# Quantum light depolarization: the phase-space perspective

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Quantum light depolarization is handled through a master equation obtained by coupling dispersively the field to a randomly distributed atomic reservoir. This master equation is solved by transforming it into a quasiprobability distribution in phase space and the quasiclassical limit is investigated.

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#### I. INTRODUCTION

Polarization of light is a key concept that has deserved a lot of attention over the years. Apart from its fundamental significance, it is also of interest in several active technological fields. In many of these applications it is crucial to determine the decrease of the degree of polarization experienced by the light when traversing an optical system: we refer to this as depolarization [1].

In classical optics, this depolarization is ascribed, broadly speaking, either to birefringence (as it usually happens, e.g., in optical fibers [2, 3, 4, 5, 6, 7, 8]) or to scattering by randomly distributed particles [9, 10, 11]. In both cases the net result is an effective anisotropy that leads to a decorrelation of the phases of the electric field vector.

In quantum optics, a sensible approach to deal with this decorrelation is through the notion of decoherence, by which we loosely understand the appearance of irreversible and uncontrollable quantum correlations when a system interacts with its environment [12]. Usually, decoherence is accompanied by dissipation, i.e., a net exchange of energy with the environment. However, we are interested in the case of pure decoherence (also known as dephasing), for which the process of energy dissipation is negligible. Models in which the populations do not change, while the coherences are strongly decaying, are at hand [13, 14, 15]. Since a good knowledge of dephasing is of utmost importance (prominently for quantum information processing, where operations completely rely on the presence of coherence), these models have been successfully applied to dephasing in systems such as quantum dots [16, 17, 18, 19], Josephson junctions [20, 21, 22, 23], or general quantum registers [24], to cite only a few relevant examples.

In the modern parlance of quantum information it is usual to call depolarizing channel a decoherence induced by an unbiased noise generating bit-flip and phase-flip errors [25]. While this terminology fits well with abstract qubits, whenever light is concerned there is an extra essential ingredient to be taken into account: quantum polarization has a quite natural su(2) invariance that leads to a structure of invariant subspaces [26]. On physical grounds, we argue that this structure must be preserved in the evolution, which makes previous approaches to fail in this case.

The purpose of this paper is twofold. First, we wish to pro-

vide a simple model that goes around this drawback and gives a picture of the mechanisms involved in depolarization. The main idea is to couple the field dispersively to a randomly distributed atomic reservoir: the resulting master equation has a quite appealing structure that complies with our requirements of su(2) invariance. Our second goal is to solve this equation in phase space, presenting then a representation of the depolarizing dynamics that makes an easy contact with the classical one on the Poincaré sphere.

# II. SU(2) POLARIZATION STRUCTURE

We assume a monochromatic plane wave propagating in the z direction, whose electric field lies in the xy plane. Under these conditions, we have a two-mode field that can be described by two complex amplitude operators. They are denoted by  $\hat{a}_+$  and  $\hat{a}_-$ , when using the basis of circular (right and left) polarizations, which we shall employ in this paper. The commutation relations of these operators are standard:

$$[\hat{a}_{\lambda}, \hat{a}_{\lambda'}^{\dagger}] = \delta_{\lambda\lambda'} \hat{\mathbb{1}}, \qquad \lambda, \lambda' \in \{+, -\}.$$
 (2.1)

The Stokes operators are then defined as the quantum counterparts of the classical variables, namely [27, 28, 29, 30, 31]

$$\hat{S}_{0} = \hat{a}_{+}^{\dagger} \hat{a}_{+} + \hat{a}_{-}^{\dagger} \hat{a}_{-} , \qquad \hat{S}_{1} = \hat{a}_{+}^{\dagger} \hat{a}_{-} + \hat{a}_{-}^{\dagger} \hat{a}_{+} ,$$

$$\hat{S}_{2} = i(\hat{a}_{+} \hat{a}_{-}^{\dagger} - \hat{a}_{+}^{\dagger} \hat{a}_{-}) , \qquad \hat{S}_{3} = \hat{a}_{+}^{\dagger} \hat{a}_{+} - \hat{a}_{-}^{\dagger} \hat{a}_{-} ,$$

