High-precision ϵ -expansions of three-loop master integrals contributing to the electron g-2 in QED

S. Laporta*

Dipartimento di Fisica, Università di Bologna, Via Irnerio 46, I-40126 Bologna, Italy

Abstract

In this paper we calculate at high-precision the Laurent expansions in $\epsilon = (4 - D)/2$ of the 17 master integrals which appeared in the analytical calculation of 3-loop QED contribution to the electron g-2, using difference and differential equations. The coefficients of the expansions so obtained are in perfect agreement with all the analytical expressions already known. The values of coefficients not previously known will be used in the high-precision calculation of the 4-loop QED contribution to the electron g-2.

PACS number(s):

12.20.Ds Specific calculations and limits of quantum electrodynamics.

Keywords: quantum electrodynamics, anomalous magnetic moment, master integrals.

^{*}E-mail: laporta@bo.infn.it

The g-2 of the electron is probably one of the most precise test of QED. Over the years, the continuous improvements in the precision of experimental determinations have demanded correspondent improvements of the theoretical predictions. Let us recall the current status of calculations of QED contribution to the electron g-2. One- and two-loop contributions are known in closed analytical form for a long time. The calculation of three-loop contribution in closed analytical form was completed more recently[1], after many years of hard work. Four-loop contribution is known at present only numerically[2], with a precision of about 2.5%, obtained using Monte-Carlo integration methods. At present this precision is adequate for comparison with current experimental determinations. Anyway, there is the need of a new independent high-precision calculation, in order to cross-check the current numerical value and to improve considerably its precision, in view of future experiments.

A technique which turned out to be useful in g-2 calculations is integration-by-parts [3, 4]. The contribution to g-2 of a graph, expressed in the form of a combination of many different integrals with different powers in the numerator and in the denominator, is reduced to a linear combination of a small set of "master integrals" by using identities obtained by integrating-by-parts in D-dimension space-time. The master integrals must be calculated analytically or numerically in the limit $D \to 4$ by some method.

In Ref.[1] this technique was applied to the analytical calculation of the contribution to the electron g-2 of the last family of three-loop graphs still not known analytically, the so-called triple-cross graphs; the contributions of all the other three-loop graphs were already obtained in analytical form by other methods. The contribution of triple-cross graphs to the g-2 was reduced to a combination of 18 master integrals, called I_1, I_2, \ldots, I_{18} . The Laurent expansions in $\epsilon = (4 - D)/2$ of the master integrals were calculated in analytical form, by direct calculation (the simpler ones) or by using identities which relate the coefficients of expansions to values of integrals in 4 dimensions already known from previous work[5, 6, 7, 8]. Subsequently, in Ref.[9], we found that the QED contribution of all three-loop graphs can be reduced to a linear combination of the same master integrals, which are therefore the only master integrals needed in the three-loop calculation. We also found that one master integral, I_{11} , is a linear combination of the integrals I_{14} and I_{18} so that only 17 master integrals are needed. The topology of the 17 master integrals is shown in Fig.1.

We plan to use the integration-by-parts method in the calculation of the 4-loop contribution. We expect about $300 \sim 400$ master integrals; an analytical calculation seems to be out of reach, so that alternatively we consider a high-precision numerical calculation. Of course, we begin by calculating the simplest master integrals, those which factorize in products of master integrals with fewer loops. At 3-loop level there are two master integrals, I_{15} and I_{16} (see Fig.1), which factorize in one 2-loop master integral multiplied by one one-loop tadpole. At 4-loop, analogously, there are 17 4-loop master integrals which factorize in the product of one 3-loop master integral I_j and the one-loop tadpole. The expansion in ϵ of the one-loop tadpole contains $1/\epsilon$, so that we need one additional term in the expansions of the 3-loop master integrals in order to obtain 4-loop master integrals expanded at the same level required in the 3-loop calculation. Unfortunately the number of terms of the expansions of I_j of Ref.[1] just suffices to obtain the 3-loop result.

