Some Generalizations of the MacMahon Master Theorem

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Abstract

We consider a number of generalizations of the β -extended MacMahon Master Theorem for a matrix. The generalizations are based on replacing permutations on multisets formed from matrix indices by partial permutations or derangements over matrix or submatrix indices.

1 Introduction

The Master Theorem due to Percy MacMahon first appeared in 1915 in his classic text Combinatory Analysis [MM]. A generalization known as the β -extended MacMahon Master Theorem was discovered in more recent times by Foata and Zeilberger [FZ]. This present paper is concerned with several further generalizations of the β -extension informed by recent results in the theory of vertex operator algebras concerning the partition and correlation functions on a genus zero and higher Riemann surface [MT1, MT2, TZ1, HT, TZ2, TZ3].

One formulation of the MacMahon Master Theorem (MMT) is the identity of $\det(I-A)^{-1}$, for a given matrix A, to an infinite weighted sum over all permanents for matrices indexed by multisets formed from the indices of A [W, KP]. The β -extended MMT relates $\det(I-A)^{-\beta}$ to a similar sum

over so-called β -extended permanents [FZ, KP]. We consider the following generalizations:

- (i) The Submatrix MMT. Here the infinite sum runs over multisets formed from the indices of a given submatrix of A.
- (ii) The Partial Permutation MMT. In this case the β -extended permanent is replaced by what we refer to as a (β, θ, ϕ) -extended partial permanent defined in terms of a sum over all partial permutations of the A-indices.
- (iii) The Derangement MMT. We replace the β -extended permanent by what we refer to as a β -extended deranged partial permanent defined in terms of a sum over the derangements of the A-indices.

We begin in Section 2 with a review of the β -extended MMT [FZ]. We provide a graph theoretic proof based on an enumeration of appropriate weights of non-isomorphic permutation graphs labelled by multisets of the indexing set for A. In particular, the connected subgraphs are cycles corresponding to permutation cycles. Section 3 describes our first generalization, the Submatrix MMT (Theorem 3.1), where the set of permutation graphs is modified to account for multisets formed from the indices of an A submatrix. In Section 4 we introduce the (β, θ, ϕ) -extended partial permanent of a matrix, a variation on the β -extended permanent involving a sum over the partial permutations of the matrix indices. The corresponding Partial Permutation MMT (Theorem 4.1) is proved by a consideration of partial permutation graphs whose connected subgraphs are cycles and open necklaces. Section 5 combines both of the previous generalizations into one general result in Theorem 5.1. Finally, in Section 6 we introduce another variation, the β -extended deranged permanent of a matrix, where we sum over the derangements (fixed point free permutations) of the matrix indices. We conclude with a Derangement MMT (Theorem 6.1) and a corresponding Submatrix Derangement MMT (Theorem 6.2) which are proved by applying the graph theory description of Sections 2 and 3 respectively, but where no 1-cycle graphs occur.

2 The β -Extended MacMahon Master Theorem

Let $A = (A_{ij})$ be an $n \times n$ matrix indexed by $i, j \in \{1, ..., n\}$. The β -extended Permanent of A is defined by [FZ], [KP]

$$\operatorname{perm}_{\beta} A = \sum_{\pi \in \Sigma_n} \beta^{C(\pi)} \prod_{i=1}^n A_{i\pi(i)}, \tag{1}$$

where $C(\pi)$ is the number of cycles in $\pi \in \Sigma_n$, the symmetric group. The permanent and determinant are the special cases:

$$\operatorname{perm} A = \operatorname{perm}_{+1} A, \qquad \det(-A) = \operatorname{perm}_{-1} A. \tag{2}$$

Let $\mathbf{r} = (r_1, \dots, r_n)$ denote an *n*-tuple of non-negative integers. Define

$$\mathbf{r}! = r_1! \dots r_n!,\tag{3}$$

and let

$$n^{\mathbf{r}} = \{1^{r_1}2^{r_2}\dots n^{r_n}\} = \{1_1,\dots,1_{r_1},\dots,n_1,\dots,n_{r_n}\},\tag{4}$$

denote the multiset of size $N = \sum_{i=1}^{n} r_i$ formed from the original index set $\{1,\ldots,n\}$ where the index i is repeated r_i times. We sometimes notate a repeated index by i_a for label $a=1,\ldots,r_i$. For an $n\times n$ matrix A, we let $A(n^{\mathbf{r}},n^{\mathbf{r}})$ denote the $N\times N$ matrix indexed by the elements of $n^{\mathbf{r}}$ and define $A(n^{\mathbf{r}},n^{\mathbf{r}})=1$ for $\mathbf{r}=(0,0,\ldots,0)$.

