The Master Field For 2D QCD On The Sphere

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Abstract

We continue our analysis of the field strength correlation functions of two-dimensional QCD on Riemann surfaces by studying the large N limit of these correlation functions on the sphere for gauge group $U(N)$. Our results allow us to exhibit an explicit master field for the field strength $F_{\mu\nu}$ in a "topological gauge", given by a single master matrix in the Lie algebra of the maximal torus of the gauge group. Field correlators are obtained from traces of products of the master field. We also obtain a master field for the gauge potential A_μ on the sphere, consistent with the master field for the field strength.

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1 Introduction

Two dimensional Yang-Mills theories have provided an arena to test many of the ideas related to non-perturbative features of gauge theories. Since the exact partition function of the theory has been computed by a variety of methods [1, 2, 8, 9], a number of interesting issues may be studied. It has been shown that the $1/N$ expansion of these theories allows them to be formally represented as a string theory, in terms of sums of maps from worldsheets to a target space Σ_g [3]-[7]. It was found by Douglas and Kazakov that the $U(N)$ theory on the sphere has a third-order phase transition at large N when $e^2(Area) = \pi^2$ [11]-[16]. Similar transitions take place on RP_2 and for other gauge groups [14].

In [18] we were able to apply the beautiful path integral methods developed by Blau and Thompson in [8, 9, 10] to study the correlation functions of the field strength in 2d gauge theories over Riemann surfaces. We showed that these correlators are essentially topological in the gauge appropriate to the abelianization methods of Blau and Thompson. In this gauge the "abelian" components of the electric field (*i.e.* in the Lie algebra of the maximal torus T of the gauge group G) carry the essential information about these correlation functions. Moreover, in the large N limit on the sphere, the correlators are dominated by the critical representation found in [11]. We studied [18] the large N limit of the two, four and six-point functions explicitly, and showed that these undergo a second-order phase transition. In fact, all $2n$ point functions have a second-order phase transition [15], while of course all odd point functions vanish.

In this paper we will study the large N limit of these correlation functions on the sphere in more detail. As Douglas as remarked [17], two-dimensional QCD on the Euclidean sphere is the case in two dimensions that may be most representative of higher dimensions, as this is the only case where the free energy behaves as N^2 for large

N, as in higher dimensions, and where a saddle point dominates the path integral. We will show that the field strength correlators factorize at large N as products of gauge invariant correlators, as expected. This, together with the consequences of the abelianization of the theory, will enable us to compute the master field [20, 21] for the electric field on the sphere. This master field is directly and simply related to the critical representation found by Douglas and Kazakov [11]. The master field for the field strength then enables us to compute the master field for the gauge potential on the sphere. These master fields are given in terms of a single master matrix in the Lie algebra of the maximal torus of the gauge group.

In section 2 we present the field strength correlators and study the two and four-point functions on the sphere in the large N limit. A general argument for the detailed structure of arbitrary correlators in the large N limit is presented. In section 3 we exhibit the master field, and in section 4 we close with some comments and conclusions.

2 The Correlators

In [18] we considered the $U(N)$ gauge theory on a Riemann surface Σ_g coupled to a source $J(x)$, and computed the partition function (here presented only for the sphere)

$$Z_{S^2}(J) = \int \mathcal{D}A_\mu \exp[\frac{-1}{2e^2}\int_{S^2} d\mu \, \mathrm{Tr}(\xi^2) + 2\int_{S^2} d\mu \, \mathrm{Tr}(J\xi)] \tag{1}$$

where the scalar fields $\xi^a(x)$ are defined by $F_{\mu\nu}(x) = \xi(x)\sqrt{g(x)}\epsilon_{\mu\nu}$, where $\xi = T^a\xi^a$ with T^a a generator of the group, $d\mu = \sqrt{g(x)}d^2x$ the riemannian measure for the metric $g_{\mu\nu}$ on S^2, and $\epsilon_{\mu\nu}$ is the usual antisymmetric tensor with $\epsilon_{01} = 1$. Functional derivatives of (1) with respect to J give the correlation functions of the field strength. Naively, one might expect the correlators to be trivial because (1) is gaussian in the field strength. However this is not the case. The correlators are non-trivial due to

