# A generalized Ramanujan Master Theorem applied to the evaluation of Feynman diagrams

<sup>a</sup>Iván González\*, <sup>b</sup>V. H. Moll<sup>†</sup>, <sup>a</sup>Iván Schmidt<sup>‡</sup>

<sup>a</sup>Departamento de Física
Universidad Técnica Federico Santa María
and
Centro Científico-Tecnológico de Valparaiso (CCTVal), Valparaiso, Chile.

<sup>b</sup>Department of Mathematics, Tulane University, New Orleans, LA 70118, USA.

#### Abstract

Ramanujan Master Theorem is a technique developed by the indian mathematician S. Ramanujan to evaluate a class of definite integrals. This technique is used here to calculate the values of integrals associated with specific Feynman diagrams.

**PACS**: 11.25.Db; 12.38.Bx.

**Keywords**: Perturbation theory; Scalar integrals; Multiloop Feynman diagrams; Schwinger parameters; Ramanujan's Master Theorem (RMT).

\*e-mail : ivan.gonzalez@usm.cl †e-mail : vhm@math.tulane.edu ‡e-mail : ivan.schmidt@usm.cl

#### 1 Introduction

Precise experimental measurements in high energy physics require, in its theoretical counterpart, the development of techniques for the evaluation of analytic objects associated with the corresponding Feynman diagrams. These techniques have lately emphasized the automatization of calculations of multiscale, multiloop diagrams.

Modern numerical methods for the evaluation of Feynman diagrams benefit from analytical techniques employed as preliminary work to detect the presence of divergences. Recent advances include a method based on the Bernstein-Tkachov theorem for the corrections of one and two loop diagrams [1], a method based on sector-decomposition, as applied in [2] to a variety of diagrams and to the evaluation of integrals in phase space, and a third example is the one developed in [3], which contains numerical evaluations of two-point functions by differential equation methods.

New analytic methods include techniques to reduce Feynman diagrams to a small number of scalar integrals, such as integration by parts [4], the use of Lorentz invariance [5] and the use of symmetry [6]. Methods for the evaluation of scalar integrals include expansion by regions [7], Mellin-Barnes transforms [8], relations among integrals in different dimensions [9] and differential equations [10].

This paper contains examples of an alternative method for the evaluation of some Feynman diagrams. The ideas are based on the classical Ramanujan Master Theorem (RMT), which is a favorite technique employed by the well-known mathematician to evaluate definite integrals. The theoretical aspects of this method are presented in [12] and a collection of modern examples is given in [13]. The application of this technique has been illustrated in [11] with the evaluation of some multidimensional integrals obtained by the Schwinger parametrization of Feynman diagrams. The goal of the present work is to use this technique to evaluate integrals associated to two and three loop diagrams. The method works for a large variety of definite integrals and our first example illustrates this by computing the Mellin transform of a Bessel function. The next example illustrates the evaluation a multidimensional integral corresponding to the massless bubble Feynman diagram. Then we provide a generalization of RMT to multiple integrals, required for multiloop calculations. Further applications will be described in future work.

## 2 Ramanujan's Master Theorem (RMT) and its generalization: The formalism

Integrals of the form  $\int_0^\infty dx \, x^{\nu-1} f(x)$  may be evaluated by one of Ramanujan's favorite tools; the so-called Ramanujan Master Theorem. It states that if f(x) admits a series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} \varphi(n) \frac{(-x)^n}{n!}$$
 (1)

in a neighborhood of x = 0, with  $f(0) = \varphi(0) \neq 0$ , then

$$\int_{-\infty}^{\infty} dx \ x^{\nu-1} f(x) = \Gamma(\nu) \varphi(-\nu). \tag{2}$$

The integral is the Mellin transform of f(x) and the term  $\varphi(-\nu)$  requires an extension of the function  $\varphi$ , initially defined only for  $\nu \in \mathbb{N}$ . Details on the natural unique extension of  $\varphi$  are given

in [13]. Observe that, for  $\nu > 0$ , the condition  $\varphi(0) \neq 0$  guarantees the convergence of the integral near x = 0. The proof of Ramanujan Master Theorem and the precise conditions for its application appear in Hardy [12]. The reader will find in [13] many other examples.

#### 2.1 The Mellin transform of a Bessel function

The first illustration of RMT involves an integral containing the classical Bessel function, given in its hypergeometric form by

$$J_{\alpha}\left(\sqrt{x}\right) = \left(\frac{\sqrt{x}}{2}\right)^{\alpha} \frac{1}{\Gamma\left(1+\alpha\right)} \,_{0}F_{1}\left(\begin{array}{cc} - \\ 1+\alpha \end{array} \middle| -\frac{1}{4}x\right). \tag{3}$$

Here  ${}_{0}F_{1}$  is the hypergeometric function defined by

$${}_{0}F_{1}\left(\begin{array}{c|c} - & x \end{array}\right) = \sum_{n=0}^{\infty} \frac{1}{(a)_{n}} \frac{x^{n}}{n!},\tag{4}$$

using the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. (5)$$

The integral to be evaluated here is

$$I = \int_{0}^{\infty} dx \ x^{\beta - 1} J_{\alpha} \left( \sqrt{x} \right), \tag{6}$$

and can be expressed as

$$I = \int_{0}^{\infty} dx \ x^{\beta-1} \left(\frac{\sqrt{x}}{2}\right)^{\alpha} \frac{1}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \frac{1}{(1+\alpha)_{n}} \frac{x^{n}}{4^{n}}$$

$$= \int_{0}^{\infty} \frac{dx}{x} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \left[\frac{1}{2^{\alpha+2n} \Gamma(1+\alpha+n)}\right] x^{n+\beta+\frac{\alpha}{2}}.$$

$$(7)$$

The data matches the statement of RMT with

$$\varphi(n) = \frac{1}{2^{\alpha + 2n} \Gamma(1 + \alpha + n)}.$$
(8)

