Some Non Perturbative Aspects of Gauge theories

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Dedicated

to

My Parents

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1. Introduction and Overview

Strong interactions are adequately described at high energies by quantum chromodynamics (QCD). At low energies, the QCD coupling is large and color (QCD charge) is confined, but a precise description of how that happens is as yet unknown. The hadron spectrum found in nature consists of color-singlet combinations of color non singlet objects: the quarks and gluons. Unlike atomic physics, where we can separate electrons from atoms, it is not possible to separate quarks from hadrons. So there is no color-charge version of ionization in hadronic physics. This problem is often referred to as 'color confinement'. The discovery of renormalization and asymptotic freedom [1, 2] of non-Abelian gauge theory established SU(3) gauge theory as the theory which describes the dynamics of quarks and gluons. The quark charge density ρ^a_{quark} is the source of static color electric field, as required by Gauss law

$$\nabla \cdot \vec{E}^a = \rho^a_{quark} - g f^{abc} A^b_k E^c_k. \tag{1.1}$$

The last term containing the structure constant of the gauge group f^{abc} and the gauge field A^a_k reflects the non-vanishing color electric charge of the gluons. However, gluons are in the adjoint representation of SU(3) whereas the quarks are in the fundamental 3 representation of SU(3). So the color electric field of an isolated quark could only end on another isolated quark or else extend out to infinity.

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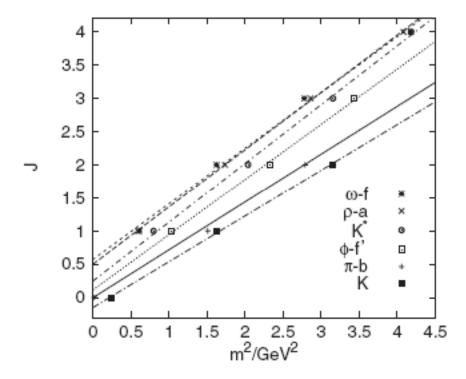
Figure 1.1.: Regge trajectories (taken from [3])

To extract a single quark from a hadron we can scatter hadrons with high energy photons. As the struck quark begins to move away from the other quarks in the hadron, it brings along the color electric field also. However, the system becomes unstable if the energy stored in the color electric field becomes large enough and it creates a light quark-antiquark pair. The final states will be highly excited two color-singlet hadrons and they decay into lighter hadrons. So at the end there is no free quark or a color-ionized hadron but only a shower of ordinary hadrons.

The hadron scattering process alone cannot fully determine the form of the confining potential. More knowledge about the potential comes from what are known as 'Regge trajectories'. Experimentally it is found that all mesons and baryons have many excited states (resonances) and when the spins of mesons (and baryons)

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are plotted against their squared masses, the mesons and baryons of given flavor quantum numbers seem to lie on nearly parallel straight lines, known as Regge trajectories. Regge trajectories are given by the equation $J = \alpha(s)$, where J is the angular momentum and $s = M^2$ (the square of energy in the center of mass frame). Resonance occurs for some values of s for which $\alpha(s)$ is a non negative integer (mesons) or a half integer (baryons).

For a fixed s the largest J is called the ' leading trajectory'. Experimentally it is seen that the leading trajectories are almost linear in s:

$$\alpha(s) = \alpha(0) + \sigma's. \tag{1.2}$$

There are also other trajectories for which

$$\alpha(s) = \alpha(0) - n + \sigma's \tag{1.3}$$

where n is an integer. $\alpha(0)$ depends on the quantum numbers such as strangeness and baryon number. The value of the Regge slope σ' is approximately $1\ GeV^{-2}$ [4] and its value is universal.

A simple model satisfies Regge trajectories. Let us suppose that a meson is constructed by a flux tube, with a quark and an anti quark attached to the ends of the tube. Then all lines of force of the color field are confined inside the tube. We shall ignore the contribution to the energy by the quarks. We can imagine a rotating flux tube as a rotating straight line whose end points are moving at a speed of light. Suppose the mass per unit length of the flux tube is σ and the length of the tube is $2R$. Let us assume that the flux tube is rotating around a axis which perpendicularly bisects the straight fluxtube of length $2R$. If we take the quarks to be mass less, the endpoints move essentially at the speed of light $c = 1$. Then we can write the mass of the rotating flux tube (spinning stick) in the center of mass

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frame as

$$m = Energy \;=\; 2\int_0^R \frac{\sigma dr}{\sqrt{1-v^2(r)}} \tag{1.4}$$

$$=\; 2\int_0^R \frac{\sigma dr}{\sqrt{1-r^2/R^2}} \tag{1.5}$$

$$=\; \pi\sigma R, \tag{1.6}$$

where $v(r)$ is the speed at distance r. The angular momentum J will be

$$J \;=\; 2\int_0^R \frac{\sigma v(r)dr}{\sqrt{1-v^2(r)}} \tag{1.7}$$

$$=\; \frac{2}{R}\int_0^R \frac{\sigma r dr}{\sqrt{1-r^2/R^2}} \tag{1.8}$$

$$=\; \frac{1}{2}\pi\sigma R^2. \tag{1.9}$$

So, we can calculate

$$\frac{J}{E^2} \;=\; \frac{1}{2\pi\sigma}. \tag{1.10}$$

$$J \;=\; \sigma' E^2, \tag{1.11}$$

here σ' is the Regge slope. From experimental data we can estimate the value of σ.

$$\sigma' \;=\; \frac{1}{2\pi\sigma} = 0.9\,GeV^{-2} \tag{1.12}$$

$$\sigma \;\approx\; 0.18\,GeV^2 \tag{1.13}$$

We can make the model more realistic by taking the flux tube as a string, instead of taking it as a rigid stick. This model can explain the existence of other trajectories also.

The idea of constructing flux tubes to explain quark confinement was first given by Nambu and Mandelstam [5, 6]. Following them it is now generally thought that

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the QCD vacuum behaves like a dual superconductor, created by condensation of magnetic monopoles, in which confinement is analogous to a dual Meissner effect [7, 8, 9]. A meson state is then formed by attaching a quark-anti-quark pair to the ends of a flux string analogous to the Abrikosov-Nielsen-Olesen (ANO) vortex string of Abelian gauge theory [10, 11]. As a consequence, the energy of the pair increases linearly with their separation and quarks are confined in hadrons. Calculation with explicit models of this type [12, 13] have been compared both with experimental data and with Monte Carlo simulation of QCD [14].

However, this model has its limitations. One difference with reality is that the static Abrikosov-Nielsen-Olesen vortex string carries magnetic field but static QCD flux strings must carry only electric field. The construction of flux tubes in field theory are formed via spontaneous symmetry breaking by scalar fields. However, it is not very clear whether or how the symmetry is broken at low energy in QCD. We can build a model for confinement by confining magnetic particles by magnetic flux tubes and try to describe the system by dual variable to compare the system with the real system. Flux strings in the Weinberg-Salam theory was suggested by Nambu [15], in which a pair of magnetic monopoles are bound by a flux string of Z condensate. Another construction of flux tubes in the Weinberg-Salam theory were also given in [16]. A different construction of flux strings, involving two adjoint scalar fields in an SU(2) gauge theory, has been discussed in [11, 17]. Recently there has been a resurgence of interest in such constructions [18, 19, 20, 9].

In this thesis I construct flux strings and write the action in terms of string variables as a dual gauge theory. I will show that, in these dual gauge theories monopoles are attached at the end of the flux strings. In chapter 2 we shall give a brief description of magnetic monopoles in electromagnetism and Proca massive electrodynamics. Here I will discuss the quantization of charge in Proca massive elec-

tromagnetic theory by quantizing angular momentum in the presence of monopoles. I will also discuss CPT symmetries in the presence of magnetic monopoles in this chapter.

In chapter 3 I will first give a review of flux string configuration in Abelian Higgs model. In the presence of these strings, I first dualize the scalar field to find the strings interacting via an antisymmetric tensor potential [21, 22, 23], while the Abelian gauge field is dualized [24, 25] to a 'magnetic' photon [26]. Next I introduce fermionic magnetic monopoles into the theory and minimally couple these to the magnetic photon. Parity conservation of Maxwell equation suggests that the monopole current may be an axial current. However, the axial current produces an anomaly and I cancel the anomaly by postulating additional species of fermionic monopoles. Then I dualize the resulting theory again, to find a theory of magnetic flux tubes interacting with a massive Abelian vector gauge field. The tubes are sealed at the ends by fermions, thus providing a toy model for quark confinement.

For non-Abelian gauge theories, the construction is a little different, as the theories themselves contain magnetic monopole solutions. In chapters 4 and 5 I consider configurations corresponding to a pair of 't Hooft - Polyakov monopole [27, 28, 29] and anti-monopole attached to the end of a flux tube in SU(2) gauge theory. This corresponds to Nambu's picture of confinement but one in which the confined monopoles are the magnetic monopoles (topological objects) in the theory.

In Chapter 4 I review spontaneous symmetry breaking and magnetic monopoles. Then I describe flux tube solutions in non Abelian gauge theory by two stage symmetry breaking. This can be done for SU(2) gauge theory by starting with two scalr fields. The first scalar field breaks the SU(2) symmetry to U(1). Breaking this U(1) produces a flux tube. The U(1) can be broken in two ways by taking the second scalar field in one of two different representations of SU(2). One way is to take it

in the adjoint representation of SU(2), and other is to have it in the fundamental representation of SU(2). The idea of two-scale symmetry breaking in SU(2), the first to produce monopoles and the second to produce strings, has appeared in [32]. Later this idea was used in a supersymmetric setting in [33, 34, 35, 36].

In Chapter 5 we start with SU(2) gauge theory with two scalar fields. One of them, call it ϕ_1, acquires a vacuum expectation value (vev) $\vec{\xi}_1$ which is a vector in internal space, and breaks the symmetry group down to U(1). The 't Hooft-Polyakov monopoles are associated with this breaking. The other scalar field also has a non-vanishing vev. This second field is in the adjoint representation and it is free to wind around $\vec{\xi}_1$ in the internal space. This winding is mapped to a circle in space, giving rise to the vortex string. We will construct two effective low energy Lagrangians with monopole and strings, one for two adjoint scalars and the other for one adjoint and one fundamental scalar. We shall see that these two Lagrangians are the same except for the values of two parameters, the coupling constant and mass of the photon. We then dualize the fields as in [22, 24, 25, 30, 31] to write the action in terms of string variables and here we shall show the attachment of monopoles at the end of flux tubes. The idea of flux matching, following Nambu [15] also appears in this thesis.

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2.1. Duality and magnetic monopoles

The equations which describe the electromagnetic field with sources $\rho(x), \vec{J}(x)$ are

$$\nabla \cdot \vec{E} = \rho \tag{2.1}$$

$$\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{J} \tag{2.2}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.3}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{2.4}$$

We can define vector potential from Eq. (2.3) as

$$\vec{B} = \nabla \times \vec{A}, \qquad B^i = \epsilon^{ijk}\partial_j A^k. \tag{2.5}$$

This is a unification procedure, because by defining the vector potential we can describe both electric and magnetic fields. From Helmholtz theorem we know that any vector field is uniquely specified by its divergence and curl. However, Eq. (2.5) is only the curl part of the field $\vec{A}$. So we can fix $\nabla \cdot \vec{A}$ by hand to uniquely specify the field $\vec{A}$. This is a process that we call gauge fixing in electrodynamics.

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The Maxwell equations without any sources are invariant under the transformation

$$\vec{E} \to \vec{B}, \qquad \vec{B} \to -\vec{E}. \tag{2.6}$$

This invariance is called the duality symmetry of the Maxwell equations. This duality symmetry breaks down when we add an electric charge current density into the equations. To restore the symmetry we have to add a magnetic charge density and a magnetic current density into the equations. If we add both magnetic and electric currents into the Maxwell equations then the equations look like

$$\nabla \cdot \vec{E} = \rho_e \tag{2.7}$$

$$\nabla \times \vec{B} = \vec{j}_e + \partial_t \vec{E} \tag{2.8}$$

$$\nabla \cdot \vec{B} = \rho_m \tag{2.9}$$

$$\nabla \times \vec{E} = -\vec{j}_m - \partial_t \vec{B}. \tag{2.10}$$

It follows from Eq. (2.6) and the Maxwell equations with electric and magnetic currents that the duality transformation for the currents are

$$\rho_e \to \rho_m, \qquad \rho_m \to -\rho_e \tag{2.11}$$

$$\vec{j}_e \to \vec{j}_m, \qquad \vec{j}_m \to -\vec{j}_e. \tag{2.12}$$

One can generalize the transformations by introducing a parameter ξ, and the transformations can be written as

$$\vec{E}' = \vec{E}\cos\xi + \vec{B}\sin\xi \quad , \quad \vec{B}' = -\vec{E}\sin\xi + \vec{B}\cos\xi \tag{2.13}$$

$$\rho_e' = \rho_e\cos\xi + \rho_m\sin\xi \quad , \quad \rho_m' = -\rho_e\sin\xi + \rho_m\cos\xi \tag{2.14}$$

$$\vec{j}_e' = \vec{j}_e\cos\xi + \vec{j}_m\sin\xi \quad , \quad \vec{j}_m' = -\vec{j}_e\sin\xi + \vec{j}_m\cos\xi. \tag{2.15}$$

It is then a matter of convention to say that a particle has magnetic charge or electric charge, because it fully depends on the value of ξ that we choose. The

question can be asked whether all particles have the same ratio of magnetic to electric charge. If they are the same, we can choose the angle ξ in the above equations so that $\rho_m = 0$, $\vec{j}_m = 0$. We then have the Maxwell equations as they are usually known. If we choose the electric and magnetic charges of an electron as $q_e = -e$, $q_m = 0$, then it is known [37] that for a proton, $q_e = +e$ (with the error limit $|q_e(electron) + q_e(proton)|/e \simeq 10^{-20}$) and $|q_m(nucleon)| < 2 \times 10^{-24}$. This limit on the magnetic charge of a proton or neutron follows directly from knowing that the average magnetic field at the surface of the earth is not more than 10^{-4} T. To a very high degree of precision we can conclude that the particles of ordinary matter possess only electric charge or, equivalently, they all have the same ratio of magnetic to electric charge. For unstable particles the question of magnetic charge is more open, but there exists no positive evidence.

2.2. Dynamical system with magnetic charge

Following the equations (2.7)-(2.10), we can write down the Maxwell equations with electric and magnetic currents (j_μ^e, j_μ^m),

$$\partial_\mu F^{\mu\nu} = j_e^\nu \,,\ \partial_\mu *F^{\mu\nu} = -j_m^\nu. \tag{2.16}$$

Here $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}$ and

$$F^{i0} = E^i, \qquad F_{ij} = \epsilon_{ijk}B^k. \tag{2.17}$$

If the currents result from point particle sources then we can write

$$j_e^\mu = \sum_i e_i \int dx_i{}^\mu \delta^4(x - x_i), \tag{2.18}$$

$$j_m^\mu = \sum_i g_i \int dx_i{}^\mu \delta^4(x - x_i), \tag{2.19}$$

where the integral over x_i is taken along the world line of the i-th particle whose electric and magnetic charges are e_i and g_i, respectively. The Lorentz force law can be generalized for a particle carrying magnetic as well as electric charge,

$$m\frac{d^2x^\mu}{d\tau^2} = (eF^{\mu\nu} + g*F^{\mu\nu})\frac{dx_\nu}{d\tau}. \tag{2.20}$$

Here τ parametrizes the world line of the particle. The dynamics of classical particle with electric and magnetic charge can be completely described by the above equations (2.16) and (2.20). The field due to a static monopole is

$$\vec{B} = \frac{g}{4\pi r^2}\hat{r}. \tag{2.21}$$

The dynamics of an electrically charged particle of charge e in a static monopole field is governed by the generalized Lorentz force Eq. (2.20),

$$m\frac{d^2\vec{r}}{dt^2} = e\frac{d\vec{r}}{dt} \times \vec{B}. \tag{2.22}$$

Though the force in (2.22) is not a central force (i.e. not directed towards the origin), angular momentum is conserved because of the spherically symmetric nature of magnetic field of Eq. (2.21). The rate of change of orbital angular momentum of the particle can be written as

$$\frac{d}{dt}(\vec{r} \times m\frac{d\vec{r}}{dt}) = \vec{r} \times m\frac{d^2\vec{r}}{dt^2} \tag{2.23}$$

$$= e\vec{r} \times (\frac{d\vec{r}}{dt} \times \vec{B}) \tag{2.24}$$

$$= \frac{eg}{4\pi r^3}\vec{r} \times (\frac{d\vec{r}}{dt} \times \vec{r}) \tag{2.25}$$

$$= \frac{d}{dt}\left(\frac{eg}{4\pi}\hat{r}\right). \tag{2.26}$$

Thus it is possible to define a conserved quantity

$$\vec{J} = \vec{r} \times m\frac{d\vec{r}}{dt} - \frac{eg}{4\pi}\hat{r}. \tag{2.27}$$

The second term in above Eq. (2.27) is actually the contribution from the electromagnetic field.

To see this, we calculate the angular momentum of the electromagnetic field for a system with one electric charge at some point $\vec{r}_e$ and one magnetic charge at the origin,

$$\vec{J}_{em} = \int d^3x\, \vec{r} \times (\vec{E} \times \vec{B}) \tag{2.28}$$

$$= \frac{g}{4\pi} \int d^3x\, \vec{r} \times (\vec{E} \times \frac{\vec{r}}{r^3}) \tag{2.29}$$

$$= \frac{g}{4\pi} \int d^3x\, \frac{1}{r}(\vec{E} - (\vec{E}\cdot\hat{r})\hat{r}). \tag{2.30}$$

Using the identity $\dfrac{\vec{E}}{r} = \dfrac{\hat{r}(\hat{r}\cdot\vec{E})}{r} + (\vec{E}\cdot\nabla)\hat{r}$, we get

$$\vec{J}_{em} = \frac{g}{4\pi} \int d^3x\, (\vec{E}\cdot\nabla)\hat{r} \tag{2.31}$$

$$= -\frac{g}{4\pi} \int d^3x\, \hat{r}\, \nabla\cdot\vec{E} + \frac{g}{4\pi} \int \hat{r}\, \vec{E}\cdot\vec{ds'}. \tag{2.32}$$

where the second integral is over a surface S' at infinity and $\vec{ds'}$ is directed along the outward normal to that surface. With $\vec{E}$ for a point charge this surface integral reduces to $\frac{e}{4\pi}\int \hat{r}d\Omega$ because the integral (2.28) for this system is invariant under a shift of origin. Since $\hat{r}$ is radially directed, it has zero angular average. Thus the second term vanishes.

The second term is a surface term at spatial boundary and all the components of the field $\vec{E}$ go to zero on the spatial boundary. So J_{em} becomes

$$\vec{J}_{em} = -\frac{g}{4\pi} \int d^3x\, \hat{r}\, \nabla\cdot\vec{E} = -\frac{g}{4\pi} \int d^3x\, \hat{r}\, \rho(\vec{r}). \tag{2.33}$$

Since there is a static electric charge e at $\vec{r}_e$, the charge density $\rho(r) = e\, \delta^3(\vec{r} - \vec{r}_e)$. Using this the above Eq. (2.33) becomes

$$\vec{J}_{em} = -\frac{g}{4\pi} \int d^3x\, \hat{r}\, \delta^3(\vec{r} - \vec{r}_e), = -\frac{ge}{4\pi}\, \hat{r}_e. \tag{2.34}$$

This result [38, 39] exactly matches the second term of Eq. (2.27) with $\hat{r} = \hat{r}_e$. So the total angular momentum of particle and electromagnetic field is

$$\vec{\tilde{J}} = \vec{J} + \vec{J}_{em} = m_e \vec{r} \times \frac{d\vec{r}}{dt} - \frac{ge}{4\pi}\hat{r}_e. \tag{2.35}$$

The component of the total angular momentum in the direction of the electric charge is

$$\vec{\tilde{J}} \cdot \hat{r}_e = -\frac{eg}{4\pi}. \tag{2.36}$$

$\vec{\tilde{J}}$ is a constant of motion, so the particle will rotate around $-\vec{\tilde{J}}$ with the angle $\cos^{-1}(eg/4\pi|\vec{\tilde{J}}|)$ (between position vector $\vec{r}_e$ and $-\vec{\tilde{J}}$) . Since the angular momentum is quantized, we can choose the z axis along $\vec{r}_e$ and find

$$J_z = \frac{eg}{4\pi} = \frac{n\hbar}{2}. \tag{2.37}$$

Eq. (2.37) is called the Dirac quantization condition. It follows that if there is at least one magnetic monopole in the universe, electric charge is quantized in multiples of some fundamental unit of charge.

2.3. Monopole gauge field and Dirac string

Dirac [40, 43] was the first to take magnetic monopoles seriously and tried to establish a theory for magnetic monopoles. The main problem was that if there are isolated magnetic charges, the vector potential cannot be smooth and differentiable everywhere. In particular, if we consider the radial magnetic field of Eq. (2.21) for any closed surface $\vec{S}$ containing the origin, then

$$g = \int_S \vec{B} \cdot d\vec{S}. \tag{2.38}$$

However, if $\vec{B} = \nabla \times \vec{A}$, the integral (2.38) would have to vanish. Thus $\vec{A}$ cannot be smooth and differentiable everywhere on S. Let us consider the field due to an

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infinitely long and thin solenoid placed along the negative z axis with one end at the origin (with total flux strength g) . Its magnetic field would be:

$$\vec{B}_{sol} = \frac{g}{4\pi r^2}\hat{r} + g\theta(-z)\delta(x)\delta(y)\hat{z}, \tag{2.39}$$

where $\hat{z}$ is a unit vector in the z direction and $\theta(\xi) = 0$ if $\xi < 0, \theta(\xi) = 1$ if $\xi > 0$. This magnetic field differs from $\vec{B}$ only by the singular magnetic flux along the solenoid but it is clearly source free, i.e, $\nabla \cdot \vec{B}_{sol}$ vanishes even at the origin. Thus we can define a vector potential $\vec{A}$ everywhere to write $\vec{B}_{sol} = \nabla \times \vec{A}$, i.e.

$$\frac{g}{4\pi}\frac{\hat{r}}{r^2} = \nabla \times \vec{A} - g\theta(-z)\delta(x)\delta(y)\hat{z}. \tag{2.40}$$

The line occupied by the solenoid is called the Dirac string. We should think of the field $\vec{B}$ as being represented not just by $\vec{A}$, but by $\vec{A}$ together with a line on which it is singular. Given our choice of the position of the negative z axis we can calculate an explicit form for $\vec{A}$ by exploiting axial symmetry. The magnetic field due to the monopole contains only the radial component in spherical polar coordinates. So from symmetry we can choose the vector potential as $\vec{A} = A(r, \theta)\hat{\varphi}$, where (r, θ, φ) are the spherical polar coordinates. The magnetic flux through a circle C (corresponding to fixed values of r and θ, and φ ranging over the values 0 to 2π) is given by solid angle subtended by C at the origin multiplied by $\frac{g}{4\pi}$ namely $\frac{1}{2}g(1 - \cos\theta)$. But we can also write the flux using Stokes' theorem as

$$\frac{1}{2}g(1 - \cos\theta) = \int_S \vec{B} \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{l} = 2\pi A(r, \theta)\, r\, \sin\theta, \tag{2.41}$$

here S is the surface enclosed by the circle C. It follows that we can write the vector potential as

$$\vec{A}(\vec{r}) = \frac{g}{4\pi}\frac{(1 - \cos\theta)}{\sin\theta}\hat{\varphi}. \tag{2.42}$$

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This vector potential (2.42) shows the anticipated singularity on the negative z axis $\theta = \pi$. There are other ways of defining this vector potential. Define a two form F in three dimension

$$F = F_{ij}\, dx^i \wedge dx^j. \tag{2.43}$$

Now if we go from Cartesian coordinates x^i to other coordinates ξ^α, we have

$$F = F_{ij}\frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial x^j}{\partial \xi^\beta}\, d\xi^\alpha \wedge d\xi^\beta. \tag{2.44}$$

Now suppose $\xi^\alpha = (r, \theta, \varphi)$.

We can write Eq. (2.44) as

$$F = \epsilon^{ijk} B^k \left[\frac{dx^i}{dr}\frac{dx^j}{d\theta} dr \wedge d\theta + \frac{dx^i}{d\theta}\frac{dx^j}{d\varphi} d\theta \wedge d\varphi + \frac{dx^i}{d\varphi}\frac{dx^j}{dr} d\varphi \wedge dr \right]. \tag{2.45}$$

Let us write $x^1 = r\sin\theta\cos\varphi$, $x^2 = r\sin\theta\sin\varphi$, $x^3 = r\cos\theta$ for spherical polar coordinate. Calculating all the derivatives we can write Eq. (2.45) as

$$F = Q_m \sin\theta\, d\theta \wedge d\varphi, \tag{2.46}$$

with the corresponding vector potential

$$A = -Q_m \cos\theta d\varphi. \tag{2.47}$$

Though the term $-Q_m \cos\theta$ is a smooth function, the vector potential is not smooth everywhere. To see the singularity let us write down vector potential in Cartesian coordinate. Using

$$\cos\theta = \frac{z}{r}, \qquad \tan\varphi = \frac{y}{x}, \tag{2.48}$$

we can write Eq. (2.47) as

$$A = -Q_m \left[-\frac{zy}{r(x^2 + y^2)}dx + \frac{zx}{r(x^2 + y^2)}dy \right]. \tag{2.49}$$

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Now we want to check the behavior near the z-axis, i.e in a region where $\dfrac{|z|}{r} \approx 1$. So for the positive z-axis, Eq. (2.49) becomes

$$A_{\frac{z}{r}\approx 1} = -Q_m \left[-\frac{y}{(x^2+y^2)}dx + \frac{x}{(x^2+y^2)}dy \right]. \tag{2.50}$$

The above expression (2.50) is singular on the positive z-axis and in the same way we can show that it is also singular on the negative z-axis. So the expression (2.47) cannot be treated as a gauge potential for the monopole. If we calculate the magnetic field for this vector potential, we will get the magnetic field of a monopole along with a singular line magnetic field along the whole z-axis.

The gauge potential (2.49) near the z-axis takes the form as

$$A_{\frac{|z|}{r}\approx 1} = -Q_m\, d\varphi, \text{ for } z > 0 \tag{2.51}$$

$$= Q_m\, d\varphi, \text{ for } z < 0. \tag{2.52}$$

If we add $\pm Q_m d\varphi$ with the expression (2.50) then the gauge potential is singular only on one of the z-axis. So the vector potential can be written as

$$A_1 = -Q_m \left(\cos\theta - 1 \right) d\varphi, \text{ singular along } \theta = \pi, \tag{2.53}$$

$$A_2 = -Q_m \left(\cos\theta + 1 \right) d\varphi, \text{ singular along } \theta = 0. \tag{2.54}$$

The vector potential A_1 gives the magnetic field of a monopole field at the origin with a singular field line (Dirac string) along the negative z-axis. For the vector potential A_2 the singular string will appear on the positive z-axis. So to get only the monopole field we have to subtract the singular field due to the string from the curl of the vector potentials A_1 and A_2.

