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A New Proof of the Garoufalidis-Lê-Zeilberger Quantum MacMahon Master Theorem

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Abstract. We propose a new proof of the quantum version of MacMahon's Master Theorem, established by Garoufalidis, Lê and Zeilberger.

Résumé. Nous proposons une nouvelle démonstration de la version quantique du Master Théorème de MacMahon, établi par Garoufalidis, Lê et Zeilberger.

1. Introduction

MacMahon's Master Theorem is still regarded as a keystone in Combinatorial Analysis. Numerous applications were already given by MacMahon himself [Ma15, vol. 1, p. 97]. Several other proofs are due to Cartier [Ca72]. As the Master Theorem is a special case of the multivariable Lagrange Inversion Formula [Ge87], as was shown by Hofbauer [Ho82], each proof of the later formula yields a proof of this theorem. Its first noncommutative version can be found in [Fo65], which was later replaced into an appropriate algebraic set-up [CF69]. Recently Garoufalidis, Lê and Zeilberger have derived another noncommutative version [GLZ05] (called "quantum Master Theorem") by using difference operator techniques developed by Zeilberger [Z80]. In this paper we propose a new proof of this quantum Master Theorem. As indicated in [FH05], the formal parameter q in the Garoufalidis-Lê-Zeilberger version plays no role; we will then let $q = 1$ in the present derivation without loss of generality.

Let r be a positive integer; the set $\mathbb{A} = \{1, 2, \ldots, r\}$ is referred to as the underlying *alphabet.* A *biword* on $\mathbb{A}$ is a $2 \times n$ matrix α ($n \geq 0$), whose entries are in $\mathbb{A}$, the first (resp. second) row being called the *top word* (resp. *bottom word*) of the biword α. The number n is the length of α; we write $\ell(\alpha) = n$. Let $\mathcal{B}$ be the set of all biwords on $\mathbb{A}$. Each biword can also be viewed as a word of *biletters* $\binom{x}{a}$ written vertically. The *product* of two biwords is just the concatenation of them viewed as two words of biletters. The biword of length 0 is denoted by $\binom{}{} = 1$.

Let $\mathbb{Z}$ be the ring of all integers. The set $\mathcal{A} = \mathbb{Z}\langle\langle\mathcal{B}\rangle\rangle$ of the formal sums $\sum_\alpha c(\alpha)\alpha$, where $\alpha \in \mathcal{B}$ and $c(\alpha) \in \mathbb{Z}$ for all $\alpha \in \mathcal{B}$, together with the above multiplication, the free addition and the free scalar product is an algebra over $\mathbb{Z}$, called the *free biword large $\mathbb{Z}$-algebra.*

The *right quantum algebra* $\mathcal{R}$ is defined to be the *associative* algebra, which is the quotient of $\mathcal{A}$ by the ideal $\mathcal{I}$ generated by the commutation relations

$$\text{(R1)} \qquad \binom{xy}{aa} = \binom{yx}{aa},$$

$$\text{(R2)} \qquad \binom{xy}{ab} = \binom{yx}{ab} + \binom{yx}{ba} - \binom{xy}{ba},$$

for all letters a, b, x, y in $\mathbb{A}$ and $x \neq y$. Notice that the associativity of the right quantum algebra is set by *definition*. In fact, the associativity is a *consequence* of the commutation relations (R1) and (R2), as was proved in [FH05]. All further calculations in the paper will be made in the right quantum algebra $\mathcal{R}$, unless explicitly indicated. The elements of $\mathcal{R}$ will be called *expressions*. If "can" is the canonical homomorphism of $\mathcal{A}$ onto $\mathcal{R}$, we identify $\operatorname{can}\mathbb{B}$ with $\mathbb{B}$ and each biword $\binom{x_1\, x_2\, \cdots\, x_m}{x_{j_1} x_{j_2} \cdots x_{j_m}}$ occurring in an expression from $\mathcal{R}$ with the product $\binom{x_{i_1}}{x_1}\binom{x_{i_2}}{x_2}\cdots\binom{x_{i_m}}{x_m}$ in the quotient algebra $\mathcal{R}$.

