UNIVERSITÀ DEGLI STUDI DI TRIESTE

Dottorato di Ricerca in Fisica

XI Ciclo

Boundary versus Bulk Dynamics of Extended Objects and the Fractal Structure Quantum Spacetime

Dottorando
Dott. Stefano Ansoldi

Tutore
Prof. Tullio Weber
Università di Trieste

Co-tutore **Dott. Euro Spallucci**Università di Trieste

Coordinatore
Prof. Paolo Schiavon
Università di Trieste

Contents	i
List of Figures	vii
List of Tables	ix
List of Definitions	xi
List of Propositions	xv
List of Notations	xxi
I Notations	xxiii
I Acknowledgments	xxv
II Introduction	1
1 Toeholds	15
1.1 Boundary Dynamics	15
1.1.1 "Shadow" Dynamics	16
2 Hamilton–Jacobi String Theory	25
9.1 Droliminarios	25

	2.2	Area Quantization Scheme: Original Formulation	28
	2.3	The Basic Action	30
	2.4	Reparametrized Schild Formulation	40
	2.5	Hamilton–Jacobi Theory	44
		2.5.1 Hamilton–Jacobi Equation: Ogielski Formulation	45
		2.5.2 Hamilton-Jacobi Equation: Reparametrized Formulation	48
	2.6	Classical Area Effective Formulation	52
	2.7	Reparametrized Canonical Formulation	56
	2.8	Covariant Schild Action	60
	G. •		20
3	Stri	ng Functional Quantization	63
	3.1	Quantum Propagator	63
	3.2	Functional Wave Equation	68
	3.3	Computing the Kernel	74
		3.3.1 Integrating the Functional Wave Equation	74
		3.3.2 Integrating the Path–Integral	79
	3.4	Summary, Comments, Highlights and More	81
4	Ger	p-branes	89
	4.1	Reparametrized Schild Action	89
	4.2	<i>p</i> -brane Hamilton Jacobi Theory	94
	4.3	<i>p</i> -brane Quantum Dynamics	97
		4.3.1 Equivalence with Nambu–Goto Dynamics	97
		4.3.2 Functional Schrödinger Equation and p -brane	101
		4.3.3 <i>p</i> -brane Quantum Propagator	102
5	Stri	ng Functional Solutions	107

	5.1	Plane Wave Solution	108
	5.2	Gaussian Loop Wave–Packet	110
6	"Mi	inisuperspace"	113
	6.1	Preliminaries	113
	6.2	The String	114
	6.3	The p -brane	122
7	Frac	etal Strings	125
	7.1	The Shape Uncertainty Principle	125
	7.2	Fractal Strings	130
		7.2.1 The Hausdorf Dimension of a Quantum String	130
		7.2.2 Classical—to—Fractal Geometric Transition	133
8	The	"Double" Classical Limit	137
	8.1	Couplings and Limits	137
	8.2	Classical Limit	138
9	Bou	indary versus Bulk Relation	145
	9.1	Overview	145
	9.2	Loop and String States	146
10	Non	standard & Speculative	15 9
	10.1	Nonstandard Functional Quantization	159
	10.2	Holographic Coordinates and $M\text{-Theory}$	165
		10.2.1 Boundary Shadow Dynamics and $M\text{-Theory}$	165
		10.2.2 Covariant "Speculation"	168
	10.3	Superconductivity and Quantum Spacetime	170

IV	Conclusion	175
\mathbf{A}	Detailed Calculations	183
	A.1 Lagrangian, Hamiltonian and Coefficients!	. 183
	A.2 ξ and π Variation of the Action for a p -brane	. 184
	A.3 Momentum Variation in Reparametrized Hamiltonian Formalism	. 187
	A.4 Hamilton Principle with one Free Boundary	. 188
	A.5 Functional Integration of the $\boldsymbol{\xi}(\boldsymbol{\sigma})$ Fields	. 191
	A.6 Functional Integration of the $\pi(\sigma)$ Fields	. 193
	A.7 Functional Integration of the $P(\sigma)$ Fields	. 194
	A.8 Saddle Point Evaluation	. 195
	A.9 Equations Satisfied by the Kernel	. 196
	A.10 Kernel Functional Wave Equation	. 197
	A.11 Holographic Coordinates: Functional Derivatives	. 198
	A.12 Functional Derivatives of the Classical Action	. 202
	A.13 Solutions for the α and β Kernel Ansatz Parameters	. 204
В	NonStandard Analysis	205
	B.1 Short Introduction	. 205
	B.1.1 Infinitesimals and Infinities	. 205
	B.1.2 Some Examples	. 211
	B.2 Ultra Euclidean Space	. 213
\mathbf{C}	Nonstandard Stochastic Processes	217
	C.1 Functional Spaces	. 217
	C.2 Diffusion Process on ${}^*\mathcal{E}_{\{e_n\}} \left(\mathbb{R}^d \to {}^*\mathbb{R} \right) \dots \dots \dots \dots \dots$	
D	Loop Derivatives.	233

D.1	Overview										 2	233
D.2	Functional and Holographic Derivatives										 2	234

List of Figures

1	History of physics shows that conflicting theories eventually merge into a broader and deeper synthesis.	Wil
2	Logical connection between chapters of the Thesis	

List of Tables

3.1	Loop Space functionals and Boundary fields	87
3.2	The Particle/String "Dictionary"	88
B.1	Computation tables with Infinity, Infinitesimal, Limited and Appreciable numb	ers: (a) sum; (b) subtraction

List of Definitions

1.1	Parameter Space	16
1.2	Boundary Space	16
1.3	Target Space	18
1.4	Tangent and Normal Vectors	18
1.5	Parameter Space Area Variation	21
1.6	Area Funcitonal Derivative	21
2.1	World–Sheet & Parametrization	26
2.2	Classical Closed Bosonic String & Parametrization	26
2.3	Holographic Coordinates	31
2.4	Holographic Functional Derivative	32
2.5	Local World–Sheet Area Velocity	33
2.6	String Area Velocity	33
2.7	Bulk Area Momentum	37
2.8	Boundary Area Momentum	38
2.9	Reparametrized Schild Lagrangian Density	42
2.10	Hamiltonian Full Reparametrized Schild Action	48
2.11	Hamiltonian Reduced Reparametrized Schild Action	50
2.12	Projected World–Sheet Area Momentum	56
2.13	Projected World-Sheet Boundary Area Momentum	56

2.14	Covariant Schild Action	60
3.1	Loop Derivative	73
3.2	Loop Space Dirac Delta Function	77
3.3	Total Area Loop Momentum Operator	85
4.1	<i>p</i> -brane Parameter Space	89
4.2	<i>p</i> -brane Boundary Space	89
4.3	World–HyperTube and p -brane	90
4.4	<i>p</i> -brane Tangent Element	90
4.5	p-brane Holographic Coordinates	91
4.6	Local World–HyperTube Volume Velocity	91
4.7	p-brane Volume Velocity	91
4.8	p-brane Reparametrized Schild Lagrangian Density	92
4.9	$p\text{-}\mathrm{brane}$ Bulk Volume Momentum	93
4.10	p-brane Boundary Volume Momentum	93
4.11	Restricted Reparametrized p -brane Action	93
4.12	p-brane Projected Boundary Area Momentum	95
4.13	p-brane HyperVolume	95
6.1	Minisuperspace Stationary Schrödinger Equation	21
7.1	<i>D</i> -measure	32
7.2	Hausdorff Dimension & Hausdorff Measure	32
7.3	DeBroglie Area of a String	33
8.1	Functional Current	40
9.1	Bulk Current	47
9.2	Loop Current	48
9.3	Boundary Current	48
94	Loop Dirac Functional	49

9.5	Wilson Factor	150
9.6	Loop Transform	150
10.1	Non Standard Wave Functional	160
B.1	Filter	206
B.2	Ultrafilter	207
В.3	Free Ultrafilter	207
B.4	Equivalence Modulo Ultrafilter	209
B.5	Nonstandard Reals	209
B.6	Non Standard Isomorphism	210
B.7	Infinite, Infinitesimal, Limited, Appreciable Numbers	211
B.8	Equivalence Relation on $\prod_{n\in\mathbb{N}}\mathbb{R}^n$	213
B.9	Ultra Euclidean Space	214
C.1	Kawabata Space	218
C.2	Scalar Product on Kawabata Space	218
C.3	Integral of an Element of Kawabata Space	219
C.4	Directional Derivative of an Element of Kawabata Space	220
C.5	Functional on Kawabata Space	220
C.6	Functional Derivative of a Functional on Kawabata Space	221
C.7	Functional Integral of a Functional on Kawabata Space	221
C.8	Operator on Kawabata Space	222
C.9	Stochastic Process on Kawabata Space	223
C.10	0 Elementary Volume Element on Kawabata Space	224
C.1	1 Non Standard Probability	224
C.12	2 Forward Drift Operator	226
C 1	3 Non Standard Expectation Value	229

Contents.	
C.14 Mean Non Standard Forward/Backward Derivative	į

List of Propositions

1.1	Boundary Variation in (t, n) Coordinates	19
1.2	Normal and Tangential Boundary Variation of the Action	22
2.1	Invariance of Holographic Coordinates	31
2.2	Schild String Equations of Motion	34
2.3	Nambu-Goto Equations of Motions	35
2.4	Schild versus Nambu-Goto Equivalence	36
2.5	Schild Hamiltonian Density	38
2.6	Schild Action In Hamiltonian Form	39
2.7	Schild Lagrangian Variation under Reparametrization	40
2.8	Conjugated Momenta in Reparametrized Formulation	43
2.9	String Area Velocity in (T, N) Coordinates	45
2.10	Ogielski Hamilton–Jacobi String Equation	46
2.11	Full Reparametrized Theory Equations of Motion	48
2.12	Restricted Reparametrized Action Boundary Variation	50
2.13	Classical Area Hamiltonian Formalism	52
2.14	Classical Area Newtonian Formalism	53
2.15	Continuity Equation	55
2.16	Projected World–Sheet Area Momentum Computation	56
2.17	Alternative Expression for the Schild Hamiltonian	57

2.18	Bulk-Boundary Interference	58
2.19	Actions in the Reparametrized Mixed Formulation $\ \ldots \ \ldots \ \ldots \ \ldots$	59
2.20	Equation of Motion from the Covariant Schild Action $\ \ldots \ \ldots \ \ldots$	60
2.21	Solution of the Equation of Motion	61
2.22	Covariant Schild Action and Nambu–Goto Action	62
3.1	Integrating out the $\pmb{\xi}$ Fields	64
3.2	Integrating out the π Fields	65
3.3	Integrating out the ${m P}$ Fields	66
3.4	Schild–Nambu Goto Quantum Equivalence	68
3.5	Kernel Variation	69
3.6	Kernel Derivatives	70
3.7	String Kernel Functional Schrödinger Equation	71
3.8	Alternative Forms of the Functional Kernel Equation	73
3.9	Amplitude and Phase Equations for the Kernel	74
3.10	Holographic Derivatives	75
3.11	String Propagation Kernel	77
3.12	Representation of Nambu–Goto String Dynamics	78
3.13	Functional Bulk Integration	79
3.14	String Propagator from Path–Integration $\ \ldots \ \ldots \ \ldots \ \ldots$	80
3.15	String Loop Schrödinger Equation	84
4.1	p-brane Energy Balance Equation	95
4.2	Boundary Variation of the p -brane Action	96
4.3	p-brane Hamilton-Jacobi Equation	97
4.4	Meaning of the Lagrange Multiplier N	98
4.5	$p\text{-}\mathrm{brane}$ Schild –Nambu Goto Quantum Equivalence	98
46	n-brane Green Function	100

4.7	p-brane Variation of the Kernel
4.8	p-brane Amplitude and Phase Kernel Equations
4.9	<i>p</i> -brane Kernel and Green Function
5.1	Stationary States Schrödinger Equation
5.2	Plane Wave Solution
5.3	Boundary Momentum Operator Eigenstate
5.4	Gaussian Wave–Packet Solution
6.1	Minisuperspace Linear Velocity Vector
6.2	Minisuperspace Holographic Derivative
6.3	Ordinary Functional Derivatives
6.4	Ordering Problems
6.5	Functional Schrödinger Equation: Covariant Formalism
6.6	Solution of Ordering Ambiguities
6.7	Hyperspherical p -brane
7.1	Holographic/Area Functional Fourier Transform
7.2	Holographic/Area Expectations: Gaussian Wavepacket
7.3	Shape Uncertainty Principle
7.4	Hausdorff Dimension of the Quantum Dynamic Process
7.5	Classical—to—Fractal Transition
8.1	Quantum Lagrangian Density I
8.2	Quantum Lagrangian Density II
8.3	Classical Lagrangian Density and Current
8.4	Gauge Theory of the String Geodesic Field
9.1	String Wave Functional: Covariant Formulation
9.2	Loop Functional State
9.3	Loop Dirac Functional and Wilson Loops

9.4 Relation between Loop and String Functionals
9.5 Bulk-Boundary Decoupling
9.6 Boundary <i>versus</i> Bulk Dynamics
10.1 Derivatives of the Non Standard Stochastic Process
10.2 Non Standard Drift Velocity
10.3 Nonstandard Osmotic Velocity
10.4 Non Standard Functional Schrödinger Wave Equation
A.1 p-brane Schild Hamiltonian Density
A.2 Standard $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ Variations
A.3 p -brane Standard $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ Variation
A.4 p-brane π Variation of the Action
A.5 Free Boundary Variation of the String Action
A.6 Free Boundary Variation of the <i>p</i> -brane Action
A.7 Functional Integration over $\boldsymbol{\xi}$ Fields
A.8 Functional Integration over the π Fields
A.9 Functional Integration over the $m{P}$ Fields
A.10 Saddle Point Evaluation
A.11 Functional Differential Equations for the Kernel
A.12 p -brane Functional Schröedinger Equation
A.13 Functional Derivatives of the Holographic Coordinates
A.14 Functional Derivatives of the <i>p</i> -brane Classical Action
B.1 Operations in $\mathbb{R}^{\mathbb{N}}$
B.2 Equivalence Relation Modulo Ultrafilter
B.3 Linearly Ordered Structure on ${}^*\mathbb{R}$
B 4 Order Preserving Isomorphism

B.5	An Infinite Number and an Infinitesimal Number	211
B.6	Linearity of $^*\mathcal{E}$	215
B.7	Euclideanity of $^*\mathcal{E}$	215
C.1	Homeomorphism between ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) and ${}^*\mathcal{E}$	219
C.2	Local Properties of Elements in Kawabata Space	219
C.3	Structure of Operators in Kawabata Space	222
C.4	Non Standard Probability Distribution	225
C.5	Mean NonStandard Forward/Backward Derivative	230
D 1	Functional and Holographic Derivatives	235

List of Notations

1.1	First Derivative
1.2	Modulus of the Boundary Space Tangent Vector
2.1	String Area Velocity
2.2	Indexing of Fields and Parameters
2.3	Linear Velocity Vector
3.1	Holographic Distance
4.1	p-brane Volume Velocity
6.1	Closed Unit Interval with End Points Identified
7.1	Fourier Transform Related Width
7.2	String Mean Square Deviations in Position and Momentum
9.1	Loop Transform and Inverse Loop Transform
9.2	Dual String Functional
B.1	Standard Entity
B.2	Sequence
В.3	$\mho, @, \pounds, \infty$
B.4	Element of $\prod_{n\in\mathbb{N}}\mathbb{R}^n$
B.5	Element of the Ultra Euclidean Space $^*\mathcal{E}$
C.1	Schwartz Space
C_2	\mathcal{L}^2 Space 217

C.3	Complete Orthonormal Set	217
C.4	nth-approximation of an Elment of Kawabata Space	218
C.5	Functional on Kawabata Space	220

I Notations

```
derivative with respect to the parameter s
            Antisymmetric Derivative on the Boundary
           Integral over a manifold without Boundary
           Functional Measure
            Antisymmetric Derivative on the Bulk
Λ
            Exterior product of Forms
\partial
            Boundary
\partial
           Partial Derivative
\delta
           Functional Derivative
δ
            Variation
\mathcal{W}
           World Sheet of a String
\mathcal{W}^{(p+1)}
           World Hypertube of a p-brane
Ξ
           Parameter Space of a String
\Xi^{(p+1)}
           Parameter Space of a p-brane
\Sigma
           Parameter Space of a String
\sum^{(p+1)}
           Parameter Space of a p-brane
\mathbb{T}
           Target Space
Γ
           Boundary of the Parameter Space
_{\mathbb{S}^{1}}^{\gamma}
           Boundary of \Xi
            Circumpherence
\mathbb{S}^D
            D-dimensional Sphere
C
           Loop in SpaceTime
\boldsymbol{B}
           Boundary of \Sigma^{(p+1)}
oldsymbol{D}^{(p)}
           p-closed Surface in SpaceTime
\mathcal{L}
           Lagrangian Density
\mathcal{H}
           Hamiltonian Density
```

 \forall for all \exists exists \in is an element of is a subset of $[expression]_{...}$ expression evaluated for . . . expressionexpression evaluated for ... $\mathop{\mathbb{M}^{D}}^{condition}$ restricted to condition D-dimensional Minkowski Space |...| Determinant of ... Bijection □...|... distinguishes internal from spacetime indices $\Delta_{\boldsymbol{g}}$ Laplacian with respect to a metric g

- Closed indices are summed over.
- Vectors tensors and forms, when the indeces are not explicitly written, are typed in **boldface**.
- Functions have arguments enclosed in round brackets.
- Functionals have arguments enclosed in square brackets.
- All the quantities for which we give an explicit definition are typed in *slanted*, after the definition.

II Acknowledgments

I would like to heartily thank Dr. Euro Spallucci for the valuable and friendly support that he gave me as co-tutor during my Ph.D..

I am also indebted with Ms. Alessandra Richetti and Ms. Rosita Glavina, System Manager and Secretary at the Department of Theoretical Physics of the University of Trieste, for their professional help in a lot of practical problems.

II Introduction

"\textit{\varphi} uture?"
"Difficult to see.
Always in motion is the future."

Subject of this thesis is the study of a closed bosonic string: the treatment will follow an approach *alternative* with respect to the traditional one and stems from a research line pursued in by Eguchi.

In particular we will follow an approach based on an action functional which is not the Nambu-Goto proper area of the string world—sheet but its "square", i.e. the Schild action. Starting from this action we formulate the dynamics of a 1-(or p-)dimensional extended object following the same procedures and techniques, which characterize the Quantum Mechanics of point particles. Of course, since we are dealing with a 1-dimensional object and not with a pointlike one, there are some problems in adapting the formalism and interpreting the results: we tried to solve these problems and we remarked the relevance of some results, which are strictly related to the extended nature of the object under study.

A second goal that we will pursue in this work is to give a mathematically consistent formulation of our alternative description of string dynamics; we will discuss in some depth the formal "tricks" needed to carry on some calculations and the underlying mathematics. This is our main motivation in presenting *detailed computation*, whenever it is possible, in order to give the reader a better understanding of the the results themselves and the way they are derived.

It is also worth pointing out that all the results, that we will get for the closed string, can be generalized to higher dimensional closed extended objects (*p-branes*) without facing

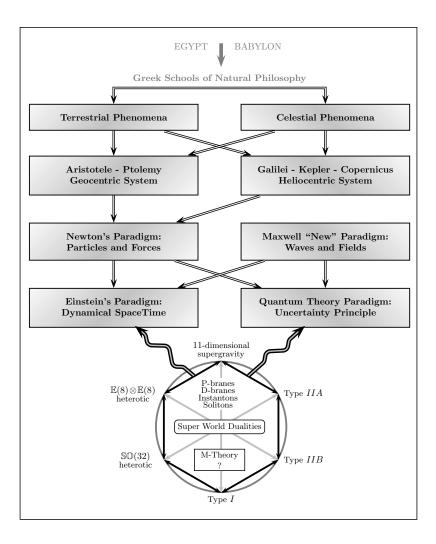


Figure 1: History of physics shows that conflicting theories eventually merge into a broader and deeper synthesis. Will M-Theory lead to a UNIQUE super synthesis of Quantum Theory, Gravity Theory and supersymmetry?

any further technical problems. This is important, since we are given the opportunity to focus the simpler 1-brane \equiv string case, without any loss of generality but with the smaller technical effort.

Shortly presented our goals are the following ones:

- to formulate a Classical Theory of relativistic extended objects as a natural generalization of the Hamilton-Jacobi Theory for point particles;
- 2. to quantize the Theory using path-integrals and deriving the *propagator* as well as the corresponding functional wave equation; in other words, we will quantize the Theory in the Schrödinger representation;
- 3. to provide some explicit solutions of the stringy Schrödinger equation, with particular emphasis on the new characteristic features following from the 1-dimensional extension of the string; these results can also be extended, of course, to higher dimensional objects;
- 4. to get a better understanding of the general results by considering the particular case of a circular string, where, some problems and their solutions are more transparent;
- 5. last but not least, to provide an alternative interpretation of the dynamics of an extended object as a *stochastic shape deformation process*.

To reach these goals we will use, among others, the following mathematical tools:

- a generalized Hamilton–Jacobi formulation of the dynamics of fields and relativistic extended objects;
- functional integral techniques;
- NonStandard Analysis techniques.

In order to provide the reader with an as self contained text as possible, the first and the third items will be treated in some detail in chapters 1-2 and in appendices B-C.

At this stage, we hope that the curious and interested reader would be asking himself the reason why we propose this non-standard formulation for string dynamics. The shortest and probably most general answer is already present in the very famous book of Green, Schwartz and Witten. so that our hope is that a non-standard approach could be not an alternative, but a complementary one. With this hope we will also devote chapter 9 to link our approach with the more traditional one, which nowadays has a central role in high energy theoretical physics. In our opinion, research in Theoretical High Energy Physics has now the meaning of investigating about the very nature of mass and energy, and ultimately about the structure of space and time themselves. It may even be argued that the whole history of Physics, to a large extent, represents the history of the ever changing notion of space and time in response to our ability to probe infinitesimally small distance scales as well as larger and larger cosmological distances.

The "flow chart" in Figure 1 summarizes the dialectic process which has led, through nearly twenty five hundred years of philosophical speculation and scientific inquiry, to the current theoretical efforts in search of a super-synthesis of the two conflicting paradigms of 20th century physics, namely, the Theory of General Relativity and Quantum Theory. In that Hegelian perspective of the history of physics, such a super synthesis is regarded by many as the "holy grail" of contemporary High Energy Physics. However, the story of the many efforts towards the formulation of that synthesis, from supergravity to superbranes, constitutes, in itself, a fascinating page in the history of theoretical physics at the threshold of the new millennium. The early '80s excitement about String Theory ("The First String Revolution") followed from the prediction that only the gauge groups SO(32) and $\mathbb{E}(8) \otimes \mathbb{E}(8)$ provide a quantum mechanically consistent, i.e., anomaly free, unified Theory which includes Gravity [17], and yet is capable, at least in principle, of reproducing the standard Electro-Weak Theory below the GUT scale. However, several fundamental questions were left unanswered. Perhaps, the most prominent one regards the choice of the compactification scheme required to bridge the gap between the multi-dimensional, near-Planckian string—world, and the low energy, 4-dimensional universe we live in [18]. Some related problems, such as the vanishing of the cosmological constant (is it really vanishing, after all?) and the breaking of supersymmetry were also left without a satisfactory answer.

The common feature of all these unsolved problems is their intrinsically non-perturbative character. More or less ten years after the First String Revolution, the second one, which is still in progress, has offered a second important clue into the nature of the superworld. The diagram in Figure 1 encapsulates the essential pieces of a vast mosaic out of which the final Theory of the superworld will eventually emerge. Among those pieces, the six surviving viable supermodels known at present, initially thought to be candidates for the role of a fundamental Theory of Everything, are now regarded as different asymptotic realizations, linked by a web of dualities, of a unique and fundamentally new paradigm of physics which goes under the name of M-Theory. The essential components of this underlying Matrix Theory appear to be string—like objects as well as other types of extendons, e.g., p-branes, D-branes, Moreover, a new computational approach is taking shape which is based on the idea of trading off the strongly coupled regime of a supermodel with the weakly coupled regime of a different model through a systematic use of dualities.

Having said that, the fact remains that M-Theory, at present, is little more than a name for a mysterious supertheory yet to be fully formulated. In particular, we have no clue as to what radical modification it will bring to the notion of spacetime in the short distance regime. In the meantime, it seems reasonable to attempt to isolate the essential elements of such non-perturbative approach to the dynamics of extended objects.

One such approach, developed over the last few years [19], [20], [33], is a refinement of an early formulation of *Quantum String Theory* by T.Eguchi [5], elaborated by following a formal analogy with a Jacobi-type formulation of the *Canonical Quantization of Gravity*. The relevance of this approach can be traced down to an intriguing similarity that we can understand between the problem of quantizing gravity, as described by General Relativity, and that of quantizing a relativistic string, or any higher dimensional relativistic extended object. In either case, one can follow one of two main routes:

- 1. a Quantum Field Theory inspired covariant quantization;
- 2. a canonical quantization of the Schrödinger type.

The basic idea underlying the covariant approach is to consider the metric tensor $g_{\mu\nu}(x)$ as an ordinary matter field and follow the standard quantization procedure, namely, to Fourier analyze small fluctuation around a classical background configuration and give the Fourier coefficients the meaning of creation/annihilation operators of the gravitational field quanta, the gravitons. In the same fashion, quantization of the string world–sheet fluctuations leads to a whole spectrum of particles with different values of mass and spin: these is a local, short scale and perturbative approach. Put briefly,

$$g_{\mu\nu}(x)={
m background}+{
m ``graviton''}$$

$$X^{\mu}(\tau,s)={
m zero-mode}+{
m particle \ spectrum} \quad .$$

Against this background, one may elect to forgo the full covariance of the Quantum Theory of Gravity in favor of the more restricted symmetry under transformations preserving the "canonical spacetime slicing" into a 1-parameter family of spacelike 3-surfaces. This splitting of space and time amounts to selecting the spatial components of the metric, modulo 3-space reparametrizations, as the gravitational degrees of freedom to be quantized. This approach focuses on the quantum mechanical description of the space itself, rather than the corpuscular content of the gravitational field. Then, the quantum state of the spatial 3-geometry is controlled by the Wheeler–DeWitt equation

$$[Wheeler - DeWitt operator] \Psi [G_3] = 0$$
 (1)

and the wave functional $\Psi[G_3]$, the wave function of the universe, assigns a probability amplitude to each allowed three geometry. We note that in the case of extended objects (strings) the main route is constituted by the previous one. So to understand in more detail the differences in the described approaches, let us concentrate for a while on the case of Gravity. In this case we can understand that the relation between the two quantization schemes is akin to the relationship in particle dynamics between first quantization, formulated in terms of single particle wave functions along with the corresponding Schrödinger equation, and the second quantization expressed in terms of creation/annihilation operators along with the corresponding field equations. Thus, covariant Quantum Gravity is, conceptually, a second quantization framework for calculating amplitudes, cross sections, mean life, etc., for any

physical process involving graviton exchange. Canonical Quantum Gravity, on the other hand, is a Schrödinger-type first quantization framework, which assigns a probability amplitude for any allowed geometric configuration of three dimensional physical space. It must be emphasized that there is no immediate relationship between the graviton field and the wave function of the universe. Indeed, even if one elevates $\Psi[G_3]$ to the role of field operator, it would create or destroy entire 3-surfaces instead of single gravitons. In a more pictorial language, the wave functional $\Psi[G_3]$ becomes a quantum operator creating/destroying full universes! Of course, as far as gravity is concerned, any quantization scheme is affected by severe problems: perturbative covariant quantization of General Relativity is not renormalizable, while the intrinsically non-perturbative Wheeler-DeWitt equation can be solved only under extreme simplification such as the mini-superspace approximation; thus probably these shortcomings provided the impetus toward the formulation of string Theory first, and Superstring Theory then, as the only consistent quantization scheme which accommodates the graviton in its (second quantized) particle spectrum. Indeed the key observation that the string position $X^{\mu}(\sigma^0, \sigma^1)$ and momentum $P^{\mu}_{\tau}(\sigma^0, \sigma^1)$ are dependent on two variables let to the conclusion that the quantization procedure, which rises X^{μ} and P^{μ} to the role of operators, results in a 2-dimensional second quantized point particle Quantum Field Theory [1]. The non-linearity of the Theory is a major difficulty in this approach since the equations of motions, arising from the Nambu-Goto action, present intractable computational difficulties. The way out from this problem is the *choice* of a gauge for our description. In this way constraints are imposed on the gauge freedom present in the Theory; their interpretation is then clarified starting from the Polyakov action and showing how they arise gauging away the parameter space metric g_{ab} . Canonical Quantization can then be performed. Thus, according to the prevalent way of thinking, there is no compelling reason, nor clear cut procedure to formulate a first quantized Theory (i.e. a Quantum Mechanics) of relativistic extended objects. In the case of strings, this attitude is also deeply rooted in the conventional interpretation of the world-sheet coordinates $X^{\mu}(\sigma^0, \sigma^1)$ as a "multiplet of scalar fields" defined over a 2-dimensional manifold, Σ , covered by the (σ^0, σ^1) coordinate mesh. According to this point of view, quantizing a relativistic string is formally equivalent to quantizing a 2-dimensional Field Theory, bypassing a preliminary quantum mechanical

formulation. However, there are at least two objections against this kind of reasoning. The first follows from the analogy between the canonical formulation of General Relativity and 3-brane dynamics, and the second objection follows from the "Schrödinger representation" of Quantum Field Theory. More specifically:

- 1. the Wheeler-DeWitt equation can be interpreted, in a modern perspective, as the wave equation for the orbit of a relativistic 3-brane. In this perspective, then, why not conceive of a similar equation for a 1-brane?
- 2. the functional Schrödinger representation of Quantum Field Theory assigns a probability amplitude to each field configuration over a spacelike slice t = const., and the corresponding wave function obeys a functional Schrödinger-type equation.

Pushing the above arguments to their natural conclusion, we are led to entertaining the interesting possibility of formulating a functional Quantum Mechanics for strings and other p-branes. This approach has received scant attention in the mainstream work about Quantum String Theory, presumably because it requires an explicit breaking of the celebrated reparametrization invariance, which is the distinctive symmetry of relativistic extended objects.

All of the above reasoning leads us to the central question that we wish to analyze, namely: is there any way to formulate a reparametrization invariant String Quantum Mechanics? As a matter of fact, a possible answer was suggested by T.Eguchi as early as 1980 [5], and our own elaboration of that quantization scheme [3] is the topic of this thesis. We will in particular focus our attention in what follows on the string as a whole, i.e. as a physical system by itself, restricting our attention to the case of **closed** strings. Moreover we try to generalize in a peculiar way what happens for a particle, i.e, unlike Superstring Theory, our formulation represents an attempt to construct a *Quantum Mechanical* Theory of (closed) strings in analogy to the familiar case of point particles. The ground work of this approach is developed extending the Hamilton–Jacobi formulation and Feynman's path–integral approach. Indeed a particle is a 0-dimensional manifold, and when it evolves in time it spans (at least classically) a 1-dimensional manifold, i.e. its world–line; the time evolution can be

described in terms of the world-line proper length. The way in which we reformulate this observation for a string is as follows: a (closed) string is a 1-dimensional closed manifold, and when it evolves in "time 1" it spans (at least classically) a 2-dimensional manifold, i.e. its world-sheet. Our implementation of the analogy with the point particle is then reflected by our unconventional choice of dynamical variables for the string, namely, the areas enclosed by the projections of the string loop onto the coordinate planes and its 2-form conjugate momentum. Also central to our own quantum mechanical approach, is the choice of "time variable", which we take to be the area of the parameter space associated with the string world-sheet, in analogy to the point particle case.

Thanks to the previous pointed out analogy between our proposal for string quantization and the canonical quantization of the Schrödinger type for Quantum Gravity, we think that our method could led to some deeper insight into the problems related to the structure of spacetime at small scales. Indeed, since the advent of Quantum Theory and General Relativity, the notion of spacetime as a preexisting manifold in which physical events take place, is undergoing a process of radical revision. Thus, reflecting on those two major revolutions in physics of this century, Edward Witten writes [2], "Contemporary developments in theoretical physics suggest that another revolution may be in progress, through which a new source of "fuzziness" may enter physics, and spacetime itself may be reinterpreted as an approximate, derived concept". The new source of fuzziness comes from String Theory, specifically from the introduction of the new fundamental constant, α' , which determines the tension of the string. Thus, at scales comparable to $(\alpha')^{1/2}$, spacetime becomes fuzzy, even in the absence of conventional quantum effects ($\hbar \simeq 0$). While the exact nature of this fuzziness is unclear, it manifests itself in a new form of Heisenberg's principle, which now depends on both α' and \hbar . Thus, in Witten's words, while "a proper theoretical framework for the [new] uncertainty principle has not yet emerged, [...] the natural framework of the [String] Theory may eventually prove to be inherently quantum mechanical".

The essence of the above remarks, at least in our interpretation, is that there may exist different degrees of fuzziness in the *making* of spacetime, which set in at various scales of length, or energy, depending on the nature and resolution of the Heisenberg microscope

¹It is worth to use with care this word!

used to probe its structure. In other words, spacetime becomes a sort of dynamical variable, responding to quantum mechanical resolution just as, in General Relativity, it responds to mass-energy. The response of spacetime to mass-energy is curvature. Its response to resolution seems to be "fractalization".

Admittedly, in the above discussion, the term "fuzziness" is loosely defined, and the primary aim of chapter 7 is to suggest a precise measure of the degree of fuzziness of the quantum mechanical path of a string. In order to achieve this objective, we need two things:

- 1. the new form of the uncertainty principle for strings;
- 2. the explicit form of the wave–packet for string loops.

Then, we will be able to compute the Hausdorff dimension of a quantum string and to identify the parameter which controls the transition from the smooth phase to the fractal phase.

It seems worth emphasizing at this introductory stage, before we embark on a technical discussion, that we are interested primarily in the analysis of the quantum fluctuations of a string loop. By quantum fluctuations, we mean a random transition, or quantum jump, between different string configurations. Since in any such process, the shape of the loop changes, we refer to it as a "shape-shifting" process. We find that any such process, random as it is, is subject to an extended form of the Uncertainty Principle (we call it Shape Uncertainty Principle), which forbids the exact, simultaneous knowledge of the string shape and its area conjugate momentum. The main consequence of the Shape Uncertainty Principle is the "fractalization" of the string orbit in spacetime. The degree of fuzziness of the string world-sheet will be measured by its Hausdorff dimension. We will also try to quantify the transition from the classical, or smooth phase, to the quantum, or fractal phase. A deeper insight into the meaning of the "fractalization" process and its physical interpretation can also be gained in our opinion, by realizing that the fractal properties of the quantum dynamics of a string can be interpreted using a stochastic process, which we will call an Areal Brownian Motion. This process describes the shape-shifting process, which is string dynamics, as a fractal modification of the form of the object due to random fluctuations of its shape.

Accordingly, the Thesis is structurted as follows².

In chapter 1 we emphasize the relation between a closed 1-dimensional manifold considered as the only Boundary of a 2-dimensional simply connected domain (Bulk). We take here the opportunity to point out the relevance of this passage from the Bulk to the Boundary defining precisely all the necessary quantities: this will be relevant in the following chapters. Moreover a short presentation of the Functional Schrödinger Quantization, that we will take later as a basis for our procedure, is given.

After these introductory, but necessary, topics we address in chapter 2 the problem of describing the Classical Dynamics of an extended object from the point of view of what we will call its *Boundary Shadow Dynamics*. After the definition of the relevant quantities we solve this problem using a Functional Hamilton–Jacobi Formulation. Two different derivations of this result are given, as a consistency check, and we see how the relevant property of our formulation turns out to be the fact that it is a Dynamics of *Area*. After the presentation of a classical effective areal equivalent formulation we also show some possible generalization of the basic concepts that will be useful later on.

Now that we have a good starting point, namely the Classical Functional theory, we perform in chapter 3 its quantization: in particular we will derive the equation for the propagtor that describes the evolution of a *closed bosonics*tring as well as the solution to this equation. Again two different procedures are shown to give the same result, which reassures us about the correctness of the result. After an interestion by–product of our procedure, a functional representation of the Nambu–Goto string propagatorhus, we turn to the problem of defining a string quantum state and the corresponding equation for the probability amplitude, which we will call the *String Functional Schrödinger Equation*.

Before proceeding on this road by finding some solutions to this equation, we present again the procedure, given for the string in the last two chapters, generalizing it to the case

²The logical interdependence among chapters is shown in figure 2.

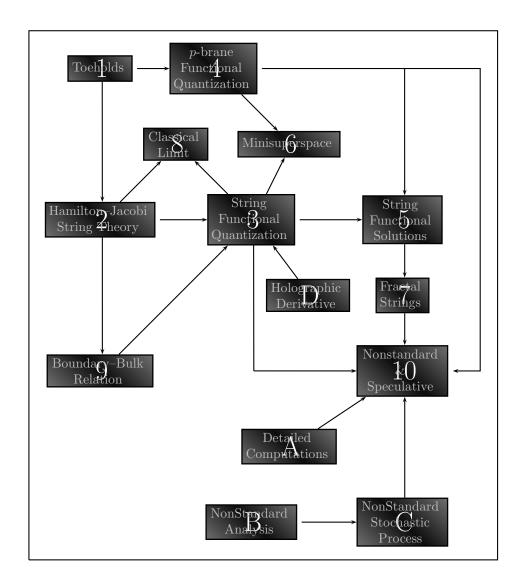


Figure 2: Logical connection between chapters of the Thesis.

of higher dimensional objects (p-branes). This is the topic of chapter (4).

Then we proceed in chapter 5 to the derivation of some solutions to the *String Schrödinger Equation*, finding the Plane Wave and the Gaussian Wave Packet. The most interesting properties of these solutions are also pointed out.

We can so spend chapter 6 to give a closer look at the meaning of the results derived before in a very special case, which, by analogy with similar procedures used in Quantum Gravity, we will call *Minisuperspace Approximation*. Here a drastic reduction of the number of degrees of freedom of the problem is performed assuming a circular symmetry for the object. This gives us the opportunity of making some comments about a very common problem in Quantum Theories, namely the problem of ordering ambiguities.

Chapter 7, which follows, is a first interesting result that we can deduce from the work done so far. From the discussion of the String Functional Schrödinger Equation and its solutions we derive the Uncertainty Principle for strings as well as its principal consequence, namely, the fractalization of the string quantum evolution. This is strongly related to the fractalization of Quantum SpaceTime.

We then begin in chapter 8 a process that will bring the procedure of String Quantization presented in this Thesis under a new light: here we clarify its relation with the gauge Theory of the String Geodesic Field, i.e. we show that in aproperly chosen limit we can recover a Field Theory equivalent to classical string dynamics.

Then in chapter 9 we show how our formulation can be given a connection to the usual Polyakov procedure of string quantization: this is an intersting result that enlightens the properties of Boundary Dynamics versus Bulk Dynamics and gives a useful hints about possible connections with Loop Quantum Gravity and large N QCD.

Finally chapter 10 is devoted to the mathematical fundation of the Functional Operations we performed above. Using an interesting mathematical tool, namely NonStandard Analysis, we show how it is possible to give rigorous definitions to all the formal steps we took in previous chapters. We also present two more speculative sections, one about possible connections with M-Theory, the other about a possible description of the structure of spacetime in terms of an effective lagrangian based on a covariant, functional extension of the Ginzburg-Landau model of superconductivity.

Four appendices follow the main tratment:

- in appendix A we present some detailed calculations which we skipped in chapter 4;
- in appendix B we give a brief introduction to NonStandard Analysis with the aim of introducing the Ultra Euclidean Space;
- in appendix C we define a Stochastic Process on aparticular space defined in terms of NonStandard Analysis;
- in appendix D we give a more detailed discussion about Functional and Holographic Derivatives, which are the principal quantities in writing our functional equations.

Eventought we tried to give an as self contained as possible treatment, we were not able to cover all the prerequisites in full depth and we refer the interested reader to the bibliography for further readings.

Chapter 1

Toeholds

A glowing fire dances in the center of the spartan, low-ceilinged room, creating a kaleidoscope of shadows on the walls.

1.1 Boundary Dynamics

In this first chapter we provide an in depth discussion of the main ideas and motivations underlying the whole Thesis. In particular, we shall explain how the Quantum Dynamics of a string (and, more in general, of a p-dimensional extended object) can be viewed as the Boundary Dynamics induced by the world–sheet (or, more in general, by the (p+1)-dimensional world–manifold) quantum vibrations. This concepts generalizes in a very straightforward way the quite similar view of a pointlike particle as the free end–point of a 1-dimensional (p=0) world–line. Now, we have to deal with the problem of a non–local quantum mechanics as a consequence of the string spatial extension. The way, which we consider more appropriate to approach such a problem, goes through the relation between the Hamilton–Jacobi formulation of Classical Mechanics and the associated Quantum formulation.

We think that the above considerations can be taken as a good evidence of the relevance of the Hamilton–Jacobi Theory in passing from the classical formulation of a system to the quantum one. This is the reason why we decided to derive the Dynamics of an extended object starting from the Hamilton–Jacobi formulation of the Classical Theory and to devote chapter 2 to a detailed exposition of this subject starting from different viewpoints.

1.1.1 "Shadow" Dynamics

A central point in our treatment is based on the observation that all physical theories rely on the solution of a (more or less intricate) system of differential equations (ordinary or partial) encoding the law of evolution of the system. The set of dynamical equations must be supplemented with appropriate Boundary conditions taking into account the "action of the environment" on the system under study. In this kind of description, Dynamics inside a given region is constrained by the conditions imposed on the Boundary of the region itself. In other world, we have a pure Bulk Dynamics while the Boundary is a non-dynamical region, where the behavior of the system is a priori fixed. In our opinion, Boundary conditions are often assigned for the only purpose of convenience without any further justification. Against this background, the Hamilton-Jacobi approach "shifts" the physical Dynamics to the Boundary, and assign a secondary role to the Dynamics of the Bulk. A good starting point to illustrate this different strategy is provided by the Ogielski's formulation of the closed, bosonic string Dynamics.¹.

Definition 1.1 (Parameter Space).

The <u>Parameter Space</u> is a compact connected two dimensional domain $\Sigma \in \mathbb{R}^2$ coordinatized by the couple of variables $(\sigma^0, \sigma^1) \equiv \sigma$.

For what concerns Ogielski's treatment we are going to assume a *simply connected Parameter Space*. Of course, as we will have the opportunity to underline in chapter 3, more complicated cases can be worth of interest in string Dynamics². No problems thus arise in the following definition

¹Here and in what follows we always assume to consider systems that admit a Lagrangian Dynamics as well as an Hamiltonian one.

²In particular, a doubly connected space is the most appropriate in treating the free propagation of a free string between two different configurations.

Definition 1.2 (Boundary Space).

The <u>Boundary Space</u> is the Boundary $\Gamma \equiv \partial \Sigma$ of the Parameter Space: it is parametrized by a parameter s, varying in the interval $S \equiv [s_0, s_1]$; since the Boundary of a Boundary is the empty set,

$$\partial \Gamma = \partial \partial \Sigma = \emptyset$$
 ,

the interval S defined above has its end points identified,

$$s_0 \equiv s_1$$
 ,

i.e. is topologically equivalent to a circle, or a 1-dimensional sphere, \mathbb{S}^1 .

We note that in more complicate situations, such as a multiply connected *Parameter Space*, it is useful consider different ways of parametrizing the *Boundary*, usually with a properly chosen set of functions. Indeed, the *Boundary* of a multiply connected space is not always connected. Thus, it can be useful to associate different parametrizations to different connected components of the *Boundary Space*.

We will often use quantities differentiated once with respect to the parameter $s \in \mathbb{S}^1$. Hence, let us introduce the following notations.

Notation 1.1 (First Derivative).

We will use a prime "'" to denote the first derivative with respect to the unique parameter, say s, of a curve: for example

$$\boldsymbol{\sigma}'(\bar{s}) \stackrel{\text{def.}}{=} \left. \frac{d\boldsymbol{\sigma}\left(s\right)}{ds} \right|_{s=\bar{s}} = \left(\frac{d\sigma^0}{ds} \right|_{s=\bar{s}}, \left. \frac{d\sigma^1}{ds} \right|_{s=\bar{s}} \right) \quad .$$

Moreover, we are also going to use the more compact

Notation 1.2 (Modulus of the Boundary Space Tangent Vector).

The modulus of the vector tangent to the Boundary Space Γ at the point $P = (\sigma^0(\bar{s}), \sigma^1(\bar{s}))$ is denoted by

$$\sqrt{\left(\boldsymbol{\sigma}'(\bar{s})\right)^2} \stackrel{\text{def.}}{=} \sqrt{\sigma'^{i}(\bar{s})\,\sigma'_{i}(\bar{s})} = \sqrt{\frac{d\sigma^{i}}{ds}\frac{d\sigma_{i}}{ds}} \bigg|_{s=\bar{s}} .$$

Suppose now that the model we are going to study is defined by the following 2-form

$$\boldsymbol{\omega} = \mathcal{L}(X^{\mu}, X^{\mu}_{,i}; \sigma^{i}) \, d\sigma^{0} \wedge d\sigma^{1} \quad , \tag{1.1}$$

where, \mathcal{L} is a sufficiently regular function (the Lagrangian density of our model), $X^{\mu} \equiv X^{\mu}(\sigma^{i})$ are the fields defined over the domain Σ , and $Y^{\mu} \equiv Y^{\mu}(s)$ their values on the Boundary given by

$$Y^{\mu}(s) \equiv X^{\mu}(\sigma^{i}(s))$$
.

The fields X and Y take their values in the below defined target space.

Definition 1.3 (Target Space).

The <u>Target Space</u>, \mathbb{T} , is the space in which the fields X^{μ} (equivalently Y^{μ}) take their values as functions on Σ .

In the target space the fields $X^{\mu}(\boldsymbol{\sigma})$ define a surface $\mathcal{W} \stackrel{\text{def.}}{=} X^{\mu}(\Sigma)$ bounded by a loop $C \stackrel{\text{def.}}{=} Y^{\mu}(\Gamma) = X^{\mu}(\sigma^{i}(\mathbb{S}^{1})) = \partial \mathcal{W}$.

It is already clear how the *Boundary* of the system is playing a central role in this description. It is the physical dynamical object rather than the locus where constraints are imposed over the classical solutions. The *Boundary* Dynamics will essentially depend from the chosen Lagrangian density \mathcal{L} , in equation (1.1), for the fields X^{μ} . The action corresponding to the Lagrangian (1.1) is

$$S = \int_{\Sigma} \boldsymbol{\omega} = \int_{\Sigma} \mathcal{L}(X^{\mu}, X^{\mu}_{,i}; \sigma^{i}) d\sigma^{0} \wedge d\sigma^{1} \quad , \tag{1.2}$$

Moreover, we can introduce the following quantities:

Definition 1.4 (Tangent and Normal Vectors).

The following vectors are the tangent and the normal to the Boundary $\Gamma \approx \mathbb{S}^1$ of the Parameter Space, and to its image C in the Target Space \mathbb{T} :

1. the tangent unit vector to the (Parameter Space) Boundary curve $\Gamma \approx \mathbb{S}^1$ at the point

defined by the value \bar{s} of the parameter $s \in \mathbb{S}^1$ is

$$\boldsymbol{t}(\bar{s}) = \boldsymbol{t}|_{s=\bar{s}} = \frac{1}{\sqrt{\left(\boldsymbol{\sigma'}(s)^{2}\right)}} \frac{d\sigma^{j}(s)}{ds} \bigg|_{s=\bar{s}} \boldsymbol{\partial}_{j} = \frac{\sigma'^{j}(\bar{s})}{\sqrt{\left(\boldsymbol{\sigma'}(\bar{s})\right)^{2}}} \boldsymbol{\partial}_{j}$$
(1.3)

and is called the (Parameter Space) Tangent Unit Vector;

2. the normalized normal vector to the Boundary Space $\Gamma \approx \mathbb{S}^1$ at the point \bar{s}

$$\boldsymbol{n}(\bar{s}) = \boldsymbol{n}|_{s=\bar{s}} = \frac{e^{jk}}{\sqrt{(\boldsymbol{\sigma}'(s))^2}} \frac{d\sigma_k(s)}{ds} \bigg|_{s=\bar{s}} \boldsymbol{\partial}_j = \frac{e^{jk}\sigma_k'(\bar{s})}{\sqrt{(\boldsymbol{\sigma}'(\bar{s}))^2}} \boldsymbol{\partial}_j$$
(1.4)

and is called the (Parameter Space) Normal Vector;

3. the tangent vector to the Target Space Boundary curve C at the point \bar{s}

$$T(\bar{s}) = T|_{s=\bar{s}} = \frac{d\sigma^{j}(s)}{ds} Y^{\mu}_{,j}(s) \bigg|_{s=\bar{s}} \partial_{\mu} = \sigma'^{j}(\bar{s}) Y^{\mu}_{,j}(\bar{s}) \partial_{\mu}$$
(1.5)

and is called the <u>Linear Velocity Vector</u> of the curve C;

4. the normal vector to the Target Space Boundary curve C at the point \bar{s}

$$\mathbf{N}(\bar{s}) = \mathbf{N}|_{s=\bar{s}} = \epsilon^{ij} \frac{d\sigma_i(s)}{ds} Y^{\mu}_{,j}(s) \Big|_{s=\bar{s}} \partial_{\mu} = \epsilon^{ij} \sigma'_i(\bar{s}) Y^{\mu}_{,j}(\bar{s}) \partial_{\mu}$$
(1.6)

and is called the <u>Acceleration</u> of the curve C.

After these definitions, we can prove the following result.

Proposition 1.1 (Boundary Variation in (t, n) Coordinates).

A variation of the domain Σ near the point $\sigma^i(\bar{s})$ of the Boundary $\Gamma \approx \mathbb{S}^1$ (in Parameter Space) can be decomposed into a tangential part, which we denote by $\delta t(\bar{s})$, and a normal part, denoted by $\delta n(\bar{s})$, given respectively by:

$$\delta t(\bar{s}) = \frac{1}{\sqrt{(\boldsymbol{\sigma}'(s))^2}} \frac{d\sigma^j(s)}{ds} \bigg|_{s=\bar{s}} \delta\sigma_j(\bar{s})$$
(1.7)

$$\delta n(\bar{s}) = \frac{\epsilon^{jk}}{\sqrt{(\sigma'(s))^2}} \frac{d\sigma_j(s)}{ds} \bigg|_{s=\bar{s}} \delta \sigma_k(\bar{s}) , \qquad (1.8)$$

where $\delta \sigma_j(s)$ is the component of the variation in the ∂_j direction.

Proof:

We are going to express the variation $\Delta \sigma(\bar{s})$ in the two different basis, (∂_0, ∂_1) and (t, n) at the given point and then we will compare the two results. Thus first we consider the variation of the Boundary at the point $\sigma^i(\bar{s})$ and decompose it into the basis formed by the vectors ∂_i , i = 1, 2,

$$\Delta \sigma(\bar{s}) = \delta \sigma^{j}(\bar{s}) \partial_{j} \quad . \tag{1.9}$$

Then we perform the same operation but with respect to the basis formed by the vectors (1.3-1.4), thus getting

$$\Delta \sigma(\bar{s}) = \delta n(\bar{s}) \, \boldsymbol{n}(\bar{s}) + \delta t(\bar{s}) \, \boldsymbol{t}(\bar{s})
= \delta n(\bar{s}) \, \frac{\epsilon^{jk}}{\sqrt{(\boldsymbol{\sigma}'(\bar{s}))^2}} \sigma'_k(\bar{s}) \, \boldsymbol{\partial}_j + \delta t(\bar{s}) \, \frac{1}{\sqrt{(\boldsymbol{\sigma}'(\bar{s}))^2}} \sigma'^j(\bar{s}) \, \boldsymbol{\partial}_j , \qquad (1.10)$$

where we used definitions (1.3) and (1.4) to express the (Parameter Space) Tangent Unit Vector and the (Parameter Space) Normal Vector in terms of the (∂_0, ∂_1) basis. Comparing then equation (1.9) with equation (1.10) we obtain

$$\delta \sigma^{j}(\bar{s}) \,\partial_{j} = \Delta n(\bar{s}) \, \frac{\epsilon^{jk}}{\sqrt{(\sigma'(\bar{s}))^{2}}} \sigma'_{k}(\bar{s}) \,\partial_{j} + \Delta t(\bar{s}) \, \frac{1}{\sqrt{(\sigma'(\bar{s}))^{2}}} \sigma'^{j}(\bar{s}) \,\partial_{j} \quad . \tag{1.11}$$

Applying both sides to the one form $\theta = \sigma'_i(\bar{s}) dx^i$ and suppressing, for the sake of notational simplicity, the point (\bar{s}) , we get

$$\delta \sigma^{j} \sigma'_{i} =$$

$$\delta \sigma^{j} \sigma'_{i} \delta^{i}_{j} =$$

$$\delta \sigma^{j} \sigma'_{i} \partial_{j} \left(dx^{i} \right) =$$

$$\delta \sigma^{j} \partial_{j} \left(\sigma'_{i} dx^{i} \right) \stackrel{3}{=} \delta n \frac{\epsilon^{jk}}{\sqrt{(\sigma')^{2}}} \sigma'_{k} \partial_{j} \left(\sigma'_{i} dx^{i} \right) + \delta t \frac{\sigma'^{j}}{\sqrt{(\sigma')^{2}}} \partial_{j} \left(\sigma'_{i} dx^{i} \right)$$

$$= \delta n \frac{\epsilon^{jk}}{\sqrt{(\sigma')^{2}}} \sigma'_{k} \sigma'_{i} \partial_{j} \left(dx^{i} \right) + \delta t \frac{1}{\sqrt{(\sigma')^{2}}} \sigma'^{j} \sigma'_{i} \partial_{j} \left(dx^{i} \right)$$

$$= \delta n \frac{\epsilon^{jk}}{\sqrt{(\sigma')^{2}}} \sigma'_{k} \sigma'_{i} \delta^{i}_{j} + \delta t \frac{1}{\sqrt{(\sigma')^{2}}} \sigma'^{j} \sigma'_{i} \delta^{i}_{j}$$

$$= \delta n \frac{\epsilon^{jk}}{\sqrt{(\sigma')^{2}}} \sigma'_{k} \sigma'_{j} + \delta t \frac{1}{\sqrt{(\sigma')^{2}}} \sigma'^{j} \sigma'_{j}$$

$$\stackrel{4}{=} 0 + \sqrt{(\sigma')^{2}} \delta t , \qquad (1.13)$$

so that from the equality of (1.12) and (1.13) we get exactly result (1.7). Proceeding in the same way but applying (1.11) to the one form $\tilde{\theta} = \epsilon_{lm} \sigma_l'(\bar{s}) dx^m$ we get

$$\delta \sigma^{j} \epsilon_{lj} \sigma^{l} =$$

$$\delta \sigma^{j} \epsilon_{lm} \sigma^{l} \delta^{m}_{j} =$$

$$\delta \sigma^{j} \epsilon_{lm} \sigma^{l} \partial_{j} (dx^{m}) =$$

$$(1.14)$$

 $^{^{3}}$ We use here equation (1.11).

⁴The first term vanishes since it has two symmetric indices contracted with two antisymmetric ones.

$$\delta\sigma^{j}\partial_{j}\left(\epsilon_{lm}\sigma^{\prime l}d\boldsymbol{x}^{m}\right) \stackrel{5}{=} \delta n \frac{\epsilon^{jk}}{\sqrt{(\boldsymbol{\sigma}^{\prime})^{2}}}\sigma_{k}^{\prime}\partial_{j}\left(\epsilon_{lm}\sigma^{\prime l}d\boldsymbol{x}^{m}\right) + \delta t \frac{1}{\sqrt{(\boldsymbol{\sigma}^{\prime})^{2}}}\sigma^{\prime j}\partial_{j}\left(\epsilon_{lm}\sigma^{\prime l}d\boldsymbol{x}^{m}\right)$$

$$= \delta n \frac{\epsilon^{jk}\epsilon_{lm}}{\sqrt{(\boldsymbol{\sigma}^{\prime})^{2}}}\sigma_{k}^{\prime}\sigma^{\prime l}\partial_{j}(d\boldsymbol{x}^{m}) + \delta t \frac{\epsilon_{lm}}{\sqrt{(\boldsymbol{\sigma}^{\prime})^{2}}}\sigma_{j}^{\prime}\sigma^{\prime l}\partial_{j}(d\boldsymbol{x}^{m})$$

$$= \delta n \frac{\epsilon^{jk}\epsilon_{lm}}{\sqrt{(\boldsymbol{\sigma}^{\prime})^{2}}}\sigma_{k}^{\prime}\sigma^{\prime l}\delta_{j}^{m} + \delta t \frac{\epsilon^{lm}}{\sqrt{(\boldsymbol{\sigma}^{\prime})^{2}}}\sigma_{l}^{\prime}\sigma_{m}^{\prime}$$

$$\stackrel{6}{=} \sqrt{(\boldsymbol{\sigma}^{\prime})^{2}}\delta n + 0 \tag{1.15}$$

getting at the end equation (1.8).

We can now define the variation of the oriented area A of the domain Σ at the point s as follows:

Definition 1.5 (Parameter Space Area Variation).

The Parameter Space Area Variation at the point s of the Boundary Γ is given by the modulus of the tangent vector multiplied by the normal variation

$$\delta A(s) = \sqrt{\left(\boldsymbol{\sigma}'(s)\right)^2} \delta n(s)$$

and can be expressed also as

$$\delta A(s) = \epsilon^{ab} \sigma_a'(s) \, \delta \sigma_b(s) \quad .$$

The functional derivative with respect to the *Parameter Space* area can be defined as well.

Definition 1.6 (Area Funcitonal Derivative).

The functional derivative with respect to a variation of the Area in the Parameter Space at the point $s \in \Gamma = \partial \Sigma$ is the rescaled normal variation ad s:

$$\frac{\delta}{\delta A(s)} = \frac{1}{\sqrt{(\boldsymbol{\sigma}')^2}} \frac{\delta}{\delta n(s)} \quad . \tag{1.16}$$

$$\epsilon^m_k \epsilon_{lm} = \delta_{kl}$$
 ;

moreover the second term vanishes as before.

 $^{^5 \}text{This}$ line is equation (1.11) applied to the 1-form $\tilde{\pmb{\theta}}.$

⁶We use the relation

We observe at this stage that, if the system is invariant under *Boundary* reparametrizations, it is insensitive to the place on the *Boundary* where the Area variation takes place and the *Area Functional Derivative* can be traded with an ordinary (partial) derivative.

After this results, we can tackle the problem of what kind of Dynamics results from the variation of the action (1.2). We observe that under the term "variation", we understand not only a variation of the fields defined on the *fixed* domain Σ , but also a variation of Σ itself. As we have already pointed out, this is related to the dynamical role which is assigned to the *Boundary* $\mathbb{S}^1 \approx \Gamma = \partial \Sigma$ in this formulation: we do not push it to infinity, or consider it as a non–dynamical, purely geometrical quantity, on the contrary, the physical, evolving object is the *Boundary* itself. We anticipate that this is a far reaching point of view. By extremizing the action integral (1.2) we get the following results:

Proposition 1.2 (Normal and Tangential Boundary Variation of the Action).

Let $X^{\mu}(\sigma_0, \sigma_1)$ be an extremal of the action functional (1.2); then the following equations for the variation of the action hold:

$$\frac{\delta S}{\delta n\left(s\right)} = \sqrt{\left(\boldsymbol{\sigma}'\right)^{2}} \left(\frac{\partial \mathcal{L}}{\partial N^{\mu}} N^{\mu} - \mathcal{L}\right) \tag{1.17}$$

$$\frac{\delta S}{\delta t(s)} = \sqrt{\left(\boldsymbol{\sigma}'\right)^2} \frac{\partial \mathcal{L}}{\partial N^{\mu}} T^{\mu} \tag{1.18}$$

$$\frac{\delta S}{\delta Y^{\mu}(s)} = (\sigma')^{2} \frac{\partial \mathcal{L}}{\partial N^{\mu}} , \qquad (1.19)$$

where the quantities \mathbf{t} and \mathbf{n} are defined in equations (1.3-1.4) respectively and T^{μ} and N^{μ} are the components of \mathbf{T} and \mathbf{N} in equations (1.5-1.6). Moreover $\delta/(\delta n(s))$ means that we perform a variation of the Σ domain at the Boundary in a direction normal to Γ , $\delta/(\delta t(s))$ has the same meaning for the tangential direction and $\delta/(\delta Y^{\mu}(s))$ takes into account variation related to the embedding in the Target Space \mathbb{T} .

It is worthwhile to remark that:

1. the vector normal to the *Boundary* in the *Target Space* \mathbb{T} is not an independent variable; it can be eliminated from the functional derivatives of the action with respect

to the normal n in Parameter Space and the tangent T in Target Space (equations (1.17), (1.19) respectively); in this way we combine these two equations to get the Hamilton-Jacobi equation for the Dynamics of the Boundary;

2. the functional derivative with respect to the tangent vector \boldsymbol{t} in Σ (equation 1.18), gives a condition for the action S to be a reparametrization invariant functional of the loop. The physical meaning is that every tangential deformation of the Boundary can be absorbed in a reparametrization of the loop itself; different parametrizations physically correspond to the same loop.

Chapter 2

Hamilton-Jacobi String Theory

"What's in there?"
"Only what
you take with you."

2.1 Preliminaries

In this section we specialise the results already obtained in subsection 1.1.1 of chapter 1 to the Dynamics of a closed bosonic string described by the following Lagrangian density:

$$\mathcal{L}_{\text{Schild}} = \frac{m^2}{4} \dot{X}^{\mu\nu} \dot{X}_{\mu\nu} \quad , \tag{2.1}$$

where

$$\dot{X}^{\mu\nu} = \epsilon^{ab} \partial_{\xi^a} X^{\mu} \partial_{\xi^b} X^{\nu} \tag{2.2}$$

and

$$X^{\mu} = X^{\mu} (\xi^0, \xi^1) \quad . \tag{2.3}$$

The pair of variables (ξ^0, ξ^1) ranges in a 2-dimensional domain Ξ , so that action 1.2 becomes

$$S = \frac{m^2}{4} \int_{\Xi} \dot{X}^{\mu\nu}(\xi) \, \dot{X}_{\mu\nu}(\xi) \, d\xi^0 \wedge d\xi^1 \quad . \tag{2.4}$$

Our main task is thus to derive the classical Hamilton–Jacobi equation, reproducing Ogielski's procedure in this particular setting; then this result will be used to derive a functional Schrödinger equation. This procedure is quite similar to the one performed to quantize a free particle. Even for a single string we need an infinite, continuous, set of indices labelling the collection of constituent points. This is the reason why the functional calculus enters the game. We would like to remark, once again, the basic difference between this approach and the traditional way of approaching string Dynamics: instead of expanding the string embedding functions in normal modes to find the excitation spectrum, we focus on the wholeness of the closed string as a geometric object. Some new features of string Dynamics can be better understood in this way. In particular it turns out that the Quantum Dynamics of the object is, consistently described as a Shadow Dynamics of areas parametrized by an areal time. Before turning to this interesting developments, it is worth to recall the work by T.Eguchi, who was the first in suggesting such a non–standard string quantization method, and give sound motivations for the choice of the Lagrangian density. This is the subject of the next two sections; after, we will apply all that to the Functional Schrödinger Quantization of the bosonic string; the possible relations between this approach and the more traditional one will be pointed out in chapter 9. To connect the mathematical description with the physical interpretation the following definitions are useful.

Definition 2.1 (World-Sheet & Parametrization).

Let Ξ be a 2-dimensional domain with boundary $\gamma = \partial \Xi$ in the 2-dimensional Minkowski space. A (Classical) World-Sheet, W, is a map of the Parameter Space Ξ into the D-dimensional Minkowski spacetime \mathbb{M}^D ,

$$\mathcal{W}:\Xi\longrightarrow\mathbb{M}^D$$
.

If the map W is described by the embedding functions

$$X^{\mu}:\Xi\longrightarrow\mathbb{M}^{D}$$

we will refer to X^{μ} as a Parametrization of the Classical World-Sheet W.

The World-Sheet is the Bulk of the Theory: then a string is the Boundary of the World-Sheet.

¹We do not make at present any assumption about the quantum nature of the string; indeed, as it will be clear in the subsequent developments, it is really impossible to attribute to the quantum object the same properties (smoothness for example) that characterize the classical one.

Definition 2.2 (Classical Closed Bosonic String & Parametrization).

Let Ξ be a 2-dimensional domain with boundary $\gamma = \partial \Xi$ in the 2-dimensional Minkowski space and W the World-Sheet defined according to the definition 2.1 and parametrized by $X^{\mu}(\xi^0, \xi^1)$. A (Classical) Bosonic String is an embedding

$$\tilde{C}: \gamma \longrightarrow \mathbb{M}^D$$

from the boundary of Ξ to the D-dimensional Minkowski space. Then the embedding functions of the String, Y^{μ} , are the embedding functions of the World-Sheet X^{μ} restricted to the boundary γ :

$$Y^{\mu} \stackrel{\text{def.}}{=} X^{\mu} \rceil_{\gamma} : \gamma = \partial \Xi \longrightarrow \tilde{C}(\gamma) \subset \mathbb{M}^{D}$$
.

Moreover, if we consider a one to one and onto map ξ , defined as

$$\begin{array}{cccc} \xi^i & : & \mathbb{S}^1 & \longrightarrow & \gamma \\ & s & \longrightarrow & \left(\xi^0\left(s\right), \xi^1\left(s\right)\right) & , \end{array}$$

i.e. a parametrization of the boundary γ , we can think a Classical Closed Bosonic String simply as a map from \mathbb{S}^1 to \mathbb{M}^D . This definition is clearly diplayed by the following notation

$$Y^{\mu}(s) = X^{\mu}(\xi^{0}(s), \xi^{1}(s)) \tag{2.5}$$

for the parametrization of the boundary. We will also call $Y^{\mu} = Y^{\mu}(s)$ a <u>Parametrization</u> of the Classical Closed Bosonic String. Moreover we set

$$C = \tilde{C} \circ \mathcal{E}$$
.

Of course, at least in the computations related to the Classical Dynamics of the objects, we always assume that all the defined maps are sufficiently regular in such a way that all the derivations and integrations can be unambiguously carried out. The problem of defining the most adequate functional class for the desciption of the Quantum Dynamics of the objects will be tackled in a more deep way in chapter 10 and in two appendices, B and C. Indeed this problem is really a non trivial one, since in view of the peculiar properties of the Quantum Dynamics it seems reasonable that, in trying to carry the classical approach toward the quantum one, some regularity requirments imposed at the classical level should be weakned.

²Please, see footnote 1 on page 26.

2.2 Area Quantization Scheme: Original Formulation

Eguchi's approach to String quantization, which we are going to briefly review, is an immediate generalization of the point–particle quantization along the guidelines of the Feynman–Schwinger method. The essential point is that reparametrization invariance is not assumed as an original symmetry of the classical action; rather, it is a symmetry of the physical Green functions to be obtained at the very end of the calculations by means of an appropriate averaging procedure. More explicitly, the basic action is not the Nambu–Goto proper area of the String World–Sheet, but the "square" of it, i.e. the Schild Lagrangian (2.1). As discussed in the next section, the corresponding (Schild) action is invariant under area preserving transformations only. Even if this is a restricted symmetry with respect to the full reparametrization invariance of the Nambu–Goto action, it allows a non–standard formulation of String Dynamics leading to a new, Jacobi–type, canonical formalism in which the proper area of the String Parameter Space plays the role of evolution parameter. In other words, the "proper time" appropriate to this problem is neither the String manifold timelike coordinate τ , nor the target space the coordinate time x^0 , but the invariant combination of String manifold coordinates provided by

$$A = \int_{\Xi} d\xi^0 \wedge d\xi^1 = \frac{1}{2} \epsilon_{ab} \int_{\Xi} d\xi^a \wedge d\xi^b \quad . \tag{2.6}$$

Once committed to this unconventional definition of time, the quantum amplitude for the transition from an initial vanishing String configuration to a final non-vanishing String configuration after a lapse of areal time A, is provided by the kernel G[Y(s); A] which satisfies the following diffusion–like equation, or imaginary area Schrödinger equation

$$\frac{1}{2} \frac{\delta^{2} K\left[\boldsymbol{Y}\left(s\right);A\right]}{\delta Y^{\mu}\left(s\right) \delta Y_{\mu}\left(s\right)} = \frac{\partial K\left[\boldsymbol{Y}\left(s\right);A\right]}{\partial A} \quad , \tag{2.7}$$

where, following definition 2.2, $Y^{\mu}(s) = X^{\mu}(\xi^{0}(s), \xi^{1}(s))$, the Parametrization of the Classical Closed Bosonic String³, represents the physical String coordinate, i.e. the only spacelike boundary of the World-Sheet. It may be worth to recall that in the Quantum Mechanics of point particles the "time" t is not a measurable quantity but an arbitrary parameter, since there does not exist a self-adjoint quantum operator with eigenvalues t.

 $^{^{3}}$ We indicate with C the image of the String in spacetime.

Similarly, since there is no self-adjoint operator corresponding to the World-Sheet area, K[Y(s); A] turns out to be explicitly dependent from an arbitrary parameter A, and cannot represent a measurable quantity. However, the Laplace transformed Green function is A-independent and corresponds to the Feynman propagator

$$G\left[\mathbf{Y}\left(s\right);E\right] \stackrel{4}{\equiv} \int_{0}^{\infty} dA K\left[\mathbf{Y}\left(s\right);A\right] \exp(-EA)$$

$$= -\frac{1}{2\left(2\pi\right)^{3/2}} \int_{0}^{\infty} \frac{dA}{A^{3/2}} \exp\left(-\frac{\bar{F}\left[C\right]}{2A} - \frac{1}{2}M^{2}A\right) \tag{2.8}$$

where
$$\bar{F}[C] = \frac{1}{4} (F^{\mu\nu}[C] \pm {}^*F^{\mu\nu}[C])^2$$
 (2.9)

and
$$F^{\mu\nu}\left[C\right] = \oint_C Y^{\mu} dY^{\nu} \quad , \tag{2.10}$$

where F stands for the self-dual (anti self-dual) area element.

Evidently, this approach is quite different from the "normal mode quantization" based on the Nambu-Goto action or the path-integral formulation a la' Polyakov. Moreover, there are some ambiguities in interpreting the formulae for the functional equation and the propagator (2.7-2.8), because at this stage it is still not clear how the dependence from the parametrization of the loop has to be interpreted. If the Green function is independent from the loop paramter, i.e. it is just a functional of the loop, then the right-hand side of equation (2.7) does not depend on s, whereas the left-hand side, does. Instead if the Green function, or the propagator, explicitly depend on s, they cannot be considered as evolution operators for the object as a whole, and this is difficult to reconcile with the invariance under area preserving transformation of the starting action. Moreover, naively, we would expect an equation for the kernel independent from the particular choice of the parametrization, at least of the loop, because, as already pointed out in the exposition of the previous chapter, the last one can be considered as the true dynamical object⁵. These problems can be solved by a closer analysis of the the propagator within a formulation derived from the classical Theory of loop Dynamics. Before facing this task, we would like to provide a stronger motivation for the choice of the lagrangian (2.1) we started from.

⁴Note the naturalness of the inferior bound, namely 0, in the area integration: indeed, an area cannot be negative!

⁵This is true, of course, at the Classical as well as at the Quantum level.

2.3 The Basic Action

As we already anticipated in the previous discussion, our starting point is somehow unconventional with respect to the well known formulation of *String* Theory. We do not start from the Nambu–Goto action:

$$S_{\rm NG} = T_{\rm str.} \int_{\Xi} d^2 \sigma \sqrt{-\frac{1}{2} \dot{X}^{\mu\nu} \dot{X}_{\mu\nu}} \quad , \tag{2.11}$$

where

$$\sqrt{-\frac{1}{2}\dot{X}^{\mu\nu}\dot{X}_{\mu\nu}} \stackrel{\text{def.}}{=} \mathcal{L}_{\text{NG}} , \qquad (2.12)$$

 $\dot{X}^{\mu\nu}(\xi^0,\xi^1)$ is defined as in (2.2), and $T_{\rm str.}$ is the string tension⁶

The action (2.11) has the remarkable geometrical meaning of being the proper area of the String World-Sheet in Target Space. For this reason, it is the most natural generalization of the relativistic point particle action, i.e. the proper length of the particle world-line. The price to pay for such a well defined geometrical meaning is the vanishing of the corresponding canonical Hamiltonian, as a consequence of the reparametrization invariance of the system. Accordingly, a first quantization of the system based over a Schröedinger like approach is washed out from the very beginning. This problem can be successfully skipped by straightforwardly switching to a second quantized, or field theoretical, framework where the embedding functions $X^{\mu}(\sigma^0, \sigma^1)$ are treated as multiplet of scalar fields living over the 2-dimensional String manifold. Such a formulation has elevated String Theory to the role of the most valuable candidate to the Theory of Everything. Notwithstanding, it seems to us to be worth exploring different formulations as well, and clarify the eventual relations among different approaches. The main reason for that stems from the fundamental developments of the last years, that deeply changed the general attitude towards String Theory. The basic result was that the five different superstring models candidate to the role of ultimate unified Theory were not really distinct, but related by an intricate web of non-perturbative duality relations! Thus, a non-perturbative formulation of String Dynamics is compelling. Is it

$$\frac{1}{2\pi\alpha'}$$

which appears usually in all the papers on String Theory.

 $^{^6{\}rm The~string~tension},\,T_{\rm str.}.$ is exactly the quantity

possible to describe the motion of a one-dimensional object without relying on an expansion into harmonic modes? Is it possible to describe the state of the system as a whole, without resolving it, from the very beginning, into a collection of pointlike constituents? And If the answer is yes, can this new formulation give at least a slightly better insight toward the non-perturbative properties that nowadays play such a central role in *String* Theory?

As a personal contribution to answer, at least partially, these questions we propose a functional generalization of the first quantization procedure of a point particle through path-integrals methods [36]. This approach:

- 1. is intrinsically non–perturbative;
- encodes the extended nature of the physical object into a mathematical formulation in terms of "functionals" of the String configuration, instead of ordinary functions.

Of course, we cannot proceed further without a well defined Hamiltonian. So, we will spend some time in considering an action functional, different from (2.1): that is, the one already proposed by Schild in 1977, the Lagrangian density of which we rewrite as:

$$\mathcal{L}_{\text{Schild}} = \frac{m^2}{4} \dot{X}^{\mu\nu} \dot{X}_{\mu\nu} \quad .$$

Before going on, we need some definitions.

Definition 2.3 (Holographic Coordinates).

