Batalin-Vilkoviski master equation and absence of anomalies in string field theory

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Abstract

We study the Batalin-Vilkovisky master equation for both open and closed string field theory with special attention to anomalies. Open string field theory is anomaly free once the minimal coupling to closed strings induced by loop amplitudes is considered. In closed string field theory the full-fledged master equation has to be solved order by order in perturbation theory. The existence of a solution implies the absence of anomaly. We briefly discuss the relation of the iterative process of solution to methods used in the first quantized formalism and comment on some possible non-perturbative corrections.

1 Introduction

Recent developments in the formulation of string field theory [1],[2] rely on the powerful formalism to quantize gauge theories introduced by BRST and extended by Batalin and Vilkovisky (BV). For a recent review of the BRST/BV formalism see [3]. Its crucial property is that it allows to consider gauge theories that have an open algebra or are reducible. This makes it convenient for string theory.

The BV formalism introduces new fields, called antifields and expresses the BRST invariance in a compact way by a master equation. The solutions of this equation are all possible gauge theories in a definite configuration space. In SFT

one looks for an action (function of the string field) that, on the one hand, reproduces the first quantized string amplitudes and, on the other, once it is second quantized must have a solution corresponding to the correct physical theory.

Unlike the situation in Witten's open string field theory (OSFT), which has cubic action [4], Sonoda and Zwiebach [5] have shown that for closed string field theory (CSFT) the condition of modular invariance, necessary to reproduce string amplitudes, can be expressed by an equation, called the geometric equation. It is solved iteratively and the solution involves an infinite number of vertices at tree level as well as an infinite number of "counterterms" for loop amplitudes.

The BV master equation was introduced by Hata [6] in cubic CSFT to cure its unitarity problems. He used an analogy with the path integral quantization of the Non-Linear Sigma Model (NLSM). In this model it is possible to show that if one uses the naive measure in field space the theory is not unitary. The reason is that the naive path integral measure is not invariant under the nonlinear symmetry and it is necessary to add a correction term to restore invariance and unitarity. In SFT one deals with BRST symmetry, an elaborate non-linear transformation. Nevertheless, Hata was able to show that the BV master equation can be solved at one-loop order in a similar way to the NLSM case and, with more effort, to higher orders. He further proved the resulting quantum action to be unitary up to three-loop order.

The similarity of the geometric equation to the perturbative expansion of the BV master equation was further clarified by Kaku [7], showing a closer relation between the lack of modular and gauge invariance, which he called anomaly. He did not use though the BRST formalism nor attempted to connect with either Sonoda-Zwiebach or Hata's work.

In this paper we make explicit the relation between modular and BRST invariance with special concern for the measure problem. We base our discussion on the BV master equation,

$$\frac{1}{2}(S, S) - i\hbar \Delta S = 0 \tag{1}$$

which we consider the fundamental equation of SFT. The relevant term for us is ΔS, which expresses the failure of BRST (or modular) invariance of the path integral measure. Nevertheless, it is possible to find a local functional (counterterm) such that its BRST variation compensates for ΔS. Hence, strictly speaking, there is no anomaly. Following Hata, we shall appeal to the NLSM to see that the basic idea is very simple. However, in CSFT we encounter an important difference. The BRST transformation is also altered by the counterterm. The new BRST transformation is again "anomalous" at two-loop order and has to be corrected, and so on ad infinitum. This recursive process represents the iterative solution of the BV master equation in the loop expansion. In this process, unlike ordinary iterative schemes the equation as well as the solution change in each order.

