## Models of Gauge Field Theory on Noncommutative Spaces

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## Abstract

Models of gauge field theories on noncommutative spaces are studied. In the first chapter we consider a canonically deformed space. Its noncommutativity is induced by means of a star product with a constant Poisson tensor. We discuss the dependence of the established gauge field theory on the choice of the star product itself: The dependence of the action is derived and it is shown that in general two arbitrary star products lead to two different actions. A large class of star products which all correspond to equal actions is determined as well. The Moyal-Weyl star product and the normal ordered star product are among this set of star products.

In the second chapter we discuss noncommutative, abelian gauge field theory on the  $E_q(2)$ -symmetric plane. Two different approaches are studied.

The first is the generalization of the theory established for a canonically deformed space. We induce the noncommutativity via a star product with a nonconstant Poisson tensor. To be able to define a gauge invariant action, we construct an integral with trace property (considerations made here may be transfered to other noncommutative spaces). The noncommutative fields are expressed in terms of the ordinary, commutative fields via the Seiberg-Witten map. This allows us to calculate explicitly the corrections to the commutative theory predicted by the noncommutative one in orders of the deformation parameter. We present the results up to first order, which shows that new interactions appear. The integral we had to introduce is not invariant with respect to  $E_q(2)$ —transformations.

In the second approach we establish a gauge field theory which is  $E_q(2)$  – covariant. We construct a  $U_q(e(2))$  – invariant integral. It turns out that this integral is not cyclic but possesses a "deformed" cyclic property. We establish an  $E_q(2)$ —covariant differential calculus. A frame for one-forms is found as well. A crucial issue are gauge transformations: In order to obtain a gauge invariant and  $E_q(2)$ —invariant action, we introduce "deformed" gauge transformations. The semi-classical limit  $q \to 1$  of those gauge transformations is examined.

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### Introduction

The idea to study noncommutative space-time is very old. It goes back to Snyder, who first suggested a noncommutative structure at small length scales to obtain this way an effective ultraviolet cutoff [1]. This in turn should help to deal with the divergences that already appeared in the early days of quantum field theory. The problem still persists and during the last two decades noncommutative spaces have been intensively studied in the hope to find a natural regularization of deformed quantum field theories.

There are different ways to introduce a noncommutativity of space-time. One is inspired by quantum mechanics, where position and momentum operators obey the Heisenberg commutation relations. Similarly, the quantized space-time coordinates  $\hat{x}^i$  are demanded to satisfy the so called canonical commutation relations

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \quad , \tag{1}$$

where  $\theta^{ij}$  are real constants. This approach became especially interesting when in the eighties the study of open strings in a magnetic background field led to a canonical noncommutativity in string theory [2]. However, by deforming the space alone the space will in general loose its classical background symmetry.

To maintain a background symmetry in the deformed setting, it is therefore in general necessary to deform both the symmetry group as well as the space. This leads to so called q—deformed spaces that admit a quantum group symmetry. Correctly spoken the algebra of functions on a manifold and the algebra of functions on a Lie group are q—deformed [3]. Such a noncommutative theory is a generalization of the commutative theory since the deformed space as well as the deformed symmetry group become in the semi-classical limit  $q \to 1$  the commutative space together with its undeformed background symmetry. The rich mathematical structure of quantum groups gives rise to the hope to get a better understanding of physics at short distances and to solve the above-mentioned problems.

Gauge field theories have been very successful for the understanding of the fundamental forces of nature. The standard model made a unification of electromagnetism, the weak and the strong force possible. Therefore a generalization of the gauge invariance principle to noncommutative spaces is of particular interest and subject matter of this thesis. The hope is that a deep understanding of gauge field theories on noncommutative spaces may make it possible to finally incorporate the last fundamental force, gravitation, in a unified theory.

This thesis is divided into two chapters. The first chapter deals with a canonically deformed space. Gauge field theory on this noncommutative space has been examined intensively in [4, 5, 6, 7, 8]. A crucial ingredient for the construction of gauge field theory on a canonically deformed space is the use of a star product that renders the originally commutative algebra of functions on the space isomorphic as algebra to the noncommutative algebra of functions. By means of Seiberg-Witten maps it then is possible to relate ordinary gauge theory and noncommutative gauge theory and to express noncommutative fields in terms of their commutative analogs. This allows us to read off explicitly the corrections to the commutative theory predicted by the noncommutative one in orders of the deformation parameter h. In the whole formalism a special star product was used: the Moyal-Weyl star product that corresponds to the symmetric ordering prescription. However, there exist infinitely many star products, all corresponding to a different ordering prescription, that cannot be distinguished from the algebraic point of view. Thus, the question arises how far a change of the star product changes the physical theory, i.e. the action. We will examine in all detail how a change of the ordering prescription explicitly affects the action. We will see that in general different star products indeed lead to different actions, but that nonetheless a large class of star products generates identical actions. The star products corresponding to normal ordering and symmetric ordering belong to this class.

The second chapter treats the case of a two-dimensional quantum space with an  $E_q(2)$ -symmetry. By  $E_q(2)$  we denote the q-deformed algebra of functions on the two-dimensional Euclidean group. We start introducing the quantum group  $E_q(2)$  and the corresponding  $E_q(2)$ -covariant plane. The underlying commutation relation of the space coordinates is

$$z\overline{z} = q^2\overline{z}z\tag{2}$$

in the basis z = x + iy,  $\overline{z} = x - iy$ . We then proceed to study two different approaches to abelian gauge field theory on this noncommutative space.

The first is a generalization of the formalism developed in the case of a constant Poisson tensor. If we substitute q by  $e^h$ , we can write down the commutation

relations of the coordinates of the  $E_q(2)$ -covariant plane in the form of (1), where  $\theta^{ij} = \theta^{ij}(z, \overline{z})$  now depends on the coordinates and is not constant. A star product for this space is introduced, as well as a noncommutative gauge field  $A^{i}$  by the concept of covariant coordinates. Moreover, we can again express the noncommutative quantities in terms of the commutative physical fields via a solution for the Seiberg-Witten maps. To get a gauge field which approaches in the semi-classical limit  $h \to 1$  the commutative gauge field, we must introduce vector fields with lower indices. Since  $\theta$  is not constant, we cannot use  $\theta$  to lower indices as this would spoil gauge covariance of the gauge field  $A^{i}$ . We have to do it covariantly using the "covariantizer"  $\mathcal{D}$  [9]. At the end of the section we calculate explicitly the field strength, the Lagrangian and the action expanded in h up to first order. As we will see, new interactions appear. Unfortunately, we will learn that the theory possesses some freedom in defining the field strength and the action. Furthermore, we will see that it is not  $E_q(2)$ —covariant: We are forced to introduce an integral with trace property in order to obtain a gauge invariant action. This action is not  $E_q(2)$ -invariant.

In the third section we try to establish an  $E_q(2)$ -covariant gauge field theory. Considering  $E_q(2)$ —covariance as a fundamental concept underlying all considerations, we introduce an  $E_q(2)$ -invariant (or equivalently a  $U_q(e(2))$  - invariant) integral. We will see that this integral is not cyclic but possesses a "deformed" cyclic property. This is a crucial result, telling us that we cannot introduce gauge transformations of gauge fields as conjugation with a unitary element as it was done in the previous section. In a second step, we establish an  $E_q(2)$ -covariant differential calculus. Furthermore, we find a frame, a basis of one-forms that commutes with all functions, and a generator  $\Theta$  of the exterior differential d. This simplifies calculations in the q-deformed space. A noncommutative gauge field A and a noncommutative field strength F that approaches the commutative field strength in the semi-classical limit  $q \to 1$  are introduced. Finally, "deformed" gauge transformations are defined. This allows us to obtain an action that is both gauge invariant and  $E_q(2)$  – invariant and thereby we get an  $E_q(2)$  –covariant gauge field theory. In the end, we examine the semi-classical limit for the deformed gauge transformations.

## Chapter 1

# Influence of the Ordering Prescription in the Case $\theta = \text{const.}$

Noncommutative spaces, especially in the case of a canonical noncommutativity, have been intensively studied in recent years. In [5] and [4], for instance, a gauge theory has been developed on such a noncommutative space. The star product formalism plays a crucial role in this theory because it gives us, together with the Seiberg-Witten map, a possibility to express the noncommutative fields entering the theory in terms of the well known commutative fields. This makes it possible to read off explicitly the corrections to the commutative theory predicted by the noncommutative one. The whole formalism was developed for a special star product, the Moyal-Weyl star product, and it is necessary to ask the question whether a different star product leads to a different physical theory. Since a star product corresponds to an ordering prescription, we are led directly to the question of how far a change of the ordering prescription affects the physical theory.

In fact, we will see that for a large class of ordering prescriptions the theories are physically equivalent in the sense that all those theories lead to the same action. In particular this is the case for the usually used star products, the Moyal-Weyl product corresponding to the symmetric ordering prescription and the normal ordered star product. Nonetheless, arbitrary star products will in general lead to different actions and thereby to different physical theories.

We do not want to repeat in all detail the formalism constructed in [5, 4], presuming that the general ideas are known. Hence, we will only present the essential ideas. Nonetheless, we want to explain in detail the concepts that will be important within the framework of this thesis starting with the notions of ordering and star product.

#### 1.1 Ordering and Star Products

In this section we want to explain what we mean by an ordering prescription, what a star product is and how a certain ordering prescription corresponds to a star product. It will be a mixture of well known definitions and facts (see for example [10] or [11]) with some new results in the application on the case of a constant Poisson tensor.

#### 1.1.1 About Ordering

Underlying all our considerations is the following noncommutative algebra of space time generated by the elements  $\hat{x}^{\mu}$ , to which we want to refer as coordinates, satisfying the relations

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = ih\theta^{\mu\nu}, \quad h, \theta^{\mu\nu} \in \mathbb{R} \quad , \tag{1.1}$$

i.e. the algebra

$$\mathcal{A}_{\rm nc} \equiv \mathbb{C}\langle\langle \hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3 \rangle\rangle / (\{[\hat{x}^\mu, \hat{x}^\nu] - ih\theta^{\mu\nu}\}) , \qquad (1.2)$$

where Greek indices are to be understood in the whole chapter as elements of the set  $\{0,1,2,3\}^1$ , if nothing else is indicated. We additionally want to assume that  $\theta^{\mu\nu}$  is non-degenerated. Furthermore, we have the well known four dimensional space of commutative coordinates  $x^{\mu}$ ,

$$[x^{\mu}, x^{\nu}] = 0$$
 , (1.3)

generating the algebra

$$\mathcal{A}_{c} \equiv \mathbb{C}[[x^{0}, x^{1}, x^{2}, x^{3}]]$$
 (1.4)

We want to call a map  $\rho$  from  $\mathcal{A}_c$  to  $\mathcal{A}_{nc}$  an ordering prescription<sup>2</sup> if

- $\rho: \mathcal{A}_{c} \to \mathcal{A}_{nc}$  is a vector space isomorphism,
- $\rho(1) = 1$  and
- $\rho$  approaches the identity for  $h \to 0$ , i.e.  $\rho = \mathrm{id} + \mathcal{O}(h)$ .

Obviously, we have a big freedom in constructing ordering prescriptions. Let us illustrate that by giving two examples: the normal ordering and the symmetric ordering.

<sup>&</sup>lt;sup>1</sup>Considerations made in this chapter are true for higher dimensions as well.

<sup>&</sup>lt;sup>2</sup>We do not want to understand an ordering prescription as simple "ordering" the coordinates in a special way but admit more general isomorphisms as well. One could have said *quantization* instead of ordering prescription, too.

#### (a) The normal ordering

We define a vector space isomorphism

$$\rho_n: \mathcal{A}_{\mathrm{c}} \to \mathcal{A}_{\mathrm{nc}}$$

on the basis of monomials by

$$(x^0)^i(x^1)^j(x^2)^k(x^3)^l \mapsto (\hat{x}^0)^i(\hat{x}^1)^j(\hat{x}^2)^k(\hat{x}^3)^l \text{ for } i, j, k, l \in \mathbb{N} . \tag{1.5}$$

This ordering prescription we want to call normal ordering.

#### (b) The symmetric ordering

We can define another vector space isomorphism

$$\rho_s: \mathcal{A}_c \to \mathcal{A}_{nc}$$

by assigning any monomial of the variables  $x^0, x^1, x^2, x^3$  to its totally symmetric counterpart, i.e. for example

This ordering prescription we want to call symmetric ordering.

Of course it is possible to construct arbitrarily many other ordering prescriptions. But in the following we will often come back to these important examples to illustrate our ideas and results.

While the noncommutative algebra  $\mathcal{A}_{nc}$  and the commutative algebra  $\mathcal{A}_{c}$  can impossibly be isomorphic as algebras, we use a vector space isomorphism  $\rho$  to transfer the multiplicative structure of  $\mathcal{A}_{nc}$  onto  $\mathcal{A}_{c}$ . We then obtain a new product, called *star product* (or  $\star$ -*product*).

#### 1.1.2 Star Products

Star products are usually defined on Poisson manifolds [11, 10]. Taking the smooth functions on a manifold we get an algebra that we can deform by means of a star product. Here we want to take an analogous way but do not want to abandon the concept of formal power series in the coordinates and do not want to speak about smooth functions on a manifold. This would direct our attention to problems that are of secondary interest for the general physical statement and intention of this thesis. Therefore we want to introduce star products for *Poisson algebras*, where a Poisson algebra is an associative algebra  $\mathcal{A}$  together with a *Poisson-bracket*. A Poisson bracket on an algebra  $\mathcal{A}$  in turn is defined as a bilinear map

$$\{\cdot,\cdot\}:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$$

satisfying

(i) 
$$\{f, g\} = -\{g, f\}$$
 for all  $f, g \in \mathcal{A}_c$  (Anti-symmetry)

(ii) 
$$\{\{f,g\},h\}+\{\{h,f\},g\}+\{\{g,h\},f\}=0$$
 (Jacobi-identity)

(iii) 
$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$
 (Leibniz-rule).

If we define

$$\{f,g\} := \theta^{\mu\nu}(\partial_{\mu}f)(\partial_{\nu}g) ,$$

it is easy to see that  $\{\cdot, \cdot\}$  is a Poisson bracket for  $\mathcal{A}_c$  in the sense of the above definition. The tensor  $\theta$  is called a *Poisson tensor* for  $\mathcal{A}_c$ . It is an antisymmetric tensor because of (i) and satisfies (ii) and (iii).

#### (a) Definition and Remarks

The deformation theory of associative algebras was first studied by Gerstenhaber [12]. A formal deformation of a Poisson algebra can be defined as follows:

**Definition 1.** Let  $(A, \{\cdot, \cdot\})$  be a Poisson algebra. We call a  $\mathbb{C}[[h]]$ -bilinear map

$$\star: \mathcal{A}[[h]] \times \mathcal{A}[[h]] \to \mathcal{A}[[h]]$$

that we can write as a formal power series in  $h^3$ ,

$$\star = \sum_{r=0}^{\infty} h^r M_r \,,$$

<sup>&</sup>lt;sup>3</sup>This is always possible.

with  $\mathbb{C}$ -bilinear maps  $M_r: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$  a star product (or  $\star$ -product) for  $(\mathcal{A}, \{\cdot, \cdot\})$  if for all  $f, g, h \in \mathcal{A}$  the following holds:

(i) 
$$f \star (g \star h) = (f \star g) \star h \qquad (associativity)$$

(ii) 
$$M_0(f,g) = fg$$
 (deformation of the commutative product)

(iii) 
$$M_1(f,g) - M_1(g,f) = i\{f,g\}$$
 (deformation in direction of  $\{\cdot,\cdot\}$ )

$$(iv) f \star 1 = f = 1 \star f .$$

A star product that satisfies for all  $f, g \in \mathcal{A}[[h]]$  the condition

$$\overline{f \star g} = \overline{g} \star \overline{f},$$

is called a hermitian star product (here the bar denotes the usual complex conjugation on  $\mathbb{C}$ ).

Similar to this definition, star products are defined on a Poisson manifold M taking as algebra the algebra of smooth functions  $C^{\infty}(M)$  [11, 10].

As we see, a star product is a deformation of the commutative product assuring that the noncommutative theory we establish approaches in the semi-classical limit  $h \to 0$  the usual commutative physical theory. The third condition gives the analog to the quantum mechanical principle of correspondence: The Poisson bracket  $\{U,V\}$  of two observables in classical mechanics corresponds to the quantum mechanical commutator  $\frac{-i}{\hbar}[\hat{U},\hat{V}]$  in the classical limit. In our setting, the noncommutative algebra of quantum mechanical observables is replaced by the noncommutative algebra of functions on noncommutative space time.

Having specified what we want to demand from a star product to assure physical interpretation, we discuss how to obtain  $\star$ -products satisfying these properties. Using an ordering prescription  $\rho: \mathcal{A}_{c} \to \mathcal{A}_{nc}$  with  $\rho(1) = 1$  (for example  $\rho_{n}$  or  $\rho_{s}$  given above), we can define a product for  $f, g \in \mathcal{A}_{c}$  by pulling back the noncommutative product of the algebra  $\mathcal{A}_{nc}$  onto the algebra  $\mathcal{A}_{c}$ :

$$f \star g := \rho^{-1}(\rho(f) \cdot \rho(g)) . \tag{1.7}$$

We want to interpret  $f \star g$  as a formal power series in the deformation parameter h

$$f \star g \in \mathcal{A}_{\mathbf{c}}[[h]]$$
,

such that we obtain by (1.7)

$$(\mathcal{A}_{c}[[h]], \star) \cong (\mathcal{A}_{nc}[[h]], \cdot) \tag{1.8}$$

as algebras.

We claim that the product  $\star$  defined in (1.7) actually is a star product on  $\mathcal{A}_{c}[[h]]$  in the sense of Definition 1. To see this, we have to check some properties:

(i) Associativity follows since multiplication in  $\mathcal{A}_{nc}$  is associative:

$$\begin{array}{lcl} f\star(g\star h) &=& \rho^{-1}(\rho(f)\cdot\rho(\rho^{-1}(\rho(g)\cdot\rho(h))))\\ &=& \rho^{-1}(\rho(f)\cdot(\rho(g)\cdot\rho(h))\\ &=& \rho^{-1}((\rho(f)\cdot\rho(g))\cdot\rho(h)) &=& (f\star g)\star h \end{array}.$$

- (ii) By definition an ordering prescription  $\rho$  approaches for  $h \to 0$  the identity such that we have  $\rho = \mathrm{id} + \mathcal{O}(h)$ . This yields with (1.7) that  $f \star g = fg + \mathcal{O}(h)$ .
- (iii) Write  $f \star g = \sum h^r M_r(f,g)$  with  $M_r : \mathcal{A}_c \times \mathcal{A}_c \to \mathcal{A}_c$  and define  $\{f,g\}' := -i(M_1(f,g) M_1(g,f))$ . We claim that  $\{\cdot,\cdot\}'$  is a Poisson bracket: It is obviously antisymmetric and to see that the Leibniz rule and the Jacobi identity are satisfied we have to use a property the maps  $M_r$  possess derived from the fact that  $\star$  is associative as shown in (i): We have

$$f \star (g \star k) = (f \star g) \star k$$

$$\Leftrightarrow f(gk) + h(M_1(f, gk) + fM_1(g, k))$$

$$+h^2(M_1(f, M_1(g, k)) + fM_2(g, k) - M_2(f, gk)) + \mathcal{O}(h^3)$$

$$= (fg)k + h(M_1(fg, k) + M_1(f, g)k)$$

$$+h^2(M_1(M_1(f, g), k) + M_2(f, g)k - M_2(fg, k)) + \mathcal{O}(h^3)$$

$$\Rightarrow M_1(f, gk) + fM_1(g, k) = M_1(fg, k) + M_1(f, g)k$$
and
$$M_1(f, M_1(g, k)) + fM_2(g, k) - M_2(f, gk)$$

$$= M_1(M_1(f, g), k) + M_2(f, g)k - M_2(fg, k)$$

Using those two properties for  $M_1$  a longer but easy calculation shows that  $\{\cdot,\cdot\}'$  satisfies the Leibniz rule and the Jacobi identity so that  $\{\cdot,\cdot\}'$  defined as above is indeed a Poisson bracket. Since all Poisson brackets on  $\mathcal{A}_c$  are of the form

$$\{f,g\} = \alpha^{\mu\nu}(x)(\partial_{\mu}f)(\partial_{\nu}g)$$

with  $\alpha^{\mu\nu}$  antisymmetric (this follows from the Leibniz rule and the Jacobi identity), we can conclude that actually  $\{f,g\}' = \alpha^{\mu\nu}(x)(\partial_{\mu}f)(\partial_{\nu}g)$  for functions  $\alpha^{\mu\nu}(x)$ . Now we use that  $\star$  is defined by means of an

ordering prescription  $\rho$ . As an ordering prescription,  $\rho$  goes to the identity for  $h \to 0$ . Therefore we have  $\rho(x^{\mu}) = \hat{x}^{\mu} + \mathcal{O}(h)$ . Moreover, we denote by  $\pi_1$  the projection to the first order of h. With that we obtain

$$h(M_1(x^{\mu}, x^{\nu}) - M_1(x^{\nu}, x^{\mu})) = \pi_1(x^{\mu} \star x^{\nu} - x^{\nu} \star x^{\mu})$$

$$= \pi_1(\rho^{-1}([\rho(x^{\mu}), \rho(x^{\nu})]))$$

$$= \pi_1(\rho^{-1}([\hat{x}^{\mu}, \hat{x}^{\nu}]))$$

$$= ih\theta^{\mu\nu}.$$

This yields

$$\{x^{\mu}, x^{\nu}\}' = \theta^{\mu\nu}$$
.

On the other hand we have

$$\{x^{\mu}, x^{\nu}\}' = \alpha^{\sigma\tau}(x)(\partial_{\sigma}x^{\mu})(\partial_{\tau}x^{\nu}) = \alpha^{\mu\nu}(x) .$$

Hence, we can conclude that  $\alpha^{\mu\nu}(x) = \theta^{\mu\nu}$  and therefore we get  $(M_1(f,g) - M_1(g,f)) = i\{f,g\}' = i\theta^{\mu\nu}(\partial_{\mu}f)(\partial_{\nu}g)$ , which is exactly what we wanted to show.

(iv) 
$$f \star 1 = \rho^{-1}(\rho(f) \cdot 1) = f = \rho^{-1}(1 \cdot \rho(f)) = 1 \star f$$
.

Let us summarize what we learned so far: Starting with an arbitrary ordering prescription  $\rho: \mathcal{A}_c \to \mathcal{A}_{nc}$  we get by (1.7) a star product  $\star$  on  $\mathcal{A}_c$  such that  $(\mathcal{A}_c, \star)$  is isomorphic as algebra to  $\mathcal{A}_{nc}$ . Then the question arises immediately, whether all existing star products correspond to an ordering prescription. To answer this question we first have to be able to classify star products. That is why we want to continue by introducing the notion of equivalence.

Remark: By the above proof we showed in particular the existence of star products for  $\mathcal{A}_{c}[[h]]$ . In the setting of a general Poisson manifold the question of existence is much more difficult. Nevertheless, it is known that there exist star products for any Poisson manifold [13].

#### (b) Equivalence of Star Products

**Definition 2.** Two star products  $\star$  and  $\star'$  for  $\mathcal{A}_{c}[[h]]$  are called equivalent if there exists a formal series

$$S = \mathrm{id} + \sum_{k=1}^{\infty} h^k S_k \tag{1.9}$$

of linear operators  $S_k: \mathcal{A}_c \to \mathcal{A}_c$  such that for all  $f, g \in \mathcal{A}_c$ 

$$f \star' g = S^{-1}(S(f) \star S(g)) \text{ and } S(1) = 1 .$$
 (1.10)

S is called equivalence transformation from  $\star$  to  $\star'$  and for hermitian star products we also require that for all  $f \in \mathcal{A}_c$ 

$$\overline{S(f)} = S(\overline{f}) .$$

Let us remark that this notion of equivalence indeed defines an equivalence relation. Moreover, we see that we can formulate equivalence in the following way, too:

Two star products  $\star$  and  $\star'$  are equivalent if and only if  $(\mathcal{A}_{c}[[h]], \star) \cong (\mathcal{A}_{c}[[h]], \star')$ . We additionally note that by giving an equivalence transformation as in (1.9) we can construct a new star product defining  $\star'$  as in (1.10), so that to a given star product there exist arbitrarily many equivalent ones. Then immediately arises the question how many equivalence classes exist. For a symplectic Poisson manifold, this is a Poisson manifolds with non-degenerated Poisson tensor, the answer is indeed known (to get a definition of equivalence in the case of a Poisson manifold M just substitute in the above definition  $\mathcal{A}_{c}[[h]]$  by  $C^{\infty}(M)$ ) [13, 14]:

Let M be a symplectic Poisson manifold. Then  $\{[\star]\} \cong H^2_{\mathrm{dR}}(M)[[h]]$  where  $[\star]$  denotes the equivalence class of the star product  $\star$ .<sup>4</sup>

In the case  $M = \mathbb{R}^n$  then follows that *all* star products are equivalent since  $H^2_{dR}(\mathbb{R}^n) = 0$ . In our example we do not consider a Poisson manifold but the Poisson algebra  $(\mathcal{A}_c, \theta^{\mu\nu})$ . Nevertheless, we can interpret the Poisson tensor  $\theta^{\mu\nu}$  as a Poisson tensor on  $C^{\infty}(\mathbb{R}^4)$  and star products for  $(\mathcal{A}_c[[h]], \theta^{\mu\nu})$  become star products for the Poisson manifold  $(\mathbb{R}^4, \theta^{\mu\nu})$ . If  $\theta^{\mu\nu}$  is non-degenerated, we have the case of a symplectic manifold and since on  $\mathbb{R}^4$  all star products are equivalent this leads us to the following conclusion:

#### Conclusion 1.

For  $(\mathcal{A}_c[[h]], \theta^{\mu\nu})$  all star products are equivalent if  $\theta^{\mu\nu}$  is non-degenerated.

Starting with a certain ordering prescription  $\rho$  and constructing the corresponding star product  $\star$  by (1.7), we now know that all star products that exist are in fact equivalent to this one. Moreover, we notice that for an arbitrary equivalence transformation S,  $\rho' := \rho \circ S$  again is an ordering prescription and obviously we have  $f \star' g := S^{-1}(S(f) \star S(g)) = S^{-1} \circ \rho^{-1}(\rho \circ S(f) \cdot \rho \circ S(g)) = \rho'^{-1}(\rho'(f) \cdot \rho'(g))$ . Thus, the answer to the question at the end of (a) is:

 $<sup>^4</sup>H_{
m dR}^2$  denotes the second de Rham cohomology.

Conclusion 2. Every star product  $\star$  for  $(\mathcal{A}_{c}[[h]], \theta^{\mu\nu})$  can be obtained by

$$f \star g = \rho^{-1}(\rho(f) \cdot \rho(g)) ,$$

where  $\rho$  is an ordering prescription.<sup>5</sup>

#### (c) Examples for Star Products

• The symmetric ordering  $\rho_s$  (1.6) leads to the star product

$$f \star_s g = \mu \circ e^{\frac{1}{2}ih\theta^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu})}(f\otimes g) . \tag{1.11}$$

where  $\mu$  is to be understood as the multiplication map  $\mu(f \otimes g) = fg$ . This star product is called *Moyal-Weyl star product* [15, 4]. Moreover, we see that  $\star_s$  is a *hermitian* star product (see Definition 1): Taking into account that  $\theta$  is antisymmetric and real, we obtain:

$$\overline{f} \star_{s} \overline{g} = \overline{\mu \circ e^{\frac{1}{2}ih\theta^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu})}(f\otimes g)} 
= \mu \circ e^{-\frac{1}{2}ih\theta^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu})}(\overline{f}\otimes \overline{g}) 
= \mu \circ e^{\frac{1}{2}ih\theta^{\nu\mu}(\partial_{\nu}\otimes\partial_{\mu})}(\overline{g}\otimes \overline{f}) 
= \overline{g} \star_{s} \overline{f} .$$
(1.12)

• The normal ordering  $\rho_n$  (1.5) leads to the star product

$$f \star_n g = \mu \circ e^{ihm^{\mu\nu}(\partial_{\mu} \otimes \partial_{\nu})} (f \otimes g) , \qquad (1.13)$$

where  $m^{\mu\nu} := a^{\mu\nu}\theta^{\mu\nu}$  (no summation!) with  $a^{\mu\nu} := \begin{cases} 0, & \mu \leq \nu \\ 1, & \mu > \nu \end{cases}$ .