Let \tilde{C} be a String, as defined in 2.2, and $Y^{\mu}(s)$, one of its possible Parametrizations. The String Holographic Coordinates are

$$Y^{\mu\nu}\left[\gamma\right] = \oint_{\gamma} ds Y^{\mu}(s) \, Y^{\prime\,\nu}(s) = \oint_{C} Y^{\mu} dY^{\nu} \quad . \label{eq:Ymu}$$

It is possible to see that these are well-defined objects for a given String thanks to the following proposition.

Proposition 2.1 (Invariance of Holographic Coordinates).

The Holografic Coordinates are invariant under String reparametrization.

Proof:

This result can be understood since the geometrical interpretation of the *Holographic Coordinates* reveals that they are nothing but the areas of the projection of the *String* onto the coordinate planes and these are invariant quantities. Moreover, we can also prove in a few lines this result observing that if

$$R: \mathbb{S}^1 \longleftrightarrow \mathbb{S}^1$$

with

$$\bar{s} = R(s) \in \mathbb{S}^1$$
 , $\forall s \in \mathbb{S}^1$

is a reparametrization of the String, i.e. a diffeomorphism of the circle such that $\mathbb{S}^1 \approx \bar{\Gamma} = R(\Gamma)$, and

$$\bar{Y}^{\mu}(\bar{s}) = Y^{\mu}(R^{-1}(s))$$
 where $S^1 \stackrel{R}{\longleftarrow} S^1 \stackrel{Y^{\mu}}{\longrightarrow} M^4$
 $\bar{s} = R(s) \stackrel{R}{\longleftarrow} s \stackrel{}{\longrightarrow} Y^{\mu}(s)$

then we have

$$\begin{split} Y^{\mu\nu}\left[C\right] &= \oint_{\bar{\Gamma}\approx\mathbb{S}^1} d\bar{s} \, \bar{Y}^{\mu}(\bar{s}) \, \frac{d\bar{Y}^{\nu}(\bar{s})}{d\bar{s}} \\ &= \oint_{\bar{\Gamma}\approx\mathbb{S}^1} d(R\left(s\right)) \, \bar{Y}^{\mu}(R\left(s\right)) \, \frac{d\bar{Y}^{\nu}(R\left(s\right))}{dR\left(s\right)} \\ &= \oint_{\Gamma\approx\mathbb{S}^1} ds \, \frac{dR\left(s\right)}{ds} Y^{\mu}\!\left(R^{-1}\!\left(R\left(s\right)\right)\right) \, \frac{dY^{\nu}\!\left(R^{-1}\!\left(R\left(s\right)\right)\right)}{ds} \, \frac{1}{dR\left(s\right)} \\ &= \oint_{\Gamma\approx\mathbb{S}^1} ds \, Y^{\mu}\left(s\right) Y'^{\nu}\left(s\right) \quad . \end{split}$$

From the right hand sides in the first and the last lines we thus get the desired result.

A functional differential operator that will be central in our description of the *String* Dynamics, will be the functional derivative with respect to the *Holographic Coordinates*: Now, we define the following quantity:

Definition 2.4 (Holographic Functional Derivative).

The <u>Holographic Functional Derivative</u> is the functional derivative with respect to the tensor density corresponding to the Holographic Coordinates; we will indicate is as

$$\frac{\delta}{\delta Y^{\mu\nu}\left(s\right)}$$

and define it implicitly in terms of the standard functional derivative as,

$$\frac{\delta}{\delta Y^{\mu}(s)} \stackrel{\text{def.}}{=} Y^{\prime \nu} \frac{\delta}{\delta Y^{\mu \nu}(s)} \quad . \tag{2.13}$$

More information about this functional operation can be found in appendix D. Here we only note that, while the *Holographic Coordinates* $Y^{\mu\nu}$ [C] are functionals of the loop C, i.e., they contain no reference to a special point of the loop, the holographic Functional Derivative $\delta/\delta\sigma^{\mu\nu}$ (\bar{s}) operates at the contact point $\bar{Y}^{\mu}=Y^{\mu}$ (\bar{s}). Thus, the area derivative of a functional is no longer a functional. Even if the functional under derivation is reparametrization invariant, its area derivative behaves as a scalar density under redefinition of the loop coordinate.

Moreover since we would like to define the Dynamics of the object we have actually to concentrate our attention on the World–Sheet of the String also, defined as in 2.1. The presence of two parameters ξ^i allows us to define the following dynamical quantities:

Definition 2.5 (Local World-Sheet Area Velocity).

Let $X^{\mu}\left(\xi^{i}\right)$ be a Parametrization of the String World–Sheet . The <u>Local Area Velocity</u> of the World–Sheet is

$$\dot{X}^{\mu\nu}\left(\xi^{i}\right)\stackrel{\mathrm{def.}}{=}\epsilon^{ab}\partial_{a}X^{\mu}\left(\xi^{i}\right)\partial_{b}X^{\nu}\left(\xi^{i}\right)\stackrel{\mathrm{def.}}{=}\left\{X^{\mu},X^{\nu}\right\}_{\xi}\quad,$$

where we use the shorthand notation

$$\partial_a X^{\mu}(\xi^i) = \frac{\partial X^{\mu}(\xi^i)}{\partial \xi^a} \quad .$$

A similar quantity can be defined for the *Boundary* of the *World–Sheet* as well:

Definition 2.6 (String Area Velocity).

Let $\dot{X}^{\mu\nu}(\xi^i)$ be the Local Area Velocity of the World–Sheet, associated with a String; the <u>String Area Velocity</u> is

$$\dot{X}^{\mu\nu}\left(s\right) \stackrel{\text{def.}}{=} \epsilon^{ab}\partial_{a}X^{\mu}\left(\xi^{c}\right)\partial_{b}X^{\nu}\left(\xi^{c}\right)\Big|_{\xi^{c}=\xi^{c}\left(s\right)} = \left\{X^{\mu}\left(\xi^{c}\right), X^{\nu}\left(\xi^{c}\right)\right\}_{\xi^{c}=\xi^{c}\left(s\right)} \quad , \tag{2.14}$$

i.e. the Local Area Velocity of the World-Sheet computed on the boundary.

We remark that the String Area Velocity implicitly encodes the information that the String is glued to a World-Sheet. This means that it is impossible to express $\dot{X}^{\mu\nu}$ (s) only in

terms of $Y^{\mu}(s)$, without any referenced to the Local Area Velocity of the World-Sheet. A String is not just a free fluctuating loop, because it is also the boundary of a World-Sheet. We will have the opportunity in chapter 9 to spend some more comments about this feature, enlightening also the relation between loop Dynamics and our formulation of String Dynamics. Momentarily, let us introduce the following notation, which is useful to distinguish between Bulk and Boundary quantities:

Notation 2.1 (String Area Velocity).

We will indicate the String Area Velocity with the symbol

$$\dot{Y}^{\mu\nu}\left(s\right)$$
 .

We stress again that this is just a convenient notation to understand at first sight if we are dealing with a *Bulk* quantity or with a *Boundary* one: but it is *NOT* possible to compute $\dot{Y}^{\mu\nu}$ with equation 2.14 by simply replacing X^{μ} with Y^{μ} , because the passage to the boundary is definitely non trivial: namely our *String* is still attached to its *World–Sheet*!

As already pointed out by Schild [21] the Lagrangian density (2.1) has some short-comings as well as some advantages. The most relevant problem is that it lacks a full reparametrization invariance, being invariant only under area preserving transformations, i.e transformations of the *Parameter Space* of the type

$$\left(\xi^{0}, \xi^{1}\right) \longrightarrow \left(\bar{\xi}^{0}, \bar{\xi}^{1}\right) \quad : \quad \left|\frac{\partial \left(\bar{\xi}^{0}, \bar{\xi}^{1}\right)}{\partial \left(\xi^{0}, \xi^{1}\right)}\right| = 1 \quad . \tag{2.15}$$

Neverthless, it is possible to prove by varying the action functional

$$S_{\text{Schild}} = \int_{\Xi} \mathcal{L}_{\text{Schild}} d\xi^1 \wedge d\xi^2$$
 (2.16)

that

Proposition 2.2 (Schild String Equations of Motion).

The equations of motion for a String (i.e a String defined by the Schild Lagrangian density)
are

$$\epsilon^{ab}\partial_a \left[\dot{X}_{\mu\nu}\partial_b X^\nu \right] = 0$$

and they correspond to motions with constant $\left(\dot{X}^{\mu\nu}\right)^2$.

Proof:

We use the same notation as before, where the String is defined on the $Parameter\ Space\ \Xi$ and thus the chosen Parametrization for the World-Sheet is denoted by $X(\xi)$. Then the equations of motion for the $Schild\ String$ are the Euler-Lagrange equations associated with the Lagrangian Density (2.1). We thus compute

$$\frac{\partial \mathcal{L}_{Schild}}{\partial X^{\alpha}} = 0$$

and

$$\frac{\partial \mathcal{L}_{\text{Schild}}}{\partial (\partial_a X^{\alpha})} = \frac{\partial \mathcal{L}_{\text{Schild}}}{\partial \dot{X}^{\rho \tau}} \frac{\partial \dot{X}^{\rho \tau}}{\partial (\partial_a X^{\alpha})}
= \frac{\partial \mathcal{L}_{\text{Schild}}}{\partial \dot{X}^{\rho \tau}} \epsilon^{mn} \left[\delta_m^a \delta_\alpha^\rho \partial_n X^{\tau} + \delta_n^a \delta_\alpha^{\tau} \partial_m X^{\rho} \right]
= \frac{m^2}{4} 2 \dot{X}_{\alpha \tau} 2 \epsilon^{an} \partial_n X^{\tau} .$$

Then

$$\partial_a \left(\frac{\partial \mathcal{L}_{\text{Schild}}}{\partial (\partial_a X^{\alpha})} \right) - \frac{\partial \mathcal{L}_{\text{Schild}}}{\partial X^{\alpha}} = 0$$

is equal to

$$\partial_a \left(m^2 \dot{X}_{\alpha \tau} \epsilon^{an} \partial_n X^{\tau} \right) = 0 \quad ,$$

which is the desired result.

At the same time with the same procedure, but starting from the Nambu–Goto Lagrangian density (2.12), we obtain

Proposition 2.3 (Nambu-Goto Equations of Motions).

The equations of motion for a Nambu-Goto String are

$$\epsilon^{ab}\partial_a \left[\frac{\dot{X}_{\mu\nu}}{\sqrt{-\frac{1}{2}\dot{X}^{\mu\nu}\dot{X}_{\mu\nu}}} \, \partial_b X^\nu \right] = 0 \quad .$$

Proof:

The procedure is as before. We now have again

$$\frac{\partial \mathcal{L}_{NG}}{\partial X^{\alpha}} = 0$$

but

$$\begin{split} \frac{\partial \mathcal{L}_{\text{NG}}}{\partial (\partial_a X^\alpha)} &= \frac{\partial \mathcal{L}_{\text{NG}}}{\partial \dot{X}^{\rho\tau}} \frac{\partial \dot{X}^{\rho\tau}}{\partial (\partial_a X^\alpha)} \\ &= \frac{\partial \mathcal{L}_{\text{NG}}}{\partial \dot{X}^{\rho\tau}} \epsilon^{mn} \left[\delta^a_m \delta^\rho_\alpha \partial_n X^\tau + \delta^a_n \delta^\tau_\alpha \partial_m X^\rho \right] \\ &= -\frac{1}{2} T_{\text{Str.}} \frac{2 \dot{X}^{\alpha\tau}}{\sqrt{-\frac{1}{2} \dot{X}^{\eta\lambda} \dot{X}_{\eta\lambda}}} 2 \epsilon^{an} \partial_n X^\tau \quad . \end{split}$$

The above equation is meaningful only if

$$\dot{X}^{\eta\lambda}\dot{X}_{n\lambda} \neq 0$$

of course. Then the Euler-Lagrange equations are

$$2T_{\rm Str.}\partial_a \left(\frac{\dot{X}_{\alpha\tau}}{\sqrt{-\frac{1}{2}\dot{X}^{\eta\lambda}\dot{X}_{\eta\lambda}}} \epsilon^{an} \partial_n X^{\tau} \right) = 0 \quad ,$$

as stated in the proposition.

Thus we can see that

Proposition 2.4 (Schild versus Nambu-Goto Equivalence).

For motions with $(\dot{X}^{\mu\nu})^2 = \text{const.} \neq 0$ the Schild String has the same Dynamics of the Nambu–Goto String.

Proof:

We firstly observe that the following identity holds:

$$\partial_{b} \left[\frac{\partial \left(-\frac{1}{4} \dot{X}^{\eta \lambda} \dot{X}_{\eta \lambda} \right)}{\partial_{a} X^{\mu}} \right] = -\partial_{b} \left[\frac{1}{4} 2 \dot{X}_{\eta \lambda} \frac{\partial \dot{X}^{\eta \lambda}}{\partial_{a} X^{\mu}} \right]$$

$$= -\frac{1}{2} \partial_{b} \left[\dot{X}_{\eta \lambda} \frac{\partial}{\partial_{a} X^{\mu}} \left(\epsilon^{mn} \partial_{m} X^{\eta} \partial_{n} X^{\lambda} \right) \right]$$

$$= -\frac{1}{2} \partial_{b} \left[\dot{X}_{\eta \lambda} \epsilon^{mn} \left(\delta_{n}^{a} \delta_{\mu}^{\lambda} \partial_{m} X^{\eta} + \delta_{m}^{a} \delta_{\mu}^{\eta} \partial_{n} X^{\lambda} \right) \right]$$

$$= \partial_{b} \left[\dot{X}_{\mu \lambda} \epsilon^{an} \partial_{n} X^{\lambda} \right] . \tag{2.17}$$

Then we can use it to see that the equation of motion for the Schild String implies

$$0 = \partial_b X^{\mu} \partial_a \left[\frac{\partial \left(-\frac{1}{4} \dot{X}^{\eta \lambda} \dot{X}_{\eta \lambda} \right)}{\partial (\partial_a X^{\mu})} \right]$$

$$= \partial_{a} \left[\partial_{b} X^{\mu} \frac{\partial \left(-\frac{1}{4} \dot{X}^{\eta \lambda} \dot{X}_{\eta \lambda} \right)}{\partial (\partial_{a} X^{\mu})} \right] - \partial_{ab}^{2} X^{\mu} \left[\frac{\partial \left(-\frac{1}{4} \dot{X}^{\eta \lambda} \dot{X}_{\eta \lambda} \right)}{\partial (\partial_{a} X^{\mu})} \right]$$

$$= \partial_{b} \left(-\frac{1}{2} \dot{X}^{\eta \lambda} \dot{X}_{\eta \lambda} \right) - \partial_{b} \left(-\frac{1}{4} \dot{X}^{\eta \lambda} \dot{X}_{\eta \lambda} \right)$$

$$= \partial_{b} \left(-\frac{1}{4} \dot{X}^{\eta \lambda} \dot{X}_{\eta \lambda} \right) ; \qquad (2.18)$$

then a Schild String motion has

$$\dot{X}^{\eta\lambda}\dot{X}_{\eta\lambda} = \text{const.}$$
 .

When this condition is satisfied, we see that the Nambu-Goto String equation is

$$2T_{\rm Str.}\partial_a \left(\dot{X}_{\alpha\tau} \epsilon^{an} \partial_n X^{\tau} \right) = 0 \quad ,$$

i.e. nothing but the Schild String equation. This completes the proof.

The similarity between the Schild String and the Nambu–Goto String is hardly surprising. Indeed, if we consider the Nambu–Goto String as a Classical Field Theory, it can be proved that, being reparametrization invariant as a dynamical system, if X^{μ} are the coordinate variables, then the Lagrangian does not depend, besides X^{μ} , on all the conjugated momenta, but only on their antisymmetric combination

$$\dot{X}^{\mu\nu} = \frac{\partial X^{[\mu}}{\partial \xi^0} \frac{\partial X^{\nu]}}{\partial \xi^1} = \frac{1}{2} \epsilon^{ab} \frac{\partial X^{\mu}}{\partial \xi^a} \frac{\partial X^{\nu}}{\partial \xi^b} \quad ,$$

i.e. on the Local World–Sheet Area Velocity, constrained in a proper way. Therefore, we can see that in the Schild Lagrangian density the main property implied, by reparametrization invariance in \mathcal{L}_{NG} of equation (2.12), i.e. the statement about which are the correct speed variables, is still there. Moreover one thing that we gain with the Schild formulation is that it allows the description of null Strings. The second one is, of course, that the lack of full reparametrization invariance leads to a well defined canonical Hamiltonian density. We first give some more definitions.

Definition 2.7 (Bulk Area Momentum).

The <u>Bulk Area momentum</u> or <u>World-Sheet Area Momentum</u> is the momentum canonically conjugated to the Local World-Sheet Area Velocity, i.e.

$$P_{\mu\nu}(\boldsymbol{\xi}) = \frac{\partial \mathcal{L}_{\text{Schild}}}{\partial \dot{X}^{\mu\nu}(\boldsymbol{\xi})} \quad .$$

Then, we extend this result from the *Bulk* to the *Boundary* as we already did in the case of the *Local World–Sheet Area Velocity*:

Definition 2.8 (Boundary Area Momentum).

The <u>Boundary Area Momentum</u> or <u>String Area Momentum</u> is the Bulk Area Momentum computed on the boundary, i.e.

$$Q_{\mu\nu}(s) = \frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu\nu}(\xi)} \bigg|_{\xi=\xi(s)} = P_{\mu\nu}(\xi) \bigg|_{\xi=\xi(s)} ,$$

where, $\boldsymbol{\xi} = \boldsymbol{\xi}(s)$ is a parametrization of the boundary $\partial \Xi$ of the domain⁷ Ξ .

After this digression, which will be useful later on, we go back to our previous assertion which is the argument of the following proposition.

Proposition 2.5 (Schild Hamiltonian Density).

The Schild String admits the Hamiltonian density

$$\mathcal{H}_{\text{Schild}} = \frac{P^{\mu\nu}(\boldsymbol{\xi}) P_{\mu\nu}(\boldsymbol{\xi})}{4m^2} \quad , \tag{2.19}$$

where $P^{\mu\nu} = P^{\mu\nu}(\xi^i)$ is the Bulk Area Momentum.

Proof:

Starting from the definition (2.7), and remembering the Schild Lagrangian Density (2.1), we firstly get

$$P_{\mu\nu}(\boldsymbol{\xi}) = \frac{\partial \mathcal{L}_{\text{Schild}}}{\partial \dot{X}^{\mu\nu}(\boldsymbol{\xi})}$$

$$= \frac{\partial}{\partial \dot{X}^{\mu\nu}(\boldsymbol{\xi})} \left[\frac{m^2}{4} \dot{X}^{\rho\sigma}(\boldsymbol{\xi}) \, \dot{X}_{\rho\sigma}(\boldsymbol{\xi}) \right]$$

$$= 2 \frac{m^2}{4} \dot{X}_{\rho\sigma}(\boldsymbol{\xi}) \frac{\partial \dot{X}^{\rho\sigma}(\boldsymbol{\xi})}{\partial \dot{X}^{\mu\nu}(\boldsymbol{\xi})}$$

$$= \frac{m^2}{2} \dot{X}_{\rho\sigma}(\boldsymbol{\xi}) \left[\delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma} \delta^{\nu}_{\rho} \right]$$

$$= m^2 \dot{X}_{\mu\nu}(\boldsymbol{\xi}) ,$$

⁷This definition is the natural counterpart of definition 2.6: again we go to the *Boundary*, which is just a curve in the $\boldsymbol{\xi}$ variables. As the *Parametrization* of the boundary of the *World–Sheet* is named with a letter \boldsymbol{Y} which is the next one after the one for the *Parametrization* of the *World–Sheet* itself, i.e. \boldsymbol{X} , in the alphabet, the standard name for the *Boundary Area Momentum*, \boldsymbol{Q} , is the next letter after the one for the *Bulk Area Momentum*, namely \boldsymbol{P} .

so that

$$\dot{X}_{\mu\nu}(\boldsymbol{\xi}) = \frac{1}{m^2} P_{\mu\nu}(\boldsymbol{\xi}) \quad .$$

Substituting the last equation in the expression for the Hamiltonian, defined as the Legendre Transform of the Lagrangian (2.1),

$$\mathcal{H}_{\text{Schild}} \stackrel{8}{=} \frac{1}{2} P^{\mu\nu}(\xi) \dot{X}_{\mu\nu}(\xi) - \mathcal{L}_{\text{Schild}}$$

$$= \frac{1}{2} P^{\mu\nu}(\xi) \frac{1}{m^2} P_{\mu\nu}(\xi) - \frac{m^2}{4} \frac{1}{m^2} P_{\mu\nu}(\xi) \frac{1}{m^2} P_{\mu\nu}(\xi)$$

$$= \frac{P^{\mu\nu}(\xi) P_{\mu\nu}(\xi)}{4m^2} ,$$
(2.20)

we get the desired result.

So we see that a quantization procedure starting from the Schild action can be performed more easily on the same guidelines of the quantization of a classical particle. In particular we can translate the Schild Action (2.16) derived from the Lagrangian (2.1) in the Hamiltonian Formalism.

Proposition 2.6 (Schild Action In Hamiltonian Form).

The Schild Action (2.1) admits the Hamiltonian form

$$S[\mathbf{X}, \mathbf{P}] = \frac{1}{2} \int_{\Xi} d^2 \boldsymbol{\xi} \left[P_{\mu\nu}(\boldsymbol{\xi}) \, \dot{X}^{\mu\nu}(\boldsymbol{\xi}) - \frac{1}{2m^2} P_{\mu\nu}(\boldsymbol{\xi}) \, P^{\mu\nu}(\boldsymbol{\xi}) \right] \quad . \tag{2.21}$$

Proof:

If we solve for the Lagrangian density in the Legendre Transform (2.20), we substitute the expression (2.19) for the Hamiltonian and take out a common factor 1/2, we obtain

$$\mathcal{L}_{\text{Schild}} = \mathcal{L}_{\text{Schild}}(\boldsymbol{X}, \boldsymbol{P})$$

$$= \frac{1}{2} \left[P_{\mu\nu} \dot{X}^{\mu\nu} - \frac{1}{2m^2} P_{\mu\nu} P^{\mu\nu} \right]$$

for the Lagrangian, which we then put in (2.16).

Neverthless, we do not like giving up reparametrization invariance without fighting!

⁸We note that the factor 1/2 in the Legendre Transform is there to take into account the antisimmetry of the contracted indices; in this way, overcounting is avoided. We adhere to the convention that in the general case the factor is 1/(p!) (see also the computations in appendix A.1).

We will present in the following section a possible way out of these seemingly problematic situation.

2.4 Reparametrized Schild Formulation

The central step in our proposal is already present in the following key remark:

Proposition 2.7 (Schild Lagrangian Variation under Reparametrization).

Under a reparametrization of the domain Ξ ,

$$\begin{array}{cccc} \mathfrak{s} & : & \Sigma & \longrightarrow & \Xi \\ & \left(\sigma^0,\sigma^1\right) & \longrightarrow & \left(\xi^0\left(\sigma^0,\sigma^1\right),\xi^1\left(\sigma^0,\sigma^1\right)\right) & , \end{array}$$

the 2-form

$$\boldsymbol{\omega} = \mathcal{L}_{\mathrm{Schild}}\!\left(X^{\mu}\!\left(\boldsymbol{\xi}^{0}, \boldsymbol{\xi}^{1}\right), \dot{X}^{\mu\nu}\!\left(\boldsymbol{\xi}^{0}, \boldsymbol{\xi}^{1}\right)\right) \boldsymbol{d}\boldsymbol{\xi}^{0} \wedge \boldsymbol{d}\boldsymbol{\xi}^{1}$$

transforms as

$$\omega \longrightarrow ilde{\omega}$$

where,

$$\begin{split} \tilde{\boldsymbol{\omega}} &= \frac{1}{|\mathcal{J}(\mathfrak{s})|} \mathcal{L}_{\mathrm{Schild}} \Big(\tilde{X}^{\mu} \big(\sigma^0, \sigma^1 \big) \,, \dot{\tilde{X}}^{\mu\nu} \big(\sigma^0, \sigma^1 \big) \Big) \, \boldsymbol{d} \boldsymbol{\sigma}^0 \wedge \boldsymbol{d} \boldsymbol{\sigma}^1 \quad, \\ \tilde{X}^{\mu} \big(\sigma^0, \sigma^1 \big) &= X^{\mu} \big(\xi^0 \big(\sigma^0, \sigma^1 \big) \,, \xi^1 \big(\sigma^0, \sigma^1 \big) \big) \quad, \\ \dot{\tilde{X}}^{\mu\nu} \big(\sigma^0, \sigma^1 \big) &= \dot{X}^{\mu\nu} \big(\xi^0 \big(\sigma^0, \sigma^1 \big) \,, \xi^1 \big(\sigma^0, \sigma^1 \big) \big) \end{split}$$

and

$$\mathcal{J}(\mathfrak{s}) \equiv \begin{pmatrix} \frac{\partial \xi^0}{\partial \sigma^0} & \frac{\partial \xi^0}{\partial \sigma^1} \\ \frac{\partial \xi^1}{\partial \sigma^0} & \frac{\partial \xi^1}{\partial \sigma^1} \end{pmatrix}$$

is the Jacobean matrix of the transformation \mathfrak{s} .

Proof:

The Lagrangian one form ω depends only from the Area Velocity $\dot{X}^{\mu\nu}$. Thus, we get

$$\dot{\tilde{X}}^{\mu\nu} = \epsilon^{ab} \frac{\partial \tilde{X}^{\mu}(\sigma)}{\partial \sigma^a} \frac{\partial \tilde{X}^{\nu}(\sigma)}{\partial \sigma^b}$$

$$= \epsilon^{ab} \frac{\partial X^{\mu}(\xi(\sigma))}{\partial \sigma^{a}} \frac{\partial X^{\nu}(\xi(\sigma))}{\partial \sigma^{b}}$$

$$= \epsilon^{ab} \frac{\partial X^{\mu}(\xi)}{\partial \xi^{A}} \frac{\partial X^{\nu}(\xi)}{\partial \xi^{B}} \frac{\partial \xi^{A}}{\partial \sigma^{a}} \frac{\partial \xi^{B}}{\partial \sigma^{b}} . \qquad (2.22)$$

Now, let us consider the quantity

$$\epsilon^{ab} \frac{\partial \xi^A}{\partial \sigma^a} \frac{\partial \xi^B}{\partial \sigma^b}$$
 :

this is a totally antisymmetric 2nd rank tensor in two dimensions, and thus it must be proportional to the Levi–Civita tensor; then,

$$\epsilon^{ab} \frac{\partial \xi^A}{\partial \sigma^a} \frac{\partial \xi^B}{\partial \sigma^b} = \epsilon^{AB} \kappa(\boldsymbol{\sigma}) \quad . \tag{2.23}$$

The expression of $\kappa(\sigma)$ can be deduced, for example, by looking at the a=0, b=1 component of the tensor in the equation above; then, equation (2.23) reduces to

$$\kappa(\boldsymbol{\sigma}) = \epsilon_{01}\kappa(\boldsymbol{\sigma}) = \epsilon^{ab} \frac{\partial \xi^{0}}{\partial \sigma^{a}} \frac{\partial \xi^{1}}{\partial \sigma^{b}}$$

$$= \frac{1}{2} \epsilon^{ab} \epsilon_{AB} \frac{\partial \xi^{A}}{\partial \sigma^{a}} \frac{\partial \xi^{B}}{\partial \sigma^{b}}$$

$$= |\mathcal{J}(\mathfrak{s})| . \tag{2.24}$$

From equation (2.22) we thus obtain

$$\dot{\tilde{X}}^{\mu\nu} = |\mathcal{J}(\mathfrak{s})| \epsilon_{AB} \frac{\partial X^{\mu}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^{A}} \frac{\partial X^{\nu}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^{B}}
= |\mathcal{J}(\mathfrak{s})| \dot{X}^{\mu\nu}$$

or, equivalently

$$\dot{X}^{\mu\nu} = \frac{\dot{\ddot{X}}^{\mu\nu}}{|\mathcal{J}(\mathfrak{s})|} \quad . \tag{2.25}$$

At the same time, we have

$$d\xi^{0} \wedge d\xi^{1} = \frac{1}{2} \epsilon_{AB} d\xi^{A} \wedge d\xi^{B}$$

$$= \frac{1}{2} \epsilon_{AB} \frac{\partial \xi^{A}}{\partial \sigma^{i}} \frac{\partial \xi^{B}}{\partial \sigma^{j}} d\sigma^{i} \wedge d\sigma^{j}$$

$$= \frac{1}{2!} \epsilon_{AB} \frac{\partial \xi^{A}}{\partial \sigma^{i}} \frac{\partial \xi^{B}}{\partial \sigma^{j}} \epsilon^{ij} d\sigma^{0} \wedge d\sigma^{1}$$

$$= \left[\frac{1}{2!} \epsilon_{AB} \epsilon^{ij} \frac{\partial \xi^{A}}{\partial \sigma^{i}} \frac{\partial \xi^{B}}{\partial \sigma^{j}} \right] d\sigma^{0} \wedge d\sigma^{1}$$

$$= \left| \frac{\partial (\xi^{0}, \xi^{1})}{\partial (\sigma^{0}, \sigma^{1})} \right| d\sigma^{0} \wedge d\sigma^{1}$$

$$= |\mathcal{J}(\mathfrak{s})| d\sigma^{0} \wedge d\sigma^{1} \qquad (2.26)$$

Then, by substituting equations (2.25) and (2.26) into the Schild Lagrangian 2-form ω , we get

$$ilde{\omega} = rac{\dot{ ilde{X}}^{\mu
u}\dot{ ilde{X}}_{\mu
u}}{|\mathcal{J}(\mathfrak{s})|}d\sigma^0\wedge d\sigma^1 \quad ,$$

the desired result. Incidentally, we observe that if we define in a natural way

$$\dot{\xi}^{AB} = \epsilon^{ab} \frac{\partial \xi^A}{\partial \sigma^a} \frac{\partial \xi^B}{\partial \sigma^b}$$

then,

$$|\mathcal{J}(\mathfrak{s})| = \frac{1}{2} \epsilon_{ab} \dot{\xi}^{ab} \quad . \tag{2.27}$$

This is the key step toward our next proposal: we can lift the variables ξ^a to the role of dynamical fields provided we introduce a new pair of coordinates, and transforming our original model into a two dimensional field Theory in six dimensions. We point out that there is no more dependence in the X^{μ} variables from the ξ^A variables, which are now independent fields: the World-Sheet coordinates, and the new ξ^M fields are both functions of the new parameters⁹ σ^a .

Notation 2.2 (Indexing of Fields and Parameters).

To keep a clear distinction between the new fields $\boldsymbol{\xi}$, and the new parameters σ^a , from now on we will use capital latin indices for the new fields: $\boldsymbol{\xi}^A$. This should also be seen as a clear distinction between the previous role of the $\boldsymbol{\xi}$ and the present one! Moreover, all the quantities already defined on Ξ , when referred after this point, are to be considered as defined on Σ and thus functions of the $\boldsymbol{\sigma}$ variables $\boldsymbol{\sigma}$.

In light of proposition 2.7 we introduce the Reparametrized Schild Lagrangian density.

Definition 2.9 (Reparametrized Schild Lagrangian Density).

The Reparametrized Schild Lagrangian Density is

$$\mathcal{L} = \mathcal{L}_{\text{Schild}}^{\text{rep.}} \stackrel{\text{def.}}{=} \frac{m^2}{2} \frac{\dot{X}^{\mu\nu}(\sigma^0, \sigma^1) \, \dot{X}_{\mu\nu}(\sigma^0, \sigma^1)}{\epsilon^{AB} \dot{\xi}_{AB}(\sigma^0, \sigma^1)}$$
(2.28)

so that, the basic dynamical object is now the 2-form

$$\Omega = \frac{m^2}{2} \frac{\dot{X}^{\mu\nu} \dot{X}_{\mu\nu}}{\epsilon^{AB} \dot{\xi}_{AB}} d\sigma^0 \wedge d\sigma^1 \quad . \tag{2.29}$$

Note that we used here relation (2.27). Some clarifications are due at this stage.

⁹Please, see notation (2.2) below for the change in the notation of the index.

¹⁰This is just a matter of convention. We are aware that changing the name of the variables should not be a problem. However, we believe that a good notation can enlighten some important aspects of the underlying concepts. Accordingly, from now on we will consistently parametrize the *World–Sheet* with $\sigma \in \Sigma$.

- 1. The Theory defined by the previous formulae is reparametrization invariant: thus in this formulation we recover the main property of the Nambu–Goto action. Neverthless, it is polynomial in the dynamical variables, because this 6-dimensional extension does not treat all the fields on the same footing. It is basically different from, say, the 6-dimensional extension that we could obtain by simply considering X^{μ} as a 6-dimensional vector, e.g. there is no Lorenz symmetry relating the ξ^{A} and X^{μ} fields.
- 2. The last, but not the least, we can connect the meaning of the additional fields with the discussion about boundary Dynamics we made in the previous chapter; we saw that assigning a dynamical role to the boundary implies the need to vary the boundary itself. This is a non-standard procedure in field Theory. In this new framework, we expect to capture the whole boundary Dynamics through the variation of the ξ^A fields. These extra fields provide a geometrical representation of the Parameter Space after an arbitrary reparametrization. Thus, they give us the opportunity to build up an "ordinary Field Theory", embodying arbitrary variations of the Parameter Space itself.

Before concluding this section we take the first results from the Reparametrized Schild Lagrangian Density, giving the Hamiltonian formulation of the Theory:

Proposition 2.8 (Conjugated Momenta in Reparametrized Formulation).

The momenta canonically conjugated to the Area Velocities are

$$P^{\mu\nu} = 2m^2 \frac{\dot{X}^{\mu\nu}}{\epsilon_{CD}\dot{\xi}^{CD}} \tag{2.30}$$

$$\pi^{AB} = -m^2 \frac{\dot{X}^{\mu\nu} \dot{X}_{\mu\nu}}{\left(\epsilon_{CD} \dot{\xi}^{CD}\right)^2} \epsilon^{AB} = -m^2 \frac{P^{\mu\nu} P_{\mu\nu}}{2} \epsilon^{AB} = -\epsilon^{AB} \mathcal{H}_{\text{Schild}}(\mathbf{P}) \quad . \quad (2.31)$$

Proof:

To get the results it is just necessary to comput the derivatives of the reparametrized Lagrangian Density, with respect to $\dot{X}^{\mu\nu}$ and $\dot{\xi}^{\dot{A}B}$. We thus have

$$\frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu\nu}} \quad = \quad \frac{m^2}{2} \frac{1}{\epsilon^{AB} \dot{\xi}_{AB}} 2 \dot{X}^{\lambda\eta} \frac{\partial \dot{X}^{\lambda\eta}}{\partial \dot{X}^{\mu\nu}}$$

$$= m^2 \frac{1}{\epsilon^{AB} \dot{\xi}_{AB}} \dot{X}^{\lambda \eta} \left[\delta^{\lambda}_{\mu} \delta^{\eta}_{\nu} - \delta^{\lambda}_{\nu} \delta^{\eta}_{\mu} \right]$$

$$= m^2 \frac{1}{\epsilon^{AB} \dot{\xi}_{AB}} 2 \dot{X}^{\mu \nu}$$

$$2 = m^2 \frac{\dot{X}^{\mu \nu}}{\epsilon^{AB} \dot{\xi}_{AB}} .$$

In the same way we can obtain the second result:

$$\frac{\partial \mathcal{L}}{\partial \dot{\xi}^{AB}} = \frac{m^2}{2} \frac{\dot{X}^{\mu\nu} \dot{X}_{\mu\nu}}{\left(\epsilon^{EF} \dot{\xi}_{EF}\right)^2} \frac{\partial \left(\epsilon_{CD} \dot{\xi}^{CD}\right)}{\partial \dot{\xi}^{AB}}$$

$$= \frac{m^2}{2} \frac{\dot{X}^{\mu\nu} \dot{X}_{\mu\nu}}{\left(\epsilon^{EF} \dot{\xi}_{EF}\right)^2} \epsilon_{CD} \frac{\partial \dot{\xi}^{CD}}{\partial \dot{\xi}^{AB}}$$

$$= \frac{m^2}{2} \frac{\dot{X}^{\mu\nu} \dot{X}_{\mu\nu}}{\left(\epsilon^{EF} \dot{\xi}_{EF}\right)^2} \epsilon_{CD} \left(\delta^{AC} \delta^{BD} - \delta^{AD} \delta^{BC}\right)$$

$$= m^2 \frac{\dot{X}^{\mu\nu} \dot{X}_{\mu\nu}}{\left(\epsilon^{EF} \dot{\xi}_{EF}\right)^2} \epsilon_{CD}$$

$$= \epsilon_{CD} \mathcal{H}_{\text{Schild}}(P_{\mu\nu}) \quad . \tag{2.32}$$

According with [34] the 1-form defining the Hamiltonian Theory will be

$$\Omega_{\rm H} = P_{\mu\nu} \, dX^{\mu} \wedge dX^{\nu} + \pi_{AB} \, d\xi^{A} \wedge d\xi^{B} + \frac{1}{2} N^{AB} \left(\pi_{AB} - \epsilon_{AB} \mathcal{H}_{\rm Schild}(\mathbf{P}) \right) d\sigma^{0} \wedge d\sigma^{1} , \qquad (2.33)$$

where $N^{AB} \equiv N^{AB}(\sigma^0, \sigma^1)$ is simply a Lagrange multiplier enforcing the relation (2.31) between the momenta $P^{\mu\nu}$ and π^{AB} . We also note that the expression $\mathcal{H}_{Schild}(\mathbf{P})$ means the quantity which has the functional dependence of \mathcal{H}_{Schild} but from the variable which is the conjugated momentum of equation (2.30).

2.5 Hamilton-Jacobi Theory

In our opinion, as already pointed out on page 15, the Hamilton–Jacobi formulation of the Dynamics of a classical system is a preferred starting point toward the quantum framework. In this section, we will derive the Hamilton–Jacobi equation for a closed *String* from two

different perspectives in order to enlighten the properties of the *Boundary* Dynamics, and to prepare in a solid way the derivation of a *Schrödinger like functional equation*.

2.5.1 Hamilton–Jacobi Equation: Ogielski Formulation

As we already pointed out at the end of subsection 1.1.1, on page 23, if we are able to eliminate the vector T from the functional derivatives (1.17, 1.19) of the action (1.2), then we can find a relation between them. This relation is nothing but the Hamilton–Jacobi equation. Then, we shall specialise this result to the Lagrangian density (2.1) for the Schild String. As a preliminary result we can write the String Area Velocity in the basis (T, N) given by the vectors tangent to the boundary and normal to the World–Sheet at the boundary.

Proposition 2.9 (String Area Velocity in (T, N) Coordinates).

The String Area Velocity expressed in terms of the vector fields normal N and tangent T to the loop C, representing the String in Target Space, is

$$\dot{X}^{\alpha\beta} = \frac{1}{\left(\sigma'\right)^2} \left(T^{\alpha} N^{\beta} - T^{\beta} N^{\alpha} \right) \tag{2.34}$$

Proof:

We start from the expression we want to prove and, remembering the definitions (1.3, 1.4), we can rewrite it as as

$$\dot{X}^{\alpha\beta} = \frac{1}{(\sigma')^2} \left(T^{\alpha} N^{\beta} - T^{\beta} N^{\alpha} \right)
= \frac{1}{(\sigma')^2} \left(\sigma'_c(s) \, \partial^c X^{\alpha} \epsilon^{bd} \sigma'_b(s) \, \partial_d X^{\beta} - \sigma'_c(s) \, \partial^c X^{\beta} \epsilon^{bd} \sigma'_b(s) \, \partial_d X^{\alpha} \right)
= \frac{1}{(\sigma')^2} \left(\sigma'_c(s) \, \partial^c X^{\alpha} \epsilon^{bd} \sigma'_b(s) \, \partial_d X^{\beta} - \sigma'_d(s) \, \partial^d X^{\beta} \epsilon^{bc} \sigma'_b(s) \, \partial_c X^{\alpha} \right)
= \frac{1}{(\sigma')^2} \left[\sigma'_c(s) \, \epsilon_{bd} \sigma'^b(s) - \sigma'_d(s) \, \epsilon_{bc} \sigma'^b(s) \right] \partial^c X^{\alpha} \partial^d X^{\beta} \quad .$$
(2.35)

Now, we consider the quantity in square brackets: it is a tensor in 2 dimensions with 2 indices which are skew–symmetric. Thus, it must be proportional to the Levi–Civita tensor in 2-dimensions, i.e. we have

$$\left[\sigma'_{c}(s)\,\epsilon_{bd}\sigma'^{b}(s) - \sigma'_{d}(s)\,\epsilon_{bc}\sigma'^{b}(s)\right] = k(s)\,\epsilon_{cd} \quad . \tag{2.36}$$

The expression k(s) can be determined by observing that if c = 0, d = 1, then we get

$$k\left(s\right) = \left(\sigma'^{\,0}\right)^{2} + \left(\sigma'^{\,1}\right)^{2} = \left(\boldsymbol{\sigma}'\right)^{2} \quad ;$$

so, by substituting this result, together with (2.36), in (2.35) we recover

$$\dot{X}^{\alpha\beta} = \frac{1}{\sigma'^{2}} \sigma'^{2} \epsilon_{cd} \, \partial^{c} X^{\alpha} \partial^{d} X^{\beta}
= \epsilon_{cd} \, \partial^{c} X^{\alpha} (\sigma(s)) \, \partial^{d} X^{\beta} (\sigma(s))
= \epsilon_{cd} \, \partial^{c} X^{\alpha} (\sigma(s)) \, \partial^{d} X^{\beta} (\sigma(s)) \Big|_{\sigma = \sigma(s)} \stackrel{\text{def.}}{=} \dot{Y}^{\alpha\beta} (s) ,$$
(2.37)

which coincides with the definition 2.6^{11} .

Again, we remember that $\dot{Y}^{\alpha\beta}(s)$ is only a notational way to make immediatly clear that we have to do with a *Boundary* quantity. By no means it is possible to compute this result relying only on the *Embedding Function* of the loop C in *Parameter Space*, because in general there is no unique normal defined for a loop in a space with more than two dimensions. Thus we see that the Dynamics of the *World–Sheet* is tightly connected with the *Shadow Dynamics* of the boundary; we shall comment again about this remark later on¹². For the sake of clarity, we also set the following notation.

Notation 2.3 (Linear Velocity Vector).

The Linear Velocity Vector of the loop C, T, will be indicated with Y' (Y'^{μ} in components).

We can now turn to the main result of this section:

Proposition 2.10 (Ogielski Hamilton-Jacobi String Equation).

The Schild String Shadow Dynamics is described by the following reparametrization invariant Hamilton–Jacobi equation:

$$\frac{\partial S}{\partial A} = \frac{1}{2m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} \frac{ds}{\sqrt{Y'^{\mu}Y'_{\mu}}} \frac{\delta S}{\delta Y^{\mu}(s)} \frac{\delta S}{\delta Y_{\mu}(s)}$$

Proof:

We will use the expression for the Area Velocity $\dot{Y}^{\mu\nu}$ of the String on the boundary Γ of the domain Σ , we derived in the previous proposition. Starting from the result (2.34), we can derive

¹¹Please, remember that with respect to this definition we now have renamed Ξ with Σ and ξ with σ .

 $^{^{12}}$ The curious reader could make a quick jump to page 146, section 9.2.

the following results for the various terms appearing in equations (1.17-1.19):

$$\frac{\partial \mathcal{L}}{\partial N^{\mu}} = \frac{m^{2}}{2} \dot{Y}_{\alpha\beta} \frac{\partial \dot{Y}^{\alpha\beta}}{\partial N^{\mu}}$$

$$= \frac{m^{2}}{2} \frac{1}{\sqrt{(\boldsymbol{\sigma}')^{2}}} (T_{\alpha}N_{\beta} - T_{\beta}N_{\alpha}) \left(T^{\alpha}\delta_{\mu}^{\beta} - T^{\beta}\delta_{\mu}^{\alpha} \right)$$

$$= \frac{m^{2}}{2} \frac{1}{\sqrt{(\boldsymbol{\sigma}')^{2}}} (T^{2}N_{\mu} - T_{\mu}(\boldsymbol{T} \cdot \boldsymbol{N}) - T_{\mu}(\boldsymbol{T} \cdot \boldsymbol{N}) + T^{2}N_{\mu})$$

$$= m^{2} \frac{1}{\sqrt{(\boldsymbol{\sigma}')^{2}}} (T^{2}N_{\mu} - T_{\mu}(\boldsymbol{T} \cdot \boldsymbol{N}))$$
(2.38)

$$\frac{\partial \mathcal{L}}{\partial N^{\mu}} N^{\mu} = m^2 \frac{1}{\sqrt{(\boldsymbol{\sigma}')^2}} \left(\boldsymbol{T}^2 \boldsymbol{N}^2 - (\boldsymbol{T} \cdot \boldsymbol{N})^2 \right)$$
 (2.39)

$$\frac{\partial \mathcal{L}}{\partial N^{\mu}} T^{\mu} = m^2 \frac{1}{\sqrt{(\boldsymbol{\sigma}')^2}} \left(\boldsymbol{T}^2 (\boldsymbol{T} \cdot \boldsymbol{N}) - (\boldsymbol{T} \cdot \boldsymbol{N}) \, \boldsymbol{T}^2 \right) \equiv 0 \tag{2.40}$$

$$\frac{\partial \mathcal{L}}{\partial N^{\mu}} N^{\mu} - \mathcal{L} = \frac{m^{2}}{\sqrt{\left(\boldsymbol{\sigma}'\right)^{2}}} \left(\boldsymbol{T}^{2} \boldsymbol{N}^{2} - \left(\boldsymbol{T} \cdot \boldsymbol{N}\right)^{2}\right) - \frac{m^{2}}{4} \frac{1}{\sqrt{\left(\boldsymbol{\sigma}'\right)^{2}}} \left(\boldsymbol{T}^{\alpha} N^{\beta} - \boldsymbol{T}^{\beta} N^{\alpha}\right) \left(T_{\alpha} N_{\beta} - T_{\beta} N_{\alpha}\right)
= \frac{m^{2}}{\sqrt{\left(\boldsymbol{\sigma}'\right)^{2}}} \left(\boldsymbol{T}^{2} \boldsymbol{N}^{2} - \left(\boldsymbol{T} \cdot \boldsymbol{N}\right)^{2}\right) - \frac{m^{2}}{4} \frac{1}{\sqrt{\left(\boldsymbol{\sigma}'\right)^{2}}} 2 \left(\boldsymbol{T}^{2} \boldsymbol{N}^{2} - \left(\boldsymbol{T} \cdot \boldsymbol{N}\right)^{2}\right)
= \frac{m^{2}}{2} \frac{1}{\sqrt{\left(\boldsymbol{\sigma}'\right)^{2}}} \left(\boldsymbol{T}^{2} \boldsymbol{N}^{2} - \left(\boldsymbol{T} \cdot \boldsymbol{N}\right)^{2}\right) .$$
(2.41)

Then, substituting these results, equations (1.17-1.19) become:

$$\frac{\delta S}{\delta n(s)} = \frac{m^2}{2} \frac{1}{\sqrt{(\sigma')^2}} \left[T^2 N^2 - (T \cdot N)^2 \right]$$
 (2.42)

$$\frac{\delta S}{\delta Y^{\mu}(s)} = -m^2 \frac{1}{(\boldsymbol{\sigma}')^2} \left[\boldsymbol{T}^2 N_{\mu} - (\boldsymbol{T} \cdot \boldsymbol{N}) T_{\mu} \right]$$
 (2.43)

$$\frac{\delta S}{\delta t(s)} = 0 \quad . \tag{2.44}$$

We see, from the last equation, that the model under investigation is invariant under reparametrization of the loop Γ (respectively C) in Parameter Space (respectively Target Space), Then, by combining results (2.42-2.43) it turns out that the local Hamilton–Jacobi equation satisfied by the system is

$$\frac{\partial S}{\partial A} = \frac{1}{2m^2} \frac{1}{(\mathbf{Y}')^2} \frac{\delta S}{\delta Y^{\mu}} \frac{\delta S}{\delta Y_{\mu}}$$

It has been a common feature of all previous works on this subject to stop at this stage and to call this one the Hamilton–Jacobi equation for the *String*. In our opinion a further step should be carried out to achieve a reparametrization invariant equation for the Dynamics of the system. For this reason, we rewrite the result above as

$$\sqrt{\left(\boldsymbol{Y}'\right)^2}\frac{\partial S}{\partial A} = \frac{1}{2m^2}\frac{1}{\sqrt{\left(\boldsymbol{Y}'\right)^2}}\frac{\delta S}{\delta Y^\mu}\frac{\delta S}{\delta Y_\mu}$$

and integrate both sides with respect to s to get

$$\frac{1}{2m^2} \left(\oint_{\Gamma \approx \mathbb{S}^1} dl(s) \right)^{-1} \oint_{\Gamma \approx \mathbb{S}^1} ds \frac{1}{\sqrt{(\mathbf{Y}')^2}} \frac{\delta S}{\delta Y^{\mu}} \frac{\delta S}{\delta Y_{\mu}} = \frac{\partial S}{\partial A} \quad , \tag{2.45}$$

which we call the String Functional Hamilton-Jacobi Equation.

2.5.2 Hamilton-Jacobi Equation: Reparametrized Formulation

We are now going to rederive the same result in the reparametrized framework that we discussed in section 2.4, i.e. starting from the Hamiltonian formulation of reparametrized Schild *String* Theory. This procedure will give us the opportunity to remark some important features of the model and to address in a more detailed way some key questions related to the quantization procedure. The first step is to have a closer look to the classical equations of motion originating from the action associated with the canonical 2-form (2.33):

Definition 2.10 (Hamiltonian Full Reparametrized Schild Action).

The <u>Hamiltonian Full Reparametrized Schild Action</u> is the action associated with the canonical 2-form (2.33), i.e.

$$S\left[X^{\mu}, P_{\mu\nu}, \xi^{A}, \pi_{AB}, N_{AB}\right] =$$

$$= \int_{\mathcal{W}} P_{\mu\nu} d\mathbf{X}^{\mu} \wedge d\mathbf{X}^{\nu} + \int_{\Xi} \pi_{AB} d\boldsymbol{\xi}^{A} \wedge d\boldsymbol{\xi}^{B} +$$

$$+ \frac{1}{2} \int_{\Sigma} N^{AB} \left(\pi_{AB} - \epsilon_{AB} \mathcal{H}_{Schild}(\boldsymbol{P})\right) d\boldsymbol{\sigma}^{0} \wedge d\boldsymbol{\sigma}^{1}$$

$$= S\left[X^{\rho}, P_{\sigma\tau}; \xi^{A}, \pi_{CD}\right] + S_{cnstr.} \left[X^{\rho}, P^{\sigma\tau}; \xi^{A}, \pi_{CD}; N^{AB}\right]$$

$$(2.47)$$

and its field variables are X, P, ξ , π , N.

We emphasized in the previous definition that the last term corresponds to the constraint which must be imposed on the conjugated momenta, i.e.

$$S_{\text{cnstr.}}\left[X^{\rho}, P^{\sigma\tau}; \xi^{A}, \pi_{CD}; N^{AB}\right] = \frac{1}{2} \int_{\Sigma} N^{AB} \left(\pi_{AB} - \epsilon_{AB} \mathcal{H}_{\text{Schild}}(\boldsymbol{P})\right) d\boldsymbol{\sigma}^{0} \wedge d\boldsymbol{\sigma}^{1} \quad . \quad (2.48)$$

We stress again, because this is a relevant point, that the dependence in \mathcal{H}_{Schild} is from the momentum of (2.30).

Proposition 2.11 (Full Reparametrized Theory Equations of Motion).

The equations of motion for the full reparametrized Theory (2.46) are

$$\dot{X}^{\mu\nu} = N^{AB} \epsilon_{AB} \frac{P^{\mu\nu}}{2m^2} \tag{2.49}$$

$$\epsilon^{ab}\partial_a P^{\mu\nu}\partial_b X^{\nu} = 0 (2.50)$$

$$\epsilon^{ab}\partial_a P^{\mu\nu}\partial_b X^{\nu} = 0 \qquad (2.50)$$

$$\pi_{AB} = \epsilon_{AB} \frac{P_{\mu\nu}P^{\mu\nu}}{4m^2} \qquad (2.51)$$

$$\epsilon^{ab}\partial_a \pi_{AB}\partial_b \xi^B = 0 (2.52)$$

$$N^{AB} = \dot{\xi}^{AB} (2.53)$$

Proof:

The procedure is as usual to consider a variation of the action (2.46). Then it is possible to extract the equation of motion thanks to the fundamental principle of the variational calculus.

Now we set up a particular, but by no means restrictive framework where the result found in the previous section can be generalised to Parameter Space topologies more general than the one assumed in Ogielski's formulation. Suppose we choose a doubly connected Parameter Space with the anulus topology. Physically speaking, the boundary corresponding, say, to the hole can be considered as the initial String configuration, whereas the rest of the boundary of the anulus can be mapped to the final String configuration. We will consider only variations of the parameter space having the part of the boundary corresponding to the initial string configuration fixed. The rest of the boundary is on the contrary free and can be varied: it is also parametrized by $s \in \mathbb{S}^1 \approx \Gamma$ and C is its image in Parameter Space. Let us call C_0 the image in Parameter Space of the initial String configuration. The steps leading to the Hamilton-Jacobi equation are as follows. First, we remove the Lagrange multiplier from the action using equation (2.53), since it is a non-dynamical quantity; moreover, thanks to equation (2.51), we can still simplify the expression for the action and get the following result:

$$S_{\mathrm{Red.}}\left[X^{\rho}\left(\boldsymbol{\sigma}\right),P^{\sigma\tau}\left(\boldsymbol{\sigma}\right),\xi^{A}\left(\boldsymbol{\sigma}\right)\right]=$$

$$= \frac{1}{2} \int_{\mathcal{W}(\sigma)} P_{\mu\nu} d\mathbf{X}^{\mu} \wedge d\mathbf{X}^{\nu} + \frac{1}{2} \epsilon_{AB} \int_{\Xi} d\boldsymbol{\xi}^{A} \wedge d\boldsymbol{\xi}^{B} \mathcal{H}_{\text{Schild}} \left(P^{\sigma\tau} \left(\boldsymbol{\xi} \right) \right) \quad . \tag{2.54}$$

Definition 2.11 (Hamiltonian Reduced Reparametrized Schild Action).

We call the expression (2.54) the <u>Hamiltonian</u>, <u>Reduced</u>, <u>Reparametrized Schild Action</u>, because it is the Hamiltonian Reparametrized Schild Action with solved constraints.

Since we assume the equation of motion for the World-Sheet are satisfied, by varying the action the contribution from the variation of the World-Sheet itself vanishes, and we obtain:

Proposition 2.12 (Restricted Reparametrized Action Boundary Variation).

The variation of the reparametrized action due to a variation in the boundary $\Gamma \approx \mathbb{S}^1$ is

$$\delta S_{\text{Red.}} = \int_{\Gamma \approx \mathbb{S}^1} q_{\mu}(s) \, \delta Y^{\mu}(s) \, ds - \mathcal{H}_{\text{Schild}} \delta A \quad , \tag{2.55}$$

where

$$q_{\mu}(s) \stackrel{\text{def.}}{=} \frac{\delta S_{\text{Red.}}}{\delta Y^{\mu}(s)} = Q_{\mu\nu}(s) Y^{\prime\nu}(s)$$
 (2.56)

$$A \stackrel{\text{def.}}{=} \frac{1}{2} \epsilon_{AB} \int_{\Xi} d\xi^A \wedge d\xi^B \tag{2.57}$$

$$\mathcal{H}_{\text{Schild}}(s) = -\frac{\delta S_{\text{Red.}}}{\delta A(s)} ,$$
 (2.58)

where $Q_{\mu\nu}$ is the Boundary Area Momentum of definition (2.8).

This result can be obtained specializing to the particular case with p=1 the computation for the p-brane given in proposition 4.2.

Proof:

Please, see proposition A.5 in appendix A.4 for the detailed computation.

Thanks to equations (2.51-2.52) the Hamiltonian is constant along a classical trajectory, then we may move the Hamiltonian outside the area integral and the variation operator. Moreover, the functional variation δA becomes an ordinary differential variation dA:

$$\mathcal{H}_{\text{Schild}} \stackrel{\text{def.}}{=} E \stackrel{\text{def.}}{=} -\frac{\partial S_{\text{Red.}}}{\partial A} \quad ;$$
 (2.59)

this is the same result obtained in subsection 2.5.1 on page 47 as a direct consequence of the restricted invariance under loop reparametrizatrions. This result, is expressed in Ogielski's procedure, as equation (2.44). Moreover, it shows up in the reparametrized formulation as the constancy of the Schild Hamiltonian along a classical trajectory, encoded in equations (2.51) and (2.52). Hence, we can see that the Jacobi variational principle in the form of equation (2.55) shows that q_{μ} is conjugated to the World–Sheet boundary variation, while $E \equiv H_{\text{Schild}}$ describes the response of the classical action to an arbitrary area variation in Parameter Space, due to a deformation of the boundary. Thus if we consider the classical Dynamics of the String from this point of view, then A and $X^{\mu}(s)$ can be interpreted as the classical "time" and "space" coordinates of the String C.

Finally we note that in this formulation E can be indentified with the energy per unit area associated with an extremal World-Sheet of the action (2.54), while $q_{\mu}(s)$ is the momentum per unit length of the String loop C. Therefore, the energy-momentum dispersion relation can be written either as an equation between densities

$$\frac{1}{2m^{2}}q^{\mu}(s) q_{\mu}(s) = \frac{1}{4m^{2}}Q_{\mu\nu}(s) Q^{\mu\nu}(s) (\mathbf{Y}(s)')^{2} = (\mathbf{Y}(s)')^{2} E$$
(2.60)

or as an integrated relation

$$\frac{1}{2m^2} \oint_{\Gamma \approx \mathbb{S}^1} \frac{ds}{\sqrt{\left(\mathbf{Y}'\right)^2(s)}} q^{\mu}(s) q_{\mu}(s) = E \oint_{\Gamma} ds \sqrt{\left(\mathbf{Y}'(s)\right)^2} \quad . \tag{2.61}$$

The above equation once written using equations (2.56, 2.59) turns exactly into (2.45), the Functional Hamilton–Jacobi Equation for the String:

$$-\frac{\partial S}{\partial A} = \frac{1}{2m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} \frac{ds}{\sqrt{Y'^{\mu}Y'_{\mu}}} \frac{\delta S}{\delta Y^{\mu}(s)} \frac{\delta S}{\delta Y_{\mu}(s)} \quad . \tag{2.62}$$

¹³As already pointed out in footnotes 1 on page 26 and 2 on page 27, we try to keep a well defined distinction between quantities related to the classical domain and quantities associated to the quantum realm.

Looking in more detail at this equation, we observe that the covariant integration over s takes into account all the possible locations of the point, along the contour C, where the variation can be applied. But, in this way, every point of C is overcounted a "number of times" equal to the String proper length. The first factor, in round parenthesis, is just the String proper length and removes such overcounting. In other words, we sum over all the possible ways in which one can deform the String loop, and then divide by the total number of them. The net result is that the left-hand side of equation (2.62) is insensitive to the choice of the point where the final String C is deformed. Therefore the right-hand side is a genuine reparametrization scalar which describes the system's response to the extent of area variation, irrespective of the way in which the deformation is implemented. With hindsight, the wave equation proposed in [5], [35] appears to be more restrictive than equation (2.62), in the sense that it requires the second variation of the line fuctional to be proportional to $(Y'(s))^2$ at any point on the String loop, in contrast to equation (2.62), which represents an integrated constraint on the String as a whole.

2.6 Classical Area Effective Formulation

It is important to observe that the functional Hamilton–Jacobi equation (2.62) can be cast in a form, in which all the derivatives are with respect to area quantities. In particular we can trade the functional derivative with the derivative with respect to the *Holographic Coordinates*, thanks to equation (D.1). In particular the Hamilton-Jacobi equation then takes the form

$$-\frac{\partial S}{\partial A} = \frac{1}{4m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{Y'^{\mu} Y'_{\mu}} \frac{\delta S}{\delta Y^{\mu\nu}(s)} \frac{\delta S}{\delta Y_{\mu\nu}(s)} \quad . \tag{2.63}$$

Starting from this result we can prove

Proposition 2.13 (Classical Area Hamiltonian Formalism).

The Hamilton-Jacobi equation (2.63) is the Hamilton-Jacobi equation associated with the following Hamiltonian:

$$\mathcal{H}_{\text{EFF}} = \frac{1}{4m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{\left(\boldsymbol{Y}' \right)^2} Q_{\mu\nu} \left(s \right) Q^{\mu\nu} \left(s \right) \quad ; \tag{2.64}$$

this Hamiltonian can be taken as defining a Theory in which the conjugate variables are the Holographic Coordinates $Y^{\mu\nu}[C;A]$ and the corresponding Conjugate Momentum

$$Q^{\mu\nu}\left[C;A\right] = \frac{1}{2m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{\left(\mathbf{Y}'\right)^2} Q_{\mu\nu}\left(s\right) \quad ,$$

which is nothing but the "average" of the Boundary Area Momentum taken over the String. Solving then equation (2.63) is the equivalent to find a solution of the following first order system of <u>Area Hamilton Equations</u>:

$$\begin{cases}
\frac{dY^{\mu\nu}\left[C;A\right]}{dA} = Q^{\mu\nu}\left[C;A\right] \\
\frac{dQ^{\mu\nu}\left[C;A\right]}{dA} = 0
\end{cases}$$
(2.65)

Proof:

To get the desired result we have to compute in first istance the following functional derivatives:

$$\frac{\delta \mathcal{H}_{\rm EFF}}{\delta Q_{\mu\nu}(t)} = \frac{1}{2m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \sqrt{(\mathbf{Y}(t))^2} Q_{\mu\nu}(t)$$

and

$$\frac{\delta \mathcal{H}_{\text{EFF}}}{\delta Y_{\mu\nu}(t)} = 0 \quad .$$

As discussed in appendix D the Holographic Derivatives are explicitly dependent from the loop parameter, t in this case. To get reparametrization invariant quantities we integrate over this variable and define

$$\frac{\delta \mathcal{H}_{\text{EFF}}}{\delta Q_{\mu\nu} [C]} \stackrel{\text{def.}}{=} \frac{1}{2m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} dt \sqrt{(\mathbf{Y}(t))^2} Q_{\mu\nu}(t) \stackrel{\text{def.}}{=} Q_{\mu\nu} [C; A] \qquad (2.66)$$

$$\frac{\delta \mathcal{H}_{\text{EFF}}}{\delta Y_{\mu\nu} [C]} \stackrel{\text{def.}}{=} 0 \quad . \qquad (2.67)$$

$$\frac{\delta \mathcal{H}_{\text{EFF}}}{\delta Y_{\mu\nu} \left[C \right]} \stackrel{\text{def.}}{=} 0 \quad . \tag{2.67}$$

This are well defined *loop* quantities, in terms of which we can write the following Hamilton equations

$$\begin{cases}
\frac{dY^{\mu\nu}[C;A]}{dA} = \frac{\delta \mathcal{H}}{\delta Q_{\mu\nu}[C]} \\
\frac{dQ^{\mu\nu}[C;A]}{dA} = -\frac{\delta \mathcal{H}}{\delta Y_{\mu\nu}[C]}
\end{cases}$$
(2.68)

From these equations, after the substitutions of (2.66) and (2.67) in the right hand sides, we get the desired result.

Proposition 2.14 (Classical Area Newtonian Formalism).

The first order system of Area Hamilton Equations (2.65) is equivalent to the following

second order <u>Area Newton Equation</u>:

$$\frac{d^2Y^{\mu\nu}[C;A]}{dA^2} = 0 . (2.69)$$

Proof:

Substituting the first equation of the system (2.65) into the second one we obtain the desired result.

In previous equations the Area Derivative d/(dA) explicitly appears and to clarify its meaning, as well as other features of this model, some remarks are worthwhile.

Firstly, we note that our procedure has singled out a very striking feature of the Dynamics of the String described starting from the Eguchi formulation: Classical String Dynamics can be viewed exactly as a 2-dimensional generalisation of particle Dynamics in the sense that it is a Dynamics of area. The coordinates of the Theory have length squared dimensions, as well as the evolution parameter. Geometrically speaking we have just raised by one the particle case, but not simply by adding an extra parameter. On the contrary, all the formulation describes the evolution of some areas (more specifically, the areas of the projection of the String on the coordinate planes) as their response to the Parameter Space (or World-Sheet) area variation.

Secondly note that the geometrical description of area naturally relies on antisymmetric 2-forms. These are the natural generalization of the tangent vector, because they can be interpreted as the tangent elements to the World-Sheet of the String.

As a last observation, we note that it is possible to set up a statistical formalism for Strings (p-branes in the general case) starting from the classical area formalism developed in this section. In particular we can take as unknown dynamical fields the action $S[Y^{\mu\nu}; A]$ and the probability density $P[Y^{\mu\nu}; A]$. The meaning of this last quantity is that it can be used to obtain the probability of finding a given loop C that has shadows onto the coordinate planes described by the Holographic Coordinates $Y^{\mu\nu}[C; A]$. Then we can take for our fields

the following action functional:

$$\mathfrak{S} = \int dA \int [\mathcal{D}C] P[C; A] \left\{ \frac{1}{4m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \cdot \oint_{\Gamma} ds \sqrt{Y'^{\mu}Y'_{\mu}} \frac{\delta S[C; A]}{\delta Y^{\mu\nu}(s)} \frac{\delta S[C; A]}{\delta Y_{\mu\nu}(s)} + \frac{\partial S[C; A]}{\partial A} \right\} \quad . \tag{2.70}$$

It can be proved that a variation with respect to P[C; A] exactly gives Hamilton–Jacobi equation (2.63). Moreover we can derive a result which will be useful in chapter 10.

Proposition 2.15 (Continuity Equation).

The variation of the action (2.70) with respect to S[C;A] results in the following equation

$$\frac{dP\left[C;A\right]}{dA} + \frac{1}{4m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{\mathbf{Y'}^2} \frac{\delta}{\delta Y^{\mu\nu}\left(s\right)} \left[P\left[C;A\right] \frac{\delta S\left[C;A\right]}{\delta Y_{\mu\nu}\left(s\right)} \right] = 0 \quad , \quad (2.71)$$

which we call the Continuity Equation for Area Dynamics.

Proof:

As a preliminary remark we underline that in this proof we are going to indicate the variation of the action S with the following notation:

$$\Delta S$$
 in place of δS

to avoid confusion with the functional derivative symbol δ . Then we see that a variation of S in (2.70) yields

$$\Delta\mathfrak{S} = \int dA \int [\mathcal{D}C] P [C; A] \left\{ \frac{1}{2m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \cdot \oint_{\Gamma} ds \sqrt{Y'^{\mu}Y'_{\mu}} \frac{\delta S [C; A]}{\delta Y^{\mu\nu} (s)} \frac{\delta(\Delta S [C; A])}{\delta Y_{\mu\nu} (s)} + \frac{\partial(\Delta S)}{\partial A} \right\} \quad . \tag{2.72}$$

Integrating by parts and dropping Boundary terms as usual¹⁴ gives

$$\Delta\mathfrak{S} = \int dA \int \left[\mathcal{D}C\right] P\left[C;A\right] \left\{ \frac{1}{2m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \cdot \oint_{\Gamma} ds \sqrt{Y'^{\mu}Y'_{\mu}} \frac{\delta}{\delta Y^{\mu\nu}\left(s\right)} \left[P\left[C;A\right] \frac{\delta \Delta S\left[C;A\right]}{\delta Y_{\mu\nu}\left(s\right)} \right] + \frac{\partial P\left[C;A\right]}{\partial A} \right\} \Delta S \quad , (2.73)$$

from which the desired result follows.

 $^{^{14}}$ This procedure gives no problems with the dA integration; in principle the meaning of an integration by parts could be much more troubleful in the functional case where we make the same absumption as in [].

2.7 Reparametrized Canonical Formulation

We will now introduce another equivalent formulation that will give a deeper insight into further developments and will be useful to understand the connection between our formulation of the *String* Dynamics and the more traditional one. This can be a good opportunity to stress again some important concepts related to the classical Dynamics and a useful premise to the quantization procedure, to be analysed in detail in the next chapter as well. We will also take the opportunity to anticipate some deep results that will become more clear in the following sections. We start with a definition:

Definition 2.12 (Projected World-Sheet Area Momentum).

The World-Sheet Area Momentum (definition 2.7) projected onto the World-Sheet directions.

$$P^{m}_{|\mu} \stackrel{\text{def.}}{=} P_{\mu\nu}(\boldsymbol{\sigma}) \, \epsilon^{mn} \partial_{n} X^{\nu}(\boldsymbol{\sigma}) \quad ,$$

is the <u>Projected World-Sheet Area Momentum</u>.

Definition 2.13 (Projected World-Sheet Boundary Area Momentum).

The World-Sheet Boundary Area Momentum of definition 2.8 projected onto the World-Sheet directions and calculated on the Boundary,

$$P^{m}{}_{|\mu} \stackrel{\text{def.}}{=} P_{\mu\nu}(\boldsymbol{\sigma}) \, \epsilon^{mn} \partial_{n} X^{\nu}(\boldsymbol{\sigma}) \big|_{\boldsymbol{\sigma} = \boldsymbol{\sigma}(s)} \quad ,$$

is the <u>Projected World-Sheet Boundary Area Momentum</u>.

Now it is possible to prove that

Proposition 2.16 (Projected World-Sheet Area Momentum Computation).

The <u>Projected World-Sheet Area Momentum</u> is the momentum conjugated to the velocity $\partial_m X^{\nu}$, i.e.

$$P^{m}{}_{|\mu} = \frac{\partial \mathcal{L}}{\partial (\partial_{n} X^{\nu})} \quad .$$

Thus, it is the canonical momentum of the String World-Sheet.

Proof:

The starting point is as natural the Schild Lagrangian (2.1) and we also remember (2.2). Then we have

$$P^{m}_{|\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{m}X^{\mu})}$$

$$= \frac{m^{2}}{4} 2\dot{X}_{\rho\tau} \frac{\partial X^{\rho\tau}}{\partial(\partial_{m}X^{\mu})}$$

$$= \frac{m^{2}}{2} \dot{X}_{\rho\tau} \epsilon^{ab} \left(\frac{\partial(\partial_{a}X^{\rho})}{\partial(\partial_{m}X^{\mu})} \partial_{b}X^{\tau} + \frac{\partial(\partial_{b}X^{\tau})}{\partial(\partial_{m}X^{\mu})} \partial_{a}X^{\rho} \right)$$

$$= \frac{m^{2}}{2} \dot{X}_{\rho\tau} \epsilon^{ab} 2 \frac{\partial(\partial_{a}X^{\rho})}{\partial(\partial_{m}X^{\mu})} \partial_{b}X^{\tau}$$

$$= m^{2} \dot{X}_{\rho\tau} \epsilon^{ab} \delta_{am} \delta^{\rho\mu} \partial_{b}X^{\tau}$$

$$= P_{\mu\nu} \epsilon^{am} \partial_{a}X^{\nu} , \qquad (2.74)$$

as required.

Moreover, the following equality holds.

Proposition 2.17 (Alternative Expression for the Schild Hamiltonian).

The Schild Hamiltonian can be written in the following equivalent forms

$$\mathcal{H}_{\text{Schild}} = \frac{P^{\mu\nu}P_{\mu\nu}}{4m^2} = \frac{P^{m|\mu}P_{m|\mu}}{2m^2}$$
 (2.75)

and the product of the Area Velocity times the World-Sheet Area Momentum as

$$P_{\mu\nu}\dot{X}^{\mu\nu} = P^{m|\mu}\partial_m X_\mu \quad . \tag{2.76}$$

All these relations are World-Sheet relations.

Proof:

The first result is just a consequence of " ϵ tensors contraction":

$$P^{m|\mu}P_{m|\mu} = \epsilon^{ma}P^{\mu\nu}\partial_{a}X_{\nu}\epsilon_{mb}P_{\mu\rho}\partial^{b}X^{\rho}$$

$$= P^{\mu\nu}P_{\mu\rho}\partial_{a}X_{\nu}\partial^{b}X^{\rho}\epsilon^{ma}\epsilon_{mb}$$

$$= \frac{1}{2}P^{\mu\nu}P_{\mu\rho}\partial_{a}X_{\nu}\partial^{b}X^{\rho}\delta^{a}_{b}$$

$$= \frac{1}{2}P^{\mu\nu}P_{\mu\rho}\partial_{a}X_{\nu}\partial^{a}X^{\rho}$$

$$= \frac{1}{2} P^{\mu\nu} P_{\mu\rho} \delta^{\rho}_{\nu}$$

$$= \frac{1}{2} P^{\mu\nu} P_{\mu\nu} . \qquad (2.77)$$

Then the result for the Hamiltonian follows. Moreover in the second case

$$P^{m|\mu}\partial_{m}X_{\mu} = \epsilon^{mn}P^{\mu\nu}\partial_{n}X_{\nu}\partial_{m}X_{\mu}$$

$$= P^{\mu\nu}\epsilon^{mn}\partial_{n}X_{\nu}\partial_{m}X_{\mu}$$

$$= P^{\mu\nu}\dot{X}_{\mu\nu} . \qquad (2.78)$$

Proposition 2.18 (Bulk-Boundary Interference).

The Holographic Coordinates of the String are the World-Sheet integral of the World-Sheet Area Velocity, i.e.

$$Y^{\mu\nu} [C] = \oint_C Y^{\mu} dY^{\nu} = \int_{\mathcal{W}} dX^{\mu} \wedge dX^{\nu} \quad ,$$

which can be written in coordinates

$$Y^{\mu\nu}\left[C
ight] = \oint_{\Gamma=\mathbb{S}^1} ds Y^{\mu}\left(s
ight) Y'^{
u}\left(s
ight) = \int_{\Sigma} d^2 oldsymbol{\sigma} \dot{X}^{\mu
u}(oldsymbol{\sigma}) \quad .$$

Proof:

We prove the result written in an exlpictly choosen coordinate system, since the one in intrinsic forms, simply comes out going to the boundary $C = \partial W$. Thus we have

$$Y^{\mu\nu} [C] = \oint_{C} Y^{\mu} dY^{\nu}$$

$$1. = \oint_{\Gamma \approx \mathbb{S}^{1}} ds Y^{\mu} (s) Y'^{\nu} (s)$$

$$2. = \oint_{\Gamma \approx \mathbb{S}^{1}} ds X^{\mu} (\boldsymbol{\sigma}(s)) \frac{dX^{\nu} (\boldsymbol{\sigma}(s))}{ds}$$

$$3. = \oint_{\partial \Sigma = \Gamma \approx \mathbb{S}^{1}} ds X^{\mu} (\boldsymbol{\sigma}(s)) \frac{\partial X^{\nu} (\boldsymbol{\sigma}(s))}{\partial \sigma^{a}(s)} \frac{d\sigma^{a}}{ds}$$

$$4. = \oint_{\partial \Sigma = \Gamma \approx \mathbb{S}^{1}} ds [X^{\mu} (\boldsymbol{\sigma}(s)) \partial_{a} X^{\nu} (\boldsymbol{\sigma}(s))] \frac{d\sigma^{a}(s)}{ds}$$

$$5. = \int_{\Sigma} d^{2} \boldsymbol{\sigma} \epsilon^{ba} \partial_{b} [X^{\mu} (\boldsymbol{\sigma}) \partial_{a} X^{\nu} (\boldsymbol{\sigma})]$$

$$6. = \int_{\Sigma} d^{2} \boldsymbol{\sigma} \epsilon^{ab} \partial_{a} X^{\mu} (\boldsymbol{\sigma}) \partial_{b} X^{\nu} (\boldsymbol{\sigma})$$

$$= \int_{\Sigma} d^{2} \boldsymbol{\sigma} \dot{X}^{\mu\nu} (\boldsymbol{\sigma})$$

$$(2.79)$$

We used the following properties:

1. we first use the boundary parametrization

$$Y^{\mu}: \Gamma \approx \mathbb{S}^1 \longrightarrow C$$

for the String, ...

