# Weil Representation, Deligne Sheaf and Proof of the Kurlberg-Rudnick Conjecture

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by

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## Abstract

Consider the two dimensional symplectic torus  $(\mathbf{T}, \omega)$  and an hyperbolic automorphism A of  $\mathbf{T}$ . The automorphism A is known to be ergodic. In 1980, using a non-trivial procedure called quantization, the physicists J. Hannay and M.V. Berry attached to this automorphism a quantum operator  $\rho_{\hbar}(A)$  acting on a Hilbert space  $\mathcal{H}_{\hbar}$ . One of the central questions of "Quantum Chaos Theory", in this model, is whether the operator  $\rho_{\hbar}(A)$  is "quantum ergodic"?

We consider the following two distributions on the algebra  $\mathcal{A} = \mathcal{C}^{\infty}(\mathbf{T})$  of smooth complex valued functions on  $\mathbf{T}$ . The first one is given by the *Haar* integral:

$$f \longmapsto \int_{\mathbf{T}} f\omega$$

and the second one is given by the Wigner distribution:

$$f \longmapsto \mathcal{W}_{\chi}(f)$$

defined as the expectation of the "quantum observable"  $\pi_{\hbar}(f)$  in the Hecke state  $v_{\chi}$ , i.e.  $\mathcal{W}_{\chi}(f) := \langle v_{\chi} | \pi_{\hbar}(f) v_{\chi} \rangle$ . Here the vector  $v_{\chi}$  is a common eigenvector, with eigencharacter  $\chi$ , of the *Hecke* group of symmetries of the quantum operator  $\rho_{\hbar}(A)$ .

The Kurlberg-Rudnick rate conjecture is a quantitative description of the behavior of the Wigner distribution attached to the ergodic automorphism A. It states that for Planck constant of the form  $\hbar = \frac{1}{p}$ , where p is a prime number, one has:

Rate Conjecture. The following bound holds:

$$\left| \mathcal{W}_{\chi}(f) - \int_{\mathbf{T}} f\omega \right| \leq \frac{C_f}{\sqrt{p}}$$

where  $C_f$  is a constant that depends only on the function f.

In the current thesis we present a proof of the Kurlberg-Rudnick conjecture. This is

carried out using new representation theoretic constructions and algebro-geometric sheaf realization of the Weil metaplectic representation, which was proposed by P. Deligne in 1982.

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## Introduction

### Hannay-Berry model

In the paper "Quantization of linear maps on the torus - Fresnel diffraction by a periodic grating", published in 1980 [HB], the physicists and J. Hannay and M.V. Berry explore a model for quantum mechanics on the two dimensional symplectic torus  $(\mathbf{T}, \omega)$ . Hannay and Berry suggested to quantize simultaneously the functions on the torus and the linear symplectic group  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ .

### Quantum chaos

One of their main motivations was to study the phenomenon of quantum chaos [R2, S2] in this model. More precisely, they considered an ergodic discrete dynamical system on the torus, which is generated by an hyperbolic automorphism  $A \in \mathrm{SL}_2(\mathbb{Z})$ . Quantizing the system, we replace: the classical phase space  $(\mathbf{T}, \omega)$  by a Hilbert space  $\mathcal{H}_{\hbar}$ , classical observables, i.e., functions  $f \in C^{\infty}(\mathbf{T})$ , by operators  $\pi_{\hbar}(f) \in \mathrm{End}(\mathcal{H}_{\hbar})$  and classical symmetries by a unitary representation  $\rho_{\hbar} : \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{U}(\mathcal{H}_{\hbar})$ . A fundamental metaquestion in the area of quantum chaos is to understand the ergodic properties of the quantum system  $\rho_{\hbar}(A)$ , at least in the semi-classical limit as  $\hbar \to 0$ .

### Hecke quantum unique ergodicity

This question was addressed in a paper by Kurlberg and Rudnick [KR1]. In this paper they formulated a rigorous definition of quantum ergodicity for the case  $\hbar = \frac{1}{p}$ . The following is a brief description of that work. The basic observation is that the representation  $\rho_h$  factors through the quotient group  $\Gamma_p \simeq \mathrm{SL}_2(\mathbb{F}_p)$ . We denote by  $\mathrm{T}_A \subset \Gamma_p$  the centralizer of the element A, now considered as an element of the quotient group  $\Gamma_p$ . The group  $\mathrm{T}_A$  is called (cf. [KR1]) the Hecke torus corresponding to the element A. The Hecke torus

acts semisimply on  $\mathcal{H}_{\hbar}$ . Therefore we have a decomposition:

$$\mathcal{H}_{\hbar} = \bigoplus_{\chi: \mathrm{T}_A \longrightarrow \mathbb{C}^*} \mathcal{H}_{\chi}$$

where  $\mathcal{H}_{\chi}$  is the Hecke eigenspace corresponding to the character  $\chi$ . Considering a unit vector  $v \in \mathcal{H}_{\chi}$ , one defines the Wigner distribution  $\mathcal{W}_{\chi}: C^{\infty}(\mathbf{T}) \longrightarrow \mathbb{C}$  by the formula  $\mathcal{W}_{\chi}(f) := \langle v | \pi_{\hbar}(f) v \rangle$ . The main statement in [KR1] asserts about an explicit bound of the semi-classical asymptotic of  $\mathcal{W}_{\chi}(f)$ :

$$\left| \mathcal{W}_{\chi}(f) - \int_{\mathbf{T}} f\omega \right| \leq \frac{C_f}{p^{1/4}}$$

where  $C_f$  is a constant that depends only on the function f. In Rudnick's lectures at MSRI, Berkeley 1999 [R1] and ECM, Barcelona 2000 [R2] he conjectured that a stronger bound should hold true, namely:

Conjecture (Rate Conjecture). The following bound holds:

$$\left| \mathcal{W}_{\chi}(f) - \int_{\mathbf{T}} f\omega \right| \leq \frac{C_f}{p^{1/2}}.$$

The basic clues suggesting the validity of this stronger bound come from two main sources. The first source is computer simulations [Ku] accomplished over the years to give extremely precise bounds for considerably large values of p. A more mathematical argument is based on the fact that for special values of p, in which the Hecke torus splits, namely  $T_A \simeq \mathbb{F}_p^*$ , one is able to compute explicitly the eigenvector  $v \in \mathcal{H}_\chi$  and as a consequence to give an explicit formula for the Wigner distribution [KR2, DGI]. More precisely, in case  $\xi \in \mathbf{T}^\vee$ , i.e., a character, the distribution  $\mathcal{W}_\chi(\xi)$  turns out to be equal to an exponential sum very much similar to the Kloosterman sum:

$$\frac{1}{p} \sum_{a \in \mathbb{F}_p^*} \psi\left(\frac{a+1}{a-1}\right) \sigma(a) \chi(a)$$

where  $\sigma$  denotes the Legendre character. In this case the classical Weil bound [W1] yields the result. In this thesis a proof of the rate conjecture, for all tori (split or inert) simultaneously, is presented. For this peropus we view things from a more abstract perspective.

### Geometric approach (Deligne sheaf)

The basic observation to be made is that the theory of quantum mechanics on the torus, in case  $\hbar = \frac{1}{p}$ , can be equivalently recast in the language of representation theory of finite

groups in characteristic p. We will endeavor to give a more precise explanation of this matter. Consider the quotient  $\mathbb{F}_p$ -vector space  $V = \mathbf{T}^{\vee}/p\mathbf{T}^{\vee}$ , where  $\mathbf{T}^{\vee}$  is the lattice of characters on  $\mathbf{T}$ . We denote by E = E(V) the Heisenberg group. The group  $\Gamma_p \cong \mathrm{SL}_2(\mathbb{F}_p)$  is naturally identified with the group of linear symplectomorphisms of V. We have an action of  $\Gamma_p$  on E. The Stone-von Neumann theorem states that there exists a unique irreducible representation  $\pi: E \longrightarrow \mathrm{GL}(\mathcal{H})$ , with the non-trivial central character  $\psi$ , for which its isomorphism class is fixed by  $\Gamma_p$ . This is equivalent to saying that  $\mathcal{H}$  is equipped with a compatible projective representation  $\rho: \Gamma_p \longrightarrow \mathrm{PGL}(\mathcal{H})$ . Noting that E and E0 are the sets of rational points of corresponding algebraic groups, it is natural to ask whether there exists an algebro-geometric object that underlies the pair  $(\pi, \rho)$ ? The answer to this question is positive. The construction is proposed in an unpublished letter that was sent in 1982 from Pierre Deligne to David Kazhdan [D1]. Parts of this letter will be published for the first time in this thesis. In one sentence, the content of this letter is a construction of Representation Sheaves  $\mathcal{K}_{\pi}$  and  $\mathcal{K}_{\rho}$  on the algebraic varieties E1 and E2 respectively. One obtains, as a consequence, the following general principle:

(\*) Motivic principle. All quantum mechanical quantities in the Hannay-Berry model are motivic in nature.

By this we mean that every quantum-mechanical quantity Q, is associated with a vector space  $V_Q$  endowed with a Frobenius action  $Fr: V_Q \longrightarrow V_Q$  s.t.:

$$Q = \operatorname{Tr}(\operatorname{Fr}_{|_{V_{\mathcal{O}}}}).$$

The main contribution of this paper is to implement this principle. In particular we show that there exists a two dimensional vector space  $V_{\chi}$ , endowed with an action  $Fr: V_{\chi} \longrightarrow V_{\chi}$  s.t.:

$$\mathcal{W}_{\chi}(\xi) = \operatorname{Tr}(\operatorname{Fr}_{|_{\operatorname{V}_{\chi}}}).$$

This, combined with a bound on the modulus of the eigenvalues of Frobenius, i.e.,

$$\left| \operatorname{e.v}(\operatorname{Fr}_{|_{\operatorname{V}_{\chi}}}) \right| \le \frac{1}{p^{1/2}},$$

completes the proof of the rate conjecture.

## Side remarks

There are two remarks we would like to make at this point:

Remark 1: Discreteness principle. "Every" quantity  $\mathcal{Q}$  that appears in the Hannay-Berry model admits discrete spectrum in the following arithmetic sense: the modulus  $|\mathcal{Q}|$  can take only values of the form  $p^{i/2}$  for  $i \in \mathbb{Z}$ . This is a consequence of principle (\*) and Deligne's weight theory [D2]. We believe that this principle can be effectively used in various situations in order to derive strong bounds out of weaker bounds. A striking example is an alternative trivial "proof" for the bound  $|\mathcal{W}_{\chi}(\xi)| \leq \frac{C_{\xi}}{p^{1/2}}$ :

$$|\mathcal{W}_{\chi}(\xi)| \leq \frac{C_{\xi}}{p^{1/4}} \Rightarrow |\mathcal{W}_{\chi}(\xi)| \leq \frac{C_{\xi}}{p^{1/2}}.$$

Kurlberg and Rudnick proved in their paper [KR1] the weak bound  $|\mathcal{W}_{\chi}(\xi)| \leq \frac{C_{\xi}}{p^{1/4}}$ . This directly implies (under certain mild assumptions) that the stronger bound  $|\mathcal{W}_{\chi}(\xi)| \leq \frac{C_{\xi}}{p^{1/2}}$  is valid.

Remark 2: Higher dimensional exponential sums. Proving the bound  $|\mathcal{W}_{\chi}(f)| \leq \frac{C_f}{\sqrt{p}}$  can be equivalently stated as bounding by  $\frac{C_f}{\sqrt{p}}$  the spectral radius of the operator  $\mathbf{Av}_{\mathrm{T}_A}(f) := \frac{1}{|\mathrm{T}_A|} \sum_{B \in \mathrm{T}_A} \rho_{h}(B) \pi_{h}(f) \rho_{h}(B^{-1})$ . This implies a bound on the  $L_N$  norms, for every  $N \in \mathbb{Z}^+$ :

$$\|\mathbf{A}\mathbf{v}_{\mathrm{T}_{A}}(f)\|_{N} \le \frac{C_{f}}{p^{N}}.\tag{1}$$

In particular for  $0 \neq f = \xi \in \mathbf{T}^{\vee}$  one can compute explicitly the left hand side of (1) and obtain:

$$\|\mathbf{A}\mathbf{v}_{T_A}(\xi)\|_N := \text{Tr}(|\mathbf{A}\mathbf{v}_{T_A}(\xi)|^N) = \frac{1}{|T_A|^{2N}} \sum_{(x_1,\dots,x_{2N})\in X} \psi(\sum_{i< j} \omega(x_i,x_j))$$

where  $X := \{(x_1, \ldots, x_{2N}) | x_i \in \mathcal{O}_{\xi}, \sum x_i = 0\}$  and  $\mathcal{O}_{\xi} := \mathrm{T}_A \cdot \xi \subset \mathrm{V}$  denotes the orbit of  $\xi$  under the action of  $\mathrm{T}_A$ . Therefore referring to (1) we obtained a non-trivial bound for a higher dimensional exponential sum. It would be interesting to know whether there exists an independent proof for this bound and whether this representation theoretic approach can be used to prove optimal bounds for other interesting higher dimensional exponential sums.

### Sato-Tate conjecture

The next level of the theory is to understand the *complete statistics* of the Hecke-Wigner distributions for different Hecke states. More precisely, let us fix a character  $\xi \in \mathbf{T}^{\vee}$ . For every character  $\chi: T_A \longrightarrow \mathbb{C}^*$  we consider the normalized value  $\tilde{\mathcal{W}}_{\chi}(\xi) := \frac{1}{2\sqrt{p}}\mathcal{W}_{\chi}(\xi)$ ,

which lies in the interval [-1, 1]. Now running over all multiplicative characters we define the following atomic measure on the interval [-1, 1]:

$$\mu_p := \frac{1}{|\mathcal{T}_A|} \sum_{\chi} \delta_{\tilde{\mathcal{W}}_{\chi}(\xi)}.$$

One would like to describe the limit measure (if it exists!). This is the content of another conjecture of Kurlberg and Rudnick [KR2]:

Conjecture (Sato-Tate Conjecture). The following limit exists:

$$\lim_{p \to \infty} \mu_p = \mu_{ST}$$

where  $\mu_{ST}$  is the projection of the Haar measure on  $S^1$  to the interval [-1,1].

We hope that by using the methodology described in this paper one will be able to gain some progress in proving this conjecture.

**Remark.** Note that the family  $\{\tilde{\mathcal{W}}_{\chi}(\xi)\}_{\chi\in T_A^*}$  runs over a non-algebraic space of parameters. Hence Deligne's equidistribution theory (cf. Weil II [D2]) can not be applied directly in order to solve the Sato-Tate Conjecture.

#### Results

- 1. **Kurlberg-Rudnick conjecture.** The main result of the current work is Theorem 2.1.1, which is the *proof* of the Kurlberg-Rudnick rate conjecture on the asymptotic behavior of the Hecke-Wigner distributions.
- 2. Weil representation and the Hannay-Berry model. We introduce two new constructions of the Weil representation over  $\mathbb{Z}$  and over the finite fields  $\mathbb{F}_q$  of characteristic  $\neq 2$ . As an application we obtained a construction of the Hannay-Berry model.
  - (a) The first construction is stated in Theorem 1.2.2, Corollary 4.1.4 and Corollary A.1.2. It is based on the Rieffel quantum torus  $\mathcal{A}_{\hbar}$ , for  $\hbar \in \mathbb{Q}$ . This approach is essentially equivalent to the classical approach (cf. [Kl, W2]) that uses the representation theory of the Heisenberg group in characteristic p. As an application we obtained (Chapters 1, 4 and Appendix A) a construction of the Hannay-Berry model of quantum mechanics for Tori in all dimensions. This is a new realization of the Hannay-Berry model. This was an important

achievement, since the original Hannay-Berry model was formulated in physical terms. In particular, using this new approach we were able to construct (Chapters 1 and 4) a slightly more general model which has a larger and algebraic group of symmetries, namely the whole symplectic group  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ . This was then generalized (Appendix A) to the higher dimensional tori and the groups  $\mathrm{Sp}(2n,\mathbb{Z})$ .

(b) Canonical Hilbert space (Kazhdan's question). The second construction uses our new "Method of Canonical Hilbert Space" (see Chapter 3). This approach is based on the following statement:

Proposition (Canonical Hilbert Space). Let  $(V, \omega)$  be a two dimensional symplectic vector space over the finite field  $\mathbb{F}_q$ . There exists a canonical Hilbert space  $\mathcal{H}_V$  attached to V.

An immediate consequence of this proposition is that all symmetries of  $(V, \omega)$  automatically act on  $\mathcal{H}_V$ . In particular, we obtain a *linear* representation of the group  $\mathrm{Sp}_{\omega} := \mathrm{Sp}_{\omega}(V, \omega)$  on  $\mathcal{H}_V$ . Probably this approach has higher dimensional generalization, for the case where V is of dimension 2n. This will be a subject of a future publication.

**Remark.** Note the main difference of our construction from the classical approach due to Weil (cf. [W2]). The classical construction proceeds in two stages. Firstly, one obtains a projective representation of  $\mathrm{Sp}_{\omega}$  and secondly using general arguments about the group  $\mathrm{Sp}_{\omega}$ , one proves the existence of a linearization. A consequence of our approach is that there exists a distinguished linear representation and its existence is not related to any group theoretic property of  $\mathrm{Sp}_{\omega}$ . We would like to mention that this approach answers, in the case of the two dimensional Heisenberg group, a question of David Kazhdan [Ka] dealing with the existence of Canonical Hilbert Spaces for co-adjoint orbits of general unipotent groups. The main motive behind our construction is the notion of oriented Lagrangian subspace. This idea was suggested to us by Joseph Bernstein [B].

- 3. Deligne's Weil representation sheaf and applications. In our work we developed  $\ell$ -adic geometric techniques for the investigation of the Weil representation in general and the Hannay-Berry model in particular. These techniques are based on Deligne's letter to Kazhdan [D1]. We include for the sake of completeness (see Chapter 3 section 3.4) a formal presentation of Deligne's letter, that places the Weil representation on a complete algebro-geometric ground. These techniques play a central rule in the proof of the Kurlberg-Rudnick conjecture.
- 4. The higher-dimensional Kurlberg-Rudnick conjecture. Lately we were able

[GH4] to extend some of our results to the higher-dimensional tori. This will a subject of a future research and publication. However, we want to note here an interesting phenomenon (see Appendix B). Namely, the two dimensional Kurlberg-Rudnick conjecture in its original formulation, that claims that any ergodic automorphism of the torus is represented by a Hecke quantum ergodic operator, is not true in higher dimensions:

**Observation.** Let **T** be the 2n dimensional torus, n > 1. There exists an element  $A \in \operatorname{Sp}(2n, \mathbb{Z})$  which acts ergodically on **T** such that the corresponding quantum operator  $\rho_h(A)$  is not Hecke ergodic.

The discussion on the *meaning* of this observation and the search for a "correct formulation" to the conjectures, that will be valid in any dimension, will be a subject of a future research in the field of quantum chaos.

#### Structure of the thesis

The paper is naturally separated into four parts:

**Part I.** Chapter **1**. In this chapter we *present* the *Hannay-Berry model*. In section 1.1 we discuss classical mechanics on the torus. In section 1.2 we discuss quantum mechanics á-la Hannay and Berry, using the Rieffel quantum torus model. This part of the paper is self-contained and consists of mainly linear algebraic considerations.

Part II. Chapter 2. This is the main part of the paper, consisting of the formulation and the proof of the Kurlberg-Rudnick conjecture. In section 2.1 we formulate the Hecke quantum unique ergodicity conjecture of Kurlberg-Rudnick (Theorem 2.1.1). In section 2.2 the proof is given in two stages. The first stage consists of mainly linear algebra manipulations to obtain a more transparent formulation of the statement, resulting in Theorem 2.2.2. In the second stage we venture into algebraic geometry. All linear algebraic constructions are replaced by sheaf theoretic objects, concluding with the Geometrization Theorem, i.e., Theorem 2.2.4. Next, the statement of Theorem 2.2.2 is reduced to a geometric statement, the Vanishing Lemma, i.e., Lemma 2.2.6. The remainder of the chapter is devoted to the proof of Lemma 2.2.6. For the convenience of the reader we include a large body of intuitive explanations for all the constructions involved. In particular, we devote some space explaining the Grothendieck's Sheaf to Function Correspondence procedure which is the basic bridge connecting sections 2.1 and 2.2.

**Part III.** Chapter **3**. In section 3.1 we describe the *method of canonical Hilbert space*. In section 3.2 we describe the Weil representation in this manifestation. In section 3.3

we relate the invariant construction to the more classical constructions, supplying explicit formulas that will be used later. In section 3.4 we give a formal presentation of Deligne's letter to Kazhdan [D1]. The main statement of this section is Theorem 3.4.2, in which the Weil representation sheaf K is introduced. We include in our presentation only the parts of that letter which are most relevant to our needs. In particular, we consider only the two dimensional case of this letter. In section 3.5 we supply proofs for all technical lemmas and propositions appearing in the previous sections of the chapter.

- Part IV. Chapter 4. Here we present the formal construction of the two dimensional Hannay-Berry model that were used in the previous chapters.
- **Part V.** Appendices **A** and **B**. In Appendix A we give the construction of the Hannay-Berry model for the higher-dimensional tori. In Appendix B we give the *example* of a symplectic automorphism  $A \in \operatorname{Sp}(2n,\mathbb{Z})$  which acts ergodically on the higher dimensional torus **T**, but is represented by a quantum operator  $\rho_h(A)$  which is not Hecke ergodic.
- **Part VI.** Appendix C. In this Appendix we supply the proofs for all statements appearing in Part I and Part II. In particular, we give the *proof* of the *Geometrization Theorem* (Theorem 2.2.4) which essentially consists of taking the *Trace* of Deligne's *Weil representation sheaf* K.

# Chapter 1

## The Hannay-Berry Model

#### 1.1 Classical Torus

Let  $(\mathbf{T}, \omega)$  be the two dimensional symplectic torus. Together with its linear symplectomorphisms  $\Gamma \simeq \mathrm{SL}_2(\mathbb{Z})$  it serves as a simple model of classical mechanics (a compact version of the phase space of the harmonic oscillator). More precisely, let  $\mathbf{T} = W/\Lambda$  where W is a two dimensional real vector space, i.e.,  $W \simeq \mathbb{R}^2$  and  $\Lambda$  is a rank two lattice in W, i.e.,  $\Lambda \simeq \mathbb{Z}^2$ . We obtain the symplectic form on  $\mathbf{T}$  by taking a non-degenerate symplectic form on W:

$$\omega: W \times W \longrightarrow \mathbb{R}$$
.

We require  $\omega$  to be integral, namely  $\omega: \Lambda \times \Lambda \longrightarrow \mathbb{Z}$  and normalized, i.e.,  $\operatorname{Vol}(\mathbf{T}) = 1$ .

Let  $\mathrm{Sp}(W,\omega)$  be the group of linear symplectomorphisms, i.e.,  $\mathrm{Sp}(W,\omega) \simeq \mathrm{SL}_2(\mathbb{R})$ . Consider the subgroup  $\Gamma \subset \mathrm{Sp}(W,\omega)$  of elements that preserve the lattice  $\Lambda$ , i.e.,  $\Gamma(\Lambda) \subseteq \Lambda$ . Then  $\Gamma \simeq \mathrm{SL}_2(\mathbb{Z})$ . The subgroup  $\Gamma$  is the group of linear symplectomorphisms of  $\mathbf{T}$ . We denote by  $\Lambda^* \subseteq W^*$  the dual lattice,  $\Lambda^* = \{\xi \in W^* | \xi(\Lambda) \subset \mathbb{Z}\}$ . The lattice  $\Lambda^*$  is identified with the lattice of characters of  $\mathbf{T}$  by the following map:

$$\xi \in \Lambda^* \longmapsto e^{2\pi i < \xi, \cdot>} \in \mathbf{T}^{\vee}$$

where  $\mathbf{T}^{\vee} := \text{Hom}(\mathbf{T}, \mathbb{C}^*)$ .

### 1.1.1 Classical mechanical system

We consider a very simple discrete mechanical system. An hyperbolic element  $A \in \Gamma$ , i.e., |Tr(A)| > 2, generates an ergodic discrete dynamical system. The *Birkhoff's Ergodic* 

Theorem states that:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(A^k x) = \int_{\mathbf{T}} f\omega$$

for every  $f \in \mathcal{S}(\mathbf{T})$  and for almost every point  $x \in \mathbf{T}$ . Here  $\mathcal{S}(\mathbf{T})$  stands for a good class of functions, for example trigonometric polynomials or smooth functions.

We fix an hyperbolic element  $A \in \Gamma$  for the remainder of the paper.

### 1.2 Quantization of the Torus

Quantization is one of the big mysteries of modern mathematics, indeed it is not clear at all what is the precise structure which underlies quantization in general. Although physicists have been using quantization for almost a century, for mathematicians the concept remains all-together unclear. Yet, in specific cases, there are certain formal models for quantization that are well justified mathematically. The case of the symplectic torus is one of these cases. Before we employ the formal model, it is worthwhile to discuss the general phenomenological principles of quantization which are surely common for all models.

Let us start with a model of classical mechanics, namely a symplectic manifold, serving as a classical phase space. In our case this manifold is the symplectic torus  $\mathbf{T}$ . Principally, quantization is a protocol by which one associates a quantum "phase" space  $\mathcal{H}$  to the classical phase space  $\mathbf{T}$ , where  $\mathcal{H}$  is a Hilbert space. In addition, the protocol gives a rule by which one associates to every classical observable, namely a function  $f \in \mathcal{S}(\mathbf{T})$ , a quantum observable  $\mathrm{Op}(f): \mathcal{H} \longrightarrow \mathcal{H}$ , an operator on the Hilbert space. This rule should send a real function into a self adjoint operator.

To be more precise, quantization should be considered not as a single protocol, but as a one parameter family of protocols, parameterized by  $\hbar$ , the Planck constant. For every fixed value of the parameter  $\hbar$  there is a protocol which associates to  $\mathbf{T}$  a Hilbert space  $\mathcal{H}_{\hbar}$  and for every function  $f \in \mathcal{S}(\mathbf{T})$  an operator  $\operatorname{Op}_{\hbar}(f) : \mathcal{H}_{\hbar} \longrightarrow \mathcal{H}_{\hbar}$ . Again the association rule should send real functions to self adjoint operators.

Accepting the general principles of quantization, one searches for a formal model by which to quantize, that is a mathematical model which will manufacture a family of Hilbert spaces  $\mathcal{H}_{\hbar}$  and association rules  $\mathcal{S}(\mathbf{T}) \leadsto \operatorname{End}(\mathcal{H}_{\hbar})$ . In this work we employ a model of quantization called the Weyl Quantization model.

#### 1.2.1 The Weyl quantization model

The Weyl quantization model works as follows. Let  $\mathcal{A}_{\hbar}$  be a one parameter deformation of the algebra  $\mathcal{A}$  of trigonometric polynomials on the torus. This algebra is known in the literature as the Rieffel torus [Ri]. The algebra  $\mathcal{A}_{\hbar}$  is constructed by taking the free algebra over  $\mathbb{C}$  generated by the symbols  $\{s(\xi) \mid \xi \in \Lambda^*\}$  and quotient out by the relation  $s(\xi + \eta) = e^{\pi i \hbar \omega(\xi, \eta)} s(\xi) s(\eta)$ . We point out two facts about the algebra  $\mathcal{A}_{\hbar}$ . First, when substituting  $\hbar = 0$  one gets the group algebra of  $\Lambda^*$ , which is exactly equal to the algebra of trigonometric polynomials on the torus. Second, the algebra  $\mathcal{A}_{\hbar}$  contains as a standard basis the lattice  $\Lambda^*$ :

$$s: \Lambda^* \longrightarrow \mathcal{A}_{\hbar}$$
.

