Solution of the master equation for Bak-Sneppen model of biological evolution in finite ecosystem

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ABSTRACT

The master equations describing processes of biological evolution in the framework of the random neighbor Bak-Sneppen model are studied. For the eqosystem of N species they are solved exactly and asymptotical behavior of this solution for large N is analyzed.

1 Introduction

The model of biological evolution proposed by Bak and Sneppen [1,2] describes mutation and natural selection of interacting species. It is the dynamical system that is defined as follows. The state of the ecosystem of N species is characterized by a set $\{x_1, ..., x_N\}$ of N number, $0 \ge x_i \ge 1$. In so doing, x_i represents the barrier toward father evolution of the species. Initially, each x_i is set to a randomly chosen value. At each time step the barrier x_i with minimal value and K-1 other barriers are replaced by K new random numbers. In the random neighbor model (RNM), which will be considered in this paper the K-1 replaced non-minimal barriers are chosen at random.

The RNM is the simplest model describing the avalanche-like processes, which are supposed by a conception of "punctuated equilibrium" in biological evolution. These processes are the most characterizing feature for self-organized criticality recently intensively investigated both numerically and analytically [3-6] The RNM is more convenient for analytical studies. The master equations obtained in [3] for RNM are very useful for this aim. In [7] the explicit solution of master equations was found for infinite ecosystem. In this paper we solve the master equations for finite number N of species in ecosystem. We restrict ourselves with the simplest case K=2.

2 Master equations for RNM

The master equations for the RNM are obtained in [3]. They are of the form:

$$P_n(t+1) = A_n P_n(t) + B_{n+1} P_{n+1}(t) + C_{n-1} P_{n-1}(t) +$$

$$D_{n+2} P_{n+2}(t) + (B_1 \delta_{n,0} + A_1 \delta_{n,1} + C_1 \delta_{n,2}) P_0(t)$$
(1)

Here, $P_n(t)$ is the probability that n at the time t is the number of barriers having values less than a fixed value λ at the time t; $0 \le n \le N$, $0 \le \lambda \le N$, $0 \le t$; $P_n(0)$

are proposed to be given. For $0 < n \le N$

$$A_n = 2\lambda(1-\lambda) + \frac{n-1}{N-1}\lambda(3\lambda - 2),$$

$$B_n = (1-\lambda)^2 + \frac{n-1}{N-1}(1-\lambda)(3\lambda - 1),$$

$$C_n = \lambda^2 - \frac{n-1}{N-1}\lambda^2, \ D_n = (1-\lambda)^2 \frac{n-1}{N-1},$$
(2)

and $A_n = B_n = C_n = D_n = 0$ for n = 0, n > N.

In virtue of the definition of $P_n(t)$,

$$P_n(t) \ge 0,\tag{3}$$

$$\sum_{n=0}^{N} P_n(t) = 1. (4)$$

Making summation in (1) over n and taking into account (2) it is easy to establish that

$$\sum_{n=0}^{N} P_n(t+1) = \sum_{n=0}^{N} P_n(t).$$
 (5)

Therefore, if $P_n(0)$ are chosen in such a way that (4) is fulfilled for t = 0, then in virtue of (5) it is the case for the solution of (1) for t > 0 too. For analysis of (1) it is convenient to introduce the generating function q(z, u):

$$q(z,u) \equiv \sum_{t=0}^{\infty} \sum_{n=0}^{N} P_n(t) z^n u^t.$$
 (6)

In virtue of (3),(4) q(z,t) is polynomial in z, analytical in u for |u| < 1 and

$$q(1,u) = \frac{1}{1-u}. (7)$$

The master equations (1) can be rewritten for the generating function q(z, u) as follows:

$$\frac{1}{u}(q(z,u) - q(z,0)) = (1 - \lambda + \lambda z)^2 \left(\frac{1}{z}\left(1 - \frac{1-z}{N-1}\left(\frac{1}{z} - \frac{\partial}{\partial z}\right)\right) \times (q(z,u) - q(0,u)) + q(0,u)\right).$$
(8)

The function $q(z,0) = \sum_{n=0}^{N} P_n(0)z^n$ in (8) is assumed to be given.

