_M^{(N)}(\mathbf{v}_1)\left[a_2\mathbf{v}_1^2 + a_4\mathbf{v}_1^4 + ..\right] = 0$$

$$\int_{U_1} d^3\mathbf{v}_1\mathbf{v}_1\rho_M^{(N)}(\mathbf{v}_1)\left[\mathbf{a}_1\cdot\mathbf{v}_1 + \mathbf{a}_3\cdot\mathbf{v}_1\mathbf{v}_1^2 + ..\right] = 0,$$

$$\int_{U_1} d^3\mathbf{v}_1\frac{m}{3}\mathbf{v}_1^2\delta\rho_1^{(N)}(\mathbf{x}_1,t)\left[a_2\mathbf{v}_1^2 + a_4\mathbf{v}_1^4 + ..\right] = 0 \quad (100)$$

It must be stressed that these constraint equations can always be satisfied in the case MMKE (31) is replaced with the homogeneous Vlasov kinetic equation, namely

$$L_1(\mathbf{F})\rho_1^{(N)}(\mathbf{x}_1, t) = 0. \qquad (101)$$

However, validity of MMKE and in particular the fact that the Master collision operator $\mathcal{C}_1\left(\rho_1^{(N)}|\rho_1^{(N)}\right)$ evaluated in the particular case (95) is necessarily non-vanishing for a perturbation $\delta\rho_1^{(N)}(\mathbf{x}_1, t)$ of the form (96) implies that nontrivial relationships must hold among the coefficients $a_i(\mathbf{r}_1, t)$ for $i \in \mathbb{N}$. Indeed, one can actually show that the same coefficients are necessarily linearly coupled, requiring

$$a_2 = a_2(\mathbf{a}_1(\mathbf{r}_1, t); \mathbf{r}_1, t);$$
$$\mathbf{a}_3 = \mathbf{a}_3(a_2(\mathbf{r}_1, t); \mathbf{r}_1, t);$$
$$...... \qquad (102)$$

This implies that the Eqs.(100) cannot generally be simultaneously satisfied. The consequence is that validity of Eqs. (92)-(94) requires that in Eq.(95) it should be $\delta\rho_1^{(N)}(\mathbf{x}_1, t) \equiv 0$. Hence under such constraint conditions it follows that the 1−body PDF must necessarily coincide with the kinetic equilibrium PDF of the type indicated above (see Eq. (87)).

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