Therefore one can identify the algebras  $\mathcal{A}_{\hbar} \simeq \mathcal{A}$  as vector spaces. Therefore, every function  $f \in \mathcal{A}$  can be viewed as an element of  $\mathcal{A}_{\hbar}$ .

For a fixed  $\hbar$  a representation  $\pi_{\hbar}: \mathcal{A}_{\hbar} \longrightarrow \operatorname{End}(\mathcal{H}_{\hbar})$  serves as a quantization protocol, namely for every function  $f \in \mathcal{A}$  one has:

$$f \in \mathcal{A} \simeq \mathcal{A}_{\hbar} \longmapsto \pi_{\hbar}(f) \in \operatorname{End}(\mathcal{H}_{\hbar}).$$

An equivalent way of saying this is:

$$f \longmapsto \sum_{\xi \in \Lambda^*} a_{\xi} \pi_{\hbar}(\xi)$$

where  $f = \sum_{\xi \in \Lambda^*} a_{\xi} \cdot \xi$  is the Fourier expansion of f.

To summarize: every family of representations  $\pi_{\hbar}: \mathcal{A}_{\hbar} \longrightarrow \operatorname{End}(\mathcal{H}_{\hbar})$  gives us a complete quantization protocol. Yet, a serious question now arises, namely what representations to choose? Is there a correct choice of representations, both mathematically, but also perhaps physically? A possible restriction on the choice is to choose an irreducible representation. Yet, some ambiguity still remains because there are several irreducible classes for specific values of  $\hbar$ .

We present here a partial solution to this problem in the case where the parameter  $\hbar$  is restricted to take only rational values (see Chapter 4 for the construction in this generality). Even more particularly, for our purpose we will take  $\hbar$  to be of the form  $\hbar = \frac{1}{p}$  where p is an odd prime number. Before any formal discussion one should recall that our classical object is the symplectic torus  $\mathbf{T}$  together with its linear symplectomorphisms  $\Gamma$ . We would like to quantize not only the observables  $\mathcal{A}$ , but also the symmetries  $\Gamma$ . Next, we are going to construct an equivariant quantization of  $\mathbf{T}$ .

#### 1.2.2 Equivariant Weyl quantization of the torus

Let  $\hbar = \frac{1}{p}$  and consider a non-trivial additive character  $\psi : \mathbb{F}_p \longrightarrow \mathbb{C}^*$ . We give here a slightly different presentation of the algebra  $\mathcal{A}_{\hbar}$ . Let  $\mathcal{A}_{\hbar}$  be the free  $\mathbb{C}$ -algebra generated by the symbols  $\{s(\xi) \mid \xi \in \Lambda^*\}$  and the relations  $s(\xi + \eta) = \psi(\frac{1}{2}\omega(\xi, \eta))s(\xi)s(\eta)$ . Here we consider  $\omega$  as a map  $\omega : \Lambda^* \times \Lambda^* \longrightarrow \mathbb{F}_p$ . The lattice  $\Lambda^*$  serves as a standard basis for  $\mathcal{A}_{\hbar}$ :

$$s: \Lambda^* \longrightarrow \mathcal{A}_{\hbar}$$
.

The group  $\Gamma$  acts on the lattice  $\Lambda^*$ , therefore it acts on  $\mathcal{A}_{\hbar}$ . It is easy to see that  $\Gamma$  acts on  $\mathcal{A}_{\hbar}$  by homomorphisms of algebras. For an element  $B \in \Gamma$ , we denote by  $f \longmapsto f^B$  the action of B on an element  $f \in \mathcal{A}_{\hbar}$ .

An equivariant quantization of the torus is a pair:

$$\pi_{\hbar}: \mathcal{A}_{\hbar} \longrightarrow \operatorname{End}(\mathcal{H}_{\hbar}),$$
 $\rho_{\hbar}: \Gamma \longrightarrow \operatorname{PGL}(\mathcal{H}_{\hbar})$ 

where  $\pi_{h}$  is a representation of  $\mathcal{A}_{h}$  and  $\rho_{h}$  is a projective representation of  $\Gamma$ . These two should be compatible in the following manner:

$$\rho_{h}(B)\pi_{h}(f)\rho_{h}(B)^{-1} = \pi_{h}(f^{B})$$
(1.2.1)

for every  $B \in \Gamma$  and  $f \in \mathcal{A}_{\hbar}$ . Equation (1.2.1) is called the *Egorov identity*.

Let us suggest now a construction of an equivariant quantization of the torus.

Given a representation  $\pi: \mathcal{A}_{\hbar} \longrightarrow \operatorname{End}(\mathcal{H})$  and an element  $B \in \Gamma$ , we construct a new representation  $\pi^B: \mathcal{A}_{\hbar} \longrightarrow \operatorname{End}(\mathcal{H})$ :

$$\pi^B(f) := \pi(f^B). \tag{1.2.2}$$

This gives an action of  $\Gamma$  on the set  $\operatorname{Irr}(\mathcal{A}_{\hbar})$  of classes of irreducible representations. The set  $\operatorname{Irr}(\mathcal{A}_{\hbar})$  has a very regular structure as a principal homogeneous space over  $\mathbf{T}$ . Moreover, every irreducible representation of  $\mathcal{A}_{\hbar}$  is finite dimensional and of dimension p. The following theorem (see Chapter 4 for the proof) plays a central role in the construction.

Theorem 1.2.1 (Canonical invariant representation) Let  $\hbar = \frac{1}{p}$ , where p is a prime. There exists a unique (up to isomorphism) irreducible representation  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  of  $\mathcal{A}_{\hbar}$  for which its equivalence class is fixed by  $\Gamma$ .

Let  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  be a representative of the fixed irreducible equivalence class. Then for every  $B \in \Gamma$  we have:

$$\pi_{\scriptscriptstyle h}^B \simeq \pi_{\scriptscriptstyle h}. \tag{1.2.3}$$

This means that for every element  $B \in \Gamma$  there exists an operator  $\rho_{\hbar}(B)$  acting on  $\mathcal{H}_{\hbar}$  which realizes the isomorphism (1.2.3). The collection  $\{\rho_{\hbar}(B) : B \in \Gamma\}$  constitutes a projective representation<sup>1</sup>:

$$\rho_{\hbar}: \Gamma \longrightarrow \mathrm{PGL}(\mathcal{H}_{\hbar}).$$
(1.2.4)

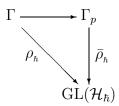
Equations (1.2.2) and (1.2.3) also imply the Egorov identity (1.2.1).

The group  $\Gamma \simeq \operatorname{SL}_2(\mathbb{Z})$  is almost a free group and it is finitely presented. A brief analysis (Chapter 4) shows that every projective representation of  $\Gamma$  can be lifted (linearized) into a true representation. More precisely, it can be linearized in 12 different ways, where 12 is the number of characters of  $\Gamma$ . In particular, the projective representation (1.2.4) can be linearized (not uniquely) into an honest representation. The next theorem asserts the existence of a canonical linearization. Let  $\Gamma_p \simeq \operatorname{SL}_2(\mathbb{F}_p)$  denotes the quotient group of  $\Gamma$  modulo p.

**Theorem 1.2.2 (Canonical linearization)** Let  $\hbar = \frac{1}{p}$ , where p is an odd prime. There exists a unique linearization:

$$\rho_{h}:\Gamma\longrightarrow \mathrm{GL}(\mathcal{H}_{\hbar})$$

characterized by the property that it factors through the quotient group  $\Gamma_p$ :



From now on  $\rho_h$  means the linearization of Theorem 1.2.2.

**Summary.** Theorem 1.2.1 confirms the existence of a unique invariant representation of  $\mathcal{A}_{\hbar}$ , for every  $\hbar = \frac{1}{p}$ . This gives a canonical equivariant quantization  $(\pi_{\hbar}, \rho_{\hbar}, \mathcal{H}_{\hbar})$ . Moreover, for p odd, by Theorem 1.2.2, the projective representation  $\rho_{\hbar}$  can be linearized in a canonical way to give an honest representation of  $\Gamma$  which factors through  $\Gamma_p^2$ . Altogether this gives a pair:

$$\pi_{\hbar}: \mathcal{A}_{\hbar} \longrightarrow \operatorname{End}(\mathcal{H}_{\hbar}), 
\rho_{\hbar}: \Gamma_{p} \longrightarrow \operatorname{GL}(\mathcal{H}_{\hbar})$$

satisfying the following compatibility condition (Egorov identity):

$$\rho_{\hbar}(B)\pi_{\hbar}(f)\rho_{\hbar}(B)^{-1}=\pi_{\hbar}(f^B)$$

<sup>&</sup>lt;sup>1</sup>This is the famous Weil representation (cf. [W2], Chapter 4 and Appendix A) of  $SL_2(\mathbb{Z})$ .

<sup>&</sup>lt;sup>2</sup>This is the Weil representation of  $SL_2(\mathbb{F}_p)$ .

for every  $B \in \Gamma_p$ ,  $f \in \mathcal{A}_\hbar$ . The notation  $\pi_{\hbar}(f^B)$  means that we take any pre-image  $\bar{B} \in \Gamma$  of  $B \in \Gamma_p$  and act by it on f, but the operator  $\pi_{\hbar}(f^{\bar{B}})$  does not depend on the choice of  $\bar{B}$ . In the following, we denote the Weil representation  $\bar{\rho}_{\hbar}$  by  $\rho_{\hbar}$  and consider  $\Gamma_p$  to be the default domain.

#### 1.2.3 Quantum mechanical system

Let  $(\pi_{\hbar}, \rho_{\hbar}, \mathcal{H}_{\hbar})$  be the canonical equivariant quantization. Let A be our fixed hyperbolic element, considered as an element of  $\Gamma_p$ . The element A generates a quantum dynamical system. For every (pure) quantum state  $v \in S(\mathcal{H}_{\hbar}) = \{v \in \mathcal{H}_{\hbar} : ||v|| = 1\}$ :

$$v \longmapsto v^A := \rho_{h}(A)v. \tag{1.2.5}$$

# Chapter 2

# The Kurlberg-Rudnick Conjecture

## 2.1 Hecke Quantum Unique Ergodicity

The main silent question of the current work is whether the system (1.2.5) is quantum ergodic. Before discussing this question, one is obliged to define a notion of quantum ergodicity. As a first approximation follow the classical definition, but replace each classical notion by its quantum counterpart. Namely, for every  $f \in \mathcal{A}_{\hbar}$  and almost every quantum state  $v \in S(\mathcal{H}_{\hbar})$ , the following holds:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \langle v | \pi_{\hbar}(f^{A^k}) v \rangle \stackrel{?}{=} \int_{\mathbf{T}} f \omega. \tag{2.1.1}$$

Unfortunately (2.1.1) is literally not true. The limit is never exactly equal to the integral for a fixed  $\hbar$ . Let us now give a true statement which is a slight modification of (2.1.1), called the *Hecke Quantum Unique Ergodicity*. First, rewrite (2.1.1) in an equivalent form. We have:

$$< v | \pi_{h}(f^{A^{k}}) v > = < v | \rho_{h}(A^{k}) \pi_{h}(f) \rho_{h}(A^{k})^{-1} v >$$
 (2.1.2)

using the Egorov identity (1.2.1).

Now, note that the elements  $A^k$  run inside the finite group  $\Gamma_p$ . Denote by  $\langle A \rangle \subseteq \Gamma_p$  the cyclic subgroup generated by A. It is easy to see, using (2.1.2), that:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \langle v | \pi_{\hbar}(f^{A^k}) v \rangle = \frac{1}{|\langle A \rangle|} \sum_{B \in \langle A \rangle} \langle v | \rho_{\hbar}(B) \pi_{\hbar}(f) \rho_{\hbar}(B)^{-1} v \rangle.$$

Altogether (2.1.1) can be written in the form:

$$\mathbf{Av}_{\langle A \rangle}(\langle v | \pi_{h}(f)v \rangle) \stackrel{?}{=} \int_{\mathbf{T}} f\omega \tag{2.1.3}$$

where  $\mathbf{Av}_{\langle A \rangle}$  denotes the average of  $\langle v | \pi_{\hbar}(f)v \rangle$  with respect to the group  $\langle A \rangle$ .

#### 2.1.1 Formulation of the Kurlberg-Rudnick conjecture

Denote by  $T_A$  the centralizer of A in  $\Gamma_p \subseteq \operatorname{SL}_2(\mathbb{F}_p)$ . The finite group  $T_A$  is an algebraic group. More particularly, as an algebraic group, it is a torus. We call  $T_A$  the *Hecke torus* (cf. [KR1]). One has,  $\langle A \rangle \subseteq T_A \subseteq \Gamma_p$ . Now, in (2.1.3) take the average with respect to the group  $T_A$  instead of the group  $\langle A \rangle$ . The precise statement of the **Kurlberg-Rudnick rate conjecture** (cf. [R1, R2]) is given in the following theorem:

Theorem 2.1.1 (Hecke Quantum Unique Ergodicity) Let  $\hbar = \frac{1}{p}$ , p an odd prime. For every  $f \in A_{\hbar}$  and  $v \in S(\mathcal{H}_{\hbar})$ , we have:

$$\left| \mathbf{A} \mathbf{v}_{\mathrm{T}_{A}}(\langle v | \pi_{h}(f)v \rangle) - \int_{\mathbf{T}} f\omega \right| \le \frac{C_{f}}{\sqrt{p}}, \tag{2.1.4}$$

where  $C_f$  is an explicit constant depending only on f.

The next section is devoted to proving Theorem 2.1.1.

## 2.2 Proof of the Kurlberg-Rudnick conjecture

The proof is given in two stages. The first stage is a preparation stage and consists mainly of linear algebra considerations. We massage statement (2.1.4) in several steps into an equivalent statement which will be better suited to our needs. In the second stage we introduce the main part of the proof. Here we invoke tools from algebraic geometry in the framework of  $\ell$ -adic sheaves and  $\ell$ -adic cohomology (cf. [M, BBD]).

### 2.2.1 Preparation stage

**Step 1.** It is enough to prove Theorem 2.1.1 for the case when f is a non-trivial character,  $\xi \in \Lambda^*$ . Because  $\int_{\mathbf{T}} \xi \omega = 0$ , statement (2.1.4) becomes:

$$\left| \mathbf{A} \mathbf{v}_{\mathrm{T}_{A}}(\langle v | \pi_{\hbar}(\xi) v \rangle) \right| \le \frac{C_{\xi}}{\sqrt{p}}. \tag{2.2.1}$$

The statement for general  $f \in \mathcal{A}_{\hbar}$  follows directly from the triangle inequality.

**Step 2.** It is enough to prove (2.2.1) in case  $v \in S(\mathcal{H}_h)$  is a *Hecke* eigenvector. To

be more precise, the *Hecke* torus  $T_A$  acts semisimply on  $\mathcal{H}_{\hbar}$  via the representation  $\rho_{\hbar}$ , thus  $\mathcal{H}_{\hbar}$  decomposes to a direct sum of character spaces:

$$\mathcal{H}_{\hbar} = \bigoplus_{\chi: T_A \longrightarrow \mathbb{C}^*} \mathcal{H}_{\chi}. \tag{2.2.2}$$

The sum in (2.2.2) is over multiplicative characters of the torus  $T_A$ . For every  $v \in \mathcal{H}_{\chi}$  and  $B \in T_A$ , we have:

$$\rho_{\scriptscriptstyle h}(B)v = \chi(B)v.$$

Taking  $v \in \mathcal{H}_{\chi}$ , statement (2.2.1) becomes:

$$|\langle v|\pi_{\hbar}(\xi)v\rangle| \le \frac{C_{\xi}}{\sqrt{p}}.$$
(2.2.3)

Here  $C_{\xi} = 2$ .

The averaged operator:

$$\mathbf{Av}_{\mathrm{T}_A}(\pi_{\scriptscriptstyle\hbar}(\xi)) := \frac{1}{|\mathrm{T}_A|} \sum_{B \in \mathrm{T}_A} \rho_{\scriptscriptstyle\hbar}(B) \pi_{\scriptscriptstyle\hbar}(\xi) \rho_{\scriptscriptstyle\hbar}(B)^{-1}$$

is essentially diagonal in the Hecke base. Knowing this, statement (2.2.1) follows from (2.2.3) by invoking the triangle inequality.

Step 3. Let  $P_{\chi}: \mathcal{H}_{\hbar} \longrightarrow \mathcal{H}_{\hbar}$  be the orthogonal projector on the eigenspace  $\mathcal{H}_{\chi}$ .

**Remark 2.2.1** For  $\chi$  other then the quadratic character of  $T_A$  we have dim  $\mathcal{H}_{\chi} = 1.^2$ 

Using Remark 2.2.1 we can rewrite (2.2.3) in the form:

$$|\mathrm{Tr}(P_{\chi}\pi_{\hbar}(\xi))| \leq \frac{2}{\sqrt{p}}.$$

The projector  $P_{\chi}$  can be defined in terms of the representation  $\rho_{\hbar}$ :

$$P_{\chi} = \frac{1}{|\mathcal{T}_A|} \sum_{B \in \mathcal{T}_A} \chi(B) \rho_{\hbar}(B).$$

<sup>&</sup>lt;sup>1</sup>This follows from Remark 2.2.1. If  $T_A$  does not split over  $\mathbb{F}_p$  then  $\mathbf{Av}_{T_A}(\pi_h(\xi))$  is diagonal in the Hecke basis. In case  $T_A$  splits then for the Legendre character  $\sigma$  we have that dim  $\mathcal{H}_{\sigma} = 2$ . However, in the later case one can prove (2.2.1) for  $v \in \mathcal{H}_{\sigma}$  by a computation of explicit eigenvectors (cf. [KR2]).

<sup>&</sup>lt;sup>2</sup>This fact, which is needed if we want to stick with the matrix coefficient formulation of the conjecture, can be proven by algebro-geometric techniques or alternatively by a direct computation (cf. [Ge]).

Now write (2.2.3):

$$\frac{1}{|\mathcal{T}_A|} \left| \sum_{B \in \mathcal{T}_A} \operatorname{Tr}(\rho_{\hbar}(B) \pi_{\hbar}(\xi)) \chi(B) \right| \le \frac{2}{\sqrt{p}}.$$
 (2.2.4)

On noting that  $|T_A| = p \pm 1$  and multiplying both sides of (2.2.4) by  $|T_A|$  we obtain the following statement:

Theorem 2.2.2 (Hecke Quantum Unique Ergodicity (Restated)) Let  $\hbar = \frac{1}{p}$ , where p is an odd prime. For every  $\xi \in \Lambda^*$  and every character  $\chi$  the following holds:

$$\left| \sum_{B \in \mathcal{T}_A} \operatorname{Tr}(\rho_{\scriptscriptstyle h}(B) \pi_{\scriptscriptstyle h}(\xi)) \chi(B) \right| \leq 2 \sqrt{p}.$$

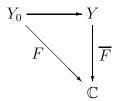
We prove the Hecke ergodicity theorem in the form of Theorem 2.2.2.

#### 2.2.2 The trace function

We prove Theorem 2.2.2 using Sheaf theoretic techniques. Before diving into geometric considerations, we investigate further the ingredients appearing in Theorem 2.2.2. Denote by F the function  $F: \Gamma_p \times \Lambda^* \longrightarrow \mathbb{C}$  defined by  $F(B,\xi) = \text{Tr}(\rho(B)\pi_h(\xi))$ . We denote by  $V:=\Lambda^*/p\Lambda^*$  the quotient vector space, i.e.,  $V\simeq \mathbb{F}_p^2$ . The symplectic form  $\omega$  specializes to give a symplectic form on V. The group  $\Gamma_p$  is the group of linear symplectomorphisms of V, i.e.,  $\Gamma_p = \text{Sp}_{\omega}(V)$ . Set  $Y_0 := \Gamma_p \times \Lambda^*$  and  $Y := \Gamma_p \times V$ . One has the quotient map:

$$Y_0 \longrightarrow Y$$
.

**Lemma 2.2.3** The function  $F: Y_0 \longrightarrow \mathbb{C}$  factors through the quotient Y.



Denote the function  $\overline{F}$  also by F and from now on Y will be considered as the default domain. The function  $F:Y\longrightarrow \mathbb{C}$  is invariant under a certain group action of  $\Gamma_p$ . To be more precise let  $S\in \Gamma_p$ . Then:

$$\operatorname{Tr}(\rho_{\hbar}(B)\pi_{\hbar}(\xi)) = \operatorname{Tr}(\rho_{\hbar}(S)\rho_{\hbar}(B)\rho_{\hbar}(S)^{-1}\rho_{\hbar}(S)\pi_{\hbar}(\xi)\rho_{\hbar}(S)^{-1}).$$

Applying the Egorov identity (1.2.1) and using the fact that  $\rho_h$  is a representation we get:

$$Tr(\rho_{h}(S)\rho_{h}(B)\rho_{h}(S)^{-1}\rho_{h}(S)\pi_{h}(\xi)\rho_{h}(S)^{-1}) = Tr(\pi_{h}(S\xi)\rho_{h}(SBS^{-1})).$$

Altogether we have:

$$F(B,\xi) = F(SBS^{-1}, S\xi). \tag{2.2.5}$$

Putting (2.2.5) in a more diagrammatic form: there is an action of  $\Gamma_p$  on Y given by the following formula:

$$\Gamma_p \times Y \xrightarrow{\alpha} Y, 
(S, (B, \xi)) \longrightarrow (SBS^{-1}, S\xi).$$
(2.2.6)

Consider the following diagram:

$$Y \xleftarrow{pr} \Gamma_n \times Y \xrightarrow{\alpha} Y$$

where pr is the projection on the Y variable. Formula (2.2.5) can be stated equivalently as:

$$\alpha^*(F) = pr^*(F)$$

where  $\alpha^*(F)$  and  $pr^*(F)$  are the pullbacks of the function F on Y via the maps  $\alpha$  and pr respectively.

#### 2.2.3 Geometrization (Sheafification)

Next, we will phrase a geometric statement that will imply Theorem 2.2.2. Moving into the geometric setting, we replace the set Y by an algebraic variety and the functions F and  $\chi$  by sheaf theoretic objects, also of a geometric flavor.

**Step 1.** The set Y is not an arbitrary finite set but it is the set of rational points of an algebraic variety  $\mathbb{Y}$  defined over  $\mathbb{F}_p$ . To be more precise,  $\mathbb{Y} \simeq \mathbb{Sp}_{\omega} \times \mathbb{V}$ . The variety  $\mathbb{Y}$  is equipped with an endomorphism:

$$\operatorname{Fr}: \mathbb{Y} \longrightarrow \mathbb{Y}$$

called Frobenius. The set Y is identified with the set of fixed points of Frobenius:

$$Y=\mathbb{Y}^{\mathrm{Fr}}=\{y\in\mathbb{Y}: \mathrm{Fr}(y)=y\}.$$

Note that the finite group  $\Gamma_p$  is the set of rational points of the algebraic group  $\mathbb{S}_{\omega}$ . The vector space V is the set of rational points of the variety V, where V is isomorphic to the affine plane  $\mathbb{A}^2$ . We denote by  $\alpha$  the action of  $\mathbb{S}_{\omega}$  on the variety Y (cf. (2.2.6)).

Having all finite sets replaced by corresponding algebraic varieties, we replace functions

by sheaf theoretic objects as shown.

**Step 2.** The following theorem proposes an appropriate sheaf theoretic object standing in place of the function  $F: Y \longrightarrow \mathbb{C}$ . Denote by  $\mathcal{D}_{c,w}^b(\mathbb{Y})$  the bounded derived category of constructible  $\ell$ -adic Weil sheaves on  $\mathbb{Y}$  (cf. [M, BBD], in addition see [BL] for equivariant sheaves theory).

**Theorem 2.2.4 (Geometrization Theorem)** There exists an object  $\mathcal{F} \in \mathcal{D}^b_{c,w}(\mathbb{Y})$  satisfying the following properties:

1. (Function) It is associated, via the *sheaf-to-function correspondence*, to the function  $F: Y \longrightarrow \mathbb{C}$ :

$$f^{\mathcal{F}} = F$$
.

2. (Weight) It is of weight:

$$w(\mathcal{F}) \leq 0.$$

3. (Equivariance) For every element  $S \in \mathbb{S}p_{\omega}$  there exists an isomorphism

$$\alpha_S^* \mathcal{F} \simeq \mathcal{F}.$$

4. (Formula) On introducing coordinates  $\mathbb{V} \simeq \mathbb{A}^2$  we identify  $\mathbb{S}p_{\omega} \simeq \mathbb{SL}_2$ . Then there exists an isomorphism:

$$\mathcal{F}_{|_{\mathbb{T}\times\mathbb{V}}}\simeq\mathscr{L}_{\psi(\frac{1}{2}\lambda\mu^{\frac{a+1}{a-1}})}\otimes\mathscr{L}_{\sigma(a)}.^3$$

Here  $\mathbb{T} := \{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \}$  stands for the standard torus and  $(\lambda, \mu)$  are the coordinates on  $\mathbb{V}$ .

We give here an *intuitive* explanation of Theorem 2.2.4, part by part, as it was stated. An object  $\mathcal{F} \in \mathcal{D}^b_{c,w}(\mathbb{Y})$  can be considered as a vector bundle  $\mathcal{F}$  over  $\mathbb{Y}$ :



The letter "w" in the notation  $\mathcal{D}_{c,w}^b$  means that  $\mathcal{F}$  is a Weil sheaf, i.e., it is equipped with a lifting of the Frobenius:

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\operatorname{Fr}} & \mathcal{F} \\
\downarrow & & \downarrow \\
\mathbb{V} & \xrightarrow{\operatorname{Fr}} & \mathbb{V}
\end{array}$$

 $<sup>^3</sup>$ By this we mean that  $\mathcal{F}_{|_{\mathbb{T}\times\mathbb{V}}}$  is isomorphic to the extension of the sheaf defined by the formula in the right-hand side.

To be even more precise, think of  $\mathcal{F}$  not as a single vector bundle, but as a complex  $\mathcal{F} = \mathcal{F}^{\bullet}$  of vector bundles over  $\mathbb{Y}$ :

$$\dots \xrightarrow{d} \mathcal{F}^{-1} \xrightarrow{d} \mathcal{F}^{0} \xrightarrow{d} \mathcal{F}^{1} \xrightarrow{d} \dots$$

The complex  $\mathcal{F}^{\bullet}$  is equipped with a lifting of Frobenius:

Here the Frobenius commutes with the differentials.

Next, we explain the meaning of property 2, i.e., the statement  $w(\mathcal{F}) \leq 0$ . Let  $y \in \mathbb{Y}^{\operatorname{Fr}} = Y$  be a fixed point of Frobenius. Denote by  $\mathcal{F}_y$  the fiber of  $\mathcal{F}$  at the point y. Thinking of  $\mathcal{F}$  as a complex of vector bundles, it is clear what one means by taking the fiber at a point. The fiber  $\mathcal{F}_y$  is just a complex of vector spaces. Because the point y is fixed by the Frobenius, it induces an endomorphism of  $\mathcal{F}_y$ :

The Frobenius acting as in (2.2.7) commutes with the differentials. Hence, it induces an action on cohomologies. For every  $i \in \mathbb{Z}$  we have an endomorphism:

$$\operatorname{Fr}: \operatorname{H}^{i}(\mathcal{F}_{y}) \longrightarrow \operatorname{H}^{i}(\mathcal{F}_{y}).$$
 (2.2.8)

Saying that an object  $\mathcal{F}$  has  $w(\mathcal{F}) \leq w$  means that for every point  $y \in \mathbb{Y}^{Fr}$  and for every  $i \in \mathbb{Z}$  the absolute value of the eigenvalues of Frobenius acting on the *i*'th cohomology (2.2.8) satisfy:

$$\left| \text{e.v}(\text{Fr} \big|_{\text{H}^i(\mathcal{F}_y)}) \right| \le \sqrt{p}^{w+i}.$$

In our case w = 0 and therefore:

$$\left| \text{e.v(Fr} \right|_{\text{H}^i(\mathcal{F}_y)}) \right| \le \sqrt{p}^i.$$
 (2.2.9)

Property 1 of Theorem 2.2.4 concerns a function  $f^{\mathcal{F}}: Y \longrightarrow \mathbb{C}$  associated to the sheaf  $\mathcal{F}$ . To define  $f^{\mathcal{F}}$ , we only have to describe its value at every point  $y \in Y$ . Let  $y \in Y = \mathbb{Y}^{\mathrm{Fr}}$ . Frobenius acts on the cohomologies of the fiber  $\mathcal{F}_y$  (cf. (2.2.8)). Now put:

$$f^{\mathcal{F}}(y) := \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(\mathrm{Fr}\big|_{\mathrm{H}^i(\mathcal{F}_y)}).$$

In words:  $f^{\mathcal{F}}(y)$  is the alternating sum of traces of Frobenius acting on the cohomologies of the fiber  $\mathcal{F}_y$ . This alternating sum is called the *Euler characteristic* of Frobenius and is denoted by:

$$f^{\mathcal{F}}(y) = \chi_{\text{Fr}}(\mathcal{F}_y).$$

Theorem 2.2.4 confirms that  $f^{\mathcal{F}}$  is the function F defined earlier. Associating the function  $f^{\mathcal{F}}$  on the set  $\mathbb{Y}^{Fr}$  to the sheaf  $\mathcal{F}$  on  $\mathbb{Y}$  is a particular case of a general procedure called *Sheaf-to-Function Correspondence* [G]. As this procedure will be used later, next we spend some space explaining it in greater details (cf. [Ga]).