For each word w let $\overline{w}$ be its *non-decreasing* rearrangement. The *Boson* is defined to be the infinite sum

$$\mathrm{Bos} := \sum_{w} \binom{\overline{w}}{w},$$

where is the sum is over all words w from the free monoid $\mathbb{A}^*$ generated by $\mathbb{A}$. The *Fermion* is defined by

$$\mathrm{Ferm} := \sum_{J \subset \mathbb{A}} (-1)^{|J|} \sum_{\sigma \in \mathcal{S}_J} (-1)^{\operatorname{inv}\sigma} \begin{pmatrix} \sigma(i_1)\, \sigma(i_2) \cdots \sigma(i_l) \\ i_1 \quad i_2 \quad \cdots \quad i_l \end{pmatrix},$$

where $J = \{i_1 < i_2 < \cdots < i_l\}$ and $\mathcal{S}_J$ is the permutation group acting on the set J. Both Boson and Fermion will be regarded as elements of $\mathcal{R}$ using the above identification. The *quantum Master Theorem* [GLZ05] is stated next.

Theorem 1. *The following identity*

$$\mathrm{Ferm} \times \mathrm{Bos} = 1$$

holds in the right quantum algebra $\mathcal{R}$.

The proof of Theorem 1 is based on specific techniques of Circuit Calculus (section 2) and determinantal identities in the context of the right quantum algebra (section 3). The end of the proof is made in section 4.

2. Real and imaginary expressions

A *circuit* is a biword whose top word is a rearrangement of its bottom word. Each formal sum $E = \sum_{\alpha} c(\alpha)\alpha \in \mathcal{A}$ is said to be *imaginary* (resp. *real*), if $c(\alpha) = 0$ for all circuits α (resp. non-circuits α). If $E, E' \in \mathcal{A}$ and $E' \equiv E \pmod{\mathcal{I}}$, then E' is imaginary (resp. real) if and only if E is imaginary (resp. real), as easily verified by considering the commutation relations (R1) and (R2). Accordingly, it makes sense to say that an expression in $\mathcal{R}$ is imaginary (resp. real). Notice that 0 is both real and imaginary. The terminology is directly inspired from Complex Analysis. The following properties, although easy to verify, are essential for the proof of the theorem.

(P1) Each expression $E \in \mathcal{R}$ can be decomposed in a unique way as a sum of a real expression $\Re(E)$ and an imaginary expression $\Im(E)$:

$$E = \Re(E) + \Im(E).$$

(P2) The real part operator $\Re$ and the imaginary part operator $\Im$ are linear. This means that for any two expresssions E and E' we have

$$\begin{aligned}\Re(E + E') &= \Re(E) + \Re(E'),\\ \Im(E + E') &= \Im(E) + \Im(E').\end{aligned}$$

(P3) The real part operator $\Re$ and the imaginary part operator $\Im$ are idempotent and orthogonal to each other. This means that for every expresssion E we have

$$\Re(\Re(E)) = \Re(E),\ \Im(\Im(E)) = \Im(E) \text{ and } \Re(\Im(E)) = \Im(\Re(E) = 0.$$

(P4) If E, E' are two nonzero expressions from $\mathcal{R}$ and if E is real, then $E \times E'$ is real (resp. imaginary) if and only if E' is real (resp. imaginary).

Lemma 2. *If E, E' are two nonzero expressions from $\mathcal{R}$ and if E is real, then*

$$\Re(E \times E') = E \times \Re(E').$$

Proof. From the above properties we have

$$\begin{aligned}\Re(E \times E') &= \Re(E \times (\Re(E') + \Im(E')))\\ &= \Re(E \times \Re(E') + E \times \Im(E'))\\ &= \Re(E \times \Re(E')) + \Re(E \times \Im(E'))\\ &= E \times \Re(E'). \quad \square\end{aligned}$$

Now, define the *universe* "Univ" to be the sum of all biwords whose top words are nondecreasing, that is,

$$\mathrm{Univ} := \sum_{u,w} \binom{u}{w},$$

where u (resp. w) runs over the set of all *nondecreasing* words (resp. all words) and where u and w are of the same length: $|u| = |w|$. Both Boson and Fermion are sums of *circuits*. Moreover, $\Re(\mathrm{Univ}) = \mathrm{Bos}$. By Lemma 2 the quantum Master Theorem is equivalent to the following theorem.