Alternatively, we can remove all singular strings by the Wu-Yang construction [41]. In this, space is covered by two coordinate patches R_a and R_b. Using spherical

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coordinates (r, θ, φ) with the monopole at the origin we choose R_a and R_b as

$$R_a: \qquad 0 \le \theta < \frac{1}{2}\pi + \delta, \qquad 0 < r, \qquad 0 \le \varphi < 2\pi \qquad (2.55)$$

$$R_b: \qquad \frac{1}{2}\pi - \delta \le \theta < \pi, \qquad 0 < r, \qquad 0 \le \varphi < 2\pi. \qquad (2.56)$$

Then the vector potential A is defined to be A_1 in R_a and A_2 in the patch R_b. In the intersection of two patches the two gauge potentials are related by gauge transformation

$$A_1 = A_2 + \frac{i}{e} G_{ab} dG_{ab}^{-1}, \qquad (2.57)$$

with $G_{ab} = e^{2ieQ_m\varphi}$. Then in order to make the gauge transformation single valued we must require the Dirac quantization condition $eQ_m = \frac{n}{2}$.

Now we will discuss the type of the singularity of magnetic field due to the vector potential A_2. The expression (2.54) is singular at the origin $r = 0$ and on the positive z-axis. The singularity at the origin reflects the singularity of the monopole field. The singularity on the positive z-axis constitute the string, which was not present in the pure monopole field. To control the singularity, we can regularize the gauge potential [42],

$$A \to A_\epsilon = Q_m \left[\frac{y}{R(R-z)}, -\frac{x}{R(R-z)}, 0 \right] \qquad (2.58)$$
$$\text{with } R = \sqrt{r^2 + \epsilon^2}.$$

Then

$$\frac{\partial}{\partial x}\left[\frac{1}{R(R-z)}\right] = -\frac{x(2R-z)}{R^3(R-z)^2} \qquad (2.59)$$

$$\frac{\partial}{\partial y}\left[\frac{1}{R(R-z)}\right] = -\frac{y(2R-z)}{R^3(R-z)^2} \qquad (2.60)$$

$$\frac{\partial}{\partial z}\left[\frac{1}{R(R-z)}\right] = \frac{1}{R^3}. \qquad (2.61)$$

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The magnetic field that we get from the vector potential (2.58) is

$$\vec{B}_\epsilon = Q_m \left[\frac{\vec{r}}{R^3} - \frac{\epsilon^2(2R-z)}{R^3(R-z)^2}\hat{k} \right].$$ (2.62)

The first term is the monopole term. Let us call the second term $\vec{B}_\epsilon^S$, it will produce the magnetic field of a singular string as we take $\epsilon \to 0$,

$$\vec{B}_\epsilon^S = -Q_m \frac{2\epsilon^2}{R^3(R-z)}\hat{k} - Q_m \frac{\epsilon^2 z}{R^3(R-z)^2}\hat{k}.$$ (2.63)

It is easy to check that $\vec{B}_\epsilon^S$ is zero everywhere other than on the positive z-axis as $\epsilon \to 0$. The first term of Eq. (2.63) gives no contribution to the singular part on the positive z-axis while on the positive z-axis the second terms becomes,

$$-Q_m \frac{1}{\epsilon^2}.$$ (2.64)

This will produce a delta function field along positive z-axis.

2.4. C,P,T symmetry and Magnetic monopoles

Experiments show that the electric current transforms as a vector under parity and time reversal

$$
\begin{aligned}
P\rho_e(x)P^{-1} &= \rho_e(x_P), & T\rho_e(x)T^{-1} &= \rho_e(-x_P) \\
P\vec{j}_e(x)P^{-1} &= -\vec{j}_e(x_P), & T\vec{j}_e(x)T^{-1} &= -\vec{j}_e(-x_P),
\end{aligned}
$$ (2.65)

where $x = (t, \vec{x})$ and $x_P = (t, -\vec{x})$. The Maxwell equations will be invariant under parity and time reversal if

$$
\begin{aligned}
P\vec{E}(x)P^{-1} &= -\vec{E}(x_P), & T\vec{E}(x)T^{-1} &= \vec{E}(-x_P) \\
P\vec{B}(x)P^{-1} &= \vec{B}(x_P), & T\vec{B}(x)T^{-1} &= -\vec{B}(-x_P).
\end{aligned}
$$ (2.66)

From these field transformations we can deduce the parity and time reversal properties of the magnetic current,

$$P\rho_m(x)P^{-1} = -\rho_m(x_P), \qquad T\rho_m(x)T^{-1} = -\rho_m(-x_P)$$
$$P\vec{j}_m(x)P^{-1} = \vec{j}_m(x_P), \qquad T\vec{j}_m(x)T^{-1} = \vec{j}_m(-x_P). \qquad (2.67)$$

If we look at the time reversal property of monopole current, we see that it does not change its sign like ordinary current. This looks peculiar, because any physical particle moving in a trajectory will reverse its motion when time reversal is applied. However, the confusion ends when we look at the magnetic charge density, which changes sign under time reversal. So under time reversal, a particle reverses not only the direction of its path but also the sign of its magnetic charge, such that the magnetic currents does not change its direction.

The discussion on charge conjugation (C), parity (P) and time reversal (T) properties of monopole currents is important, because violation of C, P and T has not been found experimentally in electromagnetism.

Now we will check whether any physical currents have the P and T properties written in Eq. (2.67). There are some beautiful theory of magnetic monopoles taking monopole currents to be classical point particle current [43, 44, 45, 46, 48, 47, 49, 50]. The classical point particle currents for monopole can be written as,

$$j^\mu(x) = \sum_i g_i \int \delta^4(x - x_i)u_i^\mu d\tau_i. \qquad (2.68)$$

Here the particle trajectories are specified by $x_i^\mu = x_i^\mu(\tau_i)$, $u_i^\mu = \dot{x}^\mu = dx_i^\mu/d\tau_i$, and τ_i. The time reversal and parity properties of the above currents are

$$j^0(x)\xrightarrow{P} j^0(x_P), \qquad j^0(x) \xrightarrow{T} j^0(-x_P)$$
$$j^i(x)\xrightarrow{P} -j^i(x_P), \qquad j^i(x)\xrightarrow{T} -j^i(-x_P). \qquad (2.69)$$

However, these parity and time reversal properties do not match with Eq. (2.67). These are exactly like those of the electric current. So this current cannot be treated

as that of a magnetic monopole. To match its behavior with that of a monopole current, one has to define a number g which must change its sign under the action of P and T. However, numbers do not change under Lorentz transformations. So in order to use this current as a monopole current we have to redefine time reversal and parity by changing the sign of g_i along with the usual time reversal operation.

Let us consider CPT in a quantum field theory of magnetic monopoles. For a complex scalar field Φ the particle current is,

$$j^\mu(x) = ig(\Phi\partial_\mu\Phi^* - \Phi^*\partial_\mu\Phi). \tag{2.70}$$

The above current (2.70) can represent a monopole if the field Φ transforms under parity and time reversal as

$$\Phi(x) \xrightarrow{T} \Phi^*(-x_P), \qquad \Phi(x) \xrightarrow{P} \Phi^*(x_P). \tag{2.71}$$

For ordinary conventional unitary representation of the Lorentz group a quantum scalar field transforms under T and P as

$$\Phi(x) \xrightarrow{T} \Phi(-x_P), \qquad \Phi(x) \xrightarrow{P} \Phi(x_P). \tag{2.72}$$

However, there is an unconventional representation [51, 52] of time reversal which converts from a particle state to an anti-particle state with reversed time, which is appropriate for a monopole. The anti-unitary property of the time reversal operator is responsible for this conversion. The time reversal matrix is block-diagonalizable under anti-unitary transformations in the particle state space. The block diagonal part of the time reversal matrix converts a particle state to an anti-particle state. However, there is no corresponding representation for parity because it is a unitary operator. So this unconventional representation is not valid for magnetic monopoles as scalar field.

2. Magnetic monopoles in Electrodynamics

For fermionic monopole currents we can write the current [48, 50],

$$j^\mu(x) = g\bar{\psi}\gamma^\mu\psi. \tag{2.73}$$

The parity and time reversal properties of this current are

$$
\begin{aligned}
j^0(x) &\xrightarrow{P} j^0(x_P), & j^0(x) &\xrightarrow{T} j^0(-x_P) \\
j^i(x) &\xrightarrow{P} -j^i(x_P), & j^i(x) &\xrightarrow{T} -j^i(-x_P).
\end{aligned}
\tag{2.74}
$$

However, the above parity and time reversal properties do not match with the P and T properties of monopole current (2.67). So this current cannot be treated as a monopole current under the usual definition of parity and time reversal.

Monopole currents are axial with respect to parity and it is possible to write down an axial current [53, 54, 55] with the fermionic fields as $\bar{\psi}\gamma_5\gamma^\mu\psi$. However, the axial fermionic current has the time reversal property of the usual vectors. So it may be possible to get time reversal property like monopole current as written in Eq. (2.67) if we use unconventional representation of time reversal. However, the axial current does not change its sign under charge conjugation,.

We have seen that all the currents written in equations (2.68), (2.70) ,(2.73) and also the axial current have some problems if they are to be thought of as monopole currents. However, if we can redefine parity and time reversal by also changing the sign of g, the magnetic coupling constant, with the usual parity and time reversal operation, then we can take some of the above currents (2.68,2.70,2.73) as monopole currents. We can generalize this idea by giving a prescription of redefinition of parity and time reversal [56]. In a theory which includes the effect of magnetic monopoles, the TCP theorem would be replaced by a TMCP theorem where T represents simple time reversal, M magnetic monopole conjugation, C electric charge conjugation and P simple inversion of space coordinate. It is of course possible to express the theorem

2. Magnetic monopoles in Electrodynamics

in various ways, such as

$$(TM) \ (CM) \ (MP) = T'C'P'. \tag{2.75}$$

Here T' indicates an extended time reversal whose definition includes magnetic pole conjugation as well, C' represents conjugation of both electric and magnetic charges, and P' represents a parity transformation includes magnetic pole conjugation as well.

In this prescription a magnetically charged scalar field transforms as

$$\Phi(x) \xrightarrow{T'} \Phi^*(-x_P) \tag{2.76}$$

$$\Phi(x) \xrightarrow{P'} \Phi^*(x_P). \tag{2.77}$$

Here we can see that the transformation property of the scalar fields under P' and T' are the same as Eq. (2.71). P' and T' also act as ordinary parity and time reversal on the other fields which have no magnetic charge. Now it is possible to construct monopole currents using ordinary vector currents, (see [45]) mentioned in equations (2.70,2.73) .

A magnetic current j_μ behaves under P' and T' as

$$
\begin{aligned}
j_0(x) \xrightarrow{P'} -j_0(x_P), & \qquad j_0(x) \xrightarrow{T'} -j_0(-x_P) \\
j_i(x) \xrightarrow{P'} j_i(x_P), & \qquad j_i(x) \xrightarrow{T'} j_i(-x_P).
\end{aligned}
\tag{2.78}
$$

The transformation (2.78) are the same as Eq.(2.67) except T and P are replaced by T' and P'.

By defining parity and time reversal in this way, we have removed the possibility of having both electric and magnetic charge of a field (dyon). In this formalism the current constituted from a field transformation either under P' and T' as in Eq. (2.78) or under P and T as in Eq. (2.65), but not both.

We can split Fock space into two spaces for all particles one for only magnetic monopoles ($\mathcal{M}$) and the other for all other particles which do not have any magnetic

charge ($\mathcal{C}$). Then we can define parity, time reversal and charge conjugation for both the spaces and then take the direct sum of operators of the two spaces. We define P, T and C for $\mathcal{C}$ and PM, TM and M for $\mathcal{M}$ (2.75) where P, T and C are ordinary parity, charge conjugation and time reversal and M is the monopole conjugation. Now if we take the direct sum of operators of the two spaces we get $C \oplus M$, $P \oplus PM$, $T \oplus TM$. These are the charge conjugation, parity and time reversal operators for the whole space.

So far we have discussed particle currents for magnetic monopoles, but there is another possible way of constructing a monopole current. One can think of a monopole as a topological defect and the corresponding monopole currents will be topological currents. A beautiful example of such a monopole is the 't Hooft-Polyakov monopole solution in non-Abelian gauge theory [27, 28] We will discuss it in detail in chapter 4. In this case the monopole current is

$$j_m^\mu = \epsilon^{\mu\nu\rho\lambda}\epsilon_{abc}\partial_\nu\phi^a\partial_\rho\phi^b\partial_\lambda\phi^c. \tag{2.79}$$

Here ϕ^i is the i-th component of an SU(2) adjoint scalar field. This is the current which has the P and T properties exactly like a magnetic monopole current (2.67) in Maxwell equation. But here the fields are in the adjoint representation of a non-Abelian gauge group and the action of time reversal operator on these fields are different from Abelian fields because the generators of the Lie-algebra contains the factor 'i', that changes under the time reversal.

In general a magnetic monopole current can be treated as a topological current and can be written as

$$j^\mu = \epsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}, \tag{2.80}$$

where $B_{\mu\nu}$ is an antisymmetric tensor field, which may be constructed out of field already present in the theory. For example $B^{\mu\nu} = \epsilon^{\mu\nu\rho\lambda}\epsilon_{abc}\phi^a\partial_\rho\phi^b\partial_\lambda\phi^c$ for the current

written in Eq. (2.79). In fact $B_{\mu\nu}$ can also be constructed by fermionic field as,

$$j^\mu(x) = g\epsilon^{\mu\nu\rho\lambda}\partial_\nu\bar{\psi}\sigma_{\mu\nu}\psi. \tag{2.81}$$

However, the above (2.81) current [53] has mass dimension four which means that it is non-renormalizable.

2.5. Charge quantization and the Proca model

In this section we will discuss the classical static monopole in Proca massive Abelian vector field theories in three space and one time dimensions. We solve the equations of motion of massive electromagnetic field for a static magnetic monopole and an static electrically charged particle. We calculate the angular momentum as well for this system to see whether its quantization leads to the quantization of charge.

2.5.1. Equations of motion and their solution

The Lagrangian density for electromagnetism with a mass term is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2 A^\mu A_\mu. \tag{2.82}$$

The field equations of motion with a point electric charge q at the origin and a point magnetic charge g at $\vec{r} = \vec{r}_g$ are

$$\nabla \cdot \vec{E} = -m^2 A^0 + q\delta^3(\vec{r}), \tag{2.83}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} + m^2 \vec{A} = 0, \tag{2.84}$$

$$\nabla \cdot \vec{B} = g\delta^3(\vec{r} - \vec{r}_g), \tag{2.85}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0. \tag{2.86}$$

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We have added the magnetic monopole source to the right hand side of Eq.(2.85), as we had done for the Maxwell theory. We have chosen the vector potential A_μ to be time independent because the sources are static. The static solution of Eq. (2.83) is the Yukawa potential

$$A^0 = q\frac{e^{-mr}}{r}, \tag{2.87}$$

$$\vec{E} = q\frac{e^{-mr}(1+mr)}{r^2}\hat{r}. \tag{2.88}$$

The electric field falls off exponentially rapidly at distances larger than m^{-1}, i.e. electric fields are screened.

Let us write the magnetic field as

$$\vec{B} = \frac{g}{4\pi}\frac{(\vec{r}-\vec{r}_g)}{|\vec{r}-\vec{r}_g|^3} + \nabla\times\vec{A}, \tag{2.89}$$

where $\vec{A}$ is non singular. We can insert the expression of magnetic field from Eq. (2.89) into Eq. (2.84) to find

$$\nabla\times\left(\nabla\times\vec{A}\right) = -m^2\vec{A}, \tag{2.90}$$

because the electric field $\vec{E}$ is time independent. This equation simplifies to

$$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = -m^2\vec{A}, \tag{2.91}$$

and the divergence of Eq. (2.91) gives

$$\nabla\cdot\vec{A} = 0. \tag{2.92}$$

It follows that Eq. (2.91) can be written as

$$\nabla^2\vec{A} - m^2\vec{A} = 0. \tag{2.93}$$

This is actually three equations for three Cartesian components. Taking the scalar product of $\vec{A}$ with the above Eq. (2.93) and integrating over the whole volume we get

$$\int [\vec{A} \cdot \nabla^2 \vec{A} - m^2 \vec{A} \cdot \vec{A}] \, d\tau \;=\; 0. \tag{2.94}$$

Integrating by parts we get

$$\int [\nabla \cdot (\vec{A} \cdot \nabla \vec{A}) - (\nabla \vec{A})^2 - (m\vec{A})^2] \, d\tau \;=\; 0. \tag{2.95}$$

Here we assume that $\vec{A}$ goes to zero on the boundary. So the equation (2.95) becomes

$$\int [(\nabla \vec{A})^2 + m^2 \vec{A}^2] \, d\tau \;=\; 0. \tag{2.96}$$

In the above expression the integrand is positive definite. So the integrand has to vanish but it is a sum of squares. So the only solution of the equation (2.96) is $\vec{A} = 0$. Then the only non singular $\vec{A}$ in Eq. (2.89) is $\vec{A} = 0$ without any electric current. In fact even in the absence of a monopole this says that static $\vec{A} = 0$ in the absence of an electric current.

2.5.2. The massive Biot-Savart law

If there is a steady current density $\vec{J}$ in the system then Eq. (2.84) becomes

$$\nabla \times \vec{B} + m^2 \vec{A} \;=\; \vec{J}. \tag{2.97}$$

Using the definition of vector potential and Eq. (2.92) we can write the above equation as

$$\nabla^2 \vec{A} - m^2 \vec{A} = -\vec{J}, \tag{2.98}$$

because $\nabla \cdot \vec{J} = 0$. These are nothing but three Helmholtz equations, one for each Cartesian component. Assuming that $\vec{J}$ goes to zero (outside a compact region), we can read off the solution

$$\vec{A}(r) = \int e^{-m|\vec{r}-\vec{r}'|} \frac{\vec{J}(r')}{|\vec{r}-\vec{r}'|} \, d\tau', \tag{2.99}$$

where $d\tau'$ is the infinitesimal volume element.

This formula gives the vector potential at a point $\vec{r} = (x, y, z)$ in terms of an integral over the current distribution $\vec{J}(x', y', z')$. The divergence and the curl are to be taken with respect to the unprimed coordinates.

$$\nabla \times \vec{A}(r) = \int \left[\left(\nabla \frac{e^{-m|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \right) \times \vec{J}(r') \right] d\tau', \tag{2.100}$$

since $\nabla \times \vec{J}(r') = 0$ (because $\vec{J}(r')$ does not depend on unprimed coordinates). so according to Eq. (2.89), the magnetic field is

$$\vec{B}(\vec{r}) = \frac{g}{4\pi} \frac{(\vec{r}-\vec{r}_g)}{|\vec{r}-\vec{r}_g|^3} + \int (1 + m|\vec{r}-\vec{r}'|) e^{-m|\vec{r}-\vec{r}'|} \frac{\vec{J}(r') \times \hat{r}}{|\vec{r}-\vec{r}'|^2} d\tau'. \tag{2.101}$$

If there is no magnetic monopole, the magnetic field becomes

$$\vec{B}(\vec{r}) = \int (1 + m|\vec{r}-\vec{r}'|) e^{-m|\vec{r}-\vec{r}'|} \frac{\vec{J}(r') \times \hat{r}}{|\vec{r}-\vec{r}'|^2} d\tau'. \tag{2.102}$$

This is the Biot-Savart law in massive electromagnetism.

2.5.3. Calculation of angular momentum and quantization condition

The energy momentum tensor for the Lagrangian of Eq. (2.82) is

$$T^{\mu\nu} = -F^{\mu}_{\ \rho} F^{\nu\rho} + \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} + m^2 (A^{\mu} A^{\nu} - \frac{1}{2} \eta^{\mu\nu} A^{\alpha} A_{\alpha}). \tag{2.103}$$

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The Hamiltonian (energy) and momentum density are T^{00} and T^{0i}, respectively, which we can write using Eq. (2.103) as

$$\text{Energy} = T^{00} \;=\; \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \frac{m^2}{2}[(A^0)^2 + (A^i)^2], \tag{2.104}$$

$$\wp^i = T^{0i} \;=\; (\vec{E} \times \vec{B})^i + m^2 A^0 A^i, \tag{2.105}$$

$$\int \vec{\wp}\, d\tau = \vec{P}, \tag{2.106}$$

where $\vec{P}$ is the total momentum. For the system we are considering, i.e. a static electric charge at the origin and a static monopole at $\vec{r}_g$, the electric current vanishes $\vec{J}(\vec{r}) = 0$. So following the equations (2.96) and (2.99) we can write $\vec{A} = 0$. Then a standard calculation [37] gives $\vec{P} = 0$.

The angular momentum for the electromagnetic field is

$$\vec{L} \;=\; \int \vec{r} \times \vec{\wp}\, d\tau. \tag{2.107}$$

Since the total momentum $\vec{P}$ vanishes for our system, we can calculate this about any origin. Then using Eq. (2.88) and Eq. (2.101) with $\vec{J}(\vec{r}) = 0$ (the fact that $\vec{A} = 0$ for $\vec{J} = 0$), we can write

$$\vec{L} \;=\; \int \vec{r} \times \left(\frac{q}{4\pi} \frac{e^{-mr}(1+mr)}{r^2}\hat{r} \times \frac{g}{4\pi} \frac{\vec{r} - \vec{r}_g}{|\vec{r} - \vec{r}_g|^3} \right) \tag{2.108}$$

$$= \int \frac{f(r)}{r}(\hat{r} \times (\hat{r} \times \vec{B}))\, d\tau, \tag{2.109}$$

$$= -\int \frac{f(r)}{r}(\vec{B} - (\vec{B} \cdot \hat{r})\hat{r})\, d\tau, \tag{2.110}$$

where $f(r) = \dfrac{q}{4\pi}e^{-mr}(1+mr)$, and the integration is over all space. Using the identity $\dfrac{\vec{B}}{r} = \dfrac{\hat{r}(\hat{r} \cdot \vec{B})}{r} + (\vec{B} \cdot \nabla)\hat{r}$ we get

$$\vec{L} \;=\; -\int f(r)(\vec{B} \cdot \nabla)\hat{r}\, d\tau, \tag{2.111}$$

$$= \int \hat{r}\nabla \cdot (f(r)\vec{B})\, d\tau - \int_S \hat{r}(f(r)\vec{B} \cdot \vec{ds}). \tag{2.112}$$

The second term of Eq. (2.112) is zero because it is a angular average of the vector $\hat{r}$ over the surface S and $f(r)$ also vanishes at infinity. So we can write Eq. (2.112) as

$$\vec{L} \;=\; \int \hat{r}(\nabla f \cdot \vec{B})\, d\tau + \int \hat{r} f(r) \nabla \cdot \vec{B}\, d\tau. \tag{2.113}$$

Using Eq. (2.85) the above equation becomes,

$$\vec{L} \;=\; \int \hat{r} \nabla f \cdot \vec{B}\, d\tau + \int \hat{r} f(r) g \delta^3(\vec{r} - \vec{r}_g)\, d\tau \tag{2.114}$$

$$= \int \hat{r} \nabla f \cdot \vec{B}\, d\tau + g f(\vec{r}_g) \hat{r}_g \tag{2.115}$$

$$= \int \hat{r} \nabla f \cdot \vec{B}\, d\tau + \frac{qg}{4\pi} e^{-mr_g}(1 + mr_g) \hat{r}_g \tag{2.116}$$

$$= -\frac{q}{4\pi} \int \vec{r} m^2 e^{-mr} (\hat{r} \cdot \vec{B})\, d\tau + \frac{qg}{4\pi} e^{-mr_g}(1 + mr_g) \hat{r}_g. \tag{2.117}$$

As mentioned earlier, the total linear momentum of the system is zero, so the angular momentum is independent of the origin of the coordinate system. For simplicity we can choose the z- axis along $\vec{r}_g$ and write $\vec{r}_g = a\hat{k}$ and the angular momentum becomes

$$\vec{L} \;=\; -\frac{q}{4\pi} \int \hat{r} m^2 r e^{-mr} \left(\hat{r} \cdot \frac{g}{4\pi} \frac{\vec{r} - a\hat{k}}{|\vec{r} - a\hat{k}|^3} \right) d\tau + \frac{qg}{4\pi}(1 + ma)e^{-ma}\hat{k}, \tag{2.118}$$

$$= -\frac{m^2 qg}{16\pi^2} \int \frac{e^{-mr}(r^2 - ar\cos\theta)}{(r^2 + a^2 - 2ar\cos\theta)^{\frac{3}{2}}} \hat{r}\, d\tau + \frac{qg}{4\pi}(1 + ma)e^{-ma}\hat{k}, \tag{2.119}$$

where $\cos\theta = \hat{r} \cdot \hat{k}$. To calculate the integral of Eq. (2.119) let us define

$$U(r) \;=\; (r^2 + a^2 - 2ar\cos\theta)^{-\frac{1}{2}}, \tag{2.120}$$

$$\frac{dU}{dr} \;=\; -\frac{(r - a\cos\theta)}{(r^2 + a^2 - 2ar\cos\theta)^{\frac{3}{2}}}, \tag{2.121}$$

so that

$$\vec{L} = \frac{m^2 qg}{16\pi^2} \int e^{-mr} r^3 \frac{dU(r)}{dr}\, dr \sin\theta\, d\theta\, d\phi \hat{r} + \frac{qg}{4\pi}(1 + ma)e^{-ma}\,\hat{k}. \tag{2.122}$$

Integrating by parts over r we get

$$
\begin{aligned}
\vec{L} &= -\frac{m^2 qg}{16\pi^2} \int \frac{(3r^2 - mr^3)e^{-mr}\sin\theta}{(r^2 + a^2 - 2ar\cos\theta)^{\frac{1}{2}}} \hat{r} \, d\theta \, d\phi \, dr + \frac{qg}{4\pi}(1 + ma)e^{-ma}\,\hat{k}, \\
&= -\frac{m^2 qg}{8\pi} \int \frac{(3r^2 - mr^3)e^{-mr}\sin\theta}{(r^2 + a^2 - 2ar\cos\theta)^{\frac{1}{2}}} \cos\theta\,\hat{k}\,d\theta\,dr \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \frac{qg}{4\pi}(1 + ma)e^{-ma}\,\hat{k}. \quad (2.123)
\end{aligned}
$$

Using the identity

$$
\frac{1}{(r^2 + a^2 - 2ar\cos\theta)^{\frac{1}{2}}} = \frac{2}{(2l+1)} \sum_{l=0}^{l=\infty} \frac{r_<^l}{r_>^{l+1}}\, P_l(\cos\theta), \qquad (2.124)
$$

where $P_l(\cos\theta)$ is the Legendre polynomial of order l, we can calculate from Eq. (2.123) that,

$$
\begin{aligned}
\vec{L} &= -\frac{m^2 qg}{8\pi} \int_0^\infty (3r^2 - mr^3)e^{-mr} \left[\int_{-1}^1 P_1(\zeta) \frac{2}{(2l+1)} \sum_{l=0}^{l=\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\zeta)\, d\zeta \right] \hat{k} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \frac{qg}{4\pi}(1 + ma)e^{-ma}\,\hat{k}, \quad (2.125) \\
&= -\frac{qg}{4\pi} \left[2\frac{e^{-ma}}{ma} + 2\frac{e^{-ma} - 1}{m^2 a^2} \right] \hat{k}, \qquad\qquad\qquad (2.126)
\end{aligned}
$$

where $\zeta = \cos\theta$. The exponential can be expanded and the result becomes

$$
\vec{L} = \frac{qg}{2\pi} \sum_{n=0}^{n=\infty} \frac{(-1)^n}{(n+2)} \frac{(ma)^n}{n!}\,\hat{k}. \qquad (2.127)
$$

The above solution gives the angular momentum of the massive electromagnetic field for a charge -magnetic monopole pair. The angular momentum depends on two continuous parameters, the position a of the monopole relative to the electric charge and the mass m of the photon and also on the product qg. Quantizing angular momentum does not lead to any simple quantization rule for the electric and magnetic charges. For $m \to 0$ we recover the angular momentum calculated by J.J.Thomson, i.e $|\vec{L}| = \frac{gq}{4\pi}$ and we can quantize electric charge by quantizing the angular momentum.