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topological considerations. Following the elegant path integral methods of Blau and Thompson [8, 9], one writes the action in terms of an auxiliary scalar field which is then conjugated to the Lie algebra t of the maximal torus T of $U(N)$. This produces, in analogy with the Weyl integral formula for integration of class functions on Lie groups, a Weyl determinant. Integrating out the non-diagonal components of the gauge field then produces an effective abelian theory for the t components of the gauge field. However, one must include all T-bundle topologies that were generated by the choice of gauge, which is the origin of the non-trivial behaviour of the electric field correlators [18]. The remaining functional integral over abelian fields can then be computed by means of the Nicolai map [8, 9]. For a clear and detailed explanation of these issues we refer the reader to [8, 9] as well as [10]. The final result is then [18],

$$Z_{S^2}(J) = \sum_l \dim(l)^2 \exp[\frac{-e^2 A(l+\rho)^2}{2}] \frac{1}{|W|} \sum_{\sigma \in W} \exp\{2ie^2 \int_{S^2} d\mu \left[\sigma(l+\rho), J^t\right]\} \quad (2)$$

where A is the area of S^2, l labels irreducible representations of $U(N)$ and ρ is the half-sum of the positive roots of $su(N)$. Notice that here we will use a nonstandard normalization of the partition function, writing $(l+\rho)^2$ instead of the usual $C_2(l)$ in the exponential in (2). This just corresponds to choosing the overall area dependent (but representation independent) multiplicative factor in Z_{S^2}, and is an allowed renormalization choice [2, 9]. This choice corresponds to the elimination of the contact term quadratic in J which is present in the usual normalization [18]. Notice that this term was responsible for the appearence of contact terms in the correlation functions in [18]. With our new choice of normalization, they are renormalized away. Nevertheless we still have,

$$\langle \mathrm{Tr}\xi^2(z) \rangle = 2e^2 \frac{d}{dA} F \quad (3)$$

relating the two-point function and the free energy F, as well as similar results for higher derivatives of the free energy. Thus, no physics is changed by our choice of normalization.

We can compute the electric field correlators from (2). As in [18] all odd-point

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functions will vanish as a consequence of symmetry. Also, notice that as an important consequence of abelianization (and renormalization choices) only the t components of the source J enter in (2), so that only those components of ξ lying in the Lie algebra of the maximal torus T will have nonvanishing correlators. Therefore, in all that follows all Lie algebra indices inside correlation functions will be in t. Differentiating (2) with respect to J at $J = 0$ produces the 2n-point functions (normalized by Z_{S^2}),

$$\langle \xi^{a_1}(z_1) \cdots \xi^{a_{2n}}(z_{2n}) \rangle = (-1)^n \frac{e^{4n}}{Z_{S^2}} \sum_l \dim(l)^2 \exp[\frac{-e^2 A(l+\rho)^2}{2}] \frac{1}{|W|} \sum_{\sigma \in W} (l+\rho)^{\sigma a_1} \cdots (l+\rho)^{\sigma a_{2n}}$$
(4)

(Points on the sphere are denoted by z_i). Notice that for $U(N)$, the Weyl group W is just the symmetric group S_N acting by permutation of the N diagonal entries of elements in t. For the two-point function we then have,

$$\langle \xi^a(z_1) \xi^b(z_2) \rangle = \frac{-e^4}{Z_{S^2}} \sum_l \dim(l)^2 \exp[\frac{-e^2 A(l+\rho)^2}{2}][p^{ab}(l+\rho)^2 + m^{ab} n^2]$$
(5)

