# Solving Virasoro Constraints on Integrable Hierarchies via the Kontsevich-Miwa Transform

A. M. Semikhatov

Theory Division, P. N. Lebedev Physics Institute 53 Leninsky prosp., Moscow 117924, Russia

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#### Abstract

We solve Virasoro constraints on the KP hierarchy in terms of minimal conformal models. The constraints we start with are implemented by the Virasoro generators depending on a background charge Q. Then the solutions to the constraints are given by the theory which has the same field content as the David-Distler-Kawai theory: it consists of a minimal matter scalar with background charge Q, dressed with an extra 'Liouville' scalar. In particular, the Virasoro-constrained tau function is related to the correlator of a product of (dressed) '21' operators. The construction is based on a generalization of the Kontsevich parametrization of the KP times achieved by introducing into it Miwa parameters which depend on the value of Q. Under the thus defined Kontsevich-Miwa transformation, the Virasoro constraints are proven to be equivalent to a master equation depending on the parameter Q. The master equation is further identified with a nullvector decoupling equation. We conjecture that  $W^{(n)}$  constraints on the KP hierarchy are similarly related to a level-n decoupling equation. We also consider the master equation for the N-reduced KP hierarchies. Several comments are made on a possible relation of the generalized master equation to scaled Kontsevich-type matrix integrals and on the form the equation takes in higher genera.

#### 1 Introduction and Discussion

The claim of the Matrix Models approach [1, 2, 3] is a non-perturbative description of twodimensional gravity (and gravity-coupled matter). The main computational tool is provided in applications [4, 5, 6] by the theory of integrable hierarchies subjected to the Virasoro constraints [7, 8, 4, 9, 10]. The relevance of the Virasoro-constrained hierarchies to the intersection theory on the moduli space of curves has also been proven in [11, 12, 13]. The Virasoro constraints thus constitute a fundamental notion of the theory and are the heart of matrix models' applications to both gravity-coupled theories and the intersection theory<sup>1</sup>. A major task is to find their general solution (see, for instance, [14]).

On the other hand, a challenging problem remains of giving a direct proof of the equivalence between the 'hierarchical' formalism and the conformal field theory description of quantum gravity [15, 16, 17]. As a 'direct proof' one would like to have something more than just the circumstantial evidence. It may seem discouraging that assuming an equivalence between the conformal-theory description of quantum gravity and the theory of (appropriately constrained) integrable hierarchies, one then has to believe that certain ingredients of conformal field theory satisfy integrable equations, while these seem to be a long way from the equations which are known to hold for conformal field theory correlators [18, 19, 20].

We will show in this paper that Virasoro constraints on the KP hierarchy are *solved* by minimal models, by virtue of the equations [18, 19] satisfied by the corelators.

As has been understood for quite some time, the only viable candidate for a 'space-time' for the conformal theory underlying integrable hierarchies to live on, could be the spectral curve associated to the hierarchy. Yet the attempts to actually build up such a theory were hindered by a problem that remained: the infinite collection of time parameters inherent to integrable hierarchies are hard to deal with within the standard conformal techniques.

Fortunately, there does exist a formalism in which the time variables are treated, in a sense, on equal footing with the spectral parameter. This is the Miwa transform used in the KP hierarchy [21, 22]. What is more, a similar construction has been introduced by Kontsevich [12] in his matrix model<sup>2</sup>. (More recently, it has been used in [13, 23, 24] in relation to the Virasoro constraints on integrable hierarchies, although in a context different from the one considered in this paper.)

It turns out that in order to relate the Virasoro constraints on the KP hierarchy to certain conformal field theory data, one needs to introduce additional parameters into the Kontsevich parametrization of the KP times. That is, the Kontsevich parametrization can be viewed as a special case of Miwa's, and the extra 'degrees of freedom' present in the Miwa transform should not be completely frozen: by varying them one moves between different (generalised) Kontsevich transformations. We will see that different Kontsevich transformations should be

<sup>&</sup>lt;sup>1</sup>The case studied in most detail is the Virasoro-constrained KdV hierarchy whose relation to the intersection theory on moduli space of Riemann surfaces has been discussed in [11].

<sup>&</sup>lt;sup>2</sup>The Kontsevich matrix model is important by itself – it provides a combinatorial model of the universal moduli space [12] and, as such, serves as an important step in demonstrating the KdV hierarchy in the intersection theory on the moduli space – and it also provides a model of quantum gravity [13]. It is not, however, of the form of the matrix models considered previously, which raises the question of its equivalence to one of the "standard" models. The crucial point in studying this equivalence is, again, the proof of the Virasoro constraints satisfied by the Kontsevich matrix integral [13, 23, 38]. Once the constraints are established, one is left with "only" the proof that they specify the model uniquely.

used depending on the operators under consideration. One thus gets a 'Miwa-parametrized set' of Kontsevich transformations, which is referred to below as the Kontsevich-Miwa transform.

For each of the generalised Kontsevich transformations, pulling back the Virasoro constraints results in relations, analogous to the "master equation" of ref.[25] (see also ref.[23]), which are satisfied [26] by correlation functions of an 'auxiliary' conformal field theory provided it contains a null-vector [18]. This conformal field theory therefore gives a solution to the Virasoro-constrained hierarchy. It consists of a 'matter' scalar  $\varphi$  with the energy-momentum tensor

$$T_{\rm m} = -\frac{1}{2}\partial\varphi\partial\varphi + \frac{i}{2}Q_{\partial}^{2}\varphi,$$

and an extra scalar  $\phi$ . The background charge Q enters in the Virasoro generators on the KP hierarchy. In order for the Kontsevich-Miwa transform of the Virasoro constraints to exist, the background charge must be related to the Miwa parameter  $n_i$  via

$$Q = \frac{1}{n_i} - 2n_i,$$

which allows us to interprete  $n_i$  as half the cosmological constant. When this condition is satisfied, the Virasoro constraints map under the Kontsevich-Miwa transform into the decoupling equation [18] for the level-2 null vector. The application of the classical technique of [18] to Virasoro-constrained hierarchies thus allows us to relate the Virasoro-constrained KP hierarchy to the formalism of refs. [16, 17].

The relation which we find, between Virasoro constraints and null-vector decoupling equations is very instructive from the point of view of string field theory. The decoupling equations acquire the rôle of the sought string field theory equations, and therefore at least for the matter central charge d < 1, "conformal models provide classical solutions to the string field theory" inasmuch as the corresponding correlators satisfy the decoupling equations.

Another implication of the identification of the Virasoro constraints with the decoupling equation has to do with recursion relations in topological theories. Recall that the recursion relations are essentially the Virasoro (or higher W-) constraints. Therefore the particular form, obtained below, of the decoupling equations serves as a generating function for the recursion relations. Reversing the argument, it is amusing to know that certain conformal field theory correlators are in fact Virasoro-constrained tau functions.

Also deserves mentioning the relation between the master equation and higher-genus Riemann surfaces. We will show that the master equation, considered on a coordinate patch of a Riemann surface, can be extended to the whole of the Riemann surface as an equation for a certain 'constituent' of the tau function.

There are several important issues which should be further elucidated. The first one is the relation of Virasoro-constrained tau functions to Kontsevich-type matrix integrals. The cases considered in the literature [12, 25, 37] seem to apply only when  $\alpha^2 = 2$ , while we would like to have matrix integrals that give rise to more general master equations. Moreover, there is an evidence that such matrix integrals pertain to the Toda lattice hierarchy, which is a 'discrete' hierarchy, and that the master equations we are considering in this paper, follow only after a certain scaling limit. This scaling limit can be viewed as an adaptation for the spectral parameter of the scaling [28] of the Virasoro-constrained hierarchies Toda $\rightarrow$ KP. Second, a possible relevance of higher Virasoro null vectors to the Virasoro-constrained KP hierarchy

points to a relation between Virasoro null vectors and W algebras, the fact observed in a different approach [39]. We conjecture that  $W^{(n)}$  constraints on the KP hierarchy give rise to a level-n decoupling equation.

Further, the KP hierarchy can be reduced to higher generalized 'KdV' hierarchies [27, 30]. In this paper we will consider only the series associated to the sl(N) Kac-Moody algebras (which correspond to the A-series minimal models [31]), and we will call these the N-KdV hierarchies<sup>3</sup>. Although neither the interpretation of Virasoro-constrained N-KdV hierarchies in terms of moduli spaces, nor the corresponding Kontsevich-type matrix integrals are known, we will show that the 'master equation' can be naturally extended to this case as well.

We thus begin in Sect. 2 with fixing our notations and recalling some basic facts about the Virasoro action on the KP hierarchy. In Sect. 3 we introduce the Kontsevich-Miwa transform and use it in order to recast the Virasoro constraints into the "master equation". The inverse transform, from the master equation to the Virasoro constraints, is also proven here. Further, to give the master equation a conformal field-theoretic interpretation, we recall in Sect. 4 the necessary elementary formulae pertaining to the decopupling of null vectors in conformal models. After that, we establish in Sect. 5 a relation between Virasoro-constrained tau functions evaluated at different values of the Miwa parameters, and the conformal field correlators. In Sect. 6 we suggest a version of the master equation for the N-reduced (generalized KdV) hierarchies. Sect. 7 presents a preliminary discussion of the relation of our formalism to matrix integrals and the scaling in the Kontsevich parametrization, as well as a possible rôle of higher null vectors <sup>4</sup>. We also show how the master equation can be given meaning on higher-genus Riemann surfaces.