Theorem 3. *The following identity*

$$\Re(\mathrm{Ferm} \times \mathrm{Univ}) = 1$$

holds in the right quantum algebra $\mathcal{R}$.

3. Determinantal Calculus

Let $A = (a_{i,j})_{1\le i,j\le r}$ be a square matrix whose entries are expressions from $\mathcal{R}$. We define the *determinant* of A to be

$$\det(A) = \sum_{\sigma} (-1)^{\mathrm{inv}\sigma} a_{\sigma_1,1} a_{\sigma_2,2} \cdots a_{\sigma_r,r}.$$

In this formula σ runs over the permutation group $\mathcal{S}_{\mathbb{A}}$ of the set $\mathbb{A}$. The ordering of the factors $a_{\sigma_1,1}$, $a_{\sigma_2,2}$, …, $a_{\sigma_r,r}$ *matters*, as the underlying algebra is noncommutative. However, several classical properties of the determinant still hold.

Property 4 (Linearity)**.** *When writing matrices as sequences of their r columns $(c_1, c_2, \ldots, c_r)$, we have*

$$\det(c_1, \ldots, c_i, \ldots, c_r) + \det(c_1, \ldots, c'_i, \ldots, c_r) = \det(c_1, \ldots, c_i + c'_i, \ldots, c_r).$$

Property 5 (Cofactor expansion)**.** *Let $A = (a_{i,j})_{1\le i,j\le r}$ be a matrix of expressions and A_{ij} be the matrix obtained from A by deleting the i-th row and j-th column. Then*

$$\det(A) = \sum_{i=1}^{r} (-1)^{r+i} \det(A_{ir}) a_{ir}.$$

In the above identity we recognize the usual expansion of $\det(A)$ by cofactors of the *rightmost* column.

Let a_i, b_i, c_i $(i = 1, 2, \dots, r)$ and x, y be scalars and form the $r \times r$-matrix

$$A = \begin{pmatrix} \cdots & a_1 + b_1\binom{1}{x} & c_1 + b_1\binom{1}{y} & \dots \\ \cdots & a_2 + b_2\binom{2}{x} & c_2 + b_2\binom{2}{y} & \dots \\ \ddots & \vdots & \vdots & \ddots \\ \cdots & a_r + b_r\binom{r}{x} & c_r + b_r\binom{r}{y} & \dots \end{pmatrix},$$

where besides the two *consecutive* columns explicitly displayed the other columns are arbitrary. Let B denote the matrix derived from A by transposing those two consecutive columns.

Property 6 (Interchanging two columns)**.** *Let A and B be the two matrices just defined. Then* $\det A = -\det B$.

Proof. In the expansion of $\det A$ the sum of the two terms

$$S_{ij} := \pm \cdots (a_i + b_i\tbinom{i}{x})(c_j + b_j\tbinom{j}{y}) \cdots \ \mp \cdots (a_j + b_j\tbinom{j}{x})(c_i + b_i\tbinom{i}{y}) \cdots$$

may be put in a one-to-one correspondence with the sum

$$T_{ij} := \pm \cdots (c_i + b_i\tbinom{i}{y})(a_j + b_j\tbinom{j}{x}) \cdots \ \mp \cdots (c_j + b_j\tbinom{j}{y})(a_i + b_i\tbinom{i}{x}) \cdots$$

