# A master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions

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We show that in four or more spacetime dimensions, the Einstein equations for gravitational perturbations of maximally symmetric vacuum black holes can be reduced to a single second-order wave equation in a two-dimensional static spacetime, irrespective of the mode of perturbations. Our starting point is the gauge-invariant formalism for perturbations in an arbitrary number of dimensions developed by the present authors, and the variable for the final second-order master equation is given by a simple combination of gauge-invariant variables in this formalism. Our formulation applies to the case of non-vanishing as well as vanishing cosmological constant  $\Lambda$ . The sign of the sectional curvature K of each spatial section of equipotential surfaces is also kept general. In the four-dimensional Schwarzschild background with  $\Lambda = 0$  and K = 1, the master equation for a scalar perturbation is identical to the Zerilli equation for the polar mode and the master equation for a vector perturbation is identical to the Regge-Wheeler equation for the axial mode. Furthermore, in the four-dimensional Schwarzschild-anti-de Sitter background with  $\Lambda < 0$  and K = 0, 1, our equation coincides with those recently derived by Cardoso and Lemos. As a simple application, we prove the perturbative stability and uniqueness of four-dimensional non-extremal spherically symmetric black holes for any  $\Lambda$ . We also point out that there exists no simple relation between scalar-type and vector-type perturbations in higher dimensions, unlike in four dimension. Although in the present paper we treat only the case in which the horizon geometry is maximally symmetric, the final master equations are valid even when the horizon geometry is described by a generic Einstein manifold, if we employ an appropriate reinterpretation of the curvature K and the eigenvalues for harmonic tensors.

#### §1. Introduction

Perturbative analyses of four-dimensional black hole spacetimes have provided useful tools for the investigation of astrophysical problems, such as gravitational wave emission from gravitational collapse and black holes,  $^{1),2),3)$  as well as for the investigation of more fundamental problems, such as the stability and uniqueness of black holes.  $^{1),4),5),6)$  In these analyses, a key role is played by the fact that the Einstein equations for perturbations can be reduced to a second-order ordinary differential equation (ODE) of the self-adjoint type through harmonic expansion. Such a master equation in the Schwarzschild background was first derived by Regge and Wheeler for the axial mode,  $^{7),8)}$  and subsequently by Zerilli for the polar mode.  $^{9),10)}$  Later, the formulation was further extended to the Kerr background by Teukolsky.  $^{11),12)}$  Recently, it was also extended to the Schwarzschild-de Sitter and Schwarzschild-

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anti-de Sitter backgrounds by Cardoso and Lemos<sup>13),14)</sup> and utilised to determine the frequencies and the behaviour of quasi-normal modes.<sup>15),16),17),18),19)</sup>

Judging from this experience in the four-dimensional case, it is expected that an extension of the formalism to higher dimensions would be quite useful for the investigation of fundamental problems as well as phenomenological problems in higher-dimensional gravity theories, which are currently subjects of intensive research. In particular, such an extension will provide an exact basic equation for the quasinormal mode analysis of higher-dimensional black holes<sup>20),21),17)</sup> motivated by the AdS/CFT issue.<sup>22)</sup>

Such extensions already exist in some limited cases. For example, the present authors developed a gauge-invariant formalism for perturbations in spacetimes of arbitrary dimension greater than three in a previous paper<sup>23</sup> (which is referred to as KIS2000 in the present paper) by extending the formulation presented in Refs. 24) and 25). There, assuming that the background spacetime of dimension n+mpossesses a spatial isometry group  $G_n$  that is isomorphic to the isometry group of an n-dimensional space  $K^n$  with constant sectional curvature K, we expanded perturbation variables in terms of harmonic tensors on  $\mathcal{K}^n$  and expressed the perturbed Einstein equations as a set of equations for gauge-invariant variables in the m-dimensional orbit space constructed from the expansion coefficients. In this formalism, the gauge-invariant variables are grouped into three types, the tensor type, vector type, and scalar type, according to the type of harmonic tensors used to expand the perturbation variables. Each of these types of variables obeys an independent closed set of equations. In particular, we showed that in the case that m=2 and the background spacetime satisfies the vacuum Einstein equations, the perturbed Einstein equations for both a vector perturbation and a tensor perturbation reduce to a single second-order wave equation—which is called a master equation in the present paper—in the two-dimensional orbit spacetime spanned by the time and radial coordinates.\*) This result is independent of the sign of K and holds for both vanishing and non-vanishing cosmological constant  $\Lambda$ .

For a four-dimensional Schwarzschild black hole background, a vector perturbation corresponds to the axial (odd) mode, and the master variable and the master equation presented in KIS2000 correspond directly to the Regge-Wheeler variable and the Regge-Wheeler equation, respectively. Hence, we already have an extension to higher dimensions of the Regge-Wheeler formalism for the axial mode. However, such an extension for a scalar perturbation, i.e., an extension to higher dimensions of the Zerilli formalism for the polar (even) mode in four dimensions, has not yet been obtained, although simple master equations have been derived for a scalar perturbation in constant curvature spacetimes using various methods. <sup>26</sup>, <sup>27</sup>, <sup>23</sup>, <sup>28</sup>, <sup>29</sup>, <sup>30</sup>, <sup>31</sup>, <sup>32</sup>, <sup>33</sup>

The main purpose of the present paper is to construct such an extension of the Zerilli formalism to spacetimes of arbitrary dimension greater than three. To be precise, we show that under the assumption of the  $G_n$  symmetry of the background

<sup>\*)</sup> The master equations for these types of perturbations were first derived by Mukohyama in Ref. 29) for the case of a maximally symmetric background. In KIS2000, it was shown that the same equations hold without the assumption of maximal symmetry if the background possesses  $G_n$  symmetry.

spacetime, the gauge-invariant vacuum Einstein equations given in KIS2000 for a scalar perturbation in a system of greater than three dimensions can be reduced to a single second-order wave equation in the two-dimensional orbit space for a master variable  $\Phi$  consisting of a simple combination of the gauge-invariant variables. This equation is identical to the Zerilli equation for the polar mode in the four-dimensional Schwarzschild background. By the generalised Birkhoff theorem, the symmetry assumption requires the background spacetime to be static, and its metric is uniquely determined by the normalized curvature K of symmetry orbits, the cosmological constant  $\Lambda$ , and a parameter M representing the black hole mass, provided that the spacetime is not of the Nariai type, 34) which is assumed in the present paper. In order to allow for the widest possible application of the present formulation, these parameters are kept arbitrary. For example, the formulation describes a scalar perturbation of a higher-dimensional Schwarzschild black hole for K=1 and  $\Lambda=0$ , while it describes a scalar perturbation of a Schwarzschild-anti-de Sitter spacetime for  $\Lambda < 0$  and  $M \neq 0$ . We also show that in the special case M = 0, the master equation derived in the present paper is essentially equivalent to the master equation derived by Mukohyama in Ref. 29). Further, for completeness, we show explicitly that vector and tensor perturbations obey master equations whose structures are the same as in the case of a scalar perturbation, and we give the corresponding effective potentials for these equations. We also point out that the simple relation between the polar mode and the axial mode, which was first obtained by Chandrasekhar and Detweiler<sup>1)</sup> for a four-dimensional Schwarzschild black hole and then extended to the four-dimensional asymptotically anti-de Sitter case by Cardoso and Lemos, <sup>13), 14)</sup> does not hold in higher dimensions.