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So, we can conclude that to quantize electric charge by quantizing angular momentum, the mass of the photon must be zero. Alternatively we can also say that magnetic monopoles in massive electrodynamics does not lead to quantization of electric charge.

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Experiments show that the angular momentum of every hadron varies linearly with the square of its mass. By some indirect experimental result and computer simulation, it is also known that the static potential varies linearly with the distance between two quarks. These two facts can be simultaneously satisfied if field lines between two quarks are confined inside a thin flux tube such as a Nielsen-Olesen string. In this section we discuss magnetic flux tube configurations in the Abelian Higgs model and show how we can get a taste of confinement by confining magnetic monopoles with these Abelian magnetic flux tubes.

3.1. Vortices in Abelian Higgs model

We start with the Abelian Higgs model,

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} |D_\mu \Phi|^2 - \lambda(|\Phi|^2 - v^2)^2. \tag{3.1}$$

Here Φ is a complex scalar field which we decompose as

$$\Phi = \rho e^{i\chi}. \tag{3.2}$$

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The equations of motion are

$$\partial_\mu F^{\mu\nu} + e\rho^2(\partial^\nu\chi + eA^\nu) = 0, \tag{3.3}$$

$$\Box\rho - \rho(eA_\mu + \partial_\mu\chi)^2 + \lambda(\rho^2 - v^2)\rho = 0, \tag{3.4}$$

$$\partial_\mu[\rho^2(\partial^\mu\chi + eA^\mu)] = 0. \tag{3.5}$$

The Hamiltonian density can be written as

$$\mathcal{H} = \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 + +\frac{1}{2}(\partial_0\rho)^2 + \frac{1}{2}(\nabla\rho)^2 + \frac{\rho^2}{2}(e\vec{A}_0 + \partial_0\chi)^2 + \frac{\rho^2}{2}(e\vec{A} - \nabla\chi)^2$$
$$+ \frac{\lambda}{4}(\rho^2 - v^2)^2. \tag{3.6}$$

We can write the vacuum configuration as

$$\rho = v \tag{3.7}$$

$$A_\mu = -\frac{1}{e}\partial_\mu\chi. \tag{3.8}$$

Instead of taking this as vacuum configuration, it is possible to choose a new gauge where we can gauge away the $\partial_\mu\chi$. The behavior of the small fluctuations around the vacuum (3.7) can be written as

$$\rho = v + \tilde{\rho} \tag{3.9}$$

$$eA_\mu = -\partial_\mu\chi + ea_\mu. \tag{3.10}$$

The Lagrangian can effectively be written as,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}e^2v^2a_\mu a^\mu + \frac{1}{2}\partial_\mu\tilde{\rho}\partial^\mu\tilde{\rho} + \lambda v^2\tilde{\rho}^2 + \text{higher order terms}, \tag{3.11}$$

and the equations of motion upto first order are

$$\Box a^\mu + e^2v^2a^\mu = 0 \tag{3.12}$$

$$\Box\tilde{\rho} + 2\lambda v^2\tilde{\rho} = 0. \tag{3.13}$$

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These are the equations of the massive electromagnetic field and a real scalar field. So the excitations a_μ and $\tilde{\rho}$ generate particles with masses ev and $v\sqrt{2\lambda}$, respectively.

Here we have not given any vacuum solution of the field $\chi(x)$. This is because we have assumed that we can gauge away all configurations of the field $\chi(x)$ at the vacuum so that it does not contribute to any physical quantity. However this is not always true. Let us consider a static solution in cylindrical coordinates where χ and $|\Phi|$ are independent of the coordinate z. It can be written as $\chi(x, y) = \chi(r, \varphi)$. Let us draw a circular loop around the z-axis and demand that

$$\Phi(r, \varphi) = \Phi(r, \varphi + 2\pi). \tag{3.14}$$

It follows from Eq. (3.14) that

$$|\Phi(r, \varphi, z)| = |\Phi(r, \varphi + 2\pi, z)|, \tag{3.15}$$

$$\chi(r, \varphi + 2\pi) = \chi(r, \varphi) + 2n\pi. \tag{3.16}$$

In other words there are solutions for which χ is not a single valued function. χ varies by $2\pi n$ ($n =$ integer) when we make a complete turn around a closed loop and χ is undefined along the z-axis. We can calculate the magnetic flux along the z-axis by integrating the vector potential $\vec{A}$ along a large loop around the z-axis. On the large loop the vector potential $\vec{A} = \frac{1}{e}\nabla\chi$, because at large distances $\vec{a}$ should vanish as it is a solution of the Helmholtz for the static situation.

$$\tilde{\Phi} = \oint \vec{A} \cdot \vec{dl} = \frac{1}{e} \oint \nabla\chi \cdot \vec{dl} = \frac{2\pi n}{e}, \tag{3.17}$$

$$\text{i.e.,} \quad \tilde{\Phi} = n\Phi_0, \ \Phi_0 = \frac{2\pi}{e}. \tag{3.18}$$

The non-trivial result of the line integral shows a singular magnetic field along z-axis and we will call it a flux tube or flux string. So in general we can decompose the field in two parts $\chi = \chi_r + \chi_s$, where the χ_r is the regular part that can be gauged out

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by the vector potential and goes to zero at very large distances, whereas the other part χ_s is the multivalued part and this part does not go to zero at large distances.

The configuration χ_s is one of a class of vacuum field configurations that are important for topological defects in field theory. These configurations do not go to zero at infinity. In the Abelian Higgs model these field configurations generate a singular magnetic field along the z-axis. We can regularize the stress-energy tensor by taking $|\Phi| = 0$ along the z-axis. However, since we need a finite energy configuration as the starting point of perturbation theory, we must have $|\Phi| = v$ at spatial infinity. We can try to get a solution for which $|\Phi|$ is constant far away from the z-axis and goes smoothly to zero on the z-axis. Far away from the z-axis, $|\Phi|$ reaches its vacuum value and the vector potential becomes

$$\vec{A} = \frac{1}{e}\nabla\chi, \tag{3.19}$$

and we can decompose the field χ in two parts $\chi = \chi_r + \chi_s$ very far away from the z-axis.

Geometrically we can say that at large distances the field $\Phi = ve^{i\chi}$ defines a mapping of a circle in space to a circle of radius v at the plane of complex Φ. The mapping of one circle to another circle can be represented by $\pi_1(S^1)$, the fundamental group of the circle in the plane of complex field. We know that $\pi_1(S^1) = Z$, so the mapping is characterized by an integer $n = 0, 1, 2, \cdots$, the winding number. The winding number determines the amount of flux through the flux tube and for any flux tube configuration $n \neq 0$.

To see the exact static behavior of the fields $\vec{A}(x)$ and $\rho(x)$ for flux tube, we have to solve the equations of motion for the static case. However, till date an exact solution has not been found for a flux tube configuration. We can solve the equations in different regions of space and then we can try to match the solutions to see the

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whole picture. Let us write down the equations of motion in cylindrical coordinates and make all the fields independent of the z coordinate. From symmetry and using the asymptotic solution, we can write

$$\vec{A} = A_\varphi \hat{\varphi}. \tag{3.20}$$

Using Eq.(3.16) we can write

$$\chi(r, \varphi) = \varphi \tag{3.21}$$

for $n = 1$. So the equations of motion become

$$-\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (rA_\varphi) + e^2 \rho^2 \left(A_\varphi - \frac{1}{r} \right) = 0, \tag{3.22}$$

$$-\frac{1}{\rho} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \rho \right) + \left(eA_\varphi - \frac{1}{r} \right)^2 \rho + \lambda \left(\rho^2 - v^2 \right) = 0. \tag{3.23}$$

An exact analytic solution of the above equations is not known. We can get a solution of Eq. (3.22) at larger r if we write

$$\rho \simeq v \tag{3.24}$$

which is valid for large r. Then a solution for A_φ is [11]

$$A_\varphi = \frac{1}{er} + \frac{c}{e} K_1(evr) \overset{r \to \infty}{\Longrightarrow} \frac{1}{er} + \frac{c}{e} \sqrt{\frac{\pi}{2evr}} e^{-evr} + \text{higher order terms}, \tag{3.25}$$

where c is a constant of integration. The magnetic field has only its z-component and can be written as

$$B_z = cv K_0(evr) \overset{r \to \infty}{\Longrightarrow} \frac{c}{e} \sqrt{\frac{v\pi}{2er}} e^{-evr} + \text{higher order}. \tag{3.26}$$

We define a characteristic length $\tilde{\lambda}$ as,

$$\tilde{\lambda} = \frac{1}{ev}, \tag{3.27}$$

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which is similar to, in fact corresponds to, the penetration length in superconductivity. $\tilde{\lambda}$ thus measures the region over which the field B_z (3.26) is appreciably different from zero. To see the configuration of ρ around the vacuum value, we write

$$\rho = v + \tilde{\rho}. \tag{3.28}$$

After substitution of Eq.(3.28) and Eq. (3.25) into Eq. (3.23) and neglecting the higher order, we see that the solution behaves as

$$\tilde{\rho} \sim e^{-v\sqrt{2\lambda}r}. \tag{3.29}$$

Here we can define a new characteristic length ξ

$$\xi = \frac{1}{v\sqrt{2\lambda}}. \tag{3.30}$$

Thus ξ measures the radial distance it takes for the field ρ to reach its vacuum value v.

From the above discussion we see that for flux tube solutions $\tilde{\lambda}$ must be greater than or equal to ξ because the magnetic field always has a decaying tail after ρ reaches its vacuum value v. At a length scale less than ξ, there is a region where photon is massless. So we can say ξ is roughly the radius of the core of flux tube. Flux tubes can be treated as strings if the core is very small. At a length scale greater than ξ but less than $\tilde{\lambda}$ fields are getting their masses, and above the length scale $\tilde{\lambda}$, there is almost nothing. However, the field χ is still non zero very far away from the string core. The integration of Eq. (3.17) showing that χ can inform us about the existence of flux tube in a region, where practically there exists no physical field.

In chapter 1 we mentioned that the angular momentum of a string is proportional to its energy squared, i.e. $J = \sigma' E^2$, where $\sigma' = \frac{1}{2\pi\sigma}$ (universal slope of Regge

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trajectory) and σ is the static mass per unit length of a strings. So for massless quarks the mass of a hadron for lower quantum mechanical levels is $M_H \sim \frac{1}{\sqrt{\sigma'}}$. It is possible to make a classical estimate of the relation of the universal slope to the three parameters, v, λ and e by calculating the energy density at rest for the vortex solution. We can make a crude estimate of the energy per unit length for the flux tube constructed form the Abelian Higgs model,

$$\frac{1}{\sigma'} = \frac{\text{Energy}}{\text{length of the string}} \sim v^2. \tag{3.31}$$

The exact ratio $v^2\sigma'$ can be computed numerically by solving the differential equations. What is important, however, is that it is of order unity.

We have seen that the two particles from the fluctuation of the fields ρ and A_μ got their masses $M_s = v\sqrt{2\lambda}$ and $M_v = ev$ by Higgs mechanism. The strong coupling limit in the Abelian Higgs model is defined by setting the coupling constant to be large, i.e.

$$e \gg 1, \qquad \lambda \gg 1. \tag{3.32}$$

In the strong coupling limit we can write using Eq. (3.31) that

$$ev\frac{1}{v} \gg 1 \Rightarrow M_v \gg \frac{1}{\sqrt{\sigma'}}, \tag{3.33}$$

$$\sqrt{2\lambda}v\frac{1}{v} \gg 1 \Rightarrow M_s \gg \frac{1}{\sqrt{\sigma'}}. \tag{3.34}$$

It follows that $M_v, M_s \gg M_H$ for strong coupling, which means that the particles corresponding to the local fields ρ, and A_μ have masses M_v and M_s much larger than the typical hadron masses. Thus in this limit low energy phenomena (low energy meaning energies of the order of $\frac{1}{\sqrt{\sigma'}}$) should be dominated by hadrons, i.e. dual strings.

3.2. Flux tubes and effective strings

In this section we will try to describe flux tubes as effective strings. In the last section we have discussed that there are two length scales ξ and $\tilde{\lambda}$. The scalar field reaches its vacuum value after a distance ξ from the axis. The magnetic field vanishes after a distance $\tilde{\lambda}$ from the axis. We also discussed flux tube solutions and there we saw that after the scalar field reaches its vacuum value there is a tail of magnetic field and we are interested in constructing an effective theory in the region where $\xi < \tilde{\lambda}$. In the strong coupling limit we can consider the core radius or ξ o be very small and the scalar field reaches its vacuum value v very quickly. Thus we can consider the flux tube as a singular line or a string. So we can construct an effective theory in the strong coupling limit by taking the fields $\Re\Phi, \Im\Phi^*$ to be zero along the string core. The two equations

$$\Re\Phi(\tilde{x}) = 0, \qquad \Im\Phi(\tilde{x}) = 0, \tag{3.35}$$

define a two dimensional worldsheet of the string. The string coordinates are the coordinates of the worldsheet on which the scalar field vanishes. We can introduce this in the generating functional by inserting the identity [57]

$$1 = J^{-1}(\Phi, \tilde{x}) \int D\tilde{x}\, \delta(Re\Phi(\tilde{x}))\delta(Im\Phi(\tilde{x})), \tag{3.36}$$

and then we can integrate over the fields Φ, Φ^*. By doing so we expect that the remaining theory will be a theory of interacting strings.

There is another way to construct the effective theory of strings and this is more geometrical than the previous one. We have seen that the flux tube solution is characterized by $\chi = \chi_s$ being an angular coordinate around the flux tube. This means that we are defining a map from a spatial circle to a circle in field space. This mapping is done by taking χ as a function of spatial azimuthal angle φ such that

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when φ changes from 0 to 2π, χ changes from 0 to $2n\pi$. The number n represents the first homotopy class of the mapping. This number is the quantum number using which we can identify the flux tubes. When ξ is zero (string core), we can take χ as a variable to describe the string. Let us consider an example where this is seen easily. We set $\chi_s = \varphi$, where φ is the polar angle on the xy plane. Then we can write

$$\tan\varphi \;=\; \frac{y}{x}, \tag{3.37}$$

$$\nabla\varphi \;=\; \frac{1}{x^2 + y^2}\left[-y\,\hat{i} + x\,\hat{j}\right]. \tag{3.38}$$

Now we calculate the magnetic field due to our asymptotic vector potential $\vec{A} = \frac{1}{e}\nabla\varphi$. Let us define a regularized vector potential,

$$\vec{A}_\epsilon = \frac{1}{e}\frac{1}{x^2 + y^2 + \epsilon^2}\left[-y\,\hat{i} + x\,\hat{j}\right]. \tag{3.39}$$

The regulated magnetic field along the z axis can be written as

$$B_z^\epsilon = \frac{1}{e}\frac{2\epsilon^2}{(x^2 + y^2 + \epsilon^2)^2}. \tag{3.40}$$

This function is zero everywhere but singular at $r = 0$ in the limit $\epsilon = 0$. The nature of the singularity can be seen if we integrate B_z^ϵ over the two dimensional space perpendicular to the string.

$$\int B_z^\epsilon\,dxdy \;=\; \int_0^\infty \frac{1}{e}\frac{2\epsilon^2\,dx\,dy}{(x^2 + y^2 + \epsilon^2)^2} \tag{3.41}$$

$$= \int_0^\infty \frac{2\epsilon^2\,2\pi\,r\,dr}{e(r^2 + \epsilon^2)^2} \tag{3.42}$$

$$= \int_0^\infty \frac{4\pi\,t\,dt}{e(t^2 + 1)^2},\ \text{for}\ t = \epsilon r \tag{3.43}$$

$$= \frac{2\pi}{e}. \tag{3.44}$$

So we can write

$$\lim_{\epsilon\to 0}\frac{2\epsilon^2}{(x^2 + y^2 + \epsilon^2)^2} = 2\pi\delta(x)\delta(y). \tag{3.45}$$

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The z component of magnetic field can be written as

$$B_z = \frac{2\pi}{e}\delta(x)\delta(y). \tag{3.46}$$

We found that the magnetic field lines corresponding to the asymptotic vector potential are confined only at a point on the two dimensional plane. This allows us to think of the flux tube as a singular string in three dimensional space. We know from the discussion of the previous section that we can consider flux tube as a singular string when the distance scale is very large or the coupling constants are very large. So we can think of this singular line as an effective long distance description of the flux tube. In 3+1 dimensions the confined magnetic field is a worldsheet and we can define the worldsheet current as,

$$\Sigma^{\mu\nu} = 2\pi n \int d\tau d\sigma \left(\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} - \frac{dx^\nu}{d\sigma}\frac{dx^\mu}{d\tau} \right) \delta^4(x - x(\sigma, \tau)) \tag{3.47}$$

where σ, τ are the parameters of the world sheet. If we take a parametrization $\tau = x^0$, $\sigma = x^3$, we can calculate for the world sheet current,

$$\Sigma^{30} = 2\pi n \delta(x)\delta(y). \tag{3.48}$$

This is nothing but the static magnetic field (3.46) along the z-axis. In fact we can write

$$\Sigma^{\mu\nu} = 2\pi n \epsilon^{\mu\nu\rho\lambda}\partial_\rho\partial_\lambda \chi_s. \tag{3.49}$$

3.3. Functional integral and Duality

In this section we will discuss the construction of a dual theory using functional integrals. Let us first consider the functional integration,

$$Z = \int \mathcal{D}A_\mu \exp\left[-i \int d^4x \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \right], \tag{3.50}$$

which can be rewritten upto normalization constant as

$$Z = \int \mathcal{D}B_{\mu\nu}\mathcal{D}A_{\mu}\, \delta\left[B_{\mu\nu} - (\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})\right] \exp\left[-i\int d^4x \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\right], \qquad (3.51)$$

where we have used the identity of functional delta

$$1 = \int \mathcal{D}\phi\, \delta\left[\phi(x) - \tilde{\phi}(x)\right]. \qquad (3.52)$$

We can exponentiate the delta functional of Eq. (3.51) by introducing an auxiliary field $F_{\mu\nu}$ and integrate over the field A_{μ} and $B_{\mu\nu}$,

$$
\begin{aligned}
Z &= \int \mathcal{D}F_{\mu\nu}\mathcal{D}B_{\mu\nu}\mathcal{D}A_{\mu}\, \exp\left[-i\int d^4x \left\{\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{1}{4}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}B_{\rho\lambda} - \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}\partial_{\rho}A_{\lambda}\right\}\right] \\
&= \int \mathcal{D}F_{\mu\nu}\mathcal{D}B_{\mu\nu}\, \delta(\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_{\nu}F_{\rho\lambda}) \exp\left[-i\int d^4x \left\{\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{1}{4}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}B_{\rho\lambda}\right\}\right] \quad (3.53) \\
&= \int \mathcal{D}F_{\mu\nu}\, \delta(\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_{\nu}F_{\rho\lambda}) \exp\left[-i\int d^4x \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]. \qquad (3.54)
\end{aligned}
$$

In Eq. (3.51) we are integrating over $B_{\mu\nu}$ for which $B_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ and then integrating over all possible A_{μ}. In other words, we are integrating over all those $B_{\mu\nu}$ for which $\epsilon^{\mu\nu\rho\lambda}\partial_{\nu}B_{\rho\lambda} = 0$ and that is exactly Eq. (3.54). So these two equations are the same up to some multiplicative constant. To get Eq. (3.50) from Eq. (3.54), we can solve the delta functional of Eq. (3.54) and write down Eq. (3.51) with $F_{\mu\nu}$ in place of $B_{\mu\nu}$, then integrate over $F_{\mu\nu}$. Here all equalities between different functional integrations are up to some multiplicative constant.

Let us introduce an auxiliary field $G_{\mu\nu}$ into Eq. (3.50) to write

$$Z = \int \mathcal{D}A_{\mu}\mathcal{D}G_{\mu\nu}\, \exp\left[i\int d^4x \left\{-\frac{1}{4}G^{\mu\nu}G_{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_{\mu}A_{\nu}G_{\rho\lambda}\right\}\right]. \qquad (3.55)$$

The action in the functional integral (3.55) is

$$I = \int d^4x \left\{-\frac{1}{4}G^{\mu\nu}G_{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_{\mu}A_{\nu}G_{\rho\lambda}\right\}. \qquad (3.56)$$

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The equations of motion for the field $G_{\mu\nu}$ are

$$G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}F_{\rho\lambda}, \tag{3.57}$$

where $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$. So we can take $G_{\mu\nu}$ as a dual field tensor according to the definition given in Eq. (2.16).

In the presence of an external electric source j^μ the functional integral (3.50) can be written as

$$Z = \int \mathcal{D}A_\mu \exp\left[i \int d^4x \left\{-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - A^\mu j_\mu\right\}\right]. \tag{3.58}$$

We can here introduce a dual field $G_{\mu\nu}$ and integrate over A_μ,

$$\begin{aligned}
Z &= \int \mathcal{D}A_\mu \mathcal{D}G_{\mu\nu} \exp\left[i \int d^4x \left\{-\frac{1}{4}G^{\mu\nu}G_{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\mu A_\nu G_{\rho\lambda} - A^\mu j_\mu\right\}\right] \tag{3.59} \\
&= \int \mathcal{D}G_{\mu\nu}\delta(\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu G_{\rho\lambda} - j^\mu) \exp\left[i \int d^4x \left\{-\frac{1}{4}G^{\mu\nu}G_{\mu\nu}\right\}\right]. \tag{3.60}
\end{aligned}$$

Eq. (3.60) can be written as a dual theory of Eq. (3.58) and Eq. (3.60) looks the same as Eq. (3.54) except the Bianchi identity, written in terms of a delta functional. So, in this case the introduction of dual vector potential is not very easy. However, it becomes easy if we write the form of the current as

$$j^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu R_{\rho\lambda}. \tag{3.61}$$

Using the above expression of the current, Eq. (3.60) can be written as

$$Z = \int \mathcal{D}G_{\mu\nu}\,\delta\left(\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu(G_{\rho\lambda} - R_{\rho\lambda})\right) \exp\left[i \int d^4x \left\{-\frac{1}{4}G^{\mu\nu}G_{\mu\nu}\right\}\right]. \tag{3.62}$$

Using the discussion after Eq. (3.54) we can introduce a dual vector potential b_μ. The generating functional becomes

$$Z = \int \mathcal{D}G_{\mu\nu}\mathcal{D}b_\mu\,\delta\left(G_{\mu\nu} - \{\partial_\mu b_\nu - \partial_\nu b_\mu + R_{\mu\nu}\}\right) \exp\left[i \int d^4x \left\{-\frac{1}{4}G^{\mu\nu}G_{\mu\nu}\right\}\right]. \tag{3.63}$$

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Now we can integrate over the field $G_{\mu\nu}$ and get

$$Z = \int \mathcal{D}b_\mu \, \exp\left[i \int d^4x \left\{ -\frac{1}{4} \left(\partial_\mu b_\nu - \partial_\nu b_\mu + R_{\mu\nu}\right)\left(\partial^\mu b^\nu - \partial^\nu b^\mu + R^{\mu\nu}\right) \right\} \right]. \quad (3.64)$$

We started with a generating functional with a Lagrangian

$$\mathcal{L}^A = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - A^\mu j_\mu \quad (3.65)$$

and we ended up with a different Lagrangian for the same generating functional (upto a multiplicative constant) , but with a dual vector potential,

$$\mathcal{L}^b = -\frac{1}{4}\left(\partial_\mu b_\nu - \partial_\nu b_\mu + R_{\mu\nu}\right)\left(\partial^\mu b^\nu - \partial^\nu b^\mu + R^{\mu\nu}\right). \quad (3.66)$$

In the first Lagrangian $(\mathcal{L}^A)$ the current (3.61) couples with the gauge field minimally, whereas in the second Lagrangian $(\mathcal{L}^b)$ the current can only be detected from the Bianchi identity. We can thus conclude that currents become topological in this dualization process. We can construct a theory of magnetic particles by dualizing a theory where the monopole current is minimally coupled with the dual vector potential. After dualization we get a Lagrangian like (3.66) with ordinary vector potential and we can minimally couple electric current with this vector potential. The Lagrangian then can be written as

$$\mathcal{L}^m = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + M_{\mu\nu}\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu + M^{\mu\nu}\right) - A^\mu j_\mu \quad (3.67)$$

where the monopole current is

$$j_m^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu M_{\rho\lambda}. \quad (3.68)$$

and j^μ is the electric current. So we can take the magnetic current as topological and electric current as Noether current to construct a theory of magnetic monopoles. We will discuss this in §3.5 and chapter 4 in more detail.

3.4. Dualization in flux tube configurations

In the last three sections we have discussed flux tube configurations in the Abelian Higgs model and dualization techniques for the electromagnetic field using functional integrals. In this section we discuss the dualization of the Abelian Higgs model with flux tube configurations. We start with the functional integral for the Abelian Higgs model in $3 + 1$ dimensions, coupled to an Abelian gauge field A_μ^e. The partition function is given by

$$Z = \int \mathcal{D}A_\mu^e \mathcal{D}\Phi \mathcal{D}\Phi^* \exp iS \,, \tag{3.69}$$

with the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^e F^{e\mu\nu} + |D_\mu\Phi|^2 - \frac{\lambda}{4}(|\Phi|^2 - v^2)^2 \right) , \tag{3.70}$$

where $D_\mu = \partial_\mu + ieA_\mu^e$, and $F_{\mu\nu}^e = \partial_\mu A_\nu^e - \partial_\nu A_\mu^e$ is the Maxwell field strength.

We change variables from Φ, Φ^* to the radial Higgs field ρ and the angular field χ, defined by $\Phi = \frac{1}{\sqrt{2}}\rho \exp(i\chi)$. Then the measure becomes, in these variables,

$$\int \mathcal{D}\Phi \mathcal{D}\Phi^* \cdots = \int \mathcal{D}\rho^2 \mathcal{D}\chi \cdots ., \tag{3.71}$$

where the dots represent the measure for any other fields and the integrand.

Remembering the discussions on the flux tube configuration in the sections (3.1) and (3.2), we have to handle the functional integral over χ carefully, since χ is not defined on the points where

$$\text{Re}\Phi = \text{Im}\Phi = 0. \tag{3.72}$$

As mentioned earlier these two equations define the two-dimensional manifold in space-time and we should integrate over all functions that are regular everywhere

except for these two-dimensional manifolds. These two dimensional singularities are the Abelian Nielsen-Olesen (ANO) string world sheets, since the Higgs field is zero at the center of the ANO string.