We now describe a generalization, due to Foata and Zeilberger [FZ], of the MacMahon Master Theorem (MMT) of classical combinatorics [MM]. We give a detailed proof based on a graph theory method which is extensively employed throughout this paper. This proof is very similar to that of Theorem 5 of [MT1] where the MMT was essentially rediscovered.

Theorem 2.1 (The β -Extended MMT)

$$\sum_{r_i \ge 0} \frac{1}{\mathbf{r}!} \operatorname{perm}_{\beta} A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \frac{1}{\det(I - A)^{\beta}}.$$
 (5)

Remark 2.2 For $\beta = 1$, Theorem 2.1 reduces to the MMT. For $\beta = -1$ we use (2) to find that only proper subsets of $\{1, \ldots, n\}$ contribute resulting in the determinant identity for B = -A e.g. [TZ1]

$$\sum_{r_i \in \{0,1\}} \det B(n^{\mathbf{r}}, n^{\mathbf{r}}) = \det(I + B).$$

Proof of Theorem 2.1. Let $\Sigma(n^{\mathbf{r}})$ denote the symmetric group of the multiset $n^{\mathbf{r}}$. For $\pi \in \Sigma(n^{\mathbf{r}})$ we define a permutation graph γ_{π} with N vertices labelled by $i \in \{1, \ldots, n\}$, and with directed edges

$$e_{ij} = i \bullet \longrightarrow \bullet j$$
,

provided $j = \pi(i)$. The connected subgraphs of $\gamma_{\pi} \in \Gamma$ are cycles arising from the cycles of π . For example, for n = 4 with $\mathbf{r} = (3, 2, 0, 1)$ and permutation $\pi = (1_1 2_1 1_2 2_2)(1_3 4_1)$ the corresponding graph has two cycles as shown in Fig. 1

$$\begin{array}{c|c}
1 & 2 \\
2 & 1 & 4
\end{array}$$

Fig. 1 γ_{π} for $\pi = (1_1 2_1 1_2 2_2)(1_3 4_1)$.

Define a weight for each edge of γ_{π} by

$$w(e_{ij}) = A_{ij},$$

and a weight for γ_{π} by

$$w(\gamma_{\pi}) = \beta^{C(\pi)} \prod_{e_{ij} \in \gamma_{\pi}} w(e_{ij}). \tag{6}$$

where $C(\pi)$ is the number of cycles in π . Note that the weight is multiplicative with respect to the cycle decomposition of π . (6) also implies

$$\operatorname{perm}_{\beta} A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \sum_{\pi \in \Sigma(n^{\mathbf{r}})} w(\gamma_{\pi}). \tag{7}$$

Let $\Lambda(\mathbf{r}) = \Sigma_{r_1} \times \ldots \times \Sigma_{r_n} \subseteq \Sigma(n^{\mathbf{r}})$ denote the label group of order $|\Lambda(\mathbf{r})| = \mathbf{r}!$ which permutes the identical elements of $n^{\mathbf{r}}$. $\Lambda(\mathbf{r})$ generates

isomorphic graphs with $\gamma_{\pi} \sim \gamma_{\lambda\pi\lambda^{-1}}$ for $\lambda \in \Lambda(\mathbf{r})$ and the automorphism group of γ_{π} is the π stabilizer $\operatorname{Aut}(\gamma_{\pi}) = \{\lambda \in \Lambda(\mathbf{r}) | \lambda\pi = \pi\lambda\} \subseteq \Lambda(\mathbf{r})$. Using the Orbit-Stabilizer theorem it follows that the number of isomorphic graphs generated by the action of $\Lambda(\mathbf{r})$ on γ_{π} is given by

$$|\Lambda(\mathbf{r})\gamma_{\pi}| = \frac{|\Lambda(\mathbf{r})|}{|\operatorname{Aut}(\gamma_{\pi})|}.$$
 (8)

(e.g. in Fig. 1, $\Lambda(\mathbf{r}) = \Sigma_2 \times \Sigma_3$ and $\operatorname{Aut}(\gamma_{\pi}) = \Sigma_2$ so that there are 6 permutations in $\Sigma(n^{\mathbf{r}})$ with graph γ_{π}). Combining (7) and (8) we find that

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \operatorname{perm}_{\beta} A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \sum_{\gamma \in \Gamma} \frac{w(\gamma)}{|\operatorname{Aut}(\gamma)|}, \tag{9}$$

where Γ denotes the set of non-isomorphic graphs.