$$(2.2)$$

and their mean values are precisely the Stokes parameters  $(\langle \hat{S}_0 \rangle, \langle \hat{\mathbf{S}} \rangle)$ , where  $\hat{\mathbf{S}} = (\hat{S}_1, \hat{S}_2, \hat{S}_3)$ . Using the relation (2.1), one immediately gets that the Stokes operators satisfy the commutation relations

$$[\hat{\mathbf{S}}, \hat{S}_0] = 0, \qquad [\hat{S}_1, \hat{S}_2] = 2i\hat{S}_3, \qquad (2.3)$$

and cyclic permutations. Since  $\hat{S}_0$  is just the operator representing the total number of photon, the first equation in (2.3) means that  $\hat{\mathbf{S}}$  is measurable in photon-counting experiments and we can treat each energy manifold separately. To bring out this point more clearly, it is advantageous to relabel the standard two-mode Fock basis in the form

$$|N,k\rangle = |k\rangle_{+} \otimes |N-k\rangle_{-}, \qquad k = 0, 1, \dots, N.$$
 (2.4)

For each fixed value of the number of photons N, these states span an invariant subspace of dimension N+1 and the operators  $\hat{\mathbf{S}}$  act therein according to

$$\hat{S}_{+} | N, k \rangle = 2\sqrt{(k+1)(N-k)} | N, k+1 \rangle , 
\hat{S}_{-} | N, k \rangle = 2\sqrt{k(N-k+1)} | N, k-1 \rangle ,$$

$$\hat{S}_{3} | N, k \rangle = 2(k-N/2) | N, k \rangle ,$$
(2.5)

where  $\hat{S}_{\pm} = \hat{S}_1 \pm i\hat{S}_2$  are raising and lowering operators. These invariant subspaces will play a key role in the following.

The quantities S are the generators of the su(2) algebra. The noncommutability of these operators precludes the simultaneous exact measurement of their physical quantities. Among other consequences, this implies that no field state (leaving aside the two-mode vacuum) can have definite nonfluctuating values of all the Stokes operators simultaneously. This is expressed by the uncertainty relation

$$(\Delta \hat{\mathbf{S}})^2 = (\Delta \hat{S}_1)^2 + (\Delta \hat{S}_2)^2 + (\Delta \hat{S}_3)^2 \ge 2\langle \hat{S}_0 \rangle.$$
 (2.6)

Contrary to what happens in classical optics, the electric vector of a monochromatic quantum field never describes a definite ellipse.

Finally, we recall that standard definition of the degree of polarization reads as

$$\mathbb{P} = \frac{|\langle \hat{\mathbf{S}} \rangle|}{\langle \hat{S}_0 \rangle} = \frac{\sqrt{\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 + \langle \hat{S}_3 \rangle^2}}{\langle \hat{S}_0 \rangle} \,. \tag{2.7}$$

Note that  $\mathbb{P}$  depends exclusively on the first moments of the Stokes operators. Higher order moments can be crucial for a full understanding of some phenomena [32, 33, 34], but (2.7) is more than enough for our purposes here.

# III. DEPOLARIZING DYNAMICS IN A NONRESONANT ATOMIC MEDIUM

Pure dephasing dynamics occurs when the system Hamiltonian  $H_{\rm sys}$  commutes (at least, approximately) with the system-reservoir interaction or for initial states that evolve very slowly under the dynamics governed by  $H_{\text{sys}}$  on the time scale of decoherence processes [35]. Various theoretical scenarios have been proposed to deal with this: apart from minor details, all of them can be modelled by a scattering process in which a reservoir quantum can be absorbed or emitted, but the number of excitations in the system is unchanged [13]. These models indeed preserve the invariant subspaces, but produce no thermalization: they merely maintain the occupation probabilities, while erasing all coherences. Therefore, they fail to describe light depolarization because depolarization not only preserves the invariant subspaces, but the steady state in each one of them must be a diagonal state [36]. We believe that these conditions are essential to ensure a correct description of any depolarizing process.