3 Asymptotic expansion of q(z, u) for big N

If the function q(z,0) has an asymptotic expansion in the region of big N of the form:

$$q(z,0) = \sum_{k=0}^{\infty} \frac{q_k(z,0)}{(N-1)^k}$$

then the equation (8) enables one to obtain the similar asymptotic expansion for q(z, u):

$$q(z, u) = \sum_{k=0}^{\infty} \frac{q_k(z, u)}{(N-1)^k}.$$

The main approximation of $q_0(z, u)$, the function q(z, u) can be found from the equation

$$(z - u(1 - \lambda + \lambda z)^2)q_0(z, u) = zq_0(z, 0) + u(1 - \lambda + \lambda z)^2(z - 1)q_0(0, u)$$
(9)

following from (8). Since $q_0(z, u)$ is analytical for |z| < 1, |u| < 1,

$$0 = \alpha q_0(\alpha, 0) + u(1 - \lambda + \lambda \alpha)^2 (\alpha - 1) q_0(0, u)$$
(10)

where

$$\alpha = \alpha(u) = \frac{1 - 2\lambda(1 - \lambda)u - (1 - 4\lambda(1 - \lambda u))^{\frac{1}{2}}}{2\lambda^{2}u}$$
(11)

is the solution of equation $\alpha - u(1 - \lambda + \lambda \alpha)^2 = 0$. Obviously, for sufficient little |u|, $|\alpha| < 1$. Thus, from equation (10) the function $q_0(0, u)$ can be found:

$$q_0(0,u) = \frac{q_0(\alpha,0)}{1-\alpha}. (12)$$

Substituting (12) in the right hand side of (9), one can find the solution in the following form:

$$q_0(z,u) = \frac{zq_0(z,0)(1-\alpha) + (z-1)u(1-\lambda+\lambda z)^2 q_0(\alpha,0)}{(z-u(1-\lambda+\lambda z)^2)(1-\alpha)},$$
(13)

where $\alpha(u)$ is defined by (10).

For k > 0 the functions $q_k(z, u)$ are defined by recurrent relations

$$q_{k}(z,u) = \frac{u(1-\lambda+\lambda z)^{2}(z-1)q_{k}(\alpha,0) + (1-\alpha)zq_{k}(z,0)}{(z-u(1-\lambda+\lambda z)^{2})(1-\alpha)} + \frac{u(1-\lambda+\lambda z)^{2}(r_{k-1}(z,u) + r_{k-1}(\alpha,u))}{z-u(1-\lambda+\lambda z)^{2}}.$$
(14)

Here,

$$r_k(z,u) \equiv z \frac{\partial}{\partial z} \frac{q_k(z,u) - q_k(0,u)}{z}.$$
 (15)

In virtue of (13), (14) the first correction to the lowest approximation (12) of q(z, u) has the form

$$q_{1}(z,u) = \frac{u(1-\lambda+\lambda z)^{2}(z-1)q_{1}(\alpha,0) + (1-\alpha)zq_{1}(z,0)}{(z-u(1-\lambda+\lambda z)^{2})(1-\alpha)} + \frac{u(1-\lambda+\lambda z)^{2}(r_{0}(z,u)+r_{0}(\alpha,u))}{z-u(1-\lambda+\lambda z)^{2}}.$$
(16)

where

$$r_0(z,u) = z \frac{\partial}{\partial z} \frac{(1-\alpha)q_0(z,u) + (u(1-\lambda+\lambda z)^2 - 1)q_0(\alpha,0)}{(z-u(1-\lambda+\lambda z)^2)(1-\alpha)}.$$
 (17)

4 Exact form of q(z, u)

Let us introduce the quantity

$$Q(z,u) \equiv \frac{q(z,u) - q(0,u)}{z}.$$
(18)

It follows from (8) that this function fulfills the relation of the form:

$$(z - u(1 - \lambda + \lambda z)^{2} - \frac{u(1 - \lambda + \lambda z)^{2}(1 - z)}{N - 1} \frac{\partial}{\partial z})Q(z, u) =$$

$$= q(z, 0) + (u(1 - \lambda + \lambda z)^{2} - 1)q(0, u). \tag{19}$$

This inhomogeneous differential equation for Q(z, u) has a special solution

$$Q(z,u) = (N-1)e^{R(z,u)} \int_{z}^{1} e^{-R(x,u)} g(x,u) dx \equiv Q_{sp}(z,u).$$
 (20)