Therefore, the evaluation

$$I = \frac{\Gamma(-n^*)}{2^{\alpha + 2n^*}\Gamma(1 + \alpha + n^*)} \tag{9}$$

is obtained directly from RMT. Here  $n^* = -\left(\beta + \frac{\alpha}{2}\right)$ . For information to be used in the latter sections, observe that  $n^*$  is the solution to

$$n + \beta + \frac{\alpha}{2} = 0. \tag{10}$$

It follows that

$$\int_{0}^{\infty} dx \ x^{\beta-1} J_{\alpha}\left(\sqrt{x}\right) = 2^{2\beta} \frac{\Gamma\left(\beta + \frac{\alpha}{2}\right)}{\Gamma\left(1 + \frac{\alpha}{2} - \beta\right)}.$$
 (11)

This result appears as entry 6.561.14 in the table of integrals [15].

#### 2.2 A second example: the Feynman diagram of a bubble

$$G = {\stackrel{p}{\overbrace{\qquad}}}$$

$$\tag{12}$$

The next example illustrates the evaluation a multidimensional integral corresponding to the massless bubble Feynman diagram shown in (12). The result is well-known [14]. In momentum space the corresponding integral is given by

$$G = \int \frac{d^{D}q}{i\pi^{D/2}} \frac{1}{\left[q^{2}\right]^{a_{1}} \left[\left(p-q\right)^{2}\right]^{a_{2}}},\tag{13}$$

where the parameters  $\{a_i\}$  are arbitrary. The Schwinger representation corresponding to this diagram produces

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_{0}^{\infty} \int_{0}^{\infty} dx \, dy \, x^{a_1 - 1} y^{a_2 - 1} \, \frac{\exp\left(-\frac{xy}{x + y} \, p^2\right)}{(x + y)^{\frac{D}{2}}}.$$
 (14)

In order to apply RMT to evaluate this integral, each part of the integrand is expanded in a Taylor series. In situations where options are available, the optimal course of action seems to be to minimize the number of expansions. At the moment this is at the heuristic level. In the present example, it is convenient to first expand the exponential function

$$\exp\left(-\frac{xy}{x+y}\ p^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{x^n y^n}{(x+y)^n},\tag{15}$$

to produce the expression

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_{0}^{\infty} \int_{0}^{\infty} dx \, dy \, x^{a_1 - 1} y^{a_2 - 1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{x^n y^n}{(x+y)^{\frac{D}{2} + n}}.$$
 (16)

The final step is to expand the binomial term  $(x+y)^{-D/2-n}$  in the form

$$\frac{1}{(x+y)^{\frac{D}{2}+n}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{D}{2} + n\right)_k x^{-D/2 - n - k} y^k, \tag{17}$$

where the Pochhammer term is

$$\left(\frac{D}{2} + n\right)_k = \frac{\Gamma\left(\frac{D}{2} + n + k\right)}{\Gamma\left(\frac{D}{2} + n\right)}.$$
(18)

Replacing in (16) gives

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_{0}^{\infty} \int_{0}^{\infty} \frac{dx}{x} \frac{dy}{y} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1)^k}{k!} \left(p^2\right)^n \left(\frac{D}{2} + n\right)_k x^{-k+a_1 - \frac{D}{2}} y^{k+n+a_2},$$
 (19)

and  $x \longrightarrow 1/x$  produces the alternative expression

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_{0}^{\infty} \int_{0}^{\infty} \frac{dx}{x} \frac{dy}{y} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1)^k}{k!} \left(p^2\right)^n \left(\frac{D}{2} + n\right)_k x^{k-a_1 + \frac{D}{2}} y^{k+n+a_2}. \tag{20}$$

There are several options to employ RMT to evaluate this integral:

a) Evaluate the integral in the x-variable and the expansion in the index k:

$$\int_{0}^{\infty} dx \dots \sum_{k=0}^{\infty} \dots \frac{(-x)^k}{k!}.$$

b) Evaluate the integral in the y-variable jointly with the series with index k:

$$\int_{0}^{\infty} dy \dots \sum_{k=0}^{\infty} \dots \frac{(-y)^k}{k!}.$$

c) Evaluate the integral in the y-variable jointly with the series with index n:

$$\int_{0}^{\infty} dy ... \sum_{n=0}^{\infty} ... \frac{(-y)^n}{n!}.$$

It is now shown that each of these options produces the same result.

#### 2.3 Solution with option (a)

The first procedure to evaluate G is given by

$$G_{(a)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \frac{dy}{y} \left[ \int_0^\infty \frac{dx}{x} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \varphi(k) \ x^{k-a_1 + \frac{D}{2}} \right], \tag{21}$$

where  $\varphi(k)$  is defined by

$$\varphi(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (p^2)^n (\frac{D}{2} + n)_k y^{k+n+a_2}.$$
 (22)

The value of (21) is given by RMT as

$$G_{(a)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \frac{dy}{y} \Gamma(-k^*) \varphi(k^*), \text{ with } k^* = -\frac{D}{2} + a_1$$

$$= \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \Gamma(\frac{D}{2} - a_1) \int_0^\infty \frac{dy}{y} \sum_{n=0}^\infty \frac{(-1)^n}{n!} (p^2)^n (\frac{D}{2} + n)_{a_1 - \frac{D}{2}} y^{a_1 + a_2 - \frac{D}{2} + n}$$

$$= \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \Gamma(\frac{D}{2} - a_1) \int_0^\infty \frac{dy}{y} \sum_{n=0}^\infty \frac{(-1)^n}{n!} (p^2)^n \frac{\Gamma(a_1 + n)}{\Gamma(\frac{D}{2} + n)} y^{n+a_1 + a_2 - \frac{D}{2}}.$$
(23)

