Quantum Dissipative Dynamics of Entanglement in the Spin-Boson Model

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We study quantum dissipative dynamics of entanglement in the spin-boson model, described by the generalized master equation. We consider the two opposite limits of pure-dephasing and relaxation models, measuring the degree of entanglement with the concurrence. When the Markovian approximation is employed, entanglement is shown to decay exponentially in both cases. On the other hand, non-Markovian contributions alter the analytic structure of the master equation, resulting in logarithmic decay in the pure dephasing model.

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The dynamical properties of an open quantum system are conveniently characterized by its decoherence times, viz, how long the system can maintain coherence in a given superposition state. It is customary and useful to distinguish further two limiting decoherence processes, relaxation and dephasing, although realistic decoherence phenomena appear with both processes admixed. In many cases of quantum information processing, a quantum device consists of two-level systems, for which rates of the two processes are designated by $1/T_1$ and $1/T_2$, respectively, in the convention of nuclear magnetic resonance. When the system can be partitioned into two parts, one can exploit another interesting property of the system, the entanglement between the two parts. As highlighted by the EPR paradox [2] and Bell's theorem [1, 3], entanglement has been regarded as one of the most fundamental properties of quantum physics. In recent decades, entanglement has attracted renewed interest as a key resource for quantum information processing such as quantum dense coding [4], quantum teleportation [5], and quantum computation [6]. Since the first experimental generation of the entangled photon pairs [7], a great number of researches have been devoted to the reliable creation and maintenance of the entanglement, e.g., between far-separated photons [8], between solid-state qubits [9, 10], and between macroscopic atomic ensembles [11].

A natural question is then how the entanglement between the two partitions evolves in time. In particular, it will be useful to examine if such dynamics of the entanglement can be characterized by a separate time scale and if so, how it is related to the decoherence times (e.g., T_1 and T_2) of the system. In this paper, we investigate the dissipative dynamics of two "spins" (two-level systems) coupled to environment within the spin-Boson model both in the Markovian and non-Markovian limits, and examine the characteristic time scale over which the entanglement decays. It is found that entanglement may

decay exponentially or logarithmically, depending on the presence of spin relaxation.

We consider two spins, coupled respectively to separate ("local") baths of harmonic oscillators[12]. Following Ref. 13, we adopt the spin-boson Hamiltonian

$$H = H_S + H_B + H_{SB},\tag{1}$$

with

$$H_S = \frac{1}{2} \sum_{j=1,2} \left(\epsilon_j \sigma_j^z + \Delta_j \sigma_j^x \right) \tag{2}$$

describing isolated spins,

$$H_B = \sum_{j=1,2} \sum_m \omega_{j,m} b_{j,m}^{\dagger} b_{j,m}$$
 (3)

baths of harmonic oscillators, and

$$H_{SB} = \sum_{j=1,2} \sum_{m} (g_{j,m} \sigma_j^z b_{j,m}^{\dagger} + g_{j,m}^* \sigma_j^{z\dagger} b_{j,m})$$
 (4)

coupling between the spins and the oscillators, where σ_j^{μ} (j=1,2) and $\mu=x,y,z)$ are Pauli matrices representing the jth spin and $b_{j,m}^{\dagger}/b_{j,m}$ are creation/annihilation operators for the oscillator of mode m coupled to the jth spin. The unit system $\hbar=k_B=1$ is used throughout this work. The dynamics of the spins at time scales longer than the bath correlation time ω_c^{-1} , which is of our concern, does not depend on the details of the coupling constants $g_{j,m}$ and the oscillator frequencies $\omega_{j,m}$. It suffices to characterize each bath collectively in terms of the bath spectral density function [13]

$$J_j(\omega) \equiv \sum_m |g_{j,m}|^2 \delta(\omega - \omega_{j,m}). \tag{5}$$

In this work, we focus on the Ohmic model of dissipation

$$J_j(\omega) = \frac{1}{2} \alpha_j \omega e^{-\omega/\omega_c},\tag{6}$$

where α_j is the dimensionless damping parameter of the bath coupled to the *j*th spin.

In spite of the simple form of the Hamiltonian, the dynamics of the spin-Boson model is highly non-trivial and has been the subject of a number of works [13]. Note in particular that the two spins in the system are completely decoupled with each other, each interacting separately with its local bath. In equilibrium, the density matrix ρ_S of the two spins is therefore separable: $\rho_S(t \to \infty) = \rho_1(\infty) \otimes \rho_2(\infty)$. However, starting from an initial state $\rho_S(0)$ with a finite amount of entanglement, the system displays non-trivial time evolution of the entanglement contents in $\rho_S(t)$. [See Refs. 7, 8, 9, 10 for initial preparation of entanglement between two spins.]

