RAMANUJAN'S MASTER THEOREM FOR STURM-LIOUVILLE OPERATOR

JOTSAROOP KAUR AND SANJOY PUSTI

Abstract. In this paper we prove an analogue of Ramanujan's master theorem in the setting of Sturm Liouville operator.

1. Introduction

Ramanujan's Master theorem ([10]) states that if a function f can be expanded around 0 in a power series of the form

$$f(x)=\sum_{k=0}^{\infty}(-1)^k a(k)x^k,$$

then

$$\int_0^\infty f(x)x^{-\lambda-1}\,dx=-\frac{\pi}{\sin\pi\lambda}a(\lambda). \tag{1.1}$$

One needs some assumptions on the function a, as the theorem is not true for $a(\lambda)=\sin\pi\lambda$. Hardy provides a rigorous statement of the theorem above as:

Let A,p,δ be real constants such that $A<\pi$ and $0<\delta\leq 1$. Let $\mathcal{H}(\delta)=\{\lambda\in\mathbb{C}\mid \Re\lambda>-\delta\}$. Let $\mathcal{H}(A,p,\delta)$ be the collection of all holomorphic functions $a:\mathcal{H}(\delta)\to\mathbb{C}$ such that

$$|a(\lambda)|\leq Ce^{-p(\Re\lambda)+A|\Im\lambda|}\text{ for all }\lambda\in\mathcal{H}(\delta).$$

Theorem 1.1 (Ramanujan's Master theorem, Hardy [10]). *Suppose $a\in\mathcal{H}(A,p,\delta)$. Then*

(1) *The power series $f(x)=\sum_{k=0}^{\infty}(-1)^k a(k)x^k$ converges for $0<x<e^p$ and defines a real analytic function on that domain.*

(2) *Let $0<\sigma<\delta$. Then for $0<x<e^p$ we have*

$$f(x)=\frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty}\frac{-\pi}{\sin\pi\lambda}a(\lambda)x^\lambda\,d\lambda.$$

The integral on the right side of the equation above converges uniformly on compact subsets of $[0,\infty)$ and is independent of σ.

(3) *Also*

$$\int_0^\infty f(x)x^{-\lambda-1}\,dx=-\frac{\pi}{\sin\pi\lambda}a(\lambda),$$

holds for the extension of f to $[0,\infty)$ and for all $\lambda\in\mathbb{C}$ with $0<\Re\lambda<\delta$.

This theorem can be thought of as an interpolation theorem, which reconstructs the values of $a(\lambda)$ from it's given values at $a(k),k\in\mathbb{N}\cup\{0\}$. In particular if $a(k)=0$ for all $k\in\mathbb{N}\cup\{0\}$, then a is identically 0. We can rewrite the theorem above in terms of Fourier series and Fourier transform as follows:

Theorem 1.2. *Suppose $a\in\mathcal{H}(A,p,\delta)$. Then*

2000 *Mathematics Subject Classification.* Primary 43A62, 43A85; Secondary 43E32, 34L10 .

Key words and phrases. Ramanujan's master theorem, compact dual, Sturm Liouville operator.

The second author is supported partially by SERB, MATRICS, MTR/2017/000235.

(1) *The Fourier series* $f(z) = \sum_{k=0}^{\infty}(-1)^k a(k)e^{-ikz}$ *converges for* $\Im z < p$ *and defines a holomorphic function on that domain.*
(2) *Let* $0 < \sigma < \delta$. *Then for* $0 < t < p$ *we have*

$$f(it) = \frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{-\pi}{\sin \pi\lambda} a(\lambda)e^{-it\lambda}\, d\lambda.$$

The integrals defined above are independent of σ *and* f *extends as a holomorphic function to a neighbourhood* $\{z \in \mathbb{C} \mid |\Re z| < \pi - A\}$ *of* $i\mathbb{R}$.
(3) *Also*

$$\int_{\mathbb{R}} f(ix)e^{i\lambda x}\, dx = -\frac{\pi}{\sin \pi\lambda} a(\lambda),$$

holds for the extension of f *to* $i\mathbb{R}$ *and for all* $\lambda \in \mathbb{C}$ *with* $0 < \Re\lambda < \delta$.

Bertram (in [2]) provides a group theoretical interpretation of the theorem in the following way: Consider $x \mapsto e^{i\lambda x}, \lambda \in \mathbb{C}$ and $x \mapsto e^{ikx}, k \in \mathbb{Z}$ as the spherical functions on $X_G = \mathbb{R}$ and $X_U = U(1)$ respectively. Both X_G and X_U can be realized as the real forms of their complexification $X_{\mathbb{C}} = \mathbb{C}$. Let $\widetilde{f}$ and $\widehat{f}$ denote the spherical transformation of f on X_G and on X_U respectively. Then equation (1.1) becomes,

$$\widetilde{f}(\lambda) = -\frac{\pi}{\sin \pi\lambda} a(\lambda), \qquad \widehat{f}(k) = (-1)^k a(k).$$

Using the duality between $X_U = U/K$ and $X_G = G/K$ inside their complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$, Bertram proved an analogue of Ramanujan's Master theorem for semisimple Riemannian symmetric spaces of rank one. This theorem was further extended to arbitrary rank semisimple Riemannian symmetric spaces by Ólafsson and Pasquale (see [16]). It was also extended for the hypergeometric Fourier transform associated to root systems by Ólafsson and Pasquale (see [17]) and also to the radial sections of line bundles over Poincaré upper half plane by Pusti and Ray ([19]) .

In this paper we prove an analogue of this theorem in the setting of Strum-Liouville operator. We consider the eigenfunction φ_λ of the Sturm-Liouville operator (2.5) and think of it as an analogue of spherical function $x \mapsto e^{i\lambda x}$ on $\mathbb{R}$. Next we consider the operator

$$L = \frac{d^2}{dt^2} + \frac{\tilde{A}'(t)}{\tilde{A}(t)}\frac{d}{dt}, \tag{1.2}$$

on $(0, \frac{\pi}{2})$ where $\tilde{A}(t) = (-i)^{2\alpha+1}A(it)$ and A is given in (2.1). The functions Ψ_j's (defined in section 3) are (countable) eigenfunctions of $-L$ with eigenvalue ν_j. These Ψ_j's are orthonormal basis of $L^2\left((0, \frac{\pi}{2}), \widetilde{A}(t)dt\right)$. We think these Ψ_j's as an analogue of $x \mapsto e^{ikx}$ on U(1). These φ_λ and Ψ_j are related by

$$\Psi_j(t) = c_j \varphi_{i\sqrt{\nu_j+\rho^2}}(it), \text{ on } (0, \frac{\pi}{2}). \tag{1.3}$$

In the non perturbed case i.e. the case when $B = 1$ in (2.1), the functions Ψ_j's reduces to Jacobi polynomials. Using relation (1.3) we state and prove an analogue of Ramanujan's Master theorem (see Theroem 5.1) for the Sturm Liouville operator. Since Ψ_j's are orthonormal basis of $L^2\left((0, \frac{\pi}{2}), \widetilde{A}(t)dt\right)$, we can think of the series (5.1) as the Fourier series corresponding to the operator $-L$. The main crux of the proof of the theorem is to find a function b for which (5.2) holds. In the Euclidean case and in the non perturbed case the function b is related to the reciprocal of sine function but here in this perturbed case reciprocal of sine function will not work. Instead, here the function b is related to inverse of some sine type function (see (5.3) for exact definition). From (5.4) we can also interpolate the values of a to continuous parameter from the discrete parameter.

The plan of the paper is as follows: In section 2 we define the necessary terminology and state some facts with references. Section 3 and 4 are devoted to developing the Fourier series analogue of the operator given by (1.2), the relation (1.3) and the corresponding function $b(\cdot)$ as mentioned above. After developing all the machinery we prove our main theorem in section 5. For the sake of completeness, in section 7 we state some standard theorems in our context and definitions of well known concepts.

2. Preliminaries

Throughout this paper we always assume that, $\alpha, \beta \geq -\frac{1}{2}$. Let $\Omega = \{t + is \mid |s| < \frac{\pi}{2}\}$. We define $A : \Omega \to \mathbb{C}$ by

$$A(z) = (\sinh z)^{2\alpha+1}(\cosh z)^{2\beta+1}B(z), \tag{2.1}$$

where $B : \Omega \to \mathbb{C} \setminus \{0\}$ is holomorphic. In this paper we assume the following (1) to (4) conditions:

(1) The function B is even on Ω, positive on $\mathbb{R}$ and $B\Big|_{\{is:\,|s|<\frac{\pi}{2}\}} > 0$.
(2) The function B has an even (with respect to $i\frac{\pi}{2}$) holomorphic extension to a neighboorhood of $i\frac{\pi}{2}$.
(3) The function $\frac{A'(t)}{A(t)}$ is non-negative decreasing function, for large t. We define
$$2\rho = \lim_{t\to\infty} \frac{A'(t)}{A(t)}.$$
Also assume that (as in [1]), there exists $\delta > 0$ such that for all t in $[t_0, \infty)$ (for some $t_0 > 0$)
$$\frac{A'(t)}{A(t)} = \begin{cases} 2\rho + e^{-\delta t}D(t) & \text{if } \rho > 0, \\ \frac{2\alpha+1}{t} + e^{-\delta t}D(t) & \text{if } \rho = 0, \end{cases} \tag{2.2}$$
where D is a smooth bounded function such that its derivatives are also bounded.
(4) The function G defined in equation (2.11) is integrable along any straight line in Ω.

The condition (2.2) above assures that for large t,

$$A(t) = \begin{cases} O(e^{2\rho t}) & \text{if} \quad \rho > 0, \\ O(|t|^{\alpha+1}) & \text{if} \quad \rho = 0. \end{cases} \tag{2.3}$$

We consider the following Sturm-Liouville operator

$$\mathcal{L} = \frac{d^2}{dt^2} + \frac{A'(t)}{A(t)}\frac{d}{dt}. \tag{2.4}$$

For each $\lambda \in \mathbb{C}$, we define φ_λ as the unique solution of

$$\mathcal{L}f + (\lambda^2 + \rho^2)f = 0, \text{ with } f(0) = 1, f'(0) = 0. \tag{2.5}$$

For the case when $B(t) = 1$ for all t, the Sturm-Liouville operator $\mathcal{L}$ is the radial part of the Laplace-Beltrami operator on the rank one symmetric spaces of noncompact type $X = G/K$ and in this case the function φ_λ (defined in (2.5)) becomes the elementary spherical function on X. We call the case $B = 1$ as the *non perturbed* case and otherwise as *perturbed case.* We remark that in the non perturbed case all the stated conditions $(1) - (4)$ are satisfied automatically.

We have the following properties of φ_λ ([3, 4]):

(1) For each $t \in \mathbb{R}$, the function $\lambda \mapsto \varphi_\lambda(t)$ is even, entire.
(2) For each $\lambda \in \mathbb{C}$, the function $t \mapsto \varphi_\lambda(t)$ is even.
(3) For $\lambda \in \mathbb{C}$ with $|\Im\lambda| \leq \rho$, $|\varphi_\lambda(t)| \leq 1$ for all $t \in \mathbb{R}$.
(4) For $\lambda \in \mathbb{C}, t \in \mathbb{R}$, $|\varphi_\lambda(t)| \leq C(1 + |t|)e^{-\rho|t|}e^{|\Im\lambda|\,|t|}$.