# 2 Virasoro action on the KP hierarchy

#### 2.1

The KP hierarchy is described in terms of  $\psi$ Diff operators [27] as an infinite set of mutually commuting evolution equations

$$\frac{\partial K}{\partial t_r} = -(KD^r K^{-1})_- K, \quad r \ge 1 \tag{2.1}$$

on the coefficients  $w_n(x, t_1, t_2, t_3, \ldots)$  of a  $\psi$ Diff operator K of the form (with  $D = \partial/\partial x$ )

$$K = 1 + \sum_{n \ge 1} w_n D^{-n} \tag{2.2}$$

The wave function and the adjoint wave function are defined by

$$\psi(t,z) = Ke^{\xi(t,z)}, \quad \psi^*(t,z) = K^{*-1}e^{-\xi(t,z)}, \quad \xi(t,z) = \sum_{r \ge 1} t_r z^r$$
(2.3)

 $<sup>^{3}</sup>$ Virasoro constraints on the N-KdV hierarchies admit a unified treatment, which is in turn a specialization of a general construction applicable to hierarchies of the r-matrix type [32].

<sup>&</sup>lt;sup>4</sup>It has been shown recently [41] that the level-3 decoupling give rise to  $W^{(3)}$  constraints on the tau function.

where  $K^*$  is the formal adjoint of K. The wave functions are related to the tau function via

$$\psi(t,z) = e^{\xi(t,z)} \frac{\tau(t-[z^{-1}])}{\tau(t)}, \quad \psi^*(t,z) = e^{-\xi(t,z)} \frac{\tau(t+[z^{-1}])}{\tau(t)}$$
(2.4)

where  $t \pm [z^{-1}] = (t_1 \pm z^{-1}, t_2 \pm \frac{1}{2}z^{-2}, t_3 \pm \frac{1}{3}z^{-3}, \ldots).$ 

#### 2.2

Now we introduce a Virasoro action on the tau function  $\tau(t)$ : The Virasoro generators read,

$$\mathsf{L}_{p>0} = \frac{1}{2} \sum_{k=1}^{p-1} \frac{\partial^{2}}{\partial t_{p-k} \partial t_{k}} + \sum_{k \geq 1} k t_{k} \frac{\partial}{\partial t_{p+k}} + (a_{0} + (J - \frac{1}{2})p) \frac{\partial}{\partial t_{p}}$$

$$\mathsf{L}_{0} = \sum_{k \geq 1} k t_{k} \frac{\partial}{\partial t_{k}} + \frac{1}{2} a_{0}^{2} - \frac{1}{2} \left(J - \frac{1}{2}\right)^{2}$$

$$\mathsf{L}_{p<0} = \sum_{k \geq 1} (k - p) t_{k-p} \frac{\partial}{\partial t_{k}} + \frac{1}{2} \sum_{k=1}^{-p-1} k (-p - k) t_{k} t_{-p-k} + (a_{0} + (J - \frac{1}{2})p) (-p) t_{-p}$$
(2.5)

They satisfy the algebra

$$[\mathsf{L}_p, \mathsf{L}_q] = (p-q)\mathsf{L}_{p+q} - \delta_{p+q,0}(p^3 - p)(J^2 - J + \frac{1}{6}) \tag{2.6}$$

which shows, in particular, the role played by the parameter J. Introducing the 'energy-momentum tensor'

$$\mathsf{T}(u) = \sum_{p \in \mathbb{Z}} u^{-p-2} \mathsf{L}_p \tag{2.7}$$

we can deform the tau function as

$$\tau(t) \mapsto \tau(t) + \delta \tau(t) = \tau(t) + \mathsf{T}(u)\tau(t) \tag{2.8}$$

This action can be translated into the space of dressing operators K. The result is [34] that K gets deformed by means of a left multiplication,

$$\delta K = -\mathfrak{T}(u)K,\tag{2.9}$$

where  $\mathfrak{T}(u)$  is the energy-momentum tensor in the guise of a pseudodifferential operator<sup>5</sup>

$$\mathfrak{T}(u) = (1 - J)\frac{\partial \psi(t, u)}{\partial u} \circ D^{-1} \circ \psi^*(t, u) - J\psi(t, u) \circ D^{-1} \circ \frac{\partial \psi^*(t, u)}{\partial u}$$
(2.10)

Thus,  $\mathfrak{T}(u)$  reproduces the structure of the energy-momentum tensor of a spin-J bc theory  $(1-J)\partial b \cdot c - Jb \cdot \partial c$  [36]. Expanding  $\mathfrak{T}(u)$  in powers of the variable u, which was introduced in (2.7) and has now acquired the role of a spectral parameter, as

$$\mathfrak{T}(u) = \sum_{p \in \mathbb{Z}} u^{-p-2} \mathfrak{L}_p \tag{2.11}$$

<sup>&</sup>lt;sup>5</sup>We have chosen the irrelevant parameter  $a_0 = J - \frac{1}{2}$ , see [34].

we arrive at the individual Virasoro generators (which are a particular case of the general construction applicable to integrable hierarchies of the r-matrix type [32])

$$\mathfrak{L}_n \equiv K(J(n+1)D^n + PD^{n+1})K^{-1})_-, \quad P \equiv x + \sum_{r>1} rt_r D^{r-1}$$
 (2.12)

These define the Virasoro-constrained KP hierarchy via  $\mathfrak{L}_{n>-1}=0$ .

Conversely, the Virasoro generators (2.5) acting on the tau function can be recovered starting from the  $\mathfrak{L}_n$ , eq.(2.12), by using the equation

$$\operatorname{res}K = -\partial \log \tau, \tag{2.13}$$

whence

$$\delta \partial \log \tau = -\text{res}\delta K = \text{res}\mathfrak{T}(u)K = \text{res}\mathfrak{T}(u)$$
 (2.14)

The (operator) residue of  $\mathfrak{T}(u)$  is immediately read off from (2.10). To the combination of the wave functions thus appearing we apply the well-known formula

$$\frac{\tau(t - [u^{-1}] + [z^{-1}])}{\tau(t)} = (u - z)e^{\xi(t,z) - \xi(t,u)} \partial^{-1} \left(\psi(t,u)\psi^*(t,z)\right)$$
(2.15)

The generators (2.5) now follow by expanding this at  $u \to z$ ,

# 3 Kontsevich-Miwa transform

The Miwa reparametrization of the KP times is accomplished by the substitution

$$t_r = \frac{1}{r} \sum_{j} n_j z_j^{-r}, \quad r \ge 1$$
 (3.1)

where  $\{z_j\}$  is a set of points on the complex plane and the parameters  $n_j$  are integer classically; we will need, however, to continue off the integer values.

This parametrization puts, in a sense, the times and the spectral parameter on equal ground. By the *Kontsevich* transform we will understand the dependence, via eq.(3.1), of  $t_r$  on the  $z_j$  for fixed  $n_j$ . Note that the way Kontsevich has used a parametrization of this type implied setting all the  $n_j$  equal to a constant which was 1 in ref.[23]. In our approach this will prove too strong a restriction. We thus proceed with the general  $n_j$  and then find how the  $n_j$  must be tuned.

Generally, the Kontsevich-Miwa parametrization turns out very inconvenient with regard to the use of the standard machinery of the KP hierarchy (instead, the Miwa parametrization has been used to construct a quite different, "discrete" formalism for the KP and related hierarchies [21, 22]). This applies also to the above Virasoro generators. That is, viewing (3.1) as

$$t_r = \frac{1}{r} \int_{\mathbb{CP}^1} d\mu(z) n(z) z^{-r}$$
(3.2)

one could define the wave functions formally as

$$\psi[n](z) = \prod_{i} \left(1 - \frac{z}{z_{i}}\right)^{-n_{i}} \frac{1}{\tau[n]} e^{-\frac{\delta}{\delta n(z)}} \tau[n]$$
(3.3)

However, using this in the energy-momentum tensor (2.10) and similar formulae would require making sense out of expressions such as  $\frac{\partial}{\partial z} \frac{\delta}{\delta n[z]}$ . Even this would not be quite satisfactory, though, as one would still have had to express the result in terms of the derivatives with respect to the Kontsevich parameters  $z_j$ : for us, the tau function must be a function  $\tau\{z_j\}$  of points scattered over  $\mathbb{CP}^1$ . The  $\partial/\partial z_j$  derivatives, however, are not easy to get hold of using the equation (3.1).