Our intention is to construct an effective theory of interacting strings from the Abelian Higgs model. This is possible because as discussed in §2.1, at large distances strongly coupled theory behaves like a string theory. So we can construct a long distance effective theory in which the mass of ρ is very large, so that it is constant everywhere except on the thin flux tubes in the effective theory. In terms of the coupling constant we can say that we are interested in the large $\lambda \to \infty$ regime. So ξ is almost zero and $\rho \simeq v$. For the calculation we will consider the theory with the radial part of the field held fixed, i.e. we will ignore the ρ-dependent part of the measure, and set $\rho = v$ (constant) in the action. The string part will be taken care of by the singularity of the field χ, as we discussed in section (3.2). The integration over the field χ must be handled carefully in the presence of flux tubes. This is because the theory has a topological winding number and we have seen that this winding number is associated with the field χ. The theory has a gauge invariance as well and gauge invariance is also related to the field χ. The gauge invariance comes in as the redefinition of the field χ every time we do a gauge transformation.

The topological winding number arises from the large gauge transformations at large distances where the fields get their vacuum configuration, i.e. the gauge transformation for which χ does not go to zero at large distances, whereas to maintain gauge invariance (small) χ must go to zero at large distances. So at least for large distances χ behaves like sum of two fields. Here we are interested to do large distance effective theory or theory with very strong coupling. As discussed earlier we can decompose the angular field χ into a regular and a singular part[30], $\chi = \chi^r + \chi^s$, where χ^s corresponds to a given magnetic flux tube configuration, and χ^r describes single

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valued fluctuations around this configuration. The singular part of the phase of the Higgs field is related to the world sheet Σ of the magnetic ANO string according to the equation

$$\epsilon^{\mu\nu\rho\lambda}\partial_\rho\partial_\lambda\chi^s = \Sigma^{\mu\nu}, \tag{3.73}$$

$$\Sigma^{\mu\nu} = 2\pi n\int_\Sigma d\sigma^{\mu\nu}(x(\xi))\,\delta^4(x-x(\xi)), \tag{3.74}$$

where $\xi = (\xi^1, \xi^2)$ are the coordinates on the world-sheet of the flux-tube, and $d\sigma^{\mu\nu}(x(\xi)) = \epsilon^{ab}\partial_a x^\mu \partial_b x^\nu$. In the above equation n is the winding number [58]. Then the partition function in the presence of flux tube reads

$$Z = \int \mathcal{D}A_\mu^e \mathcal{D}\chi^s \mathcal{D}\chi^r \exp\left[i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}^e F^{e\mu\nu}+\frac{v^2}{2}(\partial_\mu\chi + eA_\mu^e)^2\right)\right]. \tag{3.75}$$

We will dualize the action using the techniques discussed in the previous chapter. We begin by linearizing the term $\dfrac{v^2}{2}(\partial_\mu\chi + eA_\mu^e)^2$ by introducing an auxiliary field C_μ to get

$$\int \mathcal{D}\chi^r \exp[i\int d^4x \frac{v^2}{2}(\partial_\mu\chi^r + \partial_\mu\chi^s + eA_\mu^e)^2]$$

$$= \int \mathcal{D}\chi^r \mathcal{D}C_\mu \exp\left[-i\int d^4x\left\{\frac{1}{2v^2}C_\mu^2 + C^\mu(\partial_\mu\chi^r + \partial_\mu\chi^s + eA_\mu^e)\right\}\right]$$

$$= \int \mathcal{D}C_\mu \,\delta[\partial_\mu C^\mu]\exp\left[-i\int d^4x\left\{\frac{1}{2v^2}C_\mu^2 + C^\mu(\partial_\mu\chi^s + eA_\mu^e)\right\}\right]. \tag{3.76}$$

As discussed earlier we can resolve the constraint $\partial_\mu C^\mu = 0$ by introducing an antisymmetric tensor field $B_{\mu\nu}$ and writing C_μ in the form $C^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}$. Integrating over the field C_μ, we get

$$Z = \int \mathcal{D}A_\mu^e \mathcal{D}x_\mu(\xi)\mathcal{D}B_{\mu\nu}\exp\left[i\int d^4x\left\{-\frac{1}{4}F_{\mu\nu}^e F^{e\mu\nu}\right.\right.$$

$$\left.\left.+\frac{1}{12v^2}H_{\mu\nu\rho}H^{\mu\nu\rho}-\frac{1}{2}\Sigma_{\mu\nu}B^{\mu\nu}-\frac{e}{2}\epsilon^{\mu\nu\rho\lambda}A_\mu^e\partial_\nu B_{\rho\lambda}\right\}\right]. \tag{3.77}$$

Here we have written $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$, and replaced the integration over $\mathcal{D}\chi^s$ by an integration over $\mathcal{D}x_\mu(\xi)$ which represents a sum over all configurations of the worldsheet of the flux tube. Here $x_\mu(\xi)$ parametrizes the surface on

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which the field χ is singular. The Jacobian for this change of variables gives the action for the string on the background space time [30]. The string has a dynamics given by the Nambu-Goto action, plus higher order operators [59], which can be obtained from the Jacobian. We will not write the Jacobian explicitly in what follows, but of course it is necessary to include it if we want to study the dynamics of the flux tube.

Let us now integrate over the field A_μ^e. To do this we linearize $F_{\mu\nu}^e F^{e\mu\nu}$ by introducing another auxiliary field $\chi_{\mu\nu}$,

$$\int \mathcal{D}A_\mu^e \exp\left[i\int d^4x \left\{-\frac{1}{4}F_{\mu\nu}^e F^{e\mu\nu} - \frac{e}{2}\epsilon^{\mu\nu\rho\lambda}A_\mu^e \partial_\nu B_{\rho\lambda}\right\}\right]$$
$$= \int \mathcal{D}A_\mu^e \mathcal{D}\chi_{\mu\nu} \exp\left[i\int d^4x \left\{-\frac{1}{4}\chi_{\mu\nu}\chi^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\chi_{\mu\nu}\partial_\rho A_\lambda^e - \frac{e}{2}\epsilon^{\mu\nu\rho\lambda}B_{\mu\nu}\partial_\rho A_\lambda^e\right\}\right] \tag{3.78}$$

$$= \int \mathcal{D}\chi_{\mu\nu}\, \delta\left[\epsilon^{\mu\nu\rho\lambda}\partial_\nu(\chi_{\rho\lambda} - eB_{\rho\lambda})\right] \exp[i\int d^4x\{-\frac{1}{4}\chi_{\mu\nu}\chi^{\mu\nu}\}]. \tag{3.79}$$

We can integrate over $\chi_{\mu\nu}$ by introducing a vector field A_μ^m to solve the δ-functional,

$$\chi_{\mu\nu} - eB_{\mu\nu} = \partial_\mu A_\nu^m - \partial_\nu A_\mu^m. \tag{3.80}$$

A_μ^m can be thought of as a 'dual photon' because A_μ^m appears through the dualization of the vector potential A_μ^e as discussed in section §3.3. By looking at the action appearing in the path integral of Eq. (3.78), we see that the fields $\chi_{\mu\nu}$ and $B_{\mu\nu}$ transform like the tensor $*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}$ under parity and time reversal. The parity and time reversal properties of the field A_μ^m can be checked from Eq. (3.80).

$$A_0^m(x) \xrightarrow{P'} -A_0^m(x_P), \qquad A_0^m(x) \xrightarrow{T'} -A_0^m(-x_P)$$
$$A_i^m(x) \xrightarrow{P'} A_i^m(x_P), \qquad A_i^m(x) \xrightarrow{T'} A_i^m(-x_P). \tag{3.81}$$

The result of the integration is then inserted into Eq. (3.77) to give

$$Z = \int \mathcal{D}A_\mu^m \mathcal{D}x_\mu(\xi)\mathcal{D}B_{\mu\nu} \exp\left[i\int \left\{-\frac{1}{4}(eB_{\mu\nu} + \partial_{[\mu}A_{\nu]}^m)^2 + \frac{1}{12v^2}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{2}\Sigma_{\mu\nu}B^{\mu\nu}\right\}\right]. \tag{3.82}$$

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The equations of motion for the fields A_μ^m and $B_{\mu\nu}$ can be calculated from this,

$$\partial_\mu G^{\mu\nu} = 0, \tag{3.83}$$

$$\partial_\lambda H^{\lambda\mu\nu} = -\frac{m^2}{e} G^{\mu\nu} - m^2 \Sigma^{\mu\nu}, \tag{3.84}$$

where $G_{\mu\nu} = eB_{\mu\nu} + \partial_\mu A_\nu^m - \partial_\nu A_\mu^m$, and $m = ev$. We can think of $G_{\mu\nu}$ as the dual field tensor because we get Eq. (3.83) by a variation with respect to the dual gauge field A_μ^m of the action in Eq.(3.82).

Eq. (3.83) shows that there is no magnetic monopole current (dual current) present in the action. This is of course expected, since we found these equations by dualizing the Abelian Higgs model in the presence of flux tubes, but without magnetic monopoles.

Using Eq. (3.83) and (3.84) we get

$$\partial_\nu \Sigma^{\mu\nu} = 0. \tag{3.85}$$

This equation means that the vorticity tensor current $\Sigma_{\mu\nu}$ is conserved. The vector $\partial_\nu \Sigma^{\mu\nu}$ gives the current of the endpoints of the flux string. We will see later that in the presence of magnetic monopoles the right hand side of Eq. (3.83) will have the monopole current. Eq. (3.85) means that due to the conservation of magnetic flux all the flux tubes in the absence of magnetic monopoles are either closed or infinite.

3.5. Attachment of monopoles to the flux tube for confinement

We will attach magnetic monopoles at the ends of a flux tube of finite length. We will take the monopoles to be massless fermions and minimally couple the monopole

current to the magnetic or dual photon. As discussed in chapter 1, the monopole current behaves like an axial current under parity. The time reversal property of an axial current is not like a monopole current and under charge conjugation axial current also does not change its sign. However, following the parity property of magnetic monopole current from Eq. (2.67) we have taken the monopole current as axial. When these magnetic monopoles are couples are coupled to the magnetic photon discussed earlier, the resulting theory will be CPT invariant. After coupling, we will dualize the theory a second time to get back to vector gauge fields, now coupled to flux tubes.

However, a theory containing axial fermionic currents is anomalous and if we try to dualize the theory, the presence of the anomaly gives inconsistent results. We can cancel the anomaly by introducing another species of fermionic monopoles with axial charge opposite to the previous one. Let us denote the two species by q and q', with monopole charges $+g$ and $-g$, respectively. So the monopole current becomes

$$j_m^\mu \;=\; g\bar{q}\gamma_5\gamma^\mu q - g\bar{q}'\gamma_5\gamma^\mu q'. \tag{3.86}$$

The partition function of Eq. (5.80) is modified to include the fermionic monopoles, minimally coupled to the 'magnetic photon' A_μ^m, so the Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}(eB_{\mu\nu} + \partial_{[\mu}A_{\nu]}^m)^2 + \frac{1}{12v^2}H_{\mu\nu\rho}H^{\mu\nu\rho} \;-\; \frac{1}{2}\Sigma_{\mu\nu}B^{\mu\nu}$$
$$+\;\; i\bar{q}\slashed{\partial}q + i\bar{q}'\slashed{\partial}q' - A_\mu^m j_m^\mu. \tag{3.87}$$

The field equation for $G_{\mu\nu}$, Eq. (3.83) is now modefied to

$$\partial_\mu G^{\mu\nu} = j_m^\nu. \tag{3.88}$$

When we take the divergence of Eq. (3.84) and use this result, we find that

$$\frac{1}{e}\partial_\mu\Sigma^{\mu\nu}(x) + j_m^\nu(x) = 0. \tag{3.89}$$

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The above Eq. (3.89) is showing that the endpoint current of the flux tube is cancelled by monopole currents at every space-time point. So we can say that point particle monopoles are attached at the end of the flux tube. This equation can also be derived as a consequence of gauge invariance, like current conservation in electromagnetism. To see this, we take a transformation

$$
\begin{aligned}
B_{\mu\nu} &\rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu , \\
A_\mu^m &\rightarrow A_\mu^m - \frac{k}{g} \Lambda_\mu .
\end{aligned}
\tag{3.90}
$$

The second term of the Lagrangian of Eq. (3.87) is invariant under the above transformation, while the first term is made invariant by setting $eg = k$. This is related to the Dirac quantization condition as we shall see shortly.

Since the flux due to the monopoles is fully confined in the tube, the flux inside the tube must match with the flux of a monopole with magnetic charge g. The flux inside a flux tube is known from the Eq. (3.17).

Flux inside the flux tube $=$ Flux of a monopole with magnetic charge g

$$
\frac{2\pi n}{e} = 4\pi g
\tag{3.91}
$$

$$
eg = \frac{n}{2}.
\tag{3.92}
$$

So $k = \dfrac{n}{2}$, and we have the Dirac quantization condition $eg = \dfrac{n}{2}$. We can now

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write the partition function as

$$
\begin{aligned}
Z[\Lambda_\mu] &= \int \mathcal{D}A_\mu^m \mathcal{D}x_\mu(\xi)\mathcal{D}B_{\mu\nu}\mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}q'\mathcal{D}\bar{q}' \, \exp i \int d^4x \left[-\frac{1}{4}(eB_{\mu\nu} + \partial_{[\mu}A_{\nu]}^m)^2 \right. \\
&\left. +\frac{1}{12v^2}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{2}\Sigma_{\mu\nu}B^{\mu\nu} - \Sigma^{\mu\nu}\partial_\mu\Lambda_\nu + e\Lambda_\mu j_m^\mu + i\bar{q}\slashed{\partial}q + i\bar{q}'\slashed{\partial}q' - A_\mu^m j_m^\mu \right].
\end{aligned}
$$

(3.93)

Since (3.90) is only a change of variables, Z cannot depend on Λ_μ. Thus Λ_μ can be integrated out with no effect other than the introduction of an irrelevant constant factor in Z, which we ignore. After integrating over Λ_μ, we get

$$
\begin{aligned}
Z = \int \mathcal{D}A_\mu^m \cdots \delta\Big[\frac{1}{e}\partial_\mu\Sigma^{\mu\nu} + j_m^\nu\Big] \quad \exp i \int d^4x \left[-\frac{1}{4}(eB_{\mu\nu} + \partial_{[\mu}A_{\nu]}^m)^2 + \frac{1}{12v^2}H_{\mu\nu\rho}H^{\mu\nu\rho} \right. \\
\left. -\frac{1}{2}\Sigma_{\mu\nu}B^{\mu\nu} + i\bar{q}\slashed{\partial}q + i\bar{q}'\slashed{\partial}q' - A_\mu^m j_m^\mu \right], \quad (3.94)
\end{aligned}
$$

where the dots represent the measures for the other fields and x^μ. One can see from the δ-functional that the vorticity current tensor is not conserved, but is cancelled by the current of the added fermions. So the strings are open strings with fermions stuck at the ends. Now we dualize the theory a second time and get back to a vector gauge field which is something like a Maxwell field, but in the presence of monopoles and flux tubes. Introducing an auxiliary field $\chi_{\mu\nu}$ to linearize the first term of the Lagrangian, we get

$$
\begin{aligned}
Z &= \int \mathcal{D}A_\mu^m \cdots \mathcal{D}\chi_{\mu\nu}\, \delta\Big[\frac{1}{e}\partial_\mu\Sigma^{\mu\nu} + j_m^\nu\Big] \exp i \int d^4x \Big[-\frac{1}{4}\chi_{\mu\nu}\chi^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\chi_{\mu\nu}\partial_\rho A_\lambda^m \\
&+\frac{1}{4}\epsilon^{\mu\nu\rho\lambda}\chi_{\mu\nu}B_{\rho\lambda} + \frac{1}{12v^2}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{2}\Sigma_{\mu\nu}B^{\mu\nu} + i\bar{q}\slashed{\partial}q + i\bar{q}'\slashed{\partial}q' - A_\mu^m j_m^\mu \Big].
\end{aligned}
$$

(3.95)

We can now integrate out A_m^μ, and the result is

$$
\begin{aligned}
Z &= \int \mathcal{D}\chi_{\mu\nu} \cdots \delta\Big[\frac{1}{e}\partial_\mu\Sigma^{\mu\nu} + j_m^\nu\Big] \delta\Big[\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu\chi_{\rho\lambda} - j_m^\mu\Big] \exp i \int d^4x \Big[-\frac{1}{4}\chi_{\mu\nu}\chi^{\mu\nu} \\
&+\frac{e}{4}\epsilon^{\mu\nu\rho\lambda}\chi_{\mu\nu}B_{\rho\lambda} + \frac{1}{12v^2}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{2}\Sigma_{\mu\nu}B^{\mu\nu} + i\bar{q}\slashed{\partial}q + i\bar{q}'\slashed{\partial}q' \Big].
\end{aligned}
$$

(3.96)

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Both the δ-functionals must be satisfied, which requires $\frac{1}{e}\partial_\nu\Sigma^{\mu\nu} - \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu\chi_{\rho\lambda} = 0$. This can be solved by introducing a gauge field A_μ, which allows the integration over $\chi_{\mu\nu}$. Then the partition function becomes

$$
\begin{aligned}
Z &= \int \mathcal{D}x_\mu(\xi)\mathcal{D}B_{\mu\nu}\mathcal{D}A_\mu \cdots \delta\Big[\frac{1}{e}\partial_\mu\Sigma^{\mu\nu} + j_m^\nu\Big] \exp i\int d^4x\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&\quad + \frac{1}{12v^2}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{2g}\epsilon^{\mu\nu\rho\lambda}B_{\mu\nu}\partial_\rho A_\lambda + i\bar{q}\partial\!\!\!/q + i\bar{q}'\partial\!\!\!/q'\Big]\,.
\end{aligned} \tag{3.97}
$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{2e}\epsilon_{\mu\nu\sigma\lambda}\Sigma^{\sigma\lambda}$, the dots represent the fermion measure and we continue to suppress the action for the flux tube itself, as in Eq. (3.77).

The vector potential A_μ has the same parity and time reversal properties as the usual gauge potential of electromagnetism. The theory is now in the form we originally intended, and contains thin tubes of flux. The new feature is that the ends of the flux tube are *sealed* by fermions, so that no flux escapes, all flux is confined. We should not think of this as any more than a toy model of confinement, because the underlying theory is only the Abelian Higgs model and not quantum chromodynamics. Even then, some features are interesting enough to be highlighted.

There is a simple argument to calculate the length of the string. The flux confined inside the string is $4\pi g$, a constant. The radius of the string core is of the order of $1/v\sqrt{\lambda}$. From this we can calculate the energy per unit length of the tube to be $\mu \sim g^2v^2\lambda$, also a constant. Such a string, of finite length, would collapse in order to minimize the energy unless it was stabilized by its angular momentum. For a rotating string of length l, energy per unit length μ, angular momentum J, the energy function is $E = \mu l + J^2/2\mu l^3$. This has a minimum for the length $L \sim \sqrt{J/\mu}$. We see that for the stable flux tube with magnetic monopoles at the ends,

$$
\frac{J}{E^2} = constant\,, \tag{3.98}
$$

similar to the well-known Regge trajectory for mesons.

3. Flux tubes and Confinement of monopoles in Abelian theory

The gauge field A_μ is massive, with mass $m = v/g$ [62, 63]. It does not couple directly to the fermionic monopoles at the ends. Those fermions are coupled only through the δ-functional in Eq. (3.97), which guarantees that the monopoles must seal the ends of the string. However, any other gauge field, Abelian or not, axial or not, may be coupled to these fermions with charge assignments independent of their charges under A_μ^m, which has been integrated out of the theory. In particular, if we suggestively rename q and q' to u and $\bar{d}$, the allowed configurations are $u\bar{d}$, $\bar{u}d$, and $u\bar{u} \pm d\bar{d}$, which can couple to electroweak gauge fields. Note also that we could have introduced three species of fermions (with charges $1, 1, -2$, for example) in Eq. (3.86), for the purpose of anomaly cancellation, and we would again get flux tubes with ends sealed by fermions. But a single species of fermions would not produce such configurations.

In order to make the model more realistic, one would need to check if flux is truly confined in the tube or if it escapes when the tube has a finite thickness. A similar picture starting with an axial gauge field and ending with a tube of 'electric' rather than magnetic flux will be interesting as well. Further, the freedom to have other global symmetries in the theory allows in principle that the $U(1)$ producing the string here may be embedded in an $SU(N)_{global} \times U(1)_{local}$ symmetry, as in [60].

4. Monopoles and flux tubes in broken SU(2)

In this chapter we discuss some aspects of magnetic monopoles and flux tubes in SU(2) gauge theory. Under a gauge transformation, an adjoint SU(2) field transforms $\phi \to U\phi U^\dagger$, a field in the fundamental representation transforms as $\psi \to U\psi$ and the gauge field transforms as $A_\mu \to UA_\mu U^\dagger - i\partial_\mu UU^\dagger$, where $U = e^{-i\alpha(x)^i \tau^i}$ are space-time dependent SU(2) matrices. We start with a discussion about some geometrical aspects of spontaneous symmetry breaking.

4.1. Spontaneous symmetry breaking and Higgs vacuum

We start with the Lagrangian

$$L = -\frac{1}{2}\mathrm{Tr}\,[G_{\mu\nu}G^{\mu\nu}] + \mathrm{Tr}\,[\mathcal{D}_\mu\phi\mathcal{D}^\mu\phi] - \frac{\lambda}{4}\left(|\phi|^2 - \xi_1^2\right)^2, \tag{4.1}$$

where

$$\phi = \phi_i\tau_i \quad , \quad \mathcal{D}_\mu\phi = \partial_\mu\phi - ig[A_\mu, \phi] \tag{4.2}$$

$$A_\mu = A_\mu^i\tau^i \quad , \quad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{4.3}$$

4. Monopoles and flux tubes in broken SU(2)

The ϕ^i are a triplet of scalar fields and A_μ is the gauge field. ϕ transforms in the adjoint representation of SU(2). The Hamiltonian density corresponding to the Lagrangian is

$$\theta_{00} = \frac{1}{2}\mathcal{E}_i^{\,2} + \frac{1}{2}\mathcal{B}_i^{\,2} + \frac{1}{2}(\mathcal{D}_0\phi)^2 + \frac{1}{2}(\mathcal{D}_i\phi)^2 + \frac{\lambda}{4}\left(|\phi|^2 - \xi_1^2\right)^2, \tag{4.4}$$

where

$$G^{i0} = \mathcal{E}^i, \; G^{ij} = -\epsilon^{ijk}\mathcal{B}^k. \tag{4.5}$$

The energy is at minimum for $\theta_{00} = 0$, i.e. vanishes if and only if

$$G^a_{\mu\nu} = 0, \tag{4.6}$$

$$\mathcal{D}_\mu\phi = 0, \tag{4.7}$$

$$V(\phi) = 0 \Rightarrow |\phi|^2 = \xi_1^2. \tag{4.8}$$

These three equations define the vacuum and Eq. (4.8) gives the classical value of $|\phi|^2$ at the vacuum. So the vacuum expectation value (vev) of the field ϕ is non zero. At the vacuum, ϕ lies on the surface of a sphere of radius ξ_1 in the Lie algebra. For the moment let us fix the direction of ϕ in the third direction of the Lie algebra, i.e. we choose the vacuum field configuration to be $\phi_0 = \xi_1\tau^3$. If we consider fluctuations around the vacuum and also for the moment consider fluctuations only along the third direction, we can write the field as

$$\phi_0 = (\xi_1 + \rho)\tau^3. \tag{4.9}$$

The Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}G_i^{\mu\nu}G^i_{\mu\nu} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{g^2}{2}(\xi_1 + \rho)^2(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu})$$
$$-\frac{\lambda}{4}(4\xi_1^2 + \rho^4 + 4\xi_1\rho^3). \tag{4.10}$$

56

From the above Lagrangian we can see that the fields ρ, A_μ^1 and A_μ^2 have become massive. The masses are $\sqrt{\lambda}\xi_1$ and $g\xi_1$. We can take ξ_1 and λ to be very large compared to the mass scale that we are interested in which would be the momentum scale of an external particle for example. Then we can consider this scale Λ to be a vacuum for the fields ρ, A_μ^1 and A_μ^2 but in this vacuum A_μ^3 can excite particles. So at the scale Λ, an external particle will only see the interaction with A_μ^3. At this scale the Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3)^2. \tag{4.11}$$

We call this vacuum the 'Higgs vacuum'. Formally we can define the 'Higgs vacuum' by the solutions of the equations,

$$V(\phi) = 0, \qquad \mathcal{D}_\mu\phi = 0. \tag{4.12}$$

In terms of the position representation we can say that the field configurations in a certain region of space-time are in the Higgs vacuum if equations (4.7) and (4.8), but not necessarily Eq. (4.6), are satisfied at the region. In terms of scattering we can say the external particles will experience 'Higgs vacuum' upto a length scale $\sim \frac{1}{\Lambda}$.

Let us discuss some features of the Higgs vacuum. Let us define the vacuum manifold as [47]

$$\mathcal{M}_0 = \{\phi : V(\phi) = 0\}. \tag{4.13}$$

Gauge invariance of a Lagrangian requires that $V(\phi)$ is also invariant under the action of the group G. It follows that if ϕ satisfies Eq.(4.8) then so does $\mathcal{D}(g)\phi$ for all $g \in G$, where $\mathcal{D}(g)$ is a norm preserving representation of the group G under which ϕ transforms. Thus we can write

$$V(\phi) = V(\mathcal{D}(g)\phi) = 0. \tag{4.14}$$

So G acts on $\mathcal{M}_0$, i.e. every g takes any point on $\mathcal{M}_0$ to another point on $\mathcal{M}_0$. This means that if ϕ_0 is a point on $\mathcal{M}_0$ then we can go to another point simply by acting with a group element g on ϕ_0, i.e.

$$\phi_0' = \mathcal{D}(g)\phi_0. \tag{4.15}$$

However, that does not mean that G can span all $\mathcal{M}_0$ by acting on ϕ.

Two points ϕ_1, ϕ_2 which can be related by an element $g \in G$,

$$\phi_1 = \mathcal{D}(g)\phi_2, \tag{4.16}$$

are said to be on the same orbit. So the orbit of ϕ_0 is given by Eq. (4.15). If $\mathcal{D}(g)\phi_0$ spans all the points on $\mathcal{M}_0$ as g varies over G, we say that $\mathcal{M}_0$ consists of a single orbit of the gauge group G. Another way of saying it is to say that G acts transitively on $\mathcal{M}_0$. That means for every ϕ_1, ϕ_2 belonging to $\mathcal{M}_0$, there is some $g_{12} \in G$ such that

$$\phi_2 = \mathcal{D}(g_{12})\phi_1. \tag{4.17}$$

Starting from a fixed element in $\mathcal{M}_0$, say ϕ_0, we can always associate a group element g_{i0} to another element in $\mathcal{M}_0$, say ϕ_i, by using the relation

$$\phi_i = \mathcal{D}(g_{i0})\phi_0. \tag{4.18}$$

If g_{i0} is unique, ϕ_i is associated with only one group element with respect to ϕ_0, then we can say $\mathcal{M}_0$ is isomorphic to G as a manifold. If g_{i0} is not unique then we have a subgroup other than the identity which leaves ϕ invariant. Suppose G acts transitively on $\mathcal{M}_0$. Let $\phi \in \mathcal{M}_0$ and let $H_\phi \subset G$ be the subgroup leaving ϕ fixed,

$$H_\phi = \{h_\phi \in G \,|\, h_\phi \phi = \phi\}. \tag{4.19}$$

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H_ϕ is called the stability, or isotropy, or little group of ϕ. Here we have written h_ϕ instead of $\mathcal{D}(h_\phi)$ and we shall use this convention from now on for any representation of the group.