The present paper is organized as follows. In the next section, we show that the gauge-invariant perturbation equations for a scalar perturbation given in KIS2000 are equivalent to a set of first-order ODEs with a linear constraint for three gaugeinvariant variables, after carrying out the Fourier transform with respect to the time coordinate. Then, in §3, this set of equations is reduced to a single master equation of the Zerilli type in the case of a non-vanishing frequency. We further show that this master equation can be converted into a wave equation for a master field  $\Phi$  in the two-dimensional orbit space by the inverse Fourier transformation and that all gaugeinvariant variables can be represented as combinations of this field and its derivatives. A subtle point regarding static perturbations in this derivation is treated in §4. In §5, we rewrite the master equations for tensor and vector perturbations given in KIS2000 in the same form as that for a scalar perturbation. Section 6 is devoted to discussion. There, using the master equations for four dimensions, the perturbative stability and uniqueness of the Schwarzschild-de Sitter and Schwarzschild-anti de-Sitter black holes in four dimensions is proved. We also discuss the relation between the scalar and vector master variables and the extension of the formulation to the case in which the horizon geometry is described by a generic Einstein metric. Because the calculations required to derive the master equations are quite lengthy, most of them were done by symbolic computation with Maple.

#### §2. Basic equations

In this section, we show that the gauge-invariant equations for scalar perturbations given in KIS2000 reduce to a set of first-order differential equations with a linear constraint for three gauge-invariant variables in a maximally symmetric black hole background.

#### 2.1. Perturbation equations

The general form of the metric of a  $G_n$ -symmetric background spacetime considered in the present paper is given by

$$ds^2 = g_{ab}(y)dy^a dy^b + r^2(y)d\sigma_n^2, (2.1)$$

where  $g_{ab}$  is the Lorentzian metric of the two-dimensional orbit spacetime, and  $d\sigma_n^2 = \gamma_{ij}(z)dz^idz^j$  is the metric of the *n*-dimensional  $G_n$ -invariant base space  $\mathcal{K}^n$  with normalized constant sectional curvature  $K=0,\pm 1$ . Hence, the dimension of the whole spacetime is n+2, and the spherically symmetric case corresponds to K=1.

Scalar perturbations of this spacetime can be expanded in terms of the harmonic functions  $\mathbb{S}$  on  $\mathcal{K}^n$ , which satisfy the equation

$$(\hat{\triangle}_n + k^2)\mathbb{S} = 0, \tag{2.2}$$

where  $\hat{\triangle}_n$  is the Laplace-Beltrami operator on  $\mathcal{K}^n$ , and  $k^2$  is its eigenvalue. For  $K=1, k^2$  takes discrete values,

$$k^2 = l(l+n-1), l = 0, 1, 2, \dots,$$
 (2.3)

while for  $K \leq 0$ ,  $k^2$  can take any non-negative real value. Each harmonic mode of the metric perturbation  $\delta g_{MN}$  can be written

$$\delta g_{ab} = f_{ab} \mathbb{S}, \ \delta g_{ai} = r f_a \mathbb{S}_i, \ \delta g_{ij} = 2r^2 (H_L \gamma_{ij} \mathbb{S} + H_T \mathbb{S}_{ij}),$$
 (2.4)

where

$$S_i = -\frac{1}{k}\hat{D}_i S, \qquad (2.5)$$

$$S_{ij} = \frac{1}{k^2} \hat{D}_i \hat{D}_j S + \frac{1}{n} \gamma_{ij} S, \qquad (2.6)$$

and  $\hat{D}_i$  is the covariant derivative with respect to the metric  $\gamma_{ij}$  on  $\mathcal{K}^n$ . Here and hereafter we omit the indices k labelling the eigenvalues of the harmonics and the summation over them.

Note that the modes with k = 0 and  $k^2 = nK$  require special treatment. First, for the k = 0 mode,  $\mathbb{S}$  is a constant, and  $\mathbb{S}_i$  and  $\mathbb{S}_{ij}$  are not defined. This mode corresponds to a perturbation that possesses the spatial symmetry  $G_n$  of the background spacetime. Because the only perturbation with the spatial symmetry  $G_n$  satisfying the Einstein equations consists simply of a change of the background metric in the form of a shift of the mass parameter of the black hole, as a consequence of the

Birkhoff theorem, we do not have to consider this case. Next,  $k^2 = nK$  occurs only for K = 1, and this corresponds to l = 1. For such modes,  $\mathbb{S}_{ij}$  vanishes, and the variable  $H_T$  is not defined. We show below that these modes have no physical degrees of freedom.

For modes with  $k^2(k^2 - nK) \neq 0$ , the following combinations provide a basis for gauge-invariant variables for metric perturbations of the scalar type:<sup>23)</sup>

$$F = H_L + \frac{1}{r}H_T + \frac{1}{r}D^a r X_a,$$
 (2.7a)

$$F_{ab} = f_{ab} + D_a X_b + D_b X_a. (2.7b)$$

Here,  $D_a$  represents the covariant derivative with respect to the metric  $g_{ab}$  in the two-dimensional orbit space, and  $X_a$  is given by

$$X_a = \frac{r}{k} \left( f_a + \frac{r}{k} D_a H_T \right). \tag{2.8}$$

Because the Einstein tensors  $G_{MN}$  have the same structure as the metric perturbation (2·4) under the harmonic expansion, the vacuum Einstein equations for a scalar perturbation are given by the following set of equations:

$$\tilde{E}_{ab} \equiv r^{n-2} E_{ab} = 0, \ \tilde{E}_a \equiv r^{n-2} E_a = 0, \ \tilde{E}_L \equiv r^{n-2} E_L = 0, \ \tilde{E}_T \equiv r^{n-2} E_T = 0.$$
(2.9)

The explicit expressions of  $E_{ab}$ ,  $E_a$ ,  $E_L$  and  $E_T$  in terms of the gauge-invariant variables (2·7) can be obtained by specializing the general formula given by (63)–(66) in KIS2000 to the case in which the orbit space is two-dimensional (m = 2). As these expressions are long, we give them in Appendix A.

The equations in (2.9) are not independent, due to the Bianchi identities, which are written

$$\frac{1}{r^3}D_a(r^3\tilde{E}^a) - \frac{k}{r}\tilde{E}_L + \frac{n-1}{n}\frac{k^2 - nK}{kr}\tilde{E}_T = 0,$$
 (2·10a)

$$\frac{1}{r^2}D_b(r^2\tilde{E}_a^b) + \frac{k}{r}\tilde{E}_a - n\frac{D_a r}{r}\tilde{E}_L = 0.$$
 (2·10b)

Since we are only considering modes with  $k^2 > 0$ , it follows from these equations that  $E_L = 0$  and  $D_b(r^2\tilde{E}_a^b) = 0$  are automatically satisfied if  $E_a = E_T = 0$  holds. From (A·1b) and (A·1d), the latter equations are equivalent to the following set of equations:

$$\tilde{F}_a^a = -2(n-2)\tilde{F},\tag{2.11}$$

$$D_b(\tilde{F}_a^b - 2\tilde{F}\delta_a^b) = 0, \tag{2.12}$$

where

$$\tilde{F}_{ab} = r^{n-2}F_{ab}, \ \tilde{F} = r^{n-2}F.$$
 (2·13)

Note here that the equation  $E_T = 0$  is not obtained for the mode l = 1 in the spherically symmetric case, because in that case  $\mathbb{S}_{ij}$  vanishes. However, we can assume that this equation holds in this case too, by regarding it as a gauge condition, as is shown in Appendix B.