#### Grothendieck's Sheaf-to-Function Correspondence

Let X be an algebraic variety defined over  $\mathbb{F}_q$ . This means that there exists a Frobenius endomorphism:

$$\operatorname{Fr}: \mathbb{X} \longrightarrow \mathbb{X}.$$

The set  $X = \mathbb{X}^{\operatorname{Fr}}$  is called the set of rational points of  $\mathbb{X}$ . Let  $\mathcal{L} \in \mathcal{D}^b_{c,w}(\mathbb{X})$  be a Weil sheaf. One can associate to  $\mathcal{L}$  a function  $f^{\mathcal{L}}$  on the set X by the following formula:

$$f^{\mathcal{L}}(x) := \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(\mathrm{Fr}\big|_{\mathrm{H}^i(\mathcal{L}_x)}).$$

This procedure is called *Sheaf-To-Function correspondence*. Next, we describe some important functorial properties of this procedure.

Let  $\mathbb{X}_1$ ,  $\mathbb{X}_2$  be algebraic varieties defined over  $\mathbb{F}_q$ . Let  $X_1 = \mathbb{X}_1^{\operatorname{Fr}}$  and  $X_2 = \mathbb{X}_2^{\operatorname{Fr}}$  be the corresponding sets of rational points. Let  $\pi: \mathbb{X}_1 \longrightarrow \mathbb{X}_2$  be a morphism of algebraic varieties. Denote also by  $\pi: X_1 \longrightarrow X_2$  the induced map on the level of sets.

First statement. Suppose we have a sheaf  $\mathcal{L} \in \mathcal{D}^b_{c,w}(\mathbb{X}_2)$ . The following holds:

$$f^{\pi^*(\mathcal{L})} = \pi^*(f^{\mathcal{L}})$$
 (2.2.10)

where on the function level  $\pi^*$  is just the pull back of functions. On the sheaf theoretic level  $\pi^*$  is the pull-back functor of sheaves (think of pulling back a vector bundle). Equation (2.2.10) states that the *Sheaf-to-Function Correspondence* commutes with the operation of pull back.

**Second statement**. Suppose we have a sheaf  $\mathcal{L} \in \mathcal{D}^b_{c,w}(\mathbb{X}_1)$ . The following holds:

$$f^{\pi_!(\mathcal{L})} = \pi_!(f^{\mathcal{L}}) \tag{2.2.11}$$

where on the function level  $\pi_!$  means to sum up the values of the function along the fibers of the map  $\pi$ . On the sheaf theoretic level  $\pi_!$  is a compact integration of sheaves (here we

have no analogue under the vector bundle interpretation). Equation (2.2.11) states that the *Sheaf-to-Function Correspondence* commutes with integration.

**Third statement**. Suppose we have two sheaves  $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{D}^b_{c,w}(\mathbb{X}_1)$ . The following holds:

$$f^{\mathcal{L}_1 \otimes \mathcal{L}_2} = f^{\mathcal{L}_1} \cdot f^{\mathcal{L}_2}. \tag{2.2.12}$$

In words: Sheaf-to-Function Correspondence takes tensor product of sheaves to multiplication of the corresponding functions.

#### 2.2.4 Geometric statement

Fix an element  $\xi \in \Lambda^*$  with  $\xi \neq 0$ . We denote by  $i_{\xi}$  the inclusion map  $i_{\xi} : T_A \times \xi \longrightarrow Y$ . Going back to Theorem 2.2.2 and putting its content in a functorial notation, we write the following inequality:

$$\left| pr_!(i_{\xi}^*(F) \cdot \chi) \right| \le 2\sqrt{p}.$$

In words, taking the function  $F: Y \longrightarrow \mathbb{C}$  and:

- Restrict F to  $T_A \times \xi$  and get  $i_{\varepsilon}^*(F)$ .
- Multiply  $i_{\varepsilon}^* F$  by the character  $\chi$  to get  $i_{\varepsilon}^* (F) \cdot \chi$ .
- Integrate  $i_{\xi}^*(F) \cdot \chi$  to the point, this means to sum up all its values, and get a scalar  $a_{\chi} := pr_!(i_{\xi}^*(F) \cdot \chi)$ . Here pr stands for the projection  $pr : T_A \times \xi \longrightarrow pt$ .

Then Theorem 2.2.2 asserts that the scalar  $a_{\chi}$  is of an absolute value less than  $2\sqrt{p}$ .

Repeat the same steps in the geometric setting. We denote again by  $i_{\xi}$  the closed imbedding  $i_{\xi}: \mathbb{T}_A \times \xi \longrightarrow \mathbb{Y}$ . Take the sheaf  $\mathcal{F}$  on  $\mathbb{Y}$  and apply the following sequence of operations:

- Pull-back  $\mathcal{F}$  to the closed subvariety  $\mathbb{T}_A \times \xi$  and get the sheaf  $i_{\xi}^*(\mathcal{F})$ .
- Take the tensor product of  $i_{\varepsilon}^*(\mathcal{F})$  with the Kummer sheaf  $\mathscr{L}_{\chi}$  and get  $i_{\varepsilon}^*(\mathcal{F}) \otimes \mathscr{L}_{\chi}$ .
- Integrate  $i_{\varepsilon}^*(\mathcal{F}) \otimes \mathscr{L}_{\chi}$  to the point and get the sheaf  $pr_!(i_{\varepsilon}^*(\mathcal{F}) \otimes \mathscr{L}_{\chi})$  on the point.

The Kummer sheaf  $\mathcal{L}_{\chi}$  is the *character sheaf* (cf. [Ga]) associated via *Sheaf-to-Function Correspondence* to the character  $\chi$ .

The operation of *Sheaf-to-Function Correspondence* commutes both with pullback (2.2.10), with integration (2.2.11) and takes the tensor product of sheaves to the multiplication of functions (2.2.12). This means that it intertwines the operations carried out on the level of sheaves with those carried out on the level of functions. The following diagram describes pictorially what has been said so far:

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\chi_{\operatorname{Fr}}} & F \\
\downarrow_{\xi} & & \downarrow_{\xi} \\
\downarrow_{\xi}(\mathcal{F}) \otimes \mathscr{L}_{\chi} & \xrightarrow{\chi_{\operatorname{Fr}}} & i_{\xi}^{*}(F) \cdot \chi \\
\downarrow_{pr} & & \downarrow_{pr} \\
pr_{!}(i_{\xi}^{*}(\mathcal{F}) \otimes \mathscr{L}_{\chi}) & \xrightarrow{\chi_{\operatorname{Fr}}} & pr_{!}(i_{\xi}^{*}(F) \cdot \chi)
\end{array}$$

Recall  $w(\mathcal{F}) \leq 0$ . Now, the effect of functors  $i_{\xi}^*$ ,  $pr_!$  and tensor product  $\otimes$  on the property of weight should be examined.

The functor  $i_{\xi}^*$  does not increase weight. Observing the definition of weight this claim is immediate. Therefore we get:

$$w(i_{\varepsilon}^*(\mathcal{F})) \leq 0.$$

Assume we have two sheaves  $\mathcal{L}_1$  and  $\mathcal{L}_2$  with weights  $w(\mathcal{L}_1) \leq w_1$  and  $w(\mathcal{L}_2) \leq w_2$ . The weight of the tensor product satisfies  $w(\mathcal{L}_1 \otimes \mathcal{L}_2) \leq w_1 + w_2$ . This is again immediate from the definition of weight.

Knowing that the Kummer sheaf has weight  $w(\mathcal{L}_{\chi}) \leq 0$  we deduce:

$$w(i_{\varepsilon}^*(\mathcal{F})\otimes \mathscr{L}_{\chi})\leq 0.$$

Finally, one has to understand the affect of the functor  $pr_!$ . The following theorem, proposed by Deligne [D2], is a very deep and important result in the theory of weights. Briefly speaking, the theorem states that compact integration of sheaves does not increase weight. Here is the precise statement:

Theorem 2.2.5 (Deligne, Weil II [D2]) Let  $\pi: \mathbb{X}_1 \longrightarrow \mathbb{X}_2$  be a morphism of algebraic varieties. Let  $\mathcal{L} \in \mathcal{D}^b_{c,w}(\mathbb{X}_1)$  be a sheaf of weight  $w(\mathcal{L}) \leq w$  then  $w(\pi_!(\mathcal{L})) \leq w$ .

Using Theorem 2.2.5 we get:

$$w(pr_!(i_{\varepsilon}^*(\mathcal{F})\otimes\mathscr{L}_{\chi}))\leq 0.$$

Consider the sheaf  $\mathcal{G} := pr_!(i_{\xi}^*(\mathcal{F}) \otimes \mathcal{L}_{\chi})$ . It is an object in  $\mathcal{D}_{c,w}^b(pt)$ . This means it is merely a complex of vector spaces,  $\mathcal{G} = \mathcal{G}^{\bullet}$ , together with an action of Frobenius:

The complex  $\mathcal{G}^{\bullet}$  is associated by Sheaf-To-Function correspondence to the scalar  $a_{\chi}$ :

$$a_{\chi} = \sum_{i \in \mathbb{Z}} (-1)^{i} \operatorname{Tr}(\operatorname{Fr}|_{\operatorname{H}^{i}(\mathcal{G})}). \tag{2.2.13}$$

Finally, we can give the geometric statement about  $\mathcal{G}$ , which will imply Theorem 2.2.2.

Lemma 2.2.6 (Vanishing Lemma) Let  $\mathcal{G} = pr_!(i_{\xi}^*(\mathcal{F}) \otimes \mathcal{L}_{\chi})$ . All cohomologies  $H^i(\mathcal{G})$  vanish except for i = 1. Moreover,  $H^1(\mathcal{G})$  is a two dimensional vector space.

Theorem 2.2.2 now follows easily. By Lemma 2.2.6 only the first cohomology  $H^1(\mathcal{G})$  does not vanish and it is two dimensional. Having  $w(\mathcal{G}) \leq 0$  implies (cf. 2.2.9) that the eigenvalues of Frobenius acting on  $H^1(\mathcal{G})$  are of absolute value  $\leq \sqrt{p}$ . Hence, using formula (2.2.13) we get:

$$|a_{\chi}| \leq 2\sqrt{p}$$
.

The remainder of the chapter is devoted to the proof of Lemma 2.2.6.

### 2.2.5 Proof of the Vanishing Lemma

The proof will be given in several steps.

**Step 1.** All tori in  $\mathbb{S}_{p_{\omega}}$  are conjugated. On introducing coordinates, i.e.,  $\mathbb{V} \simeq \mathbb{A}^2$ , we make the identification  $\mathbb{S}_{p_{\omega}} \simeq \mathbb{SL}_2$ . In these terms there exists an element  $S \in \mathbb{SL}_2$  conjugating the *Hecke* torus  $\mathbb{T}_A \subset \mathbb{SL}_2$  with the standard torus  $\mathbb{T} = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\} \subset \mathbb{SL}_2$ , namely:

$$S\mathbb{T}_A S^{-1} = \mathbb{T}.$$

The situation is displayed in the following diagram:

$$\begin{array}{cccc}
\mathbb{SL}_{2} \times \mathbb{A}^{2} & \xrightarrow{\alpha_{S}} & \mathbb{SL}_{2} \times \mathbb{A}^{2} \\
\downarrow^{i_{\xi}} & & \downarrow^{i_{\eta}} \\
\mathbb{T}_{A} \times \xi & \xrightarrow{\alpha_{S}} & \mathbb{T} \times \eta \\
\downarrow^{pr} & & \downarrow^{pr} \\
\downarrow^{pt} & & pt
\end{array}$$

where  $\eta = S \cdot \xi$  and  $\alpha_S$  is the restriction of the action  $\alpha$  to the element S.

**Step 2.** Using the equivariance property of the sheaf  $\mathcal{F}$  (see Theorem 2.2.4, property 3) we will show that it is *sufficient* to prove the Vanishing Lemma for the sheaf  $\mathcal{G}_{st} := pr_!(i_n^*\mathcal{F} \otimes \alpha_{s!}\mathcal{L}_{\chi}).$ 

We have:

$$pr_!(i_{\varepsilon}^*\mathcal{F}\otimes\mathscr{L}_{\chi}) \simeq pr_!\alpha_{s_!}(i_{\varepsilon}^*\mathcal{F}\otimes\mathscr{L}_{\chi}).$$
 (2.2.14)

The morphism  $\alpha_s$  is an isomorphism, therefore  $\alpha_{s!}$  commutes with taking  $\otimes$ , hence we obtain:

$$pr_!\alpha_{s_!}(i_{\varepsilon}^*(\mathcal{F})\otimes\mathscr{L}_{\chi}) \simeq pr_!(\alpha_{s_!}(i_{\varepsilon}^*\mathcal{F})\otimes\alpha_{s_!}(\mathscr{L}_{\chi})).$$
 (2.2.15)

Applying base change we obtain:

$$\alpha_{s!} i_{\varepsilon}^* \mathcal{F} \simeq i_{\eta}^* \alpha_{s!} \mathcal{F}. \tag{2.2.16}$$

Now using the equivariance property of the sheaf  $\mathcal{F}$  we have the isomorphism:

$$\alpha_{s} \mathcal{F} \simeq \mathcal{F}.$$
 (2.2.17)

Combining (2.2.14), (2.2.15), (2.2.16) and (2.2.17) we get:

$$pr_!(i_{\varepsilon}^*\mathcal{F}\otimes\mathscr{L}_{\chi}) \simeq pr_!(i_{\eta}^*\mathcal{F}\otimes\alpha_{S!}\mathscr{L}_{\chi}).$$
 (2.2.18)

Therefore we see from (2.2.18) that it is sufficient to prove vanishing of cohomologies for:

$$\mathcal{G}_{st} := pr_!(i_\eta^* \mathcal{F} \otimes \alpha_{s!} \mathcal{L}_\chi). \tag{2.2.19}$$

But this is a situation over the standard torus and we can compute explicitly all the sheaves involved!

**Step 3.** The Vanishing Lemma holds for the sheaf  $\mathcal{G}_{st}$ .

We are left to compute (2.2.19). We write  $\eta = (\lambda, \mu)$ . By Theorem 2.2.4 Property 4 we have  $i_{\eta}^* \mathcal{F} \simeq \mathcal{L}_{\psi(\frac{1}{2}\lambda\mu\frac{a+1}{a-1})} \otimes \mathcal{L}_{\sigma(a)}$ , where a is the coordinate of the standard torus  $\mathbb{T}$  and  $\lambda \cdot \mu \neq 0^4$ . The sheaf  $\alpha_{s!} \mathcal{L}_{\chi}$  is a character sheaf on the torus  $\mathbb{T}$ . Hence we get that (2.2.19) is a kind of a Kloosterman-sum sheaf. A direct computation (Appendix C section C.4) proves the Vanishing Lemma for this sheaf. This completes the proof of the Hecke quantum unique ergodicity conjecture.

<sup>&</sup>lt;sup>4</sup>This is a direct consequence of the fact that  $A \in SL_2(\mathbb{Z})$  is an hyperbolic element and does not have eigenvectors in  $\Lambda^*$ .

# Chapter 3

# Metaplectique

In the first part of this chapter we give new construction of the Weil (metaplectic) representation  $(\rho, \operatorname{Sp}(V), \mathcal{H}_V)$ , attached to a two dimensional symplectic vector space  $(V, \omega)$  over  $\mathbb{F}_q$ , which appears in the body of the thesis. The difference is that here the construction is slightly more general. But even more importantly, it is obtained in completely natural geometric terms. The focal step in our approach is the introduction of a canonical Hilbert space on which the Weil representation is naturally manifested. The motivation to look for this space was initiated by a question of David Kazhdan [Ka]. The key idea behind this construction was suggested to us by Joseph Bernstein [B]. The upshot is to replace the notion of a Lagrangian subspace by a more refined notion of an oriented Lagrangian subspace  $^1$ .

In the second part of this chapter we apply a geometrization procedure to the construction given in the first part, meaning that all sets are replaced by algebraic varieties and all functions are replaced by  $\ell$ -adic sheaves. This part is based on a letter of Deligne to Kazhdan from 1982 [D1]. We extract from that work only the part that is most relevant to this thesis. Although all basic ideas appear already in the letter, we tried to give here a slightly more general and detailed account of the construction. As far as we know, the contents of this mathematical work has never been published. This might be a good enough reason for writing this part.

The following is a description of the chapter. In section 3.1 we introduce the notion of oriented Lagrangian subspace and the construction of the canonical Hilbert space. In section 3.2 we obtain a natural realization of the Weil representation. In section 3.3 we give the standard Schrödinger realization (cf. [W2]). We also include several formulas for the kernels of basic operators. These formulas will be used in section 3.4 where the

<sup>&</sup>lt;sup>1</sup>We thank A. Polishchuk for pointing out to us that this is an  $\mathbb{F}_q$ -analogue of well known considerations with usual oriented Lagrangians giving explicitly the metaplectic covering of  $\operatorname{Sp}(2n,\mathbb{R})$  (cf. [LV]).

geometrization procedure is described. In section 3.5 we give proofs of all lemmas and propositions which appear in previous sections.

For the remainder of this chapter we fix the following notations. Let  $\mathbb{F}_q$  denote the finite field of characteristic  $p \neq 2$  and q elements. Fix  $\psi : \mathbb{F}_q \longrightarrow \mathbb{C}^*$  a non-trivial additive character. Denote by  $\sigma : \mathbb{F}_q^* \longrightarrow \mathbb{C}^*$  the *Legendre* multiplicative quadratic character.

## 3.1 Canonical Hilbert space

#### 3.1.1 Oriented Lagrangian subspace

Let  $(V, \omega)$  be a 2-dimensional symplectic vector space over  $\mathbb{F}_q$ .

**Definition 3.1.1** An oriented Lagrangian subspace is a pair  $(L, \varrho_L)$ , where L is a Lagrangian subspace of V and  $\varrho_L : L \setminus \{0\} \to \{\pm 1\}$  is a function which satisfies the following equivariant property:

$$\varrho_{\text{\tiny L}}(t \cdot l) = \sigma(t)\varrho_{\text{\tiny L}}(l)$$

where  $t \in \mathbb{F}_q^*$  and  $\sigma$  the Legendre character of  $\mathbb{F}_q^*$ .

We denote by Lag° the space of oriented Lagrangians subspaces. There is a forgetful map Lag°  $\longrightarrow$  Lag, where Lag is the space of Lagrangian subspaces, Lag  $\simeq \mathbb{P}^1(\mathbb{F}_q)$ . In the sequel we use the notation L° to specify that L is equipped with an orientation.

## 3.1.2 The Heisenberg group

Let E be the Heisenberg group. As a set we have  $E = V \times \mathbb{F}_q$ . The multiplication is defined by the following formula:

$$(v,\lambda)\cdot(v',\lambda'):=(v+v',\lambda+\lambda'+\frac{1}{2}\,\omega(v,v')).$$

We have a projection  $\pi: E \longrightarrow V$ . We fix a section of this projection:

$$s: V \dashrightarrow E, \quad s(v) := (v, 0).$$
 (3.1.1)

## 3.1.3 Models of irreducible representation

Given  $L^{\circ} = (L, \varrho_L) \in Lag^{\circ}$ , we construct the Hilbert space  $\mathcal{H}_{L^{\circ}} = Ind_{\tilde{L}}^{E} \mathbb{C}_{\tilde{\psi}}$ , where  $\tilde{L} = \pi^{-1}(L)$  and  $\tilde{\psi}$  is the extension of the additive character  $\psi$  to  $\tilde{L}$  using the section s,

i.e.,  $\tilde{\psi}: \tilde{\mathbf{L}} = \mathbf{L} \times \mathbb{F}_q \longrightarrow \mathbb{C}^*$  is given by the formula:

$$\tilde{\psi}(l,\lambda) := \psi(\lambda).$$

More concretely:  $\mathcal{H}_{L^{\circ}} = \{f : E \longrightarrow \mathbb{C} \mid f(\lambda le) = \psi(\lambda)f(e)\}$ . The group E acts on  $\mathcal{H}_{L^{\circ}}$  by multiplication from the right. It is well known (and easy to prove) that the representations  $\mathcal{H}_{L^{\circ}}$  of E are irreducible and for different  $L^{\circ}$ 's they are all isomorphic. These are different models of the same irreducible representation. This is stated in the following theorem:

**Theorem 3.1.2 (Stone-von Neumann)** For an oriented Lagrangian subspace  $L^{\circ}$ , the representation  $\mathcal{H}_{L^{\circ}}$  of E is irreducible. Moreover, for any two oriented Lagrangians  $L_{1}^{\circ}, L_{2}^{\circ} \in Lag^{\circ}$  one has  $\mathcal{H}_{L_{2}^{\circ}} \simeq \mathcal{H}_{L_{2}^{\circ}}$  as representations of E.

#### 3.1.4 Canonical intertwining operators

Let  $L_1^{\circ}, L_2^{\circ} \in Lag^{\circ}$  be two oriented Lagrangians. Let  $\mathcal{H}_{L_1^{\circ}}, \mathcal{H}_{L_2^{\circ}}$  be the corresponding representations of E. We denote by  $\Theta_{L_2^{\circ}, L_1^{\circ}} := \operatorname{Hom}_{\mathbb{E}}(\mathcal{H}_{L_1^{\circ}}, \mathcal{H}_{L_2^{\circ}})$  the space of intertwining operators between the two representations. Because all representations are irreducible and isomorphic to each other we have, dim  $\Theta_{L_2^{\circ}, L_1^{\circ}} = 1$ . Next, we construct a canonical element in  $\Theta_{L_2^{\circ}, L_1^{\circ}}$ .

Let  $L_1^{\circ}=(L_1,\varrho_{L_1}),\ L_2^{\circ}=(L_2,\varrho_{L_2})$  be two oriented Lagrangian subspaces. Assume they are in general position, i.e.,  $L_1\neq L_2$ . We define the following specific element  $\theta_{L_2^{\circ},L_1^{\circ}}\in\Theta_{L_2^{\circ},L_1^{\circ}},\ \theta_{L_2^{\circ},L_1^{\circ}}:\mathcal{H}_{L_1^{\circ}}\longrightarrow\mathcal{H}_{L_2^{\circ}}$ . It is defined by the following formula:

$$\theta_{L_{2}^{\circ},L_{1}^{\circ}} = a_{L_{2}^{\circ},L_{1}^{\circ}} \cdot \tilde{\theta}_{L_{2}^{\circ},L_{1}^{\circ}}$$
(3.1.2)

where  $\tilde{\theta}_{L_2^{\circ},L_1^{\circ}}: \mathcal{H}_{L_1^{\circ}} \longrightarrow \mathcal{H}_{L_2^{\circ}}$  denotes the standard averaging operator and  $a_{L_2^{\circ},L_1^{\circ}}$  denotes the normalization factor. The formulas are:

$$\tilde{\theta}_{\mathbf{L}_{2}^{\circ},\mathbf{L}_{1}^{\circ}}(f)(e) = \sum_{l_{2} \in \mathbf{L}_{2}} f(l_{2}e)$$

where  $f \in \mathcal{H}_{\mathrm{L}_{1}^{\circ}}$ .

$$\mathbf{a}_{\mathbf{L}_{2}^{\circ},\mathbf{L}_{1}^{\circ}} = \frac{1}{q} \sum_{l_{1} \in \mathbf{L}_{1}} \psi(\frac{1}{2} \omega(l_{1},\xi_{\mathbf{L}_{2}})) \varrho_{\mathbf{L}_{1}}(l_{1}) \varrho_{\mathbf{L}_{2}}(\xi_{\mathbf{L}_{2}})$$

where  $\xi_{L_2}$  is a fixed non-zero vector in  $L_2$ . Note that  $a_{L_2^{\circ},L_1^{\circ}}$  does not depend on  $\xi_{L_2}$ .