- 2. ... expressing then the *Boundary* functions as restrictions of the *Bulk* ones through expression (2.5), where we remember that after the *reparametrization procedure* Σ substitutes Ξ and thus we have σ in place of $\boldsymbol{\xi}$;
- 3. we can now use the chain rule in computing the derivatives of the *Bulk parametrization* functions with respect to the boundary parameter and ...
- 4. ... put in evidence the Parameter Space vector $[X^{\mu}(\boldsymbol{\sigma}(s)) \partial_a X^{\nu}(\boldsymbol{\sigma}(s))]$, of which the present integral is the circulation;
- 5. we are now in the position of applying Green's Theorem to write the expression as an integration on the Parameter Space of the 2-dimensional curl of $[X^{\mu}(\sigma(s)) \partial_a X^{\nu}(\sigma(s))] \dots$
- 6. ... recognising that in computing the 2-dimensional curl, one of the two terms vanishes having two derivative symmetric indices contracted with the two totally antisymmetric indices of the Levi–Civita tensor.

This completes the proof.

Having established the results above (proposition 2.17 in particular), we can rewrite the action functionals (2.46) and (2.21) as follows

Proposition 2.19 (Actions in the Reparametrized Mixed Formulation).

In the Reparametrized Mixed Formulation we have the following expressions

$$S\left[\boldsymbol{X},\boldsymbol{P},\boldsymbol{\xi}\right] = \frac{1}{2} \int_{\Sigma} d^{2}\boldsymbol{\sigma} \left[P^{m|\mu} \partial_{m} X_{\mu} + \pi^{AB} \dot{\boldsymbol{\xi}}_{AB} + N_{AB} \left(\pi^{AB} - \frac{\epsilon^{AB}}{2m^{2}} P^{m|\mu} P_{m|\mu} \right) \right]$$

$$S\left[\boldsymbol{X},\boldsymbol{P}\right] = \frac{1}{2} \int_{\Sigma} d^{2}\boldsymbol{\sigma} \left[P^{m|\mu} \partial_{m} X_{\mu} - \frac{1}{2m^{2}} P^{m|\mu} P_{m|\mu} \right]$$

$$(2.80)$$

for the fully reparametrized action in Hamiltonian form (2.46) and for the Hamiltonian formulation of the Schild action (2.21) respectively.

Proof:

The second result immediately comes out from the action (2.21) when we use results (2.75-2.76). The same procedure can then be used to get the first one.

2.8 Covariant Schild Action

In this section we take the last step in the generalization process we started in section 2.4 with equation (2.28). We will then see in chapter 9 how this last step enlightens some very deep consequences of the formulation we are proposing. The reader is probably already able to understand what kind of generalization immediatly comes in mind observing equation (2.80). In this equation we have a quantity, the Area Momentum Projected on the World–Sheet, which is carrying a particular indexing, having one, let us call it internal, index (the greek one!) and one Parameter Space index (the lowercase latin one). We call the greek index an internal one since it just labels a multiplet of fields defined on a 2-dimensional String domain. The lowercase latin index is instead the index associated with this 2-dimensional domain, that we called the Parameter Space. When we saturated two of these indices in the previous computations, we implicitly assumed to close them in the "flat" way, but there is no reason to restrict ourself to this case; in general we can equipe the Parameter Space with an arbitrarly general metric, let us call it g^{ab} . Then the action (2.80) turns into the following one.

Definition 2.14 (Covariant Schild Action).

The covariant version of the Schild Action in Hamiltonian Form is:

$$S_{(\boldsymbol{g})}\left[\boldsymbol{X},\boldsymbol{P},\boldsymbol{g}\right] = \int_{\Sigma} d^2 \boldsymbol{\sigma} \sqrt{|\boldsymbol{g}|} g^{mn} \left[\partial_m X^{\mu}(\boldsymbol{\sigma}) P_{n|\mu}(\boldsymbol{\sigma}) - \frac{1}{2m^2} P_m^{|\mu}(\boldsymbol{\sigma}) P_{n|\mu}(\boldsymbol{\sigma}) \right] , \quad (2.81)$$

which we will call the <u>Covariant Schild Action</u> in Hamiltonian form.

It can be proved now that

Proposition 2.20 (Equation of Motion from the Covariant Schild Action).

The equation of motion for the action (2.81), $S_{(g)}$, are:

$$\partial_m \left[\sqrt{|\boldsymbol{g}|} P^{m|\mu} \right] = 0 \tag{2.82}$$

$$P_m^{|\mu} = m^2 \partial_m X^{\mu} \tag{2.83}$$

$$P_{m|\mu}\left(P_n^{|\mu} + m^2 \partial_n X^{\mu}\right) = \frac{1}{2} g_{mn} g^{ab} P_{a|\alpha} \left(P_b^{|\alpha} + m^2 \partial_b X^{\alpha}\right) . \tag{2.84}$$

Proposition 2.21 (Solution of the Equation of Motion).

The equations (2.82-2.84) have the following solutions:

$$g_{mn} = \partial_m X^{\mu} \partial_n X^{\nu} \stackrel{\text{def.}}{=} \gamma_{mn}(\boldsymbol{X})$$

$$P_{a|\mu}(\boldsymbol{\sigma}) = \bar{P}_{\mu\nu} \epsilon_a{}^n \partial_n Y^{\nu}(\boldsymbol{\sigma}) + m \partial_a \eta_{\mu}(\boldsymbol{\sigma}) . \qquad (2.85)$$

Proof:

Equation (2.84) requires the vanishing of the String energy-momentum tensor T_{mn} . Equation (2.83) allows us to solve equation (2.84) with respect to the String metric:

$$g_{mn} = \partial_m X^{\mu} \partial_n X^{\nu} \stackrel{\text{def.}}{=} \gamma_{mn}(\boldsymbol{X}) \tag{2.86}$$

Equations (2.83) and (2.86) show that the on-shell canonical momentum is proportional to the gradient of the String coordinate and the on-shell String metric matches the World-Sheet induced metric.

$$P_{\mu\nu} = \bar{P}_{\mu\nu} + mp_{\mu\nu}(\boldsymbol{\sigma})$$

$$P_{a|\mu}(\boldsymbol{\sigma}) = \bar{P}_{\mu\nu}\epsilon_a{}^n\partial_nY^{\nu}(\boldsymbol{\sigma}) + m\partial_a\eta_{\mu}(\boldsymbol{\sigma}) ,$$
where
$$\epsilon_{mn}p_{\mu\nu}\partial^nY^{\nu} = \partial_m\eta_{\mu} , \qquad (2.87)$$

 $\eta_{\mu}(\boldsymbol{\sigma})$ being a *D*-components multiplet of World–Sheet scalar fields, and $\bar{P}_{\mu\nu}$ being a constant background over the String manifold, i.e. $\bar{P}_{\mu\nu}$ is the <u>Area Momentum Zero Mode</u>:

$$\partial_m \bar{P}_{\mu\nu} = 0 \quad . \tag{2.88}$$

By averaging the η -field over the String World-Sheet, one can extract its zero frequency component

$$\begin{array}{cccc} \boldsymbol{\eta}^{\mu}(\boldsymbol{\sigma}) & = & \bar{\boldsymbol{\eta}}^{\mu} + \tilde{\boldsymbol{\eta}}^{\mu}(\boldsymbol{\sigma}) & , \\ & \bar{\boldsymbol{\eta}}^{\mu} & \stackrel{\mathrm{def.}}{=} & \frac{1}{\int_{\Sigma} d^{2}\boldsymbol{\sigma}\sqrt{|\mathbf{g}|}} \int_{\Sigma} d^{2}\boldsymbol{\sigma}\sqrt{|\boldsymbol{g}|}\boldsymbol{\eta}^{\mu}(\boldsymbol{\sigma}) & , \end{array}$$

where $\tilde{\eta}^{\mu}(\boldsymbol{\sigma})$ describes the Bulk quantum fluctuations, as measured with respect to the reference value $\bar{\eta}^{\mu}$. Confining $\tilde{\eta}^{\mu}(\boldsymbol{\sigma})$ to the Bulk of the String World–Sheet requires appropriate boundary conditions. Accordingly, we assume that both $\tilde{\eta}^{\mu}(\sigma)$ and its (normal and tangential) derivatives vanish when restricted on the boundary Γ :

$$\tilde{\eta}^{\mu}\big|_{\Gamma} = 0 \tag{2.89}$$

$$t^m \partial_m \tilde{\eta}^\mu \big|_{\Gamma} = 0 \tag{2.90}$$

$$t^{m}\partial_{m}\tilde{\eta}^{\mu}|_{\Gamma} = 0$$

$$n^{m}\partial_{m}\tilde{\eta}^{\mu}|_{\Gamma} = 0$$

$$(2.90)$$

$$(2.91)$$

¹⁵By "confinement" we mean that there is no leakage of the field current $j_m \stackrel{\text{def.}}{=} \tilde{\eta}^\mu \partial_m \tilde{\eta}_\mu$ off the World– Sheet Boundary, i.e.

$$\oint_{\Gamma} dn^a j_a = 0 \quad .$$

Moreover

Proposition 2.22 (Covariant Schild Action and Nambu-Goto Action).

The Covariant Schild Action, when computed on shell with respect to the World-Sheet metric g turns into the Nambu-Goto action for the String.

Proof:

By inserting the classical solutions into (2.81) one recovers the Nambu-Goto action.

With some hindsight, this result follows from having introduced a non-trivial metric $g_{mn}(\boldsymbol{\sigma})$ in the *String* manifold. Thus, the Schild action becomes diffeormophism invariant as the Nambu-Goto action. Accordingly, we recovered the classical equivalence showed in (2.22).

Chapter 3

String Functional Quantization

"You see,"
You can do it."
"I call it luck."
"In my experience,
there's no such thing
as luck."

3.1 Quantum Propagator

Let us briefly recall the conclusions of section 2.4 of chapter 2. After promoting the original pair of World–Sheet parameters to the role of dynamical fields through a Boundary preserving transformation, we were able to define the Dynamics for the String as a reparametrization invariant Theory in 6 dimensions; indeed we had the usual four spacetime embedding functions $X^{\mu}(\sigma^0, \sigma^1)$ as well as the new fields $\xi^A(\sigma^0, \sigma^1)$. The corresponding conjugate momenta, derived from the Reparametrized Schild Action (2.28) in proposition 2.8, are the Bulk Area Momentum, denoted by $P_{\mu\nu}(\sigma^0, \sigma^1)$, and $\pi_{AB}(\sigma^0, \sigma^1)$ respectively: it is worthwhile to remind that we implemented the relation between momentum conjugated to $\dot{\xi}^{AB}$ (relation (2.31)) in the Hamiltonian formalism through to the Lagrange multiplier N^{AB} . This is a straightforward consequence of the reparametrization invariance. Thus, our preferred starting point to treat the quantum Dynamics of the system with path–integral techniques is the Hamiltonian formulation associated to the Hamiltonian

2-form (2.33), or equivalently, action (2.46), which we rewrite here for convenience as

$$S\left[X^{\mu}, \xi^{A}; P_{\mu\nu}, \pi_{AB}; N^{AB}\right] =$$

$$= S_{P_{\mu\nu}} \left[X^{\mu}, P_{\mu\nu}; N^{AB}\right] + S_{\pi_{AB}}^{(1)} \left[\pi_{AB}; N^{AB}\right] + S_{\xi} \left[\xi^{A}, \pi_{AB}\right] , \quad (3.1)$$

where

$$S_{\xi} = \frac{1}{2} \int_{\Xi} \pi_{AB} \, d\boldsymbol{\xi}^{A} \wedge d\boldsymbol{\xi}^{B}$$

$$S_{\pi_{AB}}^{(1)} = -\frac{1}{2} \int_{\Sigma} d^{2}\boldsymbol{\sigma} \, N^{AB} \pi_{AB}$$

$$S_{P_{\mu\nu}} = \frac{1}{2} \left[\int_{W} P_{\mu\nu} d\boldsymbol{X}^{\mu} \wedge d\boldsymbol{X}^{\nu} + \epsilon_{AB} \int_{\Sigma} d^{2}\boldsymbol{\sigma} N^{AB} \mathcal{H}_{Schild} (P_{\mu\nu}) \right] . \quad (3.3)$$

Let us denote the measure in the functional space of path as

$$\left[\mathcal{D}\mu\left(\boldsymbol{\sigma}\right)\right] = \left[\mathcal{D}X^{\mu}\left(\boldsymbol{\sigma}\right)\right]\left[\mathcal{D}P_{\mu\nu}\left(\boldsymbol{\sigma}\right)\right]\left[\mathcal{D}\xi^{A}\left(\boldsymbol{\sigma}\right)\right]\left[\mathcal{D}\pi_{AB}\left(\boldsymbol{\sigma}\right)\right]\left[\mathcal{D}N^{AB}\left(\boldsymbol{\sigma}\right)\right]$$

Then, the propagator can be expressed by means of the following path-integral¹:

$$K\left[Y_{f}^{\mu}(s), Y_{i}^{\mu}(s); A\right] = \int_{Y_{i}^{\mu}(s), \zeta_{f}^{A}(s)}^{Y_{f}^{\mu}(s), \zeta_{f}^{A}(s)} \left[\mathcal{D}\mu(\boldsymbol{\sigma})\right] e^{\frac{i}{\hbar}\left\{S\left[X^{\rho}, P_{\sigma\tau}; \xi^{A}, \pi_{CD}\right] + S_{\text{cnstr.}}\left[X^{\rho}, P^{\sigma\tau}; \xi^{A}, \pi_{CD}; N^{AB}\right]\right\}} \\ = \int_{Y_{i}^{\mu}(s), \zeta_{f}^{A}(s)}^{Y_{f}^{\mu}(s), \zeta_{f}^{A}(s)} = \int_{Y_{i}^{\mu}(s), \zeta_{f}^{A}(s)}^{Y_{i}^{\mu}(s), \zeta_{f}^{A}(s)} , \qquad (3.5)$$

where, the initial and final loops are:

$$C_i$$
 : $Y_i^{\mu}(s)$ C_f : $Y_f^{\mu}(s)$.

We recall of course that

$$S_{\text{cnstr.}}\left[X^{\rho}, P^{\sigma\tau}, \xi^{A}, \pi_{AB}, N^{AB}\right] = -\frac{1}{2} \int_{\Sigma} d^{2}\boldsymbol{\sigma} N^{AB} \left(\boldsymbol{\sigma}\right) \left[\pi_{AB} - \epsilon_{AB} \mathcal{H}\left(P_{\mu\nu}\right)\right] , \quad (3.6)$$

as written in equation (2.48). Now, we are ready to start with the pat-integral computation.

$$\lim_{A \to 0} K[\boldsymbol{Y}(s), \boldsymbol{Y}_0(s); A] = \delta[\boldsymbol{Y}(s), \boldsymbol{Y}_0(s)] \quad . \tag{3.4}$$

Such a procedure effectively amounts to a renormalization of the field-dependent determinants produced by gaussian integration.

 $^{^{1}}$ The normalization constant will be fixed after all the functional integrations are carried out, by imposing the Boundary condition

Proposition 3.1 (Integrating out the ξ Fields).

Performing the path-integration over the ξ fields in (3.5) we get

$$\int_{Y_{i}^{\mu}(s), \zeta_{f}^{A}(s)}^{Y_{f}^{\mu}(s), \zeta_{f}^{A}(s)} [\mathcal{D}\mu(\boldsymbol{\sigma})] e^{\frac{i}{\hbar} \left[S_{\xi} + S_{\pi_{AB}}^{(1)} + S_{p\mu\nu} \right]} =$$

$$= \int_{Y_{i}^{\mu}(s)}^{Y_{f}^{\mu}(s)} [\mathcal{D}\tilde{\mu}(\boldsymbol{\sigma})] \delta \left[\epsilon^{m_{1}m_{2}} (\partial_{m_{1}} \pi_{a_{1}a_{2}}) (\partial_{m_{2}} \xi^{a_{2}}) \right] e^{\frac{i}{\hbar} \left[S_{\pi^{AB}}^{(2)} + S_{\pi_{AB}}^{(1)} + S_{P\mu\nu} \right]} , (3.7)$$

where we denote by $[\mathcal{D}\tilde{\mu}(\boldsymbol{\sigma})]$ the remaining measure,

$$\left[\mathcal{D}\tilde{\mu}\left(\boldsymbol{\sigma}\right)\right] = \left[\mathcal{D}X^{\mu}\left(\boldsymbol{\sigma}\right)\right]\left[\mathcal{D}P_{\mu\nu}\left(\boldsymbol{\sigma}\right)\right]\left[\mathcal{D}\pi_{AB}\left(\boldsymbol{\sigma}\right)\right]\left[\mathcal{D}N^{AB}\left(\boldsymbol{\sigma}\right)\right]$$

and

$$S_{\pi^{AB}}^{(2)} = \int_{\partial\Xi} oldsymbol{d} oldsymbol{\zeta}^B arpi_{AB} oldsymbol{\zeta}^A = \pi_{AB} \int_{\Xi} oldsymbol{d} \left(\xi^A oldsymbol{d} oldsymbol{\xi}^B \pi_{AB}
ight)$$

Proof:

To get the desired result we have to prove that,

$$\int_{\zeta_i^A}^{\zeta_f^A} \left[\mathcal{D}\xi^A(\boldsymbol{\sigma}) \right] e^{\frac{i}{\hbar}S_{\xi}} = \delta \left[\epsilon^{m_1 m_2} \partial_{m_1} \pi_{a_1 a_2} \partial_{m_2} \xi^{a_2} \right] e^{\frac{i}{2\hbar} S_{\pi^{AB}}^{(2)}}$$

$$(3.8)$$

$$S_{\pi^{AB}}^{(2)} = \pi_{AB} \int_{\Xi(\sigma)} d\left(\xi^A d\xi^B\right)$$
 (3.9)

since these is the only term depending on ξ in (3.5). We now have

$$\int_{\zeta_{i}^{A}}^{\zeta_{f}^{A}} \left[\mathcal{D}\xi^{A} (\boldsymbol{\sigma}) \right] e^{\frac{i}{\hbar}S_{\xi}} = \int_{\zeta_{i}^{A}}^{\zeta_{f}^{A}} \left[\mathcal{D}\xi^{A} (\boldsymbol{\sigma}) \right] \exp \left\{ \frac{i}{2\hbar} \int_{\Xi} \pi_{AB} d\xi^{A} \wedge d\xi^{B} \right\} \\
= \int_{\zeta_{i}^{A}}^{\zeta_{f}^{A}} \left[\mathcal{D}\xi^{A} (\boldsymbol{\sigma}) \right] \exp \left\{ \frac{i}{2\hbar} \int_{\Xi} d \left(\pi_{AB} \xi^{A} d\xi^{B} \right) - \frac{i}{2\hbar} \int_{\Xi} d \pi_{AB} \wedge d\xi^{B} \xi^{A} \right\} \\
= \int_{\zeta_{i}^{A}}^{\zeta_{f}^{A}} \left[\mathcal{D}\xi^{A} (\boldsymbol{\sigma}) \right] \exp \left\{ \frac{i}{2\hbar} \int_{\partial\Xi} \varpi_{AB} \zeta^{A} d\zeta^{B} - \frac{i}{2\hbar} \int_{\Xi} d \pi_{AB} \wedge d\xi^{B} \xi^{A} \right\} \\
= \exp \left\{ \frac{i}{2\hbar} \int_{\partial\Xi} \varpi_{AB} \zeta^{A} d\zeta^{B} \right\} \cdot \\
\cdot \int_{\zeta_{i}^{A}}^{\zeta_{f}^{A}} \left[\mathcal{D}\xi^{A} (\boldsymbol{\sigma}) \right] \exp \left\{ -\frac{i}{2\hbar} \int_{\Xi} \epsilon^{m_{1}m_{2}} \xi^{A_{1}} \partial_{m_{1}} \pi_{A_{1}A_{2}} \partial_{m_{2}} \xi^{A_{2}} \right\} \\
= \delta \left[\epsilon^{m_{1}m_{2}} \partial_{m_{1}} \pi_{A_{1}A_{2}} \partial_{m_{2}} \xi^{A_{2}} \right] e^{\frac{i}{2\hbar} S_{\pi^{AB}}^{(2)}}$$

as we desired to prove.

Now we can perform the functional integration over π_{AB} .

Proposition 3.2 (Integrating out the π Fields).

Performing the path-integration over the π fields in (3.7) we get

$$\int_{Y_{i}^{\mu}(s), \zeta_{f}^{A}(s)}^{Y_{f}^{\mu}(s), \zeta_{f}^{A}(s)} \left[\mathcal{D}\mu\left(\boldsymbol{\sigma}\right) \right] e^{\frac{i}{\hbar} \left[S_{\xi} + S_{\pi AB}^{(1)} + S_{P_{\mu\nu}} \right]} =
= \int_{0}^{\infty} dE e^{\frac{i}{\hbar} EA} \int_{Y_{i}^{\mu}(s)}^{Y_{f}^{\mu}(s)} \left[\mathcal{D}X^{\mu}(\boldsymbol{\sigma}) \right] \left[\mathcal{D}N^{AB}(\boldsymbol{\sigma}) \right] \left[\mathcal{D}P_{\mu\nu}(\boldsymbol{\sigma}) \right] e^{-\frac{i}{\hbar} \left[S_{P_{\mu\nu}}^{(2)} + S_{NAB}^{(1)} \right]} , \quad (3.10)$$

where we have

$$S_{N_{AB}}^{(1)} = E \int_{\Sigma} d^2 \boldsymbol{\sigma} \epsilon_{AB} N^{AB} \left(\boldsymbol{\sigma} \right) \quad . \tag{3.11}$$

Proof:

Of course since the only terms containing $\pi_{AB}(\sigma)$ in equation (3.7), apart from the functional Dirac delta, are $S_{\pi^{AB}}^{(1)}$ and $S_{\pi^{AB}}^{(2)}$ we have to prove that

$$\int \left[\mathcal{D}\pi_{AB} \left(\boldsymbol{\sigma} \right) \right] \delta \left[e^{m_1 m_2} \partial_{m_1} \pi_{A_1 A_2} \partial_{m_2} \xi^{A_2} \right] e^{\frac{i}{2\hbar} \left[S_{\pi^{AB}}^{(1)} + S_{\pi^{AB}}^{(2)} \right]} =$$

$$= \int_0^\infty dE e^{\frac{i}{\hbar} E A} e^{-\frac{i}{2\hbar} S_{N^{AB}}^{(1)}} . \tag{3.12}$$

This result follows since, thanks to the functional Dirac delta

$$\delta \left[\epsilon^{m_1 m_2} \partial_{m_1} \pi_{A_1 A_2} \partial_{m_2} \xi^{A_2} \right]$$

the functional integration over $[\mathcal{D}\pi_{AB}]$ is restricted to the "classical trajectory", so that we have (thanks to the classical Energy balance equation (2.52))

$$\pi_{A_1 A_2} = E \epsilon_{A_1 A_2}$$

$$\int [\mathcal{D} \pi_{A_1 A_2}] [\dots] = \int dE [\dots] .$$

Substituting the last two expressions gives the desired result.

Thus we can now write

$$K\left[Y_i^{\mu}(\boldsymbol{\sigma}), Y_f^{\mu}(\boldsymbol{\sigma}); A\right] = \int_0^\infty dE e^{\frac{i}{\hbar}EA} G\left[C_i, C_f; E\right] , \qquad (3.13)$$

where

$$G\left[C_{i},C_{f};E\right] = \int_{Y_{r}^{\mu}\left(s\right)}^{Y_{f}^{\mu}\left(s\right)} \left[\mathcal{D}X^{\mu}(\boldsymbol{\sigma})\right] \left[\mathcal{D}N^{AB}\left(\boldsymbol{\sigma}\right)\right] \left[\mathcal{D}P_{\mu\nu}\left(\boldsymbol{\sigma}\right)\right] e^{-\frac{i}{\hbar}\left[S_{p\mu\nu}+S_{N^{AB}}^{(1)}\right]} \quad . \tag{3.14}$$

The last step is now the Gaussian Functional Integral over $P_{\mu\nu}$.

Proposition 3.3 (Integrating out the P Fields).

Performing the Gaussian Functional Integration over $P_{\mu\nu}$ in equation (3.10) we get

$$\int_{Y_{i}^{\mu}(s), \xi_{f}^{A}(s)}^{Y_{f}^{\mu}(s), \xi_{f}^{A}(s)} [\mathcal{D}\mu(\boldsymbol{\sigma})] e^{\frac{i}{\hbar} \left[S_{\xi} + S_{\pi AB}^{(1)} + S_{p\mu\nu}\right]} =
= \int_{0}^{\infty} dE e^{\frac{i}{\hbar} EA} \int_{Y_{i}^{\mu}(s)}^{Y_{f}^{\mu}(s)} [\mathcal{D}X^{\mu}(\boldsymbol{\sigma})] \left[\mathcal{D}N^{AB}(\boldsymbol{\sigma})\right] e^{-\frac{i}{\hbar} \left[S_{NAB}^{(1)} + S_{NAB}^{(2)}\right]} , \quad (3.15)$$

where

$$S_{N^{AB}}^{(2)} = -\int_{\Sigma} d^2 \sigma \frac{m^2}{\epsilon_{AB} N^{AB}} \dot{X}^{\mu\nu} \dot{X}_{\mu\nu}$$
 (3.16)

.

Proof:

The desired result can be proved if we are able to prove

$$\int \left[\mathcal{D}P_{\mu\nu} \left(\boldsymbol{\sigma} \right) \right] e^{\frac{i}{\hbar} S_{P_{\mu\nu}}} = e^{\frac{i}{\hbar} S_{NAB}^{(2)}} \quad . \tag{3.17}$$

This result in turn can be derived performing a functional Gaussian integration. In more detail we have

$$\int [\mathcal{D}P_{\mu\nu}(\sigma)] e^{\frac{i}{\hbar}S_{P_{\mu\nu}}} = \int [\mathcal{D}P_{\mu\nu}(\sigma)] \exp\left\{\frac{i}{2\hbar} \left[\int_{X(\sigma)} P_{\mu\nu} dX^{\mu} \wedge dX^{\nu} + \epsilon_{AB} \int_{\mathcal{W}} d^{2}\sigma N^{AB} \mathcal{H}_{Schild}(P_{\mu\nu})\right]\right\}
= \int [\mathcal{D}P_{\mu\nu}(\sigma)] \exp\left\{\frac{i}{2\hbar} \left[\int_{\Sigma} P_{\mu\nu} \dot{X}^{\mu\nu} d^{2}\sigma + \epsilon_{AB} \int_{\Sigma} d^{2}\sigma N^{AB} \frac{P_{\mu\nu}P^{\mu\nu}}{4m^{2}}\right]\right\}
= \int [\mathcal{D}P_{\mu\nu}(\sigma)] \exp\left\{\int_{\Sigma} P_{\mu\nu} \frac{i\dot{X}^{\mu\nu}}{2\hbar} d^{2}\sigma + \int_{\Sigma} d^{2}\sigma P_{\mu\nu} \left(\frac{1}{2} \frac{i\epsilon_{AB}N^{AB}}{4\hbar m^{2}}\right) P^{\mu\nu}\right\}
= \exp\left\{\int_{\Sigma} d^{2}\sigma \frac{i\dot{X}^{\mu\nu}}{2\hbar} \left(\frac{1}{2} \frac{4\hbar m^{2}}{i\epsilon_{AB}N^{AB}}\right) \frac{i\dot{X}_{\mu\nu}}{2\hbar}\right\}
= \exp\left\{-\frac{im^{2}}{2\hbar} \int_{\Sigma} d^{2}\sigma \frac{\dot{X}^{\mu\nu}\dot{X}_{\mu\nu}}{\epsilon_{AB}N^{AB}}\right\} .$$
(3.18)

At the end we can thus rewrite $G[C_i, C_f; E]$ as the functional integral over $X^{\mu}(\sigma)$ and $N^{AB}(\sigma)$ of the exponential of i/\hbar times $S_{N^{AB}}^{(1)} + S_{N^{AB}}^{(2)}$, i.e. introducing the shorthand

notation $N\left(\boldsymbol{\sigma}\right) = \epsilon_{AB} N^{AB}\left(\boldsymbol{\sigma}\right)/2$,

$$G\left[C_{i}, C_{f}; E\right] = \int_{Y_{i}^{\mu}(s)}^{Y_{f}^{\mu}(s)} \left[\mathcal{D}Y^{\mu}\left(\boldsymbol{\sigma}\right)\right] \left[\mathcal{D}N^{AB}\left(\boldsymbol{\sigma}\right)\right] \exp\left\{-\frac{i}{\hbar} \int_{\Sigma} d^{2}\boldsymbol{\sigma} \left[-\frac{m^{2}}{4N} \dot{X}^{\mu\nu} \dot{X}_{\mu\nu} + NE\right]\right\}$$
(3.19)

Then

Proposition 3.4 (Schild-Nambu Goto Quantum Equivalence).

The saddle point evaluation of the String propagator (3.19) is

$$G\left[C_{i}, C_{f}; E\right] \simeq \int_{C_{i}}^{C_{f}} \left[\mathcal{D}\boldsymbol{X}\left(\sigma\right)\right] \exp\left\{-i\sqrt{m^{2}E} \int_{\Sigma} d^{2}\sigma \sqrt{-\frac{1}{2}\dot{X}^{\mu\nu}\dot{X}_{\mu\nu}}\right\} \quad . \tag{3.20}$$

Proof:

To determine the saddle point we have to solve the following equation

$$\frac{d}{dN}\left(-\frac{m^2}{4N}\dot{X}^{\mu\nu}\dot{X}_{\mu\nu} + NE\right) = 0 \quad ,$$

which is

$$\frac{m^2}{4N^2} \dot{X}^{\mu\nu} \dot{X}_{\mu\nu} + E = 0$$

The saddle point is given by the following value of $N(s)\sigma$

$$\hat{N}\left(\boldsymbol{\sigma}\right) = \sqrt{-\frac{m^2 \dot{X}^{\mu\nu} \dot{X}_{\mu\nu}}{4E}}$$

and the substitution of this value in equation 3.19 provides us with the result:

$$G\left[C_{i},C_{f};E\right] = \int_{Y_{f}^{\mu}\left(s\right)}^{Y_{f}^{\mu}\left(s\right)} \left[\mathcal{D}Y^{\mu}\left(\boldsymbol{\sigma}\right)\right] e^{-\frac{i}{\hbar}\sqrt{m^{2}E}\int_{\Sigma}d^{2}\boldsymbol{\sigma}\sqrt{-\frac{1}{2}\dot{X}^{\mu\nu}\dot{X}_{\mu\nu}}} \quad . \tag{3.21}$$

Then, since E has dimension of inverse length square, in natural units, we can set the *String* tension equal to m^2 , and (3.20) reproduces exactly the Nambu–Goto path–integral.

In terms of the Green function we found above the kernel can be expressed using equation (3.13).

3.2 Functional Wave Equation

The purpose of this section is to show how to derive the functional wave equation for $K[\mathbf{Y}(s), \mathbf{Y}_0(s); A]$ from the corresponding path–integral. This equation will describe how

the String responds to a variation of the final Boundary $\zeta^A = \zeta^A(s)$, just as the ordinary Schrödinger equation describes a particle reacts to a shift of the time interval end-point. As we have seen in section 2.5, the String "natural" evolution parameter is the area A of the Parameter Space, so that functional or Area Derivatives generate "translations" in loop space, or String deformations in Minkowski space. Thus, we expect the functional wave equation to be of order one in $\partial/\partial A$, and of order two in $\delta/\delta Y^{\mu}(s)$, or $\delta/\delta Y^{\mu\nu}(s)$.

The standard procedure to obtain the kernel wave equation goes through a recurrence relation satisfied by the discretized version of the Jacobi path–integral [8], [9]. However, such a construction is well defined only when the action is a polynomial in the dynamical variables. For a non–linear action such as the Nambu–Goto area functional, a lattice definition of the path–integral is much less obvious. Moreover, the continuum functional wave equation is recovered through the highly non–trivial limit of vanishing lattice spacing [8], [9]. In any case, the whole procedure seems disconnected from the classical approach to *String* Dynamics, whereas we would like to see a logical continuity between Quantum and Classical Dynamics. Against this background, it seems useful to offer an *alternative* path–integral derivation of the *String* functional wave equation, which is deeply rooted in the Hamiltonian formulation of *String* Dynamics discussed in subsection 2.5.2, and is basically derived from the same Jacobi variational principle which we have consistently adopted so far.

We first start with a result that will be useful later on

Proposition 3.5 (Kernel Variation).

The kernel variation under infinitesimal deformations of the field variables is

$$\delta K\left[\boldsymbol{Y}\left(s\right),\boldsymbol{Y}_{0}\left(s\right);A\right] = \frac{i}{\hbar} \int_{Y_{0}^{\mu}\left(s\right)}^{Y^{\mu}\left(s\right)} \int_{\zeta_{0}^{A}\left(s\right)}^{\zeta^{A}\left(s\right)} \left[D\mu(\sigma)\right] \left(\delta S_{\mathrm{Red.}}\right) \exp\left(\frac{i}{\hbar} S_{\mathrm{Red.}}\right) \quad . \tag{3.23}$$

$$\frac{\delta}{\delta Y^{\mu}\left(s\right)}=Y^{\prime\,\nu}\frac{\delta}{\delta Y^{\mu\nu}\left[C\right]}\quad,\qquad Y^{\prime\,\nu}\equiv\frac{dY^{\nu}}{ds}\quad. \tag{3.22}$$

Therefore, the functional wave equation can be written in terms of either type of derivative. Contrary to the statement in ref.[35], area derivatives are regular even when ordinary functional derivatives are not [7].

²We stress again that functional and Area Derivatives are related by [7]

Proof:

This result can be derived observing that in the kernel the field variables are contained only in the action, which in turn is exponentiated:

$$K\left[Y^{\mu}\left(s\right),Y_{0}^{\mu}\left(s\right);A\right]=\int_{Y_{0}^{\mu}\left(s\right)}^{Y^{\mu}\left(s\right)}\int_{\zeta_{0}^{A}\left(s\right)}^{\zeta^{A}\left(s\right)}\left[D\mu(\sigma)\right]\exp\left[\frac{iS_{\mathrm{Red.}}\left[\boldsymbol{Y},\boldsymbol{P};A\right]}{\hbar}\right]$$

then, applying the usual properties of the exponential, we get

$$\begin{split} \delta K\left[Y^{\mu}\left(s\right),Y_{0}^{\mu}\left(s\right);A\right] &= \delta\left[\int_{Y_{0}^{\mu}\left(s\right)}^{Y^{\mu}\left(s\right)}\int_{\zeta_{0}^{A}\left(s\right)}^{\zeta^{A}\left(s\right)}\left[D\mu(\sigma)\right]\exp\left(\frac{iS_{\mathrm{Red.}}\left[\boldsymbol{Y},\boldsymbol{P};A\right]}{\hbar}\right)\right] \\ &= \int_{Y_{0}^{\mu}\left(s\right)}^{Y^{\mu}\left(s\right)}\int_{\zeta_{0}^{A}\left(s\right)}^{\zeta^{A}\left(s\right)}\left[D\mu(\sigma)\right]\delta\left[\exp\left(\frac{iS_{\mathrm{Red.}}\left[\boldsymbol{Y},\boldsymbol{P};A\right]}{\hbar}\right)\right] \\ &= \int_{Y_{0}^{\mu}\left(s\right)}^{Y^{\mu}\left(s\right)}\int_{\zeta_{0}^{A}\left(s\right)}^{\zeta^{A}\left(s\right)}\left[D\mu(\sigma)\right]\left[\frac{i}{\hbar}\exp\left(\frac{iS_{\mathrm{Red.}}\left[\boldsymbol{Y},\boldsymbol{P};A\right]}{\hbar}\right)\delta S\left[\boldsymbol{Y},\boldsymbol{P};A\right]\right] \end{split}$$

which is the desired result.

We now observe that only *Boundary* variations will contribute to equation (3.23) if we restrict the fields to vary within the family of classical solutions corresponding to a given initial *String* configuration, $C_0: Y_i^{\mu}(s)$. Then

Proposition 3.6 (Kernel Derivatives).

The derivatives of the kernel with respect to the dynamical variables, the Areal Time A and the final shape of the loop $Y^{\mu}(s)$, are

$$\frac{\partial K\left[Y^{\mu}\left(s\right),Y_{0}^{\mu}\left(s\right);A\right]}{\partial A}=-\frac{iE}{\hbar}K\left[Y^{\mu}\left(s\right),Y_{0}^{\mu}\left(s\right);A\right]\quad,\tag{3.24}$$

$$\frac{\delta K\left[Y^{\rho}\left(s\right),Y_{0}^{\rho}\left(s\right);A\right]}{\delta Y^{\mu}\left(s\right)} = \frac{i}{\hbar} \int_{Y_{0}^{\rho}\left(s\right)}^{Y^{\rho}\left(s\right)} \int_{\zeta_{0}^{A}\left(s\right)}^{\zeta^{A}\left(s\right)} \left[\mathcal{D}\mu(\boldsymbol{\sigma})\right] Q_{\mu\nu} Y^{\prime\nu} \exp\left(\frac{iS_{\mathrm{Red.}}}{\hbar}\right) \quad . \tag{3.25}$$

Proof:

We are going to use equations (3.23) and (2.60); from the last of them we already derived in subsection (2.5.2) results (2.56, 2.58), which we now rewrite for clarity:

$$\frac{\delta S_{\text{Red.}}}{\delta Y^{\mu}(s)} = Q_{\mu\nu} Y^{\prime\nu}$$

$$\frac{\delta S_{\text{Red.}}}{\delta A} = \frac{dS_{\text{Red.}}}{dA} = E ;$$
(3.26)

here, we remembered again that the Hamiltonian H_{Schild} is constant, $H_{\text{Schild}} \equiv E$, along a classical trajectory. Thus, from (3.23)

$$\frac{\partial K\left[Y^{\mu}\left(s\right), Y_{0}^{\mu}\left(s\right); A\right]}{\partial A} =$$

$$= \frac{i}{\hbar} \int_{Y_{0}^{\mu}\left(s\right)}^{Y^{\mu}\left(s\right)} \int_{\zeta_{0}^{A}\left(s\right)}^{\zeta^{A}\left(s\right)} \left[D\mu(\sigma)\right] \exp\left(\frac{iS_{\text{Red.}}\left[\boldsymbol{Y}, \boldsymbol{P}; A\right]}{\hbar}\right) \frac{\delta S_{\text{Red.}}\left[\boldsymbol{Y}, \boldsymbol{P}; A\right]}{\delta A}$$

$$= \frac{i}{\hbar} \int_{Y_{0}^{\mu}\left(s\right)}^{Y^{\mu}\left(s\right)} \int_{\zeta_{0}^{A}\left(s\right)}^{\zeta^{A}\left(s\right)} \left[D\mu(\sigma)\right] \exp\left(\frac{iS_{\text{Red.}}\left[\boldsymbol{Y}, \boldsymbol{P}; A\right]}{\hbar}\right) \frac{dS_{\text{Red.}}\left[\boldsymbol{Y}, \boldsymbol{P}; A\right]}{dA}$$

$$= \frac{iE}{\hbar} \int_{Y_{0}^{\mu}\left(s\right)}^{Y^{\mu}\left(s\right)} \int_{\zeta_{0}^{A}\left(s\right)}^{\zeta^{A}\left(s\right)} \left[D\mu(\sigma)\right] \exp\left(\frac{iS_{\text{Red.}}\left[\boldsymbol{Y}, \boldsymbol{P}; A\right]}{\hbar}\right)$$

$$= \frac{iE}{\hbar} K\left[Y^{\mu}\left(s\right), Y_{0}^{\mu}\left(s\right); A\right] . \tag{3.27}$$

In the same way, we get

$$\frac{\partial K\left[Y^{\rho}\left(s\right), Y_{0}^{\rho}\left(s\right); A\right]}{\partial Y^{\mu}\left(s\right)} =$$

$$= \frac{i}{\hbar} \int_{Y_{0}^{\rho}\left(s\right)}^{Y^{\rho}\left(s\right)} \int_{\zeta_{0}^{A}\left(s\right)}^{\zeta^{A}\left(s\right)} \left[D\mu(\sigma)\right] \exp\left(\frac{iS_{\text{Red.}}\left[\boldsymbol{Y}, \boldsymbol{P}; A\right]}{\hbar}\right) \frac{\delta S_{\text{Red.}}\left[\boldsymbol{Y}, \boldsymbol{P}; A\right]}{\delta Y^{\mu}\left(s\right)}$$

$$= \frac{i}{\hbar} \int_{Y_{0}^{\rho}\left(s\right)}^{Y^{\rho}\left(s\right)} \int_{\zeta_{0}^{A}\left(s\right)}^{\zeta^{A}\left(s\right)} \left[D\mu(\sigma)\right] \exp\left(\frac{iS_{\text{Red.}}\left[\boldsymbol{Y}, \boldsymbol{P}; A\right]}{\hbar}\right) Q_{\mu\nu}\left(s\right) Y^{\prime\nu}\left(s\right) \quad . \tag{3.28}$$

П

After these preliminary steps the main result of this chapter follows:

Proposition 3.7 (String Kernel Functional Schrödinger Equation).

The propagation Kernel of the String satisfies the following Schrödinger-like functional equation

$$-\frac{\hbar^{2}}{2m^{2}} \left(\oint_{\Gamma \approx \mathbb{S}^{1}} dl(s) \right)^{-1} \oint_{\Gamma \approx \mathbb{S}^{1}} \frac{ds}{\sqrt{\left(\mathbf{Y}'\right)^{2}}} \frac{\delta^{2} K \left[\mathbf{Y}\left(s\right), \mathbf{Y}_{0}\left(s\right); A\right]}{\delta^{\mu} Y\left(s\right) \delta_{\mu} Y\left(s\right)} = i\hbar \frac{\partial K \left[\mathbf{Y}\left(s\right), \mathbf{Y}_{0}\left(s\right); A\right]}{\partial A} . \tag{3.29}$$

Proof:

The proof is a standard procedure, in which the classical Hamilton–Jacobi equation is assumed as the evolution equation for the mean values of the quantum operators. Mean values, according to Feynman path–integral formulation of Quantum Mechanics/Quantum Field Theory can be

computed as functional integrals. First, we remember result (3.25), which is here rewritten for convenience:

$$\frac{\delta}{\delta Y^{\mu}(s)} K[\mathbf{Y}(s), \mathbf{Y}_{0}(s); A] = \frac{i}{\hbar} \int_{\mathbf{Y}_{0}(s)}^{\mathbf{Y}(s)} \int_{\zeta_{0}(s)}^{\zeta(s)} [\mathcal{D}\mu(s)] \frac{\delta S_{\text{Red.}}}{\delta Y^{\mu}(s)} e^{\frac{i}{\hbar} S_{\text{Red.}}}$$

$$= \frac{i}{\hbar} \int_{\mathbf{Y}_{0}(s)}^{\mathbf{Y}(s)} \int_{\zeta_{0}(s)}^{\zeta(s)} [\mathcal{D}\mu(s)] q_{\mu} \exp\left(\frac{i}{\hbar} S_{\text{Red.}}\right) ; \qquad (3.30)$$

the definition of equation (2.56) has also been used. We can now take one more functional derivative of the Kernel, again with respect to Y^{μ} , to get

$$\frac{\delta^{2}}{\delta Y^{\mu}(s)\delta Y_{\mu}(s)}K\left[\boldsymbol{Y}\left(s\right),\boldsymbol{Y}_{0}\left(s\right);\boldsymbol{A}\right] = -\frac{1}{\hbar^{2}}\int_{\boldsymbol{Y}_{0}\left(s\right)}^{\boldsymbol{Y}\left(s\right)}\int_{\boldsymbol{\zeta}_{0}\left(s\right)}^{\boldsymbol{\zeta}\left(s\right)}\left[\mathcal{D}\mu\left(s\right)\right]q_{\mu}q^{\mu}\exp\left(\frac{i}{\hbar}S_{\mathrm{Red.}}\right)$$

$$\equiv -\frac{1}{\hbar^{2}}\overline{q^{\mu}q_{\mu}}, \qquad (3.31)$$

where we recognize in the right hand side the expectation value for the square of the Boundary Area Momentum. Moreover we also have equation (3.24):

$$\frac{\partial}{\partial A}K\left[\boldsymbol{Y}\left(s\right),\boldsymbol{Y}_{0}\left(s\right);A\right] = -\frac{iE}{\hbar}K\left[\boldsymbol{Y}\left(s\right),\boldsymbol{Y}_{0}\left(s\right);A\right] \quad . \tag{3.32}$$

If we now consider the Hamilton–Jacobi equation (2.62) we can interpret it as the evolution equation for the mean values of the quantum mechanical probabilities, i.e. we can substitute to $q_{\mu}q^{\mu}$ the quantum mechanical expectation value $\overline{q_{\mu}q^{\mu}}$ so that

$$\frac{1}{2m^2} \oint_{\Gamma \approx \mathbb{S}^1} \frac{ds}{\sqrt{(\mathbf{Y}')^2}} \overline{q^\mu q_\mu} = E \oint_{\Gamma \approx \mathbb{S}^1} ds \sqrt{(\mathbf{Y}')^2}$$
(3.33)

Using then equations (3.31, 3.32) to express the quantities appearing above in terms of the propagator, we get

$$-\frac{\hbar^{2}}{2m^{2}}\left(\oint dl(s)\right)^{-1}\oint_{\Gamma\approx\mathbb{S}^{1}}\frac{ds}{\sqrt{\left(\mathbf{Y}'\right)^{2}}}\frac{\delta^{2}K\left[\mathbf{Y}\left(s\right),\mathbf{Y}_{0}\left(s\right);A\right]}{\delta Y^{\mu}\left(s\right)\delta Y_{\mu}\left(s\right)}=i\hbar\frac{\partial}{\partial A}K\left[\mathbf{Y}\left(s\right),\mathbf{Y}_{0}\left(s\right);A\right]\quad,\quad(3.34)$$

i.e. the equation for the evolution of the kernel of the String.

Thus, $K[Y(s), Y_0(s); A]$ can be determined either by solving the functional wave equation (3.29), or by evaluating the path–integral (3.5).

Once equation (3.29) is given, it is straightforward to show that $G[C, C_0; m^2]$ defined in equation (3.13) satisfies the following equation

$$\left[-\hbar^{2} \left(\int_{0}^{1} ds \sqrt{(\mathbf{Y}')^{2}} \right)^{-1} \cdot \left[-\hbar^{2} \left(\int_{0}^{1} ds \sqrt{(\mathbf{Y}')^{2}} \frac{\delta^{2}}{\delta Y^{\mu}(s) \delta Y_{\mu}(s)} + m^{4} \right] G \left[C, C_{0}; m^{2} \right] = -\delta \left[C - C_{0} \right] \quad . \quad (3.35)$$

Therefore, $G[C, C_0; m^2]$ can be identified with the Green function for the String.

The functional equation (3.29) can be rewritten in different forms, which are useful to point out some specific properties. In particular it is natural to ask wether it is possible to give a formulation in which we have only reference to *Holographic* quantities or not. We thus recall the definition 2.4 as well as the comments following it on page 33. In particular we remember that the *Holographic Functional Derivative* has an explicitly dependence from the point of the loop where it is calculated. In order to recover reparametrization invariance, we have to get rid of the arbitrariness in the choice of the attachment point. This can be achieved by summing over all its possible locations along the loop, and then compensating for the overcounting of the area variation by averaging the result over the proper length of the loop. This is what we have done in equation (3.29) and in equation (2.62) (and we will do the same in (3.70)). The same prescription enables us to define any other reparametrization invariant quantity or operator. Hence, starting with the *Holographic Functional Derivative*, we introduce a simpler but more effective notation.

Definition 3.1 (Loop Derivative). The <u>Loop Derivative</u> is the average of the Holographic Derivative over the String loop:

$$\frac{\delta}{\delta C^{\mu\nu}} \equiv \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} dl(s) \frac{\delta}{\delta Y^{\mu\nu}(s)} . \tag{3.36}$$

This represents a genuine loop operation without reference to the way in which the loop is parametrized. We can now cast the functional wave equation for the Kernel (3.29) in some alternative forms.

Proposition 3.8 (Alternative Forms of the Functional Kernel Equation).

The Functional Wave Equation for the Kernel, (3.29), can be cast in the following alternative forms:

$$-\frac{\hbar^{2}}{2m^{2}}\left(\oint_{\Gamma\approx\mathbb{S}^{1}}dl(s)\right)^{-1}\oint_{\Gamma\approx\mathbb{S}^{1}}ds\sqrt{\left(\mathbf{Y}'\right)^{2}}\frac{\delta^{2}K\left[\mathbf{Y}\left(s\right),\mathbf{Y}_{0}\left(s\right);A\right]}{\delta\mathbf{Y}^{\mu\nu}\left(s\right)\delta\mathbf{Y}_{\mu\nu}\left(s\right)}=i\hbar\frac{\partial K\left[\mathbf{Y}\left(s\right),\mathbf{Y}_{0}\left(s\right);A\right]}{\partial A}$$
(3.37)

$$\frac{1}{2} \frac{\delta^{2} K \left[\boldsymbol{Y} \left(s \right), \boldsymbol{Y}_{0} \left(s \right); A \right]}{\delta C^{\mu\nu} \delta C_{\mu\nu}} = i \hbar \frac{\partial K \left[\boldsymbol{Y} \left(s \right), \boldsymbol{Y}_{0} \left(s \right); A \right]}{\partial A}$$
(3.38)

Proof:

The quickest way to derive this result is to take a look at equation (2.60). In particular we observe that the dispersion relation in integrated form can also be written as

$$\frac{1}{2m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma \approx \mathbb{S}^1} ds \sqrt{\left(\mathbf{Y}' \right)^2(s)} Q^{\mu\nu} \left(s \right) Q_{\mu\nu} \left(s \right) = E \quad .$$

Of course, as we will stress again on page 84 $Q_{\mu\nu}(s)$ is the classical local quantity conjugated to the operator $i\delta/\delta Y^{\mu\nu}(s)$, so that, using a generalized form of the Correspondence Principle we get exactly (3.37). Moreover in the notation (3.36), the *loop laplacian* reads

$$\frac{1}{2} \frac{\delta^2}{\delta C^{\mu\nu} \delta C_{\mu\nu}} \equiv \frac{1}{2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} dl(s) \frac{\delta^2}{\delta Y^{\mu\nu}(s) \delta Y_{\mu\nu}(s)} \quad . \tag{3.39}$$

so that from the previous result we recover exactly equation (3.38).

3.3 Computing the Kernel

3.3.1 Integrating the Functional Wave Equation

It is possible to compute $K[Y(s), Y_0(s); A]$ exactly in the "free" case because the Lagrangian corresponding to the Schild Hamiltonian (2.54) is quadratic with respect to the generalized velocities $\dot{X}^{\mu\nu}$. Previous experience with this class of Lagrangians suggests the following ansatz for the *String* quantum kernel.

Ansatz (Quantum Kernel): The Quantum Kernel of a String has the following functional dependence:

$$K[\boldsymbol{Y}(s), \boldsymbol{Y}_{0}(s); A] = \mathcal{N}A^{\alpha} \exp\left(\frac{i}{\hbar}I[\boldsymbol{Y}(s), \boldsymbol{Y}_{0}(s); A]\right)$$
, (3.40)

where \mathcal{N} is a normalization constant, and α a real number.

To solve the problem of determining the form of the propagator is now equivalent to determine the unknown function $I[\boldsymbol{Y}(s), \boldsymbol{Y}_{0}(s); A]$ and the real number α . We make a first step in this direction in the following

Proposition 3.9 (Amplitude and Phase Equations for the Kernel).

The exponent $\alpha \in \mathbb{R}$ and the phase $I[Y, Y_0; A]$ that appear in the ansatz (3.40) satisfy the

following two independent equations:

$$\frac{2\alpha m^2}{A} = -\left(\oint_{\Gamma} dl(s)\right)^{-1} \oint_{\Gamma} \frac{ds}{\sqrt{(\mathbf{Y}')^2}} \frac{\delta^2 I[\mathbf{Y}, \mathbf{Y}_0; A]}{\delta Y^{\mu}(s) \delta Y_{\mu}(s)} , \qquad (3.41)$$

$$2m^{2} \frac{\partial I[\boldsymbol{Y}, \boldsymbol{Y}_{0}; A]}{\partial A} = -\left(\oint_{\Gamma} dl(s)\right)^{-1} \oint_{\Gamma} \frac{ds}{\sqrt{(\boldsymbol{Y}')^{2}}} \frac{\delta I[\boldsymbol{Y}, \boldsymbol{Y}_{0}; A]}{\delta Y^{\mu}(s)} \frac{\delta I[\boldsymbol{Y}, \boldsymbol{Y}_{0}; A]}{\delta Y_{\mu}(s)}$$
(3.42)

Proof:

We substitute the ansatz (3.40) in the functional wave equation (3.29). Fist we get

$$-\frac{\hbar^{2}}{2m^{2}} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} \frac{ds}{\sqrt{(\mathbf{Y}'(s))^{2}}} \frac{\delta}{\delta Y^{\mu}(s)} \left(\frac{i}{\hbar} \frac{\delta I}{\delta Y_{\mu}(s)} e^{\frac{i}{\hbar}I} \right) \mathcal{N} A^{\alpha} =$$

$$= i\hbar \mathcal{N} \frac{A^{\alpha}}{A} e^{\frac{i}{\hbar}I} + i\hbar \mathcal{N} A^{\alpha} e^{\frac{i}{\hbar}I} \frac{i}{\hbar} \frac{\partial I}{\partial A} . \tag{3.43}$$

Rearranging this result, in particular performing the remaining functional derivative ans dividing by $\mathcal{N}A^{\alpha}$ we obtain:

$$-\frac{\hbar^{2}}{2m^{2}} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} \frac{ds}{\sqrt{\left(\mathbf{Y}'(s) \right)^{2}}} \left[\left(\frac{i}{\hbar} \frac{\delta I}{\delta Y_{\mu}(s)} \right)^{2} + \frac{i}{\hbar} \frac{\delta^{2} I}{\delta Y^{\mu}(s) \delta Y_{\mu}(s)} \right] e^{\frac{i}{\hbar} I} =$$

$$= i\hbar \frac{1}{A} e^{\frac{i}{\hbar} I} + i\hbar \mathcal{N} A^{\alpha} e^{\frac{i}{\hbar} I} \frac{i}{\hbar} \frac{\partial I}{\partial A} . \tag{3.44}$$

The final result then comes out, after dividing by $\exp\{iI/\hbar\}$, by separating the real part of the resulting equation (which gives (3.42)) from the imaginary one (which is (3.41)).

Comparing equations (3.42) and (2.62), we see that $I = S_{\text{Red.}}[\mathbf{Y}(s), \mathbf{Y}_0(s); A]$ and (3.42) is just the classical Hamilton–Jacobi equation. Therefore, the main problem is to determine the form of $S_{\text{Red.}}$ in the *String* case. We proceed by analogy with the relativistic point particle case, where $S_{\text{Red.}}$ is a functional of the world–line length element. In this case we try with a functional of the natural generalization of that concept, i.e. the *Holographic Coordinates* of definition (2.3). As a preliminary result we compute some useful derivatives.

Proposition 3.10 (Holographic Derivatives).

The following expressions for the first and second functional derivatives of the Holographic

Coordinates,

$$\frac{\delta Y^{\mu\nu}[C]}{\delta Y^{\alpha}(s)} = \delta_{\alpha}{}^{\mu}Y^{\prime\nu}(s) - \delta_{\alpha}{}^{\nu}Y^{\prime\mu}(s) \tag{3.45}$$

$$\frac{\delta Y^{\mu\nu}[C]}{\delta Y^{\alpha}(s)} = \delta_{\alpha}{}^{\mu}Y^{\prime\nu}(s) - \delta_{\alpha}{}^{\nu}Y^{\prime\mu}(s) \qquad (3.45)$$

$$\frac{\delta^{2}Y^{\mu\nu}[C]}{\delta Y^{\alpha}(s)\delta Y^{\beta}(u)} = (\delta_{\alpha}{}^{\mu}\delta_{\beta}{}^{\nu} - \delta_{\alpha}{}^{\nu}\delta_{\beta}{}^{\mu})\frac{d}{ds}\delta(s-u) , \qquad (3.46)$$

hold.

Proof:

We start from the coordinate expression of the Holographic Coordinates,

$$Y^{\mu\nu}\left[C\right] = \oint_{\Gamma \approx \mathbb{S}^1} du Y^{\mu}(u) \, \frac{dY^{\nu}(u)}{du}$$

Then the results are derived by simply applying the chain rule for the functional derivative. In particular we have

$$\frac{\delta Y^{\mu\nu} [C]}{\delta Y^{\alpha} (s)} = \oint_{\Gamma} du \left[\frac{\delta Y^{\mu}(u)}{\delta Y^{\alpha} (s)} \frac{dY^{\nu}(u)}{du} + Y^{\mu}(u) \frac{d}{du} \left(\frac{\delta Y^{\nu}(u)}{\delta Y^{\alpha} (s)} \right) \right]
= \oint_{\Gamma} du \left[\delta^{\mu}_{\alpha} \delta(u - s) \frac{dY^{\nu}(u)}{du} + \delta^{\nu}_{\alpha} Y^{\mu}(u) \frac{d\delta(u - s)}{du} \right]
= \oint_{\Gamma} du \left[\delta^{\mu}_{\alpha} \delta(u - s) \frac{dY^{\nu}(u)}{du} - \delta^{\nu}_{\alpha} \frac{dY^{\mu}(u)}{du} \delta(u - s) \right]
= \delta^{\mu}_{\alpha} Y^{\prime\nu} (s) - \delta^{\nu}_{\alpha} Y^{\prime\mu} (s) .$$
(3.47)

Then, taking one more functional derivative we obtain

$$\frac{\delta^2 Y^{\mu\nu}\left[C\right]}{\delta Y^\alpha\left(s\right)\delta Y^\beta\left(u\right)} = \left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta\right)\frac{d}{ds}\delta(s-u) \quad .$$

We are now ready to determine the complete form of the propagator, which is done in the following proposition. As in the particle case we will recognize it as the exponential of the distance squared, but this time in loop space; here the distance is given by the difference in the area of the projected shadows onto the coordinate planes, so that it is useful to use the following notation.

Notation 3.1 (Holographic Distance).

We will indicate with $\Sigma^{\mu\nu}$ $[C-C_0]$ the area distance between the two loops C and C_0 in the $\mu - \nu$ plane, i.e.

$$\Sigma^{\mu\nu} [C - C_0] = Y^{\mu\nu} [C] - Y^{\mu\nu} [C_0]$$
 ,

and call it the <u>Holographic Distance</u> of the considered loops.

Moreover, we need a suitable definition of the Loop Space Dirac Delta Function in the *Holographic Coordinates* in order to fix the normalization constant.

Definition 3.2 (Loop Space Dirac Delta Function).

The <u>Loop Space Dirac Delta Function</u> is defined as

$$\delta[C - C_0] \equiv \lim_{\epsilon \to 0} \left(\frac{1}{\pi \epsilon} \right)^{(D-1)/2} \exp\left(-\frac{1}{2\epsilon} \Sigma^{\mu\nu} [C - C_0] \Sigma_{\mu\nu} [C - C_0] \right) .$$

Then, we can prove

Proposition 3.11 (String Propagation Kernel).

The Propagation Kernel for the String is given by

$$K[\mathbf{Y}(s), \mathbf{Y}_{0}(s); A] = \left(\frac{m^{2}}{2i\pi\hbar A}\right)^{(D-1)/2} \exp\left(\frac{im^{2}}{4\hbar A} \Sigma^{\mu\nu} [C - C_{0}] \Sigma_{\mu\nu} [C - C_{0}]\right) \quad . \quad (3.48)$$

Proof:

Since we have already determined the relation between the phase of the propagator $I[Y, Y_0; A]$ and the classical action $S_{\text{Red.}}[Y, Y_0; A]$ we introduce the trial solution

$$S_{\text{Red.}}[\boldsymbol{Y}(s)), \boldsymbol{Y}_{0}(s); A] = \frac{\beta}{4A} (Y^{\mu\nu}[C] - Y^{\mu\nu}[C_{0}]) (Y_{\mu\nu}[C] - Y_{\mu\nu}[C_{0}])$$

$$\equiv \frac{\beta}{4A} \Sigma^{\mu\nu}[C - C_{0}] \Sigma_{\mu\nu}[C - C_{0}] , \qquad (3.49)$$

where, β is a second parameter to be fixed by the equations (3.41) and (3.42). By taking into account (3.45) and (3.46), we find

$$\frac{\delta S_{\text{cl.}}}{\delta Y^{\mu}(s)} = \frac{\beta}{A} \Sigma_{\mu\nu} [C - C_0] Y^{\prime \nu}(s) \quad . \tag{3.50}$$

Note that the dependence on the parameter s is only through the factor $Y'^{\nu}(s)$. Then,

$$\frac{\delta^2 S_{\text{cl.}}}{\delta Y_{\mu}(s)\delta Y^{\mu}(s)} = \frac{(D-1)\beta}{A} (\mathbf{Y}'(s))^2 \quad . \tag{3.51}$$

Equations (3.42) and (3.41) now give

$$\beta = -\frac{2\alpha}{(D-1)}m^2$$
, $\alpha = -\frac{(D-1)}{2}$ (3.52)

Finally, using definition 3.2 the kernel normalization constant is fixed by the *Boundary* condition (3.4), and we finally obtain the promised expression of the quantum kernel.

The above equation, in turn, provides us to the following representation of the Nambu–Goto propagator.

Proposition 3.12 (Representation of Nambu-Goto String Dynamics).

The Quantum Dynamics of a String can be determined thanks to the following representation of the Nambu–Goto String Propagator in terms of the Kernel in the Holographic Coordinates:

$$\int_{Y_0^{\mu}(s)}^{Y^{\mu}(s)} [DX^{\mu}(\sigma)] \exp \left\{ -\frac{im^2}{\hbar} \int_{\Sigma} d^2 \sigma \sqrt{-\frac{1}{2} \dot{X}^{\mu\nu} \dot{X}_{\mu\nu}} \right\} =
= -\int_0^{\infty} dA \, e^{-im^2 A/2\hbar} \left(\frac{m^2}{2i\pi\hbar A} \right)^{(D-1)/2} \exp \left(\frac{im^2}{4\hbar A} \Sigma^{\mu\nu} [C - C_0] \Sigma_{\mu\nu} [C - C_0] \right) \quad (3.53)$$

Proof:

From equations (3.13), (3.20) and (3.48) we have

$$\left(\frac{m^{2}}{2i\pi\hbar A}\right)^{(D-1)/2} \exp\left\{\frac{im^{2}}{4\hbar A} \Sigma^{\mu\nu} \left[C - C_{0}\right] \Sigma_{\mu\nu} \left[C - C_{0}\right]\right\} =
= \int_{0}^{\infty} dE e^{\frac{i}{\hbar}EA} \int_{Y_{0}^{\mu}(s)}^{Y^{\mu}(s)} \left[\mathcal{D}X^{\mu}(\boldsymbol{\sigma})\right] \exp\left\{-\frac{im^{2}}{\hbar} \sqrt{-\frac{1}{2} \dot{X}^{\mu\nu} \dot{X}_{\mu\nu}}\right\}$$

From this result we thus get

$$\int_{Y_0^{\mu}(s)}^{Y^{\mu}(s)} [\mathcal{D}X^{\mu}(\sigma)] \exp\left\{-\frac{im^2}{\hbar} \sqrt{-\frac{1}{2} \dot{X}^{\mu\nu} \dot{X}_{\mu\nu}}\right\} =
= -\int_0^{\infty} dA e^{-\frac{im^2 A}{2\hbar}} \left(\frac{m^2}{2i\pi\hbar A}\right) \exp\left\{\frac{im^2}{4\hbar A} \Sigma^{\mu\nu} [C - C_0] \Sigma_{\mu\nu} [C - C_0]\right\}$$

which is the desired result.

Note that, since no approximation was used to obtain equation (3.53), the above representation can also be interpreted as a new *definition* of the Nambu–Goto path–integral. This definition is based on the classical Jacobi formulation of *String* Dynamics rather than on the customary discretization procedure.

3.3.2 Integrating the Path-Integral

As a consistency check on the above result, and in order to clarify some further properties of the path–integral, it may be useful to offer an alternative derivation of equation (3.53) which is based entirely on the usual gaussian integration technique. As we have seen in the previous section, the Feynman amplitude can be written as follows

$$K[\boldsymbol{Y}(s), \boldsymbol{Y}_{0}(s); A] = \int_{Y_{0}^{\mu}(s)}^{Y^{\mu}(s)} [\mathcal{D}X^{\mu}(\boldsymbol{\sigma})] [\mathcal{D}P_{\mu\nu}(\boldsymbol{\sigma})] \times \\ \times \exp \left\{ \frac{i}{2\hbar} \int_{\mathcal{W}} P_{\mu\nu} d\boldsymbol{X}^{\mu} \wedge d\boldsymbol{X}^{\nu} - \frac{i}{2\hbar} \epsilon_{ab} \int_{\Xi} d\boldsymbol{\xi}^{a} \wedge d\boldsymbol{\xi}^{b} H(\boldsymbol{P}) \right\} \quad . \quad (3.54)$$

It is possible to evaluate the functional integral (3.54), without discretization of the variables; we have just to carefully remember the meaning of the path–integration in this case, and to distinguish between *Bulk* and *Boundary* fields. Moreover, we are going to use the following

Proposition 3.13 (Functional Bulk Integration).

The functional integration over the Bulk coordinates X restricts the functional Area Momentum integration to the classical extremal trajectories of the String, namely:

$$\int_{Y_0^{\mu}(s)}^{Y^{\mu}(s)} [\mathcal{D}X^{\mu}(\boldsymbol{\sigma})] \exp\left\{\frac{i}{2\hbar} \int_{\mathcal{W}} P_{\mu\nu} \, \boldsymbol{d}X^{\mu} \wedge \boldsymbol{d}X^{\nu}\right\} =
= \delta \left[\boldsymbol{d} \left(P_{\mu\nu} \boldsymbol{d}X^{\nu}\right)\right] \exp\left\{\frac{i}{2\hbar} \int_{C} Q_{\mu\nu} Y^{\mu} \boldsymbol{d}Y^{\nu}\right\} .$$
(3.55)

Proof:

The proof enlightens a nice feature of the path–integration procedure. The initial and final values are simply the values of the fields on the *Boundary*, which are held fixed $(Y^{\mu}(s))$ and $Y_0^{\mu}(s)$. What varies is just the field on the *Bulk* $X^{\mu}(\sigma)$. We thus firstly explicitly separate the contribution of the *Boundary* from the first exponent in equation (3.54) thanks to an integration by parts:

$$\int_{\mathcal{W}} P_{\mu\nu} dX^{\mu} \wedge dX^{\nu} = \int_{\mathcal{W}} d(X^{\mu} P_{\mu\nu} dX^{\nu}) - \int_{\mathcal{W}} X^{\mu} d(P_{\mu\nu} dX^{\nu})$$
$$= \oint_{C=\partial \mathcal{W}} Y^{\mu} Q_{\mu\nu} dY^{\nu} - \int_{\mathcal{W}} X^{\mu} d(P_{\mu\nu} dX^{\nu})$$

We can then substitute this result into the first factor of (3.54), which is the only one depending on $X(\sigma)$:

$$\int_{\boldsymbol{Y}_{0}(s)}^{\boldsymbol{Y}(s)} [\mathcal{D}X^{\mu}(\boldsymbol{\sigma})] \exp\left\{\frac{i}{2\hbar} \int_{\mathcal{W}} P_{\mu\nu} \, d\boldsymbol{X}^{\mu} \wedge d\boldsymbol{X}^{\nu}\right\} =$$

$$= \exp\left\{\frac{i}{2\hbar} \oint_{C} Y^{\mu} Q_{\mu\nu} d\boldsymbol{Y}^{\nu}\right\} \int_{\boldsymbol{Y}_{0}(s)}^{\boldsymbol{Y}(s)} [\mathcal{D}X^{\mu}(\boldsymbol{\sigma})] \exp\left\{-\frac{i}{2\hbar} \int_{\mathcal{W}} X^{\mu} d(P_{\mu\nu} d\boldsymbol{X}^{\nu})\right\}$$

$$= \int_{\boldsymbol{Y}_{0}(s)}^{\boldsymbol{Y}(s)} [\mathcal{D}X^{\mu}(\boldsymbol{\sigma})] \exp\left\{\frac{i}{2\hbar} \left[\oint_{C} Y^{\mu} Q_{\mu\nu} d\boldsymbol{Y}^{\nu} - \int_{\mathcal{W}} X^{\mu} d(P_{\mu\nu} d\boldsymbol{X}^{\nu})\right]\right\}$$

$$= \delta \left[d \left(P_{\mu\nu} d\boldsymbol{X}^{\nu}\right)\right] \exp\left\{\frac{i}{2\hbar} \oint_{C} Q_{\mu\nu} Y^{\mu} d\boldsymbol{Y}^{\nu}\right\} .$$

The functional delta function has support on the classical, extremal trajectories of the String. Therefore, the momentum integration is restricted to the classical area—momenta and the residual integration variables are the components of the Boundary Area Momentum $Q_{\mu\nu}(s)$ along the World-Sheet Boundary. Thus we obtain the following result.

Proposition 3.14 (String Propagator from Path-Integration).

Performing the functional integrations in (3.54) correctly reproduces the propagator of equation (3.48)

Proof:

As a matter of fact, Boundary conditions fix the initial and final String loops C_0 and C but not the conjugate momenta. In analogy to the point particle case, the classical equations of motion on the final World-Sheet Boundary.

$$d\left(P_{\mu\nu}dX^{\nu}\right)\right]_{C} \quad , \tag{3.56}$$

admits a constant area momentum solution $Q_{\mu\nu}(s) = P_{\mu\nu}(\boldsymbol{\sigma}(s))$ with

$$P_{\mu\nu}(\boldsymbol{\sigma})$$
 {$\boldsymbol{\sigma}=\boldsymbol{\sigma}(s)$} $T^{\nu}(s) = Q{\mu\nu}(s) Y'^{\nu}(s) = \text{const.}$.

Hence, the functional integral over the Boundary Area Momentum reduces to a (D-1)-dimensional, generalized, Gaussian integral

$$\int [DP_{\mu\nu}(\boldsymbol{\sigma})]\delta\left[\boldsymbol{d}\left(P_{\mu\nu}\boldsymbol{d}\boldsymbol{x}^{\nu}\right)\right](\ldots) = \int [dP_{\mu\nu}](\ldots) \quad . \tag{3.57}$$

Moreover, the Hamiltonian is constant over such a classical World–Sheet and can be written in terms of the Boundary Area Momentum $Q_{\mu\nu}(s)$. In such a way, the path–integral is reduced to the Gaussian integral over the D-1 components of $P_{\mu\nu}$ which generates normal deformation of the loop.

$$K[\boldsymbol{Y}(s), Y_{0}(s); A] = \mathcal{N} \int [dQ_{\rho\tau}] \exp\left\{\frac{i}{\hbar} \left[Q_{\mu\nu} \int_{C} Y^{\mu}(s) d\boldsymbol{Y}^{\nu} - \frac{A}{4m^{2}} Q_{\mu\nu} Q^{\mu\nu}\right]\right\}$$
$$= \mathcal{N} \int [dQ_{\rho\tau}] \exp\left\{\frac{i}{\hbar} \left[\frac{1}{2} Q_{\mu\nu} \Sigma^{\mu\nu} [C - C_{0}] - \frac{A}{4m^{2}} Q_{\mu\nu} Q^{\mu\nu}\right]\right\}. (3.58)$$

The integral (3.58) correctly reproduces the expression (3.53), i.e.

$$K\left[Y^{\mu}\left(s\right),Y_{0}^{\mu}\left(s\right);A\right]=\left(\frac{m^{2}}{2i\pi A}\right)^{(D-1)/2}\exp\left\{\frac{im^{2}}{4A}\left[Y^{\mu\nu}\left[C\right]-Y^{\mu\nu}\left[C_{0}\right]\right]\left[Y_{\mu\nu}\left[C\right]-Y_{\mu\nu}\left[C_{0}\right]\right]\right\}$$

3.4 Summary, Comments, Highlights and More

Before driving our discussion of the Dynamics of extended objects toward the more general case of p-branes (this is the topic of the next chapter) we try to summarize what we have done so far, reviewing briefly our procedure without the bounds that computations and proofs put on physical intuition³.

The central point of all our formulation of String Dynamics is that we try to interpret the Quantum Dynamics of a Physical System as the Dynamics of the Boundary of what we understand as its Classical History, the Bulk. The Differential Geometric approach to the subject is already in such an interpretation, and will turn out again and again in the following chapters, when we will turn to analyze in more detail the properties of the Quantum Dynamics. As a result of this geometrical interpretation, starting from a Classical Hamilton–Jacobi Theory and developing in this way the connection toward the Quantum Realm, we see that the correct "coordinates" for the description of the system appear in a natural way. In particular, as we can see from the expression for the propagator (3.48) the geometric doublet $(Y_{\mu\nu}, A)$ represents the correct set of dynamical variables in our formulation of String Quantum Dynamics. We stress again that this choice makes it possible to develop a Hamilton–Jacobi Theory of String loops which represents a natural extension of the familiar formulation of classical and quantum mechanics of point–particles [3], [4]. With hindsight, the analogy becomes transparent when one compares equation 3.48 with the amplitude for a relativistic particle of mass m to propagate from x_0^{μ} to x^{μ} in a proper-time lapse T:

$$K(x, x_0; T) = \left(\frac{m}{2\pi T}\right)^2 \exp\left[\frac{im}{2T}|x - x_0|^2\right]$$
 (3.59)

³There are only two "rigorous" exception to this claim in this section, namely proposition 3.15 and definition 3.3.