Consider the decomposition of a graph γ into cycle graphs

$$\gamma = \gamma_{\sigma_1}^{m_1} \dots \gamma_{\sigma_K}^{m_K},$$

where $\{\gamma_{\sigma_i}\}$ are non-isomorphic and γ_{σ_i} occurs m_i times. The automorphism group is

$$\operatorname{Aut}(\gamma) = \prod_{i=1}^{M} \operatorname{Aut}(\gamma_{\sigma_i}^{m_i}),$$

where $\operatorname{Aut}(\gamma_{\sigma}^{m}) = \Sigma_{m} \rtimes \operatorname{Aut}(\gamma_{\sigma})^{m}$ of order $m! |\operatorname{Aut}(\gamma_{\sigma})|^{m}$. Furthermore, since the weight is multiplicative, $w(\gamma) = \prod_{i=1}^{M} w(\gamma_{\sigma_{i}})^{m_{i}}$. Thus we find

$$\sum_{\gamma \in \Gamma} \frac{w(\gamma)}{|\operatorname{Aut}(\gamma)|} = \prod_{\gamma_{\sigma} \in \Gamma_{\sigma}} \sum_{m \geq 0} \frac{1}{m!} \left(\frac{w(\gamma_{\sigma})}{|\operatorname{Aut}(\gamma_{\sigma})|} \right)^{m}$$

$$= \exp \left(\sum_{\gamma_{\sigma} \in \Gamma_{\sigma}} \frac{w(\gamma_{\sigma})}{|\operatorname{Aut}(\gamma_{\sigma})|} \right), \tag{10}$$

where Γ_{σ} denotes the set of non-isomorphic cycle graphs. For a cycle σ of order $|\sigma| = t$ we have $\operatorname{Aut}(\gamma_{\sigma}) = \langle \sigma^s \rangle$ for some s|t with $|\operatorname{Aut}(\gamma_{\sigma})| = \frac{t}{s}$. Using the trace identity

$$\sum_{\gamma_{\sigma}, |\sigma| = t} s \ w(\gamma_{\sigma}) = \beta \operatorname{Tr}(A^{t}),$$

we find

$$\sum_{\gamma_{\sigma} \in \Gamma_{\sigma}} \frac{w(\gamma_{\sigma})}{|\operatorname{Aut}(\gamma_{\sigma})|} = \beta \sum_{t \geq 1} \frac{1}{t} \operatorname{Tr}(A^{t})$$
$$= -\beta \operatorname{Tr} \log(I - A)$$
$$= -\beta \log \det(I - A).$$

Thus

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \operatorname{perm}_{\beta} A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \det(I - A)^{-\beta}. \qquad \Box$$

Let $w_1(\gamma)$ denote the weight for γ with $\beta = 1$ in (6). Define a cycle to be primitive (or rotationless) if $|\operatorname{Aut}(\gamma_{\sigma})| = 1$. For a general cycle σ with $|\operatorname{Aut}(\gamma_{\sigma})| = k$ we have $\gamma_{\sigma} = \gamma_{\rho}^{k}$ for a primitive cycle ρ . Let Γ_{ρ} denote the set of all primitive cycles. Then

$$\sum_{\gamma_{\sigma} \in \Gamma_{\sigma}} \frac{w_{1}(\gamma_{\sigma})}{|\operatorname{Aut}(\gamma_{\sigma})|} = \sum_{\gamma_{\rho} \in \Gamma_{\rho}} \sum_{k \geq 1} \frac{1}{k} w_{1}(\gamma_{\rho})^{k}$$
$$= -\sum_{\gamma_{\rho} \in \Gamma_{\rho}} \log \det(1 - w_{1}(\gamma_{\rho})).$$

Combining this with (10) implies [MT1]

Proposition 2.3

$$\det(I - A) = \prod_{\gamma_{\rho} \in \Gamma_{\rho}} (1 - w_1(\gamma_{\rho})).$$

3 The Submatrix MMT

Our first generalization of Theorem 2.1 concerns submatrices. Consider an $(n'+n)\times(n'+n)$ matrix with block structure

$$\left[\begin{array}{cc} B & U \\ V & A \end{array}\right],\tag{11}$$

where $A = (A_{ij})$ is an $n \times n$ matrix indexed by $i, j, B = (B_{i'j'})$ is an $n' \times n'$ matrix indexed by $i', j', U = (U_{i'j})$ is an $n' \times n$ matrix and $V = (V_{ij'})$ is an $n \times n'$ matrix. For a multiset $n^{\mathbf{r}}$ of size N define the $(n' + N) \times (n' + N)$ matrix

$$\begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix}, \tag{12}$$

where, as before, $A(n^{\mathbf{r}}, n^{\mathbf{r}})$ denotes the $N \times N$ matrix indexed by $n^{\mathbf{r}}$, $U(n^{\mathbf{r}})$ is an $n' \times N$ matrix and $V(n^{\mathbf{r}})$ is an $N \times n'$ matrix. We then find

Theorem 3.1

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \operatorname{perm}_{\beta} \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix} = \frac{\operatorname{perm}_{\beta} \widetilde{B}}{\det(I - A)^{\beta}}, \quad (13)$$

for $n' \times n'$ matrix

$$\widetilde{B} = B + U(I - A)^{-1}V,$$

where $(I - A)^{-1} = \sum_{k>0} A^k$.