To proceed, we diagonalize Eq. (2), and obtain

$$H_{S} = \frac{1}{2} \sum_{j=1,2} E_{j} \tau_{j}^{z}$$

$$H_{SB} = \sum_{j=1,2} \sum_{m} \left[\tau_{j}^{z} \cos \theta_{j} \left(g_{j,m} b_{j,m}^{\dagger} + g_{j,m}^{*} b_{j,m} \right) \right]$$
(7)

$$+\tau_j^x \sin \theta_j \left(g_{j,m} b_{j,m}^\dagger + g_{j,m}^* b_{j,m} \right) \right], \quad (8)$$

where $E_j = \sqrt{\epsilon_j^2 + \Delta_j^2}$, $\theta_j = \tan^{-1}(\Delta_j/\epsilon_j)$, and τ_j^{μ} $(\mu = x, y, z)$ are the Pauli matrices in the rotated basis. Here we probe the evolution of entanglement in the two limiting cases: When $|\epsilon_j| \gg |\Delta_j|$, i.e, $\theta_j \approx 0$, each spin is not allowed to flip its direction: $[\tau_i^z, H] = 0$. In the analogy to a particle in a double-well potential [13], this corresponds to the case of an infinitely high potential barrier. The presence of the bath thus cannot change the population of spin states, merely breaking the coherent superposition of different spin states. In this sense we call this limit the "pure dephasing" model [14]. In the opposite limit $|\epsilon_i| \ll |\Delta_i|$, i.e, $\theta_i \approx \pi/2$, with the rotating wave approximation employed, coupling to the bath leads to relaxation of spin polarizations; this corresponds to the strong bias potential in the analogy to the particle in a double-well potential. We thus call this limit the "relaxation" model.

The evolution of the density matrix ρ of the total (isolated) system is unitary and follows the von Neumann equation

$$\frac{d\rho}{dt} = -i[H, \rho] \equiv \mathcal{L}\rho,\tag{9}$$

with the Liouville operator $\mathcal{L} = \mathcal{L}_S + \mathcal{L}_B + \mathcal{L}_{SB}$ corresponding to the three parts of H. The reduced density matrix $\rho_S = \text{Tr}_B \rho$ of spins, given by the trace of ρ over the bath variables B, is not unitary due to the coupling to baths. Its equation of motion, obtained from Eq. (9) by taking the trace, reads

$$\frac{d\rho_S}{dt} = \mathcal{L}_S \rho_S + \mathcal{D}\rho_S \tag{10}$$

with the dissipative part formally expressed as [15]

$$\mathcal{D}\rho_{S}(t) = -\int_{0}^{t} dt' \operatorname{Tr}_{B} \left[H_{SB}, e^{\mathcal{QL}(t-t')} \left[H_{SB}, \rho_{S}(t') \otimes \rho_{B} \right] \right],$$
(11)

where we have introduced the projection operators $\mathcal{P}\rho \equiv (\operatorname{Tr}_B\rho)\otimes\rho_B$ and $\mathcal{Q}\equiv 1-\mathcal{P}$ with the equilibrium bath density matrix $\rho_B\equiv e^{-H_B/T}/\operatorname{Tr}_B e^{-H_B/T}$ at temperature T [15]. We have also assumed $\rho(0)=\rho_S(0)\otimes\rho_B$ for the initial configuration . Further, to get analytic expressions for $\rho_S(t)$, we take the first-order Born approximation and make the replacement in Eq. (11)

$$e^{\mathcal{QL}(t-t')} \to e^{\mathcal{Q}(\mathcal{L}_S + \mathcal{L}_B)(t-t')},$$
 (12)

which is valid for sufficiently weak spin-bath coupling.

We measure the entanglement degree in the mixed state ρ_S by means of the *concurrence*

$$C(\rho_S) \equiv \max \left[\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0 \right], \quad (13)$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the eigenvalues of the matrix $\rho_S(\tau_y \otimes \tau_y)\rho_S^*(\tau_y \otimes \tau_y)$. The concurrence vanishes for a separable state and becomes unity for a maximally entangled state. While it is possible to investigate $C(\rho_S(t))$ for a general initial state $\rho_S(0)$, we shall usually assume, for clearer physical interpretation, a particular class of initial states, namely the Werner states:

$$\rho_S(0) = W(r) \equiv r|\Phi^+\rangle\langle\Phi^+| + \frac{1-r}{4}I, \qquad (14)$$

where $|\Phi^{+}\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)$ is one of the Bell states and I denotes the 4×4 identity matrix. The parameter r in the range [0,1] interpolates between the fully random state I/4 (for r=0) and the maximally entangled state $|\Phi^{+}\rangle \langle \Phi^{+}|$ (for r=1).