RMT is applied one more time to the last integral to obtain

$$G_{(a)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)}\Gamma(\frac{D}{2} - a_1)\Gamma(-n^*) \left(p^2\right)^{n^*} \frac{\Gamma(a_1 + n^*)}{\Gamma(\frac{D}{2} + n^*)},\tag{24}$$

with  $n^* = -a_1 - a_2 + \frac{D}{2}$ . Therefore, option (a) gives the value of G as

$$G_{(a)} = (-1)^{-\frac{D}{2}} \left(p^2\right)^{\frac{D}{2} - a_1 - a_2} \frac{\Gamma(a_1 + a_2 - \frac{D}{2})\Gamma(\frac{D}{2} - a_1)\Gamma(\frac{D}{2} - a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(D - a_1 - a_2)}.$$
 (25)

#### 2.4 Solution with option (b)

A similar argument now yields

$$G_{(b)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \frac{dx}{x} \left[ \int_0^\infty \frac{dy}{y} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \varphi(k) \ y^{k+n+a_2} \right]$$
 (26)

with

$$\varphi(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(p^2\right)^n \left(\frac{D}{2} + n\right)_k x^{k - a_1 + \frac{D}{2}}.$$
 (27)

Therefore

$$G_{(b)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \frac{dx}{x} \Gamma(-k^*) \varphi(k^*), \quad \text{with } k^* = -n - a_2$$

$$= \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \frac{dx}{x} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \Gamma(n+a_2) \left(p^2\right)^n \left(\frac{D}{2} + n\right)_{-n-a_2} x^{-n-a_2-a_1+\frac{D}{2}}$$

$$= \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \frac{dx}{x} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{\Gamma(n+a_2)\Gamma\left(\frac{D}{2} - a_2\right)}{\Gamma\left(\frac{D}{2} + n\right)} x^{-n-a_2-a_1+\frac{D}{2}}.$$
(28)

The change of variables  $x \mapsto 1/x$  gives

$$G_{(b)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_{0}^{\infty} \frac{dx}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{\Gamma(n+a_2)\Gamma\left(\frac{D}{2}-a_2\right)}{\Gamma\left(\frac{D}{2}+n\right)} x^{n+a_2+a_1-\frac{D}{2}}, \tag{29}$$

and RMT produces

$$G_{(b)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)}\Gamma(-n^*) \left(p^2\right)^{n^*} \frac{\Gamma(n^* + a_2)\Gamma\left(\frac{D}{2} - a_2\right)}{\Gamma\left(\frac{D}{2} + n^*\right)},\tag{30}$$

with  $n^* = -a_2 - a_1 + \frac{D}{2}$ . Therefore, the value of G obtained from option (b) is

$$G_{(b)} = (-1)^{-\frac{D}{2}} \left(p^2\right)^{\frac{D}{2} - a_2 - a_1} \frac{\Gamma(\frac{D}{2} - a_1)\Gamma(\frac{D}{2} - a_2)\Gamma(a_2 + a_1 - \frac{D}{2})}{\Gamma(a_1)\Gamma(a_2)\Gamma(D - a_2 - a_1)}.$$
 (31)

#### 2.5 Solution with option (c)

Proceeding as before produces

$$G_{(c)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \frac{dx}{x} \left[ \int_0^\infty \frac{dy}{y} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \varphi(n) y^{k+n+a_2} \right]$$
(32)

with

$$\varphi(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( p^2 \right)^n \left( \frac{D}{2} + n \right)_k x^{k - a_1 + \frac{D}{2}}.$$
 (33)

This yields

$$G_{(c)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_{0}^{\infty} \frac{dx}{x} \varphi(n^*) \Gamma(-n^*)$$
(34)

with  $n^* = -k - a_2$ . Then

$$G_{(c)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_{0}^{\infty} \frac{dx}{x} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(p^2\right)^{-k-a_2} \left(\frac{D}{2} - k - a_2\right)_k x^{k-a_1 + \frac{D}{2}} \Gamma(k+a_2)$$
(35)

with the equivalent form

$$G_{(c)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)}\Gamma(\frac{D}{2} - a_2) \int_0^\infty \frac{dx}{x} \sum_{k=0}^\infty \frac{(-1)^k}{k!} (p^2)^{-k-a_2} \frac{\Gamma(k+a_2)}{\Gamma(\frac{D}{2} - k - a_2)} x^{k-a_1 + \frac{D}{2}}.$$
 (36)

An application of RMT gives

$$G_{(c)} = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \Gamma\left(\frac{D}{2} - a_2\right) \left(p^2\right)^{-k^* - a_2} \frac{\Gamma(-k^*)\Gamma(k^* + a_2)}{\Gamma\left(\frac{D}{2} - k^* - a_2\right)} \text{ with } k^* = a_1 - \frac{D}{2},$$
 (37)

or equivalently

$$G_{(c)} = (-1)^{-\frac{D}{2}} \left(p^2\right)^{\frac{D}{2} - a_1 - a_2} \frac{\Gamma\left(\frac{D}{2} - a_2\right) \Gamma\left(\frac{D}{2} - a_1\right) \Gamma(a_1 + a_2 - \frac{D}{2})}{\Gamma\left(a_1\right) \Gamma\left(a_2\right) \Gamma\left(D - a_1 - a_2\right)}.$$
 (38)

Observe that each option produces the same value for the integral G; that is,  $G_{(a)} = G_{(b)} = G_{(c)}$ .