We begin by reviewing the BRST and BV formalisms with special attention to the role of anomalies [8]. To set the scene for the use of the master equation in SFT and for its own sake as well we regard its role in OSFT. Here one achieves a complete theory with just the cubic Witten's action plus a series of terms coupling an increasing number of open strings to one closed string. It is not necessary to introduce the master equation though it is convenient (see [9] for an introduction). The calculation of ΔS follows a line analogous to the Yang-Mills case and must yield $\Delta S = 0$ once the divergence is cured by the introduction of the closed-string terms mentioned above. From the alternative Riemman-surface point of view, this calculation is not necessary since there is a fair amount of evidence for the conjecture that Witten OSFT exactly covers moduli space [11, 12]. We proceed to CSFT; we briefly comment why there is no classical polynomial solution to the master equation before we begin with quantum corrections. We show explicitly how to obtain the one and two-loop counterterms. We discuss gauge invariance in SFT in comparison with field theory. We end with a brief discussion of the relevance of our analysis to the first quantized formalism and some issues regarding non-perturbative effects.

2   Anomalies in the BRST formalism

Anomalies can be understood as the non-invariance of the path integral measure under gauge transformations. Therefore, one should expect that in the gauge-fixed path integral they manifest themselves as lack of invariance under the BRST transformation; or in the BV formulation as the impossibility to fulfill the master equation. Recent work shows that this is indeed the case [8].

Nevertheless, it is convenient to compare first the BRST anomaly to a similar problem in a theory that is not gauge, the NLSM, following [6]. This is because the peculiar behavior of the measure under the BRST transformation δ_B is mainly due to its nonlinear character.

The quantum NLSM was found not to be invariant under the classical symmetry and, as a consequence, unitarity would be lost. The problem could be expressed as the necessity to add a term to the classical action to cancel the unwanted counterterms that break the symmetry. This can be traced back to the non-gaussian integral over momenta in the functional integral. Another interpretation of this term, more suitable to our purposes, is as the function that converts the naive path integral measure into that invariant under the nonlinear transformation (Haar measure).

The relevant transformations for a definite NLSM are the isometries (Killing vectors) of the defining manifold

$$\delta_f \phi^i = f^i(\phi), \tag{2}$$

satisfying

$$f^k_{,i}g_{kj} + f^k_{,j}g_{ik} + f^k g_{ij,k} = 0. \tag{3}$$

Multiplying by g^{ij} we obtain

$$2f^i_{,i} + f^k g^{ij}g_{ij,k} = 2f^i_{,i} + f^k \partial_k lng = 0. \tag{4}$$

where $g = detg_{ij}$. Recalling the change of the measure under diffeomorphisms

$$\delta_f lnd\phi = \frac{\partial}{\partial \phi^i}\delta_f \phi^i = f^i_{,i}, \tag{5}$$

we obtain

$$\delta_f[lnd\phi + (1/2)lng] = \delta_f[lnd\phi + (1/2)trlng_{ij}] = 0. \tag{6}$$

This correction is local as a functional, which implies the presence of $\delta^D(0)$, yielding as quantum lagrangian

$$\mathcal{L}_q = \mathcal{L} - i(\hbar/2)\delta^D(0)trlng_{ij}(\phi). \tag{7}$$

We have corrected the non-invariance of the quantum theory by a local modification of the lagrangian and hence there is no actual anomaly. Alternatively, the measure can be corrected $\prod_i[d\phi_i] \to \prod_i[d\phi_i]\sqrt{g}$, leaving the lagrangian unchanged

BRST is usually a nonlinear transformation and we expect that the naive path integral measure is not invariant under it,

$$\delta_B ln[d\phi] = \frac{\delta}{\delta \phi^i}\delta_B \phi^i \tag{8}$$

will generally be some non-null functional. The BV formalism generalizes BRST with the introduction of antifields,ϕ^*_i, and the classical action, $S(\phi^i, \phi^*_i)$, satisfies

$$\delta_B \phi^i = \frac{\delta S}{\delta \phi^*_i}, \tag{9}$$

and therefore

$$\delta_B ln[d\phi] = \frac{\delta^2 S}{\delta \phi^i \delta \phi^*_i} \equiv \Delta S. \tag{10}$$

Hence, a sufficient condition for the absence of anomalies is

$$\Delta S = 0. \tag{11}$$

However, $\Delta S = 0$ is not a necessary condition. As long as there exists a local functional[1] $M_1(\phi)$ such that

$$\Delta S = -i\delta_B M_1(\phi), \tag{12}$$

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[1]The dependence on antifields is not shown, since the gauge has to be eventually fixed and they are then removed.