For a function $f \in L^1(\mathbb{R}, A(t)\,dt)$, the (Sturm-Liouville) Fourier transform of f is defined by

$$\widehat{f}(\lambda) = \int_{\mathbb{R}} f(t)\varphi_\lambda(t)A(t)\,dt. \tag{2.6}$$

Also for suitable function f the inverse Fourier transform is given by

$$f(t) = \frac{1}{4\pi}\int_{\mathbb{R}} \widehat{f}(\lambda)\varphi_\lambda(t)|c(\lambda)|^{-2}\,d\lambda, \tag{2.7}$$

where $c(\lambda)$ is the Harish-Chandra c-function associated with the Sturm-Liouville operator. Let $C^\infty_{c,R}(\mathbb{R})$ be the space of all compactly supported smooth functions on $\mathbb{R}$ with support in $[-R, R]$. Also, let $PW_R(\mathbb{C})$ be the space of all entire functions F such that for each $N \in \mathbb{N}$,

$$\sup_{\lambda\in\mathbb{C}}(1+|\lambda|)^N|F(\lambda)|e^{-R|\Im\lambda|} < \infty. \tag{2.8}$$

We also denote $PW_R(\mathbb{C})_e$ as the space of even functions in $PW_R(\mathbb{C})$. Then we have the following Paley-Wiener theorem:

Theorem 2.1. ([5, Theorem 3]) *The (Sturm-Liouville) Fourier transform $f \mapsto \widehat{f}$ is a topological isomorphism between $C^\infty_{c,R}(\mathbb{R})$ and $PW_R(\mathbb{C})_e$.*

Definition 2.2. For $1 \le p \le 2$, the L^p-Schwartz space $\mathcal{C}^p(\mathbb{R})$ is the collection of all C^∞ functions f on $\mathbb{R}$ such that for each $N, m \in \mathbb{N}\cup\{0\}$,

$$\sup_{x\in\mathbb{R}}(1+|x|)^N\,|\mathcal{L}^m f(x)|\,e^{\frac{2}{p}\rho|x|} < \infty.$$

Using (2.3) it follows that $\mathcal{C}^p(\mathbb{R}) \subseteq L^p(\mathbb{R}, A(t)dt)$.

For $1 \le p \le 2$, let $S_p = \{\lambda \in \mathbb{C} \mid |\Im\lambda| \le (\frac{2}{p}-1)\rho\}$. Also let $\mathcal{S}_p(\mathbb{R})_e$ be the collection of all even C^∞ functions on S_p such that for each $N, m \in \mathbb{N}\cup\{0\}$,

$$\sup_{\xi\in S_p}(1+|\xi|)^N\left|\frac{d^m}{d\xi^m}f(\xi)\right| < \infty.$$

Theorem 2.3. ([1]) *The map $f \mapsto \widehat{f}$ is a topological isomorphism between $\mathcal{C}^p(\mathbb{R})$ and $\mathcal{S}_p(\mathbb{R})_e$.*

We already know that the function $x \mapsto \varphi_\lambda(x)$ is C^∞ on $\mathbb{R}$. But the following theorem states that the function has a holomorphic extension to the "crown domain" $\Omega := \{z \in \mathbb{C} \mid |\Im z| < \frac{\pi}{2}\}$.

Lemma 2.4. *The function $x \mapsto \varphi_\lambda(x)$ has holomorphic extension to Ω.*

Proof. We recall that φ_λ is the unique solution of

$$\frac{d^2 f}{dt^2} + \frac{A'(t)}{A(t)}\frac{df}{dt} + (\lambda^2+\rho^2)f = 0, \tag{2.9}$$

with $f(0) = 1, f'(0) = 0$. In [7, Theorem 2] it is proved that φ_λ has a real analytic extension on the real line around zero. The same proof also works in our case to show that φ_λ has a holomorphic extension to a neighboorhood of 0 in $\mathbb{C}$, call it Ω_0. Let $\Omega_1 = \Omega \setminus (-\infty, 0]$, $\Omega_2 = \Omega \setminus [0, \infty)$ and let $y_0 \in \Omega_1\cap\Omega_0\cap\Omega_2$. Then from Theorem 7.3 (in Appendix), there exists a unique holomorphic solution f on Ω_1 of (2.9) with initial condition $f(y_0) = \varphi_\lambda(y_0), f'(y_0) = \varphi'_\lambda(y_0)$. Similarly there exists a unique holomorphic solution f on Ω_2 of (2.9) with initial condition $f(y_0) = \varphi_\lambda(y_0), f'(y_0) = \varphi'_\lambda(y_0)$. Therefore by analytic continuation it follows that φ_λ has a holomorphic extension to Ω. $\square$

Before going further we will rewrite $\mathcal{L}$ as a perturbation of the Bessel equation to deduce some more properties of φ_λ. After applying the classical Liouville transformation i.e. $v(t) = \sqrt{A(t)}u(t)$, equation (2.9) reduces to

$$\frac{d^2v}{dt^2} - \frac{\alpha^2 - \frac{1}{4}}{t^2}v - G(t)v + \lambda^2 v = 0 \tag{2.10}$$

where

$$G(t) = \frac{1}{4}\left(\frac{A'(t)}{A(t)}\right)^2 + \frac{1}{2}\left(\frac{A'(t)}{A(t)}\right)' - \rho^2 - \frac{\alpha^2 - \frac{1}{4}}{t^2}.$$

Let

$$G_0(t) := \left((\alpha + \frac{1}{2})^2 - (\alpha + \frac{1}{2})\right)\coth^2 t + \left((\beta + \frac{1}{2})^2 - (\beta + \frac{1}{2})\right)\tanh^2 t + (\alpha + \frac{1}{2})$$
$$+(\beta + \frac{1}{2}) + 2(\alpha + \frac{1}{2})(\beta + \frac{1}{2}) - \rho^2 - \frac{\alpha^2 - \frac{1}{4}}{t^2}.$$

It is easy to check that $\coth^2 t - \frac{1}{t^2}$ is a holomorphic function on Ω, hence G_0 is a holomorphic function on Ω.

A simple computation shows that

$$G(t) = G_0(t) + (\beta + \frac{1}{2})\tanh t\frac{B'(t)}{B(t)} + (\alpha + \frac{1}{2})\coth t\frac{B'(t)}{B(t)} + \frac{1}{4}\left(\frac{B'(t)}{B(t)}\right)^2 + \frac{B''(t)}{2B(t)}. \tag{2.11}$$

The assumptions on B implies that $B'(0) = 0$, which assure that $(\alpha + \frac{1}{2})\coth t\frac{B'(t)}{B(t)}$ is a holomorphic function on Ω and therefore G is a holomorphic function on Ω.

Theorem 2.5. *Let G be integrable along any straight line in Ω. Then for $\lambda(\neq 0) \in \mathbb{C}$, there exists a polynomial P and a constant $C > 0$ such that*

$$|\varphi_\lambda(\xi)| \leq C\,|P(\xi)|\,e^{|\Im(\lambda\xi)|},$$

for all $\xi \in \Omega$.

The proof of this theorem is similar to [4, Theorem 1.2] but for the sake of completeness we give the proof of the theorem above, in the Appendix.

Remark 2.6. In the nonperturbed case (that the case when $B(t) = 1$ for all t), we have

$$A(t) = (\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1},$$

and hence

$$G(t) = \left((\alpha + \frac{1}{2})^2 - (\alpha + \frac{1}{2})\right)\coth^2 t + \left((\beta + \frac{1}{2})^2 - (\beta + \frac{1}{2})\right)\tanh^2 t + (\alpha + \frac{1}{2})$$
$$+(\beta + \frac{1}{2}) + 2(\alpha + \frac{1}{2})(\beta + \frac{1}{2}) - \rho^2 - \frac{\alpha^2 - \frac{1}{4}}{t^2}.$$

This shows that G is in L^1 in every direction in the domain Ω. Therefore the condition on G is satisfied automatically for the nonperturbed case.

Let Φ_λ be the second solution of (2.9) on $(0, \infty)$. Also it follows that $\Phi_{-\lambda}$ is a solution of (2.9) on $(0, \infty)$. The Wronskian of $\Phi_\lambda, \Phi_{-\lambda}$ is given by ([4, Corollary 1.12])

$$\mathcal{W}(\Phi_\lambda, \Phi_{-\lambda}) = -2i\lambda A(t)^{-1}.$$

Therefore, for $\lambda(\neq 0)$, the soultions $\Phi_\lambda, \Phi_{-\lambda}$ are linearly independent. Hence there is a function c such that for $\lambda \neq 0$,

$$\varphi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda}.$$

It follows from [4, Theorem 2.4] that $c(\lambda)$ is holomorphic for $\Im\lambda < 0$. We need the following improved theorem.

Theorem 2.7. *The function c is holomorphic on $\mathbb{C} \setminus \{0, \frac{m}{2}i \mid m \in \mathbb{N}\}$ if $\rho > 0$. Also the function c is homorphic on $\mathbb{C} \setminus \{\frac{m}{2}i \mid m \in \mathbb{N}\}$ if $\rho = 0$.*

Proof. For $\lambda(\neq 0) \in \mathbb{C}$, we have for $t > 0$

$$\varphi_\lambda(t) = c(\lambda)\Phi_\lambda(t) + c(-\lambda)\Phi_{-\lambda}(t).$$

Therefore, for $t > 0$

$$\varphi'_\lambda(t) = c(\lambda)\Phi'_\lambda(t) + c(-\lambda)\Phi'_{-\lambda}(t).$$

Hence,

$$c(\lambda) = \frac{\varphi_\lambda(t)\Phi'_{-\lambda}(t) - \varphi'_\lambda(t)\Phi_{-\lambda}(t)}{\mathcal{W}(\Phi_\lambda, \Phi_{-\lambda})(t)} = \frac{iA(t)}{2\lambda}\left(\varphi_\lambda(t)\Phi'_{-\lambda}(t) - \varphi'_\lambda(t)\Phi_{-\lambda}(t)\right). \tag{2.12}$$

Now we consider the case when $\rho > 0$. To prove the theorem (in this case) we shall prove that the functions

$$\lambda \mapsto \Phi_{-\lambda}(t), \Phi'_{-\lambda}(t),$$

both are holomorphic on $\mathbb{C} \setminus \{\frac{m}{2}i \mid m \in \mathbb{N}\}$. We have Φ_λ is a solution of

$$y'' + \frac{A'(t)}{A(t)}y' + (\lambda^2 + \rho^2)y = 0. \tag{2.13}$$

Let

$$A(t) = (\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}B(t).$$

Then

$$\frac{A'(t)}{A(t)} = (2\alpha + 1)\coth t + (2\beta + 1)\tanh t + \frac{B'(t)}{B(t)}.$$