There are two circumstances that save the day. First, we are interested not in all the Virasoro generators, but rather in those with non-negative (and, in addition, -1) mode numbers  $\mathfrak{L}_{n\geq -1}$  (which are used to define the Virasoro-constrained hierarchy via  $\mathfrak{L}_n=0, n\geq -1$ ). Picking these out amounts to retaining in  $\mathfrak{T}(z)$  only terms with z to negative powers, i.e., the terms vanishing at  $z\to\infty$ . This part of  $\mathfrak{T}(z)$  is singled out as

$$\mathfrak{T}^{(\infty)}(v) = \sum_{n \ge -1} v^{-n-2} \frac{1}{2\pi i} \oint dz z^{n+1} \mathfrak{T}(z) = \frac{1}{2\pi i} \oint dz \frac{1}{v-z} \mathfrak{T}(z)$$
(3.4)

where v is from a neighbourhood of the infinity and the integration contour encompasses this neighbourhood.

Second, a crucial simplification will be achieved by evaluating  $\mathfrak{T}^{(\infty)}(v)$  only at the points from the above set  $\{z_j\}$  (one has to take care that they be inside the chosen neighbourhood). We thus have to evaluate the operator  $\mathcal{T}(z_i)$  from (see eqs.(2.13),(2.14))

$$\partial \left( \mathcal{T}(z_i) \tau \right) = \frac{1}{2\pi i} \oint dz \frac{1}{z_i - z} \operatorname{res} \mathfrak{T}(z)$$
(3.5)

This will depend on the collection of the  $n_j$ , which we will indicate by a subscript. It is straightforward to find that

$$\mathcal{T}_{\{n\}}(z_{i}) = \frac{1}{2\pi i} \oint dz \frac{1}{z_{i} - z} \left\{ (J - \frac{1}{2}) \frac{1}{z} \sum_{r \geq 1} z^{-r-1} \frac{\partial}{\partial t_{r}} + \frac{1}{2} \sum_{r,s} z^{-r-s-2} \frac{\partial^{2}}{\partial t_{r} \partial t_{s}} \right. \\
+ \sum_{j} \frac{n_{j}}{z_{j} - z} \sum_{r \geq 1} z^{-r-1} \frac{\partial}{\partial t_{r}} + \frac{1}{2} \sum_{j} \frac{n_{j} + n_{j}^{2}}{(z_{j} - z)^{2}} + \frac{1}{2} \sum_{\substack{j,k \ j \neq k}} \frac{n_{j} n_{k}}{(z_{j} - z)(z_{k} - z)}$$

$$- J \sum_{j} \frac{n_{j}}{(z_{j} - z)^{2}} + (J - \frac{1}{2}) \sum_{r \geq 1} z^{-r-2} r \frac{\partial}{\partial t_{r}} \right\}$$
(3.6)

where we have substituted (3.1) for each explicit occurrence of the  $t_r$ .

However, the problem is that we need all the  $\partial/\partial t_r$ -derivatives to be expressed in terms of the  $\partial/\partial z_j$  as well, while the equation relating  $t_r$  and  $z_j$  does not allow this. It is only when we evaluate the contour integral in (3.6) that the t-derivatives will arrange into the combinations which are just the desired  $\partial/\partial z_j$ 's. As the integration contour encompasses all the points  $\{z_j\}$ , the residues at both  $z = z_i$  and  $z = z_j$ ,  $j \neq i$ , contribute to (3.6). The residue at  $z_i$  consists of the following parts: first, the terms with simple poles contribute

$$\left(J - \frac{1}{2} - \frac{1}{2n_i}\right) \frac{1}{n_i} \frac{1}{z_i} \frac{\partial}{\partial z_i} - \frac{1}{2n_i^2} \frac{\partial^2}{\partial z_i^2} + \frac{1}{n_i} \sum_{j \neq i} \frac{n_j}{z_j - z_i} \frac{\partial}{\partial z_i} - \left(J - \frac{1}{2} - \frac{1}{2n_i}\right) \sum_{r \geq 1} r z_i^{-r-2} \frac{\partial}{\partial t_r} - \frac{1}{2} \sum_{j \neq i} \frac{n_j + n_j^2 - 2Jn_j}{(z_j - z_i)^2} - \frac{1}{2} \sum_{\substack{j \neq i \\ k \neq i \\ k \neq i}} \frac{n_j n_k}{(z_j - z_i)(z_k - z_i)} \right)$$
(3.7)

where we have substituted

$$\sum_{r,s\geq 1} z_i^{-r-s-2} \frac{\partial^2}{\partial t_r \partial t_s} = \frac{1}{n_i^2} \frac{\partial^2}{\partial z_i^2} + \frac{1}{n_i^2} \frac{1}{z_i} \frac{\partial}{\partial z_i} - \frac{1}{n_i} \sum_{r\geq 1} z_i^{-r-2} r \frac{\partial}{\partial t_r},$$

$$\sum_{r\geq 1} z_i^{-r-1} \frac{\partial}{\partial t_r} = -\frac{1}{n_i} \frac{\partial}{\partial z_i}$$
(3.8)

In the third term inside the curly brackets, a second-order pole occurs when j = i, which produces

$$\frac{1}{z_i}\frac{\partial}{\partial z_i} - n_i \sum_{r>1} r z_i^{-r-2} \frac{\partial}{\partial t_r}$$
(3.9)

Next, second-order poles occur in the double sum over j, k in (3.6):

$$\frac{1}{2\pi i} \oint dz \frac{1}{z_i - z} \sum_{j \neq i} \frac{n_j n_i}{(z_j - z)(z_i - z)} = \sum_{j \neq i} \frac{n_i n_j}{(z_i - z_j)^2}$$
(3.10)

We thus see that the term  $\sum_{r\geq 1} rz_i^{-r-2} \frac{\partial}{\partial t_r}$ , which cannot be expressed locally through  $\frac{\partial}{\partial z_j}$ , enters with the coefficient  $-\left(J-\frac{1}{2}-\frac{1}{2n_i}+n_i\right)$ . We have to set this coefficient to zero; therefore  $n_i$  and J are related by

$$\frac{1}{n_i} - 2n_i = 2J - 1 \equiv Q \tag{3.11}$$

Then the contribution of the residue at  $z = z_i$  becomes

$$\mathcal{T}_{\{n\}}^{(i)}(z_i) = -\frac{1}{2n_i^2} \frac{\partial^2}{\partial z_i^2} + \frac{1}{n_i} \sum_{j \neq i} \frac{n_j}{z_j - z_i} \frac{\partial}{\partial z_i} 
- \frac{1}{2} \sum_{\substack{j \neq i \\ k \neq i \\ k \neq i}} \frac{n_j n_k}{(z_j - z_i)(z_k - z_i)} - \frac{1}{2} \sum_{j \neq i} \frac{n_j + n_j^2 - 2Jn_j - 2n_i n_j}{(z_j - z_i)^2}$$
(3.12)

Similarly, each of the residues at  $z_j$ ,  $j \neq i$ , contributes

$$\mathcal{T}_{\{n\}_{(j)}}(z_i) = -\frac{1}{z_j - z_i} \frac{\partial}{\partial z_j} + \frac{1}{z_j - z_i} \sum_{k \neq j} \frac{n_j n_k}{z_k - z_j} + \frac{1}{2} \frac{n_j + n_j^2 - 2J n_j}{(z_i - z_j)^2}$$
(3.13)

and thus, finally <sup>6</sup>,

$$\mathcal{T}_{\{n\}}(z_i) = \mathcal{T}_{\{n\}}^{(i)}(z_i) + \sum_{j \neq i} \mathcal{T}_{\{n\}}_{(j)}(z_i) 
= -\frac{1}{2n_i^2} \frac{\partial^2}{\partial z_i^2} + \frac{1}{n_i} \sum_{j \neq i} \frac{1}{z_j - z_i} \left( n_j \frac{\partial}{\partial z_i} - n_i \frac{\partial}{\partial z_j} \right)$$
(3.14)

$$\sum_{j \neq i} \sum_{k \neq i \atop k \neq j} \frac{1}{(z_j - z_i)} \frac{n_j n_k}{(z_k - z_j)} = \frac{1}{2} \sum_{j \neq i} \sum_{k \neq i \atop k \neq j} \frac{n_j n_k}{(z_j - z_i)(z_k - z_i)}$$

<sup>&</sup>lt;sup>6</sup>We have used the identity

where  $n_i$  is to be determined from (3.11).

If one wishes all the  $\mathfrak{T}^{(\infty)}(z_j)$  to carry over to the Kontsevich variables along with  $\mathfrak{T}^{(\infty)}(z_i)$ , all the  $n_i$  have to be fixed to the same value  $n_i$ . Then, one gets "symmetric" operators

$$\mathcal{T}(z_i) = -\frac{Q^2 + 4 \pm Q\sqrt{Q^2 + 8}}{4} \frac{\partial^2}{\partial z_i^2} - \sum_{j \neq i} \frac{1}{z_j - z_i} \left( \frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_i} \right)$$
(3.15)

These differential operators, of course, satisfy the centreless algebra spanned by the  $\{n \ge -1\}$ -Virasoro generators.

Clearly, if one starts with the *Virasoro-constrained* KP hierarchy, *i.e.*,  $\mathfrak{T}^{(\infty)}(z) = 0$ , one ends up in the Kontsevich parametrization with the KP Virasoro master equation (cf. ref.[25])  $\mathcal{T}(z_i).\tau\{z_j\}=0$ . In the next section we show that this is solved by certain conformal field theory correlators.