#### 2.2. Fourier transformation

Because solutions to the vacuum Einstein equations with the spatial symmetry  $G_n$  are always static as asserted by the Birkhoff theorem, our background metric can take the form

$$ds^{2} = -f(r)dt^{2} + \frac{dr^{2}}{f(r)} + r^{2}d\sigma_{n}^{2}, \qquad (2.14)$$

with

$$f(r) = K - \frac{2M}{r^{n-1}} - \lambda r^2, \tag{2.15}$$

provided that the spacetime is not of the Nariai type, which is assumed in the present paper. Here,  $\lambda$  is related to the cosmological constant  $\Lambda$  by

$$\lambda = \frac{2\Lambda}{n(n+1)}. (2.16)$$

When the metric  $(2\cdot14)$  with  $(2\cdot15)$  describes a black hole spacetime, the  $G_n$ -invariance of the spacetime implies the maximal symmetry of the spatial section of the event horizon. For this reason, we refer to the black hole described by the metric  $(2\cdot14)$  as a maximally symmetric black hole.\*). For K=1,  $\lambda=0$ ,  $(2\cdot14)$  is the standard higher-dimensional Schwarzschild metric—also referred to as the Tangherlini metric<sup>37</sup>)—with mass given in terms of the parameter M by<sup>40</sup>)

$$\frac{nM\mathcal{A}_n}{8\pi G},\tag{2.17}$$

where  $A_n = 2\pi^{(n+1)/2}/\Gamma[(n+1)/2]$  is the area of a unit *n*-sphere and *G* denotes the n+2-dimensional Newton constant.

Because the non-vanishing connection coefficients for this metric are given by

$$\Gamma_{tr}^{t} = \frac{f'}{2f}, \ \Gamma_{tt}^{r} = \frac{ff'}{2}, \ \Gamma_{rr}^{r} = -\frac{f'}{2f},$$
(2.18)

the equations  $D_b(r^2\tilde{E}_a^b) = 0$  can be written

$$r^2 \partial_t \tilde{E}_t^t + \partial_r (r^2 \tilde{E}_t^r) = 0, \tag{2.19a}$$

$$r^{2}\partial_{t}\tilde{E}_{r}^{t} + \partial_{r}(r^{2}\tilde{E}_{r}^{r}) + \frac{r^{2}f'}{2f}\tilde{E}_{r}^{r} - \frac{r^{2}f'}{2f}\tilde{E}_{t}^{t} = 0,$$
 (2·19b)

in the (t,r) coordinates. From the latter equation, we find that  $E_t^t=0$  holds if the equations  $D_b(r^2\tilde{E}_a^b)=\tilde{E}_t^r=\tilde{E}_r^r=0$  are satisfied. Hence, taking account of the argument involving the Bianchi identities in the previous subsection, we only have to consider the equations (2·11), (2·12),  $E_t^r=0$  and  $E_r^r=0$ .

<sup>\*)</sup> It is known that, in higher dimensions, the metric (2·14) with (2·15) can describe many different black hole solutions to the vacuum Einstein equations, with the simple replacement of  $d\sigma_n^2$  by the metric for any Einstein manifold.<sup>35),36)</sup> Among such solutions, the  $G_n$ -symmetric metric is the maximally symmetric solution. Our formulation also holds in the case that  $d\sigma_n^2$  is given by a generic Einstein metric, as mentioned in §6.

Next, we reduce these equations to first-order ODEs for three variables. First, let us introduce the variables X, Y and Z by

$$X = \tilde{F}_t^t - 2\tilde{F}, \ Y = \tilde{F}_r^r - 2\tilde{F}, \ Z = \tilde{F}_t^r.$$
 (2.20)

Then, from  $(2\cdot11)$ , all basic gauge-invariant variables can be expressed in terms of these three variables as

$$F_t^t = \frac{(n-1)X - Y}{nr^{n-2}}, \ F_r^r = \frac{-X + (n-1)Y}{nr^{n-2}}, \ F_t^r = \frac{Z}{r^{n-2}},$$
 (2·21a)

$$F = -\frac{X+Y}{2nr^{n-2}}. (2.21b)$$

Inserting these expressions into (2.12), we obtain

$$(2r/k)\tilde{E}_t \equiv -\partial_t X - \partial_r Z = 0, \qquad (2.22a)$$

$$(2r/k)\tilde{E}_r \equiv \frac{f'}{2f}X - \partial_r Y - \frac{f'}{2f}Y + \frac{1}{f^2}\partial_t Z.$$
 (2·22b)

Unlike  $\tilde{E}_a$ , the expressions of  $\tilde{E}_b^a$  in terms of X, Y and Z, which are given in Appendix A, are rather complicated. In order to put them into simpler forms, we utilize the Fourier transformation with respect to the time coordinate t. Then, writing the Fourier component proportional to  $e^{-i\omega t}$  as

$$X \to X, \quad Y \to Y, \quad Z \to i\omega \tilde{Z},$$
 (2.23)

we find that the Einstein equations are reduced to the following equations for  $\omega \neq 0$ :

$$X' = \frac{n-2}{r}X + \left(\frac{f'}{f} - \frac{2}{r}\right)Y + \left(\frac{k^2}{fr^2} - \frac{\omega^2}{f^2}\right)\tilde{Z},\tag{2.24a}$$

$$Y' = \frac{f'}{2f}(X - Y) + \frac{\omega^2}{f^2}\tilde{Z},$$
(2.24b)

$$\tilde{Z}' = X, \tag{2.24c}$$

$$\left[\omega^{2}r^{2} + K\lambda r^{2} + \frac{M}{r^{n-1}}\left(n(n-1)K - n(n+1)\lambda r^{2} - \frac{(n^{2}-1)M}{r^{n-1}}\right)\right]X 
+ \left[\omega^{2}r^{2} - k^{2}f + nK^{2} - (n-1)K\lambda r^{2} - \frac{4nKM}{r^{n-1}} + \frac{(n+1)^{2}M^{2}}{r^{2(n-1)}}\right]Y 
- \frac{1}{r}\left[n\omega^{2}r^{2} + \left(\lambda r^{2} - \frac{(n-1)M}{r^{n-1}}\right)k^{2}\right]\tilde{Z} = 0.$$
(2·24d)

Here, the last equation corresponds to  $-2r^2f\tilde{E}_r^r=0$ .

Now, we comment on the l=1 mode. As explained in Appendix B, the above basic equations also hold for this mode, if we regard  $E_T=0$  as a gauge condition. However, the number of the residual gauge degrees of freedom that remain after this partial gauge fixing is the same as that of the degrees of freedom of the general solution to  $(2\cdot24)$ . This implies that there exist no physical degrees of freedom for l=1. For this reason, we assume that  $l\geq 2$  from this point.

#### §3. Master equation

Because the constraint (2·24d) is linear, we can always reduce the system of first-order ODEs (2·24) to a second-order ODE for any linear combination of X, Y and Z. However, the corresponding 2nd-order ODE in general has coefficients that depend on the frequency  $\omega$  in an intricate way and therefore it is not useful. In particular, in order to allow for its use in the stability analysis of black holes, it is desirable that the master equation has the form  $A\Phi = \omega^2 \Phi$ , where A is a self-adjoint second-order ordinary differential operator independent of  $\omega$ , as does the Zerilli equation for the four-dimensional Schwarzschild black hole. Starting from the expression of the master variable for the Zerilli equation in terms of our gauge-invariant variables, after some trial and error, we have found that the following form of  $\Phi$  is the best:

$$\Phi := \frac{n\tilde{Z} - r(X+Y)}{r^{n/2-1}H}. (3.1)$$

Here, H is a function of r defined by

$$H(r) = m + \frac{1}{2}n(n+1)x; (3.2)$$

$$m = k^2 - nK, (3.3)$$

$$x = \frac{2M}{r^{n-1}}. (3.4)$$

In terms of  $\Phi$ , (2.24) is reduced to

$$-f\frac{d}{dr}\left(f\frac{d\Phi}{dr}\right) + V_S\Phi = \omega^2\Phi. \tag{3.5}$$

Here, the effective potential  $V_S$  is given by

$$V_S(r) = \frac{f(r)Q(r)}{16r^2H^2},$$
(3.6)

with

$$Q(r) = -\left[n^{3}(n+2)(n+1)^{2}x^{2} - 12n^{2}(n+1)(n-2)mx + 4(n-2)(n-4)m^{2}\right]y + n^{4}(n+1)^{2}x^{3} + n(n+1)\left[4(2n^{2} - 3n + 4)m + n(n-2)(n-4)(n+1)K\right]x^{2} - 12n\left[(n-4)m + n(n+1)(n-2)K\right]mx + 16m^{3} + 4Kn(n+2)m^{2},$$
(3.7)

where

$$y = \lambda r^2. (3.8)$$

It is easy to see that this equation is identical to the Zerilli equation<sup>9)</sup> for n=2, K=1 and  $\lambda=0$  and identical to the equation for even modes derived by Cardoso and Lemos<sup>13),14)</sup> for n=2, K=0,1 and  $\lambda<0$ .