This result is related to Theorem 10 of [MT1] when $\beta = 1$ and Theorem 2 of [TZ1] for $\beta = -1$.

Proof. Let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$ and let $\mathbf{n}' \cup n^{\mathbf{r}}$ denote the multiset indexing the block matrix (12). Define a permutation graph γ_{π} with weight $w(\gamma_{\pi})$ for each $\pi \in \Sigma(\mathbf{n}' \cup n^{\mathbf{r}})$ as follows. Each vertex is labelled by an element of \mathbf{n} or \mathbf{n}' which we refer to as \mathbf{n} -vertex or \mathbf{n}' -vertex respectively. For $l = \pi(k)$ with $k, l \in \mathbf{n}' \cup n^{\mathbf{r}}$ we define an edge $e_{kl} = k \bullet \longrightarrow \bullet l$ with weight

$$w(e_{kl}) = \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix}_{kl}.$$

Define a weight for γ_{π} by

$$w(\gamma_{\pi}) = \beta^{C(\pi)} \prod_{e_{kl} \in \gamma_{\pi}} w(e_{kl}),$$

where $C(\pi)$ is the number of cycles in π . As before, we find

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \operatorname{perm}_{\beta} \begin{bmatrix} B & U(\mathbf{k}) \\ V(\mathbf{k}) & A(\mathbf{k}, \mathbf{k}) \end{bmatrix} = \sum_{\gamma \in \widehat{\Gamma}} \frac{w(\gamma)}{|\operatorname{Aut}(\gamma)|},$$

where $\widehat{\Gamma}$ denotes the set of non-isomorphic graphs. Each $\gamma \in \widehat{\Gamma}$ has a decomposition into cycles γ_{σ_a} which contain **n**-vertices only and cycles $\gamma_{\sigma'_b}$ which contain at least one **n**'-vertex:

$$\gamma = \gamma_{\sigma_1}^{m_1} \dots \gamma_{\sigma_K}^{m_K} \gamma_{\sigma_1'} \dots \gamma_{\sigma_L'},$$

with weight

$$w(\gamma) = \prod_{a} w(\gamma_{\sigma_a})^{m_a} \prod_{b} w(\gamma_{\sigma'_b}).$$

The set of non-isomorphic γ_{σ_a} cycle graphs labelled by **n** is equivalent to Γ_{σ} introduced in the proof of Theorem 2.1. Since each **n**'-vertex occurs exactly once in γ , each $\gamma_{\sigma'_b}$ cycle occurs at most once and has trivial automorphism group. Hence

$$|\operatorname{Aut}(\gamma)| = \prod_{a} |\operatorname{Aut}(\gamma_{\sigma_a})|^{m_a} m_a!,$$

as before. Thus the sum over weights of all graphs decomposes into the product

$$\sum_{\gamma \in \widehat{\Gamma}} \frac{w(\gamma)}{|\operatorname{Aut}(\gamma)|} = \sum_{\gamma_{\sigma'}} w(\gamma_{\sigma'}) \prod_{\gamma_{\sigma} \in \Gamma_{\sigma}} \sum_{m \geq 0} \frac{w(\gamma_{\sigma})^m}{|\operatorname{Aut}(\gamma_{\sigma})|^m m!}$$

$$= \frac{\sum_{\gamma_{\sigma'}} w(\gamma_{\sigma'})}{\det(I - A)^{\beta}},$$

using Theorem 2.1 and where $\gamma_{\sigma'}$ ranges over non-isomorphic cycles in $\widehat{\Gamma}$ containing at least one \mathbf{n}' -vertex.

It remains to compute $\sum_{\gamma_{\sigma'}} w(\gamma_{\sigma'})$. Let $\sigma' \in \Sigma_{n'}$ denote the permutation cycle corresponding to the cyclic sequence of \mathbf{n}' -vertices in a $\gamma_{\sigma'}$ -cycle (for arbitrary intermediate \mathbf{n} -vertices). The total edge weight coming from all subgraphs, illustrated in Fig. 2, joining two \mathbf{n}' -vertices, i' and j', summed over all intermediate \mathbf{n} -vertices is

$$B_{i'j'} + (UV)_{i'j'} + (UAV)_{i'j'} + (UA^{2}V)_{i'j'} + \dots$$

$$= (B + U(I - A)^{-1}V)_{i'j'} = \widetilde{B}_{i'j'}.$$

$$i'$$
 j' , i' k j' , ...