• Let me introduce the following more general star product:

$$f \star_k g := \mu \circ e^{ihm^{\mu\nu}(\partial_{\mu} \otimes \partial_{\nu})} (f \otimes g) , \qquad (1.14)$$

where  $m^{\mu\nu}:=k^{\mu\nu}\theta^{\mu\nu}$  with  $k^{\mu\mu}=0$  and  $k^{\mu\nu}+k^{\nu\mu}=1$  for  $\mu\neq\nu$ .

It is easy to check that  $\star_k$  is a star product in the sense of Definition 1. As we know from Conclusion 2, we can obtain  $\star_k$  by an ordering prescription.<sup>6</sup>

 $<sup>^5</sup>$ In particular, every star product renders  $\mathcal{A}_c$  isomorphic to  $\mathcal{A}_{nc}$ . In the general case of a non-constant Poisson tensor this is not true.

<sup>&</sup>lt;sup>6</sup>In general this is not unique.

To get an impression of how such an ordering prescription looks like, we calculate explicitly for  $\mu \neq \nu$ :

$$x^{\mu} \star_{k} x^{\nu} = x^{\mu} x^{\nu} + i h m^{\mu\nu}$$
 and  $x^{\nu} \star_{k} x^{\mu} = x^{\nu} x^{\mu} + i h m^{\nu\mu}$ 

and therefore

$$k^{\nu\mu}(x^{\mu} \star_{k} x^{\nu}) + k^{\mu\nu}(x^{\nu} \star_{k} x^{\mu}) = (k^{\mu\nu} + k^{\nu\mu})x^{\mu}x^{\nu} + ihk^{\mu\nu}k^{\nu\mu}(\theta^{\mu\nu} + \theta^{\nu\mu})$$
$$= x^{\mu}x^{\nu} , \qquad (1.15)$$

where the last equality follows because  $\theta$  is antisymmetric and because  $k^{\mu\nu} + k^{\nu\mu} = 1$ .

With (1.15) we find that an ordering prescription  $\rho_k : \mathcal{A}_c \to \mathcal{A}_{nc}$  which starts with the assignments

$$\begin{array}{ccc}
1 & \mapsto & 1 \\
x^{\mu} & \mapsto & \hat{x}^{\mu} \\
x^{\mu}x^{\nu} & \mapsto & k^{\nu\mu}\hat{x}^{\mu}\hat{x}^{\nu} + k^{\mu\nu}\hat{x}^{\nu}\hat{x}^{\mu}
\end{array} \tag{1.16}$$

leads to the star product  $\star_k$ . Below we will derive an equivalence transformation  $T_k$  from  $\star_s$  to  $\star_k$  such that the entire ordering prescription is given by  $\rho_k = \rho_s \circ T_k$ . We see that we must require  $k^{\mu\nu} + k^{\nu\mu} = 1$  such that  $\rho_k$  equals the identity map for h = 0. The star product  $\star_k$  is obviously a generalization of the first two examples  $\star_s$  and  $\star_n$ , since we get  $\star_s$  for  $k^{\mu\nu} := \begin{cases} \frac{1}{2}, & \mu \neq \nu \\ 0, & \mu = \nu \end{cases}$  and  $\star_n$  for  $k^{\mu\nu} = a^{\mu\nu}$  as in (1.13).

#### (d) Example for Equivalence

Can we find an explicit equivalence transformation  $T_k$  (see Definition 2) to pass over from the usually used Moyal-Weyl product  $\star_s$  to the more general star product  $\star_k$ ? We already know that such a  $T_k$  exists because of Conclusion 1 and the following Lemma provides it explicitly.

**Lemma 1.** The star products  $\star_s$  defined in (1.11) and  $\star_k$  defined in (1.14) are equivalent and the equivalence transformation from  $\star_s$  to  $\star_k$  is given by

$$T_k = e^{-\frac{i}{2}hm^{\mu\nu}\partial_{\mu}\partial_{\nu}} , \qquad (1.17)$$

where  $m^{\mu\nu} := k^{\mu\nu}\theta^{\mu\nu}$  (no summation!).

*Proof.* We have to prove that  $T_k(f \star_k g) = T_k(f) \star_s T_k(g)$  for all f, g. For this purpose we have to know how to commute the multiplication map  $\mu$  with the operator  $T_k$ . In this regard the following equation, nothing else than the Leibniz rule written in tensor notation, turns out to be very useful:

$$m^{\mu\nu}\partial_{\mu}\partial_{\nu}\circ\mu=\mu\circ(m^{\mu\nu}\partial_{\mu}\partial_{\nu}\otimes\mathrm{id}+\mathrm{id}\otimes m^{\mu\nu}\partial_{\mu}\partial_{\nu}+m^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu})+m^{\mu\nu}(\partial_{\nu}\otimes\partial_{\mu}))\ .$$

Using this commutation relation we can write

$$T_{k}(f \star_{k} g) = e^{-\frac{i}{2}hm^{\mu\nu}\partial_{\mu}\partial_{\nu}} \circ \mu \circ e^{ihm^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu})}(f \otimes g)$$

$$= \mu \circ e^{-\frac{i}{2}h(m^{\mu\nu}\partial_{\mu}\partial_{\nu}\otimes id + id \otimes m^{\mu\nu}\partial_{\mu}\partial_{\nu} + m^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu}) + m^{\mu\nu}(\partial_{\nu}\otimes\partial_{\mu}))}$$

$$\circ e^{ihm^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu})}(f \otimes g)$$

$$= \mu \circ e^{-\frac{i}{2}hm^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu} + \partial_{\nu}\otimes\partial_{\mu}) + ihm^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu})}(T_{k}(f) \otimes T_{k}(g))$$

$$= \mu \circ e^{\frac{i}{2}hm^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu}) - \frac{i}{2}hm^{\mu\nu}(\partial_{\nu}\otimes\partial_{\mu})}(T_{k}(f) \otimes T_{k}(g))$$

$$= \mu \circ e^{\frac{i}{2}hm^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu}) - \frac{i}{2}h\theta^{\mu\nu}(\partial_{\nu}\otimes\partial_{\mu}) + \frac{i}{2}hk^{\nu\mu}\theta^{\mu\nu}(\partial_{\nu}\otimes\partial_{\mu})}(T_{k}(f) \otimes T_{k}(g))$$

$$= \mu \circ e^{\frac{i}{2}hm^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu}) + \frac{i}{2}h\theta^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu}) - \frac{i}{2}hk^{\nu\mu}\theta^{\nu\mu}(\partial_{\nu}\otimes\partial_{\mu})}(T_{k}(f) \otimes T_{k}(g))$$

$$= \mu \circ e^{\frac{i}{2}h\theta^{\mu\nu}(\partial_{\mu}\otimes\partial_{\nu})}(T_{k}(f) \otimes T_{k}(g))$$

$$= T_{k}(f) \star_{s} T_{k}(g) ,$$

where we used in the third line that partial derivatives commute with each other, in the fifth that  $m^{\mu\nu} = k^{\mu\nu}\theta^{\mu\nu} = (1 - k^{\nu\mu})\theta^{\mu\nu}$  since  $k^{\mu\nu} + k^{\nu\mu} = 1$  if  $\mu \neq \nu$  and in the next to last line again that  $m^{\mu\nu} = k^{\mu\nu}\theta^{\mu\nu}$ . Furthermore, we used that  $\theta$  is an antisymmetric tensor. The last equation follows directly from the explicit expression of  $\star_s$  given in (1.11). This proves the lemma.

We have finished our discussion of orderings and star products and proceed to study how far the gauge theory developed in [4, 5, 7] for the star product  $\star_s$  is affected if we take a different star product.

<sup>&</sup>lt;sup>7</sup>Compare with [10] where quantum mechanics is considered as an example for deformation quantization.

# 1.2 Gauge Field Theory in the Case of a constant Poisson Tensor $\theta$

A detailed description of gauge field theory in the case of a constant Poisson structure can be found in [5, 4, 7, 16, 17]. We assume the general ideas to be known and present in the first subsection just a brief summary of results that are important in our context. Then we will rather concentrate on what has to be modified in the theory if we change the star product: We will discuss the issue of involution and study how the Seiberg-Witten maps depend on the choice of the star product. Finally, in the last subsection we will discuss how far physical considerations confine the full freedom in choosing a star product.

#### 1.2.1 Noncommutative Gauge Field Theory

Matter fields are functions in  $\mathcal{A}_c$  and an infinitesimal noncommutative gauge transformations of a matter field  $\hat{\psi}$  is defined as [4]:<sup>8</sup>

$$\delta\hat{\psi}(x) = i\hat{\Lambda}(x) \star_s \hat{\psi}(x) \tag{1.18}$$

and

$$\delta x^{\mu} = 0 .$$

Since multiplication of  $\hat{\psi}$  on the left by coordinates is not a covariant operation, covariant coordinates  $\hat{X}^{\mu}$  are introduced:

$$\hat{X}^{\mu} := x^{\mu} + \hat{A}^{\mu} \quad , \tag{1.19}$$

where  $\hat{A}^{\mu}$  has to transform as

$$\delta \hat{A}^{\mu} = -i[x^{\mu} \, \stackrel{\star}{,} \hat{\Lambda}] + i[\hat{\Lambda} \, \stackrel{\star}{,} \hat{A}^{\mu}] \tag{1.20}$$

to render  $\hat{X}^{\mu}$  covariant. If  $\theta$  is non-degenerated, we can use  $\theta$  to upper and lower indices and can define

$$\hat{A}^{\mu} =: h\theta^{\mu\nu}\hat{A}_{\nu} .$$

The gauge transformation of  $\hat{A}_{\nu}$  is then given by

$$\delta \hat{A}_{\nu} = \partial_{\nu} \hat{\Lambda} + i [\hat{\Lambda}^{\star,s} \hat{A}_{\nu}] , \qquad (1.21)$$

<sup>&</sup>lt;sup>8</sup>Functions with a hat are indeed to be understood as elements of  $\mathcal{A}_c$  and not as elements of  $\mathcal{A}_{nc}$ . We add the hat to distinguish noncommutative fields and ordinary commutative fields for the noncommutative ones will be expressed by the commutative ones (1.28).

<sup>&</sup>lt;sup>9</sup>This means  $\delta \hat{X}^{\mu} = i[\hat{\Lambda}^{\star,s} \hat{X}^{\mu}].$ 

since we have

$$-i[x^{\mu} \stackrel{\star_s}{,} f] = h\theta^{\mu\nu}\partial_{\nu}f \tag{1.22}$$

for all f. Additionally, a covariant field strength and a covariant derivative are introduced:

$$\hat{F}_{\mu\nu} = \partial_{\mu}\hat{A}_{\nu} - \partial_{\nu}\hat{A}_{\mu} - i[\hat{A}_{\mu} *_{s} \hat{A}_{\nu}]$$

$$(1.23)$$

$$\hat{D}_{\mu}\hat{\psi} = \partial_{\mu}\hat{\psi} - i\hat{A}_{\mu} \star_{s} \hat{\psi} \tag{1.24}$$

with

$$\delta \hat{F}_{\mu\nu} = i[\hat{\Lambda} \stackrel{\star_s}{\cdot} \hat{F}_{\mu\nu}] \tag{1.25}$$

and

$$\delta \hat{D}_{\mu} \hat{\psi} = i \hat{\Lambda} \star_s \hat{D}_{\mu} \hat{\psi} . \tag{1.26}$$

In the general case we want to describe non-abelian gauge field theories based on a Lie algebra

$$[T^a, T^b] = i f^{ab}{}_c T^c .$$

In the noncommutative setting, however,  $\hat{\Lambda}$  has to lie in the enveloping algebra, i.e.

$$\hat{\Lambda}(x) = \hat{\Lambda}_a(x)T^a + \hat{\Lambda}_{ab}^1(x) : T^aT^b : + \dots + \hat{\Lambda}_{a_1\dots a_n}^{n-1} : T^{a_1}\dots T^{a_n} : + \dots , \quad (1.27)$$

with

$$: T^{a} : = T^{a}$$

$$: T^{a}T^{b} : = \frac{1}{2} \{T^{a}, T^{b}\}$$

$$: T^{a_{1}} \dots T^{a_{n}} : = \frac{1}{n!} \sum_{\pi \in S_{n}} T^{a_{\pi(1)}} \dots T^{a_{\pi(n)}}.$$

This is because the star commutator of two Lie algebra valued transformation parameters doesn't close in the Lie algebra anymore [6]. On first glance this seems to lead to infinitely many parameters  $\hat{\Lambda}_{a_1...a_n}^{n-1}$ , but the gauge transformations can be restricted to those which permit to express these infinitely many parameters by the finitely many commutative ones. This is done by means of the so called Seiberg-Witten maps

$$\hat{\Lambda} = \hat{\Lambda}_{\alpha}[A_{\mu}]$$

$$\hat{A}_{\mu} = \hat{A}_{\mu}[A_{\mu}]$$

$$\hat{\psi} = \hat{\psi}[\psi, A_{\mu}].$$
(1.28)

Here,  $\alpha$  denotes the commutative gauge parameter,  $A_{\mu}$  the commutative gauge field and  $\psi$  the commutative matter field. To calculate the explicit dependence of the noncommutative fields on the commutative ones we assume that it is possible to expand  $\hat{\Lambda}_{\alpha}[A_{\mu}]$ ,  $\hat{A}_{\mu}[A_{\mu}]$ ,  $\hat{\psi}[\psi, A_{\mu}]$  in the formal deformation parameter h:

$$\hat{\Lambda}_{\alpha}[A_{\mu}] = \alpha + h\hat{\Lambda}_{\alpha}^{1}[A_{\mu}] + h^{2}\hat{\Lambda}_{\alpha}^{2}[A_{\mu}] + \dots 
\hat{A}_{\mu}[A_{\mu}] = A_{\mu} + h\hat{A}_{\mu}^{1}[A_{\mu}] + h^{2}\hat{A}_{\mu}^{2}[A_{\mu}] + \dots 
\hat{\psi}[\psi, A_{\mu}] = \psi + h\hat{\psi}^{1}[\psi, A_{\mu}] + h^{2}\hat{\psi}^{2}[\psi, A_{\mu}] + \dots$$
(1.29)

Finally, we get the explicit dependence on the commutative fields by requiring the following *consistency condition* 

$$(\delta_{\alpha}\delta_{\beta} - \delta_{\beta}\delta_{\alpha})\hat{\psi} = \delta_{-i[\alpha,\beta]}\hat{\psi}$$
  

$$\Leftrightarrow i\delta_{\alpha}\hat{\Lambda}_{\beta} - i\delta_{\beta}\hat{\Lambda}_{\alpha} + [\hat{\Lambda}_{\alpha} *_{s} \hat{\Lambda}_{\beta}] = i\hat{\Lambda}_{-i[\alpha,\beta]}$$
(1.30)

and by requiring that the noncommutative gauge transformations are induced by the commutative gauge transformations of the commutative fields the noncommutative ones depend on:

$$\hat{\Lambda}_{\alpha}[A_{\mu}] + \delta \hat{\Lambda}_{\alpha}[A_{\mu}] = \hat{\Lambda}_{\alpha}[A_{\mu} + \delta A_{\mu}]$$

$$\hat{A}_{\mu}[A_{\mu}] + \delta \hat{A}_{\mu}[A_{\mu}] = \hat{A}_{\mu}[A_{\mu} + \delta A_{\mu}]$$

$$\hat{\psi}[\psi, A_{\mu}] + \delta \hat{\psi}[\psi, A_{\mu}] = \hat{\psi}[\psi + \delta \psi, A_{\mu} + \delta A_{\mu}] .$$
(1.31)

These equations were explicitly solved in [5] up to second order in h for the star product  $\star_s$ .

Remark: In [6] it is shown that for  $\hat{\Lambda}$  exists a solution where the *n*-th order of *h* corresponds to the n+1-th order of  $T^a$ .

# 1.2.2 Noncommutative Gauge Field Theory for an arbitrary star product

Instead of the Moyal-Weyl product  $\star_s$  we can also use every other star product that is equivalent to this one. Let T be an equivalence transformation from  $\star_s$  to  $\star$ , i.e. we have

$$f \star g = T^{-1}(T(f) \star_s T(g)) .$$

Matter fields  $\hat{\psi}'$  are still elements of  $\mathcal{A}_c$  with the following gauge transformation law:

$$\delta\hat{\psi}'(x) = i\hat{\Lambda}'(x) \star \hat{\psi}'(x) . \qquad (1.32)$$

We can again introduce covariant coordinates

$$\hat{X}'^{\mu} := x^{\mu} + \hat{A}'^{\mu}$$

where  $\hat{A}'$  must transform under gauge transformations as

$$\delta \hat{A}^{\prime\mu} = -i[x^{\mu} \uparrow \hat{\Lambda}^{\prime}] + i[\hat{\Lambda}^{\prime} \uparrow \hat{A}^{\prime\mu}] .$$

If  $\theta$  is non-degenerated, we can again lower indices using  $\theta$  as we did in the case of the Moyal-Weyl product and can define

$$\hat{A}^{\prime\mu} = \theta^{\mu\nu} \hat{A}^{\prime}_{\nu} .$$

The transformation law of  $\hat{A}'_{\nu}$  is given by

$$\theta^{\mu\nu}\delta\hat{A}'_{\nu} = -i[x^{\mu} \stackrel{\star}{,} \hat{\Lambda}'] + i\theta^{\mu\nu}[\hat{\Lambda}' \stackrel{\star}{,} \hat{A}'_{\nu}] \ .$$

New is that in the case of an arbitrary star product we do *not* have that  $-i[x^{\mu} \, {}^{\star}, f] = h\theta^{\mu\nu}\partial_{\nu}f$  as it is the case for  $\star_s$ . The reason is that in general the usual partial derivative  $\partial_{\nu}$  is *not* a derivation for  $(\mathcal{A}_{c}, \star)$ , i.e. in general  $\partial_{\nu}(f\star g) \neq \partial_{\nu}(f)\star g + f\star\partial_{\nu}(g)$ , whereas the commutator  $-i[x^{\mu}, f]$  is a derivation. Nonetheless, we have:

**Remark 1.** Let  $\star$  be an arbitrary star product equivalent to  $\star_s$  and let T be the equivalence transformation from  $\star_s$  to  $\star$ . Then

$$\partial_{\mu}' := T^{-1} \circ \partial_{\mu} \circ T \tag{1.33}$$

is a derivation for  $(A_c, \star)$ .

*Proof.* Since T is an equivalence transformation from  $\star_s$  to  $\star$ , we have for arbitrary functions f, g

$$\begin{array}{lcl} \partial'_{\mu}(f\star g) & = & \partial'_{\mu}(T^{-1}(T(f)\star_s T(g))) \\ & = & T^{-1}\circ\partial_{\mu}(T(f)\star_s T(g)) \ . \end{array}$$

As  $\partial_{\mu}$  is a derivation for  $(\mathcal{A}_{c}, \star_{s})$ , we obtain

$$= T^{-1}(\partial_{\mu}(T(f)) \star_{s} T(g) + T(f) \star_{s} \partial_{\mu}(T(g)))$$

$$= T^{-1}(T(\partial'_{\mu}(f) \star_{s} T(g) + T(f) \star_{s} T(\partial'_{\mu}(g))$$

$$= \partial'_{\mu}(f) \star_{g} f + f \star_{g} \partial'_{\mu}(g) .$$

Furthermore, let us consider only those star products, which correspond to an ordering prescription that maps all coordinates  $x^{\mu}$  to their noncommutative analogs  $\hat{x}^{\mu}$ . Until now, we just demanded  $\rho(x^{\mu}) = \hat{x}^{\mu} + \mathcal{O}(h)$  (Subsection 1.1.1). In this case we have

$$T(x^{\mu}) = x^{\mu}$$

and we indeed obtain

$$-i[x^{\mu} , f] = h\theta^{\mu\nu}\partial'_{\nu}f .$$

*Proof.* We know from Conclusion 1 and Conclusion 2 that  $\star$  corresponds to an ordering prescription  $\rho$ . One possibility is given by  $\rho = \rho_s \circ T$ , where  $\rho_s$  is the symmetric ordering prescription (1.6), since obviously

$$\rho^{-1}(\rho(f) \circ \rho(g)) = T^{-1} \circ \rho_s^{-1}(\rho_s(T(f)) \circ \rho_s(T(g))) = T^{-1}(T(f) \star_s T(g)) = f \star g.$$

If we require as above that  $\rho_s \circ T(x^{\mu}) = \rho(x^{\mu}) = \hat{x}^{\mu} = \rho_s(x^{\mu})$ , this yields  $T(x^{\mu}) = x^{\mu}$ .

In this case we have because of (1.22) and (1.33)

$$-i[x^{\mu} * f] = -iT^{-1}[T(x^{\mu}) *_{s} T(f)] = -iT^{-1}[x^{\mu} *_{s} T(f)]$$
$$= T^{-1}(\theta^{\mu\nu}\partial_{\nu}T(f)) = \theta^{\mu\nu}\partial'_{\nu}f.$$

Thus, we finally derive the gauge transformation law for  $\hat{A}'_{\nu}$ :

$$\delta \hat{A}'_{\nu} = \partial'_{\nu} \hat{\Lambda}' + i[\hat{\Lambda}' \, \dot{\hat{\Lambda}} \, \hat{A}'_{\nu}] \quad . \tag{1.34}$$

Analogously, the field strength and the covariant derivative for an arbitrary star product  $\star$  read

$$\hat{F}'_{\mu\nu} = \partial'_{\mu}\hat{A}'_{\nu} - \partial'_{\nu}\hat{A}'_{\mu} - i[\hat{A}'_{\mu} * \hat{A}'_{\nu}] 
\hat{D}'_{\mu}\hat{\psi}' = \partial'_{\mu}\hat{\psi}' - i\hat{A}'_{\mu} * \hat{\psi}' ,$$
(1.35)

with

$$\delta \hat{F}'_{\mu\nu} = i[\hat{\Lambda}' \, \stackrel{*}{,} \hat{F}'_{\mu\nu}] \tag{1.36}$$

and

$$\delta \hat{D}'_{\mu}\hat{\psi}' = i\hat{\Lambda}' \star \hat{D}'_{\mu}\hat{\psi}' . \qquad (1.37)$$

Now we can proceed as done for the star product  $\star_s$  in the previous subsection and express the noncommutative fields in terms of the commutative ones by means of the Seiberg-Witten maps for the star product  $\star$ . A solution for those Seiberg-Witten maps is given later.

 $<sup>^{10}</sup>$ If we assign to coordinates the dimension of length, this is a physical requirement which guarantees that the dimension is preserved.

#### 1.2.3 Noncommutative Gauge Theory and Involution

#### **Definition and Examples**

A \*-involution (Don't mix up \* and  $\star$ !) of an algebra  $\mathcal{A}$  is an anti-linear map such that for all  $a \in \mathcal{A}$ 

$$(ab)^* = b^*a^*$$
 and  $(a^*)^* = a$ .

Now we make an important observation:

 $\mathcal{A}_{c}$  together with the symmetric star product  $\star_{s}$  admits the usual complex conjugation as involution. We have already given the proof in Subsection 1.1.2 (c), where we checked that  $\star_{s}$  is a hermitian star product (see (1.12)). On the other hand it can be immediately shown that for  $(\mathcal{A}_{c}, \star_{n})$  the complex conjugation is not an involution (the complex conjugation is not even an involution for  $(\mathcal{A}_{c}, \star_{k})$  whenever  $\star_{k} \neq \star_{s}$  (cf. (1.14))).

The question arises how we can find an involution  $i_{\star}$  for an arbitrary star product  $\star$ . The following proposition gives the answer.

**Proposition 1.** We get an involution for  $(A_c, \star)$  by defining

$$i_{\star} := T^{-1} \circ \overline{\cdot} \circ T \quad , \tag{1.38}$$

where  $\overline{\phantom{a}}$  denotes the ordinary complex conjugation and T is the equivalence transformation from  $\star_s$  to  $\star$  (whose existence is guaranteed by Conclusion 1).

*Proof.* Let be  $f, g \in \mathcal{A}_c$  arbitrary. With the definition of  $i_{\star}$  (1.38), by using that T is an equivalence transformation and because  $\overline{\cdot}$  is an involution for  $\star_s$  we obtain

$$\begin{array}{lll} i_{\star}(f\star g) & = & T^{-1}\circ\overline{\cdot}\circ T(T^{-1}(T(f)\star_s T(g))) \\ & = & T^{-1}(\overline{T(f)\star_s T(g)}) \\ & = & T^{-1}(\overline{T(g)\star_s T(f)}) \\ & = & T^{-1}(T(T^{-1}\circ\overline{\cdot}\circ T(g))\star_s T(T^{-1}\circ\overline{\cdot}\circ T(f))) \\ & = & i_{\star}(g)\star i_{\star}(f) \ . \end{array}$$

Obviously  $i_{\star}^2(f) = f$  is satisfied, too and  $i_{\star}$  is an involution in the sense of the definition given above.

#### The Gauge Invariant Matter Field Term

We still haven't defined a gauge invariant action. For the Moyal-Weyl product  $\star_s$  this was done in [8]. A noncommutative Yang-Mills action is defined in this case as follows:

$$S = \int d^4x \,\mathcal{L} = \int d^4x \, \left[ -\frac{1}{4} \operatorname{tr} \hat{F}_{\mu\nu} \star_s \hat{F}^{\mu\nu} + \overline{\hat{\psi}} \star_s (i\gamma^{\mu} \hat{D}_{\mu} - m) \hat{\psi} \right] , \qquad (1.39)$$

where  $\overline{\psi} := \hat{\psi}^{\dagger} \gamma^0$  denotes the adjoint matter field. Let us look at this action: In analogy to the commutative case we want to obtain a gauge invariant expression

$$\overline{\hat{\psi}} \star_s \hat{\psi} . \tag{1.40}$$

Since

$$\delta \overline{\hat{\psi}} = -i\overline{\hat{\psi}} \star_s \hat{\Lambda}^{\dagger} \tag{1.41}$$

we have

$$\delta(\overline{\hat{\psi}} \star_s \hat{\psi}) = -i\overline{\hat{\psi}} \star_s \hat{\Lambda}^{\dagger} \star_s \hat{\psi} + i\overline{\hat{\psi}} \star_s \hat{\Lambda} \star_s \hat{\psi}$$
$$= i\overline{\hat{\psi}} \star_s (\hat{\Lambda} - \hat{\Lambda}^{\dagger}) \star_s \hat{\psi}$$

and therefore

$$\begin{array}{rcl} \delta(\overline{\hat{\psi}} \star_s \hat{\psi}) & = & 0 \\ \Leftrightarrow & \hat{\Lambda} & = & \hat{\Lambda}^{\dagger} \end{array}$$

This may seem to be trivial but in fact (1.41) is only true because  $\star_s$  is a hermitian star product (1.12) and thereby

$$(\hat{\Lambda} \star_s \hat{\psi})^{\dagger} = \hat{\psi}^{\dagger} \star_s \hat{\Lambda}^{\dagger} . \tag{1.42}$$

We want to demand  $\hat{\Lambda}$  to be hermitian, i.e.

$$\hat{\Lambda} = \hat{\Lambda}^{\dagger} \quad , \tag{1.43}$$

in analogy to the commutative case such that (1.40) is indeed gauge invariant. Let us recall that in contrast to the commutative case  $\hat{\Lambda}$  does not lie in the Lie algebra but in the enveloping algebra. If we use a basis of the enveloping algebra that is totally symmetric with respect to the hermitian Lie-algebra generators  $T^a$  and write  $\hat{\Lambda}$  as in (1.27), we apparently obtain

$$\frac{\hat{\Lambda}}{\hat{\Lambda}_{a_1...a_n}^{n-1}} = \hat{\Lambda}^{\dagger}$$

$$\Leftrightarrow \hat{\Lambda}_{a_1...a_n}^{n-1} = \hat{\Lambda}_{a_1...a_n}^{n-1}$$

for all n, where the bar denotes the usual complex conjugation. Thus,  $\hat{\Lambda}$  is hermitian if all coefficient functions are real.