To solve these difficulties we take another route: we assume that the field mode of frequency  $\omega$  propagates through a material medium made of a collection of two-level atoms with

transition frequencies  $\omega_a$ . In consequence, our basic system is represented by (in units  $\hbar=1$  that we shall use throughout all this paper)

$$\hat{H}_{\text{sys}} = \hat{H}_{\text{f}} + \hat{H}_{\text{a}} + \hat{H}_{\text{fa}},$$
 (3.1)

where

$$\hat{H}_{f} = \sum_{\lambda=\pm} \omega \, \hat{a}_{\lambda}^{\dagger} \hat{a}_{\lambda} \,,$$

$$\hat{H}_{a} = \sum_{a} \frac{1}{2} \omega_{a} \, \hat{\sigma}_{a}^{z} \,,$$

$$\hat{H}_{fa} = \sum_{a} \sum_{\lambda=\pm} (g_{a\lambda} \, \hat{\sigma}_{a}^{-} \hat{a}_{\lambda}^{\dagger} + g_{a\lambda}^{*} \, \hat{\sigma}_{a}^{+} \hat{a}_{\lambda}) \,.$$
(3.2)

Here, as is normal in practice, we have described each individual atom in terms of the standard Pauli operators. The sum over a runs over all the atoms and the interaction  $\hat{H}_{\rm fa}$  is written in the dipolar and rotating-wave approximations. In addition, the atoms are randomly distributed so the coupling constants  $g_{a\lambda}$  have random phases: we thus write them as  $g_{a\lambda} = |g_a|e^{i\lambda\varphi_a/2}$ , for  $\lambda = \pm$ . This random-phase approximation [37] is used in many areas of physics to estimate response functions and it works properly in the longwavelength limit, a hypothesis implicit in the form of the Hamiltonian (3.2).

It is a well-known fact that atoms decay irreversibly. This is usually assigned to their interaction with the continuum of modes of an additional thermal electromagnetic environment. When we take this point into account, the density matrix for the system (3.2) evolves according to

$$\dot{\hat{\varrho}}_{\text{sys}}(t) = -i[\hat{H}_{\text{sys}}, \hat{\varrho}_{\text{sys}}] 
+ \sum_{a} \frac{\gamma_{a}}{2} \{ (\bar{n}_{a} + 1) \mathcal{L}[\hat{\sigma}_{a}^{-}] \hat{\varrho}_{\text{sys}} + \bar{n}_{a} \mathcal{L}[\hat{\sigma}_{a}^{+}] \hat{\varrho}_{\text{sys}} \},$$
(3.3)

where  $\mathcal{L}[\hat{C}]$  are Lynblad superoperators [38]

$$\mathcal{L}[\hat{C}]\,\hat{\varrho} = 2\hat{C}\,\hat{\varrho}\hat{C}^{\dagger} - \{\hat{C}^{\dagger}\hat{C},\hat{\varrho}\}\,,\tag{3.4}$$

and  $\gamma_a$  is the decay constant of the ath atom due to its coupling to the thermal environment with  $\bar{n}_a$  excitations. Note that, by introducing a different bosonic reservoir for each atom, we avoid the occurrence of any collective effect. Whereas a collective dephasing can introduce remarkable dynamical changes [39, 40, 41, 42, 43, 44], we expect it plays no relevant role in understanding the quasiclassical limit of depolarizing dynamics we are interested here [45].

The properties of a random medium are well reproduced when  $\bar{n}_a \gg 1$  [46, 47]. In this high-temperature limit the effect of spontaneous emission can be disregarded in comparison with stimulated processes. Emission into the reservoir and absorption from the reservoir therefore become identical; i. e., they balance each other in the stationary state: the emission and absorption processes depend solely on the initial population of the atomic state. Consequently, the steady-state reduced density operator is approximately given by a mixture

of equally populated atomic states and the density matrix of the *a*th atom becomes thus diagonal  $(\hat{\varrho}_a = \frac{1}{2}\hat{\mathbb{1}})$ .

If  $\Delta_a = \omega_a - \omega$  is the detuning, we consider the far off-resonant regime

$$|g_a| \ll |\Delta_a| \,. \tag{3.5}$$

In this limit we can adiabatically eliminate the atomic variables and obtain a master equation that, after averaging over the random phases, reads as (see Ref. [48] for technical details on the derivation)

$$\dot{\hat{\varrho}} = -i\omega[\hat{S}_0, \hat{\varrho}] + 2\gamma \mathcal{L}[\hat{S}_0] \,\hat{\varrho} + \gamma \mathcal{L}[\hat{S}_+] \,\hat{\varrho} + \gamma \mathcal{L}[\hat{S}_-] \,\hat{\varrho} \,, \quad (3.6)$$

where  $\hat{\varrho}(t) = \operatorname{Tr}_{\mathrm{at}}[\hat{\varrho}_{\mathrm{sys}}(t)]$  is the reduced density operator for the field mode and the decoherence rate  $\gamma$  is

$$\gamma = \sum_{a} \frac{|g_a|^4}{\gamma_a \, \Delta^2 \, \bar{n}_a} \,. \tag{3.7}$$