Conversely, let us also see how, given the master equation  $\mathcal{T}_{\{n\}}(z_i).\tau\{z_j\}=0$ , one can recover the usual form of the Virasoro constraints. The required transformation is inverse to the one we have just performed, and its less trivial part is to get rid of the explicit occurences of the  $z_j$ . The derivatives  $\partial/\partial z_j$ , on the other hand, are straightforwardly replaced with  $\partial/\partial t_r$  according to (3.8). We thus get

$$\mathcal{T}_{\{n\}}(z_i) = -\frac{1}{2n_i^2} \left( n_i^2 \sum_{r,s \ge 1} z_i^{-r-s-2} \frac{\partial^2}{\partial t_r \partial t_s} + n_i \sum_{r \ge 1} z_i^{-r-2} (r+1) \frac{\partial}{\partial t_r} \right) + \sum_{j \ne ir \ge 1} n_j \frac{z_j^{-r-1} - z_i^{-r-1}}{z_j - z_i} \frac{\partial}{\partial t_r}$$

$$(3.16)$$

In the last term, we divide by  $z_j - z_i$  and get  $-\sum_{j \neq i} n_j \sum_{r \geq 1} \sum_{s=1}^{r+1} z_j^{-s} z_i^{-r-s+2} \frac{\partial}{\partial t_r}$ . Now the sum over all j gives  $st_s$  according to the Miwa transform (3.1); the missing term with j = i, which is to be added and subtracted, combines with the term of the same structure from (3.16). We thus recover the  $(\geq -1)$  Virasoro generators (2.5) for J given by (3.11).

# 4 Trivialities on conformal field theory

Now, to prepare the presentation in the next section, consider the subject which is apparently quite different from what we have had so far. Introduce a conformal theory of a U(1) current and an energy-momentum tensor:

$$j(z) = \sum_{n \in \mathbb{Z}} j_n z^{-n-1}, \quad T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$
 (4.1)

$$[j_{m}, j_{n}] = km\delta_{m+n,0}$$

$$[L_{m}, L_{n}] = (m-n)L_{m+n} + \frac{d+1}{12}(m^{3}-m)\delta_{m+n,0}$$

$$[L_{m}, j_{n}] = -nj_{m+n}$$

$$(4.2)$$

(We have parametrized the central charge as d+1). Let  $\Psi$  be a primary field with conformal dimension  $\Delta$  and U(1) charge q. Then, by a slight variation of  $[18]^7$ , we find that the level-2

<sup>&</sup>lt;sup>7</sup>We extend the standard system of [18] by introducing a current, but then 'compensate' the extra freedom by suppressing a term (the one proportional to  $j_{-1}^2$ ) in the general form of the decoupling equation.

state

$$|\Upsilon\rangle = \left(\alpha L_{-1}^2 + L_{-2} + \beta j_{-2} + \gamma j_{-1} L_{-1}\right) |\Psi\rangle$$
 (4.3)

is primary provided

$$\alpha = \frac{k}{2q^2} , \quad \beta = -\frac{q}{k} - \frac{1}{2q} , \quad \gamma = -\frac{1}{q} , \quad \Delta = -\frac{q^2}{k} - \frac{1}{2}$$
 (4.4)

with q given by,

$$\frac{q^2}{k} = \frac{d - 13 \pm \sqrt{(25 - d)(1 - d)}}{24} \tag{4.5}$$

and, accordingly,

$$\Delta = \frac{1 - d \mp \sqrt{(25 - d)(1 - d)}}{24} \ . \tag{4.6}$$

Factoring out the state  $|\Upsilon\rangle$  leads in the usual manner to the equation

$$\left\{ \frac{k}{2q^2} \frac{\partial^2}{\partial z^2} - \frac{1}{q} \sum_j \frac{1}{z_j - z} \left( q \frac{\partial}{\partial z_j} - q_j \frac{\partial}{\partial z} \right) + \frac{1}{q} \sum_j \frac{q \Delta_j - q_j \Delta}{(z_j - z)^2} \right\} \langle \Psi(z) \Psi_1(z_1) \dots \Psi_n(z_n) \rangle = 0$$
(4.7)

where  $\Psi_j$  are primaries of dimension  $\Delta_j$  and U(1) charge  $q_j$ . In particular,

$$\left\{ \frac{k}{2q^2} \frac{\partial^2}{\partial z_i^2} + \sum_{j \neq i} \frac{1}{z_i - z_j} \left( \frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_i} \right) \right\} \langle \Psi(z_1) \dots \Psi(z_n) \rangle = 0$$
(4.8)

These equations will be crucial for comparison with the KP hierarchy in Sect. 5.

Writing the Hilbert space as (matter)  $\otimes$  (current)  $\equiv \mathcal{M} \otimes \mathcal{C}$ ,  $|\Psi\rangle = |\psi\rangle \otimes |\widetilde{\Psi}\rangle$ , we introduce the matter Virasoro generators  $l_n$  by,

$$L_n = l_n + \widetilde{L}_n \equiv l_n + \frac{1}{2k} \sum_{m \in \mathbb{Z}} : j_{n-m} j_m :$$

$$\tag{4.9}$$

They then have central charge d. Now, using in (4.3) that

$$\widetilde{L}_{-1}|\widetilde{\Psi}\rangle = \frac{q}{k}j_{-1}|\widetilde{\Psi}\rangle, \quad \widetilde{L}_{-2}|\widetilde{\Psi}\rangle = \left(\frac{q}{k}j_{-2} + \frac{1}{2k}j_{-1}^2\right)|\widetilde{\Psi}\rangle, \quad \widetilde{L}_{-1}^2|\widetilde{\Psi}\rangle = \left(\frac{q}{k}j_{-2} + \frac{q^2}{k^2}j_{-1}^2\right)|\widetilde{\Psi}\rangle \tag{4.10}$$

we find

$$|\Upsilon\rangle = \left(\frac{k}{2q^2}l_{-1}^2 + l_{-2}\right)|\Psi\rangle \tag{4.11}$$

Therefore we are left with a null vector in the matter Hilbert space  $\mathcal{M}$ . The dimension of  $|\psi\rangle$  in the matter sector is found from

$$L_0|\Psi\rangle = \left(l_0 + \frac{1}{2k}j_0^2\right)|\Psi\rangle \tag{4.12}$$

and equals

$$\delta = \Delta - \frac{1}{2k}q^2 = \frac{5 - d \mp \sqrt{(1 - d)(25 - d)}}{16} \tag{4.13}$$

which for the appropriate values of d is of course the dimension of the '21' operator of the minimal model with central charge d.

# 5 A la récherche de Liouville perdu

#### 5.1 Ansatz for the Virasoro-constrained tau function

A contact between sections 4 and 3, i.e., between conformal field theory formalism and the KP hierarchy is suggested by the above derivation of the 'master' operators (3.14),(3.15), in which the  $z_i$  were viewed as coordinates on the spectral curve.

For the Virasoro-constrained tau function in the Kontsevich parametrization we assume the ansatz

$$\tau\{z_j\} = \lim_{n \to \infty} \langle \Psi(z_1) \dots \Psi(z_n) \rangle \tag{5.1}$$

Then, comparing the decoupling equation (4.8) with the master equation (3.15), we find

$$n_i^2 = \frac{q^2}{-k} \tag{5.2}$$

and therefore, taking into account (3.11) and (4.5),

$$Q = \sqrt{\frac{1-d}{3}} \equiv Q_{\rm m} \tag{5.3}$$

The Miwa parameter  $n_i$  is determined in terms of the central charge d as (with  $\sigma$  being a conventional sign factor,  $\sigma^2 = 1$ )

$$n_i = \sigma \frac{\mp Q + \sqrt{Q^2 + 8}}{4} \equiv -\sigma \frac{-Q_L \pm Q_m}{4} = -\frac{\sigma}{2} \alpha_{\pm}$$
 (5.4)

where  $Q_{\rm L}$  and  $Q_{\rm m}$  are recognized as the Liouville and the matter central charge respectively, and  $\alpha$  is the cosmological constant.

Note that the energy-momentum tensor T(z) from (4.1) appears to have a priori nothing to do with  $\mathfrak{T}(z)$  (or, which is the same, with  $\mathsf{T}^{(\infty)}(z_i)$ , see (2.7)) we have started with. In terms of the latter tensor, the master operator (3.14) comprises contributions of all the positive-moded Virasoro generators, while out of T(z) only  $L_{-1}$  and  $L_{-2}$  are used in the construction of  $|\Upsilon\rangle$ , eq.(4.3).