The master equation for ρ_S , given by Eqs. (10)–(12), takes the form of an integro-differential equation, and the state $\rho_S(t)$ at time t depends on the history $\rho_S(t')$ with $t' \leq t$. We first discuss the Markovian limit, where bath correlations decay sufficiently fast compared with the relaxation time of the spins. Mathematically, it corresponds to replacing $\rho_S(t')$ in the integrand of Eq. (11) by $\rho_S(t)$. With this Markovian approximation, the master equation (10) reduces to a linear differential equation with time-independent coefficients. Then expected is exponential behavior of $\rho_S(t)$ and accordingly of the concurrence $C[\rho_S(t)]$, from which one can extract the characteristic time scale of the entanglement dynamics.

For the pure dephasing model ($\Delta_j = 0$), the dissipative part given by Eq. (11) reads

$$\mathcal{D}\rho_S(t) = \sum_j \gamma_j \left[\tau_j^z \rho_S(t) \tau_j^z - \rho_S(t) \right], \qquad (15)$$

where $\gamma_j = 2\pi\alpha_j T$ is the dephasing rate of the jth spin. Given an initial state $\rho_S(0) = W(r)$, it is straightforward to find the solution $\rho_S(t)$ of Eqs. (10) and (15). It leads to the concurrence

$$C(r,t) = \Theta\left(r - \frac{1}{3}\right) \left[re^{-4(\gamma_1 + \gamma_2)t} - \frac{1-r}{2}\right],$$
 (16)

where $\Theta(x)$ is the unit step function. In particular, for the maximally entangled initial state (r = 1), the concurrence becomes

$$C(r=1,t) = e^{-4(\gamma_1 + \gamma_2)t},$$
 (17)

which decays exponentially with the rate proportional to the dephasing rate $\gamma_1 + \gamma_2$. Notice that the concurrence does not decay at zero temperature, where the dephasing rates vanishes.

For the relaxation model, Eq. (11) reduces to

$$\mathcal{D}\rho_{S}(t) = \sum_{j=1,2} \left\{ i\delta E_{j} \tau_{j}^{z} + \gamma_{j} [1 + n(E_{j})] \left[\tau_{j}^{+} \rho(t) \tau_{j}^{-} - \frac{1}{2} \left\{ \tau_{j}^{-} \tau_{j}^{+}, \rho_{S}(t) \right\} \right] + \gamma_{j} n(E_{j}) \left[\tau_{j}^{-} \rho(t) \tau_{j}^{+} - \frac{1}{2} \left\{ \tau_{j}^{+} \tau_{j}^{-}, \rho_{S}(t) \right\} \right] \right\}$$
(18)

where $\gamma_j \equiv 2\pi J_j(E_j)$ is the dephasing rate,

$$\delta E_j \equiv \Pr \int_0^\infty dE \, \coth\left(\frac{E}{2T}\right) \, \frac{J_j(E)}{E_j - E}$$
 (19)

is the environment-induced level shift [15], and $n(E) \equiv [e^{E/T} - 1]^{-1}$ is the Bose distribution function. It is again straightforward to find the solution $\rho_S(t)$ of Eqs. (10) and (18), which gives the concurrence (for $\gamma_1 = \gamma_2 \equiv \gamma$ and T = 0)

$$C(r,t) = \Theta(r-1/3)e^{-2\gamma t} \left[e^{\gamma t}r - e^{\gamma t} + \frac{1+r}{2} \right].$$
 (20)

It is observed that the concurrence again decays exponentially and the rate is determined by the dephasing time of a separate spin. For the maximally entangled initial state, it is simply given by

$$C(r=1,t) = e^{-2\gamma t}.$$
 (21)

We now go beyond the Markovian approximation (still within the first Born approximation). In the non-Markovian case, it is convenient to take the Laplace transform

$$\tilde{\rho}_S(z) = \int_0^\infty dt \ e^{-zt} \rho_S(t) \tag{22}$$

and solve the resulting algebraic equations in the place of Eqs. (10) and (11). Then the dissipative dynamics of the system is determined by the pole structure of $\tilde{\rho}_S(z)$ on the complex z-plane [16]: Whereas an isolated simple pole at $z=z_k$ with residue $r_k/2\pi i$ contributes an exponential part $r_k e^{z_k t_k}$ to the evolution of $\rho_S(z)$, a branch cut of $\tilde{\rho}(z)$ produces non-exponential behavior.