The master equation (3.6) preserves the N-photon subspaces and the steady state in each one of them is a completely random state

$$\hat{\varrho}(t \to \infty) = \frac{1}{N+1} \,\hat{\mathbb{1}} \,, \tag{3.8}$$

which is an important benefit of our approach. The depolarization rates (3.7) are very small when compared with other typical system parameters (observe the dependence on  $\Delta_a^{-2}$ ), in agreement with experimental observations.

The terms  $\mathcal{L}[\hat{S}_{\pm}]$  describe depolarization in each invariant subspace, meanwhile the action of  $\mathcal{L}[\hat{S}_0]$  therein is trivial. Nevertheless,  $\mathcal{L}[\hat{S}_0]$  is responsible for the relative phase decay between blocks of the density matrix corresponding to different excitation numbers.

As an illustrative example, we focus on one-photon states. We are then within a two-dimensional subspace and the density matrix can be expressed as

$$\hat{\varrho} = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} = \frac{1}{2} (\mathbb{1} + \mathbf{r} \cdot \hat{\boldsymbol{\sigma}}), \qquad (3.9)$$

where  $\mathbf{r}=2\,\mathrm{Tr}\,(\hat{\varrho}\hat{\boldsymbol{\sigma}}),\,\hat{\boldsymbol{\sigma}}$  being the Pauli matrices. The vector  $\mathbf{r}$  satisfies  $|\mathbf{r}|\leq 1$ , with the equality for pure states (that are all them SU(2) coherent states [49]).

In this subspace the master equation (3.6) can be recast in terms of  $\mathbf{r}(t)$  and one obtains the solution as

$$x(t) = x(0)e^{-\gamma t},$$
  
 $y(t) = y(0)e^{-\gamma t},$   
 $z(t) = z(0)e^{-2\gamma t}.$ 
(3.10)

If we apply the definition (2.7) to this case, we get

$$\mathbb{P}(t) = \frac{1}{2} |\mathbf{r}(t)| e^{-\gamma t}. \tag{3.11}$$

This degree tends then to zero with a typical time scale  $\gamma^{-1}$ , which, as commented before, is exceedingly large.

## IV. THE QUASICLASSICAL REGIME

The first term in the master equation (3.6) produces only free evolution and can be dropped in what follows. In consequence, we consider the depolarizing equation

$$\dot{\hat{\varrho}} = \Gamma \mathcal{L}[\hat{S}_0] \,\hat{\varrho} + \gamma \mathcal{L}[\hat{S}_+] \,\hat{\varrho} + \gamma \mathcal{L}[\hat{S}_-] \,\hat{\varrho} \,, \tag{4.1}$$

where, for the sake of generality, we have considered different decoherence rates for  $\hat{S}_0$  and  $\hat{S}_\pm$ . To work out the solutions of (4.1) we resort to phase-space methods, since they provide the most suitable approach to study the quasiclassical limit. According to the discussion in Sect. II, one could think in using quasiprobability distributions on the sphere [50, 51, 52, 53]. However, although naturally related to the SU(2) group, they are not so widely employed by the quantum-optics community, so we prefer to take a more standard approach. Therefore, we shall use the *s*-parametrized quasiprobability distributions introduced by Cahill and Glauber [54], which for our two-mode fields read as

$$W^{(s)}(\alpha_{+}, \alpha_{-}) = \frac{1}{\pi^{2}} \int d^{2}\beta_{+} d^{2}\beta_{-} \chi^{(s)}(\beta_{+}, \beta_{-})$$

$$\times \exp \left[ \sum_{\lambda = \pm} (\alpha_{\lambda} \beta_{\lambda}^{*} - \alpha_{\lambda}^{*} \beta_{\lambda}) \right], \quad (4.2)$$

where  $\chi^{(s)}(\beta_+,\beta_-)$  is the s-ordered characteristic function

$$\chi^{(s)}(\beta_+, \beta_-) = \text{Tr}[\hat{\varrho} \, \hat{D}^{(s)}(\beta_+, \beta_-)],$$
(4.3)

and  $\hat{D}^{(s)}(\beta_+,\beta_-)$  is the two-mode displacement operator

$$\hat{D}^{(s)}(\beta_+, \beta_-) = \prod_{\lambda = +} e^{s|\beta_\lambda|^2/2} \exp(\beta_\lambda \hat{a}_\lambda^\dagger - \beta_\lambda^* \hat{a}_\lambda). \quad (4.4)$$