(Note we have the normalization $v^2 = \frac{1}{2} \sum_{i=0}^N v^i v^i$, for $v \in t$). From [18]

$$\frac{1}{|W|} \sum_\sigma (l+\rho)^{\sigma_a}(l+\rho)^{\sigma_b} = p^{ab}(l+\rho)^2 + m^{ab} n^2$$
(6)

where

$$p^{ab} = \begin{cases} \frac{-2}{N(N-1)} & \text{if } a \neq b \\ \frac{2}{N} & \text{if } a = b \end{cases} \quad \text{and} \quad m^{ab} = \begin{cases} \frac{1}{N(N-1)} & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$
(7)

and $n = \sum_i l^i$ is the total number of boxes in the Young tableau, of row lengths $l^1 \geq l^2 \geq \cdots \geq l^N$, defined by l. In the large N limit we can use continuum variables,

$$x = \frac{i}{N} \qquad \text{with} \qquad 0 \leq x \leq 1$$

$$l(x) = \frac{l^i}{N} \qquad \text{and} \qquad \sum_i = N \int_0^1 dx$$
(8)

It is then useful to define,

$$h(x) = -l(x) + x - \frac{1}{2} = -(l+\rho)(x)$$
(9)

In this limit the partition function is dominated by the critical representation $\bar{l}$, which solves the saddle point equation for the effective action $S_{eff}(h)$. This is an integral equation for the density of eigenvalues $u(h) = \frac{\partial x(h)}{\partial h} \leq 1$ (so that $\int dx f(x) = \int dh u(h) f(h)$). At weak coupling ($e^2 Area < \pi^2$) this is given by the semi-circle law, while for strong coupling ($e^2 Area > \pi^2$) one has more complicated expressions involving elliptic functions [11]. In both phases of the theory, the critical representation is self-conjugate, so that one has $\bar{n} = n(\bar{l}) = 0$. Therefore, with $\lambda = e^2 N$ held fixed in the large N limit, the two-point function becomes,

$$\langle \xi^i(z_1) \xi^j(z_2) \rangle \longrightarrow -\lambda^2 \delta^{ij} \int_0^1 \bar{h}^2(x) dx$$

$$\text{as } N \to \infty \tag{10}$$

We see that in the large N limit only the diagonal term with $i = j$ case gives a contribution, and that this correlator is proportional to $\text{Tr}\bar{h}^2 = \int_0^1 dx \bar{h}^2(x)$.

The four-point function is,

$$\langle \xi^i(z_1) \xi^j(z_2) \xi^k(z_3) \xi^l(z_4) \rangle = \frac{e^8}{Z_{S^2}} \sum_l \dim(l)^2 \exp[\frac{-e^2 A(l+\rho)^2}{2}] \cdot$$

$$\frac{1}{|W|} \sum_\sigma (l+\rho)^{\sigma_i}(l+\rho)^{\sigma_j}(l+\rho)^{\sigma_k}(l+\rho)^{\sigma_l} \tag{11}$$

The Weyl sum is [18]

$$\frac{1}{|W|} \sum_\sigma (l+\rho)^{\sigma_i}(l+\rho)^{\sigma_j}(l+\rho)^{\sigma_k}(l+\rho)^{\sigma_l} =$$

$$\{a^{ijkl} C_4(l) + b^{ijkl}[(l+\rho)^2]^2 + c^{ijkl} n C_3(l) + d^{ijkl} n^2 (l+\rho)^2 + e^{ijkl} n^4\} \tag{12}$$

where the coefficents of the various W invariant terms are completely symmetric in the indices i,j,k,l. We define Casimir operators by

$$C_k(l) = \sum_i [(l+\rho)^i]^k \qquad \text{for } k \geq 2 \tag{13}$$

(Notice that the usual normalization for $k = 2$ is $C_2(l) = [(l+\rho)^2 - \rho^2]$). With

$$\varepsilon^{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

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we have (with *no sum* on repeated indices in (14)) [18]:

$$
a^{ijkl} = \begin{cases} \frac{-6(N-4)!}{N!} & i,j,k,l \text{ all different} \\ \frac{2(N-3)!}{N!} & \delta^{ij}\varepsilon^{ik}\varepsilon^{il}\varepsilon^{kl} \\ \frac{-(N-2)!}{N!} & \delta^{ij}\delta^{jk}\varepsilon^{il} \\ \frac{-(N-2)!}{N!} & \delta^{ij}\delta^{kl}\varepsilon^{ik} \\ \frac{(N-1)!}{N!} & i=j=k=l \end{cases} \qquad b^{ijkl} = \begin{cases} \frac{12(N-4)!}{N!} & i,j,k,l \text{ all different} \\ \frac{-4(N-3)!}{N!} & \delta^{ij}\varepsilon^{ik}\varepsilon^{il}\varepsilon^{kl} \\ 0 & \delta^{ij}\delta^{jk}\varepsilon^{il} \\ \frac{4(N-2)!}{N!} & \delta^{ij}\delta^{kl}\varepsilon^{ik} \\ 0 & i=j=k=l \end{cases}
$$