# 3 Generalization of RMT to multiple integrals. Multiloop calculations

This section discusses a generalization of RMT to multidimensional integrals of the form

$$I = \int_{0}^{\infty} dx_1 \, x_1^{\nu_1 - 1} \dots \int_{0}^{\infty} dx_N \, x_N^{\nu_N - 1} \, f(x_1, \dots, x_N) \,. \tag{39}$$

As in the one-dimensional case, the function  $f = f(x_1, ..., x_N)$  is assumed to admit a Taylor expansion given by

$$f(x_1, ..., x_N) = \sum_{l_1=0}^{\infty} ... \sum_{l_N=0}^{\infty} \frac{(-1)^{l_1}}{l_1!} ... \frac{(-1)^{l_N}}{l_N!} \varphi(l_1, ..., l_N)$$
(40)

$$\times \ x_1^{a_{11}l_1+\ldots+a_{1N}l_N+b_1}\ldots \ x_N^{a_{N1}l_1+\ldots+a_{NN}l_N+b_N},$$

so that I is expressed as

$$I = \int_{0}^{\infty} \frac{dx_{1}}{x_{1}} \dots \int_{0}^{\infty} \frac{dx_{N}}{x_{N}} \sum_{l_{1}=0}^{\infty} \dots \sum_{l_{N}=0}^{\infty} \frac{(-1)^{l_{1}}}{l_{1}!} \dots \frac{(-1)^{l_{N}}}{l_{N}!} \varphi(l_{1}, \dots, l_{N})$$

$$\times x_{1}^{a_{11}l_{1}+\dots+a_{1N}l_{N}+\widetilde{b}_{1}} \dots x_{N}^{a_{N1}l_{1}+\dots+a_{NN}l_{N}+\widetilde{b}_{N}}$$

$$(41)$$

with  $\widetilde{b}_i = \nu_i + b_i \ (i = 1, ..., N)$ .

Applying RMT in iterative manner produces

$$I = \frac{1}{|\det(\mathbf{A})|} \Gamma(-l_1^*) \dots \Gamma(-l_N^*) \varphi(l_1^*, \dots, l_N^*)$$
(42)

where  $\det(\mathbf{A})$  is the determinant

$$\det\left(\mathbf{A}\right) = \begin{vmatrix} a_{11} & \dots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \dots & a_{NN} \end{vmatrix},\tag{43}$$

and the variables  $l_i^*$  (i = 1, ..., N) are solutions of the linear system

$$\begin{cases} a_{11}l_1 + \dots + a_{1N}l_N + \tilde{b}_1 &= 0 \\ \vdots & \vdots \\ a_{N1}l_1 + \dots + a_{NN}l_N + \tilde{b}_N &= 0. \end{cases}$$
(44)

Details of the proof of this result appear in [13]. This procedure will be called the Generalized Ramanujan's Master Theorem (GRMT).

### 4 Applications

#### 4.1 Massive sunset diagram

The integral evaluated first is associated to the diagram shown in the figure. In momentum space, the integral is given by

$$G = \int \frac{d^{D}q_{1}}{i\pi^{D/2}} \frac{d^{D}q_{2}}{i\pi^{D/2}} \frac{1}{\left[q^{2} - M^{2}\right]^{a_{1}}} \frac{1}{\left[\left(q_{1} - q_{2}\right)^{2}\right]^{a_{2}}} \frac{1}{\left[\left(p + q_{2}\right)^{2}\right]^{a_{3}}}.$$
(45)

$$G = \frac{\rho}{\rho}$$
 (46)

In terms of the Schwinger parametrization, G becomes

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty d\overrightarrow{x} \frac{\exp(x_1 M^2) \exp(-\frac{x_1 x_2 x_3}{x_1 x_2 + x_1 x_3 + x_2 x_3} p^2)}{(x_1 x_2 + x_1 x_3 + x_2 x_3)^{\frac{D}{2}}}, \quad (47)$$

where  $d\overrightarrow{x} = dx_1 dx_2 dx_3 x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1}$ . In order to illustrate the power of the GRMT method, the special physical case  $p^2 = M^2$  is considered. The integral is now written as

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_{0}^{\infty} \int_{0}^{\infty} d\overrightarrow{x} \frac{\exp\left[\frac{x_1^2(x_2 + x_3)}{x_1(x_2 + x_3) + x_2x_3}M^2\right]}{[x_1(x_2 + x_3) + x_2x_3]^{\frac{D}{2}}}.$$
 (48)

The expansion of the exponential function gives

$$\exp\left(\frac{x_1^2(x_2+x_3)}{x_1(x_2+x_3)+x_2x_3}M^2\right) = \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \left(-M^2\right)^{n_1} \frac{x_1^{2n_1}(x_2+x_3)^{n_1}}{\left[x_1(x_2+x_3)+x_2x_3\right]^{n_1}}$$
(49)

and this produces

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} d\overrightarrow{x} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \left(-M^2\right)^{n_1} \frac{x_1^{2n_1} (x_2 + x_3)^{n_1}}{\left[x_1 (x_2 + x_3) + x_2 x_3\right]^{\frac{D}{2} + n_1}}.$$
 (50)

The binomial theorem is employed next to expand the integrand. This can be done via the hypergeometric representation

$$(1+x)^a = {}_1F_0 \left( \begin{array}{c} -a \\ - \end{array} \middle| -x \right), \tag{51}$$

or by the explicit formula

$$(x+y)^{a} = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \frac{\Gamma(-a+n)}{\Gamma(-a)} x^{a-n} y^{n}.$$
 (52)

The result is

$$\frac{1}{\left[x_1\left(x_2+x_3\right)+x_2x_3\right]^{\frac{D}{2}+n_1}} = \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{n_2!} \frac{\Gamma\left(\frac{D}{2}+n_1+n_2\right)}{\Gamma\left(\frac{D}{2}+n_1\right)} \times x_1^{-\frac{D}{2}-n_1-n_2} \left(x_2+x_3\right)^{-\frac{D}{2}-n_1-n_2} x_2^{n_2} x_3^{n_2},$$
(53)

and (50) is now written as

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} d\vec{x} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \frac{(-1)^{n_2}}{n_2!} \left(M^2\right)^{n_1} \times \frac{\Gamma\left(\frac{D}{2} + n_1 + n_2\right)}{\Gamma\left(\frac{D}{2} + n_1\right)} x_1^{n_1 - \frac{D}{2} - n_2} x_2^{n_2} x_3^{n_2} \left(x_2 + x_3\right)^{-\frac{D}{2} - n_2}.$$
(54)