we can absorb the BRST variation of the measure in that of the action and the path integral as a whole will be invariant,

$$\delta_B[ln[d\phi] + i(S + M_1)] = 0. \tag{13}$$

We can write (12) with the use of the antibracket as

$$\Delta S = (M_1, S). \tag{14}$$

However, the addition of M_1 to the classical action modifies the BRST transformation to the quantum one-loop BRST transformation

$$\delta_B^q \alpha = (\alpha, S + M_1), \tag{15}$$

which in turn implies that at two-loop order

$$\delta_B^q[ln[d\phi] + i(S + M_1)] = \Delta M_1 + i\frac{1}{2}(M_1, M_1), \tag{16}$$

using the previous equations. In order not to have anomaly at this order, there must exist a function $M_2(\phi)$ that accounts for this variation,

$$\Delta M_1 + i\frac{1}{2}(M_1, M_1) = -i\delta_B^q M_2(\phi), \tag{17}$$

and the whole argument repeats itself.

Proceeding this construction, we get the loop expansion of the full BV master equation, which expresses in a compact form the invariance of the path integral under the BRST transformation, or equivalently the no anomaly condition:

$$2i\hbar\Delta W - (W, W) = 0, \tag{18}$$
$$W = S + \hbar M_1 + \hbar^2 M_2 + \cdots.$$

If this equation were to be violated we would definitely have an anomaly. This violation usually manifest itself in perturbation theory as impossibility to find the functions M_i but it could also appear in a non-perturbative manner, for example, as a failure of the perturbative series for W to converge. It has been proposed that this problem might arise in string theory and we shall comment on it below. The analysis of the NLSM suggests that it may be simpler to correct the measure than to correct the lagrangian. The final result is, however, the same.

3   Quantization of OSFT.

The question of BRST invariance can be divided into a classical part, namely to find an S that fulfills $(S, S) = 0$, and a quantum part that begins with checking whether S satisfies $\Delta S = 0$. If it does, the full quantum theory is BRST

invariant and the iterative process ends. The actual calculation of ΔS involves divergences and requires the introduction of a regulator. In field theory there are several regularization methods available, which we separate in two, dimensional and cutoff-like regularization. The former is not convenient since it hides some divergences (e.g. quadratic), for example, yielding a vanishing quantum contribution in Eq.(7). The second is thus more adequate; in particular, a variant of the Pauli-Villars scheme that has been proposed in [8]. This method has been applied to Yang-Mills (YM) gauge theory [14]; the value of ΔS to be regularized is

$$\delta^D(0) \, f_{ba}^a c^b = \infty \times 0.$$

It is nonvanishing in a general gauge but its value can be absorbed by local counterterms, not surprisingly since pure YM is a renormalizable anomaly-free theory.

In Witten's OSFT the computation of ΔS was undertaken by Thorn but his result is inconclusive [9]. It has been later argued by Kaku that it must be zero[2] [7]. The argument relies on an analogy with YM: The gauge group of OSFT can be formulated in a similar form to an ordinary gauge group and thus its functional structure constants are antisymmetric as well [13]. The value of ΔS is

$$\prod_\sigma \delta[X(\sigma) - X(\sigma)] \, f_{ji}^i \Phi^j.$$

It vanishes provided that one disregards the divergence. However, it can occur as in YM, namely, that a careful computation does not yield zero. We actually know that this one-loop divergence can be associated with coupling to a closed string. Therefore, we must take into account the interaction of a closed string with an arbitrary number of open strings which is originated in this way,

$$S_{int-oc} = \int \Psi(\Phi + \Phi^2 + \Phi^3 + \cdots). \tag{19}$$

The OSFT that includes these interactions satisfies the classical master equation [10].

Similar conclusions can be reached from the Rieman-surface point of view, seeking covering of moduli space. There is sufficient evidence by now that amplitudes formed with Witten vertices correctly fill the relevant moduli spaces, even there is no general mathematical proof [11, 12]. Thus the full quantum theory is modular (hence BRST) invariant and Witten's cubic action, including the closed string interaction (19), does not need further addition of quantum counterterms.