Fig. 2

Thus the total weight of all $\gamma_{\sigma'}$ cycles for a given \mathbf{n}' -vertex cycle $\sigma' = (i'_1 \dots i'_p)$ is $\beta \prod_l \widetilde{B}_{i'_l \sigma'(i'_l)}$. Altogether, it follows that

$$\sum_{\gamma_{\sigma'}} w(\gamma_{\sigma'}) = \sum_{\pi' \in \Sigma_{n'}} \beta^{C(\pi')} \prod_{i'} \widetilde{B}_{i'\pi'(i')}$$
$$= \operatorname{perm}_{\beta} \widetilde{B}. \quad \Box$$

Lemma 3.2 For $\beta = -1$ Theorem 3.1 implies

$$\sum_{r_i \in \{0,1\}} \det \left[\begin{array}{cc} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{array} \right] = \det \left[\begin{array}{cc} B & -U \\ -V & I+A \end{array} \right].$$

Proof. For $\beta = -1$ the right hand side of (13) gives

$$\operatorname{perm}_{-1} \widetilde{B} \det(I - A) = (-1)^{n'} \det(B + U(I - A)^{-1}V) \det(I - A)$$
$$= \det \begin{bmatrix} -B & U \\ V & I - A \end{bmatrix},$$

by means of the matrix identity

$$\left[\begin{array}{cc} -B & U \\ V & I-A \end{array} \right] = \left[\begin{array}{cc} -I' & U(I-A)^{-1} \\ 0 & I \end{array} \right] \left[\begin{array}{cc} B+U(I-A)^{-1}V & 0 \\ V & I \end{array} \right] \left[\begin{array}{cc} I' & 0 \\ 0 & I-A \end{array} \right],$$

where I and I' are respectively $n \times n$ and $n' \times n'$ identity matrices. The result follows on replacing A, B, U, V by -A, -B, -U, -V. \square

4 The Partial Permutation MMT

The next generalization of Theorem 2.1 is concerned with replacing permutations by partial permutations with a suitable generalization of the notions of permanent and β -extended permanent. Let Ψ denote the set of partial permutations of the set $\{1,\ldots,n\}$ i.e. injective partial mappings from $\{1,\ldots,n\}$ to itself. For $\psi \in \Psi$ we let $\operatorname{dom} \psi$ and $\operatorname{im} \psi$ denote the domain and image respectively and let π_{ψ} denote the (possibly empty) permutation of $\operatorname{dom} \psi \cap \operatorname{im} \psi$ determined by ψ .

We introduce the Partial Permanent of an $n \times n$ matrix $A = (A_{ij})$ indexed by $i, j \in \{1, ..., n\}$ as follows

$$pperm A = \sum_{\psi \in \Psi} \prod_{i \in \text{dom } \psi} A_{i\psi(i)}, \tag{14}$$

with unit contribution for the empty map. Let $\theta = (\theta_i)$, $\phi = (\phi_i)$ be *n*-vectors and define the (β, θ, ϕ) -extended Partial Permanent by

$$\operatorname{pperm}_{\beta\theta\phi} A = \sum_{\psi \in \Psi} \beta^{C(\pi_{\psi})} \prod_{i \in \operatorname{dom} \psi} A_{i\psi(i)} \prod_{j \notin \operatorname{im} \psi} \theta_j \prod_{k \notin \operatorname{dom} \psi} \phi_k, \tag{15}$$

where $C(\pi_{\psi})$ is the number of cycles in π_{ψ} e.g.

$$\operatorname{pperm}_{\beta\theta\phi} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \theta_1\phi_1\theta_2\phi_2 + \beta(A_{11}\theta_2\phi_2 + A_{22}\theta_1\phi_1) + A_{12}\theta_1\phi_2 + A_{21}\theta_2\phi_1 + \beta^2A_{11}A_{22} + \beta A_{12}A_{21}.$$

A recent application of an extended partial permanent appears in [HT].

Let $A(n^{\mathbf{r}}, n^{\mathbf{r}})$ denote the $N \times N$ matrix indexed by a multiset $n^{\mathbf{r}}$ as before. We also let $\operatorname{pperm}_{\beta\theta\phi}A(n^{\mathbf{r}}, n^{\mathbf{r}})$ denote the corresponding partial permanent with N-vectors $(\theta_{1_1}, \ldots, \theta_{n_{r_n}})$ and $(\phi_{1_1}, \ldots, \phi_{n_{r_n}})$. We then find

Theorem 4.1

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \operatorname{pperm}_{\beta\theta\phi} A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \frac{e^{\theta(I-A)^{-1}\phi^{T}}}{\det(I-A)^{\beta}}, \tag{16}$$

where ϕ^T denotes the transpose of the row vector ϕ .

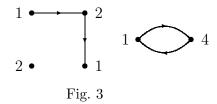
This result is related to Theorem 11 of [MT1] for $\beta = 1$.