Only the binomial  $(x_2 + x_3)^{-\frac{D}{2} - n_2}$  needs to be expanded. This is done as before to produce

$$(x_2 + x_3)^{-\frac{D}{2} - n_2} = \sum_{n_3 = 0}^{\infty} \frac{(-1)^{n_3}}{n_3!} \frac{\Gamma\left(\frac{D}{2} + n_2 + n_3\right)}{\Gamma\left(\frac{D}{2} + n_2\right)} x_2^{-\frac{D}{2} - n_2 - n_3} x_3^{n_3}.$$
 (55)

The point has been reached for a direct application of GRMT to evaluate

$$G = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \frac{dx_{1}}{x_{1}} \frac{dx_{2}}{x_{2}} \frac{dx_{3}}{x_{3}}$$

$$\times \sum_{n_{1}=0}^{\infty} \sum_{n_{2}=0}^{\infty} \sum_{n_{3}=0}^{\infty} \frac{(-1)^{n_{1}}}{n_{1}!} \frac{(-1)^{n_{2}}}{n_{2}!} \frac{(-1)^{n_{3}}}{n_{3}!} x_{1}^{a_{1} - \frac{D}{2} + n_{1} - n_{2}} x_{2}^{a_{2} - \frac{D}{2} - n_{3}} x_{3}^{a_{3} + n_{2} + n_{3}}$$

$$\times \frac{(-1)^{-D}}{\Gamma(a_{1}) \Gamma(a_{2}) \Gamma(a_{3})} \frac{\Gamma(\frac{D}{2} + n_{1} + n_{2})}{\Gamma(\frac{D}{2} + n_{1})} \frac{\Gamma(\frac{D}{2} + n_{2} + n_{3})}{\Gamma(\frac{D}{2} + n_{2})} (-M^{2})^{n_{1}}.$$

$$(56)$$

It follows that the integral G is given by

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\Gamma(-n_1^*)\Gamma(-n_2^*)\Gamma(-n_3^*)\frac{\Gamma(\frac{D}{2} + n_1^* + n_2^*)}{\Gamma(\frac{D}{2} + n_1^*)}\frac{\Gamma(\frac{D}{2} + n_2^* + n_3^*)}{\Gamma(\frac{D}{2} + n_2^*)}\left(-M^2\right)^{n_1^*},$$
(57)

where the values assigned to the indices  $\{n_i^*\}$  are the unique solution to the linear system obtained from (56). In detail,

$$\begin{cases}
 n_1 - n_2 = -a_1 + \frac{D}{2}, \\
 n_3 = a_2 - \frac{D}{2}, \\
 n_2 + n_3 = -a_3,
\end{cases}$$
(58)

with solution

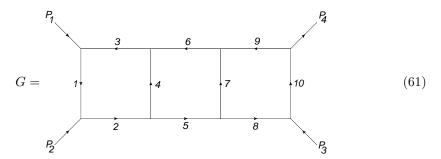
$$\begin{cases}
 n_1^* = -a_1 - a_2 - a_3 + D, \\
 n_2^* = -a_2 - a_3 + \frac{D}{2}, \\
 n_3^* = -\frac{D}{2} + a_2.
\end{cases} (59)$$

Replacing in (57) yields

$$G = (-1)^{-D} \frac{\Gamma(a_1 + a_2 + a_3 - D) \Gamma(a_2 + a_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - a_2)}{\Gamma(a_1) \Gamma(a_2) \Gamma(a_3)} \times \frac{\Gamma(\frac{D}{2} - a_3) \Gamma(2D - a_1 - 2a_2 - 2a_3)}{\Gamma(\frac{3D}{2} - a_1 - a_2 - a_3) \Gamma(D - a_2 - a_3)} (-M^2)^{D - a_1 - a_2 - a_3}.$$
(60)

#### 4.2 Massless three loops ladder diagram

The next example gives the evaluation of the integral associated to the diagram seen in Eq. (61). To illustrate the method in a relatively simple situation, the conditions  $P_i^2 = 0$  for  $1 \le i \le 4$  and s = 0 are imposed.



The parametric representation of this diagram is given by the integral

$$G = \frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_1) \dots \Gamma(a_{10})} \int_{0}^{\infty} \dots \int_{0}^{\infty} d\overrightarrow{x} \frac{\exp\left(-\frac{x_1 x_4 x_7 x_{10}}{U} t\right)}{U^{\frac{D}{2}}},$$
 (62)

where  $d\overrightarrow{x} = \prod_{j=1}^{10} dx_j \ x_j^{a_j-1}$  and the polynomial U, written in a form adapted to the application of GRMT, is given by

$$U = x_5 (x_7 + \mathbf{f}_1) (\mathbf{f}_2 + x_4) + x_6 (x_7 + \mathbf{f}_1) (\mathbf{f}_2 + x_4) + x_4 (x_7 + \mathbf{f}_1) \mathbf{f}_2 + x_7 (\mathbf{f}_2 + x_4) \mathbf{f}_1,$$
(63)

with

$$\begin{cases}
\mathbf{f}_1 = x_8 + x_9 + x_{10}, \\
\mathbf{f}_2 = x_1 + x_2 + x_3.
\end{cases}$$
(64)