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[2]In fact, Kaku considers instead the jacobian of the gauge transformation but its vanishing is equivalent to the vanishing of ΔS

4 Quantization of CSFT

It has long been known that the SFT program and, in particular, Witten's cubic action encounter problems when adapted to closed strings [16]. As a partial solution, it was proposed to introduce new vertices, originating from a nonpolynomial action [17, 18, 19] that constitutes a modular invariant theory at tree level. This expansion is analogous to the infinite expansion of the Einstein- Hilbert action $\sqrt{g}R$ around a particular background. In fact, the mode expansion of the theory contains the expansion of $\sqrt{g}R$. It is therefore not surprising that an infinite number of vertices is necessary. Loop amplitudes were later analysed [5, 20]. From the requirement of single covering of moduli space, Sonoda and Zwiebach concluded with the necessity to add an infinite number of vertices with increasing number of external legs for each genus. In so doing they arrived to a geometric equation as a consistency condition,

Figure 1. Pictorial description of the geometric equation.
a is the maximal allowed value of the sewing parameter t.

This equation can be cast in a different form [21], reinterpreting it as a Wegner-Polchinski renormalization group equation for CSFT,

$$a \sum_{N \geq 2} \partial \mathcal{V}_{G,N} = a \frac{dS_{int}}{da} = \frac{\delta S_{int}}{\delta \Psi} \frac{\delta S_{int}}{\delta \Psi} - \frac{\delta^2 S_{int}}{\delta \Psi^2}. \tag{20}$$

The left hand side of this equation is the variation of the interaction action due to an infinitesimal change of a stub boundary. Alternatively, it can be regarded as the derivative w.r.t. the maximal lenght of the sewing parameter, which acts as a short-distance world-sheet cutoff in this picture. Interestingly, it can be written as the linear (first-quantized) BRST variation of the interaction,

$$a \frac{dS_{int}}{da} = Q S_{int} = \frac{\delta S_{int}}{\delta \Psi} Q \Psi = \frac{\delta S_{int}}{\delta \Psi} \frac{\delta S_2}{\delta \Psi}. \tag{21}$$

Hence, the geometric equation simplifies to OA

$$\frac{\delta S}{\delta \Psi} \frac{\delta S}{\delta \Psi} - \frac{\delta^2 S}{\delta \Psi^2} = 0. \tag{22}$$

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This is Hata's version of the BV equation for CSFT, as was already realized in [21, 22]. It embodies BRST invariance in just the same way as modular invariance in the original SZ picture, showing the equivalence of both concepts.

However, the one-loop calculation takes very different forms whether it is made in the BRST or Riemman-surface formalisms. In the latter the calculation is performed in ref. [23] (see also [20]) and the divergence appears because of the necessary inclusion of an infinite number of torus modular regions. BRST loop calculations were made by Hata for the purely cubic theory [6].

Hata's one-loop computation resembles somewhat the one in standard QFT, though the divergence is more severe due to the propagation of an infinite number of modes round the loop. The essential part is the (one-loop) one-leg counterterm $M_{1,1}$, solution of the corresponding part of Eq. (22),

$$M_{1,1}^i Q\Psi_i \equiv QM_{1,1} = \frac{\delta^2 S_3}{\delta\Psi_i \delta\Psi_i}, \tag{23}$$

which is the simplest version of (12). Multiplying by the propagator Q^{-1},

$$M_{1,1} = Q^{-1}\frac{\delta^2 S_3}{\delta\Psi_i \delta\Psi_i} = Tr[Q^{-1}\Psi*]. \tag{24}$$

The divergence is exposed by explicitly writing the propagator as a sum of modes. The trace is equivalent to performing the integration over momenta round the loop.