If G acts transitively on $\mathcal{M}_0$ then any $\phi_1, \phi_2 \in \mathcal{M}_0$ are related by some $g_{21} \in G$,

$$\phi_2 = g_{21}\phi_1. \tag{4.20}$$

On the other hand, if $h_{\phi_1} \in H_{\phi_1}$ and $h_{\phi_2} \in H_{\phi_2}$ are elements of the little groups of ϕ_1 and ϕ_2 respectively,

$$h_{\phi_1}\phi_1 = \phi_1, \; h_{\phi_2}\phi_2 = \phi_2. \tag{4.21}$$

Using Eq. (4.20) we can write

$$h_{\phi_1}\phi_1 = g_{21}^{-1}\phi_2, \tag{4.22}$$

$$g_{21}h_{\phi_1}g_{21}^{-1}g_{21}\phi_1 = h_{\phi_2}\phi_2, \tag{4.23}$$

$$g_{21}h_{\phi_1}g_{21}^{-1}\phi_2 = h_{\phi_2}\phi_2. \tag{4.24}$$

Since Eq. (4.24) is true for any h_{ϕ_1} and h_{ϕ_2},

$$g_{21}H_{\phi_1}g_{21}^{-1} = H_{\phi_2}. \tag{4.25}$$

Thus H_ϕ varies within G by conjugation and consequently different H_ϕ are isomorphic. Using Eq.s (4.20) and (4.21) we can write

$$\phi_2 = g_{21}h_{\phi_1}\phi_1, \tag{4.26}$$

$$\phi_2 = h_{\phi_2}\phi_2 = h_{\phi_2}g_{21}h_{\phi_1}\phi_1. \tag{4.27}$$

So it is not only the element g_{21} which takes ϕ_1 to ϕ_2, but all elements of the set $H_{\phi_2}g_{21}H_{\phi_1}$ also do this, and using Eq. (4.25) we can write the elements as

$$H_{\phi_2}g_{21}H_{\phi_1} = g_{21}H_{\phi_1}g_{21}^{-1}g_{21}H_{\phi_1} \tag{4.28}$$

$$= g_{21}H_{\phi_1} \in G/H_{\phi_1} \tag{4.29}$$

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So it is clear from the above equations that the group elements that take ϕ_1 to ϕ_2 are elements of the left coset space with respect to the little group of ϕ_1. Since all little groups are isomorphic in this case, we can write down the coset spaces as G/H.

We have defined the vacuum manifold $\mathcal{M}_0$ by Eq. (4.13). However, the Higgs vacuum was defined by the two equations (4.7) and (4.8), and it is not necessary that if ϕ is a solution of the equation $V(\phi) = 0$, it will always be a solution of the equation $\mathcal{D}_\mu \phi = 0$. We know that all solutions of $V(\phi) = 0$ must lie on the vacuum manifold $\mathcal{M}_0$. If ϕ_0 is a solution of $V(\phi) = 0$ then all the ϕ's that are related to ϕ_0 by a gauge transformation by the gauge group G must lie on $\mathcal{M}_0$ because $V(\phi) = V(\mathcal{D}(g)\phi)$. However, it is not necessary that all the points on $\mathcal{M}_0$ must be related by a gauge transformations of the group G. Then what we have is a non-transitive action on $\mathcal{M}_0$ by the group G. In that case the vacuum manifold can be divided into several orbits, each of which consists of points that are related by gauge transformations among themselves.

Each orbit is a homogeneous space of the group G. So each orbit must be isomorphic to G or a subspace of G. As we have discussed earlier that in this case there is a subgroup H of G that makes ϕ invariant and the orbit is isomorphic to the coset space G/H. If $\mathcal{M}_0$ is isomorphic to G then $H = e$, the identity element.

When the system is at the vacuum, i.e. $V(\phi) = 0$, we can write locally $G \simeq H \times \mathcal{M}_0$. However, this may not always be the global structure of G. We can say that vacuum manifold breaks the global structure of the symmetry group G. For example, for SU(2) adjoint scalars we can write $V(\phi) = (\phi^i \phi^i - 1)^2 = 0$. So the vacuum manifold $\mathcal{M}_0 \sim S^2$. Here we can write the group manifold S^3 as $S^2 \times S^1$ but this is not possible globally. The fundamental representation of SU(3), at the vacuum $V(\psi) = (\psi^\dagger \psi - 1)^2 = 0$, breaks the group manifold to $S^3 \times S^5$ which cannot be written globally. So we can say that the equation $V(\phi) = 0$ breaks the gauge

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group G to some local product form.

If $\mathcal{M}_0$ is not a single orbit of ϕ, then we have to choose any one of the orbits of ϕ where we can fix our vacuum. For example if we choose the i-th orbit, say $\mathcal{O}^i$, in which ϕ lies at the vacuum, then we can write the group manifold $G \simeq H^i \times \mathcal{O}^i$. Here H^i is the isotropy group of $\phi^i \in O^i$ and the coset space is O^i. There may be some bigger symmetry group $\tilde{G}$ that makes $V(\phi)$ invariant and acts transitively on $\mathcal{M}_0$. The gauge group G is a subgroup of the group $\tilde{G}$ ($G \subset \tilde{G}$). So the total symmetry space can be written as a local product form $\mathcal{O}^i \times H^i \times \tilde{G}/G$. If $\tilde{G}/G$ is a group then this group will be the remaining global symmetry of the theory.

Let us illustrate this with an example. In a SU(2) gauge theory we can write SU(2) fundamental representation as,

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{4.30}$$

We consider the potential

$$V(\Psi) = \frac{\lambda}{4}(|\Psi|^2 - v^2)^2 \tag{4.31}$$

$$= \frac{\lambda}{4}\left((\Re\psi_1)^2 + (\Im\psi_2)^2 + (\Re\psi_2)^2 + (\Im\psi_2)^2 - v^2\right)^2, \tag{4.32}$$

$V(\Psi)$ has a global $\tilde{G} = \mathrm{SO}(4)$ invariance since we can think of $(\Re\psi_1, \Im\psi_2, \Re\psi_2, \Im\psi_2)$ as a four vector. On the other hand

$$(\Re\psi_1)^2 + (\Im\psi_2)^2 + (\Re\psi_2)^2 + (\Im\psi_2)^2 - v^2 = 0 \tag{4.33}$$

is the equation of a 3-sphere, so the vacuum manifold $\mathcal{M}_0 \sim S^3$. The group $\mathrm{SO}(4)$ can be written in a local product form $SU_l(2) \times SU_g(2)$. We can identify the gauge group $G = SU_l(2)$ as the local gauge group. Then the quotient group $SU_g(2) \sim SO(4)/SU_l(2)$ is the global symmetry which survives after the full $SU_l(2)$ symmetry group is broken [8].

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Let us discuss the significance of the equation $\mathcal{D}_\mu \phi = 0$ in symmetry breaking. We can write $\phi = |\phi|\hat{\phi}$, where $\hat{\phi}$ is a unit vector in some representation of the gauge group G. The equation $V(\phi) = 0$ generally gives $|\phi| = \xi_1$ where ξ_1 is some predefined constant. $\mathcal{D}_\mu \phi$ can be written as

$$\mathcal{D}_\mu \phi = \partial_\mu |\phi|\hat{\phi} + |\phi|\mathcal{D}_\mu \hat{\phi}. \tag{4.34}$$

The first term on the right hand side of Eq. (4.34) is automatically zero on the vacuum manifold $\mathcal{M}_0$ because $|\phi|$ is a constant, but the second term transforms like ϕ under a gauge transformation. So it is not necessarily zero on the vacuum manifold. However, if it is zero for a $\hat{\phi}$ in the vacuum then it will be zero for the whole orbit. $\mathcal{D}_\mu \hat{\phi}$ can be made zero by fixing the gauge at the vacuum. For example, in Abelian Higgs model $\hat{\phi}$ can be written as $e^{-i\chi}$. $\mathcal{D}_\mu e^{-i\chi}$ will be zero at vacuum if we take $A_\mu = -\frac{1}{e}\partial_\mu \chi$. The gauge fixing may be 'partial' because there may be some subgroup H of G under which Eq. (4.34) remains invariant. For example, in SU(2) gauge theory with ϕ as an adjoint scalar filed, $\mathcal{D}_\mu \hat{\phi}$ is invariant under the action of the group elements $e^{-i\chi\hat{\phi}}$ and here $H = U(1)$. So in this process it is not possible to fix the gauge at the vacuum for the components of the gauge field that lies in L_H, the algebra of H. Under this gauge fixing procedure the components of the Yang-Mills tensor $G_{\mu\nu}$ that lies in the subspace $L_G - L_H$ becomes zero. If we look at the fluctuations of the gauge fields along the subspace $L_G - L_H$ around $\mathcal{M}_0$, we find that they have mass $g\xi_1$, where g is the coupling constant. However, the components of the Yang-Mills tensor which belong to L_H are non-zero on the vacuum manifold, so $\mathcal{M}_0$ is not a vacuum for the gauge fields that belong to L_H. The Lagrangian only has the gauge fields that belong to L_H on $\mathcal{M}_0$, as in Eq. (4.11) and we say that symmetry is broken down from G to H.

4.2. Monopoles in SU(2) scalar gauge theory

In the last section we have seen that at the 'Higgs vacuum' the gauge group G can be written as $G \sim G/H \times H$, where H is the stabilizer (isotropy or little group) and the coset space G/H is the vacuum manifold $\mathcal{M}_0$. The fields $|\phi|$ and $A_\mu \in L_G - L_H$ are massive around this vacuum. The L_H valued components of the gauge field A_μ are non-zero in this vacuum, so $\mathcal{M}_0$ is not the vacuum for these components of A_μ. It is also possible to have non-zero components of $A_\mu \in L_G - L_H$ in the vacuum but it is completely determined by the elements of G/H and the components of Yang-Mills tensor $F_{\mu\nu} \in L_G - L_H$ are always zero at the vacuum manifold $\mathcal{M}_0$. This is like the case of $|\phi|$, whose vacuum expectation value is non-zero and there is no dynamics of $|\phi|$ at the vacuum $\mathcal{M}_0$. These vacuum configurations of the $L_G - L_H$ valued components of A_μ often correspond to some configurations which are not particles but stable against decay to the " trivial solution." They are truly distinct, and maintain their integrity, even in the face of extremely powerful forces. These configurations are called solitons in gauge theory.

Suppose we consider SU(2) gauge theory coupled to an adjoint scalar in 3+1 dimensions. In this case the solitons are the magnetic monopoles. All these soliton solutions can be constructed by giving a large gauge transformation to the gauge field A_μ whose $L_G - L_H$ valued components are zero in the vacuum at large distances. This is equivalent to saying that we have to write down A_μ for a constant ϕ on $\mathcal{M}_0$ and then give a large gauge transformation in the vacuum at large distances.

We can give an example of this kind of large gauge transformation in the construction of quantized magnetic flux lines of Abelian Higgs model which we have discussed in chapter 2. In this case there is a 'kernel' [27] in the form of a tube outside which all physical fields decrease exponentially to their vacuum configurations.

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The vector potential A_μ becomes pure gauge outside the kernel . The flux through the tube is an integer times a constant. The integer is called the winding number of the configuration. The winding numbers for flux tubes are just the Π_1 (fundamental group) of the vacuum manifold, which is S^1, as we have seen in chapter 2. If we take the Higgs field to be a constant then the vector potential becomes zero outside the kernel and there is no flux tube solution, or we could say that there are flux tubes with zero winding number. To get flux tubes we have to make a large gauge transformation to make the gauge field to be a pure gauge solution outside the kernel or in the vacuum at large distances. Magnetic monopole solutions in SU(2) gauge theory with a adjoint scalar can be found using the same technique. The Lagrangian was described at Eq. (4.1). From the structure of the potential it is clear that the scalar field gets a non-zero vacuum expectation value,

$$\langle \phi^i \phi^i \rangle = \xi_1^2.$$

So the symmetry is broken spontaneously. The vector fields acquire a mass $g\xi_1$ and the Higgs has a mass $\sqrt{\lambda}\xi_1$ near the Higgs vacuum. The Higgs vacuum is defined as

$$|\phi|^2 = \xi_1^2, \tag{4.35}$$

$$D_\mu \phi = \partial_\mu \phi - ig[A_\mu, \phi] = 0. \tag{4.36}$$

From Eq.(4.35) we can see that the vacuum manifold $\mathcal{M}_0$ is S^2 and the little group H corresponds to rotation around a point on S^2. So $H = U(1)$ and we can write $\mathcal{M}_0 = SU(2)/U(1) \sim S^2$. The gauge group $SU(2)$ is transitive on the vacuum manifold S^2 and this transitivity makes the theory independent of the direction of ϕ_1 at the vacuum as discussed in §4.1.

It is possible to write a general solution of Eq.(4.36). Using the discussion in §A.3 Eq. (4.36) can be written as

$$\partial_\mu \vec{\phi} + g\vec{A}_\mu \times \vec{\phi} = 0. \tag{4.37}$$

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Here we have written the gauge field as

$$A_\mu = \vec{A}_\mu \cdot \vec{\tau} \tag{4.38}$$

and $\vec{\tau}$ are the Pauli matrices multiplied by half. Taking the cross product of $\vec{\phi}$ with the above equation we get

$$\vec{\phi} \times \partial_\mu \vec{\phi} + g\vec{\phi} \times \vec{A}_\mu \times \vec{\phi} = 0 \tag{4.39}$$

$$\text{i.e.,} \quad \vec{\phi} \times \partial_\mu \vec{\phi} + g|\phi|^2 \vec{A}_\mu - g\vec{\phi}\vec{A}_\mu \cdot \vec{\phi} = 0. \tag{4.40}$$

At the vacuum $|\phi|^2 = \xi_1^2$, so we can write Eq. (4.40) as

$$\vec{A}_\mu = \left(\vec{A}_\mu \cdot \hat{\phi}\right)\hat{\phi} - \frac{1}{g}\hat{\phi} \times \partial_\mu \hat{\phi}, \tag{4.41}$$

where

$$\hat{\phi} = \frac{\vec{\phi}}{\xi_1}. \tag{4.42}$$

In terms of matrices, we can write

$$A_\mu = \frac{1}{\xi_1^2} 2\text{Tr}\,(A_\mu \phi)\phi + \frac{i}{g\xi_1^2}[\phi, \partial_\mu \phi]. \tag{4.43}$$

This is the configuration of the gauge field in the vacuum manifold. The first term on the right hand side of Eq. (4.43) is the component that lies along ϕ. It is the massless part of the gauge field and it is expected to be non-zero on $\mathcal{M}_0$. The other components are zero on $\mathcal{M}_0$, except the part that is fully describable by the vacuum configurations of the field ϕ. The value of the Higgs field is known at the Higgs vacuum, so the second part of the right side of the equation is fully known at the Higgs vacuum. We say that the SU(2) symmetry is spontaneously broken to U(1). We can also say that at the Higgs vacuum the gauge symmetry is partially fixed up to U(1). The second term on the right hand side of Eq. (4.43) is responsible for

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monopole solutions. To fix the gauge at the vacuum, we fix the field ϕ along the radial direction of the S^2. So at the vacuum we can write

$$\phi^i = \xi_1 \frac{r^i}{r}. \tag{4.44}$$

Using this Eq. (4.44) we can write the second term of the right hand side of the equation (4.43) as

$$A^i_\mu(\text{only the second term}) = -\frac{1}{g\xi_1^2}\epsilon^{ijk}\phi^j\partial_\mu\phi^k, \tag{4.45}$$

$$= -\frac{1}{g}\epsilon_{\mu ij}\frac{r^j}{r^2}. \tag{4.46}$$

Now if we define

$$F_{\mu\nu} = 2\text{Tr}\,[\hat{\phi}G_{\mu\nu}] = \frac{i}{g\xi_1^3}2\text{Tr}\,(\phi[\partial_\mu\phi,\partial_\nu\phi]) \tag{4.47}$$

$$= -\frac{1}{g}\epsilon_{\mu\nu i}\frac{r^i}{r^3}, \tag{4.48}$$

we can write the magnetic field as,

$$\vec{B} = Q_m\frac{\hat{r}}{r^2}. \tag{4.49}$$

Here $Q_m = \dfrac{1}{g}$ and the arrow indicates a vector in the usual three dimensional space.

The flux for the above field is $\dfrac{4\pi}{g}$ and a quantization condition for $n = 1$ can be written $Q_mg = 1$. However, this radial gauge fixing procedure gives a singular magnetic field at the position of the monopole. This singularity can be regularized by choosing a smooth configuration of the Higgs field $|\phi|$ such that $|\phi| = 0$ at the position of the monopole. On the other hand, in the vacuum manifold $\mathcal{M}_0$, we know that $|\phi| = \xi_1$. So, some regions of space-time cannot be at the vacuum configuration of the Higgs field. These regions were called the "kernel" by 't Hooft [27]. The existence of the kernel gives a finite size to a magnetic monopole, because $|\phi|$ takes

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some distance to reach its vacuum value from zero. Here we always assume that a kernel always exists if we have a Higgs field for non zero winding number.

In the Abelian Higgs model one can construct the vacuum by choosing the Higgs field as constant and also choosing the gauge field to be zero. A flux tube can be constructed by a large gauge transformation at the vacuum (outside the kernel). In other words, if we take the Higgs field to be constant, we get the flux tube solution with zero winding number. Non-zero winding number solutions can be found by applying appropriate large gauge transformation to the vacuum solutions. The story is the same for monopoles as well. We shall see that a monopole solution can be constructed by giving a large gauge transformation to the gauge field solution for a constant value of the Higgs field. The construction is as follows. Let us fix ϕ over the vacuum manifold defined by Eqs. [4.35, 4.36] by setting

$$\phi = \xi_1 \tau^3. \tag{4.50}$$

It follows that

$$\partial_\mu \phi = 0. \tag{4.51}$$

Then (4.43) gives

$$A_\mu = A_\mu^3 \tau^3, \tag{4.52}$$

$$G_{\mu\nu} = [\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3]\tau^3. \tag{4.53}$$

The above A_μ and $G_{\mu\nu}$ do not have any monopole solutions. So these are zero winding number solutions. Now we can make a gauge transformation by $U(x) \in G$ to get a general gauge field on this vacuum solution.

$$A_\mu = A_\mu^3 U \tau^3 U^\dagger - \frac{i}{g}\partial_\mu U U^\dagger. \tag{4.54}$$

If this U is a large gauge transformation, we will have a solution with a non-zero winding number. For this solution we have

$$A_\mu^3 = 2\text{Tr}\,(A_\mu U\tau^3 U^\dagger) + \frac{i}{g}2\text{Tr}\,[\partial_\mu UU^\dagger(U\tau^3 U^\dagger)], \qquad (4.55)$$

so that using Eq.(4.55) we can rewrite Eq.(4.54) as

$$
\begin{aligned}
A_\mu = \; & 2\text{Tr}\,(A_\mu U\tau^3 U^\dagger)U\tau^3 U^\dagger \\
& -\frac{i}{g}[\partial_\mu UU^\dagger - 2\text{Tr}\,[\partial_\mu UU^\dagger(U\tau^3 U^\dagger)]U\tau^3 U^\dagger].
\end{aligned}
\qquad (4.56)
$$

If we define

$$\hat\phi = U\tau^3 U^\dagger, \qquad (4.57)$$

then Eq. (4.56) can be written as

$$A_\mu = 2\text{Tr}\,(A_\mu\hat\phi)\hat\phi - \frac{i}{g}\left[\partial_\mu UU^\dagger - 2\text{Tr}\,(\partial_\mu UU^\dagger\hat\phi)\hat\phi\right] \qquad (4.58)$$

which in turn can be written as

$$A_\mu = 2\text{Tr}\,(A_\mu\hat\phi)\hat\phi + \frac{i}{g}[\hat\phi, \partial_\mu\hat\phi]. \qquad (4.59)$$

Eq. (4.59) is nothing but Eq. (4.43).

So we see that the second term of the right hand side of Eq. (4.58) is responsible for the monopole solution. Let us define

$$2\text{Tr}\,[\hat\phi A_\mu] = B_\mu\,, \qquad (4.60)$$

then Eq.(4.41) becomes,

$$A_\mu = B_\mu\hat\phi - \frac{1}{g}\hat\phi \times \partial_\mu\hat\phi. \qquad (4.61)$$

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This is the field at the vacuum, and using this A_μ we can calculate $G_{\mu\nu}$

$$
\begin{aligned}
\vec{G}_{\mu\nu} &= \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \vec{A}_\mu \times \vec{A}_\nu && (4.62) \\
&= \partial_\mu \left(B_\nu \hat{\phi} - \frac{1}{g} \hat{\phi} \times \partial_\nu \hat{\phi} \right) - \partial_\nu \left(B_\nu \hat{\phi} - \frac{1}{g} \hat{\phi} \times \partial_\nu \hat{\phi} \right) \\
&\quad + g \left(B_\mu \hat{\phi} - \frac{1}{g} \hat{\phi} \times \partial_\mu \hat{\phi} \right) \times \left(B_\nu \hat{\phi} - \frac{1}{g} \hat{\phi} \times \partial_\nu \hat{\phi} \right) && (4.63) \\
&= \left(\partial_\mu B_\mu - \partial_\nu B_\mu + \frac{1}{g} \hat{\phi} \cdot \partial_\mu \hat{\phi} \times \partial_\nu \hat{\phi} \right) \hat{\phi} - \frac{2}{g} \partial_\mu \hat{\phi} \times \partial_\nu \hat{\phi} \\
&\quad - \frac{1}{g} \hat{\phi} \times [\partial_\mu, \partial_\nu] \hat{\phi}, && (4.64)
\end{aligned}
$$

using the fact that $\hat{\phi} \cdot \hat{\phi} = \mathrm{Tr}\,(\hat{\phi}\hat{\phi}) = \frac{1}{2}$. It is easy to show that

$$
\hat{\phi} \times \left(\partial_\mu \hat{\phi} \times \partial_\nu \hat{\phi} \right) = 0, \tag{4.65}
$$

and following this we can write

$$
\partial_\mu \hat{\phi} \times \partial_\nu \hat{\phi} = \left(\hat{\phi} \cdot \partial_\mu \hat{\phi} \times \partial_\nu \hat{\phi} \right) \hat{\phi}. \tag{4.66}
$$

Using this relation $\vec{G}_{\mu\nu}$ can be written as

$$
\vec{G}_{\mu\nu} = \left(\partial_\mu B_\mu - \partial_\nu B_\mu - \frac{1}{g} \hat{\phi} \cdot \partial_\mu \hat{\phi} \times \partial_\nu \hat{\phi} \right) \hat{\phi} - \frac{1}{g} \hat{\phi} \times [\partial_\mu, \partial_\nu] \hat{\phi}. \tag{4.67}
$$

The last term in $G_{\mu\nu}$ written above is a string term and we shall see that this is an unstable string configuration. The string part can be written as

$$
-\frac{1}{g} \hat{\phi} \times [\partial_\mu, \partial_\nu] \hat{\phi} = -\frac{1}{g} \sin\theta\, \hat{\theta}\, [\partial_\mu, \partial_\nu] \varphi, \tag{4.68}
$$

where we have written in the internal three dimensional space

$$
\begin{aligned}
\hat{\phi} &= \cos\theta\, \tau^3 + \sin\theta \cos\varphi\, \tau^1 + \sin\theta \sin\varphi\, \tau^2, && (4.69) \\
\hat{\theta} &= -\sin\theta\, \tau^3 + \cos\theta \cos\varphi\, \tau^1 + \cos\theta \sin\varphi\, \tau^2. && (4.70)
\end{aligned}
$$

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Here θ and ϕ are the parameters of the field orbit which is a sphere. A flux string can be constructed if there is a map from any spatial loop to any loop on the orbit. However, we know that $\Pi_1(S^2) = 0$. This means that here any loop on the sphere can be shrunk to a point by a suitable gauge transformation. For a loop at the equator the flux of the string is $\dfrac{2\pi}{g}$. However, we can make the loop disappear by taking it to the pole where the flux is zero and this can be done by setting $\theta = 0$ using just a gauge transformation. So we can ignore the last term of the right hand side of the equation (4.67). In terms of matrices, $G_{\mu\nu}$ can be written as

$$G_{\mu\nu} = \left[\partial_\mu B_\nu - \partial_\nu B_\mu + \frac{2i}{g} \mathrm{Tr}\left(\hat{\phi}[\partial_\mu \hat{\phi}, \partial_\nu \hat{\phi}] \right) \right] \hat{\phi}, \tag{4.71}$$

where $\hat{\phi} = \dfrac{\phi}{\xi_1}$. We can write down an effective Lagrangian at the vacuum as

$$\begin{aligned} L &= -\frac{1}{2} \mathrm{Tr}\left[G_{\mu\nu} G^{\mu\nu} \right] \\ &= -\frac{1}{4} \left[F_{\mu\nu} F^{\mu\nu} \right], \end{aligned} \tag{4.72}$$

where we have defined

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \frac{2i}{g} \mathrm{Tr}\left[\hat{\phi}[\partial_\mu \hat{\phi}, \partial_\nu \hat{\phi}] \right]. \tag{4.73}$$

Let us discuss the form of the field ϕ when ϕ is an SU(2) adjoint scalar and can be written as $\phi(x) = |\phi(x)|\hat{\phi}(x)$ where $x \equiv \vec{x}$ and under gauge transformations $\hat{\phi}$ has a trajectory on S^2. Since ϕ is in the adjoint of SU(2), we can always write ϕ as

$$\phi(x) = |\phi(x)|g(x)\tau^3 g^{-1}(x) = |\phi(x)|\hat{\phi}(x), \tag{4.74}$$

for some $g(x) \in$ SU(2). Then for a given $\phi(x)$, we can locally decompose $g(x)$ as $g(x) = h(x)U(x)$, with $h(x) = \exp(-i\xi(x)\hat{\phi}(x))$, and we can write

$$\phi(x) = |\phi(x)|U(\varphi(x), \theta(x))\tau^3 U^\dagger(\varphi(x), \theta(x)). \tag{4.75}$$

4. Monopoles and flux tubes in broken SU(2)

Here $\xi(x), \varphi(x), \theta(x)$ are angles on $S^3 \simeq \mathrm{SU}(2)$. The matrix U rotates $\hat{\phi}(x)$ in the internal space, and is an element of $\mathrm{SU}(2)/\mathrm{U}(1)$, where the U(1) is the one generated by h. If $|\phi|$ is zero at the origin and $|\phi|$ goes smoothly to its vacuum value ξ_1 on the sphere at infinity, the field ϕ defines a map from space to the vacuum manifold such that the second homotopy group of the mapping is Z, the set of integers. Equating $\hat{\phi}$ with the unit radius vector of a sphere we can solve for $U(\theta(x), \varphi(x))$,

$$U = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{-i\varphi} \\ \sin\frac{\theta}{2}e^{i\varphi} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{4.76}$$

In other words, an 't Hooft-Polyakov monopole [27, 28] (in the point approximation, or as seen from infinity) at the origin is described by

$$U = \cos\frac{\theta}{2}\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} + \sin\frac{\theta}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{4.77}$$

where $0 \le \theta(\vec{x}) \le \pi$ and $0 \le \varphi(\vec{x}) \le 2\pi$ are two parameters on the group manifold. Both choices, Eq. (4.76) and Eq. (4.77), lead to the field configuration

$$\vec{\phi} = \xi_1\frac{r^i}{r}\tau_i \tag{4.78}$$

upon using Eq. (4.75) with $|\phi| = \xi_1$. For this monopole, $Q_m g = 1$, as we mentioned earlier. A monopole of charge n/g is obtained by making the replacement $\varphi \to n\varphi$ in Eq.s (4.76) or (4.77).

$$U_n = \cos\frac{\theta}{2}\begin{pmatrix} e^{in\varphi} & 0 \\ 0 & e^{-in\varphi} \end{pmatrix} + \sin\frac{\theta}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{4.79}$$
$$n = \pm 1, \pm 2, \pm 3, \ldots$$

The integer n labels the homotopy class, $\pi_2(SU(2)/U(1)) \sim \pi_2(S^2) \sim Z$, of the scalar field configuration. Other choices of $U(\vec{x})$ can give other configurations. For

example, a monopole-antimonopole pair located on the z axis [67] is given by the choice

$$U = \sin\frac{(\theta_1 - \theta_2)}{2}\begin{pmatrix} 0 & -e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix} + \cos\frac{(\theta_1 - \theta_2)}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.80}$$

For our purposes, we will need to consider a ϕ-vacuum configuration with $U(\vec{x}) \in SU(2)$ corresponding to a monopole-anti-monopole pair separated from each other by a distance $\gg 1/\xi_1$. Then the total magnetic charge vanishes, but each monopole (or anti-monopole) can be treated as a point particle.