Therefore, for this reason, in this paper we have calculated at high precision (30 digits) deeper expansions in ϵ of the 17 master integrals I_i :

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I_1C^{-1} = -.74727427517503872293442889217 - 5.3830601530185743669980288649\epsilon 
 - 26.213893908510814642139484925\epsilon^2 - 105.674804179662879301749904804\epsilon^3 
 - 385.27172279141197066520099021\epsilon^4 - 1318.9119175947583164248019535\epsilon^5 
 - 4337.082974360622762021390292\epsilon^6 + O(\epsilon^7) , (1)
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I_2C^{-1} = 2.40411380631918857079947632302\epsilon^{-1} - 5.33950114034530650865549677321
    +25.5217203762232230112382243421\epsilon - 73.5271290561422487313093251595\epsilon^{2}
    +249.044154178834576505276496985\epsilon^{3} -753.473335692793671251435382464\epsilon^{4}
    +2346.37248004908075896840609088\epsilon^{5} - 7109.19379761589640023765834979\epsilon^{6} + O(\epsilon^{7}),
                                                                                         (2)
+88.8774427983382280511485935554\epsilon + 95.6785746389971388003677314582\epsilon^{2}
    +626.528599797078129746832638577\epsilon^{3} + 210.711024315707077431312202546\epsilon^{4}
    +4305.95682691978475295985357628\epsilon^{5} -1994.41925210304147109855938156\epsilon^{6} + O(\epsilon^{7})
I_4C^{-1} = 2.40411380631918857079947632302\epsilon^{-1} - 1.88228069596232589686771921947
    \phantom{+}+24.9738222767976352300663548079\epsilon-40.2430832265114989313433432375\epsilon^2
    +\ 216.453265450196273372723759103\epsilon^3 -\ 490.48300355255485916860004085\epsilon^4
    +1877.77603630120111667601080315\epsilon^{5} -4980.53077825841233320374729215\epsilon^{6} + O(\epsilon^{7}),
+5.87679853297021379372183633337\epsilon^{-1} + 10.2027383130242843587260466742
    +49.4062940751534383431822667245\epsilon +38.7587897454906373636992141483\epsilon^{2}
    +355.54075314113673706462779833\epsilon^{3} - 12.8313984708255340018628252229\epsilon^{4}
    +2544.39228074929353310098747381\epsilon^{5} - 2170.06912961044909561870930877\epsilon^{6} + O(\epsilon^{7})
+10.3333333333333333333333333333336^{-1} + 29.7659868661137435310832928686
    +\ 102.438098206123109827196396063\epsilon +\ 229.885864095858993939826105646\epsilon^2
    +762.395106594671638134114238656\epsilon^{3}+1423.01797639907075045462241083\epsilon^{4}
    +5052.27961041516425974606724499\epsilon^5 + 7498.83181107570646929366512712\epsilon^6 + O(\epsilon^7),
+5.876798532970213793721836333337\epsilon^{-1} + 18.0613667204622021855023607236
    +\ 55.4913080622726427983915540249\epsilon +\ 134.432560982177142077741730996\epsilon^2
    +394.304047315939025666187053151\epsilon^{3} + 824.935548393665344225617774097\epsilon^{4}
    +2505.10730543388776893266213278\epsilon^{5} + 4469.58551067612956096036609952\epsilon^{6} + O(\epsilon^{7}),
-\ 43.9148312632524344127591663433 - 154.918028662961860140720725441\epsilon
    -374.094185333355168755155126113\epsilon^2 - 1436.67271253544219472250003758\epsilon^3
    -3281.94043631920573984798601407\epsilon^4 - 13100.0652782910379543889228475\epsilon^5
                                        -29197.1252330882109400594603379\epsilon^6 + O(\epsilon^7),
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-11.956534800363119539611497\epsilon^{-1} - 44.5995196208454844578584038609
    -125.546733075809915579874065329\epsilon -442.029796112593386030203112952\epsilon^{2}
    -\ 1165.46537028123770794611984172\epsilon^3 - 4078.5492471997365803005187143\epsilon^4
    -10533.3774099326183047654759014\epsilon^5 - 36961.4052223824089107170672131\epsilon^6 + O(\epsilon^7).
-10.57973626739290574588966066666\epsilon^{-1} -30.1135700965916539174115430661
    \phantom{-} - 136.273953266205829743326933622\epsilon - 333.735442456113204996932778211\epsilon^2
    -1376.89949509222838398713383632\epsilon^3 -3194.567378327629536037718821\epsilon^4
      -12862.3364927028759414511191045\epsilon^{5} - 29234.6901535778480407834346202\epsilon^{6} + O(\epsilon^{7}),
                                                                                     (10)
I_{12}C^{-1} = \epsilon^{-3} + 3.5\epsilon^{-2} + 7.02777777777777777777777778\epsilon^{-1}
    +\ 11.5787037037037037037037037037037 - 24.3219708028082785734403310608\epsilon
    -121.961009709497309747947635538\epsilon^2 - 819.989946014746295714157758046\epsilon^3
    -2404.86316062636537480556898164\epsilon^4 -10421.8794385530161287164813459\epsilon^5
                                      -27852.0785333053790606284709671\epsilon^6 + O(\epsilon^7),
-184.230005105298483421085631284\epsilon^2 - 661.110586153353833254708841052\epsilon^3
    -3685.05477938169880569422657784\epsilon^4 - 10050.975403938380587540545542\epsilon^5
                                      -42319.9745464132618093606637193\epsilon^6 + O(\epsilon^7),
                                                                                     (12)
\phantom{-} - 21.7478477083774449677944881844\epsilon - 84.9528451829928094897709758796\epsilon^2
    -520.302648810090620535897692307\epsilon^{3} -1504.0311391147000649985117616\epsilon^{4}
      -6482.74483328181932000540551865\epsilon^5 - 17315.8330540574013027834356625\epsilon^6 + O(\epsilon^7).
                                                                                     (13)
I_{15}C^{-1} = 1.5\epsilon^{-3} + 5.75\epsilon^{-2} + 13.125\epsilon^{-1} + 30.3469725347858114917793213332
    +59.9061795885052938207577452894\epsilon + 128.784074758885330304917817688\epsilon^{2}
   +\,247.030559404244292769216590768\epsilon^{3}+522.536074129783764428998974715\epsilon^{4}
      +995.526049616742468296298492052\epsilon^{5} + 2097.54543056218400478046531901\epsilon^{6} + O(\epsilon^{7})
                                                                                     (14)
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 $I_{16}C^{-1} = 0.5\epsilon^{-3} + 1.75\epsilon^{-2} + 6.41486813369645287294483033329\epsilon^{-1}$ $+ 16.0102660805759621969058588126 + 42.0560659585935725800136421336\epsilon$ $+ 91.5580001461564241262070311808\epsilon^{2} + 211.940358797491166303979686218\epsilon^{3}$ $+ 432.923773655861990482605036606\epsilon^{4} + 946.511285468009699310821028163\epsilon^{5}$ $+ 1874.61717426917014415736822215\epsilon^{6} + O(\epsilon^{7}), \quad (15)$