in the expansion of $\det B$. But

$$\begin{aligned} S_{ij} = \pm \cdots \Big(& a_i c_j + a_i b_j\tbinom{j}{y} + c_j b_i\tbinom{i}{x} + b_i b_j\tbinom{ij}{xy} \\ & - a_j c_i - a_j b_i\tbinom{i}{y} - c_i b_j\tbinom{j}{x} - b_i b_j\tbinom{ji}{xy}\Big) \cdots \\ T_{ij} = \pm \cdots \Big(& c_i a_j + c_i b_j\tbinom{j}{x} + b_i a_j\tbinom{i}{y} + b_i b_j\tbinom{ij}{yx} \\ & - c_j a_i - c_j b_i\tbinom{i}{x} - b_j a_i\tbinom{j}{y} - b_i b_j\tbinom{ji}{yx}\Big) \cdots . \end{aligned}$$

Because of the identity $\binom{ij}{xy} - \binom{ji}{xy} = -(\binom{ij}{yx} - \binom{ji}{yx})$ and the fact that the commutation relations can be made at *any* position within each biword (using the associativity property of the right quantum algebra) we conclude that $S_{ij} = -T_{ij}$. $\square$

It is worth noticing that Property 6 is only stated for very special matrices. In general, the column interchanging property does *not* hold for arbitrary matrices with entries in $\mathcal{R}$.

Consider the $r \times r$-matrix $\mathbf{B}^{(r)} = \big(\binom{i}{j}\big)_{1 \le i,j \le r}$. Notice that every entry in $\mathbf{B}^{(r)}$ is a biletter. Define the *fermion-matrix* to be the matrix $\mathbf{F}^{(r)} = \mathbf{I} - \mathbf{B}^{(r)}$. Using Property 4 (linearity) we have

$$\mathrm{Ferm} = \det(\mathbf{F}^{(r)}) = \det \begin{pmatrix} 1 - \binom{1}{1} & -\binom{1}{2} & \dots & -\binom{1}{r} \\ -\binom{2}{1} & 1 - \binom{2}{2} & \dots & -\binom{2}{r} \\ \vdots & \vdots & \ddots & \vdots \\ -\binom{r}{1} & -\binom{r}{2} & \dots & 1 - \binom{r}{r} \end{pmatrix}.$$

Let $i \leq r-1$ and replace the rightmost column in $\mathbf{F}^{(r)}$ by the i-th column of $\mathbf{F}^{(r)}$. Let $\mathbf{F}_i$ be the resulting matrix, so that $\mathbf{F}_i$ has two identical columns.

Lemma 7. *We have:* $\det(\mathbf{F}_i) = 0$.

Proof. Permute columns i and $i+1$, then columns $i+1$ and $i+2$, $\cdots$, finally columns $r-2$ and $r-1$, We obtain a matrix A whose rightmost two columns are identical. Property 6 implies that $\det(A) = 0$ and also $\det(\mathbf{F}_i) = \pm\det(A) = 0$. $\square$

4. The proof

First, define

$$S_i = S_i^{(r)} := \binom{i}{1} + \binom{i}{2} + \cdots + \binom{i}{r};$$
$$K_i = K_i^{(r)} := \frac{1}{1 - S_i} = \sum_{n \geq 0} S_i^n.$$

Lemma 8. *The universe defined in Section 2 is equal to*

$$\mathrm{Univ} = K_1 K_2 \cdots K_r.$$

Proof. By definition of "Univ" we have

$$\begin{aligned}\mathrm{Univ} &= \sum_{u,w} \binom{u}{w} = \sum_{w_1, w_2, \cdots, w_r} \binom{11\cdots1}{w_1}\binom{22\cdots2}{w_2}\cdots\binom{rr\cdots r}{w_r}\\ &= \sum_{w_1}\binom{11\cdots1}{w_1}\sum_{w_2}\binom{22\cdots2}{w_2}\cdots\sum_{w_r}\binom{rr\cdots r}{w_r}\\ &= K_1K_2\cdots K_r. \quad \square\end{aligned}$$

Lemma 9. *We have:* $S_iS_j = S_jS_i$ *and* $K_iK_j = K_jK_i$.