This comparison suggests the following correspondence between dynamical variables for particles and Strings: a particle position in D-dimensional spacetime is labeled by D real numbers x^{μ} which represent the projection of the particle position vector along the coordinate axes. In the String case, the conventional choice is to consider the position vector for each constituent point, and then follow their individual dynamical evolution in terms of the coordinate time X^0 , or the proper time τ . As a matter of fact, the canonical String quantization is usually implemented in the proper time gauge $X^0 = \tau$. However, this choice explicitly breaks the reparametrization invariance of the Theory, whereas in the Hamilton-Jacobi formulation of String Dynamics, we have insisted that reparametrization invariance be manifest at every stage. The form (3.48) of the String propagator reflects that requirement. Moreover, we called $Y^{\mu\nu}$ [C] the Holographic Coordinates of the String: they play the same role as the position vector in the point-particle case. Indeed, the six components of $Y^{\mu\nu}$ [C] are areas and represent the area of the projections of the loop onto the coordinate planes in spacetime. Likewise, the reparametrization invariant evolution parameter for the String turns out to be again an area, i.e. the proper area A of the whole String Parameter Space. Thus, as particle Dynamics is the Dynamics of "lengths", String Dynamics turns out to be, in our formulation, a Dynamics of areas. As a matter of fact, the String World-Sheet is the spacetime image, through the embedding $X^{\mu} = X^{\mu}(\sigma^0, \sigma^1)$, of a two dimensional manifold Σ of coordinates (σ^0, σ^1) . Thus, just as the proper time τ is a measure of the timelike distance between the final and initial position of a point particle, the proper area A of Parameter Space is a measure of the timelike, or parametric distance between C and C_0 , i.e., the final and the initial String configurations, or the Boundary of the World-Sheet in Target Space. In our discussion we always kept fixed the initial loop C_0 , quantizing then the other free end loop C; in this perspective

$$(Y^{\mu\nu} [C] - Y^{\mu\nu} [C_0])^2 = (\Sigma^{\mu\nu} [C - C_0])^2$$

represents the "spatial distance squared" between C and C_0 , and A represents the classical time lapse for the *String* to change its shape from C_0 to C.

The Quantum Formulation of String Dynamics based on the non canonical variables that

⁴The connection between the conventional approach to *String* Theory and the formulation presented here is deepened in chapter 9.

⁵ neither the coordinate time, nor the proper time of the constituent points, but

we carried out through the evaluation of the String kernel can now be completed toward the derivation of the Schrödinger loop equation. Presently, we are interested in the quantum fluctuations of a loop. By this, we mean a shape changing transition, and we would like to assign a probability amplitude to any such process⁶. Suppose now that the shape of the initial String is approximated by the loop configuration $C_0: Y^{\mu} = Y_0^{\mu}(s)$. The corresponding "wave packet" $\Psi[Y_0;0]$ will be concentrated around C_0 . As the areal time increases, the initial String evolves, sweeping a World-Sheet which is the image of a Parameter Space of proper area A. Once $C_0: Y^{\mu} = Y_0^{\mu}(s)$ and A are assigned, the final String $Y^{\mu} = Y^{\mu}(s)$ can attain any of the different shapes compatible with the given initial condition and with the extension of the World-Sheet Parameter Space. Each geometric configuration corresponds to a different "point" in the Holographic Representation of the loop space provided by the $Y^{\mu\nu}$ coordinates [4]. Then⁷, $\Psi[C;A]$ will represent the probability amplitude to find a String of shape $C: Y^{\mu} = Y^{\mu}(s)$ as the final Boundary of the World-Sheet of Parameter Space area A, originating from C_0 . From this vantage point, the quantum String evolution is a random shape-shifting process which corresponds, mathematically, to the spreading of the initial wave packet $\Psi[C_0; 0]$ throughout loop space. The wave functional $\Psi[C; A]$ can be obtained by means of the amplitude (3.48), summing over all the initial String configurations. This amounts to integrate over all the allowed loop configurations $Y^{\mu\nu}$ [C₀]:

$$\Psi [C; A] = \sum_{C_0} K [\mathbf{Y} \equiv C, \mathbf{Y}_0 \equiv C_0; A] \Psi [C_0; 0]
= \int_{C_0} [\mathcal{D}C_0] K [\mathbf{Y} \equiv C, \mathbf{Y}_0 \equiv C_0; A] \Psi [C_0; 0]
= \left(\frac{m^2}{2i\pi A}\right)^{3/2} [\mathcal{D}Y^{\mu\nu} [C_0]] \exp \left[\frac{im^2}{4A} (Y^{\mu\nu} [C] - Y^{\mu\nu} [C_0])^2\right] \Psi [C_0; 0] \quad (3.61)$$

Thanks to previous expressions we can see that the String Functional Schrödinger Equation for the propagator (3.29) is equivalent to the following Functional Schrödinger Equation for the wave functional $\Psi[C; A]$:

$$-\frac{1}{2m^{2}}\left(\oint_{\Gamma\approx\mathbb{S}^{1}}dl(s)\right)^{-1}\oint_{\Gamma\approx\mathbb{S}^{1}}\frac{ds}{\sqrt{\left(\mathbf{Y}'\right)^{2}}}\frac{\delta^{2}\Psi\left[C;A\right]}{\delta Y^{\mu}\left(s\right)\delta Y_{\mu}\left(s\right)}=i\frac{\partial\Psi\left[C;A\right]}{\partial A}\quad.\tag{3.62}$$

⁶We underline, and after previous comments this should be deeply motivated, that in order to do this, we make use of "areal, or loop, derivatives", as described so far and developed, for instance, by Migdal [6]; as we already pointed out they are reviewed in appendix D.

⁷Maintaining the promise we made at the beginning of this section we give an intuitive treatment in this section, avoiding mathematical problems which are deferred until chapter 10.

We turn now to the only proposition of this section.

Proposition 3.15 (String Loop Schrödinger Equation).

The String Wave Functional $\Psi[C;A]$ can be obtained by solving the loop Schrödinger equation

$$-\frac{1}{4m^2} \left(\oint_{\Gamma \approx \mathbb{S}^1} dl(s) \right)^{-1} \oint_{\Gamma \approx \mathbb{S}^1} dl(s) \frac{\delta^2 \Psi\left[C; A\right]}{\delta Y^{\mu\nu}\left(s\right) \delta Y_{\mu\nu}\left(s\right)} = i \frac{\partial \Psi\left[C; A\right]}{\partial A} \quad . \tag{3.63}$$

Proof:

To get this equation we take equation 3.37 multiply both sides by the initial loop wave functional $\Psi[C_0]$ and functionally integrate over $[\mathcal{D}C_0]$:

$$\int \left[\mathcal{D}C_{0}\right] \frac{\hbar^{2}}{2m^{2}} \left(\oint_{\Gamma} dl(s)\right)^{-1} \oint_{\Gamma} ds \sqrt{\left(\mathbf{Y}'\left(s\right)\right)^{2}} \frac{\delta^{2} K\left[Y^{\mu}, Y_{0}^{\mu}; A\right]}{\delta Y^{\mu\nu}\left(s\right) \delta Y_{\mu\nu}\left(s\right)} \Psi\left[C_{0}\right] =$$

$$= \int \left[\mathcal{D}C_{0}\right] i\hbar \frac{\partial K\left[Y^{\mu}, Y_{0}^{\mu}; A\right]}{\partial A} \Psi\left[C_{0}\right] \quad . \tag{3.64}$$

This equation can be written also as

$$\frac{\hbar^{2}}{2m^{2}} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{\left(Y'(s) \right)^{2}} \frac{\delta^{2}}{\delta Y^{\mu\nu}(s)} \frac{\delta^{2}}{\delta Y^{\mu\nu}(s)} \left[\int \left[\mathcal{D}C_{0} \right] K \left[Y^{\mu}, Y_{0}^{\mu}; A \right] \Psi \left[C_{0} \right] \right] = \int \left[\mathcal{D}C_{0} \right] i\hbar \frac{\partial}{\partial A} \Psi \left[C_{0} \right] \left[\int \left[\mathcal{D}C_{0} \right] K \left[Y^{\mu}, Y_{0}^{\mu}; A \right] \Psi \left[C_{0} \right] \right] , \tag{3.65}$$

where we recognize on both sides that $\Psi[C;A]$ appears thanks to equation (3.60); thus we get the desired result.

We can easily see that Equation (3.63) is the quantum transcription, through the Correspondence Principle,

$$H \rightarrow -i\frac{\partial}{\partial A} \tag{3.66}$$

$$Q_{\mu\nu}(s) \rightarrow i \frac{\delta}{\delta Y^{\mu\nu}(s)}$$
 (3.67)

of the classical relation between the Area Hamiltonian H and the loop momentum density $Q_{\mu\nu}(s)$ [4]:

$$H[C] = \left(\oint_{\Gamma \approx \mathbb{S}^{1}} dl(s) \right)^{-1} \oint_{\Gamma \approx \mathbb{S}^{1}} ds \sqrt{\left(\mathbf{Y}'\right)^{2}} \mathcal{H}_{Schild}.$$

$$= \frac{1}{4m^{2}} \left(\oint_{\Gamma \approx \mathbb{S}^{1}} dl(s) \right)^{-1} \oint_{\Gamma \approx \mathbb{S}^{1}} ds \sqrt{\left(\mathbf{Y}'\right)^{2}} Q_{\mu\nu}(s) Q^{\mu\nu}(s) \quad . \tag{3.68}$$

Once again, we note the analogy between equation (3.68) and the familiar energy momentum relation for a point particle, $H = p^2/(2m)$. We shall comment on the "non-relativistic" form of equation (3.68) at the end of this section. Presently, we limit ourselves to note that the difference between the point–particle case and the *String* case, stems from the spatial extension of the loop, and is reflected in equation (3.68) by the averaging integral of the momentum squared along the loop itself. Equation (3.68) represents the total loop energy instead of the energy of a single constituent *String* bit. Just as the particle linear momentum gives the direction along which a particle moves and the rate of position change, so the loop momentum describes the deformation in the loop shape and the rate of shape change. The corresponding Hamiltonian describes the energy variation as the loop area varies, irrespective of the actual point along the loop where the deformation takes place.

Accordingly, the Hamiltonian (3.66) represents the generating operator of the loop area variations, and the momentum density (3.67) represents the generator of the deformations in the loop shape at the point $Y^{\mu}(s)$.

Note that we can define (in the same way as we did in equation (3.68), passing from the Hamiltonian density \mathcal{H} to the integrated quantity H) an integrated $Area\ Loop\ Momentum$

$$Q^{\mu\nu}\left[C\right] = i\left(\oint_{\Gamma \approx \mathbb{S}^1} dl(s)\right)^{-1} \oint_{\Gamma \approx \mathbb{S}^1} ds \sqrt{\left(\mathbf{Y}'\right)^2} Q^{\mu\nu}\left(s\right) \tag{3.69}$$

and through the correspondence (3.67) we obtain the

Definition 3.3 (Total Area Loop Momentum Operator).

The <u>Total Area Loop Momentum Operator</u> obtained according to equations (3.69) and (3.67) is

$$\mathfrak{Q}^{\mu\nu}\left[C\right] = i\left(\oint_{\Gamma \approx \mathbb{S}^1} dl(s)\right)^{-1} \oint_{\Gamma \approx \mathbb{S}^1} ds \sqrt{\left(\mathbf{Y}'\right)^2} \frac{\delta}{\delta Y^{\mu\nu}\left(s\right)} \quad . \tag{3.70}$$

Note that we use the name of the image of the loop in Target Space, i.e. C, to label the Total Area Loop Momentum Operator as well as the Total Area Loop Momentum.

From the above discussion, we are led to deduce that:

1. deformations may occur randomly at any point on the loop;

2. the antisymmetry in the indices μ , ν guarantees that $Q_{\mu\nu}(s) Y'^{\mu}(s)$ generates only physical deformations, i.e. deformations which orthogonal to the loop itself

$$Q_{\mu\nu}(s) Y^{\prime\mu}(s) Y^{\prime\nu}(s) \equiv 0 \quad ;$$

- 3. a shape changes cost energy because of the *String* tension and the addition of a small loop, or "petal", increases the total length of the *String*;
- 4. the energy balance condition is provided by equation (3.68) at the classical level and by equation (3.63) at the quantum level; in both cases the global energy variation per unit proper area is obtained by a loop average of the double deformation at single point.

The above deductions represent the distinctive features of the String quantum shape—shifting phenomenon. Moreover it should be clear now how the described quantization program is essentially a sort of quantum mechanics formulated in a space of String loops, i.e., a space in which each point represents a possible geometrical configuration of a closed String. We also stress again that all the procedure takes fully advantage from the enlargement of the canonical phase space that we obtain promoting the original World—Sheet coordinates ξ to the role of $dynamical\ variables^8$ and and introducing π_{AB} , the momentum conjugate to ξ^A into the Hamiltonian form of the action⁹. This enlargement endows the Schild action with the full reparametrization invariance under the transformation $\sigma^a \longrightarrow \tilde{\sigma}^a(\sigma)$, while preserving the polynomial structure in the dynamical variables, which is a necessary condition to solve the path—integral. In this way we were able to derive without approximations expressions (3.13) and (3.48), which show the explicit relation between the fixed area string propagator $K[Y,Y_0;A]$, and the fixed "energy" string propagator $G[C,C_0;E]$ without recourse to any $ad\ hoc\ averaging\ prescription\ in order to eliminate the <math>A$ parameter dependence. Moreover result (3.20) allows us to establish the following facts:

1. our approach, which directly stems from Eguchi's proposal, corresponds to quantizing

⁸They now represent fields $\boldsymbol{\xi}(\boldsymbol{\sigma})$ defined over the String manifold $\boldsymbol{\sigma}$.

⁹Please, refer to table 3.1 for a list of all the relevant dynamical quantities in loop space.

 $^{^{10}\}mathrm{Or}$ Green Function

$H[C] = (4m^2l_C)^{-1} \oint_C dl(s) Q_{\mu\nu}(s) Q^{\mu\nu}(s)$	(Schild) Loop Hamiltonian
$\mathcal{H}_{\text{Schild}}(s) = \left(4m^2 l_C\right)^{-1} Q_{\mu\nu}(s) Q^{\mu\nu}(s)$	(Schild) String Hamiltonian
$dl(s) \equiv \sqrt{\left(\mathbf{Y}'(s)\right)^2} Y'^{\mu}, Y' \equiv \frac{dY^{\mu}(s)}{ds}$	Loop Invariant Measure
$l_{C} \stackrel{\text{def.}}{=} \oint_{C} dl \left(s \right)$	Loop Proper Length
$Q_{\mu\nu}(s) = m^{2} \epsilon^{mn} \partial_{m} X_{\mu}(\boldsymbol{\sigma}) \partial_{n} X_{\nu}(\boldsymbol{\sigma}) \big _{\boldsymbol{\sigma} = \boldsymbol{\sigma}(s)}$	Area Momentum Density
$Q_{\mu\nu}\left[C\right] \equiv l_C^{-1} \oint_C dl\left(s\right) Q_{\mu\nu}\left(s\right)$	Loop Momentum

Table 3.1: Loop Space functionals and *Boundary* fields.

a *String* by keeping fixed the *area* of the *String Parameter Space* in the path–integral, and then taking the average over the *String* tension values;

- the Nambu–Goto approach, on the other hand, corresponds to quantizing a String by keeping fixed the String tension and then taking the average over the Parameter Space Areas;
- 3. the two quantization schemes are equivalent in the saddle point approximation 11.

In particular we obtain through equations (3.13), (3.53) a non-perturbative definition of the Nambu-Goto propagator (3.20).

¹¹This is the content of proposition 3.4.

Physical Quantity	Particle –	\rightarrow String
<u>Object</u>	Massive Point-Particle	Non-Vanishing
		String Tension
Mathematical Model	Point P	SpaceLike Loop C
	in \mathbb{R}^D	in \mathbb{R}^{D+1}
Topological Meaning	Boundary (=Endpoint)	Boundary
	of a Line	of an Open Surface
<u>Coordinates</u>	$\left\{x^1, x^2, x^3\right\}$	Area Element
		$Y^{\mu\nu}\left[C\right]=\oint_C Y^\mu dY^\nu$
<u>Trajectory</u>	1-parameter family	1-parameter family
	of points $\{x(t)\}$	of loops $\{Y^{\mu}(s;A)\}$
Evolution Parameter	"Time" t	Area A of the String
		Parameter Space
<u>Translations Generators</u>	Spatial Shifts:	Shape Deformations:
	$rac{\partial}{\partial x^i}$	$\frac{\delta}{\delta Y^{\mu\nu}(s)}$
Evolution Generator	Time Shifts:	Proper Area Variations:
	$rac{\partial}{\partial t}$	$rac{\partial}{\partial A}$
<u>Topological Dimension</u>	Particle Trajectory	String Trajectory
	D = 1	D=2
<u>Distance</u>	$\left(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_0\right)^2$	$(Y^{\mu\nu} [C] - Y^{\mu\nu} [C_0])^2$
<u>Linear Momentum</u>	Rate of Change of	Rate of Change of
	Spatial Position	String Shape
<u>Hamiltonian</u>	Time Conjugate	Area Conjugate
	Canonical Variable	Canonical Variable

Table 3.2: The Particle/String "Dictionary".

Chapter 4

Generalization: p-branes

\$\mathcal{O}k\$, "it's good to see you" at work again.

4.1 Reparametrized Schild Action

The reparametrized formulation we presented in section (2.4) can be formulated also for higher dimensional objects without stumble into more troubles. Hence, we start extending some definitions to the case of a p-dimensional, relativistic object in D-dimensional spacetime:

Definition 4.1 (p-brane Parameter Space).

The <u>Parameter Space</u> of a p-brane is a compact connected (p+1)-dimensional domain $\Xi^{(p+1)} \subset \mathbb{M}^{(p+1)}$, coordinatized by the (p+1)-ple of variables $\boldsymbol{\xi} = (\xi^0, \dots, \xi^p)$.

Definition 4.2 (p-brane Boundary Space).

The <u>Boundary Space</u> of a p-brane is the Boundary $\mathbf{B} \stackrel{\text{def.}}{=} \partial \Xi^{(p+1)}$ of the Parameter Space: it is parametrized by the p-ple $\mathbf{s} = (s^1, \dots, s^p)$; we remember that, since \mathbf{B} is already a Boundary, it has no Boundary

$$\partial \boldsymbol{B} = \emptyset$$
 .

In first instance, we suppose that our model is defined by the following (p+1)-form

$$\boldsymbol{\omega}^{(p+1)} = \mathcal{L}(X^{\mu}, X^{\mu}_{.i}; \xi^{i}) \, d\boldsymbol{\xi}^{0} \wedge \ldots \wedge d\boldsymbol{\xi}^{p}$$
 .

We will call $X^{\mu} \stackrel{\text{def.}}{=} X^{\mu}(\xi^{i}) = X^{\mu}(\xi)$ the fields on the domain $\Xi^{(p+1)}$: these are the World-HyperTube Embedding Functions. Then, we will use the following definition

Definition 4.3 (World-HyperTube and p-brane).

The image of the p-brane Parameter Space in the Target Space¹ \mathbb{T} is the <u>World-HyperTube</u>, $\mathcal{W}^{(p+1)}$, of the p-brane:

$$\mathcal{W}^{(p+1)} = X^{\mu} \Big(\Xi^{(p+1)} \Big)$$

The <u>p-brane</u>, $\mathbf{D}^{(p)}$, is the only Boundary of the World-HyperTube if this last one is simply connected with a connected Boundary. If the Boundary is composed by more than one connected component, then we are in a multiple p-brane configuration. In any case the p-brane is a manifold without Boundary. We will call p-brane also the Boundary Space \mathbf{B} , by extension, in view of the following equalities:

$$X^{\mu}(\mathbf{B}) = \mathbf{D}^{(p)} \stackrel{\text{def.}}{=} \partial \mathcal{W}^{(p+1)} = \partial X^{\mu} \Big(\Xi^{(p+1)} \Big)$$
.

On the p-brane, i.e. on the Boundary B, the World-HyperTube Embedding Functions (they are our fields) reduce to

$$Y^{\mu}(s) \stackrel{\text{def.}}{=} X^{\mu}(\xi^{i}(s))$$
 ,

which are the <u>p-brane Embedding Functions</u>; we will always call with different letters fields on the Bulk and their values on the Boundary, as we already did in the String case, to avoid confusion.

It is possible to generalize also the definitions 2.3 and 2.6 of Holographic Coordinates and Area Velocity, but first we have to generalize the Linear Velocity Vector of a String, C. For a p-brane we will use the totally antisymmetric p-dimensional volume element at the Boundary of the (p+1)-dimensional World-HyperTube.

¹This is the space, \mathbb{M}^D , where the p-brane embedding functions take their values.

Definition 4.4 (p-brane Tangent Element).

The <u>Tangent Element</u> to the p-brane World-HyperTube is the totally antisymmetric tensor

$$Y'^{\mu_1\dots\mu_p} = \epsilon^{A_1\dots A_p} \frac{\partial Y^{\mu_1}}{\partial \xi^{A_1}} \dots \frac{\partial Y^{\mu_p}}{\partial \xi^{A_p}} \quad . \tag{4.1}$$

Now we can pass directly to the Holographic Coordinates.

Definition 4.5 (p-brane Holographic Coordinates).

Let **B** be a p-brane, and $Y^{\mu}(s)$ one of its possible Parametrizations.

The <u>Holographic Coordinates</u> of the p-brane are

$$Y^{\mu_0\dots\mu_p} \stackrel{\text{def.}}{=} \oint_{\mathcal{B}} d^p s \, Y^{\mu_0}(s) \, Y'^{\mu_1\dots\mu_p}(s)$$

$$= \oint_{\mathcal{D}^{(p)} = \partial \mathcal{W}^{(p+1)}} Y^{\mu_0} dY^{\mu_1} \wedge \dots \wedge dY^{\mu_p}$$

$$= \int_{\mathcal{W}^{(p+1)}} dY^{\mu_0} \wedge \dots \wedge dY^{\mu_p} , \qquad (4.2)$$

where $Y'^{\mu_1...\mu_p}$ is the p-dimensional Tangent Element to the p-brane of definition (4.4).

We note incidentally that in this case, even if we are using the same notation as in the *String* case, the "'" symbol has a more complex meaning than the one assumed in notation 1.1.

Definition 4.6 (Local World-HyperTube Volume Velocity).

Let $X^{\mu}(\xi^{j})$, j = 0, ..., p be a Parametrization of the World-HyperTube of a p-brane. The <u>Local Volume Velocity</u> of the World-HyperTube is

$$\dot{X}^{\mu_0\dots\mu_p}(\boldsymbol{\xi}) \stackrel{\text{def.}}{=} \epsilon^{a_0\dots a_p} \frac{X^{\mu_0}(\boldsymbol{\xi})}{\partial \xi^{a_0}} \cdot \dots \cdot \frac{X^{\mu_p}(\boldsymbol{\xi})}{\partial \xi^{a_p}} .$$

Definition 4.7 (p-brane Volume Velocity).

Let $\dot{X}^{\mu_0...\mu_p}(\boldsymbol{\xi})$ be the Local Volume Velocity of the World-HyperTube $\mathcal{W}^{(p+1)}$ associated with a p-brane defined on

$$\mathbf{B} = \partial \Xi^{(p+1)}$$
:

the <u>p-brane Volume Velocity</u> is the Local Volume Velocity of the World-HyperTube $W^{(p+1)}$ computed on the Boundary:

$$\dot{X}^{\mu_0\dots\mu_p}(s) \stackrel{\text{def.}}{=} \epsilon^{a_0\dots a_p} \frac{X^{\mu_0}(\boldsymbol{\xi})}{\partial \xi^{a_0}} \cdot \dots \cdot \frac{X^{\mu_p}(\boldsymbol{\xi})}{\partial \xi^{a_p}} \bigg|_{\boldsymbol{\xi} = \boldsymbol{\xi}(s)}$$

To have a clearer distinction between *Bulk* and *Boundary* quantities we are going to adopt the following convention.

Notation 4.1 (p-brane Volume Velocity).

We will denote the p-brane Volume Velocity with the following symbol:

$$\dot{Y}^{\mu_0...\mu_p}(s) \stackrel{ ext{def.}}{=} \dot{X}^{\mu_0...\mu_p}(oldsymbol{\sigma})igg|_{oldsymbol{\sigma} = oldsymbol{\sigma}(s)}$$

Now, we would like to set up a reparametrization invariant formalism for a p-brane, but always with a Schild-type action. This can be done as in the String case: the Reparametrized Schild Lagrangian Density for a p-brane is the natural generalization of the Lagrangian (2.28). Of course all the procedures we developed in chapter 2 to motivate this choice are still valid for higher dimensional objects. Again, we promote the $\boldsymbol{\xi}$ fields to the role of dynamical variables. They explicitly take into account possible variations of the Boundary of the Parameter Space, the Boundary Space \boldsymbol{B} . From now on, all the fields, the new $\boldsymbol{\xi}$ as well as the \boldsymbol{X} ones, are defined on a (p+1)-dimensional manifold that we will call $\boldsymbol{\Sigma}^{(p+1)}$, coordinatized by (p+1)-variables, that in all possible cases we will call $\boldsymbol{\sigma} = (\sigma^0, \dots, \sigma^p)$. All the quantities defined so far on $\boldsymbol{\Xi}^{(p+1)}$ are to be interpreted as defined on $\boldsymbol{\Sigma}^{(p+1)}$, since this is now the new name for the Parameter Space. Then, we can give the following

Definition 4.8 (p-brane Reparametrized Schild Lagrangian Density).

The <u>Reparametrized Schild Lagrangian Density</u> for a p-brane is

$$\mathcal{L}_{\text{Schild}}^{(p)\text{rep.}} = \mathcal{L}^{(p)} = \frac{\rho_p}{2(p+1)!} \frac{\dot{X}^{\mu_0\dots\mu_p}(\boldsymbol{\sigma}) \, \dot{X}_{\mu_0\dots\mu_p}(\boldsymbol{\sigma})}{\epsilon^{A_0\dots A_p} \dot{\xi}_{A_0\dots A_p}(\boldsymbol{\sigma})} \quad . \tag{4.3}$$

Again, we adopt the convention to use uppercase latin indices for the new $\xi^A(\sigma^i)$ fields, and lowercase latin indices for the Parameter Space variables σ^a . Moreover, the quantity $\dot{\xi}$ is

the natural generalization of the same quantity already defined for the *String* (and also for the $X(\sigma^a)$ fields):

$$\dot{\xi}^{A_0...A_p}(oldsymbol{s}) \stackrel{\mathrm{def.}}{=} \epsilon^{a_0...a_p} rac{\partial \xi^{A_0}(oldsymbol{\sigma})}{\partial \sigma_{a_0}} \cdot \ldots \cdot rac{\partial \xi^{A_p}(oldsymbol{\sigma})}{\partial \sigma^{a_p}} \quad .$$

Now, we can proceed to the generalization of the Schild action in Hamiltonian form. As we already did in section (2.3) on page 37 for the *String*, we firstly give the definition of momenta:

Definition 4.9 (p-brane Bulk Volume Momentum).

The <u>p-brane Bulk Volume Momentum</u> or <u>p-brane World-HyperTube Volume Momentum</u> is the momentum canonically conjugated to the Local World-HyperTube Volume Velocity:

$$P_{\mu_0...\mu_p} = \frac{\partial \mathcal{L}^p}{\partial \dot{X}^{\mu_0...\mu_p}(\boldsymbol{\sigma})} \quad .$$

Again, we can extend, or better restrict (!!!), this definition from the Bulk to the Boundary.

Definition 4.10 (p-brane Boundary Volume Momentum).

The <u>p-brane Boundary Volume Momentum</u> or <u>p-brane Volume Momentum</u> is the p-brane Bulk Volume Momentum computed on the Boundary, i.e.

$$Q_{\mu_0...\mu_p}(s) \stackrel{\text{def.}}{=} \frac{\partial \mathcal{L}^p}{\partial \dot{X}^{\mu_0...\mu_p}(\boldsymbol{\sigma})} \bigg|_{\boldsymbol{\sigma} = \boldsymbol{\sigma}(s)} = P_{\mu_0...\mu_p}(\boldsymbol{\sigma}) \big|_{\boldsymbol{\sigma} = \boldsymbol{\sigma}(s)} \quad ,$$

where, $\sigma = \sigma(s)$ is a parametrization of the Boundary $B = \partial \Sigma^{(p+1)}$ of the domain $\Sigma^{(p+1)}$.

Then, we can express the reparametrized p-brane action as

Definition 4.11 (Restricted Reparametrized *p*-brane Action).

The <u>Restricted Reparametrized Action</u> for a p-brane is

$$S\left[\boldsymbol{X}(\boldsymbol{\sigma}), \boldsymbol{P}(\boldsymbol{\sigma}), \boldsymbol{\xi}(\boldsymbol{\sigma})\right] =$$

$$= \frac{1}{(p+1)!} \int_{\mathcal{W}^{(p+1)}} P_{\mu_0 \dots \mu_p} d\boldsymbol{X}^{\mu_0} \wedge \dots \wedge d\boldsymbol{X}^{\mu_p} +$$

$$- \frac{1}{(p+1)!} \epsilon_{A_0 \dots A_p} \int_{\Xi} d\boldsymbol{\xi}^{A_0} \wedge \dots \wedge d\boldsymbol{\xi}^{A_p} \mathcal{H}(\boldsymbol{P}) \quad , \tag{4.4}$$

i.e. the reparametrization of the Hamiltonian form of the Schild Action.

The Hamiltonian $\mathcal{H}(\mathbf{P})$ is given by²

$$\mathcal{H}(\mathbf{P}) = \frac{1}{2\rho_p (p+1)!} P_{\beta_0 \dots \beta_p} P^{\beta_0 \dots \beta_p}$$

$$\tag{4.5}$$

and the complete derivation of this result is given in appendix A.1. The Schild Hamiltonian, in the reparametrized formulation of the model, is proportional to the momenta conjugated to the $\boldsymbol{\xi}$ fields; we stress again, that this is due to the fact that in the reparametrized formulation the modifications of the *Boundary* are encoded in the variations of the new $\boldsymbol{\xi}$ fields. For the sake of completeness, we write the expressions for the momenta, which are

$$P_{\mu_0...\mu_p} = \frac{\rho_p \dot{X}_{\mu_0...\mu_p}}{\epsilon_{A_0...A_p} \dot{\xi}^{A_0...A_p}}$$

$$\pi^{A_0...A_p} = \epsilon^{A_0...A_p} \mathcal{H}(\mathbf{P}) . \tag{4.6}$$

In terms of a Lagrange multiplier $N^{B_0...B_p}$ the action (4.4) can also be written in terms of the (p+1)-form

$$\Omega^{(p+1)} = \frac{1}{(p+1)!} P_{\mu_0 \dots \mu_p} dX^{\mu_0} \wedge \dots \wedge dX^{\mu_p} + \frac{1}{(p+1)!} \pi_{A_0 \dots A_p} d\xi^{A_0} \wedge \dots \wedge d\xi^{A_p} + \frac{1}{(p+1)!} N^{A_0 \dots A_p} \left[\pi_{A_0 \dots A_p} - \epsilon_{A_0 \dots A_p} \mathcal{H}(P) \right] ds^0 \wedge \dots \wedge ds^p ,$$
(4.7)

which is the Hamiltonian form of the reparametrized Lagrangian (4.3): here the momentum (4.6) enters explicitly and, again, the relation between the momenta P and π is enforced as a constraint by the Lagrange multiplier N.

4.2 p-brane Hamilton Jacobi Theory

Having lifted the p+1 coordinates ξ to the role of dynamical variables, we achieved a reparametrization invariant formulation to start from, in order to derive the functional Hamilton–Jacobi equation. The procedure is quite similar to the one we already used in the

²We set $\rho_p = m^{p+1}$: then in the particle case, p = 0, $\rho_1 = m$, the prefactor becomes correctly 1/(2m); for the String p = 1, $\rho_1 = m^2$ give $1/(4m^2)$, and so on.

String case. By noting that the embedding functions X, as well as the ξ fields, are now functions of the σ 's, we could derive the equation of motion for the p-brane. However, we prefer to restrict our attention to the most important of them, which is the one obtained by variation of the action with respect to the new ξ fields. We emphasize the relevance of this result in performing the following computation.

Proposition 4.1 (p-brane Energy Balance Equation).

The variation of the action (4.4) with respect to the $\xi(\sigma)$ fields gives the following equation:

$$\frac{1}{p!} \epsilon_{A_0...A_p} \epsilon^{m_1...m_p m} \left(\partial_{m_1} \xi^{A_1} \right) \cdot ... \cdot \left(\partial_{m_p} \xi^{A_p} \right) \partial_m \mathcal{H}(\mathbf{P}) = 0 \quad .$$

Proof:

See appendix A.2 for details.

The result has of course the same meaning as in the *String* case: it is the energy-balance equations, which again states that the Hamiltonian is constant along a classical solution. We can now perform a variation corresponding to a deformation of the *future Boundary* of the *World-HyperTube* in the functional class of fields that solve the equation of motion for the *Bulk*. Before that, let us give some definitions.

Definition 4.12 (p-brane Projected Boundary Area Momentum).

The <u>Projected Boundary Area Momentum</u> of the p-brane is the Boundary Area Momentum projected in the "direction" of the Tangent Element to the p-brane World-HyperTube, i.e.

$$q_{\mu}(\mathbf{s}) = Q_{\mu\mu_1...\mu_p}(\mathbf{s}) Y'^{\mu_1...\mu_p}(\mathbf{s})$$
 (4.8)

As a natural generalization for the area of the Parameter Space of the String we now take the (p+1)-dimensional HyperVolume of the World-HyperTube of the p-brane, V:

Definition 4.13 (p-brane HyperVolume).

The <u>Hypervolume</u> of a p-brane,

$$V \stackrel{\text{def.}}{=} \frac{1}{(p+1)!} \epsilon_{A_0 \dots A_p} \int_{\Xi(\boldsymbol{\sigma})} d\boldsymbol{\xi}^{A_0} \wedge \dots \wedge d\boldsymbol{\xi}^{A_p} \quad , \tag{4.9}$$

is the volume associated with its World-HyperTube.

Now, we are ready for the following

Proposition 4.2 (Boundary Variation of the p-brane Action).

The variation of the action (4.4) among the functional class of field configurations satisfying the classical equation of motion for the Bulk is

$$\delta S = \frac{1}{p!} \int_{\mathbf{D}^{(p)}} Q_{\mu_0 \dots \mu_p} d\mathbf{Y}^{\mu_1} \wedge \dots \wedge d\mathbf{Y}^{\mu_p} \delta Y_{(f)}^{\mu_0} - (\delta V) \mathcal{H}$$

$$= \frac{1}{p!} \int_{\mathbf{B}} d^p \mathbf{s} \, q_{\mu}(\mathbf{s}) \, \delta Y_{(f)}^{\mu}(\mathbf{s}) - (\delta V) \mathcal{H} , \qquad (4.10)$$

where, $\delta Y^{\mu}_{(f)}$ indicates that we performed a variation of the embedding functions of the final closed p-brane, which is the future Boundary of the considered World-HyperTube.

Proof:

The detailed computation is performed in appendix A.4.

From equation (4.10) in proposition 4.2 we can see that

$$\frac{\delta S}{\delta Y^{\mu}_{(f)}(s)} = q_{\mu}(s) \tag{4.11}$$

$$\mathcal{H} \equiv E = -\frac{\delta S}{\delta V} = -\frac{\partial S}{\partial V} \quad . \tag{4.12}$$

Moreover, the dispersion relation for the p-brane has exactly the same form of the one we already derived for the String (compare with equation (2.60) on page 51)

$$\frac{1}{2\rho_{p}p!}q_{\mu}q^{\mu} = \frac{1}{2\rho_{p}(p+1)!}Q_{\mu\mu_{1}...\mu_{p}}Q^{\mu\mu_{1}...\mu_{p}}Y'_{\nu_{1}...\nu_{p}}Y'^{\nu_{1}...\nu_{p}}$$

$$= \frac{(\mathbf{Y}')^{2}Q_{\mu_{0}...\mu_{p}}Q^{\mu_{0}...\mu_{p}}}{2\rho_{p}(p+1)!} = (\mathbf{Y}')^{2}E . \tag{4.13}$$

We can of course rewrite it in integrated form to regain reparametrization invariance,

$$\frac{1}{2\rho_p p!} \oint_{\boldsymbol{B}} \frac{d^p s}{\sqrt{(\boldsymbol{Y}')^2}} p_{\mu} p^{\mu} = E \oint_{\boldsymbol{B}} d^p s \sqrt{(\boldsymbol{Y}')^2} \quad . \tag{4.14}$$

Then, we obtain the following result.

Proposition 4.3 (p-brane Hamilton-Jacobi Equation).

The Classical Dynamics of a p-brane is regulated by the following equation

$$\frac{1}{2\rho_p p!} \left(\oint_{\mathbf{B}} d^p \mathbf{s} \sqrt{(\mathbf{Y}')^2} \right)^{-1} \oint_{\mathbf{B}} \frac{d^p \mathbf{s}}{\sqrt{(\mathbf{Y}')^2}} \frac{\delta S}{\delta Y^{\mu}(\mathbf{s})} \frac{\delta S}{\delta Y_{\mu}(\mathbf{s})} = -\frac{\partial S}{\partial V} \quad , \tag{4.15}$$

which is nothing but the functional Hamilton-Jacobi equation for the p-brane.

Proof:

The hard work has already been done. Thus, we can start with equation (4.14) and use the correspondences (4.11) and (4.12) to obtain the functional Hamilton-Jacobi equation for the *p*-brane (4.14).

Note how in our formulation the equation has the same form as in the *String* case!!! The only difference is the larger number of indices carried by the *Holographic Derivative*. Anyway, the physical interpretation is the same: we have a *Boundary Dynamics* of *HyperVolume Elements* (which are the *Holographic Coordinates*) as *shadows* of a *Bulk Dynamics* in one more dimension. We note that equation (4.15) can be expressed in terms of the *p-brane Holographic Coordinates* as

$$\frac{1}{2\rho_p} \left(\oint_{\mathbf{B}} d^p \mathbf{s} \sqrt{(\mathbf{Y}')^2} \right)^{-1} \oint_{\mathbf{B}} d^p \mathbf{s} \sqrt{(\mathbf{Y}')^2} \frac{\delta S}{\delta Y^{\mu_0 \dots \mu_p}(\mathbf{s})} \frac{\delta S}{\delta Y_{\mu_0 \dots \mu_p}(\mathbf{s})} = -\frac{\partial S}{\partial V} \quad . \tag{4.16}$$

4.3 p-brane Quantum Dynamics

4.3.1 Equivalence with Nambu–Goto Dynamics

Having obtained the Functional Hamilton–Jacobi equation for the p-brane, we can now turn to the problem of deriving its propagation kernel. As we have already seen in section 3.1,

a convenient starting point is the action for the Reparametrization invariant Hamiltonian Theory with the Lagrange multiplier N, which is (4.8),

$$S\left[\boldsymbol{X}(\boldsymbol{\sigma}), \boldsymbol{P}(\boldsymbol{\sigma}), \boldsymbol{\xi}(\boldsymbol{\sigma}), \boldsymbol{N}(\boldsymbol{\sigma}); V\right] =$$

$$= \frac{1}{(p+1)!} \int_{\mathcal{W}^{(p+1)}} P_{\mu_0 \dots \mu_p} d\boldsymbol{Y}^{\mu_0} \wedge \dots \wedge d\boldsymbol{Y}^{\mu_p} +$$

$$+ \frac{1}{(p+1)!} \int_{\Xi^{(p+1)}} \pi_{A_0 \dots A_p} d\boldsymbol{\xi}^{A_0} \wedge \dots \wedge d\boldsymbol{\xi}^{A_p} +$$

$$- \frac{1}{(p+1)!} \int_{\Sigma^{(p+1)}} d^2 \boldsymbol{\sigma} N^{A_0 \dots A_p} \left[\pi_{A_0 \dots A_p} - \epsilon_{A_0 \dots A_p} H(\boldsymbol{P}) \right] .$$

$$(4.17)$$

An interesting observation about the Lagrange multiplier N can be reported at this stage:

Proposition 4.4 (Meaning of the Lagrange Multiplier N).

The Lagrange multiplier N is the Levi-Civita tensor in the ξ variables 3 .

Proof:

This result can be obtained varying the action (4.17) with respect to the π field momentum, the one conjugated to ξ . Then, we get the following result (which is explicitly derived in section A.3) for $N^{A_0...A_p}$,

$$N^{A_0 \dots A_p}(\boldsymbol{\sigma}) = \dot{\boldsymbol{\xi}}^{A_0 \dots A_p} = \epsilon^{m_0 \dots m_p} \left(\partial_{m_0} \boldsymbol{\xi}^{A_0} \right) \cdot \dots \cdot \left(\partial_{m_p} \boldsymbol{\xi}^{A_p} \right) \quad , \tag{4.18}$$

which is the desired result.

Now, we can find for the *p*-brane the result corresponding to proposition 3.4.

Proposition 4.5 (p-brane Schild-Nambu Goto Quantum Equivalence).

The quantum propagation of a reparametrized Schild p-brane in the saddle point approximation and with energy

$$E = \frac{\rho_p}{2}$$

is equivalent to the quantum propagation of a Nambu-Goto p-brane.

³Which now, we remember, are fields!

Proof:

To get the desired result we will compute the propagation kernel $K[Y(s), Y_0(s); V]$, starting from the Schild Reparametrized Formulation expressed in the action (4.17) by means of the following path-integral

$$K[\mathbf{Y}(\hat{\mathbf{s}}), \mathbf{Y}_{0}(\hat{\mathbf{s}}); V] = \int_{\mathbf{Y}_{0}(\mathbf{s})}^{\mathbf{Y}(\mathbf{s})} \int_{\zeta_{0}(\mathbf{s})}^{\zeta(\mathbf{s})} [\mathcal{D}\mathbf{X}(\boldsymbol{\sigma})] [\mathcal{D}\boldsymbol{\xi}(\boldsymbol{\sigma})] [\mathcal{D}\boldsymbol{P}(\boldsymbol{\sigma})] [\mathcal{D}\boldsymbol{\pi}(\boldsymbol{\sigma})] [\mathcal{D}\boldsymbol{N}(\boldsymbol{\sigma})] \cdot \\ \cdot \exp\left\{\frac{i}{\hbar} S[\mathbf{X}(\boldsymbol{\sigma}), \boldsymbol{P}(\boldsymbol{\sigma}), \boldsymbol{\xi}(\boldsymbol{\sigma}), \boldsymbol{N}(\boldsymbol{\sigma}); V]\right\} = \\ = \int_{\mathbf{Y}_{0}(\mathbf{s})}^{\mathbf{Y}(\mathbf{s})} \int_{\zeta_{0}(\mathbf{s})}^{\zeta(\mathbf{s})} [\mathcal{D}\mathbf{X}(\boldsymbol{\sigma})] [\mathcal{D}\boldsymbol{P}(\boldsymbol{\sigma})] [\mathcal{D}\boldsymbol{\pi}(\boldsymbol{\sigma})] [\mathcal{D}\boldsymbol{N}(\boldsymbol{\sigma})] \cdot \\ \cdot \exp\left\{\frac{i}{(p+1)!} \int_{\mathcal{W}^{(p+1)}} P_{\mu_{1} \dots \mu_{p+1}} d\boldsymbol{x}^{\mu_{1}} \wedge \dots \wedge d\boldsymbol{x}^{\mu_{p+1}} + \right. \\ \left. + \frac{i}{(p+1)!} \int_{\Xi^{(p+1)}} \pi_{A_{1} \dots A_{p+1}} d\boldsymbol{\xi}^{A_{1}} \wedge \dots \wedge d\boldsymbol{\xi}^{A_{p+1}} + \right. \\ \left. - \frac{i}{(p+1)!} \int_{\Sigma^{(p+1)}} d^{p+1} \boldsymbol{\sigma} N^{A_{1} \dots A_{p+1}} \left[\pi_{A_{1} \dots A_{p+1}} - \epsilon_{A_{1} \dots A_{p+1}} \mathcal{H}(\boldsymbol{P}) \right] \right\} (4.19)$$

through the following four steps:

1. integrate out the $\xi(\sigma)$ fields using the energy balance equation and obtaining a functional Dirac Delta, which meets the requirement that the momentum $\pi(\sigma)$ satisfies the classical equation of motion;

$$K[Y(s), Y_{0}(s); V] =$$

$$= \int_{Y_{0}(s)}^{Y(s)} [\mathcal{D}X(\sigma)][\mathcal{D}P(\sigma)][\mathcal{D}\pi(\sigma)][\mathcal{D}N(\sigma)] \cdot \\
\cdot \delta \left[\epsilon^{m_{1} \dots m_{p+1}} \left(\partial_{m_{p+1}} \pi_{a_{1} \dots a_{p+1}} \right) \cdot \left(\partial_{m_{1}} \xi^{a_{1}} \right) \cdot \dots \cdot \left(\partial_{m_{p}} \xi^{a_{p}} \right) \right] \cdot \\
\cdot \exp \left\{ \frac{i}{(p+1)!} \int_{\mathcal{W}^{(p+1)}} P_{\mu_{1} \dots \mu_{p+1}} dX^{\mu_{1}} \wedge \dots \wedge dX^{\mu_{p+1}} + \right. \\
+ \frac{i}{(p+1)!} \int_{\Xi^{(p+1)}} d \left(\xi^{a_{p+1}} \pi_{a_{1} \dots a_{p+1}} d\xi^{a_{1}} \wedge \dots \wedge d\xi^{a_{p}} \right) + \\
- \frac{1}{(p+1)!} \int_{\Sigma^{(p+1)}} d^{p+1} \sigma N^{a_{i} \dots a_{p+1}} \left[\pi_{a_{1} \dots a_{p+1}} - \epsilon_{a_{1} \dots a_{p+1}} \mathcal{H}(P) \right] \right\}$$
(4.20)

2. take advantage of this to integrate out the momentum $\pi(\sigma)$ also;

$$K[\boldsymbol{Y}(\boldsymbol{s}), \boldsymbol{Y}_{0}(\boldsymbol{s}); V] =$$

$$= \int_{0}^{\infty} dE e^{iEV} \int_{\boldsymbol{Y}_{0}(\boldsymbol{s})}^{\boldsymbol{Y}(\boldsymbol{s})} [\mathcal{D}\boldsymbol{X}(\boldsymbol{\sigma})] [\mathcal{D}\boldsymbol{P}(\boldsymbol{\sigma})] [\mathcal{D}\boldsymbol{N}(\boldsymbol{\sigma})] \cdot$$

$$\cdot \exp \left\{ \frac{i}{(p+1)!} \int_{\mathcal{W}^{(p+1)}} P_{\mu_{1} \dots \mu_{p+1}} d\boldsymbol{X}^{\mu_{1}} \wedge \dots \wedge d\boldsymbol{X}^{\mu_{p+1}} + \frac{i}{(p+1)!} \epsilon_{a_{1} \dots a_{p+1}} \int_{\Sigma^{(p+1)}} d^{p+1} \boldsymbol{\sigma} N^{a_{1} \dots a_{p+1}} [E - \mathcal{H}(\boldsymbol{P})] \right\} ; (4.21)$$

3. remembering now the form of the Hamiltonian $\mathcal{H}(P)$ and defining

$$N(\boldsymbol{\sigma}) = \frac{1}{(p+1)!} \epsilon_{A_1 \dots A_{p+1}} N^{A_1 \dots A_{p+1}} , \qquad (4.22)$$

we can cast the last expression into the following one,

$$K[\mathbf{Y}(\mathbf{s}), \mathbf{Y}_{0}(\mathbf{s}); V] =$$

$$= \int_{0}^{\infty} dE e^{iEV} \int_{\mathbf{Y}_{0}(\mathbf{s})}^{\mathbf{Y}(\mathbf{s})} [\mathcal{D}\mathbf{X}(\boldsymbol{\sigma})] [\mathcal{D}\mathbf{P}(\boldsymbol{\sigma})] [\mathcal{D}\mathbf{N}(\boldsymbol{\sigma})] \cdot$$

$$\cdot \exp \left\{ i \int_{\Sigma^{(p+1)}} d^{p+1} \boldsymbol{\sigma} EN \right\} \cdot$$

$$\cdot \exp \left\{ i \int_{\Sigma^{(p+1)}} d^{p+1} \boldsymbol{\sigma} \left[\frac{N}{2\rho_{p} (p+1)!} P_{\mu_{1} \dots \mu_{p+1}} P^{\mu_{1} \dots \mu_{p+1}} + P_{\mu_{1} \dots \mu_{p+1}} \frac{\dot{X}^{\mu_{1} \dots \mu_{p+1}}}{(p+1)!} \right] \right\} , \quad (4.23)$$

where, we recognize a functional gaussian integral in $P(\sigma)$ which gives

$$K[\mathbf{Y}(s), \mathbf{Y}_{0}(s); V] =$$

$$= \int_{0}^{\infty} dE e^{iEV} \int_{\mathbf{Y}_{0}(s)}^{\mathbf{Y}(s)} [\mathcal{D}\mathbf{X}(\boldsymbol{\sigma})] [\mathcal{D}\mathbf{N}(\boldsymbol{\sigma})] \cdot$$

$$\cdot \exp \left\{ -i \int_{\Sigma^{(p+1)}} d^{p+1} \boldsymbol{\sigma} \left[-\frac{\rho_{p}}{2N(p+1)!} \dot{X}_{\mu_{1} \dots \mu_{p+1}} \dot{X}^{\mu_{1} \dots \mu_{p+1}} + NE \right] \right\} \quad (4.24)$$

4. ... calculate the result around the saddle point $\bar{N}(\sigma) = \sqrt{-\rho_p \left(\dot{X}\right)^2 / \left[2E\left(p+1\right)!\right]}$ to get rid of the $N(\sigma)$ dependence in the result, obtaining

$$K[\mathbf{Y}(\mathbf{s}), \mathbf{Y}_{0}(\mathbf{s}); V] = \int_{0}^{\infty} dE e^{iEV} \int_{\mathbf{Y}_{0}(\mathbf{s})}^{\mathbf{Y}(\mathbf{s})} [\mathcal{D}\mathbf{X}(\boldsymbol{\sigma})] \cdot \exp \left\{ -i \left(2E\rho_{p}\right)^{\frac{1}{2}} \int_{\Sigma^{(p+1)}} d^{p+1}\boldsymbol{\sigma} \sqrt{-\frac{\rho_{p}}{(p+1)!} \left(\dot{\mathbf{X}}\right)^{2}} \right\} \quad . \quad (4.25)$$

Once the propagator is known, we get

Proposition 4.6 (p-brane Green Function).

The p-brane Green Function in the saddle point approximation is

$$G[\boldsymbol{B}, \boldsymbol{B}_0; E] = \int_{\boldsymbol{Y}_0(\boldsymbol{s})}^{\boldsymbol{Y}(\boldsymbol{s})} [\mathcal{D}\boldsymbol{X}(\boldsymbol{\sigma})] \exp \left\{ -i \left(2E\rho_p \right)^{\frac{1}{2}} \int_{\Sigma^{(p+1)}} d^{p+1} \boldsymbol{\sigma} \sqrt{-\frac{\rho_p}{(p+1)!} \left(\dot{\boldsymbol{X}} \right)^2} \right\}$$

Proof:

As a matter of fact, there is nothing to prove: thanks to the relation

$$K[\mathbf{Y}(s), \mathbf{Y}_0(s); V] = \int_0^\infty dE e^{iEV} G[\mathbf{B}, \mathbf{B}_0; E]$$
(4.26)

we have just to read the correct result in equation (4.25), which gives the desired result,

Functional Schrödinger Equation and p-brane 4.3.2

The next step is to find the functional wave equation for the kernel $K[Y(s), Y_0(s); V]$, a task which we will perform by means of a path-integral technique.

Proposition 4.7 (p-brane Variation of the Kernel).

The variation of the kernel under an infinitesimal variation of the fields is

$$\delta K[\boldsymbol{Y}(\boldsymbol{s}), \boldsymbol{Y}_{0}(\boldsymbol{s}); V] =$$

$$= \frac{i}{\hbar} \int_{\boldsymbol{Y}(\boldsymbol{s}_{0})}^{\boldsymbol{Y}(\boldsymbol{s})} \int_{\boldsymbol{\zeta}(\boldsymbol{s}_{0})}^{\boldsymbol{\zeta}(\boldsymbol{s})} [\mathcal{D}\mu(\boldsymbol{\sigma})] \delta S[\boldsymbol{X}, \boldsymbol{P}, \boldsymbol{\xi}, \boldsymbol{N}; V] \exp \left\{ \frac{i}{\hbar} S[\boldsymbol{X}, \boldsymbol{P}, \boldsymbol{\xi}, \boldsymbol{N}; V] \right\} , (4.27)$$

where, we collected all the functional integrations in $[\mathcal{D}\mu(\boldsymbol{\sigma})]$,

$$[\mathcal{D}\mu(\sigma)] = [\mathcal{D}X(\sigma)][\mathcal{D}\xi(\sigma)][\mathcal{D}P(\sigma)][\mathcal{D}\pi(\sigma)][\mathcal{D}N(\sigma)]$$
.

Proof:

We start from the expression for the Kernel in equation (4.19), noting that a variation only affects the exponential, i.e. the action in the exponent. The result then follows by the chain rule applied to that exponential, i.e. we use

$$\delta\left(e^{\frac{i}{\hbar}S}\right) = \frac{i}{\hbar}e^{\frac{i}{\hbar}S}(\delta S)$$

Now we remember that we again restrict the field variation within the family of classical solutions; this is exactly the same computation we performed in appendix A.4, from which we keep the result of formula (A.22):

$$\delta S_{\text{cl.}}[\boldsymbol{B}; V] = \frac{1}{p!} \int_{\partial \mathcal{W}^{(p+1)}} p_{\mu_1 \dots \mu_{p+1}} \boldsymbol{dY}^{\mu_2} \wedge \dots \wedge \boldsymbol{dY}^{\mu_{p+1}} \delta Y^{\mu_1} - E \delta V$$
(4.28)

where, we only substituted E for \mathcal{H} . The resulting equations for the kernel, derived in appendix A.9 are

$$\frac{\partial K\left[\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{B}_{0}(\boldsymbol{s}); V\right]}{\partial V} = -\frac{iE}{\hbar} K\left[\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{B}_{0}(\boldsymbol{s}); V\right]$$
(4.29)

$$\frac{\partial K\left[\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{B}_{0}(\boldsymbol{s}); V\right]}{\partial V} = -\frac{iE}{\hbar} K\left[\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{B}_{0}(\boldsymbol{s}); V\right] \qquad (4.29)$$

$$\frac{\delta K\left[\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{B}_{0}(\boldsymbol{s}); V\right]}{\delta Y^{\mu}(\boldsymbol{s})} = \frac{i}{\hbar} \int_{\boldsymbol{Y}_{0}(\boldsymbol{s})}^{\boldsymbol{Y}(\boldsymbol{s})} \int_{\boldsymbol{\zeta}_{0}(\boldsymbol{s})}^{\boldsymbol{\zeta}(\boldsymbol{s})} \left[\mathcal{D}\mu(\boldsymbol{\sigma})\right] P_{\mu\mu_{1}\dots\mu_{p}} Y'^{\mu_{1}\dots\mu_{p}} \exp\left(\frac{iS}{\hbar}\right) \qquad (4.30)$$

and we recall the comment which follows formula (A.24) about the definition of \mathbf{Y}' . By comparing the last two equations with the functional Jacobi equation for the p-brane (4.15), we obtain the functional wave equation for the kernel (see appendix A.10):

$$-\frac{\hbar^{2}}{2\rho_{p}(p)!} \left(\oint_{\mathbf{B}} d^{p} s \sqrt{(\mathbf{Y}')^{2}} \right)^{-1} \cdot \left(\oint_{\mathbf{B}} \frac{d^{p} s}{\sqrt{(\mathbf{Y}')^{2}}} \frac{\delta^{2} K \left[\mathbf{B}(\mathbf{s}), \mathbf{B}_{0}(\mathbf{s}); V \right]}{\delta Y^{\mu}(\mathbf{s}) \delta Y_{\mu}(\mathbf{s})} = i\hbar \frac{\partial K \left[\mathbf{B}(\mathbf{s}), \mathbf{B}_{0}(\mathbf{s}); V \right]}{\partial V} , \quad (4.31)$$

By Fourier transforming, we get

$$\left[-\frac{\hbar^2}{p!} \left(\oint_{\boldsymbol{B}} d^p \boldsymbol{s} \sqrt{\left(\boldsymbol{Y}' \right)^2} \right)^{-1} \oint_{\boldsymbol{B}} \frac{d^p \boldsymbol{s}}{\sqrt{Y'^2}} \frac{\delta^2}{\delta Y^{\mu}(\boldsymbol{s}) \delta Y_{\mu}(\boldsymbol{s})} + \rho_p^2 \right] G\left[\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{B}_0(\boldsymbol{s}); V \right] = -\delta \left[\boldsymbol{B} - \boldsymbol{B}_0 \right]$$

and we can identify $G[B(s), B_0(s); V]$ with the Green function for the *p*-brane.

4.3.3 p-brane Quantum Propagator

The kernel wave equation can be solved then through the following ansatz:

Ansatz (p-brane Quantum Kernel): The Quantum Kernel of a p-brane has the following functional dependence:

$$K[\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{B}_0(\boldsymbol{s}); V] = \mathcal{N}V^{\alpha} \exp\left\{\frac{i}{\hbar}I[\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{B}_0(\boldsymbol{s}); V]\right\},$$
 (4.32)

where \mathcal{N} is the normalization constant, and α a real number.

Proposition 4.8 (p-brane Amplitude and Phase Kernel Equations).

The exponent α and the phase $I[Y, Y_0; V]$ satisfy two equations, namely

$$2\rho_{p}(p)! \frac{\alpha}{V} = -\left(\oint_{\mathbf{B}} d^{p} \mathbf{s} \sqrt{\mathbf{Y}'^{2}}\right)^{-1} \oint_{\mathbf{B}} \frac{d^{p} \mathbf{s}}{\sqrt{Y'^{2}}} \frac{\delta^{2} I\left[\mathbf{B}, \mathbf{B}_{0}, V\right]}{\delta Y^{\mu}(\mathbf{s}) \delta Y_{\mu}(\mathbf{s})}$$
(4.33)
$$2\rho_{p}(p)! \frac{\partial I\left[\mathbf{B}, \mathbf{B}_{0}, V\right]}{\partial V} = -\left(\oint_{\mathbf{B}} d^{p} \mathbf{s} \sqrt{\left(\mathbf{Y}'\right)^{2}}\right)^{-1} \oint_{\mathbf{B}} \frac{d^{p} \mathbf{s}}{\sqrt{\left(\mathbf{Y}'\right)^{2}}} \cdot \frac{\delta I\left[\mathbf{B}, \mathbf{B}_{0}, V\right]}{\delta Y^{\mu}(\mathbf{s})} \frac{\delta I\left[\mathbf{B}, \mathbf{B}_{0}, V\right]}{\delta Y_{\mu}(\mathbf{s})} ,$$
(4.34)

in exact analogy with the String case (see proposition 3.9.)

Proof:

Following the same procedure as in chapter 3, we first insert the ansatz (4.32) into (4.31). The results for the second functional derivative, after the calculation of

$$\frac{\delta K}{\delta Y^{\mu}(t)} = \mathcal{N} \frac{i}{\hbar} V^{\alpha} e^{\frac{i}{\hbar}I} \frac{\delta I}{\delta Y^{\mu}(t)} \quad , \label{eq:delta-K}$$

is

$$\frac{\delta^2 K}{\delta Y^\mu(t)\,\delta Y_\mu(t)} = \mathcal{N} V^\alpha e^{\frac{i}{\hbar}I} \left[-\frac{1}{\hbar^2} \left(\frac{\delta I}{\delta Y^\mu(t)} \right)^2 + \frac{i}{\hbar} \frac{\delta^2 I}{\delta Y^\mu(t)\,\delta Y_\mu(t)} \right]$$

Moreover the first derivative with respect to V is

$$\frac{\partial K}{\partial V} = \mathcal{N}V^{\alpha}e^{\frac{i}{\hbar}I}\left(\frac{\alpha}{V} + \frac{i}{\hbar}\frac{\partial I}{\partial V}\right) \quad .$$

Substituting the last two equations in (4.31) and separating the real and imaginary part, we obtain the desired result.

Now, using the formulae in appendix A, the following result can be proved.

Proposition 4.9 (p-brane Kernel and Green Function).

The Propagation Kernel and the Green Function of a p-brane are given by

$$K[\mathbf{Y}(s), \mathbf{Y}_0(s); V] = \left(\frac{\rho_p}{2i\pi\hbar V}\right)^{(D-p)/2} \exp\left(\frac{i\rho_p}{4\hbar V} \Sigma^{\mu\nu} [\mathbf{B} - \mathbf{B}_0] \Sigma_{\mu\nu} [\mathbf{B} - \mathbf{B}_0]\right) \quad . \quad (4.35)$$

$$G[\boldsymbol{B}, \boldsymbol{B}_0; V] = \int_0^\infty dV e^{-i\rho_p V/2\hbar} \exp\left(\frac{i\rho_p}{4\hbar V} \Sigma^{\mu\nu} [\boldsymbol{B} - \boldsymbol{B}_0] \Sigma_{\mu\nu} [\boldsymbol{B} - \boldsymbol{B}_0]\right)$$
(4.36)

respectively.

Proof:

As a first step, from the comparison of equations (4.34) and (4.15) we understand that the phase $I[\mathbf{B}, \mathbf{B}_0, V]$ is nothing but the classical action

$$I[\mathbf{Y}, \mathbf{Y}_0, V] = S_{\text{cl.}}[\mathbf{Y}, \mathbf{Y}_0, V] \quad , \tag{4.37}$$

which we shall evaluate in analogy with the *String* case. The generalization of the *String Holographic Coordinates* ⁴ are of course the *p-brane Holographic Coordinates* of definition 4.5, $Y^{\mu_1...\mu_{p+1}}[\boldsymbol{B}]$, which we rewrite here for convenience:

$$Y^{\mu_1 \dots \mu_{p+1}}[B] \equiv \oint_{D^{(p)}} Y^{\mu_1} dY^{\mu_2} \wedge \dots \wedge dY^{\mu_{p+1}} = \oint_B d^p s \, Y^{\mu_1}(s) Y'^{\mu_2 \dots \mu_{p+1}} \quad . \tag{4.38}$$

⁴Which in turn are the generalization of the point particle line element.

We quote the results, calculated in appendix A.11, for the first and second functional derivatives of the p-brane $Holographic\ Coordinates$, which are:

$$\frac{\delta Y^{\mu_1 \dots \mu_{p+1}}[B]}{\delta Y^{\alpha}(\bar{s})} = \delta_{\alpha}^{\mu_1} Y'^{\mu_2 \dots \mu_{p+1}}(\bar{s}) - \sum_{i}^{2,p+1} \delta_{\alpha}^{\mu_i} Y'^{\mu_2 \dots \mu_{i-1} \bar{\mu}_i \mu_1 \mu_{i+1} \dots \mu_{p+1}}$$
(4.39)

$$\frac{\delta Y^{\mu_1 \dots \mu_{p+1}}[B]}{\delta Y^{\alpha}(\bar{s}) \delta Y^{\beta}(\tilde{s})} = \sum_{j}^{2,p+1} \delta_{\alpha\beta}^{\mu_1 \mu_j} \epsilon^{a_2 \dots a_{p+1}} \left(\partial_{a_2} Y^{\mu_2} \right) \cdot \dots \cdot \left(\partial_{a_j} Y^{\mu_j} \right) \cdot \dots \cdot \left(\partial_{a_{p+1}} Y^{\mu_{p+1}} \right) + \\
- \sum_{i,j}^{2,p+1} \delta_{\alpha}^{\mu_i} \delta_{\beta}^{\mu_j} \epsilon^{a_2 \dots a_i \dots a_j \dots a_{p+1}} \cdot \left(\partial_{a_2} Y^{\mu_2} \right) \cdot \dots \cdot (\partial_{a_2} Y^{\mu_2}) \cdot \dots$$

$$\cdot \ldots \cdot \left(\partial_{a_i} Y^{\mu_1}\right) \cdot \ldots \cdot \left(\partial_{a_{p+1}} Y^{\mu_{p+1}}\right) \partial_{a_j} \delta\left(\bar{s} - \tilde{s}\right) \quad . \tag{4.40}$$

Then we can write our "guess" for the action, or equivalently, for the phase (4.37):

$$S_{\text{cl.}}[\boldsymbol{B}(\boldsymbol{s}), \boldsymbol{B}_{0}(\boldsymbol{s}); V] = \frac{\beta}{2(p+1)V} \left(Y^{\mu_{1} \dots \mu_{p+1}}[\boldsymbol{B}] - Y^{\mu_{1} \dots \mu_{p+1}}[\boldsymbol{B}_{0}] \right) \cdot \left(Y_{\mu_{1} \dots \mu_{p+1}}[\boldsymbol{B}] - Y_{\mu_{1} \dots \mu_{p+1}}[\boldsymbol{B}_{0}] \right) \qquad (4.41)$$

$$\equiv \frac{\beta}{2(p+1)V} \Sigma^{\mu_{1} \dots \mu_{p+1}}[\boldsymbol{B} - \boldsymbol{B}_{0}] \Sigma_{\mu_{1} \dots \mu_{p+1}}[\boldsymbol{B} - \boldsymbol{B}_{0}] \quad , \quad (4.42)$$

where

$$\Sigma^{\mu_1 \dots \mu_{p+1}} = Y^{\mu_1 \dots \mu_{p+1}}[B] - Y^{\mu_1 \dots \mu_{p+1}}[B_0] \quad . \tag{4.43}$$

Starting from (4.42) we compute in appendix (A.12) the following equations expressing the functional derivatives of the classical action:

$$\frac{\delta S_{\text{cl.}}}{\delta Y^{\mu_1}(\mathbf{s})} = \frac{\beta}{V} \Sigma^{\mu_1 \dots \mu_{p+1}} [\mathbf{B} - \mathbf{B}_0] Y'_{\mu_2 \dots \mu_{p+1}}$$
(4.44)

$$\frac{\delta^2 S_{cl.}}{\delta Y^{\mu_1}(s)\delta Y_{\mu_1}(s)} = (D-p)\frac{\beta}{V}(Y'(s))^2 . \tag{4.45}$$

Then, by taking into account (4.37), and substituting into equations (4.33, 4.34), we obtain the solutions

$$\alpha = -\frac{D-p}{2(p+1)} \tag{4.46}$$

$$\beta = \rho_p(p)! (4.47)$$

If we define

$$\delta\left[\boldsymbol{B}-\boldsymbol{B}_{0}\right]=\lim_{\epsilon\to0}\left(\frac{1}{\pi\epsilon}\right)^{(D-p)/2}\exp\left(-\frac{1}{(p+1)!\epsilon}\Sigma^{\mu_{1}...\mu_{p+1}}[\boldsymbol{B}-\boldsymbol{B}_{0}]\Sigma_{\mu_{1}...\mu_{p+1}}[\boldsymbol{B}-\boldsymbol{B}_{0}]\right)$$

then, the kernel turns out to be

$$K[\boldsymbol{Y}(s),\boldsymbol{Y}_{0}(s);V] = \left(\frac{\rho_{p}}{2i\pi\hbar V}\right)^{(D-p)/2} \exp\left(\frac{i\rho_{p}}{4\hbar V}\Sigma^{\mu_{1}...\mu_{p+1}}[\boldsymbol{B}-\boldsymbol{B}_{0}]\Sigma_{\mu_{1}...\mu_{p+1}}[\boldsymbol{B}-\boldsymbol{B}_{0}]\right) , \tag{4.48}$$

and the corresponding Green Function is

$$G[\boldsymbol{B}, \boldsymbol{B}_0; V] = \int_0^\infty dV e^{-i\rho_p V/2\hbar} \exp\left(\frac{i\rho_p}{4\hbar V} \Sigma^{\mu_1 \dots \mu_{p+1}} [\boldsymbol{B} - \boldsymbol{B}_0] \Sigma_{\mu_1 \dots \mu_{p+1}} [\boldsymbol{B} - \boldsymbol{B}_0]\right) \quad . \tag{4.49}$$

These are the desired results.

As a consistency check, it is possible to derive the same quantities by applying gaussian integration techniques to

$$K[\boldsymbol{Y}, \boldsymbol{Y}_0; V] = \int_{\boldsymbol{Y}_0(\boldsymbol{s})}^{\boldsymbol{Y}(\boldsymbol{s})} [\mathcal{D}\boldsymbol{X}^{\mu}(\boldsymbol{\sigma})] [\mathcal{D}P_{\mu\nu}(\boldsymbol{\sigma})] \exp\left\{iS\left[\boldsymbol{X}(\boldsymbol{\sigma}), \boldsymbol{P}(\boldsymbol{\sigma}); \boldsymbol{\xi}(\boldsymbol{\sigma})\right]\right\} , \qquad (4.50)$$

which is the usual path–integral expression for the kernel with $S[Y(\sigma), P(\sigma); \xi(\sigma)]$ given by formula (4.4).

Chapter 5

String Functional Solutions

"Well, see what you can do with" it.

After the higher dimensional digression of the previous chapter we return to our main task: the next step in our program to set up a (spacetime covariant) functional quantum mechanics of closed Strings, is to find the basic solutions of equation (3.63). As in the quantum mechanics of point particles we can factorize explicitly the A dependence, which gives nothing but an exponential factor. We note that from now on we will work in 4-dimensional spacetime: anyway, no complications arise in keeping an arbitrary dimension.

Proposition 5.1 (Stationary States Schrödinger Equation).

A solution of the wave equation (3.63) can be always written in the form

$$\Psi\left[C;A\right] = \Phi\left[C\right]e^{iEA/\hbar}$$

provided that the functional of loops $\Phi[C]$ satisfies

$$-\frac{\hbar^{2}}{4m^{2}}\left(\oint_{\Gamma}dl(s)\right)^{-1}\oint_{\Gamma}\sqrt{\left(\mathbf{Y}'\left(s\right)\right)^{2}}ds\frac{\delta^{2}\Psi\left[C\right]}{\delta Y^{\mu\nu}\left(s\right)\delta Y_{\mu\nu}\left(s\right)}=E\Psi\left[C\right]\quad,\tag{5.1}$$

the <u>Stationary States Schrödinger Equation</u>, which reads, in the more compact notation of equation (3.36),

$$-\frac{\hbar^2}{4m^2}\frac{\delta^2\Psi\left[C\right]}{\delta C^{\mu\nu}\delta C_{\mu\nu}}=E\Psi\left[C\right]\quad.$$

Proof:

We compute

$$\frac{\partial \Psi \left[C;A\right]}{\partial A} = \Phi \left[C\right] \frac{\partial \left(e^{-\frac{i}{\hbar}EA}\right)}{\partial A}$$

$$= \Phi \left[C\right] \left(-\frac{iE}{\hbar}\right)e^{-\frac{i}{\hbar}EA}$$

$$= -\frac{i}{\hbar}E\Psi \left[C;A\right] ; \tag{5.2}$$

substituting then (5.2) in (3.63) and simplifying by $\exp(-iEA/\hbar)$ we obtain (5.1) (or its compact form using (3.39) also).

5.1 Plane Wave Solution

Now we can turn to the task of determining some particular solutions of equation (3.63), or better of its stationary version (5.1).

Proposition 5.2 (Plane Wave Solution).

The Plane Wave

$$\Phi_{\mathbf{p}}[C] = \mathcal{N} \exp\left\{\frac{i}{2} \oint_{C} Q_{\mu\nu} Y^{\mu} dY^{\nu}\right\}$$

$$= \mathcal{N} \exp\left\{\frac{i}{2} \oint_{\Gamma \equiv \mathbb{S}^{1}} ds \, Q_{\mu\nu}(s) \, Y^{\mu}(s) \, \frac{dY^{\nu}(s)}{ds}\right\} \tag{5.4}$$

is a solution of equation (5.1) if the Boundary Area Momentum $Q^{\mu\nu}$ (s) satisfies the classical dispersion relation. \mathcal{N} is a normalization constant.

Proof:

We need to evaluate the second functional variation of $\Phi_p[C]$ corresponding to the addition of two petals to C, say at the point $\bar{Y}^{\mu} = Y^{\mu}(\bar{s})$. The first variation of $\Psi_p[C]$ can be obtained from Eq.(5.4)

$$\delta_{\mathbf{p}}\Phi\left[C\right] = \frac{1}{2}Q_{\mu\nu}(\bar{s})\,\delta Y^{\mu\nu}(\bar{s})\,\Phi_{\mathbf{p}}\left[C\right] \quad . \tag{5.5}$$

Reopening the loop C at the same contact point and adding a second infinitesimal loop, we arrive at the second variation of $\Phi_{\rm p}\left[C\right]$,

$$\delta^{2} \Phi_{p} [C] = \frac{1}{4} Q_{\mu\nu}(\bar{s}) Q_{\rho\tau}(\bar{s}) \delta Y^{\mu\nu}(\bar{s}) \delta Y^{\rho\tau}(\bar{s}) \Phi_{p} [C] . \qquad (5.6)$$

Then, Eq.(5.4) solves equation (3.63) if

$$E = \frac{1}{4m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} dl(s) Q_{\mu\nu}(s) Q^{\mu\nu}(s) \quad , \tag{5.7}$$

which, as claimed in the proposition, is exactly the classical dispersion relation between String energy and momentum.

Proposition 5.3 (Boundary Momentum Operator Eigenstate).

The wave functional (5.4) represents an eigenstate of the loop total momentum operator with eigenvalue $\bar{Q}^{\mu\nu} = \mathbb{E} \langle Q^{\mu\nu} [C] \rangle$, i.e. the momentum average of the loop:

$$\bar{Q}^{\mu\nu}\left[C\right] \stackrel{\text{def.}}{=} \mathsf{E}\left\langle Q^{\mu\nu}\left[C\right]\right\rangle = \left(\oint_{\Gamma} dl(s)\right)^{-1} \oint_{\Gamma} dl\left(s\right) Q_{\mu\nu}\left(s\right) \quad . \tag{5.8}$$

Proof:

We have just to apply the total loop momentum operator (3.70) to our plane wave

$$\Phi_{\rm p}\left[C\right] = \mathcal{N} \exp\left\{\frac{i}{2} \oint_C Q_{\mu\nu} Y^{\mu} dY^{\nu}\right\}$$

to get

$$i\left(\oint_{\Gamma} dl(s)\right)^{-1} \oint_{\Gamma} dl\left(s\right) \frac{\delta\Phi_{p}\left[C\right]}{\delta\sigma^{\mu\nu}\left(s\right)} = \left\{\left(\oint_{\Gamma} dl(s)\right)^{-1} \oint_{\Gamma} dl\left(s\right) Q_{\mu\nu}\left(s\right)\right\} \Phi_{p}\left[C\right]$$
$$= \bar{Q}^{\mu\nu}\left[C\right] \Phi_{p}\left[C\right], \tag{5.9}$$

where, $\bar{Q}^{\mu\nu}\left[C\right]$ is exactly the momentum $Q^{\mu\nu}\left(s\right)$ averaged over the loop, as defined in equation (5.8) above.