- $-5.87783109665941583590779329626 -23.4899848806977618982607310895\epsilon$
- $-97.5806595141990146514480659561\epsilon^2 330.869679035399716046952939963\epsilon^3$
- $-1181.19322772402976499602304607\epsilon^4 3694.74722374791467371175473492\epsilon^5$ $-12281.2292376715157620002229398\epsilon^6 + O(\epsilon^7) , (16)$

$$I_{18}C^{-1} = -\epsilon^{-3} - 3\epsilon^{-2} - 6\epsilon^{-1} - 10 - 15\epsilon - 21\epsilon^{2} - 28\epsilon^{3} - 36\epsilon^{4} - 45\epsilon^{5} - 55\epsilon^{6} + O(\epsilon^{7}) .$$
(17)

The integrals I_j are defined as[1]

$$I_j = \left(\frac{-i}{\pi^{D-2}}\right)^3 \int d^D k_1 d^D k_2 d^D k_3 \frac{P_j}{Q_j} , \qquad P_1 = p \cdot k_2 , \quad P_j = 1 \text{ if } j \ge 2 ,$$

 Q_j contains the product of the denominators of the corresponding j-th graph of Fig.1. The normalization factor is $C = (\pi^{\epsilon}\Gamma(1+\epsilon))^3$. The number of terms of the expansions of Eqs.(1)-(17) suffices for the use in g-2 calculations at four and even more loops (actually, we have calculated much deeper expansions, not completely listed here for lack of space).

Eqs.(1)-(17) are to be compared with the corresponding analytical results of Ref.[1]. Due to an unfortunate misprint (see [9]), in Ref.[1] the terms containing the constants C_1 , C_2 are missing in the r.h.s. of the integrals I_2 , I_3 , I_4 , I_5 , I_6 , I_7 and I_{11} ; all the results which follow are however correct, as C_1 , C_2 cancel out systematically in the final results of [1]. For the ease of the reader we list here the correct expressions of the integrals I_2 , I_3 , ..., I_7 (already appeared in [9]); the remaining integrals are listed in Ref.[1]:

$$I_{2}C^{-1} = 2\frac{\zeta(3)}{\epsilon} - \frac{13}{90}\pi^{4} - \frac{1}{3}\pi^{2} + 10\zeta(3) + \epsilon \left(\frac{385}{2}\zeta(5) - \frac{85}{6}\pi^{2}\zeta(3) - \frac{7}{15}\pi^{4} - 82\zeta(3) - 4\pi^{2}\ln 2 + 16\pi^{2} - 2C_{1} + 6C_{2}\right) + O(\epsilon^{2}), \quad (18)$$