Proof. Let $\Psi(n^{\mathbf{r}})$ denote the partial permutations of $n^{\mathbf{r}}$. Define a partial permutation graph γ_{ψ} labelled by $\{1, \ldots, n\}$ for each $\psi \in \Psi(\mathbf{k})$ with edges

$$e_{ij} = i \bullet \longrightarrow \bullet j$$
,

for $j = \psi(i)$ with $i \in \text{dom } \psi$ and $j \in \text{im } \psi$. Let v_i denote the vertex of γ_{ψ} with label i. If $i \notin \text{dom } \psi$ then either $\deg v_i = 0$ or $\deg v_i = \text{indeg } v_k = 1$ whereas if $i \notin \text{im } \psi$ then either $\deg v_i = 0$ or $\deg v_i = \text{outdeg } v_i = 1$. In all other cases

 $\deg v_i=2$ with indeg $v_i=0$ utdeg $v_i=1$. The connected subgraphs in this case consist of cycles and open necklaces i.e. graphs with two end points of degree one. We regard a graph consisting of a single degree zero vertex as a degenerate necklace. For example, for n=4, $\mathbf{r}=(3,2,0,1)$ and partial permutation $\psi=\begin{pmatrix} 1_1 & 1_2 & 1_3 & 2_1 & 2_2 & 4_1 \\ 2_1 & 4_1 & 1_2 & 1_3 \end{pmatrix}$ then γ_ψ is shown in Fig. 3. In this case dom $\psi=\{1_1,1_3,2_1,4_1\}$ and im $\psi=\{1_2,1_3,2_1,4_1\}$ and $\pi_\psi=(1_34_1)$.



Define an edge weight as before by $w(e_{ij}) = A_{ij}$ and introduce a vertex weight

$$w(v_k) = \begin{cases} 1, & \deg v_k = 2, \\ \theta_k, & \deg v_k = \text{outdeg } v_k = 1, \\ \phi_k, & \deg v_k = \text{indeg } v_k = 1, \\ \theta_k \phi_k, & \deg v_k = 0. \end{cases}$$

The weight of a graph γ_{ψ} is defined by

$$w(\gamma_{\psi}) = \beta^{C(\pi_{\psi})} \prod_{e_{ij}} w(e_{ij}) \prod_{v_k} w(v_k),$$

where $C(\pi_{\psi})$ is the number of cycles in π_{ψ} . The weight is multiplicative with respect to the cycle and necklace decomposition. We find again that

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \operatorname{pperm}_{\beta\theta\phi} A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \sum_{\gamma \in \widetilde{\Gamma}} \frac{w(\gamma)}{|\operatorname{Aut}(\gamma)|},$$

where $\widetilde{\Gamma}$ denotes the set of non-isomorphic graphs. Each $\gamma \in \widetilde{\Gamma}$ has a decomposition into connected cycle graphs γ_{σ_a} and open necklaces ν_b :

$$\gamma = \nu_1^{l_1} \dots \nu_L^{l_1} \gamma_{\sigma_1}^{m_1} \dots \gamma_{\sigma_K}^{m_K},$$

with weight

$$w(\gamma) = \prod_b w(\nu_b)^{l_b} \cdot \prod_a w(\gamma_{\sigma_a})^{m_a}.$$

Each necklace has trivial automorphism group but can have multiple occurrences. Hence we find that

$$|\operatorname{Aut}(\gamma)| = \prod_{b} l_b! \cdot \prod_{a} |\operatorname{Aut}(\gamma_{\sigma_a})|^{m_a} m_a!.$$

Thus the sum over weights of all graphs decomposes into the product

$$\sum_{\gamma \in \widetilde{\Gamma}} \frac{w(\gamma)}{|\operatorname{Aut}(\gamma)|} = \prod_{\nu \in \Gamma_{\nu}} \sum_{l \ge 0} \frac{w(\nu)^{l}}{l!} \cdot \prod_{\gamma_{\sigma} \in \Gamma_{\sigma}} \sum_{m \ge 0} \frac{w(\gamma_{\sigma})^{m}}{|\operatorname{Aut}(\gamma_{\sigma})|^{m} m!}$$
$$= \exp\left(\sum_{\nu \in \Gamma_{\nu}} w(\nu)\right) \frac{1}{\det(I - A)^{\beta}},$$

where Γ_{ν} denotes the set of non-isomorphic open necklaces and using Theorem 2.1 again. Finally, the sum over the weights of connected necklaces, such as depicted in Fig. 4, is

$$\sum_{\nu \in \Gamma_{\nu}} w(\nu) = \theta \phi^{T} + \theta A \phi^{T} + \theta A^{2} \phi^{T} + \dots$$
$$= \theta (I - A)^{-1} \phi^{T}. \quad \Box$$

$$i$$
, i j , k , ...