$$
c^{ijkl} = \begin{cases} \frac{8(N-4)!}{N!} & i,j,k,l \text{ all different} \\ \frac{-2(N-3)!}{N!} & \delta^{ij}\varepsilon^{ik}\varepsilon^{il}\varepsilon^{kl} \\ \frac{(N-2)!}{N!} & \delta^{ij}\delta^{jk}\varepsilon^{il} \\ 0 & \text{otherwise} \end{cases} \qquad d^{ijkl} = \begin{cases} \frac{-12(N-4)!}{N!} & i,j,k,l \text{ all different} \\ \frac{2(N-3)!}{N!} & \delta^{ij}\varepsilon^{ik}\varepsilon^{il}\varepsilon^{kl} \\ 0 & \text{otherwise} \end{cases}
$$

$$
e^{ijkl} = \begin{cases} \frac{(N-4)!}{N!} & i,j,k,l \text{ all different} \\ 0 & \text{otherwise} \end{cases} \tag{14}
$$

Therefore the four-point function is given by

$$
\langle \xi^i(z_1)\xi^j(z_2)\xi^k(z_3)\xi^l(z_4) \rangle = \frac{e^8}{Z_{S^2}} \sum_l \dim(l)^2 \exp[\frac{-e^2 A(l+\rho)^2}{2}] \cdot
$$
$$
\{a^{ijkl}C_4(l) + b^{ijkl}[(l+\rho)^2]^2 + c^{ijkl}nC_3(l) + d^{ijkl}n^2(l+\rho)^2 + e^{ijkl}n^4\} \tag{15}
$$

The large N limit of the Casimir operators of (13) for the critical representation are given by

$$
C_{2k}(\bar{l}) = N^{2k+1} \int_0^1 dx \bar{h}^{2k}(x) \tag{16}
$$

(Odd degree Casimirs will vanish because $\bar{l}$ is self-conjugate). Therefore, in the large N limit only the a^{ijkl} and b^{ijkl} terms in (15) will contribute. Thus

$$
\langle \xi^i(z_1)\xi^j(z_2)\xi^k(z_3)\xi^l(z_4) \rangle \longrightarrow \lambda^4\{(\delta^{ij}\delta^{jk}\delta^{kl} \int_0^1 dx \bar{h}^4(x)) + (\delta^{ij}\delta^{kl}\varepsilon^{ik} + \text{permut.})(\int_0^1 dx \bar{h}^2)^2\}
$$
$$
\text{as } N \to \infty \tag{17}
$$

where repeated indices are *not* summed. Contract equation (17) with the generators of the maximal torus, $E_{ii}/\sqrt{2}$ (E_{ii} is the diagonal matrix with 1 in the ith position of the diagonal and zeroes everywhere else). The first term in (17) produces $\langle \mathrm{Tr}\xi^4 \rangle$,

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while the second term will give $\langle \mathrm{Tr}\xi^2 \mathrm{Tr}\xi^2 \rangle$. We will now argue that this structure holds for all higher-point functions. The large N limit correlators will vanish unless each t-index appears an even number of times in the correlator, therefore with no essential loss of information, we restrict ourselves to traced correlators (*i.e.* with all indices contracted with the generators of T, as in the example above).