The two-leg equation

$$M_{1,1}^i \frac{\delta S_3}{\delta\Psi_i} + M_{1,2}^i Q\Psi_i = \frac{\delta^2 S_4}{\delta\Psi_i \delta\Psi_i}, \tag{25}$$

differs from Hata's in the presence of a term on the right-hand side coming from the quartic vertex, which he neglects. It can be solved to yield

$$M_{1,2} = -M_{1,1}^i Q^{-1}\frac{\delta S_3}{\delta\Psi_i} + Q^{-1}\frac{\delta^2 S_4}{\delta\Psi_i \delta\Psi_i}. \tag{26}$$

It contains a quantum correction to the kinetic term.

We can pursue this procedure for greater number of legs but the equations and hence their solutions become increasingly complex. Restricting to the cubic vertex only, Hata gives a closed expression for the one-loop action much in the style of standard QFT [6]. He also proceeds to higher loops. Unfortunately, his expressions are formal because they are divergent and he does not introduce any regularization. Therefore, one cannot read off the values of the M_n from them. For the same reason, it is impossible to read off M_n from Kaku's computations [7].

To see concrete solutions we have to appeal to the work done in the Riemman-surface formalism. The vertices $\mathcal{V}_{G,N}$, solution of the Sonoda-Zwiebach equation, are to be identified with $M_{n,N}$ ($n = G$). However, the corresponding calculations, comprising analysis of the divergences, have been carried out to a much lesser extent; namely, only one-loop calculations are available and only for one or two punctures [24, 20].

A comparison with the first quantized formalism is in order here. In [21] and in [25] the relation between the second quantized formalism based on solving the classical BV equation and the first quantized formalism based on solving the conditions of conformal invariance was explained (This was done in a limited framework, involving only the massless modes and in the limit $a \to 0$). The main result was that a solution to the conformal invariance conditions automatically solves the classical BV equation. The situation is different when the full quantum BV equation is considered. In the limit $a \to 0$, the dependence of the linear term ΔS on a is completely different than the one coming from (S, S) and therefore both terms cannot be treated on the same footing. Because of that the contribution coming from the "non-dividing pinch" [26], was largely ignored. Note that the divergences coming from the term ΔS are not those known as Fischler-Susskind divergences [25]. As realized in [27] unitarity dictates certain analytic continuation that turn the divergence into an imaginary part. From the BV equation for SFT, it is clear, however, that both terms have to be included. In fact, it has long been conjectured [26],[28] that these terms are responsible for the anomalous $(2G)!$ growth of large order perturbation series in string theory [29]. A proof of this conjecture should be helpful in deciding whether there are any non-perturbative obstructions ("anomalies") to finding a complete solution.

5   Conclusions and Discussion

We have studied anomalies for both OSFT and CSFT in the BV formalism, spelling out the equivalence of BV master equation with Sonoda-Zwiebach geometric equation. We have shown how to solve this equation to obtain the quantum correction to the measure. The concrete solution is not very illuminating and the important point is whether it can be solved, which implies the absence of anomalies. An analogy with first-quantized string theory is appropriate: In this case, the BRST charge Q is nilpotent except for the appearance of the conformal anomaly. In the second-quantized theory one pursues as well $\delta_B^2 = 0$. We know that this condition is equivalent to the master equation $(S, S) = 0$. This equation is solved in OSFT by Witten's interaction. In CSFT the classical master equation is complemented by quantum corrections, hence $\delta_B^2 \neq 0$. Nevertheless, there is no anomaly provided we can solve the quantum BV master equation, thus finding a

nilpotent quantum BRST transformation.

With regard to anomalies our answer is that they seem to be absent on the formal level of calculations presented here. We would like to add however some words of caution. It is obvious that the situation in CSFT is complicated, since the infinity of vertices in the non-polynomial action is corrected at each loop order. It is desirable to further investigate the issue and to make the absence of anomalies manifest. One way of achieving this goal would be to find a gauge in which $\Delta S = 0$. We know that the solution of the BV master equation is gauge dependent and we might expect that for some yet unknown gauge the equation $\Delta S = 0$ would be satisfied. This gauge would be particularly useful in separating the question of background independence and invariance of the measure. Another way of making the absence of anomalies manifest would be to find a string field redefinition for which the jacobian exactly cancelled ΔS. The covariant formalism of Schwarz [30] as presented in [31], is useful in stating the problem clearly. In this formalism this condition can be expressed as an equation for the measure ρ, $\Delta_\rho^2 = 0$, $\frac{1}{2}(\log \rho, S) + \Delta_1 S = 0$, where Δ_1 is the naive measure.