$$I_3C^{-1} = \frac{1}{3\epsilon^3} + \frac{7}{3\epsilon^2} + \frac{31}{3\epsilon} - \frac{2}{15}\pi^4 - \frac{4}{3}\zeta(3) + \frac{103}{3} + \epsilon \left(95\zeta(5) - \frac{25}{3}\pi^2\zeta(3) - \frac{1}{15}\pi^4 - \frac{184}{3}\zeta(3) - 8\pi^2\ln 2 + \frac{44}{3}\pi^2 + \frac{235}{3} + 4C_2\right) + O(\epsilon^2) , \quad (19)$$

$$I_4C^{-1} = 2\frac{\zeta(3)}{\epsilon} - \frac{7}{90}\pi^4 + 2\zeta(3) + \frac{1}{3}\pi^2 + \epsilon \left(\frac{385}{2}\zeta(5) - \frac{85}{6}\pi^2\zeta(3) - \frac{7}{15}\pi^4 - 82\zeta(3) - 4\pi^2\ln 2 + 16\pi^2 - 2C_1 + 4C_2\right) + O(\epsilon^2) , \quad (20)$$

$$I_5C^{-1} = \frac{1}{6\epsilon^3} + \frac{3}{2\epsilon^2} + \frac{1}{\epsilon} \left(-\frac{1}{3}\pi^2 + \frac{55}{6} \right) - \frac{4}{45}\pi^4 - \frac{14}{3}\zeta(3) - \frac{7}{3}\pi^2 + \frac{95}{2} + \epsilon \left(-\frac{2}{9}\pi^4 - 44\zeta(3) - \frac{29}{3}\pi^2 + \frac{1351}{6} + 2C_1 \right) + O(\epsilon^2) , \quad (21)$$

$$I_{6}C^{-1} = \frac{1}{3\epsilon^{3}} + \frac{7}{3\epsilon^{2}} + \frac{31}{3\epsilon} - \frac{4}{45}\pi^{4} + \frac{2}{3}\zeta(3) + \frac{1}{3}\pi^{2} + \frac{103}{3} + \epsilon \left(\frac{45}{2}\zeta(5) - \frac{7}{2}\pi^{2}\zeta(3) + \frac{11}{45}\pi^{4} + \frac{14}{3}\zeta(3) - 4\pi^{2}\ln 2 + \frac{14}{3}\pi^{2} + \frac{235}{3} + 2C_{1}\right) + O(\epsilon^{2}), \quad (22)$$

$$I_7C^{-1} = \frac{1}{6\epsilon^3} + \frac{3}{2\epsilon^2} + \frac{1}{\epsilon} \left(-\frac{1}{3}\pi^2 + \frac{55}{6} \right) - \frac{1}{15}\pi^4 - \frac{8}{3}\zeta(3) - 2\pi^2 + \frac{95}{2} + \epsilon \left(\frac{45}{2}\zeta(5) - \frac{17}{6}\pi^2\zeta(3) - \frac{7}{9}\pi^4 - 50\zeta(3) - 4\pi^2\ln 2 + \frac{1}{3}\pi^2 + \frac{1351}{6} + 2C_2 \right) + O(\epsilon^2) , \quad (23)$$

where $\zeta(p) = \sum_{n=1}^{\infty} 1/n^p$, $a_4 = \sum_{n=1}^{\infty} \frac{1}{2^n n^4}$, and C_1 , C_2 are the coefficients of the ϵ term in the expansion of the integrals

$$M_{1} = \left(\frac{-i}{\pi^{D-2}}\right)^{3} \int \frac{d^{D}k_{1}d^{D}k_{2}d^{D}k_{3} (p.k_{1})}{D_{1}D_{3}D_{4}D_{5}D_{6}D_{7}D_{8}} = C\left(-\frac{2}{45}\pi^{4} + \zeta(3) + \epsilon C_{1} + O(\epsilon^{2})\right),$$

$$M_{2} = \left(\frac{-i}{\pi^{D-2}}\right)^{3} \int \frac{d^{D}k_{1}d^{D}k_{2}d^{D}k_{3} (p.k_{3})}{D_{2}D_{3}D_{4}D_{5}D_{6}D_{7}D_{8}} = C\left(-\frac{1}{30}\pi^{4} - \frac{1}{3}\pi^{2} + 4\zeta(3) + \epsilon C_{2} + O(\epsilon^{2})\right),$$

$$D_1 = (p - k_1)^2 + 1 - i0$$
, $D_2 = (p - k_1 - k_2)^2 + 1 - i0$, $D_3 = (p - k_1 - k_2 - k_3)^2 + 1 - i0$, $D_4 = (p - k_2 - k_3)^2 + 1 - i0$, $D_5 = (p - k_3)^2 + 1 - i0$, $D_6 = k_1^2 - i0$, $D_7 = k_2^2 - i0$, $D_8 = k_3^2 - i0$, $p^2 = -1$.