Let

$$\frac{B'(t)}{B(t)} = \sum_{j=0}^{\infty} 2b_j e^{-jt}.$$

Then $\rho = \alpha + \beta + 1 + b_0$. Let

$$y(t) = \sum_{n=0}^{\infty} \Gamma_n(\lambda)e^{(i\lambda-\rho-n)t}$$

be a solution of (2.13). Then putting this in the equation (2.13) and comparing coefficients we get

$$\Gamma_1(\lambda) = \frac{-2b_1(i\lambda - \rho)}{1 - 2i\lambda}\Gamma_0(\lambda)$$

$$\Gamma_2(\lambda) = \frac{1}{4(1 - i\lambda)}\left(-2(2\alpha - 2\beta + b_2)(i\lambda - \rho)\Gamma_0(\lambda) - 2b_1(i\lambda - \rho - 1)\Gamma_1(\lambda)\right),$$

and

$$3(3 - 2i\lambda)\Gamma_3(\lambda) = -2(2\alpha - 2\beta + b_2)(i\lambda - \rho - 1)\Gamma_1(\lambda) - 2b_3(i\lambda - \rho)\Gamma_0(\lambda) - 2b_1(i\lambda - \rho - 2)\Gamma_2(\lambda),$$

and continued in this way. This shows that each Γ_n is holomorphic on $\mathbb{C} \setminus \{-i\frac{m}{2} \mid m \in \mathbb{N}\}$. As in the classical case of symmetric spaces (see [11, Ch. IV, Lemma 5.3]), it is easy to check that for any $t_0 > 0$,

$$|\Gamma_n(\lambda)| \leq K_{\lambda,t_0}e^{nt_0}. \tag{2.14}$$

From equation (2.14) it follows that

$$\Phi_\lambda(t) = e^{(i\lambda-\rho)t}\sum_{n=0}^{\infty}\Gamma_n(\lambda)e^{-nt}, t > 0 \tag{2.15}$$

is well defined and converges absolutely and uniformly for $t > 0$ and for $\lambda \in \mathbb{C} \setminus \{-i\frac{m}{2} \mid m \in \mathbb{N}\}$. This shows that $\lambda \mapsto \Phi_\lambda(t), \Phi'_\lambda(t)$ are holomorphic on $\mathbb{C} \setminus \{-i\frac{m}{2} \mid m \in \mathbb{N}\}$.

Now if $\rho = 0$, it follows from the expression above that $\Gamma_n(0) = 0$ for $n = 1, 2, \cdots$. Therefore we have

$$\Phi_0(t) = \Gamma_0(0) \text{ for all } t > 0.$$

Then the function

$$\lambda \mapsto \varphi_\lambda(t)\Phi'_{-\lambda}(t) - \varphi'_\lambda(t)\Phi_{-\lambda}(t)$$

at the point $\lambda = 0$ reduces to $-\Gamma_0(0)\varphi'_0(t)$ which is equal to 0, as in this ($\rho = 0$) case, $\varphi'_0(t) = 0$. Hence from (2.12) it follows that the function c is holomorphic at $\lambda = 0$. Rest is similar to above case. This completes the proof. ☐

We have the following corollary:

Corollary 2.8. (1) *The c-function has a simple pole at $\lambda = 0$ if $\rho > 0$ but it is holomorphic at $\lambda = 0$ if $\rho = 0$.*

(2) *Let $\delta < \frac{1}{2}$. Then for $|\Im\lambda| \leq \delta$ and for all λ outside a neighbourhood of 0*

$$|c(\lambda)| \leq Cp(|\lambda|),$$

for some polynomial p.

Proof. First part follows from (2.12). Second part follows from the series expansion (2.15) of Φ_λ and noting that each Γ_n is a rational function. ☐

3. Compact Case

In order to obtain an analogue of Ramanujan's Master theorem, Theorem 1.2 for the Sturm Liouville operator, we need to develop the corresponding Fourier series for $\mathcal{L}$ when restricted to, $z = it, t \in (0, \pi/2)$. In this section we will define a positive, symmetric densely defined differential operator $-L$ on $(0, \pi/2)$ on a suitable Hilbert space such that the following holds

$$-Lw(t) = \mathcal{L}u(z)_{z=it}, \tag{3.1}$$

where $w(t) = u(it)$ for u twice differentiable defined in $\Omega \setminus \{0\}$. We will study the spectral decomposition of $-L$ on $(0, \pi/2)$ under suitable boundary conditions such that we obtain the eigenfunctions of L as the restriction of the eigenfunctions of $\mathcal{L}$ on $z = it$. This will be in direct analogy of the functions $\{e^{i\lambda z}\}$ as discussed in the introduction. Let us recall that

$$A(t) = (\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}B(t),$$

where B is a non zero, even holomorphic function on Ω. We also have $B(it) > 0$ when $t \in (-\pi/2, \pi/2)$. We define

$$\tilde{A}(t) = (-i)^{2\alpha+1}A(it).$$

Let $\tilde{B}(t) = B(it)$. Indeed

$$\tilde{A}(t) = (\sin t)^{2\alpha+1}(\cos t)^{2\beta+1}\tilde{B}(t).$$

Clearly $\tilde{A} > 0$ on $(0, \pi/2)$. We define the Sturm Liouville operator corresponding to $\tilde{A}$ on $(0, \pi/2)$ as

$$L = \frac{d^2}{dt^2} + \frac{\tilde{A}'(t)}{\tilde{A}(t)}\frac{d}{dt}. \tag{3.2}$$

It is easy to verify that this choice of L satifies (3.1) above. We define $\mathcal{D}(L)$ to be the space of $f \in L^2\left((0, \pi/2), \tilde{A}(t)dt\right)$ such that f and $\tilde{A}(\cdot)f'$ are absolutely continuous on any compact subinterval of $(0, \pi/2)$ and $L(f) \in L^2\left((0, \pi/2), \tilde{A}(t)dt\right)$.

The operator $-L$ is a densely defined operator from $\mathcal{D}(L)$ to $L^2\left((0, \pi/2), \tilde{A}(t)dt\right)$. Let

$$\mathcal{D}(L)_0 := \{f \in \mathcal{D}(L) : f \text{ is compactly supported on } (0, \pi/2)\}.$$

Let us denote $-L$ restricted on $\mathcal{D}(L)_0$ as $-L_0$. An integration by parts argument shows that $-L_0$ is a positive and symmetric operator on $\mathcal{D}(L)_0$. We need to obtain a self adjoint extension of $-L_0$ on $L^2((0,\pi/2),\tilde{A}(t)dt)$ with suitable boundary conditions so that the eigenfunctions are restriction of φ_λ to $z = it, t \in (0,\pi/2)$ for λ related to the spectrum of the self adjoint extension of $-L_0$.

Let u be an eigenfunction of $-L$ with eigenvalue μ. After applying the classical Liouville transformation i.e. $v(t) = \sqrt{\tilde{A}(t)}u(t)$ we get another differential operator $-l$ such that v is an eigenfunction of $-l$ with eigenvalue μ. One can explicitly write the expression of $-l$ as follows:

$$-l = -\frac{d^2}{dt^2} + q(t),$$

where

$$q(t) = \frac{1}{4}\left(\frac{\tilde{A}'(t)}{\tilde{A}(t)}\right)^2 + \frac{1}{2}\left(\frac{\tilde{A}'(t)}{\tilde{A}(t)}\right)'.$$

In fact

$$q(t) = \left((\alpha^2 - \frac{1}{4})\cot^2 t + (\beta^2 - \frac{1}{4})\tan^2 t - \chi(t)\right), \tag{3.3}$$

where

$$\chi(t) = \left(\beta + \frac{1}{2}\right)\frac{\tilde{B}'(t)}{\tilde{B}(t)}\tan t - \left(\alpha + \frac{1}{2}\right)\frac{\tilde{B}'(t)}{\tilde{B}(t)}\cot t + \frac{1}{4}\left(\frac{\tilde{B}'(t)}{\tilde{B}(t)}\right)^2 - \frac{1}{2}\frac{\tilde{B}''(t)}{\tilde{B}(t)}.$$

It follows from the assumptions on $\tilde{B}$ that χ is a smooth function on $[-\frac{\pi}{2},\frac{\pi}{2}]$. Therefore q has singularities only at 0 and $\pi/2$. A simple evaluation gives that

$$\sqrt{\tilde{A}(t)}Lu(t) = lv(t), \tag{3.4}$$

where $v(t) = \sqrt{\tilde{A}(t)}u(t)$. The unbounded operator $l : \mathcal{D}(l) \subset L^2(0,\pi/2) \to L^2(0,\pi/2)$ is defined on

$$\mathcal{D}(l) = \left\{f \text{ and } f' \text{ AC on any compact subinterval of } (0,\pi/2),\ f, l(f) \in L^2(0,\pi/2)\right\}.$$

Here AC stands for absolutely continuous. We observe that $\tilde{A} > 0$ and $\sqrt{\tilde{A}}$ is an AC function on any compact sub interval of $(0,\pi/2)$ and therefore it follows that

$$\mathcal{D}(l) = \sqrt{\tilde{A}(\cdot)}\mathcal{D}(L). \tag{3.5}$$

Let

$$\mathcal{D}(l)_0 := \{f \in \mathcal{D}(l) : f \text{ is compactly supported on } (0,\pi/2)\}.$$

A similar identity (3.5) also holds for $\mathcal{D}(l)_0$ and $\mathcal{D}(L)_0$. We denote $-l$ restricted on $\mathcal{D}(l)_0$ as $-l_0$. Let $v_1, v_2 \in \mathcal{D}(l)_0$. The following holds:

$$(-L_0u_1, u_2)_{L^2((0,\pi/2),\tilde{A}(t)dt)} = (-l_0v_1, v_2)_{L^2(0,\pi/2)},$$

where $v_i(t) = \sqrt{\tilde{A}(t)}u_i(t)$ for $i = 1,2$. The last identity just uses the relation between u_i and $v_i, i = 1,2$ and equation (3.4). This shows that $-l_0$ is a positive and symmetric operator on $\mathcal{D}(l)_0$. In fact if u is an eigenfunction of $-L$ with eigenvalue μ, the equation (3.4) gives that $\sqrt{A(t)}u(t)$ is an eigenfunction of $-l$ with the same eigenvalue and vice versa too. Therefore the eigenfunctions of $-L$ and $-l$ are in one to one correspondence by Liouville transformation. A simple computation gives

$$\|v\|_{L^2(0,\pi/2)} = \|u\|_{L^2((0,\pi/2),\tilde{A}(t)dt)}.$$

In fact the map $u \to \sqrt{A(\cdot)}u(.)$ is an isometry from $L^2((0,\pi/2), \tilde{A}(t)dt))$ onto $L^2(0,\pi/2)$. In view of the above relation between $-L$ and $-l$, in order to obtain self adjoint extension of $-L_0$ on $L^2((0,\pi/2), \tilde{A}(t)dt)$, it is enough to obtain a self adjoint extension of $-l_0$ on $L^2(0,\pi/2)$.