We have mentioned in section 2 the possibility of non-perturbative violation of Eq.(18), what we should call a non-perturbative anomaly. It can occur in two related ways: The first, that the sum does not converge, typically because terms grow too fast. The second, that a piece is missing and cannot be reached by perturbation theory. It would be interesting to obtain from the quantum BV equation recursion relations that will shed light on this issue.

6   Acknowledgments

References

[1] B. Zwiebach, Nucl. Phys. **B390** (1993) 33.

[2] E. Witten, Phys.Rev.D46 (1992) 5467.

[3] M. Henneaux, Nucl. Phys. **B18A** (Proc. Suppl.) (1990) 47.

[4] E. Witten, Nucl. Phys. **B268** (1986) 253.

[5] H. Sonoda and B. Zwiebach, Nucl. Phys. **B331** (1990) 592.

[6] H. Hata, Nucl. Phys. **B329** (1990) 698.

[7] M. Kaku, Phys. Lett. **B250** (1990) 64.

[8] W.Troost, P. Van Nieuwenhuizen and A. Van Proeyen, Nucl. Phys. **B333** (1990) 727.

[9] C.B. Thorn, Phys. Rep. **175** (1989) 1.

[10] B. Zwiebach, Phys. Lett. **B256** (1991) 22.

[11] S.B. Giddings, E. Martinec and E. Witten, Phys. Lett. **B176** (1986) 362.

[12] S. Samuel, Nucl. Phys. **B341** (1990) 513.

[13] M. Kaku, *Introduction to Superstrings*, Graduate Texts in contemporary Physics (Springer, New-York, 1988).

[14] F. De Jonghe, R. Siebelink, W. Troost, S. Vandoren, P. Van Nieuwenhuizen and A. Van Proeyen, Phys. Lett. **B289** (1992) 354.

[15] J. Zinn-Justin, in *Trends in Elementary Particle Theory*, Springer Lecture Notes in Physics **37**, eds. H. Rollnik and K. Dietz (Springer, Berlin, 1975).

[16] S.B. Giddings and E. Martinec, Nucl. Phys. **B278** (1986) 256.

[17] T. Kugo, H. Kunimoto and K. Suehiro, Phys. Lett. **B226** (1989) 48.

[18] M. Kaku, Phys.Rev.**D38** (1988) 3052.

[19] M. Saadi and B. Zwiebach, Ann. Phys. **192** (1989) 213.

[20] L. Hua and M. Kaku, Phys. Lett. **B250** (1990) 56.

[21] R. Brustein and S. De Alwis, Nucl. Phys. **B352** (1991) 451.

[22] B. Zwiebach, Mod. Phys. Lett. **A5** (1990) 2753.

[23] G. Zemba and B. Zwiebach, Jour. Math. Phys. bf 30 (1989) 2388.

[24] M. Saadi, Mod. Phys. Lett. **A5** (1990) 551.
M. Saadi, Int. Jour. Mod. Phys. **A6** (1991) 1003.

[25] R. Brustein and K. Roland, Nucl. Phys. **B372** (1991) 201.

[26] R, Brustein and S. De Alwis, Phys. Lett. **B247** (1990) 31.

[27] N. Marcus, Phys. Lett. **B219** (1989) 265.

[28] S. Shenker, Private communication.

[29] S. Shenker, Proceedings of Cargese Workshop on Random Surfaces, Quantum Gravity and Strings, Cargese, France (1990) 191.

[30] A. Schwarz, Geometry of Batalin-Vilkovisky Quantization, UC Davis preprint, Print-92-0192, 1992.

[31] H. Hata and B. Zwiebach, MIT preprint, MIT-CTP-2184, 1993.