Fig. 4

Example. Consider n = 1 with A = z and $\theta_1 = \phi_1 = \sqrt{\alpha z}$. Then we find

$$pperm_{\beta\theta\phi}A(1^r, 1^r) = p_r(\alpha, \beta)z^r,$$

where $p_r(\alpha, \beta) = \sum_{s,t} p_{rst} \alpha^s \beta^t$ is the generating polynomial for p_{rst} the number of graphs with r identically labelled vertices, s open necklaces and t cycles. Theorem 4.1 provides the exponential generating function for $p_r(\alpha, \beta)$ [HT]

$$\sum_{r>0} \frac{p_r(\alpha,\beta)}{r!} z^r = \frac{\exp\left(\frac{\alpha z}{1-z}\right)}{(1-z)^{\beta}}.$$

5 The Submatrix Partial Permutation MMT

We can combine the two generalizations above into one theorem concerning partial permutations of submatrices of the $(n'+n) \times (n'+n)$ block matrix (11). Let $\theta' = (\theta'_{i'})$ and $\phi' = (\phi'_{i'})$ be n'-vectors and $\theta = (\theta_i)$ and $\phi = (\phi_i)$ be n-vectors. For a multiset $n^{\mathbf{r}}$ of size N and block matrix (12) labelled by $\mathbf{n}' = \{1', \ldots, n'\}$ and $n^{\mathbf{r}}$, we let $\operatorname{pperm}_{\beta\theta\phi} \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix}$ denote the (β, θ, ϕ) -extended partial permanent with (n'+N)-vectors $(\theta'_{1'}, \ldots, \theta'_{n'}, \theta_{1_1}, \ldots, \theta_{n_{r_n}})$ and $(\phi'_{1'}, \ldots, \phi'_{n'}, \phi_{1_1}, \ldots, \phi_{n_{r_n}})$ respectively. We then find

Theorem 5.1

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \operatorname{pperm}_{\beta\theta\phi} \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix} = \frac{e^{\theta(I-A)^{-1}\phi^{T}} \cdot \operatorname{pperm}_{\beta\widetilde{\theta}\widetilde{\phi}}\widetilde{B}}{\det(I-A)^{\beta}}, (17)$$

for

$$\begin{split} \widetilde{B} &= B + U(I-A)^{-1}V, \\ \widetilde{\theta} &= \theta' + \theta(I-A)^{-1}V, \\ \widetilde{\phi}^T &= \phi'^T + U(I-A)^{-1}\phi^T. \end{split}$$

This result is related to Theorem 13 of [MT1] for $\beta = 1$.

Proof. We sketch the proof since it runs along very similar lines to the preceding ones. Define a partial permutation graph γ_{ψ} for each partial permutation ψ of $\mathbf{n}' \cup n^{\mathbf{r}}$. In this case, the connected subgraphs consist of cycle graphs Γ_{σ} and open necklaces Γ_{ν} containing only **n**-vertices, and cycles and open necklaces containing at least one \mathbf{n}' -vertex. Define a graph weight $w(\gamma_{\psi})$ as a product of edge weights, vertex weights and cycle factors as before. This results in

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \operatorname{pperm}_{\beta\theta\phi} \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix} = \frac{e^{\theta(I-A)^{-1}\phi^{T}}}{\det(I-A)^{\beta}} \sum_{\gamma' \in \Gamma'} w(\gamma'),$$

where the sum is over all graphs Γ' containing at least one \mathbf{n}' -vertex. The remaining terms arise as before.

Each $\gamma' \in \Gamma'$ canonically determines a partial permutation $\psi' \in \Psi(\mathbf{n}')$ described by the corresponding ordered sequences of \mathbf{n}' -vertices (for any intermediate \mathbf{n} -vertices). As before, the total edge weight coming from all

subgraphs joining two \mathbf{n}' -vertices i' and j' with intermediate \mathbf{n} -vertices is $\widetilde{B}_{i'j'}$. The total weight arising from the subgraphs of all necklaces joining \mathbf{n} -vertices to an \mathbf{n}' -vertex i' with intermediate \mathbf{n} -vertices as depicted in Fig. 5 is

$$\theta'_{i'} + (\theta V)_{i'} + (\theta A V)_{i'} + \dots$$

= $(\theta' + \theta (I - A)^{-1} V)_{i'} = \widetilde{\theta}_{i'}.$

$$i'$$
, j i' , j k i' , ...