If we want to use an arbitrary star product we have to adjust the hermiticity condition since the complex conjugation in general will not be an involution and therefore (1.42) will not hold. Let us define the *star-dagger* 

$$\dagger_{\star} := (i_{\star}(\cdot))^{tr} , \qquad (1.44)$$

where  $i_{\star}$  denotes the involution given in Conclusion 1 that corresponds to an arbitrary star product  $\star$ . Let us denote by  $\hat{\psi}'$  the matter field for the case where we use  $\star$ . Then, we want to define the *adjoint matter field* 

$$\mathrm{ad}_{\star}(\hat{\psi}') := (\hat{\psi}')^{\dagger_{\star}} \gamma^{0} = \left(i_{\star}(\hat{\psi}')\right)^{tr} \gamma^{0} . \tag{1.45}$$

Furthermore,  $ad_{\star}(\hat{\psi}') \star \hat{\psi}'$  is to be gauge invariant for an arbitrary star product, i.e.

$$\begin{split} \delta(\mathrm{ad}_{\star}(\hat{\psi}')\star\hat{\psi}') &= 0\\ \Leftrightarrow & \delta\mathrm{ad}_{\star}(\hat{\psi}')\star\hat{\psi}'+\mathrm{ad}_{\star}(\hat{\psi}'\star\delta\hat{\psi}') &= 0\\ \Leftrightarrow & -i\,\mathrm{ad}_{\star}(\hat{\psi}')\star(\hat{\Lambda}')^{\dagger\star}\star\hat{\psi}'+i\,\mathrm{ad}_{\star}(\hat{\psi}')\star\hat{\Lambda}'\star\hat{\psi}' &= 0\\ \Leftrightarrow & i\,\mathrm{ad}_{\star}(\hat{\psi}')\star(\hat{\Lambda}'-(\hat{\Lambda}')^{\dagger\star})\star\hat{\psi}' &= 0\\ \Leftrightarrow & \hat{\Lambda}' &= (\hat{\Lambda}')^{\dagger\star} \end{split}.$$

Hence, we demand the hermiticity condition

$$\hat{\Lambda}' = (\hat{\Lambda}')^{\dagger_{\star}} \quad . \tag{1.46}$$

Can we find a connection between  $\hat{\Lambda}'$  and  $\hat{\Lambda}$ , the gauge parameter for  $\star_s$ ? Having  $\hat{\Lambda}$  with  $\hat{\Lambda}^{\dagger} = \hat{\Lambda}$  (1.43), we can write

$$\begin{array}{rcl} \hat{\Lambda}^{\dagger} & = & \hat{\Lambda} \\ \Leftrightarrow & \left( (T^{-1} \circ \overline{\,\cdot\,} \circ T) T^{-1} \hat{\Lambda} \right)^{tr} & = & T^{-1} \hat{\Lambda} \\ \Leftrightarrow & \left( i_{\star} (T^{-1} \hat{\Lambda}) \right)^{tr} & = & T^{-1} \hat{\Lambda} \\ \Leftrightarrow & (T^{-1} \hat{\Lambda})^{\dagger_{\star}} & = & T^{-1} \hat{\Lambda} \end{array} ,$$

where we used the definition of the involution  $i_{\star}$  for an arbitrary star product given in Proposition 1. Defining

$$\hat{\Lambda}' := T^{-1}\hat{\Lambda} \quad , \tag{1.47}$$

 $\hat{\Lambda}'$  actually satisfies (1.46) and therefore  $\mathrm{ad}_{\star}(\hat{\psi}') \star \hat{\psi}'$  becomes gauge invariant. Remark: The gauge parameter  $\hat{\Lambda}'$ can be expressed in terms of the commutative fields by solving the Seiberg-Witten equations (1.31), which in fact depend on the choice of the star product, too. In the next subsection we will show that (1.47) is actually consistent with the results we get this way.

#### 1.2.4 Seiberg-Witten Map for Arbitrary Star Products

It is convenient to start determining  $\hat{\Lambda}_{\alpha}[A_{\mu}]$  by means of the consistency condition (1.30). We can put the expansion (1.29) in the consistency relation (1.30) and solve the equation order by order. We see that the consistency condition (1.30) contains a  $\star$ . It therefore depends on the special choice of a star product (remember that  $\hat{\Lambda}$  lies in the enveloping Lie algebra so that  $[\hat{\Lambda}_{\alpha} \stackrel{\star}{,} \hat{\Lambda}_{\beta}]$  depends on the star product  $\star$  even to first order in h). The solution to zeroth order is the commutative gauge parameter  $\alpha$  whereas for higher orders we obtain differential equations that depend on the choice of the star product.

For the Moyal-Weyl product the equation and the solution for the first order is given in [5]. We are not interested in the explicit form of equations and solutions but just want to expand the solution in orders of h

$$\hat{\Lambda}_{\alpha}^{s}[A_{\mu}] = \alpha + h\hat{\Lambda}_{\alpha}^{s,1}[A_{\mu}] + h^{2}\hat{\Lambda}_{\alpha}^{s,2}[A_{\mu}] + \dots ,$$

where s stands for "symmetric". We don't want to specify these equations but treat the problem generally. Of course it is possible to solve the equations explicitly order by order if one fixes a star product, but with the knowledge we have gathered so far we can give a solution a lot more elegantly and in full generality. So let  $\star$  in (1.30) be an arbitrary star product. As we know,  $\star$  is equivalent to the Moyal-Weyl product  $\star_s$  (Conclusion 1) and we want to denote the equivalence transformation from  $\star_s$  to  $\star$  by T. Thus,

$$f \star g = T^{-1}(T(f) \star_s T(g))$$
 (1.48)

With this input we get:

**Proposition 2.** If  $\hat{\Lambda}^s$  is a solution of the consistency condition (1.30) for  $\star_s$  then  $T^{-1}(\hat{\Lambda}^s)$  is a solution of the consistency condition for  $\star$ .

*Proof.* To show that  $T^{-1}(\hat{\Lambda}^s)$  satisfies the consistency condition for  $\star$  given in (1.30), we will reduce the problem to the consistency condition for  $\star_s$  such that we can make use of the fact that  $\hat{\Lambda}^s$  is a solution. This can be done using that T

is an equivalence transformation from  $\star$  to  $\star_s$  which yields (1.48). This explicitly reads

$$\begin{split} & i\delta_{\alpha}T^{-1}(\hat{\Lambda}_{\beta}^{s}) - i\delta_{\beta}T^{-1}(\hat{\Lambda}_{\alpha}^{s}) + [T^{-1}(\hat{\Lambda}_{\alpha}^{s}) * T^{-1}(\hat{\Lambda}_{\beta}^{s})] \\ = & i\delta_{\alpha}T^{-1}(\hat{\Lambda}_{\beta}^{s}) - i\delta_{\beta}T^{-1}(\hat{\Lambda}_{\alpha}^{s}) + T^{-1}([T(T^{-1}(\hat{\Lambda}_{\alpha}^{s})) * T(T^{-1}(\hat{\Lambda}_{\beta}^{s}))] \\ = & T^{-1}(i\delta_{\alpha}\hat{\Lambda}_{\beta}^{s} - i\delta_{\beta}\hat{\Lambda}_{\alpha}^{s} + [\hat{\Lambda}_{\alpha}^{s} * \hat{\Lambda}_{\beta}^{s}]) \\ = & T^{-1}(i\hat{\Lambda}_{-i[\alpha,\beta]}^{s}) \\ = & iT^{-1}(\hat{\Lambda}_{-i[\alpha,\beta]}^{s}) \end{split}$$

which proves the claim.

We emphasize that this result is exactly what we got at the end of the previous subsection to guarantee an invariant matter field term in the Lagrangian (cf. (1.47)).

This is a nice result: Obviously we can immediately give a solution for the consistency condition (1.30) for an arbitrary star product if we know it for  $\star_s$  (this is known up to second order) and if we know T (whereas this indeed can be a problem).

Let us continue with searching a solution for  $\hat{\psi}$  and  $\hat{A}_{\mu}$ . If we write out the left hand sides of the Seiberg-Witten equations introduced in (1.31), we get with the gauge transformation laws (1.32) and (1.34) the following star product-dependent equations

$$\hat{\psi}[\psi, A_{\mu}] + \delta \hat{\psi}[\psi, A_{\mu}] = \hat{\psi}[\psi, A_{\mu}] + i\hat{\Lambda}_{\alpha}[A_{\mu}] \star \hat{\psi}[\psi, A_{\mu}] 
\hat{A}_{\mu}[A_{\mu}] + \delta \hat{A}_{\mu}[A_{\mu}] = \hat{A}_{\mu}[A_{\mu}] + \partial'_{\mu}\hat{\Lambda}_{\alpha}[A_{\mu}] + i[\hat{\Lambda}_{\alpha}[A_{\mu}] \star \hat{A}_{\mu}[A_{\mu}]] ,$$
(1.49)

where  $\partial'_{\mu}$  was introduced in Remark 1. Again we can take the expansions of  $\hat{\psi}[\psi, A_{\mu}]$  respectively  $\hat{A}_{\mu}[A_{\mu}]$  in orders of h (1.29) and solve the Seiberg-Witten equations order by order for a fixed star product. This was also done up to second order in [5] for  $\star_s$ . With this solution we then obtain:

**Proposition 3.** If  $\hat{\psi}^s$  and  $\hat{A}^s_{\mu}$  are a solution of the Seiberg-Witten equations (1.49) for  $\star_s$  then  $T^{-1}(\hat{\psi}^s)$  and  $T^{-1}(\hat{A}^s_{\mu})$  are a solution for the Seiberg-Witten equations for  $\star$ , where T denotes the equivalence transformation from  $\star_s$  to  $\star$ .

*Proof.* We have

$$T^{-1}(\hat{\psi}^s[\psi, A_{\mu}]) + \delta T^{-1}(\hat{\psi}^s[\psi, A_{\mu}]) = T^{-1}(\hat{\psi}^s[\psi, A_{\mu}]) + T^{-1}(\delta \hat{\psi}^s[\psi, A_{\mu}])$$
$$= T^{-1}(\hat{\psi}^s[\psi, A_{\mu}])$$

$$+T^{-1}(i\hat{\Lambda}_{\alpha}^{s}[A_{\mu}] \star_{s} \hat{\psi}^{s}[\psi, A_{\mu}])$$

$$= T^{-1}(\hat{\psi}^{s}[\psi, A_{\mu}])$$

$$+iT^{-1}(\hat{\Lambda}_{\alpha}^{s}[A_{\mu}]) \star T^{-1}(\hat{\psi}^{s}[\psi, A_{\mu}]) .$$

We used as in the previous proof that T is an equivalence transformation from  $\star_s$  to  $\star$  and that the Seiberg-Witten equation is satisfied by  $\hat{\psi}^s[\psi, A_{\mu}]$  for the star product  $\star_s$ . With Proposition 2 we see that  $T^{-1}(\hat{\psi}^s[\psi, A_{\mu}])$  is a solution for (1.49).

For the vector field we obtain

$$\begin{split} T^{-1}(\hat{A}_{\mu}^{s}[A_{\mu}]) + \delta T^{-1}(\hat{A}_{\mu}^{s}[A_{\mu}]) &= T^{-1}(\hat{A}_{\mu}^{s}[A_{\mu}]) \\ &+ T^{-1}(\hat{A}_{\mu}^{s}[A_{\mu}] + \partial_{\mu}\hat{\Lambda}_{\alpha}^{s}[A_{\mu}] + i[\hat{\Lambda}_{\alpha}^{s}[A_{\mu}] \stackrel{\star,s}{,} \hat{A}_{\mu}^{s}[A_{\mu}]]) \\ &= T^{-1}(\hat{A}_{\mu}^{s}[A_{\mu}]) + \partial'_{\mu}T^{-1}(\hat{\Lambda}_{\alpha}^{s}[A_{\mu}]) \\ &+ i[T^{-1}(\hat{\Lambda}_{\alpha}^{s}[A_{\mu}]) \stackrel{\star}{,} T^{-1}(\hat{A}_{\mu}^{s}[A_{\mu}])] \end{split} ,$$

where we used in the last step that for  $\partial'_{\mu}$  (1.33) holds that  $T^{-1} \circ \partial_{\mu} = \partial'_{\mu} \circ T^{-1}$ . Thus, if we define  $\hat{A}'^{\mu} = T^{-1}\hat{A}^{\mu}_{s}$  we obtain with Proposition 2 and the gauge transformation law (1.34) that  $\hat{A}'_{\mu}[A_{\mu}] + \delta \hat{A}'_{\mu}[A_{\mu}] = \hat{A}'_{\mu}[A_{\mu} + \delta A_{\mu}]$ .

Having studied star products and its properties enabled us to get easily these general results for the solution of the Seiberg-Witten map. Knowing that, we can now treat the problem of how far "physics" changes with the change of the star product.

#### 1.2.5 Physics and the Choice of the Star Product

#### The Action

The noncommutative gauge field theory permits to write down a gauge-covariant Lagrangian and an invariant action (1.39) using the Moyal-Weyl product  $\star_s$ .<sup>11</sup> In the case of an arbitrary star product we obtain with (1.35) and (1.45)

$$S' = \int d^4x \, \mathcal{L}' = \int d^4x \, \left[ -\frac{1}{4} \text{tr} \hat{F}'_{\mu\nu} \star \hat{F}'^{\mu\nu} + \text{ad}_{\star}(\hat{\psi}') \star (i\gamma^{\mu} \hat{D}'_{\mu} - m) \hat{\psi}' \right] . \quad (1.50)$$

The crucial question is whether this action differs from (1.39). If the action does not change, the equations of motion will not change and physics will be unaffected by a change of the star product. But if it changes we will get different physical predictions. The following conclusion states how the action is affected.

<sup>&</sup>lt;sup>11</sup>See below for gauge invariance.

Conclusion 3. Let  $S^s$  and  $\mathcal{L}^s$  be a noncommutative Yang-Mills action and Lagrangian that correspond to a solution of the Seiberg-Witten maps for the Moyal-Weyl product  $\star_s$ . Let  $\star$  be another star product equivalent to  $\star_s$  by means of the equivalence transformation T (from  $\star_s$  to  $\star$ ). Then we get a Yang-Mills action and a Lagrangian, denoted by S' and  $\mathcal{L}'$  respectively, that correspond to a solution of the Seiberg-Witten maps for  $\star$  by:

$$S' = \int d^4x \, \mathcal{L}' = \int d^4x \, T^{-1} \mathcal{L}^s \ . \tag{1.51}$$

*Proof.* We want to understand any matter field and field strength written with a prime as a solution of the Seiberg-Witten maps for the star product  $\star$ . We obtain because of  $f \star g = T^{-1}(T(f) \star_s T(g))$ , because of Proposition 2 and Proposition 3 and with the definitions for the field strength resp. the covariant derivative (1.35) that

$$\hat{F}'_{\mu\nu} = T^{-1}\hat{F}^s_{\mu\nu}$$

and

$$\hat{D}'_{\mu}\hat{\psi} = T^{-1}\hat{D}^{s}_{\mu}\hat{\psi}$$
.

The equation (1.51) then follows from the definition of the action (1.39) respectively (1.50) using once again that  $f \star g = T^{-1}(T(f) \star_s T(g))$ .

Let us comment on this result: As an equivalence transformation, T (and therefore  $T^{-1}$  as well) starts in zeroth order with the identity. Therefore we always recover for any star product to zeroth order the commutative theory. Changes can only occur in higher orders of h. To get a better feeling for what happens, let us consider for instance  $\star_k$  (1.14) that we introduced in Subsection 1.1.2 (c) with the equivalence transformation  $T_k$  from  $\star_s$  to  $\star_k$  given in (1.17). We get with equation (1.51):

$$S^{k} = \int d^{4}x T_{k}^{-1}(\mathcal{L}^{s})$$

$$= \int d^{4}x e^{-\frac{i}{2}hm^{\mu\nu}\partial_{\mu}\partial_{\nu}}(\mathcal{L}^{s})$$

$$= \int d^{4}x \mathcal{L}^{s} + \sum_{k=1}^{\infty} \int d^{4}x \frac{(-ih)^{k}}{k!} (m^{\mu\nu}\partial_{\mu}\partial_{\nu})^{k}(\mathcal{L}^{s})$$

$$= \int d^{4}x \mathcal{L}^{s} = S^{s} ,$$

$$(1.52)$$

since for non-zero orders we integrate partial derivatives of  $\mathcal{L}^s$  over the whole space which, assuming that fields vanish in infinity, equals zero.

Usually, an equivalence transformation is defined to be a differential operator in all orders of h. In this case the two actions S' and  $S^s$  in Conclusion 3 are

actually always equal if the equivalence transformations T consists in all orders of h of purely  $\mathbb{C}$ -linear combinations of partial derivatives (a non-constant coefficient would spoil this property). In this case, as a matter of fact, physics does not change. But it is not difficult to find an ordering prescription for which the corrections of the action to non-zeroth order in h do not vanish. We just have to take any ordering prescription that in a non-zeroth order contains non-constant coefficients. Nevertheless, there is a physical reason why we shall not allow star products in full arbitrariness for our theory: the gauge invariance principle.

#### Gauge Invariance

Gauge invariance is a fundamental concept of our theory. Knowing the covariant transformation laws (1.26) and (1.25) respectively (1.36) and (1.37), invariance of the Yang-Mills action is only guaranteed if the integral satisfies the so called *trace* property, i.e. if the integral is cyclic

$$\int d^4x \, f \star g = \int d^4x \, g \star f \tag{1.53}$$

for all f, g. In the usually considered case, where the Moyal-Weyl product is used, this property is given. We even have

$$\int d^4x f \star_s g = \int d^4x fg = \int d^4x g \star_s f .$$

This can be seen by means of partial integration. Again by partial integration it is not difficult to check that all the star products  $\star_k$  satisfy

$$\int d^4x f \star_k g = \int d^4x g \star_k f ,$$

too.

Obviously, if we do not want to loose the concept of an invariant action, we are forced to accept only star products that satisfy the trace property (1.53). Those star products we want to call star products with trace property.

Let T be an equivalence transformation from  $\star_s$  to a star product  $\star$ . Let us summarize which conditions we need to get equivalent physical theories in the end:

(i)  $\star$  must be a star product with trace property, i.e.:  $\int d^4x f \star g = \int d^4x g \star f$  for all f, g.

(ii) The corresponding actions have to be equal, i.e. 
$$S' = \int d^4x \, \mathcal{L}' = \int d^4x \, T^{-1}(\mathcal{L}^s) \stackrel{!}{=} \int d^4x \, \mathcal{L}^s = S^s$$
 (cf. Conclusion 3).

If the equivalence transformation T consists in all orders of purely  $\mathbb{C}$ -linear combinations of partial derivatives, the property (ii) is satisfied as we saw above. In this case we obtain, assuming as always that functions vanish at infinity, that also the first property is satisfied:

$$\int d^4x f \star g = \int d^4x T^{-1}(T(f) \star_s T(g))$$

$$= \int d^4x T(f) \star_s T(g)$$

$$= \int d^4x T(g) \star_s T(f)$$

$$= \int d^4x T^{-1}(T(g) \star_s T(f))$$

$$= \int d^4x g \star f ,$$

where we used in the third line that  $\star_s$  is a star product with trace property.

In particular, the transformations  $T_k$ , which we introduced in Lemma 1, are of this type. Hence, in particular the normal ordered star product, that is sometimes used instead of the symmetric ordered one, satisfies both conditions.

However, the first condition, which reflects a physical demand on the set of star products, does not imply the second one. Thus, the requirement of trace property is not sufficient to guarantee that all considered star products lead to the same action.

The question remains, whether there exists another "physical reason" that confines the set of arbitrary star products to those that finally obey (ii) as well. An answer may be found by restricting the set of allowed ordering prescriptions. Until now, an ordering prescription was just defined as a vector space isomorphism with  $\rho(1)=1$  that approaches the identity for  $h\to 0$ . We recall that we already demanded  $\rho(x^\mu)=\hat{x}^\mu$  in Subsection 1.2.2. Nevertheless, in this generality ordering prescriptions still are not "physical". Giving the coordinates the dimension of length, we would like to obtain the same dimensions for the preimages and images of  $\rho$ . This implies  $\rho$  to be homogeneous. The presumption is, that if we demand an ordering prescription  $\rho$  to be additionally homogeneous in all coordinates, that means  $\rho$  is to preserve the degree of all coordinates, the corresponding equivalence transformation T with  $\rho=\rho_s\circ T$  then ends up to consist in all orders of h of

C-linear combinations of partial derivatives.<sup>12</sup> Ordering prescriptions that we would understand to be "physical" then are actually those which just "order" the coordinates in a special way. As we discussed above, star products corresponding to those ordering prescriptions then lead all to the same action as the symmetric star product does.

<sup>&</sup>lt;sup>12</sup>The idea is to use the fact that for a monomial f ordering  $\rho(f)$  such that we get a linear combination of symmetric ordered monomials is done by commuting coordinates. But commuting two coordinates  $\hat{x}^{\mu}$  and  $\hat{x}^{\nu}$  in turn can be achieved by applying  $\theta^{\mu\nu}\partial_{\mu}\partial_{\nu}$  such that T will consist of linear combinations of such contributions

# Chapter 2

# Gauge Field Theory on the $E_q(2)$ -Symmetric Plane

In this chapter we will treat a noncommutative space with  $E_q(2)$ —symmetry. By  $E_q(2)$  we denote the q-deformed Euclidean group on the two dimensional space (or more precisely the q-deformed algebra of functions on the two dimensional Euclidean group E(2)). This is a simple example which allows us to study how gauge field theory can be implemented on q-deformed spaces, discussing the general difficulties we meet and giving possible solutions. The considerations in this chapter are kept general as far as possible so that this work may be generalized to other q-deformed spaces as well.

This chapter is divided into three sections. In the first section, we briefly introduce the quantum group  $E_q(2)$  and the  $E_q(2)$ -symmetric plane that is underlying the following considerations. In the second section, we develop gauge field theories on the  $E_q(2)$ -symmetric plane based on a generalization of the theory developed for the case of a constant Poisson structure  $\theta$  (cf. Chapter 1 and references within). We consider infinitesimal gauge transformations of matter fields in full analogy to the commutative theory, i.e. taking the transformation law of commutative matter fields and replacing the ordinary, commutative multiplication by a convenient star product that reflects the algebraic properties of the noncommutative  $E_q(2)$ -symmetric plane. In a second step, we introduce covariant coordinates and a gauge field A. The problem arises that we cannot simply lower indices using  $\theta$  as in Chapter 1, since a non-constant  $\theta$  in general spoils covariance. A solution is found by introducing the "covariantizer"  $\mathcal{D}$  [9]. Furthermore, this approach allows us to apply the concept of the Seiberg-Witten maps on our noncommutative plane. Thus, even for this q-deformed space, the noncommutative functions

can be expressed in terms of the commutative ones. This makes it possible to calculate explicitly the corrections to the commutative theory in the first order of the deformation parameter, which this noncommutative theory implies. This is done at the end of the second section. Unfortunately, it turns out that the approach we have chosen in this section leads to a theory that is not covariant with respect to  $E_q(2)$ -transformations of the space. The problem is the choice of the integral: the invariance principle forces us to introduce an integral with trace property that is not invariant under  $E_q(2)$ -transformations. But, starting with a q-deformed plane possessing a quantum group symmetry, we would like to set up a theory that is covariant with respect to this symmetry transformations. We try to establish such a theory in the third section, this time putting priority on the  $E_q(2)$ -symmetry. First we construct an  $E_q(2)$ -invariant integral. As we will see, this integral in turn looses the trace property. In a second step we introduce an  $E_q(2)$ —covariant differential calculus, making it possible to speak about covariant one- and two-forms. Difficulties appear when introducing gauge transformations: As the integral is not cyclic anymore, gauge transformations cannot be defined by usual conjugation with an unitary element but we have to adjust them somehow. We have to define "q-deformed" gauge transformations that permit to get a gauge invariant action. Regrettably, we loose in this approach the concept of Seiberg-Witten map, at least at the moment we do not know how to convey it to this setting. Nevertheless, future work could go in this direction, establishing a Seiberg-Witten map for the  $E_q(2)$  – covariant theory...

# 2.1 $E_q(2)$ and the $E_q(2)$ -Symmetric Plane

In classical physics we have a Lie group acting on a vector space. Since a Lie group cannot be deformed (the set of semi-simple Lie groups is a discrete set), one has to consider the algebra of functions on the considered vector space. For the action of a Lie group on a space is equivalent to considering the algebra of functions on the Lie group coacting on the algebra of functions on the space. Those algebras of functions, both on the Lie algebra and on the space, can be deformed. The resulting objects are Hopf algebras. First non-trivial examples of Hopf algebras were for instance introduced by Faddeev, Reshetikhin and Takhtadjan [18]. Thus, in the noncommutative realm, the action of a Lie group becomes the coaction of the corresponding deformed Hopf algebra on the deformed algebra of functions on the space.

We start introducing the generators  $n, v, \bar{n}, \bar{v}$  of the quantum group  $E_q(2)$  with their defining relations and structure maps [19]

$$v\bar{v} = \bar{v}v = 1 \qquad n\bar{n} = \bar{n}n \qquad vn = qnv$$

$$n\bar{v} = q\bar{v}n \qquad v\bar{n} = q\bar{n}v \qquad \bar{n}\bar{v} = q\bar{v}\bar{n}$$

$$\Delta(n) = n \otimes \bar{v} + v \otimes n \qquad \Delta(v) = v \otimes v \qquad \Delta(\bar{n}) = \bar{n} \otimes v + \bar{v} \otimes \bar{n}$$

$$\Delta(\bar{v}) = \bar{v} \otimes \bar{v} \qquad \varepsilon(n) = \varepsilon(\bar{n}) = 0 \qquad \varepsilon(v) = \varepsilon(\bar{v}) = 1$$

$$S(n) = -q^{-1}n \qquad S(v) = \bar{v} \qquad S(\bar{n}) = -q\bar{n} \qquad S(\bar{v}) = v$$

$$(2.1)$$

where  $q \in \mathbb{R}$ .

If we define new operators  $\theta, \bar{\theta}, t, \bar{t}$  by [19]

$$v = e^{\frac{i}{2}\theta}$$
  $\bar{\theta} = \theta$   $t = nv$   $\bar{t} = \bar{v}\bar{n}$  (2.2)

(note that v is unitary and can therefore be parametrized by a hermitian element) the coproduct of t and  $\bar{t}$  reads

$$\Delta(t) = t \otimes 1 + e^{i\theta} \otimes t \qquad \Delta(\bar{t}) = \bar{t} \otimes 1 + e^{-i\theta} \otimes \bar{t} \qquad (2.3)$$

A Hopf algebra H coacting on an algebra A means that algebra is a left (or right) H-comodule algebra:

**Definition 3.** A left coaction of a Hopf algebra H on an algebra A is a linear mapping  $\rho$ 

$$\rho: A \longrightarrow H \otimes A$$
(2.4)

satisfying

$$(id \otimes \rho) \circ \rho = (\Delta \otimes id) \circ \rho \quad and \quad (\varepsilon \otimes id) \circ \rho = id$$

$$\rho(ab) = \rho(a)\rho(b) \quad (m: A \otimes A \longrightarrow A \text{ is an } A\text{-comodule homomorphism})$$

$$\rho(1) = 1 \otimes 1 \quad (\eta: \mathbb{C} \longrightarrow A \text{ is an } A\text{-comodule homomorphism}) \quad .$$

$$(2.5)$$

In Sweedler notation we write [20, p. 32]:

$$\rho(a) =: a_{(-1)} \otimes a_{(0)}$$
.

An algebra A with a left coaction of a Hopf algebra H is called a left H-comodule algebra.

This definition implies that every Hopf algebra H admits a comodule structure on itself in virtue of its comultiplication

$$\Delta: H \longrightarrow H \otimes H \quad . \tag{2.6}$$

Therefore we can interpret the algebra generated by  $t, \bar{t}$ , which we want to rename by  $z, \bar{z}$  to distinguish them from the elements in  $E_q(2)$ , as the q-deformed algebra of functions on the  $E_q(2)$ -symmetric plane. We want to write  $\mathbb{C}_q^2$  for the algebra generated by  $z, \bar{z}$ . From (2.3) we get the following left  $E_q(2)$ -coaction on  $\mathbb{C}_q^2$ 

$$\rho(z) = e^{i\theta} \otimes z + t \otimes 1 
\rho(\overline{z}) = e^{-i\theta} \otimes \overline{z} + \overline{t} \otimes 1 .$$
(2.7)

We note that it follows from (2.1) (resp. by requiring  $m:A\otimes A\longrightarrow A$  to be a  $E_q(2)$ -comodule homomorphism) that

$$z\overline{z} = q^2 \overline{z}z \quad , \tag{2.8}$$

so that we can write  $\mathbb{C}_q^2 = \mathbb{C}\langle z, \bar{z}\rangle/(z\bar{z}-q^2\bar{z}z)$ . We want to extend the coaction of  $E_q(2)$  to the bigger algebra of formal power series

$$\mathbb{C}_q^{2,\text{ext}} := \mathbb{C}\langle\langle z, \bar{z}\rangle\rangle/(z\bar{z} - q^2\bar{z}z)$$
(2.9)

which we call the algebra of functions on the  $E_q(2)$ -symmetric plane. From now on functions are considered to lie in this algebra. It is covariant under the  $E_q(2)$ -coaction as described above.