Now we extend the definition of  $\theta_{L_2^{\circ},L_1^{\circ}}$  to the case where  $L_1 = L_2$ . Define:

$$\theta_{\scriptscriptstyle \mathrm{L}_2^\circ, \scriptscriptstyle \mathrm{L}_1^\circ} = \begin{cases} \mathrm{I}, & \varrho_{\scriptscriptstyle \mathrm{L}_1} &= \varrho_{\scriptscriptstyle \mathrm{L}_2} \\ -\mathrm{I}, & \varrho_{\scriptscriptstyle \mathrm{L}_1} &= -\varrho_{\scriptscriptstyle \mathrm{L}_2} \end{cases}$$

The main claim is that the collection  $\{\theta_{L_2^{\circ},L_1^{\circ}}\}_{L_1^{\circ},L_2^{\circ} \in Lag^{\circ}}$  is associative. This is formulated in the following theorem:

**Theorem 3.1.3 (Associativity)** Let  $L_1^{\circ}, L_2^{\circ}, L_3^{\circ} \in Lag^{\circ}$  be a triple of oriented Lagrangian subspaces. The following associativity condition holds:

$$\theta_{\mathbf{L}_3^{\circ},\mathbf{L}_2^{\circ}} \circ \theta_{\mathbf{L}_2^{\circ},\mathbf{L}_1^{\circ}} = \theta_{\mathbf{L}_3^{\circ},\mathbf{L}_1^{\circ}}.$$

#### 3.1.5 Canonical Hilbert space

Define the canonical Hilbert space  $\mathcal{H}_V \subset \bigoplus_{L^\circ \in Lag^\circ} \mathcal{H}_{L^\circ}$  as the subspace of compatible systems of vectors, namely:

$$\mathcal{H}_{\mathrm{V}} := \{ (f_{_{\mathrm{L}^{\circ}}})_{_{\mathrm{L}^{\circ} \in \mathrm{Lag}^{\circ}}}; \quad \theta_{_{\mathrm{L}^{\circ}_{2},\mathrm{L}^{\circ}_{1}}}(f_{_{\mathrm{L}^{\circ}_{1}}}) = f_{_{\mathrm{L}^{\circ}_{2}}} \}.$$

## 3.2 The Weil representation

In this section we construct the Weil representation using the Hilbert space  $\mathcal{H}_{V}$ . We denote by  $\mathrm{Sp}_{\omega} := \mathrm{Sp}(\mathrm{V}, \omega)$  the group of linear symplectomorphisms of V. Before giving any formulas, note that the space  $\mathcal{H}_{V}$  was constructed out of the symplectic space  $(\mathrm{V}, \omega)$  in a complete canonical way. This immediately implies that all the symmetries of  $(\mathrm{V}, \omega)$  automatically act on  $\mathcal{H}_{V}$ . In particular, we obtain a *linear* representation of the group  $\mathrm{Sp}_{\omega}$  in the space  $\mathcal{H}_{V}$ . This is the famous Weil representation of  $\mathrm{Sp}_{\omega}$  and we denote it by  $\rho: \mathrm{Sp}_{\omega} \longrightarrow \mathrm{GL}(\mathcal{H}_{V})$ . It is given by the following formula:

$$\rho(g)[(f_{L^{\circ}})] := (f_{L^{\circ}}^g). \tag{3.2.1}$$

Let us elaborate on this formula. The group  $\operatorname{Sp}_{\omega}$  acts on the space  $\operatorname{Lag}^{\circ}$ . Any element  $g \in \operatorname{Sp}_{\omega}$  induces an automorphism  $g : \operatorname{Lag}^{\circ} \longrightarrow \operatorname{Lag}^{\circ}$  defined by:

$$(\mathbf{L},\varrho_{\scriptscriptstyle{\mathbf{L}}})\longmapsto (g\mathbf{L},\varrho_{\scriptscriptstyle{\mathbf{L}}}^g)$$

where  $\varrho_L^g(l) = \varrho_L(g^{-1}l)$ . Moreover, g induces an isomorphism of vector spaces  $g : \mathcal{H}_{L^{\circ}} \longrightarrow \mathcal{H}_{gL^{\circ}}$  defined by the following formula:

$$f_{\text{\tiny L}^{\circ}} \longmapsto f_{\text{\tiny L}^{\circ}}^g, \ \ f_{\text{\tiny L}^{\circ}}^g(e) := f_{\text{\tiny L}^{\circ}}(g^{-1}e) \eqno(3.2.2)$$

where the action of  $g \in \operatorname{Sp}_{\omega}$  on  $e = (v, \lambda) \in \operatorname{E}$  is given by  $g(v, \lambda) = (gv, \lambda)$ . It is easy to verify that the action (3.2.2) of  $\operatorname{Sp}_{\omega}$  commutes with the canonical intertwining

operators, i.e., for any two  $L_1^{\circ}, L_2^{\circ} \in Lag^{\circ}$  and any element  $g \in Sp_{\omega}$  the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{H}_{\mathrm{L}_{1}^{\circ}} & \xrightarrow{\theta_{\mathrm{L}_{2}^{\circ},\mathrm{L}_{1}^{\circ}}} & \mathcal{H}_{\mathrm{L}_{2}^{\circ}} \\ g \Big\downarrow & & g \Big\downarrow \\ \\ \mathcal{H}_{g\mathrm{L}_{1}^{\circ}} & \xrightarrow{\theta_{g\mathrm{L}_{2}^{\circ},g\mathrm{L}_{1}^{\circ}}} & \mathcal{H}_{g\mathrm{L}_{2}^{\circ}} \end{array}$$

From this we deduce that formula (3.2.1) indeed gives the action of  $Sp_{\omega}$  on  $\mathcal{H}_{V}$ .

## 3.3 Realization and formulas

In this section we give the standard Schrödinger realization of the Weil representation. Several formulas for the kernels of basic operators are also included.

#### 3.3.1 Schrödinger realization

Fix  $V = V_1 \oplus V_2$  to be a Lagrangian decomposition of V. Fix  $\varrho_{V_2}$  to be an orientation on  $V_2$ . Denote by  $V_2^{\circ} = (V_2, \varrho_{V_2})$  the oriented space. Using the system of canonical intertwining operators we identify  $\mathcal{H}_V$  with a specific representative  $\mathcal{H}_{V_2^{\circ}}$ . Using the section  $s: V \dashrightarrow E$  (cf. 3.1.1) we further make the identification  $s: \mathcal{H}_{V_2^{\circ}} \simeq \mathcal{S}(V_1)$ , where  $\mathcal{S}(V_1)$  is the space of complex valued functions on  $V_1$ . We denote  $\mathcal{H} := \mathcal{S}(V_1)$ . In this realization the Weil representation,  $\rho: \mathrm{Sp}_{\omega} \longrightarrow \mathrm{GL}(\mathcal{H})$ , is given by the following formula:

$$\rho(g)(f) = \theta_{\mathbf{V}_2^{\circ}, g\mathbf{V}_2^{\circ}}(f^g)$$

where  $f \in \mathcal{H} \simeq \mathcal{H}_{V_2^{\circ}}$  and  $g \in \mathrm{Sp}_{\omega}$ .

## 3.3.2 Formulas for the Weil representation

First we introduce a basis  $e \in V_1$  and the dual basis  $e^* \in V_2$  normalized so that  $\omega(e, e^*) = 1$ . In terms of this basis we have the following identifications:  $V \simeq \mathbb{F}_q^2$ ,  $V_1, V_2 \simeq \mathbb{F}_q$ ,  $\operatorname{Sp}_{\omega} \simeq \operatorname{SL}_2(\mathbb{F}_p)$  and  $\operatorname{E} \simeq \mathbb{F}_q^2 \times \mathbb{F}_q$  (as sets). We also have  $\mathcal{H} \simeq \mathcal{S}(\mathbb{F}_q)$ .

For every element  $g \in \operatorname{Sp}_{\omega}$  the operator  $\rho(g) : \mathcal{H} \longrightarrow \mathcal{H}$  is represented by a kernel  $K_g : \mathbb{F}_q^2 \longrightarrow \mathbb{C}$ . The multiplication of operators becomes convolution of kernels. The collection  $\{K_g\}_{g \in \operatorname{Sp}_{\omega}}$  gives a single function of "kernels" which we denote by  $K_{\rho} : \operatorname{Sp}_{\omega} \times \mathbb{F}_q^2 \longrightarrow \mathbb{C}$ . For every element  $g \in \operatorname{Sp}_{\omega}$  the kernel  $K_{\rho}(g)$  is of the form:

$$K_{\rho}(g, x, y) = a_g \cdot \psi(R_g(x, y))$$

where  $a_g$  is a certain normalizing coefficient and  $R_g : \mathbb{F}_q^2 \longrightarrow \mathbb{F}_q$  is a quadratic function supported on some linear subspace of  $\mathbb{F}_q^2$ . Next, we give an explicit description of the kernels  $K_{\rho}(g)$ .

Consider the (opposite) Bruhat decomposition  $Sp_{\omega} = BwB \cup B$  where:

$$B := \begin{pmatrix} * \\ * & * \end{pmatrix}$$

and  $w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  is the standard Weyl element.

If  $g \in BwB$  then:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where  $b \neq 0$ . In this case we have:

$$a_g = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(\frac{b}{2}t) \sigma(t),$$

$$R_g(x,y) = \frac{-b^{-1}d}{2} x^2 + \frac{b^{-1} - c + ab^{-1}d}{2} xy - \frac{ab^{-1}}{2} y^2.$$

Altogether we have:

$$K_{\rho}(g,x,y) = a_g \cdot \psi(\frac{-b^{-1}d}{2}x^2 + \frac{b^{-1}-c+ab^{-1}d}{2}xy - \frac{ab^{-1}}{2}y^2).$$

If  $g \in B$  then:

$$g = \begin{pmatrix} a & 0 \\ r & a^{-1} \end{pmatrix}.$$

In this case we have:

$$a_g = \sigma(a),$$
  
 $R_g(x,y) = \frac{-ra^{-1}}{2}x^2 \cdot \delta_{y=a^{-1}x}.$ 

Altogether we have:

$$K_{\rho}(g, x, y) = a_g \cdot \psi(\frac{-ra^{-1}}{2}x^2)\delta_{y=a^{-1}x}.$$
(3.3.1)

## 3.3.3 Formulas for the Heisenberg representation

On  $\mathcal{H}$  we also have a representation of the Heisenberg group E. We denote it by  $\pi: \to GL(\mathcal{H})$ . For every element  $e \in E$  we have a kernel  $K_e: \mathbb{F}_q^2 \longrightarrow \mathbb{C}$ . We

denote by  $K_{\pi}: \mathbb{E} \times \mathbb{F}_q^2 \longrightarrow \mathbb{C}$  the function of kernels. For an element  $e \in \mathbb{E}$  the kernel  $K_{\pi}(e)$  has the form  $\psi(R_e(x,y))$  where  $R_e$  is an affine function which is supported on a certain one dimensional subspace of  $\mathbb{F}_q^2$ . Here are the exact formulas:

For an element  $e = (q, p, \lambda)$  we have:

$$R_e(x,y) = (\frac{pq}{2} + px + \lambda)\delta_{y=x+q},$$

$$K_{\pi}(e, x, y) = \psi(\frac{pq}{2} + px + \lambda)\delta_{y=x+q}. \tag{3.3.2}$$

#### 3.3.4 Formulas for the representation of the semi-direct product

The representations  $\rho: \mathrm{Sp}_{\omega} \longrightarrow \mathrm{GL}(\mathcal{H})$  and  $\pi: \mathrm{E} \longrightarrow \mathrm{GL}(\mathcal{H})$  combine together to give a representation of the semi-direct product  $\mathrm{D} = \mathrm{Sp}_{\omega} \ltimes \mathrm{E}$ . We denote the total representation by  $\rho \ltimes \pi: \mathrm{D} \longrightarrow \mathrm{GL}(\mathcal{H}), \ \rho \ltimes \pi(g,e) = \rho(g) \cdot \pi(e)$ . The representation  $\rho \ltimes \pi$  is given by a kernel  $K_{\rho \ltimes \pi}: \mathrm{D} \times \mathbb{F}_q^2 \longrightarrow \mathbb{C}$ . We denote this kernel simply by K.

We give an explicit formula for the kernel K only in the case  $(g, e) \in BwB \times E$ , i.e.,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where  $b \neq 0$  and  $e = (q, p, \lambda)$ . In this case:

$$R(g, e, x, y) = R_g(x, y - q) + R_e(y - q, y),$$
 (3.3.3)

$$K(g, e, x, y) = a_g \cdot \psi(R_g(x, y - q) + R_e(y - q, y)).$$
 (3.3.4)

## 3.4 Deligne's letter

In this section we geometrize (Theorem 3.4.2) the total representation  $\rho \ltimes \pi : D \longrightarrow \mathcal{H}$ . First, we realize all finite sets as rational points of certain algebraic varieties. Vector spaces:  $V = \mathbb{V}(\mathbb{F}_q)$ ,  $V_1 = \mathbb{V}_1(\mathbb{F}_q)$ . Groups:  $E = \mathbb{E}(\mathbb{F}_q)$ , where  $\mathbb{E} = \mathbb{V} \times \mathbb{G}_a$ ,  $\operatorname{Sp}_{\omega} = \operatorname{Sp}_{\omega}(\mathbb{F}_q)$  and finally  $D = \mathbb{D}(\mathbb{F}_q)$ , where  $\mathbb{D} = \operatorname{Sp}_{\omega} \times \mathbb{E}$ . The second step is to replace the kernel  $K := K_{\rho \ltimes \pi} : D \times \mathbb{F}_q^2 \longrightarrow \mathbb{C}$  (see (3.3.4)) by sheaf theoretic object. Recall that the kernel K is a representation kernel, namely it is a kernel of a representation and hence satisfies the convolution property:

$$m^*K = K * K \tag{3.4.1}$$

where  $m: D \times D \longrightarrow D$  denotes the multiplication map and \* means convolution of kernels, i.e., matrix multiplication. We replace the kernel K by Deligne's Weil representation

sheaf [D1]. This is an object  $\mathcal{K} \in \mathcal{D}^b_{c,w}(\mathbb{D} \times \mathbb{A}^2)$  that satisfies<sup>2</sup> the analogue (to (3.4.1)) convolution property:

$$m^*\mathcal{K} \cong \mathcal{K} * \mathcal{K}$$

and its function is:

$$f^{\mathcal{K}} = K$$
.

Here  $m: \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}$  denotes the multiplication morphism and \* means convolution of sheaves.

#### 3.4.1 Uniqueness and existence of the Weil representation sheaf

#### The strategy

The method of constructing the Weil representation sheaf  $\mathcal{K}$  is close in spirit to the construction of an analytic function via analytic continuation. In the realm of perverse sheaves one uses the operation of perverse extension (cf. [BBD]). The main idea is to construct, using formulas, an explicit irreducible perverse sheaf  $\mathcal{K}_{\mathbb{Q}}$  on a "good" open subvariety  $\mathbb{Q} \subset \mathbb{D} \times \mathbb{A}^2$  and then to obtain the sheaf  $\mathcal{K}$  by perverse extension of  $\mathcal{K}_{\mathbb{Q}}$  to the whole variety  $\mathbb{D} \times \mathbb{A}^2$ .

#### Explicit construction on open subvariety

On introducing coordinates we get:  $\mathbb{V} \simeq \mathbb{A}^2$ ,  $\mathbb{V}_1 \simeq \mathbb{A}^1$  and  $\mathbb{S}p_{\omega} \simeq \mathbb{S}\mathbb{L}_2$ . We denote by  $\mathbb{O}$  the open subvariety:

$$\mathbb{O} := \mathbb{O}_w \times \mathbb{E} \times \mathbb{A}^2$$

where  $\mathbb{O}_w$  denotes the (opposite) big Bruhat cell  $\mathbb{B}w\mathbb{B} \subset \mathbb{SL}_2$ .

Let us fix some standard notations from the theory of  $\ell$ -adic sheaves (see [BBD] for the notions of  $\ell$ -adic sheaves). We denote by  $\mathcal{L}_{\psi}$  the Artin-Schreier sheaf on the group  $\mathbb{G}_a$  that corresponds to the character  $\psi$ . We denote by  $\mathcal{L}_{\sigma}$  the Kummer sheaf on the multiplicative group  $\mathbb{G}_m$  that corresponds to the Legendre character  $\sigma$ . In the sequel we will frequently make use of the *character* property (cf. [Ga]) of the sheaves  $\mathcal{L}_{\psi}$  and  $\mathcal{L}_{\sigma}$ , namely:

$$s^* \mathcal{L}_{\psi} \simeq \mathcal{L}_{\psi} \boxtimes \mathcal{L}_{\psi},$$
 (3.4.2)

$$m^* \mathcal{L}_{\sigma} \simeq \mathcal{L}_{\sigma} \boxtimes \mathcal{L}_{\sigma}$$
 (3.4.3)

<sup>&</sup>lt;sup>2</sup>However, in this work we will prove a weaker property (see Theorem 3.4.2) which is sufficient for our purposes.

where  $s: \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$  and  $m: \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$  denote the addition and multiplication morphisms correspondingly, and  $\boxtimes$  means exterior tensor product of sheaves.

At last, given a sheaf  $\mathcal{L}$  we use the notation  $\mathcal{L}[i]$  for the translation functors and the notation  $\mathcal{L}(i)$  for the i'th Tate twist.

#### The construction of the explicit sheaf $\mathcal{K}_{\mathbb{O}}$

We sheafify the kernel  $K_{\rho \ltimes \pi}$  of the total representation, when restricted to the set  $O := O_w \times E \times \mathbb{F}_q^2$ , using the formula (3.3.4). We obtain a sheaf on the open subvariety  $\mathbb{O}_w \times \mathbb{E} \times \mathbb{A}^2$ , which we denote by  $\mathcal{K}_{\mathbb{O}}$ . We define:

$$\mathcal{K}_{\mathbb{O}}:=\mathcal{A}_{\mathbb{O}}\otimes ilde{\mathcal{K}}_{\mathbb{O}}$$

where  $\tilde{\mathcal{K}}_{\mathbb{O}}$  is the sheaf of the non-normalized kernels and  $\mathcal{A}_{\mathbb{O}}$  is the sheaf of the normalization coefficients. The sheaves  $\tilde{\mathcal{K}}_{\mathbb{O}}$  and  $\mathcal{A}_{\mathbb{O}}$  are constructed as follows: define the morphism  $R: \mathbb{O}_w \times \mathbb{E} \times \mathbb{A}^2 \longrightarrow \mathbb{A}^1$  by formula (3.3.3) and let  $pr: \mathbb{O}_w \times \mathbb{E} \times \mathbb{A}^2 \longrightarrow \mathbb{O}_w$  be the projection morphism. Now take:

$$\tilde{\mathcal{K}}_{\mathbb{O}} := \mathbb{R}^* \mathscr{L}_{\psi}, 
\mathcal{A}_{\mathbb{O}} := pr^* \mathcal{A}_{\mathbb{O}_w}$$

where:

$$\mathcal{A}_{\mathbb{O}_w} := pr_{1!}(\nu^* \mathscr{L}_{\psi} \otimes pr_2^* \mathscr{L}_{\sigma})[2](1).$$

Here  $\nu: \mathbb{O}_w \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$  is the morphism defined by  $\nu(g,t) = \frac{1}{2}bt$ , where  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , and  $pr_1: \mathbb{O}_w \times \mathbb{A}^1 \longrightarrow \mathbb{O}_w$ ,  $pr_2: \mathbb{O}_w \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$  denote the natural projections on the first and second coordinates correspondingly. From the construction we deduce:

Corollary 3.4.1 The sheaf  $K_{\mathbb{O}}$  is perverse irreducible of pure weight zero and its function agrees with K on O, i.e.,  $f^{K_{\mathbb{O}}} = K_{|_{\mathcal{O}}}$ .

#### Main theorem

We are now ready to state and prove the main theorem of this section:

Theorem 3.4.2 (Weil representation sheaf [D1]) There exists a unique (up to isomorphism) object  $K \in \mathcal{D}_{c.w}^b(\mathbb{D} \times \mathbb{A}^2)$  which satisfies the following properties:

1. (Restriction) There exists an isomorphism  $\mathcal{K}_{|_{\mathbb{O}}} \simeq \mathcal{K}_{\mathbb{O}}$ .

2. (Convolution property) For every element  $g \in \mathbb{D}$  there exists isomorphisms:

$$\mathcal{K}_{|g} * \mathcal{K} \simeq L_g^*(\mathcal{K})$$

and:

$$\mathcal{K} * \mathcal{K}_{|_{q}} \simeq R_{q}^{*}(\mathcal{K}).$$

Here  $\mathcal{K}_{|g}$  denotes the restriction of  $\mathcal{K}$  to the subvariety  $g \times \mathbb{A}^2$ , \* means convolution of sheaves and  $R_g, L_g : \mathbb{D} \longrightarrow \mathbb{D}$  are the morphisms of right and left multiplication by g respectively.

Corollary 3.4.3 The Weil representation sheaf K has the following two properties:

- a. (Function)  $f^{\mathcal{K}} = K$ .
- b. (Weight)  $\omega(\mathcal{K}) = 0$ .

*Proof* (of Theorem 3.4.2). **Uniqueness.** Let  $\mathcal{K}, \mathcal{K}'$  be two sheaves satisfying properties 1 and 2. By property 1 there exists isomorphisms:

$$\begin{array}{cccc} \mathcal{K}_{\mid_{\mathbb{O}}} & \simeq & \mathcal{K}_{\mathbb{O}}, \\ \mathcal{K}'_{\mid_{\mathbb{O}}} & \simeq & \mathcal{K}_{\mathbb{O}}. \end{array}$$

This implies that  $\mathcal{K}_{|_{\mathbb{O}}}$  and  $\mathcal{K}'_{|_{\mathbb{O}}}$  are two isomorphic irreducible perverse sheaves. Moreover, dim  $\text{Hom}(\mathcal{K}_{|_{\mathbb{O}}}, \mathcal{K}'_{|_{\mathbb{O}}}) = 1$ . Applying property 2 for the Weyl element  $w \in \mathbb{SL}_2$ , we obtain the following isomorphisms:

$$\begin{array}{cccc} \mathcal{K}_{|_{w\mathbb{O}}} & \simeq & \mathcal{K}_w * L_{w^{-1}}^* \mathcal{K}_{|_{\mathbb{O}}}, \\ \mathcal{K}'_{|_{w\mathbb{O}}} & \simeq & \mathcal{K}_w * L_{w^{-1}}^* \mathcal{K}'_{|_{\mathbb{O}}} \end{array}$$

where  $\mathcal{K}_w := \mathcal{K}_{\mathbb{O}_{|w}}$ . Convolving with  $\mathcal{K}_w$  is basically applying the Fourier transform, therefore it takes irreducible perverse sheaves into irreducible perverse sheaves (cf. [KL]). Hence, we get that  $\mathcal{K}_{|w\mathbb{O}}$  and  $\mathcal{K}'_{|w\mathbb{O}}$  are two isomorphic irreducible perverse sheaves and in particular dim  $\operatorname{Hom}(\mathcal{K}_{|w\mathbb{O}}, \mathcal{K}'_{|w\mathbb{O}}) = 1$ . Having that  $\mathbb{D} \times \mathbb{A}^2 = \mathbb{O} \cup w\mathbb{O}$  we are left to show that one can choose  $\theta_{|\mathbb{O}} : \mathcal{K}_{|\mathbb{O}} \longrightarrow \mathcal{K}'_{|\mathbb{O}}$  and  $\theta_{|w\mathbb{O}} : \mathcal{K}_{|w\mathbb{O}} \longrightarrow \mathcal{K}'_{|w\mathbb{O}}$  that agree on the intersection  $\mathbb{O} \cap w\mathbb{O}$ . This can be done since the restrictions  $\mathcal{K}_{|\mathbb{O} \cap w\mathbb{O}}$  and  $\mathcal{K}'_{|\mathbb{O} \cap w\mathbb{O}}$  are again two isomorphic irreducible perverse sheaves. Hence we obtained the required isomorphism.

**Existence.** The construction is immediate. We take our sheaf to be the *perverse extension*:

$$\mathcal{K} := j_{!*}(\mathcal{K}_{\mathbb{O}})$$

where j stands for the open imbedding  $j:\mathbb{O}\longrightarrow\mathbb{D}\times\mathbb{A}^2$ .

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Claim 3.4.4 The sheaf K satisfies properties 1 and 2.

This completes the proof of Theorem 3.4.2.

From the construction we learn:

Corollary 3.4.5 The sheaf K is irreducible perverse.

## 3.5 Proofs

In this section we give the proofs for all technical facts that appeared in Chapter 3.

**Proof of Theorem 3.1.3**. Before giving the proof, we introduce a structure which is inherent to configurations of triple Lagrangian subspaces. Let  $L_1, L_2, L_3 \subset V$  be three Lagrangian subspaces which are in a general position. In our case, these are just three different lines in the plane. Then the space  $L_j$  induces an isomorphism  $r_{L_i,L_k}: L_k \longrightarrow L_i$ ,  $i \neq j \neq k$ , which is given by the rule  $r_{L_i,L_k}(l_k) = l_i$  where  $l_k + l_i \in L_j$ .

The actual proof of the theorem will be given in two parts. In the first part we deal with the case where the three lines  $L_1, L_2, L_2 \in Lag$  are in a general position. In the second part we deal with the case when two of the three lines are equal to each other.

**Part 1.** (General position) Let  $L_1^{\circ}, L_2^{\circ}, L_3^{\circ} \in Lag^{\circ}$  be three oriented lines in a general position. Using the presentation (3.1.2) we can write:

$$\begin{array}{rcl} \theta_{L_{3}^{\circ},L_{2}^{\circ}} \circ \theta_{L_{2}^{\circ},L_{1}^{\circ}} & = & a_{L_{3}^{\circ},L_{2}^{\circ}} \cdot a_{L_{2}^{\circ},L_{1}^{\circ}} \cdot \tilde{\theta}_{L_{3}^{\circ},L_{2}^{\circ}} \circ \tilde{\theta}_{L_{2}^{\circ},L_{1}^{\circ}}, \\ \\ \theta_{L_{3}^{\circ},L_{1}^{\circ}} & = & a_{L_{3}^{\circ},L_{1}^{\circ}} \cdot \tilde{\theta}_{L_{3}^{\circ},L_{1}^{\circ}}. \end{array}$$

The result for Part 1 is a consequence of the following three simple lemmas:

**Lemma 3.5.1** The following equality holds:

$$\tilde{\theta}_{L_3^{\circ},L_2^{\circ}} \circ \tilde{\theta}_{L_2^{\circ},L_1^{\circ}} = C \cdot \tilde{\theta}_{L_3^{\circ},L_1^{\circ}}$$

where 
$$C = \sum_{l_2 \in L_2} \psi(\frac{1}{2}\omega(l_2, r_{L_3, L_2}(l_2)))$$

**Lemma 3.5.2** The following equality holds:

$$a_{L_3^{\circ}, L_2^{\circ}} \cdot a_{L_2^{\circ}, L_1^{\circ}} = D \cdot a_{L_3^{\circ}, L_1^{\circ}}$$

where D = 
$$\frac{1}{q} \sum_{l_2 \in L_2} \psi(-\frac{1}{2} \omega(l_2, r_{L_3, L_2}(\xi_{L_2}))) \varrho_{L_2}(l_2) \varrho_{L_2}(\xi_{L_2})$$

**Lemma 3.5.3** The following equality holds:

$$D \cdot C = 1$$
.

Part 2. (Non-general position) It is enough to check the following equalities:

$$\theta_{\mathbf{L}_{1}^{\circ},\mathbf{L}_{2}^{\circ}} \circ \theta_{\mathbf{L}_{2}^{\circ},\mathbf{L}_{1}^{\circ}} = \mathbf{I}, \tag{3.5.1}$$

$$\theta_{L_{1}^{\circ},L_{2}^{\circ}}^{1/2} \circ \theta_{L_{2}^{\circ},L_{1}^{\circ}}^{1/2} = -I$$
 (3.5.2)

where  $L_1^{\bar{o}}$  has the opposite orientation to  $L_1^{\circ}$ . We verify equation (3.5.1). The verification of (3.5.2) is done in the same way, therefore we omit it.