We have the following Theorem:

Theorem 3.1. *The operator $-L_0$ has a self adjoint extension (with abuse of notation call it $-L$) on $\tilde{D} := \left(\sqrt{\tilde{A}}\right)^{-1}\mathcal{D}$ (where $\mathcal{D} \subset L^2(0,\pi/2)$ is as defined in Proposition 3.2). The spectrum of $-L$ is purely discrete, bounded below and all eigenvalues are simple. The eigenvalues can be ordered by*

$$\nu_0 < \nu_1 < \ldots < \nu_n < \ldots$$

with $\nu_n \to +\infty$. The eigenfunctions $\{\Psi_j\}$ corresponding to the eigenvalues $\{\nu_j\}$ form an orthonormal basis of $L^2((0,\pi/2), \tilde{A}(t)dt)$ and there exist constants c_j with a polynomial growth in j such that

$$\Psi_j(t) = c_j \varphi_{i\sqrt{\nu_j+\rho^2}}(it)$$

for all $j \geq j_0$, where j_0 is the least natural number for which $\nu_j + \rho^2 > 0$. When $\nu_j + \rho^2 \leq 0$, let $\beta_j := \sqrt{-(\nu_j+\rho^2)}$. Then for $j = 0, 1, 2, ...j_0 - 1$ we have

$$\Psi_j(t) = c_j \varphi_{\beta_j}(it).$$

When $\alpha, \beta > 0$, the operator $-L$ is the Friedrich's extension of $-L_0$ and $\nu_j' s$ are non negative.

The proof of the above theorem is given at the end of this section.

Relation between $\mathcal{L}$ and $-L$: Let u be a twice differentiable function on Ω. Define $w(t) := u(it)$. Using the relation between $\tilde{A}$ and A it is easy to see that the relation (3.1) holds, more precisely

$$\mathcal{L}u(z)_{z=it} = -Lw(t)$$

We define $w_\mu(t) = \varphi_{i\sqrt{\mu+\rho^2}}(it)$, where $\mu + \rho^2 \in \mathbb{C} \setminus (-\infty, 0]$. We know that

$$\mathcal{L}\varphi_{i\sqrt{\mu+\rho^2}}(z) = \mu\varphi_{i\sqrt{\mu+\rho^2}}(z),$$

for all $z \in \Omega \setminus \{0\}$ (in particular when $z = it, t \in (0,\pi/2)$). Therefore we have

$$-Lw_\mu(t) = \mu w_\mu(t), t \in (0,\pi/2).$$

In the case when $\mu + \rho^2 \leq 0$, define $\beta = \sqrt{-(\mu+\rho^2)}$. By the same principle as above we can check that $-Lu_\beta(t) = \beta u_\beta(t)$, where $u_\beta(t) = \varphi_\beta(it)$. We define

$$\tilde{\varphi}_{i\sqrt{\mu+\rho^2}}(t) = \sqrt{\tilde{A}(t)}\varphi_{i\sqrt{\mu+\rho^2}}(it)$$

for $t \in (0,\pi/2)$. By the correspondence between $-L$ and $-l$ (3.4), we also have

$$-l\tilde{\varphi}_{i\sqrt{\mu+\rho^2}}(t) = \mu\tilde{\varphi}_{i\sqrt{\mu+\rho^2}}(t), t \in (0,\pi/2).$$

Similarly when $\mu + \rho^2 \leq 0$, define $\beta = \sqrt{-(\mu+\rho^2)}$. We get that $-L\tilde{\varphi}_\beta = \mu\tilde{\varphi}_\beta$.

Spectral Decomposition of $-l$: Let us recall that

$$-l = -\frac{d^2}{dt^2} + q(t),$$

where $q(t)$ is defined as in equation (3.3). It is known that 0 and $\pi/2$ are non-oscillatory end points of $-l$, i.e. there exist solutions of $-lu_i = \mu u_i, i = 1, 2$ such that u_i is non zero in $(0,\epsilon)$ and $(\pi/2 - \epsilon, \pi/2)$ respectively for $i = 1, 2$ (see section 2 [21]) where $u_i' s$ are defined on $(0,\pi/2)$. (See Appendix for further details.)

Let us fix $\mu \geq 0$. We know that $-l\tilde{\varphi}_{i\sqrt{\mu+\rho^2}}(t) = \mu\tilde{\varphi}_{i\sqrt{\mu+\rho^2}}(t), t \in (0, \pi/2)$. As stated in the last section, $\varphi_{i\sqrt{\mu+\rho^2}}(0) = 1, \varphi'_{i\sqrt{\mu+\rho^2}}(0) = 0$ and $\sqrt{\tilde{A}(t)} \sim t^{\alpha+1/2}, \sqrt{\tilde{A}'(t)} \sim t^{\alpha-1/2}$ near $0+$. Therefore $\tilde{\varphi}_{i\sqrt{\mu+\rho^2}}(t) \sim t^{\alpha+1/2}$ and $\tilde{\varphi}'_{i\sqrt{\mu+\rho^2}}(t) \sim t^{\alpha-1/2}$ in some right neighbourhood of 0. In fact one can also construct another solution of $-lu = \mu u$ in $(0, \pi/2)$ such that it satisfies $u(t) \sim t^{-\alpha+1/2}$ and and $u'(t) \sim t^{-\alpha-1/2}$ and it is also linearly independent to $\tilde{\varphi}_{i\sqrt{\mu+\rho^2}}$ (see section 2 [21]). Similarly, given $\tilde{\mu}$, we can also find two linearly independent eigenfunctions $W_{\widetilde{\mu},\pm\beta}$ defined on $(0, \frac{\pi}{2})$ satisfying

$$-lW_{\widetilde{\mu},\pm\beta}(t) = \tilde{\mu}W_{\widetilde{\mu},\pm\beta}(t), t \in (0, \frac{\pi}{2})$$

such that $W_{\widetilde{\mu},\pm\beta}(t) \sim (\frac{\pi}{2} - t)^{\pm\beta+\frac{1}{2}}$ and $W'_{\widetilde{\mu},\pm\beta}(t) \sim (\frac{\pi}{2} - t)^{\pm\beta-\frac{1}{2}}$ (see section 2 [21]). For the $\beta \leq \alpha$ we have the following proposition. The other case i.e. $\alpha < \beta$ follows similarly. By [15, Theorem 4.2] [6, Theorem 50, page 1478] and [8, Proposition 9], the following holds:

Proposition 3.2. *The operator $-l_0$ has a self adjoint extension (call it $-l$) on*

$$\mathcal{D} = \begin{cases} \left\{y \in \mathcal{D}(l) : \left[y, \tilde{\varphi}_{i\sqrt{\mu+\rho^2}}\right]_l (0) = \left[y, W_{\widetilde{\mu},\beta}\right]_l (\pi/2) = 0\right\} & \text{for } -1/2 < \beta \leq \alpha < 1 \\ \left\{y \in \mathcal{D}(l) : \left[y, W_{\mu,\beta}\right]_l (\pi/2) = 0\right\} & \text{for } -1/2 < \beta < 1 \leq \alpha \\ \mathcal{D}(l) & \text{for } 1 \leq \beta \leq \alpha. \end{cases}$$

Here

$$[y, u]_l (0) = \lim_{t\to 0+} \left(y(t)\overline{u'(t)} - y'(t)\overline{u(t)}\right)$$
$$[y, u]_l (\pi/2) = \lim_{t\to \pi/2-} \left(y(t)\overline{u'(t)} - y'(t)\overline{u(t)}\right).$$

The operator $-l$ is bounded from below in $L^2(0, \pi/2)$ and the domain is independent of the choice of μ and $\widetilde{\mu}$. The spectrum of $-l$ is purely discrete, bounded below and all eigenvalues are simple. The eigenvalues can be ordered by

$$\nu_0 < \nu_1 < \ldots < \nu_n < \ldots$$

with $\nu_n \to +\infty$.

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. The existence of a self adjoint extension of $-L_0$ on $\tilde{D}$ follows from the equation (3.4) and the relation between $\mathcal{D}(L)_0$ and $\mathcal{D}(l)_0$. In fact the eigenvalues of $-L$ are same as that of $-l$ as explained earlier and the eigenfunctions are related by the classical Liouville transformation.

When $\alpha, \beta > 0$, it is clear from the above estimates that $\tilde{\varphi}_{i\sqrt{\mu+\rho^2}}$ and $W_{\widetilde{\mu},\beta}$ are principal solutions respectively at 0 and $\pi/2$ (See Appendix). In fact the boundary conditions in the above proposition correspond to that of with respect to the principal solution as in [15, Theorem 4.2]. By uniqueness of the principal solutions upto constant multiples the boundary conditions coincide with that of the Friedrich's extension.

Therefore when $\alpha, \beta > 0$ we obtain the Friedrich's extension of $-L$ on $\tilde{D}$. The lower bound of the Friedrich's extension is same as that of $-L_0$. Therefore the self adjoint extension considered above is also non negative when $\alpha, \beta > 0$. This implies that ν_j's are non negative. Let $\{u_j\}_{j\geq 0}$ be the eigenfunctions of $-l$ such that $-lu_j = \nu_j u_j$. We further assume that $\|u_j\|_{L^2(0.\pi/2)} = 1$ for all $j \geq 0$. It is clear that $\{u_j\}_{j\geq 0}$ form an orthonormal basis of $L^2(0, \pi/2)$. By Proposition 3.2, it follows that there exist constants c_j and d_j such that for all $t \in (0, \pi/2)$,

$$u_j(t) = c_j\tilde{\varphi}_{i\sqrt{\nu_j+\rho^2}}(t) = d_j W_{\nu_j,\beta}(t).$$

This is clearly true for $\alpha < 1$ due to the boundary conditions of Proposition 3.2, and in the other cases it follows from the uniqueness of $\tilde{\varphi}_{i\sqrt{\nu_j+\rho^2}}$ among the solutions of the equation $-lu = \nu_j u$ which are in $L^2\left((0,\varepsilon)\right)$, and of $W_{\nu_j,\beta}$ in $L^2\left(\pi/2-\varepsilon,\pi/2\right)$. The constant c_j has polynomial growth in j (see [8] Lemma 13, Page 21).

We define $\Psi_j(t) = \left(\sqrt{\tilde{A}(t)}\right)^{-1} u_j(t), t \in (0,\pi/2)$. Clearly $\{\Psi_j\}_{j\geq 1}$ forms an orthonormal basis of $L^2((0,\pi/2),\tilde{A}(t)dt)$ satisfying $-L\Psi_j(t) = \nu_j\Psi_j(t), t \in (0,\pi/2)$.

Therefore we have

$$\Psi_j(t) = c_j \varphi_{i\sqrt{\nu_j+\rho^2}}(it)\,.$$

□

Corollary 3.3. *For each $m \geq 0$, the function Ψ_m satisfies the following inequality:*

$$|\Psi_m(t+is)| \leq C_m Q(|\sqrt{\nu_m+\rho^2}|)\,|P(t+is)|\,e^{|s|\sqrt{\nu_m+\rho^2}}, \tag{3.6}$$

for some polynomials P and Q.