4.3. Flux tubes in SU(2) scalar gauge theory

In the Abelian Higgs model we can construct a flux tube configuration by U(1) symmetry breaking. The asymptotic form of the gauge field is pure gauge, and if we map an angle of a loop around the flux tube to the gauge parameter of the group U(1) we get a flux tube solution. Geometrically we can say that for a stable flux tube solution in gauge theory, there has to be a non-trivial mapping from a spatial loop in space to the vacuum manifold $\mathcal{M}_0 = G/H$. That means $\pi_1(G/H) \neq 0$, where H is a subgroup of the group G and G/H is the coset space.

We have seen in last section that SU(2) symmetry can be broken down to U(1) by an adjoint scalar. It is possible to construct a flux tube solution by breaking this U(1) symmetry. However, unlike the Abelian Higgs model there are two ways by which we can construct flux tubes from this U(1) theory. We can break U(1) by using another SU(2) adjoint scalar or we can use a fundamental SU(2) scalar. As we will discuss, the energy scale of this symmetry breaking must be different from the one that breaks SU(2) down to U(1). This is a requirement for stability and we

will take the difference between energy scales to be very high. The scales determine the masses of the two Higgs particles.

4.3.1. Flux tubes with a second adjoint scalar

For the theory with two adjoint scalar fields, we can write down the Lagrangian as

$$
\begin{aligned}
L &= -\frac{1}{2} \mathrm{Tr}\ (G_{\mu\nu} G^{\mu\nu}) + \mathrm{Tr}\ (D_\mu \phi_1 D^\mu \phi_1) + \mathrm{Tr}\ (D_\mu \phi_2 D^\mu \phi_2) \\
&\quad - \frac{\lambda_1}{4} \left(|\phi_1|^2 - \xi_1^2 \right)^2 - \frac{\lambda_2}{4} \left(|\phi_2|^2 - \xi_2^2 \right)^2. \quad (4.81)
\end{aligned}
$$

Let us suppose that the SU(2) symmetry is broken to U(1) at a scale ξ_1 and the U(1) is broken at a scale ξ_2. Since ϕ_1 is in the adjoint representation of SU(2), we can also think of this as SO(3) being broken down to SO(2), which is subsequently broken. We assume that $\xi_1 \gg \xi_2$ for the stability reason. We choose the vacuum ϕ_1 along the third axis, i.e.

$$
\phi_1 = \xi_1 \tau^3. \quad (4.82)
$$

Below this vacuum scale there is only one massless gauge field present and that is A_μ^3 as discussed in §4.1. The only gauge transformation that is allowed on the field ϕ_2 is then

$$
\phi_2' = e^{-i\chi\tau^3} \phi_2 e^{i\chi\tau^3}. \quad (4.83)
$$

The general form of ϕ_2 in the ϕ_1-vacuum is

$$
\phi_2 = |\phi_2| e^{-i\chi\tau^3} \hat{\rho} e^{i\chi\tau^3}, \quad (4.84)
$$

where $\hat{\rho}$ is some unit vector in the SU(2) Lie algebra. Then the covariant derivative becomes

$$
\begin{aligned}
\mathcal{D}_\mu \phi_2 &= \partial_\mu \phi_2 - i A_\mu^3 [\tau^3, \phi_2] \quad (4.85) \\
&= e^{-i\chi\tau^3} \left(\partial_\mu |\phi_2| \hat{\rho} + |\phi_2| \partial_\mu \hat{\rho} - i|\phi_2|(g A_\mu^3 + \partial_\mu \chi) \left[\tau^3, \hat{\rho}\right] \right) e^{i\chi\tau^3}. \quad (4.86)
\end{aligned}
$$

So the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3)^2 + \frac{1}{2}(\partial_\mu |\phi_2|)^2 + |\phi_2|^2 \mathrm{Tr}\left(\partial_\mu \hat{\rho} - i(gA_\mu^3 + \partial_\mu \chi)\left[\tau^3, \hat{\rho}\right]\right)^2$$
$$-\frac{\lambda_2}{4}\left(|\phi_2|^2 - \xi_2^2\right)^2 \quad (4.87)$$

It followes from the above Lagrangian that further symmetry breaking is possible if

$$[\hat{\rho}, \tau^3] \neq 0, \quad (4.88)$$

because if $[\hat{\rho}, \tau^3] = 0$, there will be no interaction between A_μ^3 and $\hat{\rho}$. If symmetry breaking happens then we can write

$$A_\mu^3 = -\frac{1}{g}\partial_\mu \chi, \qquad |\phi_2| = \xi_2 \quad (4.89)$$

in the vacuum. A flux tube through the origin can be constructed if χ is at least isomorphic to the angle around the flux tube. So the flux of a tube through the origin (along the z- axis) can be calculated by taking the line integral of A_μ^3 around a loop far away from the flux tube. We consider a distant loop because there is a kernel near the origin which smooths out the line singularity and makes a real flux tube. However, far away from the flux tube core it looks like a singular line, and the flux can be calculated by integrating $\vec{B}_3$ over a surface $\vec{\Omega}$ encircled by a loop C,

$$\mathrm{Flux} = \int_\Omega \vec{B}_3 \cdot d\vec{\Omega} \quad (4.90)$$
$$= \oint_C \vec{A}_3 \cdot d\vec{l} \quad (4.91)$$
$$= \oint_C \frac{1}{g}\nabla\chi \cdot d\vec{l} \quad (4.92)$$
$$= \frac{2\pi n}{g}. \quad (4.93)$$

Here n is the winding number of the homotopy class of the mapping from the spatial angle of the loop C to χ. For $n = 1$ this mapping is an isomorphism. To see the kernel of the flux tube we need the solutions of the equations of motion. No exact solution is known till date but asymptotic solutions do exist.

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4.3.2. Flux tubes with one adjoint and one fundamental scalar

In §4.3.1 we have seen that we can construct a flux tube configuration by using two adjoint scalars. However, instead of using an adjoint scalar to break the U(1) symmetry, we can have a fundamental scalar which causes that. In this case the starting Lagrangian is

$$
\begin{aligned}
L &= -\frac{1}{2}\mathrm{Tr}\,(G_{\mu\nu}G^{\mu\nu}) + \mathrm{Tr}\,(D_\mu\phi D^\mu\phi) + \mathrm{Tr}\,(\mathcal{D}_\mu\psi)^2 \\
&\qquad - \frac{\lambda_1}{4}\left(|\phi_1|^2 - \xi_1^2\right)^2 - \frac{\lambda_2}{4}\left(|\psi|^2 - \zeta_2^2\right)^2. \quad (4.94)
\end{aligned}
$$

Here the field $\psi(x)$ is in the fundamental representation of SU(2) with a covariant derivative defined by

$$
\mathcal{D}_\mu\psi = \partial_\mu\psi - igA_\mu\psi, \qquad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}. \quad (4.95)
$$

As we discussed in the last section the first symmetry is broken by the vacuum expectation value of the field ϕ from SU(2) to U(1). As in the last section, here also we fix the vacuum as $\phi = \xi_1\tau^3$, and call this the ϕ-vacuum. As explained in the previous section, two of the gauge fields become massive at the scale ξ_1, which we take to be large compared to the other scale ζ_2, $\xi_1 \gg \zeta_2$. We will look at the theory below the scale ζ_2, and ignore the fields which have masses of the order ξ_1. Then the Lagrangian for the remaining fields is given by

$$
\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3)^2 + \mathrm{Tr}\,(\mathcal{D}_\mu\psi)^2 - \frac{\lambda_2}{4}\left(|\psi|^2 - \zeta_2^2\right)^2. \quad (4.96)
$$

The equation

$$
\mathrm{Re}\,(\psi_1)^2 + \mathrm{Im}\,(\psi_1)^2 + \mathrm{Re}\,(\psi_2)^2 + \mathrm{Im}\,(\psi_2)^2 = \zeta_2^2 \quad (4.97)
$$

defines a three sphere (S^3). To fix any point on this S^3 as vacuum, we have to fix all the three parameters of the group SU(2). So any $\psi(x)$ that satisfies Eq. (4.97)

can break the residual U(1) of the theory because two parameters are fixed already by the vacuum expectation value of the field ϕ. If ϕ is fixed along τ^3, then the only allowed gauge transformations are represented by the element $e^{-i\xi(x)\tau^3}$. So the gauge transformation on the field ψ in the ϕ-vacuum can be written as

$$\psi \to \psi'(x) = e^{-i\zeta(x)\tau^3}\psi(x). \tag{4.98}$$

We choose a form

$$\psi(x) = |\psi(x)|e^{-\xi(x)\tau^3}\begin{pmatrix}1\\0\end{pmatrix}. \tag{4.99}$$

Any ψ can be written like this for some ξ. Then we can write the Lagrangian as

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3)^2 + \frac{1}{2}(\partial_\mu|\psi|)^2 + \frac{1}{2}(gA_\mu^3 + \partial_\mu\xi)^2 - \frac{\lambda}{4}\left(|\psi|^2 - \zeta_2^2\right)^2. \tag{4.100}$$

At the vacuum we can write

$$|\psi| = \zeta_2, \qquad A_\mu^3 = -\frac{1}{g}\partial_\mu\xi(x). \tag{4.101}$$

We can construct flux tubes in the same way as we did for two adjoint scalars, by mapping ξ onto a spatial circle. However, there is a problem of uniqueness here. To construct a flux tube we have to write down a mapping from a spatial angular variable, say χ, to the variable ξ and the homotopy class of this mapping must be non trivial. The form of ψ we have written in above Eq. (4.99) is periodic in ξ with periodicity 4π,

$$\psi(\xi) = \psi(\xi + 4\pi). \tag{4.102}$$

Since $\psi(x)$ is a physical scalar field on space time, it must be single valued on the space time points,

$$\psi(\xi(\chi)) = \psi(\xi(\chi + 2\pi)). \tag{4.103}$$

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For flux tube solutions we have seen that there is a mapping from χ to ξ. The minimum nontrivial flux tube configuration was constructed for an isomorphism $\chi \to \xi$. However, when ψ is in the fundamental representation, the two equations (4.102) and (4.103) are inconsistent if we take $\chi = \xi$. So for the consistency of the two equations (4.102) and (4.103) we have to set

$$2\chi = \xi(x). \tag{4.104}$$

The flux of this flux tube will be

$$\text{Flux} \quad = \quad \int \vec{B}_3 \cdot d\vec{\Omega} \tag{4.105}$$

$$= \quad \oint \vec{A}_3 \cdot \vec{dl} \tag{4.106}$$

$$= \quad \frac{1}{g} \oint \nabla \xi \cdot \vec{dl} \tag{4.107}$$

$$= \quad \frac{2}{g} \oint \nabla \chi \cdot \vec{dl} \tag{4.108}$$

$$= \quad \frac{4\pi n}{g}. \tag{4.109}$$

The minimum non-zero flux for the flux tube constructed by one adjoint scalar and one fundamental scalar is thus twice the flux of the flux tube constructed by two adjoint scalars.

5. Confinement of monopoles in broken SU(2)

In this chapter we will construct an effective Lagrangian from the original SU(2) Lagrangian with two scalar fields. We should clarify the meaning of the word "effective". The theory that we are going to discuss is a theory of two stage symmetry breaking. In other words, the theory has two different energy scales at which the symmetry is broken. We shall consider a very high energy scale ξ_1 and a low energy scale ξ_2. We will assume that $\xi_1 \gg \xi_2$, specific values of ξ_1, ξ_2 can be determined if this theory is embedded in a larger theory but we will not do so. Above the high energy scale there is no isolated monopole and below the low energy scale we find confinement. As discussed in the previous chapter, spontaneous symmetry breaking at the scale ξ_1, creates magnetic monopoles and breaks the SU(2) down to U(1). This U(1)is then broken at the scale ξ_2 and strings (flux tubes) are produced. We will show that below the scale ξ_2 there can be flux tubes of finite length (open strings) with magnetic monopoles and anti-monopoles attached to their ends. However, to show the attachment of monopoles at the end of confining strings and the interactions of confining strings, we should consider the theory near the confining scale ξ_2. At this scale many of the original degrees of freedom are frozen out. The remaining degrees of freedom, which are the propagating degrees at this scale, have

a description in terms of dual variables. These make it convenient to see that the magnetic monopoles are indeed attached at the ends of the string, as was in the case of the Abelian Higgs model with external monopole in chapter 2. Here we will construct a Lagrangian to describe, in terms of these dual variables, only the physics near the length scale ξ_2^{-1}. This Lagrangian is the one we call " effective ".

5.1. Long distance Effective action by two adjoint scalars

We start from a theory with SU(2) symmetry and a pair of adjoint scalars, as discussed in §3.2 . The non-zero vacuum expectation value of the field ϕ_1 breaks the symmetry to U(1) at a scale ξ_1. Below ξ_1 the theory is effectively an Abelian theory with magnetic monopoles.

The Lagrangian for this system is

$$L = -\frac{1}{2}\text{Tr}\ (G_{\mu\nu}G^{\mu\nu}) + \text{Tr}\ (D_\mu\phi_1 D^\mu\phi_1) + \text{Tr}\ (D_\mu\phi_2 D^\mu\phi_2)$$
$$-\frac{\lambda_1}{4}\left(|\phi_1|^2 - \xi_1^2\right)^2 - \frac{\lambda_2}{4}\left(|\phi_2|^2 - \xi_2^2\right)^2. \tag{5.1}$$

The vacuum can be chosen according to the eqs. (4.35, 4.36). Here our plan will be to construct an effective Lagrangian from the Lagrangian (5.1). The effective Abelian Lagrangian with monopole was written in Eq. (4.72). For two adjoint scalars the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\vec{\phi}_2 \cdot D^\mu\vec{\phi}_2 - \frac{\lambda_2}{4}(|\phi_2|^2 - \xi_2^2)^2, \tag{5.2}$$

where, similar to Eq. (4.73) and Eq.(4.61),

$$F_{\mu\nu} \ = \ \partial_\mu B_\nu - \partial_\nu B_\mu - \frac{1}{g}\hat{\phi}_1 \cdot \partial_\mu\hat{\phi}_1 \times \partial_\nu\hat{\phi}_1, \tag{5.3}$$

5. Confinement of monopoles in broken SU(2)

$$2\mathrm{Tr}\,[\hat{\phi}_1 A_\mu] \;=\; B_\mu\,, \qquad A_\mu = B_\mu\hat{\phi}_1 - \frac{1}{g}\hat{\phi}_1 \times \partial_\mu\hat{\phi}_1. \tag{5.4}$$

One important difference with the general discussion on spontaneous symmetry breaking earlier is that here we did not fix the field ϕ_1 at a constant internal direction everywhere. Since the gauge group $SU(2)$ is transitive on the vacuum manifold S^2, this transitivity makes the symmetry breaking independent of the direction of ϕ_1. There is a little group U(1) in the theory which leaves ϕ_1 invariant on the vacuum and this little group becomes the remaining symmetry of the theory. This little group is defined at every point on the vacuum. For the symmetry breaking from $SU(2)$ to U(1), the little group action is the rotation around a point on the vacuum manifold S^2. That is why the little group is the same for every point on S^2 and the little group is U(1). However, ϕ_2 is also in the adjoint representation of SU(2). It has three real scalar components, out of which three, one component can be chosen along the direction of the field ϕ_1. Then the other two will rotate on a two dimensional plane normal to ϕ_1 under the action of the little group U(1). Since the little group of ϕ_1 is a subgroup of the gauge group, we can say that a gauge transformation rotates ϕ_2 around ϕ_1. Flux tubes will be produced when this $U(1)$ symmetry is spontaneously broken down to Z_2. It is natural to take the $U(1)$ breaking scale ξ_2 to be very small compared to the $SU(2)$ symmetry breaking scale, $\xi_2 \ll \xi_1$.

In order to find string configurations, we write the covariant derivative of ϕ_2 using Eq. (5.4),

$$
\begin{aligned}
D_\mu\vec{\phi}_2 &= \partial_\mu\vec{\phi}_2 + g\vec{A}_\mu \times \vec{\phi}_2, \\
&= \partial_\mu\vec{\phi}_2 + g\left[B_\mu\hat{\phi}_1 - \frac{1}{g}\hat{\phi}_1 \times \partial_\mu\hat{\phi}_1\right] \times \vec{\phi}_2, \\
&= \partial_\mu\vec{\phi}_2 + gB_\mu\hat{\phi}_1 \times \vec{\phi}_2 + \left[\hat{\phi}_1\left(\partial_\mu\hat{\phi}_1 \cdot \vec{\phi}_2\right) - \partial_\mu\hat{\phi}_1\left(\hat{\phi}_1 \cdot \vec{\phi}_2\right)\right].
\end{aligned}
\tag{5.5}
$$

This is of course in the ϕ_1 vacuum.

5. Confinement of monopoles in broken SU(2)

For string configurations, ϕ_2 has to approach its vacuum value far away from the string. The ϕ_2 vacuum is defined by

$$|\vec{\phi_2}|^2 = \xi_2^2 \,, \tag{5.6}$$

$$D_\mu \vec{\phi_2} = 0. \tag{5.7}$$

These equations are taken in the ϕ_1 vacuum, so in particular we use Eq. (5.5) in the left hand side of Eq. (5.7). If we now dot Eq. (5.7) with $\hat{\phi}_1$, we get

$$\hat{\phi}_1 \cdot \partial_\mu \phi_2 + \phi_2 \cdot \partial_\mu \hat{\phi}_1 = \partial_\mu(\vec{\phi}_1 \cdot \vec{\phi}_2) = 0. \tag{5.8}$$

So in the ϕ_2 vacuum (which by definition is embedded in the ϕ_1 vacuum), the component of $\vec{\phi}_2$ along $\vec{\phi}_1$ remains constant.

As mentioned above, we can decompose ϕ_2 (not necessarily in the ϕ_2 vacuum) into a component along ϕ_1 and another component normal to ϕ_1 in the internal space,

$$\vec{\phi}_2 = (\hat{\phi}_1 \cdot \vec{\phi}_2)\hat{\phi}_1 + \vec{K} \,, \tag{5.9}$$

with

$$\vec{K} \cdot \hat{\phi} \; = \; 0. \tag{5.10}$$

Then

$$\partial_\mu \vec{K} \cdot \hat{\phi}_1 \; = \; -\vec{K} \cdot \partial_\mu \hat{\phi}_1. \tag{5.11}$$

The form of $\vec{K}$ will be important to write string degrees of freedom and it will be discussed in next section. Now we can calculate $\mathcal{D}_\mu \phi_2$ using the above expressions.

$$
\begin{aligned}
\mathcal{D}_\mu \phi_2 \; &= \; \partial_\mu \phi_2 + \vec{A} \times \phi_2 \\
&= \; \partial_\mu \left((\hat{\phi}_1 \cdot \vec{\phi}_2)\hat{\phi}_1 + \vec{K} \right)
\end{aligned}
\tag{5.12}
$$

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$$+ \left(B_\mu \hat{\phi} - \frac{1}{g} \hat{\phi} \times \partial_\mu \hat{\phi} \right) \times \left((\hat{\phi}_1 \cdot \vec{\phi}_2) \hat{\phi}_1 + \vec{K} \right) \quad (5.13)$$

$$= \ \partial_\mu (\hat{\phi}_1 \cdot \vec{\phi}_2) \hat{\phi}_1 + (\hat{\phi}_1 \cdot \vec{\phi}_2) \partial_\mu \hat{\phi}_1 + \partial_\mu \vec{K} + g B_\mu \hat{\phi}_1 \times \vec{K}$$

$$-(\hat{\phi}_1 \cdot \vec{\phi}_2)(\phi_1 \times \partial_\mu \phi_1) \times \phi_1 - (\phi_1 \times \partial_\mu \phi_1) \times \vec{K} \quad (5.14)$$

$$= \ \hat{\phi}_1 \partial_\mu (\hat{\phi}_1 \cdot \vec{\phi}_2) + \partial_\mu \vec{K} + \hat{\phi}_1 (\vec{K} \cdot \partial_\mu \hat{\phi}_1) + g B_\mu \hat{\phi}_1 \times \vec{K}\,, \quad (5.15)$$

$$= \ \hat{\phi}_1 \partial_\mu (\hat{\phi}_1 \cdot \vec{\phi}_2) + \partial_\mu \vec{K} - \hat{\phi}_1 (\hat{\phi}_1 \cdot \partial_\mu \vec{K}) + g B_\mu \hat{\phi}_1 \times \vec{K}\,. \quad (5.16)$$

To write down the Lagrangian let us calculate

$$(D_\mu \vec{\phi}_2)^2 = \ \left[\partial_\mu (\hat{\phi}_1 \cdot \vec{\phi}_2) \right]^2 + \left[\left(\partial_\mu \vec{K} - \hat{\phi}_1 (\hat{\phi}_1 \cdot \partial_\mu \vec{K}) \right) + g B_\mu \hat{\phi}_1 \times \vec{K} \right]^2, \quad (5.17)$$

$$= \ \left[\partial_\mu (\hat{\phi}_1 \cdot \vec{\phi}_2) \right]^2$$
$$+ \left[\left(\partial_\mu \vec{K} \right)^2 - \left(\partial_\mu \vec{K} \cdot \hat{\phi}_1 \right)^2 + 2g B^\mu \partial_\mu \vec{K} \cdot \hat{\phi}_1 \times \vec{K} + g^2 B^\mu B_\mu |\vec{K}|^2 \right]$$
$$(5.18)$$

$$= \ \left[\partial_\mu (\hat{\phi}_1 \cdot \vec{\phi}_2) \right]^2 + \left(\partial_\mu |\vec{K}| \right)^2$$
$$+ |\vec{K}|^2 \left[\left(\partial_\mu \hat{k} \right)^2 - \left(\partial_\mu \hat{k} \cdot \hat{\phi}_1 \right)^2 + 2g B^\mu \partial_\mu \hat{k} \cdot \hat{\phi}_1 \times \hat{k} + g^2 B^\mu B_\mu \right] (5.19)$$

$$= \ \left[\partial_\mu (\hat{\phi}_1 \cdot \vec{\phi}_2) \right]^2 + \left(\partial_\mu |\vec{K}| \right)^2$$
$$+ |\vec{K}|^2 \left[\left(\partial_\mu \hat{k} \times \hat{\phi}_1 \right)^2 + 2g B^\mu \partial_\mu \hat{k} \cdot \hat{\phi}_1 \times \hat{k} + g^2 B^\mu B_\mu \right], \quad (5.20)$$

where we have defined $\hat{k} = \dfrac{\vec{K}}{|\vec{K}|}$. The above expression of $(\mathcal{D}_\mu \phi_2)^2$ can be simplified if we use the identity

$$\left(\partial_\mu \hat{k} \times \phi_1 \cdot \hat{k} \right) \left(\partial_\mu \hat{k} \times \phi_1 \cdot \hat{k} \right) - \left(\partial_\mu \hat{k} \times \phi_1 \right) \cdot \left(\partial_\mu \hat{k} \times \phi_1 \right)$$

$$= \ \left(\partial_\mu \hat{k} \times \phi_1 \cdot \hat{k} \right) \left(\partial_\mu \hat{k} \times \phi_1 \cdot \hat{k} \right) - \left(\partial_\mu \hat{k} \times \phi_1 \right) \cdot \left(\partial_\mu \hat{k} \times \phi_1 \right) \hat{k} \cdot \hat{k}$$

$$= \ \left([\partial_\mu \hat{k} \times \phi_1] \times \hat{k} \right) \cdot \left([\partial_\mu \hat{k} \times \phi_1] \times \hat{k} \right)$$

$$= \ [(\hat{k} \cdot \partial_\mu \hat{k}) \hat{\phi}_1 - \partial_\mu \hat{k} (\hat{\phi}_1 \cdot \hat{k})]^2 = 0, \quad (5.21)$$

which holds because

$$\hat{k} \cdot \partial_\mu \hat{k} = 0 = \hat{\phi}_1 \cdot \hat{k} \quad (5.22)$$

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by the definition of $\hat{k}$. Using the identity of Eq. (5.21) we can write

$$
\begin{aligned}
(D_\mu \vec{\phi}_2)^2 &= \left[\partial_\mu(\hat{\phi}_1 \cdot \vec{\phi}_2)\right]^2 + \left(\partial_\mu |\vec{K}|\right)^2 \\
&\quad + |\vec{K}|^2 \left[\left(\hat{k} \cdot \partial_\mu \hat{k} \times \hat{\phi}_1\right)^2 + 2gB^\mu \, \hat{k} \cdot \partial_\mu \hat{k} \times \hat{\phi}_1 + g^2 B^\mu B_\mu\right] \\
&= \left[\partial_\mu(\hat{\phi}_1 \cdot \vec{\phi}_2)\right]^2 + \left(\partial_\mu |\vec{K}|\right)^2 + |\vec{K}|^2 \left[\hat{k} \cdot \partial_\mu \hat{k} \times \hat{\phi}_1 + gB_\mu\right]^2 . \quad (5.23)
\end{aligned}
$$

We put this expression into the Lagrangian of Eq. (5.2). Then in order to extract the string variables, we note that at large distances away from the string, ϕ_2 approaches its vacuum value $|\phi_2| \to \xi_2$. Further, according to Eq. (5.8), $\hat{\phi}_1 \cdot \vec{\phi}_2$ also approaches a constant, so using Eq. (5.9) we see that $|\vec{K}|$ should also approach a constant. Then the first two terms of Eq. (5.23) disappear at infinity, as does the last term of Eq. (5.2), and the Lagrangian at infinity behaves as

$$
\mathcal{L} = \frac{|\vec{K}|^2}{2} \left[\hat{k} \cdot \partial_\mu \hat{k} \times \hat{\phi}_1 + gB_\mu\right]^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad (5.24)
$$

where now $|\vec{K}|$ is a constant.