In the three special cases s=0,+1, and -1, one easily recognizes the Husimi Q, the Wigner W, and the Glauber-Sudarshan P functions, respectively. In terms of  $W^{(s)}$ , Eq. (4.1) can be transformed into a differential equation using the following rules of mapping [55]

$$\hat{a}_{\lambda}\hat{\varrho} \mapsto \left(\alpha_{\lambda} + \frac{1-s}{2} \frac{\partial}{\partial \alpha_{\lambda}^{*}}\right) W^{(s)}, 
\hat{a}_{\lambda}^{\dagger}\hat{\varrho} \mapsto \left(\alpha_{\lambda}^{*} - \frac{1+s}{2} \frac{\partial}{\partial \alpha_{\lambda}}\right) W^{(s)}, 
\hat{\varrho}\hat{a}_{\lambda} \mapsto \left(\alpha_{\lambda} - \frac{1+s}{2} \frac{\partial}{\partial \alpha_{\lambda}^{*}}\right) W^{(s)}, 
\hat{\varrho}\hat{a}_{\lambda}^{\dagger} \mapsto \left(\alpha_{\lambda}^{*} + \frac{1-s}{2} \frac{\partial}{\partial \alpha_{\lambda}}\right) W^{(s)}.$$
(4.5)

For definiteness, we choose to work with the P function, which will facilitate the following calculations. Applying the rules (4.5) for each mode, we get that Eq. (4.1) can be equivalently written as

$$\dot{P}(\alpha_{+}, \alpha_{-}) = -\Gamma \left( \alpha_{+} \partial_{\alpha_{+}} - \alpha_{+}^{*} \partial_{\alpha_{+}^{*}} - \alpha_{-} \partial_{\alpha_{-}} + \alpha_{-}^{*} \partial_{\alpha_{-}^{*}} \right)^{2} P(\alpha_{+}, \alpha_{-}) 
+ 2\gamma \left[ |\alpha_{-}|^{2} \partial_{\alpha_{+}} \partial_{\alpha_{+}^{*}} + |\alpha_{+}|^{2} \partial_{\alpha_{-}} \partial_{\alpha_{-}^{*}} - \alpha_{+} \alpha_{-} \partial_{\alpha_{+}} \partial_{\alpha_{-}} - \alpha_{+}^{*} \alpha_{-}^{*} \partial_{\alpha_{+}^{*}} \partial_{\alpha_{+}^{*}} \right. 
\left. - \frac{1}{2} \left( \alpha_{+} \partial_{\alpha_{+}} + \alpha_{+}^{*} \partial_{\alpha_{+}^{*}} + \alpha_{-} \partial_{\alpha_{-}} + \alpha_{-}^{*} \partial_{\alpha_{-}^{*}} \right) \right] P(\alpha_{+}, \alpha_{-}).$$
(4.6)

At first glance, this equation looks very intricate to be of any practical value. However, let us introduce the following differential operators

$$\hat{S}_{0} = \alpha_{-} \partial_{\alpha_{-}} - \alpha_{-}^{*} \partial_{\alpha_{-}^{*}} + \alpha_{+} \partial_{\alpha_{+}} - \alpha_{+}^{*} \partial_{\alpha_{+}^{*}}, 
\hat{S}_{\pm} = \alpha_{\mp} \partial_{\alpha_{\pm}} - \alpha_{\pm}^{*} \partial_{\alpha_{\mp}^{*}}, 
\hat{S}_{3} = \alpha_{-} \partial_{\alpha_{-}} - \alpha_{-}^{*} \partial_{\alpha_{-}^{*}} - \alpha_{+} \partial_{\alpha_{+}} + \alpha_{+}^{*} \partial_{\alpha_{-}^{*}},$$
(4.7)

which are nothing but a differential realization of the Stokes operators (i.e., of the u(2) algebra). It turns out that Eq. (4.6) reduces to the very simple factorized form

$$\dot{P}(\alpha_+, \alpha_-) = -4 \left[ \Gamma \hat{\mathcal{S}}_0^2 + \gamma (\hat{\mathcal{S}}_+ \hat{\mathcal{S}}_- + \hat{\mathcal{S}}_- \hat{\mathcal{S}}_+) \right] P(\alpha_+, \alpha_-). \tag{4.8}$$