Proof. It follows from Theorem 2.5, that

$$|\varphi_{\lambda_1+i\lambda_2}(t+is)| \leq CQ(|\lambda|)|P(t+is)|e^{|\lambda_1 s+\lambda_2 t|}.$$

Hence the required inequality will follow from Theorem 3.1. □

4. Sine type function

In this section we consider the case when $\alpha,\beta > 0$. We recall that ν_n's $(n = 0,1,2,\cdots)$ are eigenvalues of $-L$ with eigenfunctions Ψ_n respectively. In this case (i.e. for $\alpha,\beta > 0$), $\nu_0 \geq 0$ and $\nu_n > 0$ for $n = 1,2,\cdots$. We define the following function:

$$S(z) = \begin{cases} \pi z \prod_{n=0}^{\infty}\left(1+\frac{z^2}{\nu_n+\rho^2}\right) & \text{if} \quad \nu_0+\rho^2 > 0, \\ \pi z^3 \prod_{n=1}^{\infty}\left(1+\frac{z^2}{\nu_n+\rho^2}\right) & \text{if} \quad \nu_0+\rho^2 = 0. \end{cases} \tag{4.1}$$

The asymptotic expansion of $\nu_n' s$ is known. More precisely (see [8] and Theorem 2_I[9])

$$\sqrt{\nu_n} = 2n+1+\alpha+\beta-\frac{\Theta}{4n}+O(n^{-2}), \tag{4.2}$$

where $\Theta = \alpha^2+\beta^2-1/2+\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\chi(t)dt$ for $n \to \infty$. The function $S(z)$ has zeros exactly at $\{\pm i\sqrt{\nu_n+\rho^2}\}_{n\geq 0}$ and 0. Clearly $\sqrt{\nu_n+\rho^2}$ is also a perturbation of $2n+1+\alpha+\beta$ for n large enough. The function $S(iz)$ is a function of sine type (see [13]).

In order to obtain the main Theorem we need to find a uniform bound on the residue of $S(z)^{-1}$ at $\{i\sqrt{\nu_n+\rho^2}\}_{n\geq 0}$. We prove the following theorem:

Theorem 4.1. *Let*

$$S_1(z) = \frac{z^2}{S(z)}. \tag{4.3}$$

Let d_n be the residue of S_1 at $i\sqrt{\nu_n+\rho^2}$. Then the following holds:

(1)

$$|S_1(z)| \asymp \begin{cases} |z|^2 e^{-\frac{\pi}{2}|\Re z|} & \text{if} \quad \nu_0+\rho^2 > 0, \\ e^{-\frac{\pi}{2}|\Re z|} & \text{if} \quad \nu_0+\rho^2 = 0. \end{cases} \tag{4.4}$$

(2)

$$|d_n| \leq c|\nu_n+\rho^2| \text{ for all } n \text{ large}. \tag{4.5}$$

Proof. Let us put $\rho_0 = 1 + \alpha + \beta$. By using equation (4.2) we can write

$$\sqrt{\nu_n + \rho^2} = 2n + \rho_0 + \frac{2\rho^2 - \Theta}{4n} + O(n^{-2}),$$

for $n \geq M$, M large enough. We first deal with the case $\nu_0 + \rho^2 > 0$. Let $\mu_n := \sqrt{\nu_n + \rho^2}$. We define

$$M(z) = \frac{S(iz)}{\pi z}.$$

Then M has zeros of order one precisely on the set $\{\pm\mu_n\}_{n\geq 0}$. The residue of S_1 at $i\mu_n$ is given by

$$d_n := \lim_{z \to i\mu_n} (z - i\mu_n) S_1(z).$$

Since $S_1(z) = z^2(-i\pi z M(-iz))^{-1}$, it is easy to see that

$$d_n = -\mu_n^2 \lim_{-iz \to \mu_n} \frac{(-iz - \mu_n)}{-\pi z M(-iz)}.$$

As μ_n is a zero of $M(z)$,

$$d_n = \frac{\mu_n^2}{i\pi\mu_n M'(\mu_n)}.$$

In order to show $d_n \times \mu_n^{-2}$ is bounded for large n, it is enough to show that $(zM'(z))_{z=\mu_n}$ is bounded below as $n \to \infty$. We define

$$M_0(z) = \frac{2\sin(\pi(z - \rho_0)/2)}{\pi(z - \rho_0)}.$$

The function M_0 is a sine type function and exponential of type $\pi/2$. Clearly the zeros of M are the perturbation of the zeros of M_0. Indeed from (4.2) it is clear that the perturbation is of order 2. Using the notation of [13, p.86], let $\lambda_k = 2k + \rho_0$ and $\psi_k = -\frac{2\rho^2-\Theta}{4k} + O(k^{-2})$. By [13, Theorem 2, p.86] with $n = 2$ we have the asymptotic expansion of M in terms of M_0

$$M(z) = c_0 M_0(z) + z^{-1}(c_1 M_0(z) + c_2 M_0'(z)) + z^{-2}(d_0 M_0(z) + d_1 M_0'(z) + M_0''(z)) + z^{-2} f_2(z), \quad (4.6)$$

where f_2 is a holomorphic function of exponential type $\leq \frac{\pi}{2}$ and $c_i, d_i, i = 0, 1, 2$ are constants. For getting lower bound on $zM'(z)$ at $\mu_n's$ for n large we use the above expression of M in terms of M_0. On calculating we get that

$$M_0'(z) = \frac{\cos(\pi(z - \rho_0/2))}{z - \rho_0} - \frac{2\sin(\pi(z - \rho_0)/2}{\pi(z - \rho_0)^2}.$$

Using the fact that μ_n is a very small perturbation of $2n + \rho_0$, it is clear that $(zM_0'(z))_{z=\mu_n}$ is bounded from below for n large enough. On computing the derivative of M from equation (4.6) it is easy to see that the most dominating term of $zM'(z)$ at $z = \mu_n, n \geq M$ is $\mu_n M_0'(\mu_n)$ and the rest of the terms are of the order μ_n^{-1} which can be made as small as possible for large M. More precisely

$$|\mu_n M'(\mu_n)| \asymp |\mu_n M_0'(\mu_n)| + O(\mu_n^{-1})$$

As $\mu_n M_0'(\mu_n)$ is bounded below for $n \to \infty$, it implies that $(zM'(z))_{z=\mu_n}$ is bounded from below as $n \to \infty$. The pointwise estimate (4.4) of S_1 is clear from the asymptotic expansion (4.6) of M above. When $\nu_0 + \rho^2 = 0$ we define $M(z) = \frac{S(iz)}{z^3}$ and then follow the above proof in a similar manner to get the desired results. $\square$

5. Main theorem

In this section we prove Ramanujan's Master theorem for the case when $\alpha, \beta > 0$. Let A, p, δ be real numbers such that $A < \pi/2$, $p > 0$ and $0 < \delta \leq 1$. Let

$$\mathcal{H}(\delta) = \{\lambda \in \mathbb{C} \mid \Im\lambda > -\delta\},$$

and

$$\mathcal{H}(A, p, \delta) = \left\{a : \mathcal{H}(\delta) \to \mathbb{C} \text{ holomorphic } \mid |a(\lambda)| \leq C\, e^{-p(\Im\lambda)+A|\Re\lambda|} \text{ for all } \lambda \in \mathcal{H}(\delta)\right\}.$$

We recall that the function S_1 (defined in (4.3)) has simple poles at $\lambda = i\sqrt{\nu_m + \rho^2}, m = 0, 1, 2, \cdots$ where $\nu_0 \geq 0$ and $\nu_m > 0$ for all $m = 1, 2, \cdots$. We also recall that d_m is the residue of S_1 at $\lambda = i\sqrt{\nu_m + \rho^2}$ for $m = 0, 1, 2, \cdots$.

Theorem 5.1. *Let $\alpha, \beta > 0$ and let $a \in \mathcal{H}(A, p, \delta)$. Then the following holds:*

(1) *The Fourier series*

$$f(t) = 2\pi i \sum_{m=0}^{\infty} \frac{d_m}{c_m}\, a(i\sqrt{\nu_m + \rho^2})\Psi_m(t), \tag{5.1}$$

converges uniformly on compact subsets of $\Omega_p := \{t \in \mathbb{C} \mid |\Re t| < \frac{\pi}{2}, |\Im t| < p\}$ and hence holomorphic there.

(2) *Let $0 \leq \sigma < \delta$. Then the function f can also be expressed in the integral form for $t \in \mathbb{R}$ with $|t| < p$ as*

$$f(it) = \int_{-\infty-i\sigma}^{\infty-i\sigma} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda))\, \varphi_\lambda(t)\, \frac{d\lambda}{c(\lambda)c(-\lambda)}, \tag{5.2}$$

where the function b is defined by

$$\frac{b(\lambda)}{c(\lambda)c(-\lambda)} = S_1(\lambda). \tag{5.3}$$

The integrals defined above are independent of σ and extends as a holomorphic function to a neighboorhood $\{z \in \mathbb{C} \mid |\Re z| < \frac{\pi}{2} - A\}$ of $i\mathbb{R}$.

(3) *The extension of f to $i\mathbb{R}$ satisfies, for all $\lambda \in \mathbb{R}$*

$$\int_{\mathbb{R}} f(it)\, \varphi_\lambda(t) A(t)\, dt = a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda). \tag{5.4}$$

Proof. We shall first prove the theorem for the case when $\nu_0 + \rho^2 > 0$. Proof of the other case, i.e. the case when $\nu_0 + \rho^2 = 0$ will follow similarly.

So let us assume that $\nu_0 + \rho^2 > 0$. We first prove (1). Using Corollary 3.3 and equation (4.5) we get that

$$\begin{aligned} & \textstyle\sum_{m=0}^{\infty} \frac{|d_m|}{|c_m|} |a(i\sqrt{\nu_m + \rho^2})|\, |\Psi_m\left((t + is)\right)| \\ \leq \quad & C \textstyle\sum_{m=0}^{\infty} (\nu_m + \rho^2) Q(|\sqrt{\nu_m + \rho^2}|) e^{-p\sqrt{\nu_m+\rho^2}} e^{|s|\sqrt{\nu_m+\rho^2}} |P(t + is)| \\ < \quad & \infty \text{ if } |s| < p. \end{aligned}$$

Here we have used the fact that $c_m \to \infty$ as $m \to \infty$.

Now we shall prove (2). Let $R, \sigma > 0$. Let Γ_1 be the straight line joining $(-R, -\sigma)$ and $(R, -\sigma)$, Γ_2 be a straight line joining $(R, -\sigma)$ and (R, R), Γ_3 be a strainght line joining (R, R) and $(-R, R)$, Γ_4 be a strainght line joining $(-R, R)$ and $(-R, -\sigma)$. Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ be the rectangle with anticlockwise direction.