# 5.2 More general correlators

To reconstruct matter theory field operators, consider the form the  $\mathsf{L}_{n\geq -1}$ -Virasoro constraints take for the wave function of the hierarchy,  $w(t,z_k)\equiv e^{-\xi(t,z_k)}\psi(t,z_k)$ , which should now become a function of the  $z_j$ ,  $w\{z_j\}(z_k)$ . More precisely, consider the 'unnormalized' wave function  $\overline{w}\{z_j\}(z_k) = \tau\{z_j\}w\{z_j\}(z_k)$ . Then the use of the Kontsevich transform at the Miwa point  $n_j = \alpha/2$ ,  $j \neq k$  and  $n_k = -1$ , gives<sup>8</sup>

$$\overline{w}\{z_j\}(z_k) = \left\langle \prod_{j \neq k} \Psi(z_j) \cdot \Xi(z_k) \right\rangle \tag{5.5}$$

<sup>&</sup>lt;sup>8</sup>To obtain the insertion into the correlation function (5.5) at the point  $z_k$  of the operator we are interested in by itself, rather than its fusion with the 'background'  $\Psi$ , we use the Kontsevich transform at the value of the Miwa parameter  $n_k = -1$  instead of  $n_i - 1$ . This means that we are in fact considering  $\overline{w}\{z_j\}_{j\neq k}(z_k)$ . Similar remarks apply to other correlation functions considered below. Of course, the conceptual difference between the tau function and the 'unnormalized' wave functions  $\overline{w}(z_k)$  disappears in the Miwa parametrization.

where  $\Xi$  is a primary field with the U(1) charge  $q/n_i$  and dimension  $\Delta/n_i = -\sigma \frac{2\Delta}{\alpha}$ . This implies in turn that its dimension in the matter sector equals

$$-\sigma \frac{2\Delta}{\alpha} - \frac{1}{2k} \left(\frac{q}{n_i}\right)^2 = (\mp) \frac{1}{2} Q + \frac{1}{2} \equiv \begin{cases} 1 - J \\ J \end{cases}$$
 (5.6)

Thus the wave function is related to, say, (depending on the sign conventions) the *b*-field of the bc system <sup>9</sup>. The adjoint wave function is then similarly related to the corresponding c field: for instance, the function  $\tau(t-[z_k^{-1}]+[z_l^{-1}])$  is annihilated by the operator

$$-\frac{2}{\alpha^2}\frac{\partial^2}{\partial z_i^2} - \sum_{\substack{j \neq i, \ j \neq k \\ j \neq l}} \frac{1}{z_j - z_i} \left( \frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_i} \right) + \frac{2}{\alpha} \left( \frac{1}{z_l - z_i} - \frac{1}{z_k - z_i} \right) \frac{\partial}{\partial z_i}$$
(5.7)

Again, we interpret this as a decoupling equation which accounts for the effect of certain insertions at  $z_k$  and  $z_l$ . We thus find that the tau function  $\tau(t - [z_k^{-1}] + [z_l^{-1}])$  is proportional to the correlation function

$$\left\langle \prod_{\substack{j \neq k \ j \neq l}} \Psi(z_j) \exp\left(\frac{q}{kn_i} \int^{z_k} j\right) b(z_k) \exp\left(-\frac{q}{kn_i} \int^{z_l} j\right) c(z_l) \right\rangle$$

$$= \left\langle \prod_{\substack{j \neq k \ j \neq l}} \Psi(z_j) (z_k - z_l) \exp\left(\frac{q}{kn_i} \int_{z_l}^{z_k} j\right) B(z_k) C(z_l) \right\rangle$$
(5.8)

where we have used

$$-\frac{q^2}{k^2 n_i^2} j(z) j(w) \sim -\frac{q^2}{k^2 n_i^2} k \ln(z - w) = \ln(z - w).$$

Note that, although it is tempting to take in (5.8) the limit  $z_k \to z_l$ , this cannot be done naively, as it would affect the whole construction of the Kontsevich-Miwa transform!

As a cross-check, it is interesting to compare (5.8) with the identity (2.15) (which is valid for a general (i.e., not necessarily Virasoro-constrained) KP tau-function). In the Kontsevich parametrization we have

$$e^{\xi(t,z)-\xi(t,u)} = \prod_{j} \left(\frac{z_{j}-u}{z_{j}-z}\right)^{n_{j}}$$
 (5.9)

On the other hand, fusing the exponential in (5.8) with the product of the  $\Psi(z_j)$ , gives the factor

$$\prod_{\substack{j \neq k \\ j \neq l}} \left( \frac{z_j - z_k}{z_j - z_l} \right)^{\frac{q}{k} \left( -\frac{q}{kn_i} \right) \cdot k} = \prod_{\substack{j \neq k \\ j \neq l}} \left( \frac{z_j - z_k}{z_j - z_l} \right)^{n_i}$$

$$(5.10)$$

which agrees with the above now that the  $n_j$  have been set in (5.1) equal to  $n_i$ . This suggests extending the Kontsevich-Miwa transform 'off-shell', i.e., off the Virasoro constraints. Both of the two classes of objects, the tau function etc., and the theory in  $\mathcal{M} \otimes \mathcal{C}$ , exist by themselves,

 $<sup>\</sup>overline{\phantom{a}}^9$ For the values of J that we will actually need (which are not half-integer nor even rational), the bc system would be purely formal. We will not keep it in the 'bc' form for long and bosonize it shortly.

while we have seen that imposing the Virasoro constraints on the one end results in factoring over a submodule on the other.

By bosonizing the formal bc system one gets a matter scalar  $\varphi$  with the familiar energy-momentum tensor ([18, 19, 35])

$$T_{\rm m} = -\frac{1}{2}\partial\varphi\partial\varphi + \frac{i}{2}Q_{\rm m}\partial^2\varphi. \tag{5.11}$$

Note that for the unitary series

$$d = 1 - \frac{6}{p(p+1)},\tag{5.12}$$

the Miwa parameter  $n_i$  is determined as

$$n_i^2 = \frac{1}{2} \left( \frac{p+1}{p} \right)^{\mp 1}. \tag{5.13}$$

Further, as to the theory in  $\mathcal{C}$ , recall that we have

$$[j_m, j_n] = km\delta_{m+n,0}, \quad j_{n>0}|\Psi\rangle = 0, \quad j_0|\Psi\rangle = q|\Psi\rangle$$
(5.14)

with negative  $q^2/k^{-10}$ . To see what the current corresponds to in the KP theory, consider the correlation function with an extra insertion of an operator which depends on only j:

$$\left\langle \prod_{\substack{j \neq k \\ j \neq l}} \Psi(z_j) \exp\left( (\pm) \frac{Q}{\sqrt{-k}} \int_{z_l}^{z_k} j \right) \right\rangle \tag{5.15}$$

The decoupling equation states that this is annihilated by the operator

$$\mathcal{T}(z_i) + \frac{1}{2}Q(Q \pm Q_L) \left(\frac{1}{z_k - z_i} - \frac{1}{z_l - z_i}\right) \frac{\partial}{\partial z_i}$$
 (5.16)

and therefore coincides, up to a constant, with the Virasoro-constrained tau function  $\tau(t)$  evaluated at the Miwa point

$$n_{j} = \begin{cases} -\frac{\sigma}{2}\alpha, & j \neq k, \quad j \neq l \\ Q, & j = k \\ -Q, & j = l \end{cases}$$

$$(5.17)$$

i.e., this is

$$\tau \left( -\frac{\sigma}{2} \alpha \sum_{\substack{j \neq k \\ j \neq l}} [z_j^{-1}] + Q_[z_k^{-1}] - Q_[z_l^{-1}] \right)$$

This is another illustration of how the Kontsevich-Miwa transform works: establishing the relation to different conformal field operators  $\mathcal{O}(z_k)$  requires fixing different values of  $n_k$ .

<sup>&</sup>lt;sup>10</sup>for d < 1. For d > 25, on the other hand,  $q^2/k$  is positive, but then one has to consider the hierarchy with imaginary  $n_i$ ! It appears that the matter and the Liouville theory then take place of one another, and  $n_i = \sqrt{-1}\alpha/2$  with  $\alpha$  being the cosmological constant.

#### 5.3 The dressing prescription

The balance of dimensions and U(1) charges of both the  $\Psi$  and  $\Xi$  operators follows a particular general pattern. That is, as there are no  $1/(z_i - z_j)^2$ -terms in the master equation, we have to ensure that these terms be absent from the decoupling equation (4.7). Therefore we can only consider operators from a special sector, i.e., those whose dimensions  $\Delta_j$  and U(1) charges  $q_j$  satisfy (see (4.7))<sup>11</sup>,

$$\Delta_j = \Delta \frac{q_j}{q} = (\pm) \frac{1}{2} Q n_j \tag{5.18}$$

(Clearly, the U(1) charges are related to the Miwa parameters via  $q_j/\sqrt{-k} = n_j$ .) With this condition, the decoupling equation (4.7) takes the form,

$$\left\{ -\frac{2}{\alpha^2} \frac{\partial^2}{\partial z_i^2} + \sum_{j \neq i} \frac{1}{z_j - z_i} \left( -\frac{2\sigma}{\alpha} n_j \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) \right\} \left\langle \Psi(z_i) \prod_{j \neq i} \Psi_j(z_j) \right\rangle = 0$$
(5.19)

and we are still able, as before, to relate this to the Virasoro constraints, since the operator on the LHS is precisely the general form of the 'master' operator (3.14) in which only one value, that of  $n_i$ , out of the  $n_j$ , has been fixed. Let us repeat once again that evaluating the tau function at different Miwa parameters  $\{n_j\}$  corresponds to different operator insertions in the conformal field theory language.