But  $\hat{S}^2 = \hat{S}_3^2 + (\hat{S}_+ \hat{S}_- + \hat{S}_- \hat{S}_+)/2$  is the Casimir operator for su(2), which reduces to  $S(S+1)\hat{\mathbb{I}}$  in each invariant subspace (S taking only integer or half-odd integer values). Then, this master equation has the formal solution

$$P(\alpha_{+}, \alpha_{-}|t) = \exp\{-4[\Gamma \hat{S}_{0}^{2} + 2\gamma(\hat{S}^{2} - \hat{S}_{3}^{2})]t\} \times P(\alpha_{+}, \alpha_{-}|t=0).$$
(4.9)

Because of the properties of the P representation, the average value of any observable  $\hat{O}(t)$  can be recast as

$$\langle \hat{O}(t) \rangle = \int d^2 \alpha_+ d^2 \alpha_- P(\alpha_+, \alpha_- | t = 0)$$

$$\times \exp\{-4[\Gamma \hat{S}_0^2 + 2\gamma (\hat{S}^2 - \hat{S}_3^2)] \langle \alpha_+ \alpha_- | \hat{O} | \alpha_+ \alpha_- \rangle,$$
(4.10)

where by  $\hat{O}$  we mean the operator at t=0. The problem for a practical application of this equation is that we need to give meaning to the action of the exponential. To this end, we first introduce the parametrization

$$\alpha_{+} = r e^{-i(\phi + \psi)/2} \cos(\theta/2),$$

$$\alpha_{-} = r e^{i(\phi - \psi)/2} \sin(\theta/2),$$
(4.11)

where  $r^2=|\alpha_+|^2+|\alpha_-|^2$  is a radial variable related with the global intensity. The angles  $\theta$  and  $\phi$  can be interpreted as the polar and azimuthal angles, respectively, on the Poincaré sphere, while  $\psi$  is a common phase. These coordinates reflect the fact that polarization needs only three independent quantities to be fully characterized: the amplitudes of each mode and the relative phase between them.

If we concentrate in the evolution of the components of the Bloch vector  $\langle \hat{\mathbf{S}} \rangle$ , we have that

$$\langle \alpha_{+}, \alpha_{-} | \hat{S}_{3} | \alpha_{+}, \alpha_{-} \rangle = r^{2} \cos \theta = r^{2} D_{00}^{1}(\phi, \theta, \psi),$$

$$\langle \alpha_{+}, \alpha_{-} | \hat{S}_{\pm} | \alpha_{+}, \alpha_{-} \rangle = r^{2} e^{\mp i\phi} \sin \theta = \sqrt{2} r^{2} D_{+10}^{1}(\phi, \theta, \psi),$$
(4.12)

where  $D_{mm'}^S$  are Wigner D-functions [56], which constitute an orthogonal basis of complex-valued functions defined on the irreducible carrier subspaces of su(2), so that the action of the operators (4.7) on them is standard. Then a direct application of (4.10) yields

$$\langle \hat{S}_{3}(t) \rangle = \exp(-8\gamma t) \langle \hat{S}_{3}(0) \rangle,$$

$$\langle \hat{S}_{\pm}(t) \rangle = \exp[-(2\gamma + \Gamma)t] \langle \hat{S}_{\pm}(0) \rangle.$$
(4.13)

The procedure can be applied much in the same way for the field amplitudes; we merely quote the simplest results:

$$\langle \hat{a}_{\pm}(t) \rangle = \exp[-(\gamma + \Gamma/4)t] \langle \hat{a}_{\pm}(0) \rangle,$$

$$\langle \hat{a}_{+}^{2}(t) \rangle = \exp[-2(\gamma + \Gamma)t] \langle \hat{a}_{+}^{2}(0) \rangle.$$
(4.14)