We continue by discussing two different approaches to establish gauge field theory on  $\mathbb{C}_q^{2,\text{ext}}$  starting with a generalization of the concepts we got to know in Chapter 1.

# 2.2 Generalization of the Case $\theta = \text{const.}$

Let us adjust our notation to the notation we used in Chapter 1, especially in the section about orderings and star products. Henceforth, we will use the following abbreviations:

$$\mathcal{A}_{c} := \mathbb{C}[[z, \overline{z}]] \text{ and } \mathcal{A}_{nc} := \mathbb{C}_{q}^{2, \text{ext}} .$$
 (2.10)

In this section we generalize the ideas we studied in Chapter 1. We start doing so by constructing a star product for the algebra  $\mathcal{A}_{c}[[h]]$  such that it becomes isomorphic as algebra to  $\mathcal{A}_{nc}[[h]]$  in full analogy to what we have done in the first chapter. Since a star product is a deformation in direction of a Poisson structure, we will first deduce the corresponding Poisson structure.

#### 2.2.1 The Poisson Structure for $A_c$

To distinguish commutative coordinates and noncommutative ones we want to denote in this subsection the former by  $z, \overline{z}$  and the latter by  $\hat{z}, \overline{\hat{z}}$ . In the following subsections we will only need commutative coordinates and therefore they cannot be confounded with noncommutative ones.

We recall that we introduced in the first chapter a formal deformation parameter h. In this case we want to do so, too. We take

$$h := \ln q \tag{2.11}$$

as formal deformation parameter and our aim is to study deviations from the commutative theory in orders of this parameter h. Thus, the commutation relations (2.8) in  $\mathcal{A}_{nc}$  become

$$\hat{z}\overline{\hat{z}} = e^{2h}\overline{\hat{z}}\hat{z} = (1 + 2h + 2h^2 + \ldots)\overline{\hat{z}}\hat{z}$$
.

This yields

$$[\hat{z}, \overline{\hat{z}}] = 2h\overline{\hat{z}}\hat{z} + \mathcal{O}(h^2)$$
.

We want to construct a star product that renders the commutative algebra  $\mathcal{A}_c$  isomorphic to  $\mathcal{A}_{nc}$ . A star product is a deformation in direction of a Poisson structure. Therefore we need to know which Poisson structure we have to attribute to  $A_c$ .

Let us suppose we had an arbitrary star product  $\star$  (star products exist as we learned in Chapter 1) reflecting the algebraic properties of  $\mathcal{A}_{\rm nc}$ , i.e.  $(\mathcal{A}_{\rm c}, \star) \cong \mathcal{A}_{\rm nc}$ . Let  $\rho: \mathcal{A}_{\rm c} \to \mathcal{A}_{\rm nc}$  be the corresponding algebra isomorphism. We want to assume additionally that  $\mathcal{A}_{\rm nc} \to \mathcal{A}_{\rm c}$  for  $h \to 0$  such that  $\rho = \mathrm{id} + \mathcal{O}(h)$  (cf. Chapter 1). Thus, we have  $\rho(z^i) = \hat{z}^i + \mathcal{O}(h)$  and since by definition a star product always starts with the identity in zeroth order, we obtain for the star-commutator of the coordinates

$$[z * \overline{z}] = 2hz\overline{z} + \mathcal{O}(h^2) . \tag{2.12}$$

As we know, the Poisson bracket on  $\mathcal{A}_{c}[[h]]$  corresponds to the first order term in h of the star commutator multiplied by -i (Definition 1), such that we can derive from (2.12) the Poisson structure for  $\mathcal{A}_{c}[[h]]$ :

$$\{f,g\} = -2iz\overline{z}((\partial_z f)(\partial_{\overline{z}}g) - (\partial_{\overline{z}}f)(\partial_z g)) = -2i\varepsilon^{ij}z\overline{z}(\partial_i f)(\partial_j g) . \qquad (2.13)$$

<sup>&</sup>lt;sup>1</sup>These are the star products that we are interested in. A priory star products that correspond to a certain Poisson structure do not lead to isomorphic algebras. This is only the case if the star products lie in the same equivalence class. Thus, we need star products that lie in the equivalence class which consists of exactly those star products that render  $\mathcal{A}_c$  isomorphic as algebra to  $\mathcal{A}_{nc}$ .

Thus, the Poisson tensor is given by

$$\theta^{ij} = -2i\varepsilon^{ij}z\overline{z} = -2i\begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} z\overline{z} . \qquad (2.14)$$

*Remark:* If we write down the commutation relations in terms of the basis x, y, where z = x + iy and  $\overline{z} = x - iy$ , we find with  $h = \ln q$ :

$$[\hat{x}, \hat{y}] = -i\frac{1 - q^2}{1 + q^2}(\hat{x}^2 + \hat{y}^2) = ih(\hat{x}^2 + \hat{y}^2) + \mathcal{O}(h^2) .$$

Hence, we get for a star product  $\star$  with the above properties:

$$[x , y] = ih(x^2 + y^2) + \mathcal{O}(h^2)$$
.

Therefore the Poisson structure in the basis x, y reads

$$\{\tilde{f}, \tilde{g}\} = (x^2 + y^2)\varepsilon^{\alpha\beta}(\partial_{\alpha}\tilde{f})(\partial_{\beta}\tilde{g})$$

and we get the following Poisson tensor:

$$\tilde{\theta}(x,y) = (x^2 + y^2)\varepsilon^{\alpha\beta} . {2.15}$$

Notation: We will always write functions depending on x, y with tilde and use Greek indices if we consider the basis x, y, whereas we will use no tilde for functions depending on  $z, \overline{z}$  and Latin indices, for example  $i \in \{z, \overline{z}\}$ , for the basis  $z, \overline{z}$ . We will also often write  $z^i$  meaning  $z^1 = z$  and  $z^2 = \overline{z}$ .

# 2.2.2 A Star Product for $\mathcal{A}_{\mathrm{c}}[[h]]$

Again we have a big freedom in choosing a special ordering prescription  $\rho$  since every ordering prescription leads to a star product by defining  $f \star g = \rho^{-1}(\rho(f) \cdot \rho(g))$  (cf. Subsection 1.1.1). Here, we do not want to repeat all the discussion of orderings and corresponding star products but just remark that a star product corresponding to the normal ordering (all z to the left and all  $\overline{z}$  to the right) is given by (see for example [4] or also [21] where the Manin plane is discussed)

$$f \star_n g = \mu \circ e^{-2h(\overline{z}\partial_{\overline{z}}\otimes z\partial_z)}(f \otimes g)$$
.

For us the following star product will be of primary interest

$$f \star_{q} g := \mu \circ e^{h(z\partial_{z} \otimes \overline{z}\partial_{\overline{z}} - \overline{z}\partial_{\overline{z}} \otimes z\partial_{z})} (f \otimes g) . \tag{2.16}$$

First of all it can be easily verified, checking all requirements in Definition 1, that  $\star_q$  defined in (2.16) is indeed a star product for  $\mathcal{A}_c[[h]]$ . Let us think about which ordering prescription  $\rho_q$  corresponds to  $\star_q$ : We have

$$z \star_q \overline{z} = qz\overline{z}$$
 and  $\overline{z} \star_q z = q^{-1}z\overline{z}$ .

Therefore we get

$$\frac{q^{-1}z \star_q \overline{z} + q\overline{z} \star_q z}{2} = z\overline{z} .$$

We assume that  $\star_q$  is given by an ordering prescription  $\rho_q$ , i.e. we assume that  $f \star_q g = \rho_q^{-1}(\rho_q(f) \cdot \rho_q(g))$ . Than we find with the short calculation given above that an ordering prescription  $\rho_q : \mathcal{A}_c \to \mathcal{A}_{nc}$  that starts with the assignments<sup>2</sup>

$$\begin{array}{ccc} z & \mapsto & \hat{z} \\ \overline{z} & \mapsto & \overline{\hat{z}} \\ z\overline{z} & \mapsto & \frac{q^{-1}\hat{z}\overline{\hat{z}} + q\overline{\hat{z}}\hat{z}}{2} \end{array}$$

leads to  $\star_q$ . We want to call such an ordering prescription q-symmetric ordering<sup>3</sup>. In particular we see that  $\star_q$  neither corresponds to the symmetric ordering prescription nor to the normal ordering.

We note that  $\star_q$  is indeed a *hermitian* star product. Thus, complex conjugation is an involution for  $(\mathcal{A}_{c}[[h]], \star_q)$ .

Remark: In the real basis x, y (2.16) becomes

$$\tilde{f} \star_q \tilde{g} = \mu \circ e^{\frac{1}{4}h((x+iy)(\partial_x - i\partial_y) \otimes (x-iy)(\partial_x + i\partial_y) - (x-iy)(\partial_x + i\partial_y) \otimes (x+iy)(\partial_x - i\partial_y))} (\tilde{f} \otimes \tilde{g}) .$$
(2.17)

If we want to determine the corresponding ordering prescription in this basis things are not so easy. We find for example

$$x^{2} + y^{2} \to \frac{q^{-1}(\hat{x} + i\hat{y})(\hat{x} - i\hat{y}) + q(\hat{x} - i\hat{y})(\hat{x} + i\hat{y})}{2}.$$

Obviously in this basis the ordering prescription as well as the star product become more complicated. Therefore it is reasonable to take the complex basis.

<sup>&</sup>lt;sup>2</sup>Of course this is not the entire isomorphism  $\rho_q$  but we get an impression of how it looks like.

<sup>&</sup>lt;sup>3</sup>This notion goes back to Peter Schupp.

Nevertheless, we want to give all results for the usually used and more familiar real basis as well.

In the next chapter we will learn about gauge field theory on this noncommutative space. Later we will see why we want to choose especially  $\star_q$  to establish abelian gauge field theory: Actually,  $\star_q$  together with a modification of the common integral leads to an integral with trace property (see (1.53)), which is necessary to obtain an invariant action (cf. Subsection 1.2.5).

# 2.2.3 Noncommutative Abelian Gauge Field Theory

We treat the case of abelian, noncommutative gauge field theory on  $(\mathcal{A}_{c}[[h]], \theta)$ . This may be generalized to non abelian gauge field theories in a second step.

Infinitesimal gauge transformations of a matter field  $\hat{\psi}(z, \overline{z}) \in \mathcal{A}_{c}$  are introduced as in Subsection 1.2.1, i.e.

$$\delta\psi(z,\overline{z}) = i\Lambda(z,\overline{z}) \star \psi(z,\overline{z})$$
.

The concept of covariant coordinates  $Z^i := z^i + A^i$  with  $\delta Z^i = i[\Lambda \, \, , \, Z^i]$  leads again to a gauge field  $A^i$  transforming as

$$\delta A^i = -i[z^i \uparrow \Lambda] + i[\Lambda \uparrow A^i] .$$

One difference to the case where  $\theta$  is constant arises now: we cannot simply use  $\theta$  to lower indices as we did in Subsection 1.2.1. Since  $\theta$  is not constant, this would not lead to a gauge covariant field. Thus, we will stay with the gauge fields  $A^i$  and will try to analyze and to solve the emerging problems step by step. Let us continue by giving a solution for the Seiberg-Witten map (1.28).

#### Seiberg-Witten map

The solution for the Seiberg-Witten map for the gauge field  $A^i$  in the case of abelian gauge field theory is given by:

$$A^{i}(z,\overline{z}) = h\theta^{ij}a_{j} + h^{2}\frac{1}{2}\theta^{kl}a_{l}(\partial_{k}(\theta^{ij}a_{j}) - \theta^{ij}f_{jk}) + \dots$$
 (2.18)

Compare with the publication [9]<sup>4</sup>, where a solution for an arbitrary Poisson structure  $\theta$  up to second order was derived using the Kontsevich star product. The solution found there was taken and we checked that it is indeed a solution for  $\star_q$ , too.

 $<sup>\</sup>overline{{}^{4}\text{Mind that we want to contribute to every }\theta}$  an order in h.

By  $a_i = a_i(z, \overline{z})$  and  $f_{ij}(z, \overline{z})$  we denote the commutative gauge field and field strength written in the basis  $z, \overline{z}$ . Expressed in terms of the ordinary, commutative gauge field  $\tilde{a}_{\alpha}(x, y)$  and field strength  $\tilde{f}_{\alpha\beta}(x, y) = \partial_{\alpha}\tilde{a}_{\beta} - \partial_{\beta}\tilde{a}_{\alpha}$ , we have (Appendix A)

$$a_{z}(z,\overline{z}) = \frac{1}{2} \left\{ \tilde{a}_{1}(\phi(z,\overline{z})) - i\tilde{a}_{2}(\phi(z,\overline{z})) \right\}$$

$$a_{\overline{z}}(z,\overline{z}) = \frac{1}{2} \left\{ \tilde{a}_{1}(\phi(z,\overline{z})) + i\tilde{a}_{2}(\phi(z,\overline{z})) \right\}$$

and furthermore the commutative field strength  $\tilde{f}_{\alpha\beta}(x,y)$  becomes in the basis  $z, \overline{z}$  (Appendix A)

$$\frac{1}{2}i\tilde{f}_{ij}(\phi(z,\overline{z})) = f_{ij}(z,\overline{z}) = \partial_i a_j(z,\overline{z}) - \partial_j a_i(z,\overline{z}) .$$

# 2.2.4 The Field Strength

Let us think about the principles that we want to impose on a field strength F:

1. We want the field strength F to be gauge covariant, that means that the transformation law of F is to be given by

$$\delta F^{ij} = i[\Lambda \star F^{ij}] . \tag{2.19}$$

2. In the semi-classical limit  $q \to 1$  resp.  $h \to 0$  we want to obtain the commutative field strength  $f_{ij} = \partial_i a_j - \partial_j a_i$ .

We start treating the first requirement. For this purpose we write the commutation relations (2.8) in the following form introducing the  $\hat{R}$ -matrix:

$$P^{ij}{}_{kl}z^k \star z^l := (\delta^i_k \delta^j_l - \hat{R}^{ij}{}_{kl})z^k \star z^l = 0$$
 (2.20)

where

$$\hat{R}^{ij}{}_{kl} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - q^{-4} & q^{-2} & 0 \\ 0 & q^{-2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Here the upper indices number the rows in the sequence  $\{11, 12, 21, 22\}$  and the lower indices the columns in the same sequence. Defining

$$F^{ij} := -iP^{ij}{}_{kl}Z^k \star_q Z^l , \qquad (2.21)$$

we get a covariant expression since  $Z^i$  are covariant coordinates. Thereby (2.21) can be considered a good candidate for a field strength.

Let us calculate the first nontrivial order of  $F^{ij}$  to study its semi-classical limit. To do so, we first have to expand the matrix P in h

$$P^{ij}_{kl} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(h^2) \quad (2.22)$$

$$=: P_0 + hP_1 + \mathcal{O}(h^2) . \quad (2.23)$$

We introduce the notation

$$f \star g =: fg + h(f \star g)_1 + h^2(f \star g)_2 + \mathcal{O}(h^3)$$

such that we can write, using that  $Z^i = z^i + A^i$ , the commutation relations (2.20) and the Seiberg-Witten map up to first order in h for  $A^i$  given in (2.18)

$$F^{ij} = -iP^{ij}{}_{kl}(z^k \star_q A^l + A^k \star_q z^l + A^k \star_q A^l + z^k \star_q z^l)$$

$$= -ih^2 P^{ij}{}_{0kl}((z^k \star_q (\theta^{lm} a_m))_1 + ((\theta^{km} a_m) \star_q z^l)_1 + \theta^{km} a_m \theta^{lm} a_m)$$

$$-ih^2 P^{ij}{}_{kl}(z^k \theta^{lm} a_m + \theta^{km} a_m z^l) + \mathcal{O}(h^3) .$$

We split the calculation and start with the first two terms on the right hand side for i = z and  $j = \overline{z}$ . Since  $P_0^{z\overline{z}}{}_{kl} = \varepsilon_{kl}$  (see above), we get immediately

$$P_0^{z\overline{z}}{}_{kl}((z^k \star_q (\theta^{lm} a_m))_1 + ((\theta^{km} a_m) \star_q z^l)_1) = ([z \star_q^* \theta^{\overline{z}m} a_m])_1 + ([\theta^{zm} a_m \star_q^* \overline{z}])_1.$$

As the star-commutator always starts to first order in h by i times the Poisson structure (cf. Definition 1), we find

$$= i\theta^{z\overline{z}}\partial_{\overline{z}}(\theta^{\overline{z}z}a_z) + i\theta^{z\overline{z}}\partial_z(\theta^{z\overline{z}}_{nc}a_{\overline{z}})$$

$$= i\theta^{z\overline{z}}\theta^{z\overline{z}}(\partial_{\overline{z}}a_z - \partial_z a_{\overline{z}}) + i\theta^{z\overline{z}}(\partial_z(\theta^{z\overline{z}})a_{\overline{z}} - \partial_{\overline{z}}(\theta^{z\overline{z}})a_z)$$

$$= i\theta^{z\overline{z}}\theta^{z\overline{z}}f_{\overline{z}z} + i\theta^{z\overline{z}}(\partial_z(\theta^{z\overline{z}})a_{\overline{z}} - \partial_{\overline{z}}(\theta^{z\overline{z}})a_z) ,$$

where we used that  $\theta$  is antisymmetric, too.

We continue with the following term:

$$P_0^{ij}{}_{kl}\theta^{km}a_m\theta^{lm}a_m$$

It is easy to see that this equals zero noting that  $P_0^{ij}{}_{kl}$  is antisymmetric in k, l whereas  $\theta^{km} a_m \theta^{lm} a_m$  is symmetric in k, l. The last contribution reads

$$P_1^{z\overline{z}}{}_{kl}(z^k\theta^{lm}a_m + \theta^{km}a_mz^l) = -4(z\theta^{\overline{z}z}a_z + \theta^{z\overline{z}}a_{\overline{z}}\overline{z}) + 2(\overline{z}\theta^{z\overline{z}}a_{\overline{z}} + \theta^{\overline{z}z}a_zz)$$
$$= -2\theta^{z\overline{z}}(\overline{z}a_{\overline{z}} - za_z)$$

such that we find, putting all contributions together multiplied by -i as in the definition of F and taking into account that  $\theta^{z\overline{z}} = -2iz\overline{z}$ ,

$$F^{z\overline{z}} = h^{2}\theta^{z\overline{z}}\theta^{z\overline{z}}(\partial_{z}a_{\overline{z}} - \partial_{\overline{z}}a_{z}) + h^{2}(\partial_{z}\theta^{z\overline{z}})a_{\overline{z}} - (\partial_{\overline{z}}\theta^{z\overline{z}})a_{z} + 2ih^{2}\theta^{z\overline{z}}(\overline{z}a_{\overline{z}} - za_{z}) + \mathcal{O}(h^{3}) = h^{2}\theta^{z\overline{z}}\theta^{z\overline{z}}(\partial_{z}a_{\overline{z}} - \partial_{\overline{z}}a_{z}) + \mathcal{O}(h^{3}) = h^{2}\theta^{z\overline{z}}f_{\overline{z}z}\theta^{z\overline{z}} + \mathcal{O}(h^{3}) .$$

It can be easily checked that  $F^{\overline{z}z} = -h^2 \theta^{zi} f_{ij} \theta^{j\overline{z}} + \mathcal{O}(h^3)$  and we can write

$$F^{kl} = h^2 \theta^{ki} \theta^{jl} f_{ij} + \mathcal{O}(h^3) . \tag{2.24}$$

Let us remark that the calculation made so far for the semi-classical limit of F is *independent* of the choice of a star product, since until now only the star-commutator up to first order in h entered into the calculation.

While  $F^{ij}$  defined above transforms covariantly, the semi-classical limit does not exactly lead to the commutative field strength  $f_{ij}$ . Therefore  $F^{ij}$  cannot be taken like this as a noncommutative field strength. Nevertheless, we can modify  $F^{ij}$ : We want to get rid of the two  $\theta$ 's without loosing covariance. This can be done by multiplying  $F^{ij}$  by a covariant function with the property that in the semi-classical limit we actually obtain  $f_{ij}$ . In [9] we get a hint how we can do so: Here, covariant functions are introduced generated by applying a mapping  $\mathcal{D}$ , called "covariantizer", to an arbitrary function f

$$\mathcal{D}: f \mapsto \mathcal{D}f = f + f_{\mathcal{A}}$$

(applying  $\mathcal{D}$  to coordinates leads to covariant coordinates). The covariantizer is defined by requiring

$$\delta \mathcal{D}(f) = i[\Lambda * \mathcal{D}(f)] . \tag{2.25}$$

We can determine  $\mathcal{D}$  up to first order in h. With  $\Lambda = \alpha + \mathcal{O}(h)$ , where  $\alpha$  is the commutative gauge parameter, we see that  $\mathcal{D} = \mathrm{id} + h\theta^{ij}a_j\partial_i + \mathcal{O}(h^2)$  obeys the conditional equation (2.25). Here  $a_j$  denotes the commutative gauge field with the gauge transformation law  $\delta a_j = \partial_j \alpha$ . Thus, in our two dimensional case we obtain<sup>5</sup>

$$\mathcal{D} = \mathrm{id} + h\theta^{z\overline{z}}(a_{\overline{z}}\partial_z - a_z\partial_{\overline{z}}) + \dots$$
 (2.26)

We define

$$F'_{ij} := -\frac{1}{h^2} \mathcal{D}(\theta^{-1})_{ik} \star_q F^{kl} \star_q \mathcal{D}(\theta^{-1})_{lj}$$
 (2.27)

<sup>&</sup>lt;sup>5</sup>The existence of  $\mathcal{D}$  to all orders in h was derived in [9].

where

$$(\theta^{-1})_{kl} := -i\frac{1}{2z\overline{z}}\varepsilon_{kl} = -i\frac{1}{2z\overline{z}}\begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix}$$
 (2.28)

and

$$\mathcal{D}(\theta^{-1})_{kl} = -\frac{1}{2}i\mathcal{D}(\frac{1}{z\overline{z}})\varepsilon_{kl}$$

$$= -\frac{1}{2}i\varepsilon_{kl}(\frac{1}{z\overline{z}} + h\theta^{12}(a_{\overline{z}}\partial_z - a_z\partial_{\overline{z}})(\frac{1}{z\overline{z}}) + \mathcal{O}(h^2)$$

$$= -\frac{1}{2}i\varepsilon_{kl}\frac{1}{z\overline{z}}(1 - h\frac{\theta^{12}}{z\overline{z}}(a_{\overline{z}}\overline{z} - a_zz)) + \mathcal{O}(h^2)$$

$$= -\frac{1}{2}i\varepsilon_{kl}\frac{1}{z\overline{z}} + h\frac{(a_{\overline{z}}\overline{z} - a_zz)}{z\overline{z}}\varepsilon_{kl} + \mathcal{O}(h^2)$$
(2.29)

(the explicit contribution to first order in h will be used later when we expand the action in h). We see that because of (2.24)

$$F'_{ij} = f_{ij} + \mathcal{O}(h)$$

and therefore F' admits the right semi-classical limit. Moreover,  $\mathcal{D}(\theta^{-1})$  transforms covariantly ( $\mathcal{D}$  is the "covariantizer"). Hence, we can conclude that

- $F'_{ij}$  is gauge covariant and that
- $F'_{ij} \to f_{ij}$  for  $h \to 0$

such that  $F'_{ij}$  satisfies all we demanded above from a field strength. Hence, we consider  $F'_{ij}$  to be the noncommutative abelian field strength.

Now we can write down a Lagrangian for a noncommutative, free photon field

$$\mathcal{L} = \frac{1}{4} \delta^{ik} \delta^{lj} F'_{kl} \star_q F'_{ij} .$$

*Remark:* In the basis of real coordinates we want to define in analogy to what we studied for the basis  $z, \overline{z}$ :

$$\tilde{F}^{\alpha\beta} := \frac{1}{2} P^{\alpha\beta}{}_{12} (X + iY) \star_q (X - iY) + \frac{1}{2} P^{\alpha\beta}{}_{21} (X - iY) \star_q (X + iY) \tag{2.30}$$

where P is given in (2.20) and  $X := x + A^1$  resp.  $Y := y + A^2$  are the covariant coordinates (cf. (1.19)). Then  $\tilde{F}^{\alpha\beta}$  is gauge-covariant. If we undertake a similar

<sup>&</sup>lt;sup>6</sup>See the remark below to get an explanation for the factor  $\frac{1}{2}$ .

calculation as done above for determining the semi-classical limit of F, we get for the first non vanishing order

$$\tilde{F}^{\alpha\beta} = h^2 \tilde{\theta}^{\alpha\gamma} \tilde{\theta}^{\delta\beta} \tilde{f}_{\gamma\delta} + \mathcal{O}(h^3)$$

with  $\tilde{\theta}^{\alpha\beta} = (x^2 + y^2)\varepsilon^{\alpha\beta}$ . Again we want to get rid of the two  $\tilde{\theta}$  that appear without loosing gauge covariance so that we multiply by  $\mathcal{D}(\tilde{\theta}^{-1})$ . Since

$$(\tilde{\theta}^{-1})_{\alpha\beta} := -\frac{1}{x^2 + y^2} \varepsilon_{\alpha\beta} \tag{2.31}$$

we get with the definition of  $\mathcal{D}$  given in (2.26) up to first order in h that

$$\mathcal{D}(\tilde{\theta}^{-1})_{\alpha\beta} = -\mathcal{D}(\frac{1}{x^2 + y^2})\varepsilon_{\alpha\beta}$$

$$= -\varepsilon_{\alpha\beta}(\frac{1}{x^2 + y^2} + h\tilde{\theta}^{\gamma\delta}\tilde{a}_{\delta}\partial_{\gamma}(\frac{1}{x^2 + y^2}) + \mathcal{O}(h^2)$$

$$= -\varepsilon_{\alpha\beta}\frac{1}{x^2 + y^2}(1 - 2h\frac{\tilde{\theta}^{\gamma\delta}}{x^2 + y^2}\tilde{a}_{\delta}x_{\gamma})$$

$$= -\varepsilon_{\alpha\beta}\frac{1}{x^2 + y^2}(1 - 2h\varepsilon^{\gamma\delta}x_{\gamma}\tilde{a}_{\delta}) . \tag{2.32}$$

Finally, we define the field strength in real coordinates by

$$\tilde{F}'_{\alpha\beta} := -\frac{1}{h^2} \mathcal{D}(\tilde{\theta}^{-1})_{\alpha\gamma} \star_q \tilde{F}^{\gamma\delta} \star_q \mathcal{D}(\tilde{\theta}^{-1})_{\delta\beta} = \tilde{f}_{\alpha\beta} + \mathcal{O}(h)$$
 (2.33)

and the Lagrangian as

$$\tilde{\mathcal{L}} := \frac{1}{4} \delta^{\alpha \gamma} \delta^{\beta \delta} \tilde{F}'_{\gamma \delta} \star_q \tilde{F}'_{\alpha \beta} . \tag{2.34}$$

Remark: We found a field strength and a Lagrangian both, in the real and the complex basis. In both cases we got the right semi-classical limit and of course it is possible to come from one to the other case by basis transformation. But we want to underline the following: If we compare the definitions (2.21) of  $F^{ij}$  resp. (2.30) of  $\tilde{F}^{\alpha\beta}$  made above, we see that  $F^{ij}(\phi^{-1}(x,y)) = -2i\tilde{F}^{ij}(x,y)$  (for the definition of the basis transformation  $\phi(z,\bar{z})$  see Appendix A). Both definitions obviously differ by the factor -2i. Where does it come from? We need the factor to guarantee in both cases the right semi-classical limit.  $F^{z\bar{z}}$  starts in the first non-vanishing order with  $h^2\theta^{z\bar{z}}f_{\bar{z}z}\theta^{z\bar{z}}$  (see (2.24)). If we write this expression in terms of the real basis x,y we have to use that  $\theta^{z\bar{z}}(\phi^{-1}(x,y)) = -2i\tilde{\theta}^{12}(x,y)$  (cf. (2.14) and (2.15)) as well as the fact that  $f_{z\bar{z}} = \frac{1}{2}i\tilde{f}_{12}(\phi(z,\bar{z}))$  (A.3). Together this

gives  $F^{ij}(\phi^{-1}(x,y)) = (-2i)(\frac{1}{2}i)(-2i)\tilde{F}^{ij}(x,y) = -2i\tilde{F}^{ij}(x,y)$ . This factor must not be forgotten if translating results from the case with complex basis to the case where we use the real basis. Therefore definitions made here are reasonable (best proven by the fact that they lead to the right semi-classical limit as they should).