Having determined a set of solutions to the stationary part of equation (3.63), the complete solutions describing the Quantum evolution from an initial state $\Psi[C_0, 0] = \Phi_p[C_0]$ ($\Phi_p[C_0]$) being the Plane Wave solution determined above) to a final state $\Psi[C; A]$ can be obtained by means of the amplitude (3.48), using equation (3.61)

$$\Psi [C; A] = \sum_{C_0} K [C, C_0; A] \Psi [C_0; 0]$$

$$= \left(\frac{m^2}{2i\pi A}\right)^{3/2} \int \left[\mathcal{D}Y^{\alpha\beta}(C_0)\right] \exp \left[\frac{im^2}{4A} \left(Y^{\mu\nu} [C] - Y^{\mu\nu} [C]\right)^2\right] \Phi_{\mathbf{p}} [C_0]$$

$$= \frac{1}{(2\pi)^{3/2}} \exp \frac{i}{2} \left[\oint_C Q_{\mu\nu} Y^{\mu} dY^{\nu} - \frac{A}{2m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} dl(s) Q_{\mu\nu} Q^{\mu\nu} \right]$$

$$= \frac{1}{(2\pi)^{3/2}} \exp \frac{i}{2} \left[\oint_C Q_{\mu\nu} Y^{\mu} dY^{\nu} - EA \right] . \tag{5.10}$$

As one would expect, the solution (5.10) represents a "monochromatic *String* wave train" extending all over loop space.

5.2 Gaussian Loop Wave–Packet

The quantum state represented by equation (5.10) is completely de-localized in loop space, which means that all the *String* shapes are equally probable, or that the *String* has no definite shape at all. Thus, even though the wave functional (5.10) is a solution of equation (3.63), it does not have an immediate physical interpretation: at most, it can be used to describe a flux in loop space rather than to describe a single physical object. Physically acceptable one–*String* states are obtained by a suitable superposition of "elementary" plane wave solutions. The quantum state closest to a classical *String* will be described by a *Gaussian Wave-Packet*.

Proposition 5.4 (Gaussian Wave-Packet Solution).

The functional

$$\Phi_{G}[C] = \left[\frac{1}{2\pi \left(\Delta\sigma\right)^{2}}\right]^{3/4} \exp\left(\frac{i}{2} \oint_{C} Y^{\mu} dY^{\nu} Q_{\mu\nu}\right) \exp\left[-\frac{1}{4 \left(\Delta\sigma\right)^{2}} \left(\oint_{C} Y^{\mu} dY^{\nu}\right)^{2}\right] , \quad (5.11)$$

is a gaussian loop wave function solution of the Functional Stationary Scröedinger equation (3.63). Its areal evolution spreads throughout loop space and its width $\Delta \sigma$ broadens as

$$\Delta\sigma(A) = \Delta\sigma \left[1 + \frac{A^2}{4m^4(\Delta\sigma)^4}\right]^{1/2}$$

Proof:

By inserting Eq.(5.11) into Eq.(3.61), and integrating out $Y^{\mu\nu}[C]$, we find

$$\Psi[C; A] = \left[\frac{1}{2\pi (\Delta \sigma)^2}\right]^{3/4} \frac{1}{\left(1 + iA/m^2 (\Delta \sigma)^2\right)^{3/2}} \exp\left\{\frac{1}{\left(1 + iA/m^2 (\Delta \sigma)^2\right)}\right\}$$

$$\cdot \left[-\frac{1}{4(\Delta\sigma)^{2}} Y^{\mu\nu} \left[C \right] Y_{\mu\nu} \left[C \right] + \frac{i}{2} \oint_{C} Y^{\mu} dY^{\nu} Q_{\mu\nu} \left(x \right) - \frac{iA}{4m^{2}} Q_{\mu\nu} \left[C \right] Q^{\mu\nu} \left[C \right] \right] \right\} \quad . \quad (5.12)$$

The wave functional represented by equation (5.12) spreads throughout loop space in conformity with the laws of quantum mechanics. In particular, the center of the Wave–Packet moves according to the stationary phase principle, i.e.

$$Y^{\mu\nu} [C] - \frac{A}{m^2} Q^{\mu\nu} [C] = 0$$
 (5.13)

and the width broadens as A increases

$$\Delta\sigma(A) = \Delta\sigma \left(1 + A^2/4m^4 \left(\Delta\sigma\right)^4\right)^{1/2} \quad . \tag{5.14}$$

Some comment are in order to clarify the physical meaning of the various quantities. $\Delta \sigma$ represents the width, or position uncertainty in loop space, corresponding to an uncertainty in the physical shape of the loop. Thus, as discussed previously, A represents a measure of the "timelike" distance between the initial and final String loop. Then, $m^2 (\Delta \sigma)^2$ represent the wavepacket mean life. As long as $A \ll m^2 (\Delta \sigma)^2$, the wavepacket maintains its original width $\Delta \sigma$. However, as A increases with respect to $m^2 (\Delta \sigma)^2$, the wavepacket becomes broader and the initial String shape decays in the background space. Notice that, for a sharp initial Wave–Packets, i.e., for $(\Delta \sigma) \ll 2\pi \alpha'$, the shape shifting process is more "rapid" than for larger Wave–Packets. Hence, Strings with a well defined initial shape will sink faster into the sea of quantum fluctuations than broadly defined String loops.

Chapter 6

"Minisuperspace"

"Jcan't even see. How am I supposed to ..." "Your eyes deceive you. Don't trust them."

6.1 Preliminaries

We are now going to give a brief exposition of a very particular setting in which it is possible to analize a specific realization of the concepts associated to the Functional Schrödinger Equation that we derived in chapters 3 and 4 for the String and the p-brane respectively. The reason that motivates such an approach is only slightly related to the clarification of the concepts we already explained in previous chapters. On the contrary we can see how we can use them, althought in a very indirect way, to tackle a completely different problem, which often appears in quantizing geometric theories, namely the problem of ordering ambiguities. The reason why the treatment we gave in previous sections can be helpful in this situation is that we have a problem, even if with an infinte number of degrees of freedom, expressed in different formulations. In particular we have seen that the functional equation can be considered in both, form (3.62) as well as form (3.63). These equations are again two different ways of writing the equation of motion for a system which involves an infinte number of degrees of freedom: this situation is somewhat far beyond the possibilities of any possible exact computation of the solutions in very general case. Moreover, as we alread

showed in chapter 5, even the simplest possible solutions to the Functional Schrödinger Equation for the String involve some functional steps that can offer difficulties in grasping the physical meaning of the computations and obscure some points, which are instead worth to be pointed out. In this chapter we will thus pass from the general formulation, to a more particular one, that we will call, by analogy to what is often done in General Relativity to tackle similar problems, the Minisuperspace Approximation. The next section presents the String case, whereas the p-brane one is exposed in the one after the next.

6.2 The String

We limit ourself to the case with in which we have a *circular String* in 2 + 1 dimensions; moreover, since we want to have the simplest possible case, we take the circular *String* lying in the 1 - 2 plane, i.e. without extension in the 0 direction. A possible parametrization for the image of the *Boundary Space* in the *Target Space* \mathbb{T} is thus:

$$Y^{\mu}(s) = (0, R\cos(2\pi s), R\sin(2\pi s)) \quad , \tag{6.1}$$

where $s \in \mathbb{S}^1$ is the parameter labeling different points of the *String*. For convenience we use the representation of \mathbb{S}^1 consisting of the closed interval $[0,1] \in \mathbb{R}$ with end point identified.

Notation 6.1 (Closed Unit Interval with End Points Identified).

The closed unit real interval with end points identified is denoted by

$$\stackrel{\circ}{\mathbb{I}} \stackrel{(1)}{\stackrel{\text{def.}}{=}} \left[0, 1^{\equiv 0}\right] \quad . \tag{6.2}$$

It is then possible to derive the expression for the Target Space tangent vector to the loop $C = Y^{\mu}(\mathring{\mathbb{I}}^{(1)})$.

Proposition 6.1 (Minisuperspace Linear Velocity Vector).

The Linear Velocity Vector to the String C is given by

$$Y^{\prime\mu}(s) = \frac{\partial Y^{\mu}}{\partial s} = (0, -2\pi R \sin(2\pi s), 2\pi R \cos(2\pi s))$$

$$(6.3)$$

and the only nonvanishing Holographic Coordinate is $Y_{12}[C] = \pi R^2$. Moreover we also quote the result

$$\left(\mathbf{Y}'\right)^2 = 4\pi^2 R^2 \quad , \tag{6.4}$$

so that the normalization is

$$\left(\oint_{\tilde{\mathbb{T}}^{(1)}} dl(s)\right)^{-1} = 2\pi R \quad .$$

Proof:

Directly from the definition, we can calculate

$$Y^{\mu\nu} = \oint_{\stackrel{\circ}{\mathbb{T}}(1)} Y^{\mu} dY^{\nu} \quad : \tag{6.5}$$

it is a 3×3 matrix, totally antisymmetric, so that it has 3 independent entries. Two of them vanish because the *String* lies in the 1-2 plane, so it does not extend in the 0 one:

$$Y_{10} = \oint_{\mathbb{I}^{(1)}} Y_{1} dY_{0}$$

$$= \int_{0}^{1} ds Y_{1} Y'_{0}'$$

$$= 0$$

$$Y_{20} = \oint_{\mathbb{I}^{(1)}} Y_{2} dY_{0}$$

$$= \int_{0}^{1} ds Y_{2} Y'_{0}'$$

$$= 0 .$$
(6.6)

The only nonzero component is Y_{12}

$$Y_{12} = \oint_{\mathbb{I}^{(1)}} Y_1 dY_2$$

$$= \int_0^1 ds Y_1 Y_2'$$

$$= \int_0^1 ds 2\pi R^2 \cos^2(2\pi s)$$

$$= \pi R^2 . \tag{6.8}$$

Then the other two results follow quickly: we first compute the modulus of the vector Y'^{μ}

$$Y'^{\mu}\left(s\right)Y'_{\mu}\left(s\right) = 0 + (2\pi R)^{2}\sin^{2}(2\pi s) + (2\pi R)^{2}\cos^{2}(2\pi s) = (2\pi R)^{2} = 4\pi^{2}R^{2}$$

and then its square root:

$$\sqrt{\left(\mathbf{Y}'\left(s\right)\right)^{2}} = \sqrt{Y'^{\mu}\left(s\right)Y'_{\mu}\left(s\right)} = 2\pi R \quad .$$

The integration over the loop parameter is trivial because the chosen parametrization implies no parameter dependence in the integrand, so that

$$\int_{\mathbb{I}^{(1)}}^{\circ} ds \sqrt{(Y'(s))^2} = \int_{0}^{1} ds 2\pi R = 2\pi R \quad .$$

The Holographic Derivative can also be calculated using the last result.

Proposition 6.2 (Minisuperspace Holographic Derivative).

The only non-vanishing Holographic Derivative is the one with respect to Y_{12} :

$$\frac{\delta}{\delta Y^{12}} = \frac{1}{2\pi R} \frac{d}{dR} \quad . \tag{6.9}$$

Proof:

The result follows by translating the results about the $Holographic\ Derivative$ in infinitesimal form. The only non–vanishing is the one related to Y^{12} so that we first get

$$dY^{12} = \delta Y^{12}(s) = 2\pi R dR \quad ,$$

where thanks to the circular symmetry there is no dependence from the loop parameter s anymore. Then the functional derivative is just an ordinary derivative with respect to R:

$$\frac{\delta}{\delta Y^{12}(s)} = \frac{d}{d(\pi R^2)} = \frac{1}{2\pi R} \frac{d}{dR} \quad ; \tag{6.10}$$

this is the desired result.

We can now remember relation (2.13) between *Holographic Derivatives* and *Ordinary Functional Derivatives*:

$$\frac{\delta}{\delta Y^{\mu}(s)} \sim Y^{\prime\nu}(s) \frac{\delta}{\delta Y^{\mu\nu}(s)}$$
 (6.11)

Proposition 6.3 (Ordinary Functional Derivatives).

The Ordinary Functional Derivatives with respect to the loop shape Y^{μ} are

$$\frac{\delta}{\delta Y^1(s)} \sim \cos(2\pi s) \frac{d}{dR}$$
 (6.12)

$$\frac{\delta}{\delta Y^2(s)} \sim \sin(2\pi s) \frac{d}{dR} \tag{6.13}$$

$$\frac{\delta}{\delta Y^0(s)} \sim 0 \quad , \tag{6.14}$$

where the symbol \sim is used since the equality holds only for reparametrization invariant functional in an integrated way ¹.

Proof:

We compute these quantities using relation (D.1) and the expressions for the *Holographic Derivative* that we derived in the proposition above. Then we have

$$\begin{split} \frac{\delta}{\delta Y^{1}(s)} &= Y'^{0}(s) \frac{\delta}{\delta Y^{10}(s)} + Y'^{1}(s) \frac{\delta}{\delta Y^{11}(s)} + Y'^{2}(s) \frac{\delta}{\delta Y^{12}(s)} \\ &= 0 + 0 + R \left(\sin (2\pi s) \right)' \frac{1}{2\pi R} \frac{d}{dR} \\ &= \cos (2\pi s) \frac{d}{dR} \quad ; \qquad (6.15) \\ \frac{\delta}{\delta Y^{2}(s)} &= Y'^{0}(s) \frac{\delta}{\delta Y^{20}(s)} + Y'^{1}(s) \frac{\delta}{\delta Y^{21}(s)} + Y'^{2}(s) \frac{\delta}{\delta Y^{22}(s)} \\ &= \sin (2\pi s) \frac{d}{dR} \quad ; \qquad (6.16) \\ \frac{\delta}{\delta Y^{0}(s)} &= Y'^{0}(s) \frac{\delta}{\delta Y^{00}(s)} + Y'^{1}(s) \frac{\delta}{\delta Y^{01}(s)} + Y'^{2}(s) \frac{\delta}{\delta Y^{02}(s)} = 0 \quad . \qquad (6.17) \end{split}$$

These are the desired results.

As we will see in the sequel, the functional derivatives are in some sense misleading objects. As explained in appendix D this objects are not reparametrization invariant: in our opinion reparametrization invariance is an important property that should be preserved by any type of physically meaningful operator. Thus we can trace to the non invariance of the functional derivatives the following (and **negative**) proposition.

Proposition 6.4 (Ordering Problems).

The functional laplacian computed in terms of the Holographic Derivatives is not consistent with the same operator computed in terms of the Ordinary Functional Derivatives.

¹Please see appendix D for a more detailed explanation.

Proof:

We first calculate the second order operator which is the functional laplacian, in terms of the functional derivatives; we find

$$\frac{\delta^2}{\delta Y^{\mu}(s)\delta Y_{\mu}(s)} = \frac{\delta^2}{\delta Y^0(s)\delta Y_0(s)} + \frac{\delta^2}{\delta Y^1(s)\delta Y_1(s)} + \frac{\delta^2}{\delta Y^2(s)\delta Y_2(s)}$$

$$= \cos(2\pi s) \frac{d}{dR} \left(\cos(2\pi s) \frac{d}{dR}\right) + \sin(2\pi s) \frac{d}{dR} \left(\sin(2\pi s) \frac{d}{dR}\right)$$

$$= \frac{d^2}{dR^2} , \qquad (6.18)$$

so that the corresponding integral operator is

$$\int_{0}^{1} \frac{ds}{\sqrt{(Y')^{2}}} \frac{\delta^{2}}{\delta Y^{\nu}(s)\delta Y_{\nu}(s)} = \int_{0}^{1} \frac{ds}{2\pi R} \frac{d^{2}}{dR^{2}} = \frac{ds}{2\pi R} \frac{d^{2}}{dR^{2}} \quad . \tag{6.19}$$

Now we will compute the same quantity using the *Holographic Derivatives*; note that since we have relation (2.13) between functional and holographic derivatives, it might be that ordering ambiguities could arise in defining this second oreder operator, because of the $Y'^{\mu}(s)$ factor. We have

$$\frac{1}{2} \int_{0}^{1} ds \sqrt{(\mathbf{Y}')^{2}} \frac{\delta^{2}}{\delta Y^{\nu\tau}(s) \delta Y_{\nu\tau}(s)} = \frac{1}{2} \int_{0}^{1} ds \sqrt{(\mathbf{Y}')^{2}} 2 \frac{1}{2\pi R} \frac{d}{dR} \left(\frac{1}{2\pi R} \frac{d}{dR} \right)$$

$$= \int_{0}^{1} \frac{ds}{2\pi} \frac{d}{dR} \left(\frac{1}{R} \frac{d}{dR} \right)$$

$$= \frac{1}{2\pi} \frac{d}{dR} \left(\frac{1}{R} \frac{d}{dR} \right) . \tag{6.20}$$

This result is in disagreement with (6.18), because we have

$$\int_{0}^{1} \frac{ds}{\sqrt{\left(\mathbf{Y}'\right)^{2}}} \frac{\delta^{2}}{\delta Y^{\mu}\left(s\right) \delta Y_{\mu}\left(s\right)} = \frac{1}{2} \int_{0}^{1} ds \sqrt{\left(\mathbf{Y}'\right)^{2}} \frac{\delta^{2}}{\delta Y^{\mu\nu}\left(s\right) \delta Y_{\mu\nu}\left(s\right)}$$

but of course

$$\frac{ds}{2\pi R}\frac{d^2}{dR^2} \neq \frac{1}{2\pi}\frac{d}{dR}\left(\frac{1}{R}\frac{d}{dR}\right)$$

Some light into this problem can be given by the observation [16] that the functional differential operator does not transform *covariantly* under a reparametrization of the loop. We thus make the following absumption:

Axiom (Quantization Procedure): When we quantize a Theory having some invariance ² we have to promote to the role of Quantum Operators only quantities that respect the invariance of the Theory and transform covariantly.

 $^{^2}$ Reparametrization invariance is the interesting case for the Theory of Extended Objects and General Relativity

This can be euristically understood, since good quantum numbers are to be associated with properties that do not change under a symmetry transformation, and so must then be the corresponding operators. In the case of a symmetry under reparametrization we can easily construct invariant functional quantities from covariant ones, simply by integration with the natural "volume element":

$$\oint_{\Gamma} ds \sqrt{(\mathbf{Y}')^2} \quad ;$$

the analogy with Quantum Gravity is transparent at this stage.

Going back to our particular case we see that a functional differential operator that transforms *covariantly*, is

$$\frac{\Delta}{\Delta Y^{\mu}(s)} = \frac{1}{\sqrt{(\mathbf{Y}')^{2}}} \frac{\delta}{\delta Y^{\mu}(s)} \quad . \tag{6.21}$$

Moreover we now observe that all the procedure we performed in chapter 3 to get the functional equation (3.62) can be repeated using the operator of equation (6.21) and getting the following result.

Proposition 6.5 (Functional Schrödinger Equation: Covariant Formalism).

In terms of the functional differential operator (6.21) the functional equation (3.62) can be rewritten as

$$-\frac{1}{2m^{2}}\left(\oint_{\Gamma\approx\mathbb{S}^{1}}dl(s)\right)^{-1}\oint_{\Gamma\approx\mathbb{S}^{1}}ds\sqrt{\left(\mathbf{Y}'\right)^{2}}\frac{\Delta^{2}\Psi\left[C;A\right]}{\Delta Y^{\mu}\left(s\right)\Delta Y_{\mu}\left(s\right)}=i\frac{\partial\Psi\left[C;A\right]}{\partial A}\quad.\tag{6.22}$$

Proof:

To derive the desired result we recall the classical dispersion relation written in integrated form (2.61). If we define

$$\tilde{q}^{\mu}\left(s\right) = \frac{q^{\mu}\left(s\right)}{\sqrt{\boldsymbol{Y}'\left(s\right)^{2}}} = \frac{1}{\sqrt{\left(\boldsymbol{Y}'\right)^{2}\left(s\right)}} \frac{\delta S_{\mathrm{Red.}}}{\delta Y^{\mu}\left(s\right)} = \frac{\Delta S_{\mathrm{Red.}}}{\Delta Y^{\mu}\left(s\right)}$$

i.e. a sort of normalized Boundary Linear Momentum that equation can be rewritten as

$$\frac{1}{2m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma \approx \mathbb{S}^1} ds \sqrt{\left(\mathbf{Y}' \right)^2(s)} \tilde{q}^{\mu}(s) \, \tilde{q}_{\mu}(s) = E \quad . \tag{6.23}$$

Then the first and second functional derivatives of the kernel (cf. equations (3.30) and (3.31)) expressed in terms of the covariant functional derivative are

$$\frac{\Delta}{\Delta Y^{\mu}(s)} K[\mathbf{Y}(s), \mathbf{Y}_{0}(s); A] = \frac{i}{\hbar} \int_{\mathbf{Y}_{0}(s)}^{\mathbf{Y}(s)} \int_{\mathbf{\zeta}_{0}(s)}^{\mathbf{\zeta}(s)} [\mathcal{D}\mu(s)] \frac{q_{\mu}}{\sqrt{(\mathbf{Y}'(s))^{2}}} \exp\left(\frac{i}{\hbar} S_{\text{Red.}}\right)$$

$$= \frac{i}{\hbar} \int_{\mathbf{Y}_{0}(s)}^{\mathbf{Y}(s)} \int_{\mathbf{\zeta}_{0}(s)}^{\mathbf{\zeta}(s)} [\mathcal{D}\mu(s)] \tilde{q}_{\mu} \exp\left(\frac{i}{\hbar} S_{\text{Red.}}\right) \qquad (6.24)$$

$$\frac{\Delta^{2}}{\Delta Y^{\mu}(s) \Delta Y_{\mu}(s)} K[\mathbf{Y}(s), \mathbf{Y}_{0}(s); A] = -\frac{1}{\hbar^{2}} \int_{\mathbf{Y}_{0}(s)}^{\mathbf{Y}(s)} \int_{\mathbf{\zeta}_{0}(s)}^{\mathbf{\zeta}(s)} [\mathcal{D}\mu(s)] \tilde{q}_{\mu} \tilde{q}^{\mu} \exp\left(\frac{i}{\hbar} S_{\text{Red.}}\right)$$

$$\equiv -\frac{1}{\hbar^{2}} \tilde{q}^{\mu} \tilde{q}_{\mu} \qquad (6.25)$$

Now the last form of the Classical Dispersion Relation, (6.23), can be interpreted as the evolution equation for the mean values of quantum operators, so that substituting in it (6.25) as well as (2.59), which is unchanged, we get

$$-\frac{1}{2m^{2}}\left(\oint_{\Gamma\approx\mathbb{S}^{1}}dl(s)\right)^{-1}\oint_{\Gamma\approx\mathbb{S}^{1}}ds\sqrt{\left(\mathbf{Y}'\right)^{2}}\frac{\Delta^{2}\Psi\left[C;A\right]}{\Delta Y^{\mu}\left(s\right)\Delta Y_{\mu}\left(s\right)}=i\frac{\partial\Psi\left[C;A\right]}{\partial A}\quad,\tag{6.26}$$

the desired result.

Moreover using this form of the equation all ordering ambiguities disappear since

Proposition 6.6 (Solution of Ordering Ambiguities).

The operator

$$\frac{\Delta}{\Delta Y^{\mu}\left(s\right)}$$

in the minisuperspace approximation is given by

$$\frac{1}{2\pi R} \frac{d}{dR}$$

so that the second order operator

$$\frac{1}{2m^{2}}\left(\oint_{\mathbb{I}^{(1)}}dl(s)\right)^{-1}\oint_{\mathbb{I}^{(1)}}ds\sqrt{\left(\mathbf{Y}'\right)^{2}}\frac{\Delta^{2}\Psi\left[C;A\right]}{\Delta Y^{\mu}\left(s\right)\Delta Y_{\mu}\left(s\right)}$$

coincides with

$$\frac{1}{2m^{2}} \left(\oint_{\mathbb{I}^{(1)}} dl(s) \right)^{-1} \oint_{\mathbb{I}^{(1)}} ds \sqrt{\left(\mathbf{Y}'\right)^{2}} \frac{\delta^{2} \Psi\left[C; A\right]}{\delta Y^{\mu\nu}\left(s\right) \delta Y_{\mu\nu}\left(s\right)}$$

Proof:

Since we have

$$\frac{\Delta}{\delta Y^{\mu}\left(s\right)}=\left(0,\frac{\cos(2\pi s)}{2\pi R}\frac{d}{dR},\frac{\sin(2\pi s)}{2\pi R}\frac{d}{dR}\right)$$

then we can compute

$$\begin{split} \frac{\Delta^2}{\Delta Y^{\mu}\left(s\right)\Delta Y_{\mu}\left(s\right)} &= \frac{\Delta^2}{\Delta Y^0\left(s\right)\Delta Y_0\left(s\right)} + \frac{\Delta^2}{\Delta Y^1\left(s\right)\Delta Y_1\left(s\right)} + \frac{\Delta^2}{\Delta Y^2\left(s\right)\Delta Y_2\left(s\right)} \\ &= 0 + \cos^2(2\pi s) \, \frac{1}{2\pi R} \frac{d}{dR} \left(\frac{1}{2\pi R} \frac{d}{dR}\right) + \sin^2(2\pi s) \, \frac{1}{2\pi R} \frac{d}{dR} \left(\frac{1}{2\pi R} \frac{d}{dR}\right) \\ &= \frac{1}{4\pi^2 R} \frac{d}{dR} \left(\frac{1}{R} \frac{d}{dR}\right) \quad . \end{split}$$

At the same time we have

$$\frac{\delta^{2}}{\delta Y^{\mu\nu}\left(s\right)\Delta Y_{\mu\nu}\left(s\right)} = \frac{1}{4\pi^{2}R}\frac{d}{dR}\left(\frac{1}{R}\frac{d}{dR}\right) \tag{6.27}$$

and comparing the last two results we conclude the proof.

Now we have a completely non–ambiguous expression for the functional Laplacian, i.e. the fundamental quantity in determining the functional wave equation for the *String* in the minisuperspace approximation. We also hope, at least, to have brought to the light some problems that can be present in the functional formulation and that make it absolutely non-trivial. We can now conclude this section:

Definition 6.1 (Minisuperspace Stationary Schrödinger Equation).

The minisuperspace approximation to the Stationary States Functional Wave Equation for a circular String lying in the 1-2 plane of the 2+1 dimensional Target Space is the following differential equation:

$$\frac{1}{2m^2} \frac{1}{2\pi R} \frac{d}{dR} \left(\frac{1}{2\pi R} \frac{d\Phi(R)}{dR} \right) = E\Phi(R) \quad . \tag{6.28}$$

Proof:

This result simply follows substituting, the result for the Holographic Derivative (say, equation (6.27)) into the Stationary States Schrödinger Equation (5.1). Of course now the loop C is described only by its radius R so that functionals of C become ordinary functions of the real variable R in the same way as Functional and Holographic Derivative converted to ordinary derivatives with respect to R.

6.3 The p-brane

We will here shortly give the equation corresponding to (6.28) in the more general case of a p-brane living in a D+1>p+1 dimensional Target Space. The main reason for this short digression is to translate the abstract formulation given in the previous chapters, and its very simple representation presented in the section above, into a slightly more complicated case, in which we can see general ideas at work but not in the simplest possible case. To avoid spending too much paper for this part, we rely on a geometrical reinterpretation of the results of previous section. In particular the circular String is a 1-sphere, \mathbb{S}^1 , and the normalization integral is the circumpherence, i.e. the surface of the 1-sphere. Moreover since the String is in 1-2 plane, then the areas of its projections onto the 0-1 and 0-2 planes are vanishing, as well as the corresponding Holographic Coordinates, Y^{01} and Y^{02} . On the contrary Y^{12} is the area of the shadow on the 1-2 plane of the of the circle, i.e. a disk, so that we obtained the natural result

$$Y^{12} = \pi R^2$$
.

We now turn these considerations into the more general case of a p-brane in (D + 1)dimensions.

Proposition 6.7 (Hyperspherical *p*-brane).

Let us consider an Hyperspherical p-brane, i.e. a p-brane which is a p-sphere, \mathbb{S}^p , contained in a (p+1)-dimensional Hypersubspace cooresponding to the $1, 2, \ldots, p+1$ cartesian coordinates, $X^{(1)}$, $X^{(2)}$, ..., $X^{(p+1)}$. Let us call R its radius; the Stationary States Functional Schrödinger Equation for this extended object is given by

$$\frac{\hbar^2 p}{2\rho_p} \left[\frac{\Gamma\left(\frac{p+1}{2}\right)}{\left(2\pi\right)^{\frac{p+1}{2}}} \right] \frac{1}{R^p} \frac{d}{dR} \left(\frac{1}{R^p} \frac{d\Psi^{(\mathrm{p})}}{dR} \right) = E\Psi^{(\mathrm{p})} \quad .$$

Proof:

We will derive previous equation using the natural generalizations of the concept of circunference of a circle and area of a disk. In particular in the case of a *p-brane* the normalization factor

$$\oint_{\mathbb{S}^p} d^p s \sqrt{\left(\boldsymbol{Y}'(\boldsymbol{s}) \right)^2}$$

is the generalization of the length of a one sphere, \mathbb{S}^1 : thus it is the hypersurface of a p sphere, \mathbb{S}^p , i.e.

$$\oint_{\mathbb{S}^p} d^p s \sqrt{(Y'(s))^2} = \frac{(2\pi)^{\frac{p+1}{2}}}{\Gamma(\frac{p+1}{2})} R^p$$
(6.29)

We can now safely turn to the holographic coordinates; since we are considering our *p*-brane contained in the hyperplane defined by the coordinates with indices $1, \ldots, p+1$, and the *Holographic Coordinates* have p+1 totally antisymmetric indices varying in the range $1, \ldots, D$, we have that the only non-zero components are those indexed by a permutation of $1, \ldots, p+1$; there are (p+1)! possible combination of the indices and thus (p+1)! non vanishing $Y^{\mu_0 \dots \mu_p}$. Moreover their value is just the volume of \mathbb{S}^p , i.e. we get for all the nonvanishing *Holographic Coordinates*:

$$Y^{\gamma(1)\dots\gamma(p+1)} = \frac{(p+1)(2\pi)^{\frac{p+1}{2}}}{\Gamma(\frac{p+1}{2})} R^{p+1} \quad , \qquad \forall \gamma \in \mathfrak{S}_{p+1} \quad ,$$

 \mathfrak{S} being the symmetric group of order p+1. From the above we directly get the expression for the nonvanishing *Hypervolume Derivatives*, which are

$$\frac{\delta}{\delta Y^{\gamma(1)\dots\gamma(p+1)}} = \frac{\Gamma\left(\frac{p+1}{2}\right)}{(2\pi)^{\frac{p+1}{2}} R^p} \frac{d}{dR} \quad , \qquad \forall \gamma \in \mathfrak{S}_{p+1} \quad . \tag{6.30}$$

Substituting then equations (6.30) and (6.29) into the stationary equation derived from (4.16), we get the desired result.

Chapter 7

Fractal Strings

"You're going to find that many of the truths we cling do depend greatly on our own point of view."

7.1 The Shape Uncertainty Principle

In this section we discuss the new form that the Uncertainty Principle takes in the functional Theory of *String* loops.

Notation 7.1 (Fourier Transform Related Width). The width in the Momentum Representation ΔQ is related to the width in Holographic Coordinates Representation by

$$\Delta Q = \frac{1}{2\Delta\sigma} \quad , \tag{7.1}$$

qhich with our convention is the usual relation between (Functional) Fourier transformed quantities.

The Uncertainty Principle in Quantum Mechanics gives a definite lower bound to the accuracy about the simultaneous knowledge of the position and momentum of a pointlike particle. In our description the role of the position is played by the *Holographic Coordinates* of the loop, $Y^{\mu\nu}$ [C], and the role of the momentum is played by the *Boundary Area Momentum* $Q^{\mu\nu}$ [C]. We first prove a useful

Proposition 7.1 (Holographic/Area Functional Fourier Transform).

Given the Gaussian Wave Packet (5.11) in the "Holographic Coordinates Representation", which is

$$\Phi_{G}\left[C\right] = \left[\frac{1}{2\pi \left(\Delta\sigma\right)^{2}}\right]^{3/4} \exp\left(\frac{i}{2} \oint_{C} Y^{\mu} dY^{\nu} Q_{\mu\nu}\right) \exp\left[-\frac{1}{4 \left(\Delta\sigma\right)^{2}} \left(\oint_{C} Y^{\mu} dY^{\nu}\right)^{2}\right] ,$$

$$(7.2)$$

its functional Fourier transform, i.e. the corresponding wave functional in the "Area Momentum Representation" is

$$\tilde{\Phi}_{G}[Q] = \frac{1}{\left[\pi(\Delta Q)^{2}\right]^{3/4}} \exp\left[-\frac{1}{4(\Delta Q)^{2}} \left(\oint dl(s)\right)^{-1} \oint_{C} dl(s) \left(Q_{\mu\nu}(s) - K_{\mu\nu}\right)^{2}\right] , \quad (7.3)$$

where the relation between $\Delta \sigma$ and ΔQ is given in notation 7.1.

Proof:

The Functional Fourier Transform of expression (7.2) is

$$\tilde{\Phi}_{G}[C] = \frac{1}{(2\pi)^{3/2}} \int [\mathcal{D}Y^{\mu\nu}[C]] \Phi_{G}[C] \exp\left[-\frac{i}{2} \oint_{C} Y^{\mu} dY^{\nu} K_{\mu\nu}\right]
= \frac{1}{(2\pi)^{3/2}} \left[\frac{1}{2\pi (\Delta\sigma)^{2}}\right]^{3/4} \int [\mathcal{D}Y^{\mu\nu}[C]] \exp\left[\frac{i}{2} \oint_{C} Y^{\mu} dY^{\nu} (Q_{\mu\nu} - K_{\mu\nu})\right] \cdot \exp\left\{-\frac{(\Delta\sigma)^{2}}{4} \left[\oint_{C} Y^{\mu} dY^{\nu}\right]^{2}\right\}$$

and the final result can be computed performing the path integration:

$$\frac{1}{(2\pi)^{3/2}} \left[\frac{1}{2\pi(\Delta\sigma)^2} \right]^{3/4} \int [\mathcal{D}Y^{\mu\nu} [C]] \exp\left[\frac{i}{2} \oint_C Y^{\mu} dY^{\nu} (Q_{\mu\nu} - K_{\mu\nu}) \right] \cdot \\
\cdot \exp\left\{ -\frac{(\Delta\sigma)^2}{4} \left[\oint_C Y^{\mu} dY^{\nu} \right]^2 \right\} \\
= \frac{1}{(2\pi)^{3/2}} \left[\frac{1}{2\pi(\Delta\sigma)^2} \right]^{3/4} \int [\mathcal{D}Y^{\mu\nu} [C]] \cdot \\
\cdot \exp\left\{ \oint_C dl(s) Y^{\mu} Y^{\prime\nu} \frac{i(Q_{\mu\nu} - K_{\mu\nu})}{2} - \frac{1}{2} \frac{(\Delta\sigma)^2}{2} \left[\oint_C dl(s) Y^{\mu} Y^{\prime\nu} \right]^2 \right\} \\
= \frac{1}{(2\pi)^{3/2}} \left(\frac{(\Delta\sigma)^2}{2} \right)^{3/2} \left[\frac{1}{2\pi(\Delta\sigma)^2} \right]^{3/4} \cdot \\
\cdot \int [\mathcal{D}Y^{\mu\nu} [C]] \exp\left\{ -\frac{1}{2} \frac{2}{(\Delta\sigma)^2} \oint_C dl(s) \left[\frac{i(Q_{\mu\nu} - K_{\mu\nu})}{2} \right]^2 \right\} + \\
= \frac{1}{(2\pi)^{3/2}} \left[\frac{1}{2\pi(\Delta\sigma)^2} \right]^{3/4} \exp\left\{ \left(\oint_\Gamma dl(s) \right)^{-1} \frac{(\Delta\sigma)^2}{4} \left[\oint_C dl(s) (Q_{\mu\nu} - K_{\mu\nu})^2 \right]^2 \right\} \quad (7.4)$$

This is again a Gaussian wave packet, whose "center of mass" moves in loop space with a momentum $K_{\mu\nu}$.

We are now ready to prove the following results:

Proposition 7.2 (Holographic/Area Expectations: Gaussian Wavepacket).

For a Gaussian wavepacket of the form (5.11) in the "Holographic Coordinates Representation" the expectation values of the "position in loop space" and of the "position squared in loop space" are:

$$\mathbf{E} \langle Y^{\mu\nu} \left[C \right] \rangle = 0 \tag{7.5}$$

$$\mathbb{E} \langle Y^{\mu\nu} [C] Y_{\mu\nu} [C] \rangle = 3(\Delta \sigma)^2 = \frac{3}{2(\Delta Q)^2}$$
 (7.6)

Moreover the corresponding quantities for the momentum are

$$\mathbb{E}\left\langle Q^{\mu\nu}\left[C\right]\right\rangle = K_{\mu\nu}\left[C\right] \tag{7.7}$$

$$\mathbb{E}\langle Q^{\mu\nu}[C]Q_{\mu\nu}[C]\rangle = \frac{1}{2}K^{\mu\nu}K_{\mu\nu} + 6(\Delta Q)^2$$
 (7.8)

Proof:

To compute this expectation values we have to compute the following functional integrals respectively,

$$\mathbb{E}\left\langle Y^{\mu\nu}\left[C\right]\right\rangle = \int \left[\mathcal{D}Y^{\lambda\rho}\left(s\right)\right]Y^{\mu\nu}\left[C\right]\left|\Psi\left[C;A\right]\right|^{2} \tag{7.9}$$

$$\mathbb{E}\left\langle Y^{\mu\nu}\left[C\right]Y_{\mu\nu}\left[C\right]\right\rangle \quad = \quad \int \left[\mathcal{D}Y^{\lambda\rho}\left(s\right)\right]Y^{\mu\nu}\left[C\right]Y_{\mu\nu}\left[C\right]\left|\Psi\left[C;A\right]\right|^{2} \quad ; \tag{7.10}$$

since the dependence from the area A in $\Psi[C;A]$ resides in the exponential, it goes away computing the modulus square so that at the end we get

$$\mathbb{E}\left\langle Y^{\mu\nu}\left[C\right]\right\rangle = \int \left[\mathcal{D}Y^{\lambda\rho}\left(s\right)\right]Y^{\mu\nu}\left[C\right]\left|\Phi_{G}\left[C\right]\right|^{2} \tag{7.11}$$

$$\mathbb{E}\left\langle Y^{\mu\nu}\left[C\right]Y_{\mu\nu}\left[C\right]\right\rangle \quad = \quad \int \left[\mathcal{D}Y^{\lambda\rho}\left(s\right)\right]Y^{\mu\nu}\left[C\right]Y_{\mu\nu}\left[C\right]\left|\Phi_{G}\left[C\right]\right|^{2} \quad , \tag{7.12}$$

where we can see from (7.2) that the loop probability density still has a Gaussian form

$$\left| \Phi_{G} [C] \right|^{2} = \left[\frac{(\Delta Q)^{2}}{2\pi} \right]^{3/2} \exp \left[-\frac{(\Delta Q)^{2}}{4} Y^{\mu\nu} [C] Y_{\mu\nu} [C] \right]$$

$$\equiv \left[\frac{1}{4\pi (\Delta \sigma)^{2}} \right]^{3/2} \exp \left[-\frac{1}{4 (\Delta \sigma)^{2}} Y^{\mu\nu} [C] Y_{\mu\nu} [C] \right]$$
(7.13)

centered around the vanishing loop with a dispersion given by $\Delta \sigma$. Now after integrating by parts, we can use the known result for functional gaussian integration to get the desired expressions. In particular

$$\mathbf{E} \langle Y^{\mu\nu} [C] \rangle = \int \left[\mathcal{D} Y^{\lambda\rho} (s) \right] Y^{\mu\nu} [C] \left| \Phi_G [C] \right|^2 \\
= \int \left[\mathcal{D} Y^{\lambda\rho} (s) \right] Y^{\mu\nu} [C] \left(\frac{1}{4\pi (\Delta\sigma)^2} \right)^{3/2} \exp \left\{ -\frac{1}{4(\Delta\sigma)^2} Y^{\mu\nu} [C] Y_{\mu\nu} [C] \right\} \\
= 0 \tag{7.14}$$

since the integrand is odd. With the same procedure, but a bit of more effort we also have

$$\mathbb{E}\left\langle Y^{\mu\nu}\left[C\right]Y_{\mu\nu}\left[C\right]\right\rangle = \int \left[\mathcal{D}Y^{\lambda\rho}\left(s\right)\right]Y^{\mu\nu}\left[C\right]Y_{\mu\nu}\left[C\right]\left|\Phi_{G}\left[C\right]\right|^{2} = 3(\Delta\sigma)^{2} \quad . \tag{7.15}$$

The same procedure can be followed for the expectation values related to the $Boundary\ Area\ Momentum.$

Let us introduce the following notation:

Notation 7.2 (String Mean Square Deviations in Position and Momentum).

 $\Delta\Sigma_V^2$ is the <u>Mean Square Deviation in the Position of the String</u>¹.

 $\Delta\Sigma_Q^2$ is the <u>Mean Square Deviation in the Momentum of the String</u>².

The following proposition is now just a matter of applying previous results.

Proposition 7.3 (Shape Uncertainty Principle).

The Quantum Shadow Dynamics of the String is such that it is impossible to have at the Areal Time A an exact knowledge of both the Holographic Position of the object and of its conjugated Boundary Area Momentum; in particular

$$\Delta\Sigma_Y \Delta\Sigma_Q = \frac{3}{\sqrt{2}}$$
 , $(\hbar = 1 \quad units)$. (7.16)

Proof:

From the definition of mean square deviation of a stochastic variable Z and the equality

$$\operatorname{Var} Z = \operatorname{E} \langle (Z - \operatorname{E} \langle Z \rangle)^2 \rangle = \operatorname{E} \langle Z \rangle^2 - \operatorname{E} \langle Z^2 \rangle$$
 (7.17)

¹Or, equivalently, the <u>String Position Uncertainty</u>.

²Or, equivalently, the <u>String Momentum Uncertainty</u>.

we can just substitute results (7.5-7.6) in the following equation

$$\Delta \Sigma_{Y}^{2} = \frac{1}{2} \mathrm{E} \left\langle Y^{\mu\nu} \left[C \right] Y_{\mu\nu} \left[C \right] \right\rangle - \mathrm{E} \left\langle Y^{\mu\nu} \left[C \right] \right\rangle \mathrm{E} \left\langle Y_{\mu\nu} \left[C \right] \right\rangle$$

to get

$$\Delta \Sigma_Y^2 = \frac{3}{2(\Delta Q)^2} \quad . \tag{7.18}$$

In the same way starting from (7.7-7.8) and

$$\Delta \Sigma_Q^2 = \frac{1}{2} \mathbf{E} \left\langle Q^{\mu\nu} \left[C \right] Q_{\mu\nu} \left[C \right] \right\rangle - \mathbf{E} \left\langle Q^{\mu\nu} \left[C \right] \right\rangle \mathbf{E} \left\langle Q_{\mu\nu} \left[C \right] \right\rangle$$

the String momentum mean square deviation, or uncertainty squared, is

$$\Delta \Sigma_Q^2 = 3 \left(\Delta Q \right)^2 \quad . \tag{7.19}$$

Then, comparing equation (7.18) with equation (7.19), we find that the uncertainties are related by

$$\Delta \Sigma_Y \Delta \Sigma_Q = \frac{3}{\sqrt{2}} \quad .$$

Equation (7.16) represents the new form that the Heisenberg Principle takes when String Quantum Mechanics is formulated in terms of diffusion in loop space, or quantum shape shifting. Just as a pointlike particle cannot have a definite position in space and a definite linear momentum at the same time, a physical String cannot have a definite shape and a definite rate of shape changing at a given areal time. In other words, a String loop cannot be totally at rest neither in physical nor in loop space: it is subject to a zero-point motion characterized by

$$\langle Q_{\mu\nu} [C] \rangle = 0 \tag{7.20}$$

$$\langle Q_{\mu\nu} [C] \rangle = 0 \qquad (7.20)$$

$$\left[\frac{1}{2} \langle Q_{\mu\nu} [C] Q^{\mu\nu} [C] \rangle \right]^{1/2} = \sqrt{3} (\Delta Q) = \Delta \Sigma_Q \qquad (7.21)$$

In such a state a physical String undergoes a zero-point shape shifting, and the loop momentum attains its minimum value compatible with an area resolution $\Delta \sigma$.

To keep ourselves as close as possible to Heisenberg's seminal idea, we interpret the lack of a definite shape as follows: as we increase the resolution of the "microscope" used to probe the structure of the String, more and more quantum petals will appear along the loop. The picture emerging out of this is that of a classical line turning into a fractal object as we move from the classical domain of physics to the quantum realm of quantum fluctuations. If so, two questions immediately arise:

- 1. a classical bosonic String is a closed line of topological dimension one. Its spacetime image consists of a smooth, timelike, two-dimensional world-sheet. Then, if a quantum String is a fractal object, which Hausdorff dimension should be assigned to it?
- 2. Is there any critical scale characterizing the classical-to-fractal geometrical transition?

These two questions will be addressed in the next section.

7.2 Fractal Strings

In the path-integral formulation of quantum mechanics, Feynman and Hibbs noted that the trajectory of a particle is continuous but nowhere differentiable. We extend this result to the quantum mechanical path of a relativistic *String* and find that the "trajectory", in this case, is a fractal surface with Hausdorff dimension three. Depending on the resolution of the detecting apparatus, the extra dimension is perceived as "fuzziness" of the *String World-Sheet*. We give an interpretation of this phenomenon in terms of a new form of the uncertainty principle for *Strings* and study the transition from the smooth to the fractal phase.

As pointed out in appendix D the functional area derivative introduces a non differentiable process. Non–differentiability is the hallmark of fractal objects. Thus, anticipating one of our results, quantum loop fluctuations, interpreted as singular shape–changing transitions resulting from "petal addition", are responsible for the fractalization of the *String*. Evidently, in order to give substance to this idea, we must formulate the shape uncertainty principle for loops, and the centerpiece of this whole discussion is again the loop wave functional $\Psi[C; A]$, whose precise meaning we already discussed.

7.2.1 The Hausdorf Dimension of a Quantum String

One of the major achievements of Feynman's formulation of quantum mechanics was to restore the particle's trajectory concept at the quantum level. However, the dominant contribution to the "sum over histories" is provided by trajectories which are nowhere differentiable [10]. Non differentiability is the hallmark of fractal lines. In fact, it seems that Feynman and Hibbs were aware that the quantum mechanical path of a particle is inherently fractal. This idea was revisited and further explored by Abbott and Wise in the case of a free nonrelativistic particle [11], and the extension to relativistic particles was carried out by several authors, but without a general agreement [12], [13]. This is one of the reasons for setting up a quantum mechanical, rather than a field theoretical, framework, even for relativistic objects. It enables us to adapt the Abbott-Wise discussion to the String case, with the following basic substitution: the point particle, erratically moving through euclidean space, is replaced by the String configuration whose representative point randomly drifts through loop space. We have shown in the previous chapters that "flow of time" for particles is replaced by area variations for Strings. Hence, the image of an abstract linear "trajectory" connecting the two "points" C_0 and C in a lapse of time A, corresponds, physically, to a family of closed lines stacked into a two-dimensional surface of proper area A. Here is where the quantum mechanical aspect of our approach and our choice of dynamical variables seem to have a distinct advantage over the more conventional relativistic description of String Dynamics. The conventional picture of a String World-Sheet, consisting of a collection of world-lines associated with each constituent point, is replaced by a World-Sheet "foliation" consisting of a stack of closed lines labeled by the internal parameter A. In other words, we interpret the String World-Sheet as a sequence of "snapshots" of single closed lines ordered with respect to the area of the Parameter Space associated with the World-Sheet formed by them. Then, the randomness of the "motion" of a point in loop space is a reflection of the non-differentiability of the String World-Sheet, which, in turn, is due to zero-point quantum fluctuations: the random addition of petals to each loop results in a fuzziness, or graininess of the world surface, by which the String stack acquires an effective thickness. One can expect that this graininess becomes apparent only when one can resolve the surface small irregularities.

The technical discussion on which this picture is based, follows closely the analysis by Abbott and Wise [11]. Thus, let us divide the *String* internal coordinates domain, which is the Parameter Space, into N strips of area ΔA . Accordingly, the *String* stack is approximated

by the discrete set of the N+1 loops,

$$Y^{\mu}_{(n)}(s) = Y^{\mu}(s; n\Delta A)$$
 , $n = 0, 1, \dots, N$.

Suppose now that we take a snapshot of each one of them and measure their "area in Parameter Space". If the cross section of the emulsion grains is $\Delta \sigma$, then we have an area indeterminacy greater or equal to $\Delta \sigma$. Then, the total area of the surface subtended by the last and first loop in the stack will be given by

$$\langle S \rangle = N \langle \Delta S \rangle \tag{7.22}$$

where, $\langle \Delta S \rangle$ is the average area variation in the interval ΔA ,

$$\langle \Delta S \rangle \equiv \left[\int \left[\mathcal{D} Y^{\lambda \rho} \right] Y^{\mu \nu} \left[C \right] Y_{\mu \nu} \left[C \right] \left| \Psi \left[C ; \Delta A \right] \right|^2 \right]^{1/2}. \tag{7.23}$$

The finite resolution in $Y^{\mu\nu}$ is properly taken into account by choosing for $\Psi[C; \Delta A]$ a gaussian wave functional of the type (5.11). After these preliminary remarks we can give a central definition for the contents of this chapter.

Definition 7.1 (*D*-measure).

The <u>D-measure</u> S_D of the previously considered stack of fluctuating loops is

$$S_D \stackrel{\text{def.}}{=} N \langle \Delta S \rangle (\Delta \sigma)^{D-2} \tag{7.24}$$

where $D \in \mathbb{R}$ and we call $\Delta \sigma$ the <u>resolution</u> of the measure³.

Of course, not all values of D give equally useful D-measures; in particular we can single out a particular value of D thanks to the following definition.

Definition 7.2 (Hausdorff Dimension & Hausdorff Measure).

The <u>Hausdorff Dimension</u> of the stack of fluctuating loops is the value $D = D_H$ such that its D_H -measure is independent from the resolution $\Delta \sigma$. Then the D_H -measure is called the <u>Hausdorff Measure</u> of the stack and denoted with S_H

³Since we have in mind physical application of this concept we are willing to leave this ambiguous joke between different meanings of the word *measure*, in particular between the more rigorous mathematical one and the more "phenomenological" one.

Now, we have at our disposal all the tools that we need to determine the *Hausdorff Dimension* associated to the loop *Shadow Dynamics*. In particular, we can show that

Proposition 7.4 (Hausdorff Dimension of the Quantum Dynamic Process).

The Quantum Shadow Dynamics is characterized by the Hausdorff dimension $D_H = 3$.

Proof:

To get the desired result we represent the quantum state of the String by the loop functional

$$\Psi\left[C\right] = \frac{1}{\left(2\pi\right)^{3/2}} \int \left[\mathcal{D}P_{\mu\nu}\left(s\right)\right] \tilde{\Phi}_{G}\left[P\right] \exp\left\{\frac{i}{2} \oint_{C} Q_{\mu\nu} Y^{\mu} dY^{\nu}\right\}$$
(7.25)

where $\tilde{\Phi}_{G}(P)$ is a gaussian momentum distribution centered around a vanishing *String* average momentum $K_{\mu\nu} = 0$, i.e. we consider a free loop subject only to zero-point fluctuations. Then,

$$\langle \Delta S \rangle \propto \frac{\Delta A}{4m^2 \Delta \sigma} \sqrt{1 + \left(\frac{4m^2 (\Delta \sigma)^2}{\Delta A}\right)^2} ,$$
 (7.26)

and to determine the String fractal dimension we keep ΔA fixed and take the limit $\Delta \sigma \rightarrow 0$:

$$S_H \approx \frac{N\Delta A}{4m^2\Delta\sigma} \left(\Delta\sigma\right)^{D_H-2}.$$
 (7.27)

Hence, in order to eliminate the dependence on $\Delta \sigma$, $D_H = 3$.

As in the point particle case, quantum fluctuations increase by one unit the dimension of the *String* classical path. As discussed in the following subsection, the appearance of an extra dimension is perceived as fuzziness of the *String* manifold and the next question to be addressed is which parameter, in our quantum mechanical approach, controls the transition from classical to fractal geometry.

7.2.2 Classical-to-Fractal Geometric Transition

The role of area resolution, as pointed out in the previous section, leads us to search for a critical area characterizing the transition from Classical to Fractal Geometry of the String stack. We can define

Definition 7.3 (DeBroglie Area of a String).

The <u>DeBroglie Area</u>, Λ_{DB} , of a String is the critical area characterizing the transition between the Classical and the Quantum behaviour of the String.

Now, we would like to understand the features related to such a transition. In particular we will characterize in terms of the *Hausdorff Dimension* the transition, so that the distinction between the Classical and the Quantum behavior will be associated to the distinction between the standard and the fractal geometry of the *String* stack. This is the content of the following

Proposition 7.5 (Classical-to-Fractal Transition).

Let us consider a String with a non-vanishing average momentum $K_{\mu\nu}$ along the loop C. The classical behavior of the String is characterized by the Hausdorff Dimension $D_H=2$ for the stack of Strings that represents its classical evolution, whereas the quantum behavior is characterized by the Hausdorff Dimension $D_H=3$. Moreover, the De Broglie Area of the String can be identified with the quantity

$$\left[\frac{1}{2}K_{\mu\nu}\left[C\right]K^{\mu\nu}\left[C\right]\right]^{1/2}$$

Proof:

Since we consider the case in which the *String* possesses a non vanishing average momentum we may represent it with a Gaussian wave packet of the form (5.12) remembering that $K_{\mu\nu}$ is constant along the loop. The *A*-dependent wave functional, in the Holographic Coordinate Representation, corresponding to an initial wave packet of the form (5.12) is thus

$$\Psi_{K}[C;A] = \frac{\left[(\Delta\sigma)^{2} / (2\pi) \right]^{3/4}}{\left[(\Delta\sigma)^{2} + iA/(2m^{2}) \right]^{3/2}} \cdot \exp \left\{ -\frac{\left(Y^{\mu\nu}Y_{\mu\nu} - 2i(\Delta\sigma)Y^{\mu\nu}K_{\mu\nu} + iA(\Delta\sigma)^{2}K_{\mu\nu}K^{\mu\nu} / (4m^{2}) \right)}{2\left[(\Delta\sigma)^{2} + iA/(2m^{2}) \right]} \right\} , \quad (7.28)$$

where we have used equation (7.1) to exchange ΔQ with $\Delta \sigma$. The corresponding probability density "evolves" as follows

$$\left|\Psi_{K}\left[C;A\right]\right|^{2} = \frac{(2\pi)^{-3/2}}{\left[\left(\delta\sigma\right)^{2} + A^{2}/\left(4\left(\delta\sigma\right)^{2}m^{4}\right)\right]^{3/2}} \exp\left\{-\frac{\left[Y^{\mu\nu}\left[C\right] - AK^{\mu\nu}/\left(2\left(\delta\sigma\right)m^{2}\right)\right]^{2}}{\left[\left(\delta\sigma\right)^{2} + A^{2}/\left(4\left(\delta\sigma\right)^{2}m^{4}\right)\right]}\right\}$$
(7.29)

Therefore, the average area variation $\langle \Delta S \rangle$, when the loop wave packet drifts with a momentum $K_{\mu\nu}[C]$, is

$$\langle \Delta S \rangle \equiv \left[\int \left[\mathcal{D} Y^{\lambda \rho} \right] Y^{\mu \nu} \left[C \right] Y_{\mu \nu} \left[C \right] \left| \Psi_K \left[C ; \Delta A \right] \right|^2 \right]^{1/2} . \tag{7.30}$$

For our purpose, there is no need to compute the exact form of the mean value (7.30), but only its dependence on $\Delta \sigma$. This can be done in three steps:

1. introduce the adimensional integration variable

$$y^{\mu\nu} \left[C \right] \equiv \frac{Y^{\mu\nu} \left[C \right]}{\Delta \sigma} \quad ; \tag{7.31}$$

2. shift the new integration variable as follows:

$$y^{\mu\nu}[C] \to \bar{y}^{\mu\nu}[C] \equiv y^{\mu\nu}[C] - \frac{(\Delta A)K^{\mu\nu}}{2m^2(\Delta\sigma)^2}$$
; (7.32)

3. rescale the integration variable as

$$\bar{y}^{\mu\nu}[C] \to Z^{\mu\nu}[C] \equiv \bar{y}^{\mu\nu}[C] \left[1 + \frac{(\Delta A)^2}{4m^4(\Delta \sigma)^4} \right]^{1/2}$$
 (7.33)

Then, we obtain

$$\langle \Delta S \rangle = \frac{\Delta A}{\sqrt{2} \Lambda_{DB} 2m^2 (2\pi)^{3/4}} \left[\int [\mathcal{D}Z] \left(\frac{\Lambda_{DB} Z^{\mu\nu}}{(\Delta \sigma)} \sqrt{1 + \beta^{-2}} + \Lambda_{DB} K^{\mu\nu} \right)^2 e^{-Z^{\mu\nu} Z_{\mu\nu}/2} \right]^{1/2} , \tag{7.34}$$

where

$$\Lambda_{DB}^{-1} \equiv \sqrt{\frac{1}{2}K^{\mu\nu}K_{\mu\nu}} \tag{7.35}$$

$$\beta \equiv \frac{\Delta A}{2m^2 \left(\Delta\sigma\right)^2} \quad . \tag{7.36}$$

The parameter β measures the ratio of the "temporal" to "spatial" uncertainty, while the area Λ_{DB} sets the scale of the surface variation at which the *String* momentum is $K_{\mu\nu}$. Therefore, with the particle analogy in mind, we see that Λ_{DB} can be assigned the role of *loop De Broglie Area*. Let us assume, for the moment, that ΔA is independent of $\Delta \sigma$, so that either quantity can be treated as a free parameter in the Theory. A notable exception to this hypothesis will be discussed shortly. Presently, we note that taking the limit $(\Delta \sigma) \to 0$, affects only the first term of the integral (7.34) and that its weight with respect to the second term is measured by the ratio $\Lambda_{DB}/(\Delta \sigma)$. If the area resolution is much larger than the loop De Broglie area, then the first term is negligible: $\langle \Delta S \rangle$ is independent of $\Delta \sigma$ and $\langle S \rangle$ scales as

$$\Lambda_{DB} \ll (\Delta \sigma) : \mathcal{S}_H \approx (\Delta \sigma)^{D_H - 2} .$$
 (7.37)

In this case, independence of $(\Delta \sigma)$ is achieved by assigning $D_H = 2$. As one might have anticipated, the detecting apparatus is unable to resolve the graininess of the *String* stack, which therefore appears as a smooth two dimensional surface.

The fractal, or quantum, behavior manifests itself below Λ_{DB} , when the first term in (7.34) provides the leading contribution

$$\Lambda_{DB} \gg (\Delta \sigma) \quad : \quad \mathcal{S}_{H} \approx \frac{N(\Delta A)}{\Delta \sigma} (\Delta \sigma)^{D_{H}-2} \sqrt{1 + \frac{4m^{4} (\Delta \sigma)^{4}}{(\Delta A)^{2}}} \quad .$$
(7.38)

This expression is less transparent than the relation (7.37), as it involves also the ΔA resolution. However, one may now consider two special sub cases in which the Hausdorff dimension can be assigned a definite value.

In the first case, we keep ΔA fixed and scale $\Delta \sigma$ down to zero. Then,

$$\langle \Delta S \rangle \propto (\Delta \sigma)^{-1}$$

diverges, because of larger and larger shape fluctuations, and

$$S_H \approx \frac{A}{\Delta \sigma} \left(\Delta \sigma\right)^{D_H - 2} \tag{7.39}$$

requires $D_H = 3$.

The same result can be obtained also in the second subcase, in which both $\Delta \sigma$ and ΔA scale down to zero, but in such a way that their ratio remains constant,

$$\left[\frac{2m^2 (\Delta \sigma)^2}{(\Delta A)}\right]_{\Delta \sigma \to 0} = \text{const.} \equiv \frac{1}{b} \quad . \tag{7.40}$$

The total area of the Parameter Space $A = N(\Delta A)$ is kept fixed. Therefore, as $\Delta A \sim (\Delta \sigma)^2 \to 0$, then $N \to \infty$ in order to keep A finite. Then,

$$\langle \Delta S \rangle \propto \frac{\Delta A}{\Delta \sigma} \sqrt{1 + \frac{1}{b^2}} \propto \Delta \sigma$$
 (7.41)

and

$$S_H \propto A \left(\Delta\sigma\right)^{D_H - 2} \frac{1}{\Delta\sigma} \sqrt{1 + \frac{1}{b^2}} \quad ,$$
 (7.42)

which leads to $D_H = 3$ again. In the language of fractal geometry, this interesting subcase corresponds to *self-similarity*. Thus, the condition (7.41) defines a special class of *self-similar loops* characterized by an average area variation which is proportional to $\Delta \sigma$ at any scale.

Chapter 8

The "Double" Classical Limit

"It's all right, you can trust him."

8.1 Couplings and Limits

$$T = \frac{c}{\pi \alpha'} \quad . \tag{8.1}$$

is the tension of a classical String. In the limit of vanishing Regge slope $\alpha' \to 0$ the String tension diverges $T \to \infty$.

At the quantum level the classical action is measured in \hbar units. Thus, one can introduce an effective coupling constant

$$T' \equiv \frac{c}{\hbar\pi\alpha'} \quad . \tag{8.2}$$

Accordingly, the Classical Limit $\hbar \to 0$ is usually identified with the infinite tension limit. However, we shall keep distinct the two limiting procedures in what follows. To avoid any confusion, we shall maintain the term "classical limit" in the original form, i.e. $\hbar \to 0$ with α' fixed, while in the infinite tension, or pointlike limit, \hbar is fixed while $\alpha' \to 0$.

8.2 Classical Limit

In this chapter we will see how, starting from the String Functional Wave Equation, it is possible to recover, in what we will call the Classical Limit, a very interesting Classical Field Theory of Extended Objects [24]. In this formulation an extended object (we will concentrate on 1-dimensional ones, i.e. Strings, but the same is true for membranes and p-branes in complete generality) is conveniently described through a current, which is seen as a source of the object itself and has support only over the World-Sheet (or in general the World-HyperTube).

As a first step in taking this limiting procedure we see how the Functional Schrödinger Equation can be derived starting from a Lagrangian written in terms of the String Wave Functional, Ψ [C].

Proposition 8.1 (Quantum Lagrangian Density I).

The Lagrangian density

$$\mathcal{L}(\Psi, \Psi^*) = \frac{2\pi\alpha'}{4} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{Y'^2} \frac{\delta\Psi^*}{\delta Y^{\mu\nu}(s)} \frac{\delta\Psi}{\delta Y_{\mu\nu}(s)} + \Psi^* i \partial_A \Psi$$
 (8.3)

is a Lagrangian Density associated with the Functional Schrödinger Equation (3.63).

Proof:

To get the desired result we have to make a variation of, say, the $\Psi[C;A]$ functional and compute the corresponding variation of the action

$$S = \int \left[\mathcal{D}C \right] \int dA \mathcal{L} \left[\Psi, \Psi^* \right] \quad .$$

After integration by parts this gives the desired ressult.

Now we specialize the result (8.3) above to the case of tension eigenstates

$$\Psi = \exp\left(-\frac{i}{2\pi\alpha'}A\right)\Phi \quad . \tag{8.4}$$

Then, we obtain a Lagrangian Density for the "stationary", i.e. A independent, wave functional

$$\mathcal{L}(\Phi, \Phi^*) = \frac{1}{4} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{Y'^2} \frac{\delta \Phi^*}{\delta Y^{\mu\nu}(s)} \frac{\delta \Phi}{\delta Y_{\mu\nu}(s)} + \left(\frac{1}{2\pi\alpha'} \right)^2 \Phi^* \Phi \quad , \tag{8.5}$$

where $\Phi \equiv \Phi[C]$ is a functional of the loop C, $C: Y^{\mu} = Y^{\mu}(s)$. To carry on the limit procedure we introduce the following *Parametrization* of the loop C

$$Y^{\mu}(s) = y^{\mu} + \sqrt{2\pi\alpha'}\kappa^{\mu}(s) \quad . \tag{8.6}$$

where, y is the loop center of mass, or "zero mode" defined as:

$$y^{\mu} = \left(\oint_{\Gamma} dl(s)\right)^{-1} \oint_{\Gamma} ds \sqrt{\left(\mathbf{Y}'\right)^{2}} Y^{\mu}(s) \tag{8.7}$$

Accordingly, we can write

$$\Phi\left[C\right] = \Phi\left[y^{\mu} + \sqrt{\alpha'}\kappa^{\mu}\left(s\right)\right] \quad , \tag{8.8}$$

which is an appropriate form to expand the String field in powers of $\sqrt{\alpha'}$.

On the other hand, any complex functional can be written in terms of modulus and a phase. The novelty is that we can express the phase as a contour integral and write the Loop Functional as

$$\Psi\left[C\right] = \sqrt{P\left[C\right]} \exp\left\{\frac{i}{\hbar} \oint_{C} A_{\mu} dY^{\mu}\right\}$$

where

$$P\left[C\right] = |\Psi\left[C\right]|^2$$

and

$$\exp\left\{\frac{i}{\hbar}\oint_C A_\mu dY^\mu\right\}$$

is the Abelian Wilson Loop associated to a fictitious point charge traveling along the closed contour C. The Stokes Theorem allows one to express the Wilson factor in term of the flux of the field strength of A across any surface bounded by C

$$\Phi [C] = \sqrt{P[C]} \exp \left\{ \frac{i}{\hbar} \oint_{C=\partial \mathcal{W}} A_{\mu} dY^{\mu} \right\}
= \sqrt{P[C]} \exp \left\{ \frac{i}{2\hbar} \int_{\mathcal{W}} \partial_{[\mu} A_{\nu]} dX^{\mu} \wedge dX^{\nu} \right\}
= \sqrt{P[C]} \exp \left\{ \frac{i}{2\hbar} \int_{\mathcal{W}} B_{\mu\nu} dX^{\mu} \wedge dX^{\nu} \right\} ,$$
(8.9)

where we defined the field strength of A_{μ} as $B_{\mu\nu} = \partial_{[\mu}A_{\nu]}$

Using expression (8.9) for the Wave Functional we can re-express the Lagrangian (8.5) in terms of P[C] and the field strength $B_{\mu\nu}$.

Proposition 8.2 (Quantum Lagrangian Density II).

The expression of the Lagrangian Density (8.5) in terms of P[C] and $B_{\mu\nu}$ is

$$L = -\frac{\hbar^2}{4} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{Y'^2} \left[\frac{\delta \sqrt{P[C]}}{\delta Y^{\mu\nu}(s)} \frac{\delta \sqrt{P[C]}}{\delta Y_{\mu\nu}(s)} + \frac{P[C]}{4\hbar^2} B^{\mu\nu} B_{\mu\nu} \right] + \left(\frac{1}{2\pi\alpha'} \right)^2 P[C] \quad . \tag{8.10}$$

Proof:

We first compute the Holographic Derivative of Φ

$$\frac{\delta\Phi\left[C\right]}{\delta Y^{\mu\nu}\left(s\right)} = \left[\frac{\delta\sqrt{P\left[C\right]}}{\delta Y^{\mu\nu}\left(s\right)} + \frac{i}{2\hbar}\sqrt{P\left[C\right]}B_{\mu\nu}\left(s\right)\right] \tag{8.11}$$

where, $B_{\mu\nu}(s)$ is a shorthand notation for the $B_{\mu\nu}(x)$ field evaluated at the point $x^{\mu} = Y^{(s)}$ along the loop. The modulus square of (8.11) is

$$\frac{\delta\Phi^{*}\left[C\right]}{\delta Y^{\mu\nu}\left(s\right)} \frac{\delta\Phi\left[C\right]}{\delta Y_{\mu\nu}\left(s\right)} = \left[\left(\frac{\delta\sqrt{P\left[C\right]}}{\delta Y^{\mu\nu}\left(s\right)}\right)^{2} + \left(\frac{\sqrt{P\left[C\right]}B_{\mu\nu}}{2\hbar}\right)^{2} \right] \quad . \tag{8.12}$$

Then we can substitute this results into expression (8.5) together with equation (8.9) for $\Phi[C]$ to get the desired result.

As a last step towards the main result of this chapter, we give the following definition.

Definition 8.1 (Functional Current).

The <u>Functional Current</u>, which is the "Probability Current", associated with the Theory described by the Lagrangian Density (8.3) is

$$J_{\mu\nu}\left[C\right] = \frac{\hbar}{2i} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{Y^{\prime 2}} \left[\Psi^* \left[C \right] \frac{\delta \Phi \left[C \right]}{\delta Y^{\mu\nu} \left(s \right)} - \Psi \left[C \right] \frac{\delta \Phi^* \left[C \right]}{\delta Y^{\mu\nu} \left(s \right)} \right]$$
$$= \frac{\hbar}{2i} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{Y^{\prime 2}} \left[\Psi^* \left[C \right] \frac{\delta \Psi \left[C \right]}{\delta Y^{\mu\nu} \left(s \right)} - \Psi \left[C \right] \frac{\delta \Psi^* \left[C \right]}{\delta Y^{\mu\nu} \left(s \right)} \right] \quad . (8.13)$$

Now we are ready to compute the *Classical Limit* and see what is the resulting Theory. As a first step we give the following proposition.

Proposition 8.3 (Classical Lagrangian Density and Current).

In what we will call the classical limit, i.e. the limit where we neglect $\mathcal{O}(\hbar)$ terms and keep only first order quantities in the inverse String tension, the Lagrangian Density (8.10) and the current (8.13) turn into the following local quantities:

$$L \longrightarrow L_{\text{cl.}} = -\frac{1}{4}P(y)\,\partial_{[\mu}A_{\nu]}\partial^{[\mu}A^{\nu]} + \left(\frac{1}{2\pi\alpha'}\right)^2 P(y)$$

$$J_{\mu\nu} \longrightarrow P(y)\,B_{\mu\nu}(y) \quad . \tag{8.14}$$

Proof:

To obtain the desired results we first see the consequences of taking the classical limit in the two quantities that appear in the expression (8.9) for the wave functional, $\Phi[C]$ In particular the expansion 8.6 in P[C] gives

$$P[C] = P\left[y^{\mu} + \sqrt{2\pi\alpha'}\kappa^{\mu}(s)\right] \longrightarrow P(y^{\mu})$$
(8.15)

and in $B_{\mu\nu} = B_{\mu\nu}(s) = B_{\mu\nu}(Y^{\rho}(s))$ results in

$$B_{\mu\nu}(s) = B_{\mu\nu} \left(y^{\mu} + \sqrt{2\pi\alpha'} \kappa^{\mu}(s) \right) \longrightarrow B_{\mu\nu}(y) = \partial_{[\mu} A_{\nu]}(y) \quad .$$
 (8.16)

Note how taking the limit of large String tension amounts to squeezing the String to a single point, which is the String center of mass. At larger scales its extension is no more detectable and we recover from functionals, ordinary Quantum Fields. Substituting the last results in the expression (8.10) for the Lagrangian, we get in the limit $\sqrt{2\pi\alpha'} \to 0$,

$$L \longrightarrow -\frac{1}{4}P(y)\,\partial_{[\mu}A_{\nu]}\partial^{[\mu}A^{\nu]} + \left(\frac{1}{2\pi\alpha'}\right)^2 P(y) + \mathcal{O}\left(\hbar^2\right) \quad . \tag{8.17}$$

Moreover if we let also $\hbar \to 0$, we remain with the Lagrangian Density

$$\mathcal{L}_{\text{cl.}} = -\frac{1}{4} P(y) \, \partial_{[\mu} A_{\nu]} \partial^{[\mu} A^{\nu]} + \left(\frac{1}{2\pi\alpha'}\right)^2 P(y) \tag{8.18}$$

or, equivalently, with the action

$$S_{\text{cl.}} = \int d^4 y P(y) \left(-\frac{1}{4} \partial_{[\mu} A_{\nu]} \partial^{[\mu} A^{\nu]} + \left(\frac{1}{2\pi\alpha'} \right)^2 \right) \quad . \tag{8.19}$$

The action (8.19) can now be varied with respect to P(y),

$$\frac{\delta S_{\text{cl.}}}{\delta P(y)} = 0 \quad \Rightarrow \quad -\frac{1}{4} B_{\mu\nu} (y) B^{\mu\nu} (y) + \frac{1}{(2\pi\alpha')^2} = 0 \quad , \tag{8.20}$$

and, with respect to A_{μ} :

$$\frac{\delta S_{\text{cl.}}}{\delta \mathcal{A}_{\mu}(y)} = 0 \quad \Rightarrow \quad \partial_{\nu} \left[P(y) \, \partial^{[\mu} A^{\nu]} \right] = 0 \quad . \tag{8.21}$$

We can now follow the same procedure for the functional current Inserting again expression (8.11) together with expansions (8.15-8.16) in (8.13) we get

$$J_{\mu\nu}\left[C\right] \longrightarrow J_{\mu\nu}(y) = \frac{\hbar}{2i} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{Y^{\prime 2}} \left[\frac{2i}{\hbar} P(y) \, \partial_{[\mu} A_{\nu]}(y) \right]$$
$$= P(y) \, B_{\mu\nu}(y) \quad . \tag{8.22}$$

Now we can turn to the final result, i.e. to recover the Classical Field Formulation for a Closed *String* described in [24].

Proposition 8.4 (Gauge Theory of the String Geodesic Field). The classical limit of the Theory described by the Lagrangian (8.3) is the Classical Formulation of the Gauge Theory for the String Geodesic Field¹.