The rest of the moments can be obtained in similar fashion. On the other hand, in terms of (4.11) the operators  $\hat{S}_0$  and  $\hat{S}_3$  take the suggestive form [56]

$$\hat{\mathcal{S}}_0 = -i\partial_{\psi}\,,\qquad \hat{\mathcal{S}}_3 = -i\partial_{\phi}\,,$$
 (4.15)

and the exact solution of (4.8) can be expressed as

$$P(\phi, \theta, \psi|t) = \sum_{S}^{\infty} \sum_{m,m'=-S}^{S} c_{mm'}^{S} D_{mm'}^{S}(\phi, \theta, \psi) \times \exp\{[-8\gamma S(S+1) + 8\gamma m^{2} - 4\Gamma {m'}^{2}]t\},$$
(4.16)

where the coefficients  $c_{nn^\prime}^S$  are determined by the initial state:

$$c_{mm'}^{S} = \frac{8\pi^{2}}{2S+1} \int_{0}^{2\pi} d\psi \int_{0}^{\pi} d\theta \sin\theta \int_{0}^{2\pi} d\phi D_{mm'}^{S^{*}}(\phi, \theta, \psi) \times P(\phi, \theta, \psi | t = 0).$$
(4.17)

We can proceed to integrate over the physically irrelevant global phase  $\psi$ , obtaining in this way the P function over the

unit sphere. The result is

$$P(\phi, \theta|t) = \sum_{S}^{\infty} \sqrt{\frac{4\pi}{2S+1}} \sum_{m=-S}^{S} c_{m0}^{S} Y_{Sm}(\theta, \phi) \times \exp\{-8\gamma [S(S+1) - m^{2}]t\}, \quad (4.18)$$

where  $Y_{Sm}(\theta,\phi)$  are spherical harmonics. This is a closed formula that can be considered as our major result: it allows to trace the depolarizing dynamics on the Poincaré sphere for arbitrary states. For example, for (quadrature) coherent states, we have

$$P(\alpha_+, \alpha_-|t=0) = \delta^2(\alpha_+ - r_0)\delta^2(\alpha_- - r_0), \quad (4.19)$$

where we have assumed that both modes have the same real amplitude  $r_0$ . We then get

$$P(\phi, \theta, \psi | t = 0) = 8\sqrt{2} \frac{\delta(r - r_0)}{r^3} \delta(\psi) \delta(\phi) \delta(\theta - \pi/2),$$
(4.20)

and so

$$c_{mm'}^{S} = \frac{64\sqrt{2}\pi^2}{2S+1} D_{mm'}^{S*}(0, \pi/2, 0) \frac{\delta(r-r_0)}{r^3}.$$
 (4.21)

From here we can immediately compute the dynamics of this state.

In view of the delta functions appearing in (4.20), one can argue that the P representation results in a distribution with singularities. We can look instead at smoother quasidistributions, such as, e.g., the Q function. Curiously enough, when we repeat the calculations, we end up with the fact that the evolution equation for the Q function is exactly the same as for the P function. In consequence, the time evolution of the Q function can be expressed as in Eq. (4.16) [obviously, the

coefficients  $c_{mm'}^S$  are determined now as in (4.17), but with P replaced by Q].

This can be used to determine any function of Q. In fact, it has been argued [57] that, for a given field, the distance between its Q function and the Q function for unpolarized light can be taken as a proper degree of polarization in phase space: this distance is, except from constant factors, proportional to  $Q^2$ . Therefore, we can take as an unnormalized depolarization measure

$$\mathbb{D}(t) = \int_0^{2\pi} d\psi \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \, Q^2(\phi, \theta, \psi | t), \quad (4.22)$$

and then we obtain

$$\mathbb{D}(t) = \sum_{S}^{\infty} \frac{4\pi}{2S+1} \sum_{m,m'=-S}^{S} |c_{mm'}^{S}|^{2}$$

$$\times \exp[-8(2\gamma S(S+1) - 2\gamma m^{2} + \Gamma {m'}^{2})] (4.23)$$

Again we can ascertain the corresponding time evolution for arbitrary states.

### V. CONCLUDING REMARKS

In this paper we have reported a comprehensive, simple theory of quantum light depolarization. In our model the field couples dispersively to a randomly distributed atomic reservoir, and the resulting master equation has unique properties that we have explored in detail.

We have solved this master equation resorting to a simple phase-space formalism for polarization on the Poincaré sphere, based on s-ordered quasidistributions for the two basic polarization modes. These results may have interesting consequences to implement experimental procedures for determining polarization properties.

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