Now, the dimension of the matter part of  $\Psi_j$  is equal to

$$\delta_j = \Delta_j - \frac{q_j^2}{2k} = \Delta \frac{q_j}{q} - \frac{q_j^2}{2k}$$
 (5.20)

On the other hand, dimensions of the matter field operators  $e^{i\gamma\varphi}$  are fixed by the energy-momentum tensor (5.11). It is crucial for consistency that the two formulae agree: as the term linear in  $q_j/\sqrt{-k}$  enters in (5.20) with the coefficient (sign) $\frac{1}{2}Q$ , eq.(5.20) will always be satisfied for the matter operators  $e^{i\gamma\varphi}$  provided  $q_j/\sqrt{-k}=(\pm)\left\{\begin{matrix} \gamma\\ Q-\gamma\end{matrix}$ . Therefore, the prescription for the 'dressing' inherited from the Virasoro-constrained KP hierarchy says that the coefficients in front of the two scalars  $\varphi$  and  $\varphi$  in the exponents coincide up to the reflection  $\gamma\mapsto Q_{\rm m}-\gamma$  in the matter sector (and, to be precise, up to the usual overall factor of i). Thus, although the field content is the same as in ref.[16], it is not quite the David-Distler-Kawai formalism that follows directly from the KP hierarchy<sup>12</sup>. Our 'dressing exponent'

$$\frac{q_j}{\sqrt{-k}} = -\frac{\sqrt{-k}\Delta}{q} \pm \frac{1}{2\sqrt{3}}\sqrt{1 - d + 24\delta_j} \tag{5.21}$$

differs from eq.(3.12) of [16]

$$\beta_j = -\frac{1}{2}Q_{\rm L} \pm \frac{1}{2\sqrt{3}}\sqrt{1 - d + 24\delta_j} \tag{5.22}$$

<sup>&</sup>lt;sup>11</sup>we continue to denote by  $\Delta$  and q the dimension and U(1) charge from (4.2) – (4.6), i.e., those of  $\Psi$ .

<sup>&</sup>lt;sup>12</sup>This can be seen also by noticing that the dimensions in  $\mathcal{M}$  and  $\mathcal{C}$  do not add up to 1; nor is the central charge equal to 26. This is not a surprise, since the current j is not anomalous.

by the cosmological constant  $\alpha$ .

Equivalently, the 'bulk' dimensions  $\Delta_j$ , rather than being equal to 1, are related to the gravitational scaling dimensions of fields. Indeed, evaluating the gravitational scaling dimension of  $\psi$  according to [17, 16, 15],

$$\hat{\delta}_{\pm} = \frac{\pm\sqrt{1 - d + 24\delta} - \sqrt{1 - d}}{\sqrt{25 - d} - \sqrt{1 - d}}$$
(5.23)

one would find

$$\hat{\delta}_{+} = \frac{3}{8} \pm \frac{d - 4 - \sqrt{(1 - d)(25 - d)}}{24} \tag{5.24}$$

with the sign on the RHS corresponding to that in (4.5) and the subsequent formulae. In particular, choosing the *lower* signs throughout, we have  $\hat{\delta}_{+} = \Delta + \frac{1}{2}$ . More generally, the gravitational scaling dimensions corresponding to (5.20) equal

$$\hat{\delta}_{j+} = -\frac{q_j q}{k} = \Delta_j + \frac{1}{2} \frac{q_j}{q} = \Delta_j - \frac{\sigma}{\alpha} \frac{q_j}{\sqrt{-k}}$$
(5.25)

and thus are given by the  $\Delta_j$  'corrected' by the terms linear in the charge.

The combination  $-n_k \sum_{r\geq 1} z_k^{-r-1} \partial/\partial t_r = \partial/\partial z_k$  of the  $\partial/\partial t_r$ -derivatives applied to the decoupling equation gives rise to the recursion relations for correlators of  $\sum_{r\geq 1} z_k^{-r-1} \sigma_r$ . Interestingly, it is therefore the decoupling equation that serves as a generating relation for the recursion relations.

# 6 The N-reduced equations

In this section we show how the master equation can be obtained for the N-reduced case. The KP hierarchy can be reduced to generalized N-KdV hierarchies [27] by imposing the constraint

$$Q^N \equiv L \in \text{Diff} \tag{6.1}$$

requiring that the  $N^{\text{th}}$  power of the Lax operator be purely differential. Then, in a standard manner, the evolutions along the times  $t_{Nk}$ ,  $k \geq 1$ , drop out and these times may be set to zero. The rest of the  $t_n$  are conveniently relabelled as  $t_{a,i} = t_{Ni+a}$ ,  $i \geq 0$ ,  $a = 1, \ldots, N-1$ .

As to the Virasoro generators, only  $\mathfrak{L}_{Nj}$  out of the generators (2.12) are compatible with the reduction in the sense that they remain symmetries of the reduced hierarchy without imposing further constraints [28]. The value of J can be set to zero [28], and thus we arrive at the generators

$$\mathfrak{L}_{j}^{[N]} = \frac{1}{N} \left( K \left( x + \sum_{a,i} (Ni + a) t_{a,i} D^{N(i+j)+a} \right) K^{-1} \right)_{-}$$
 (6.2)

which span a Virasoro algebra of their own. To construct the corresponding energy-momentum tensor, recall that the spectral parameter of the N-KdV hierarchy is  $\zeta = z^N$ . Then

$$\mathfrak{T}^{[N]}(\zeta)(d\zeta)^{2} \equiv \sum_{j\in\mathbb{Z}} \zeta^{-j-2} \mathfrak{L}_{j}^{[N]}(d\zeta)^{2}$$

$$= N \left( K \sum_{b,j} (Nj+b) t_{b,j} D^{Nj+b} \frac{1}{z^{2}} \delta(D^{N}, z^{N}) K^{-1} \right)_{-} (dz)^{2}$$
(6.3)

Now,  $\delta(z, D)$  is a projector onto an eigenspace of D with the eigenvalue z, and thus

$$\delta(D^N, z^N) = \frac{1}{N} \sum_{c=0}^{N-1} \delta(z^{(c)}, D), \quad z^{(c)} = \omega^c z, \quad \omega = \exp\left(\frac{2\pi\sqrt{-1}}{N}\right)$$
 (6.4)

Using this we bring the above energy-momentum tensor to the form

$$\mathfrak{T}^{[N]} = \sum_{c=0}^{N-1} \omega^c \frac{\partial \psi(t, z^{(c)})}{\partial z} \circ D^{-1} \circ \psi^*(t, z^{(c)}) = \frac{1}{N} \sum_{c=0}^{N-1} \omega^{2c} \mathfrak{T}(z^{(c)})$$
(6.5)

where we have used that the spectral parameter of an N-KdV hierarchy lies on a complex curve defined in  $\mathbb{C}^2 \ni (z, E)$  by an equation  $z^N = P(E)$ . Then,  $\psi$  and  $\psi^*$  are defined on this curve, and after the projection onto  $\mathbb{CP}^1$  yield N wave functions  $\psi^{(a)}(t, E)$ , distinct away from the branch points. That is, we have defined

$$\psi^{(a)}(t,E) = Ke^{\xi(t,z^{(a)})} \equiv w(t,z^{(a)})e^{\xi(t,z^{(a)})}, \qquad \xi(t,z^{(a)}) = \sum_{j,b} t_{b,j}(z^{(a)})^{Nj+b}$$
(6.6)

Note a similarity between (6.5) and the energy-momentum tensor of conformal theories on  $\mathbb{Z}_N$ -curves [33].

The Virasoro action on the tau-function of the N-reduced hierarchy can be recovered using the eqs. (2.13) – (2.15), combined with taking the appropriate average over  $\mathbb{Z}_N$ . In this way we arrive at the Virasoro generators

$$n > 0: \qquad \mathsf{L}_{n}^{[N]} = \frac{1}{N} \frac{1}{2} \sum_{a=1}^{N-1} \sum_{i=0}^{n-1} \frac{\partial^{2}}{\partial t_{a,i} \partial t_{N-a,n-i-1}} + \frac{1}{N} \sum_{a=1}^{N-1} \sum_{i \geq 0} (Ni+a) t_{a,i} \frac{\partial}{\partial t_{a,i+n}},$$

$$\mathsf{L}_{0}^{[N]} = \frac{1}{N} \sum_{a=1}^{N-1} \sum_{i \geq 0} (Ni+a) t_{a,i} \frac{\partial}{\partial t_{a,i}},$$

$$n < 0: \qquad \mathsf{L}_{n}^{[N]} = \frac{1}{N} \frac{1}{2} \sum_{a=1}^{N-1} \sum_{i=0}^{n-1} (Ni+a) (-N(i+n)-a) t_{a,i} t_{N-a,-i-n-1}$$

$$+ \frac{1}{N} \sum_{a=1}^{N-1} \sum_{i \geq -n} (Ni+a) t_{a,i} \frac{\partial}{\partial t_{a,i+n}}$$

$$(6.7)$$

Now, we have learnt from the above derivation of the matter operator (3.14) that  $z_i$  is nothing but a value taken by the spectral parameter. Therefore the trick with averaging over  $\mathbb{Z}_N$  as in (6.4) can be carried over to the Kontsevich parametrization. That is, to perform the reduction to an N-KdV hierarchy, it suffices to substitute

$$z_i \mapsto \omega^c z_i$$
 (6.8)

and then sum over  $\mathbb{Z}_N$  as in (6.5). Indeed, having defined the reduced  $\mathcal{T}$ -operator as

$$\partial \left( \mathcal{T}_i^{[N]} \tau \right) = \frac{1}{2\pi i} \oint dz \frac{1}{z_i - z} \frac{1}{N} \sum_{c=0}^{N-1} \omega^{2c} \operatorname{res} \mathfrak{T}(\omega^c z), \tag{6.9}$$

one continues this as

$$= \frac{1}{2\pi i} \frac{1}{N} \sum_{c=0}^{N-1} \oint \frac{dz \omega^{-c}}{z_i - \omega^{-c} z} \omega^{2c} \operatorname{res} \mathfrak{T}(z) = \frac{1}{2\pi i} \frac{1}{N} \sum_{c=0}^{N-1} \omega^{2c} \oint \frac{dz}{\omega^c z_i - z} \operatorname{res} \mathfrak{T}(z)$$

$$= \partial \left( \frac{1}{N} \sum_{c=0}^{N-1} \omega^{2c} \mathcal{T}(\omega^c z_i) \tau \right)$$
(6.10)