Proof:

Starting from equation (8.22) we see that

$$B^{\mu\nu}(y) = \frac{J^{\mu\nu}(y)}{P(y)} \tag{8.23}$$

and we can use this result to rewrite the classical action (8.19) in terms of the classical current; substituting we get

$$I_{\text{cl.eq.}} = \int d^4 y \left[-\frac{1}{4} \frac{J^{\mu\nu}(y) J_{\mu\nu}(y)}{P(y)} + \left(\frac{1}{2\pi\alpha'}\right)^2 P(y) + B_{\nu} \partial_{\mu} J^{\mu\nu} \right] , \qquad (8.24)$$

where the last term is necessary to assure that the current $J^{\mu\nu}$ is divergence less and B_{μ} is a non-dynamical Lagrange multiplier enforcing the previous condition. Varying with respect to P(x) we obtain

$$\frac{\delta I_{\text{cl.eq.}}}{\delta P(y)} = 0 \quad \Rightarrow \quad \frac{1}{4} \frac{J^{\mu\nu}(y) J_{\mu\nu}(y)}{P(y)} + \left(\frac{1}{2\pi\alpha'}\right)^2 = 0$$

$$\Rightarrow \quad P(y)^2 = -\left(2\pi\alpha'\right)^2 \frac{J^{\mu\nu}(y) J_{\mu\nu}(y)}{4}$$

$$\Rightarrow \quad P(y) = \pm \left(2\pi\alpha'\right) \sqrt{\frac{J^{\mu\nu}(y) J_{\mu\nu}(y)}{4}} \quad . \tag{8.25}$$

¹Please, look at the work cited above for details on this formulation

Substituting then this solution in the action we get

$$I_{\text{cl.eq.red.}} = \int d^4x \left[\frac{1}{2\pi\alpha'} \sqrt{-\frac{1}{4} J^{\mu\nu}(x) J_{\mu\nu}(x)} + A_{\mu} \partial_{\nu} J^{\mu\nu} \right] ,$$
 (8.26)

which is the classical action for the Gauge Theory of the String geodesic field!

Chapter 9

Boundary versus Bulk Relation

"This is totally different."
"No! No different!
Only different in your mind.
You must unlearn
what you have learned."

9.1 Overview

In this section we recover a general representation for the quantum state of a relativistic closed line (loop) in terms of String degrees of freedom. The general form of the loop functional splits into the product of the Eguchi functional, encoding the Holographic Quantum Dynamics, times the Polyakov path-integral, taking into account the full Bulk Dynamics, times a loop effective action, which is needed to renormalize Boundary ultraviolet divergences. The Polyakov String action is derived as an effective action from the Covariant Schild Action (2.81) by functionally integrating out the World-Sheet coordinates. The Holographic Coordinates description of the Boundary Shadow Dynamics, is shown to be induced by the "zero mode" of the Bulk quantum fluctuations. Finally, we briefly comment about a "unified, fully covariant" description of points, loops and Strings in terms of Matrix Coordinates.

Non perturbative effects in Quantum Field Theory are usually difficult to study because of the missing of appropriate mathematical tools. In the few cases where some special invariance, like electric/magnetic duality or supersymmetry, allows to open a window over the strong coupling regime of the Theory, one usually is faced with a new kind of solitonic excitations describing extended field configurations, which are "dual" to the pointlike states of the perturbative regime. Remarkable examples of extended structures of this sort ranges from the Dirac magnetic monopoles [28] up to the loop states of *Quantum Gravity*.

The Dynamics of these extended objects is usually formulated in terms of an "Effective Theory" of Relativistic *Strings*, i.e. one switches from a Quantum Field theoretical framework to a different description by taking for granted that there is some relation between the two. Such a kind of relation has recently shown up in the context of Superstring Theory as a web of dualities among different phases of different superstring models. On the other hand, there is still no explicit way to connect pointlike and stringy phases of non–supersymmetric Field Theories like QCD and Quantum Gravity.

The main purpose of this section is to recover a general String Representation of a loop wave functional in terms of the Bulk and Boundary wave functional of a Quantum String. The loop wave functional can describe the quantum state of a 1-dimensional, closed excitation of some Quantum Field Theory, while the corresponding String functional is induced by the quantum fluctuations of an open World-Sheet. We shall find a quite general relation linking these two different objects. The matching between a quantum loop and a quantum String being provided by a Functional Fourier transform over an abelian vector field.

As a byproduct of this new *Loops/Strings connection*, we shall clarify the interplay between the *Bulk Quantum Dynamics*, encoded into the Polyakov path–integral [29] and the induced *Boundary* Dynamics, which, as we already saw, is the *Holographic Description* of the *Boundary* quantum fluctuations.

9.2 Loop and String States

To accomplish the task, that we briefly outlined above, we will use the Covariant Schild Action of equation (2.81) that we already introduced in section 2.8 using the canonical formalism described in section 2.7. The quantum state of the Closed Bosonic String is described as in chapters 5 and 7 by $\Psi[C]$, the complex functional of the Target Space

Boundary $C = \partial W$ or, equivalently, of the Parameter Space Boundary $\Gamma = \partial \Sigma$. After the results we gained in the previous chapters, the following proposition should not be a surprise:

Proposition 9.1 (String Wave Functional: Covariant Formulation).

The quantum state of a Closed Bosonic String can be expressed as a "phase space" path—integral, namely

$$\Psi[C] = \int_{\partial \Sigma = \Gamma} [\mathcal{D}X^{\mu}] [\mathcal{D}P_{m|\mu}] [\mathcal{D}g_{ab}] \exp\left(i \int_{\Sigma} d^2 \boldsymbol{\sigma} \sqrt{|\boldsymbol{g}|} \mathcal{L}_{(\boldsymbol{g})}(\boldsymbol{X}, \boldsymbol{P}, g_{ab})\right) , \qquad (9.1)$$

where $\mathcal{L}_{(q)}$, the Covariant Schild Lagrangian Density, can be read out of equation (2.81) as

$$\mathcal{L}_{(g)}(\boldsymbol{Y}, \boldsymbol{P}, g_{ab}) = g^{mn} \left[\partial_m X^{\mu}(\boldsymbol{\sigma}) P_{n|\mu}(\boldsymbol{\sigma}) - \frac{1}{2m^2} P_m^{|\mu}(\boldsymbol{\sigma}) P_{n|\mu}(\boldsymbol{\sigma}) \right]$$

and in the path-integral (9.1) we sum over all the the String World-Sheets having the closed curve Γ in Parameter Space (equivalently C in Target Space) as the only Boundary.

Proof:

We apply here the same concepts that we developed in chapter 3. There we saw that the propagator for an extended object (string) can be interpreted as the probability amplitude to obatin a shape modification from an initial one, described by a loop C_0 , to a final one, described by a loop C. We wrote the propagator $K[Y^{\mu}, Y_0^{\mu}; A]$ to fully emphasize this situation. Now the quantity which we call $\Psi[C]$ is equivalent to $K[Y^{\mu}, \emptyset; A]$, where with \emptyset we denote a vanishing initial string configuration. Note that, since we are using the *Covariant Schild Action*, we have also to functionally integrate over the *World-Sheet* metric, g^{ab} , since now it is an additional dynamical variable.

We note now that to a closed line and an open 2-surface can be given different geometrical characterizations. In this case we are going to use a formulation slightly different from the one used extensively before, which resembles more the contents of chapter 8: we will use the associated currents (the *Boundary* as well as the *Bulk* one) which physically represent the sources of our extended objects.

Definition 9.1 (Bulk Current).

The Bulk Current $J^{\mu\nu}(x;\mathcal{W})$ is a rank 2 antisymmetric tensor distribution having non van-

[9.2].147

ishing support over the two dimensional World-Sheet W, parametrized by $X^{\mu}(\sigma)$:

$$J^{\mu\nu}(x;\mathcal{W}) \stackrel{\text{def.}}{=} \int_{\Sigma} d^2 \boldsymbol{\sigma} \dot{X}^{\mu\nu}(\boldsymbol{\sigma}) \, \delta^D \left[x - \boldsymbol{X}(\boldsymbol{\sigma}) \right] \quad . \tag{9.2}$$

Definition 9.2 (Loop Current).

The Loop Current $J^{\mu}(y;L)$ is a vector distribution with support over a closed line L which represents a loop and is parametrized by $l^{\mu}(s)$:

$$J^{\mu}(x;L) \stackrel{\text{def.}}{=} \oint_{\Gamma} ds \frac{dl^{\mu}(s)}{ds} \delta\left[y - \boldsymbol{l}(s)\right] \quad . \tag{9.3}$$

Definition 9.3 (Boundary Current).

The Boundary Current is the divergence of the Bulk Current:

$$J^{\mu}(x;\partial \mathcal{W}) \stackrel{\text{def.}}{=} \partial_{\lambda} J^{\lambda \mu}(x;\mathcal{W}) \quad . \tag{9.4}$$

Please, note that at this stage the Loop Current is not linked in any way to the Bulk Current, i.e. the loop can be considered free: it is not the Boundary of the World-Sheet. Of course we need to implement such a constraint, because our String is the only free Boundary of its history, and a natural way to link a closed line to a surface is by "appending" the surface to the assigned loop [30]. This matching condition can be formally written by identifying the loop current with the Boundary current:

$$J^{\mu}(x;L) = J^{\mu}(x;\partial \mathcal{W} = C) = \partial_{\nu}J^{\nu\mu}(x;\mathcal{W}) \quad : \tag{9.5}$$

this is the mathematical way of requiring $L \equiv C = \partial W$. We remark again that equation (9.5) defines $J^{\mu}(x; L \equiv C)$ as the current associated to the Boundary C of the surface W. In the absence of (9.5) $J^{\mu}(x; L)$ is a loop current with no reference to any surface. In this sense, equation (9.5) is a formal description of the "gluing operation" between the surface W and the closed line C.

Accordingly, one would relate loops states to String states through a functional relation of the type

$$\bar{\Psi}[L] = \int [\mathcal{D}C] \,\bar{\delta}[L - C] \,\Psi[C] \quad , \tag{9.6}$$

which in a more explicit way can be written

$$\bar{\Psi}[L] = \int [\mathcal{D}C] \,\bar{\delta}[L - \partial \mathcal{W}] \,\Psi[\partial \mathcal{W}] \quad ; \tag{9.7}$$

here, we introduced a Loop Dirac Functional which picks up the assigned loop configuration L among the (infinite) family of all the allowed String Boundary configurations $\partial W = C$: the definition of such an object is definitely non trivial and one of its possible representations, that we will take as definition, is given below; in particular a more appropriate definition of such a Loop Dirac Functional can be offered by the "current representation" of extended objects (provided by (9.2), (9.3), (9.4)). To make rigorous the previous formal step, we thus give the following definition.

Definition 9.4 (Loop Dirac Functional).

The <u>Loop Dirac Functional</u> (formally written in equation (9.6)) can be given a suitable "Fourier" form as a functional integral over a vector field $A_{\mu}(x)$:

$$\bar{\delta} [L - \partial \mathcal{W}] = \delta [J^{\mu}(x; L) - J^{\mu}(x; \partial \mathcal{W})]$$

$$= \int [\mathcal{D}A_{\mu}(x)] \exp \left\{ -i \int d^{D}x A_{\mu}(x) [J^{\mu}(x; L) - J^{\mu}(x; \partial \mathcal{W})] \right\} . (9.8)$$

This gives us the possibility of rewriting the Loop Functional in a clearer form.

Proposition 9.2 (Loop Functional State).

The loop functional (9.6) can be written as

$$\bar{\Psi}[L] = \int [\mathcal{D}C] [\mathcal{D}X^{\mu}] [\mathcal{D}P_{m|\mu}] [\mathcal{D}g_{ab}] [\mathcal{D}A_{\mu}] \exp \left\{ i \int_{\Sigma} d^{2}\boldsymbol{\sigma} \sqrt{|\boldsymbol{g}|} \mathcal{L}_{(\boldsymbol{g})}(\boldsymbol{X}, \boldsymbol{P}, g_{ab}) \right\} \cdot \exp \left\{ -i \int d^{D}x A_{\mu}(x) [J^{\mu}(x; L) - J^{\mu}(x; \partial \mathcal{W})] \right\}$$
(9.9)

Proof:

The result follows by substituting equation (9.1) and applying definition 9.4 in equation (9.7).

These seemingly harmless manipulations are definitely non trivial. A proper implementation of the *Boundary* conditions introduces an *abelian vector field* coupled both to the loop and the *Boundary* currents. The first integral in (9.8) is the *circulation* of A along the loop L:

$$\int d^D y \, A_{\mu}(y) \, J^{\mu}(y; L) = \oint_L dl^{\mu} A_{\mu}(l) \quad ; \tag{9.10}$$

in the same way the second represents the circulation along $\partial \mathcal{W} = C$:

$$\int d^D x A_{\mu}(x) J^{\mu}(x; \partial \mathcal{W}) = \oint_{\partial \mathcal{W}} dY^{\mu} A_{\mu}(Y) \quad . \tag{9.11}$$

Now, let us recall the usual definition of the Wilson factor.

Definition 9.5 (Wilson Factor).

The Wilson Factor associated with a loop L is

$$W\left[A_{\mu}, L\right] \stackrel{\text{def.}}{=} \exp\left[-i \oint_{L} dl^{\mu} A_{\mu}(l)\right] \quad . \tag{9.12}$$

In terms of the Wilson Factor we can define the Loop Transform of a Loop Functional $\Psi[\tilde{L}]$ of a loop \tilde{L} :

Definition 9.6 (Loop Transform).

The <u>Loop Transform</u> of the Loop Functional $\Psi[\tilde{C}]$ is the functional integral over all the possible Loops of the product of the Wilson Factor times the Loop Functional:

$$\psi [A_{\mu}(x)] \stackrel{\text{def.}}{=} \int [\mathcal{D}C]W[A_{\mu}, C]\Psi[C]$$
.

Note that, if C is parametrized by $Y^{\mu}\left(s\right)$, we can more clearly express the equation above as

$$\psi \left[A_{\mu}(x) \right] \stackrel{\text{def.}}{=} \int [\mathcal{D}Y^{\mu}(s)] W[A_{\mu}, C] \Psi \left[C \right] .$$

Notation 9.1 (Loop Transform and Inverse Loop Transform).

We will use the following shorthand for the Loop Transform that exchanges the loop \tilde{L} with

the abelian vector field $A_{\mu}(x)$:

$$\mathfrak{L}_{A,\tilde{L}}$$
 : $\psi[A_{\mu}(x)] = \mathfrak{L}_{A,\tilde{L}}\Psi[\tilde{L}]$,

as well as the following

$$\mathfrak{L}^{-1}$$
 : $\Psi[L] = \mathfrak{L}_{L,\mathbf{A}}^{-1} \psi[A_{\mu}(x)]$,

for its inverse, which exchanges the abelian vector field $A_{\mu}(x)$ with its support L.

We will give a natural name to $\psi[A_{\mu}]$:

Notation 9.2 (Dual String Functional).

Let $\Psi[C]$ be a String Functional. We will call Loop Transformed String Functional, $\psi[A_{\mu}]$, the <u>Dual Loop Functional</u>.

We also note the Loop Dirac Functional can be expressed in terms of Wilson Loops.

Proposition 9.3 (Loop Dirac Functional and Wilson Loops).

Using the definition 9.5 for the Wilson Factor the Loop Dirac Functional can be written as

$$\bar{\delta}[C - L] = \int [\mathcal{D}A_{\rho}] W^{-1}[A_{\mu}, C] W[A_{\mu}, L] \qquad (9.13)$$

Proof:

We have

$$W^{-1}[A_{\mu}, C] = \exp\left[i\oint_C d\mathbf{Y}^{\mu}A_{\mu}(\mathbf{Y})\right] = \exp\left[i\int d^Dx A_{\mu}(x)J^{\mu}(x;\partial\mathcal{W})\right]$$

and

$$W\left[A_{\mu},L
ight]=\exp\left[-i\oint_{L}dm{l}^{\mu}A_{\mu}(m{l})
ight]=\exp\left[-i\int d^{D}xA_{\mu}(x)\,J^{\mu}(x;L)
ight]$$

Then

$$W^{-1}[A_{\mu}, C]W[A_{\mu}, L] = \exp\left\{-i \int d^{D}x A_{\mu}(x)[J^{\mu}(x; \partial L) - J^{\mu}(x; \partial W)]\right\}$$

so that

$$\int [\mathcal{D}A_{\rho}] W^{-1} [A_{\mu}, C] W [A_{\mu}, L] = \int [\mathcal{D}A_{\rho}] \exp \left\{ -i \int d^{D}x A_{\mu}(x) [J^{\mu}(x; \partial L) - J^{\mu}(x; \partial W)] \right\}$$

$$= \bar{\delta} [C - L] , \qquad (9.14)$$

where we used the equality $C = \partial \mathcal{W}$.

In this way we can see that there is a natural relation between a *Loop Functional* and the corresponding *String Functional*:

Proposition 9.4 (Relation between Loop and String Functionals).

Let $\Psi[C]$ be a String Functional. The corresponding Loop Functional $\bar{\Psi}[L]$ can be written as

$$\bar{\Psi}[L] = \int [\mathcal{D}A_{\mu}] W^{-1}[A_{\mu}, L] \psi[A_{\mu}] \quad , \tag{9.15}$$

where $\psi[A_{\mu}]$ is the Dual String Functional.

Proof:

We can use the representation of the $Dirac\ Fuction$ in terms of Wilson Loop Factors. Then starting from (9.7) we get

$$\bar{\psi}[L] = \int [\mathcal{D}C] \,\bar{\delta}[L - \partial \mathcal{W}] \,\Psi[\partial \mathcal{W}]$$

$$= \int [\mathcal{D}C] \,[\mathcal{D}A_{\rho}] \,W^{-1}[A_{\mu}, L] \,W[A_{\mu}, C] \,\Psi[\partial \mathcal{W}]$$

$$= \int [\mathcal{D}A_{\rho}] \,W^{-1}[A_{\mu}, L] \int [\mathcal{D}C] \,W[A_{\mu}, C] \,\Psi[\partial \mathcal{W}]$$

$$= \int [\mathcal{D}A_{\rho}] \,W^{-1}[A_{\mu}, L] \int [\mathcal{D}C] \,\psi[A_{\mu}] \quad .$$
(9.16)

Hence, The vector field $A_{\mu}(x)$ is the Fourier conjugate variable to the *String Boundary* configuration and the wanted result is obtained by projecting $\phi[A_{\mu}(x)]$ along the loop C. The whole procedure can be summarized as follows:

$$\mathfrak{L}_{\pmb{C},\pmb{A}}^{-1}: \ \Psi\left[C\right] \ String \ Functional \ \longrightarrow \ \psi\left[A_{\mu}\right] \ Dual \ String \ Functional$$

$$\mathfrak{L}_{\pmb{A},\pmb{L}}: \ \psi\left[A_{\mu}\right] \ Dual \ String \ Functional \ \longrightarrow \ \bar{\Psi}\left[L\right] \ Loop \ Functional \ .$$

The loop has thus been glued to the *Boundary* of the *World–Sheet*, which in the formulation in terms of currents means that the *Loop Current* has been identified with the *Boundary Current*, i.e. the divergence of the *Bulk Current*.

Let us proceed by unraveling the information contained in the *String* functional $\Psi[C]$. We already pointed out at the beginning of this section that we choose as the classical *String*

action the "covariant" Schild action of equation (2.81). We now would like to push forward the functional integration and as a first result we get that:

Proposition 9.5 (Bulk-Boundary Decoupling).

In the Loop Functional (9.1) the Bulk Dynamics decouples from the Boundary Dynamics.

Proof:

To begin with, it is instrumental to extract a pure Boundary term from the first integral in (2.81)

$$\frac{1}{2} \int_{\mathcal{W}} d\mathbf{X}^{\mu} \wedge d\mathbf{X}^{\nu} P_{\mu\nu} = \frac{1}{2} \int_{\mathcal{W}} d\left(Y^{\mu} d\mathbf{Y}^{\nu} P_{\mu\nu}\right) - \frac{1}{2} \int_{\mathcal{W}} Y^{\mu} d\mathbf{P}_{\mu\nu} \wedge d\mathbf{Y}^{\nu}$$

$$= \frac{1}{2} \oint_{C} Y^{\mu} d\mathbf{Y}^{\nu} Q_{\mu\nu}(Y) - \frac{1}{2} \int_{\Sigma} d^{2} \boldsymbol{\sigma} Y^{\mu}(\boldsymbol{\sigma}) \, \epsilon^{mn} \partial_{[m} P_{n]|\mu} \quad . \quad (9.17)$$

Then, we recognize that $X^{\mu}(\sigma)$ appears in the path–integral only in the last term of (9.17) through a linear coupling to the left hand side of the classical equation of motion (2.82). Accordingly, to integrate over the *String* coordinates is tantamount to integrate over a *Lagrange multiplier* enforcing the canonical momentum to satisfy the classical equation of motion (2.82):

$$\int \left[\mathcal{D}X^{\mu}(\boldsymbol{\sigma}) \right] \exp \left[\frac{1}{2} \int_{\Sigma} d^2 \boldsymbol{\sigma} Y^{\mu}(\boldsymbol{\sigma}) \, \epsilon^{mn} \partial_{[m} P_{n]|\mu} \right] = \delta \left[\partial_{[m} P_{n]|\mu} \right] \quad . \tag{9.18}$$

Once the String coordinates have been integrated out, the resulting path-integral reads

$$\Psi\left[C\right] = \int \left[\mathcal{D}g_{mn}\right] \left[\mathcal{D}P_{m|\mu}\right] \delta \left[\partial_{[m}P_{n]|\mu}\right] \cdot \\ \cdot \exp\left(\frac{i}{2} \oint_{C} Y^{\mu} dY^{\nu} Q_{\mu\nu}(Y) - \frac{i}{2\mu_{0}} \int_{\Sigma} d^{2}\sigma \sqrt{|g|} g^{mn} P_{m|\mu} P_{n}^{|\mu}\right) \quad . \tag{9.19}$$

Equation (9.19) shows that we have sum only over classical momentum trajectories. Such a restricted integration measure spans the subset of momentum trajectories we found in proposition 2.21. Thanks to them we can now give a definite meaning to the integration measure over the classical solutions, namely

$$\int \left[\mathcal{D}P_{m|\mu} \right] \delta \left[\partial_{[m} P_{n|\mu} \right] = \int d^D \bar{\eta} \int \left[d\bar{P}_{\mu\nu} \right] \int \left[\mathcal{D}\tilde{\eta}_{\mu}(\boldsymbol{\sigma}) \right] , \qquad (9.20)$$

where, we remark that the first two integrations are "over numbers" and not over functions. We have to sum over all possible constant values of $\bar{P}_{\mu\nu}$ and and $\bar{\eta}^{\mu}$. The constant mode of the *Bulk Momentum* does not mix with the other modes in the on–shell Hamiltonian because the cross term vanish identically

$$\bar{P}_{[\mu\nu]}g^{(mn)}\partial_{(m}Y^{[\nu}\partial_{n)}\eta^{\mu]} \equiv 0$$

$$\delta^{[mn]}\partial_{[m}Y^{\mu}\partial_{n]}\eta_{\mu} \equiv 0 . \tag{9.21}$$

Accordingly, Boundary Dynamics decouples from the Bulk Dynamics¹:

$$\frac{1}{2} \oint_{\gamma} \overline{y}^{\mu} d\overline{y}^{\nu} P_{\mu\nu}(\overline{y}) = \frac{1}{4} \overline{P}_{\mu\nu} \oint_{\gamma} d\sigma^{m} \overline{y}^{[\mu} \partial_{m} \overline{y}^{\nu]} \equiv \frac{1}{2} \overline{P}_{\mu\nu} \sigma^{\mu\nu}(\gamma) \tag{9.22}$$
erm

$$\frac{1}{2\sqrt{\mu_0}}\bar{P}_{\mu\nu}\oint_{\Gamma}dt^m\eta^{[\mu}\partial_mX^{\nu]} + \frac{1}{2}\oint_{\Gamma}dn^m\eta^{\mu}\partial_m\eta_{\mu}$$

vanishes because of the Boundary conditions (2.89), (2.90), (2.91).

¹A boundary term

$$-\frac{1}{2\mu_{0}}\int_{\Sigma}d^{2}\sigma\sqrt{|g|}g^{mn}P_{m|\mu}P_{n}^{|\mu} = -\frac{1}{4\mu_{0}}\bar{P}_{\mu\nu}\bar{P}^{\mu\nu}\int_{\Sigma}d^{2}\sigma\sqrt{|g|} - \frac{1}{2}\int_{\Sigma}d^{2}\sigma\sqrt{|g|}\eta_{\mu}\Delta_{g}\eta^{\mu} (9,23)$$

where, Δ_g is the covariant, World–Sheet D'Alembertian. This is the desired result.

The proof above shows as the dynamical areas, we introduced in previous chapters as the relevant dynamical variables to describe what we called the Shadow Dynamics of the Boundary, are selected by the system itself. We remark how the Holographic Coordinate appears as the canonical partner of the zero mode Bulk Area Momentum $\bar{P}_{\mu\nu}$. The zero mode Bulk Area Momentum transfers to the Boundary the World–Sheet vibrations. In this way we can see how the Functional Schrödinger Equation for the String is related to the propagation of the classical modes: a Quantum Dynamics for the Boundary turns out to be an induced effect of the "Classical" Dynamics of the Bulk. Moreover the Area Time

$$\int_{\Sigma} d^2 \sigma \sqrt{|g|} \equiv A \tag{9.24}$$

provides an intrinsic evolution parameter for the system. We take here the opportunity to develop a bit further this concept at the Quatum Level. Indeed, a quantum World-Sheet has not a definite area in Parameter Space, the metric in (9.24) being itself a quantum operator. Thus, the left hand side of the definition (9.24) has to be replaced by the corresponding quantum expectation value. Then, we can split the sum over the String metrics into a sum over metrics h_{mn} , with fixed quantum expectation value of the proper area, times an ordinary integral over all the values of the area quantum average:

$$\int \left[\mathcal{D}g_{mn}(\boldsymbol{\sigma}) \right] (\dots) = \int_0^\infty dA \exp\left(i\lambda A\right) \int \left[\mathcal{D}h_{mn}(\boldsymbol{\sigma}) \right] \exp\left(-i\lambda \int_{\Sigma} d^2 \boldsymbol{\sigma} \sqrt{|\boldsymbol{h}|} \right) (\dots)$$
(9.25)

The λ parameter enters the path–integral as a constant external source enforcing the condition that the quantum average of the proper area operator is A. From a physical point of view it represents the World–Sheet cosmological constant, or vacuum energy density.

We thus arrive at the fundamental result of this section.

Proposition 9.6 (Boundary versus Bulk Dynamics).

The Boundary and Bulk Dynamics encoded in the Loop Functional can be factorized as

$$\bar{\Psi}[L] \equiv \Psi[Y^{\mu}(s), Y^{\mu\nu}[C]]$$

$$= \int d^{D}\bar{\eta} \left[\exp\left(iS^{\text{eff}}[C]\right) \right] \int_{0}^{\infty} dA \exp\left(i\lambda A\right) \Psi[Y[C]; A] Z_{\text{BULK}}^{A} . \tag{9.26}$$

The Bulk Quantum Physics is encoded into the Polyakov partition function [29]

$$Z_{\text{BULK}}^{A} = \int \left[\mathcal{D}h_{mn}(\boldsymbol{\sigma}) \right] \left[\mathcal{D}\tilde{\eta}_{\mu}(\boldsymbol{\sigma}) \right] \exp \left[-\frac{i}{2} \int_{\Sigma} d^{2}\boldsymbol{\sigma} \sqrt{|\boldsymbol{h}|} \tilde{\eta}_{\mu} \Delta_{h} \tilde{\eta}^{\mu} - i \int_{\Sigma} d^{2}\boldsymbol{\sigma} \sqrt{|\boldsymbol{h}|} \left(\kappa R + \lambda \right) \right]$$

$$(9.27)$$

for a Scalar Field Theory covariantly coupled to 2D gravity on a disk² whereas the Boundary Dynamics appears as a solution of the Functional Schrödinger Equation (3.37)

$$\Psi\left[Y^{\mu\nu}\left[C\right];A\right] \equiv \int [d\bar{P}_{\mu\nu}] \exp\left[\frac{i}{2}\bar{P}_{\mu\nu}Y^{\mu\nu}\left[C\right] - i\left(\frac{\bar{P}_{\mu\nu}\bar{P}^{\mu\nu}}{4\mu_0}\right)A\right] \quad : \tag{9.28}$$

this wave functional encodes the holographic quantum mechanics of the String Boundary resulting from a superposition of the classical World–Sheet solutions driven by the zero mode area momentum onto the Boundary. Finally, $S^{\text{eff}}[C]$ is the effective action induced by the quantum fluctuation of the String World–Sheet. It is a local quantity written in terms of the "counterterms" needed to cancel the Boundary ultraviolet divergent terms³.

Proof:

Starting from the result of proposition 9.5 we can write then the String functional as

$$\Psi[C] = \int d^{D} \bar{\eta} \int_{0}^{\infty} dA \exp(i\lambda A) \int [d\bar{P}_{m|\mu}] \int [\mathcal{D}h_{mn}(\boldsymbol{\sigma})] [\mathcal{D}\tilde{\eta}^{\mu}(\boldsymbol{\sigma})] \cdot \\ \cdot \exp\left(\frac{i}{2}\bar{P}_{\mu\nu}Y^{\mu\nu} [C] - \frac{i}{4\mu_{0}}\bar{P}_{\mu\nu}\bar{P}^{\mu\nu}A\right) \cdot \\ \cdot \exp\left(-\frac{i}{2}\int_{\Sigma} d^{2}\boldsymbol{\sigma}\sqrt{|\boldsymbol{h}|}\tilde{\eta}_{\mu}\Delta_{\boldsymbol{g}}\tilde{\eta}^{\mu} - i\lambda\int_{\Sigma} d^{2}\boldsymbol{\sigma}\sqrt{|\boldsymbol{h}|}\right) \quad . \tag{9.29}$$

The World-Sheet Scalar Field Theory contains geometry dependent, ultraviolet divergent quantities. This 2-dimensional Quantum Field Theory on a Riemannian manifold can be renormalized by introducing suitable Bulk and Boundary "counter terms". These new terms absorb the ultraviolet

²Our result refers to the disk topology. The extension to more complex World–Sheet topology is straightforward: the single, Bulk, partition functional has to be replaced by a sum over definite genus path–integrals, i.e. $Z_{\rm BULK}^A \longrightarrow \sum_{g} Z_{\rm BULK}^{(g),A}$. This is the starting point for introducing topology changing quantum processes in the framework of String Theory.

 $^{^{3}}$ The required counterterms are proportional to the loop proper length and extrinsic curvature.

divergencies, and represent induced weight factors in the functional integration over the metric, $[\mathcal{D}h_{mn}]^4$, and the *Boundary* shape $[\mathcal{D}C]$.

The most part of current investigations in Quantum String Theory starts from the Polyakov path-integral and elaborate String Theory as a Scalar Field Theory defined over a Riemann surface. String Perturbation Theory come from this term as an expansion in the genus of the Riemann surface. Against this background, we assumed the phase space, covariant path-integral for the Schild String as the basic quantity encoding the whole information about String Quantum behavior, and we recovered the Polyakov "partition functional" as an effective path-integral, after integrating out the String coordinates and factorizing out the Boundary Shadow Dynamics. It is worth to recall that the Conformal Anomaly and the critical dimension are encoded into the Polyakov path-integral. Accordingly, they are Bulk effects. But, this is not the end of the story. Our approach provides the Boundary Shadow Dynamics as well. The fluctuations of the $\bar{\eta}^{\mu}$ field induce the non-local part of the effective action for $Y^{\mu}(s)$, while the World-Sheet vibrations induce the local, geometry dependent terms. Furthermore, the Eguchi String Wave Functional $\Psi[C;A]$ encodes the Quantum Holographic Dynamics of the Boundary in terms of area coordinates. Ψ gives the amplitude to find a closed String, with area tensor $Y^{\mu\nu}$ as the only Boundary of a World-Sheet with Parameter Space of proper area A, as it was originally introduced in the areal formulation of "String Quantum Mechanics". This overlooked approach implicitly broke the accepted "dogma" that String Theory is intrinsically a "Second Quantized Field Theory", which cannot be given a First Quantized, or Quantum Mechanical, formulation. This claim is correct as far as it is referred to the infinite vibration modes of the World-Sheet Bulk. On the other hand, Boundary vibrations are induced by the 1-dimensional field living on it and by the constant zero mode of the area momentum $P_{\mu\nu}$. From this viewpoint, the Eguchi approach appears as a sort of minisuperspace approximation of the full String Dynamics in momentum space: all the infinite Bulk modes, except the constant one, has been frozen out. The "Field Theory" of this single mode collapses into a generalized

⁴This functional integration measure has to be factored out by the orbit of the diffeomorphism group, and in the critical dimension by the Weyl group, as well.

"Quantum Mechanics", where both spatial and timelike coordinates are replaced by area tensor and scalar respectively:

Second Quantized Bulk Dynamics
$$\longrightarrow$$
 First Quantized Boundary Dynamics . (9.30)

In the absence of external interactions, the quantum state of the "free World-Sheet Boundary" is represented by a Gaussian Wave Functional

$$\Psi[\sigma; A] \propto \left(\frac{m^2}{A}\right)^{(D-1)/2} \exp\left(-i\mu_0 \frac{Y^{\mu\nu}[C]Y_{\mu\nu}[C]}{4A}\right)$$
(9.31)

solving the corresponding "Schrödinger equation" [33]:

$$-\frac{1}{4m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{\left(\mathbf{Y}'(s) \right)^2} \frac{\delta^2 \Psi[C; A]}{\delta Y^{\mu\nu}(s) \, \delta Y_{\mu\nu}(s)} = i \frac{\partial}{\partial A} \Psi[C; A] \quad . \tag{9.32}$$

Our formulation enlightens the complementary role played by the Bulk and Boundary formulation of String Quantum Dynamics and the subtle interplay between the two. The wave equation (9.32) displays one of the most intriguing aspects of Eguchi formulation: the role of spatial coordinates is played by the area tensor while the area A is the evolution parameter. Such an unusual Dynamics is now explained as an induced effect due to the Bulk zero mode $\bar{P}^{\mu\nu}$. In this way a bare Quantum Loop is dressed with the degrees of freedom carried by a Quantum String.

Chapter 10

Nonstandard & Speculative

We "[...] must have the deepest commitment, the most serious mind."

10.1 Nonstandard Functional Quantization

As we have seen in chapter 2, section 2.6 it is possible to express the fundamental equation of motion for the *Boundary* of a domain Σ in terms of *Area Dynamics* as the *second derivative* of the loop *Holographic Coordinates* with respect to the area A in *Parameter Space* (cf. equation (2.69), which we report here for convenience):

$$\frac{d^2Y^{\mu\nu} [C; A]}{dA^2} = 0 . (10.1)$$

Moreover, the functional equation of continuity can be written as (cf. equation (2.71))

$$\frac{dP\left[C;A\right]}{dA} + \frac{1}{4m^2} \left(\oint_{\Gamma} dl(s) \right)^{-1} \oint_{\Gamma} ds \sqrt{\mathbf{Y}'^2} \frac{\delta}{\delta Y^{\mu\nu}(s)} \left[P\left[C;A\right] \frac{\delta S\left[C;A\right]}{\delta Y_{\mu\nu}(s)} \right] = 0 \quad . \quad (10.2)$$

Now we observe that all the results presented in appendix C can be generalized cases more general than that of a single scalar field. In particular we are going to use those results for a multiplet of scalar fields. The classical quantities are exactly the *Holographic Coordinates* $Y^{\mu\nu}$, so that the corresponding *NonStandard Stochastic Process* is now denoted by $\hat{\mathbf{Y}}^{\mu\nu}$. Now, we state a central

Axiom (Stochastic Area Evolution). The areal evolution of a Quantum String is a stochastic process in the variable A.

According to this axiom we can now substitute in the Area Newton Equation above (10.1) the second derivative with what we call the NonStandard Stochastic Acceleration Operator, as follows

$$\frac{d^2 Y^{\mu\nu} \left[C;A\right]}{dA^2} \longrightarrow \frac{1}{2} \left(\mathcal{D}^+ \mathcal{D}^- + \mathcal{D}^- \mathcal{D}^+\right) \hat{\boldsymbol{Y}}^{\mu\nu} \left(A\right) \quad . \tag{10.3}$$

The NonStandard Mean Forward/Backward Derivative is defined in appendix C as well. Hence, the <u>Stochastic Equation of Motion</u> for the <u>String</u> turns out to be

$$\frac{1}{2} \left(\mathcal{D}^+ \mathcal{D}^- + \mathcal{D}^- \mathcal{D}^+ \right) \hat{\mathbf{Y}}^{\mu\nu} \left(A \right) = 0 \quad , \tag{10.4}$$

where $\hat{\boldsymbol{Y}}^{\mu\nu}$ is a tensorial *Stochastic Process* on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) defined in a way analogous to that of definition C.9. We first give the following definition, since this will be central in the conclusions of this chapter:

Definition 10.1 (Non Standard Wave Functional).

The Functional Nonstandard Loop Wavefunctional is

$$\hat{\mathbf{\Lambda}}\left[C,A\right] \stackrel{\text{def.}}{=} \sqrt{\check{\mathcal{P}}\left[C,A\right]} e^{i\check{\mathcal{S}}\left[C,A\right]} = e^{\check{\mathcal{R}}\left[C,A\right] + i\check{\mathcal{S}}\left[C,A\right]} \quad , \tag{10.5}$$

where we rename the probability density in the following way

$$\sqrt{\check{\mathcal{P}}[C,A]} \stackrel{\mathrm{def.}}{=} e^{\check{\mathcal{R}}[C,A]}$$

i.e.

$$\frac{1}{2}\log\check{\mathcal{P}}\left[C,A\right] = \check{\mathcal{R}}\left[C,A\right] \quad . \tag{10.6}$$

Starting from the previous results, we can get some useful relations to be used later on.

Proposition 10.1 (Derivatives of the Non Standard Stochastic Process).

The Area Derivative of $\hat{\Lambda}$ is given by

$$\frac{\partial \hat{\mathbf{\Lambda}} [C, A]}{\partial A} = \left(\frac{\partial \check{\mathcal{R}} [C, A]}{\partial A} + i \frac{\partial \check{\mathcal{S}} [C, A]}{\partial A} \right) \hat{\mathbf{\Lambda}} [C, A] \quad ; \tag{10.7}$$

moreover the first and second functional derivatives with respect to the Nonstandard Holographic Coordinates $\hat{\mathbf{Y}}^{\mu\nu}(s)$, respectively, are

$$\frac{\delta \hat{\mathbf{\Lambda}} [C, A]}{\delta \hat{\mathbf{Y}}^{\mu\nu} (u)} = \left[\frac{\delta}{\delta \hat{\mathbf{Y}}^{\mu\nu} (u)} (\check{\mathcal{K}} + i\check{\mathcal{S}}) \right] \hat{\mathbf{\Lambda}} [C, A] \qquad (10.8)$$

$$\frac{\delta^2 \hat{\mathbf{\Lambda}} [C, A]}{\delta \hat{\mathbf{Y}}^{\mu\nu} (u) \delta \hat{\mathbf{Y}}_{\mu\nu} (u)} = \left[\frac{\delta^2}{\delta \hat{\mathbf{Y}}^{\mu\nu} (u) \delta \hat{\mathbf{Y}}_{\mu\nu} (u)} (\check{\mathcal{K}} + i\check{\mathcal{S}}) \right] \hat{\mathbf{\Lambda}} [C, A] + \left[\frac{\delta}{\delta \hat{\mathbf{Y}}^{\mu\nu} (u)} (\check{\mathcal{K}} + i\check{\mathcal{S}}) \right]^2 \hat{\mathbf{\Lambda}} [C, A] \qquad (10.9)$$

Proof:

The first result is just the application of the chain rule to the exponential and is thus evident. The same is true for the second one.

$$\frac{\delta^{2} \hat{\boldsymbol{\Lambda}} [C, A]}{\delta \hat{\boldsymbol{Y}}^{\mu\nu} (u) \delta \hat{\boldsymbol{Y}}_{\mu\nu} (u)} = \frac{\delta}{\delta \hat{\boldsymbol{Y}}^{\mu\nu} (u)} \left\{ \left[\frac{\delta}{\delta \hat{\boldsymbol{Y}}_{\mu\nu} (u)} (\check{\mathcal{K}} + i\check{\mathcal{S}}) \right] \hat{\boldsymbol{\Lambda}} [C, A] \right\}
= \left[\frac{\delta^{2}}{\delta \hat{\boldsymbol{Y}}^{\mu\nu} (u) \delta \hat{\boldsymbol{Y}}_{\mu\nu} (u)} (\check{\mathcal{K}} + i\check{\mathcal{S}}) \right] \hat{\boldsymbol{\Lambda}} [C, A] + \left[\frac{\delta}{\delta \hat{\boldsymbol{Y}}^{\mu\nu} (u)} (\check{\mathcal{K}} + i\check{\mathcal{S}}) \right]^{2} \hat{\boldsymbol{\Lambda}} [C, A] \quad . \tag{10.10}$$

Now we can turn to the stochastic process in Kawabata space that we defined for a scalar field in section C.2. In particular we can analyse more closely how the *Forward and Backward Drift Operators* are related with classical dynamical quantities. This is done in the following way.

Proposition 10.2 (Non Standard Drift Velocity).

The average of the Forward and Backward Drift Operators is proportional to the first functional derivative of the classical action, i.e.

$$\hbar \frac{\delta \check{\mathbf{S}} \left[C, A\right]}{\delta \hat{\mathbf{Y}}^{\mu\nu} \left(u\right)} = \frac{1}{2} \left(\mathbf{A}_{A}^{+} + \mathbf{A}_{A}^{-}\right) \hat{\mathbf{Y}}^{\mu\nu} \left(u\right) \quad . \tag{10.11}$$

Proof:

We start from the forward and backward Fokker Planck equations that can be written by analogy with those presented in appendix C in the case of a single scalar field; then we take their average. This gives the following result:

$$\frac{\partial \check{\mathcal{P}}\left[C,A\right]}{\partial A} + \int ds \frac{\delta}{\delta \hat{\boldsymbol{Y}}^{\mu\nu}\left(s\right)} \left[\left(\frac{1}{2} \left(\boldsymbol{\mathcal{A}}_{A}^{+} + \boldsymbol{\mathcal{A}}_{A}^{-} \right) \hat{\boldsymbol{Y}}^{\mu\nu}\left(s\right) \right) \check{\mathcal{P}}\left[C,A\right] \right] = 0 \tag{10.12}$$

Thus, we see that the final result has exactly the same form of the continuity equation (2.71), i.e. we can identify

$$\frac{1}{2} \left(\mathcal{A}_A^+ + \mathcal{A}_A^- \right) \hat{Y}^{\mu\nu} \left(s \right) \tag{10.13}$$

with the drift velocity. So, this expression is a functional gradient, because thanks to equation (2.56) we have

$$\hbar \frac{\delta \check{\mathcal{S}}\left[C,A\right]}{\delta \hat{\boldsymbol{Y}}^{\mu\nu}\left(u\right)} = Q_{\mu\nu} = \frac{1}{2} \left(\mathcal{A}_{A}^{+} + \mathcal{A}_{A}^{-} \right) \hat{\boldsymbol{Y}}_{\mu\nu}\left(u\right)$$

where $\check{\mathcal{S}}$ is the classical action!

After identifying the immaginary part of the exponent in expression (10.5) with the classical action, we can also express the difference of the Forward and Backward Drift Operators in terms of the redefined form (10.6) for the probability density. This gives the following result:

Proposition 10.3 (Nonstandard Osmotic Velocity).

The difference of the Forward and Backward Drift Operators, applied on the NonStandard Tensorial Stochastic Process in Kawabata Space, can be expressed as the following gradient,

$$\frac{1}{2} \left(\mathbf{A}_A^+ - \mathbf{A}_A^- \right) \hat{\mathbf{Y}}^{\mu\nu} \left(u \right) = \frac{\hbar}{2} \frac{\delta \check{\mathbf{R}} \left[C, A \right]}{\delta \hat{\mathbf{Y}}^{\mu\nu} \left(u \right)} \quad . \tag{10.14}$$

Proof:

Again we start from the forward and backward Fokker Planck equations multiplying by 1/2 their difference to get:

$$\oint_{\Gamma} ds \frac{\delta}{\delta \hat{\boldsymbol{Y}}^{\mu\nu}(s)} \left[\left(\frac{1}{2} \left(\boldsymbol{\mathcal{A}}_{A}^{+} - \boldsymbol{\mathcal{A}}_{A}^{-} \right) \hat{\boldsymbol{Y}}^{\mu\nu}(s) \right) \check{\mathcal{P}} \left[\hat{\boldsymbol{Y}}, A \right] \right] = \frac{\hbar}{2} \oint_{\Gamma} ds \frac{\delta \check{\mathcal{P}} \left[\hat{\boldsymbol{Y}}, A \right]}{\delta \hat{\boldsymbol{Y}}^{\mu\nu}(s)^{2}} , \qquad (10.15)$$

which we can rewrite

$$\oint_{\Gamma} ds \frac{\delta}{\delta \hat{\boldsymbol{Y}}_{\mu\nu}(s)} \left\{ \left[\left(\frac{1}{2} \left(\boldsymbol{\mathcal{A}}_{A}^{+} - \boldsymbol{\mathcal{A}}_{A}^{-} \right) \hat{\boldsymbol{Y}}^{\mu\nu}(s) \right) \check{\boldsymbol{\mathcal{P}}} \left[\hat{\boldsymbol{Y}}, A \right] \right] - \frac{\hbar}{2} \frac{\delta \check{\boldsymbol{\mathcal{P}}} \left[\hat{\boldsymbol{Y}}, A \right]}{\delta \hat{\boldsymbol{Y}}^{\mu\nu}(s)} \right\} = 0 \quad . \tag{10.16}$$

Thanks to the redefinition (10.6) the previous equation becomes

$$\oint_{\Gamma} ds \frac{\delta}{\delta \hat{\boldsymbol{Y}}_{\mu\nu}(s)} \left\{ \left[\left[\left(\frac{1}{2} \left(\boldsymbol{\mathcal{A}}_{A}^{+} - \boldsymbol{\mathcal{A}}_{A}^{-} \right) \hat{\boldsymbol{Y}}^{\mu\nu}(s) \right) \right] - \frac{\hbar}{2} \frac{\delta \check{\boldsymbol{\mathcal{R}}} \left[\hat{\boldsymbol{Y}}, A \right]}{\delta \hat{\boldsymbol{Y}}^{\mu\nu}(s)} \right] \check{\boldsymbol{\mathcal{P}}} \left[\hat{\boldsymbol{Y}}, A \right] \right\} = 0 \quad . \tag{10.17}$$

A sufficient condition to satisfy the above is that

$$\frac{1}{2} \left(\mathcal{A}_{A}^{+} - \mathcal{A}_{A}^{-} \right) \hat{\boldsymbol{Y}}^{\mu\nu} \left(s \right) = -\frac{\hbar}{2} \frac{\delta \check{\mathcal{R}} \left[\hat{\boldsymbol{Y}}, A \right]}{\delta \hat{\boldsymbol{Y}}^{\mu\nu} \left(s \right)}$$

which is the desired result.

We can now turn to the central result of this chapter:

Proposition 10.4 (Non Standard Functional Schrödinger Wave Equation).

The NonStandard Loop WaveFunctional in Kawabata Space, $\hat{\mathbf{\Lambda}}$, satisfies the functional wave equation

$$\frac{\hbar^2}{2} \oint ds \frac{\delta^2 \hat{\mathbf{\Lambda}} [C, A]}{\delta \hat{\mathbf{Y}}^{\mu\nu} (s) \delta \hat{\mathbf{Y}}_{\mu\nu} (s)} = i\hbar \frac{\partial \hat{\mathbf{\Lambda}} [C, A]}{\partial A} \quad . \tag{10.18}$$

Proof:

We start the proof from the forward and backward Fokker–Planck equations using them to express equation (10.4):

$$-\frac{\partial}{\partial A} \left(\frac{1}{2} \left(\mathcal{A}_{A}^{+} + \mathcal{A}_{A}^{-} \right) \hat{\mathbf{Y}}^{\mu\nu}(t) \right) = -\hbar \int ds \frac{\delta^{2}}{\delta \hat{\mathbf{Y}}^{\alpha\beta} (s) \delta \hat{\mathbf{Y}}_{\alpha\beta} (s)} \left[\left(\mathcal{A}_{A}^{+} - \mathcal{A}_{A}^{-} \right) \hat{\mathbf{Y}}^{\mu\nu}(t) \right]$$

$$-\frac{1}{4} \int ds \left\{ \left[\left(\mathcal{A}_{A}^{+} - \mathcal{A}_{A}^{-} \right) \hat{\mathbf{Y}}^{\alpha\beta} (s) \right] \frac{\delta}{\delta \hat{\mathbf{Y}}^{\alpha\beta} (s)} \left[\left(\mathcal{A}_{A}^{+} - \mathcal{A}_{A}^{-} \right) \hat{\mathbf{Y}}^{\mu\nu}(t) \right] + \right.$$

$$- \left[\left(\mathcal{A}_{A}^{+} + \mathcal{A}_{A}^{-} \right) \hat{\mathbf{Y}}^{\alpha\beta} (s) \right] \frac{\delta}{\delta \hat{\mathbf{Y}}^{\alpha\beta} (s)} \left[\left(\mathcal{A}_{A}^{+} + \mathcal{A}_{A}^{-} \right) \hat{\mathbf{Y}}^{\mu\nu}(t) \right] \right\} , \qquad (10.19)$$

which we can express in terms of the quantities $\check{\mathcal{S}}$ and $\check{\mathcal{R}}$ using results 10.11 and 10.14 as,

$$-\hbar \frac{\partial}{\partial A} \left(\frac{\delta \check{\mathbf{S}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \right) = -\frac{\hbar^2}{2} \int ds \frac{\delta^2}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)^2} \left(\frac{\delta \check{\mathbf{X}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \right) + \\ -\hbar^2 \int ds \left\{ \frac{\delta \check{\mathbf{K}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \frac{\delta}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \left(\frac{\delta \check{\mathbf{K}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \right) - \frac{\delta \check{\mathbf{S}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \frac{\delta}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \left(\frac{\delta \check{\mathbf{S}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \right) \right\}$$
(10.20)

We will use these result in a while. Indeed at the same time we have also relation (10.12), which using equation (10.11) we can rewrite as,

$$\frac{\partial \check{\mathcal{P}}\left[C,A\right]}{\partial A} = -\hbar \oint_{\Gamma} ds \frac{\delta}{\delta \hat{\boldsymbol{Y}}^{\alpha\beta}\left(s\right)} \left[\frac{\delta \check{\mathcal{S}}}{\delta \hat{\boldsymbol{Y}}^{\alpha\beta}\left(s\right)} \check{\mathcal{P}} \right]$$
(10.21)

We can then substitute the definition of $\check{\mathcal{P}}[C,A]$ given in equation (10.6), and performing the functional derivatives, we get¹

$$\frac{\partial \check{\mathcal{R}}}{\partial A} = -\frac{\hbar}{2} \int ds \frac{\delta^2 \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta} (s)^2} - \hbar \int ds \frac{\delta \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta} (s)} \frac{\delta \check{\mathcal{R}}}{\delta \hat{\mathbf{Y}}_{\nu} (s)} \quad . \tag{10.22}$$

Multiplying now by \hbar and taking the functional derivative with respect to $\hat{\boldsymbol{Y}}^{\mu\nu}(t)$, results in

$$\hbar \frac{\partial}{\partial A} \left(\frac{\delta \tilde{\mathbf{X}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \right) = -\frac{\hbar^2}{2} \int ds \frac{\delta}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)^2} \frac{\delta \tilde{\mathbf{S}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} + \\
-\hbar^2 \int ds \left\{ \frac{\delta \tilde{\mathbf{S}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \frac{\delta}{\delta \hat{\mathbf{Y}}_{\mu\nu}(t)} \frac{\delta \tilde{\mathbf{X}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} - \frac{\delta \tilde{\mathbf{X}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \frac{\delta}{\delta \hat{\mathbf{Y}}_{\mu\nu}(t)} \frac{\delta \tilde{\mathbf{S}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \right\} \quad . \quad (10.23)$$

We can now add equation (10.20) to i times equation (10.23) to get

$$i\hbar \frac{\partial}{\partial A} \left(\frac{\delta \check{\mathcal{R}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \right) - \hbar \frac{\partial}{\partial A} \left(\frac{\delta \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \right) =$$

$$= -i\frac{\hbar^{2}}{2} \int ds \frac{\delta}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)^{2}} \frac{\delta \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} - \frac{\hbar^{2}}{2} \int ds \frac{\delta^{2}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)^{2}} \frac{\delta \check{\mathcal{R}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} +$$

$$-i\hbar^{2} \int ds \left\{ \frac{\delta \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \frac{\delta}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \frac{\delta \check{\mathcal{R}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} - \frac{\delta \check{\mathcal{R}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \frac{\delta}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \frac{\delta \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \right\} +$$

$$-\hbar^{2} \int ds \left\{ \frac{\delta \check{\mathcal{R}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \frac{\delta}{\delta \hat{\mathbf{Y}}_{\alpha\beta}(s)} \frac{\delta \check{\mathcal{R}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} - \frac{\delta \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \frac{\delta}{\delta \hat{\mathbf{Y}}_{\alpha\beta}(s)} \frac{\delta \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \right\} . (10.24)$$

Rearranging this result we then get

$$i\hbar \frac{\partial}{\partial A} \left\{ \frac{\delta \check{\mathcal{R}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} + i \frac{\delta \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \right\} =$$

$$= -\frac{\hbar^2}{2} \int ds \frac{\delta}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)^2} \left\{ \frac{\delta \check{\mathcal{R}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} + i \frac{\delta \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \right\} +$$

$$-i\hbar^2 \int ds \left[\frac{\delta \check{\mathcal{R}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} + i \frac{\delta \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \right] \frac{\delta}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \left[\frac{\delta \check{\mathcal{R}}}{\delta \hat{\mathbf{Y}}_{\alpha\beta}(s)} + i \frac{\delta \check{\mathcal{S}}}{\delta \hat{\mathbf{Y}}_{\alpha\beta}(s)} \right]$$
(10.25)

and at the end

$$i\hbar \frac{\delta}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \left\{ \frac{\partial}{\partial A} \left[\check{\mathbf{X}} + i\check{\mathcal{S}} \right] \right\} =$$

$$= -\frac{\hbar^2}{2} \frac{\delta}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \int ds \frac{\delta}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)^2} \left\{ \check{\mathbf{X}} + i\check{\mathcal{S}} \right\} +$$

$$-i\frac{\hbar^2}{2} \frac{\delta}{\delta \hat{\mathbf{Y}}^{\mu\nu}(t)} \int ds \left\{ \frac{\delta}{\delta \hat{\mathbf{Y}}^{\alpha\beta}(s)} \left[\check{\mathbf{X}} + i\check{\mathcal{S}} \right] \right\}^2 . \tag{10.26}$$

We thus see that a total functional gradient can be integrated away and after multiplying by $\hat{\mathbf{\Lambda}}$ the result we get

$$i\hbar \frac{\partial}{\partial A} \left[\tilde{\mathcal{R}} + i\tilde{\mathcal{S}} \right] \hat{\mathbf{\Lambda}} =$$

$$= \frac{\hbar^2}{2} \int ds \left\{ -\frac{\delta^2 \tilde{\mathcal{R}}}{\delta \hat{\mathbf{Y}}^{\mu\nu} (s)^2} + i \frac{\delta^2 \tilde{\mathcal{S}}}{\delta \hat{\mathbf{Y}}^{\mu\nu} (s)^2} + \left[\frac{\delta}{\delta \hat{\mathbf{Y}}^{\mu\nu} (s)} (\tilde{\mathcal{R}} + i\tilde{\mathcal{S}}) \right] \right\} \hat{\mathbf{\Lambda}}$$
(10.27)

¹Without problems we can simplify by $2\check{\mathcal{P}}[C,A]$.

which, remembering results (10.7-10.9), gives exactly

$$\frac{\hbar^2}{2} \oint ds \frac{\delta^2 \hat{\mathbf{\Lambda}} [C, A]}{\delta \hat{\mathbf{Y}}^{\mu\nu} (s) \delta \hat{\mathbf{Y}}_{\mu\nu} (s)} = i\hbar \frac{\partial \hat{\mathbf{\Lambda}} [C, A]}{\partial A}$$
(10.28)

This result is very similar to the *String Functional Wave Equation* that we derived in chapter 3 and, choosing in a proper way a weight for the NonStandard Stochastic Quantities, can be cast in exactly the same form. Thus we have a completely rigorous derivation of that equation in terms of NonStandard Analysis.

10.2 Holographic Coordinates and M-Theory

10.2.1 Boundary Shadow Dynamics and M-Theory

One of the most enlightening features of the Eguchi approach is the formal correspondence it establishes between the Quantum Mechanics of point particles and String loops. Such a relationship is summarized in the translation code displayed in the Table 3.2. Instrumental to this correspondence is the replacement of the canonical String coordinates $Y^{\mu}(s)$ with the reparametrization invariant Holographic Coordinates $Y^{\mu\nu}[C]$. We observed in previous chapters that, surprising as it may appear at first sight, the new coordinates $Y^{\mu\nu}[C]$ are just as "natural" as the old $Y^{\mu}(s)$ for the purpose of defining the String "position". As a matter of fact, a Classical Gauge Field Theory of Relativistic Strings was proposed several years $ago^2[23]$, but only recently it was extended to generic p-branes including their coupling to (p+1)-forms and Gravity[24].

Now that we have established the connection between *Holographic* Quantization and the path–integral formulation of Quantum *Strings*, it seems almost compelling to ponder about the relationship, if any, between the $Y^{\mu\nu}$ [C] matrix coordinates and the matrix coordinates which, presumably, lie at the heart of the M-Theory formulation of superstrings. Since the general framework of M-Theory is yet to be discovered, it seems reasonable to focus on a

²We already saw in chapter 8 the relation between our present "quantum" proposal and that classical framework

specific matrix model recently proposed for *Type IIB* superstrings³ [26]. The Dynamics of this model is encoded into a simple Yang-Mills type action

$$S_{\text{IKKT}} = -\frac{\alpha}{4} \text{Tr} \left[[A^{\mu}, A^{\nu}]^2 \right] + \beta \text{Tr} \left[\mathbb{I} \right] + \text{fermionic part} \quad ,$$
 (10.29)

where the A^{μ} variables are represented by $N \times N$ hermitian matrices and \mathbb{I} is the unit matrix. The novelty of this formulation is that it identifies the ordinary spacetime coordinates with the eigenvalues of the non-commuting Yang-Mills matrices. In such a framework, the emergence of Classical spacetime occurs in the large-N limit, i.e., when the commutator goes into a Poisson bracket⁴ and the matrix trace operation turns into a double continuous sum over the row and column indices, which amounts to a two dimensional invariant integration. Put briefly,

$$\lim_{N \to \infty} \text{"Tr"} \to -i \int d^2 \sigma \sqrt{|g|}$$
 (10.30)

$$-i\lim_{N\to\infty} \left[A^{\mu}, A^{\nu}\right] \quad \to \quad \{Y^{\mu}, Y^{\nu}\} \tag{10.31}$$

$$\sqrt{|\boldsymbol{g}|} \left\{ Y^{\mu}, Y^{\nu} \right\} \quad \equiv \quad \dot{Y}^{\mu\nu} \quad . \tag{10.32}$$

What interests us is that, in such a classical limit, the IKKT action (10.29) turns into the Schild action in Equation (3.53) once we make the identifications

$$\alpha \longleftrightarrow -m^2$$

$$\beta \longleftrightarrow \mathcal{E}$$
 (10.33)

$$N\left(\boldsymbol{\sigma}\right) \longleftrightarrow \sqrt{\left|\boldsymbol{g}\right|} , \qquad (10.34)$$

while the trace of the Yang-Mills commutator turns into the oriented surface element

$$\lim_{N\to\infty} \text{Tr}\left[\left[A^{\mu}, A^{\nu}\right]\right] \longrightarrow \int_{\Sigma} d^{2}\boldsymbol{\sigma} \dot{X}^{\mu\nu}(\boldsymbol{\sigma}) = Y^{\mu\nu}\left(C\right) ,$$
 (10.35)

C being the image of $\Gamma = \partial \Sigma$ in Target Space, as usual: $C = \mathbf{Y}(\Gamma)$. This formal relationship can be clarified by considering a definite case. As an example let us consider a static D-String configuration both in the classical Schild formulation and in the corresponding matrix

³A similar matrix action for the Type IIA model has been conjectured in [25].

 $^{^4}$ This is not the canonical Poisson bracket which is replaced by the Quantum Mechanical commutator. Rather, it is the World–Sheet symplectic structure which is replaced by the (classical) matrix commutator for finite N.

description. It is straightforward to prove that a length L static String stretched along the x^1 direction, i.e.

$$X^{\mu} = \sigma^{0} T \delta^{\mu 0} + \frac{L \sigma^{1}}{2\pi} \delta^{\mu 1} \quad , \quad 0 \le \sigma^{0} \le 1 \quad , \quad 0 \le \sigma^{1} \le 2\pi$$
 (10.36)

$$X^{\mu} = 0 , \quad \mu \neq 0, 1$$
 (10.37)

solves the classical equations of motion

$$\{X_{\mu}, \{X^{\mu}, X^{\nu}\}\} = 0$$
 (10.38)

During a time lapse T the String sweeps a timelike World–Sheet in the 0-1 plane characterized by the Holographic Coordinates

$$Y^{\mu\nu}(L,T) = \int_0^1 d\sigma^0 \int_0^{2\pi} d\sigma^1 \sqrt{|g|} \quad , \qquad \{X^{\mu}, X^{\nu}\} = TL\delta^{0[\mu}\delta^{\nu]1} \quad . \tag{10.39}$$

Equation (10.39) gives both the area and the orientation of the rectangular loop which is the *Boundary* of the *String World–Sheet*. The corresponding matrix solution, on the other hand, must satisfy the equation

$$[A_{\mu}, [A^{\mu}, A^{\nu}]] = 0 \quad . \tag{10.40}$$

Consider, then, two hermitian, $N \times N$ matrices \hat{q} , \hat{p} with an approximate c-number commutation relation $[\hat{q}, \hat{p}] = i$, when $N \gg 1$. Then, a solution of the Classical Equation of Motion (10.40), corresponding to a *solitonic* state in *String Theory*, can be written as

$$A^{\mu} = T\delta^{\mu 0}\hat{q} + \frac{L}{2\pi}\delta^{\mu 1}\hat{p} \quad . \tag{10.41}$$

In the large-N limit we find

$$-i[A^{\mu}, A^{\nu}] = -i\frac{LT}{2\pi} \delta^{0[\mu} \delta^{\nu]1} [\hat{q}, \hat{p}] \approx \frac{LT}{2\pi} \delta^{0[\mu} \delta^{\nu]1}$$
(10.42)

and

$$-i\text{Tr}\left[\left[A^{\mu}, A^{\nu}\right]\right] \approx \int_{0}^{1} d\sigma^{0} \int_{0}^{2\pi} d\sigma^{1} \sqrt{|\boldsymbol{g}|} \left\{X^{\mu}, X^{\nu}\right\} = Y^{\mu\nu} \left(L, T\right) \quad . \tag{10.43}$$

These results, specific as they are, point to a deeper connection between the loop space description of *String* Dynamics and matrix models of superstrings which, in our opinion, deserves a more detailed investigation.

10.2.2 Covariant "Speculation"

We complete the observations of the previous section by speculating about a "fully covariant" formulation of the Boundary Wave Equation (9.32) and its physical consequences. The spacelike and timelike character of the Holographic Coordinates $Y^{\mu\nu}$ and of the Areal Time A shows up in the Schrödinger, "non–relativistic" form, of the wave equation (9.32) which is second order in th Holographic Functional Derivatives $\delta/\delta Y^{\mu\nu}$ (s), and first order in $\partial/\partial A$. Therefore, it is intriguing to ponder about the form and the meaning of the corresponding Klein–Gordon equation. The first step towards the "relativistic" form of (9.32) is to introduce an appropriate coordinate system where $Y^{\mu\nu}$ and A can play a physically equivalent role. Let us introduce a matrix coordinate \mathbb{X}^{MN} , where certain components are $Y^{\mu\nu}$ and A. There is a great freedom in the choice of \mathbb{X}^{MN} . However, an interesting possibility would be

$$\mathbb{X}^{MN} = \begin{pmatrix} m Y^{\mu\nu} & \delta^{\mu(\nu)} Y_{(\nu)}^{C.M.} \\ \delta^{\mu(\nu)} Y_{(\nu)}^{C.M.} & m A_{\mu\nu} \end{pmatrix} , \qquad (10.44)$$

where

$$A_{\mu\nu} = \begin{pmatrix} \mathbb{O} & \mathbb{A} \\ -\mathbb{A} & \mathbb{O} \end{pmatrix} \tag{10.45}$$

with

$$\mathbb{A} = \left(\begin{array}{cc} 0 & A \\ 0 & 0 \end{array}\right) \quad : \tag{10.46}$$

here we arranged the String center of mass coordinate $Y_{\nu}^{C.M.}$ inside off-diagonal sub-matrices and build-up an antisymmetric proper area tensor in oder to endow A with the same tensorial character as $Y^{\mu\nu}$. The String length scale, 1/m, has been introduced to provide the block diagonal area sub-matrices the coordinate canonical dimension of a length, in natural units. The most remarkable feature for this choice of \mathbb{X}^{MN} is that if we let μ, ν to range over four values, then \mathbb{X}^{MN} is an 8×8 antisymmetric matrix with eleven independent entries. Inspired by the recent progress in non-commutative geometry, where point coordinates are described by non-commuting matrices, we associate to each matrix (10.44) a representative point in an eleven dimensional space which is the product of the 4-dimensional spacetime \times the 6-dimensional Holographic Loop Space \times the 1-dimensional Area Time axis. Eleven dimensional spacetime is the proper arena of M-Theory, and we do not believe this is a mere

coincidence.

According with the assignment of the eleven entries in \mathbb{X}^{MN} the corresponding "point" can represent different physical objects:

1. a pointlike particle:

$$\{Y^{\mu\nu} = 0, A = 0, Y_{\nu}^{C.M.}\}$$
;

2. a loop with center of mass in $Y_{\nu}^{C.M.}$ and Holographic coordinates $Y^{\mu\nu}$:

$$\{Y^{\mu\nu}(\gamma), A=0, Y^{C.M.}_{\nu}\}$$
;

3. an open surface of proper area A, Boundary Holographic Coordinates $Y^{\mu\nu}$ and center of mass in $Y^{C.M.}_{\nu}$, i.e. a real String:

$$\{Y^{\mu\nu}(\gamma), A, Y^{C.M.}_{\nu}\}$$
;

4. a closed surface with Parameter Space of proper area A, i.e. a virtual String,

$$\{Y^{\mu\nu} = 0, A, Y^{C.M.}_{\nu}\}$$
.

It is appealing to conjecture that "Special Relativity" in this enlarged space will transform one of the above objects into another by a reference frame transformation! From this vantage viewpoint particles, loops, real and virtual Strings would appear as the same basic object as viewed from different reference frames. Accordingly, a quantum field $\Phi(X)$ would create and destroy the objects listed above, or, a more basic object encompassing all of them. A unified Quantum Field Theory of points, loops and Srings, and its relation, if any, with M-Theory or non–commutative geometry, is an issue which deserves a throughly, future, investigation: of particolar relevance are the possible consequences about the nature of spacetime. This is a very deep and fundamental problem, that needs a careful investigation and a better comprehension of the empirical meaning of these highly theoretical formulation. Neverthless we give, in the next section, a proposal, without pretend to be exhaustive, but only to see how it could be possible to set up a convenient framework to address this topis starting from the concepts we presented so far.

10.3 Superconductivity and Quantum Spacetime

To set up a correct environment, it is worth to recall that Quantum *Strings*, or more generally ?-branes of various kind, are currently viewed as the fundamental constituents of everything: not only every matter particle or gauge boson must be derived from the *String* vibration spectrum, but spacetime itself is built out of them.

If spacetime is no longer preassigned, then logical consistency demands that a matrix representation of p-brane Dynamics cannot be formulated in any given background spacetime. The exact mechanism by which M-Theory is supposed to break this circularity is not known at present, but Loop Quantum Mechanics points to a possible resolution of that paradox. If one wishes to discuss Quantum Strings on the same footing with point particles and other p-branes, then their Dynamics is best formulated in Loop Space rather than in physical spacetime. As we have repeatedly stated throughout this paper, our emphasis on String shapes rather than on the String constituent points, represents a departure from the canonical formulation and requires an appropriate choice of dynamical variables, namely the String Holographic Coordinates and the Areal Time. Then, at the Loop Space level, where each "point" is representative of a particular loop configuration, our formulation is purely Quantum Mechanical, and there is no reference to the background spacetime. At the same time, the functional approach leads to a precise interpretation of the fuzziness of the underlying Quantum spacetime in the following sense: when the resolution of the detecting apparatus is smaller than a particle De Broglie wavelength, then the particle Quantum trajectory behaves as a Fractal curve of Hausdorff dimension two. Similarly we have concluded on the basis of the Shape Uncertainty Principle that the Hausdorff dimension of a Quantum String World-Sheet is three, and that two distinct phases (smooth and fractal phase) exist above and below the loop De Broglie area. Now, if particle world-lines and String World-Sheets behave as fractal objects at small scales of distance, so does the World-HyperTube of a generic p-brane including spacetime itself [13], and we are led to the general expectation that a new kind of Fractal Geometry may provide an effective dynamical arena for physical phenomena near the String or Planck scale in the same way that a smooth Riemannian geometry provides an effective dynamical arena for physical phenomena at large distance

scales.

Once committed to that point of view, one may naturally ask, "what kind of physical mechanism can be invoked in the framework of Loop Quantum Mechanics to account for the transition from the fractal to the smooth geometric phase of spacetime?". A possible answer consists in the phenomenon of p-brane condensation. In order to illustrate its meaning, let us focus, once again, on String loops. Then, we suggest that what we call "classical spacetime" emerges as a condensate, or String vacuum similar to the ground state of a superconductor. The large scale properties of such a state are described by an "effective Riemannian geometry". At a distance scale of order $(\alpha')^{1/2}$, the condensate "evaporates", and with it, the very notion of Riemannian spacetime. What is left behind, is the fuzzy stuff of Quantum spacetime.

Clearly, the above scenario is rooted in the Functional Quantum Mechanics of String loops discussed in the previous sections, but is best understood in terms of a model which mimics the Ginzburg-Landau Theory of superconductivity. Let us recall once again that one of the main results of the Functional approach to Quantum Strings is that it is possible to describe the evolution of the system without giving up reparametrization invariance. In that approach, the clock that times the evolution of a closed bosonic String is the Area of the Parameter Space associated with the World-Sheet, which we rewrite here:

$$A = \frac{1}{2} \epsilon_{ab} \int_{\Xi} d\xi^a \wedge d\xi^b \quad .$$

The choice of such a *fiducial* surface is arbitrary, corresponding to the freedom of choosing the initial instant of time, i.e., a fiducial reference area. Then one can take advantage of this arbitrariness in the following way. In Particle Field Theory an arbitrary lapse of euclidean, or Wick rotated, imaginary time between initial and final field configurations is usually given the meaning of *inverse temperature*

$$i\Delta t \longrightarrow \tau \equiv \frac{1}{\kappa_B T}$$

and the resulting Euclidean Field Theory provides a *finite temperature* statistical description of vacuum fluctuations.

Following the same procedure, we suggest to analytically extend A to imaginary values, $iA \longrightarrow a$, on the assumption that the resulting finite area Loop Quantum Mechanics will

provide a statistical description of the *String* vacuum fluctuations. This leads us to consider the following effective (euclidean) lagrangian of the Ginzburg–Landau type,

$$L(\Psi, \Psi^*) = \Psi^* \frac{\partial}{\partial a} \Psi - \frac{1}{4m^2} \left(\oint dl(s) \right)^{-1} \oint_C dl(s) \left| \left(\frac{\delta}{\delta \sigma^{\mu\nu}(s)} - igA_{\mu\nu}(x) \right) \Psi \right|^2 + V\left(|\Psi|^2 \right) - \frac{1}{2 \cdot 3!} H^{\lambda\mu\nu} H_{\lambda\mu\nu}$$

$$(10.47)$$

$$V(|\Psi|^{2}) \equiv \mu_{0}^{2} \left(\frac{a_{c}}{a} - 1\right) |\Psi|^{2} + \frac{\lambda}{4} |\Psi|^{4}$$
(10.48)

$$H_{\lambda\mu\nu} = \partial_{[\lambda} A_{\mu\nu]} \quad . \tag{10.49}$$

Here, the String field is coupled to a Kalb–Ramond gauge potential $A_{\mu\nu}(x)$ and a_c represents a critical area such that, when $a \leq a_c$ the potential energy is minimized by the ordinary vacuum $\Psi[C] = 0$, while for $a > a_c$ Strings condense into a superconducting vacuum. In other words,

$$|\Psi[C]|^2 = \begin{cases} 0 & \text{if } a \le a_c \\ \frac{\mu_0^2}{\lambda} (1 - a_c/a) & \text{if } a > a_c \end{cases}$$
 (10.50)

Evidently, we are thinking of the String condensate as the large scale, background spacetime. On the other hand, as one approaches distances of the order $(\alpha')^{1/2}$ Strings undergo more and more shape—shifting, transitions which destroy the long range correlation of the String condensate. As we have discussed earlier, this signals the transition from the smooth to the fractal phase of the String World—Sheet. On the other hand, the Quantum Mechanical approach discussed in this paper is in no way restricted to String—like objects. In principle⁵, it can be extended to any quantum p-brane and the limiting value of the corresponding fractal dimension is $D_H = p + 2$. Then, if the above over all picture is correct, spacetime fuzziness acquires a well defined meaning. Far from being a smooth, 4-dimensional manifold assigned "ab initio", spacetime is, rather, a "process in the making", showing an ever changing fractal structure which responds dynamically to the resolving power of the detecting apparatus. At a distance scale of the order of Planck's length, i.e., when

$$a_c = G_N \tag{10.51}$$

then the whole of spacetime boils over and no trace is left of the large scale condensate of either Strings or p-branes.

 $^{^5}$ We have already seen in chapter 4 and in section 6.3 how the results given for the *String* can be generalized without further technical problems to the more general case of p-branes.

As a final remark, it is interesting to note that since the original paper by A.D.Sakarov about gravity as spacetime elasticity, G_N has been interpreted as a phenomenological parameter describing the large scale properties of the gravitational vacuum. Eq.(10.51) provides a deeper insight into the meaning of G_N as the critical value corresponding to the transition point between an "elastic" Riemannian-type condensate of extended objects and a Quantum phase which is just a Planckian foam of Fractal objects.