Fig. 5

Likewise, the total weight arising from all subgraphs joining an \mathbf{n}' -vertex j' to \mathbf{n} -vertices with intermediate \mathbf{n} -vertices is $\widetilde{\phi}_{j'}$. Combining these results we find that

$$\sum_{\gamma' \in \Gamma'} w(\gamma') = \operatorname{pperm}_{\beta \, \widetilde{\theta} \, \widetilde{\phi}} \widetilde{B}. \quad \Box$$

6 The Derangement MMT

Let $\Delta_n \subset \Sigma_n$ denote the derangements of the set $\{1, \ldots, n\}$ i.e. each $\pi \in \Delta_n$ contains no cycles of length 1. We introduce the β -extended Deranged Permanent of an $n \times n$ matrix A by

$$\operatorname{dperm}_{\beta} A = \sum_{\pi \in \Delta_n} \beta^{C(\pi)} \prod_i A_{i\pi(i)}. \tag{18}$$

Using the same multiset notation as before we find

Theorem 6.1

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \operatorname{dperm}_{\beta} A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \frac{e^{-\beta \operatorname{Tr} A}}{\det(I - A)^{\beta}}.$$
 (19)

Proof. Following the proof of Theorem 2.1 we find

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \mathrm{dperm}_{\beta} A(n^{\mathbf{r}}, n^{\mathbf{r}}) = \exp \left(\sum_{\gamma_{\sigma} \in \Gamma_{\sigma}, |\sigma| \ge 2} \frac{w(\gamma_{\sigma})}{|\mathrm{Aut}(\gamma_{\sigma})|} \right),$$

where cycles of length one are excluded. Using

$$\sum_{\gamma_{\sigma} \in \Gamma_{\sigma}, |\sigma| \ge 2} \frac{w(\gamma_{\sigma})}{|\operatorname{Aut}(\gamma_{\sigma})|} = \beta \sum_{s \ge 1} \frac{1}{s} \operatorname{Tr}(A^{s}) - \beta \operatorname{Tr} A$$
$$= -\beta \operatorname{Tr} \log(I - A) - \beta \operatorname{Tr} A,$$

the result follows. \square

Example. Consider n=1 with A=z. Then for multisets $\{1^r\}$ we find

$$\mathrm{dperm}_{\beta} A(1^r, 1^r) = d_r(\beta) z^r,$$

where $d_r(\beta) = \sum_s d_{rs}\beta^s$ is the generating polynomial for d_{rs} the number of derangements of r labels with s cycles. From Theorem 6.1 the exponential generating function for $d_r(\beta)$ is [HT]

$$\sum_{r>0} \frac{1}{r!} z^r d_r(\beta) = \left(\frac{e^{-z}}{1-z}\right)^{\beta}.$$

Finally, we can further generalize Theorem 6.1 to deranged permanents of submatrices as in Theorem 3.1. Using the notation of (11) and (12) we find using similar techniques that

Theorem 6.2

$$\sum_{\mathbf{r}} \frac{1}{\mathbf{r}!} \operatorname{dperm}_{\beta} \begin{bmatrix} B & U(n^{\mathbf{r}}) \\ V(n^{\mathbf{r}}) & A(n^{\mathbf{r}}, n^{\mathbf{r}}) \end{bmatrix} = \frac{e^{-\beta \operatorname{Tr} A} \cdot \operatorname{perm}_{\beta} \widehat{B}}{\det(I - A)^{\beta}}, \quad (20)$$

for $n' \times n'$ matrix

$$\widehat{B} = B - \operatorname{diag} B + U(I - A)^{-1}V,$$

where diag $B_{i'j'} = B_{i'i'}\delta_{i'j'}$. \square

References

- [FZ] Foata, D. and Zeilberger, D.: Laguerre polynomials, weighted derangements and positivity, SIAM J.Disc.Math. 1 (1988) 425–433.
- [HT] Hurley, D. and Tuite, M.P.: Virasoro correlation functions for vertex operator algebras, arXiv:1111.2170.
- [KP] Konvalinka, M. and Pak, I.: Non-commutative extensions of the MacMahon master theorem, Adv.Math. **216** (2007) 29-61.
- [MM] MacMahon, P.A.: Combinatory Analysis, Vols. 1 and 2, Cambridge University Press, (Cambridge 1915); reprinted by Chelsea (New York, 1955).
- [MT1] Mason, G. and Tuite, M.P.: Free bosonic vertex operator algebras on genus two Riemann surfaces I, Commun.Math.Phys. **300** (2010) 673–713.
- [MT2] Mason, G. and Tuite, M.P.: Free bosonic vertex operator algebras on genus two Riemann surfaces II, arXiv:1111.2264.
- [TZ1] Tuite, M.P. and Zuevsky, A.: Genus two partition and correlation functions for fermionic vertex operator superalgebras I, Commun.Math.Phys. 306 (2011) 419–447.
- [TZ2] Tuite, M.P. and Zuevsky, A.: Genus two partition and correlation functions for fermionic vertex operator superalgebras II, to appear.
- [TZ3] Tuite, M.P. and Zuevsky, A.: The bosonic vertex operator algebra on a genus g Riemann surface, to appear.
- [W] Wenchang, C: Determinant, permanent, and MacMahon's Master Theorem, Lin.Alg.andAppl. **255** (1997) 171-183.