### 2.2.5 The Integral with Trace Property

In this section we treat the problem of how to find an integral with trace property for non-constant Poisson structures (for a constant Poisson structure the usual integral with an appropriate star product admits the trace property as we saw in Chapter 1). We will treat the special example  $\theta^{ij} = -2iz\overline{z}\varepsilon^{ij}$ , the Poisson tensor for  $\mathcal{A}_{\rm c}$  (2.14), but considerations made here can be transferred to other, more complicated Poison structures as well. Thus, the aim is not just to give a solution for our example but also to discuss two approaches how it may be possible to find solutions for more general Poisson structures.

#### **General Remarks**

We want to find a measure  $\mu(z, \overline{z})$  for the usual integral such that for a given star product  $\star$  the integral admits the trace property

$$\int \mu(z,\overline{z})dzd\overline{z}\,f\star g = \int \mu(z,\overline{z})dzd\overline{z}\,g\star f \tag{2.35}$$

for all functions f, g. Let us emphasize that this is a combination of two problems:

- find a star product and
- find a measure corresponding to this star product.

In general it surely is not possible to start with an arbitrary star product for the considered quantum space in the hope to find a measure giving the integral the above trace property to all orders in h.

#### 1. Calculation

Of course it is possible to start with a special star product that seems to be "good" in order to try to calculate a corresponding measure. The problems that arise are that

• in general it is difficult to find a solution valid to all orders in h

• we have to start with a special choice of a star product without knowing if this concrete star product permits a solution, i.e. if there exists a measure corresponding to this star product that renders the usual integral cyclic in the above sense.

In our example things can actually be done by calculation: First we have to fix a star product. The star product  $\star_q$  (2.16) seems to be a good choice since the symmetry of  $\star_q$  tells us that  $f \star_q g$  and  $g \star_q f$  are equal for all even orders of h such that we do not have to bother about even orders anymore. If one starts first to consider the condition (2.35) up to first order in h, a short calculation leads to the result  $\frac{1}{z\overline{z}}$  for the measure. We can even show that  $\frac{1}{z\overline{z}}$  is a measure that satisfies (2.35) to all orders in h:

**Proposition 4.** For all functions f, g we have

$$\int \frac{dz d\overline{z}}{z\overline{z}} f \star_q g = \int \frac{dz d\overline{z}}{z\overline{z}} f g = \int \frac{dz d\overline{z}}{z\overline{z}} g \star_q f .$$

*Proof.* We have

$$\int \frac{dz d\overline{z}}{z\overline{z}} f \star_q g = \int \frac{dz d\overline{z}}{z\overline{z}} f g + \int \frac{dz d\overline{z}}{z\overline{z}} \mu \circ \sum_{n=1}^{\infty} \frac{h^n}{n!} \left( \sum_{i_1,j_1=1}^2 \varepsilon^{i_1 j_1} z^{i_1} \frac{\partial}{\partial z^{i_1}} \otimes z^{j_1} \frac{\partial}{\partial z^{j_1}} \right)$$
$$\left( \sum_{i_2,i_2=1}^2 \varepsilon^{i_2 j_2} z^{i_2} \frac{\partial}{\partial z^{i_2}} \otimes z^{j_2} \frac{\partial}{\partial z^{j_2}} \right) \dots \left( \sum_{i_n,i_n=1}^2 \varepsilon^{i_n j_n} z^{i_n} \frac{\partial}{\partial z^{i_n}} \otimes z^{j_n} \frac{\partial}{\partial z^{j_n}} \right) (f \otimes g) .$$

Let us consider the n -th term of the sum on the right hand side

$$\int \frac{dz d\overline{z}}{z\overline{z}} \frac{h^n}{n!} \mu \circ \left( \sum_{i_1,j_1=1}^2 \varepsilon^{i_1j_1} z^{i_1} \frac{\partial}{\partial z^{i_1}} \otimes z^{j_1} \frac{\partial}{\partial z^{j_1}} \right) \left( \sum_{i_2,j_2=1}^2 \varepsilon^{i_2j_2} z^{i_2} \frac{\partial}{\partial z^{i_2}} \otimes z^{j_2} \frac{\partial}{\partial z^{j_2}} \right) \cdots \left( \sum_{i_n,j_n=1}^2 \varepsilon^{i_nj_n} z^{i_n} \frac{\partial}{\partial z^{i_n}} \otimes z^{j_n} \frac{\partial}{\partial z^{j_n}} \right) (f \otimes g) .$$

We introduce the short hand notation

$$f^{'} \otimes g^{'} := (\sum_{i_2, j_2 = 1}^{2} \varepsilon^{i_2 j_2} z^{i_2} \frac{\partial}{\partial z^{i_2}} \otimes z^{j_2} \frac{\partial}{\partial z^{j_2}}) \dots (\sum_{i_n, j_n = 1}^{2} \varepsilon^{i_n j_n} z^{i_n} \frac{\partial}{\partial z^{i_n}} \otimes z^{j_n} \frac{\partial}{\partial z^{j_n}}) (f \otimes g)$$

and with that the n-th term of the sum can be written as

$$\int \frac{dzd\overline{z}}{z\overline{z}} \frac{h^n}{n!} \mu \circ \left( \sum_{i_1,j_1=1}^2 \varepsilon^{i_1j_1} z^{i_1} \frac{\partial}{\partial z^{i_1}} \otimes z^{j_1} \frac{\partial}{\partial z^{j_1}} \right) (f' \otimes g')$$

$$= \frac{h^n}{n!} \int dzd\overline{z} \sum_{i_1,i_1=1}^2 \varepsilon^{i_1j_1} \frac{\partial}{\partial z^{i_1}} (f') \frac{\partial}{\partial z^{j_1}} (g') .$$

For n > 0 this leads after partial integration (we assume always that functions considered here vanish at infinity) to

$$-\frac{h^{n}}{n!} \int dz d\overline{z} \sum_{i_{1}, j_{1}=1}^{2} \varepsilon^{i_{1}j_{1}} f' \frac{\partial}{\partial z^{i_{1}}} \frac{\partial}{\partial z^{j_{1}}} (g') = 0 .$$

This is valid for any summand corresponding to n > 0 and only the zeroth order term does not vanish such that at the end we find what we claimed:

$$\int \frac{dz d\overline{z}}{z\overline{z}} f \star_q g = \int \frac{dz d\overline{z}}{z\overline{z}} f g .$$

#### 2. Using a Theorem from Felder and Shoikhet

We will use the following theorem from Felder and Shoikhet published in [22]:

**Theorem 1.** Let M be a Poisson manifold with the bivector field  $\pi$  and let  $\Omega$  be any volume form on M such that  $\operatorname{div}_{\Omega}\pi = 0$ . Then there exists a star product on  $C^{\infty}(M)$  such that for any two functions f, g one has

$$\int_{M} \Omega \left( f \star g \right) = \int_{M} \Omega f g .$$

Let me remark on how we have to understand  $\operatorname{div}_{\Omega}\pi$ . In [22] it is defined as follows:

$$\operatorname{div}: T_{\operatorname{poly}}^{k}(M) \xrightarrow{\Omega} \Omega^{d-k-1}(M) \xrightarrow{\operatorname{d}} \Omega^{d-k}(M) \xrightarrow{\Omega^{-1}} T_{\operatorname{poly}}^{k-1}(M)$$

where  $d = \dim M$ ,  $T_{\text{poly}}^k(M)$  is the set of k+1 - polyvector fields and d denotes the exterior differential. In have the following setting:  $\pi = -2iz\overline{z}\varepsilon^{ij}\partial_i \wedge \partial_j$  and we write for a general volume form  $\Omega(z,\overline{z}) = \omega(z,\overline{z})dz \wedge d\overline{z}$ . Then we have

$$\Omega(\pi) = \omega(z, \overline{z})(-2i)z\overline{z}.$$

Now we see that if we choose  $\omega(z,\overline{z}) = \frac{1}{z\overline{z}}$  we get  $\Omega(\pi) = -2i = \text{const.}$  such that in this case surely  $d \circ \Omega(\pi) = 0$ . With that

$$\operatorname{div}_{\Omega}\pi = 0$$
.

What did we achieve this way? We easily could find a volume form  $\Omega$  such that  $\operatorname{div}_{\Omega}\pi=0$ . The above theorem then assures the existence of a star product such that the integral over that volume form is cyclic (even more: The integral of two functions' star multiplication is equal to the integral of the ordinarily multiplied functions). Of course we still do not know which star product has together with that volume form this nice property. Nonetheless, the *existence* at least is assured and for usual physical applications this can be sufficient: usually we do only need to know the star product explicitly up to second order and these orders can be, if the volume form is known, determined by calculation. In our example the above calculation shows that  $\star_q$  is the right star product corresponding to the volume form  $\frac{1}{z\overline{z}}dz \wedge d\overline{z}$ .

We want to remark that for other, more complicated poisson structures where a solution cannot be found easily by calculation, this approach could be the more systematic one. It should also be possible for more complicated Poisson structures to determine a volume form  $\Omega$  satisfying the condition  $\operatorname{div}_{\Omega}\pi=0$  guaranteeing then by Felder's and Shoikhet's theorem the existence of a star product leading to the trace property. This star product then hopefully can be determined explicitly in a second step (but in any case at least the first orders of it).

# 2.2.6 The Action and the Integral

Having found an integral with trace property for the star product  $\star_q$  as well as a gauge covariant field strength with right semi-classical limit, we can write down an invariant action:

$$S = \int \frac{dz d\overline{z}}{z\overline{z}} \mathcal{L} = \int \frac{dz d\overline{z}}{z\overline{z}} \frac{1}{4} \delta^{ik} \delta^{lj} F'_{kl} \star_q F'_{ij} .$$

A new problem arises immediately: Let us determine the semi-classical limit for S. We know that  $\frac{1}{4}F'_{ij} \star_q F'^{ij} \to \frac{1}{4}f_{ij}f^{ij}$  for  $h \to 0$  because of (2.33) such that we obtain

$$S = \int \frac{dz d\overline{z}}{z\overline{z}} \frac{1}{4} f_{ij} f^{ij} + \mathcal{O}(h) .$$

This is not exactly what we want. In the semi-classical limit we do not find the free action for the flat plane, since the measure we had to introduce to guarantee

the trace property of the integral does not disappear. Again we have to make a modification, similar to what we already did for the field strength in (2.27): we have to multiply by a covariant term that in the semi-classical limit reduces to  $z\overline{z}$ . Then, for  $h \to 0$  we get the usual integral over  $z, \overline{z}$ . Let me remark that we do not loose the gauge invariance of the action if the additional term we introduce is gauge covariant. We want to multiply by the covariant coordinates and finally define

$$S' := \int \frac{dz d\overline{z}}{z\overline{z}} Z \star_q \overline{Z} \star_q \mathcal{L} = \int \frac{dz d\overline{z}}{z\overline{z}} \frac{1}{4} Z \star_q \overline{Z} \star_q \frac{1}{4} \delta^{ik} \delta^{lj} F'_{kl} \star_q F'_{ij}$$

as the free action. By this definition we get that

- S' is gauge invariant as well as that
- $S' = \int dz d\overline{z} \frac{1}{4} f_{ij} f^{ij} + \mathcal{O}(h)$ , i.e. it admits the right semi-classical limit.

(The reader might note that we have some freedom in choosing this additional, covariant term and also in choosing the position where to place it under the integral. We come back to this topic later.)

#### **Expanded Field Strength and Action**

We want to expand this action in h to be able to read off directly the corrections we get to the commutative theory in first order. We start calculating the first order term in h for  $F'_{z\overline{z}} = -\frac{1}{h^2} \mathcal{D}(\theta^{-1})_{z\overline{z}} \star_q F^{\overline{z}z} \star_q \mathcal{D}(\theta^{-1})_{z\overline{z}}$  using the expression for  $\mathcal{D}(\theta^{-1})_{z\overline{z}}$  up to first order in h (see (2.32)) and the second order term of  $F^{\overline{z}z}$  given in (2.24). Additionally we have to determine the third order term of  $F^{\overline{z}z}$  and must put all the results together. This was done making use of "Mathematica" leading to the following results:

$$F'_{z\overline{z}} = f_{z\overline{z}} + h \left\{ -f_{z\overline{z}} - 2iz\overline{z}f_{z\overline{z}}^2 + 2iz\overline{z}(a_z\partial_{\overline{z}}f_{z\overline{z}} - a_{\overline{z}}\partial_z f_{z\overline{z}}) \right\} + \mathcal{O}(h^2)$$

$$= f_{z\overline{z}} + h \left\{ -f_{z\overline{z}} + \theta^{z\overline{z}}f_{z\overline{z}}^2 - \theta^{z\overline{z}}(a_z\partial_{\overline{z}}f_{z\overline{z}} - a_{\overline{z}}\partial_z f_{z\overline{z}}) \right\} + \mathcal{O}(h^2) \quad (2.36)$$

and

$$F'_{\overline{z}z} = f_{\overline{z}z} + h \left\{ -3f_{\overline{z}z} + 2iz\overline{z}f_{z\overline{z}}^2 + 2iz\overline{z}(a_z\partial_{\overline{z}}f_{\overline{z}z} - a_{\overline{z}}\partial_z f_{\overline{z}z}) \right\} + \mathcal{O}(h^2)$$

$$= f_{\overline{z}z} + h \left\{ -3f_{\overline{z}z} - \theta^{z\overline{z}}f_{z\overline{z}}^2 - \theta^{z\overline{z}}(a_z\partial_{\overline{z}}f_{\overline{z}z} - a_{\overline{z}}\partial_z f_{\overline{z}z}) \right\} + \mathcal{O}(h^2) . (2.37)$$

<sup>&</sup>lt;sup>7</sup>I want to thank Peter Schupp who put forward this idea.

<sup>&</sup>lt;sup>8</sup>Special thanks to Fabian Bachmaier who made this possible.

Expansion of the Lagrangian  $\mathcal{L} := \frac{1}{4} \delta^{ik} \delta^{lj} F'_{kl} \star_q F'_{ij}$  yields:

$$\mathcal{L} = \frac{1}{4} f_{ij} f^{ij} + h \left\{ -2 f_{z\overline{z}}^2 - 2iz\overline{z} f_{z\overline{z}} (a_{\overline{z}} \partial_z f_{z\overline{z}} - a_z \partial_{\overline{z}} f_{z\overline{z}} + f_{z\overline{z}}^2) \right\} + \mathcal{O}(h^2)$$

$$= \frac{1}{4} f_{ij} f^{ij} \qquad (2.38)$$

$$+ h \left\{ -2 f_{z\overline{z}}^2 + \theta^{z\overline{z}} f_{z\overline{z}} (a_{\overline{z}} \partial_z f_{z\overline{z}} - a_z \partial_{\overline{z}} f_{z\overline{z}} + f_{z\overline{z}}^2) \right\} + \mathcal{O}(h^2) .$$

Finally, multiplying the action with the covariant term  $Z \star_q \overline{Z}$  to guarantee the right semi-classical limit (see above) we find for the action:

$$S' := \int \frac{dz d\overline{z}}{z\overline{z}} Z \star_{q} \overline{Z} \star_{q} \mathcal{L}_{free}$$

$$= \int dz d\overline{z} \frac{1}{4} f_{ij} f^{ij}$$

$$+ h \left\{ -\frac{3}{2} f_{z\overline{z}}^{2} + (iza_{z} - i\overline{z}a_{\overline{z}}) f_{z\overline{z}}^{2} + f_{z\overline{z}} (z\partial_{z} f_{z\overline{z}} - \overline{z}\partial_{\overline{z}} f_{z\overline{z}}) - z\overline{z} (2if_{z\overline{z}}^{3} - 2if_{z\overline{z}} (a_{z}\partial_{\overline{z}} f_{z\overline{z}} - a_{\overline{z}}\partial_{z} f_{z\overline{z}}) \right\} + \mathcal{O}(h^{2}) \qquad (2.39)$$

$$= \int dz d\overline{z} \frac{1}{4} f_{ij} f^{ij}$$

$$+ h \left\{ -\frac{3}{2} f_{z\overline{z}}^{2} - \frac{1}{2} (\partial_{\overline{z}} (\theta^{z\overline{z}}) a_{z} - \partial_{z} (\theta^{z\overline{z}}) a_{\overline{z}}) f_{z\overline{z}}^{2} + \frac{i}{2} f_{z\overline{z}} (\partial_{\overline{z}} (\theta^{z\overline{z}}) \partial_{z} f_{z\overline{z}} - \partial_{z} (\theta^{z\overline{z}}) \partial_{z} f_{z\overline{z}}) + \theta^{z\overline{z}} f_{z\overline{z}}^{3} - \theta^{z\overline{z}} f_{z\overline{z}} (a_{z}\partial_{\overline{z}} f_{z\overline{z}} - a_{\overline{z}}\partial_{z} f_{z\overline{z}}) \right\} + \mathcal{O}(h^{2}) .$$

#### Expanded Field Strength and Action in the Real Basis x, y

The field strength  $\tilde{F}'$  in the basis x, y was already introduced in (2.33). If we now use the star product  $\star_q$  in the basis x, y given in (2.17), we obtain the following results up to first order in h:

$$\tilde{F}'_{\alpha\beta} = -\frac{1}{h^2} \mathcal{D}(\theta^{-1})_{\alpha\gamma} \star_q \tilde{F}^{\gamma\delta} \star_q \mathcal{D}(\theta^{-1})_{\delta\beta}$$

gives

$$\tilde{F}'_{12} = \tilde{f}_{12} + h \left\{ -\tilde{f}_{12} + (x^2 + y^2) \tilde{f}_{12}^2 - (x^2 + y^2) (\tilde{a}_1 \partial_y \tilde{f}_{12} - \tilde{a}_2 \partial_x \tilde{f}_{12}) \right\} + \mathcal{O}(h^2) 
= \tilde{f}_{12} + h \left\{ -\tilde{f}_{12} + \tilde{\theta}^{12} \tilde{f}_{12}^2 - \tilde{\theta}^{12} (\tilde{a}_1 \partial_y \tilde{f}_{12} - \tilde{a}_2 \partial_x \tilde{f}_{12}) \right\} + \mathcal{O}(h^2)$$
(2.40)

and

$$\tilde{F}'_{21} = \tilde{f}_{21} + h \left\{ -3\tilde{f}_{21} - (x^2 + y^2)\tilde{f}_{12}^2 - (x^2 + y^2)(\tilde{a}_1\partial_y\tilde{f}_{21} - \tilde{a}_2\partial_x\tilde{f}_{21}) \right\} + \mathcal{O}(h^2)$$

$$= \tilde{f}_{21} + h \left\{ -3\tilde{f}_{21} - \tilde{\theta}^{12}\tilde{f}_{12}^2 - \tilde{\theta}^{12}(\tilde{a}_1\partial_y\tilde{f}_{21} - \tilde{a}_2\partial_x\tilde{f}_{21}) \right\} + \mathcal{O}(h^2) . \tag{2.41}$$

The Lagrangian introduced in (2.34) has the following form up to first order in h:

$$\tilde{\mathcal{L}} = \frac{1}{4} \tilde{f}_{\alpha\beta} \tilde{f}^{\alpha\beta} 
+ h \left\{ -2\tilde{f}_{12}^2 + (x^2 + y^2) (\tilde{f}_{12} (\tilde{a}_2 \partial_x \tilde{f}_{12} - \tilde{a}_1 \partial_y \tilde{f}_{12}) + \tilde{f}_{12}^3) \right\} + \mathcal{O}(h^2) 
= \frac{1}{4} \tilde{f}_{\alpha\beta} \tilde{f}^{\alpha\beta} 
+ h \left\{ -2\tilde{f}_{12}^2 + \tilde{\theta}^{12} (\tilde{f}_{12} (\tilde{a}_2 \partial_x \tilde{f}_{12} - \tilde{a}_1 \partial_y \tilde{f}_{12}) + \tilde{f}_{12}^3) \right\} + \mathcal{O}(h^2) .$$
(2.42)

To get the right classical limit for the integral we must again multiply by a covariant term. We take  $X \star_q X + Y \star_q Y$  (note that this is not equal to  $(X + iY) \star_q (X - iY)$ !) and finally find for the modified field strength

$$\tilde{S}' = \int dx dy \frac{1}{4} \tilde{f}_{\alpha\beta} \tilde{f}^{\alpha\beta} 
+ h \left\{ 2 \tilde{f}_{12}^2 - (x^2 + y^2) \tilde{f}_{12}^3 - i(x \tilde{a}_2 \partial_y \tilde{f}_{12} - y \tilde{a}_1 \partial_x \tilde{f}_{12}) \tilde{f}_{12} 
- (x^2 + y^2) \tilde{f}_{12} (\tilde{a}_2 \partial_x \tilde{f}_{12} - \tilde{a}_1 \partial_y \tilde{f}_{12}) + (y \tilde{a}_1 - x \tilde{a}_2) \tilde{f}_{12}^2 \right\} + \mathcal{O}(h^2) 
= \int dx dy \frac{1}{4} \tilde{f}_{\alpha\beta} \tilde{f}^{\alpha\beta} 
+ h \left\{ 2 \tilde{f}_{12}^2 - \tilde{\theta}^{12} \tilde{f}_{12}^3 - \frac{i}{2} (\partial_x (\tilde{\theta}^{12}) \tilde{a}_2 \partial_y \tilde{f}_{12} - \partial_y (\tilde{\theta}^{12}) \tilde{a}_1 \partial_x \tilde{f}_{12}) \tilde{f}_{12} 
- \tilde{\theta}^{12} \tilde{f}_{12} (\tilde{a}_2 \partial_x \tilde{f}_{12} - \tilde{a}_1 \partial_y \tilde{f}_{12}) + \frac{1}{2} (\partial_y (\tilde{\theta}^{12}) \tilde{a}_1 - \partial_x (\tilde{\theta}^{12}) \tilde{a}_2) \tilde{f}_{12}^2 \right\} + \mathcal{O}(h^2) .$$

Again we find the right semi-classical limit, i.e. the theory we established approaches for both cases, complex and real basis, the commutative theory in the limit  $h \to 0$ . Moreover, we can now read off the correction to the commutative theory this noncommutative theory leads to up to first order in h. Let us comment on the results:

- First we see that we get after transformation in both bases the same results for the field strength F respectively  $\tilde{F}$  and the Lagrangian  $\mathcal{L}$  respectively  $\tilde{\mathcal{L}}$  providing us a correctness check.
- The two results for S' respectively  $\tilde{S}$  are not identically equal. The reason is that we multiplied in the case of the real basis by the additional covariant term  $X \star_q X + Y \star_q Y$  to compensate the measure of the integral in the limit  $q \to 1$ , whereas we took in the case of the complex basis the covariant term  $Z \star_q \overline{Z} = (X + iY) \star_q (X iY) \neq X \star_q X + Y \star_q Y$ . This was done on purpose to show that the results we get here depend on the special choice of the additional term (we just required gauge-covariance and a special semi-classical limit but these requirements do not uniquely determine such an additional term). We will comment on this in the following subsection.
- The results we get lead us to the following interpretation: In first order of h we obtain a correction to the common photon propagator (quadratic term in f). Additionally, we get, if we interpret  $\theta$  as a background field, some interaction term between three photons and the  $\theta$ -field (the rest of the action's first-order term).
- Matter fields could easily be introduced, too, leading to more new interactions in the first order of the deformation parameter h.

# 2.2.7 Freedom of the Theory

We established in all detail a gauge field theory on  $\mathcal{A}_{nc}$ , the  $E_q(2)$ -symmetric space. We defined a field strength, introduced a cyclic integral and got finally an invariant action with right semi-classical limit. At the end the expansion of field strength, Lagrangian and action in the deformation parameter h was calculated up to first order. We can read off explicitly how far the theory differs from the commutative theory.

Nevertheless, it is our obligation to look back and to examine whether all the assumptions and the definitions we made are dictated by principles we want to require or how far our considerations actually admit freedom in defining the introduced physical quantities such as field strength, Lagrangian and action. The reader might already have noticed that in the definition of F' and of  $\mathcal{L}$  (cf. (2.27) and (2.34)) at least the position where to put the appearing  $\mathcal{D}(\theta^{-1})$  terms is arbitrary. There is no conceptual criterion that distinguishes one special choice of position. All possible choices of position lead to the same semi-classical limit

and give covariant expressions and this is all we required. The hope was, that nevertheless physics might be unaffected of the choice of position: We hoped that at least the first order term of the expanded quantities would not depend on the choice of position. Unfortunately, calculation shows that for example  $\frac{1}{h^2}\mathcal{D}(\theta^{-1})_{z\overline{z}} \star_q F^{\overline{z}z} \star_q \mathcal{D}(\theta^{-1})_{z\overline{z}}$  and  $\frac{1}{h^2}\mathcal{D}(\theta^{-1})_{z\overline{z}} \star_q \mathcal{D}(\theta^{-1})_{z\overline{z}} \star_q F^{\overline{z}z}$ , two equivalent, possible ways to define F', differ in first order of h by

$$-2h(\overline{z}\partial_{\overline{z}}f_{z\overline{z}}-z\partial_{z}f_{z\overline{z}})$$

such that the choice of position of the intervening  $\mathcal{D}(\theta^{-1})$  terms influences the physical results.

Moreover, when we defined the action we had to introduce a covariant expression that compensates in the semi-classical limit the measure  $\frac{1}{z\overline{z}}$ , which in turn we had to introduce to guarantee the trace property of the integral (see (2.19) and Proposition 4). At least if we want to treat gauge field theory on the flat space we cannot do without this additional term. We added  $Z \star_q \overline{Z}$  but a lot of other terms could have been chosen. For example  $\mathcal{D}(z\overline{z})$  admits the same semi-classical limit as  $Z \star_q \overline{Z}$  and is also covariant. None of both can be regarded the better one. In the way we established the theory, no conceptual criterion exists to get rid of this freedom.

The other general problem we have with this approach is the following: Our considerations did not include in all points the background symmetry of the space, the q – deformed two-dimensional Euclidean group. We see that the integral with the measure we had to choose to get the trace property is certainly not  $E_q(2)$  -invariant. Nevertheless, having a background symmetry we should try to keep it and try to set up an  $E_q(2)$  - covariant theory.

That is why we study in the next section an alternative approach to the problem. We will establish a theory where the  $E_q(2)$ — symmetry stands in the foreground of all considerations.

# 2.3 $E_q(2)$ - Covariant Abelian Gauge Field Theory

A covariant, abelian gauge theory was already established on another noncommutative space: the q-deformed fuzzy sphere that is covariant with respect to the coaction of  $SU_q(2)$  [23, 24, 25]. Guided by these publications, we set up a gauge theory based on  $E_q(2)$ -covariance. At the beginning, we construct an  $E_q(2)$ -invariant integral. As we will see, it is not cyclic. In a second step an  $E_q(2)$ -covariant differential calculus is introduced. We define an exterior differential, q-one-forms, q-two-forms and q-deformed derivatives. Moreover, a

generator of the exterior differential as well as a frame, a basis of one-forms that commutes with arbitrary functions, are derived. We additionally introduce a gauge field A and a field strength F that in the semi-classical limit  $q \to 1$  becomes the commutative field strength. Problems arise when trying to define gauge transformations: Since the integral does not possess the trace property, we cannot introduce gauge transformations of gauge fields as we did in the previous section. By means of an algebra homomorphism  $\alpha: U_q(e(2)) \to \mathcal{A}_{nc}$ , we define gauge transformations of one forms in a way that allows us to speak about a gauge invariant action.

For a detailed discussion of quantum groups the reader is referred to [26, 20, 27].

# 2.3.1 An Invariant Integral on the $E_q(2)$ - Symmetric Plane

We need an  $E_q(2)$  - invariant integral to define an action. Considerations made in this subsection are guided by [28], where a Haar-functional on  $E_q(2)$  extended to formal power series is derived. Nevertheless, we use another, physically more intuitive basis of the dual of  $E_q(2)$  and present the necessary calculations. Whereas we introduced in Section 2.1 the quantum group  $E_q(2)$  and the  $E_q(2)$ -symmetric plane, we proceed to introduce at this point the quantum dual of  $E_q(2)$ .

## The Quantum Universal Enveloping Algebra $U_q(e(2))$ and Duality

In many cases it is convenient to consider the action of the quantum groups dual instead of the coaction of the quantum group itself. Therefore we introduce  $U_q(e(2))$ , the dual of  $E_q(2)$ .