The original fundamental variables  $X,\,Y$  and Z are expressed in terms of the master variable  $\varPhi$  as

$$X = r^{n/2-2} \left[ \left( \frac{\omega^2 r^2}{f} - \frac{P_X}{16H^2} \right) \Phi + \frac{Q_X}{4H} r \partial_r \Phi \right], \tag{3.9a}$$

$$Y = r^{n/2-2} \left[ -\left(\frac{\omega^2 r^2}{f} + \frac{P_Y}{16H^2}\right) \Phi + \frac{Q_Y}{4H} r \partial_r \Phi \right], \tag{3.9b}$$

$$Z = i\omega r^{n/2-1} \left[ \frac{P_Z}{4H} \Phi - fr \partial_r \Phi \right]. \tag{3.9c}$$

Here, the coefficients  $P_X, Q_X, P_Y, Q_Y$ , and  $P_Z$  are functions of r expressed in terms of  $x = 2M/r^{n-1}$  and  $y = \lambda r^2$  as

$$P_{X}(r) = 4(n-1)[n^{2}(n+1)x - 2(n-2)m]my$$

$$+n^{3}(n+1)^{3}x^{3} + 2n(n+1)[2(n^{2}+n+2)m - n(n-2)(n+1)K]x^{2}$$

$$-4n[(n-11)m + n(n+1)(n-3)K]mx + 16m^{3} + 8Km^{2}n^{2}, \quad (3.10a)$$

$$Q_{X}(r) = 4(n-1)my + n(n+1)^{2}x^{2} + 2[(3n-1)m - n(n+1)K]x$$

$$-4Knm, \quad (3.10b)$$

$$P_{Y}(r) = [2n^{4}(n+1)^{2}x^{2} - 4n^{2}(n+1)(n-3)mx + (-8n+16)m^{2}]y$$

$$+n^{3}(n-1)(n+1)^{2}x^{3} + 2n(n^{2}-1)[4m - n(n-2)(n+1)K]x^{2}$$

$$+4n(n-1)[3m + n(n+1)K]mx, \quad (3.10c)$$

$$Q_{Y}(r) = 2[n^{2}(n+1)x + 2m]y + n(n-1)(n+1)x^{2}$$

$$-2(n-1)[m+n(n+1)K]x, \quad (3.10d)$$

$$P_{Z}(r) = [-n^{2}(n+1)x + 2(n-2)m]y + n(n+1)x^{2}$$

$$+[2(2n-1)m+n(n+1)(n-2)K]x - 2Knm. \quad (3.10e)$$

Although we have utilized the Fourier transformation with respect to the time coordinate to derive the master equation (3.5), we can apply the inverse Fourier transform to it and thereby obtain the master equation in the form of a wave equation for a master field  $\Phi(t,r)$  in the two-dimensional orbit space with the coordinates (t,r). This master wave equation is obtained simply replacing  $\omega$  by  $i\partial_t$  in (3.5). This yields

$$\Box \Phi - \frac{V_S}{f} \Phi = 0, \tag{3.11}$$

where  $\Box$  is the d'Alembertian operator in the two-dimensional orbit space with the metric  $g_{ab}dy^ady^b$ . Furthermore, because the right-hand sides of (3·9) are polynomials in  $\omega$ , these expressions for the original gauge-invariant variables in terms of the master variable can be transformed into expressions for X(t,r), Y(t,r) and Z(t,r) in terms of  $\Phi(t,r)$  through the same replacement:

$$X = r^{n/2-2} \left( -\frac{r^2}{f} \partial_t^2 \Phi - \frac{P_X}{16H^2} \Phi + \frac{Q_X}{4H} r \partial_r \Phi \right), \tag{3.12a}$$

$$Y = r^{n/2-2} \left( \frac{r^2}{f} \partial_t^2 \Phi - \frac{P_Y}{16H^2} \Phi + \frac{Q_Y}{4H} r \partial_r \Phi \right), \tag{3.12b}$$

$$Z = r^{n/2-1} \left( -\frac{P_Z}{4H} \partial_t \Phi + f r \partial_r \partial_t \Phi \right). \tag{3.12c}$$

If we introduce the variable  $\tilde{\Omega}$  defined by

$$\tilde{\Omega} = r^{n/2} H \Phi, \tag{3.13}$$

these expressions can be put into the covariant form\*)

$$\tilde{F} = \frac{1}{4nr^2H} \left[ (2H - nrf')\tilde{\Omega} + 2nrDr \cdot D\tilde{\Omega} \right], \tag{3.14a}$$

$$\tilde{F}_{ab} + (n-2)\tilde{F}g_{ab} = \frac{1}{H} \left( D_a D_b \tilde{\Omega} - \frac{1}{2} \Box \tilde{\Omega} g_{ab} \right). \tag{3.14b}$$

We found by symbolic computations that when these expressions are inserted into  $\tilde{E}_a$ ,  $\tilde{E}_L$ ,  $\tilde{E}_T$  and  $\tilde{E}_{ab}$ , the latter take forms consisting of linear combinations of the master wave equation (3·11) and its derivatives. This suggests that the set of equations (3·11) and (3·12) is equivalent to the Einstein equations even if the Fourier transforms of the gauge-invariant variables with respect to the time coordinate do not exist. In fact, we can confirm this by the following argument. First, by inspecting the procedure leading to the master equation, we find that if we take  $\partial_t X(t,r)$ ,  $\partial_t Y(t,r)$  and Z(t,r) as basic variables and define  $\Phi(t,r)$  by

$$\partial_t \Phi(t,r) = -\frac{nZ(t,r) + \partial_t X(t,r) + \partial_t Y(t,r)}{r^{n/2-1}H},$$
(3.15)

we can obtain the time derivative of the master wave equation,

$$\partial_t \left( \Box \Phi - \frac{V_S}{f} \Phi \right) = 0, \tag{3.16}$$

from an algebraic combination of  $\tilde{E}_a$ ,  $\tilde{E}_t^r$ ,  $\tilde{E}_r^r$  and their derivatives. The expressions for  $\partial_t X$ ,  $\partial_t Y$  and Z in terms of  $\partial_t \Phi$  corresponding to (3·12) can be also obtained with the same method. This implies that if the perturbed Einstein equations had a solution that could not be expressed in terms of the master wave equation (3·11) with (3·12), it must be static. This is consistent with the fact that original definition of  $\Phi$  in terms of the Fourier transform, (3·1), becomes singular for  $\omega = 0$ . However, as we show in the next section, static solutions are also solutions to (3·11) with (3·12). Therefore, these equations are equivalent to the original perturbed Einstein equations.