Here,

$$R(z,u) = \frac{N-1}{u} (\ln(1-\lambda+\lambda z) - (1-u)\ln(1-z) + \frac{1-\lambda}{\lambda(1-\lambda+\lambda z)}),$$
 (21)

$$g(x,u) = \frac{q(x,0) + (u(1-\lambda+\lambda z)^2 - 1)q(0,u)}{u(1-\lambda+\lambda z)^2(1-x)}$$
(22)

and the derivative of R(z, u) with respect to z has the form

$$\frac{\partial R(z,u)}{\partial z} = \frac{(N-1)(1-(1-\lambda+\lambda z)^2)}{u(1-\lambda+\lambda z)^2(1-z)}$$
(23)

General solution of the corresponding to (19) homogeneous equation

$$(z - u(1 - \lambda + \lambda z)^2 - \frac{u(1 - \lambda + \lambda z)^2 (1 - z)}{N - 1} \frac{\partial}{\partial z})S(z, u) = 0$$
 (24)

is of the form

$$S(z,u) = F(u)e^{R(z,u)}. (25)$$

Here, F(u) is an arbitrary function of u. Hence, it follows from (19),(20),(25) that the function Q(z,t) can be represented as follows:

$$Q(z,u) = F(u)e^{R(z,u)} + Q_{sp}(z,u).$$
(26)

In virtue of initial condition (7) for q(z, u)

$$Q(1,u) = \frac{1}{1-u} - q(0,u) \tag{27}$$

For $0 < u < 1, z \rightarrow 1$, S(z, u) diverges and $S_{sp}(z, u)$ has the finite limit:

$$\lim_{z \to 1} Q_{sp}(z, u) = \frac{1}{1 - u} - q(0, u) \tag{28}$$

Hence, F(u) = 0 in (25) and this representation for Q(z, u) can be rewritten in the form:

$$Q(z,u) = (N-1)e^{R(z,u)} \int_{\frac{\lambda-1}{\lambda}}^{1} e^{-R(x,u)} g(x,u) dx +$$

$$+ (N-1)e^{R(z,u)} \int_{z}^{\frac{\lambda-1}{\lambda}} e^{-R(x,u)} g(x,u) dx$$
(29)

It follows from (18) that

$$Q(\frac{\lambda - 1}{\lambda}, u) = \frac{\lambda(q(\frac{\lambda - 1}{\lambda}, u) - q(0, u))}{\lambda - 1}$$

For the terms in the right hand side of (29) we have for $0 < u < 1, 0 < \lambda < 1$:

$$\lim_{z \to \frac{\lambda - 1}{\lambda} + 0} e^{R(z, u)} = +\infty, \tag{30}$$

$$\lim_{z \to \frac{\lambda - 1}{\lambda} + 0} (N - 1)e^{R(z, u)} \int_{z}^{\frac{\lambda - 1}{\lambda}} e^{-R(x, u)} g(x, u) dx = \frac{\lambda(q(\frac{\lambda - 1}{\lambda}, u) - q(0, u)}{\lambda - 1}.$$
(31)

Therefore (29) can represent the function Q(z, u) with necessary analytical properties only if

$$\int_{\frac{\lambda-1}{\lambda}}^{1} e^{-R(x,u)} g(x,u) dx = 0$$
(32)

This equation defines the function q(0, u):

$$q(0,u) = \frac{\int_{\frac{\lambda-1}{\lambda}}^{1} e^{-R(x,u)} \frac{q(x,0)dx}{(1-\lambda+\lambda x)^{2}(1-x)}}{\int_{\frac{\lambda-1}{\lambda}}^{1} e^{-R(x,u)} \frac{(1-u(1-\lambda+\lambda x)^{2})dx}{(1-\lambda+\lambda x)^{2}(1-x)}}$$
(33)

Thus, we obtain from (29) the solution of equation (8) in the following form:

$$q(z,u) = z \frac{N-1}{u} e^{R(z,u)} \int_{z}^{\frac{\lambda-1}{\lambda}} e^{-R(x,u)} \frac{q(x,0)dx}{(1-\lambda+\lambda x)^{2}(1-x)} + q(0,u)(1+z\frac{N-1}{u}e^{R(z,u)} \int_{z}^{\frac{\lambda-1}{\lambda}} e^{-R(x,u)} \frac{(1-u(1-\lambda+\lambda x)^{2})dx}{(1-\lambda+\lambda x)^{2}(1-x)}),$$
(34)

where q(0, u) is defined by (33).

5 Conclusion

We constructed the solution of the master equation (8) for the finite number N of species in the ecosystem. It can be proven that the main term (13) of its asymptotic

for large N coincides with the one obtained in [7]. Using (34) one can obtain all the known analytical results for RNM. One can hope that it helps to understand better the most important properties of the self-organized criticality processes.

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