**Definition 4.** Let  $(H, m, \eta, \Delta, \varepsilon, S)$  and  $(H', m', \eta', \Delta', \varepsilon', S')$  be two Hopf algebras. We say that H and H' are in duality if there exists a bilinear form, called dual pairing,

$$\langle \cdot, \cdot \rangle : H \otimes H' \to \mathbb{C}$$

satisfying

$$\langle gh, x \rangle = \langle g \otimes h, \Delta'(x) \rangle \qquad \langle h, xy \rangle = \langle \Delta(h), x \otimes y \rangle$$

$$\langle 1, x \rangle = \varepsilon'(x) \qquad \langle h, 1 \rangle = \varepsilon(h)$$

$$(2.44)$$

$$\langle S(h), x \rangle = \langle h, S'(x) \rangle$$

for all  $g, h \in H$  and  $x, y \in H'$ , where  $\langle g \otimes h, x \otimes y \rangle := \langle g, x \rangle \langle h, y \rangle$  is the extension of  $\langle \cdot, \cdot \rangle$  on tensor products.

In [19] the dual to  $E_q(2)$ , called  $U_q(e(2))$ , is constructed. With the following identifications for the generators  $\mu, \nu, \xi$  introduced in this publication,

$$\mu \equiv T - q^2 \nu \equiv \overline{T} \quad \xi \equiv J$$
,

we then get that  $U_q(e(2))$  is generated by  $T, \overline{T}, J$  with the following commutation relations and structure maps

$$T\overline{T} = q^{2}\overline{T}T \qquad [J,T] = iT \qquad [J,\overline{T}] = -i\overline{T}$$

$$\Delta(T) = T \otimes q^{2iJ} + 1 \otimes T \qquad \Delta(\overline{T}) = \overline{T} \otimes q^{2iJ} + 1 \otimes \overline{T} \qquad (2.45)$$

$$\Delta(J) = J \otimes 1 + 1 \otimes J \qquad \varepsilon(T) = \varepsilon(\overline{T}) = \varepsilon(J) = 0$$

$$S(T) = -Tq^{-2iJ} \qquad S(\overline{T}) = -\overline{T}q^{-2iJ} \qquad S(J) = -J \quad ,$$

where the dual pairing on the generators is given by

$$\langle T, \theta^i t^j \overline{t}^k \rangle = \delta_{0i} \delta_{1j} \delta_{0k}, \quad \langle \overline{T}, \theta^i t^j \overline{t}^k \rangle = -q^2 \delta_{0i} \delta_{0j} \delta_{1k}, \quad \langle J, \theta^i t^j \overline{t}^k \rangle = \delta_{1i} \delta_{0j} \delta_{0k} .$$
(2.46)

Moreover, we have

$$\overline{J} = -J \quad . \tag{2.47}$$

The generators  $T, \overline{T}$  as duals of  $t, \overline{t}$  can be interpreted as the q-analogs of the generators of translations whereas J, the dual of  $\theta$ , can be viewed as the generator of rotations (rotations are not deformed). After having introduced the dual, we can now use the dual pairing to pass over from the coaction of  $E_q(2)$  to an action of the dual  $U_q(e(2))$  on  $\mathbb{C}_q^2$ .

# The Action of $U_q(e(2))$ on $\mathcal{A}_{nc}$

First of all we want to define what me mean by the action of a Hopf algebra on an algebra.

**Definition 5.** We say that a Hopf algebra H is acting on an algebra A from the right (left) if A is a right (left) H-module such that  $m: A \otimes A \longrightarrow A$  and  $\eta: \mathbb{C} \longrightarrow A$  are right (left) H-module homomorphisms, that means if holds

$$ab \triangleleft h = (a \otimes b) \triangleleft \Delta(h) = (a \triangleleft h_{(1)})(b \triangleleft h_{(2)})$$
 and  $1 \triangleleft h = \varepsilon(h)1$  (2.48)

respectively

$$h \triangleright ab = \Delta(h) \triangleright (a \otimes b) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$$
 and  $h \triangleright 1 = \varepsilon(h)1$ ,

for any  $h \in H$  and  $a, b \in A$ . An algebra A with a Hopf algebra H acting on it from the right (left) is called a right (left) H-module algebra.

**Lemma 2.** Given a dual pairing between Hopf algebras H and H' and a left coaction  $\rho$  of H' on an algebra A, we get a right action of H on A,  $\triangleleft : A \otimes H \longrightarrow A$ , by defining

$$f \triangleleft X := (\langle X, \cdot \rangle \otimes id) \circ \rho(f) = \langle X, f_{(-1)} \rangle f_{(0)}, \qquad X \in H, f \in A . \tag{2.49}$$

*Proof.* We use Sweedler notation.

If we define as in (2.49) we get from the properties of the dual pairing (Definition 4) that A is a right H-module. For example we have

$$i) \quad (f \triangleleft X) \triangleleft Y = (\langle X, f_{(-1)} \rangle f_{(0)}) \triangleleft Y$$

$$= \langle X, f_{(-1)} \rangle \langle Y, f_{(0)_{(-1)}} \rangle f_{(0)_{(0)}}$$

$$= \langle X, f_{(-2)} \rangle \langle Y, f_{(-1)} \rangle f_{(0)}$$

$$= \langle X \otimes Y, \Delta f_{(-1)} \rangle f_{(0)}$$

$$= \langle XY, f_{(-1)} \rangle f_{(0)}$$

$$= f \triangleleft (XY) .$$

We prove the remaining requirements again using Definition 4:

$$\begin{array}{lll} ii) & fg \triangleleft X & = & \langle X, (fg)_{(-1)} \rangle (fg)_{(0)} \\ & = & \langle X_{(1)}, f_{(-1)} \rangle \langle X_{(2)}, g_{(-1)} \rangle f_{(0)} g_{(0)} \\ & = & (f \triangleleft X_{(1)}) (g \triangleleft X_{(2)}) \end{array}$$

$$\begin{array}{rcl} iii) & 1 \triangleleft X & = & \langle X, 1 \rangle 1 \\ & = & \varepsilon(X) 1 & . \end{array}$$

<sup>&</sup>lt;sup>9</sup>Similarly we a get a left action via a dual pairing from a right coaction.

Thus, with the dual pairing given in (2.46) and with the coaction of  $z, \overline{z}$  given in (2.7) we can define an action of  $U_q(e(2))$  on  $\mathbb{C}_q^2$  in analogy to (2.49). We obtain for the action of J, T and  $\overline{T}$  on the generators  $z, \overline{z}$ :

$$z \triangleleft \overline{T} = \langle \overline{T}, e^{i\theta} \rangle z + \langle \overline{T}, t \rangle = 1$$

$$z \triangleleft \overline{T} = \langle \overline{T}, e^{i\theta} \rangle z + \langle \overline{T}, t \rangle = 0$$

$$z \triangleleft J = \langle J, e^{i\theta} \rangle z + \langle J, t \rangle = iz$$

$$\overline{z} \triangleleft \overline{T} = \langle \overline{T}, e^{i\theta} \rangle \overline{z} + \langle \overline{T}, \overline{t} \rangle = 0$$

$$\overline{z} \triangleleft \overline{T} = \langle \overline{T}, e^{-i\theta} \rangle \overline{z} + \langle \overline{T}, \overline{t} \rangle = -q^{2}$$

$$\overline{z} \triangleleft J = \langle J, e^{-i\theta} \rangle \overline{z} + \langle J, \overline{t} \rangle = -i\overline{z}.$$

$$(2.50)$$

Now we want to extend the action of  $U_q(e(2))$  on the algebra of formal power series in  $z, \overline{z}$  respecting the commutation relation (2.8). This algebra we denoted by  $\mathcal{A}_{nc}$  (2.10). Before giving explicit formulas, let us remark that any formal power series  $f(z, \overline{z}) \in \mathcal{A}_{nc}$  can be decomposed as follows:

$$f(z,\overline{z}) = f_1(z,z\overline{z}) + f_2(\overline{z},z\overline{z}) \tag{2.51}$$

where  $f_1, f_2$  are formal power series. To see that, we have to split a formal power series depending of  $z, \bar{z}$  into two series, the one consisting of all summands where the exponent of z is bigger or equal to that of  $\bar{z}$  and the other consisting of the remaining part. Taking into account that

$$z^{k}\bar{z}^{k} = q^{k(k-1)}(z\bar{z})^{k} \qquad , \tag{2.52}$$

which can be verified using the commutation relation (2.8) and that  $\sum_{i=0}^{k-1} i = \frac{1}{2}k(k-1)$ , we get a decomposition of the above type (2.51).  $f_1$  and  $f_2$  in turn can be decomposed in sums consisting of summands of the following type:

$$z^k f(z\bar{z})$$
 respectively  $\bar{z}^k f(z\bar{z})$  . (2.53)

Moreover, positive powers of  $\bar{z}$  can be rewritten in terms of negative powers of z by using

$$\bar{z}^k = q^{k(k-1)} z^{-k} (z\bar{z})^k$$
 (2.54)

This leads to the following remark:

**Remark 2.** Any formal power series  $f(z,\bar{z})$  can be written in the following form:

$$f(z,\overline{z}) = \sum_{m \in \mathbb{Z}} z^m f_m(z\overline{z})$$
 (2.55)

where  $f_m$  are formal power series in  $z\overline{z}$ .

To know the action on  $\mathcal{A}_{nc}$  we now just have to define it on a summand

$$z^k f(z\bar{z})$$
 ,  $k \in \mathbb{Z}$  (2.56)

and extend it linearly to power series (2.55). Using the structure maps of  $U_q(e(2))$  given in (2.45) it is possible to determine the right-action of  $J, T, \overline{T}$  on such a summand. This is done in Appendix B leading to the following results:

$$z^{k} f(z\overline{z}) \triangleleft T = \frac{z^{k-1}}{1 - q^{-2}} (f(q^{2}z\overline{z}) - q^{-2k}f(z\overline{z}))$$

$$z^{k} f(z\overline{z}) \triangleleft \overline{T} = \frac{q^{4}}{1 - q^{2}} z^{k+1} \frac{f(z\overline{z}) - f(q^{-2}z\overline{z})}{z\overline{z}}$$

$$z^{k} f(z\overline{z}) \triangleleft J = i^{k} z^{k} f(z\overline{z}) .$$

$$(2.57)$$

We see that the action of  $U_q(e(2))$  on a function  $z^k f(z\overline{z})$  closes in  $\mathcal{A}_{nc}$ , i.e. we have

$$f \in \mathcal{A}_{nc} \implies f \triangleleft X \in \mathcal{A}_{nc} \text{ for all } X \in U_q(e(2))$$
. (2.58)

Remark: A priori it is not clear whether an extension of the action to formal power series is possible. But the special form of the action on summands as we got it in (2.57) shows, that in our case this indeed can be done: The action on z to the power of k multiplied by a function  $f(z\overline{z})$  leads in all cases to the powers k-1,k or k+1. Thus, the action on a formal power series, where also negative powers of z are allowed, again leads in all cases to finitely many contributions in every power of z and therefore the linear extension is actually well-defined.

#### The $U_q(e(2))$ Invariant Integral

We will construct a  $U_q(e(2))$  – invariant integral following the methods given in [28] starting by defining what we want to call an invariant integral.

**Definition 6.** We call an integral (i.e. a linear functional) invariant with respect to the right action of  $U_q(e(2))$  if it satisfies the following invariance condition:

$$\int_{-1}^{q} f(z, \overline{z}) \, dX = \varepsilon(X) \int_{-1}^{q} f(z, \overline{z}) \tag{2.59}$$

for all f and  $X \in U_q(e(2))$ .

Here the action of X on  $f(z, \bar{z})$  is to be understood as in (2.57). Since  $\varepsilon$  is an algebra homomorphism and  $\triangleleft$  an algebra action, it is sufficient to check the condition (2.59) for  $T, \overline{T}$  and J (recall that  $\{J^kT^l\overline{T}^m|k\in\mathbb{Z}, l, m\in\mathbb{Z}_+\}$  is a basis of  $U_q^{\text{ext}}(e(2))$ , the extension of  $U_q(e(2))$  to formal power series). Let us first consider functions of the type

$$z^m f(z\bar{z}), \qquad m \neq 0. \tag{2.60}$$

In this case it is apparently true that

$$\int^{q} z^{m} f(z\overline{z}) := 0 \text{ for } m \neq 0$$
(2.61)

is invariant if for  $z^m f(z\bar{z}) \triangleleft X$  holds that the exponent of z still is not equal to 0 (where  $X \in \{J, T, \overline{T}\}$ ). The only cases where this is not guaranteed are (cf. (2.57))

$$zf(z\overline{z}) \triangleleft T = \frac{1}{1 - q^{-2}} (f(q^2 z\overline{z}) - q^{-2} f(z\overline{z}))$$

and

$$z^{-1}f(z\overline{z}) \triangleleft \overline{T} = \frac{q^4}{1-q^2} \frac{f(z\overline{z}) - f(q^{-2}z\overline{z})}{z\overline{z}} = \frac{q^4}{1-q^2} \left( \frac{f(z\overline{z})}{z\overline{z}} - q^{-2} \frac{f(q^{-2}z\overline{z})}{q^{-2}z\overline{z}} \right) \ .$$

Since  $\varepsilon(T) = \varepsilon(\overline{T}) = 0$ , we have to require that  $\int_{-\infty}^{q} z f(z\overline{z}) dT$  and  $\int_{-\infty}^{q} z^{-1} f(z\overline{z}) dT$  equal zero if the integral is to satisfy the invariance condition (2.59). This implies

$$\int_{0}^{q} f(q^2 z \overline{z}) - q^{-2} f(z \overline{z}) = 0 \tag{2.62}$$

respectively

$$\int^{q} \tilde{f}(z\overline{z}) - q^{-2}\tilde{f}(q^{-2}z\overline{z}) = 0 , \qquad (2.63)$$

where  $\tilde{f}(z\overline{z}) - q^{-2}\tilde{f}(q^{-2}z\overline{z}) := \frac{f(z\overline{z})}{z\overline{z}} - q^{-2}\frac{f(q^{-2}z\overline{z})}{q^{-2}z\overline{z}}$ . If we define

$$\int^{q} f(z\overline{z}) := \sum_{k=-\infty}^{\infty} q^{2k} f(q^{2k}r_0^2) ,$$

where  $r_0^2q^{2k}$ ,  $k \in \mathbb{Z}$  are the eigenvalues of  $z\bar{z}$  in a representation of  $\mathbb{C}_q^2$  (see for example [29]), we see that this integral satisfies (2.62) and (2.63). Together with (2.61), we can now define an invariant integral. Let us summarize the result in the following Lemma:

**Lemma 3.** For all functions  $f \in \mathcal{A}_{nc}$ 

$$\int^{q} z^{m} f(z\overline{z}) := \delta_{m,0} r_{0}^{2} (q^{2} - 1) \sum_{k=-\infty}^{\infty} q^{2k} f(q^{2k} r_{0}^{2})$$
 (2.64)

is an invariant integral under the action of  $U_q(e(2))$  in the sense of Definition 6.

The factor  $r_0^2(q^2-1)$  was added to guarantee the right semi-classical limit of the integral.

#### Cyclic property of the Invariant Integral

Let us examine whether the integral (2.64) is cyclic as it is in the undeformed case.

First of all we provide a commutation relation which we will need later on.

**Remark 3.** For any formal power series  $f(z\overline{z})$  we have:

$$f(z\overline{z})z^m = z^m f(q^{-2m}z\overline{z}) \tag{2.65}$$

and

$$f(z\overline{z})\overline{z}^m = \overline{z}^m f(q^{2m} z\overline{z}) \tag{2.66}$$

for  $m \in \mathbb{Z}$ .

*Proof.* Using (2.8) we immediately get

$$(z\bar{z})z = q^{-2}z(z\bar{z})$$

implying

$$(z\bar{z})z^m = q^{-2m}z^m(z\bar{z})$$

Extending this to formal power series, we get the first equation in the Remark. The second follows similarly.

Let us undertake the following calculation to examine the cyclic property of the q-integral (2.64) using in the first step the commutation relation (2.65):

$$\begin{split} & \int^q z^m f_m(z\bar{z}) z^n g_n(z\bar{z}) \\ & = \int^q z^{m+n} f_m(q^{-2n}z\bar{z}) g_n(z\bar{z}) \\ & = \delta_{m+n,0} r_0^2(q^2-1) \sum_{k=-\infty}^{+\infty} q^{2k} f_m(q^{2k}q^{-2n}r_0^2) g_n(q^{2k}r_0^2) \\ & = \begin{cases} r_0^2(q^2-1) \sum_{k=-\infty}^{+\infty} q^{2k} f_m(q^{2k}q^{2m}r_0^2) g_{-m}(q^{2k}r_0^2) & \text{if } n+m=0 \\ 0 & \text{if } n+m\neq 0 \end{cases} \\ \stackrel{(k\to k+m)}{=} \begin{cases} r_0^2(q^2-1) \sum_{k=-\infty}^{+\infty} q^{-2m}q^{2k} f_m(q^{2k}r_0^2) g_{-m}(q^{2k}q^{-2m}r_0^2) & \text{if } n+m=0 \\ 0 & \text{if } n+m\neq 0 \end{cases} \\ & = \delta_{m+n,0} r_0^2(q^2-1) \sum_{k=-\infty}^{+\infty} q^{-2m}q^{2k} g_{-m}(q^{2k}q^{-2m}r_0^2) f_m(q^{2k}r_0^2) \\ & = q^{-2m} \int^q z^n g_n(z\bar{z}) z^m f_m(z\bar{z}) \end{cases} . \end{split}$$

If we now define an operator

$$\mathcal{D}(z^m) := q^{-2m} z^m \qquad \mathcal{D}(\overline{z}^m) := q^{2m} \overline{z}^m \tag{2.67}$$

and extend it linearly on the entire algebra  $\mathcal{A}_{nc}$ , we can write:

**Lemma 4.** For any two functions  $f, g \in \mathcal{A}_{nc}$  we have

$$\int_{-q}^{q} fg = \int_{-q}^{q} g\mathcal{D}(f) . \tag{2.68}$$

A cyclic property of this kind for invariant integrals of functions on q—deformed spaces was first derived by Harold Steinacker in [30], where a  $SO_q(N)$ —covariant space is considered.

We see that the integral we defined in (2.64) is not cyclic. Obviously the demand for  $U_q(e(2))$ —invariance spoils the trace property, whereas the demand for a cyclic integral implies loosing  $U_q(e(2))$ —invariance (as we saw in Subsection 2.2.5). We will comment on this "dilemma" later. The considerations made here are important if we want to write down an action that is invariant under gauge transformations later on. If we define the gauge transformation of a field strength  $f \in \mathcal{A}_{nc}$  in the usual way, i.e. by

$$f \mapsto \Omega f \Omega^{-1}$$
,  $\Omega$  unitary, (2.69)

we will now in the deformed case not get an invariant action because of (2.68). The only way out without abandoning the demand for an invariant integral is to define new gauge transformations, to "deform" them in a way that compensates the property (2.68). How this can be done will be treated later.

# 2.3.2 Covariant Differential Calculus on the $E_q(2)$ - Symmetric Plane

Differential calculus on quantum groups was first studied by Woronowicz [31]. For more details the reader is also referred to [32] and [33]. Calculi on noncommutative spaces that are covariant with respect to the coaction of a quantum group have been studied by Wess and Zumino in [34], where the quantum hyper plane, which is covariant with respect to the coaction of the quantum group  $GL_q(n)$ , is discussed. We will develop such a differential calculus that is covariant with respect to the coaction of  $E_q(2)$ .

#### q-One-Forms and q-Derivatives

We start with introducing variables dz and  $d\overline{z}$ , the q-differentials of z and  $\overline{z}$ . These are noncommutative differentials which do not commute with the space coordinates  $z, \overline{z}$ , either. How can we find the corresponding commutation relations? Certainly, the commutation relations have to be covariant with respect to the coaction of  $E_q(2)$  on  $\mathcal{A}_{nc}$  given in (2.7). Let me explain how this is to be understood: We get the coaction of  $E_q(2)$  on the q-differentials  $dz, d\overline{z}$  by applying d on the coaction of  $E_q(2)$  on  $z, \overline{z}$ , where we want to assume that d applied on elements of  $E_q(2)$  equals zero. In other words

$$\begin{array}{rcl} \rho(z) & = & e^{i\theta} \otimes z + t \otimes 1 \\ \rho(\overline{z}) & = & e^{-i\theta} \otimes \overline{z} + \overline{t} \otimes 1 \end{array}$$

leads to

$$\rho(dz) = e^{i\theta} \otimes dz 
\rho(d\overline{z}) = e^{-i\theta} \otimes d\overline{z} .$$

The requirement of covariance implies the commutation relations between coordinates and their differentials in the following sense: If we want to extend the algebra  $\mathcal{A}_{nc}$  by generating elements dz,  $d\overline{z}$  with the above coaction and if we want the newly generated algebra to be a left  $E_q(2)$ —comodule algebra, then  $\rho$  has to be a left coaction in the sense of Definition 3 on the extended algebra. One property of a coaction on an algebra is that the multiplication in the algebra has to be a

comodule homomorphism, i.e. it has to satisfy  $\rho(ab) = \rho(a)\rho(b)$  for all algebra elements a, b.<sup>10</sup> In our setting, it therefore has to be true that

$$\rho(zdz) = \rho(z)\rho(dz) = (e^{i\theta} \otimes z + t \otimes 1)(e^{i\theta} \otimes dz) = e^{2i\theta} \otimes zdz + te^{i\theta} \otimes dz.$$

On the other hand we find with the same argument that

$$\rho(dzz) = \rho(dz)\rho(z) = (e^{i\theta} \otimes dz)(e^{i\theta} \otimes z + t \otimes 1) = e^{2i\theta} \otimes dzz + e^{i\theta}t \otimes dz$$

has to be true, too. But in the quantum group  $E_q(2)$  we have  $te^{i\theta} = q^{-2}e^{i\theta}t$  (see commutation relations (2.1)). Thus,  $\rho$  as defined above is only well-defined if

$$zdz = q^{-2}dzz .$$

Analogously follow the remaining commutation relations:

$$\begin{array}{rcl}
 zdz & = & q^{-2}dzz & \overline{z}dz & = & q^{-2}dz\overline{z} \\
 zd\overline{z} & = & q^2d\overline{z}z & \overline{z}d\overline{z} & = & q^2d\overline{z}\overline{z} .
 \end{array}$$
(2.70)

We see that for  $q \to 1$  coordinates and differentials commute. Let us remark that these considerations do not lead to any information about the commutation relations of dz and  $d\overline{z}$  themselves, since  $e^{-i\theta}e^{i\theta}=e^{i\theta}e^{-i\theta}$ . How to find them anyway will be discussed below.

Now we can define an exterior differential  $d: \mathcal{A}_{nc} \to \Omega_q^1$  on the entire algebra  $\mathcal{A}_{nc}$ , where by  $\Omega_q^1$  we want to denote the space of q-deformed one-forms, by demanding

$$d(\text{const.}) = 0$$
,

as well as the Leibniz rule

$$d(fg) = (df)g + f(dg) (2.71)$$

for all elements f and g in  $\mathcal{A}_{nc}$ . To see that d is indeed well-defined we have to check whether it respects the commutation relations of the algebra, i.e. we need to examine whether

$$d(z\overline{z} - q^2\overline{z}z) \stackrel{!}{=} 0 . {(2.72)}$$

This follows immediately from (2.70) and (2.71).

In the next step we can introduce q-deformed partial derivatives by defining in analogy to the noncommutative case

$$d =: dz^i \partial_i . (2.73)$$

<sup>&</sup>lt;sup>10</sup>The remaining properties can be easily checked, too.

In (2.72) we saw that d is well-defined such that the partial derivatives  $\partial_z$  and  $\partial_{\overline{z}}$  are well-defined, too.

Remark about notation: We want to distinguish notationally on the one hand applying partial derivatives to a function and on the other hand understanding  $\partial_z$ ,  $\partial_{\overline{z}}$  as elements of the algebra and multiplying them by functions in the algebra itself. When applying  $\partial_z$ ,  $\partial_{\overline{z}}$  to a function, we always want to put the function the derivatives are applied on in brackets, i.e.

$$\partial_z(f)$$
 and  $\partial_{\overline{z}}(f)$ ,

whereas we won't use brackets when we interpret  $\partial_z$ ,  $\partial_{\overline{z}}$  as part of the algebra itself.

According to definition (2.73), we find for example for the q-deformed partial derivatives applied to the coordinates

$$\begin{array}{rcl}
\partial_z(z) &=& 1 & \partial_{\overline{z}}(z) &=& 0 \\
\partial_z(\overline{z}) &=& 0 & \partial_{\overline{z}}(\overline{z}) &=& 1
\end{array}$$
(2.74)

which is just what we expected. Nevertheless, the derivatives  $\partial_z$ ,  $\partial_{\overline{z}}$  do not satisfy the usual Leibniz rule but a modified one that we want to call the q-Leibniz rule. It can be derived from the Leibniz rule of the exterior differential together with the commutation relations of differentials and coordinates (2.70):

$$d(fg) = (df)g + f(dg)$$

$$= dz^{i}\partial_{i}(f)g + fdz^{i}\partial_{i}(g)$$

$$= dz^{i}\partial_{i}(f)g + dzf(q^{-2}z, q^{-2}\overline{z})\partial_{z}(g) + d\overline{z}f(q^{2}z, q^{2}\overline{z})\partial_{\overline{z}}(g) \quad (2.75)$$

where we have used the following commutation relations

$$\begin{array}{rcl} f(z,\overline{z})dz & = & dz f(q^{-2}z,q^{-2}\overline{z}) \\ f(z,\overline{z})d\overline{z} & = & d\overline{z} f(q^{2}z,q^{2}\overline{z}) \end{array},$$

which we want to prove:

*Proof.* From (2.58) we get that  $f(z,\overline{z}) = \sum_{m \in \mathbb{Z}} z^m f_m(z\overline{z})$ , where the  $f_m$  are formal power series for all m. Since  $z\overline{z}dz = q^{-4}dz(z\overline{z})$ , we deduce that  $f_m(z\overline{z})dz = dz f_m(q^{-4}z\overline{z})$ . Moreover,  $z^m dz = dz (q^{-2}z)^m$ , such that together follows  $f(z,\overline{z})dz = dz f(q^{-2}z,q^{-2}\overline{z})$ , the first claim. A similar calculation proves the second claim.

On the other hand we have by definition

$$d(fg) = dz\partial_z(fg) + d\overline{z}\partial_{\overline{z}}(fg) ,$$

such that with the result in (2.75) we conclude that

$$\partial_z(fg) = \partial_z(f)g + f(q^{-2}z, q^{-2}\overline{z})\partial_z(g) 
\partial_{\overline{z}}(fg) = \partial_{\overline{z}}(f)g + f(q^2z, q^2\overline{z})\partial_{\overline{z}}(g) .$$
(2.76)

Obviously, we obtain for  $q \to 1$  the usual commutative Leibniz rule just as expected. With the q-Leibniz rule it is now possible to derive the commutation relations of partial derivatives and space coordinates. Applying the q-Leibniz rule (2.76) on the function z = zf resp.  $\overline{z} = \overline{z}f$  and using the results in (2.74) we find

$$\begin{array}{rclcrcl} \partial_z(zf) & = & 1f + q^{-2}z\partial_z(f) & & \partial_z(\overline{z}f) & = & q^{-2}\overline{z}\partial_z(f) \\ \partial_{\overline{z}}(zf) & = & q^2z\partial_{\overline{z}}(f) & & \partial_{\overline{z}}(\overline{z}f) & = & 1f + q^2\overline{z}\partial_{\overline{z}}(f) \end{array}$$

which yields the following commutation relations

$$\begin{array}{rcl}
\partial_z z &=& 1 + q^{-2} z \partial_z & \partial_z \overline{z} &=& q^{-2} \overline{z} \partial_z \\
\partial_{\overline{z}} z &=& q^2 z \partial_{\overline{z}} & \partial_{\overline{z}} \overline{z} &=& 1 + q^2 \overline{z} \partial_{\overline{z}} \end{array} .$$
(2.77)

Furthermore, we find, applying  $\partial_z \partial_{\overline{z}}$  on the function  $z\overline{z}$  using the q-Leibniz rule, that

$$\partial_z \partial_{\overline{z}}(z\overline{z}) = \partial_z(q^2z) = q^2$$

whereas

$$\partial_{\overline{z}}\partial_z(z\overline{z}) = \partial_{\overline{z}}(\overline{z}) = 1$$
,

such that we derive the following commutation relation for the q-derivatives:

$$\partial_z \partial_{\overline{z}} = q^2 \partial_{\overline{z}} \partial_z \quad . \tag{2.78}$$

Again for  $q \to 1$  these commutation relations become those of the commutative theory. We still do not know the commutation relations for the q-differentials. To derive them, we need to extend the exterior differential to one-forms: we want to demand for the exterior differential d in addition to the Leibniz rule (2.71) that in analogy to the commutative case

$$d^2 \equiv 0$$

as well as

$$d(\alpha\beta) = (d\alpha)\beta + (-1)^{\deg\alpha}\alpha(d\beta) \tag{2.79}$$

for forms  $\alpha, \beta$ , where deg $\alpha$  denotes the degree of  $\alpha$ . Using this and applying d on the commutation relation  $dz\overline{z} = q^2\overline{z}dz$  (cf. 2.70), we find

$$dzd\overline{z} = -q^2d\overline{z}dz \quad , \tag{2.80}$$

becoming for  $q \to 1$  the usual commutation rule of differentials. Applying d to  $zdz = q^{-2}dzz$  respectively  $\overline{z}d\overline{z} = q^2d\overline{z}\overline{z}$  (2.70), we obtain

$$(dz)^2 = (d\overline{z})^2 = 0 .$$

To see that this extension of d is indeed well-defined, we remark that applying d to the remaining equation of (2.70), namely  $\overline{z}dz = q^{-2}dz\overline{z}$ , leads to (2.80), too.