IV Conclusion

"You know, I did feel something. I could almost see the remote." "That's good. You have taken your first step into a larger world."

In the ten chapters before this conclusion we have touched a lot of arguments, some of them are very far away one from another. Even if the main subject has been a particular formulation of the Classical and Quantum Dynamics, of Relativistic Extended Objects, the main idea was to show how to develop, using quite natural concepts, an organic Theory without assuming to much preexisting structures behind. Thus it seems important to us to summarize the results we obtained, to remark the motivations which support our work, and the future developments of this line of research. Indeed the long road that brought after years of work and reflection to the results we gave here is not at the end: perhaps at the beginning, since it would be important to apply in a more concrete way the concepts developed so far.

In particular, we have in mind the difficulties that have been found so far in trying to approach the problem of quantizing Gravity: this is a problem that refused any proposed solution; the purely geometrical approach of General Relativity was not able to describe short distance spacetime structure in a consistent way; no better results came from the purely Quantum Mechanical approach, or from Quantum Field Theory. The problem of Quantizing Geometry remains still with us and there is still no satisfactory answer.

In our opinion all the difficulties are very deeply related to the fact that, until now, Gravity Theory and Quantum Theory have been viewed and interpreted as pertaining to very different physical realms: in connection to large scale problems, the first, and related to short range effects the second one. Thus a Quantum Theory of Gravity should be something that pertains to both fields: it should be something able to relate the physics at very different order of magnitude in length scale, or energy scale, which is the same. If we assume that, after all, purely quantum gravitational effects are really part of our world, we see only three possible roads toward the goal of finding a Theory able to describe them, i.e. to give some phenomenological prediction capable of being experimentally tested:

- 1. in first instance it is possible to investigate quantum effects on scales macroscopic with respect to, say, the atomic scale: far from being a new research field we think we are simply speaking of the *Quantum Theory of Measurement*, where Quantum Effects are brought to our everyday scale by a detecting apparatus;
- 2. in second instance we could also think to find on very small scales, say the scales of elementary particles, effects related to the large scale structure of spacetime: this topics are strongly related with research in AstroParticle Physics;
- 3. as a third possibility, we can start more or less by scratch, trying to see if there is some theoretical formulation capable to give a consistent framework to short as well as long range effects.

By the way, it is clear that we adhered in this work to the third possibility, which is the one developed in *String* Theory, but trying to give it a more Geometrical flavor, instead than a Quantum Field Theoretical one. In our point of view the Theory of extended objects has a main advantage in tackling problems associated with the presence of different scales because an extended object is composed by the *Bulk*, its history⁶, but also by the *Boundary*. Effects on the *Bulk* can be in some sense *local*, but can have some *global* influence on the *Boundary* too. This is the reason why we introduced the *Boundary Shadow Dynamics* of the *String* as a way to keep track of both effects. Moreover, the formulation we gave in terms

⁶The World-Sheet for a String.

of Hamilton–Jacobi equation is motivated by the fact that this is a key tool in deriving a connection between the Classical and the Quantum Theory, already at the point particle level.

The main result of our formulation is that it is the same Theory that singles out, given the Parametrization for the Bulk, the most appropriate canonical variables for the Boundary Quantum Dynamics which is encoded in our Functional Wave Equation⁷: these are the Holographic Coordinates. Moreover, also the evolution parameter is given as a product of our description of the Boundary Dynamics and is again an areal quantity, namely the area of the Parameter Space, the fiducial manifold on which the String and World-Sheet Embedding Functions are defined.

Now, it is not difficult to imagine that a complete Quantum Theory of Gravity would probably have relevant effects on our view of the world, which for a physicist, probably, means spacetime. Let us explain in more detail our ideas about this issue:

- the main revolution of General Relativity has been to recognize that the presence of
 mass and energy can influence the causal structure of our world and, at the same time,
 be driven from it: we are backreacted in some way;
- 2. on the other hand, Quantum Theory gave us the very deep result that, thanks to the Uncertainty Principle, not all situations, that we could imagine starting from our classical background, are practically realizable: we are constrained in some other way.

Thus, in particular, it is a logical conclusion that in a would be Theory of Quantum Gravity the distribution of mass and energy should give a constrained backreacting quantum interaction that only in the average, taken at some particular scale, reproduces our classical world. Moreover, not all the possible configurations, or, which is the same, spacetime structures⁸, empirically detectable as true in this classical average, are realizable in the Quantum Gravitational Realm.

 $^{^{7}}$ Incidentally we observe the striking similarity between our equation and the Wheeler-deWitt equation for the *Wave Function of the Universe*.

⁸Or, again, mass and energy distribution, i.e. what we perceive as motion, flow, and so on.

We like to think that as happens in Atomic Physics, the Uncertainty Principle could be advocated to stabilize systems that, otherwise, would undergo collapse. That is to say that we like to think the Uncertainty Principle as the physical reason for the existence of extension at small scales, because pointlikness explicitly violates such a fundamental principle. Thus in a Quantum Theory of Gravity to assume the *point* as the basic constituent of spacetime is not consistent: the rules of Quantum Mechanics forbids pointlike situations, because we are uncertainty bounded in our empirical knowledge. In this way the reader can see how we have reach by a completely different point of view the same conclusion already present in contemporary String Theory: the elementary constituents of a Quantum Gravitational spacetime must be extended objects, not point at all. This is the ground of our proposal of describing the Quantum Dynamics of Extended Objects with a first quantized functional formulation: if points becomes extended, let us say Strings, but way not membranes or other extendons, then a Quantum Theory of spacetime is a Theory defined in a space of loops, membranes, ..., p-branes in general. Even if Classical String Dynamics is based on the simple and intuitive notion that the World-Sheet of a Relativistic String consists of a smooth, 2-dimensional manifold embedded in a preexisting spacetime, switching from Classical to Quantum Dynamics changes this picture in a fundamental way. Indeed, already in the path-integral approach to particle Quantum Mechanics, Feynman and Hibbs were first to point out that the trajectory of a particle is continuous but nowhere differentiable [10]. This is again the Uncertainty Principle at work⁹: when a particle is more and more precisely located in space, its trajectory becomes more and more erratic. Abbott and Wise were next to point out that a particle trajectory, appearing as a smooth line of topological dimension one, turns into a fractal object of Hausdorff dimension two, when the resolution of the detecting apparatus is smaller than the particle De Broglie wavelength [11]. Now in our point of view extended objects, not points, are the constituents of spacetime at very short scales. This is the reason why we extended the path-integral approach to the String case: in this case one is taking into account the coherent contributions from all the World-Sheets satisfying some preassigned Boundary conditions and one finds that a String Quantum World-Sheet is a fractal, non-differentiable surface. The way we explained in this

⁹In this sense, we stress that again, trying to combine Quantum Theory with a Theory of spacetime, points directly toward a formulation in terms of extended objects.

Thesis to give a quantitative support to this expectation takes advantage by working with a first quantized formulation of String Dynamics: in this way we are quantizing the String motion not through the displacement of each point on the String, but through the String shape. The outcome of this novel approach is a Quantum Mechanics of Loops, describing quantum shape shifting transitions, i.e., the Shadow Dynamics of the String. The emphasis on String shapes, rather than points, meets the interpretation we gave a few lines above of Strings as constituents of spacetime. We think worth to note that this departure from the canonical formulation which requires an appropriate choice of dynamical variables, namely, the String Holographic Coordinates Y[C] and the areal time A, immediately present itself with a very striking and relevant property. In particular, the Shadow Dynamics is not able to completely single out a precise loop in the Quantum String First Quantized Dynamics. From the information encoded in the Holographic Coordinates we can not say that our String, i.e. extended spacetime point, is a square, or a circle, ..., but only that there is only one parallelogram in space that has on the coordinate planes the projections such that their areas coincide with the components of the Holographic Coordinates Tensor. Thus our functional approach enables us to extend the Quantum Mechanical discussion to spacetime itself: but we have to adhere to the following principle: events cannot be localized with infinite precision, because spacetime is no more made of points; its elementary constituents are extended object (say Strings) and we thus can only say that we are able to find a "plaquette", with the form of a parallelogram, of which we know the projection onto those object that classically are the coordinate planes and in which our event takes place.

The Dynamics of such extended spacetime elements could be a bit confusing at first sight, if not contradictory: following all the procedure the reader could object that we deal with a relativistic system in a Quantum Mechanical, i.e., non relativistic framework, as opposed to a quantum field theoretical framework [14]. Even if this is not new in theoretical physics [15], in any case, one has to realize that there are two distinct levels of discussion in our approach. The *spacetime level*, where the actual deformations of the *String*, thought as a constituent element, take place, and where the formulation is fully relativistic, as witnessed by the covariant structure of our equations with respect to the Lorentzian indices. However,

at the loop space level, where each "point" is representative of a particular loop configuration, our formulation is Quantum Mechanical, in the sense that the String coordinates $Y^{\mu\nu}$ and A are not treated equally, as it is manifest, for instance, in the loop Schrödinger equation (3.63). As a matter of fact, this is the very reason for referring to that equation as the "Schrödinger equation" of String Dynamics: the timelike variable A enters the equation through a first order partial derivative, as opposed to the functional "laplacian", which is of second order with respect to the spacelike variables $Y^{\mu\nu}$. Far from being an artifact of our formulation, we emphasize that this spacetime covariant, Quantum Mechanics of loops, is a direct consequence of the Hamilton–Jacobi formulation of Classical String Dynamics. Moreover, since with our present quantum mechanical formulation we have given a concrete meaning to the fractalization of a String in terms of the Shape Uncertainty Principle, taking it as a fundamental constituent of spacetime transfers this property to it as well. We have concluded that the Hausdorff dimension of a quantum String is three, and that two distinct geometric phases exist above and below the loop De Broglie area. As a matter of fact, the Quantum String is literally "fuzzy" to a degree which depends critically on a well defined parameter, the ratio between temporal and spatial resolution. Then, if the above over all picture is correct, String as well as p-brane fuzziness not only acquires a well defined meaning, but points to a fundamental change in our perception of physical spacetime. It is also fractal, since its constituent are fractal objects. Far from being a smooth, fourdimensional manifold assigned "ab initio", spacetime is, rather, a "process in the making", showing an ever changing fractal structure which responds dynamically to the resolving power of the detecting apparatus. It is important to point out that Classical String Theory can be recovered, as we have shown in chapter 8. Of course we want that at traditional energy scales the world appears exactly like we know it. But as shown in chapter 9 at very high energy Bulk and Boundary effects take place: enlightening is in particular the relation between our formulation and the traditional way of looking at String Theory as the Polyakov Bulk Theory.

Let us conclude our work with just a curious observation. With our interpretation it is evident that the Quantum Gravitational spacetime has a Fractal Structure since, as we al-

ready pointed out, its constituents elements are extended and fractal objects. It is thus very appealing to see that our functional equation can be recovered using an interesting mathematical tool, namely NonStandard Analysis: the problem of infinities and infinitesimals disappears, thanks to it, from the mathematical computations and thanks to our Functional Wave Equation, that can be rigorously defined using it, the Fractal Structure of Quantum Gravitational spacetime looses all the divergence problems related to pointlike localization. Moreover, NonStandard Analysis looks a good tool in dealing problems associated to Fractals, where the usual notion of differentiability breaks down. We believe that NonStandard Analysis will be the natural framework to solve other problems, and attack more fundamental questions, in both Quantum Gravity and other different sectors of theoretical Physics, even if we cannot give it for granted.

Finally, it is very tempting to conjecture a relation between our *NonStandard* formulation and NonCommutative Geometry that is, nowadays, perhaps the best candidate to describe the short distance structure of quantum spacetime: but ... this is another story.

Appendix A

Detailed Calculations

A.1 Lagrangian, Hamiltonian and Coefficients!

We devote this section to the standard derivation of the Hamiltonian density from the Lagrangian density in the case of a p-brane, in order to motivate the values of the numerical factors that appear as coefficients. With ρ_p we indicate the mass in the particle case, m^2 in the String case and in general for the p-brane we will have $\rho_p = m^{p+1}$, where m is a constant with dimension of mass. The factorial in front of the Schild Lagrangian density avoids the overcounting due to the p+1 saturated antisymmetric indices of the kinetic term.

Proposition A.1 (p-brane Schild Hamiltonian Density).

The Schild Hamiltonian density $\mathcal{H}(\mathbf{P})$ is given by

$$\mathcal{H}(\mathbf{P}) = \frac{1}{2\rho_p (p+1)!} P_{\beta_1 \dots \beta_{p+1}} P^{\beta_1 \dots \beta_{p+1}}$$
(A.1)

where, $P_{\mu_1...\mu_{p+1}}$ is the momentum canonically conjugated to the World–HyperTube Volume Velocity $\dot{X}^{\nu_1...\nu_{p+1}}$.

Proof:

We start from the Schild Lagrangian density

$$\mathcal{L}(X, \dot{X}) = \frac{\rho_p}{2(p+1)!} \dot{X}^{\mu_1 \dots \mu_{p+1}} \dot{X}_{\mu_1 \dots \mu_{p+1}} \quad ; \tag{A.2}$$

moreover we defined the *p*-brane Bulk Area Momentum as the variable canonically conjugated to $\dot{X}^{\beta_1...\beta_{p+1}}$, so that we get as usual:

$$P_{\alpha_{1}...\alpha_{p+1}} = \frac{\partial \mathcal{L}}{\partial \dot{X}^{\alpha_{1}...\alpha_{p+1}}}$$

$$= \frac{\rho_{p}}{2(p+1)!} \frac{\partial \left(\dot{X}^{\mu_{1}...\mu_{p+1}} \dot{X}_{\mu_{1}...\mu_{p+1}}\right)}{\partial \dot{X}^{\alpha_{1}...\alpha_{p+1}}}$$

$$= \frac{\rho_{p}}{(p+1)!} (p+1)! \dot{X}^{\alpha_{1}...\alpha_{p+1}}$$

$$= \rho_{p} \dot{X}^{\alpha_{1}...\alpha_{p+1}} . \tag{A.3}$$

Then, Legendre transforming, we get for the Hamiltonian the result:

$$\mathcal{H} = \frac{1}{(p+1)!} P_{\mu_{1} \dots \mu_{p+1}} \dot{X}^{\mu_{1} \dots \mu_{p+1}} - \mathcal{L}$$

$$= \frac{1}{\rho_{p} (p+1)!} P_{\mu_{1} \dots \mu_{p+1}} P^{\mu_{1} \dots \mu_{p+1}} - \frac{1}{2\rho_{p} (p+1)!} P_{\mu_{1} \dots \mu_{p+1}} P^{\mu_{1} \dots \mu_{p+1}}$$

$$= \frac{1}{2\rho_{p} (p+1)!} P_{\mu_{1} \dots \mu_{p+1}} P^{\mu_{1} \dots \mu_{p+1}} . \tag{A.4}$$

We observe that this general result correctly reproduces the *Bulk Area Momentum* in the *String* case (since p=1 gives $\rho_p=m^2$) and also the momentum of a non relativistic particle, which is a p=0-brane (so that¹) $\rho_p=m$).

A.2 ξ and π Variation of the Action for a p-brane

We will derive, the energy balance equation both for Strings and p-branes. This is the equation of motion associated with a variation of the $\xi(\sigma)$ fields in the actions (2.21) and (4.4) respectively. The interested reader can compare this derivation with the procedure used in proposition 2.11 to relate the two methods. To clear the way, we start from the case with less floating indices, i.e. the String case.

Proposition A.2 (Standard ξ and π Variations).

The equations of motion induced by variations of the ξ and π fields in the action (2.21) are given by

$$\epsilon^{mn}\partial_m \pi_{AB}\partial_n \xi^B = 0 \tag{A.5}$$

¹Of course we have also to *formally* understand under the "dot · " a time derivative.

$$\pi_{AB} = \epsilon_{AB} \mathcal{H}(\mathbf{P}) \quad . \tag{A.6}$$

Proof:

The first result (equation (A.5)) requires a longer procedure. The only term contributing is the second one in equation 2.46; we thus get:

$$\delta_{\xi} \left(\frac{1}{2} \int_{\Xi} d\xi^{A} \wedge d\xi^{B} \pi_{AB} \right) =$$

$$1. \qquad = \frac{1}{2} \int_{\Xi} \delta_{\xi} \left(d\xi^{A} \wedge d\xi^{B} \right) \pi_{AB}$$

$$2. \qquad = \frac{1}{2} \int_{\Xi} d \left(\delta \xi^{A} \right) \wedge d\xi^{B} \pi_{AB} + \frac{1}{2} \int_{\Xi} d\xi^{A} \wedge d \left(\delta \xi^{B} \right) \pi_{AB}$$

$$3. \qquad = \frac{1}{2} \int_{\Xi} d \left(\delta \xi^{A} \right) \wedge d\xi^{B} \pi_{AB} - \frac{1}{2} \int_{\Xi} d \left(\delta \xi^{B} \right) \wedge d\xi^{A} \pi_{AB}$$

$$4. \qquad = \int_{\Xi} d \left(\delta \xi^{A} \right) \wedge d\xi^{B} \pi_{AB}$$

$$5. \qquad = \int_{\Xi} d \left[\delta \xi^{A} d\xi^{B} \pi_{AB} \right] - \int_{\Xi} \left\{ d\xi^{B} \wedge d\pi_{AB} \right\} \delta \xi^{A}$$

$$6. \qquad = 0 - \int_{\Sigma} \left\{ d\sigma^{m} \wedge d\sigma^{n} \partial_{m} \xi^{B} \partial_{n} \pi_{AB} \right\} \delta \xi^{A}$$

$$(A.7)$$

We used the following properties in the various steps:

- 1. the variation affects only the Ξ volume form, since in the reparametrization invariant action we have $P = P(\sigma)$;
- 2. distributive property of the \land -product with respect to the variation δ ;
- 3. symmetry property of the \land -product;
- 4. symmetry property of π_{AB} ;
- 5. integration by parts and nihilpotency of the exterior derivative;
- 6. the first contribution vanishes, because if we go to the Boundary, there the ξ variations vanish; then we have

$$d\xi^{B} = \frac{\partial \xi^{B}}{\partial \sigma^{m}} d\sigma^{m} \equiv \partial_{m} \xi^{b} d\sigma^{m}$$
(A.8)

and

$$d\mathcal{H}(P) = \frac{\partial \mathcal{H}(P)}{\partial \sigma^n} d\sigma^n \equiv \partial_n \mathcal{H}(P) d\sigma^n \quad . \tag{A.9}$$

Remembering now that $d\sigma^m \wedge d\sigma^n$ is a 2-form in 2 dimensions we can write $d\sigma^m \wedge d\sigma^n = \epsilon^{mn} d^2\sigma$, so that from equation (A.7) we obtain

$$\epsilon^{mn}\partial_m \xi^B \partial_n \pi_{AB} = 0 \quad . \tag{A.10}$$

The second equation has a simpler derivation since no integration by parts is needed. The following chain of equalities leads to the result:

(A.11)

Now, it is a routine to follow the same steps for the case of the *p-brane*, because the generalizations of the previous properties work in the same way.

Proposition A.3 (p-brane Standard ξ and π Variation).

The equations of motion induced by variations of the ξ and π fields in the action (4.4) are given by

$$\frac{1}{p!} \epsilon^{m_1 \dots m_p m} \left(\partial_{m_1} \xi^{A_1} \right) \cdot \dots \cdot \left(\partial_{m_p} \xi^{a_p} \right) \partial_m \pi_{A_0 \dots A_p} = 0 \tag{A.12}$$

$$\pi_{A_0...A_p} = \epsilon_{A_0...A_p} \mathcal{H}(\mathbf{P}) \quad . \tag{A.13}$$

Proof:

Of course the first result needs again a longer computation:

$$\delta_{\boldsymbol{\xi}} \left(\frac{1}{(p+1)!} \int_{\Xi^{(p+1)}} d\boldsymbol{\xi}^{A_0} \wedge \dots \wedge d\boldsymbol{\xi}^{A_p} \pi_{A_0 \dots A_p} \right) =$$

$$1. \qquad = \frac{1}{(p+1)!} \int_{\Xi^{(p+1)}} \delta_{\boldsymbol{\xi}} \left(d\boldsymbol{\xi}^{A_0} \wedge \dots \wedge d\boldsymbol{\xi}^{A_p} \right) \pi_{A_0 \dots A_p}$$

$$2. \qquad = \frac{1}{(p+1)!} \sum_{i}^{1,p+1} \int_{\Xi^{(p+1)}} d\boldsymbol{\xi}^{A_0} \wedge \dots \wedge d\left(\delta \boldsymbol{\xi}^{A_i} \right) \wedge \dots \wedge d\boldsymbol{\xi}^{A_p} \pi_{A_0 \dots A_i \dots A_p}$$

$$3. \qquad = \frac{p+1}{(p+1)!} \int_{\Xi^{(p+1)}} d\left(\delta \boldsymbol{\xi}^{A_0} \right) \wedge d\boldsymbol{\xi}^{A_1} \wedge \dots \wedge d\boldsymbol{\xi}^{A_p} \pi_{A_0 \dots A_p}$$

$$4. \qquad = \frac{1}{p!} \int_{\Xi^{(p+1)}} d\left[d\boldsymbol{\xi}^{A_1} \wedge \dots \wedge d\boldsymbol{\xi}^{A_p} \left(\delta \boldsymbol{\xi}^{A_0} \right) \pi_{A_0 \dots A_p} \right] +$$

$$-\frac{1}{p!} \int_{\Xi^{(p+1)}} \left[d\boldsymbol{\xi}^{A_1} \wedge \dots \wedge d\boldsymbol{\xi}^{A_p} \wedge \pi_{A_0 \dots A_p} \right] \delta \boldsymbol{\xi}^{A_0}$$

$$5. \qquad = 0 - \int_{\Sigma^{(p+1)}} \left\{ \frac{1}{p!} d\sigma^{m_2} \wedge \dots \wedge d\sigma^{m_{p+1}} \wedge d\sigma^{m} \cdot \left(\partial_{m_2} \boldsymbol{\xi}^{A_2} \right) \cdot \dots \cdot \left(\partial_{m_{p+1}} \boldsymbol{\xi}^{A_{p+1}} \right) \partial_m \pi_{A_1 \dots A_{p+1}} \right\} \delta \boldsymbol{\xi}^{A_1} \quad . \quad (A.14)$$

The main computational steps are:

- 1. as before the variation affects only the integration volume in $\Xi^{(p+1)}$ space, which is now (p+1)-dimensional;
- 2. by distributing the variation to the single basis 1-forms, we obtain a sum of (p+1)-terms, which . . .
- 3. . . . can be summed over by means of the antisymmetry property of the \land -product and of the (p+1)-dimensional ϵ tensor;

- 4. integrating by parts (using of course $d^2 = 0$) and
- 5. taking into account that the ξ fields are fixed on the *Boundary*, $\partial \Xi^{(p+1)}$, gives the vanishing of the first term; for the second one we can then proceed expressing the differentials in terms of the new σ variables.

Again we remember that a (p+1)-form in (p+1)-dimensions is proportional to the ϵ tensor to conclude from equation (A.14) that

$$\frac{1}{p!} \epsilon_{A_1 \dots A_{p+1}} \epsilon^{m_2 \dots m_{p+1} m} \left(\partial_{m_2} \xi^{A_2} \right) \cdot \dots \cdot \left(\partial_{m_{p+1}} \xi^{A_{p+1}} \right) \partial_m \pi_{A_1 \dots A_{p+1}} = 0 \quad . \tag{A.15}$$

A.3 Momentum Variation in Reparametrized Hamiltonian Formalism

Let us perform the variation of the action for the full reparametrized Theory (4.17) with respect to π to get equation (4.18). The computation is shown in the general case. The String case can be derived by setting p = 1.

Proposition A.4 (p-brane π Variation of the Action).

The equation of motion obtained by the π variation of the action (4.17) (respectively (2.46) for the String) are given by

$$N^{A_1...A_{p+1}} = \dot{\xi}^{A_1...A_{p+1}} \quad . \tag{A.16}$$

Proof:

There are no problems due to integrations by parts here. Then

$$\delta_{\pi}S = \frac{1}{(p+1)!} \delta \left\{ \int_{\Xi^{(p+1)}} \pi_{a_{1}...a_{p+1}} d\xi^{a_{1}} \wedge ... \wedge d\xi^{a_{p+1}} \right\} + \\
- \frac{1}{(p+1)!} \delta \left\{ \int_{\Sigma^{(p+1)}} d^{p+1} \sigma N^{A_{1}...A_{p+1}} \left[\pi_{A_{1}...A_{p+1}} - \epsilon_{A_{1}...A_{p+1}} \mathcal{H}(\mathbf{P}) \right] \right\} \\
= \frac{1}{(p+1)!} \delta \left\{ \int_{\Sigma^{(p+1)}} d^{p+1} \sigma \left[\pi_{A_{1}...A_{p+1}} \left(\dot{\xi}^{A_{1}...A_{p+1}} - N^{A_{1}...A_{p+1}} \right) - \epsilon_{A_{1}...A_{p+1}} \mathcal{H}(\mathbf{P}) \right] \right\} \\
= \frac{1}{(p+1)!} \int_{\Sigma^{(p+1)}} d^{p+1} \sigma \left[\dot{\xi}^{A_{1}...A_{p+1}} - N^{A_{1}...A_{p+1}} \right] \delta \pi_{A_{1}...A_{p+1}} , \qquad (A.17)$$

where,

$$\dot{\xi}^{A_1\dots A_{p+1}} = \epsilon^{m_1\dots m_{p+1}} \left(\partial_{m_1} \xi^{A_1}\right) \cdot \dots \cdot \left(\partial_{m_{p+1}} \xi^{A_{p+1}}\right) \quad . \tag{A.18}$$

The result of equation (4.18) (respectively 2.53) is then obtained from the stationarity condition $\delta_{\pi}S = 0$

$$\Rightarrow \quad \dot{\xi}^{A_1...A_{p+1}} = N^{A_1...A_{p+1}} \quad .$$

A.4 Hamilton Principle with one Free Boundary

In this section we perform the variation of the action letting the future Boundary vary freely. As we pointed out in subsection 2.5.2 and in section 4.2, this is an important kind of variational procedure, since it gives the opportunity to identify the canonical momentum conjugated to the Boundary and define the energy associated to the Boundary Dynamics. It turns out that the energy is now conjugated to the proper area of the String Parameter Space or, equivalently, to the hypervolume associated with the p-brane Parameter Space when we consider higher dimensional objects in D-dimensional spacetime. In this more general framework the Boundary is a (p-1)-brane. Now, we have to consider variations of the fields, which satisfy the classical equations of motion on the Bulk, i.e. on the World-HyperTube swept by the evolution of the object itself.

As in the previous section, we start from the simpler String case.

Proposition A.5 (Free Boundary Variation of the String Action).

The free Boundary variation of the action (2.54) with respect to the fields \mathbf{Y} is given by:

$$\delta_{\mathbf{Y}_{(f)}} S = \int_{\partial \Sigma} ds \, Q_{\mu\nu} Y^{\prime\nu} \delta Y^{\mu}_{(f)} - \mathcal{H} \delta A \quad . \tag{A.19}$$

Proof:

The computation is of standard type; we have only to perform in the proper way the passage from the Bulk to the Boundary, i.e. from the Parameter Space Σ to the Boundary Space $\Gamma = \partial \Sigma$.

$$\delta_{\mathbf{Y}}S = \frac{1}{2} \int_{\mathcal{W}} (\delta P_{\mu\nu} d\mathbf{X}^{\mu} \wedge d\mathbf{X}^{\nu} + P_{\mu\nu} d(\delta X^{\mu}) \wedge d\mathbf{X}^{\nu} + P_{\mu\nu} d\mathbf{X}^{\mu} \wedge d(\delta X^{\nu})) + \\ -\delta \left(\frac{1}{2} \epsilon_{AB} \int_{\Xi} d\boldsymbol{\xi}^{A} \wedge d\boldsymbol{\xi}^{B} \right) \mathcal{H}(\mathbf{P}) - \frac{1}{2} \epsilon_{AB} \int_{\Xi} d\boldsymbol{\xi}^{A} \wedge d\boldsymbol{\xi}^{B} \frac{1}{2m^{2}} P_{\mu\nu} \delta P^{\mu\nu}$$

1.
$$= \int_{\Sigma} d^{2}\sigma \left(\frac{\dot{X}^{\mu\nu}}{2} - \frac{\epsilon_{AB}\dot{\xi}^{AB}}{4m^{2}} P^{\mu\nu} \right) \delta P_{\mu\nu} - \mathcal{H}\delta A + \int_{\mathcal{W}} P_{\mu\nu} d \left(\delta X^{\mu} \right) \wedge dX^{\nu}$$
2.
$$= \int_{\Sigma} d^{2}\sigma \left(\frac{\dot{X}^{\mu\nu}}{2} - \frac{\epsilon_{AB}\dot{\xi}^{AB}}{4m^{2}} P^{\mu\nu} \right) \delta P_{\mu\nu} - \mathcal{H}\delta A +$$

$$+ \int_{\mathcal{W}} d \left[P_{\mu\nu} dX^{\nu} \delta X^{\mu} \right] - \int_{\mathcal{W}} d \left[P_{\mu\nu} dX^{\nu} \right] \delta X^{\mu}$$
3.
$$= \int_{\Sigma} d^{2}\sigma \left(\frac{\dot{X}^{\mu\nu}}{2} - \frac{\epsilon_{AB}\dot{\xi}^{AB}}{4m^{2}} P^{\mu\nu} \right) \delta P_{\mu\nu} - \mathcal{H}\delta A +$$

$$+ \int_{\partial \mathcal{W} = C} P_{\mu\nu} dY^{\nu} \delta Y^{\mu}_{(f)} - \int_{\mathcal{W}} dP_{\mu\nu} \wedge dX^{\nu} \delta X^{\mu}$$
4.
$$= \int_{\Sigma} d^{2}\sigma \left(\frac{\dot{X}^{\mu\nu}}{2} - \frac{\epsilon_{AB}\dot{\xi}^{AB}}{4m^{2}} P^{\mu\nu} \right) \delta P_{\mu\nu} - \mathcal{H}\delta A +$$

$$+ \int_{\partial \Sigma = \Gamma} ds Q_{\mu\nu} Y'^{\nu} \delta Y^{\mu}_{(f)} - \int_{\Sigma} d^{2}\sigma \epsilon^{AB} \partial_{A} P_{\mu\nu} \partial_{B} X^{\nu} \delta X^{\mu}$$
5.
$$= \int_{\Gamma} ds Q_{\mu\nu} Y'^{\nu} \delta Y^{\mu}_{(f)} - \mathcal{H}\delta A$$

$$(A.20)$$

- 1. we put together the first and the last term expressing all the quantities in terms of the new variables, σ ; then, taking into account the antisymmetry of both $P_{\mu\nu}$ and the \wedge -product, we can add together the second and third terms; as far as the fourth term is concerned, we recall that the Hamiltonian is constant along a classical trajectory and so can be moved outside the integral;
- 2. we integrate by parts the last term;
- 3. we compute the first term on the second line on the Boundary, and apply the properties of the exterior derivative to the second term on the second line;
- 4. we express the first integral on the second line in terms of the variable parametrizing the loop $(s \in \Gamma)$; the notation $\delta Y_{(f)}^{\mu}$ wants to remark that we are using a variational principle with free end loop; the same procedure is applied to the last integral;
- 5. we select only variations that satisfy the classical equations of motion on the *Bulk* (actually, we did it already as we took $\mathcal{H}(\mathbf{P})$ outside the integral as \mathcal{H}); then the first and last term vanish.

Following the same line we can now perform the same computation for the p-brane in D-dimensions. The steps are somewhat more tedious due to the increased number of dimensions, but the computation is basically the same.

Proposition A.6 (Free Boundary Variation of the *p*-brane Action).

The one free Boundary variation of the action (4.4) with respect to the fields X is given by:

$$\delta_{\mathbf{Y}_{(f)}} S = \frac{1}{p!} \int_{\Gamma(s)} d^p s \, Q_{\mu_1 \dots \mu_{p+1}} Y'^{\mu_2 \dots \mu_{p+1}} \delta Y^{\mu_1} - \mathcal{H} \delta V \tag{A.21}$$

Proof:

The proof follows the same steps as the previous one, with just a little bit of more troubles due to the proliferation of indices. We thus get

$$\delta S = \frac{1}{(p+1)!} \int_{\mathcal{W}(p+1)} \left[\delta P_{\mu_{1} \dots \mu_{p+1}} dX^{\mu_{1}} \wedge \dots \wedge dX^{\mu_{p+1}} + \sum_{i}^{1,p+1} \left(P_{\mu_{1} \dots \mu_{p+1}} dX^{\mu_{1}} \wedge \dots \wedge \delta (dX^{\mu_{i}}) \wedge \dots \wedge dX^{\mu_{p+1}} \right) \right] + \\ + \sum_{i}^{1,p+1} \left(P_{\mu_{1} \dots \mu_{p+1}} dX^{\mu_{1}} \wedge \dots \wedge \delta (dX^{\mu_{i}}) \wedge \dots \wedge dX^{\mu_{p+1}} \right) \right] + \\ - \delta \left(\frac{1}{(p+1)!} \epsilon_{a_{1} \dots a_{p+1}} \int_{\Xi^{(p+1)}} d\xi^{a_{1}} \wedge \dots \wedge d\xi^{a_{p+1}} \right) \mathcal{H}(P) + \\ - \frac{1}{(p+1)!} \epsilon_{a_{1} \dots a_{p+1}} \int_{\Xi^{(p+1)}} d\xi^{a_{1} \wedge \dots \wedge d\xi^{a_{p+1}}} P_{\mu_{1} \dots \mu_{p+1}} \delta P^{\mu_{1} \dots \mu_{p+1}} \right]$$

$$1. = \int_{\Sigma^{(p+1)}} d^{p+1} \sigma \left(\frac{\dot{X}_{\mu_{1} \dots \mu_{p+1}}}{(p+1)!} - \frac{1}{\rho_{p} (p+1)!} \epsilon_{a_{1} \dots a_{p+1}} \dot{\xi}^{a_{1} \dots a_{p+1}} P_{\mu_{1} \dots \mu_{p+1}} \right) \delta P^{\mu_{1} \dots \mu_{p+1}} + \\ - \mathcal{H}\delta V + \frac{1}{p!} \int_{\mathcal{W}^{(p+1)}} P_{\mu_{1} \dots \mu_{p+1}} d(\delta X^{\mu_{1}}) \wedge dX^{\mu_{2}} \wedge \dots \wedge dX^{\mu_{p+1}} \right] + \\ - \mathcal{H}\delta V + \frac{1}{p!} \int_{\mathcal{W}^{(p+1)}} d\left[P_{\mu_{1} \dots \mu_{p+1}} dX^{\mu_{2}} \wedge \dots \wedge dX^{\mu_{p+1}} \dot{\xi}^{a_{1} \dots a_{p+1}} P_{\mu_{1} \dots \mu_{p+1}} \right) \delta P^{\mu_{1} \dots \mu_{p+1}} + \\ - \frac{1}{p!} \int_{\mathcal{W}^{(p+1)}} d\left[P_{\mu_{1} \dots \mu_{p+1}} dX^{\mu_{2}} \wedge \dots \wedge dX^{\mu_{p+1}} \right] \delta X^{\mu_{1}}$$

$$3. = \int_{\Sigma^{(p+1)}} d^{p+1} \sigma \left(\frac{\dot{X}_{\mu_{1} \dots \mu_{p+1}}}{(p+1)!} - \frac{1}{\rho_{p} (p+1)!} \epsilon_{a_{1} \dots a_{p+1}} \dot{\xi}^{a_{1} \dots a_{p+1}} p_{\mu_{1} \dots \mu_{p+1}} \right) \delta p^{\mu_{1} \dots \mu_{p+1}} + \\ - \mathcal{H}\delta V + \frac{1}{p!} \int_{\partial \mathcal{W}^{(p+1)}} D^{(p)} P_{\mu_{1} \dots \mu_{p+1}} dY^{\mu_{2}} \wedge \dots \wedge dY^{\mu_{p+1}} \delta Y^{\mu_{1}} + \\ - \frac{1}{p!} \int_{\Sigma^{(p+1)}} d^{p+1} \sigma \epsilon^{a_{1} \dots a_{p+1}} \partial_{a_{1}} P_{\mu_{1} \dots \mu_{p+1}} \partial_{a_{2}} X^{\mu_{2}} \dots \partial_{a_{p+1}} X^{\mu_{p+1}} \delta X^{\mu_{1}}$$

$$4. = \frac{1}{p!} \oint_{D^{(p)}} Q_{\mu_{1} \dots \mu_{p+1}} dY^{\mu_{2}} \wedge \dots \wedge dY^{\mu_{p+1}} \delta Y^{\mu_{1}} - \mathcal{H}\delta V$$

$$(A.22)$$

- 1. we collect together the terms corresponding to the previous (String) case, i.e. those containing the variation of the momentum δP , expressing them as functions of the new variables σ ; the terms in the \sum can be summed over, as usually, thanks to the properties of the \wedge -product and of the ϵ tensor (third term), whereas the second term is obtained by implementing the constancy of the Hamiltonian along a classical trajectory;
- 2. we integrate by parts the last integral obtaining the last two ones ...
- 3. . . . then, we go to the *Boundary* in the first of them and switch to the σ variables in the last one:
- 4. now, we restrict ourself to variations which solve the classical equations of motions on the Bulk...
- $5.\ldots$ and turn to the new variables in the first integral.

We stress again that in the quantity

$$Y'^{\mu_1\dots\mu_p} = \epsilon^{a_1\dots a_p} \partial_{a_1} Y^{\mu_1} \dots \partial_{a_p} Y^{\mu_p} \quad , \tag{A.24}$$

quoted also in definition 4.5 and in equation (4.8), the derivation involves only the p Boundary variables $s = (s_1, \ldots, s_p)$; i.e., \mathbf{Y}' is the tangent element to the p-brane Boundary. So, one is led to the interpretation of

$$q_{\mu} = Q_{\mu\mu_1\dots\mu_p} Y^{\prime\mu_1\dots\mu_p} \tag{A.25}$$

as the projection of the momentum on the Boundary configuration, in the case of the String as well as of the p-brane.

A.5 Functional Integration of the $\xi(\sigma)$ Fields

In this section we perform the functional integration of the ξ fields in the case of the pbrane. The main reason to devote a section for this computation is to point out the way
in which, using a Functional Fourier Transform, it is possible to get a Functional Integral
Representation of a Functional Dirac Delta over the classical equation of motion. Moreover
we stress again the relevance of the passage to the Boundary and we carefully choose the
notation to underline that.

Proposition A.7 (Functional Integration over ξ Fields).

The functional integration over the ξ fields singles out a functional Dirac delta over the classical equation of motion of the Bulk $W^{(p+1)}$ times a term dependent only from the fields on the Boundary:

$$\int_{\boldsymbol{\zeta}_{0}(\boldsymbol{s})}^{\boldsymbol{\zeta}(\boldsymbol{s})} [D\boldsymbol{\xi}(\boldsymbol{\sigma})] \exp \left\{ \frac{i}{(p+1)!} \int_{\Xi(\boldsymbol{\sigma})} \pi_{A_{1}...A_{p+1}} d\boldsymbol{\xi}^{A_{1}} \wedge ... \wedge d\boldsymbol{\xi}^{A_{p+1}} \right\} =$$

$$= \exp \left\{ \frac{i}{(p+1)!} \int_{\partial\Xi(\boldsymbol{\sigma})} \zeta^{A_{p+1}} \pi_{A_{1}...A_{p+1}} d\zeta^{A_{1}} \wedge ... \wedge d\zeta^{A_{p}} \right\} \cdot$$

$$\cdot \delta \left[\epsilon^{m_{1}...m_{p+1}} \partial_{m_{p+1}} \pi_{A_{1}...A_{p+1}} \partial_{m_{1}} \xi^{A_{1}} ... \partial_{m_{p}} \xi^{A_{p}} \right] . \quad (A.26)$$

Proof:

The first step to perform the integration is to use the following identity

$$\int_{\Xi} \pi_{A_1 \dots A_{p+1}} d\boldsymbol{\xi}^{A_1} \wedge \dots \wedge d\boldsymbol{\xi}^{A_{p+1}} =
= \int_{\Xi} d \left(\boldsymbol{\xi}^{A_{p+1}} \pi_{A_1 \dots A_{p+1}} d\boldsymbol{\xi}^{A_1} \wedge \dots \wedge d\boldsymbol{\xi}^{A_p} \right) +
- \int_{\Xi} \boldsymbol{\xi}^{A_{p+1}} d \left(\pi_{A_1 \dots A_{p+1}} d\boldsymbol{\xi}^{A_1} \wedge \dots \wedge d\boldsymbol{\xi}^{A_p} \right) ,$$
(A.27)

which becomes, after an integration by parts of the first term,

$$\int_{\Xi} \pi_{A_1...A_{p+1}} d\boldsymbol{\xi}^{A_1} \wedge ... \wedge d\boldsymbol{\xi}^{A_{p+1}} =$$

$$= \int_{\partial\Xi} \zeta^{A_{p+1}} \pi_{A_1...A_{p+1}} d\zeta^{A_1} \wedge ... \wedge d\zeta^{A_p} +$$

$$- \int_{\Xi} \xi^{A_{p+1}} d\pi_{A_1...A_{p+1}} \wedge d\xi^{A_1} \wedge ... \wedge d\xi^{A_p} , \qquad (A.28)$$

where, the variable ζ indicates the fields on the Boundary. Let us quickly recall the general setting where $\Xi(\sigma)$ is a parametrization of the domain of variation of the ξ fields, defined on the Parameter Space Σ , and $\Sigma(s)$ is a parametrization of the Boundary Γ of Σ . Then, the fields ξ , restricted to the Boundary Γ , will vary in the Boundary $\partial \Xi$ of the domain Ξ , and will have the induced parametrization

$$oldsymbol{\zeta}(s) \stackrel{ ext{def.}}{=} oldsymbol{\xi}(oldsymbol{\sigma}(s))$$

Hence, we find

$$\int_{\zeta_{0}(s)}^{\zeta(s)} [D\xi(\sigma)] \exp\left\{\frac{i}{(p+1)!} \int_{\Xi} \pi_{A_{1}...A_{p+1}} d\xi^{A_{1}} \wedge ... \wedge d\xi^{A_{p+1}}\right\} =$$

$$= \int_{\zeta_{0}(s)}^{\zeta(s)} [D\xi(\sigma)] \exp\left\{\frac{i}{(p+1)!} \int_{\partial\Xi} \zeta^{A_{p+1}} \pi_{A_{1}...A_{p+1}} d\zeta^{A_{1}} \wedge ... \wedge d\zeta^{A_{p}} + \frac{i}{(p+1)!} \int_{\Xi} \xi^{A_{p+1}} d\pi_{A_{1}...A_{p+1}} d\xi^{A_{1}} \wedge ... \wedge d\xi^{A_{p}}\right\}$$

$$1. = \exp\left\{\frac{i}{(p+1)!} \int_{\partial\Xi} \zeta^{A_{p+1}} \pi_{A_{1}...A_{p+1}} d\zeta^{A_{1}} \wedge ... \wedge d\zeta^{A_{p}}\right\} \cdot \cdot \int_{\zeta_{0}(s)}^{\zeta(s)} [D\xi(\sigma)] \exp\left\{-\frac{i}{(p+1)!} \int_{\Xi} \xi^{A_{p+1}} d\pi_{A_{1}...A_{p+1}} \wedge d\xi^{A_{1}} \wedge ... \wedge d\xi^{A_{p}}\right\}$$

$$2. = \exp\left\{\frac{i}{(p+1)!} \int_{\partial\Xi} \zeta^{A_{p+1}} \pi_{A_{1}...A_{p+1}} d\zeta^{A_{1}} \wedge ... \wedge d\zeta^{A_{p}}\right\} \cdot \cdot \int_{\zeta_{0}(s)}^{\zeta(s)} [D\xi(\sigma)] \exp\left\{-\frac{i}{(p+1)!} \int_{\Xi} d^{p+1} \sigma \epsilon^{m_{1}...m_{p+1}} \cdot ... \cdot \left(\partial_{m_{p}} \xi^{A_{p}}\right)\right\}$$

$$3. = \exp\left\{\frac{i}{(p+1)!} \int_{\partial\Xi} \zeta^{A_{p+1}} \pi_{A_{1}...A_{p+1}} d\zeta^{A_{1}} \wedge ... \wedge d\zeta^{A_{p}}\right\} \cdot \cdot \delta\left[\epsilon^{m_{1}...m_{p+1}} (\partial_{m_{p+1}} \pi_{A_{1}...A_{p+1}}) (\partial_{m_{1}} \xi^{A_{1}}) \cdot ... \cdot \left(\partial_{m_{p}} \xi^{A_{p}}\right)\right\}$$

$$\delta\left[\epsilon^{m_{1}...m_{p+1}} (\partial_{m_{p+1}} \pi_{A_{1}...A_{p+1}}) (\partial_{m_{1}} \xi^{A_{1}}) \cdot ... \cdot \left(\partial_{m_{p}} \xi^{A_{p}}\right)\right]$$

$$(A.29)$$

1. the integral in the exponent, which is calculated on the *Boundary*, is constant with respect to the path–integration and can be safely taken outside it;

- 2. we change variables in the second integral introducing the σ ones and ...
- 3. ... we perform the path–integral taking into account the functional integral representation of the functional Dirac $\delta[\ldots]$ function.

A.6 Functional Integration of the $\pi(\sigma)$ Fields

This integration with respect to the momenta has always to be performed when we start with a *Hamiltonian path-integral*. In this case the process is non ambiguous, because we have just to compute a functional integral which, restricted by a *Functional Dirac Delta*, collapses into an ordinary integral of one real variable.

Proposition A.8 (Functional Integration over the π Fields).

The functional integration over the momentum conjugated to the ξ fields, π , collapses into an ordinary integral over all the possible values of the classical energy of the system, i.e. we have

$$\int [\mathcal{D}\pi_{A_{1}...A_{p+1}}]\delta\left[\epsilon^{m_{1}...m_{p+1}}\partial_{m_{p+1}}\pi_{A_{1}...A_{p+1}}\left(\partial_{m_{1}}\xi^{A_{1}}\right)\cdot\ldots\cdot\left(\partial_{m_{p}}\xi^{A_{p}}\right)\right]\cdot$$

$$\cdot\exp\left\{\frac{i}{(p+1)!}\left[\int_{\Xi(\boldsymbol{\sigma})}\boldsymbol{d}\left(\xi^{A_{p+1}}\pi_{A_{1}...A_{p+1}}\boldsymbol{d}\xi^{A_{1}}\wedge\ldots\wedge\boldsymbol{d}\xi^{A_{p}}\right)+\right.$$

$$\left.\left.-\int_{\Sigma}\boldsymbol{d}^{p+1}\boldsymbol{\sigma}N^{A_{1}...A_{p+1}}\pi_{A_{1}...A_{p+1}}\right]\right\}=$$

$$=\int_{0}^{\infty}\boldsymbol{d}Ee^{iEV}\exp\left\{-\frac{iE}{(p+1)!}\left(\int_{\Sigma}\boldsymbol{d}^{p+1}\boldsymbol{\sigma}N^{A_{1}...A_{p+1}}\epsilon_{A_{1}...A_{p+1}}\right)\right\}(A.30)$$

Proof:

As we already briefly pointed out a few lines above we can use the fact that the functional Dirac delta in front of the integrand, requires $\pi_{a_1...a_{p+1}}$ to satisfy the classical equation of motion; but at the classical level the energy balance equation implies that the Hamiltonian $\mathcal{H}(\sigma)$ is independent of the σ 's, i.e. $H(\sigma) \equiv E \equiv \text{const.}$; thus, we can conclude

$$\pi_{A_1...A_{p+1}} = \epsilon_{A_1...A_{p+1}} \mathcal{H}(\boldsymbol{\sigma}) = E\epsilon_{A_1...A_{p+1}} \quad , \tag{A.31}$$

and the functional integration turns into an ordinary integration over all the possible values of E. Accordingly, our functional integral reduces to the following intermediate expression

$$\int_0^\infty dE \exp\left\{\frac{iE}{(p+1)!} \left[\int_{\Xi} \epsilon_{a_1...a_{p+1}} d\left(\xi^{a_{p+1}} d\xi^{a_1} \wedge \ldots \wedge d\xi^{a_p}\right) + \right. \right.$$

$$-\int_{\Sigma} d^{p+1} \boldsymbol{\sigma} N^{a_1 \dots a_{p+1}} \epsilon_{a_1 \dots a_{p+1}} \bigg] \bigg\} \quad (A.32)$$

where, we recognize the first term as the definition of the differential element associated to the World-HyperTube swept by the p-brane; indeed, we have

$$V = \frac{\epsilon_{a_1...a_{p+1}}}{(p+1)!} \int_{\Xi} d\boldsymbol{\xi}^{a_1} \wedge \ldots \wedge d\boldsymbol{\xi}^{a_{p+1}} = \frac{\epsilon_{a_1...a_{p+1}}}{(p+1)!} \int_{\Xi} d\left(\boldsymbol{\xi}^{a_{p+1}} d\boldsymbol{\xi}^{a_1} \wedge \ldots \wedge d\boldsymbol{\xi}^{a_p}\right) , \qquad (A.33)$$

which, inserted in formula (A.32), gives the final result

$$\int_0^\infty dE e^{iEV} \exp\left\{-\frac{iE}{(p+1)!} \left(\int_{\Sigma} d^{p+1} \boldsymbol{\sigma} N^{a_1 \dots a_{p+1}} \epsilon_{a_1 \dots a_{p+1}}\right)\right\} \quad . \tag{A.34}$$

A.7 Functional Integration of the $P(\sigma)$ Fields

Since the functional integration over the P fields is gaussian it can be performed exactly; we will not care about the determinant from the integration, because it will be re-adsorbed into an overall normalization constant. Then, we find the following result:

Proposition A.9 (Functional Integration over the P Fields).

If we set $N = \epsilon_{A_1...A_{p+1}} N^{A_1...A_{p+1}}$ then, the functional integration over the momentum $[\mathcal{D}P]$ gives the following result:

$$\int \left[\mathcal{D}P^{\alpha\beta} \right] \exp \left\{ \frac{i}{(p+1)!} \int d^{p+1} \sigma \left[P_{\mu_1 \dots \mu_{p+1}} \dot{X}^{\mu_1 \dots \mu_{p+1}} + \frac{\epsilon_{A_1 \dots A_{p+1}} N^{A_1 \dots A_{p+1}}}{2\rho_p (p+1)!} P_{\mu_1 \dots \mu_{p+1}} P^{\mu_1 \dots \mu_{p+1}} \right] \right\} \simeq \\
\simeq \exp \left\{ -i \int d^{p+1} \sigma \left[-\frac{\rho_p}{2N (p+1)!} \dot{X}^2 \right] \right\} (A.35)$$

Proof:

This is a gaussian integration which can be performed exactly; nevertheless we avoid the computation of the functional determinant that will be adsorbed in an overall normalization constant and this is the meaning of the *proportionality* symbol in the last two steps. Moreover, we use the *Minkowski-space result*; a more appropriate mathematical procedure would require an analytic continuation to Euclidean space, where the resulting integral would be convergent, and a *counter*

Wick rotation at the end of the calculations to get the final result. Apart from these technicalities, the calculation goes as follows:

- 1. we adopted the convention written in the proposition dots
- $2.\ \dots$ using the $\mathit{Minkowski-space}$ result for the Gaussian integration and
- 3. symplifing the result.

A.8 Saddle Point Evaluation

To understand the physical meaning of equation (4.24) we have to eliminate the Lagrange multiplier $N^{A_1...A_{p+1}}$ which appears in the combination $N = \epsilon_{A_1...A_{p+1}} N^{A_1...A_{p+1}}$.

Proposition A.10 (Saddle Point Evaluation).

The exponent in equation (4.24) is proportional to the Nambu–Goto Lagrangian density when estimated at the saddle point, i.e. we have

$$-\frac{\rho_p}{2N(p+1)!}\dot{\mathbf{X}}^2 + NE = (2E\rho_p)^{\frac{1}{2}} \left[-\frac{\dot{X}^2}{(p+1)!} \right]^{\frac{1}{2}} .$$

Proof:

The phase stationarity point is characterized by the vanishing of the first derivative with respect to N, i.e. we have to solve

$$\frac{d}{dN} \left[-\frac{\rho_p}{2N(p+1)!} \dot{X}^2 + NE \right]_{N=\hat{N}} = 0 \quad , \tag{A.37}$$

where \hat{N} is the stationarity point. So, we find

$$\frac{\rho_p}{2\hat{N}^2 (p+1)!} \dot{\mathbf{X}}^2 + E = 0 \quad \Rightarrow \quad \hat{N}^2 = -\frac{\rho_p}{2E (p+1)!} \dot{\mathbf{X}}^2 . \tag{A.38}$$

Thus the exponent turns out to be

$$-\frac{\rho_{p}}{2N(p+1)!}\dot{\mathbf{X}}^{2} + NE\Big|_{N=\hat{N}} = \frac{1}{\hat{N}} \left[-\frac{\rho_{p}}{2(p+1)!}\dot{\mathbf{X}}^{2} - \hat{N}^{2}E \right]
= \frac{(2E)^{\frac{1}{2}}}{\left(-\frac{\rho_{p}}{(p+1)!}\dot{\mathbf{X}}^{2} \right)^{\frac{1}{2}}} \left[-\frac{\rho_{p}}{2(p+1)!}\dot{\mathbf{X}}^{2} - \frac{\rho_{p}}{2(p+1)!}\dot{\mathbf{X}}^{2} \right]
= (2E\rho_{p})^{\frac{1}{2}} \left[-\frac{\dot{\mathbf{X}}^{2}}{(p+1)!} \right]^{\frac{1}{2}} .$$
(A.39)

A.9 Equations Satisfied by the Kernel

This section is devoted to the derivation of the equations for the kernel

$$K[\boldsymbol{B}(\boldsymbol{\sigma}),\boldsymbol{B}(\boldsymbol{\sigma}_0);V]$$
 .

Proposition A.11 (Functional Differential Equations for the Kernel).

The Propagation Kernel of a p-brane satisfies the following equations:

$$\frac{\partial K\left[\boldsymbol{B}(\boldsymbol{\sigma}),\boldsymbol{B}_{0}(\boldsymbol{\sigma});V\right]}{\partial V} = -\frac{iE}{\hbar}K\left[\boldsymbol{Y}\boldsymbol{Y}_{0}V\right] \qquad (A.40)$$

$$\frac{\delta K\left[\boldsymbol{Y}\boldsymbol{Y}_{0}V\right]}{\delta Y^{\mu}(\boldsymbol{s})} = \frac{i}{\hbar}\int_{\boldsymbol{Y}_{0}(\boldsymbol{s})}^{\boldsymbol{Y}(\boldsymbol{s})}\int_{\zeta_{0}(\boldsymbol{s})}^{\zeta(\boldsymbol{s})}\left[D\mu(\boldsymbol{\sigma})\right]Q_{\mu\mu_{1}...\mu_{p}}Y'^{\mu_{1}...\mu_{p}}\exp\left(\frac{iS}{\hbar}\right)(A.41)$$

Proof:

The result can be obtained starting from equations (4.27–4.28). The left hand side of equation (4.27), i.e. the total variation of the Kernel with respect to the physical coordinates, which are the ones singled out by Hamilton–Jacobi Theory, i.e. V and Y, can be rewritten as

$$\delta K[\boldsymbol{Y}, \boldsymbol{Y}_{0}; V] = \frac{\partial K[\boldsymbol{Y}, \boldsymbol{Y}_{0}; V]}{\partial V} dV + \oint_{\Sigma} \frac{d^{p} \boldsymbol{s}}{\sqrt{\boldsymbol{Y}^{\prime 2}}} \frac{\delta K[\boldsymbol{Y}, \boldsymbol{Y}_{0}; V]}{\delta Y^{\mu}(\boldsymbol{s})} \delta Y^{\mu}(\boldsymbol{s}) \quad ; \tag{A.42}$$

then, inserting this equality, as well as formula (4.28) into equation (4.27) we obtain

$$\frac{\partial K[\boldsymbol{Y},\boldsymbol{Y}_{0};\boldsymbol{V}]}{\partial \boldsymbol{V}}d\boldsymbol{V} + \oint_{\Sigma} \frac{d^{p}s}{\sqrt{\boldsymbol{Y}'^{2}}} \frac{\delta K[\boldsymbol{Y},\boldsymbol{Y}_{0};\boldsymbol{V}]}{\delta \boldsymbol{Y}^{\mu}(s)} \delta \boldsymbol{Y}^{\mu}(s) = \\
= i \int_{\boldsymbol{Y}(s_{0})}^{\boldsymbol{Y}(s)} \int_{\boldsymbol{\zeta}(s_{0})}^{\boldsymbol{\zeta}(s)} [D\mu(\boldsymbol{\sigma})] \oint_{C} \frac{1}{p!} Q_{\mu_{1}...\mu_{p+1}} d\boldsymbol{Y}^{\mu_{2}} \wedge ... \wedge d\boldsymbol{Y}^{\mu_{p+1}} \delta \boldsymbol{Y}^{\mu_{1}} \cdot \\
\cdot \exp\left\{iS[\boldsymbol{X},\boldsymbol{P},\boldsymbol{\xi},\boldsymbol{N};\boldsymbol{V}]\right\} + \\
-iE \int_{\boldsymbol{Y}(s_{0})}^{\boldsymbol{Y}(s)} \int_{\boldsymbol{\zeta}(s_{0})}^{\boldsymbol{\zeta}(s)} [D\mu(\boldsymbol{\sigma})] \exp\left\{iS[\boldsymbol{X},\boldsymbol{P},\boldsymbol{\xi},\boldsymbol{N};\boldsymbol{V}]\right\} d\boldsymbol{V} \\
= i \int_{\boldsymbol{Y}(s_{0})}^{\boldsymbol{Y}(s)} \int_{\boldsymbol{\zeta}(s_{0})}^{\boldsymbol{\zeta}(s)} [D\mu(\boldsymbol{\sigma})] \oint_{\Sigma} \frac{d^{p}s}{\sqrt{\boldsymbol{Y}'^{2}}} \frac{1}{p!} Q_{\mu_{1}...\mu_{p+1}} \boldsymbol{Y}^{\prime\mu_{2}...\mu_{p+1}} \delta \boldsymbol{Y}^{\mu_{1}} \cdot \\
\cdot \exp\left\{iS[\boldsymbol{X},\boldsymbol{P},\boldsymbol{\xi},\boldsymbol{N};\boldsymbol{V}]\right\} + \\
-iEK[\boldsymbol{Y},\boldsymbol{Y}_{0};\boldsymbol{V}] d\boldsymbol{V} ; \tag{A.43}$$

comparing the first and the last quantities, equating the terms corresponding to dV and δX^{μ_1} yields the desired result, i.e.

$$\frac{\partial K[\mathbf{Y}\mathbf{Y}_0V]}{\partial V} = -\frac{iE}{\hbar}K[\mathbf{Y},\mathbf{Y}_0;V] \tag{A.44}$$

$$\frac{\partial V}{\partial V} = -\frac{1}{\hbar} K[Y, Y_0, V] \qquad (A.44)$$

$$\frac{\delta K[Y, Y_0; V]}{\delta Y^{\mu}(s)} = \frac{i}{\hbar} \int_{Y_0(s)}^{Y(s)} \int_{\zeta_0(s)}^{\zeta(s)} [D\mu(\sigma)] Q_{\mu\mu_1\dots\mu_p} Y'^{\mu_1\dots\mu_p} \exp\left(\frac{iS}{\hbar}\right) \qquad (A.45)$$

A.10 Kernel Functional Wave Equation

As already pointed out in the text, the Kernel for an extended object considered as the only dynamical *Boundary* of its history can be obtained by solving a Schrödinger-like functional equation. This is the main result we get by starting from the Hamilton–Jacobi description of the *Quantum Dynamics of the Boundary*. We already derived this result according with the procedure of section 3.2. Now, we will apply the same procedure for a *p-brane*. The result is not surprising, since, apart from some slightly different notation, we get exactly the same formula as (3.29).

Proposition A.12 (p-brane Functional Schröedinger Equation).

The propagation Kernel of the p-brane satisfies the following Schrödinger-like functional equation:

$$-\frac{\hbar}{2\rho_{p}p!}\left(\oint_{\boldsymbol{B}}d^{p}\boldsymbol{v}(s)\right)^{-1}\oint_{\boldsymbol{B}}\frac{d^{p}\boldsymbol{s}}{\sqrt{Y'^{2}}}\frac{\delta^{2}}{\delta Y^{\mu}(\boldsymbol{s})}\frac{\delta^{2}}{\delta Y_{\mu}(\boldsymbol{s})}K\left[\boldsymbol{Y}(\boldsymbol{s})\,,\boldsymbol{Y}_{0}(\boldsymbol{s})\,;\boldsymbol{V}\right]=$$

$$= i\hbar \frac{\partial}{\partial V} K[\mathbf{Y}(\mathbf{s}), \mathbf{Y}_0(\mathbf{s}); V]$$
 (A.46)

Proof:

The proof reproduces exactly the one we gave for the *String*. Thus, we just briefly recall the various steps; firstly

$$\frac{\delta^{2}K\left[\boldsymbol{Y}\left(\boldsymbol{s}\right),\boldsymbol{Y}_{0}\left(\boldsymbol{s}\right);V\right]}{\delta Y^{\mu}\left(\boldsymbol{s}\right)\delta Y_{\mu}\left(\boldsymbol{s}\right)} = -\frac{1}{\hbar^{2}}\int_{\boldsymbol{Y}_{0}\left(\boldsymbol{s}\right)}^{\boldsymbol{Y}\left(\boldsymbol{s}\right)}\int_{\boldsymbol{\zeta}_{0}\left(\boldsymbol{s}\right)}^{\boldsymbol{\zeta}\left(\boldsymbol{s}\right)}\left[\mathcal{D}\mu\left(\boldsymbol{\sigma}\right)\right]q_{\mu}q^{\mu}e^{\frac{i}{\hbar}S}$$
(A.47)

$$\frac{\partial}{\partial A}K\left[\boldsymbol{Y}\left(\boldsymbol{s}\right),\boldsymbol{Y}_{0}\left(\boldsymbol{s}\right);V\right] = -\frac{iE}{\hbar}K\left[\boldsymbol{Y}\left(\boldsymbol{s}\right),\boldsymbol{Y}_{0}\left(\boldsymbol{s}\right);V\right] , \qquad (A.48)$$

which are the relations equivalent to (3.31) and (3.32). Now, of course q_{μ} is the momentum projected on the *p*-dimensional *Boundary*, i.e.

$$q_{\mu} = Q_{\mu\mu_{1}...\mu_{p}} Y^{\prime\mu_{1}...\mu_{p}} \quad . \tag{A.49}$$

The final step requires to identify equation (A.47) with the expectation value of the square of the Boundary Momentum, and by substituting (A.47,A.48) in the Hamilton–Jacobi equation. Thus, we get the final result

$$\frac{\hbar^{2}}{2\rho_{p}p!}\left(\oint_{\boldsymbol{B}}d^{p}\boldsymbol{v}(s)\right)^{-1}\oint_{\boldsymbol{B}}\frac{d^{p}\boldsymbol{s}}{\sqrt{\boldsymbol{Y'}^{2}}}\frac{\delta^{2}K\left[\boldsymbol{Y},\boldsymbol{Y}_{0};\boldsymbol{V}\right]}{\delta Y^{\mu}\left(\boldsymbol{s}\right)\delta Y_{\mu}\left(\boldsymbol{s}\right)}=i\hbar\frac{\partial K\left[\boldsymbol{Y},\boldsymbol{Y}_{0};\boldsymbol{V}\right]}{\partial V}\quad.\tag{A.50}$$

A.11 Holographic Coordinates: Functional Derivatives

In this section we compute the first and second functional derivatives of the *Holographic* coordinates, which are the most appropriate coordinates of the Quantized Theory for a p-brane.

Proposition A.13 (Functional Derivatives of the Holographic Coordinates).

If we consider the coordinates of the quantized p-brane,

$$Y^{\mu_1...\mu_{p+1}}[B] \equiv \oint_B Y^{\mu_1} dY^{\mu_2} \wedge ... \wedge dY^{\mu_{p+1}} = \oint_B ds Y^{\mu_1}(s) Y'^{\mu_2...\mu_{p+1}}(s)$$
(A.51)

where, for the reader's convenience, we recall the expression for the Tangent Element to a p-brane (4.38),

$$Y'^{\mu_2\dots\mu_{p+1}} = \epsilon^{a_2\dots a_{p+1}} \frac{\partial Y^{\mu_2}}{\partial \sigma_{a_2}} \cdot \dots \cdot \frac{\partial Y^{\mu_{p+1}}}{\partial \sigma_{a_{p+1}}} \quad ,$$

Accordingly, the first and second functional derivatives are given by

$$\frac{\delta Y^{\mu_{1}\dots\mu_{p+1}}[\boldsymbol{B}]}{\delta Y^{\alpha}(\bar{\boldsymbol{\sigma}})} = \delta_{\alpha}^{\mu_{1}}Y^{\prime\mu_{2}\dots\mu_{p+1}}(\bar{\boldsymbol{s}}) - \sum_{i}^{2,p+1} \delta_{\alpha}^{\mu_{i}}Y^{\prime\mu_{2}\dots\mu_{i-1}\check{\mu}_{i}\mu_{1}\mu_{i+1}\dots\mu_{p+1}} \qquad (A.52)$$

$$\frac{\delta^{2}Y^{\mu_{1}\dots\mu_{p+1}}[\boldsymbol{B}]}{\delta Y^{\alpha}(\bar{\boldsymbol{\sigma}})\delta Y^{\beta}(\tilde{\boldsymbol{\sigma}})} = \sum_{j}^{2,p+1} \delta_{\alpha\beta}^{\mu_{1}\mu_{j}} \epsilon^{a_{2}\dots a_{p+1}} \partial_{a_{2}}Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} \delta(\bar{\boldsymbol{s}} - \tilde{\boldsymbol{s}}) \cdot \dots \cdot \partial_{a_{p+1}}Y^{\mu_{p+1}} + \\
- \sum_{\substack{i \neq j \\ i,j}} \delta_{\alpha}^{\mu_{i}} \delta_{\beta}^{\mu_{j}} \epsilon^{a_{2}\dots a_{i}\dots a_{j}\dots a_{p+1}} \partial_{a_{2}}Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}}Y^{\mu_{1}} \cdot \\
\cdot \dots \cdot \partial_{a_{j}} \delta(\bar{\boldsymbol{s}} - \tilde{\boldsymbol{s}}) \cdot \dots \cdot \partial_{a_{p+1}}Y^{\mu_{p+1}} . \qquad (A.53)$$

Proof:

We have to be very careful with indices in this computation, because in contrast to the *String* case, in the general case we do not want to explicitly write down the chain rule for all term. Hence, we proceed as follows:

$$\begin{split} \frac{\delta \sigma^{\mu_1 \dots \mu_{p+1}}[\underline{B}]}{\delta Y^{\alpha}(\bar{\mathbf{s}})} &= & \oint_{B} d^{p} s \frac{\delta Y^{\mu_{1}}(s)}{\delta Y^{\alpha}(\bar{\mathbf{s}})} Y'^{\mu_{2} \dots \mu_{p+1}}(s) + \oint_{B} d^{p} s Y^{\mu_{1}}(s) \frac{\delta Y'^{\mu_{2} \dots \mu_{p+1}}(s)}{\delta Y^{\alpha}(\bar{\mathbf{s}})} \\ &1. &= & \oint_{B} d^{p} s \delta_{\alpha}^{\mu_{1}} \delta \left(s - \bar{s}\right) Y'^{\mu_{2} \dots \mu_{p+1}}(s) + \\ & & & + \oint_{B} d^{p} s Y^{\mu_{1}}(s) \sum_{i}^{2, p+1} \epsilon^{a_{2} \dots a_{p+1}} \cdot \\ & & & \frac{\partial Y^{\mu_{2}}(s)}{\partial s_{a_{2}}} \cdot \dots \cdot \frac{\delta}{\delta Y^{\alpha}(\bar{s})} \left(\frac{\partial Y^{\mu_{i}}(s)}{\partial s_{a_{i}}}\right) \cdot \dots \cdot \frac{\partial Y^{\mu_{p+1}}(s)}{\partial s_{a_{p+1}}} \\ &2. &= & \delta_{\alpha}^{\mu_{1}} Y'^{\mu_{2} \dots \mu_{p+1}}(\bar{s}) + \\ & & & + \oint_{B} d^{p} s Y^{\mu_{1}}(s) \sum_{i}^{2, p+1} \epsilon^{a_{2} \dots a_{p+1}} \cdot \\ & & & \frac{\partial Y^{\mu_{2}}(s)}{\partial s_{a_{2}}} \cdot \dots \cdot \partial_{s_{a_{i}}} \left[\delta_{\alpha}^{\mu_{i}} \delta \left(s - \bar{s}\right)\right] \cdot \dots \cdot \frac{\partial Y^{\mu_{p+1}}(s)}{\partial s_{a_{p+1}}} \\ &3. &= & \delta_{\alpha}^{\mu_{1}} Y'^{\mu_{2} \dots \mu_{p+1}}(\bar{s}) + \\ & & & + \oint_{B} d^{p} s \sum_{i}^{2, p+1} \partial_{s_{a_{i}}} \left[Y^{\mu_{1}}(s) \epsilon^{a_{2} \dots a_{p+1}} \frac{\partial Y^{\mu_{2}}(s)}{\partial s_{a_{2}}} \cdot \dots \cdot \frac{\partial Y^{\mu_{p+1}}(s)}{\partial s_{a_{p+1}}}\right] + \\ & & - \oint_{B} d^{p} s \delta \left(s - \bar{s}\right) \sum_{i}^{2, p+1} \delta_{\alpha}^{\mu_{i}} \partial_{s_{a_{i}}} \left[Y^{\mu_{1}}(s) \epsilon^{a_{2} \dots a_{p+1}} \frac{\partial Y^{\mu_{2}}(s)}{\partial s_{a_{i}}} \cdot \dots \cdot \frac{\partial Y^{\mu_{p+1}}(s)}{\partial \sigma_{a_{p+1}}}\right] \\ &4. &= & \delta_{\alpha}^{\mu_{1}} Y'^{\mu_{2} \dots \mu_{p+1}}(\bar{s}) + \\ & & - \sum_{i}^{2, p+1} \delta_{\alpha}^{\mu_{i}} \left[\epsilon^{a_{2} \dots a_{p+1}} \frac{\partial Y^{\mu_{2}}(\bar{s})}{\partial s_{s_{0}}} \dots \frac{\partial Y^{\bar{\mu}_{i}}(\bar{s})}{\partial s_{a}} \frac{\partial Y^{\mu_{1}}(\bar{s})}{\partial s_{a}} \dots \frac{\partial Y^{\mu_{p+1}}(\bar{s})}{\partial s_{a}}\right] \end{aligned}$$

5. =
$$\delta_{\alpha}^{\mu_1} Y'^{\mu_2 \dots \mu_{p+1}}(\bar{s}) - \sum_{i}^{2,p+1} \delta_{\alpha}^{\mu_i} Y'^{\mu_2 \dots \mu_{i-1} \tilde{\mu}_i \mu_1 \mu_{i+1} \dots \mu_{p+1}}$$
 (A.54)

The main steps goes as follows; we first apply the *chain* rule, letting the functional derivative act on Y and Y' respectively;

- 1. the action on Y singles out a Dirac delta on the *brane parameters* times a Kronecker delta over field components, whereas, we must be more careful acting on Y' since the result is still differentiated once with respect to the *brane* parameters²; in this way ...
- 2. ... we can use the Dirac delta in the first term to kill the integration over $d^p s$ and integration by parts in the second term; this procedure ...
- 3. ... singles out a *Boundary* term (the second one) and a term where we find again a Dirac delta over the parameters; now ...
- 4. ... the first term vanishes because Γ has no *Boundary*; integration in the second term is *killed* again by the Dirac delta; moreover we observe that applying the *chain rule* to the partial derivative, only acting on Y^{μ_1} we get a non-vanishing result: in the other cases we have only second derivatives which, being symmetric, give a vanishing result when the indices are saturated with the totally anti symmetric Levi-Civita tensor;
- 5. using the suppressed "" notation we can then write the result in more compact form.