Since $|\vec{\phi}_2|$, and the component of $\vec{\phi}_2$ along $\hat{\phi}_1$, both approach constant values at infinity, and so does $|\vec{K}|$, the only degree of freedom remaining in $\vec{\phi}_2$ at infinity is an angle χ which parametrizes the rotation of $\vec{\phi}_2$ around $\hat{\phi}_1$. The first term inside the brackets in Eq. (5.24) provides the $\partial_\mu \chi$ as we will see below. This is the angle which is mapped onto a circle at infinity to produce a flux string. Further, the system is in the ϕ_1-vacuum, i.e. $\vec{\phi}_1$ is in a vacuum configuration given by Eqs. (4.35) and (4.41). So in particular we can choose this vacuum to contain 't Hooft-Polyakov monopoles as discussed after Eq. (4.47).

5.2. Monopoles and Strings from the effective action

With the above in mind, let us parametrize the ϕ_1-vacuum as discussed in Eq. (4.75).

$$\vec{\phi}_1 = \xi_1 U(\vec{x})\tau_3 U(\vec{x})^\dagger, \qquad \text{with } U(\vec{x}) \in SU(2)/U(1). \tag{5.25}$$

Appropriate choices of $U(\vec{x})$ provide different monopole configurations, some examples were given in Eqs. (4.76), (4.77, (4.79), (4.80).

For our purposes, we will need to consider a ϕ_1-vacuum configuration with $U(\vec{x}) \in SU(2)/U(1)$ corresponding to a monopole-anti-monopole pair separated from each other by a distance $\gg 1/\xi_1$ [67]. Then the total magnetic charge vanishes, but the monopole and anti-monopole can be treated as point particles.

We also need to choose the form of the vector $\vec{K}$ as in Eq. (5.9), so that it is orthogonal to $\hat{\phi}_1 = \vec{\phi}_1/\xi_1$ in the internal space and rotates around $\hat{\phi}_1$. Let us write $\hat{k}$ in terms of matrices as

$$\hat{k} \equiv \hat{k}(\vec{x})^i \tau^i = e^{-i\chi(\vec{x})\hat{\phi}_1(\vec{x})} U(\vec{x})\tau_2 U^\dagger(\vec{x}) e^{i\chi(\vec{x})\hat{\phi}_1(\vec{x})}. \tag{5.26}$$

We have used τ_2 to write $\hat{k}$ here but we can substitute any constant vector orthogonal to τ_3 without affecting the results below. The $\hat{\phi}_1(\vec{x}) = \dfrac{\vec{\phi}_1}{\xi_1}$ used here is constructed according to Eq. (5.25) with $U(\vec{x})$ as described above. Then $\chi(\vec{x})$ is the angle by which the vector $U(\vec{x})\tau_2 U^\dagger(\vec{x})$ is rotated in the group.

To get string and monopole terms in the Lagrangian we have to calculate the term $\hat{k} \cdot \partial_\mu \hat{k} \times \hat{\phi}_1$. Let us first calculate $\partial_\mu \hat{k}$. Using Eq. (5.26) we can write

$$\partial_\mu \hat{k} = \partial_\mu(g\tau^2 g^\dagger), \tag{5.27}$$

where $g = hU$ and

$$h(x) = e^{-i\chi(\vec{x})\hat{\phi}_1(\vec{x})}. \tag{5.28}$$

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Then

$$\partial_\mu \hat{k} = \left[\partial_\mu g g^\dagger, \hat{k}\right] \tag{5.29}$$

$$= \left[\partial_\mu h h^\dagger + h \partial_\mu U U^\dagger h^\dagger, \hat{k}\right]. \tag{5.30}$$

For the sake of convenience, we write Eq. (5.30) as a vector equation,

$$\partial_\mu \hat{k} = i\vec{R}_\mu \times \hat{k}, \tag{5.31}$$

where $\vec{R}_\mu = \partial_\mu h h^\dagger + h \partial_\mu U U^\dagger h^\dagger$. It follows from this that

$$\hat{k} \cdot \partial_\mu \hat{k} \times \hat{\phi}_1 \equiv \hat{k} \times \partial_\mu \hat{k} \cdot \hat{\phi}_1$$

$$= i\hat{k} \times \left(\vec{R}_\mu \times \hat{k}\right) \cdot \hat{\phi}_1$$

$$= i\left(\vec{R}_\mu - \hat{k}(\hat{k} \cdot \vec{R}_\mu)\right) \cdot \hat{\phi}_1$$

$$= i\vec{R}_\mu \cdot \hat{\phi}_1, \tag{5.32}$$

where we have used the fact that $\hat{\phi}_1 \cdot \hat{k} = 0$. Using Eq. (5.28) and the expression of R_μ written above, we can write

$$\hat{k} \cdot \partial_\mu \hat{k} \times \hat{\phi}_1 = i2\text{Tr}\left[\partial_\mu h h^\dagger \hat{\phi}_1\right] + i2\text{Tr}\left[\partial_\mu U U^\dagger \hat{\phi}_1\right] \tag{5.33}$$

$$= i2\text{Tr}\left[\partial_\mu(Ue^{-i\chi\tau^3}U^\dagger)Ue^{i\chi\tau^3}U^\dagger \hat{\phi}_1\right] + i2\text{Tr}\left[\partial_\mu U U^\dagger \hat{\phi}_1\right] \tag{5.34}$$

$$= i2\text{Tr}\left[\partial_\mu(Ue^{-i\chi\tau^3}U^\dagger)Ue^{i\chi\tau^3}\tau^3 U^\dagger\right] + i2\text{Tr}\left[\partial_\mu U U^\dagger \hat{\phi}_1\right] \tag{5.35}$$

$$= i2\text{Tr}\left[U^\dagger \partial_\mu U \tau^3 + \partial_\mu U^\dagger U \tau^3\right] + \partial_\mu \chi + i2\text{Tr}\left[\partial_\mu U U^\dagger \hat{\phi}_1\right] \tag{5.36}$$

$$= \partial_\mu \chi + i2\text{Tr}\left[\partial_\mu U U^\dagger \hat{\phi}_1\right] \tag{5.37}$$

$$= \partial_\mu \chi(\vec{x}) + gN_\mu(\vec{x}), \tag{5.38}$$

where the vector N_μ is given by

$$N_\mu = \frac{2i}{g}\text{Tr}\left[\partial_\mu U U^\dagger \hat{\phi}_1\right]. \tag{5.39}$$

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χ is the angle which is mapped onto a circle in space to exhibit the flux tube. As we will see now, N_μ is the (Abelian) field corresponding to magnetic monopoles.

Let us calculate the field strength tensor for N_μ,

$$\partial_\mu N_\nu - \partial_\nu N_\mu = -\frac{1}{g}\hat{\phi}_1 \cdot \partial_\mu \hat{\phi}_1 \times \partial_\nu \hat{\phi}_1 + \frac{2i}{g}\mathrm{Tr}\left[(\partial_{[\mu}\partial_{\nu]}U)U^\dagger\hat{\phi}_1\right]. \tag{5.40}$$

If we use the $U(\vec{x})$ of Eq. (4.77), the first term on the right hand side of this equation is the field strength of a magnetic monopole at the origin, while the second term is a gauge dependent line singularity, commonly known as a Dirac string. In this case,

$$N_\mu = -\frac{1}{g}(1 + \cos\theta)\partial_\mu\psi. \tag{5.41}$$

If θ and ψ are mapped to the polar and the azimuthal angles, N_μ is the familiar 4-potential of a magnetic monopole with a Dirac string [40]. For the $U(\vec{x})$ of the monopole-anti-monopole pair of Eq. (4.80), the first term of Eq. (5.40) gives the Abelian magnetic field of a monopole-anti-monopole pair, while the second term again contains a Dirac string.

The Dirac string is a red herring, and we are going to ignore it, for the following reason. The singular Dirac string appears because we have used a $U(\vec{x})$ which is appropriate for a point monopole. If we look at the system from far away, the monopoles will look like point objects, and it would seem that we should find Dirac strings attached to each of them. However, we know that the 't Hooft-Polyakov monopoles are actually not point objects, and their near magnetic field is not describable by an Abelian four-potential N_μ, so if we could do our calculations without the far field approximation, we would not find a Dirac string.

There is another way of confirming that the Dirac string will not appear in any calculations. In the far field approximation, we have written the Lagrangian of

Eq. (4.72) as Eq. (5.24), which we can rewrite using Eq. (5.38) as

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu B_\nu - \partial_\nu B_\mu + M_{\mu\nu}\right)^2 + \frac{|\vec{K}|^2}{2}\left(gB_\mu + \partial_\mu\chi + gN_\mu\right)^2 , \qquad (5.42)$$

where $|\vec{K}|$ is a constant as mentioned earlier, and $M_{\mu\nu}$ is the monopole field,

$$M_{\mu\nu} = -\frac{1}{g}\hat{\phi}_1 \cdot \partial_\mu\hat{\phi}_1 \times \partial_\nu\hat{\phi}_1 . \qquad (5.43)$$

The second term of the Lagrangian (5.42) is the one which exhibits a flux tube or a 'physical' string, as opposed to the unphysical Dirac string, which is an artifact of the far field approximation and can be relocated by a gauge transformation. An exactly analogous term appears in the Abelian Higgs model, where instead of $|\vec{K}|$ we get the physical Higgs field. This model also exhibits a flux string, and just like in the Abelian Higgs model, we know that the flux string here will appear along the zeroes of $|\vec{K}|$, even though Eq. (5.42) is written in the far field approximation, where $|\vec{K}|$ is a constant. The Dirac string is also an artifact of the far field approximation, and we can get rid of it by choosing $U(\vec{x})$ such that the Dirac string lies along the zeroes of $|\vec{K}|$, i.e., along the core of the flux string. Then the troublesome line singularity, which appears in the second term of Eq. (5.42), is always multiplied by zero, and we can ignore it for the rest of the calculations.

5.3. Low energy effective action with one adjoint and one fundamental scalars

In this section we will first construct an effective Lagrangian from the SU(2) Lagrangian with one adjoint scalar and one fundamental scalar field. Like the last section here we shall also consider two mass scales ξ_1, ξ_2 with $\xi_1 \gg \xi_2$. At the length scale shorter than ξ_1^{-1} there is no isolated monopole. At scales larger than

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the long distance scale ξ_2^{-1} we can see confinement. Like the last two sections here also we will construct a Lagrangian which only describes the physics near the length scale ξ_2^{-1} . We consider an SU(2) gauge theory coupled to an adjoint scalar field as well as a fundamental scalar field. The two fields break the symmetry at two scales. At the higher energy scale the adjoint scalar breaks the symmetry down to U(1) and produces 't Hooft-Polyakov magnetic monopoles [27, 28, 29]. The fundamental scalar breaks the remaining U(1) symmetry at a lower energy scale and produces a flux string.

We start with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\text{Tr } (G_{\mu\nu}G^{\mu\nu}) + \text{Tr } (D_\mu\phi D^\mu\phi) + \frac{1}{2}(D_\mu\psi)^\dagger(D^\mu\psi)$$
$$- \frac{\lambda_1}{4}(|\phi|^2 - \xi_1^2)^2 - \frac{\lambda_2}{4}(\psi^\dagger\psi - \xi_2^2)^2. \quad (5.44)$$

Here ϕ is in the adjoint representation of $SU(2)$, $\phi = \phi^i\tau^i$ with real ϕ^i and ψ is a fundamental scalar complex doublet of $SU(2)$. The $SU(2)$ generators τ^i satisfy $\text{Tr }(\tau^i\tau^j) = \frac{1}{2}\delta^{ij}$. The covariant derivatives D_μ and the Yang-Mills field strength tensor $G_{\mu\nu}$ are defined as

$$(D_\mu\phi)^i = \partial_\mu\phi^i + g\epsilon^{ijk}A_\mu^j\phi^k, \qquad (D_\mu\psi)_\alpha = \partial_\mu\psi_\alpha - igA_\mu^i\tau_{\alpha\beta}^i\psi_\beta, \quad (5.45)$$
$$G_{\mu\nu}^i = \partial_\mu A^i_{\ \nu} - \partial_\nu A^i_{\ \mu} + g\epsilon^{ijk}A^j_{\ \mu}A^k_{\ \nu} .. \quad (5.46)$$

The adjoint scalar ϕ acquires a vacuum expectation value $\vec{\xi_1}$ which is a vector in internal space, and breaks the symmetry group down to U(1). The 't Hooft-Polyakov monopoles are associated with this breaking. As in the Eqs. (4.35) and (4.36), the vacuum is defined by

$$|\vec{\phi}|^2 = \xi_1^2, \qquad D_\mu\vec{\phi} = 0. \quad (5.47)$$

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Like the equation (4.96) here also we can write the Lagrangian in the ϕ-vacuum

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\psi^\dagger)(D^\mu\psi) - \frac{\lambda_2}{4}(\psi^\dagger\psi - \xi_2^2)^2, \qquad (5.48)$$

where

$$F_{\mu\nu} \;=\; \partial_\mu B_\nu - \partial_\nu B_\mu - \frac{1}{g}\hat{\phi}\cdot\partial_\mu\hat{\phi}\times\partial_\nu\hat{\phi}, \qquad (5.49)$$

$$2\mathrm{Tr}\,[\hat{\phi}A_\mu] \;=\; B_\mu, \qquad A_\mu = B_\mu\hat{\phi} - \frac{1}{g}\hat{\phi}\times\partial_\mu\hat{\phi}. \qquad (5.50)$$

The last term of Eq. (5.49) is the 'monopole term' as discussed in §3.2.

After the original SU(2) is broken down to U(1) in the ϕ-vacuum, the only remaining gauge symmetry of the SU(2) doublet ψ is a transformation by the little group U(1). We will find flux tubes when this U(1) symmetry is spontaneously broken down to nothing. The elements of this U(1) are $h(x) = \exp[-i\xi(x)\hat{\phi}(x)]$, rotations by an angle $\xi(x)$ around the direction of $\phi(x)$ at any point in space. This U(1) will be broken by the vacuum configuration of ψ. Like the quation (5.25) here also we write

$$\vec{\phi} = \xi_1 U(\vec{x})\tau_3 U(\vec{x})^\dagger, \qquad \text{with } U(\vec{x}) \in SU(2)/U(1). \qquad (5.51)$$

Although in principle this process is the same for the fundamental scalar as it was for an adjoint scalar, there are some important differences in the construction, as we will see.

Let us then define the ψ-vacuum by

$$\psi^{*i}\psi^i = \xi_2^2 \qquad (5.52)$$

$$D_\mu\psi = 0, \qquad (5.53)$$

where D_μ is defined using A_μ in the ϕ-vacuum, as in Eq. (5.50). Multiplying Eq. (5.53) by $\psi^\dagger\hat{\phi}$ from the left, its adjoint by $\hat{\phi}\psi$ from the right, and adding the

results, we get

$$
\begin{aligned}
0 &= \psi^\dagger \hat\phi D_\mu \psi + (D_\mu \psi)^\dagger \hat\phi \psi \\
&= \psi^\dagger \hat\phi \partial_\mu \psi + (\partial_\mu \psi^\dagger)\hat\phi\psi - ig\psi^\dagger \left[A_\mu, \hat\phi \right] \psi && (5.54) \\
&= \psi^\dagger \hat\phi \partial_\mu \psi + (\partial_\mu \psi^\dagger)\hat\phi\psi + \psi^\dagger \partial_\mu \hat\phi \psi && (5.55) \\
&= \partial_\mu \left[\psi^\dagger \hat\phi \psi \right], && (5.56)
\end{aligned}
$$

from which it follows that

$$
\psi^\dagger \hat\phi \psi = \text{constant}, \tag{5.57}
$$

or explicitly in terms of the components,

$$
\psi_i^\dagger \tau_{ij}^\alpha \psi_j \hat\phi^\alpha = \text{Tr}\left[\psi^\dagger{}_i \sigma_{ij}^\alpha \psi_j \tau_\alpha \hat\phi \right] = \text{constant}. \tag{5.58}
$$

Since $\psi^\dagger \psi = \text{constant}$, it follows that the components of the adjoint vector $\psi^\dagger{}_i \sigma_{ij}^\alpha \psi_j \tau_\alpha$ parallel and orthogonal to ϕ are both constants. Then we can decompose

$$
\psi^\dagger{}_i \sigma_{ij}^\alpha \psi_j \tau_\alpha = \xi_2^2 \cos\theta_c \hat\phi + \xi_2^2 \sin\theta_c \hat k, \tag{5.59}
$$

where $\hat k$ is a vector in the adjoint, orthogonal to $\hat\phi$. Following the equation (5.26) we can also write $\hat k$ as

$$
\hat k = hU\tau^2 U^\dagger h^\dagger, \tag{5.60}
$$

where h and U are as defined in §3.2 .

Using the identity $\sigma^\alpha{}_{ij}\sigma^\alpha{}_{kl} = \delta_{il}\delta_{kj} - \frac{1}{2}\delta_{ij}\delta_{kl}$, we find that ψ is an eigenvector of the expression on the left hand side of Eq. (5.59) (see Appendix A.2). Then writing the right hand side of that equation in terms of h and U, we find that ψ can be written as

$$
\psi = \xi_2 hU \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \tag{5.61}
$$

where ρ_1 and ρ_2 are constants. Keeping U fixed, we vary ξ and find the periodicity

$$\psi(\xi) = \psi(\xi + 4\pi). \tag{5.62}$$

This ξ is the angle parameter of the residual $U(1)$ gauge symmetry and in the presence of a string solution, this ξ is mapped onto a circle around the string. In order to make ψ single valued around the string, we need $\xi = 2\chi$, where χ is the angular coordinate for a loop around the string. Next let us calculate the Lagrangian of the scalar field ψ. We have, writing ρ for the constant doublet of Eq.(5.60),

$$
\begin{aligned}
D_\mu \psi &= \partial_\mu \psi - ig A_\mu \psi && (5.63) \\
&= \partial_\mu(hU\rho) - ig \left[B_\mu \hat{\phi} + ig \left[\hat{\phi}, \partial_\mu \hat{\phi} \right] \right] hU\rho && (5.64) \\
&= \partial_\mu(Uh_0\rho) - ig \left[B_\mu \hat{\phi} + ig \left[\hat{\phi}, \partial_\mu \hat{\phi} \right] \right] Uh_0\rho && (5.65) \\
&= \partial_\mu Uh_0\rho - i(2\partial_\mu\chi + gB_\mu)Uh_0\tau^3\rho + \left[\hat{\phi}, \left[\partial_\mu UU^\dagger, \hat{\phi} \right] \right] Uh_0\rho && (5.66) \\
&= \partial_\mu Uh_0\rho - i(2\partial_\mu\chi + gB_\mu)Uh_0\tau^3\rho + \hat{\phi}2\mathrm{Tr}\left(\hat{\phi}\partial_\mu UU^\dagger \right) Uh_0\rho - \partial_\mu Uh_0\rho \\
& && (5.67) \\
&= -iUh_0\tau^3\rho \left[2\partial_\mu\chi + g\left(B_\mu + N_\mu \right) \right], && (5.68)
\end{aligned}
$$

where $h_0 = e^{-i2\chi\tau_3}$, $\rho^i\rho^i = \xi_2{}^2$, and we have used the identity $U^\dagger h U = \exp(-2i\chi\tau^3)$. We have also introduced the Abelian 'monopole field' N_μ as defined in Eq. (5.39). The term N_μ reproduces the magentic field of the monopole configuration with the Dirac string. As in the earlier construction, this singular string is a red herring, and we are going to ignore it. We have discussed the reason for this at the end of §4.2. Then we can write our effective Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\xi_2{}^2}{2}\left(\partial_\mu\chi + e\left(B_\mu + N_\mu \right) \right)^2. \tag{5.69}$$

Here we have defined the electric charge as $e = \frac{g}{2}$ and written the magnetic charge as $Q_m = \frac{1}{2e}$. Then $Q_m e = \frac{1}{2}$.

5.4. Dual theory and confinement

Let us now dualize the low energy effective action in order to express the theory in terms of the macroscopic string variables. The functional integration Z can be written using Eq. (5.42) or Eq.(5.69). The functional integration for the Lagrangian of Eq. (5.69) can be recovered if we replace the terms $|\vec{K}|$ and g of the first equation into ξ_2 and e of the second equation. So we have decided to use Eq.(5.69) for dualization.

$$Z = \int \mathcal{D}B_\mu \mathcal{D}\chi \exp i \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\xi_2^2}{2}\left(eB_\mu + \partial_\mu\chi + eN_\mu\right)^2 \right]. \quad (5.70)$$

We know that the field χ cannot be treated as a regular field that reaches to zero very fast at large distances because in the presence of topological defects it has non zero values at very large distances. Following the discussion in §2.4, in the presence of flux tubes here also we decompose the angle χ into a part χ^s which measures flux in the tube, and a part χ^r describing single valued fluctuations around this configuration, $\chi = \chi^r + \chi^s$.

Now we have integrations over both χ^r and χ^s, and the second term in the action can be linearized by introducing an auxiliary field C_μ,

$$\int \mathcal{D}\chi^r \exp\left[i \int d^4x \frac{\xi_2^2}{2}\left(eB_\mu + \partial_\mu\chi_s + \partial_\mu\chi_r + eN_\mu\right)^2 \right]$$
$$= \int \mathcal{D}\chi^r \mathcal{D}C_\mu \exp\left[-i \int d^4x \left\{ \frac{1}{2\xi_2^2}C_\mu^2 + C^\mu(eN_\mu + eB_\mu + \partial_\mu\chi^r + \partial_\mu\chi^s) \right\} \right].$$
$$(5.71)$$

The integration over χ_r can be replaced by a functional integration over a regular vector field f_μ by introducing a delta functional in Z. So the integration over χ_r in Eq. (5.71),

$$\int \mathcal{D}\chi^r \exp\left[i \int d^4x C^\mu \partial_\mu\chi^r \right] = \int \mathcal{D}f_\mu \delta\left(\partial_\mu f_\nu - \partial_\nu f_\mu\right) \exp\left[i \int d^4x C^\mu f_\mu \right] \quad (5.72)$$

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$$= \int \mathcal{D}f_\mu \mathcal{D}B_{\mu\nu} \exp\left[i \int d^4 x C^\mu f_\mu - \frac{\xi_2}{2} \epsilon^{\mu\nu\rho\lambda} \partial_\mu f_\nu B_{\rho\lambda}\right] \tag{5.73}$$

$$= \int \mathcal{D}f_\mu \mathcal{D}B_{\mu\nu} \exp\left[i \int d^4 x C^\mu f_\mu - \frac{\xi_2}{2} \epsilon^{\mu\nu\rho\lambda} f_\mu \partial_\nu B_{\rho\lambda}\right] \tag{5.74}$$

$$= \int \mathcal{D}B_{\mu\nu} \delta\left(C^\mu - \frac{\xi_2}{2} \epsilon^{\mu\nu\rho\lambda} \partial_\nu B_{\rho\lambda}\right), \tag{5.75}$$

where we have introduced a second rank tensor field $B_{\mu\nu}$ by exponentiating the delta functional.

Integrating over the field C_μ, we get the partition function,

$$Z = \int \mathcal{D}B_\mu \quad \mathcal{D}x_\mu(\xi) \mathcal{D}B_{\mu\nu} \exp\left[i \int d^4 x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}\right.\right.$$
$$\left.\left. -\frac{\xi_2}{2} \Sigma_{\mu\nu} B^{\mu\nu} - \frac{e\xi_2}{4} \epsilon^{\mu\nu\rho\lambda} M_{\mu\nu} B_{\rho\lambda} - \frac{e\xi_2}{2} \epsilon^{\mu\nu\rho\lambda} B_\mu \partial_\nu B_{\rho\lambda}\right\}\right], \tag{5.76}$$

here we have written $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + M_{\mu\nu}$, defined $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$, and also written $M_{\mu\nu} = (\partial_\mu N_\nu - \partial_\nu N_\mu)$. $\Sigma_{\mu\nu}$ is defined as

$$\epsilon^{\mu\nu\rho\lambda} \partial_\rho \partial_\lambda \chi^s = 2\pi n \int_\Sigma d\sigma^{\mu\nu}(x(\xi)) \, \delta^4(x - x(\xi)) \equiv \Sigma^{\mu\nu}, \tag{5.77}$$

if χ winds around the tube n times. Here $\xi = (\xi^1, \xi^2)$ are the coordinates on the worldsheet and $d\sigma^{\mu\nu}(x(\xi)) = \epsilon^{ab} \partial_a x^\mu \partial_b x^\nu$.

Let us now integrate over the field B_μ which we can do by introducing an auxiliary field $\chi_{\mu\nu}$,

$$\int \mathcal{D}B_\mu \exp\left[i \int d^4 x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{e\xi_2}{2} \epsilon^{\mu\nu\rho\lambda} B_\mu \partial_\nu B_{\rho\lambda} - \frac{e\xi_2}{4} \epsilon^{\mu\nu\rho\lambda} M_{\mu\nu} B_{\rho\lambda}\right\}\right]$$
$$= \int \mathcal{D}B_\mu \mathcal{D}\chi_{\mu\nu} \exp\left[i \int d^4 x \left\{ -\frac{1}{4} \chi_{\mu\nu} \chi^{\mu\nu} + \frac{1}{4} \epsilon^{\mu\nu\rho\lambda} \chi_{\mu\nu} F_{\rho\lambda}\right.\right.$$
$$\left.\left. -\frac{e\xi_2}{2} \epsilon^{\mu\nu\rho\lambda} B_{\mu\nu} \partial_\rho B_\lambda - \frac{e\xi_2}{4} \epsilon^{\mu\nu\rho\lambda} M_{\mu\nu} B_{\rho\lambda}\right\}\right]$$
$$= \int \mathcal{D}\chi_{\mu\nu} \, \delta\left[\epsilon^{\mu\nu\rho\lambda} \partial_\nu (\chi_{\rho\lambda} - e\xi_2 B_{\rho\lambda})\right]$$

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$$\exp\left[i\int d^4x\left\{-\frac{1}{4}\chi_{\mu\nu}\chi^{\mu\nu}+\frac{1}{4}\epsilon^{\mu\nu\rho\lambda}(\chi_{\mu\nu}-e\xi_2 B_{\mu\nu})M_{\rho\lambda}\right\}\right]\,. \tag{5.78}$$

We can integrate over $\chi_{\mu\nu}$ by solving the δ-functional in the same way as before as

$$\chi_{\mu\nu}=e\xi_2 B_{\mu\nu}+\partial_\mu A_\nu^m-\partial_\nu A_\mu^m\,, \tag{5.79}$$

and thus dualizing the vector potential B_μ to a theory of a magnetic photon A_μ^m. The result of the integration is then inserted into Eq. (5.76) to give

$$Z=\int\mathcal{D}A_\mu^m\mathcal{D}x_\mu(\xi)\mathcal{D}B_{\mu\nu}$$
$$\exp\left[i\int\left\{-\frac{1}{4}\left(e\xi_2 B_{\mu\nu}+\partial_{[\mu}A_{\nu]}^m\right)^2+\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}-\frac{\xi_2}{2}\Sigma_{\mu\nu}B^{\mu\nu}-j_m^\mu A^{m\mu}\right\}\right]\,. \tag{5.80}$$

Here $j_m^\mu=-\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu M_{\rho\lambda}$ is the current of magnetic monopoles.