For completeness we want to derive the commutation relations for q-differentials and q-derivatives. For this purpose we assume that

$$\partial_{z^i} dz^j = c_i^j dz^j \partial_{z^i}$$
 (no summation)

for some constants  $c_j^i$ . Multiplying for example  $\partial_z dz - c_1^1 dz \partial_z = 0$  by z from the right and commuting z through to the left using the above commutation relations (2.70) and (2.77) leads to the following:

$$0 = (\partial_z dz - c_1^1 dz \partial_z)z = q^2 \partial_z z dz = q^2 (1 + q^{-2} z \partial_z) dz - c_1^1 dz (1 + q^{-2} z \partial_z)$$
$$= (q^2 - c_1^1) dz + (1 - c_1^1 q^{-2}) z \partial_z dz$$

such that we conclude

$$c_1^1 = q^2 .$$

Similar calculations provide the remaining constants  $c_j^i$  and we get as result:

$$\begin{array}{rcl}
\partial_z dz & = & q^2 dz \partial_z & \partial_z d\overline{z} & = & q^{-2} d\overline{z} \partial_z \\
\partial_{\overline{z}} dz & = & q^2 dz \partial_{\overline{z}} & \partial_{\overline{z}} d\overline{z} & = & q^{-2} d\overline{z} \partial_{\overline{z}} .
\end{array} (2.81)$$

As we see, the q-differentials and q-derivatives become the classical differentials resp. derivatives in the semi-classical limit  $q \to 1$ .

Remark: Above we derived all possible types of commutation relations between coordinates, differentials and derivatives. Nevertheless, we still have to check if all these relations are consistent with each other. If we extend the algebra by  $dz, d\overline{z}$  with the relations (2.70) and (2.80), these relations have to be consistent with the commutation relation of the space coordinates  $z\overline{z} = q^2\overline{z}z$ , i.e. multiplying them

by coordinates from the right and commuting the coordinates to the left must not lead to any inconsistency. For example we have:

$$(zdz - q^{-2}dzz)z = z(q^2dzz - zdz)$$

$$(zdz - q^{-2}dzz)\overline{z} = \overline{z}(q^4zdz - q^2dzz)$$

obviously not leading to any inconsistency. Multiplying the remaining relations of (2.70) or (2.80) by coordinates from the right and commuting them through to the left, does not lead to inconsistencies, either. The reason is that the commutation relations are of the following type: Commuting two arbitrary elements does again reproduce exactly those two elements multiplied by some power of q. Then, multiplying any relation from the right by an element of the algebra and commuting it through to the left, must lead to the same relation we started with multiplied by a power of q and therefore never produces any inconsistency. In a second step we introduced the partial derivatives  $\partial_z$ ,  $\partial_{\overline{z}}$  with the relations (2.77),(2.78) and (2.81). A lot of the commutation relations are again of the type that commuting an arbitrary derivative with a differential or a coordinate reproduces exactly the same elements multiplied by a power of q. By the same consideration as above, at least all cases, in which the commutation relations  $\partial_z z = 1 + q^{-2} z \partial_z$ or  $\partial_{\overline{z}}\overline{z} = 1 + q^2\overline{z}\partial_{\overline{z}}$  do not enter the calculation, don't lead to inconsistencies. If these two commutation relations enter in the calculation, an additional term +1appears. Those cases remain to check. We easily verify

$$\begin{array}{rcl} (\partial_z z - 1 - q^{-2} z \partial_z) z & = & z q^{-2} (\partial_z z - 1 - q^{-2} z \partial_z) \\ (\partial_z z - 1 - q^{-2} z \partial_z) \overline{z} & = & \overline{z} (\partial_z z - 1 - q^{-2} z \partial_z) \\ (\partial_z z - 1 - q^{-2} z \partial_z) dz & = & dz (\partial_z z - 1 - q^{-2} z \partial_z) \\ (\partial_z z - 1 - q^{-2} z \partial_z) d\overline{z} & = & d\overline{z} (\partial_z z - 1 - q^{-2} z \partial_z) \end{array} .$$

A short calculation shows that we don't get inconsistencies for the other equation  $\partial_{\overline{z}} \overline{z} = 1 + q^2 \overline{z} \partial_{\overline{z}}$ , either. The last equation,  $\partial_z \partial_{\overline{z}} = q^2 \partial_{\overline{z}} \partial_z$ , still has to be multiplied by coordinates from the right commuting them through to the left since again +1 terms will enter:

$$(\partial_z \partial_{\overline{z}} - q^2 \partial_{\overline{z}} \partial_z) z = q^2 \partial_{\overline{z}} + \partial_z \partial_{\overline{z}} - q^2 \partial_{\overline{z}} - q^2 z \partial_{\overline{z}} \partial_z = z (\partial_z \partial_{\overline{z}} - q^2 \partial_{\overline{z}} \partial_z)$$

$$(\partial_z \partial_{\overline{z}} - q^2 \partial_{\overline{z}} \partial_z) \overline{z} = \partial_z + \overline{z} \partial_{\overline{z}} \partial_z - \partial_z - q^2 \overline{z} \partial_{\overline{z}} \partial_z = \overline{z} (\partial_z \partial_{\overline{z}} - q^2 \partial_{\overline{z}} \partial_z) .$$

All together we have seen that the calculus established above withstands all consistency checks.

The above considerations give in all detail an  $E_q(2)$ -covariant differential calculus on  $\mathcal{A}_{nc}$ . We saw that differentials and functions do not commute. This makes calculations complicated and therefore we will try in a next step to find a basis  $\theta^i$  of the space of one-forms, called *frame*, that commutes with all functions (Frames exist on other noncommutative spaces as well (cf. for example [25, 3]) and  $\theta^i$  is the usually used notation that is not to be mixed up with the Poisson structure  $\theta^{ij}$ !).

#### A Nicer Basis of One-Forms: A Frame

We are looking for a new basis  $\theta =: \theta^z, \overline{\theta} =: \theta^{\overline{z}}$  of  $\Omega_q^1$  commuting with all functions. Let us consider

$$\frac{\theta}{\overline{\theta}} := z^{-1}\overline{z}dz 
\overline{\theta} := d\overline{z}z\overline{z}^{-1}.$$
(2.82)

Then:

Lemma 5. We have

$$[\theta, f] = [\overline{\theta}, f] = 0 \tag{2.83}$$

for all functions  $f \in \mathcal{A}_{nc}$  and

$$\theta \overline{\theta} = -q^2 \overline{\theta} \theta$$
.

*Proof.* Using the commutation relations (2.8),(2.70), we get

$$\theta z = z^{-1} \overline{z} dz z = q^2 z^{-1} \overline{z} z dz = z \theta$$

and

$$\theta \overline{z} = z^{-1} \overline{z} dz \overline{z} = q^2 z^{-1} \overline{z} \overline{z} dz = \overline{z} \theta .$$

 $\theta\overline{\theta}=-q^2\overline{\theta}\theta$  follows making use of the same commutation relations.

It is even possible to find a one-form  $\Theta$  that generates the exterior differential.

#### A Generator for the Exterior Differential

In the following lemma we introduce a one-form  $\Theta$  that generates the exterior differential d. Later on,  $\Theta$  will play an important role when considering gauge fields and gauge transformations.

Lemma 6. Consider

$$\Theta := \theta^i \lambda_i := \theta \frac{1}{1 - q^{-2}} \overline{z}^{-1} - \overline{\theta} \frac{1}{1 - q^{-2}} z^{-1}$$

Then

$$df = [\Theta, f] = [\lambda_i, f]\theta^i$$

for all functions f and

$$d\Theta = \Theta^2 = 0 .$$

*Proof.* We remember that the  $\theta^i$  commute with all functions and with that we get

$$[\Theta, f] = \theta[\frac{1}{1 - q^{-2}}\overline{z}^{-1}, f] - \overline{\theta}[\frac{1}{1 - q^{-2}}z^{-1}, f]$$
.

Plugging in the explicit expressions (2.82) for  $\theta^i$  we find

$$\begin{aligned} [\Theta, f] &= z^{-1} \overline{z} dz \left[ \frac{1}{1 - q^{-2}} \overline{z}^{-1}, f \right] - d\overline{z} z \overline{z}^{-1} \left[ \frac{1}{1 - q^{-2}} z^{-1}, f \right] \\ &= dz z^{-1} \overline{z} \left[ \frac{1}{1 - q^{-2}} \overline{z}^{-1}, f \right] - d\overline{z} z \overline{z}^{-1} \left[ \frac{1}{1 - q^{-2}} z^{-1}, f \right] , \end{aligned}$$

where in the last step we used the commutation relations (2.70). Taking f = z and  $f = \overline{z}$  we get with the commutation relations of the space coordinates

$$z^{-1}\overline{z}\left[\frac{1}{1-q^{-2}}\overline{z}^{-1},z\right] = \frac{1}{1-q^{-2}} - \frac{q^{-2}}{1-q^{-2}} = 1$$
 (2.84)

and

$$z\overline{z}^{-1}[\frac{1}{1-q^{-2}}z^{-1},z]=0$$
 (2.85)

Thus,  $[\Theta, z] = dz$  and analogously follows  $[\Theta, \overline{z}] = d\overline{z}$ . Hence, on the generators of the algebra of functions the claim is true. Since  $[\Theta, f]$  is a derivation, we can now conclude that

$$\mathit{df} = [\Theta, f]$$

for all functions f.

The second claim,  $d\Theta = \Theta^2 = 0$ , is shown by the following calculations: First

$$\begin{split} ((1-q^{-2})\Theta)^2 &= (\theta \overline{z}^{-1} - \overline{\theta}z^{-1})^2 = (q^{-2}z^{-1}dz - \overline{z}^{-1}d\overline{z})^2 \\ &= -q^{-2}z^{-1}dz\overline{z}^{-1}d\overline{z} - \overline{z}^{-1}d\overline{z}q^{-2}z^{-1}dz \\ &= -q^{-4}z^{-1}\overline{z}^{-1}dzd\overline{z} - \overline{z}^{-1}z^{-1}d\overline{z}dz = 0 \ . \end{split}$$

where we have used the commutation relations (2.7), (2.70) and (2.80), and secondly

$$(1 - q^{-2})d\Theta = d(q^{-2}z^{-1}dz - \overline{z}^{-1}d\overline{z}) = -q^{-4}z^{-2}dzdz + q^{2}\overline{z}^{-2}d\overline{z}d\overline{z} = 0$$

where we used  $d(z^{-1}) = -q^{-2}z^{-2}dz$  and  $d(\overline{z}^{-1}) = -q^{2}\overline{z}^{-2}d\overline{z}$  which follows with the q-Leibniz rule applied on  $0 = d1 = d(zz^{-1}) = d(\bar{z}\bar{z}^{-1})$ .

Remark: We notice that

$$\Theta = \theta^i \lambda_i = q^{-2} z^{-1} dz - \overline{z}^{-1} d\overline{z}$$

is not invariant under translations. It is only invariant under rotations.

#### q-Two-Forms

We can also write the extension of the exterior differential d to one-forms in a nice form:

**Lemma 7.** Let  $\Theta$  be defined as in Lemma 6. Then

$$d\alpha = \{\Theta, \alpha\}$$

for any one-form  $\alpha$ . Here  $\{\cdot,\cdot\}$  denotes the anti-commutator.

Proof. We have  $\{\Theta, \alpha f\} = \{\Theta, \alpha\}f - \alpha[\Theta, f]$  and  $\{\Theta, f\alpha\} = [\Theta, f]\alpha + f\{\Theta, \alpha\}$  for arbitrary functions f and arbitrary one-forms  $\alpha$ . Thus,  $\{\Theta, \cdot\}$  satisfies the Leibniz-rule (2.79) which implies that  $d\alpha = \{\Theta, \alpha\}$ .

This finishes the discussion of the differential calculus and we will now attend to gauge fields and the action.

### 2.3.3 Gauge Fields and Action

We consider only the easiest case: noncommutative abelian gauge fields. They are one-forms  $B \in \Omega_q^1$ . We can express B in the basis  $\theta^i$  (found in the last subsection) that commutes with arbitrary functions:

$$B = B_i \theta^i .$$

Furthermore, we write [23]

$$B = \Theta + A \quad , \tag{2.86}$$

where, as we will see, A is the analog of the commutative gauge field, and define

$$F := B^2 (2.87)$$

the noncommutative field strength. With (2.86) and since  $\Theta^2 = 0$  (Lemma 6) and  $\{\Theta, A\} = dA$  (Lemma 7) we get

$$F = A^2 + dA (2.88)$$

In terms of components with respect to the frame  $\theta^i$  we have

$$F = (A_i \theta^i)^2 + d(A_i \theta^i)$$

$$= A_i A_j \theta^i \theta^j + \{\lambda_i \theta^i, A_j \theta^j\}$$

$$= A_1 A_2 \theta \overline{\theta} + A_2 A_1 \overline{\theta} \theta + (\lambda_1 A_2 + A_1 \lambda_2) \theta \overline{\theta} + (A_2 \lambda_1 + \lambda_2 A_1) \overline{\theta} \theta$$

$$= (A_1 A_2 - q^{-2} A_2 A_1) \theta \overline{\theta} + ((\lambda_1 A_2 - q^{-2} A_2 \lambda_1) - (q^{-2} \lambda_2 A_1 - A_1 \lambda_2)) \theta \overline{\theta}$$

where we used in the last step that  $\theta \overline{\theta} = -q^2 \overline{\theta} \theta$  (Lemma 5). If we additionally denote the components of A in the basis  $dz, d\overline{z}$  by  $A_z$  and  $A_{\overline{z}}$ , i.e.

$$A = A_1\theta + A_2\overline{\theta} = A_zdz + A_{\overline{z}}d\overline{z}$$

and therefore

$$A_1\theta = A_z dz$$

$$A_2\overline{\theta} = A_{\overline{z}}d\overline{z} ,$$

we can write

$$F = (A_1 A_2 - q^{-2} A_2 A_1) \theta \overline{\theta} + (\theta (\lambda_1 A_{\overline{z}} d\overline{z} - q^{-2} A_{\overline{z}} d\overline{z} \lambda_1) - (q^{-2} \lambda_2 A_z dz - A_z dz \lambda_2)) \overline{\theta}.$$

The explicit expressions of  $\lambda_i$  given in Lemma 6 and the commutation relations (2.70) yield  $\lambda_1 d\overline{z} = q^{-2} d\overline{z} \lambda_1$  and  $dz\lambda_2 = q^{-2} \lambda_2 dz$ . Thus, we obtain

$$F = (A_1 A_2 - q^{-2} A_2 A_1) \theta \overline{\theta} + ((\lambda_1 A_{\overline{z}} - A_{\overline{z}} \lambda_1) \theta d \overline{z} - q^{-2} (\lambda_2 A_z - A_z \lambda_2) dz \overline{\theta}$$

By Lemma 6 we get that

$$[\lambda_1, f]\theta = dz\partial_z(f)$$
 and

$$[\lambda_2, f]\overline{\theta} = d\overline{z}\partial_{\overline{z}}(f) .$$

Together with  $dz\overline{\theta} = -q^2\overline{\theta}dz$  we finally have

$$F = (A_1 A_2 - q^{-2} A_2 A_1) \theta \overline{\theta} + dz \partial_z (A_{\overline{z}}) d\overline{z} + d\overline{z} \partial_{\overline{z}} (A_z) dz .$$

Thereby the above expression for F approaches in the semi-classical limit  $q \to 1$  the following expression:

$$F \xrightarrow{q \to 1} (\partial_z A_{\overline{z}} - \partial_{\overline{z}} A_z) dz d\overline{z} . \tag{2.89}$$

Hence, we see that in fact the field strength defined as above becomes the commutative field strength (cf. Appendix A) in the semi-classical limit and A admits the interpretation of a noncommutative gauge field. Therefore (2.87) seems to be in this regard a reasonable definition for a noncommutative field strength. Of course gauge covariance is another important requirement for F. This issue will be treated in the after next subsection.

At this point we want to continue introducing the action. For this purpose we consider the  $Hodge\ dual\ *_HF\ of\ F$ : In our two-dimensional case we can express any two-form F in the basis  $\theta\overline{\theta}=z^{-1}\overline{z}dzd\overline{z}z\overline{z}^{-1}=q^{-2}dzd\overline{z}$ :

$$F = f\theta \overline{\theta} = q^{-2} f dz d\overline{z}$$

for some coefficient function f. Then we define  $*_H$  on two-forms as

$$*_H F := \frac{1}{2}f$$
 (2.90)

With the integral found in Subsection 2.3.1 we can now write down an action:

$$S := \int^{q} F(*_{H}F) = \int^{q} \frac{1}{2} f^{2} \theta \overline{\theta} .$$

It admits the right semi-classical limit for  $q \to 1$ :

$$S \stackrel{q \to 1}{\longrightarrow} \int \frac{1}{2} (\partial_z A_{\overline{z}} - \partial_{\overline{z}} A_z)^2 dz d\overline{z} .$$

We can even assign to S a gauge-invariance property for special "deformed" gauge transformations, as we will see later, but first we need the following subsection as preparation.

## 2.3.4 The Algebra Homomorphism $\alpha: U_q(e(2)) \to \mathcal{A}_{nc}$

We construct an algebra homomorphism  $\alpha: U_q(e(2)) \to \mathcal{A}_{nc}$  which we will need in the following subsection when defining gauge transformations in analogy to [23], where such a homomorphism is introduced, too (even though in a different way). Nevertheless, we do not look for an arbitrary algebra homomorphism. It shall have the additional property that the right-action  $x \triangleleft u$  of an arbitrary element  $u \in U_q(e(2))$  on an element  $x \in \mathcal{A}_{nc}$  is given by

$$\alpha(S(u_{(1)}))x\alpha(u_{(2)}) = x \triangleleft u .$$

This property we will need to obtain a gauge invariant action later on. For this purpose it is convenient to consider the cross product algebra  $U_q(e(2)) \ltimes \mathcal{A}_{nc}$  (cf. for example [20]) which is as vector space isomorphic to the tensor space  $U_q(e(2)) \otimes \mathcal{A}_{nc}$  and where multiplication is defined as follows:

$$(a \otimes x) \cdot (b \otimes y) := ab_{(1)} \otimes (x \triangleleft b_{(2)})y . \tag{2.91}$$

It is common to omit the tensor signs and to identify  $1 \otimes x \equiv x$  and  $a \otimes 1 \equiv a$ , leading with the multiplication as defined above to the so called commutation relation

$$x \cdot a = a_{(1)}(x \triangleleft a_{(2)})$$
.

In our case this reads, if we use the explicit form for coproduct and right-action of  $U_q(e(2))$  given in (2.50):

1. 
$$z \cdot T = T(z \triangleleft q^{2iJ}) + z \triangleleft T = q^{-2}Tz + 1$$

2. 
$$z \cdot \overline{T} = \overline{T}(z \triangleleft q^{2iJ}) + z \triangleleft \overline{T} = q^{-2}\overline{T}z$$

3. 
$$z \cdot J = Jz + z \triangleleft J = Jz + iz$$

$$4. \ \overline{z} \cdot T = T(\overline{z} \triangleleft q^{2iJ}) + \overline{z} \triangleleft T = q^2 T \overline{z}$$

5. 
$$\overline{z} \cdot \overline{T} = \overline{T}(\overline{z} \triangleleft q^{2iJ}) + \overline{z} \triangleleft \overline{T} = q^2 \overline{T} \overline{z} - q^2$$

6. 
$$\overline{z} \cdot J = J\overline{z} + \overline{z} \triangleleft J = J\overline{z} - i\overline{z}$$

Let us try to find an algebra homomorphism  $\alpha': U_q(e(2)) \ltimes \mathcal{A}_{nc} \to \mathcal{A}_{nc}$  that on  $\mathcal{A}_{nc}$  is the identity (in a second step such a  $\alpha'$  can be restricted to  $U_q(e(2))$  leading to an algebra homomorphism  $\alpha$  with all the demanded properties). Knowing the

above commutation relations we can explicitly construct  $\alpha'$ . Being an algebra homomorphism, for example implies because of 1. above:

$$z\alpha'(T) = \alpha'(z)\alpha'(T) \stackrel{!}{=} \alpha'(z \cdot T) = q^{-2}\alpha'(T)z + 1$$

which is equivalent to

$$z\alpha'(T) - q^{-2}\alpha'(T)z = 1 . (2.92)$$

Defining

$$\alpha'(T) := \frac{z^{-1}}{1 - q^{-2}}$$

we see that (2.92) is indeed satisfied. Similarly we find how to define  $a'(\overline{T})$ :

$$a'(\overline{T}) := \frac{\overline{z}^{-1}}{1 - q^{-2}} .$$

Finally, 3. above yields, if  $\alpha'$  is to be an algebra homomorphism with  $\alpha'|_{\mathcal{A}_{nc}} = \mathrm{id}|_{\mathcal{A}_{nc}}$ ,

$$z(\alpha'(J) - i) = \alpha'(J)z$$

implying

$$\alpha'(q^{-2iJ})z = q^{-2}z\alpha'(q^{-2iJ})$$

which is satisfied for

$$\alpha'(q^{-2iJ}) := z\overline{z} .$$

To show that  $\alpha'$  defined on the generators of  $U_q(e(2))$  as suggested above really gives an algebra homomorphism  $U_q(e(2)) \ltimes \mathcal{A}_{nc} \to \mathcal{A}_{nc}$ , we have to prove that the defining commutation relations of the generators in  $U_q(e(2))$ ,

$$\begin{split} T\overline{T} &= q^2\overline{T}T \\ [J,T] &= iT \iff q^{-2iJ}T = q^2Tq^{-2iJ} \\ [J,\overline{T}] &= -i\overline{T} \iff q^{-2iJ}\overline{T} = q^{-2}\overline{T}q^{-2iJ} \enspace, \end{split}$$

are respected, too. This can easily be checked using the commutation relation of the coordinates  $z\overline{z}=q^2\overline{z}z$ . For example

$$\alpha(T)\alpha(\overline{T}) := \frac{z^{-1}\overline{z}^{-1}}{1 - q^{-2}} = q^2 \frac{\overline{z}^{-1}z^{-1}}{1 - q^{-2}} = q^2 \alpha(\overline{T})\alpha(T)$$

and the rest follows similarly. Anyhow, as explained above, our aim is to find a homomorphism  $\alpha: U_q(e(2)) \to \mathcal{A}_{nc}$ . By restricting  $\alpha'$  to  $U_q(e(2))$  we obtain such an  $\alpha$ :

#### Lemma 8. The assignments

$$T \mapsto \frac{z^{-1}}{1 - q^{-2}}$$

$$\overline{T} \mapsto \frac{\overline{z}^{-1}}{1 - q^{-2}}$$

$$q^{-2iJ} \mapsto z\overline{z}$$

$$(2.93)$$

define an algebra homomorphism

$$\alpha: U_q(e(2)) \to \mathcal{A}_{\rm nc}$$

with the property

$$\alpha(S(u_{(1)}))x\alpha(u_{(2)}) = x \triangleleft u \tag{2.94}$$

for all  $u \in U_q(e(2))$  and  $x \in \mathcal{A}_{nc}$ .

*Proof.* As restriction of the algebra homomorphism  $\alpha'$  we get that  $\alpha$  is an algebra homomorphism. Moreover, the property  $\alpha(S(u_{(1)}))x\alpha(u_{(2)}) = x \triangleleft u$  follows directly because in the cross algebra  $U_q(e(2)) \ltimes \mathcal{A}_{nc}$  the right-action is given by

$$x \triangleleft u = S(u_{(1)}) \cdot x \cdot u_{(2)}$$

(just use the definition of the multiplication in the cross algebra (2.91) to verify this). Taking into account that  $\alpha'$  is an algebra homomorphism with  $\alpha'|_{\mathcal{A}_{nc}} = \mathrm{id}|_{\mathcal{A}_{nc}}$  we then have

$$x \triangleleft u = \alpha'(x \triangleleft u) = \alpha'(S(u_{(1)}) \cdot x \cdot u_{(2)}) = \alpha'(S(u_{(1)}))x\alpha'(u_{(2)}) = \alpha(S(u_{(1)}))x\alpha(u_{(2)}).$$

Remark: The element

$$T\overline{T}q^{-2iJ}$$

is a Casimir operator of the quantum group  $U_q(e(2))$ . This may give another hint how to define the algebra homomorphism  $\alpha$  mapping it on a constant in  $\mathcal{A}_{nc}$ , the only Casimir operator of  $\mathcal{A}_{nc}$ .

With the tools gathered so far, we are now ready to treat the topic gauge transformation and gauge invariance.

### 2.3.5 Gauge Transformations and Gauge Invariance

As we have seen in Subsection 2.3.1 the q-integral is not cyclic anymore. Therefore the usual gauge transformation  $B \to UBU^{-1}$  does not lead to a gauge invariance principle for  $q \neq 1$ . Thus, we have to modify gauge transformations so that for q = 1 we get the usual, commutative gauge transformation on the one hand and a reasonable gauge invariance principle for  $q \neq 1$  on the other hand.

#### Gauge Transformations

In analogy to the procedure in [23] we introduce the following gauge transformation of a one-form B:

$$B \to \alpha(S(\gamma_{(1)}))B\alpha(\gamma_{(2)})$$
 , (2.95)

with  $\gamma \in \mathcal{G}$ , where

$$\mathcal{G} := \{ \gamma \in U_q(e(2)) : \varepsilon(\gamma) = 1, \, \overline{\gamma} = S(\gamma) \}$$
 (2.96)

and  $\alpha: U_q(e(2)) \to \mathcal{A}_{nc}$  is the algebra homomorphism found in Subsection 2.3.4.  $\mathcal{G}$  is closed under multiplication and to draw analogy to the commutative case we write

$$\mathcal{G} = e^{\mathcal{H}}$$

where

$$\mathcal{H} := \{ \gamma \in U_q(e(2)) : \varepsilon(\gamma) = 0, \, \overline{\gamma} = S(\gamma) \}$$
.

The image  $\alpha(\mathcal{H})$  can be interpreted as a sub-algebra of the algebra of functions  $\mathcal{A}_{nc}$ .  $\overline{\gamma} = S(\gamma)$  reflects the unitarity condition of a commutative unitary gauge parameter and an arbitrary gauge parameter can be written as

$$\alpha(\gamma)(z,\overline{z}) = e^{i\alpha(h)(z,\overline{z})}$$
 with  $\overline{h} = -S(h)$ .

On account of the property (2.94) and since the  $\theta^i$  commute with arbitrary functions (2.83), the gauge transformation for the components reads

$$B_i \to \alpha(S(\gamma_{(1)}))B_i\alpha(\gamma_{(2)}) = B_i \triangleleft \gamma$$
.