Expanding the exponential term produces

$$G = \frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_1) \dots \Gamma(a_{10})} \int_{0}^{\infty} \dots \int_{0}^{\infty} d\overrightarrow{x} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} (t)^{n_1} \frac{x_1^{n_1} x_4^{n_1} x_7^{n_1} x_{10}^{n_1}}{U^{\frac{D}{2} + n_1}},$$
(65)

and the polynomial U is expanded using the multinomial theorem

$$(x_1 + \dots + x_{k-1} + x_k)^a = \sum_{n_1 = 0}^{\infty} \dots \sum_{n_{k-1} = 0}^{\infty} \frac{1}{n_1! \dots n_{k-1}!} \frac{\Gamma(1+a)}{\Gamma(1+a-n_1 - \dots - n_{k-1})} \times x_1^{n_1} \dots x_{k-1}^{n_{k-1}} x_k^{a-n_1 - \dots - n_{k-1}},$$
(66)

written in a form adapted to GRMT

$$(x_1 + \dots + x_{k-1} + x_k)^a = \sum_{n_1 = 0}^{\infty} \dots \sum_{n_{k-1} = 0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \dots \frac{(-1)^{n_{k-1}}}{n_{k-1}!} \frac{\Gamma(-a + n_1 + \dots + n_{k-1})}{\Gamma(-a)} \times x_1^{n_1} \dots x_{k-1}^{n_{k-1}} x_k^{a - n_1 - \dots - n_{k-1}}.$$

$$(67)$$

These expansions produce

$$U^{-\frac{D}{2}-n_{1}} = \sum_{n_{2}=0}^{\infty} \sum_{n_{3}=0}^{\infty} \sum_{n_{4}=0}^{\infty} \frac{(-1)^{n_{2}}}{n_{2}!} \frac{(-1)^{n_{3}}}{n_{3}!} \frac{(-1)^{n_{4}}}{n_{4}!} \frac{\Gamma\left(\frac{D}{2}+n_{1}+n_{2}+n_{3}+n_{4}\right)}{\Gamma\left(\frac{D}{2}+n_{1}\right)} \times \mathbf{f}_{1}^{-\frac{D}{2}-n_{1}-n_{2}-n_{3}-n_{4}} \mathbf{f}_{2}^{n_{2}} \left(x_{7}+\mathbf{f}_{1}\right)^{n_{2}+n_{3}+n_{4}} \left(x_{4}+\mathbf{f}_{2}\right)^{-\frac{D}{2}-n_{1}-n_{2}} \times x_{4}^{n_{2}} x_{5}^{n_{3}} x_{6}^{n_{4}} x_{7}^{-\frac{D}{2}-n_{1}-n_{2}-n_{3}-n_{4}}.$$

$$(68)$$

Similarly,

$$(x_7 + \mathbf{f}_1)^{n_2 + n_3 + n_4} = \sum_{n_5 = 0}^{\infty} \frac{(-1)^{n_5}}{n_5!} \frac{\Gamma(-n_2 - n_3 - n_4 + n_5)}{\Gamma(-n_2 - n_3 - n_4)} x_7^{n_5} \mathbf{f}_1^{n_2 + n_3 + n_4 - n_5}, \tag{69}$$

and

$$(x_4 + \mathbf{f}_2)^{-\frac{D}{2} - n_1 - n_2} = \sum_{n_6 = 0}^{\infty} \frac{(-1)^{n_6}}{n_6!} \frac{\Gamma\left(\frac{D}{2} + n_1 + n_2 + n_6\right)}{\Gamma\left(\frac{D}{2} + n_1 + n_2\right)} x_4^{n_6} \mathbf{f}_2^{-\frac{D}{2} - n_1 - n_2 - n_6}.$$
 (70)

The result is

$$G = \frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_{1}) \dots \Gamma(a_{10})} \int_{0}^{\infty} \dots \int_{0}^{\infty} d\overrightarrow{x} \sum_{n_{1}=0}^{\infty} \dots \sum_{n_{6}=0}^{\infty} \frac{(-1)^{n_{1}}}{n_{1}!} \dots \frac{(-1)^{n_{6}}}{n_{6}!} (t)^{n_{1}}$$

$$\times \frac{\Gamma\left(\frac{D}{2} + n_{1} + n_{2} + n_{3} + n_{4}\right)}{\Gamma\left(\frac{D}{2} + n_{1}\right)} \frac{\Gamma\left(-n_{2} - n_{3} - n_{4} + n_{5}\right)}{\Gamma\left(-n_{2} - n_{3} - n_{4}\right)} \frac{\Gamma\left(\frac{D}{2} + n_{1} + n_{2} + n_{6}\right)}{\Gamma\left(\frac{D}{2} + n_{1} + n_{2}\right)}$$

$$\times x_{1}^{n_{1}} x_{4}^{n_{1}+n_{2}+n_{6}} x_{5}^{n_{3}} x_{6}^{n_{4}} x_{7}^{-\frac{D}{2}-n_{2}-n_{3}-n_{4}+n_{5}} x_{10}^{n_{1}} \mathbf{f}_{1}^{-\frac{D}{2}-n_{1}-n_{5}} \mathbf{f}_{2}^{-\frac{D}{2}-n_{1}-n_{6}}.$$

$$(71)$$

Finally, the powers of  $\mathbf{f}_1$  and  $\mathbf{f}_2$  are expanded in the form

$$\mathbf{f}_{1}^{-\frac{D}{2}-n_{1}-n_{5}} = \sum_{n_{7}=0}^{\infty} \sum_{n_{8}=0}^{\infty} \frac{(-1)^{n_{7}}}{n_{7}!} \frac{(-1)^{n_{8}}}{n_{9}!} \frac{\Gamma\left(\frac{D}{2}+n_{1}+n_{5}+n_{7}+n_{8}\right)}{\Gamma\left(\frac{D}{2}+n_{1}+n_{5}\right)} \times x_{8}^{n_{7}} x_{9}^{n_{8}} x_{10}^{-\frac{D}{2}-n_{1}-n_{5}-n_{7}-n_{8}},$$