Finally, we note that in the special case M=0, the master variable  $\Phi$  considered in the present paper is related to the master variable  $\Phi_{(S)}$  in Ref. 29) and  $\Omega$  in KIS2000 for scalar perturbations in a constant-curvature background spacetime as

$$\Phi_{(S)} = \Omega = r^{n/2}\Phi,\tag{3.17}$$

as is shown in Appendix C.

## §4. Static perturbations

As noted in the previous section, the master variable  $\Phi$  defined by (3·1) is illdefined for  $\omega = 0$ , and the derivation of the master equation for a scalar perturbation

<sup>\*)</sup> We would like to thank an anonymous referee for recommending us to look for these covariant expressions.

given in the previous section does not apply to static perturbations. In this section, we show that in spite of this, the master wave equation (3·11) also describes static scalar perturbations.

For static perturbations, from  $(2\cdot22a)$ ,  $(2\cdot22a)$ ,  $(A\cdot2a)$  and  $(A\cdot2b)$ , we find that the equations  $E_a = 0$ ,  $E_t^r = 0$  and  $E_r^r = 0$  can be written

$$E_t: Z' = 0, (4.1a)$$

$$E_r: Y' + \frac{f'}{2f}(Y - X) = 0,$$
 (4·1b)

$$E_t^r : \frac{k^2}{r^2} Z = 0,$$
 (4·1c)

$$\begin{split} E_r^r : -\frac{f'}{2}X' + \left(\frac{n-1}{r^2}(f-K) + \frac{2(n+1)\lambda}{n} + \frac{(n+2)f'}{2r} + \frac{f''}{n}\right)X \\ - \left(\frac{f'}{2} + \frac{nf}{r}\right)Y' - \left(\frac{n-1}{r^2}K + \frac{f}{r^2} + \frac{2(n^2-1)\lambda}{n} + \frac{(3n-2)f'}{2r} + \frac{n-1}{n}f'' - \frac{k^2}{r^2}\right)Y = 0. \end{split} \tag{4.1d}$$

Hence, Z vanishes and the basic equations are given by the following set of first-order ODEs for X and Y:

$$f'X' = \left(\frac{2(n-1)}{r^2}(f-K) + \frac{4(n+1)\lambda}{n} + \frac{2f'}{r} - \frac{(f')^2}{2f} + \frac{2f''}{n}\right)X$$

$$-\left(\frac{2(f-K)}{r^2} + \frac{4(n^2-1)}{n}\lambda + \frac{2(n-1)}{r}f' - \frac{(f')^2}{2f}\right)$$

$$+\frac{2(n-1)}{n}f'' - \frac{2(k^2 - nK)}{r^2}Y, \tag{4.2a}$$

$$Y' = \frac{f'}{2f}(X - Y), \tag{4.2b}$$

$$Z = 0. (4.2c)$$

Because this set consists of first-order ODEs for the two variables X and Y, it is always possible to reduce it to a single second-order ODE for any linear combination of X and Y. Hence, if we adopt a combination that is consistent with (3·12) in the static case, it is expected that we can obtain a master equation that coincides with (3·11) without the time derivative term. In fact, there is a unique such choice, given by

$$\Phi(r) := \frac{Q_Y(r)X(r) - Q_X(r)Y(r)}{2k^2r^{n/2-1}f'(r)H(r)}.$$
(4.3)

For this choice, if we express X and Y in terms of  $\Phi$  and  $\partial_r \Phi$  with the help of (4·2), we find that they are represented by (3·12a) and (3·12b) with  $\partial_t \Phi = 0$ . Further, insertion of these expressions into (4·2) gives (3·11) without the time derivative term. Thus, it is found that the same master equation holds for both non-static and static perturbations.

Although it is preferable for the investigation of general cases (such as the stability analysis of black holes) that every perturbation be described by a single master equation, the potential in the master equation (3·11) is rather complicated and is not always useful for analysis of static perturbations. Because the behaviour of static perturbations is important with regard to the issue of black hole uniqueness, it would be useful if we could obtain a master equation with a simpler potential.

Now, we show that such a master equation is indeed obtained if we adopt Y as the master variable. First,  $(4\cdot2)$  leads to the following second-order ODE for Y:

$$Y'' + \alpha Y' + \beta Y = 0, (4.4)$$

where

$$r^{2}ff'\alpha = (n-1)(n+2)Kx + 2nK\lambda r^{2} + (n-1)(n-4)x^{2}$$
$$-(n^{2} + 11n - 10)x\lambda r^{2} - 2(n-4)\lambda^{2}r^{4},$$
(4·5a)
$$r^{2}f\beta = -k^{2} + nK + (n-2)(K-f).$$
(4·5b)

In order to rewrite this equation in a formally self-adjoint form, we introduce the new variable  $\tilde{Y}$  defined by

$$\tilde{Y} := SY; \tag{4.6}$$

$$S = f^{-1/2} \exp\left(\int \frac{\alpha}{2} dr\right) = \frac{f^{1/2}}{r^{n/2 - 1} f'}.$$
 (4.7)

Then, the above equation for Y is transformed into

$$-f\frac{d}{dr}\left(f\frac{d\tilde{Y}}{dr}\right) + \tilde{V}\tilde{Y} = 0, \tag{4.8}$$

where

$$\tilde{V}(r) = \frac{f}{r^2}k^2 + \frac{\tilde{Q}}{4r^2(rf')^2}; \tag{4.9}$$

$$\tilde{Q}(r) = -(2n-1)(n-1)^2x^4 + 2(n-1)[n(n-1)K + (5n^2 + 23n + 6)y]x^3 + [n(n-2)(n-1)^2K^2 - 2(n-1)(n+3)(n^2 + 10n + 6)Ky + (n^4 + 28n^3 + 61n^2 - 66n - 28)y^2]x^2 + [4(n-1)(3n^2 + 10n + 4)K^2 - 8(3n^3 + 7n^2 - 5n - 6)Ky + 4(3n^3 + 7n^2 - 12n - 4)y^2]yx + 4n(n-2)K^2y^2 - 8(n^2 - 2n - 2)Ky^3 + 4n(n-2)y^4. \tag{4.10}$$

The dependence on  $k^2$  of this effective potential is simpler than that of V. Furthermore, although it is still rather complicated for  $\lambda \neq 0$ , it becomes quite simple for  $\lambda = 0$ :

$$\tilde{V} = \frac{4k^2(1-x) + n(n-2) + 2nx - (2n-1)x^2}{4r^2}.$$
(4·11)

#### §5. Vector and tensor perturbations

It was shown in KIS2000 that vector and tensor perturbations in the background spacetimes considered in the present paper are described by master equations of the wave equation type in the two-dimensional orbit space. In this section, for completeness, we show that these equations can also be written in the form of (3.11).

#### 5.1. Tensor perturbations

Perturbations of the tensor type can be expanded in terms of the tensor-type harmonic tensors  $\mathbb{T}_{ij}$  satisfying

$$(\hat{\Delta}_n + k_T^2) \mathbb{T}_{ij} = 0, \tag{5.1a}$$

$$\mathbb{T}^i_{\ i} = 0, \quad \hat{D}_i \mathbb{T}^j_{\ i} = 0. \tag{5.1b}$$

The eigenvalues  $k_T^2$  are all positive. They form a continuous set for  $K \leq 0$  and a discrete set,

$$k_T^2 = l(l+n-1) - 2, \quad l = 1, 2, \cdots,$$
 (5.2)

for K=1.