The functional integration (5.80) can be calculated from the partition function(3.70). in a straightforward way. The integration over χ in Eq. (3.70) becomes integrations over both χ^r and χ^s. However χ^r is a single-valued field, so it can be absorbed into the gauge field B_μ by a redefinition, or gauge transformation, $B_\mu\to B_\mu+\partial_\mu\chi^r$. We can linearize the action by introducing auxiliary fields $C_\mu, B_{\mu\nu}$ and A_μ^m,

$$Z=\int\mathcal{D}B_\mu\mathcal{D}C_\mu\mathcal{D}\chi_s\mathcal{D}B_{\mu\nu}\mathcal{D}A_\mu^m$$
$$\exp i\int d^4x\left[-\frac{1}{4}G^{\mu\nu}G_{\mu\nu}+\frac{1}{4}\epsilon^{\mu\nu\rho\lambda}G_{\mu\nu}F_{\rho\lambda}-\frac{1}{2\xi_2^2}C_\mu^2-C^\mu(eB_\mu+eN_\mu+\partial_\mu\chi_s)\right]\,, \tag{5.81}$$

where we have written $G_{\mu\nu}=\partial_\mu A_\nu^m-\partial_\nu A_\mu^m+e\xi_2 B_{\mu\nu}$ and $F_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu+M_{\mu\nu}$. Now we can integrate over B_μ easily to get

$$Z=\int\mathcal{D}C_\mu\mathcal{D}\chi_s\mathcal{D}B_{\mu\nu}\mathcal{D}A_\mu^m\delta\left(C^\mu-\frac{\xi_2}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}\right)\exp i\int d^4x$$
$$\left[-\frac{1}{4}G^{\mu\nu}G_{\mu\nu}+\frac{e\xi_2}{4}\epsilon^{\mu\nu\rho\lambda}B_{\mu\nu}M_{\rho\lambda}-A_\mu^m j_m^\mu-\frac{1}{2\xi_2^2}C_\mu^2-C^\mu(eN_\mu+\partial_\mu\chi_s)\right]\,. \tag{5.82}$$

Here $j_m^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu M_{\rho\lambda}$ is the magnetic monopole current. Integrating over C_μ we get

$$Z = \int \mathcal{D}\chi_s \mathcal{D}B_{\mu\nu}\mathcal{D}A_\mu^m \exp i \int d^4x \left[-\frac{1}{4}G^{\mu\nu}G_{\mu\nu} + \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} - \frac{\xi_2}{2}\Sigma_{\mu\nu}B^{\mu\nu} - A_\mu^m j_m^\mu\right].$$

(5.83)

Here we have defined $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$, used Eq. (5.77) and also used $M_{\mu\nu} = (\partial_\mu N_\nu - \partial_\nu N_\mu)$.

As discussed before, the integration over $\mathcal{D}\chi^s$ can be replaced by an integration over $\mathcal{D}x_\mu(\xi)$, representing a sum over all the flux tube worldsheets, where $x_\mu(\xi)$ parametrizes the surface of singularities of χ. The Jacobian for this change of variables gives the action for the string on the background space time [30, 68]. The string has a dynamics given by the Nambu-Goto action, plus higher order operators [59], which can be obtained from the Jacobian. Since we are not investigating the dynamics of the string here, we will simply assume that this has been done.

$$Z = \int \mathcal{D}x_\mu(\xi)\mathcal{D}B_{\mu\nu}\mathcal{D}A_\mu^m \exp i \int d^4x \left[-\frac{1}{4}G^{\mu\nu}G_{\mu\nu} + \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} - \frac{\xi_2}{2}\Sigma_{\mu\nu}B^{\mu\nu} - A_\mu^m j_m^\mu\right],$$

(5.84)

The equations of motion for the field $B_{\mu\nu}$ and A^μ can be calculated from this to be

$$\partial_\lambda H^{\lambda\mu\nu} = -m\,G^{\mu\nu} - \frac{m}{e}\Sigma^{\mu\nu},$$

(5.85)

$$\partial_\mu G^{\mu\nu} = j_m^\mu,$$

(5.86)

where $G_{\mu\nu} = mB_{\mu\nu} + \partial_\mu A_\nu^m - \partial_\nu A_\mu^m$, and $m = e\xi_2$. Combining Eq. (5.85) and Eq. (5.86) we find that

$$\frac{1}{e}\partial_\mu\Sigma^{\mu\nu}(x) + j_m^\mu(x) = 0.$$

(5.87)

The same equation that we found in the theory with two adjoint scalars. As in that case, it follows that a vanishing magnetic monopole current implies $\partial_\mu\Sigma^{\mu\nu}(x) = 0$,

5. Confinement of monopoles in broken SU(2)

or in other words if there is no monopole in the system, the flux tubes will be either closed or or infinite. There is however an important difference between this and the previous construction. Here the magnetic flux through the tube is $\dfrac{2n\pi}{e}$, while the total magnetic flux of the monopole is $4m\pi Q_m$, where n, m are integers. Since $eQ_m = \dfrac{1}{2}$ for the fundamental SU(2) scalar, it follows that we can have a string that confines a monopole and anti-monopole pair for every integer n. However, this was not true for the construction involving two adjoint scalars because in that case $e = g$ and $gQ_m = 1$ which makes $n = 2m$. This means for confinement of monpoles with $m = 1$ will be possible if $n = 2$.

Although these string configuration could be broken by creating a monopole-anti-monopole pair, there is a hierarchy of energy scales $\xi_1 \gg \xi_2$, which are respectively proportional to the mass of the monopole and the energy scale of the string. So this hierarchy can be expected to prevent string breakage by pair creation.

The conservation law of Eq. (5.87) also follows directly from Z in Eq. (5.84) and this can be shown by making a gauge transformation as discussed in chapter 2. However, this can also be derived by introducing a variable $B'_{\mu\nu} = B_{\mu\nu} + \frac{1}{m}(\partial_\mu A^m_\nu - \partial_\nu A^m_\mu)$ and integrating over the field A^m_μ. If we do so we get

$$Z = \int \mathcal{D}x_\mu(\xi)\mathcal{D}B'_{\mu\nu} \quad \delta\left[\frac{1}{e}\partial_\mu \Sigma^{\mu\nu}(x) + j^\nu_m(x)\right]$$
$$\exp\left[i\int\left\{\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}m^2 B'^2_{\mu\nu} - \frac{m}{2e}\Sigma_{\mu\nu}B'^{\mu\nu}\right\}\right],$$

$$(5.88)$$

with the delta functional showing the conservation law (5.87). Thus these strings are analogous to the confining strings in three dimensions [69]. There is no A^m_μ, the only gauge field which is present is $B'_{\mu\nu}$. This $B'_{\mu\nu}$ field mediates the direct interaction between the confining strings.

6. Conclusion

In this thesis we have studied the static and dynamical properties of magnetic monopoles and flux tubes. Then we have studied configuration which carry both magnetic monopoles and flux tubes. For the system with both monopole and flux tube we have showed the confinement of monopoles by flux tubes. By confinement we mean that when monopoles are attached at the ends of a flux tube, no flux escapes from the tube to far away. Thus all flux is "confined". To show this we have calculated three different cases. For the Abelian Higgs model first we have constructed a long distance effective theory of flux tubes and we described the system by dual variables. Then we have shown that if we add monopoles in the system externally then they confine themselves by attaching to the ends of the flux tubes to seal the flux. The second case that we have considered is an SU(2) gauge theory with two adjoint scalar fields and the last case is an SU(2) gauge theory with one adjoint and one fundamental scalar fields. For all the three cases the attachment of monopoles at the end of flux tubes have been shown by a delta functional. The delta functional enforces that at every point of space-time, the monopole current cancels the currents of the end points of flux tube. So the monopole current must be non-zero only at the end of the flux tube. The last functional integration Eq. (5.88) does not carry Abelian gauge field A_μ^m, only a massive second rank tensor gauge field is present. All these confirm the permanent attachment of monopoles at the

6. Conclusion

end of the flux tube which does not allow gauge flux to escape out of the flux tubes. There are important differences between the results from the construction involving two adjoint scalar fields and the one involving one adjoint and one fundamental scalar. In the first case the mass of the Abelian photon will be zero if the two vevs are aligned in the same direction. However, this can never happen for the second case where the scalars are in different representations. Also, in the first case, the flux inside the tube for $n = 1$ is only $\dfrac{2\pi}{g}$, whereas if the second scalar is in the fundamental representation then the flux inside the tube for $n = 1$ will be $\dfrac{4\pi}{g}$. So when two adjoint scalar fields are used, monopole confinement is not possible with a single $n = 1$ flux tube with winding number $n = 1$. However, it is possible with a single $n = 2$ flux tube construction or with two $n = 1$ flux tubes attached with two oppositely charged monopoles. There may be a possibility in which two $n = 1$ flux tubes can attach to a monopole from opposite directions.

A. A review on SU(2) and the rotation group

Here we shall talk about mainly the rotation group O(3) and relation of O(3) with a group of special complex 2×2 matrices (SU(2)).

A.1. Rotation group

A general spatial rotation is defined by a transformation

$$r'_i = R_{ij}r_j, \tag{A.1}$$

where R is a rotation matrix. Since rotations preserve distances from the origin,

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2. \tag{A.2}$$

It follows that

$$R^T R = 1, \tag{A.3}$$

and all these R form a group called $O(3)$. If we restrict $\det R = 1$ then we call the resulting group $SO(3)$. The matrix for rotation around z, x, y can be written as

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \qquad (A.4)$$

$$R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix} \qquad (A.5)$$

These group elements have generators(angular momentum),

$$J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad (A.6)$$

These generators form a basis for a vector space with anti symmetric product

$$\left[J^i, J^j\right] = i\epsilon^{ijk} J^k. \qquad (A.7)$$

This vector space is called Lie Algebra of SO(3).

A general rotation matrix in three dimensions can be written as

$$R_{Euler} = R_z(\psi) R_x(\theta) R_z(\phi), \qquad (A.8)$$

here ψ, ϕ, θ are called Euler's angles.

A.2. A representation of rotation by Cayley- Klein parameters or SU(2) group

Let us consider a vector

$$\Psi = \begin{pmatrix} u \\ v \end{pmatrix} \tag{A.9}$$

in a complex two dimensional vector space. A transformation of this vector

$$u' = \alpha u + \beta v \tag{A.10}$$

$$v' = \gamma u + \delta v \tag{A.11}$$

or

$$\Psi' = g\,\Psi, \text{ where} \tag{A.12}$$

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{A.13}$$

will leave the norm $\psi^\dagger \psi = uu^* + vv^*$ invariant if

$$g^\dagger g = 1, \qquad \det g = 1. \tag{A.14}$$

i.e

$$\alpha\alpha^* + \beta\beta^* = 1, \ \gamma\gamma^* + \delta\delta^* = 1 \tag{A.15}$$

$$\alpha\gamma^* + \beta\delta^* = 0, \ \gamma\alpha^* + \delta\beta^* = 0 \tag{A.16}$$

$$\alpha\delta - \beta\gamma = 1 \tag{A.17}$$

Above Eq.(A.14) suggests that

$$\delta = \alpha^*, \gamma = -\beta^*, \tag{A.18}$$

$$g = \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \tag{A.19}$$

A. A review on SU(2) and the rotation group

These parameters α, γ are called Cayley-Klein parameters. Using Eq. (A.18) we can write down Eq. (A.10) as

$$u' = \alpha u - \gamma^* v \tag{A.20}$$

$$v' = \gamma u + \alpha^* v, \tag{A.21}$$

and this can also be written as

$$-v^{*\prime} = \alpha(-v^*) - \gamma^* u^* \tag{A.22}$$

$$u^{*\prime} = \gamma(-v^*) + \alpha^* u^*. \tag{A.23}$$

So we can also define a vector

$$\Psi_c = \begin{pmatrix} -v^* \\ u^* \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Psi^*, \tag{A.24}$$

which transforms the same way as Ψ and is orthogonal to ψ or $\Psi_c^\dagger \Psi = 0$. These two vectors Ψ, Ψ_c are called SU(2) spinors. All the g's together with the spinor's, represent a group of Special Unitary 2×2 complex matrices (SU(2)). Now we shall see that the SU(2) group also represents spatial rotations.

The Lie algebra of SU(2) consists of 2×2 Hermitian matrices with trace zero. One can consider this a three dimensional real vector space (i.e., it is closed under multiplication by real numbers). A basis is given by the Pauli matrices multiplied by half,

$$\tau^i = \frac{1}{2} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right], \, i = 1, 2, 3. \tag{A.25}$$

This is an orthogonal basis if we define the inner product appropriately,

$$\tau^i \cdot \tau^j = 2\text{Tr}\left(\tau^i \tau^j\right) = \delta^{ij}, \tag{A.26}$$

and the commutation relations can be written as

$$\left[\tau^i, \tau^j\right] = i\epsilon^{ijk}\tau^k. \tag{A.27}$$

The SU(2) Lie algebra is isomorphic to the SO(3) Lie algebra (A.7) and there is a homomorphism from SU(2) onto SO(3). These groups are then locally "the same": The proof is an application of the Frobenius theorem. We can exhibit the classical homomorphism as

$$Ad : SU(2) \rightarrow SO(3), \tag{A.28}$$

here the adjoint action is defined by $Ad(g)Y = gYg^{-l}$ for $g \in$ SU(2). The adjoint representation of SU(2) on its 3-dimensional Lie algebra yields the standard representation of SO(3) on R^3. To show this explicitly, let us define a map

$$* \quad : \quad R^3 \rightarrow L_G(\mathrm{SU}(2)), \qquad X \mapsto X_* \tag{A.29}$$

$$X_* \quad = \quad x^i\tau^i = \frac{1}{2}\begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \tag{A.30}$$

where L_G is the Lie Algebra of SU(2). This linear transformation maps R^3 onto the space of traceless hermitian matrices and has inverse given by

$$x = 2\mathrm{Tr}\left(X_*\tau^1\right), \qquad y = 2\mathrm{Tr}\left(X_*\tau^2\right), \qquad z = 2\mathrm{Tr}\left(X_*\tau^3\right). \tag{A.31}$$

Every Lie group G acts on its Lie algebra L_G by the adjoint action

$$Ad : G \rightarrow L_G(g), \qquad Ad(g)X = gXg^{-1}, \qquad g \in G. \tag{A.32}$$

Under this adjoint action we can write down

$$X'_* = gX_*g^{-1}, \qquad \forall g \in \mathrm{SU}(2), \tag{A.33}$$

here with each 2×2 $g \in \mathrm{SU}(2)$ we associate a 3×3 matrix.

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We also note that

$$X'_* \cdot X'_* = X_* \cdot X_* = x^2 + y^2 + z^2 \qquad (A.34)$$

So this transformation (A.33) represents rotations in R^3. To see different rotations we have to write down different values of the parameters α, γ in Eq. (A.19) and use Eq. (A.33).

For example a rotation around the z-axis is represented by

$$g_\varphi = \begin{pmatrix} e^{\frac{i}{2}\varphi} & 0 \\ 0 & e^{-\frac{i}{2}\varphi} \end{pmatrix} \qquad (A.35)$$

With this g_φ Eq. (A.33) can be written as

$$
\begin{aligned}
X'_* &= x'_i \tau^i & (A.36) \\
&= g_\varphi X_* g_\varphi^\dagger & (A.37) \\
&= g_\varphi (x\tau_1 + y\tau_2 + z\tau_3) g_\varphi^\dagger & (A.38) \\
&= g_\varphi x^i \tau_i g_\varphi^\dagger & (A.39) \\
&= x^i g_\varphi \tau_i g_\varphi^\dagger & (A.40) \\
&= R_z^{ij}(\varphi) x_j \tau^i, & (A.41)
\end{aligned}
$$

where $\tau^i = \frac{1}{2}\sigma^i$ (Pauli matrices) and $R_z(\varphi)$ is defined by Eq. (A.4). Since τ^i are linearly independent

$$x'_i = R_{ij} x_j, \qquad (A.42)$$

and this is same as Eq.(A.1). So we have seen that it is possible to represent rotation by SU(2) transformation of matrices of the form $X_* = x^i \tau_i$. X_* in Eq.(A.30) is called an adjoint vector of SU(2) and Eq.(A.33) is the transformation of the adjoint vector under SU(2). That means for a given $g \in SU(2)$ we can always write down a single

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$R \in SO(3)$. However, the converse is not true, because to get a single rotation matrix R we could have used both g and $\tilde{g} = hg$ in Eq. (A.33), where $[h, X'_*] = 0$ and $h \in SU(2)$. In fact we can check that $g = \{1, -1\}$ map to $R = 1$. So the kernel of this mapping is Z_2. So the mapping from SU(2) to SO(3) is not an isomorphism but a 2 to 1 homomorphism. So we have shown that rotation group SO(3) can be represented by SU(2) adjoint representations.

Now we shall show that rotations can also be represented by SU(2) spinors. Let us define two eigenvectors of τ^3 as

$$\tau^3 |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle \tag{A.43}$$

$$\tau^3 |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle \tag{A.44}$$

We have defined two SU(2) spinors in Eq.s (A.9),(A.24) as

$$\Psi = \begin{pmatrix} u \\ v \end{pmatrix} = u |\uparrow\rangle + v |\downarrow\rangle \tag{A.45}$$

$$\Psi_c = \begin{pmatrix} -v^* \\ u^* \end{pmatrix} = -v^* |\uparrow\rangle + u^* |\downarrow\rangle \tag{A.46}$$

$$|u|^2 + |v|^2 = 1 \tag{A.47}$$

Now let us calculate the quantity $\Psi^\dagger \phi \Psi$ where ϕ is an adjoint vector of SU(2).

$$\Psi^\dagger \phi \Psi = \Psi^\dagger \phi^i \tau^i \Psi \tag{A.48}$$

$$= \phi^i \Psi^\dagger \tau^i \Psi = \mathrm{Tr}\,(\phi\Phi), \tag{A.49}$$

where $\Phi = \Psi^\dagger \sigma^i \Psi \tau^i$. Using the definitions of a spinor from Eq. (A.45)we can write

$$\Phi = (|u|^2 - |v|^2)\tau^3 + (v^*u + u^*v)\tau^1 + i(v^*u - u^*v)\tau^2 \tag{A.50}$$

$$= U(u, v)\tau^3 U^\dagger(u, v) \tag{A.51}$$

where

$$U(u, v) = \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix}. \tag{A.52}$$

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This is the same as Eq. (A.18) and the spinors can be written as

$$\Psi = U(u,v)|\uparrow\rangle, \tag{A.53}$$

$$\Psi_c = U(u,v)|\downarrow\rangle. \tag{A.54}$$

In Eq.(A.50) the components of Φ are real numbers and Φ represent an $SO(3)$ Lie algebra vector. Now we are going to calculate the spinors that represents unit vector in 3D space. To do that we first calculate $U(u,v)$ by comparing the components of Φ and components of $\hat{r}$ and we get

$$|u|^2 - |v|^2 = \cos\theta \tag{A.55}$$

$$v^*u + u^*v = \sin\theta\cos\varphi \tag{A.56}$$

$$i(v^*u - u^*v) = \sin\theta\sin\varphi. \tag{A.57}$$

Using the above equations we can write $\Phi = \hat{r}^i\tau^i$. There are many solutions of these equations, We write down two possibilities, solutions.

$$u = \cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}} \quad , \qquad v = \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}} \tag{A.58}$$

$$u = \cos\frac{\theta}{2} \quad , \qquad v = \sin\frac{\theta}{2}e^{i\varphi}. \tag{A.59}$$

The corresponding SU(2) elements according to Eq.(A.52) are then

$$U_1(\theta,\varphi) = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}} & -\sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}} & \cos\frac{\theta}{2}e^{i\frac{\varphi}{2}} \end{pmatrix}, \tag{A.60}$$

$$U_2(\theta,\varphi) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{-i\varphi} \\ \sin\frac{\theta}{2}e^{i\varphi} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{A.61}$$

Using equations (A.53) and (A.54) we also find the spinor

$$\Psi_1(\theta,\varphi) = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}} \end{pmatrix} \quad , \quad \Psi_1^c(\theta,\varphi) = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}} \\ \cos\frac{\theta}{2}e^{i\frac{\varphi}{2}} \end{pmatrix} \tag{A.62}$$

$$\Psi_2(\theta,\varphi) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix} \quad , \quad \Psi_2^c(\theta,\varphi) = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{A.63}$$

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We can write down a group element $h = exp(-i\varphi\Phi)$ for which $U_1(\theta, \varphi) = h(\varphi, \Phi)U_2(\theta, \varphi)$. So we have seen that there is no unique way to write down a SU(2) element for a SO(3) element. In general we can write the group elements as

$$G(\psi, \theta, \varphi) = h(\hat{r}, \chi)U_1(\theta, \varphi). \tag{A.64}$$

It suggests that there is a class of spinors for a single unit SO(3) vector. This class is a circular orbit which maps to a point on S^2, whereas the S^2 is the orbit of a unit SO(3) vector. The reason can be understood very easily. The orbit of a unit spinor is S^3 and locally $S^3 \simeq S^1 \times S^2$. So the mapping $S^3 \rightarrow S^2$ is possible if we manage to map S^1 to a point on S^2 and a unit adjoint vector does the job. This geometry will be useful when we discuss spontaneous symmetry breaking. So far we have shown how to construct an SO(3) vector from SU(2) spinors.

Now we shall show the converse. In Eq. (A.49) we have seen that an SU(2) adjoint vector Φ can be constructed by SU(2) spinors as $\Phi = \Psi_i^\dagger \sigma_{ij}^\alpha \Psi_j \tau^\alpha$. The squared length of the vector Φ can be calculated.

$$\begin{aligned}
|\Phi|^2 &= 2\mathrm{Tr}\,(\Phi\Phi) & \text{(A.65)} \\
&= \Phi^\alpha \Phi^\alpha & \text{(A.66)} \\
&= \Psi_i^\dagger \sigma_{ij}^\alpha \Psi_j \Psi_k^\dagger \sigma_{kl}^\alpha \Psi_l & \text{(A.67)} \\
&= (2\delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl})\Psi_i^\dagger \Psi_j \Psi_k^\dagger \Psi_l & \text{(A.68)} \\
&= \Psi_i^\dagger \Psi_i \Psi_j^\dagger \Psi_j = |\Psi|^4. & \text{(A.69)}
\end{aligned}$$

Φ is a matrix and we can multiply it from the left to the spinors. So we can multiply the adjoint vector Φ with the spinor Ψ by which it was constructed.

$$\begin{aligned}
\Phi_{ij}\Psi_j &= \Psi_k^\dagger \sigma_{kl}^\alpha \Psi_l \tau_{ij}^\alpha \Psi_j & \text{(A.70)} \\
&= \frac{1}{2}\sigma_{ij}^\alpha \sigma_{kl}^\alpha \Psi_k^\dagger \Psi_l \Psi_j & \text{(A.71)}
\end{aligned}$$

$$= (\delta_{il}\delta_{kj} - \frac{1}{2}\delta_{ij}\delta_{kl})\Psi_k^\dagger\Psi_l\Psi_j \tag{A.72}$$

$$= \frac{1}{2}|\Psi|^2\Psi_i \tag{A.73}$$

$$= \frac{1}{2}|\Phi|\Psi_i, \tag{A.74}$$

and

$$\Phi_{ij}\Psi_j^c = \Psi_k^\dagger\sigma_{kl}^\alpha\Psi_l\tau_{ij}^\alpha\Psi_j^c \tag{A.75}$$

$$= \frac{1}{2}\sigma_{ij}^\alpha\sigma_{kl}^\alpha\Psi_k^\dagger\Psi_l\Psi_j^c \tag{A.76}$$

$$= (\delta_{il}\delta_{kj} - \frac{1}{2}\delta_{ij}\delta_{kl})\Psi_k^\dagger\Psi_l\Psi_j^c \tag{A.77}$$

$$= -\frac{1}{2}|\Psi|^2\Psi_i^c \tag{A.78}$$

$$= -\frac{1}{2}|\Phi|\Psi_i, \tag{A.79}$$

here we have used the identity $\delta_{il}\delta_{kj} = \frac{1}{2}\delta_{ij}\delta_{kl} + \frac{1}{2}\sigma_{ij}^\alpha\sigma_{kl}^\alpha$. So we see that spinors are the eigenvectors of an adjoint vector Φ with eigenvalues $\pm\frac{1}{2}|\Phi|$. However, as we have seen before, the spinors are not unique because they can be multiplied by a matrix that commutes with Φ. The non trivial SU(2) element that commute with Φ is $\exp(-i\chi\hat{\Phi})$ for all χ. So we have seen that for every adjoint (SO(3)) vector there are two classes of SU(2) spinors with eigenvalues $\pm\frac{1}{2}|\Phi|$.

A.3. Some useful notation for SU(2) fields

Any 2×2 complex matrix X can be written as

$$X = X_0\mathbf{1} + X_i\sigma_i, \tag{A.80}$$

where

$$X_0 = \frac{1}{2}\mathrm{Tr}\,(X), \qquad X_i = \frac{1}{2}\mathrm{Tr}\,(X\sigma^i). \tag{A.81}$$

A. A review on SU(2) and the rotation group

Substitute Eq. (A.81) into Eq. (A.80) we get

$$X_{ij} = \frac{1}{2}X_{kk}\delta_{ij} + \frac{1}{2}X_{lk}\sigma_{kl}^{\alpha}\sigma_{ij}^{\alpha}. \tag{A.82}$$

We compare the coefficients of X_{lk} both sides and get

$$\delta_{il}\delta_{kj} = \frac{1}{2}\delta_{ij}\delta_{kl} + \frac{1}{2}\sigma_{ij}^{\alpha}\sigma_{kl}^{\alpha}. \tag{A.83}$$

Any two fields A and B in the adjoint representation of SU(2) can be written as

$$A = A^i\tau^i, \qquad B = B^i\tau^i. \tag{A.84}$$

Using the orthogonality relation of the algebra we can calculate the trace.

$$2\mathrm{Tr}\,(AB) = A^iB^i = \vec{A}\cdot\vec{B} \tag{A.85}$$

So the trace can be written (upto a factor of 2) as dot product between two vectors $\vec{A}$ and $\vec{B}$. Let us calculate the commutator of the fields A and B.

$$[A,B] = A^jB^k\left[\tau^j,\tau^k\right] = i\epsilon^{ijk}\tau^iA^jB^k \tag{A.86}$$
$$= i\left(\vec{A}\times\vec{B}\right)^i\tau^i \tag{A.87}$$

We can invert the above equation by using orthogonality relations and write the cross product between two vectors $\vec{A}$ and $\vec{B}$ as,

$$\left(\vec{A}\times\vec{B}\right)^i = -2i\mathrm{Tr}\,\left(\tau^i\,[A,B]\right). \tag{A.88}$$

The commutator of three fields can be written as

$$[A,[B,C]] = i\epsilon^{klm}B_lC_m\left[A,\tau^k\right] = -\epsilon^{ijk}\tau^iA_j\epsilon^{klm}B_lC_m \tag{A.89}$$
$$= -\left(\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}\right)\tau^iA_jB_lC_m \tag{A.90}$$
$$= C\,2\mathrm{Tr}\,(AB) - B\,2\mathrm{Tr}\,(AC) \tag{A.91}$$

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1)* C. Chatterjee and A. Lahiri, Flux dualization in broken SU(2), JHEP **1002**, 033 (2010) [arXiv:0912.2168 [hep-th]].

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5)* C. Chatterjee and A. Lahiri, Monopole confinement by flux tube, Europhys. Lett. **76**, 1068 (2006) [arXiv:hep-ph/0605107].

The papers with $*$ are included in this thesis.