Let

$$\begin{aligned} I &= \int_\Gamma a(\lambda)b(\lambda)\varphi_\lambda(t)\frac{1}{c(\lambda)c(-\lambda)}\,d\lambda \\ &= \int_{\Gamma_1} a(\lambda)\varphi_\lambda(t)\frac{b(\lambda)}{c(\lambda)c(-\lambda)}\,d\lambda + \int_{\Gamma_2} a(\lambda)\varphi_\lambda(t)\frac{b(\lambda)}{c(\lambda)c(-\lambda)}\,d\lambda \\ &+ \int_{\Gamma_3} a(\lambda)\varphi_\lambda(t)\frac{b(\lambda)}{c(\lambda)c(-\lambda)}\,d\lambda + \int_{\Gamma_4} a(\lambda)\varphi_\lambda(t)\frac{b(\lambda)}{c(\lambda)c(-\lambda)}\,d\lambda. \end{aligned}$$

We claim that

$$\int_{\Gamma_i} a(\lambda)\varphi_\lambda(t)\frac{b(\lambda)}{c(\lambda)c(-\lambda)}\,d\lambda \to 0$$

as $R\to\infty$, for $i=2,3,4$. Suppose that the claim is true. We observe that the function

$$\frac{b(\lambda)}{c(\lambda)c(-\lambda)} = S_1(\lambda),$$

has simple poles at $\lambda = i\sqrt{\nu_m+\rho^2}$ for $m=0,1,2,\cdots$ in side the rectangle Γ. Therefore from Cauchy's theorem it follows that

$$\begin{aligned} \int_{-\infty-i\sigma}^{\infty-i\sigma} a(\lambda)b(\lambda)\varphi_\lambda(t)\frac{1}{c(\lambda)c(-\lambda)}\,d\lambda &= 2\pi i\sum_{m=0}^\infty a\left(i\sqrt{\nu_m+\rho^2}\right)\varphi_{i\sqrt{\nu_m+\rho^2}}(t)\ \mathrm{Res}_{\lambda=i\sqrt{\nu_m+\rho^2}}S_1(\lambda) \\ &= 2\pi i\sum_{m=0}^\infty \frac{d_m}{c_m}\, a\left(i\sqrt{\nu_m+\rho^2}\right)\Psi_m(it). \end{aligned}$$

We shall prove the above claim. Using (4.4) we have for $|t|<p$,

$$\begin{aligned} \int_{\Gamma_2}\left|a(\lambda)\varphi_\lambda(t)\frac{b(\lambda)}{c(\lambda)c(-\lambda)}\,d\lambda\right| &= \int_{s=-\sigma}^R |a(R+is)|\,|\varphi_{R+is}(t)|\,|S_1(R+is)|\,ds \\ &\le \int_{s=-\sigma}^R e^{-ps+AR}e^{|st|}e^{-\frac{\pi}{2}R}(R^2+s^2)\,ds \\ &= e^{(A-\pi/2)R}\int_{s=-\sigma}^R e^{-sp+|s|\,|t|}(R^2+s^2)\,ds \\ &\to 0 \text{ as } R\to\infty. \end{aligned}$$

The last line follows as $A<\pi/2$. On Γ_3 for $|t|<p$,

$$\begin{aligned} \int_{\Gamma_3}\left|a(\lambda)\varphi_\lambda(t)\frac{b(\lambda)}{c(\lambda)c(-\lambda)}\,d\lambda\right| &= \int_{s=-R}^R |a(s+iR)|\,|\varphi_{s+iR}(t)|\,|S_1(s+iR)|\,ds \\ &\le \int_{s=-R}^R e^{-pR+A|s|}e^{R|t|}e^{-\pi/2|s|}(R^2+s^2)\,ds \\ &= e^{-R(p-|t|)}\int_{s=-R}^R e^{(A-\pi/2)|s|}(R^2+s^2)\,ds \\ &\to 0 \text{ as } R\to\infty. \end{aligned}$$

For Γ_4, for $|t|<p$,

$$\begin{aligned} \int_{\Gamma_4}\left|a(\lambda)\varphi_\lambda(t)\frac{b(\lambda)}{c(\lambda)c(-\lambda)}\,d\lambda\right| &= \int_{s=-\sigma}^R |a(-R+is)|\,|\varphi_{-R+is}(t)|\,|S_1(-R+is)|\,ds \\ &\le \int_{s=-\sigma}^R e^{-ps+AR}e^{|st|}e^{-\pi/2R}(R^2+s^2)\,ds \\ &= e^{(A-\pi/2)R}\int_{s=-\sigma}^R e^{-sp+|s|\,|t|}(R^2+s^2)\,ds \\ &\to 0 \text{ as } R\to\infty. \end{aligned}$$

Using the same method we can show that the right hand side of (5.2) is independent of $0<\sigma<\delta$.

Now we consider

$$\begin{aligned} \int_{-\infty-i\sigma}^{\infty-i\sigma}\left|a(\lambda)b(\lambda)\varphi_\lambda(t+is)\frac{d\lambda}{c(\lambda)c(-\lambda)}\right| &= \int_{-\infty-i\sigma}^{\infty-i\sigma}|a(\lambda)\varphi_\lambda(t+is)S_1(\lambda)\,d\lambda| \\ &\le \int_{\mathbb{R}}\left|a(y-i\sigma)\varphi_{y-i\sigma}(t+is)S_1(y-i\sigma)(y^2+\sigma^2)\,dy\right| \\ &\le \int_{\mathbb{R}} e^{p\sigma+A|y|}e^{|ys-t\sigma|}e^{-\pi/2|y|}(y^2+\sigma^2)\,dy. \end{aligned}$$

This shows that the integral defined above is finite if $|s|<\pi/2-A$ and hence holomorphic when $|s|<\pi/2-A$.

We observe that the equation (5.2) is true for every $0 < \sigma < \delta$ and the right hand side of (5.2) is independent of σ. Hence using the fact that $c(-\lambda) = \overline{c(\lambda)}, \lambda \in \mathbb{R}$ we have

$$f(it) = \int_{\mathbb{R}} \left(a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)\right) \varphi_\lambda(t)|c(\lambda)|^{-2}\, d\lambda. \tag{5.5}$$

To prove (3) if we prove that the map

$$\lambda \mapsto a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda),$$

is in $\mathcal{S}_2(\mathbb{R})_e$, then using the inversion formula we will have (5.4). To show that the map

$$\lambda \mapsto a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda),$$

is in $\mathcal{S}_2(\mathbb{R})_e$, it is enough to show (using Cauchy's theorem) that the map is holomorphic on $\mathbb{R} + i[-\delta, \delta]$ and for each $N \in \mathbb{N}$,

$$\sup_{\lambda \in \mathbb{R}+i[-\delta,\delta]} (1 + |\lambda|)^N |a(\lambda)b(\lambda)| < \infty, \tag{5.6}$$

for some $\delta > 0$. We have

$$b(\lambda) = c(\lambda)c(-\lambda)S_1(\lambda).$$

From the definition of S_1 and Corollary 2.8 it follows that b has simple pole at $\lambda = 0$. Also we have $b(-\lambda) = -b(\lambda)$. Therefore the function

$$a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda) = b(\lambda)\left(a(\lambda) - a(-\lambda)\right)$$

is holomorphic around origin, as b has simple pole at $\lambda = 0$ and $a(\lambda) - a(-\lambda)$ has zero at $\lambda = 0$. Hence it follows from Theorem 2.7 that the map

$$\lambda \mapsto a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda)$$

is holomorphic on $\mathbb{R} + i[-\delta, \delta]$ for some $\delta < \frac{1}{2}$. Using Corollary 2.8 and (4.4) we get that

$$\sup_{\lambda \in \mathbb{R}+i[-\delta,\delta] \setminus \{\text{ a nbd. of origin}\}} (1 + |\lambda|)^N |a(\lambda)b(\lambda)| < \infty,$$

for some $\delta > 0$. This completes the proof. $\square$

Remark 5.2. We now consider the non-perturbed case, i.e. the case when $B(t) \equiv 1$. In this case $\tilde{A}(t) = (\sin t)^{2\alpha+1}(\cos t)^{2\beta+1}$ and $A(t) = (\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}$. The corresponding Sturm Liouville operators for A and $\tilde{A}$ are well studied on $\mathbb{R}^+$ and $(0, \pi/2)$ respectively. Indeed the full spectral decomposition of $\mathcal{L}$ and $-L$ is known. Let $P_n^{\alpha,\beta}$ be a Jacobi polynomial of order (α, β) of degree n.

The operator $-l$ can be explicitly written as

$$-l = -\frac{d^2}{dt^2} + (\alpha^2 - \frac{1}{4})\cot^2 t + (\beta^2 - \frac{1}{4})\tan^2 t.$$

It is known (see [20, p.67, sec4.24]) that $u_n(t) = \sqrt{\tilde{A}(t)}P_n^{\alpha,\beta}(\cos 2t)$ are the eigenfunctions of $-l$ with eigenvalues $\nu_n = (2n + \rho)^2 - \alpha^2 - \beta^2 + \frac{1}{2}$, where $\rho = \alpha + \beta + 1$ and $n \geq 0$. Therefore in this case the eigenfunction Ψ_n of $-L$ reduces to a normalising factor times the Jacobi polynomial $P_n^{\alpha,\beta}(\cos 2t)$ with eigenvalue $\nu_n = (2n + \rho)^2 - \alpha^2 - \beta^2 + \frac{1}{2}$. Hence the relation (1.3) becomes

$$P_n^{\alpha,\beta}(\cos 2t) = c_n' \varphi_{i\sqrt{\nu_n+\rho^2}}(it), \text{ on } (0, \frac{\pi}{2}), \tag{5.7}$$

with $\rho = \alpha + \beta + 1$. Then we define the function b as in (5.3) and state the Ramanujan master's theorem as in Theorem 5.1. We conclude here the function b doesnot come out to be exactly $P(\lambda)\left(\sin \frac{\pi}{2}(\lambda - \rho)\right)^{-1}$ (where P is a polynomial) as in [17, Theorem 5.1] (when restricted to rank one case) is because $\mathcal{L}$ differs from the Laplace Beltrami operator considered in [17] by a constant dependent on α, β times I (Identity operator), which makes $\nu_n + \rho^2$ a complete square for all n in their case.

6. The case when α or $\beta \leq 0$

Let us consider the case when α or $\beta \in (-\frac{1}{2}, 0]$. We know from Theorem 3.1 that the eigenvalues ν_n of $-L$ goes to $+\infty$ as $n \to \infty$. In the case of $\alpha, \beta > 0$, all the eigenvalues ν_n's are non-negative. But in general (i.e. for α or $\beta \leq 0$) all of the eigenvalues may not be non-negative. Let n_0 be the smallest non negative integer such that ν_m is nonnegative for all $m \geq n_0$. Let m_0 be the largest non negative integer such that $\nu_{m_0} + \rho^2 < 0$. It is easy to see that we can write $\sqrt{-(\nu_m + \rho^2)} = \beta_m$, $\beta_m > 0$ for $m \leq m_0$.

We define the following functions:

$$S(z) = \begin{cases} \pi z \prod_{n=n_0}^{\infty} \left(1 + \frac{z^2}{\nu_n + \rho^2}\right) & \text{if} \quad \nu_{n_0} + \rho^2 > 0, \\ \pi z^3 \prod_{n=n_0}^{\infty} \left(1 + \frac{z^2}{\nu_n + \rho^2}\right) & \text{if} \quad \nu_{n_0} + \rho^2 = 0. \end{cases} \tag{6.1}$$

and

$$S_1(z) = \frac{z^2}{S(z)}. \tag{6.2}$$

Then the conclusion of the Theorem 4.3 will remain same. Now we state Ramanujan's master theorem in this case as follows and its proof is similar to the proof of Theorem 5.1.

Theorem 6.1. *Let α or $\beta \in (-\frac{1}{2}, 0]$ and let $a \in \mathcal{H}(A, p, \delta)$. Then the following holds:*

(1) *The Fourier series*

$$f(t) = 2\pi i \sum_{m=0}^{\infty} \frac{d_m}{c_m} a(i\sqrt{\nu_m + \rho^2})\Psi_m(t), \tag{6.3}$$

converges uniformly on compact subsets of $\Omega_p := \{t \in \mathbb{C} \mid |\Re t| < \frac{\pi}{2}, |\Im t| < p\}$ and hence holomorphic there.