Each harmonic component of the metric perturbation is expressed by  $(2\cdot 4)$ , with  $f_{ab} = f_a = H_L = 0$  and  $\mathbb{S}_{ij}$  replaced by  $\mathbb{T}_{ij}$ .  $H_T$  is gauge-invariant by itself, and in vacuum, it satisfies the following wave equation in the two-dimensional orbit space:

$$\Box H_T + \frac{n}{r} Dr \cdot DH_T - \frac{k_T^2 + 2K}{r^2} H_T = 0.$$
 (5.3)

If we introduce the master variable  $\Phi$  by

$$\Phi = r^{n/2} H_T, \tag{5.4}$$

this equation can be rewritten in the same form as (3.11):

$$\Box \Phi - \frac{V_T}{f} \Phi = 0, \tag{5.5}$$

with the effective potential

$$V_T(r) = \frac{f}{r^2} \left[ k_T^2 + 2K + \frac{n(n-2)}{4} K - \frac{n(n+2)}{4} \lambda r^2 + \frac{n^2 M}{2r^{n-1}} \right].$$
 (5.6)

This potential is identical to that derived by Gibbons and Hartnoll<sup>38)</sup> for the more general case in which  $d\sigma_n^2$  in (2·14) is given by an Einstein metric, if we use the relation  $\lambda_L = k_T^2 + 2nK$  that is valid for maximally symmetric black holes, where  $\lambda_L$  is the eigenvalue of the Lichnerowicz operator.\*

<sup>\*)</sup> The authors thank Sean Hartnoll for discussion of this point<sup>39)</sup>

#### 5.2. Vector perturbations

Perturbations of the vector type can be expanded in terms of the vector-type harmonic tensors  $V_i$  satisfying

$$(\hat{\Delta}_n + k_V^2) \mathbb{V}_i = 0, \tag{5.7a}$$

$$\hat{D}_j \mathbb{V}^j = 0. (5.7b)$$

As in the case of the tensor-type harmonics, the eigenvalues  $k_V^2$  are all positive, forming a continuous set for  $K \leq 0$  and a discrete set for K = 1. However, this discrete spectrum is shifted by unity with respect to that in the tensor case:

$$k_V^2 = l(l+n-1) - 1, \quad l = 1, 2, \cdots$$
 (5.8)

The expression for the metric perturbation is now given by  $(2\cdot 4)$  with  $\mathbb{S}_i$  replaced by  $\mathbb{V}_i$  and  $\mathbb{S}_{ij}$  by

$$\mathbb{V}_{ij} = -\frac{1}{2k_V}(\hat{D}_i \mathbb{V}_j + \hat{D}_j \mathbb{V}_i), \tag{5.9}$$

and the non-vanishing harmonic coefficients are  $f_a$  and  $H_T$ . For  $k^2 > (n-1)K$ ,

$$F_a = f_a + \frac{r}{k_V} D_a H_T \tag{5.10}$$

is gauge invariant and represents a natural basic variable. In our vacuum background,  $F^a$  is expressed in terms of a field  $\Omega(t,r)$  in the two-dimensional orbit space satisfying the wave equation

$$\Box \Omega - \frac{n}{r} Dr \cdot D\Omega - \frac{k_V^2 - (n-1)K}{r^2} \Omega = 0$$
 (5·11)

as

$$r^{n-1}F^a = \epsilon^{ab}D_b\Omega, \tag{5.12}$$

where  $\epsilon_{ab}$  is the Levi-Civita tensor of the two-dimensional orbit space.

If we introduce the master variable  $\Phi$  by

$$\Phi = r^{-n/2}\Omega,\tag{5.13}$$

(5.11) takes the form

$$\Box \Phi - \frac{V_V}{f} \Phi = 0, \tag{5.14}$$

with

$$V_V(r) = \frac{f}{r^2} \left[ k_V^2 + K + \frac{n(n-2)K}{4} - \frac{n(n-2)}{4} \lambda r^2 - \frac{3n^2M}{2r^{n-1}} \right].$$
 (5.15)

This equation is identical to the Regge-Wheeler equation for n=2, K=1 and  $\lambda=0$  and identical to the equation for odd modes derived by Cardoso and Lemos for n=2, K=0,1 and  $\lambda<0.^{13),14)}$ 

For  $k_V^2 = (n-1)K$ , i.e., K = 1 and l = 1,  $V_{ij}$  vanishes, and the perturbation variable  $H_T$  loses meaning. In this case, the following becomes a basic gauge-invariant variable:

$$F_{ab}^{(1)} = rD_a(f_b/r) - rD_b(f_a/r). (5.16)$$

In our vacuum background, from the Einstein equations, it follows that  $F_{ab}^{(1)}$  is expressed as

$$F_{ab}^{(1)} = \frac{L\epsilon_{ab}}{r^{n+1}},\tag{5.17}$$

where L is an arbitrary constant. This simply represents a rotational perturbation of the black hole, and in the spherically symmetric case, L corresponds to the angular-momentum parameter in the Myers-Perry solution.<sup>40)</sup>

### §6. Discussion

In this paper, we have derived master equations of the wave equation type that describe gravitational perturbations of maximally symmetric black hole spacetimes in higher dimensions. For scalar-type and vector-type perturbations, respectively, they represent extensions of the Zerilli equation for polar perturbations and of the Regge-Wheeler equation for axial perturbations of the four-dimensional Schwarzschild black hole to higher dimensions as well as to the case of a non-vanishing cosmological constant and to quasi-black hole spacetimes in which constant-time sections of equipotential surfaces have non-positive sectional curvatures. Hence, these master equations are expected to be useful in a wide variety of higher-dimensional gravity problems.

Our formulation also gives extensions of the Zerilli and the Regge-Wheeler formalisms to the four-dimensional Schwarzschild-de Sitter and Schwarzschild-anti-de Sitter backgrounds, which coincide with those given by Cardoso and Lemos recently.<sup>13)</sup> Here, we show that we can prove the stability of four-dimensional Schwarzschild-de Sitter and Schwarzschild-anti-de Sitter black holes using our formulation. For a spherically symmetric spacetime of four dimensions, i.e., for n = 2 and K = 1, Q in (3·7) becomes

$$Q = 288x^{2}f(r) + 432x^{3} + (288 + 144l_{2}^{2} + 720l_{2})x^{2} + (l_{2} + 1)^{2}(l_{2} + 4)^{2}[48x + 16(l_{2} + 3)(l_{2} + 2)],$$
(6·1)

where  $l_2 = l - 2$ . Hence, the effective potential  $V_S$  in (3·5) is positive definite. This suggests that  $\omega^2$  in (3·5) is always positive and that the black hole is stable with respect to scalar perturbations. In fact, this is actually the case for a non-extremal Schwarzschild-de Sitter black hole, if we consider the region bounded by the black hole horizon and the cosmological horizon, because the range of  $r_* = \int dr/f$  is  $(-\infty, +\infty)$ , and  $\Phi$  becomes convex for  $\omega^2 \leq 0$  (changing the sign of  $\Phi$  if necessary).

This simple argument does not hold for the Schwarzschild-anti-de Sitter black hole, as  $r_*$  has a finite limit for  $r \to \infty$  in this case. This is closely related to the fact that the differential operator on the left-hand side of (3.5) does not have a unique

self-adjoint extension in the  $L^2$ -space with respect to the inner product

$$(\Phi_1, \Phi_2)_{L^2} = \int dr_* \Phi_1^*(r) \Phi_2(r).$$
 (6.2)

In this case, we must impose a boundary condition at  $r = \infty$  to make the problem well-posed. The most natural choice is to require that  $\Phi$  vanish at  $r = \infty$ , if we are concerned with the local stability of the black hole. Under this boundary condition, we can apply the same reasoning as for the Schwarzschild-de Sitter black hole, and we thereby find that the Schwarzschild-anti-de Sitter black hole is stable.