The analytical values of C_2 and C_1 were calculated, respectively, in Ref.[9] and Ref.[10]

$$C_2 = -\frac{173}{4}\zeta(5) + \frac{53}{12}\pi^2\zeta(3) - \frac{2}{15}\pi^4 + 18\zeta(3) + 2\pi^2\ln 2 - 3\pi^2,$$

$$C_1 = -\frac{49}{4}\zeta(5) + \frac{25}{12}\pi^2\zeta(3) - \frac{49}{180}\pi^4 + \zeta(3) + 2\pi^2 \ln 2 - \frac{\pi^2}{3}.$$

Eqs.(1)-(17) agree perfectly with the analytical expressions of Eqs.(18)-(23) and of Refs.[1, 9], and with the deeper expansions of $I_{10}[10]$, I_{13} , $I_{15}[11]$, I_{16} and I_{17} .

Now we sketch the method used for obtaining Eqs.(1)-(17). We have used in this calculation the new method of calculation of master integrals based on solution of difference equations in exponents

developed and described in detail in Refs.[12, 13]. This method consists in the construction and numerical solution of systems of difference equations between the master integrals I_j (with polynomials in D and n as coefficients), seen as functions of the exponent n of one denominator. The difference equation for a given master integral contains in the r.h.s. only master integrals with less denominators, which are simpler; therefore the equations of the system can be resolved one at once, beginning with that corresponding to the simplest master integral and ending with that corresponding to the most complex. Suitably boundary conditions at $n \to \infty$ must be provided for a proper solution of the difference equations. This implies the calculation of integrals with one loop less, which can be obtained by solving other systems of difference equations. The amount of calculations needed to work out and solve the systems of difference equations is rather high, so that the practical application of the method relies on the use of an automatic tool, the program SYS described in Ref.[12].

Because of some actual limitations of the program SYS, we have found it convenient to give a mass $\lambda \neq 0$ to the photon lines, so that the master integrals become functions $I_j(\lambda)$. Then, we have used the above described method to calculate the value of master integrals $I_j(1)$ (actually, these values were already calculated using this method in Refs.[14]; here we have repeated the same calculation increasing the numerical precision and the order of expansions in ϵ).

Subsequently, always by means of the program SYS[15], we have built a system of differential equations in λ for $I_j(\lambda)$ (the approach based on the use of differential equations in masses was introduced in Ref.[16]), and we have integrated it, using as initial conditions the values (and, when needed, the derivatives w.r.t. λ) at $\lambda = 1$, obtaining the values $I_j(0)$, that is, Eqs.(1)-(17).

We note that the systems of difference and differential equations for $\lambda \neq 0$ are much more complicated (more master integrals, higher order equations, higher degree coefficients) than the system of difference equations for the original master integrals with zero masses $I_j(0)$, so that the solution of the latter would be preferable for calculating 4-loop master integrals which do not factorize. We plan to overcome this limitation in near future.

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Figure Captions
Figure 1: Topologies of the master integrals. Dotted lines are massless.
(S. Laporta, High-precision ϵ -expansions of three-loop master integrals contributing to the electron g -2 in QED)

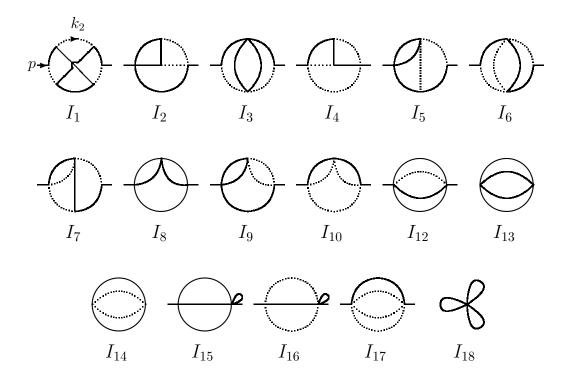


Figure 1:

(S. Laporta, High-precision ϵ -expansions of three-loop master integrals contributing to the electron g-2 in QED)