(2) *Let $0 \leq \sigma < \delta$. Then the function f can also be expressed in the integral form for $t \in \mathbb{R}$ with $|t| < p$ as*

$$\begin{aligned} f(it) = &\int_{-\infty - i\sigma}^{\infty - i\sigma} (a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda))\, \varphi_\lambda(t)\, \frac{d\lambda}{c(\lambda)c(-\lambda)} \\ &+ 2\pi i \sum_{m=m_0+1}^{n_0-1} \frac{d_m}{c_m} a(i\sqrt{\nu_m + \rho^2})\Psi_m(it) + 2\pi i \sum_{m=0}^{m_0} \frac{d_m}{c_m} a(\beta_m)\Psi_m(it), \end{aligned}$$

where the function b is defined by

$$\frac{b(\lambda)}{c(\lambda)c(-\lambda)} = S_1(\lambda). \tag{6.4}$$

The integrals defined above are independent of σ and extends as a holomorphic function to a neighbourhood $\{z \in \mathbb{C} \mid |\Re z| < \frac{\pi}{2} - A\}$ of $i\mathbb{R}$.

(3) *The extension of f to $i\mathbb{R}$ satisfies, for all $\lambda \in \mathbb{R}$*

$$\begin{aligned} \int_{\mathbb{R}} \left(f(it) - 2\pi i \sum_{m=m_0+1}^{n_0-1} \frac{d_m}{c_m} a(i\sqrt{\nu_m + \rho^2})\Psi_m(it) - 2\pi i \sum_{m=0}^{m_0} \frac{d_m}{c_m} a(\beta_m)\Psi_m(it) \right) \varphi_\lambda(t) A(t)\, dt \\ = a(\lambda)b(\lambda) + a(-\lambda)b(-\lambda). \end{aligned}$$

7. Appendix

Proof of Theorem 2.5: In this subsection we prove Theorem 2.5. The proof is similar to [4, Theorem 1.2]. To prove the theorem we need the following preliminaries:

Let J_α, Y_α be the Bessel functions of first and second kind respectively. Also let $H^{(1)}_\alpha$ and $H^{(2)}_\alpha$ be the Hankel functions defined by

$$H^{(1)}_\alpha(x) = J_\alpha(x) + iY_\alpha(x), H^{(2)}_\alpha(x) = J_\alpha(x) - iY_\alpha(x).$$

Also let

$$\mathcal{J}_\alpha(x) = \sqrt{x}J_\alpha(x), \mathcal{Y}_\alpha(x) = \sqrt{x}Y_\alpha(x), \mathcal{H}^{(1)}_\alpha(x) = \sqrt{x}H^{(1)}_\alpha(x), \mathcal{H}^{(2)}_\alpha(x) = \sqrt{x}H^{(2)}_\alpha(x). \tag{7.1}$$

Let $\Omega = \{t + is \mid |s| < \pi/2\}$. Also let

$$\langle x\rangle = \frac{|x|}{1+|x|}.$$

To prove the Theorem 2.5, the following two theorems are needed:

Theorem 7.1. ([18, Theorem 10.1, p. 219]) *Let $\alpha, \beta \in \mathbb{C}$ and let $\mathcal{P}$ be a finite chain of R_2 arcs in complex plane joining α and β. Let*

$$h(\xi) = \int_\alpha^\xi K(\xi, v)\left(\psi_0(v)h(v) + \varphi(v)J(v)\right) dv, \tag{7.2}$$

where

(1) *the path of integration lies on $\mathcal{P}$,*
(2) *the real/complex valued functions $J(v), \varphi(v), \psi_0(v)$ are continuous except for a finite number of discontinuity,*
(3) *the real/complex valued kernel $K(\xi, v)$ and its first two partial ξ derivatives are continuous function on two variables $\xi, v \in \mathcal{P}$, (here all differentiation with respect to ξ are perfomed along $\mathcal{P}$).*
(4) *The kernel satisfies*
    (a) $K(\xi, \xi) = 0$,
    (b) $|K(\xi, v)| \le p_0(\xi)q(v)$,
    (c) $\left|\frac{\partial K}{\partial \xi}(\xi, v)\right| \le p_1(\xi)q(v)$,
    (d) $\left|\frac{\partial^2 K}{\partial \xi^2}(\xi, v)\right| \le p_2(\xi)q(v)$,

    for all $\xi \in \mathcal{P}$ and $v \in (\alpha, \xi)_{\mathcal{P}}$, for some continuous functions p_0, p_1, p_2, q. Here $(\alpha, \xi)_{\mathcal{P}}$ denotes the part of $\mathcal{P}$ lying between α and ξ.
(5) *Also let the functions*

$$\Phi(\xi) = \int_\alpha^\xi |\varphi(v)\, dv|, \Psi_0(\xi) = \int_\alpha^\xi |\psi_0(v)\, dv|$$

    converges.
(6) *Let*

$$\kappa = \sup_{\xi\in\mathcal{P}}\{q(\xi)|J(\xi)|\} \text{ and } \kappa_0 = \sup_{\xi\in\mathcal{P}}\{p_0(\xi)q(\xi)\}$$

    are finite.

Then the integral equation (7.2) has a unique solution h which is continuously differentiable in $\mathcal{P}$ and satisfies

$$\frac{|h(\xi)|}{p_0(\xi)}, \frac{|h'(\xi)|}{p_1(\xi)} \le \kappa\Phi(\xi)\exp\left(\kappa_0\Psi_0(\xi)\right). \tag{7.3}$$

Theorem 7.2. ([4, Appendix, Lemma A.1]) *Let u_1 and u_2 be two linearly independent solutions of the equation*

$$u'' + p_1 u' + p_2 u = 0,$$

on $\mathbb{R}$. If φ is a C^2 solution of the integral equation

$$u(t) = -\int_{t_0}^{t} \frac{u_1(t)u_2(s) - u_2(t)u_1(s)}{u_1(s)u_2'(s) - u_2(s)u_1'(s)} \left(\psi_0(s)u(s) + J(s)\phi(s)\right) ds$$

then φ is a solution of

$$u'' + p_1 u' + p_2 u = \psi_0 u + J\phi.$$

Proof of Theorem 2.5. We first assume that $\lambda \neq 0$. From the asymptotic expansion of φ_λ (see [4, p. 219]) we have

$$\varphi_\lambda(t) = \sum_{m=0}^{M} a_m(t) \frac{\mathcal{J}_{\alpha+m}(\lambda t)}{\sqrt{A(t)}\lambda^{m+\alpha+\frac{1}{2}}} + \frac{R_M(\lambda, t)}{\sqrt{A(t)}}, \tag{7.4}$$

where a_m are holomorphic and $M \geq 0$.

The function $R_M(\lambda, t)$ satisfies (see [4, (1.3)])

$$\frac{d^2 R_M}{dt^2} + \left(\lambda^2 - \frac{\alpha^2 - \frac{1}{4}}{t^2}\right) R_M = G(t)R_M + 2a'_{M+1}(t)\frac{\mathcal{J}_{\alpha+M}(\lambda t)}{\lambda^{M+\alpha+\frac{1}{2}}}. \tag{7.5}$$

Let $\Omega_1 = \Omega \backslash (-\infty, 0]$. From Theorem 7.2, it follows that a solution of the following integral equation

$$R_M(\lambda, t) = -\pi \int_0^t \frac{\mathcal{J}_\alpha(\lambda t)\mathcal{Y}_\alpha(\lambda s) - \mathcal{J}_\alpha(\lambda s)\mathcal{Y}_\alpha(\lambda t)}{2\lambda} \left(G(s)R_M(\lambda, s) + 2a'_{M+1}(s)\frac{\mathcal{J}_{\alpha+M}(\lambda s)}{\lambda^{M+\alpha+\frac{1}{2}}}\right) ds \tag{7.6}$$

also satisfies (7.5). As shown in [4, p. 222] $R_M(\lambda, t)$ is the solution of the above integral equation (7.6), which satisfies the required Cauchy Condition. Let $t_0 \in (0, \infty)$. Then

$$R_M(\lambda, t) = -\pi \int_{t_0}^t \frac{\mathcal{J}_\alpha(\lambda t)\mathcal{Y}_\alpha(\lambda s) - \mathcal{J}_\alpha(\lambda s)\mathcal{Y}_\alpha(\lambda t)}{2\lambda} \left(G(s)R_M(\lambda, s) + 2a'_{M+1}(s)\frac{\mathcal{J}_{\alpha+M}(\lambda s)}{\lambda^{M+\alpha+\frac{1}{2}}}\right) ds + R_M(\lambda, t_0).$$

Since both side of the equation above is holomorphic in Ω_1, we have for all $\xi \in \Omega_1$,

$$R_M(\lambda, \xi) = -\pi \int_{t_0}^\xi \frac{\mathcal{J}_\alpha(\lambda \xi)\mathcal{Y}_\alpha(\lambda s) - \mathcal{J}_\alpha(\lambda s)\mathcal{Y}_\alpha(\lambda \xi)}{2\lambda} \left(G(s)R_M(\lambda, s) + 2a'_{M+1}(s)\frac{\mathcal{J}_{\alpha+M}(\lambda s)}{\lambda^{M+\alpha+\frac{1}{2}}}\right) ds + R_M(\lambda, t_0).$$

Let

$$\begin{aligned} K(\xi, s) &= -\pi \tfrac{\mathcal{J}_\alpha(\lambda\xi)\mathcal{Y}_\alpha(\lambda s) - \mathcal{J}_\alpha(\lambda s)\mathcal{Y}_\alpha(\lambda\xi)}{2\lambda} \\ &= -\pi i \tfrac{\mathcal{H}^{(1)}_\alpha(\lambda\xi)\mathcal{H}^{(2)}_\alpha(\lambda s) - \mathcal{H}^{(1)}_\alpha(\lambda s)\mathcal{H}^{(2)}_\alpha(\lambda\xi)}{4\lambda}. \end{aligned}$$

Using the estimates of Bessel and Hankel functions (as in [4]) we get,

$$|K(\xi, s)| \leq \begin{cases} \frac{C}{|\lambda|}\langle\lambda\xi\rangle^{|\alpha|+\frac{1}{2}}\langle\lambda s\rangle^{-|\alpha|+\frac{1}{2}} e^{|\Im(\lambda\xi) - \Im(\lambda s)|} & \text{for} \quad \alpha \neq 0, \\ \frac{C}{|\lambda|}\langle\lambda\xi\rangle^{\frac{1}{2}}\langle\lambda s\rangle^{\frac{1}{2}} \log(\frac{2}{\langle\lambda s\rangle}) e^{|\Im(\lambda\xi) - \Im(\lambda s)|} & \text{for} \quad \alpha = 0. \end{cases}$$

Also we have

$$\left|\frac{\partial}{\partial\xi}K(\xi, s)\right| \leq \begin{cases} C\langle\lambda\xi\rangle^{|\alpha|-\frac{1}{2}}\langle\lambda s\rangle^{-|\alpha|+\frac{1}{2}} e^{|\Im(\lambda\xi) - \Im(\lambda s)|} & \text{for} \quad \alpha \neq 0, \\ C\langle\lambda\xi\rangle^{-\frac{1}{2}}\langle\lambda s\rangle^{\frac{1}{2}} \log(\frac{2}{\langle\lambda s\rangle}) e^{|\Im(\lambda\xi) - \Im(\lambda s)|} & \text{for} \quad \alpha = 0. \end{cases}$$

For $\Im(\lambda\xi) > \Im(\lambda s) > 0$, we let

(1) $p_0(\xi) = \frac{C}{|\lambda|}\langle\lambda\xi\rangle^{|\alpha|+\frac{1}{2}}e^{\Im(\lambda\xi)}$,

(2) $q(s) = \begin{cases} \langle\lambda s\rangle^{-|\alpha|+\frac{1}{2}}e^{-\Im(\lambda s)} & \text{for } \alpha \neq 0 \\ \\ \langle\lambda s\rangle^{\frac{1}{2}}\log(\frac{2}{\langle\lambda s\rangle})e^{-\Im(\lambda s)} & \text{for } \alpha = 0 \end{cases}$

(3) $p_1(\xi) = \langle\lambda\xi\rangle^{|\alpha|-\frac{1}{2}}e^{\Im(\lambda\xi)}$,

(4) $\psi_0(s) = G(s)$, $J(s) = \frac{1}{q(s)}$ and $\varphi(s) = 2a'_{M+1}(s)\frac{\mathcal{J}_{\alpha+M}(\lambda s)}{\lambda^{M+\alpha+\frac{1}{2}}}q(s)$.