We can also undertake a similar analysis for vector perturbations of these fourdimensional black holes, although the situation in this case is slightly subtle. Here, the effective potential becomes

$$V_V = \frac{f}{r^2} \left( -\frac{6M}{r} + 6 + l_2(l_2 + 5) \right). \tag{6.3}$$

For  $\Lambda \geq 0$ , this is positive definite for  $l \geq 2$  outside the horizon from  $1 - 2M/r = f + \lambda r^2 > 0$ . Hence, the argument given above again applies. In contrast, for  $\Lambda < 0$ ,  $V_V$  becomes negative near the horizon for large  $|\lambda|$ , and the above argument does not apply. Nevertheless, we can prove using the somewhat sophisticated argument given in Ref. 41) that there exists no unstable mode in this case as well.

From the above considerations, we have established the perturbative stability of spherically symmetric black holes in four dimensions, irrespective of the sign of the cosmological constant. The above argument also proves the uniqueness of these black holes in a perturbative sense, because there should exist a regular static solution to the perturbation equation with  $l \geq 1$  if there exists a continuous family of regular static black hole solutions with the same mass that contains the spherically symmetric solutions.

In spacetimes of dimension greater than four, the stability of spherically symmetric black holes has not yet been investigated, even in the asymptotically flat case. It seems that our formulation would be useful in the investigation of this problem. For example, using our formulation, we can prove the stability of higher-dimensional Schwarzschild black holes, as will be shown in a separate paper.<sup>41)</sup> This is not as trivial as in the four-dimensional case discussed above, because the effective potentials in the master equations are not positive definite in general for vector and scalar perturbations. Nevertheless, it can be shown that the differential operators on the left-hand sides of (3.5) and (5.14) have unique positive self-adjoint extensions. A similar technique might be used to analyse the asymptotically de Sitter and anti-de Sitter cases in higher dimensions as well, although the behaviour of the potential in these cases is much more complicated. This problem is now under investigation.

Next, we comment on the relation between the master variable for a scalar perturbation and that for a vector perturbation, which we here denote  $\Phi_S$  and  $\Phi_V$ , respectively. In the four-dimensional Schwarzschild background with  $\Lambda=0$  and K=1, Chandrasekhar and Detweiler found that they are related by the simple relation

$$\Phi_S = p\Phi_V + q\Phi_V', \tag{6.4}$$

where p and q are appropriate functions of r that are independent of the frequency  $\omega$  if and only if the corresponding effective potentials are expressed in terms of a single function F of r independent of  $\omega$  as

$$V_S, V_V = \pm f \frac{dF}{dr} + F^2 + cF, \tag{6.5}$$

with some constant c. This relation with  $c = k^2(k^2 - 2K)/(3M)$  holds for any values of  $\lambda$  and K in four dimensions, as pointed out by Cardoso and Lemos. <sup>13),14)</sup> However, we have found that there does not exist a function F satisfying (6·5), and hence, there is no simple relation of the form (6·4) between the two types of perturbations in higher dimensions. This result may be closely related to the fact that a tensor-type perturbation appears as a new mode in higher dimensions, and it implies that the spectral analysis of gravitational perturbations may not reduce to that for a vector-type perturbation with a simpler effective potential, unlike in four dimensions.

Finally, we comment on an extension of our formulation. As mentioned in §2, the background metric (2·14) with (2·15) satisfies the Einstein equations even if the metric  $d\sigma_n^2$  for  $\mathcal{K}^n$  is replaced by an arbitrary Einstein metric satisfying

$$\hat{R}_{ij} = (n-1)K\gamma_{ij}. (6.6)$$

In this case, the Weyl curvature  $\hat{C}^i{}_{jkl}$  of  $\mathcal{K}^n$  no longer vanishes, and it provides a non-trivial background tensor distinct from  $\gamma_{ij}$ . However, because the Weyl curvature is trace-free and of second order with respect to the spatial derivative, it can couple only to a tensor perturbation in the linear theory. Hence, the same perturbation equations hold in this generalized case for scalar and vector perturbations. As a consequence, the master equations for these types of perturbations derived in this paper are also valid in the case that  $\mathcal{K}^n$  is Einstein, if we replace the eigenvalues  $k_S^2$  and  $k_V^2$  by the corresponding ones for the Laplacian in the Einstein space. Further, for tensor perturbations,  $\hat{C}^i{}_{jkl}$  appears in the perturbation equation only through the combination called the Lichnerowicz operator.

$$(\hat{\Delta}_L h)_{ij} = -\hat{\Delta}_n h_{ij} - 2\hat{R}_{ikjl} h^{kl} + 2(n-1)Kh_{ij}, \tag{6.7}$$

where  $h_{ij}$  is a perturbation of the metric  $\delta g_{ij}$  of  $\mathcal{K}^n$ , as shown by Gibbons and Hartnoll.<sup>38)</sup> Hence, if we expand the tensor perturbation in terms of the eigentensors with respect to  $\hat{\Delta}_L$  instead of  $\hat{\Delta}_n$  and replace the eigenvalue  $\lambda_L$  of  $\hat{\Delta}_L$  by  $k_T^2 + 2nK$ , we obtain the same equation for tensor perturbations, as mentioned in §5.1. More detailed explanation of this extension will be given in a separate paper.

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Appendix A

The expressions of 
$$E_{ab}$$
,  $E_a$ ,  $E_L$  and  $E_T$  ——

We have the following:

$$2E_{ab} = -\Box F_{ab} + D_{a}D_{c}F_{b}^{c} + D_{b}D_{c}F_{a}^{c} + n\frac{D^{c}r}{r}(-D_{c}F_{ab} + D_{a}F_{cb} + D_{b}F_{ca})$$

$$+ \left(\frac{k^{2}}{r^{2}} + 2R^{(2)} - \frac{4}{n}\Lambda\right)F_{ab} - D_{a}D_{b}F_{c}^{c}$$

$$-2n\left(D_{a}D_{b}F + \frac{1}{r}D_{a}rD_{b}F + \frac{1}{r}D_{b}rD_{a}F\right)$$

$$- \left[D_{c}D_{d}F^{cd} + \frac{2n}{r}D^{c}rD^{d}F_{cd} + \left(\frac{2n}{r}D^{c}D^{d}r + \frac{n(n-1)}{r^{2}}D^{c}rD^{d}r\right)F_{cd}\right]$$

$$-2n\Box F - \frac{2n(n+1)}{r}Dr \cdot DF + 2(n-1)\frac{k^{2} - nK}{r^{2}}F$$

$$-\Box F_{c}^{c} - \frac{n}{r}Dr \cdot DF_{c}^{c} + \left(\frac{k^{2}}{r^{2}} + \frac{1}{2}R^{(2)}\right)F_{c}^{c}\right]g_{ab}, \qquad (A\cdot 1a)$$

$$\frac{2r}{k}E_{a} = -\frac{1}{r^{n-2}}D_{b}(r^{n-2}F_{a}^{b}) + rD_{a}\left(\frac{1}{r}F_{b}^{b}\right) + 2(n-1)D_{a}F, \qquad (A\cdot 1b)$$

$$2E_{L} = -D_{a}D_{b}F^{ab} - \frac{2(n-1)}{r}D^{a}rD^{b}F_{ab} - \frac{n-1}{r^{2}}\left((n-2)D^{a}rD^{b}r + 2rD^{a}D^{b}r\right)F_{ab}$$

$$+\Box F_{c}^{c} + \frac{n-1}{r}Dr \cdot DF_{c}^{c} + \left(-\frac{n-1}{n}\frac{k^{2}}{r^{2}} + \frac{1}{2}R^{(2)}\right)F_{c}^{c}$$

$$+2(n-1)\Box F + \frac{2n(n-1)}{r}Dr \cdot DF - \frac{2(n-1)(n-2)(k^{2}-nK)}{nr^{2}}F, \qquad (A\cdot 1c)$$

$$\frac{2r^{2}}{k^{2}}E_{T} = -2(n-2)F - F_{a}^{a}. \qquad (A\cdot 1d)$$

Here,  $\Lambda$  is the cosmological constant and  $R^{(2)}$  is the scalar curvature of the two-dimensional metric  $g_{ab}$ .