Write:

$$\tilde{\theta}_{L_1^{\circ}, L_2^{\circ}}(\tilde{\theta}_{L_2^{\circ}, L_1^{\circ}}(f))(e) = \sum_{l_1 \in L_1} \sum_{l_2 \in L_2} f(l_2 l_1 e)$$
(3.5.3)

where  $f \in \mathcal{H}_{L_1}$  and  $e \in E$ . Both sides of (3.5.1) are self intertwining operators of  $\mathcal{H}_{L_1}$  therefore they are proportional. Hence it is sufficient to compute (3.5.3) for a specific function f and specific element  $e \in E$ . We take e = 0 and  $f = \delta_0$ , where  $\delta_0(\lambda le) := \psi(\lambda)$  if e = 0 and equals 0 otherwise. We get:

$$\sum_{l_1 \in L_1} \sum_{l_2 \in L_2} f(l_2 l_1 e) = q.$$

Now write:

$$\mathbf{a}_{\mathbf{L}_{1}^{\circ},\mathbf{L}_{2}^{\circ}} \cdot \mathbf{a}_{\mathbf{L}_{2}^{\circ},\mathbf{L}_{1}^{\circ}} = \frac{1}{q^{2}} \sum_{l_{1} \in \mathbf{L}_{1}, \, l_{2} \in \mathbf{L}_{2}} \psi(\frac{1}{2}\omega(l_{2},\xi_{\mathbf{L}_{1}}) + \frac{1}{2}\omega(l_{1},\xi_{\mathbf{L}_{2}})) \varrho_{\mathbf{L}_{2}}(l_{2}) \varrho_{\mathbf{L}_{1}}(\xi_{\mathbf{L}_{1}}) \varrho_{\mathbf{L}_{2}}(\xi_{\mathbf{L}_{2}}).$$

We identify L<sub>2</sub> and L<sub>1</sub> with the field  $\mathbb{F}_q$  by the rules  $s \cdot 1 \longmapsto s \cdot \xi_{L_2}$  and  $t \cdot 1 \longmapsto t \cdot \xi_{L_1}$  correspondingly. In terms of these identifications we get:

$$\mathbf{a}_{\mathbf{L}_{1}^{\circ},\mathbf{L}_{2}^{\circ}} \cdot \mathbf{a}_{\mathbf{L}_{2}^{\circ},\mathbf{L}_{1}^{\circ}} = \frac{1}{q^{2}} \sum_{t, s \in \mathbb{F}_{q}} \psi(\frac{1}{2} s\omega(\xi_{\mathbf{L}_{2}}, \xi_{\mathbf{L}_{1}}) + \frac{1}{2} t\omega(\xi_{\mathbf{L}_{1}}, \xi_{\mathbf{L}_{2}})) \sigma(t) \sigma(s). \tag{3.5.4}$$

Denote by  $a=\omega(\xi_{\text{\tiny L}_2},\xi_{\text{\tiny L}_1}).$  The right-hand side of (3.5.4) is equal to:

$$\frac{1}{q^2} \sum_{s \in \mathbb{F}_q} \psi(\frac{1}{2} as) \sigma(s) \cdot \sum_{t \in \mathbb{F}_q} \psi(-\frac{1}{2} at) \sigma(t) = \frac{q}{q^2} = \frac{1}{q}.$$

All together we get:

$$\theta_{{\scriptscriptstyle L}_1^{\circ}, {\scriptscriptstyle L}_2^{\circ}} \circ \theta_{{\scriptscriptstyle L}_2^{\circ}, {\scriptscriptstyle L}_1^{\circ}} = I.$$

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**Proof of Lemma 3.5.1.** The proof is by direct computation. Write:

$$\tilde{\theta}_{L_3^{\circ}, L_1^{\circ}}(f)(e) = \sum_{l_3 \in L_3} f(l_3 e),$$
(3.5.5)

$$\tilde{\theta}_{L_3^{\circ}, L_2^{\circ}}(\tilde{\theta}_{L_2^{\circ}, L_1^{\circ}}(f))(e) = \sum_{l_3 \in L_3} \sum_{l_2 \in L_2} f(l_2 l_3 e)$$
(3.5.6)

where  $f \in \mathcal{H}_{L_1}$  and  $e \in E$ . Both (3.5.5) and (3.5.6) are intertwining operators from  $\mathcal{H}_{L_1}$  to  $\mathcal{H}_{L_3}$ , therefore they are proportional. In order to compute the proportionality coefficient C it is enough to compute (3.5.5) and (3.5.6) for specific f and specific e. We take e = 0 and  $f = \delta_0$  where  $\delta_0(q, p, \lambda) := \psi(\lambda)$ . We get:

$$\tilde{\theta}_{L_3^\circ, L_1^\circ}(\delta_0) = 1,$$

$$\tilde{\theta}_{L_3^{\circ}, L_2^{\circ}} (\tilde{\theta}_{L_2^{\circ}, L_1^{\circ}} (\delta_0))(0) = \sum_{l_2 + l_3 \in L_1} \psi(\frac{1}{2} \omega(l_2, l_3)). \tag{3.5.7}$$

But the right-hand side of (3.5.7) is equal to:

$$\sum_{l_2 \in L_2} \psi(\frac{1}{2} \omega(l_2, r_{L_3, L_2}(l_2))).$$

**Proof of Lemma 3.5.2.** The proof is by direct computation. Write:

$$a_{L_3^{\circ}, L_1^{\circ}} = \frac{1}{q} \sum_{l_1 \in L_1} \psi(\frac{1}{2} \omega(l_1, \xi_{L_3})) \varrho_{L_1}(l_1) \varrho_{L_2}(\xi_{L_3}),$$

$$\mathbf{a}_{\mathbf{L}_{3}^{\circ},\mathbf{L}_{2}^{\circ}} \cdot \mathbf{a}_{\mathbf{L}_{2}^{\circ},\mathbf{L}_{1}^{\circ}} = \frac{1}{q^{2}} \sum_{l_{1} \in \mathbf{L}_{1}, \ l_{2} \in \mathbf{L}_{2}} \psi(\frac{1}{2}\omega(l_{1},\xi_{\mathbf{L}_{2}}) + \frac{1}{2}\omega(l_{2},\xi_{\mathbf{L}_{3}})) \varrho_{\mathbf{L}_{1}}(l_{1}) \varrho_{\mathbf{L}_{2}}(\xi_{\mathbf{L}_{2}}) \varrho_{\mathbf{L}_{2}}(l_{2}) \varrho_{\mathbf{L}_{3}}(\xi_{\mathbf{L}_{3}}).$$

$$(3.5.8)$$

The term  $\psi(\frac{1}{2}\omega(l_1,\xi_{L_2}) + \frac{1}{2}\omega(l_2,\xi_{L_3}))$  is equal to:

$$\psi(\frac{1}{2}\omega(l_1,\xi_{L_2}-\xi_{L_3})+\frac{1}{2}\omega(l_2,\xi_{L_3}))\cdot\psi(\frac{1}{2}\omega(l_1,\xi_{L_3})). \tag{3.5.9}$$

We are free to choose  $\xi_{L_3}$  such that  $\xi_{L_2} - \xi_{L_3} \in L_1$ . Therefore using (3.5.9) we get that the right-hand side of (3.5.8) is equal to:

$$\frac{1}{q} \sum_{l_2 \in L_2} \psi(\frac{1}{2} \omega(l_2, \xi_{L_3})) \varrho_{L_2}(\xi_{L_2}) \varrho_{L_2}(l_2) \cdot a_{L_3^{\circ}, L_1^{\circ}}.$$
(3.5.10)

Now, substituting  $\xi_{L_3} = -r_{L_3,L_2}(\xi_{L_2})$  in (3.5.10) we obtain:

$$\frac{1}{q} \sum_{l_2 \in \mathcal{L}_2} \psi(-\frac{1}{2} \, \omega(l_2, \mathcal{r}_{\mathcal{L}_3, \mathcal{L}_2}(\xi_{\mathcal{L}_2}))) \varrho_{\mathcal{L}_2}(l_2) \varrho_{\mathcal{L}_2}(\xi_{\mathcal{L}_2}) \cdot \mathcal{a}_{\mathcal{L}_3^\circ, \mathcal{L}_1^\circ}.$$

**Proof of Lemma 3.5.3.** Identify L<sub>2</sub> with  $\mathbb{F}_q$  by the rule  $t \cdot 1 \longmapsto t \cdot \xi_{L_2}$ . In terms of this identification we get:

$$D = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-\frac{1}{2} \omega(t\xi_{L_2}, r_{L_3, L_2}(\xi_{L_2}))) \varrho(t),$$

$$C = \sum_{t \in \mathbb{F}_q} \psi(\frac{1}{2} \omega(t\xi_{L_2}, r_{L_3, L_2}(t\xi_{L_2}))).$$

Denote by  $a:=\omega(\xi_{{\scriptscriptstyle \mathrm{L}}_2},\mathbf{r}_{{\scriptscriptstyle \mathrm{L}}_3,{\scriptscriptstyle \mathrm{L}}_2}(\xi_{{\scriptscriptstyle \mathrm{L}}_2})).$  Then:

$$D = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-\frac{1}{2} at) \varrho(t),$$

$$C = \sum_{t \in \mathbb{F}_q} \psi(\frac{1}{2} at^2).$$

Now, we have the following remarkable equality:

$$\sum_{t \in \mathbb{F}_q} \psi(\frac{1}{2} at) \varrho(t) = \sum_{t \in \mathbb{F}_q} \psi(\frac{1}{2} at^2).$$

This, combined with  $C \cdot \overline{C} = q$ , gives the result.

This completes the proof of Part 2 and of Theorem 3.1.3

**Proof of Claim 3.4.4.** Property 1 follows immediately from the construction. We give the proof of property 2 with respect to left multiplication (the proof of the equivariance property with respect to the right multiplication is the same). In the course of the proof we are going to use the following auxiliary sheaves:

• We sheafify the kernel  $K_{\pi}$  using the formula (3.3.2) and obtain a sheaf on  $\mathbb{E} \times \mathbb{A}^2$ , which we denote by  $\mathcal{K}_{\pi}$ . Define the morphisms  $R : \mathbb{E} \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$  and  $i : \mathbb{E} \times \mathbb{A}^1 \longrightarrow \mathbb{E} \times \mathbb{A}^2$  by the formulas  $R((q, p, \lambda), x) = \frac{1}{2}pq + px + \lambda$  and  $i((q, p, \lambda), x) = ((q, p, \lambda), (x, x + q))$ . Now take:

$$\mathcal{K}_{\mathbb{E}} := i_! \mathrm{R}^* \mathscr{L}_{\psi}.$$

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• We sheafify the kernel  $K_{\rho}$  when restricted to the set  $B \times \mathbb{F}_q^2$  using the formula (3.3.1) and obtain a sheaf on the variety  $\mathbb{B} \times \mathbb{A}^2$ , which we denote by  $\mathcal{K}_{\mathbb{B}}$ . We define:

$$\mathcal{K}_{\mathbb{B}}:=\mathcal{A}_{\mathbb{B}}\otimes ilde{\mathcal{K}}_{\mathbb{B}}$$

where the sheaf  $\tilde{\mathcal{K}}_{\mathbb{B}}$  stands for the non-normalized kernels and the sheaf  $\mathcal{A}_{\mathbb{B}}$  stands for the normalization coefficients. The sheaves  $\tilde{\mathcal{K}}_{\mathbb{B}}$  and  $\mathcal{A}_{\mathbb{B}}$  are constructed as follows: define the morphisms  $R: \mathbb{B} \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ ,  $\nu: \mathbb{B} \longrightarrow \mathbb{A}^1$  and  $i: \mathbb{B} \times \mathbb{A}^1 \longrightarrow \mathbb{B} \times \mathbb{A}^2$  by the formulas  $R(b, x) = \frac{1}{2} r a^{-1} x^2$ ,  $\nu(b) = a$  and  $i(b, x) = (b, x, a^{-1}x)$  correspondingly, where  $b = \begin{pmatrix} a & 0 \\ r & a^{-1} \end{pmatrix}$ . Now take:

$$\tilde{\mathcal{K}}_{\mathbb{B}} := i_! \mathrm{R}^* \mathscr{L}_{\psi}$$

and:

$$\mathcal{A}_{\mathbb{B}} := \nu^* \mathscr{L}_{\sigma}.$$

• We will frequently make use of several other sheaves obtained by restrictions from  $\mathcal{K}_{\mathbb{O}}$ ,  $\mathcal{A}_{\mathbb{O}}$ ,  $\mathcal{K}_{\mathbb{B}}$  and  $\mathcal{A}_{\mathbb{B}}$ . Suppose  $\mathbb{X} \subset \mathbb{O}_w \times \mathbb{E}$  is a subvariety. Then we define  $\mathcal{K}_{\mathbb{X}} := \mathcal{K}_{\mathbb{O}_{|_{\mathbb{X} \times \mathbb{A}^2}}}$  and  $\mathcal{A}_{\mathbb{X}} := \mathcal{A}_{\mathbb{O}_{|_{\mathbb{X} \times \mathbb{A}^2}}}$ . The same when  $\mathbb{X} \subset \mathbb{B}$ . Finally, we denote by  $\delta_0$  the *sky-scraper* sheaf on  $\mathbb{A}^1$  which corresponds to the delta function at zero.

**Proof of property 2.** The proof will be given in several steps.

**Step 1.** It is sufficient to prove the equivariance property separately for the Weyl element w, an element  $b \in \mathbb{B}$  and an element  $e \in \mathbb{E}$ . Indeed, Step 1 follows from the Bruhat decomposition, Corollary 3.5.7 below and the following decomposition lemma:

**Lemma 3.5.4** There exist isomorphisms:

$$\mathcal{K}_{\mathbb{O}} \simeq \mathcal{K}_{\mathbb{B}} * \mathcal{K}_{w} * \mathcal{K}_{\mathbb{U}} * \mathcal{K}_{\mathbb{E}}$$

where  $\mathbb{U}$  denote the unipotent radical of  $\mathbb{B}$  and w is the Weyl element.

**Step 2.** We prove property 2 for the Weyl element, i.e., g = w. We want to construct an isomorphism:

$$\mathcal{K}_{|_w} * \mathcal{K} \simeq L_w^* \mathcal{K}. \tag{3.5.11}$$

Noting that both sides of (3.5.11) are irreducible (shifted) perverse sheaves, it is sufficient to construct an isomorphism on the open set  $\mathcal{U} := \mathbb{O} \cap w\mathbb{O}$ . The advantage of working

with this subvariety is that over  $\mathcal{U}$  we have formulas for  $\mathcal{K}$ , and moreover,  $L_w$  maps  $\mathcal{U}$  into itself. On  $\mathcal{U}$  we have two coordinate systems:

$$\mathcal{U} \simeq \mathbb{U}^{\circ \times} \times \mathbb{B} \times \mathbb{E},$$
 (3.5.12)  
 $\mathcal{U} \simeq \mathbb{U}^{\times} w \times \mathbb{B} \times \mathbb{E}$ 

where  $\mathbb{U}^{\times} := \mathbb{U} \setminus \{I\}$  and  $\mathbb{U}^{\circ}$  denotes the standard unipotent radical. With respect to these coordinate systems we have the following decompositions:

#### Claim 3.5.5 There exists isomorphisms:

- 1.  $\mathcal{K}_{\mathcal{U}}(u^{\circ}be) \simeq \mathcal{K}_{\mathbb{U}^{\circ \times}}(u^{\circ}) * \mathcal{K}_{\mathbb{B}}(b) * \mathcal{K}_{\mathbb{E}}(e)$ .
- 2.  $\mathcal{K}_{\mathcal{U}}(uwbe) \simeq \mathcal{K}_{\mathbb{U}^{\times}w}(uw) * \mathcal{K}_{\mathbb{B}}(b) * \mathcal{K}_{\mathbb{E}}(e)$ .

Now, restricting to  $\mathcal{U}$  and using the coordinate system (3.5.12) our *main statement* is the existence of an isomorphism:

$$\mathcal{K}_w * \mathcal{K}_{\mathcal{U}}(u^{\circ}be) \simeq \mathcal{K}_{\mathcal{U}}(wu^{\circ}be).$$
 (3.5.13)

Indeed, on developing the right-hand side of (3.5.13) we obtain:

$$\mathcal{K}_{\mathcal{U}}(wu^{\circ}be) = \mathcal{K}_{\mathcal{U}}(u^{\circ w}wbe) 
\simeq \mathcal{K}_{\mathbb{U}^{\times}w}(u^{\circ w}w) * \mathcal{K}_{\mathbb{B}}(b) * \mathcal{K}_{\mathbb{E}}(e) 
\simeq \mathcal{K}_{w} * (\mathcal{K}_{\mathbb{U}^{\circ}}(u^{\circ}) * \mathcal{K}_{\mathbb{B}}(b) * \mathcal{K}_{\mathbb{E}}(e)) 
\simeq \mathcal{K}_{w} * \mathcal{K}_{\mathcal{U}}(u^{\circ}be)$$

where  $u^{\circ w} := wu^{\circ}w^{-1}$ . The first and third isomorphisms are applications of Claim 3.5.5 parts 2 and 1 respectively. The second isomorphism is a result of associativity of convolution and the following *central lemma*:

**Lemma 3.5.6** There exists an isomorphism:

$$\mathcal{K}_{\mathbb{U}^{\times}w}(u^{\circ w}w) \simeq \mathcal{K}_w * \mathcal{K}_{\mathbb{U}^{\circ \times}}(u^{\circ}). \tag{3.5.14}$$

The following is a consequence of (3.5.11).

Corollary 3.5.7 There exists an isomorphism:

$$\mathcal{K}_{|_{\mathbb{B}\times\mathbb{E}}} \simeq \mathcal{K}_{\mathbb{B}} * \mathcal{K}_{\mathbb{E}}. \tag{3.5.15}$$

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*Proof*. On developing the left hand side of (3.5.15) we obtain:

$$\mathcal{K}_{|_{\mathbb{B}\times\mathbb{E}}} \simeq \mathcal{K}_{w} * \mathcal{K}_{|w^{-1}\mathbb{B}\times\mathbb{E}} 
\simeq \mathcal{K}_{w} * (\mathcal{K}_{w^{-1}} * \mathcal{K}_{\mathbb{B}} * \mathcal{K}_{\mathbb{E}}) 
\simeq (\mathcal{K}_{w} * \mathcal{K}_{w^{-1}}) * \mathcal{K}_{\mathbb{B}} * \mathcal{K}_{\mathbb{E}} 
\simeq \mathcal{K}_{\mathbb{B}} * \mathcal{K}_{\mathbb{E}}.$$

The first isomorphism is a consequence of (3.5.11). The second isomorphism is a consequence of Lemma 3.5.4. The third isomorphism is the associativity property of convolution. The last isomorphism is a property of the Fourier transform [KL], namely  $\mathcal{K}_w * \mathcal{K}_{w^{-1}} \simeq \mathcal{I}$ , where  $\mathcal{I}$  is the kernel of the identity operator.

**Step 3.** We prove property 2 for element  $b \in \mathbb{B}$ . Using corollary 3.5.7 we have  $\mathcal{K}_{|_b} \simeq \mathcal{K}_{\mathbb{B}|_b} := \mathcal{K}_b$ . We want to construct an isomorphism:

$$\mathcal{K}_b * \mathcal{K} \simeq L_b^* \mathcal{K}. \tag{3.5.16}$$

Since both sides of (3.5.16) are irreducible (shifted) perverse sheaves, it is enough to construct an isomorphism on the open set  $\mathbb{O} := \mathbb{O}_w \times \mathbb{E} \times \mathbb{A}^2$ . Write

$$\mathcal{K}_b * \mathcal{K}_{|_{\mathbb{O}}} \simeq \mathcal{K}_b * \mathcal{K}_{\mathbb{O}} \simeq \mathcal{K}_b * (\mathcal{K}_{\mathbb{B}} * \mathcal{K}_w * \mathcal{K}_{\mathbb{U}} * \mathcal{K}_{\mathbb{E}}) \simeq (\mathcal{K}_b * \mathcal{K}_{\mathbb{B}}) * (\mathcal{K}_w * \mathcal{K}_{\mathbb{U}} * \mathcal{K}_{\mathbb{E}}).$$

The first isomorphism is by construction. The second isomorphism is an application of Lemma 3.5.4. The third isomorphism is the associativity property of the convolution operation between sheaves. From the last isomorphism we see that it is enough to construct an isomorphism  $\mathcal{K}_b * \mathcal{K}_{\mathbb{B}} \simeq L_b^*(\mathcal{K}_{\mathbb{B}})$ , where  $L_b : \mathbb{B} \longrightarrow \mathbb{B}$ . The construction is an easy consequence of formula (3.3.1) and the character sheaf property (3.4.2) of  $\mathcal{L}_{\psi}$ .

**Step 4.** We prove property 2 for an element  $e \in \mathbb{E}$ . We want to construct an isomorphism:

$$\mathcal{K}_{|_e} * \mathcal{K} \simeq L_e^* \mathcal{K}. \tag{3.5.17}$$

Both sides of (3.5.17) are irreducible (shifted) perverse sheaves, therefore it is sufficient to construct an isomorphism on the open set  $\mathbb{O}$ . This is done by a direct computation, similar to what has been done before, hence we omit it. This completes the proof of Claim 3.4.4.

**Proof of Lemma 3.5.4**. We will prove the Lemma in two steps.

**Step 1.** We prove that  $\mathcal{K}_{\mathbb{O}} \simeq \mathcal{K}_{\mathbb{O}_w} * \mathcal{K}_{\mathbb{E}}$ . In a more explicit form we want to show:

$$\mathcal{A}_{\mathbb{O}} \otimes \tilde{\mathcal{K}}_{\mathbb{O}} \simeq \mathcal{A}_{\mathbb{O}_w} \otimes \tilde{\mathcal{K}}_{\mathbb{O}_w} * \tilde{\mathcal{K}}_{\mathbb{E}}. \tag{3.5.18}$$

It is sufficient to show the existence of an isomorphism  $\tilde{\mathcal{K}}_{\mathbb{O}} \simeq \tilde{\mathcal{K}}_{\mathbb{O}_w} * \tilde{\mathcal{K}}_{\mathbb{E}}$ . On developing the left-hand side of (3.5.18) we obtain:

$$\tilde{\mathcal{K}}_{\mathbb{O}}(g, e, x, y) := \mathscr{L}_{\psi(R(g, e, x, y))}.$$

On developing the right-hand side we obtain:

$$\begin{split} \tilde{\mathcal{K}}_{\mathbb{O}_{w}} * \tilde{\mathcal{K}}_{\mathbb{E}}(\ (g,e)\ ,x,y) &:= \int\limits_{z \in \mathbb{A}^{1}} \tilde{\mathcal{K}}_{\mathbb{O}_{w}}(g,x,z) \otimes \tilde{\mathcal{K}}_{\mathbb{E}}(e,z,y) \\ &:= \int\limits_{\mathbb{A}^{1}} \mathcal{L}_{\psi(R_{g}(x,z))} \otimes \mathcal{L}_{\psi(R_{e}(z,y))} \otimes \delta_{y=z-\mathbf{q}} \\ &\simeq \mathcal{L}_{\psi(R_{g}(x,y-\mathbf{q}))} \otimes \mathcal{L}_{\psi(R_{e}(y-\mathbf{q},y))} \\ &\simeq \mathcal{L}_{\psi(R_{g}(x,y-\mathbf{q})+R_{e}(y-\mathbf{q},y))} \\ &= \mathcal{L}_{\psi(R(g,e,x,y))}. \end{split}$$

The only non-trivial isomorphism is the last one and it is a consequence of the Artin-Schreier sheaf being a character sheaf on the additive group  $\mathbb{G}_a$ .

**Step 2.** We prove that  $\mathcal{K}_{\mathbb{O}_w} \simeq \mathcal{K}_{\mathbb{B}} * \mathcal{K}_w * \mathcal{K}_{\mathbb{U}}$ . In a more explicit form we want to show:

$$\mathcal{A}_{\mathbb{O}_w} \otimes \tilde{\mathcal{K}}_{\mathbb{O}_w} \simeq \mathcal{A}_{\mathbb{B}} \otimes \mathcal{A}_w \otimes \mathcal{A}_{\mathbb{U}} \otimes \tilde{\mathcal{K}}_{\mathbb{B}} * \tilde{\mathcal{K}}_w * \tilde{\mathcal{K}}_{\mathbb{U}}.$$

We will separately show the existence of two isomorphisms:

$$\tilde{\mathcal{K}}_{\mathbb{O}_w} \simeq \tilde{\mathcal{K}}_{\mathbb{B}} * \tilde{\mathcal{K}}_w * \tilde{\mathcal{K}}_{\mathbb{U}},$$
(3.5.19)

$$\mathcal{A}_{\mathbb{O}_{w}} \simeq \mathcal{A}_{\mathbb{B}} \otimes \mathcal{A}_{w} \otimes \mathcal{A}_{\mathbb{U}}. \tag{3.5.20}$$

**Isomorphism (3.5.19).** Introduce the coordinate system  $\mathbb{O}_w \cong \mathbb{B} \times w \times \mathbb{U}$ . Let  $b = \begin{pmatrix} a & 0 \\ r & a^{-1} \end{pmatrix}$  and  $u = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$  be general elements in the groups  $\mathbb{U}$  and  $\mathbb{B}$  respectively. In terms of the coordinates (a, r, s) a general element in  $\mathbb{O}_w$  is of the form  $g = \begin{pmatrix} as \\ rs-a^{-1} & r \end{pmatrix}$ . Developing the left-hand side of (3.5.19) in terms of the coordinates (a, r, s) we obtain:

$$\tilde{\mathcal{K}}_{\mathbb{O}_w}(bwu, x, y) := \mathcal{L}_{\psi(-\frac{1}{2}a^{-1}rx^2 + a^{-1}xy - \frac{1}{2}sy^2)}.$$

On developing the right-hand side of (3.5.19) we obtain:

$$\tilde{\mathcal{K}}_{\mathbb{B}} * \tilde{\mathcal{K}}_{w} * \tilde{\mathcal{K}}_{\mathbb{U}}(bwu, x, y) := \int_{z, z' \in \mathbb{A}^{1}} \tilde{\mathcal{K}}_{\mathbb{B}}(b, x, z) \otimes \tilde{\mathcal{K}}_{w}(w, z, z') \otimes \tilde{\mathcal{K}}_{\mathbb{U}}(u, z', y) 
:= \int_{z, z' \in \mathbb{A}^{1}} \mathcal{L}_{\psi(-\frac{1}{2}ra^{-1}x^{2})} \otimes \delta_{x=az} \otimes \mathcal{L}_{\psi(zz')} \otimes \mathcal{L}_{\psi(-\frac{1}{2}sz'^{2})} \otimes \delta_{y=z'} 
\simeq \mathcal{L}_{\psi(-\frac{1}{2}ra^{-1}x^{2})} \otimes \mathcal{L}_{\psi(a^{-1}xy)} \otimes \mathcal{L}_{\psi(-\frac{1}{2}sy^{2})} 
\simeq \mathcal{L}_{\psi(-\frac{1}{2}ra^{-1}x^{2}+a^{-1}xy-\frac{1}{2}sy^{2})}.$$

The last isomorphism is a consequence of the fact that the Artin-Schreier sheaf is a character sheaf (3.4.2). Altogether we obtained isomorphism (3.5.19).

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**Isomorphism (3.5.20).** On developing the left-hand side of (3.5.20) in terms of the coordinates (a, r, s) we obtain:

$$\mathcal{A}_{\mathbb{O}_{w}}(bwu) := \mathcal{G}(\psi_{a}, \sigma)[2](-1) 
\simeq \mathcal{G}(\psi, \sigma_{a^{-1}})[2](-1) 
\simeq \mathcal{L}_{\sigma(a^{-1})} \otimes \mathcal{G}(\psi, \sigma)[2](-1) 
\simeq \mathcal{L}_{\sigma(a)} \otimes \mathcal{G}(\psi, \sigma)[2](-1) 
=: \mathcal{A}_{\mathbb{B}} \otimes \mathcal{A}_{w} \otimes \mathcal{A}_{\mathbb{U}}(bwu)$$

where  $\mathcal{G}(\psi_s, \sigma_a) := \int_{\mathbb{A}^1} \mathcal{L}_{\psi(\frac{1}{2}sz)} \otimes \mathcal{L}_{\sigma(az)}$  denotes the quadratic Gauss-sum sheaf. The second isomorphism is a change of coordinates  $z \mapsto az$  under the integration. The third isomorphism is a consequence of the Kummer sheaf  $\mathcal{L}_{\sigma}$  being a character sheaf on the multiplicative group  $\mathbb{G}_m$  (3.4.3). The fourth isomorphism is a specific property of the Kummer sheaf which is associated to the quadratic character  $\sigma$ . This completes the construction of isomorphism (3.5.20).

**Proof of claim 3.5.5.** Carried out in exactly the same way as the proof of the decomposition Lemma 3.5.4. Namely, using the explicit formulas of the sheaves  $\mathcal{K}_{\mathbb{O}}$ ,  $\mathcal{K}_{\mathbb{B}}$ ,  $\mathcal{K}_{\mathbb{E}}$  and the character sheaf property of the sheaves  $\mathcal{L}_{\psi}$  and  $\mathcal{L}_{\sigma}$ .

**Proof of Lemma 3.5.6.** First, we write isomorphism (3.5.14) in a more explicit form:

$$\mathcal{A}_{\mathbb{U}^{\times}w} \otimes \tilde{\mathcal{K}}_{\mathbb{U}^{\times}w}(u^{\circ w}w) \simeq \mathcal{A}_{w} \otimes \mathcal{A}_{\mathbb{U}^{\circ}} \otimes \tilde{\mathcal{K}}_{w} * \tilde{\mathcal{K}}_{\mathbb{U}^{\circ}} (u^{\circ}). \tag{3.5.21}$$

Let  $u^{\circ} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in \mathbb{U}^{\circ \times}$  be a non-trivial unipotent,. Then  $u^{\circ w}w = wu^{\circ} = \begin{pmatrix} 0 & 1 \\ -1 & -s \end{pmatrix}$ . On developing the left-hand side of (3.5.21) we obtain:

$$\tilde{\mathcal{K}}_{\mathbb{U}^{\times}w}(u^{\circ w}w, x, y) := \mathcal{L}_{\psi(\frac{1}{2}sx^{2} + xy)}, 
\mathcal{A}_{\mathbb{U}^{\times}w}(u^{\circ w}w) := \mathcal{G}(\psi, \sigma)[2](1)$$

where  $\mathcal{G}(\psi_s, \sigma_a) := \int_{\mathbb{A}^1} \mathscr{L}_{\psi(\frac{1}{2}sz)} \otimes \mathscr{L}_{\sigma(az)}$ .