Therefore

(1) $\kappa_0 := \sup\{p_0(\xi)q(\xi)\} = \frac{C}{|\lambda|}$,

(2) $\kappa := \sup\{q(\xi)J(\xi)\} = 1$.

We have

$$|\varphi(s)| \leq \begin{cases} C|a'_{M+1}(s)|\,|\lambda|^{-M-\alpha-\frac{1}{2}}\langle\lambda s\rangle^{\alpha-|\alpha|+M+1} & \text{for} \quad \alpha \neq 0 \\ \\ C|a'_{M+1}(s)|\,|\lambda|^{-M-\frac{1}{2}}\langle\lambda s\rangle^{M+1}\log(\frac{2}{\langle\lambda s\rangle}) & \text{for} \quad \alpha = 0 \end{cases}$$

We know that $|a'_{M+1}(s)| \leq Cs^M$ for $s < 1$ and $a'_{M+1} \in L^1((1,\infty))$. Therefore, we have

$$|\Phi(\xi)| \leq \begin{cases} C|\lambda|^{-M-\alpha-\frac{1}{2}}\langle\lambda\xi\rangle^{\alpha-|\alpha|+M+1}\langle\xi\rangle^M & \text{for} \quad \alpha \neq 0 \\ \\ C|\lambda|^{-M-\frac{1}{2}}\langle\lambda\xi\rangle^{M+1}\log(\frac{2}{\langle\lambda\xi\rangle})\langle\xi\rangle^M & \text{for} \quad \alpha = 0 \end{cases}$$

Hence by Theorem 7.1 we have

$$\begin{aligned} |R_M(\lambda,\xi)| &\leq C|\lambda|^{-M-\alpha-\frac{3}{2}}\langle\lambda\xi\rangle^{M+\alpha+\frac{3}{2}}e^{\Im(\lambda\xi)}\exp\{\left(\tfrac{C}{|\lambda|}\left|\textstyle\int_0^\xi G(s)\,ds\right|\right)\} \\ &\leq C|P(\xi)|\,e^{\Im(\lambda\xi)}\exp\left(\tfrac{C}{|\lambda|}\left|\textstyle\int_0^\xi G(s)\,ds\right|\right), \end{aligned}$$

for some polynomial P for the case $\alpha \neq 0$. Then as in the argument [4, Remark 1.3] we can improve the inequality above as

$$|R_M(\lambda,\xi)| \leq C|P(\xi)|\,e^{\Im(\lambda\xi)}\exp\left(\frac{C|\xi|}{1+|\lambda\xi|}\left|\int_0^\xi G(s)\,ds\right|\right), \tag{7.7}$$

Similar estimate also holds for $\alpha = 0$. Therefore if G is in L^1 in every direction, it follows that for all $\xi \in \Omega_1$

$$|R_M(\lambda,\xi)| \leq C|P(\xi)|\,e^{\Im(\lambda\xi)}.$$

We can do the similar technique to the domain $\Omega_2 = \Omega \setminus [0,\infty)$ and get the similar estimate for the domain Ω_2. Hence we have from (7.4) that

$$|\varphi_\lambda(\xi)| \leq C\,|P(\xi)|\,e^{|\Im(\lambda\xi)|},$$

for all $\xi \in \Omega$. □

Holomorphic ODE: We state the following theorem about the holomorphicity of solutions of a differential equation ([14, Theorem 1.4]). This is used in the proof of the Lemma 2.4.

Theorem 7.3. *Let Ω be a simply connected region in $\mathbb{C}$ and $z_0 \in \mathbb{C}$. Also let $a_1, a_2, \cdots, a_n$ be holomorphic functions on Ω. For any complex numbers $y_0, y_1, \cdots, y_n$, there exists a unique holomorphic function y on Ω such that*

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0, \tag{7.8}$$

and

$$y(z_0) = y_0, y^{(1)}(z_0) = y_1, \cdots, y^{(n-1)}(z_0) = y_{n-1}. \tag{7.9}$$

Singular Sturm Liouville operator: In this subsection we give few well known preliminaries of Sturm-Liouville's operator.

Let us define a singular Sturm Liouville operator on an interval $I = (a, b)$ by

$$My := \frac{1}{w} \left(-(py')' + qy \right), \tag{7.10}$$

where $p, q, w : I \to \mathbb{R}$, $p, w > 0$ a.e. and $\frac{1}{p}, q, w \in L^1_{\text{loc}}(I)$. The operator M is called *non-oscillatory* at a if there exists a solution $My = \lambda y$ such that $y \neq 0$ in the $(a, a + \delta)$ for some $\delta > 0$ and some $\lambda \in \mathbb{R}$. Similar definition is for the other end point b.

For non oscillatory end points, Niessen and Zettl (in [15]) have completely characterised all the self adjoint extensions of the Sturm Liouville operator M on $L^2((a, b), w(t)dt)$ with explicit boundary conditions at a and b. In [15, Theorem 4.2], Niessen and Zettl obtain a Friedrich's extension of a Sturm Liouville operator M as defined in 7.10 on (a, b) by defining the boundary conditions in terms of the principal solution at both end points. More precisely if u_a and u_b are any two principal solutions at a and b respectively, satisfying $Mu_j = \lambda_j u_j$ for $j = a, b$, M is self adjoint extension on $L^2((a, b), w(t)dt)$ defined on the domain

$$\mathcal{D} = \{ y \in \mathcal{D}(M) : [y, u_a]_M (a) = [y, u_b]_M (b) = 0 \}$$

The domain is independent of u_j and $\lambda_j, j = a, b$.

We say u_a is a *principal solution* at a if u_a is non zero in a right neighbourhood of a and for any other solution y of $My = \lambda_a y$ on (a, b), $u_a(t) = o(y(t))$ as $t \to a^+$. It is known that a principle solution at a of the equation $My = \lambda_a y$ is unique upto multiplicative constant. When M is non-oscillatory at a and b, principal solutions do exist at a and b respectively.

If M is a limit point case at a i.e. only one solution of $Mu = \lambda u$ lies in $L^2(a, a + \epsilon)$ for some $\epsilon > 0$ then we don't require any boundary condition at a. This classification is independent of λ. For further details see [15].

References

[1] Ben Said, Salem; Boussen, Asma; Sifi, Mohamed *L^p-Fourier analysis for certain differential-reflection operators.* Adv. Pure Appl. Math. 8 (2017), no. 1, 43–63.

[2] Bertram, Wolfgang *Ramanujan's master theorem and duality of symmetric spaces.* J. Funct. Anal. 148 (1997), no. 1, 117–151.

[3] Bloom, Walter R.; Xu, Zeng Fu *The Hardy-Littlewood maximal function for Chbli-Trimche hypergroups.* Applications of hypergroups and related measure algebras, 45–70, Contemp. Math., 183, Amer. Math. Soc., Providence, RI, 1995.

[4] Brandolini, Luca; Gigante, Giacomo *Equiconvergence theorems for Chébli-Trimèche hypergroups.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8 (2009), no. 2, 211265.

[5] Chebli, Houcine *Sur un théorème de Paley-Wiener associé à la décomposition spectrale d'un opérateur de Sturm-Liouville sur* $]0, \infty[$. J. Functional Analysis 17 (1974), 447–461.

[6] Dunford, N.; Schwartz, J. *Linear Operators Part II Spectral Thoery* Wiley New York, 1963.

[7] Fitouhi, A.; Hamza, M. M. *A uniform expansion for the eigenfunction of a singular second-order differential operator.* SIAM J. Math. Anal. 21 (1990), no. 6, 1619-1632.

[8] Gigante, G.; Jotsaroop, K. *Equiconvergence of perturbed Jacobi expansions* Preprint.

[9] Gilbert, John, E. *Maximal Theorems for some Orthogonal Series II* Journal of Math. Analysis and Applications 31, 349-368 (1970).

[10] Hardy, G. H. *Ramanujan: Twelve Lectures on subjects suggested by his life and work.* Chelsea Publishing, New York (1959).

[11] Helgason, Sigurdur *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions.* Mathematical Surveys and Monographs, 83. American Mathematical Society, Providence, RI, 2000.

[12] Koornwinder, T. H. *Jacobi functions and analysis on noncompact semisimple Lie groups.* Special functions: group theoretical aspects and applications, 1–85, Math. Appl., Reidel, Dordrecht, 1984.

[13] Levin, B. Ja.; Ostrovskiĭ, Ĭ. V. *Small perturbations of the set of roots of sine-type functions.* Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 1, 87–110, 238.

[14] Miličić, Dragan *Lectures on differential equations on complex domain*, https://www.math.utah.edu/ milicic/Eprints/de.pdf

[15] Niessen, H.-D.; Zettl, A. *Singular Sturm-Liouville problems: the Friedrichs extension and comparison of eigenvalues.* Proc. London Math. Soc. (3) 64 (1992), no. 3, 545–578.

[16] Ólafsson, Gestur; Pasquale, Angela *Ramanujan's master theorem for Riemannian symmetric spaces.* J. Funct. Anal. 262 (2012), no. 11, 4851–4890.

[17] Ólafsson, G.; Pasquale, A. *Ramanujan's master theorem for the hypergeometric Fourier transform associated with root systems.* J. Fourier Anal. Appl. 19 (2013), no. 6, 1150–1183.

[18] Olver, Frank W. J. *Asymptotics and special functions.* AKP Classics. A K Peters, Ltd., Wellesley, MA, 1997.

[19] Pusti, Sanjoy; Ray Swagato K. *Ramanujan's master theorem for radial sections of line bundle over noncompact symmetric spaces* arXiv:1808.10165

[20] Szegő, Gábor *Orthogonal polynomials.* Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975.

[21] Freiling, G. Yurko, V.*Boundary Value Problems with Singular Boundary conditions* International Journal of Mathematics and Mathematical Sciences, Vol. 2005, Issue 9, 1481-1495

Department of Mathematics, Indian Institute of Science Education and Reseach, Mohali, India.
E-mail address: `jotsaroop@iisermohali.ac.in`

Department of Mathematics, Indian Institute of Technology, Bombay, India.
E-mail address: `sanjoy@math.iitb.ac.in`