$$(72)$$

and

$$\mathbf{f}_{2}^{-\frac{D}{2}-n_{1}-n_{6}} = \sum_{n_{9}=0}^{\infty} \sum_{n_{10}=0}^{\infty} \frac{(-1)^{n_{9}}}{n_{9}!} \frac{(-1)^{n_{10}}}{n_{10}!} \frac{\Gamma\left(\frac{D}{2}+n_{1}+n_{6}+n_{9}+n_{10}\right)}{\Gamma\left(\frac{D}{2}+n_{1}+n_{6}\right)} \times x_{1}^{n_{9}} x_{2}^{n_{10}} x_{2}^{-\frac{D}{2}-n_{1}-n_{6}-n_{9}-n_{10}}.$$

$$(73)$$

At this point the expression for G is in the form required to apply GRMT:

$$G = \frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_{1})...\Gamma(a_{10})} \int_{0}^{\infty} \frac{dx_{1}}{x_{1}}...\int_{0}^{\infty} \frac{dx_{10}}{x_{10}} \sum_{n_{1}=0}^{\infty} ...\sum_{n_{10}=0}^{\infty} \frac{(-1)^{n_{1}}}{n_{1}!}...\frac{(-1)^{n_{10}}}{n_{10}!} \varphi(n_{1},...,n_{10})$$

$$\times x_{1}^{a_{1}+n_{1}+n_{9}} x_{2}^{a_{2}+n_{10}} x_{3}^{a_{3}-\frac{D}{2}-n_{1}-n_{6}-n_{9}-n_{10}} x_{4}^{a_{4}+n_{1}+n_{2}+n_{6}} x_{5}^{a_{5}+n_{3}} x_{6}^{a_{6}+n_{4}} x_{7}^{a_{7}-\frac{D}{2}-n_{2}-n_{3}-n_{4}+n_{5}}$$

$$\times x_{8}^{a_{8}+n_{7}} x_{9}^{a_{9}+n_{8}} x_{10}^{a_{10}-\frac{D}{2}-n_{5}-n_{7}-n_{8}},$$

$$(74)$$

with the notation

$$\varphi(n_1, ..., n_{10}) = \frac{\Gamma\left(\frac{D}{2} + n_1 + n_2 + n_3 + n_4\right)}{\Gamma\left(\frac{D}{2} + n_1\right)} \frac{\Gamma\left(-n_2 - n_3 - n_4 + n_5\right)}{\Gamma\left(-n_2 - n_3 - n_4\right)} \frac{\Gamma\left(\frac{D}{2} + n_1 + n_2 + n_6\right)}{\Gamma\left(\frac{D}{2} + n_1 + n_2\right)} \times \frac{\Gamma\left(\frac{D}{2} + n_1 + n_5 + n_7 + n_8\right)}{\Gamma\left(\frac{D}{2} + n_1 + n_5\right)} \frac{\Gamma\left(\frac{D}{2} + n_1 + n_6 + n_9 + n_{10}\right)}{\Gamma\left(\frac{D}{2} + n_1 + n_6\right)} (t)^{n_1}.$$

The value of G is now obtained as a direct application of GRMT as

$$G = \frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_1) \dots \Gamma(a_{10})} \frac{1}{|\det(\mathbf{A})|} \varphi(n_1^*, \dots, n_{10}^*) \prod_{j=1}^{10} \Gamma(-n_j^*)$$
(75)

where the variables  $\{n_i^*\}$  are solutions of the linear system

The determinant of this matrix is 1 and the indices  $n^*$  become

$$\begin{cases}
n_1^* = 3D/2 - a_1 - a_2 - a_3 - a_4 - a_5 - a_6 - a_7 - a_8 - a_9 - a_{10}, \\
n_2^* = -D + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}, \\
n_3^* = -a_5, \\
n_4^* = -a_6, \\
n_5^* = -D/2 + a_8 + a_9 + a_{10}, \\
n_6^* = -D/2 + a_1 + a_2 + a_3, \\
n_7^* = -a_8, \\
n_8^* = -a_9, \\
n_9^* = -3D/2 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}, \\
n_{10}^* = -a_2,
\end{cases}$$
(77)

that yields the value of the diagram as

$$G = (-1)^{-\frac{3D}{2}} \frac{\Gamma\left(\frac{D}{2} - a_{89,10}\right) \Gamma\left(\frac{D}{2} - a_{123}\right) \Gamma\left(\frac{3D}{2} - a_{23456789,10}\right) \Gamma\left(\frac{3D}{2} - a_{123456789}\right)}{\Gamma\left(a_{1}\right) \Gamma\left(a_{4}\right) \Gamma\left(a_{7}\right) \Gamma\left(a_{10}\right) \Gamma\left(2D - a_{123456789,10}\right)} \times \frac{\Gamma\left(a_{123456789,10} - \frac{3D}{2}\right) \Gamma\left(D - a_{56789,10}\right) \Gamma\left(D - a_{123456}\right) \Gamma\left(\frac{D}{2} - a_{7}\right) \Gamma\left(\frac{D}{2} - a_{4}\right)}{\Gamma\left(D - a_{789,10}\right) \Gamma\left(D - a_{1234}\right) \Gamma\left(\frac{3D}{2} - a_{1234567}\right) \Gamma\left(\frac{3D}{2} - a_{456789,10}\right)} \times t^{\frac{3D}{2} - a_{123456789,10}},$$

$$(78)$$

with the notation

$$a_{ijk...} = a_i + a_j + a_k + \dots (79)$$

employed above.