Let k be an even integer, and consider

$$\frac{1}{N!} \sum_\sigma (l + \rho)^{\sigma_{i_1}} \cdots (l + \rho)^{\sigma_{i_k}} \tag{18}$$

in the large N limit. Equation (16) shows that the Casimirs behave at large N as $C_r \sim N^{r+1}$. Further from (4), the correlation function of k electric fields has a factor of $e^{2k} = \frac{\lambda^k}{N^k}$. Therefore, for large N, the nonvanishing terms will be of the form

$$\langle \xi_1 \cdots \xi_k \rangle \sim \alpha \frac{\lambda^k}{N^k} C_{r_1}(\bar{l}) \cdots C_{r_p}(\bar{l}) \tag{19}$$

where $\sum_{i=1}^p r_i = k$, with the numerical coefficent α of order $N^{-p} \sim \frac{(N-p)!}{N!}$. The Weyl group average produces a coefficient of order N^{-p}, for p distinct Lie algebra indices in the set $\{i_1, \ldots, i_k\}$ of indices in the correlator $\langle \xi^{i_1} \cdots \xi^{i_k} \rangle$. Necessarily $p \leq N$. That is, $\{i_1, \ldots i_k\}$ is decomposed in p groups, with r_i elements each, with the same t-index. On the other hand, for such a set of indices, the Weyl group average produces a linear combination of Casimirs of the critical representation $\bar{l}$, with total degree k. Because of the extra power of N that each Casimir carries in (16), only the term with p Casimirs survives in the large N limit. This will be the term with the maximum number of Casimirs. Notice that in this limit one must have each r_i even for a non-vanishing result, since $\bar{l}$ is self-conjugate. Therefore the large N correlators will vanish unless the same index appears an even number of times. When one contracts with the generators of T, $E_{ii}/\sqrt{2}$, each one of the p Casimirs can be identified as coming from a trace. The correlator will be the expectation value of a product of traces of powers of ξ, which will factorize in the expected way (see for example [19]) as the

product of the correlators of the individual traces, *i.e.*

$$\langle \frac{1}{N} \text{Tr} \xi^{r_1} \cdots \frac{1}{N} \text{Tr} \xi^{r_p} \rangle = \langle \frac{1}{N} \text{Tr} \xi^{r_1} \rangle \cdots \langle \frac{1}{N} \text{Tr} \xi^{r_p} \rangle \tag{20}$$

Thus, one only needs to compute the expectation value of a trace of a power of ξ, $\langle \frac{1}{N} \text{Tr} \xi^r \rangle$, which we do in the next section.

3   The Master Field

The fact that only those elements of the field ξ lying in t contribute to the correlation functions makes it very easy to calculate traced correlators. Since $E_{ii} E_{jj} = \delta^{ij} E_{ii}$,

$$\langle \text{Tr} \xi(z_1) \cdots \xi(z_{2k}) \rangle = \delta^{i_1 i_2} \delta^{i_2 i_3} \cdots \delta^{i_{2k-1} i_{2k}} \, 2^{-k} \, \langle \xi^{i_1}(z_1) \cdots \xi^{i_{2k}}(z_{2k}) \rangle \tag{21}$$

where there is *no summation* over repeated indices for the δ^{ij}'s. That is, we must have all indices equal $i_1 = i_2 = \cdots = i_{2k}$. In this case the Weyl group average for the representation l will give precisely a factor of $C_{2k}(l)$. Thus,

$$\langle \text{Tr} \xi(z_1) \cdots \xi(z_{2k}) \rangle = \frac{1}{Z_{S^2}} \frac{(-1)^k e^{4k}}{2^k} \sum_l \dim(l)^2 \exp[\frac{-e^2 A(l+\rho)^2}{2}] C_{2k}(l) \tag{22}$$

In the large N limit this becomes,

$$\frac{1}{N} \langle \text{Tr} \xi(z_1) \ldots \xi(z_{2k}) \rangle \longrightarrow \frac{(-1)^k \lambda^{2k}}{2^k} \int_0^1 dx \bar{h}^{2k}(x) \tag{23}$$

where (16) has been used. Thus in the large N limit, the traced correlators are given by the traces of powers of $\bar{h} = -(\bar{l}+\rho)$, here viewed as a matrix in t, the Lie algebra of the maximal torus. Moreover, for correlation functions of products of p traces, upon contraction with the generators $E_{ii}/\sqrt{2}$, the term that survives in the large N limit is precisely the one given by the product of p Casimirs of the critical representation $\bar{l}$ in such a way that the expectation value factorizes into the product of the expectation values of the individual traces, as in (20).