Proof. Grouping the biwords by pairs if necessary we have:

$$\begin{aligned}S_iS_j &= \sum_{a<b}(\binom{ij}{ab} + \binom{ij}{ba}) + \sum_a \binom{ij}{aa}\\ &= \sum_{a<b}(\binom{ji}{ab} + \binom{ji}{ba}) + \sum_a \binom{ji}{aa}\\ &= S_jS_i. \quad \square\end{aligned}$$

Proof of Theorem 3. Let $\mathbf{M}$ be the matrix obtained from the fermion-matrix $\mathbf{F}^{(r)}$ by adding all the leftmost $r-1$ columns to the rightmost column:

$$\mathbf{M} = \begin{pmatrix} 1-\binom{1}{1} & -\binom{1}{2} & \dots & 1-S_1 \\ -\binom{2}{1} & 1-\binom{2}{2} & \dots & 1-S_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\binom{r}{1} & -\binom{r}{2} & \dots & 1-S_r \end{pmatrix}$$

By Property 4 and Lemma 7: $\det(\mathbf{M}) = \det(\mathbf{F}^{(r)}) = \mathrm{Ferm}$. Let

$$E^{(r)} := \mathrm{Ferm} \times \mathrm{Univ} = \det(\mathbf{M}) \times \mathrm{Univ}$$

By Lemmas 8 and 9

$$\begin{aligned} E^{(r)} &= \det \begin{pmatrix} 1-\binom{1}{1} & -\binom{1}{2} & \dots & (1-S_1)\times \mathrm{Univ} \\ -\binom{2}{1} & 1-\binom{2}{2} & \dots & (1-S_2)\times \mathrm{Univ} \\ \vdots & \vdots & \ddots & \vdots \\ -\binom{r}{1} & -\binom{r}{2} & \dots & (1-S_r)\times \mathrm{Univ} \end{pmatrix} \\ &= \det \begin{pmatrix} 1-\binom{1}{1} & -\binom{1}{2} & \dots & K_2K_3\cdots K_r \\ -\binom{2}{1} & 1-\binom{2}{2} & \dots & K_1K_3\cdots K_r \\ \vdots & \vdots & \ddots & \vdots \\ -\binom{r}{1} & -\binom{r}{2} & \dots & K_1K_2\cdots K_{r-1} \end{pmatrix}. \end{aligned}$$

Now applying Property 5 to the above determinant yields

$$\begin{aligned} E^{(r)} =& (-1)^{r+1}\det(\mathbf{M}_{1r})K_2K_3\cdots K_r \\ & (-1)^{r+2}\det(\mathbf{M}_{2r})K_1K_3\cdots K_r + \cdots \\ & + \det(\mathbf{M}_{rr})K_1K_2\cdots K_{r-1}, \end{aligned}$$

where, as in Property 5, $\mathbf{M}_{ij}$ denotes the minor at position (i,j). For each biword occurring in $F_1 := \det(\mathbf{M}_{1r})K_2K_3\cdots K_r$ there is a letter 1 in the bottom, but none in the top. This means that F_1 does not contain any circuit. Thus $\Re(F_1) = 0$. This argument is also valid for the other terms in the above summation of $E^{(r)}$, except for the last term. When taking the real part of the $E^{(r)}$ we obtain:

$$\begin{aligned} \Re(E^{(r)}) &= \Re(\det(\mathbf{M}_{rr})K_1K_2\cdots K_{r-1}) \\ &= \Re(\det(\mathbf{F}^{(r-1)})K_1^{(r-1)}K_2^{(r-1)}\cdots K_{r-1}^{(r-1)}) \\ &= \Re(E^{(r-1)}). \end{aligned}$$

Then, by iteration

$$\Re(E^{(r)}) = \Re(E^{(r-1)}) = \cdots = \Re(E^{(1)}) = \Re((1-\binom{1}{1})K_1^{(1)}) = 1. \quad \square$$

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