With  $B = \Theta + A$  follows then for the gauge transformation of A:

$$\Theta + A \to \alpha(S(\gamma_{(1)}))(\Theta + A)\alpha(\gamma_{(2)})$$

and

$$\alpha(S(\gamma_{(1)}))(\Theta + A)\alpha(\gamma_{(2)}) = \alpha(S(\gamma_{(1)}))\alpha(\gamma_{(2)})\Theta + \alpha(S(\gamma_{(1)}))[\Theta, \alpha(\gamma_{(2)})]$$

$$+\alpha(S(\gamma_{(1)}))A\alpha(\gamma_{(2)})$$

$$= \alpha(\varepsilon(\gamma))\Theta + \alpha(S(\gamma_{(1)}))A\alpha(\gamma_{(2)}) + \alpha(S(\gamma_{(1)}))d\alpha(\gamma_{(2)})$$

$$= \Theta + \alpha(S(\gamma_{(1)}))A\alpha(\gamma_{(2)}) + \alpha(S(\gamma_{(1)}))d\alpha(\gamma_{(2)})$$

$$=: \Theta + A'$$

where we have used the fact that  $\Theta$  generates the exterior differential d (Lemma 6), that  $\alpha$  is an algebra homomorphism (Lemma 8) and that  $S(\gamma_{(1)})\gamma_{(2)} = \varepsilon(\gamma) = 1$  for  $\gamma \in \mathcal{G}$ . Hence, we conclude:

$$A \to A' = \alpha(S(\gamma_{(1)}))A\alpha(\gamma_{(2)}) + \alpha(S(\gamma_{(1)}))d(\alpha(\gamma_{(2)})) . \tag{2.97}$$

#### Gauge Invariance

The field strength defined in (2.87) as  $F := B^2$  transforms "gauge-covariantly":

$$F \rightarrow \alpha(S(\gamma_{(1)_{(1)}}))B\alpha(\gamma_{(1)_{(2)}})\alpha(S(\gamma_{(2)_{(1)}}))B\alpha(\gamma_{(2)_{(2)}})$$

$$= \alpha(S(\gamma_{(1)}))B\varepsilon(\gamma)B\alpha(\gamma_{(2)})$$

$$= \alpha(S(\gamma_{(1)}))F\alpha(\gamma_{(2)}) ,$$

since  $\varepsilon(\gamma) = 1$  for  $\gamma \in \mathcal{G}$ . If we write  $F = f\theta\overline{\theta}$  this yields, since  $\theta$  and  $\overline{\theta}$  commute with all functions,

$$f \to \alpha(S(\gamma_{(1)}))f\alpha(\gamma_{(2)})$$
 (2.98)

We already saw in (2.89) that F has the right semi-classical limit such that we really gave a reasonable definition for F.

Moreover, the action  $S := \int^q F(*_H F)$  is gauge invariant: As defined in (2.90),  $*_H F = \frac{1}{2}f$ . Therefore, we get with (2.98) that

$$*_H F \to \alpha(S(\gamma_{(1)}))(*_H F)\alpha(\gamma_{(2)})$$

for gauge transformations. Thus,

$$S \to \int^{q} \alpha(S(\gamma_{(1)})) F(*_{H}F) \alpha(\gamma_{(2)})$$
$$= \int^{q} \alpha(S(\gamma_{(1)})) \frac{1}{2} f^{2} \alpha(\gamma_{(2)}) \theta \overline{\theta} .$$

Since the homomorphism  $\alpha$  has the property  $\alpha(S(u_{(1)}))x\alpha(u_{(2)}) = x \triangleleft u$  for all  $x \in \mathcal{A}_{nc}$  and  $u \in U_q(e(2))$  (Lemma 8) and the integral is invariant with respect to the action of  $U_q(e(2))$  (cf. Subsection 2.3.1), we obtain

$$S \to \int^{q} (\frac{1}{2} f^{2} \triangleleft \gamma) q^{-2} dz d\overline{z} = \varepsilon(\gamma) \int^{q} \frac{1}{2} f^{2} \theta \overline{\theta} = \int^{q} F(*_{H}F) = S .$$

Therefore, the action is indeed gauge invariant.

#### The Semi-classical Limit $q \rightarrow 1$

We consider the first contribution of a gauge transformation of a one-form A given in (2.97):

$$\begin{array}{lcl} \alpha(S(\gamma_{(1)}))A\alpha(\gamma_{(2)}) & = & \alpha(S(\gamma_{(1)}))\alpha(\gamma_{(2)})A + \alpha(S(\gamma_{(1)}))[A,\alpha(\gamma_{(2)})] \\ & = & \varepsilon(\gamma)A + \alpha(S(\gamma_{(1)}))[A,\alpha(\gamma_{(2)})] = A + \alpha(S(\gamma_{(1)}))[A,\alpha(\gamma_{(2)})] \end{array}$$

for  $\gamma \in \mathcal{G}$ . In the limit  $q \to 1$ , the commutator vanishes and we conclude

$$\alpha(S(\gamma_{(1)}))A\alpha(\gamma_{(2)}) \xrightarrow{q \to 1} A$$
.

On the other hand, the second term in the transformation of A reads with  $df = dz^i \partial_{z^i}(f)$  (Lemma 6):

$$\alpha(S(\gamma_{(1)}))d(\alpha(\gamma_{(2)})) = \alpha(S(\gamma_{(1)}))dz^{i}\partial_{z^{i}}(\alpha(\gamma_{(2)})) . \qquad (2.99)$$

Let us calculate an example to get a better understanding of these deformed gauge transformations. If we take  $\gamma = q^{2J}$ , we first see that  $\varepsilon(\gamma) = 1$  and  $\overline{\gamma} = q^{-2J} = S(\gamma)$  (see (2.45) and (2.47)). Thus,  $\gamma$  is an element of  $\mathcal{G}$ . Using the structure maps of J given in (2.45), we get:

$$(S \otimes 1)\Delta \gamma = q^{-2J} \otimes q^{2J} . {2.100}$$

Since  $\alpha(q^{-2iJ}) = z\overline{z}$ , we obtain with  $q = e^h$  that  $\alpha(-2ihJ) = \ln(z\overline{z})$ . Hence

$$\alpha(q^{2J}) = e^{i\ln(z\overline{z})}$$

and we finally obtain with (2.100):

$$\alpha(S(\gamma_{(1)}))d(\alpha(\gamma_{(2)})) = e^{-i\ln(z\overline{z})}d(e^{i\ln(z\overline{z})})$$
.

For  $q \to 1$  this becomes

$$id(\ln(z\overline{z})) = i\partial_{z^i}(\ln(z\overline{z}))dz^i$$
.

Altogether we see that, if we write  $\lambda := \ln(z\overline{z})$ , the gauge transformation of A becomes in the semi-classical limit  $q \to 1$ :

$$A \to A' = A + \partial_{z^i} \lambda(z, \overline{z}) dz^i$$

and this is indeed a commutative gauge transformation. This gives us an impression of the semi-classical of the q-deformed gauge transformations.

Nevertheless, this is not fully satisfying. We still have to examine whether the same result follows for an arbitrary gauge parameter  $\lambda$ . Unfortunately, it was not possible to give a general answer in the framework of this thesis. Further considerations at this point are missing and could be part of ongoing research on this topic.

However, we want to add some general considerations for the interested reader. The gauge group we considered so far is not the most general choice. It is possible to make the following generalization for the "gauge group"  $\mathcal{G}$  [35]:

Let  $\mathcal{G}$  be a subset of a Hopf algebra H such that the following requirements are satisfied:

1. There exists an algebra homomorphism

$$\alpha: H \to \mathcal{A}_{nc}$$
.

2. For all  $U \in \mathcal{G}$  we have

$$\mathcal{D}(\alpha(U)) = \alpha(S^{-2}(U)) , \qquad (2.101)$$

with  $\mathcal{D}$  as defined in (2.67).

3. For all  $U \in \mathcal{G}$  we have  $\varepsilon(U) = 1$ .

Let us point out that in particular  $H = U_q(e(2))$  and  $\mathcal{G}$  as defined in (2.96) satisfy those demands. We just have to use the homomorphism  $\alpha$  constructed in Subsection 2.3.4 to check that the second condition is satisfied: It is easy to verify that

$$S^{2}(U) = q^{2iJ}Uq^{-2iJ} \iff S^{-2}(U) = q^{-2iJ}Uq^{2iJ}$$

for all  $U \in U_q(e(2))$ . Thus, we have to check whether

$$\alpha(S^{-2}(U)) = \alpha(q^{-2iJ})\alpha(U)\alpha(q^{2iJ}) \stackrel{!}{=} \mathcal{D}(\alpha(U)) .$$

Since  $\alpha(q^{-2iJ}) = z\overline{z}$  and  $z\overline{z}f(z,\overline{z})(z\overline{z})^{-1} = f(q^{-2}z,q^2\overline{z}) = \mathcal{D}(f(z,\overline{z}))$  for all  $f \in \mathcal{A}_c$ , this indeed is true.

Defining gauge transformations again as the adjoint action of an element  $U \in \mathcal{G}$ , i.e.

$$B \to \alpha(S(U_{(1)}))B\alpha(U_{(2)})$$
,

we obtain that the action  $S := \int^q F(*_H F)$  is gauge invariant:

$$S \rightarrow \int^{q} \alpha(S(\gamma_{(1)}))F(*_{H}F)\alpha(\gamma_{(2)})$$

$$= \int^{q} F(*_{H}F)\alpha(\gamma_{(2)})\mathcal{D}(\alpha(S(\gamma_{(1)})))$$

$$= \int^{q} F(*_{H}F)\alpha(\gamma_{(2)})(\alpha(S^{-1}(\gamma_{(1)})))$$

$$= \varepsilon(\gamma) \int^{q} F(*_{H}F) = S ,$$

where we used in the second line the cyclic property of the q-integral (Lemma 4) and in the last but one step the second property from above.

The hope is to find by this generalization a gauge group  $\mathcal{G}$  that is big enough together with an adequate homomorphism  $\alpha$  such that we get in the semi-classical limit  $q \to 1$  classical gauge transformations even for arbitrary gauge parameters.

## 2.3.6 Summary and Last Remarks

In this section we established an  $E_q(2)$ —covariant gauge field theory. First we found a  $U_q(e(2))$ —invariant integral. This integral is uniquely determined by the invariance requirement. We saw that unfortunately this integral does not admit a trace property (cf. (2.68)) such that we already concluded at this point, that defining gauge transformations of a field strength as usual by conjugation with an unitary element, will not lead to an invariant action. We further studied the properties of the  $E_q(2)$ —symmetric space developing a covariant differential calculus in all detail. To make calculations easier, a convenient basis of one forms, a frame, was introduced as well as a generator for the q—deformed exterior differential. This made it possible to talk about one- and two-forms in a nice

language, finally enabling us to define a gauge field A and a field strength F that becomes the commutative field strength in the semi-classical limit  $q \to 1$ . At this point gauge transformations still have not been introduced but after having established an algebra homomorphism  $\alpha: U_q(e(2)) \to \mathcal{A}_{\mathrm{nc}}$  with the property  $f \triangleleft X = \alpha(S(X_{(1)}))f\alpha(X_{(2)})$  for all  $X \in U_q(e(2))$  and  $f \in \mathcal{A}_{nc}$  we defined gauge transformations as the action of elements of a certain sub-algebra of  $U_q(e(2))$ , that we considered as the gauge group, on functions in  $\mathcal{A}_{nc}$  (see 2.95). The homomorphism  $\alpha$  gave the gauge group an interpretation as functions in  $\mathcal{A}_{\rm nc}$  and the way of defining gauge transformations itself led to the notion of an gauge invariant action that is  $E_q(2)$  – invariant as well. At the end, we tried to give the q – deformed gauge transformations a classical interpretation in the limit  $q \to 1$ . Although the explanation is surely not fully satisfying it leads us to the assumption that we indeed introduced reasonable gauge transformations which become ordinary gauge transformations for  $q \to 1$ . Nevertheless, more detailed considerations are certainly missing at this point and give incentive to ongoing research on this topic. Moreover, future work could be to develop some analogous of the Seiberg-Witten maps we considered in Section 2.2. The aim would be to express noncommutative quantities in terms of the commutative ones in such a way, that commutative gauge transformations of the commutative quantities induce the "deformed" gauge transformations we considered in this section. If such a Seiberg-Witten map exists, it should be possible, together with the star-product formalism developed in the previous section, to express all noncommutative gauge fields in terms of the commutative ones and expand them in the deformation parameter similarly as we did in Section 2.2. This would allow us to read off explicitly order by order the corrections this noncommutative theory predicts for the commutative theory, just as in the previous section but this time for a theory that respects the background symmetry of the space.

However, before solving this problem we surely have to understand better the nature of the "deformed" symmetry transformations we introduced. Since elements in  $U_q(e(2))$  are in general not group-like, we get that for example the gauge transformation of a product of functions implies transforming each factor differently because of  $fg \triangleleft X = (f \triangleleft X_{(1)})(g \triangleleft X_{(2)})$ . This curious transformation property has to be reconciled with the physical interpretation.

## Chapter 3

## Conclusion

We studied in the first chapter how far a change of a star product changes the resulting action in the case of canonical deformation of the space, i.e. in the case of a constant Poisson tensor  $\theta^{ij}$ . The result is, that in general two different star products will lead to two different actions but that for a large class of star products the corresponding actions indeed end up to be equal. The presumption is that a physical reason implies a restriction of all allowed ordering prescriptions and that we are obliged to consider only particular star products that fit in this physical requirement. For those star products, the actions would all be equal. Thus, the gauge field theory developed using the Moyal-Weyl star product would actually be independent of the choice of star product within this set.

In the second part of this thesis two approaches to gauge field theory on the  $E_q(2)$ —symmetric plane were established. The first, as generalization of the theory developed in the case of a constant Poisson structure, admits the expression of the noncommutative physical fields in terms of the commutative ones. An explicit expansion of the action in orders of the deformation parameter h leads to new interactions that do not exist in the commutative theory. However, a freedom in defining the gauge field and the field strength could not be avoided. Moreover, the fact that we introduced infinitesimal gauge transformations of a gauge field by the star commutator with the gauge parameter, forced us to introduce an integral with trace property which in turn is not  $E_q(2)$ —invariant.

In a second approach we set up a theory that is  $E_q(2)$ —covariant constructing an  $E_q(2)$ —invariant integral and an  $E_q(2)$ —differential calculus. We defined a noncommutative field strength which approaches the classical field strength in the semi-classical limit  $q \to 1$ . It is a result of the demand for  $E_q(2)$ —covariance of the theory that the integral is not cyclic. That is why we were forced to 3. Conclusion

introduce q—deformed gauge transformations. We found a gauge invariant action and discussed the semi-classical limit of gauge transformations.

The conclusion of the work in the second chapter is the following: If we establish gauge field theories on quantum spaces we encounter the following dilemma: Either we introduce gauge transformations of gauge fields as conjugation with an unitary element. In this case we have to introduce a cyclic integral, which will in general not be invariant under the quantum group symmetry, to get a gauge invariant action. Or we consider the quantum group covariance as the fundamental concept. We then have to construct an invariant integral which at the end is in general not cyclic, forcing us to define deformed gauge transformations that are difficult to interpret.

In this thesis we exemplified this dilemma discussing two approaches to establish gauge field theory on the  $E_q(2)$  – covariant plane. We pointed out the conceptual problems we meet in each case and presented possible solutions.

## Appendix A

## Change of basis $z, \overline{z} \leftrightarrow x, y$

Let us define a change of basis by

$$\phi(z,\overline{z}) = \begin{pmatrix} \frac{1}{2}(z+\overline{z}) \\ \frac{1}{2}(z-\overline{z}) \end{pmatrix} =: \begin{pmatrix} x \\ y \end{pmatrix} . \tag{A.1}$$

Then one-forms transform as follows:

$$A := \tilde{a}_{i}(x,y)dx^{i}$$

$$= \tilde{a}_{i}(\phi(z,\overline{z}))\frac{\partial \phi^{i}}{\partial z^{j}}dz^{j}$$

$$= \frac{1}{2}(\tilde{a}_{1}(\phi(z,\overline{z})) - i\tilde{a}_{2}(\phi(z,\overline{z}))dz + \frac{1}{2}(\tilde{a}_{1}(\phi(z,\overline{z})) + i\tilde{a}_{2}(\phi(z,\overline{z}))d\overline{z}$$

$$=: a_{z}dz + a_{\overline{z}}d\overline{z}$$

such that we conclude:

$$a_{z}(z,\overline{z}) = \frac{1}{2} \{ \tilde{a}_{1}(\phi(z,\overline{z})) - i\tilde{a}_{2}(\phi(z,\overline{z})) \} a_{\overline{z}}(z,\overline{z}) = \frac{1}{2} \{ \tilde{a}_{1}(\phi(z,\overline{z})) + i\tilde{a}_{2}(\phi(z,\overline{z})) \} .$$
(A.2)

Two-forms transform in the following way:

$$F := \tilde{f}_{ij}(x,y)dx^{i} \wedge dx^{j}$$

$$= \tilde{f}_{ij}(\phi(z,\overline{z}))\frac{\partial \phi^{i}}{\partial z^{k}}dz^{k} \wedge \frac{\partial \phi^{j}}{\partial z^{l}}dz^{l}$$

$$= (\frac{1}{4}i\tilde{f}_{12}(\phi(z,\overline{z})) - \frac{1}{4}i\tilde{f}_{21}(\phi(z,\overline{z})))dz \wedge d\overline{z} + (-\frac{1}{4}i\tilde{f}_{12}(\phi(z,\overline{z})) + \frac{1}{4}i\tilde{f}_{21}(\phi(z,\overline{z})))$$

$$=: f_{z\overline{z}}dz \wedge d\overline{z} + f_{\overline{z}z}d\overline{z} \wedge dz .$$

Therefore we get

$$\begin{array}{rcl} f_{z\overline{z}} & = & \frac{1}{4}i\tilde{f}_{12}(\phi(z,\overline{z})) - \frac{1}{4}i\tilde{f}_{21}(\phi(z,\overline{z})) \\ f_{\overline{z}z} & = & -\frac{1}{4}i\tilde{f}_{12}(\phi(z,\overline{z})) + \frac{1}{4}i\tilde{f}_{21}(\phi(z,\overline{z})) \end{array}.$$

This becomes, if  $\tilde{f}_{ij}$  is antisymmetric

$$f_{z\overline{z}} = \frac{1}{2}i\tilde{f}_{12}(\phi(z,\overline{z}))$$
  

$$f_{\overline{z}z} = -\frac{1}{2}i\tilde{f}_{12}(\phi(z,\overline{z})) .$$
(A.3)

In Chapter 2, we need to know  $f_{z\overline{z}}$  for the case where  $\tilde{f}$  is the commutative, abelian field strength, i.e.  $\tilde{f}_{12}(x,y) = \partial_x \tilde{a}_2 - \partial_y \tilde{a}_1$ . To calculate  $f_{z\overline{z}}$  we first want to express the partial derivatives  $\partial_x$  and  $\partial_y$  in terms of partial derivatives with respect to z = x + iy and  $\overline{z} = x - iy$ :

$$\partial_x = \frac{\partial z}{\partial x} \partial_z + \frac{\partial \overline{z}}{\partial x} \partial_{\overline{z}} = \partial_z + \partial_{\overline{z}} \\ \partial_y = \frac{\partial z}{\partial y} \partial_z + \frac{\partial \overline{z}}{\partial y} \partial_{\overline{z}} = i \partial_z - i \partial_{\overline{z}} .$$

Using this we can calculate

$$f_{z\overline{z}} = \frac{1}{2}i\tilde{f}_{12}(\phi(z,\overline{z}))$$

$$= \frac{1}{2}i((\partial_z + \partial_{\overline{z}})\tilde{a}_2(\phi(z,\overline{z})) - (i\partial_z - i\partial_{\overline{z}})\tilde{a}_1(\phi(z,\overline{z})))$$

$$= i((\partial_z + \partial_{\overline{z}})\tilde{a}_2(\phi(z,\overline{z})) + i(\partial_z - \partial_{\overline{z}})\tilde{a}_1(\phi(z,\overline{z})))$$

$$= \frac{1}{2}\partial_z(\tilde{a}_1(\phi(z,\overline{z})) + i\tilde{a}_2(\phi(z,\overline{z}))) + \frac{1}{2}\partial_{\overline{z}}(\tilde{a}_1(\phi(z,\overline{z})) - i\tilde{a}_2(\phi(z,\overline{z})))$$

$$= \partial_z a_{\overline{z}} - \partial_{\overline{z}} a_z .$$
(A.4)

## Appendix B

# The Right-Action of $U_q(e(2))$ on $\mathcal{A}_{nc}$

Knowing the structure maps for  $J, T, \overline{T} \in U_q(e(2))$  (see (2.45)) and their action on  $z, \overline{z}$  (see (2.50)) we can determine the action of  $J, T, \overline{T}$  on arbitrary functions using  $(xy) \triangleleft A = (x \triangleleft A_{(1)})(y \triangleleft A_{(2)})$  for arbitrary  $x, y \in \mathcal{A}_{nc}$ ,  $A \in U_q(e(2))$  (cf. Definition 5). Since an arbitrary function  $f(z, \overline{z}) \in \mathcal{A}_{nc}$  can be decomposed in  $f(z, \overline{z}) = \sum_{k \in \mathbb{Z}} z^k f_k(z\overline{z})$  (see (2.55)) it is sufficient to know the action on a summand

$$z^k f(z\overline{z})$$
,

where f is a formal power series in  $z\overline{z}$ . We will derive the formulas even for negative powers of  $z\overline{z}$ , i.e.  $f(z\overline{z}) = \sum_{l \in \mathbb{Z}} a_l(z\overline{z})^l$ . We start with the action on  $z^k$ :

Claim 1. For  $k \in \mathbb{Z}$  we have

$$z^{k} \triangleleft T = \frac{1 - q^{-2k}}{1 - q^{-2}} z^{k-1}$$

$$z^{k} \triangleleft \overline{T} = 0$$

$$z^{k} \triangleleft J = i^{k} z^{k} .$$
(B.1)

*Proof.* The first equation can be shown by induction. Let us start to prove it for k > 0:

- We have  $z \triangleleft T = 1$  such that the claim is true for k = 1.
- Supposing the claim to be true for k we find for k+1, using  $\Delta(T) = T \otimes q^{2iJ} + 1 \otimes T$  as well as the actions of T and J on z given in (2.50), we

get

$$\begin{split} z^{k+1} \triangleleft T &= (z^k \triangleleft T)(z \triangleleft q^{2iJ}) + z^k(z \triangleleft T) \\ &= \frac{1 - q^{-2k}}{1 - q^{-2}} z^{k-1} q^{-2} z + z^k \\ &= \frac{1 - q^{-2k+1}}{1 - q^{-2}} z^k \ . \end{split}$$

Thus, the claim is proved for k > 0. But for k = 0 we get  $1 \triangleleft T = 0 = \frac{1-q^0}{1-q^{-2}}$  and the claim is true in this case, too. For k < 0 we first of all need to know how T acts on  $z^{-1}$ . We have

$$0 = 1 \, \triangleleft \, T = (z^{-1}z) \, \triangleleft \, T = (z^{-1} \, \triangleleft \, T)(z \, \triangleleft \, q^{2iJ}) + z^{-1}(z \, \triangleleft \, T) = (z^{-1} \, \triangleleft \, T)q^{-2}z + z^{-1}$$

such that we conclude

$$z^{-1} \triangleleft T = -q^2 z^{-2}$$
.

This is consistent with (B.1) since in fact  $-q^2 = \frac{1-q^2}{1-q^{-2}}$ . An induction as done for the case k > 0 finally proves the claim for k < 0, too and with that follows the claim for  $k \in \mathbb{Z}$ .

The last two equations finally follow immediately with  $z \triangleleft \overline{T} = 0$ ,  $z \triangleleft J = iz$  and  $\Delta(\overline{T}) = \overline{T} \otimes q^{2iJ} + 1 \otimes \overline{T}$  first for k > 0 and then with  $1 \triangleleft \overline{T} = 0 = z^{-1} \triangleleft \overline{T}$  also for  $k \leq 0$ .

To determine the action on  $f(z\overline{z}) = \sum_{l \in \mathbb{Z}} a_l(z\overline{z})^l$  we start considering the action on a summand  $(z\overline{z})^l$ :

### Claim 2. For $l \in \mathbb{Z}$ we have

$$(z\overline{z})^{l} \triangleleft T = q^{2} \frac{1 - q^{-2l}}{1 - q^{-2}} (z\overline{z})^{l-1} \overline{z}$$

$$(z\overline{z})^{l} \triangleleft \overline{T} = -q^{2} \frac{1 - q^{2l}}{1 - q^{2}} (z\overline{z})^{l-1} z$$

$$(z\overline{z})^{l} \triangleleft J = (z\overline{z})^{l}.$$
(B.2)

*Proof.* The last equation follows immediately with  $z \triangleleft J = iz$ ,  $\overline{z} \triangleleft J = -i\overline{z}$  and  $\Delta(J) = J \otimes 1 + 1 \otimes J$ . The first equation follows again by induction. We start treating the case l > 0:

- We have  $(z\overline{z}) \triangleleft T = (z \triangleleft T)(\overline{z} \triangleleft q^{2iJ}) + z(\overline{z} \triangleleft T) = q^2\overline{z}$  and the claim is proved for l = 1.
- Supposing the claim to be true for l we get, using as in the previous proof that  $\Delta(T) = T \otimes q^{2iJ} + 1 \otimes T$  as well as the actions of T and J on z given in (2.50),

$$\begin{split} (z\overline{z})^{l+1} \triangleleft T &= ((z\overline{z})^l \triangleleft T)((z\overline{z}) \triangleleft q^{2iJ}) + (z\overline{z})^l ((z\overline{z}) \triangleleft T) \\ &= q^2 \frac{1 - q^{-2l}}{1 - q^{-2}} (z\overline{z})^{l-1} \overline{z} (z\overline{z}) + (z\overline{z})^l q^2 \overline{z} \\ &= q^2 \frac{1 - q^{-2l-2}}{1 - q^{-2}} (z\overline{z})^l \overline{z} \end{split}$$

such that the claim follows for l+1.

If l = 0, then  $1 \triangleleft T = 0$ , which is consistent with the claim. To derive the action of T on  $(z\overline{z})^{-1}$  we calculate

$$0 = ((z\overline{z})^{-1}(z\overline{z})) \triangleleft T = ((z\overline{z})^{-1} \triangleleft T) z\overline{z} + (z\overline{z})^{-1} ((z\overline{z}) \triangleleft T) = ((z\overline{z})^{-1} \triangleleft T) z\overline{z} + (z\overline{z})^{-1} q^2 \overline{z}$$

such that we can conclude

$$(z\overline{z})^{-1} \triangleleft T = -(z\overline{z})^{-2}\overline{z}$$

which is consistent with (B.2), too. For l < 0 the claim then follows by induction similar as for the case l > 0.

Finally, the second equation follows by a similar induction as done for the first equation.

Putting these results together and using  $f(z\overline{z}) = \sum_{l \in \mathbb{Z}} a_l(z\overline{z})^l$  we obtain

$$\begin{split} z^k f(z\overline{z}) \, \triangleleft \, T &= (z^k \, \triangleleft \, T)(f(z\overline{z}) \, \triangleleft \, q^{2iJ}) + z^k (f(z\overline{z}) \, \triangleleft \, T) \\ &= \frac{1 - q^{-2k}}{1 - q^{-2}} z^{k-1} f(z\overline{z}) + z^k \sum_{l \in \mathbb{Z}} a_l q^2 \frac{1 - q^{-2l}}{1 - q^{-2}} (z\overline{z})^{l-1} \overline{z} \\ &= \frac{1 - q^{-2k}}{1 - q^{-2}} z^{k-1} f(z\overline{z}) + z^{k-1} \sum_{l \in \mathbb{Z}} a_l q^2 \frac{1 - q^{-2l}}{1 - q^{-2}} q^{2(l-1)} (z\overline{z})^l \\ &= \frac{z^{k-1}}{1 - q^{-2}} ((1 - q^{-2k}) f(z\overline{z}) + \sum_{l \in \mathbb{Z}} a_l (q^{2l} - 1) (z\overline{z})^l) \\ &= \frac{z^{k-1}}{1 - q^{-2}} (f(q^2 z\overline{z}) - q^{-2k} f(z\overline{z})) , \end{split}$$

where we used the commutation relation  $z\overline{z}=q^2\overline{z}z$ , too. A similar calculation finally leads to the following results:

$$z^{k} f(z\overline{z}) \triangleleft T = \frac{z^{k-1}}{1 - q^{-2}} (f(q^{2}z\overline{z}) - q^{-2k}f(z\overline{z}))$$

$$z^{k} f(z\overline{z}) \triangleleft \overline{T} = \frac{q^{4}}{1 - q^{2}} z^{k+1} \frac{f(z\overline{z}) - f(q^{-2}z\overline{z})}{z\overline{z}}$$

$$z^{k} f(z\overline{z}) \triangleleft J = i^{k} z^{k} f(z\overline{z}) .$$
(B.3)

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# Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und unter ausschließlicher Verwendung der angegebenen Quellen und Hilfsmittel verfasst habe.