A relevant special case is when all powers of propagators are 1; that is,  $a_i = 1$  for  $i = 1, \dots, 10$ . This is given by

$$G = (-1)^{-\frac{3D}{2}} \frac{\Gamma\left(10 - \frac{3D}{2}\right)\Gamma\left(\frac{D}{2} - 3\right)^{2}\Gamma\left(\frac{3D}{2} - 9\right)^{2}\Gamma\left(D - 6\right)^{2}\Gamma\left(\frac{D}{2} - 1\right)^{2}}{\Gamma\left(2D - 10\right)\Gamma\left(D - 4\right)^{2}\Gamma\left(\frac{3D}{2} - 7\right)^{2}} t^{\frac{3D}{2} - 10}.$$
 (80)

#### 5 Conclusions

This paper introduces a technique (GRMT) for the evaluation of a large variety of Feynman diagrams. The advantage over previous methods is that the evaluation of diagrams is reduced to series expansions of the integrand, coupled with the solution of a linear system of equations.

The method is illustrated here in diagrams where the number of series appearing in the process is the same as the dimension of the integrals involved. Future publications will describe examples where this condition is not present.

**Acknowledgments** The work of the second author was partially funded by NSF-DMS 0070567. The first and third authors also acknowledge support from Centro Científico-Tecnológico de Valparaiso, CCTVal, Chile.

#### References

- G. Passarino, Nucl. Phys. B 619 (2001) 257. (hep-ph/0108252). G. Passarino and S. Uccirati,
   Nucl. Phys. B 629 (2002) 97. (hep-ph/0112004). A. Ferroglia, M. Passera, G. Passarino and
   S. Uccirati, Nucl. Phys. B 650 (2003) 162. (hep-ph/0209219). A. Ferroglia, G. Passarino, S.
   Uccirati and M. Passera, Nucl. Instrum. Meth. A 502 (2003) 391. A. Ferroglia, M. Passera,
   G. Passarino and S. Uccirati, Nucl. Phys. B 680 (2004) 199. (hep-ph/0311186). S. Actis, A.
   Ferroglia, G. Passarino, M. Passera and S. Uccirati. (hep-ph/0402132).
- [2] T. Binoth and G. Heinrich, Nucl. Phys. B 585 (2000) 741. (hep-ph/0004013). T. Binoth, G. Heinrich and N. Kauer, Nucl. Phys. B 654 (2003) 277. (hep-ph/0210023). T. Binoth and G. Heinrich, Nucl. Phys. B 680 (2004) 375. (hep-ph/0305234). T. Binoth and G. Heinrich, Nucl. Phys. B 693 (2004) 134. (hep-ph/0402265). T. Binoth. (hep-ph/0407003). G. Heinrich. (hep-ph/0406332).
- [3] M. Caffo, H. Czyz and E. Remiddi, Nucl. Phys. B 634 (2002) 309. (hep-ph/0203256). M. Caffo,
   H. Czyz and E. Remiddi, Nucl. Instrum. Meth. A 502 (2003) 613. (hep-ph/0211171). M. Caffo,
   H. Czyz, A. Grzelinska and E. Remiddi, Nucl. Phys. B 681 (2004) 230. (hep-ph/0312189).
- [4] F. V. Tkachov, Phys. Lett. B 100 (1981) 65. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B 192 (1981) 159.
- [5] T. Gehrmann and E. Remiddi, Nucl. Phys. B 580 (2000) 485. (hep-ph/9912329).
- [6] R. Bonciani, P. Mastrolia and E. Remiddi, Nucl. Phys. B 661 (2003) 289. (hep-ph/0301170).
- [7] V. A. Smirnov, in Proc. of the 5th International Symposium on Radiative Corrections (RAD-COR 2000) ed. Howard E. Haber. (hep-ph/0101152). V. A. Smirnov, "Applied asymptotic expansions in momenta and masses". Berlin, Germany: Springer 2002 (Springer tracts in modern physics. 177).
- [8] E. E. Boos and A. I. Davydychev, Theor. Math. Phys. 89 (1991) 1052 (Teor. Mat. Fiz. 89 (1991) 56).
   V. A. Smirnov. (hep-ph/0406052).
   J. Blümlein, S. Klein, C. Schneider and F. Stan, DESY Technical Report 10-185, 2010 (arXiv:1011.2656).
- [9] O. V. Tarasov, Phys. Rev. D 54 (1996) 6479. (hep-th/9606018).
- [10] A. V. Kotikov, Phys. Lett. B 254 (1991) 158. A. V. Kotikov, Phys. Lett. B 259 (1991) 314.
  A. V. Kotikov, Phys. Lett. B 267 (1991) 123. E. Remiddi, Nuovo Cim. A 110 (1997) 1435 (hep-th/9711188). M. Caffo, H. Czyz, S. Laporta and . Remiddi, Acta Phys. Polon. B 29 (1998) 2627. (hep-th/9807119). M. Caffo, H. Czyz, S. Laporta and E. Remiddi, Nuovo Cim. A 111 (1998) 365. (hep-th/9805118).
- [11] I. Gonzalez, Nuclear Physics B (Proc. Suppl.) 205–206 (2010) 141–146. (arXiv:1008.2148).
- [12] G. H. Hardy. Ramanujan. Twelve Lectures on subjects suggested by his life and work. Chelsea Publishing Company, New York, N.Y., 3rd edition, 1978.
- [13] T. Amdeberhan, O. Espinosa, I. Gonzalez, M. Harrison, V. Moll, A. Straub, submitted for publication, 2011.

- [14] A. I. Davydychev, J. Math. Phys. 32 (1991) 1052, J. Math. Phys. 33 (1992) 358.
- [15] I. S. Gradshteyn and I. M. Ryzhik. Table of Integrals, Series and Products. Academic Press, 2007.
- [16] I. Gonzalez and V. Moll. Adv. Appl. Math 45 (2010) 50-73. I. Gonzalez, V. Moll and A. Straub. Contemporary Mathematics 517 (2010) 157-171. (arXiv:0812.3356).