We thus arrive at

$$\mathcal{T}_i^{[N]} = -\frac{Q^2}{2} \frac{\partial^2}{\partial z_i^2} - \sum_{j \neq i} \frac{1}{z_j^N - z_i^N} \left( z_j z_i^{N-2} \frac{\partial}{\partial z_j} - z_i^{N-1} \frac{\partial}{\partial z_i} \right)$$
(6.11)

Recall that  $z^N \equiv \zeta$  can be viewed as a spectral parameter of the N-KdV hierarchy, as the N-KdV Lax operator L (see (6.1)) satisfies  $L\psi(t,z)=z^N\psi(t,z)$ . In terms of these variables, the operator (6.11) becomes, up to an overall factor,

$$-\frac{N}{2}\zeta_i \frac{\partial^2}{\partial \zeta_i^2} - \frac{(N-1)}{2} \frac{\partial}{\partial \zeta_i} + \sum_{j \neq i} \frac{1}{\zeta_j - \zeta_i} \left( \zeta_j \frac{\partial}{\partial \zeta_j} - \zeta_i \frac{\partial}{\partial \zeta_i} \right)$$
(6.12)

(we have set  $Q^2 = 1$ ). When imposing Virasoro constraints on the N-reduced hierarchy, it is these  $\zeta_i$  that are candidates for eigenvalues of the "source" matrix in a Kontsevich-type matrix integral.

## 7 An outlook

# 7.1 Generalized master equations from scaled Kontsevich matrix integrals?

Various aspects of the conversion of Virasoro constraints into decoupling equations deserve more study from the 'Liouville' point of view. The Kontsevich-type matrix integrals whose Ward identities coincide with the generalized master equation, may provide a discretized definition of the Liouville theory. More precisely, consider the matrix integral (see [12, 25, 37, 38])

$$\mathcal{F}(\Lambda) = \int DX e^{-\text{tr}X^3 + \text{tr}\Lambda X}$$
 (7.1)

where  $\Lambda$  is a 'source' matrix. Then, as emphasized in the papers cited above, the Ward identity assumes the form,

$$\left(\sum_{j} \frac{\partial}{\partial \Lambda_{ij}} \frac{\partial}{\partial \Lambda_{jk}} - \frac{1}{3} \Lambda_{ki}\right) \mathcal{F}(\Lambda) = 0 \tag{7.2}$$

Further, the Kontsevich integral (7.1) does in fact depend only on the eigenvalues  $\lambda_i$  of  $\Lambda$ . We use this to evaluate the second-order derivative and then restrict to a diagonal  $\Lambda$ . This results is [23]

$$\left(\frac{\partial^2}{\partial \lambda_i^2} + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \left(\frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j}\right) - \frac{1}{3}\lambda_i\right) \mathcal{F}\{\lambda\} = 0$$
(7.3)

Comparing this to the master equation  $\mathcal{T}(z_i)\tau = 0$  (see (3.15)), one notices in (7.3) a puzzling term linear in  $\lambda$ . The presence of this term, clearly, implies that  $\lambda_j$  are dimensionless, and therefore so would be the time parameters constructed out of the  $\lambda_j$  according to the Miwa formula. This is in contrast with the fact that the KP times are naturally assigned dimensions  $t_r \sim (\text{lengh})^r$ , which implies  $z \sim (\text{lengh})^{-1}$ .

A useful analogy is provided by the relation between the (Virasoro-constrained) Toda and KP hierarchies; the former is a 'discrete' hierarchy with dimensionless times  $x_r$ , while the dimensionful KP times follow via a scaling ansatz [28]

$$x_r = \frac{1}{r} \sum_{q \ge r} {q \choose r} (-1)^{q+r} (q+1) \frac{t_{q+1}}{\epsilon^{q+1}}, \quad r \ge 1.$$
 (7.4)

Taking  $\epsilon$  of dimension of length then endows the  $t_q$  with the desired dimensions. Scaling implies taking  $\epsilon \to 0$ ; however, the above ansatz (7.4) is then very singular, as it contains arbitrarily large negative powers of  $\epsilon$ . One might therefore imagine the series (7.4) defined first for sufficiently large  $\epsilon$ , and then continued to  $\epsilon \to 0$ . What we will need of this scaling ansatz, is the form it takes for the Miwa-transformed variables: defining<sup>13</sup>

$$x_r = \frac{1}{r} \sum_j n_j \lambda_j^{-r-1} \tag{7.5}$$

we see that the equation (3.1) for the Miwa transform of the KP times will be recovered provided

$$\lambda_j = 1 + \epsilon z_j \tag{7.6}$$

and  $\lambda_j^{-r-1}$  are expanded in *negative* powers of  $\epsilon$  (i.e., formally, for  $\epsilon \to \infty$ , as noted above), with the result

$$x_r = \frac{1}{r} \sum_{j} n_j \sum_{q \ge r} (-1)^{q+r} \binom{q}{r} (\epsilon z_j)^{-q-1}$$
 (7.7)

From the above digression into the scaling limit of the Toda hierarchy, we borrow the expression (7.6) for the  $\lambda_j^{-14}$ . Substituting it into (7.3), we see that the linear term does not survive in the  $\epsilon \to 0$  limit, while the other terms behave nicely and scale into the operator (3.15) for  $\alpha^2 = 2$ ,

$$\frac{\partial^2}{\partial z_i^2} + \sum_{i \neq i} \frac{1}{z_j - z_i} \left( \frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_i} \right) \tag{7.8}$$

Still, having  $\alpha^2 = 2$  is very restrictive, and one would like to relax this condition. It seems very encouraging in this respect that the desired most general master operator differs from the simplest one, (7.8), by introducing *integer* coefficients in front of its various terms. That is, for

<sup>&</sup>lt;sup>13</sup>the power -r-1, instead of -r, is in the Toda case due to the presence of the 'discrete time' s, which also does scale along with the  $x_r$  and gives rise to the first (the lowest) time of the KP hierarchy.

<sup>&</sup>lt;sup>14</sup>Note, however, that the idea that the Kontsevich integral (7.1) has a direct relevance to the Toda hierarchy, is supported by the observation that one can introduce the discrete time s into it, simply by inserting  $X^s$  as a pre-exponential factor into the integrand.

a (p', p) minimal model, the correlation functions of a product of the dressed (as in Sect. 5) primary fields  $\Phi_{m'_i m_j}(z_j)$ ,

$$\Phi_{m'_{j}m_{j}} = e^{i\alpha_{m'_{j}m_{j}}\varphi}, \quad 1 \le m \le p-1, \quad 1 \le m' \le p'-1$$

$$\alpha_{m'm} = \frac{1-m}{2}\sqrt{\frac{2p'}{p}} - \frac{1-m'}{2}\sqrt{\frac{2p}{p'}}$$
(7.9)

are annihilated by the master operator

$$\frac{2}{\alpha^{2}} \frac{\partial^{2}}{\partial z_{i}^{2}} + \sum_{j \neq i} \frac{1}{z_{j} - z_{i}} \left( \frac{\partial}{\partial z_{j}} - \frac{\alpha_{m'_{j}m_{j}}}{\alpha_{21}} \frac{\partial}{\partial z_{i}} \right)$$

$$= \frac{1}{p} \left\{ p' \frac{\partial^{2}}{\partial z_{i}^{2}} + \sum_{j \neq i} \frac{1}{z_{j} - z_{i}} \left( p \frac{\partial}{\partial z_{j}} + \left[ m_{j}p' - m'_{j}p + p - p' \right] \frac{\partial}{\partial z_{i}} \right) \right\} \tag{7.10}$$

(of course, the insertion at  $z_i$  is fixed to be  $\Phi_{21}$ ). Remarkably, the operator inside the curly brackets contains only integer coefficients! It remains to be shown whether by arranging the multiplicities of the  $\Lambda$  eigenvalues, one can match the coefficients in (7.10).