On developing the right-hand side of (3.5.21) we obtain:

$$\tilde{\mathcal{K}}_{w} * \tilde{\mathcal{K}}_{\mathbb{U}^{\circ} \times}(u^{\circ}, x, y) := \int_{z \in \mathbb{A}^{1}} \tilde{\mathcal{K}}_{w}(x, z) \otimes \tilde{\mathcal{K}}_{\mathbb{U}^{\circ} \times}(u, z, y) 
:= \int_{\mathbb{A}^{1}} \mathcal{L}_{\psi(xz)} \otimes \mathcal{L}_{\psi(-\frac{1}{2}s^{-1}z^{2} + s^{-1}zy - \frac{1}{2}s^{-1}y^{2})} 
\simeq \int_{\mathbb{A}^{1}} \mathcal{L}_{\psi(xz - \frac{1}{2}s^{-1}z^{2} + s^{-1}zy - \frac{1}{2}s^{-1}y^{2})} 
\simeq \int_{\mathbb{A}^{1}} \mathcal{L}_{\psi(-\frac{1}{2}s^{-1}(z - sx - y)^{2})} \otimes \mathcal{L}_{\psi(\frac{1}{2}sx^{2} + xy)} 
\simeq \int_{\mathbb{A}^{1}} \mathcal{L}_{\psi(-\frac{1}{2}s^{-1}z^{2})} \otimes \mathcal{L}_{\psi(\frac{1}{2}sx^{2} + xy)}.$$

By applying change of coordinates  $z \mapsto sz$  under the last integration we obtain:

$$\int_{\mathbb{A}^1} \mathcal{L}_{\psi(-\frac{1}{2}s^{-1}z^2)} \otimes \mathcal{L}_{\psi(\frac{1}{2}sx^2 + xy)} \simeq \int_{z \in \mathbb{A}^1} \mathcal{L}_{\psi(-\frac{1}{2}sz^2)} \otimes \mathcal{L}_{\psi(\frac{1}{2}sx^2 + xy)}. \tag{3.5.22}$$

Now write:

$$\mathcal{A}_w \otimes \mathcal{A}_{\mathbb{I}^{0}}(u^\circ) := \mathcal{G}(\psi, \sigma)[2](1) \otimes \mathcal{G}(\psi_s, \sigma)[2](1). \tag{3.5.23}$$

Combining (3.5.22) and (3.5.23) we obtain that the right-hand side of (3.5.21) is isomorphic to:

$$\left(\mathcal{G}(\psi_s,\sigma)[2](1)\otimes \int_{\mathbb{A}^1}\mathscr{L}_{\psi(-\frac{1}{2}sz^2)}\right)\otimes \left(\mathcal{G}(\psi,\sigma)[2](1)\otimes \mathscr{L}_{\psi(\frac{1}{2}sx^2+xy)}\right).$$

The main argument is the existence of the following isomorphism:

$$\mathcal{G}(\psi_s, \sigma)[2](1) \otimes \int_{\mathbb{A}^1} \mathscr{L}_{\psi(-\frac{1}{2}sz^2)} \simeq \overline{\mathbb{Q}}_{\ell}.$$

It is a direct consequence of the following lemma:

**Lemma 3.5.8** (Main lemma) There exists a canonical isomorphism of sheaves on  $\mathbb{G}_m$ :

$$\int_{\mathbb{A}^1} \mathscr{L}_{\psi(\frac{1}{2}sz)} \otimes \mathscr{L}_{\sigma(z)} \simeq \int_{\mathbb{A}^1} \mathscr{L}_{\psi(\frac{1}{2}sz^2)}$$

where  $s \in \mathbb{G}_m$ .

Proof. The parameter s does not play any essential role in the argument, therefore it is sufficient to prove:

$$\int_{\mathbb{A}^1} \mathscr{L}_{\psi(z)} \otimes \mathscr{L}_{\sigma(z)} \simeq \int_{\mathbb{A}^1} \mathscr{L}_{\psi(z^2)}.$$
 (3.5.24)

Define the morphism  $p: \mathbb{G}_m \longrightarrow \mathbb{G}_m$ ,  $p(x) = x^2$ . The morphism p is an ètale double cover. We have  $p_*\overline{\mathbb{Q}}_\ell \simeq \mathscr{L}_\sigma \oplus \overline{\mathbb{Q}}_\ell$ . Now on developing the left-hand-side of (3.5.24) we obtain:

$$\int_{\mathbb{A}^1} \mathscr{L}_{\psi(z)} \otimes \mathscr{L}_{\sigma(z)} := \pi_! (\mathscr{L}_{\psi} \otimes \mathscr{L}_{\sigma}) \simeq \pi_! (\mathscr{L}_{\psi} \otimes (\mathscr{L}_{\sigma} \oplus \overline{\mathbb{Q}}_{\ell})).$$

The first step is just a translation to conventional notations, where  $\pi$  stands for the projection  $\pi: \mathbb{G}_m \longrightarrow pt$ . The second isomorphism uses the fact that  $\pi_! \mathscr{L}_{\psi} \simeq 0$ . Next:

$$\pi_!(\mathscr{L}_{\psi}\otimes(\mathscr{L}_{\sigma}\oplus\overline{\mathbb{Q}}_{\ell}))\simeq\pi_!(\mathscr{L}_{\psi}\otimes p_*\overline{\mathbb{Q}}_{\ell})\simeq\pi_!p^*\mathscr{L}_{\psi}=:\int_{\mathbb{A}^1}\mathscr{L}_{\psi(z^2)}.$$

This completes the proof of proposition 3.5.6.

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**Proof of Corollary 3.4.3.** We have  $f^{\mathcal{K}_{|_{\mathbb{O}}}} = K_{|_{\mathcal{O}}}$ . Applying the convolution property for the Weyl element  $w \in \mathbb{SL}_2$  we obtain the following isomorphism:

$$\mathcal{K}_{|_{w\mathbb{O}}} \simeq \mathcal{K}_w * L_{w^{-1}}^* \mathcal{K}_{|_{\mathbb{O}}}. \tag{3.5.25}$$

Hence  $f^{\mathcal{K}|_{w\mathbb{O}}} = K_{|w\mathbb{O}}$ . Having that  $\mathbb{D} \times \mathbb{A}^2 = \mathbb{O} \cup w\mathbb{O}$  we conclude that  $f^{\mathcal{K}} = K$ . In addition we want to show that the sheaf  $\mathcal{K}$  has weight zero. Since weight is a local property it is enough to prove the weight property for the sheaves  $\mathcal{K}_{|\mathbb{O}}$  and  $\mathcal{K}_{|w\mathbb{O}}$ . By corollary 3.4.1 we have  $\omega(\mathcal{K}_{|\mathbb{O}}) = 0$ . Moreover, these two sheaves are related essentially via the Fourier transform (3.5.25) which preserves weight [KL]. Hence  $\omega(\mathcal{K}_{|w\mathbb{O}}) = 0$ . This completes the proof of the corollary.

# Chapter 4

# The Two Dimensional Hannay-Berry Model

The main goal of this chapter is to construct the Hannay-Berry model of quantum mechanics, on a two dimensional symplectic torus, that were used in the previous chapters. However, for the convenience of the reader we give a self-contained presentation of this chapter which is independent from the rest of the paper. We construct a simultaneous quantization of the algebra of functions and the linear symplectic group  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ . We obtain the quantization via an action of  $\Gamma$  on the set of equivalence classes of irreducible representations of Rieffel's quantum torus  $\mathcal{A}_{\hbar}$ . For  $\hbar \in \mathbb{Q}$  this action has a unique fixed point. This gives a canonical projective equivariant quantization. There exists a Hilbert space on which both  $\Gamma$  and  $\mathcal{A}_{\hbar}$  act equivariantly. Combined with the fact that every projective representation of  $\Gamma$  can be lifted to a linear representation, we also obtain linear equivariant quantization.

## 4.1 Introduction

#### 4.1.1 Motivation

In the paper "Quantization of linear maps on the torus - Fresnel diffraction by a periodic grating", published in 1980 (cf. [HB]), the physicists J. Hannay and M.V. Berry explore a model for quantum mechanics on the 2-dimensional torus. Hannay and Berry suggested to quantize simultaneously the functions on the torus and the linear symplectic group  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ . They found (cf. [HB],[Me]) that the theta subgroup  $\Gamma_{\Theta} \subset \Gamma$  is the largest that one can quantize and asked (cf. [HB],[Me]) whether the quantization of  $\Gamma$  satisfy a multiplicativity property (i.e., is a linear representation of the group). In this chapter we

want to construct the Hannay-Berry's model for the bigger group of symmetries, i.e., the whole symplectic group  $\Gamma$ . The central question is whether there exists a Hilbert space on which a deformation of the algebra of functions and the linear symplectic group  $\Gamma$  both act in a compatible way.

#### 4.1.2 Results

In this chapter we give an affirmative answer to the existence of the quantization procedure. We show a construction (Theorem 4.1.3, Corollary 4.1.4 and Theorem 4.1.5) of the canonical equivariant quantization procedure for rational Planck constants. It is unique as a projective quantization (see definitions below). We show that the projective representation of  $\Gamma$  can be lifted in exactly 12 different ways to a linear representation (to obey the multiplicativity property). These are the first examples of such equivariant quantization for the whole symplectic group  $\Gamma$ . Our construction slightly improves the known constructions [HB, Me, KR1] for which the group of quantizable elements is  $\Gamma_{\Theta} \subset \Gamma$  and gives a positive answer to the Hannay-Berry question on the linearization of the projective representation of the group of quantizable elements. (cf. [HB], [Me]). Previously it was shown by Mezzadri and Kurlberg-Rudnick (cf. [Me], [KR1]) that one can construct an equivariant quantization for the theta subgroup, in case when the Planck constant is of the form  $\hbar = \frac{1}{N}$ ,  $N \in \mathbb{N}$ .

#### Classical torus

Let  $(\mathbf{T}, \omega)$  be the two dimensional symplectic torus. Together with its linear symplectomorphisms  $\Gamma \simeq \mathrm{SL}_2(\mathbb{Z})$  it serves as a simple model of classical mechanics (a compact version of the phase space of the harmonic oscillator). More precisely, let  $\mathbf{T} = W/\Lambda$  where W is a two dimensional real vector space, i.e.,  $W \simeq \mathbb{R}^2$  and  $\Lambda$  is a rank two lattice in W, i.e.,  $\Lambda \simeq \mathbb{Z}^2$ . We obtain the symplectic form on  $\mathbf{T}$  by taking a non-degenerate symplectic form on W:

$$\omega: W \times W \longrightarrow \mathbb{R}$$
.

We require  $\omega$  to be integral, namely  $\omega: \Lambda \times \Lambda \longrightarrow \mathbb{Z}$  and normalized, i.e.,  $\operatorname{Vol}(\mathbf{T}) = 1$ .

Let  $\mathrm{Sp}(W,\omega)$  be the group of linear symplectomorphisms, i.e.,  $\mathrm{Sp}(W,\omega) \simeq \mathrm{SL}_2(\mathbb{R})$ . Consider the subgroup  $\Gamma \subset \mathrm{Sp}(W,\omega)$  of elements that preserve the lattice  $\Lambda$ , i.e.,  $\Gamma(\Lambda) \subseteq \Lambda$ . Then  $\Gamma \simeq \mathrm{SL}_2(\mathbb{Z})$ . The subgroup  $\Gamma$  is the group of linear symplectomorphisms of  $\mathbf{T}$ . We denote by  $\Lambda^* \subseteq W^*$  the dual lattice,  $\Lambda^* = \{\xi \in W^* | \xi(\Lambda) \subset \mathbb{Z}\}$ . The lattice  $\Lambda^*$  is identified with the lattice  $\mathbf{T}^{\vee} := \mathrm{Hom}(\mathbf{T}, \mathbb{C}^*)$  of characters of  $\mathbf{T}$  by the following map:

$$\xi \in \Lambda^* \longmapsto e^{2\pi i < \xi, \cdot>} \in \mathbf{T}^{\vee}.$$

The form  $\omega$  allows us to identify the vector spaces W and W\*. For simplicity we will denote the induced form on W\* also by  $\omega$ .

#### Equivariant quantization of the torus

We will construct a particular type of quantization procedure for the functions. Moreover this quantization will be equivariant with respect to the action of the "classical symmetries"  $\Gamma$ :

**Definition 4.1.1** By Weyl quantization of  $\mathcal{A}$  we mean a family of  $\mathbb{C}$ -linear, \*- morphisms  $\pi_{\hbar}: \mathcal{A} \longrightarrow End(\mathcal{H}_{\hbar}), \ \hbar \in \mathbb{R}$ , where  $\mathcal{H}_{\hbar}$  is a Hilbert space, s.t. the following property holds:

$$\pi_{\scriptscriptstyle\hbar}(\xi+\eta) = e^{\pi i \hbar w(\xi,\eta)} \pi_{\scriptscriptstyle\hbar}(\xi) \pi_{\scriptscriptstyle\hbar}(\eta)$$

for all  $\xi, \eta \in \Lambda^*$  and  $\hbar \in \mathbb{R}$ .

This type of quantization procedure will obey the "usual" properties (cf. [D4]):

$$\begin{split} ||\pi_{{\scriptscriptstyle\hbar}}(fg)-\pi_{{\scriptscriptstyle\hbar}}(f)\pi_{{\scriptscriptstyle\hbar}}(g)||_{{\mathcal H}_{\hbar}} & \longrightarrow & 0, \quad as \; \hbar \to 0, \\ ||\frac{i}{\hbar}[\pi_{{\scriptscriptstyle\hbar}}(f),\pi_{{\scriptscriptstyle\hbar}}(g)]-\pi_{{\scriptscriptstyle\hbar}}(\{f,g\})||_{{\mathcal H}_{\hbar}} & \longrightarrow & 0, \quad as \; \hbar \to 0. \end{split}$$

where  $\{,\}$  is the Poisson brackets on functions.

**Definition 4.1.2** By equivariant quantization of  $\mathbf{T}$  we mean a quantization of  $\mathcal{A}$  with additional maps  $\rho_{\hbar}: \Gamma \longrightarrow \mathrm{U}(\mathcal{H}_{\hbar})$  s.t. the following equivariant property (called Egorov's identity) holds:

$$\rho_{h}(B)^{-1}\pi_{h}(f)\rho_{h}(B) = \pi_{h}(f \circ B) \tag{4.1.1}$$

for all  $\hbar \in \mathbb{R}$ ,  $f \in \mathcal{A}$  and  $B \in \Gamma$ . Here  $U(\mathcal{H}_{\hbar})$  is the group of unitary operators on  $\mathcal{H}_{\hbar}$ . If  $(\rho_{\hbar}, \mathcal{H}_{\hbar})$  is a projective (respectively linear) representation of the group  $\Gamma$  then we call the quantization projective (respectively linear).

The idea of the construction is as follows: We use a "deformation" of the algebra  $\mathcal{A}$  of functions on  $\mathbf{T}$ . We define an algebra  $\mathcal{A}_{\hbar}$ , usually called the two dimensional non-commutative torus (cf. [Ri]). If  $\hbar = \frac{M}{N} \in \mathbb{Q}$ , then we will see that all irreducible representations of  $\mathcal{A}_{\hbar}$  have dimension N. We denote by  $\operatorname{Irr}(\mathcal{A}_{\hbar})$  the set of equivalence classes of irreducible algebraic representations of the quantized algebra. We will see that  $\operatorname{Irr}(\mathcal{A}_{\hbar})$  is a set "equivalent" to a torus.

The group  $\Gamma$  naturally acts on a quantized algebra  $\mathcal{A}_{\hbar}$  and hence on the set  $\operatorname{Irr}(\mathcal{A}_{\hbar})$ . Let  $\hbar = \frac{M}{N}$  with  $\gcd(M, N) = 1$ . The following holds: **Theorem 4.1.3 (Canonical equivariant representation)** There exists a unique (up to isomorphism) N-dimensional irreducible representation  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  of  $\mathcal{A}_{\hbar}$  for which its equivalence class is fixed by  $\Gamma$ .

This means that:

$$\pi_{\scriptscriptstyle\hbar} \backsimeq \pi_{\scriptscriptstyle\hbar}^B$$

for all  $B \in \Gamma$ .

Since the canonical representation  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  is irreducible, by Schur's lemma we get the canonical projective representation of  $\Gamma$  compatible with  $\pi_{\hbar}$ :

Corollary 4.1.4 (Canonical projective representation) There exists a unique projective representation  $\rho_p: \Gamma \longrightarrow \mathrm{PGL}(\mathcal{H}_h)$  s.t:

$$\rho_{\mathbf{p}}(B)^{-1}\pi_{\hbar}(f)\rho_{\mathbf{p}}(B) = \pi_{\hbar}(f \circ B)$$

for all  $f \in \mathcal{A}$  and  $B \in \Gamma$ .

**Remark.** Corollary 4.1.4 is an improvement to the known constructions (cf. [HB, Me, KR1]) which has the group  $\Gamma_{\Theta} := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ab = cd = 0 \ (2) \}$  as the group of quantizable elements.

Using a result of Coxeter-Moser [CM] about the structure of the group  $\Gamma$  we get:

**Theorem 4.1.5 (Linearization)** The projective representation  $\rho_p$  can be lifted to a linear representation in exactly 12 different ways.

**Remark.** The existence of the linear representation  $\rho_h$  in Theorem 4.1.5 answers Hannay-Berry's question (cf. [HB, Me]) on the multiplicativity of the map  $\rho_h$ .

**Summary.** For  $\hbar \in \mathbb{Q}$  let  $(\rho_{\hbar}, \pi_{\hbar}, \mathcal{H}_{\hbar})$  be the canonical (projective) equivariant quantization of  $\mathbf{T}$ . We can endow the space  $\mathcal{H}_{\hbar}$  with a canonical unitary structure s.t  $\pi_{\hbar}$  is a \*-representation and  $\rho_{\hbar}$  is unitary. This "family" of \*-representations of  $\mathcal{A}_{\hbar}$  is by definition a Weyl quantization of the functions on the torus. The above results show the existence of a canonical projective equivariant quantization of the torus, and the existence of a linear equivariant quantization of the torus.

## 4.2 Construction

We consider the algebra  $\mathcal{A} := C^{\infty}(\mathbf{T})$  of smooth complex valued function on the torus and the dual lattice  $\Lambda^* := \{ \xi \in V^* | \xi(\Lambda) \subset \mathbb{Z} \}$ . Let <,> be the pairing between W and

W\*. The map  $\xi \mapsto s(\xi)$  where  $s(\xi)(x) := e^{2\pi i \langle x, \xi \rangle}$ ,  $x \in \mathbf{T}$  and  $\xi \in \Lambda^*$  defines a canonical isomorphism between  $\Lambda^*$  and the group  $\mathbf{T}^{\vee} := \operatorname{Hom}(\mathbf{T}, \mathbb{C}^*)$  of characters of  $\mathbf{T}$ .

#### 4.2.1 The quantum tori

Fix  $\hbar \in \mathbb{R}$ . The Rieffel's quantum torus (cf. [Ri]) is the non-commutative algebra  $\mathcal{A}_{\hbar}$  defined over  $\mathbb{C}$  by generators  $\{s(\xi), \xi \in \Lambda^*\}$ , and relations:

$$s(\xi + \eta) = e^{\pi i \hbar \omega(\xi, \eta)} s(\xi) s(\eta)$$

for all  $\xi, \eta \in \Lambda^*$ .

Note that the lattice  $\Lambda^*$  serves, using the map  $\xi \mapsto s(\xi)$ , as a basis for the algebra  $\mathcal{A}_{\hbar}$ . This induces an identification of vector spaces  $\mathcal{A}_{\hbar} \subseteq \mathcal{A}$  for every  $\hbar$ . We will use this identification in order to view elements of the (commutative) space  $\mathcal{A}$  as members of the (non-commutative) space  $\mathcal{A}_{\hbar}$ .

#### 4.2.2 Weyl quantization

To get a Weyl quantization of  $\mathcal{A}$  we use a specific one-parameter family of representations (see subsection 4.2.4 below) of the quantum tori. This defines an operator  $\pi_{\hbar}(\xi)$  for every  $\xi \in \Lambda^*$ . We extend the construction to every function  $f \in \mathcal{A}$  using the Fourier theory. Suppose:

$$f = \sum_{\xi \in \Lambda^*} a_\xi \cdot \xi$$

is its Fourier expansion. Then we define its Weyl quantization by:

$$\pi_{\hbar}(f) := \sum_{\xi \in \Lambda^*} a_{\xi} \pi_{\hbar}(\xi).$$

The convergence of the last series is due to the rapid decay of the Fourier coefficients of the function f.

## 4.2.3 Projective equivariant quantization

The group  $\Gamma = \operatorname{SL}_2(\mathbb{Z})$  acts on  $\Lambda$  preserving  $\omega$ . Hence  $\Gamma$  acts on  $\mathcal{A}_{\hbar}$  and the formula of this action is  $s^B(\xi) := s(B\xi)$ . Given a representation  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  of  $\mathcal{A}_{\hbar}$  and an element  $B \in \Gamma$ , define  $\pi_{\hbar}^B(s(\xi)) := \pi_{\hbar}(s^{B^{-1}}(\xi))$ . This formula induces an action of  $\Gamma$  on the set  $\operatorname{Irr}(\mathcal{A}_{\hbar})$  of equivalence classes of irreducible algebraic representations of  $\mathcal{A}_{\hbar}$ .

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**Lemma 4.2.1** All irreducible representations of  $A_{\hbar}$  are N-dimensional.

Now, suppose  $(\pi_h, \mathcal{A}_h, \mathcal{H}_h)$  is an irreducible representation for which its equivalence class is fixed by the action of  $\Gamma$ . This means that for any  $B \in \Gamma$  we have  $\pi_h \subseteq \pi_h^B$ , so by definition there exists an operator  $\rho_h(B) \in GL(\mathcal{H}_h)$  such that:

$$\rho_{\scriptscriptstyle h}(B)^{-1}\pi_{\scriptscriptstyle h}(\xi)\rho_{\scriptscriptstyle h}(B) = \pi_{\scriptscriptstyle h}(B\xi)$$

for all  $\xi \in \Lambda^*$ . This implies the Egorov identity (4.1.1) for any function . Now, since  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  is an irreducible representation then by Schur's lemma for every  $B \in \Gamma$  the operator  $\rho_{\hbar}(B)$  is uniquely defined up to a scalar. This implies that  $(\rho_{\hbar}, \mathcal{H}_{\hbar})$  is a projective representation of  $\Gamma$ .

#### 4.2.4 The canonical equivariant quantization

In what follows we consider only the case  $\hbar \in \mathbb{Q}$ . We write  $\hbar$  in the form  $\hbar = \frac{M}{N}$  with gcd(M, N) = 1.

**Proposition 4.2.2** There exists a unique  $\pi_h \in Irr(\mathcal{A}_h)$  which is a fixed point for the action of  $\Gamma$ .

#### 4.2.5 Unitary structure

Note that  $\mathcal{A}_{\hbar}$  becomes a \*- algebra using the formula  $s(\xi)^* := s(-\xi)$ . Let  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  be the canonical representation of  $\mathcal{A}_{\hbar}$ .

**Remark 4.2.3** There exists a canonical (unique up to scalar) unitary structure on  $\mathcal{H}_{\hbar}$  for which  $\pi_{\hbar}$  is a \*-representation.

## 4.3 Proofs

#### 4.3.1 Proof of Lemma 4.2.1

Suppose  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  is an irreducible representation of  $\mathcal{A}_{\hbar}$ .

Step 1. First we show that  $\mathcal{H}_{\hbar}$  is finite dimensional.  $\mathcal{A}_{\hbar}$  is a finite module over  $Z(\mathcal{A}_{\hbar}) = \{s(N\xi), \xi \in \Lambda^*\}$  which is contained in the center of  $\mathcal{A}_{\hbar}$ . Because  $\mathcal{H}_{\hbar}$  has at most countable dimension (as a quotient space of  $\mathcal{A}_{\hbar}$ ) and  $\mathbb{C}$  is uncountable then by Kaplansky's trick (cf. [MR])  $Z(\mathcal{A}_{\hbar})$  acts on  $\mathcal{H}_{\hbar}$  by scalars. Hence dim  $\mathcal{H}_{\hbar} < \infty$ .

Step 2. We show that  $\mathcal{H}_{\hbar}$  is N-dimensional. Choose a basis  $(e_1, e_2)$  of  $\Lambda^*$  s.t.  $\omega(e_1, e_2) = 1$ . Suppose  $\lambda \neq 0$  is an eigenvalue of  $\pi_{\hbar}(e_1)$  and denote by  $\mathcal{H}_{\lambda}$  the corresponding eigenspace. We have the following commutation relation  $\pi_{\hbar}(e_1)\pi_{\hbar}(e_2) = \gamma \pi_{\hbar}(e_2)\pi_{\hbar}(e_1)$  where  $\gamma := e^{-2\pi i \frac{M}{N}}$ . Hence  $\pi_{\hbar}(e_2) : \mathcal{H}_{\gamma^j \lambda} \longrightarrow \mathcal{H}_{\gamma^{j+1} \lambda}$ , and because  $\gcd(M, N) = 1$  then  $\mathcal{H}_{\gamma^i \lambda} \neq \mathcal{H}_{\gamma^j \lambda}$  for  $0 \leq i \neq j \leq N-1$ . Now, let  $v \in \mathcal{H}_{\lambda}$  and recall that  $\pi_{\hbar}(e_2)^N = \pi_{\hbar}(Ne_2)$  is a scalar operator. Then the space  $\operatorname{span}\{v, \pi_{\hbar}(e_2)v, \dots, \pi_{\hbar}(e_2)^{N-1}v\}$  is N-dimensional  $\mathcal{A}_{\hbar}$ -invariant subspace hence it equals  $\mathcal{H}_{\hbar}$ .

#### 4.3.2 Proof of Proposition 4.2.2

Let us show the existence of a unique fixed point for the action of  $\Gamma$  on  $Irr(\mathcal{A}_{\hbar})$ .

Suppose  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  is an irreducible representation of  $\mathcal{A}_{\hbar}$ . By Schur's lemma for every  $\xi \in \Lambda^*$  the operator  $\pi_{\hbar}(N\xi)$  is a scalar operator, i.e.,  $\pi_{\hbar}(N\xi) = q_{\pi_{\hbar}}(\xi) \cdot I$ . We have  $\pi_{\hbar}(0) = I$  and hence  $q_{\pi_{\hbar}}(\xi) \neq 0$  for all  $\xi \in \Lambda^*$ . Thus to any irreducible representation we have attached a scalar function  $q_{\pi_{\hbar}}: \Lambda^* \longrightarrow \mathbb{C}^*$ . Consider the set  $Q_{\hbar}$  of twisted characters of  $\Lambda^*$ :

$$Q_{\mathbf{h}} := \{q : \Lambda^* \longrightarrow \mathbb{C}^*, \ q(\xi + \eta) = (-1)^{MNw(\xi, \eta)} q(\xi) q(\eta)\}.$$

The group  $\Gamma$  acts naturally on this space by  $q^B(\xi) := q(B^{-1}\xi)$ . It is easy to see that we have defined a map  $\mathbf{q} : \operatorname{Irr}(\mathcal{A}_{\hbar}) \longrightarrow Q_{\hbar}$  given by  $\pi_{\hbar} \mapsto q_{\pi_{\hbar}}$  and it is obvious that this map is compatible with the action of  $\Gamma$ . We use the space of twisted characters in order to give a description for the set  $\operatorname{Irr}(\mathcal{A}_{\hbar})$ :

**Lemma 4.3.1** The map  $\pi_{h} \mapsto q_{\pi_{h}}$  is a  $\Gamma$ -equivariant bijection:

$$\mathbf{q}: \mathrm{Irr}(\mathcal{A}_{\hbar}) \longrightarrow Q_{\hbar}.$$

Now, Proposition 4.2.2 follows from the following claim:

Claim 4.3.2 There exists a unique  $q_o \in Q_h$  which is a fixed point for the action of  $\Gamma$ .