The dynamics of $\rho_S(t)$ in the non-Markovian limit is governed by the correlation function

$$C_{j}(t) = \int_{0}^{\infty} d\omega \ J_{j}(\omega) \left[\coth\left(\frac{\omega}{2T}\right) \cos(\omega t) + i \sin(\omega t) \right]$$
(23)

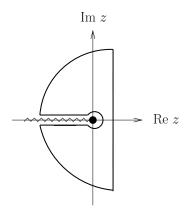


FIG. 1: Contour for the dephasing model. The branch cut has been chosen for computational convenience.

of the quantum Langevin force $\sum_{m}[g_{j,m}b_{j,m}^{\dagger}+g_{j,m}^{*}b_{j,m}]$. In particular, the solution $\tilde{\rho}_{S}(z)$ involves the respective Laplace transforms $\tilde{C}'_{j}(z)$ and $\tilde{C}'''_{j}(z)$ of the real and imaginary parts of $C_{j}(t)$, which are given explicitly by

$$\begin{bmatrix} \tilde{C}'_j(z) \\ \tilde{C}''_j(z) \end{bmatrix} = -\frac{\alpha_j z}{2} \begin{bmatrix} \cos(z/\omega_c) & +\sin(z/\omega_c) \\ \sin(z/\omega_c) & -\cos(z/\omega_c) \end{bmatrix} \begin{bmatrix} \operatorname{Ci}(z/\omega_c) \\ \operatorname{Si}(z/\omega_c) \end{bmatrix}.$$
(24)

Here Si and Ci are the sine- and the cosine-integral, respectively.

For the pure dephasing model, the solution $\tilde{\rho}(z)$ involves only $\tilde{C}'_j(z)$. Note that the cosine integral $\mathrm{Ci}(z)$ in Eq. (24) behaves as $\ln z$ in the limit $z \to 0$ while the sine integral is analytic. This leads to a branch cut with no isolated simple poles on the complex z plane, as shown in Fig. 1. It is remarkable that the solution has extra structure not present in the Markovian approximation. In particular, the absence of isolated poles at T=0 indicates that there is no exponential decay in $\rho_S(t)$ and the branch cut contribution gives the leading correction.

Figure 2 shows the time evolution of the concurrence for several different initial states (r = 1/2, 3/4, and 1). The intermediate and long time behaviors are given by

 $\ln \omega_c t$ and $(\ln \omega_c t)^{-1}$, respectively. Specifically, for the maximally entangled initial state, we have

$$C(r,t) = \frac{1}{8\pi\alpha} \left\{ \left| (4\alpha + 1)\pi - 2\tan^{-1} \left[\frac{1 - 2\alpha \left(\gamma - \ln(\omega_c t)\right)}{2\pi\alpha} \right] \right| - \left| (4\alpha - 1)\pi + 2\tan^{-1} \left[\frac{1 - 2\alpha \left(\gamma - \ln(\omega_c t)\right)}{2\pi\alpha} \right] \right| \right\}. \tag{25}$$

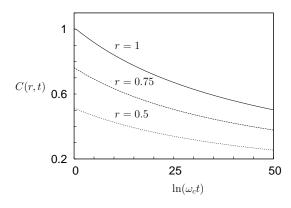


FIG. 2: Time evolution of the concurrence C(r,t) for the pure dephasing model.

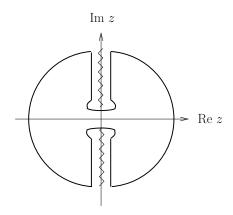


FIG. 3: Contour for the relaxation model.

Note that unlike the Markovian case, the concurrence decays, albeit slowly, even at zero temperature.

For the relaxation model, the solution $\tilde{\rho}_S(z)$ involves $\tilde{C}_j''(z)$ as well as $\tilde{C}_j'(z)$. This leads to two branch cuts on the complex z plane, as shown in Fig. 3. Another difference from the pure dephasing model is the existence of an isolated pole. A closer investigation reveals that in the long-time limit contributions from the branch cuts are negligibly small compared with those from the isolated pole [16]. Accordingly, $\rho_S(t)$ in the time domain decays exponentially, and the behavior of the concurrence is similar to that in the Markovian limit.

In conclusion, we have used the generalized master equation to study quantum dissipative dynamics of entanglement in the spin-boson model. The pure dephasing model and the relaxation one have been considered, and entanglement for both models has been shown to decay exponentially in the Markovian limit. When non-Markovian contributions are taken into account, on the other hand, logarithmic decay has been revealed for the dephasing model.

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