#### 7.2 W-constraints and higher decoupling equations

If the matter central charge d is fixed to the minimal-models series, then factoring out the null-vector leads to a bona fide minimal model. Now, thinking in terms of the minimal models, how can one 'unkontsevich' the *higher* null-vector decoupling equations? A non-trivial realization of the relation, noted in a somewhat different context in [39], between null-vector and W-algebra structures, seems to emerge in the present approach as well. Recall that the symmetries of the KP hierarchy are the implemented most easily on the dressing operators by (see [28])

$$\delta K = \left( K e^{\varepsilon P} \delta(v, D + \varepsilon J) K^{-1} \right)_{-} K \tag{7.11}$$

(with P defined in eq.(2.12)). This can be rewritten in a form which stresses the 'bilocal' structure,

$$\delta K = \psi(t, v + (1 - J)\varepsilon) \circ D^{-1} \circ \psi^*(t, v - J\varepsilon)K$$
(7.12)

whence it is immediate to derive the corresponding variation of the tau function,

$$\delta \tau = \frac{1}{\varepsilon} \exp \left\{ \xi(t, v + (1 - J)\varepsilon) - \xi(t, v - J\varepsilon) \right\}$$

$$\times \exp \left\{ \sum_{k \ge 1} \frac{\varepsilon^k}{k} \left( J^k - (J - 1)^k \right) \sum_{r \ge 1} v^{-r - k} \binom{k + r - 1}{r} \frac{\partial}{\partial t_r} \right\} \tau$$
(7.13)

Now, the set of the higher constraints implied by the Virasoro constraints reads

$$\frac{1}{2\pi i} \oint \frac{dv}{v - z} \left( K \left( e^{\varepsilon P} \delta(v, D + \varepsilon J) - \delta(v, D) \right) K^{-1} \right)_{-} = 0.$$
 (7.14)

Repeating the steps (7.12) and (7.13), and performing the substitution (3.1) in the  $\xi$ -factor, we arrive at,

$$\frac{1}{2\pi i} \oint \frac{dv}{v - z_i} \frac{1}{\varepsilon} \left( \exp\left\{ -\sum_{k \ge 1} \frac{\varepsilon^k}{k} \left( J^k - (J - 1)^k \right) \sum_j \frac{n_j}{(v - z_j)^k} \right\} \right) \times \exp\left\{ \sum_{k \ge 1} \frac{\varepsilon^k}{k} \left( J^k - (J - 1)^k \right) \sum_{r \ge 1} v^{-r - k} \binom{k + r - 1}{r} \frac{\partial}{\partial t_r} \right\} - 1 \right) \tau = \sum_{r \ge 1} z_i^{-r - 1} \frac{\partial}{\partial t_r} \tau \tag{7.15}$$

The conjecture is that by expanding an equation of the type of (7.15) in powers of  $\varepsilon$  (this equation by itself seems too naive to be *the* one needed), and for certain J-dependent values of  $n_i$ , one would arrive at a set of the higher decoupling equations. From these, expressions for the null vectors could in turn be extracted. Besides the level-2/Virasoro case considered in this paper, a version of the level-3 decoupling equation was shown recently [41] to correspond via the Kontsevich-Miwa transform to  $W^{(3)}$  constraints on the KP hierarchy.

#### 7.3 Master equations on Riemann surfaces

A comment is in order concerning the 'global' structure (or, the 'boundary conditions') of the Kontsevich-transformed tau functions. The master operator and hence eq.(5.19), with the characteristic  $(z_i - z_j)$  denominators, are written down in a given coordinate system. The coordinate patch must cover the neighbourhood of the 'infinity' which contains all the points  $z_j$ . Otherwise, it may be an arbitrary neighbourhood on a Riemann surface. That is, 'closing up' the neighbourhood to the Rimann sphere  $\mathbb{CP}^1$ , we imply certain boundary conditions on the tau function subjected to the master equation. If, however, the equation can be *covariantly* carried over to the whole of a Riemann surface glued to the patch, then it should be possible to impose the corresponding boundary conditions on  $\tau$ , i.e, consider it as a solution on the Riemann surface.

Indeed, let S be a Riemann surface of genus g and E(P,Q) its associated prime form [40]. For P and Q both in the coordinate neighbourhood, one has

$$\ln E(z,y) = \ln(z-y) + \sum_{m,n\geq 1} Q_{mn} \frac{z^{-m}y^{-n}}{mn},$$
(7.16)

(the coordinate system is centered at an  $R \in S$ ,  $z^{-1}(R) = 0$ .) Choose  $(g-1)Q \equiv (g-1)(2J-1)$  points  $P_{\alpha} \in S$ . Let  $\tau(t)$  denote, as before, a KP tau function constrained with the help of the Virasoro generators  $\mathsf{L}_{\geq -1}$ , eq.(2.5). To separate the factors that carry a dependence on the details (such as the center etc.) of the coordinate neighbourhood, define a function  $\hat{\tau}$  by (cf. [29]),

$$\tau(t) = \mathcal{N}e^{\frac{1}{2}\sum_{r,s,\geq 1}Q_{rs}t_{r}t_{s}} \prod_{\alpha=1}^{(g-1)Q} e^{-\sum_{r\geq 1}rt_{r}\int_{z}^{P_{\alpha}}\omega^{(r)}} \\
\times \exp Q\left\{\frac{1}{2\pi i}\sum_{r\geq 1}rt_{r}\sum_{i=1}^{g}\oint_{a_{i}}\omega_{i}(u)\int_{z}^{u}\omega^{(r)} + \sum_{r\geq 1}t_{r}\left(-z^{r} + \sum_{s\geq 1}Q_{rs}\frac{z^{-s}}{s}\right)\right\} \cdot \widehat{\tau} \tag{7.17}$$

where  $\omega^{(r)}$  are the meromorphic differentials with a pole at R,  $\omega^{(r)}(z) \equiv \omega_R^{(r)}(z)$  where, more generally,

 $\omega_a^{(r)}(z) = \frac{1}{r!} d_z \frac{\partial^r}{\partial a^r} \ln E(a, z)$  (7.18)

is the meromorphic differential with a pole at a of order r+1 and holomorphic everywhere else on S. Further,  $\omega_i$ ,  $i=1,\ldots,g$  are the holomorphic differentials normalized by  $\oint_{a_i} \omega_i = 0$  where  $a_i$  are the a-cycles in homology. The RHS of (7.17) is independent of a point with coordinate z. Finally,  $\mathcal{N}$  is a normalization factor,

$$\mathcal{N} = \prod_{\alpha < \beta} E(P_{\alpha}, P_{\beta}) \prod_{\beta} \sigma(P_{\beta})^{Q}, \tag{7.19}$$

where

$$\sigma(u) = \exp \frac{-1}{2\pi i} \sum_{i=1}^{g} \oint_{a_i} \omega_i(y) \ln E(y, u)$$

$$(7.20)$$

is Fay's g/2-differential.

Performing in (7.17) the substitution (3.1), we bring it to the form,

$$\tau\{z_{j}\} = \prod_{j < k} \left(\frac{E(z_{j}, z_{k})}{z_{j} - z_{k}}\right)^{n_{j}n_{k}} \prod_{\alpha = 1}^{(g-1)Q} \prod_{j} \frac{E(R, P_{\alpha})^{n_{j}} \sigma(R)^{QN_{j}}}{E(z_{j}, P_{\alpha})^{n_{j}} \sigma(z_{j})^{Qn_{j}}} \prod_{\alpha < \beta} E(P_{\alpha}, P_{\beta}) \prod_{\beta} \sigma(P_{\beta})^{Q} \cdot \widehat{\tau}\{z_{j}\}$$
(7.21)

The dependence on the reference point R drops out for the 'neutral' sets  $\sum_{j} n_{j} = 0$ . Then the points  $P_{\alpha}$  can be viewed as an addition to the  $\{z_{j}\}$ , entering with prescribed Miwa coefficients

$$n_{\alpha} = -1, \quad \alpha = 1, \dots, (g-1)(2J-1).$$
 (7.22)

Thus, the Miwa transform associated to a given Riemann surface and a coordinate neighbourhood chosen on it, reads

$$t_r = \frac{1}{r} \sum_j n_j z_j^{-r}, \quad \sum_j n_j = -(g-1)(2J-1).$$
 (7.23)

The number (g-1)(2J-1) is of course the RHS of the Riemann-Roch theorem (and should therefore be modified accordingly for g=0, 1 and Q=1).

In the master equation which holds for  $\hat{\tau}\{z_j\}$  all the 'bare'  $(z_i - z_j)^{-1}$  get replaced by the corresponding meromorphic differentials constructed out of  $E(z_i, z_j)$  (other terms appear as well, among them the projective connection on S [40]). This equation being in this sense covariant, means that  $\hat{\tau}\{z_j\}$  may be extended to the rest of S.

We have thus shown that the master equation can be extended onto higher-genus Riemann surfaces. A field-theoretic construction providing a solution on such a surface is still to be clarified. As to the *equation* alone, we have seen that, as expected, "Riemann surfaces of different genera solve equally well the integrable equations".

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