**Proof of Lemma 4.3.1. Step 1.** The map  $\mathbf{q}$  is surjective. Denote by  $\mathbb{T} := \operatorname{Hom}(\Lambda^*, \mathbb{C}^*)$  the group of complex characters of  $\Lambda^*$ . We define an action of  $\mathbb{T}$  on  $\operatorname{Irr}(\mathcal{A}_{\hbar})$  and on  $Q_{\hbar}$  by  $\pi_{\hbar} \mapsto \chi \pi_{\hbar}$  and  $q \mapsto \chi^N q$ , where  $\chi \in \mathbb{T}$ ,  $\pi_{\hbar} \in \operatorname{Irr}(\mathcal{A}_{\hbar})$  and  $q \in Q_{\hbar}$ . The map  $\mathbf{q}$  is clearly a  $\mathbb{T}$ -equivariant map with respect to these actions. Since  $\mathbf{q}$  is  $\mathbb{T}$ -equivariant, it is enough to show that the action of  $\mathbb{T}$  on  $Q_{\hbar}$  is transitive. Suppose  $q_1, q_2 \in Q_{\hbar}$ . By definition there exists a character  $\chi_1 \in \mathbb{T}$  for which  $\chi_1 q_1 = q_2$ . Let  $\chi$  be one of the N's roots of  $\chi_1$  then  $\chi^N q_1 = q_2$ .

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Step 2. The map  $\mathbf{q}$  is one to one. Suppose  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  is an irreducible representation of  $\mathcal{A}_{\hbar}$ . It is easy to deduce from the proof of Lemma 4.2.1 (Step 2) that for  $\xi \notin N\Lambda^*$  we have  $\operatorname{tr}(\pi_{\hbar}(\xi)) = 0$ . But we know from character theory that an isomorphism class of a finite dimensional irreducible representation of an algebra is recovered from its character. This completes the proof of Lemma 4.3.1.

**Proof of Claim 4.3.2.** Uniqueness. Fix  $q \in Q_h$ . The map  $\chi \mapsto \chi q$  give a bijection of  $\mathbb{T}$  with  $Q_h$ . But the trivial character  $\mathbf{1} \in \mathbb{T}$  is the unique fixed point for the action of  $\Gamma$  on  $\mathbb{T}$ .

Existence. Choose a basis  $(e_1, e_2)$  of  $\Lambda^*$  s.t  $\omega(e_1, e_2) = 1$ . This allows to identify  $\Lambda^*$  with  $\mathbb{Z} \oplus \mathbb{Z}$ . It is easy to see that the function:

$$q_o(m,n) = (-1)^{MN(mn+m+n)}$$

is a twisted character which is fixed by  $\Gamma$ . This completes the proof of Claim 4.3.2 and of Proposition 4.2.2.

#### 4.3.3 Proof of Theorem 4.1.5

The theorem follows from the following proposition:

**Proposition 4.3.3** Fix a projective representation  $\rho_p: \Gamma \longrightarrow GL(\mathcal{H}_{\hbar})$ . Then it can be lifted to a linear representation in exactly 12 ways.

*Proof.* Existence. We want to find constants c(B) for every  $B \in \Gamma$  s.t.  $\rho_h := c(\cdot)\rho_p$  is a linear representation of  $\Gamma$ . This is possible to carry out due to the following fact:

**Lemma 4.3.4** ([CM]) The group  $\Gamma$  is isomorphic to the group generated by three letters S, B and Z subjected to the relations:  $Z^2 = 1$  and  $S^2 = B^3 = Z$ .

Lemma 4.3.4  $\Rightarrow$  Existence. We need to find constants  $c_Z, c_B, c_S$  so that the operators  $\rho_{\hbar}(Z) := c_Z \rho_{\rm p}(Z), \ \rho_{\hbar}(B) := c_B \rho_{\rm p}(B), \ \rho_{\hbar}(S) := c_S \rho_{\rm p}(S)$  will satisfy the identities:

$$\rho_{\hbar}(Z)^2 = I, \ \rho_{\hbar}(B)^3 = \rho_{\hbar}(Z), \ \rho_{\hbar}(S)^2 = \rho_{\hbar}(Z).$$

This can be done by taking appropriate scalars.

Now, fix one lifting  $\rho_0$ . Then for the collection of operators  $\rho_{\hbar}(B)$  which lifts  $\rho_{\rm p}$  define a function  $\chi(B)$  by  $\rho_{\hbar}(B) = \chi(B)\rho_0(B)$ . It is obvious that  $\rho_{\hbar}$  is a representation if and only if  $\chi$  is a character. Thus liftings corresponds to characters. By Lemma 4.3.4 the group of characters  $\Gamma^{\vee} := \operatorname{Hom}(\Gamma, \mathbb{C}^*)$  is isomorphic to  $\mathbb{Z}/12\mathbb{Z}$ .

# Appendix A

# The Higher Dimensional Hannay-Berry Model

The aim of this chapter is to construct the Hannay-Berry model of quantum mechanics on a 2n-dimensional symplectic torus. We construct a simultaneous quantization of the algebra  $\mathcal{A}$  of functions on the torus and the linear symplectic group  $\Gamma = \operatorname{Sp}(2n, \mathbb{Z})$ . In the construction we use the quantum torus  $\mathcal{A}_{\varepsilon,\hbar}$ , which is a deformation of  $\mathcal{A}$ , together with a  $\Gamma$ -action on it. We obtain the quantization via the action of  $\Gamma$  on the set of equivalence classes of irreducible representations of  $\mathcal{A}_{\varepsilon,\hbar}$ . For  $\hbar \in \mathbb{Q}$  this action has a unique fixed point. This gives an equivariant quantization. There exists a Hilbert space on which both  $\Gamma$  and  $\mathcal{A}_{\varepsilon,\hbar}$  act in a compatible way.

## A.1 Introduction

#### A.1.1 Motivation

In this chapter we want to extend our construction (see Chapter 4), of the two-dimensional Hannay-Berry model, to the higher dimensional tori. The central question is whether there exists a vector space on which a deformation of the algebra of functions and the linear symplectic group  $\operatorname{Sp}(2n,\mathbb{Z})$ , both act in a compatible way. This construction is the first step toward the investigation of quantum chaos questions in the higher dimensional model. This study will be a subject of future publication.

Previously it was shown by Bouzouina and De Bievre [BDB] that one can quantize simultaneously the functions on the torus and one ergodic element  $A \in \operatorname{Sp}(2n, \mathbb{Z})$  in case where the Planck constant is of the form  $\hbar = \frac{1}{N}, \ N \in \mathbb{N}$ .

#### A.1.2 Definitions

#### Classical torus

Let  $(\mathbf{T}, \omega)$  be the 2n-dimensional symplectic torus. Together with its linear symplectomorphisms  $\Gamma \cong \operatorname{Sp}(2n, \mathbb{Z})$  it serves as a simple model of classical mechanics (a compact version of the phase space of the n-dimensional harmonic oscillator). More precisely, let  $\mathbf{T} := \mathbf{W}/\Lambda$  where W is a 2n-dimensional real vector space, i.e.,  $\mathbf{W} \cong \mathbb{R}^{2n}$  and  $\Lambda$  is a rank 2n lattice in W, i.e.,  $\Lambda \cong \mathbb{Z}^{2n}$ . We obtain the symplectic form on  $\mathbf{T}$  by taking a non-degenerate symplectic form on W:

$$\omega: W \times W \longrightarrow \mathbb{R}.$$

We require  $\omega$  to be integral, namely  $\omega: \Lambda \times \Lambda \longrightarrow \mathbb{Z}$  and normalized, i.e.,  $\operatorname{Vol}(\mathbf{T}) = 1$ .

Let  $\operatorname{Sp}(W,\omega)$  be the group of linear symplectomorphisms, i.e.,  $\operatorname{Sp}(W,\omega) \simeq \operatorname{Sp}(2n,\mathbb{R})$ . Consider the subgroup  $\Gamma \subset \operatorname{Sp}(W,\omega)$  of elements that preserve the lattice  $\Lambda$ , i.e.,  $\Gamma(\Lambda) \subseteq \Lambda$ . Then  $\Gamma \simeq \operatorname{SL}_2(\mathbb{Z})$ . The subgroup  $\Gamma$  is the group of linear symplectomorphisms of  $\mathbf{T}$ . We denote by  $\Lambda^* \subseteq W^*$  the dual lattice,  $\Lambda^* := \{\xi \in W^* | \xi(\Lambda) \subset \mathbb{Z}\}$ . The lattice  $\Lambda^*$  is identified with the lattice  $\mathbf{T}^{\vee} := \operatorname{Hom}(\mathbf{T}, \mathbb{C}^*)$  of characters of  $\mathbf{T}$  by the following map:

$$\xi \in \Lambda^* \longmapsto e^{2\pi i < \xi, \cdot>} \in \mathbf{T}^{\vee}.$$

The form  $\omega$  allows us to identify the vector spaces W and W\*. For simplicity we will denote the induced form on W\* also by  $\omega$ .

Consider the algebra  $\mathcal{A} := C^{\infty}(\mathbf{T})$  of smooth complex valued functions on  $\mathbf{T}$ . By Fourier theory the lattice  $\Lambda^*$  serves as a basis of  $\mathcal{A}$ .

#### Equivariant quantization of the torus

We will construct a particular type of quantization procedure for the functions. Moreover this quantization will be equivariant with respect to the action of the group of "classical symmetries"  $\Gamma$ :

1. By Weyl quantization of  $\mathcal{A}$  we mean a family of  $\mathbb{C}$ -linear morphisms  $\pi_h : \mathcal{A} \longrightarrow \operatorname{End}(\mathcal{H}_h)$ , where  $h \in \mathbb{R}$  and  $\mathcal{H}_h$  is a Hilbert space s.t. the following property hold:

$$\pi_{\scriptscriptstyle\hbar}(\xi)\pi_{\scriptscriptstyle\hbar}(\eta)=e^{2\pi i\hbar\omega(\eta,\xi)}\pi_{\scriptscriptstyle\hbar}(\eta)\pi_{\scriptscriptstyle\hbar}(\xi)$$

for all  $\xi, \eta \in \Lambda^*$  and  $\hbar \in \mathbb{R}$ .

2. By equivariant quantization of **T** we mean a quantization of  $\mathcal{A}$  with additional maps  $\rho_{\hbar}: \Gamma \longrightarrow \operatorname{GL}(\mathcal{H}_{\hbar})$  s.t. the following equivariant property (called "Egorov identity") holds:

$$\rho_{\hbar}(B)^{-1}\pi_{\hbar}(f)\rho_{\hbar}(B) = \pi_{\hbar}(f \circ B) \tag{A.1.1}$$

for all  $\hbar \in \mathbb{R}$ ,  $f \in \mathcal{A}$  and  $B \in \Gamma$ .

#### A.1.3 Results

In this chapter we give an affirmative answer to the existence of the quantization procedure. We show a construction (Theorem A.1.1 and Corollary A.1.2) of the quantization procedure for rational Planck constants. As far as we know this is the first construction of equivariant quantization for higher dimensional tori, together with the whole linear symplectic group  $\operatorname{Sp}(2n,\mathbb{Z})$ .

The idea of the construction is as follows: We use a "deformation" of the algebra  $\mathcal{A}$  of functions on  $\mathbf{T}$ . We define (see A.2.1) two algebras  $\mathcal{A}_{\varepsilon,\hbar}$ ,  $\varepsilon = 0,1$ . The algebra  $\mathcal{A}_{0,\hbar}$  is the usual Rieffel's quantum torus (see [Ri]) and  $\mathcal{A}_{1,\hbar}$  is some twisted version of it. If  $\hbar = \frac{M}{N} \in \mathbb{Q}$ , then we will see that all irreducible representations of  $\mathcal{A}_{\varepsilon,\hbar}$  have dimension  $N^n$ . We denote by  $Irr(\mathcal{A}_{\varepsilon,\hbar})$  the set of equivalence classes of irreducible algebraic representations of the quantized algebra. We will see that  $Irr(\mathcal{A}_{\varepsilon,\hbar})$  is a set "equivalent" to a torus.

The group  $\Gamma$  naturally acts on a quantized algebra  $\mathcal{A}_{\varepsilon,\hbar}$  and hence on the set  $\operatorname{Irr}(\mathcal{A}_{\varepsilon,\hbar})$ . Let  $\hbar = \frac{M}{N}$  with  $\gcd(M,N) = 1$ . Set  $\varepsilon := MN \pmod{2}$ . Then:

**Theorem A.1.1 (Equivariant representation)** There exists a unique (up to isomorphism) irreducible representation  $(\pi_h, \mathcal{H}_h)$  of  $\mathcal{A}_{\varepsilon,h}$  for which its equivalence class is fixed by  $\Gamma$ .

This means that:

$$\pi_{\scriptscriptstyle\hbar} \backsimeq \pi_{\scriptscriptstyle\hbar}^B$$

for every  $B \in \Gamma$ .

Since the canonical representation  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  is irreducible then by Schur's lemma we get the canonical projective representation of  $\Gamma$  compatible with  $\pi_{\hbar}$ :

Corollary A.1.2 (Canonical projective representation) For every  $B \in \Gamma$  there exists an operator  $\rho_{\hbar}(B)$  on  $\mathcal{H}_{\hbar}$  s.t.:

$$\rho_{\scriptscriptstyle h}(B)^{-1}\pi_{\scriptscriptstyle h}(f)\rho_{\scriptscriptstyle h}(B)=\pi_{\scriptscriptstyle h}(f\circ B)$$

for all  $f \in \mathcal{A}$ . Moreover the correspondence  $B \mapsto \rho_{h}(B)$  constitutes a projective representation of  $\Gamma$ .

**Remark A.1.3** The family  $(\rho_h, \pi_h, \mathcal{H}_h)$ ,  $h \in \mathbb{Q}$  presented in Corollary A.1.2 gives an equivariant Weyl quantization of the torus. We can endow (see A.2.3) the space  $\mathcal{H}_h$  with a canonical unitary structure s.t.  $\pi_h$  is a \*-representation and  $\rho_h$  unitary. This answers the question whether this quantization is also unitarizable and hence fits to the idea that quantum mechanics should be realized on a Hilbert space.

#### A.2 Construction

Consider the algebra  $\mathcal{A} := C^{\infty}(\mathbf{T})$  of smooth complex valued functions on the torus and the dual lattice  $\Lambda^* := \{ \xi \in W^* | \xi(\Lambda) \subset \mathbb{Z} \}$ . Let <,> be the pairing between W and W\*. The map  $\xi \mapsto s(\xi)$  where  $s(\xi)(x) := e^{2\pi i < x, \xi>}$ ,  $x \in \mathbf{T}$ ,  $\xi \in \Lambda^*$  defines a canonical isomorphism between  $\Lambda^*$  and the group  $\mathbf{T}^{\vee} := \operatorname{Hom}(\mathbf{T}, \mathbb{C}^*)$  of characters of  $\mathbf{T}$ .

#### A.2.1 The quantum tori

Fix  $\hbar \in \mathbb{R}$ . Define two algebras (see also [Ri] and [GH1]))  $\mathcal{A}_{\varepsilon,\hbar}$ ,  $\varepsilon = 0, 1$  as follows. The algebra  $\mathcal{A}_{\varepsilon,\hbar}$  is defined over  $\mathbb{C}$  by generators  $\{s(\xi), \xi \in \Lambda^*\}$ , and relations:

$$s(\xi + \eta) = \varepsilon(\xi, \eta)e^{\pi i\hbar\omega(\xi, \eta)}s(\xi)s(\eta)$$

where  $\varepsilon(\xi,\eta):=(-1)^{\varepsilon\omega(\xi,\eta)}$  and  $\xi,\eta\in\Lambda^*$ .

Note that the lattice  $\Lambda^*$  serves, using the map  $\xi \mapsto s(\xi)$ , as a basis for the algebra  $\mathcal{A}_{\hbar}$ . This induces an identification of vector spaces  $\mathcal{A}_{\hbar} \subseteq \mathcal{A}$  for every  $\hbar$ .

## A.2.2 Weyl quantization

To get a Weyl quantization of  $\mathcal{A}$  we use a specific one-parameter family of representations (see subsection A.2.4 below) of the quantum tori. This defines an operator  $\pi_{\hbar}(\xi)$  for every  $\xi \in \Lambda^*$ . We extend the construction to every function  $f \in \mathcal{A}$  using Fourier theory. Suppose:

$$f = \sum_{\xi \in \Lambda^*} a_\xi \cdot \xi$$

is its Fourier expansion. Then we define its Weyl quantization by:

$$\pi_{{\scriptscriptstyle\hbar}}(f) = \sum_{\xi \in \Lambda^*} a_{\xi} \pi_{{\scriptscriptstyle\hbar}}(\xi).$$

The convergence of the last series is due to the rapid decay of the fourier coefficients of f.

#### A.2.3 Equivariant quantization

We describe a strategy how to get an equivariant quantization of **T**. The group  $\Gamma$  acts on  $\Lambda$  preserving  $\omega$ . Hence  $\Gamma$  acts on  $\mathcal{A}_{\varepsilon,\hbar}$  by automorphisms of algebras. Suppose  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  is a representation of  $\mathcal{A}_{\varepsilon,\hbar}$ . For an element  $B \in \Gamma$ , define  $\pi_{\hbar}^{B}(\xi) := \pi_{\hbar}(B^{-1}\xi)$ . This formula defines an action of  $\Gamma$  on the set  $Irr(\mathcal{A}_{\varepsilon,\hbar})$  of equivalence classes of irreducible algebraic representations of  $\mathcal{A}_{\varepsilon,\hbar}$ .

**Lemma A.2.1** All irreducible representations of  $\mathcal{A}_{\varepsilon,\hbar}$  are N<sup>n</sup>-dimensional.

Now, suppose  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  is a representation for which its equivalence class is fixed by the action of  $\Gamma$ . This means that for any  $B \in \Gamma$  we have  $\pi_{\hbar} \cong \pi_{\hbar}^{B}$  and hence there exist an operator  $\rho_{\hbar}(B)$  on  $\mathcal{H}_{\hbar}$  s.t.:

$$\rho_{\scriptscriptstyle h}(B)^{-1}\pi_{\scriptscriptstyle h}(\xi)\rho_{\scriptscriptstyle h}(B)=\pi_{\scriptscriptstyle h}(B\xi)$$

for all  $\xi \in \Lambda^*$ . This implies the Egorov identity (A.1.1) for any function. Suppose in addition that  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  is an irreducible representation. Then by Schur's lemma for every  $B \in \Gamma$  the operator  $\rho_{\hbar}(B)$  is uniquely defined up to a scalar. This implies that  $(\rho_{\hbar}, \mathcal{H}_{\hbar})$  is a projective representation of  $\Gamma$ .

## A.2.4 The equivariant quantization

In what follows we consider only the case  $\hbar \in \mathbb{Q}$ . We write  $\hbar$  in the form  $\hbar = \frac{M}{N}$  with  $\gcd(M, N) = 1$ . Set  $\varepsilon = MN \pmod{2}$ .

**Proposition A.2.2** There exists a unique  $\pi_h \in \operatorname{Irr}(\mathcal{A}_{\varepsilon,h})$  which is a fixed point for the action of  $\Gamma$ .

## A.2.5 Unitary structure

Note that  $\mathcal{A}_{\varepsilon,\hbar}$  becomes a \*- algebra by the formula  $s(\xi)^* := s(-\xi)$ . Let  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  be the canonical representation of  $\mathcal{A}_{\varepsilon,\hbar}$ .

**Remark A.2.3** There exists a canonical (unique up to scalar) unitary structure on  $\mathcal{H}_{\hbar}$  for which  $\pi_{\hbar}$  is a \*-representation.

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#### A.3 Proofs

#### A.3.1 Proof of Lemma A.2.1

Suppose  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  is an irreducible representation of  $\mathcal{A}_{\varepsilon,\hbar}$ .

Step 1. First we show that  $\mathcal{H}_{\hbar}$  is finite dimensional. The algebra  $\mathcal{A}_{\varepsilon,\hbar}$  is a finite module over  $Z(\mathcal{A}_{\varepsilon,\hbar}) = \{s(N\xi), \xi \in \Lambda^*\}$  which is contained in the center of  $\mathcal{A}_{\varepsilon,\hbar}$ . Because  $\mathcal{H}_{\hbar}$  has at most countable dimension (as a quotient space of  $\mathcal{A}_{\varepsilon,\hbar}$ ) and  $\mathbb{C}$  is uncountable then by Kaplansky's trick (See [MR])  $Z(\mathcal{A}_{\varepsilon,\hbar})$  acts on  $\mathcal{H}_{\hbar}$  by scalars. Hence dim  $\mathcal{H}_{\hbar} < \infty$ .

Step 2. We show that  $\mathcal{H}_{\hbar}$  is N<sup>n</sup>-dimensional. Choose a basis  $(e_1, \ldots, e_n, e'_1, \ldots, e'_n)$  of  $\Lambda^*$  s.t.  $\omega(e_i, e_j) = \omega(e'_i, e'_j) = 0$  and  $\omega(e_i, e'_j) = \delta_{ij}$  the Kronecker's delta. Denote by E the commutative subalgebra of  $\mathcal{A}_{\varepsilon,\hbar}$  generated by  $\{s(e_i)\}_1^d$ . Suppose  $\lambda \in E^*$  is an eigencharacter of E and denote by  $\mathcal{H}_{\lambda} := \mathcal{H}_{(\lambda_1, \ldots, \lambda_n)}$  the corresponding eigenspace,  $\lambda_i := \lambda(e_i)$ . We have the following commutation relation  $\pi_{\hbar}(e_i)\pi_{\hbar}(e'_j) = \gamma^{\delta_{ij}}\pi_{\hbar}(e'_j)\pi_{\hbar}(e_i)$  where  $\gamma := e^{-2\pi i \frac{M}{N}}$ . Hence  $\pi_{\hbar}(e'_j) : \mathcal{H}_{(\lambda_1, \ldots, \gamma^k \lambda_j, \ldots, \lambda_n)} \longrightarrow \mathcal{H}_{(\lambda_1, \ldots, \gamma^{k+1} \lambda_j, \ldots, \lambda_n)}$ . Since  $\gcd(M, N) = 1$  the eigencharacters  $(\gamma^{k_1}\lambda_1, \ldots, \ldots, \gamma^{k_n}\lambda_n)$ ,  $0 \le k_j \le N - 1$ , are all different. Let  $0 \ne v \in \mathcal{H}_{\lambda}$  and recall that  $\pi_{\hbar}(e'_j)^N = \pi_{\hbar}(Ne'_j)$  is a scalar operator. Then the space  $\operatorname{span}\{\pi_{\hbar}(e'_j)^k v\}$  is N<sup>n</sup>-dimensional  $\mathcal{A}_{\varepsilon,\hbar}$ -invariant subspace hence it equals  $\mathcal{H}_{\hbar}$ .

## A.3.2 Proof of Proposition A.2.2

Suppose  $(\pi_h, \mathcal{H}_h)$  is an irreducible representation of  $\mathcal{A}_{\varepsilon,h}$ . By Schur's lemma for every  $\xi \in \Lambda^*$  the operator  $\pi_h(N\xi)$  is a scalar operator  $\pi_h(N\xi) = \chi_{\pi_h}(\xi) \cdot I$ . We have  $\pi_h(0) = I$ , hence  $\chi_{\pi_h}(\xi) \neq 0$  for all  $\xi \in \Lambda^*$ . Thus to any irreducible representation we have attached a scalar function  $\chi_{\pi_h}: \Lambda^* \longrightarrow \mathbb{C}^*$ . It is easy to see that  $\chi_{\pi_h}(\xi + \eta) = \chi_{\pi_h}(\xi)\chi_{\pi_h}(\eta)$ . Consider the group  $\mathbb{T} := \text{Hom}(\Lambda^*, \mathbb{C}^*)$  of complex characters of  $\Lambda^*$ . We have defined a map  $\text{Irr}(\mathcal{A}_{\varepsilon,h}) \longrightarrow \mathbb{T}$  given by  $\pi_h \mapsto \chi_{\pi_h}$ . This map is obviously compatible with the action of  $\Gamma$ , where the group  $\Gamma$  acts on characters by  $\chi^B(\xi) := \chi(B^{-1}\xi)$ .

**Lemma A.3.1** The map  $\pi_h \mapsto \chi_{\pi_h}$  gives a  $\Gamma$ -equivariant bijection:

$$\operatorname{Irr}(\mathcal{A}_{\varepsilon,\hbar}) \longrightarrow \mathbb{T}.$$
 (A.3.1)

From Lemma A.3.1 we easily deduce that there exists a unique equivalence class  $\pi_h \in \operatorname{Irr}(\mathcal{A}_{\varepsilon,h})$  which is fixed by the action of  $\Gamma$ . This is the one that corresponds to the trivial character  $\mathbf{1} \in \mathbb{T}$  which is the unique fixed point for the action of  $\Gamma$  on  $\mathbb{T}$ .

**Proof of Lemma A.3.1. Step 1.** The map  $\pi_h \mapsto \chi_{\pi_h}$  is onto. We define an action of  $\mathbb{T}$  on  $\operatorname{Irr}(\mathcal{A}_{\varepsilon,h})$  and on itself by  $\pi_h \mapsto \chi \cdot \pi_h$  and  $\psi \mapsto \chi^N \cdot \psi$ , where  $\chi \in \mathbb{T}$ ,  $\pi_h \in \operatorname{Irr}(\mathcal{A}_{\varepsilon,h})$  and  $\psi \in \mathbb{T}$ . The map (A.3.1) is clearly a  $\mathbb{T}$ -equivariant map with respect to these actions. The claim follows since the above action of  $\mathbb{T}$  on itself is transitive.

Step 2. We show that the map  $\pi_h \mapsto \chi_{\pi_h}$  is one to one. Suppose  $(\pi_h, \mathcal{H}_h)$  is an irreducible representation of  $\mathcal{A}_{\varepsilon,h}$ . It is easy to deduce from the proof of Lemma A.2.1 (Step 2) that for  $\xi \notin N\Lambda^*$  we have  $\operatorname{tr}(\pi_h(\xi)) = 0$ . But we know from character theory that an isomorphism class of a finite dimensional irreducible representation of an algebra is recovered from its character.

## Appendix B

# Two higher-dimensional (counter) examples

In this section we consider two examples that show the need for new ideas, already at the level of formulation, for the quantum chaos statement.

**Example 1.** The following is an example of an ergodic automorphism of the 4-dimensional torus  $\mathbf{T} := \mathbf{W}/\Lambda$  (this example works in every dimension 2n, n > 1 under appropriate modifications) which is not Hecke quantum ergodic.