We now have enough information to construct the master field for the field strength. (For some early references see [19]). Since in this gauge all correlation functions are independent of the positions where the fields ξ are inserted, one needs only *one* master matrix for ξ. The traces of powers of the master matrix, $\hat{\xi}$, should reproduce the correlation functions above, so that

$$\lim_{N \to \infty} \langle \frac{1}{N} \text{Tr} \xi^{2k_1} \cdots \frac{1}{N} \text{Tr} \xi^{2k_p} \rangle = \frac{1}{N} \text{Tr} \hat{\xi}^{2k_1} \cdots \frac{1}{N} \text{Tr} \hat{\xi}^{2k_p} \tag{24}$$

However because of factorization and (23), this is given by

$$(-1)^{k_1} \frac{\lambda^{2k_1}}{2^{k_1}} \int_0^1 dx \bar{h}^{2k_1}(x) \cdots (-1)^{k_p} \frac{\lambda^{2k_p}}{2^{k_p}} \int_0^1 dx \bar{h}^{2k_p}(x)$$

and therefore the master matrix $\hat{\xi}$ is given by

$$\hat{\xi}(x) = i \frac{\lambda}{\sqrt{2}} \bar{h}(x) \qquad \text{for } 0 \leq x \leq 1 \tag{25}$$

where x is a diagonal Lie algebra coordinate in the basis given by the $E_{ii}/\sqrt{2}$. We see that as a remarkable consequence of abelianization, the master matrix lies in t, the Lie algebra of the maximal torus T.

One can find a master field for the gauge field A_μ, which satisfies $D_{\hat{A}} \hat{A}(z) = \hat{F}(z) = \sqrt{g(z)} \hat{\xi}$. Since $\hat{\xi}$ is in t, this equation has solutions with $\hat{A}_\mu$ also in t. Such a solution cannot be globally defined on the sphere as a consequence of the presence of the non-trivial T bundles. (This can also be seen from the fact that $\hat{F}$ is proportional to the riemannian measure on the sphere). This is analogous to what happens with the Dirac monopole. In fact, for the usual riemannian measure on the sphere given by $\sqrt{g} = \frac{A}{4\pi} \sin \theta$, $\hat{F}$ and $\hat{A}$ are given by the solutions for the Dirac monopole tensored with the constant matrix $\hat{\xi}$ in t, *i.e.*

$$\hat{F} = \frac{A}{4\pi} \sin \theta \, d\theta \wedge d\phi \cdot \hat{\xi}$$

$$\hat{A}^+ = \frac{A}{4\pi} (1 - \cos \theta) \, d\phi \cdot \hat{\xi} \qquad \text{on the northern hemisphere}$$

$$\hat{A}^- = -\frac{A}{4\pi} (1 + \cos \theta) \, d\phi \cdot \hat{\xi} \qquad \text{on the southern hemisphere} \tag{26}$$

where A is the area of the sphere.

The topological nature of the gauge invariant correlators is a consequence of the invariance of the theory under area preserving diffeomorphisms. (For non gauge-invariant correlation functions, the fact that they *are* position independent is a consequence of the choice of gauge). This means that the master field on the sphere can be chosen to be position independent, so that one has one master matrix. This is also possible for the plane [20]. Although we have only one master matrix and therefore we didn't use the concept of free random variables of [20, 21, 22], one could still build a Fock space representation for this master matrix as in [20, 21].

4 Conclusions

In this paper we studied the large N limit of the field strength correlation functions for 2d QCD on the sphere. The use of the powerful abelianization methods of [9, 10] provided us with a gauge where the correlators are topological, and where essentially only the abelian components of the fields contribute. This leads to a master field which is position independent, and takes values in t, the Lie algebra of the maximal torus T of the gauge group. It should be stressed that although the non-trivial T bundles on S^2 give an obstruction to the existence of a smooth global *diagonalizing* gauge transformation map for ξ, the *diagonalized* ξ field exists globally on S^2 (see [10]). Therefore in this gauge, the master field for the field strength and the master field for the gauge potential have the same geometry as the Dirac monopole.

A number of applications of the results presented in (26) are presently under investigation.

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