For the metric (2·14), the quantities  $\tilde{E}_b^a$  are expressed in terms of the variables X, Y and Z defined in (2·20) as

$$2\tilde{E}_{t}^{r} = \left[\frac{k^{2}}{r^{2}} - f'' - \frac{nf'}{r} - 2(n+1)\lambda\right] Z$$

$$+ f\partial_{t}\partial_{r}Y + \left(\frac{2f}{r} - \frac{f'}{2}\right)\partial_{t}Y$$

$$+ f\partial_{t}\partial_{r}X - \left(\frac{(n-2)f}{r} + \frac{f'}{2}\right)\partial_{t}X,$$

$$2\tilde{E}_{r}^{r} = \frac{1}{f}\partial_{t}^{2}X - \frac{f'}{2}\partial_{r}X$$
(A·2a)

$$+ \left[ \frac{n-1}{r^2} (f - K) + \frac{2(n+1)}{n} \lambda + \frac{(n+2)f'}{2r} + \frac{f''}{n} \right] X$$

$$+ \frac{1}{f} \partial_t^2 Y - \left( \frac{f'}{2} + \frac{nf}{r} \right) \partial_r Y$$

$$+ \left[ \frac{K - f}{r^2} - \frac{2(n^2 - 1)}{n} \lambda - \frac{3n - 2}{2r} f' - \frac{n - 1}{n} f'' + \frac{k^2 - nK}{r^2} \right] Y$$

$$+ \frac{2n}{rf} \partial_t Z, \tag{A.2b}$$

$$\begin{split} 2\tilde{E}_{t}^{t} &= -f\partial_{r}^{2}X + \left(\frac{n-4}{r}f - \frac{f'}{2}\right)\partial_{r}X \\ &- \left[\frac{n-1}{r^{2}}K - \frac{(2n-3)f}{r^{2}} + \frac{2(n^{2}-1)}{n}\lambda + \frac{n-2}{2r}f' + \frac{n-1}{n}f'' - \frac{k^{2}}{r^{2}}\right]X \\ &- f\partial_{r}^{2}Y - \left(\frac{f'}{2} + \frac{4f}{r}\right)\partial_{r}Y \\ &- \left[\frac{n-1}{r^{2}}K - \frac{n-3}{r^{2}}f - \frac{2(n+1)}{n}\lambda + \frac{(n-2)f'}{2r} - \frac{f''}{n}\right]Y. \end{split} \tag{A.2c}$$

Here,  $\lambda = \frac{2A}{n(n+1)}$ . For completeness, we also give the corresponding expression for  $\tilde{E}_L$ :

$$\tilde{E}_{L} = \frac{1}{2f} \partial_{t}^{2} X + \frac{f'}{4} \partial_{r} X 
+ \left[ \frac{(n-1)(n-2)(f-K)}{2nr^{2}} + \frac{(6n-4-n^{2})f'}{4nr} + \frac{f''}{2n} \right] X 
- \frac{f}{2} \partial_{r}^{2} Y - \left( \frac{3f'}{4} + \frac{f}{r} \right) \partial_{r} Y 
+ \left[ \frac{(n-1)(n-2)(f-K)}{2nr^{2}} + \frac{(-n^{2}+2n-4)f'}{4nr} - \frac{(n-1)f''}{2n} \right] Y 
+ \left( \frac{1}{rf} - \frac{f'}{2f^{2}} \right) \partial_{t} Z + \frac{1}{f} \partial_{t} \partial_{r} Z.$$
(A·3)

#### Appendix B

—— The l=1 mode in the spherically symmetric case——

For the mode with l=1 in the spherically symmetric case, the perturbation variable  $H_T$  and, correspondingly, the component  $E_T$  of the Einstein equations do not exist, since  $\mathbb{S}_{ij} = 0$ . However, by introducing the variables  $F_{ab}$  and F using (2·7) with  $H_T = 0$ , we can recover the equation  $E_T = 0$  as a gauge condition. To see this, first note that  $F_{ab}$  and F are no longer gauge invariant, and from the equations (48)–(52) in KIS2000, they transform under the gauge transformation

$$\bar{\delta}y^a = T^a \mathbb{S}, \ \bar{\delta}z^i = L \mathbb{S}^i,$$
 (B·1)

as

$$\bar{\delta}F_{ab} = -\frac{2r}{k} \left( rD_a D_b L + D_a r D_b L + D_b r D_a L \right), \tag{B-2a}$$

$$\bar{\delta}F = -\frac{r}{k} \left( Dr \cdot DL + \frac{(n-2)k^2}{nr} L \right). \tag{B-2b}$$

In particular,  $E_T$  transforms as

$$\frac{1}{k}\bar{\delta}E_T = \frac{1}{r^n}D_a(r^nD^aL) + \frac{(n-2)k^2}{nr^2}L.$$
 (B·3)

From this, it follows that we can always make  $E_T$  vanish by choosing an appropriate gauge. This requirement does not fix the gauge completely, and there remain residual gauge degrees of freedom parametrized by two arbitrary functions of the space coordinate r in the two-dimensional orbit spacetime (t, r).

## Appendix C

— The relation to the master variable in the case M=0 —

The master variable  $\Phi_{(S)}$  in Ref. 29) and  $\Omega = \Phi_{(S)}$  in KIS2000 for scalar perturbations are related to the gauge-invariant variables by

$$\tilde{F} = \frac{1}{2n} (\Box + 2\lambda) \Omega,$$
 (C·1a)

$$\tilde{F}_{ab} = D_a D_b \Omega - \left[ \left( 1 - \frac{1}{n} \right) \Box + \left( 1 - \frac{2}{n} \right) \lambda \right] \Omega g_{ab},$$
 (C·1b)

and they satisfy the wave equation

$$\Box \Omega - \frac{n}{r} Dr \cdot D\Omega - \left(\frac{k^2 - nK}{r^2} + (n - 2)\lambda\right) \Omega = 0.$$
 (C·2)

Z is expressed in terms of  $\Omega$  as

$$Z = D^r D_t \Omega = f^{3/2} \partial_r \left( \frac{\partial_t \Omega}{\sqrt{f}} \right). \tag{C.3}$$

Also, for M=0, (3·12c) becomes

$$Z = f^{3/2} \partial_r \left( \frac{r^{n/2} \partial_t \Phi}{\sqrt{f}} \right). \tag{C-4}$$

Hence, we have

$$\Omega = r^{n/2}\Phi + C(t)\sqrt{f}.$$
 (C·5)

Because it is easily checked that  $\Omega = r^{n/2}\Phi$  satisfies (C·2), this relation implies that  $C(t)\sqrt{f}$  also satisfies this equation. This condition is expressed as

$$\ddot{C} = \left[ (m+K)\lambda - \frac{mK}{r^2} \right] C. \tag{C-6}$$

From this, it follows that K = 0 if  $C \not\equiv 0$ . However, for K = 0, the expression for X in terms of  $\Omega$ ,

$$X = -\frac{1}{f}D_t D_t \Omega - (\Box + \lambda)\Omega, \tag{C.7}$$

is identical to (3.12a) only when

$$\ddot{C} = -\frac{1}{2}n(n-1)\lambda^2 r^2 C. \tag{C.8}$$

Because  $\lambda \neq 0$  for K = 0, this implies that  $C \equiv 0$ . Hence,  $\Omega$  is related to  $\Phi$  by (3·17).

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