We fix an element  $B \in GL_2(\mathbb{Z}) \setminus SL_2(\mathbb{Z})$  with eigenvalues which are not roots of unity. Take:

$$A := \left( \begin{smallmatrix} B & 0 \\ 0 & {}^tB^{-1} \end{smallmatrix} \right)$$

where t denotes the transpose operation. It is well known that  $A \in \operatorname{Sp}(4, \mathbb{Z})$  is an ergodic automorphism of  $\mathbf{T}$ . Now, we can choose<sup>1</sup> a character  $0 \neq \xi \in \Lambda^* \simeq \mathbf{T}^\vee$  that belongs to an A-invariant Lagrangian sublattice  $L^* \subset \Lambda^*$ . Fix  $\hbar = \frac{1}{p}$  and denote by  $V := L^*/pL^* \simeq \mathbb{F}_p^2$  the quotient lattice. Quantizing the system we attach to A the quantum operator:

$$\rho_{\hbar}(A): \mathcal{H}_{\hbar} \longrightarrow \mathcal{H}_{\hbar}$$

where  $\mathcal{H}_{\hbar} := \mathcal{S}(V)$  is the space of functions on V.

In this realization elements of the form  $\begin{pmatrix} B & 0 \\ 0 & t_{B^{-1}} \end{pmatrix}$  acts on functions by:

$$\left[ \left( \begin{smallmatrix} B & 0 \\ 0 & tB^{-1} \end{smallmatrix} \right) \varphi \right] (x) = \sigma(\det B) \varphi(B^{-1}x)$$

where  $\sigma$  is the Legendre character. Hence the function  $\varphi \equiv \mathbf{1}$  is a common eigenfunction for the Hecke torus  $T_A \subset \operatorname{Sp}(4, \mathbb{F}_p)$ . Recall that we have chosen  $\xi \in V$  and hence the

<sup>&</sup>lt;sup>1</sup>This property does not hold in the 2-dimensional case.

operator  $\pi_h(\xi)$  acts on functions via translation by  $\xi$ , so we obtain:

$$\langle arphi | \pi_{\scriptscriptstyle\hbar}(\xi) arphi 
angle = \mathbf{1} 
eq \mathbf{0} = \int_{\mathbf{T}} \xi \omega.$$

**Example 2.** There exists ergodic automorphisms  $A \in \operatorname{Sp}(2n, \mathbb{Z})$  of **T** for which the centralizer  $C_A \subset \operatorname{Sp}(2n, \mathbb{F}_p)$  is not a torus (or even a commutative group). If in Example 1 above we take  $B \in \operatorname{SL}_2(\mathbb{Z})$  hyperbolic then we have  $C_A \subseteq \operatorname{GL}(2, \mathbb{F}_p)$ . Moreover, the element  $A \in \operatorname{Sp}(2n, \mathbb{F}_p)$  might belongs to several non-isomorphic maximal commutative subgroups of  $\operatorname{Sp}(2n, \mathbb{F}_p)$ . We see that in this case it is not clear what should be the statement of Hecke quantum ergodicity.

## Appendix C

# Proofs for Chapters 1 and 2

For the remainder of this chapter we fix the following notations. Let  $\hbar = \frac{1}{p}$ , where  $p \geq 5$  is a fixed prime. Consider the lattice  $\Lambda^*$  of characters of the torus  $\mathbf{T}$  and the quotient vector space  $V := \Lambda^*/p\Lambda^*$ . The integral symplectic form on  $\Lambda^*$  is specialized to give a symplectic form on V, i.e.,  $\omega : V \times V \longrightarrow \mathbb{F}_p$ . Fix  $\psi : \mathbb{F}_p \longrightarrow \mathbb{C}^*$  a non-trivial additive character. Let  $\mathcal{A}_{\hbar}$  be the algebra of functions on the quantum torus and  $\Gamma \simeq \mathrm{SL}_2(\mathbb{Z})$  its group of symmetries.

#### C.1 Proof of Theorem 1.2.2

Basic set-up: let  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  be a representation of  $\mathcal{A}_{\hbar}$ , which is a representative of the unique irreducible class which is fixed by  $\Gamma$  (cf. Theorem 1.2.1). Let  $\rho_{\hbar} : \Gamma \longrightarrow \mathrm{PGL}(\mathcal{H}_{\hbar})$  be the associated projective representation. Here we give a proof that  $\rho_{\hbar}$  can be linearized in a unique way which factors through the quotient group  $\Gamma_p \simeq \mathrm{SL}_2(\mathbb{F}_p)$ :

$$\begin{array}{ccc}
\Gamma & \longrightarrow \Gamma_p \\
\hline
\rho_{\hbar} & & \\
\hline
GL(\mathcal{H}_{\hbar})
\end{array}$$

The proof will be given in several steps.

**Step 1.** Uniqueness. The uniqueness of the linearization follows directly from the fact that the group  $SL_2(\mathbb{F}_p)$ ,  $p \geq 5$ , has no characters.

Step 2. Existence. The main technical tool in the proof of existence is a construction of a finite dimensional quotient of the algebra  $\mathcal{A}_{\hbar}$ . Let  $\mathcal{A}_{p}$  be the algebra generated over  $\mathbb{C}$  by the symbols  $\{s(u) \mid u \in V\}$  and quotient out by the relations:

$$s(u+v) = \psi(\frac{1}{2}\omega(u,v))s(u)s(v). \tag{C.1.1}$$

The algebra  $A_p$  is non-trivial and the vector space V gives on it a standard basis. These facts will be proven in the sequel. We have the following map:

$$s: V \longrightarrow \mathcal{A}_p, \ v \mapsto s(v).$$

The group  $\Gamma_p$  acts on  $\mathcal{A}_p$  by automorphisms through its action on V. We have a homomorphism of algebras:

$$q: \mathcal{A}_{\hbar} \longrightarrow \mathcal{A}_{n}.$$
 (C.1.2)

The homomorphism (C.1.2) respects the actions of the group of symmetries  $\Gamma$  and  $\Gamma_p$  respectively. This is summarized in the following commutative diagram:

$$\begin{array}{cccc}
\Gamma \times \mathcal{A}_{\hbar} & \longrightarrow & \mathcal{A}_{\hbar} \\
(p,q) \downarrow & & q \downarrow \\
\Gamma_{p} \times \mathcal{A}_{p} & \longrightarrow & \mathcal{A}_{p}
\end{array} \tag{C.1.3}$$

where  $p:\Gamma\longrightarrow\Gamma_p$  is the canonical quotient map.

**Step 3.** Next, we construct an explicit representation of  $\mathcal{A}_p$ :

$$\pi_n: \mathcal{A}_n \longrightarrow \operatorname{End}(\mathcal{H}).$$

Let  $V = V_1 \bigoplus V_2$  be a Lagrangian decomposition of V. In our case V is two dimensional, therefore  $V_1$  and  $V_2$  are linearly independent lines in V. Take  $\mathcal{H} := \mathcal{S}(V_1)$  to be the vector space of complex valued functions on  $V_1$ . For an element  $v \in V$  define:

$$\pi_p(v) = \psi(\frac{1}{2}\omega(v_1, v_2))\mathbf{L}_{v_1}\mathbf{M}_{v_2}$$
 (C.1.4)

where  $v = v_1 + v_2$  is a direct decomposition  $v_1 \in V_1$ ,  $v_2 \in V_2$ ,  $\mathbf{L}_{v_1}$  is the translation operator defined by  $v_1$ :

$$\mathbf{L}_{v_1}(f)(x) = f(x + v_1), \quad f \in \mathcal{S}(V_1)$$

and  $\mathbf{M}_{v_2}$  is a notation for the operator of multiplication by the function  $\mathbf{M}_{v_2}(x) = \psi(\omega(v_2, x))$ . Next, we show that the formulas given in (C.1.4) satisfy the relations (C.1.1) and thus constitute a representation of the algebra  $\mathcal{A}_p$ . Let  $u, v \in V$ . We have to show:

$$\pi_p(u+v) = \psi(\frac{1}{2}\omega(u,v))\pi_p(u)\pi_p(v).$$

Compute:

$$\pi_p(u+v) = \pi_p((u_1+u_2) + (v_1+v_2))$$

where  $u = u_1 + u_2$  and  $v = v_1 + v_2$  are decompositions of u and v correspondingly.

Then:

$$\pi_p((u_1+v_1)+(u_2+v_2)) = \psi(\frac{1}{2}\omega(u_1+v_1,u_2+v_2))\mathbf{L}_{u_1+v_1}\mathbf{M}_{u_2+v_2}.$$
 (C.1.5)

This is by definition of  $\pi_p$  (cf. (C.1.4)). Now use the following formulas:

$$\mathbf{L}_{u_1+v_1} = \mathbf{L}_{u_1}\mathbf{L}_{v_1},$$
  
 $\mathbf{L}_{v_1}\mathbf{M}_{u_2} = \mathbf{M}_{u_2}(v_1)\mathbf{L}_{v_1}$ 

to obtain that the right-hand side of (C.1.5) is equal to:

$$\psi(\frac{1}{2}\omega(u_1+v_1,u_2+v_2)+\omega(u_2,v_1))\mathbf{L}_{u_1}\mathbf{M}_{u_2}\mathbf{L}_{v_1}\mathbf{M}_{v_2}.$$

Now use:

$$\frac{1}{2}\omega(u_1+v_1,u_2+v_2)+\omega(u_2,v_1)=\frac{1}{2}\omega(u_1+u_2,v_1+v_2)+\frac{1}{2}\omega(u_1,u_2)+\frac{1}{2}\omega(v_1,v_2)$$

to obtain the result:

$$\psi(\frac{1}{2}\omega(u,v))\pi_p(u)\pi_p(v)$$

which completes the argument.

As a consequence of constructing  $\pi_p$  we automatically proved that  $\mathcal{A}_p$  is non-trivial. It is well known that all linear operators on  $\mathcal{S}(V_1)$  are linear combinations of translation operators and multiplication by characters, therefore  $\pi_p : \mathcal{A}_p \longrightarrow \operatorname{End}(\mathcal{H})$  is surjective, but  $\dim(\mathcal{A}_p) \leq p^2$  therefore  $\pi_p$  is a bijection. This means that  $\mathcal{A}_p$  is isomorphic to a matrix algebra  $\mathcal{A}_p \simeq M_p(\mathbb{C})$ .

Step 4. Completing the proof of existence. The group  $\Gamma_p$  acts on  $\mathcal{A}_p$  therefore it acts on the category of its representations. But  $\mathcal{A}_p$  is isomorphic to a matrix algebra, therefore it has unique irreducible representation, up to isomorphism. This is the standard representation being of dimension p. But  $\dim(\mathcal{H}) = p$ , therefore  $\pi_p$  is an irreducible representation and its isomorphism class is fixed by  $\Gamma_p$  meaning that we have a pair:

$$\pi_p: \mathcal{A}_p \longrightarrow \operatorname{End}(\mathcal{H}),$$

$$\rho_p: \Gamma_p \longrightarrow \operatorname{PGL}(\mathcal{H})$$

satisfying the Egorov identity:

$$\rho_{\scriptscriptstyle p}(B)\pi_p(v)\rho_{\scriptscriptstyle p}(B^{-1})=\pi_p(Bv)$$

where  $B \in \Gamma_p$  and  $v \in \mathcal{A}_p$ .

It is a well known general fact (attributed to I. Schur) that the group  $\Gamma_p$ , where p is an odd prime, has no non-trivial projective representations. This means that  $\rho_p$  can be linearized<sup>1</sup> to give:

$$\rho_p:\Gamma_p\longrightarrow \mathrm{GL}(\mathcal{H}).$$

Now take:

$$\begin{aligned} \mathcal{H}_{\hbar} &:= & \mathcal{H}, \\ \pi_{_{\hbar}} &:= & \pi_{p} \circ q, \\ \rho_{_{\hbar}} &:= & \rho_{_{p}} \circ p. \end{aligned}$$

Because q intertwines the actions of  $\Gamma$  and  $\Gamma_p$  (cf. diagram (C.1.3)) we see that  $\pi_h$  and  $\rho_h$  are compatible, namely the Egorov identity is satisfied:

$$\rho_{h}(B)\pi_{h}(f)\rho_{h}(B^{-1})=\pi_{h}(f^{B})$$

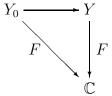
where  $B \in \Gamma_p$  and  $f \in \mathcal{A}_{\hbar}$ . Here the notation  $\pi_{\hbar}(f^B)$  means to apply any preimage  $\overline{B} \in \Gamma$  of  $B \in \Gamma_p$  on f. In particular, this implies that the isomorphism class of  $\pi_{\hbar}$  is fixed by  $\Gamma$ . Knowing that such representation  $\pi_{\hbar}$  is unique up to an isomorphism, (Theorem 1.2.1), our desired object has been obtained.

### C.2 Proof of Lemma 2.2.3

Basic set-up: let  $(\pi_{\hbar}, \mathcal{H}_{\hbar})$  be a representation of  $\mathcal{A}_{\hbar}$ , which is a representative of the unique irreducible class which is fixed by  $\Gamma$  (cf. Theorem 1.2.1). Let  $\rho_{\hbar}: \Gamma_{p} \longrightarrow \mathrm{GL}(\mathcal{H}_{\hbar})$  be the associated honest representation of the quotient group  $\Gamma_{p}$  (see Theorem 1.2.2 and Proof C.1). Recall the notation  $Y_{0} = \Gamma_{p} \times \Lambda^{*}$ . We consider the function  $F: Y_{0} \longrightarrow \mathbb{C}$  defined by the following formula:

$$F(B,\xi) = \text{Tr}(\rho_{h}(B)\pi_{h}(\xi)) \tag{C.2.1}$$

where  $\xi \in \Lambda^*$  and  $B \in \Gamma_p$ . We want to show that F factors through the quotient set  $Y = \Gamma_p \times V$ :



<sup>&</sup>lt;sup>1</sup>see Chapter 3 for an independent proof based on "The method of canonical Hilbert space".

The proof is immediate, taking into account the construction given in section C.1. Let  $\pi_p$  be the unique (up to isomorphism) representation of the quotient algebra  $\mathcal{A}_p$ . As was stated in C.1,  $\pi_h$  is isomorphic to  $\pi_p \circ q$ , where  $q : \mathcal{A}_h \longrightarrow \mathcal{A}_p$  is the quotient homomorphism between the algebras. This means that  $\pi_h(\xi) = \pi_p(q(\xi))$  depends only on the image  $q(\xi) \in V$ , and formula (C.2.1) solves the problem.

#### C.3 Proof of the Geometrization Theorem

Basic set-up: in this section we use the notations of section C.1 and Chapter 3. Set  $Y := \operatorname{Sp}_{\omega} \times \operatorname{V}$  and let  $\alpha : \operatorname{Sp}_{\omega} \times Y \longrightarrow Y$  denotes the associated action map. Let  $F : Y \longrightarrow \mathbb{C}$  be the function appearing in the statement of Theorem 2.2.4, i.e.,  $F(B, v) := \operatorname{Tr}(\rho_p(B)\pi_p(v))$ , where  $B \in \operatorname{Sp}_{\omega}$  and  $v \in \operatorname{V}$ . We use the notations  $\mathbb{V}$ ,  $\mathbb{Sp}_{\omega}$  and  $\mathbb{V}$  to denote the corresponding algebraic varieties. For the convenience of the reader we repeat here the formulation of the statement:

Theorem 2.2.4 (Geometrization Theorem). There exists an object  $\mathcal{F} \in \mathcal{D}^b_{c,w}(\mathbb{Y})$  satisfying the following properties:

- 1. (Function)  $f^{\mathcal{F}} = F$ .
- 2. (Weight)  $w(\mathcal{F}) < 0$ .
- 3. (Equivariance) For every element  $S \in \mathbb{S}p_{\omega}$  there exists an isomorphism  $\alpha_S^* \mathcal{F} \simeq \mathcal{F}$ .
- 4. (Formula) On introducing coordinates  $\mathbb{V} \simeq \mathbb{A}^2$  and identifying  $\mathbb{S}p_{\omega} \simeq \mathbb{SL}_2$ , there exists an isomorphism

$$\mathcal{F}_{|_{\mathbb{T} \times \mathbb{V}}} \simeq \mathscr{L}_{\psi(\frac{1}{2}\lambda\mu^{\frac{a+1}{a-1}})} \otimes \mathscr{L}_{\sigma(a)}.$$

Here  $\mathbb{T} = \{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \}$  stands for the standard torus and  $(\lambda, \mu)$  are the coordinates on  $\mathbb{V}$ .

Construction of the sheaf  $\mathcal{F}$ . We use the notations of Chapter 3. Let E be the Heisenberg group. As a set we have  $E = V \times \mathbb{F}_q$ . The group structure is given by the multiplication rule  $(v, \lambda) \cdot (v', \lambda') := (v + v', \lambda + \lambda' + \frac{1}{2}\omega(v, v'))$ . We fix a section  $s : V \dashrightarrow E$ , s(v) = (v, 0). The group of linear symplectomorphisms  $\mathrm{Sp}_{\omega}$  acts on E by the formula  $g \cdot (v, \lambda) := (gv, \lambda)$ . Let  $D = \mathrm{Sp}_{\omega} \ltimes E$ . We have a map  $Y \longrightarrow D$  defined using the section s. We use the notations  $\mathbb{E}$  and  $\mathbb{D}$  to denote the corresponding algebraic varieties.

Let K be the Weil representation sheaf (see Theorem 3.4.2). Define:

$$\mathcal{F}:=\mathrm{Tr}(\mathcal{K}_{|_{\mathbb{Y}}})$$

where Tr is the  $trace\ functor\ (C.3.1)$ .

We prove that  $\mathcal{F}$  satisfies properties 1 - 4.

**Property 1**. The collection of operators  $\{\pi_p(v)\}_{v\in V}$  extends to a representation of the group E. The representation  $\rho_p \ltimes \pi_p$  of the group D is isomorphic to the analogue representation  $\rho \ltimes \pi$  constructed in Chapter 3. Hence:

$$f^{\mathcal{F}} = f^{\operatorname{Tr}(\mathcal{K}_{|_{\mathbb{Y}}})} = \operatorname{Tr}(f^{\mathcal{K}_{|_{\mathbb{Y}}}}) = \operatorname{Tr}(K_{|_{\mathbb{Y}}}) =: F.$$

This proves Property 1.

**Property 2.** The sheaf K is of pure weight 0 (Corollary 3.4.3). The restriction to the subvariety Y does not increase weight. The operation of Tr also does not increase weight (using Theorem 2.2.5). It is defined as a composition:

$$Tr := pr_{1!} \Delta_{12}^{1*} \tag{C.3.1}$$

where  $\Delta_{12}^1: \mathbb{A}^1 \longrightarrow \mathbb{A}^2$  is the diagonal morphism  $\Delta_{12}^1(x) = (x, x)$  and  $pr_1$  is projection on the  $\mathbb{Y}$  coordinate:

$$Y \times \mathbb{A}^2$$

$$\Delta_{12}^1 \uparrow$$

$$Y \times \mathbb{A}^1$$

$$pr_1 \downarrow$$

$$V$$

Putting everything together we obtained Property 2.

**Property 3.** Basically follows from the convolution property of the sheaf K (see Theorem 3.4.2 property 2). More precisely, using the convolution property we obtain the following isomorphism:

$$\mathcal{K}_{|_S} * \mathcal{K} * \mathcal{K}_{|_{S^{-1}}} \simeq L_S^* R_{S^{-1}}^* \mathcal{K}$$

where  $L_S$ ,  $R_{S^{-1}}$  denotes left multiplication by S and right multiplication by  $S^{-1}$  on the group  $\mathbb{D}$  respectively. We write:

$$\alpha_S^*\mathcal{F} \simeq \operatorname{Tr}(\alpha_S^*\mathcal{K}_{|_{\mathbb{Y}}}) \simeq \operatorname{Tr}(L_S^*R_{S^{-1}}^*\mathcal{K}_{|_{\mathbb{Y}}}) \simeq \operatorname{Tr}(\mathcal{K}_{|_S} * \mathcal{K}_{|_{\mathbb{Y}}} * \mathcal{K}_{|_{S^{-1}}}).$$

Now use the following facts:

$$\mathrm{Tr}(\mathcal{K}_{|_S} * \mathcal{K}_{|_{\mathbb{Y}}} * \mathcal{K}_{|_{S^{-1}}}) \ \simeq \ \mathrm{Tr}(\mathcal{K}_{|_{S^{-1}}} * \mathcal{K}_{|_S} * \mathcal{K}_{|_{\mathbb{Y}}})$$

and:

$$\mathcal{K}_{|_{S^{-1}}} * \mathcal{K}_{|_{S}} \simeq \mathcal{I}.$$

The first isomorphism is the basic tracial property. Its proof is completely formal and we omit it. The second isomorphism is a consequence of the convolution property of K. This completes the proof of Property 3.

**Property 4.** It is directly verified using the explicit formulas appearing in 3.3 which are used to construct the sheaf  $\mathcal{K}$ . More-precisely, we need to compute the formula for the Trace of the Weil representation sheaf restricted to  $\mathbb{T} \setminus \{I\} \times \mathbb{A}^2 \times 0$ . We have:

$$\mathcal{F}(a,\lambda,\mu,0) := \operatorname{Tr}(\mathcal{K}(a,\lambda,\mu,0)) = \mathscr{L}_{\psi(\frac{1}{2}\lambda\mu\frac{a+1}{a-1})} \otimes \mathscr{L}_{\sigma(a)}, \quad a \neq 1.$$

Here  $\mathscr{L}_{\sigma}$  is the Legendre character sheaf on  $\mathbb{G}_m$ .

This completes the proof of the Geometrization Theorem.

## C.4 Computations for the Vanishing Lemma

In the computations we use some finer technical tools from the theory of  $\ell$ -adic cohomology. The interested reader can find a systematic study of this material in [K, KW].

We identify the standard torus  $\mathbb{T} \subset \mathbb{SL}_2$  with the group  $\mathbb{G}_m$ . Fix a non-trivial character sheaf<sup>2</sup>  $\mathscr{L}_{\chi}$  on  $\mathbb{G}_m$ . Consider the variety  $\mathbb{X} := \mathbb{G}_m - \{1\}$ , the sheaf:

$$\mathcal{E} := \mathcal{L}_{\psi(\frac{1}{2}\lambda\mu\frac{a+1}{a-1})} \otimes \mathcal{L}_{\chi} \tag{C.4.1}$$

on  $\mathbb{X}$  and the canonical projection  $pr: \mathbb{X} \longrightarrow pt$ . Note that  $\mathcal{E}$  is a non-trivial 1-dimensional local system on  $\mathbb{X}$ . The proof of the Lemma will be given in several steps:

Step 1. Vanishing. We want to show that  $H^i(pr_!\mathcal{E}) = 0$  for i = 0, 2.

By definition:

$$H^0(pr_!\mathcal{E}) := \Gamma(\mathbb{Y}, j_!\mathcal{E})$$

Namely, a 1-dimensional local system on  $\mathbb{G}_m$  that satisfies the property  $m^*\mathcal{L}_\chi \cong \mathcal{L}_\chi \boxtimes \mathcal{L}_\chi$ , where  $m: \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$  is the multiplication morphism.

where  $j: \mathbb{X} \hookrightarrow \mathbb{Y}$  is the imbedding of  $\mathbb{X}$  into a compact curve  $\mathbb{Y}$ . The statement follows since:

$$\Gamma(\mathbb{Y}, j_! \mathcal{E}) = \operatorname{Hom}(\overline{\mathbb{Q}}_{\ell}, j_! \mathcal{E})$$

and it is easy to see that any non-trivial morphism  $\overline{\mathbb{Q}}_{\ell} \to j_! \mathcal{E}$  should be an isomorphism, hence  $\operatorname{Hom}(\overline{\mathbb{Q}}_{\ell}, j_! \mathcal{E}) = 0$ .

For the second cohomology we have:

$$\mathrm{H}^2(pr_!\mathcal{E}) = \mathrm{H}^{-2}(Dpr_!\mathcal{E})^* = \mathrm{H}^{-2}(pr_*D\mathcal{E})^* = \Gamma(\mathbb{X}, D\mathcal{E}[-2])^*$$

where D denotes the Verdier duality functor and [-2] means translation functor. The first equality follows from the definition of D, the second equality is the Poincaré duality and the third equality easily follows from the definitions. Again, since the sheaf  $D\mathcal{E}[-2]$  is a non-trivial 1-dimensional local system on X then:

$$\Gamma(\mathbb{X}, D\mathcal{E}[-2]) = \text{Hom}(\overline{\mathbb{Q}}_{\ell}, D\mathcal{E}[-2]) = 0.$$

Step 2. Dimension. We claim that dim  $H^1(pr_!\mathcal{E}) = 2$ .

The (topological) Euler characteristic  $\chi(pr_!\mathcal{E})$  of the sheaf  $pr_!\mathcal{E}$  is the integer defined by the formula:

$$\chi(pr_!\mathcal{E}) := \sum_i (-1)^i \text{dim } H^i(pr_!\mathcal{E}).$$

Hence from the vanishing of cohomologies (Step 1) we deduce:

Substep 2.1. It is enough to show that  $\chi(pr_!\mathcal{E}) = -2$ .

The actual computation of the Euler characteristic  $\chi(pr_!\mathcal{E})$  is done using the Ogg-Shafarevich-Grothendieck formula [D3]:

$$\chi(pr_!\mathcal{E}) - \chi(pr_!\overline{\mathbb{Q}}_\ell) = \sum_{y \in \mathbb{Y} \setminus \mathbb{X}} \operatorname{Swan}_y(\mathcal{E})$$
(C.4.2)

Where  $\overline{\mathbb{Q}}_{\ell}$  denotes the constant sheaf on  $\mathbb{X}$ , and  $\mathbb{Y}$  is some compact curve containing  $\mathbb{X}$ . In words, this formula expresses the difference of  $\chi(pr_!\mathcal{E})$  from  $\chi(pr_!\overline{\mathbb{Q}}_{\ell})$  as a sum of local contributions.

Next, we take  $\mathbb{Y} := \mathbb{P}^1$ . Having that  $\chi(pr_!\overline{\mathbb{Q}}_\ell) = -1$  and using formula (C.4.2) we get:

Substep 2.2. It is enough to show that:  $\operatorname{Swan}_0(\mathcal{E}) + \operatorname{Swan}_1(\mathcal{E}) + \operatorname{Swan}_{\infty}(\mathcal{E}) = -1$ .

Now, using the explicit formula (C.4.1) of the sheaf  $\mathcal{E}$  we see that:

$$Swan_{1}(\mathcal{E}) = Swan_{\infty}(\mathcal{L}_{\psi}),$$
  

$$Swan_{\infty}(\mathcal{E}) = Swan_{\infty}(\mathcal{L}_{\chi}),$$
  

$$Swan_{0}(\mathcal{E}) = Swan_{0}(\mathcal{L}_{\chi}).$$

Applying the Ogg-Shafarevich-Grothendieck formula to the Artin-Schreier sheaf  $\mathcal{L}_{\psi}$  on  $\mathbb{A}^1$  and the projection  $pr: \mathbb{A}^1 \longrightarrow pt$  we find that:

$$\operatorname{Swan}_{\infty}(\mathcal{L}_{\psi}) = \chi(pr_{!}\mathcal{L}_{\psi}) - \chi(pr_{!}\overline{\mathbb{Q}}_{\ell}) = 0 - 1 = -1. \tag{C.4.3}$$

Finally we apply the formula (C.4.2) to the sheaf  $\mathcal{L}_{\chi}$  on  $\mathbb{G}_m$  and the projection  $pr: \mathbb{G}_m \longrightarrow pt$  and conclude:

$$\operatorname{Swan}_{\infty}(\mathscr{L}_{\chi}) + \operatorname{Swan}_{0}(\mathscr{L}_{\chi}) = \chi(pr_{!}\mathscr{L}_{\chi}) - \chi(pr_{!}\overline{\mathbb{Q}}_{\ell}) = 0 - 0 = 0. \tag{C.4.4}$$

In (C.4.3) and (C.4.4) we use the fact that  $pr_!\mathcal{L}_{\psi}$  and  $pr_!\mathcal{L}_{\chi}$  are the 0-objects in  $\mathcal{D}_{c,w}^b(pt)$ .

This completes the computations of the Vanishing Lemma.

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