Along the same lines one computes the second functional derivative; but, it is better to derive, as a preliminary result, the first functional derivative of the *p-brane Tangent Element*:

$$\frac{\delta Y'^{\mu_{2}\dots\mu_{i-1}\check{\mu}_{i}\mu_{1}\mu_{i+1}\dots\mu_{p+1}}(\bar{s})}{\delta Y^{\beta}(\tilde{s})} =$$

$$= \epsilon^{a_{2}\dots a_{p+1}} \frac{\delta \left(\partial_{a_{2}}Y^{\mu_{2}}(\bar{s}) \cdot \dots \cdot \partial_{a_{i}}Y^{\mu_{1}}(\bar{s}) \cdot \dots \cdot \partial_{a_{p+1}}Y^{\mu_{p+1}}(\bar{s})\right)}{\delta Y^{\beta}(\tilde{s})}$$

$$= \sum_{\substack{j \neq i \\ j \neq i}} \epsilon^{a_{2}\dots a_{i}\dots a_{j}\dots a_{p+1}} \partial_{a_{2}}Y^{\mu_{2}}(\bar{s}) \cdot \dots \cdot \partial_{a_{i}}Y^{\mu_{1}}(\bar{s}) \cdot \dots \cdot \partial_{a_{i}}Y^{\mu_{1}}(\bar{s}) \cdot \dots \cdot \partial_{a_{j}} \left[\delta^{\mu_{j}}_{\beta}\delta(\bar{s}-\tilde{s})\right] \cdot \dots \cdot \partial_{a_{p+1}}Y^{\mu_{p+1}}(\bar{s}) +$$

$$+\epsilon^{a_{2}\dots a_{p+1}} \cdot \partial_{a_{2}}Y^{\mu_{2}}(\bar{s}) \cdot \dots \cdot \partial_{a_{p+1}}Y^{\mu_{p+1}}(\bar{\sigma}) \quad . \tag{A.55}$$

By neglecting the complication of the suppressed/inserted indices, the formula reads

$$\frac{\delta Y'^{\mu_2 \dots \mu_{p+1}}(\bar{\sigma})}{\delta Y^{\beta}(\tilde{\sigma})} = \sum_{j}^{2,p+1} \epsilon^{a_2 \dots a_{p+1}} \partial_{a_2} Y^{\mu_2}(\bar{s}) \dots \partial_{a_j} \left[\delta^{\mu_j}_{\beta} \delta(\bar{s} - \tilde{s}) \right] \dots \partial_{a_{p+1}} Y^{\mu_{p+1}}(\bar{s}) \quad . \tag{A.56}$$

Then, applying the *chain rule* to result (A.56),we can write

$$\frac{\delta^{2}Y^{\mu_{1}\dots\mu_{p+1}}[\boldsymbol{B}]}{\delta Y^{\alpha}(\bar{\boldsymbol{s}})\delta Y^{\beta}(\tilde{\boldsymbol{s}})} = \delta^{\mu_{1}}_{\alpha} \frac{\delta Y'^{\mu_{1}\dots\mu_{p+1}(\bar{\boldsymbol{s}})}}{\delta Y^{\beta}(\tilde{\boldsymbol{s}})} + \\
- \sum_{i}^{2,p+1} \delta^{\mu_{i}}_{\alpha} \frac{\delta Y'^{\mu_{1}\dots\mu_{i-1}\check{\mu}_{i}\mu_{1}\mu_{i+1}\dots\mu_{p+1}(\bar{\boldsymbol{s}})}}{\delta Y^{\beta}(\tilde{\boldsymbol{s}})} \tag{A.57}$$

²We assume that we can exchange functional and ordinary derivatives.

and the second functional derivative results to be

$$\frac{\delta^{2}Y^{\mu_{1}\dots\mu_{p+1}}[B]}{\delta Y^{\alpha}(\bar{s})\,\delta Y^{\beta}(\bar{s})} = \delta_{\alpha}^{\mu_{1}} \sum_{j}^{2,p+1} \epsilon^{a_{2}\dots a_{p+1}} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \delta_{\beta}^{\mu_{j}} \partial_{a_{j}} \delta(\bar{s} - \bar{s}) \cdot \dots \cdot \partial_{a_{p+1}} Y^{\mu_{p+1}} + \\
- \sum_{i}^{2,p+1} \delta_{\alpha}^{\mu_{i}} \epsilon^{a_{2}\dots a_{p+1}} \sum_{j \neq i}^{2,p+1} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} Y^{\mu_{1}} \cdot \\
- \dots \cdot \delta_{\beta}^{\mu_{j}} \partial_{a_{j}} \delta(\bar{s} - \bar{s}) \cdot \dots \cdot \partial_{a_{p+1}} Y^{\mu_{p+1}} + \\
- \sum_{i}^{2,p+1} \delta_{\alpha}^{\mu_{i}} \delta_{\beta}^{\mu_{1}} \epsilon^{a_{2}\dots a_{p+1}} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} \delta(\bar{s} - \bar{s}) \cdot \dots \cdot \partial_{a_{p+1}} Y^{\mu_{p+1}} + \\
= \sum_{j}^{2,p+1} \left(\delta_{\alpha}^{\mu_{1}} \delta_{\beta}^{\mu_{1}} - \delta_{\alpha}^{\mu_{i}} \delta_{\beta}^{\mu_{1}} \right) \epsilon^{a_{2}\dots a_{p+1}} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} \delta(\bar{s} - \bar{s}) \cdot \dots \cdot \partial_{a_{p+1}} Y^{\mu_{p+1}} + \\
- \sum_{i \neq j}^{2,p+1} \delta_{\alpha}^{\mu_{i}} \delta_{\beta}^{\mu_{j}} \epsilon^{a_{2}\dots a_{i}\dots a_{j}\dots a_{p+1}} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} Y^{\mu_{1}} \cdot \\
- \dots \cdot \partial_{a_{j}} \delta(\bar{s} - \bar{s}) \cdot \dots \cdot \partial_{a_{p+1}} Y^{\mu_{p+1}} + \\
- \sum_{i \neq j}^{2,p+1} \delta_{\alpha}^{\mu_{1}} \delta_{\beta}^{\mu_{2}} \epsilon^{a_{2}\dots a_{i}\dots a_{j}\dots a_{p+1}} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} Y^{\mu_{1}} \cdot \\
- \sum_{i \neq j}^{2,p+1} \delta_{\alpha}^{\mu_{1}} \delta_{\beta}^{\mu_{2}} \epsilon^{a_{2}\dots a_{i}\dots a_{j}\dots a_{p+1}} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} Y^{\mu_{1}} \cdot \\
- \sum_{i \neq j}^{2,p+1} \delta_{\alpha}^{\mu_{1}} \delta_{\beta}^{\mu_{2}} \epsilon^{a_{2}\dots a_{i}\dots a_{j}\dots a_{p+1}} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} Y^{\mu_{1}} \cdot \\
- \sum_{i \neq j}^{2,p+1} \delta_{\alpha}^{\mu_{1}} \delta_{\beta}^{\mu_{2}} \epsilon^{a_{2}\dots a_{i}\dots a_{j}\dots a_{p+1}} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} Y^{\mu_{1}} \cdot \\
- \sum_{i \neq j}^{2,p+1} \delta_{\alpha}^{\mu_{1}} \delta_{\beta}^{\mu_{2}} \epsilon^{a_{2}\dots a_{i}\dots a_{j}\dots a_{p+1}} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} Y^{\mu_{1}} \cdot \\
- \sum_{i \neq j}^{2,p+1} \delta_{\alpha}^{\mu_{1}} \delta_{\beta}^{\mu_{2}} \epsilon^{a_{2}\dots a_{i}\dots a_{j}\dots a_{p+1}} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} Y^{\mu_{1}} \cdot \\
- \sum_{i \neq j}^{2,p+1} \delta_{\alpha}^{\mu_{1}} \delta_{\beta}^{\mu_{2}} \epsilon^{a_{2}\dots a_{i}\dots a_{j}\dots a_{p+1}} \partial_{a_{2}} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} Y^{\mu_{1}} \cdot \\
- \sum_{i \neq j}^{2,p+1} \delta_{\alpha}^{\mu_{1}} \delta_{\beta}^{\mu_{2}} \epsilon^{a_{2}\dots a_{i}\dots a_{j}\dots a_{p+1}} \partial_{\alpha} Y^{\mu_{2}} \cdot \dots \cdot \partial_{a_{i}} Y^{\mu_{1}} \cdot \\
- \sum_{i \neq j}^{2,p+1} \delta_{\alpha}^{\mu_{1}} \delta_{\beta}^{\mu_{2}} \delta_{\beta}^{\mu_{2}} \delta_{\alpha}^{$$

We can check the last result by setting p = 1, the String case:

$$\frac{\delta^2 Y^{\mu_1 \mu_2}}{\delta Y^{\alpha}(s) \delta Y^{\beta}(s)} = \left(\delta^{\mu_1}_{\alpha} \delta^{\mu_2}_{\beta} - \delta^{\mu_2}_{\alpha} \delta^{\mu_1}_{\beta}\right) \frac{d}{ds} \delta\left(s - \bar{s}\right) = \delta^{\mu_1 \mu_2}_{\alpha \beta} \frac{d}{ds} \delta\left(s - \bar{s}\right) \quad , \tag{A.59}$$

where we took into account $s \equiv s$ and $\bar{s} \equiv \bar{s}$, because a *String* is one dimensional object! In the membrane case, p = 2, we have

$$\frac{\delta^{2}\sigma^{\mu_{1}\mu_{2}\mu_{3}}}{\delta Y^{\alpha}(s)\delta Y^{\beta}(\bar{s})} = \left(\delta_{\alpha}^{\mu_{1}}\delta_{\beta}^{\mu_{2}} - \delta_{\alpha}^{\mu_{2}}\delta_{\beta}^{\mu_{1}}\right)\epsilon^{a_{2}a_{3}}\left(\partial_{a_{3}}x^{\mu_{3}}\right)\partial_{a_{2}}\delta\left(s - \bar{s}\right) + \\
+ \left(\delta_{\alpha}^{\mu_{1}}\delta_{\beta}^{\mu_{3}} - \delta_{\alpha}^{\mu_{3}}\delta_{\beta}^{\mu_{1}}\right)\epsilon^{a_{2}a_{3}}\left(\partial_{a_{2}}Y^{\mu_{2}}\right)\partial_{a_{3}}\delta\left(s - \bar{s}\right) + \\
+ \left(\delta_{\alpha}^{\mu_{2}}\delta_{\beta}^{\mu_{3}} - \delta_{\alpha}^{\mu_{3}}\delta_{\beta}^{\mu_{2}}\right)\epsilon^{a_{2}a_{3}}\left(\partial_{a_{2}}Y^{\mu_{1}}\right)\partial_{a_{2}}\delta\left(s - \bar{s}\right) + \\
= \epsilon^{a_{2}a_{3}}\partial_{a_{2}}\delta\left(s - \bar{s}\right) \cdot \\
\cdot \left[\delta_{\alpha\beta}^{\mu_{1}\mu_{2}}\partial_{a_{3}}Y^{\mu_{3}} + \delta_{\alpha\beta}^{\mu_{3}\mu_{1}}\partial_{a_{3}}Y^{\mu_{2}} + \delta_{\alpha\beta}^{\mu_{1}\mu_{3}}\partial_{a_{3}}Y^{\mu_{1}}\right] \quad (A.60)$$

A.12 Functional Derivatives of the Classical Action

In this appendix we compute the functional derivatives of the classical action (4.42) in order to find a solution of the equation (4.31).

Proposition A.14 (Functional Derivatives of the *p*-brane Classical Action).

The first and second functional derivatives of the classical action

$$S_{\text{cl.}}\left[\boldsymbol{B}(\boldsymbol{\sigma}),\boldsymbol{B}_{0}(\boldsymbol{\sigma});V\right] = \frac{\beta}{2\left(p+1\right)V} \Sigma^{\mu_{1}\dots\mu_{p+1}}\left[\boldsymbol{B}-\boldsymbol{B}_{0}\right] \Sigma_{\mu_{1}\dots\mu_{p+1}}\left[\boldsymbol{B}-\boldsymbol{B}_{0}\right]$$

are

$$\frac{\delta S_{\text{cl.}}}{\delta Y^{\mu_1}(\boldsymbol{s})} = \frac{\beta}{V} \Sigma^{\mu_1 \dots \mu_{p+1}} \left[\boldsymbol{B} - \boldsymbol{B}_0 \right] Y'_{\mu_2 \dots \mu_{p+1}}$$
(A.61)

$$\frac{\delta^2 S_{\text{cl.}}}{\delta Y^{\mu_1}(\mathbf{s}) \, \delta Y_{\mu_1}(\mathbf{s})} = (D - p + 1) \frac{\beta}{V} \mathbf{Y}^{\prime 2}(\mathbf{s}) \quad , \tag{A.62}$$

where $\Sigma^{\mu_1...\mu_{p+1}}[B - B_0]$ is given by (4.43).

Proof:

To get the desired results we refer to equations (A.52-A.53) and we find

$$\frac{\delta S_{\text{cl.}}}{\delta Y^{\alpha}(s)} = \frac{\beta}{2V(p+1)} 2\Sigma_{\mu_{1}...\mu_{p+1}} [B - B_{0}] \frac{\delta Y^{\mu_{1}...\mu_{p+1}} [B]}{\delta Y^{\alpha}(s)}
= \frac{\beta}{V(p+1)} \left[\Sigma^{\mu_{1}...\mu_{p+1}} [B - B_{0}] \delta_{\alpha}^{\mu_{1}} Y'^{\mu_{2}...\mu_{p+1}(\bar{s})} + \right.
\left. - \Sigma_{\mu_{1}...\mu_{p+1}} [B - B_{0}] \sum_{i}^{2,p+1} \delta_{\alpha}^{\mu_{i}} Y'^{\mu_{2}...\mu_{i-1}\tilde{\mu}_{i}\mu_{1}\mu_{i+1}...\mu_{p+1}} \right]
= \frac{\beta}{V(p+1)} \left[\Sigma_{\mu_{1}...\mu_{p+1}} [B - B_{0}] \delta_{\alpha}^{\mu_{1}} Y'^{\mu_{2}...\mu_{p+1}(\bar{s})} + \right.
\left. + p\Sigma_{\mu_{1}...\mu_{p+1}} [B - B_{0}] \delta_{\alpha}^{\mu_{1}} Y'^{\mu_{2}...\mu_{i-1}\tilde{\mu}_{i}\mu_{i}\mu_{i+1}...\mu_{p+1}} \right]
= \frac{\beta}{V} \Sigma_{\alpha\mu_{2}...\mu_{p+1}} [B - B_{0}] Y'^{\mu_{2}...\mu_{p+1}(\bar{s})} .$$
(A.63)

From the first line of the set of equalities above we get

$$\frac{\delta^{2} S_{\text{cl.}}}{\delta Y^{\alpha}(s) \, \delta Y^{\beta}(\bar{s})} = \frac{\beta}{V(p+1)} \frac{\delta}{\delta Y^{\beta}(s)} \left[\Sigma_{\mu_{1} \dots \mu_{p+1}} \left[\boldsymbol{B} - \boldsymbol{B}_{0} \right] \frac{\delta Y^{\mu_{1} \dots \mu_{p+1}} \left[\boldsymbol{B} \right]}{\delta Y^{\alpha}(s)} \right] \\
= \frac{\beta}{V(p+1)} \left[\left(\frac{\delta Y^{\mu_{1} \dots \mu_{p+1}} \left[\boldsymbol{B} \right]}{\delta Y^{\alpha}(s)} \right)^{2} + \frac{\delta^{2} Y^{\mu_{1} \dots \mu_{p+1}} \left[\boldsymbol{B} \right]}{\delta Y^{\alpha}(s) \, \delta Y^{\beta}(\bar{s})} \Sigma_{\mu_{1} \dots \mu_{p+1}} \right] (A.64)$$

Now, we saturate both the discrete indices α and beta and the continuous ones, i.e s and \bar{s} . From equation (A.59) follows that the second term in the sum has zero trace since it is skew symmetric in α and β . Then, we remain with the first term in the square brackets of (A.64):

$$\frac{\delta^2 S_{cl}}{\delta Y^{\alpha}(s)} \delta Y_{\alpha}(s) = \frac{\beta}{V(p+1)} \frac{\delta Y^{\mu_1 \dots \mu_{p+1}}[B]}{\delta Y^{\alpha}(s)} \frac{\delta Y^{\mu_1 \dots \mu_{p+1}}[S]}{\delta Y^{\alpha}(s)}$$

$$= \frac{\beta}{V(p+1)} \left(\delta_{\alpha}^{\mu_1} Y'^{\mu_2 \dots \mu_{p+1}} - \sum_{i} \delta_{\alpha}^{\mu_i} Y'^{\mu_2 \dots \mu_{i-1}} \hat{b}_{i} \mu_1 \mu_{i+1} \dots \mu_{p+1}} \right) \cdot \cdot \left(\delta_{\alpha \mu_1} Y'_{\mu_2 \dots \mu_{p+1}} - \sum_{j} \delta_{\mu_j \alpha} Y'^{\mu_2 \dots \mu_{j-1}} \hat{b}_{j} \mu_1 \mu_{j+1} \dots \mu_{p+1}} \right)$$

$$= \frac{\beta}{V(p+1)} \left[\delta_{\alpha}^{\alpha} Y'^{\mu_2 \dots \mu_{p+1}} Y'_{\mu_2 \dots \mu_{p+1}} + \frac{2.p+1}{V_{p+1}} \delta_{\mu_1} Y'^{\mu_2 \dots \mu_{j-1}} \hat{b}_{j} \mu_1 \mu_{j+1} \dots \mu_{p+1}} \cdot \frac{\beta}{V(p+1)} \left[\delta_{\alpha}^{\mu_1} Y'^{\mu_2 \dots \mu_{j-1}} \hat{b}_{j} \mu_1 \mu_{j+1} \dots \mu_{p+1}} + \frac{2.p+1}{V_{\alpha}^{\mu_1} \dots \mu_{p+1}} \sum_{j} \delta_{\mu_j \alpha} Y'^{\mu_2 \dots \mu_{j-1}} \hat{b}_{j} \mu_1 \mu_{j+1} \dots \mu_{p+1}} + \frac{-\delta_{\alpha \mu_1} Y'_{\mu_2 \dots \mu_{p+1}}}{V(p+1)} \left[\delta_{\alpha}^{\mu_1} Y'^{\mu_2 \dots \mu_{p+1}} Y'_{\mu_2 \dots \mu_{p+1}} + \frac{2.p+1}{V_{\alpha}^{\mu_2} \dots \mu_{p+1}} \delta_{\alpha}^{\mu_1} Y'^{\mu_2 \dots \mu_{p+1}} + \frac{2.p+1}{V_{\alpha}^{\mu_2} \dots \mu_{p+1}} \right]$$

$$= \frac{\beta}{V(p+1)} \left[\delta_{\alpha}^{\mu_1} Y'^{\mu_2 \dots \mu_{p+1}} Y'_{\mu_2 \dots \mu_{p+1}} + \frac{2.p+1}{V_{\alpha}^{\mu_2} \dots \mu_{p+1}} Y'^{\mu_2 \dots \mu_{p+1}} Y'^{\mu_2 \dots \mu_{p+1}} + \frac{2.p+1}{V_{\alpha}^{\mu_2} \dots \mu_{p+1}} \right]$$

$$= \frac{\beta}{V(p+1)} \left[\delta_{\alpha}^{\mu_1} (Y')^2 + \frac{2.p+1}{V_{\alpha}^{\mu_2} \dots \mu_{p-1}} \hat{b}_{\beta}^{\mu_1} \mu_1 \mu_{i+1} \dots \mu_{p+1} Y'^{\mu_2 \dots \mu_{p+1}} + \frac{2.p+1}{V_{\alpha}^{\mu_2} \dots \mu_{p+1}} \right]$$

$$= \frac{\beta}{V(p+1)} \left[\delta_{\alpha}^{\mu_1} (Y')^2 + \frac{2.p+1}{V_{\alpha}^{\mu_2} \dots \mu_{p-1}} \hat{b}_{\beta}^{\mu_1} \mu_1 \mu_{i+1} \dots \mu_{p+1} Y'^{\mu_2 \dots \mu_{p+1}} + \frac{2.p+1}{V_{\alpha}^{\mu_2} \dots \mu_{p+1}} \right]$$

$$= \frac{\beta}{V(p+1)} \left[\delta_{\alpha}^{\mu_1} (Y')^2 + \frac{2.p+1}{V_{\alpha}^{\mu_2} \dots \mu_{p-1}} \hat{b}_{\beta}^{\mu_1} \mu_1 \mu_{i+1} \dots \mu_{p+1} Y'^{\mu_2} \dots \mu_{p+1}} \right]$$

$$= \frac{\beta}{V(p+1)} \left[\delta_{\alpha}^{\mu_1} (Y')^2 - (p^2 - p) (Y')^2 + \frac{\beta}{V_{\alpha}^{\mu_1} (Y')^2} \right]$$

$$-p(\mathbf{Y}')^{2} + \\
-p(\mathbf{Y}')^{2} \Big]$$

$$= \frac{\beta}{V(p+1)} [D + Dp - p(p-1) - 2p] (\mathbf{Y}')^{2}$$

$$= \frac{\beta}{V(p+1)} (p+1)(D-p) (\mathbf{Y}')^{2}$$

$$= \frac{\beta}{V} (D-p) (\mathbf{Y}')^{2}$$
(A.65)

A.13 Solutions for the α and β Kernel Ansatz Parameters

To determine the parameters α , β , we insert (A.62), (A.65) into (4.33) and (4.34). The results are:

e:
$$\begin{cases}
2\rho_{p}\frac{\alpha}{V}p! = -\left(\oint dl(s)\right)^{-1}\oint \frac{d^{p}s}{\sqrt{Y'^{2}}}\left(D - p + 1\right)\frac{\beta}{V}Y'^{2} \\
2\rho_{p}\frac{\beta p!}{2\left(p + 1\right)V^{2}}\left(-\right)\Sigma^{2} = -\left(\oint dl(s)\right)^{-1} \\
\cdot \oint \frac{d^{p}\sigma}{\sqrt{Y'^{2}}}\frac{\beta^{2}}{V^{2}}\Sigma^{\mu_{1}\mu_{2}\dots\mu_{p+1}}Y'_{\mu_{2}\dots\mu_{p+1}}\Sigma_{\mu_{1}\nu_{2}\dots\nu_{p+1}}Y'^{\nu_{2}\dots\nu_{p+1}} \\
\begin{cases}
\alpha = -\frac{\left(D - p + 1\right)\beta}{2\rho_{p}\left(p\right)!} \\
\frac{\rho_{p}\left(p\right)!}{\left(p + 1\right)}\Sigma^{2} = -\left(\oint_{B}d^{p}v(s)\right)^{-1}\oint \frac{d^{p}\sigma}{\sqrt{Y'^{2}}}\frac{\left(p\right)!}{\left(p + 1\right)!}\Sigma^{2}Y'^{2}\beta \\
\begin{cases}
\alpha = -\frac{\left(D - p + 1\right)\beta}{2\rho_{p}p!} \\
\beta = \rho_{p}p! \\
\beta = \rho_{p}\left(p\right)!
\end{cases}
\end{cases}$$

$$\begin{cases}
\alpha = -\frac{\left(D - p + 1\right)\beta}{2} \\
\beta = \rho_{p}\left(p\right)!
\end{cases}$$
(A.66)

Appendix B

NonStandard Analysis

B.1 Short Introduction

In view of the developments that we describe in chapter 10 we think it is important to give here a brief account of an interesting branch of mathematics, namely NonStandard Analysis, which could play a deep role (still to be discovered!) in Relativistic Quantum Field Theory and Quantum Gravity. We used this technical apparatus to derive the String Functional Wave Equation as an independent check of the path-integral derivation in chapter 3. NonStandard Analysis can be applied to many different type of problems, but it is not yet very popular among theoretical physicists. We don't subscribe to this attitude because NonStandard Analysis provides a powerful tool to give a rigorous treatment of infinitesimal and infinities, which are ubiquitous especially in High Energy Theoretical Physics. We will give in this appendix more information than the ones strictly needed to follow the material in appendix C and chapter 10. We hope to excite the reader's interest on this subject.

B.1.1 Infinitesimals and Infinities

In our opinion the main reason why physicist should be interested in the study of *Nonstan-dard Analysis* is the key observation that most of the more difficult problems in Physics are in a more or less direct way related to the existence of singularities, i.e. infinities, of various

Nonstandard Analysis.

kind. Infinity is a strange concept that need a careful handling in mathematical proofs. For example, when the concept of limit to Infinity is introduced in a course of calculus, one must face the failure of the usual ϵ - and δ -neighborhood procedure. The same problem shows up when dealing with infinitesimal quantities: the product of a quantity that, in some limit, is divergent, times one that, in the same limit, is infinitesimal is not well defined in general. The usual way of dealing these problems is very far away from the original concept of infinitesimal and infinity as introduced by Leibniz. NonStandard Analysis recovers a more intuitive point of view and gives to it a proper foundation. Approaching NonStandard Analysis from a (beautyfull) purely mathematical side would require a long digression about logic, propositions, quantifiers, On the contrary, we would like to give the reader only the more immediate, and useful, definitions and results pertaining to the so called the NonStandard Reals.

NonStandard Entities represent an extension of the corresponding ones in standard analysis. This means that NonStandard Entities include the standard ones but there is something more. Such a "something more" is obtained employing a single new word, namely standard, as a predicate so that every mathematical entity about which we can speak can be standard or can be non standard.

Notation B.1 (Standard Entity).

To assert that the entity x is standard we will say

x IS STANDARD

or in symbols

st(x) .

This new predicate will not change the numbers we already know, but will give us a suitable definition of expressions like *infinitely large* and *infinitely small*. This is achieved through the following procedure.

Definition B.1 (Filter).

Let \mathbb{K} be a countable set. Let $\mathcal{P}(\mathbb{K})$ be the power set of \mathbb{K} , i.e. the set of all subsets of \mathbb{K} ; consider \mathcal{U} , $\emptyset \subset \mathcal{U} \subset \mathcal{P}(\mathbb{K})$ a set of subsets of \mathbb{K} , which satisfies the following properties:

- 1. $\emptyset \notin \mathcal{U}$;
- 2. $A \in \mathcal{U} \land B \in \mathcal{U} \implies A \cup B \in \mathcal{U}$;
- 3. $A \in \mathcal{U} \quad \land \quad B \in \mathcal{P}(\mathbb{K}) \quad \land \quad A \subseteq B \implies B \in \mathcal{U}$

U is called a <u>Filter</u>.

Definition B.2 (Ultrafilter).

Let \mathbb{K} be a countable set and \mathcal{U} a Filter on \mathbb{K} . If in addition \mathcal{U} satisfies

4.
$$B \in \mathcal{P}(\mathbb{K}) \implies B \in \mathcal{U} \quad \lor \quad \mathbb{K} \backslash B \in \mathcal{U}$$

it is called an <u>Ultrafilter</u> on \mathbb{K} .

Definition B.3 (Free Ultrafilter).

Let \mathbb{K} be a countable set and \mathcal{U} an Ultrafilter on \mathbb{K} . If in addition to the properties of an Ultrafilter \mathcal{U} satisfies

5.
$$\forall U, U \subset \mathbb{K} \quad \land \quad \exists n \in \mathbb{N} | \#U = n \implies U \notin \mathcal{U}$$

then it is called a <u>Free Ultrafilter</u>.

Notes:

- property 1. states that \mathcal{U} is a proper Filter;
- property 2. is called the *finite intersection property*;

- property 3. is called the *superset property*;
- property 4. requires the maximality for \mathcal{U} .

Even if these definitions makes perfectly sense, it is by no means obvious that a Free Ultrafilter exists. It is thus easier to assume this property:

Axiom (Ultrafilter): Let \mathcal{F} be a filter on \mathbb{K} ; there is an ultrafilter \mathcal{U} on \mathbb{K} which contains¹ \mathcal{F} .

Now we have all the basic elements to build an extension of \mathbb{R} in which infinitesimal and infinities will be properly defined. Of course we will loose something in the process, namely the new $field^2$ will be non Archimedean. To proceed toward this result we first set up some notations.

Notation B.2 (Sequence).

We will indicate a sequence by

$$\{a_n\}_{n\in\mathbb{N}}$$
 .

Moreover, we will indicate with \mathcal{U} a Free Ultrafilter on \mathbb{N} and with $\mathbb{R}^{\mathbb{N}}$ the set of all sequence of real numbers.

Proposition B.1 (Operations in $\mathbb{R}^{\mathbb{N}}$).

Let us consider $\{a_i\}_{i\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$. Defining

$$\exists: \quad \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \longrightarrow \quad \mathbb{R}^{\mathbb{N}} \\ \left(\left\{ a_{i} \right\}_{i \in \mathbb{N}}, \left\{ b_{j} \right\}_{j \in \mathbb{N}} \right) \quad \longrightarrow \quad \left\{ s_{k} \right\}_{k \in \mathbb{N}} \stackrel{\text{def.}}{=} \left\{ a_{k} + b_{k} \right\}_{k \in \mathbb{N}}$$

and

$$\boxtimes : \quad \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \quad \longrightarrow \quad \mathbb{R}^{\mathbb{N}} \\ \left(\left\{ a_{i} \right\}_{i \in \mathbb{N}}, \left\{ b_{j} \right\}_{j \in \mathbb{N}} \right) \quad \longrightarrow \quad \left\{ p_{k} \right\}_{k \in \mathbb{N}} \stackrel{\text{def.}}{=} \left\{ a_{k} \times b_{k} \right\}_{k \in \mathbb{N}}$$

 $(\mathbb{R}^{\mathbb{N}}, \boxplus, \boxtimes)$ becomes a commutative ring with unit element (namely the sequence $\{u_i\}_{i\in\mathbb{N}} | \forall i \in \mathbb{N}, u_i = 1$) and zero element (namely the sequence $\{z_j\}_{j\in\mathbb{N}} | \forall j \in \mathbb{N}, z_j = 0$). Furthermore, this ring is not a field since it has zero divisors.

¹It is possible to prove that this axiom follows from the axiom of choice, i.e. Zorn's lemma.

²Beware, we are not speaking of physical fields!

We thus see that this naive way of trying to construct an extension of the real numbers fails; there are too many elements in $\mathbb{R}^{\mathbb{N}}$. A standard way to cut out some unwanted presence is to construct an equivalence relation and consider the quotient of our structure modulo this relation with, the hope that the quotient structure turns out to have some richer structure. In our case we can define the following relation:

Definition B.4 (Equivalence Modulo Ultrafilter).

Consider \mathbb{N} and a Free Ultrafilter \mathcal{U} on \mathbb{N} . Moreover let us have $\{a_i\}_{i\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$. We will say that the two sequences are equal <u>Almost Everywhere</u> with respect to \mathcal{U} and we will write

$$\{a_i\}_{i\in\mathbb{N}} \equiv_{\mathcal{U}} \{b_j\}_{j\in\mathbb{N}}$$

if and only if

$${n \in \mathbb{N} \mid a_n = b_n} \in \mathcal{U}$$
.

It can be now proved, using the properties which define an Ultrafilter, that

Proposition B.2 (Equivalence Relation Modulo Ultrafilter).

 $\equiv_{\mathcal{U}}$ is an equivalence relation on $\mathbb{R}^{\mathbb{N}}$.

Thus, the following definition makes completely sense now.

Definition B.5 (Nonstandard Reals).

We consider all the equivalence classes $[\{a_i\}_{i\in\mathbb{N}}]$ of elements of \mathbb{R} generated by the relation $\equiv_{\mathcal{U}}$

$$\left[\left\{a_{i}\right\}_{i\in\mathbb{N}}\right]\overset{\mathrm{def.}}{=}\left\{\left\{x_{m}\right\}_{m\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}\left|\right.\left\{x_{m}\right\}_{m\in\mathbb{N}}\equiv_{\mathcal{U}}\left\{a_{i}\right\}_{i\in\mathbb{N}}\right\}$$

and define $*\mathbb{R}$ as

$$*\mathbb{R} \stackrel{\text{def.}}{=} \left\{ \left[\left\{ a_i \right\}_{i \in \mathbb{N}} \right] \mid \left\{ a_i \right\}_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \right\} = \left(\mathbb{R}^{\mathbb{N}} \right) / \equiv_{\mathcal{U}}$$

This is the <u>Ultrapower</u> generated by \mathcal{U} and its elements $[\{a_i\}_{i\in\mathbb{N}}]\in {}^*\mathbb{R}$ are called <u>Hyperreals</u> or <u>NonStandard Reals</u>.

Proposition B.3 (Linearly Ordered Structure on \mathbb{R}).

Consider the structure $(*\mathbb{R}, +_{\mathcal{U}}, \times_{\mathcal{U}}, <_{\mathcal{U}})$ defined as:

$$+\mu: \quad {}^*\mathbb{R} \times {}^*\mathbb{R} \quad \longrightarrow \quad {}^*\mathbb{R} \\ \left(\left[\left\{a_i\right\}_{i \in \mathbb{N}}\right], \left[\left\{b_j\right\}_{j \in \mathbb{N}}\right]\right) \quad \longrightarrow \quad \left[\left\{s_k\right\}_{k \in \mathbb{N}}\right] \stackrel{\mathrm{def.}}{=} \left[\left\{a_i\right\}_{i \in \mathbb{N}} \boxplus \left\{b_j\right\}_{j \in \mathbb{N}}\right] \\ \times \mu: \quad {}^*\mathbb{R} \times {}^*\mathbb{R} \quad \longrightarrow \quad {}^*\mathbb{R} \\ \left(\left[\left\{a_i\right\}_{i \in \mathbb{N}}\right], \left[\left\{b_j\right\}_{j \in \mathbb{N}}\right]\right) \quad \longrightarrow \quad \left[\left\{p_k\right\}_{k \in \mathbb{N}}\right] \stackrel{\mathrm{def.}}{=} \left[\left\{a_i\right\}_{i \in \mathbb{N}} \boxtimes \left\{b_j\right\}_{j \in \mathbb{N}}\right] \\ <\mu: \quad \left[\left\{a_i\right\}_{i \in \mathbb{N}} < \mu \left[\left\{b_j\right\}_{j \in \mathbb{N}}\right]\right] \quad \Longleftrightarrow \quad n \in \mathbb{N} | a_n < b_n \in \mathcal{U} \\ \le \mu: \quad \left[\left\{a_i\right\}_{i \in \mathbb{N}} \le \mu \left[\left\{b_j\right\}_{j \in \mathbb{N}}\right]\right] \quad \Longleftrightarrow \quad \left[\left\{a_i\right\}_{i \in \mathbb{N}} < \mu \left[\left\{b_j\right\}_{j \in \mathbb{N}}\right]\right] \lor \left[\left\{a_i\right\}_{i \in \mathbb{N}} = \mu \left[\left\{b_j\right\}_{j \in \mathbb{N}}\right]\right] \\ It is a linearly ordered field.$$

In such a way, one constructs a field with the same *operations* as the reals \mathbb{R} . It would be very nice if, as the name ${}^*\mathbb{R}$ already anticipate, we could also prove that it is an extension of the reals, i.e. that it is possible to find an *order preserving isomorphism* of \mathbb{R} into ${}^*\mathbb{R}$. After the following definition:

Definition B.6 (Non Standard Isomorphism).

Let us consider $x \in \mathbb{R}$. We define the map * as follows:

*:
$$\mathbb{R} \longrightarrow {}^*\mathbb{R}$$
 $x \longrightarrow \left[\{\chi_i\}_{i \in \mathbb{N}} \right]$
where $\chi_i = x$, $\forall i \in \mathbb{N}$

The map * is called the <u>NonStandard Isomorphism</u>.

It is possible to prove that the map defined in the above definition has the property we would like to require in order to embed \mathbb{R} in ${}^*\mathbb{R}$.

Proposition B.4 (Order Preserving Isomorphism).

The NonStandard Isomorphism * is an Order Preserving Isomorphism of fields.

We described the construction of ${}^*\mathbb{R}$ because it is a standard procedure we will use in what follow to define some other spaces, which we are going to use in our application to the derivation of the functional wave equation for String Dynamics. Moreover, with

a few examples we will see in the next subsection that this extension of the reals, \mathbb{R} , to the NonStandard Reals, $^*\mathbb{R}$, is not just a mathematical exercise, but can be really useful in treating in a more handy, but nevertheless perfectly rigorous way, infinitesimals and infinities

B.1.2 Some Examples

A key step in the outlined direction can be done by observing that ${}^*\mathbb{R}$ contains elements, which are bigger than any real number, as well as positive elements, which are smaller than any real positive number.

Proposition B.5 (An Infinite Number and an Infinitesimal Number).

The element $\omega = \left[\left\{ n \right\}_{n \in \mathbb{N}} \right] \in {}^*\mathbb{R}$ is bigger than any real number. The element $\varepsilon = \left[\left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}} \right]$ is positive and smaller than any positive number.

Then, the following definitions will introduce some useful names for some interesting subsets of $*\mathbb{R}$.

Definition B.7 (Infinite, Infinitesimal, Limited, Appreciable Numbers).

We define four subsets of elements in \mathbb{R} .

• A number $\omega = [\{o_k\}_{k \in \mathbb{N}}] \in {}^*\mathbb{R}$ is <u>Infinitely Large</u> or <u>Infinite</u> or <u>Unlimited</u> if and only if

$$\forall^* x \mid st(^*x) \ Longrightarrow ^*x <_{\mathcal{U}} |\omega|$$
.

• A number $\varepsilon = [\{i_k\}_{k \in \mathbb{N}}] \in {}^*\mathbb{R}$ is <u>Infinitely Small</u> or <u>Infinitesimal</u> if and only if

$$\forall n \in N \implies |\varepsilon| <_{\mathcal{U}}^* \left(\frac{1}{n}\right)$$
.

- A number $\rho = [\{r_k\}_{k \in \mathbb{N}}] \in {}^*\mathbb{R}$ is <u>Limited</u> or <u>Finite</u> if and only if it is not Infinite.
- A number $\rho = [\{r_k\}_{k \in \mathbb{N}}] \in {}^*\mathbb{R}$ is <u>Appreciable</u> if and only if it is not Infinite nor Infinitesimal.

+		Ω	@	£	∞
Ω		Ω	@	£	∞
@			£	£	∞
£	,			£	∞
\propto)				?
			(a)		
			(a)		
×		Ω	(a) @	£	∞
×		υ υ			∞
			@	£	

_	Ω	@	£	∞	
Ω	Ω	@	£	∞	
@		£	£	∞	
£			£	∞	
∞				?	
(b)					

×	Ω	@	£	∞		
Ω	Ω	Ω	Ω	?		
@		@	£	∞		
£			£	?		
∞				∞		
(c)						

/	Ω	@	£	∞	
Ω	?	∞	?	∞	
@	Ω	@	£	∞	
£	?	?	?	∞	
∞	Ω	Ω	Ω	?	
(d)					

Table B.1: Computation tables with Infinity, Infinitesimal, Limited and Appreciable numbers: (a) sum; (b) subtraction, row - column; (c) product; (d) quotient, row/column.

Moreover, for easier writing, it turns useful to introduce the following notation:

Notation B.3 (\mho , @, £, ∞).

The following symbols are used to characterize Infinitesimal, Appreciable, Limited and Infinite numbers:

- \mho represents an Infinitesimal number;
- @ represents an Appreciable number;
- £ represents a Limited number;
- ∞ represents an Infinite number.

Then, it is possible to use the tables B.1, for example to perform operations, where in standard calculus, longer limit procedures should be performed.

This example explains, although in a very simple and *poor* way, why³ NonStandard Analysis can be helpful in tackling with the problem of Infinitesimals and Infinities and this is the main reason that leads us, in this thesis work, to find a NonStandard procedure to motivate on a rigorous way the results obtained with the formal manipulations of chapter 3.

B.2 Ultra Euclidean Space

It is possible to perform the UltraPower construction in the previous section in a slightly more involved case. We remember that in all our discussion \mathcal{U} denotes a Free Ultrafilter over the natural numbers \mathbb{N} .

We then proceed along the same line that we have employed in the previous section, namely we define an equivalence relation, denoted with $\simeq_{\mathcal{U}}$, thanks to the Free Ultrafilter \mathcal{U} : this time the space we consider is the following one

$$\prod_{n\in\mathbb{N}}\mathbb{R}^r$$

Thus, sequences are now sequences of vectors in spaces of increasing dimension. In particular

Notation B.4 (Element of $\prod_{n\in\mathbb{N}}\mathbb{R}^n$).

We will denote an element in the infinite sequence of products of \mathbb{R}^n as n ranges in $1, \ldots, \infty$ with German letters

$$\mathfrak{a} = \left\{ a^{(n)}
ight\}_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathbb{R}^n$$
 .

Then, the equivalence relation modulo the Free Ultrafilter identifies elements of $\prod_{n\in\mathbb{N}}\mathbb{R}^n$, i.e. sequences whose component vectors agree for a set of spaces indexed by an element which is in the Free Ultrafilter.

Definition B.8 (Equivalence Relation on $\prod_{n\in\mathbb{N}}\mathbb{R}^n$).

³If not *how*; indeed the subject is much more deep and vast, and the account we gave here is just to explain the flavor of what we think is a really interesting branch of mathematics.

On $\prod_{n\in\mathbb{N}}\mathbb{R}^n$ the Free Ultrafilter \mathcal{U} defines the following relation

$$\mathfrak{a} \simeq_{\mathcal{U}} \mathfrak{b} \quad \Longleftrightarrow \quad \left\{ n \in N \, | \, \boldsymbol{a}^{(n)} = \boldsymbol{b}^{(n)} \right\} \in \mathcal{U} \quad ,$$

where we assume

$$\mathfrak{a} = \left\{ \left(a_1^{(1)} \right), \left(a_1^{(2)}, a_2^{(2)} \right), \dots, \left(a_1^{(i)}, a_2^{(i)}, \dots, a_i^{(i)} \right), \dots \right\}
\mathfrak{b} = \left\{ \left(b_1^{(1)} \right), \left(b_1^{(2)}, a_2^{(2)} \right), \dots, \left(b_1^{(i)}, b_2^{(i)}, \dots, b_i^{(i)} \right), \dots \right\} .$$
(B.1)

This relation is of course an equivalence relation so that it is meaningful to define a partition of $\mathbb{R}^{\mathbb{N}}$ into disjoint equivalence classes of elements; this leads directly to the following

Definition B.9 (Ultra Euclidean Space).

We call Ultra Euclidean Space the quotient

$$^*\mathcal{E} = \left(\prod_{n \in \mathbb{N}} \mathbb{R}^n\right) / \simeq_{\mathcal{U}} . \tag{B.2}$$

We remark that the basic idea of the above procedure is simply a *chinese box construction* with some identifications. Indeed, we stress again that a generic element in $(\prod_{n\in\mathbb{N}}\mathbb{R}^n)$ is simply an *infinite sequence of finite sequences of increasing length*,

$$\mathfrak{a} = \left\{ \left(a_1^{(1)} \right), \left(a_1^{(2)}, a_2^{(2)} \right), \dots, \left(a_1^{(i)}, a_2^{(i)}, \dots, a_i^{(i)} \right), \dots \right\} \in \prod_{n \in \mathbb{N}} \mathbb{R}^n$$
 (B.3)

and

$$\mathbf{a}^{(n)} = \left(a_1^{(n)}, a_2^{(n)}, \dots, a_n^{(n)}\right) \in \mathbb{R}^n$$
 (B.4)

An element of the *Ultra Euclidean Space* is thus an equivalence class in the relation defined by the *Ultrafilter* \mathcal{U} .

Notation B.5 (Element of the Ultra Euclidean Space ${}^*\mathcal{E}$).

We will denote an element of the Ultra Euclidean Space as

$$[\mathfrak{a}] = \left[\left\{oldsymbol{a}^{(n)}
ight\}_{n\in\mathbb{N}}
ight] \in {}^*\mathcal{E}$$
 .

Some properties of the *Ultra Euclidean Space* $^*\mathcal{E}$ can be easily proved. The proofs can be an useful exercise in applying simple properties of *NonStandard Analysis*.

Proposition B.6 (Linearity of ${}^*\mathcal{E}$).

* \mathcal{E} is a <u>Linear Space</u> over * \mathbb{R} .

Moreover,

Proposition B.7 (Euclideanity of $^*\mathcal{E}$).

* $\mathcal E$ is an <u>Euclidean Space</u> with scalar product defined by

$$\langle \mathfrak{a}, \mathfrak{b} \rangle_{*\mathcal{E}} = \left\langle \left[\left\{ a^{(n)} \right\}_{n \in \mathbb{N}} \right], \left[\left\{ b^{(n)} \right\}_{n \in \mathbb{N}} \right] \right\rangle_{*\mathcal{E}}$$

$$\stackrel{\text{def.}}{=} \left[\left\{ \left\langle \boldsymbol{a}^{(n)}, \boldsymbol{b}^{(n)} \right\rangle_{\mathbb{R}^{n}} \right\}_{n \in \mathbb{N}} \right] = \left[\left\{ \sum_{i}^{1, n} a_{i}^{(n)} b_{i}^{(n)} \right\}_{n \in \mathbb{N}} \right] . \tag{B.5}$$

Let us note that, in the definition of the scalar product,

$$\left\langle \boldsymbol{a}^{(n)}, \boldsymbol{b}^{(n)} \right\rangle_{\mathbb{R}^n}$$
 (B.6)

is the usual scalar product in \mathbb{R}^n , so that actually $\left\{\left\langle \boldsymbol{a}^{(n)}, \boldsymbol{b}^{(n)}\right\rangle_{\mathbb{R}^n}\right\}_{n\in\mathbb{N}}$ is a sequence, i.e. is in $\mathbb{R}^{\mathbb{N}}$; then the *Ultrafilter Equivalence Relation* sets its equivalence class in ${}^*\mathbb{R}$.

Appendix C

Nonstandard Stochastic Processes

C.1 Functional Spaces

We quote the following usual definitions and notations:

Notation C.1 (Schwartz Space).

The Schwartz space over \mathbb{R}^d , is denoted by

 $\mathbb{S}(\mathbb{R}^d)$.

Notation C.2 (\mathcal{L}^2 Space).

The \mathcal{L}^2 Space over \mathbb{R}^d , is denoted by

 $\mathcal{L}^2(\mathbb{R}^d)$.

Notation C.3 (Complete Orthonormal Set).

A complete orthonormal set, belonging to $S(\mathbb{R}^d)$, which is complete orthonormal in $\mathcal{L}^2(\mathbb{R}^d)$, is denoted by

 $\{e_n\}_{n\in\mathbb{N}}$.

From the results of section B.2 we can now proceed to some central definitions: the principal space for the procedure of section 10.1 and some other tool that are used therein.

Definition C.1 (Kawabata Space).

Consider $\{e_n\}_{n\in\mathbb{N}}$ as in notation C.3. The <u>Kawabata Space</u> is a set of \mathbb{R} valued functions and is defined as

$${}^{*}\mathcal{E}_{\{e_{n}\}}\left(\mathbb{R}^{d} \to {}^{*}\mathbb{R}\right) \stackrel{\text{def.}}{=} \left\{\hat{\boldsymbol{\phi}} \mid \hat{\boldsymbol{\phi}} = \left[\left\{\phi_{n}\right\}_{n \in \mathbb{N}}\right]\right.$$

$$= \left[\left\{\sum_{i=1}^{n} a_{i}^{(n)} e_{i}\right\}_{n \in \mathbb{N}}\right], \ \forall \mathfrak{a} = \left[\left\{\boldsymbol{a}^{(n)}\right\}_{n \in \mathbb{N}}\right] \in {}^{*}\mathcal{E}\right\} \qquad . \tag{C.1}$$

Notation C.4 (nth-approximation of an Elment of Kawabata Space).

We will call ϕ_n , in the definition above, an <u>Approximation</u>¹ of $\hat{\phi}$.

Some explanations are worth to clarify the construction:

- 1. it is important to remember that $e_i = e_i(-)$ is a function on \mathbb{R}^d , which assigns to each $\mathbf{v} \in \mathbb{R}^d$ the number $e_i(\mathbf{v}) \in \mathbb{R}$;
- 2. moreover for each n, $a^{(n)}$ is a vector in \mathbb{R}^n ; we then use its components $a_j^{(n)}$ as coefficients to write an n-th order expansion $\phi_n(-)$, in terms of the first n elements of the base $\{e_j(-)\}_{j\in\mathbb{N}}$;
- 3. thus $\forall j \in \mathbb{N}$ we obtain an approximation of $\phi(-)$; suppose now to apply $\forall j \in \mathbb{N}$ the approximation $\phi_j(-)$ to a vector $\mathbf{v} \in \mathbb{R}^d$; this results is the sequence of real numbers $\{\phi_j(\mathbf{v})\}_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, which can be uniquely associated throught the equivalence relation defined by the *Ultrafilter* \mathcal{U} to the *NonStandard* real $\left[\{\phi_j(\mathbf{v})\}_{j\in\mathbb{N}}\right] \in {}^*\mathbb{R}$: this is actually the value of $\hat{\phi}(\mathbf{v})$.

We now define an appropriate structure on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$).

 $^{^{1}}$ Or more specifically the *n*th-Approximation.

Definition C.2 (Scalar Product on Kawabata Space).

Consider $\hat{\phi}, \hat{\psi} \in {}^*\mathcal{E}_{\{e_n\}} (\mathbb{R}^d \to {}^*\mathbb{R})$. The scalar product of these two elements is defined as follows:

$$\left\langle \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\psi}} \right\rangle_{*\mathcal{E}_{\{e_n\}}(\mathbb{R}^d \to *\mathbb{R})} = \left\langle \left[\left\{ \sum_{i}^{1,n} a_i^{(n)} e_i \right\}_{n \in \mathbb{N}} \right], \left[\left\{ \sum_{i}^{1,n} a_i^{(n)} e_i \right\}_{n \in \mathbb{N}} \right] \right\rangle_{*\mathcal{E}_{\{e_n\}}(\mathbb{R}^d \to *\mathbb{R})}$$

$$\stackrel{\text{def.}}{=} \left[\left\{ \sum_{i,j}^{1,n} a_i^{(n)} b_j^{(n)} \left\langle e_i, e_j \right\rangle_{\mathcal{S}(\mathbb{R}^d)} \right\}_{n \in \mathbb{N}} \right]$$

$$= \left[\left\{ \sum_{i,j}^{1,n} a_i^{(n)} b_i^{(n)} \right\}_{n \in \mathbb{N}} \right]$$

$$= \left\langle \left[\left\{ \boldsymbol{a}^{(n)} \right\}_{n \in \mathbb{N}} \right], \left[\left\{ \boldsymbol{b}^{(n)} \right\}_{n \in \mathbb{N}} \right] \right\rangle_{*\mathcal{E}}$$

$$= \left\langle \mathbf{a}, \mathbf{b} \right\rangle_{*\mathcal{E}}, \qquad (C.3)$$

where of course $\mathfrak{a} = \left[\left\{ \boldsymbol{a}^{(n)} \right\}_{n \in \mathbb{N}} \right]$ and $\mathfrak{b} = \left[\left\{ \boldsymbol{b}^{(n)} \right\}_{n \in \mathbb{N}} \right]$.

Then two results follows.

Proposition C.1 (Homeomorphism between ${}^*\mathcal{E}_{\{e_n\}}\left(\mathbb{R}^d \to {}^*\mathbb{R}\right)$ and ${}^*\mathcal{E}$). $\left({}^*\mathcal{E}_{\{e_n\}}\left(\mathbb{R}^d \to {}^*\mathbb{R}\right), \langle -, -\rangle_{{}^*\mathcal{E}_{\{e_n\}}(\mathbb{R}^d \to {}^*\mathbb{R})}\right)$ is homeomorphic to $({}^*\mathcal{E}, \langle -, -\rangle_{{}^*\mathcal{E}})$.

Proof:

This result follows immediatly comparing the left hand side of equation (C.2) with (C.3):

$$\left\langle \hat{\phi}, \hat{\psi} \right\rangle_{*\mathcal{E}_{\{e_n\}}\left(\mathbb{R}^d \to *\mathbb{R}\right)} = \langle \mathfrak{a}, \mathfrak{b} \rangle_{*\mathcal{E}} \quad .$$

Proposition C.2 (Local Properties of Elements in Kawabata Space).

 $\forall \, \hat{\phi} \in {}^*\mathcal{E}_{\{e_n\}} \left(\mathbb{R}^d \to {}^*\mathbb{R} \right), \, \hat{\phi} \, \, \text{is locally integrable and locally differentiable.}$

In particular we can define

Definition C.3 (Integral of an Element of Kawabata Space).

The integral of $\hat{\phi}$ is defined ad

$$\int_{\mathbb{R}^d} d^d x \, \hat{\boldsymbol{\phi}}(x) \stackrel{\text{def.}}{=} \left[\left\{ \sum_{i=1}^{n} a_i^{(n)} \int_{\mathbb{R}^d} d^d x \, e_i(x) \right\}_{n \in \mathbb{N}} \right] \in {}^*\mathbb{R} .$$

Moreover we also have

Definition C.4 (Directional Derivative of an Element of Kawabata Space).

The directional derivative of $\hat{\phi}$ in the direction of x is

$$\left(\partial_j \hat{\boldsymbol{\phi}}\right) \stackrel{\text{def.}}{=} \left[\left\{ \sum_{i=1}^{n} a_i^{(n)} \partial_j e_i(x) \right\}_{n \in \mathbb{N}} \right] \in {}^*\mathbb{R} .$$

Definition C.5 (Functional on Kawabata Space).

Let $\hat{\psi} = \left[\left\{ \sum_{i=1}^{n} a_i^{(n)} e_i \right\}_{n \in \mathbb{N}} \right] \in {}^*\mathcal{E}_{\{e_n\}} \left(\mathbb{R}^d \to {}^*\mathbb{R} \right)$. Consider $\forall n \in \mathbb{N}$ the n-th approximation of $\hat{\psi}$, and the function

$$F^{(n)}: \mathbb{R}^n \longrightarrow \mathbb{R}$$

 $\boldsymbol{a}^{(n)} \longrightarrow F^{(n)}(a^{(n)})$

Then the map

$$\check{\mathcal{F}} : {}^*\mathcal{E}_{\{e_n\}} \left(\mathbb{R}^d \to {}^*\mathbb{R} \right) \longrightarrow {}^*\mathbb{R} \\
\hat{\psi} \longrightarrow \left[\left\{ F^{(n)} \left(\boldsymbol{a}^{(n)} \right) \right\}_{n \in \mathbb{N}} \right]$$

is a <u>Functional</u> on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$).

Notation C.5 (Functional on Kawabata Space).

We will write a functional on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) with its argument enclosed in square brackets, i.e. as $\check{\mathcal{F}}$ $\left[\hat{\psi}\right]$.

This notation has the following idea behind: since usually it is possible to confuse themselves with notations, we use no *oversymbols* for elements in ${}^*\mathbb{R}$. Then, since elements on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) have a hat " ${}^{\hat{}}$ " over them and functionals on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) have a check " * " which is something like an *inverse hat*, we see that acting with a functional on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) on an element of ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) the check and the hat annihilates, so we can immediatly understand that we just end in ${}^*\mathbb{R}$ with a *NonStandard Real Number*.

To see that this procedure is consistent, we note that:

- 1. $\hat{\psi}$ is completely defined by the coefficients of its expansion in terms of the complete orthonormal set $e_i(-)$; in terms of the definition of ${}^*\mathcal{E}$ we can thus consider the associated element $\mathfrak{a} = \left[\left\{a^{(n)}\right\}_{n\in\mathbb{N}}\right]$;
- 2. since $F^{(n)}$ values are in \mathbb{R} , the sequence $\{F^{(n)}(\boldsymbol{a}^{(n)})\}_{n\in\mathbb{N}}$ is a sequence of real numbers and we can apply to it the usual identification procedure *modulo* the *Ultrafilter* \mathcal{F} , so that the result $[\{F^{(n)}(\boldsymbol{a}^{(n)})\}_{n\in\mathbb{N}}]$ is in \mathbb{R} ;
- 3. since a function f, even if we think at the usual situation, in which no NonStandard elements appear, is completely determined by its coefficients in an expansion in terms of a suitably choosen functional basis, it is sensible that a functional of f is completely defined in terms of its action on the coefficients that determine f.

Keeping in mind these observations, we can easily understand the forms of the NonStandard definitions of functional derivative and functional integral. We stress again that an element of ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) is a function from \mathbb{R}^d to \mathbb{R} , which is clearly indicated by the notation of the following definitions.

Definition C.6 (Functional Derivative of a Functional on Kawabata Space).

Let
$$\check{\mathcal{F}}\left[\hat{\psi}\right] = \left[\left\{F^{(n)}\left(x^{(n)}\right)\right\}_{n\in\mathbb{N}}\right]$$
 be a functional of $\hat{\psi}(-) = \left[\left\{\sum_{i=1}^{n} x_{i}^{(n)} e_{i}(-)\right\}_{n\in\mathbb{N}}\right]$ on $^{*}\mathcal{E}_{\left\{e_{n}\right\}}\left(\mathbb{R}^{d} \to ^{*}\mathbb{R}\right)$. The functional derivative of $\check{\mathcal{F}}\left[\hat{\gamma}\right]$ with respect to $\hat{\psi}(-)$ is

$$\frac{\delta \check{\mathcal{F}}\left[\hat{\psi}\right]}{\delta \hat{\psi}(-)} = \left[\left\{ \sum_{j}^{1,n} \frac{\partial F\left(\boldsymbol{x}^{(n)}\right)}{\partial x_{j}^{(n)}} e_{j}(-) \right\}_{n \in \mathbb{N}} \right] \in {}^{*}\mathcal{E}_{\left\{e_{n}\right\}}\left(\mathbb{R}^{d} \to {}^{*}\mathbb{R}\right) .$$

Note that as is natural, if $x \in \mathbb{R}^d$, then

$$\frac{\delta \check{\mathcal{F}} \left[\hat{\psi} \right]}{\delta \hat{\psi}(x)} \in {}^*\mathbb{R} \quad . \tag{C.4}$$

Moreover we have

Definition C.7 (Functional Integral of a Functional on Kawabata Space).

Let $\check{\mathcal{F}}\left[\hat{\psi}\right] \stackrel{\text{def.}}{=} \left[\left\{F^{(n)}\left(\boldsymbol{x}^{(n)}\right)\right\}_{n\in\mathbb{N}}\right]$ be a functional of $\hat{\psi}(-) = \left[\left\{\sum_{i=1}^{1,n} x_{i}^{(n)} e_{i}(-)\right\}_{n\in\mathbb{N}}\right]$ on $*\mathcal{E}_{\{e_{n}\}}\left(\mathbb{R}^{d} \to *\mathbb{R}\right)$. The functional integral of $\check{\mathcal{F}}\left[\hat{\gamma}\right]$ with respect to $\hat{\psi}(-)$ is

$$\int_{*\mathcal{E}_{\{e_n\}}(\mathbb{R}^d \to *\mathbb{R})} \check{\mathcal{F}} \left[\hat{\boldsymbol{\psi}} \right] \stackrel{\text{def.}}{=} \left[\left\{ \int F^{(n)} \left(\boldsymbol{x}^{(n)} \right) \right\}_{n \in \mathbb{N}} \right] \delta \hat{\boldsymbol{\psi}} \in \mathbb{R} .$$

We end this section with the definition of operators on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) and some remarks.

Definition C.8 (Operator on Kawabata Space).

Consider $\left\{ f^{(n)}(-) \right\}_{n \in \mathbb{N}}$ a sequence of functions

$$f^{(n)}(-)$$
 : $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ $x \longrightarrow f^{(n)}(x)$.

An operator on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) is an application \mathcal{F} such that

$$\mathcal{F}(-) : \quad {}^*\mathcal{E}_{\{e_n\}} \left(\mathbb{R}^d \to {}^*\mathbb{R} \right) \longrightarrow \quad {}^*\mathcal{E}_{\{e_n\}} \left(\mathbb{R}^d \to {}^*\mathbb{R} \right)$$
$$\hat{\psi} = \left[\left\{ \sum_{i}^{1,n} x_i^{(n)} e_i \right\}_{n \in \mathbb{N}} \right] \longrightarrow \quad \mathcal{F}\hat{\psi} = \left[\left\{ \sum_{j}^{1,n} f_j^{(n)} \left(\boldsymbol{x}^{(n)} \right) e_j \right\}_{n \in \mathbb{N}} \right]$$

This definition recalls the same ideas of the previous ones. Moreover we observe that all the given definitions can be generalized adding some parametric dependence on one or more continuous or discrete parameters. Then we note that

Proposition C.3 (Structure of Operators in Kawabata Space).

Let

$$\mathcal{A}\left[\hat{\ }\right] = \left[\left\{ \sum_{j=0}^{1,n} a_j^{(n)}(-) e_j \right\}_{n \in \mathbb{N}} \right]$$

be an element in the set of all operators on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$); consider now a functional

$$\check{\mathcal{F}}\left[\hat{\pmb{\psi}}\right] = \left[\left\{F^{(n)}\left(\pmb{x}^{(n)}\right)\right\}_{n\in\mathbb{N}}\right]$$

on ${}^*\mathcal{E}_{\{e_n\}}$ $(\mathbb{R}^d \to {}^*\mathbb{R})^2$. We will give the following meaning to the composition between an

²We remember that both, an operator as well as a functional, act on elements of ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$); the first associates to this elements another element of ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) whereas the secon associates a NonStandard Real.

operator and a functional: by definition it is again an operator $\tilde{\mathcal{F}}$ defined as

$$\tilde{\mathcal{F}}\left[\hat{\ }\right] \stackrel{\text{def.}}{=} \mathcal{A}\left[\hat{\ }\right] \check{\mathcal{F}}\left[\hat{\ }\right] \stackrel{\text{def.}}{=} \left[\left\{ \sum_{j=1}^{n} a_{j}^{(n)}(-) F^{(n)}(-) e_{j} \right\}_{n \in \mathbb{N}} \right] . \tag{C.5}$$

That this is a meaningful definition can be seen since

$$\tilde{\mathcal{F}}\left[\hat{\boldsymbol{\psi}}\right] = \left[\left\{\sum_{j=1}^{n} a_{j}^{(n)}\left(\boldsymbol{x}^{(n)}\right) F^{(n)}\left(\boldsymbol{x}^{(n)}\right) e_{j}(-)\right\}_{n \in \mathbb{N}}\right] \in {}^{*}\mathcal{E}_{\{e_{n}\}}\left(\mathbb{R}^{d} \to {}^{*}\mathbb{R}\right) .$$

is again an elemen of ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$).

$ext{C.2} \quad ext{Diffusion Process on } ^*\mathcal{E}_{\{e_n\}} \left(\mathbb{R}^d ightarrow ^*\mathbb{R} ight)$

The procedure to properly define a diffusion process on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) is similar to the one used in the various definitions above. We pick up a family of ordinary diffusion processes

$$\left\{ \boldsymbol{X}^{(n)}(t) \right\}_{n}^{1,\infty} \tag{C.6}$$

and construct with them an element of ${}^*\mathcal{E}$ for each t, i.e. a diffusion process in ${}^*\mathcal{E}$. This is a sequence of diffusion processes in euclidean spaces of increasing dimension. More explicitly this means that we consider

$$\left\{\boldsymbol{X}^{(1)}(t), \boldsymbol{X}^{(2)}(t), \dots, \boldsymbol{X}^{(i)}(t), \dots\right\}$$

where each $X^{(j)}(t)$ is a random variable in \mathbb{R}^j . Once we have this sequence, by means of the usual equivalence relation defined thanks to the *Ultrafilter* \mathcal{U} , we obtain the definition of the stochastic process in ${}^*\mathcal{E}_{\{e_n\}}(\mathbb{R}^d \to {}^*\mathbb{R})$.

Definition C.9 (Stochastic Process on Kawabata Space).

Let $\left\{ \boldsymbol{X}^{(n)}(t) \right\}_n^{1,\infty}$ be a family of stochastic processes, with $\boldsymbol{X}^{(n)}$ an n-dimensional random variable. Then $\hat{\boldsymbol{\Psi}}(t)$ defined as

$$\hat{\boldsymbol{\Psi}}(t) \stackrel{\text{def.}}{=} \left[\left\{ \sum_{j}^{1,n} X_{j}^{(n)}(t) e_{j} \right\}_{n \in \mathbb{N}} \right] \in {}^{*}\mathcal{E}_{\{e_{n}\}} \left(\mathbb{R}^{d} \to {}^{*}\mathbb{R} \right)$$

is a <u>Stochastic Process</u> on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) or a <u>Nonstandard Stochastic Process</u>

We note that of course the Stochastic Process on Kawabata Space is a one parameter \mathbb{R} valued family of functions on \mathbb{R}^d .

Now we would be really glad if we were able to obtain a complete generalization of the properties of stochastic processes in \mathbb{R}^d to Kawabata space. In particulare we need probability distributions for our processes and a notion of volume is the first step toward this goal. The following definition is thus natural, in the light of all the work done so far. Indeed we need a definition of volume element which allows the standard definition of probability in each component \mathbb{R}^i in which the *i*-dimensional random process $X^{(i)}$ is living in. Then it is natural to define the following sequence

$$\left\{d^{n}x^{(n)}\right\}_{n\in\mathbb{N}} = \left\{dx_{1}^{(1)}, dx_{2}^{(1)}dx_{2}^{(2)}, \dots, dx_{i}^{(1)}dx_{i}^{(2)}\dots dx_{i}^{(i)}, \dots\right\}$$
(C.7)

and use the $Ultrafilter\ F$ to obtain the usual identification.

Definition C.10 (Elementary Volume Element on Kawabata Space).

The elementary volume element on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) is

$$\delta \hat{\psi} \stackrel{\text{def.}}{=} \left[\left\{ d^n x^{(n)} \right\}_{n \in \mathbb{N}} \right] .$$

Then we can define the NonStandard Probability Distribution associated with the NonStandard Stochastic Process. To motivate the following definition, we stress again that the starting point is always the family of stochastic processes C.6, the first of them living in \mathbb{R} , the second in \mathbb{R}^2 , the ... in ..., the *i*-th in \mathbb{R}^i and so on. Each of this processes has an associated probability distribution $p^{(j)}(\mathbf{x}^{(j)},t)$, defined on \mathbb{R}^j , from which it is possible to compute the probability $P^{(j)}(\mathbf{x}^{(j)})$ of finding a value of the random variable $\mathbf{X}^{(j)}$ in a small volume element $d^j x^{(j)}$ around $\mathbf{x}^{(j)}$ at time t:

$$P^{(j)} = \text{Prob}\left\{ \boldsymbol{X}^{(j)}(t) \in d^{j}x^{(j)} \right\} = p^{(j)}\left(\boldsymbol{x}^{(j)}, t\right) dx_{1}^{(j)} \cdots dx_{j}^{(j)} \quad . \tag{C.8}$$

We can generalize this definition to a NonStandard–Real Probability on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) as follows:

Definition C.11 (Non Standard Probability).

Let $\hat{\Psi}$ be a NonStandard Stochastic Process on ${}^*\mathcal{E}_{\{e_n\}}\left(\mathbb{R}^d \to {}^*\mathbb{R}\right)$ with i-th approximation

 $\sum_{j}^{1,i} \boldsymbol{X}_{j}^{(i)} e_{i} \text{ having probability } P^{(i)} \big(\boldsymbol{x}^{(i)} \big) \text{ of being found in a volume element } d^{i} x^{(i)} \text{ around } \boldsymbol{x}^{(i)}.$ The NonStandard Probability of finding the process $\hat{\boldsymbol{\Psi}}$ in a NonStandard Volume Element $\delta \hat{\boldsymbol{\psi}}$ around $\hat{\boldsymbol{\psi}} = \left[\left\{ \sum_{l}^{1,k} x_{l}^{(k)} e_{l} \right\}_{k \in \mathbb{N}} \right]$ is defined as

*Prob.
$$\left\{ \hat{\boldsymbol{\psi}}(t) \in \delta \hat{\boldsymbol{\psi}} \right\} = \text{*Prob.} \left\{ \left[\left\{ \sum_{i}^{1,n} \boldsymbol{X}_{i}^{(n)}(t) e_{i} \right\}_{n \in \mathbb{N}} \right] \in \left[\left\{ d^{n} \boldsymbol{x}^{(n)} \right\}_{n \in \mathbb{N}} \right] \right\}$$

$$\stackrel{\text{def.}}{=} \left[\left\{ P^{(n)} \right\}_{n \in \mathbb{N}} \right] . \tag{C.9}$$

Then we can prove the following result.

Proposition C.4 (Non Standard Probability Distribution).

The NonStandard Probability Distribution of finding the NonStandard Stochastic Process $\hat{\Psi}$ in a NonStandard Volume Element $\delta\hat{\psi}$ around $\hat{\psi}$ can be expressed as

*Prob.
$$\left\{\hat{\pmb{\Psi}}(t) \in \delta\hat{\pmb{\psi}}\right\} = \check{\mathcal{P}}\left[\hat{\pmb{\psi}},t\right]\delta\hat{\pmb{\psi}}$$

with

$$\check{\mathcal{P}}\left[\hat{\psi},t\right] = \left[\left\{p^{(n)}\left(x^{(n)},t\right)\right\}_{n\in\mathbb{N}}\right]$$
.

Proof:

The result follows from definitions C.10, C.11 and from the definition of product between numbers of ${}^*\mathbb{R}$. Indeed we have starting from equation (C.9) in definition C.11

$$\begin{split} ^* \text{Prob.} \left\{ \hat{\pmb{\Psi}}(t) \in \delta \hat{\pmb{\psi}} \right\} &= \left[\left\{ P^{(n)} \right\}_{n \in \mathbb{N}} \right] \\ &= \left[\left\{ \text{Prob} \left\{ X^{(j)}(t) \in d^j x^{(j)} \right\} \right\}_{j \in \mathbb{N}} \right] \\ &= \left[\left\{ p^{(j)} \left(x^{(j)}, t \right) d^j x^{(j)} \right\}_{j \in \mathbb{N}} \right] \\ &= \left[\left\{ p^{(j)} \left(x^{(j)}, t \right) \right\}_{j \in \mathbb{N}} \right] \left[\left\{ d^j x^{(j)} \right\}_{j \in \mathbb{N}} \right] \\ &= \check{\mathcal{P}} \left[\hat{\pmb{\psi}}, t \right] \delta \hat{\pmb{\psi}} \quad . \end{split}$$

Note that the NonStandard Probability Distribution $\check{\mathcal{P}}\left[\hat{\boldsymbol{\psi}},t\right]$ is a continuos family of functionals on ${}^*\mathcal{E}_{\{e_n\}}\left(\mathbb{R}^d\to{}^*\mathbb{R}\right)$. Now each of the stochastic processes which is used to define the *n*-approximation of the NonStandard Stochastic Process $\hat{\boldsymbol{\Psi}}$, being a process in \mathbb{R}^n , has a probability density, which we called $p^{(n)}\left(\boldsymbol{x}^{(n)},t\right)$, that satisfies the Fokker-Plank equation,

$$\frac{\partial p^{(n)}}{\partial t} = -\sum_{i}^{1,n} \frac{\partial}{\partial x_{i}^{(n)}} \left[a_{i}^{(n)} p^{(n)} \right] + \beta \sum_{i}^{1,n} \frac{\partial^{2} p^{(n)}}{\partial x_{i}^{(n)} 2}$$
 (C.10)

or equivalently the corresponding Langevin equation

$$d\mathbf{X}^{(n)}(t) = \mathbf{a}^{(n)} \left(\mathbf{X}^{(n)}(t), t \right) dt + \mathbf{B}^{(n)}(dt)$$
 (C.11)

 $a^{(n)}(x^{(n)},t)$ being the *drift function*. Define now the following family of operators on ${}^*\mathcal{E}_{\{e_n\}}(\mathbb{R}^d \to {}^*\mathbb{R}).$

Definition C.12 (Forward Drift Operator).

Let us have as usual $\hat{\psi} = \left[\left\{ \sum_{i=1}^{1,n} x_i^{(n)} e_i \right\}_{n \in \mathbb{N}} \right]$. Then the <u>Forward Drift Operator</u> is

where we observe that

$$\left[\mathbf{\mathcal{A}}_t^+ \hat{\mathbf{\psi}} \right] (x) = \left[\left\{ \sum_i^{1,n} a_i^{(n)} \left(x^{(n)}, t \right) e_i(x) \right\}_{n \in \mathbb{N}} \right] \in {}^*\mathbb{R} \quad .$$

In the same way the *Backward Drift Operator* \mathcal{A}_t^- can be defined and it turns out to be related to the *Forward* one by

$$\left(\mathcal{A}_{t}^{+} - \mathcal{A}_{t}^{-}\right)\hat{\psi}(y) = \frac{\beta}{2} \frac{\delta \log \check{\mathcal{P}}\left[\hat{\psi}, t\right]}{\delta \hat{\psi}(y)} \quad . \tag{C.12}$$

Consider now:

$$\left[\mathcal{A}_{t}^{+\hat{-}}\right](\circ)\,\check{\mathcal{P}}\left[\hat{-},t\right] = \left[\left\{\sum_{i=1}^{1,n}a_{i}^{(n)}(-,t)\,e_{i}(y)\right\}_{n\in\mathbb{N}}\right]\left[\left\{p^{(m)}(-,t)\right\}_{m\in\mathbb{N}}\right] \quad . \tag{C.13}$$

This is a sensible expression, which we can easily read as the product of an operator on ${}^*\mathcal{E}_{\{e_n\}} \left(\mathbb{R}^d \to {}^*\mathbb{R}\right)$ times a functional on ${}^*\mathcal{E}_{\{e_n\}} \left(\mathbb{R}^d \to {}^*\mathbb{R}\right)$, which we gave a meaning in

proposition C.3 to. Note that we have indicated the possible arguments in this expression with two different symbols: \circ indicates that there is a dependence from a point in \mathbb{R}^d and - indicates the dependence from the elements of ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$). If we just saturate the - entry we get an element of ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) because the probability distribution gives a NonStandard Real Number and the Operator an element of ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$), which is a vector space; so

$$\left[\mathcal{A}_{t}^{+}\hat{\psi}\right](\circ)\check{\mathcal{P}}\left[\hat{\psi},t\right] = \left[\left\{\sum_{i=1}^{1,n} a_{i}^{(n)}\left(\boldsymbol{x}^{(n)},t\right)e_{i}(y)\right\}_{n\in\mathbb{N}}\right]\cdot\left[\left\{p^{(m)}(\boldsymbol{x}^{n},t)\right\}_{m\in\mathbb{N}}\right] \in {}^{*}\mathcal{E}_{\left\{e_{n}\right\}}\left(\mathbb{R}^{d} \to {}^{*}\mathbb{R}\right)$$
(C.14)

where we assumed that $\hat{\psi}$ is as in the definition above. On the contrary, if we saturate the \circ entry, we get a functional on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$), because when we apply it to $\hat{\psi}$ the result is in ${}^*\mathbb{R}$. Then, since

$$\left[\mathcal{A}_{t}^{+\hat{-}}\right](y)\,\check{\mathcal{P}}\left[\hat{-},t\right] = \left[\left\{\sum_{i}^{1,n}a_{i}^{(n)}(-,t)\,e_{i}(y)\right\}_{n\in\mathbb{N}}\right]\left[\left\{p^{(m)}(-,t)\right\}_{m\in\mathbb{N}}\right] \tag{C.15}$$

is a functional on ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$) it is a complete meaningful operation to apply to it the functional derivative operation of definition C.6. If we first compute the product of the functional $\check{\mathcal{P}}$ with the operator \mathcal{A} , following equation (C.5), we then get

$$\frac{\delta}{\delta\hat{\psi}(y)} \left\{ \left[\mathcal{A}_{t}^{+} \hat{\psi} \right](y) \check{\mathcal{P}} \left[\hat{\psi}, t \right] \right\}$$

$$= \frac{\delta}{\delta\hat{\psi}(y)} \left\{ \left[\left\{ \sum_{i}^{1,n} a_{i}^{(n)} \left(\boldsymbol{x}^{(n)}, t \right) p^{(m)} \left(\boldsymbol{x}^{(m)}, t \right) e_{i}(y) \right\}_{n \in \mathbb{N}} \right] \right\}$$

$$= \left[\left\{ \sum_{j}^{1,n} \frac{\partial}{\partial x_{i}^{(n)}} \left(\sum_{i}^{1,n} a_{i}^{(n)} \left(\boldsymbol{x}^{(n)}, t \right) p^{(n)} \left(\boldsymbol{x}^{(n)}, t \right) e_{j}(y) \right) e_{j}(y) \right\}_{n \in \mathbb{N}} \right]$$

$$= \left[\left\{ \sum_{j}^{1,n} \sum_{i}^{1,n} \frac{\partial}{\partial x_{i}^{(n)}} \left(a_{i}^{(n)} \left(\boldsymbol{x}^{(n)}, t \right) p^{(n)} \left(\boldsymbol{x}^{(n)}, t \right) \right) e_{i}(y) e_{j}(y) \right\}_{n \in \mathbb{N}} \right] . \quad (C.16)$$

Now we can write

$$\begin{split} & \int d^d y \frac{\delta}{\delta \hat{\psi}(y)} \left\{ \left[\mathbf{\mathcal{A}}_t^+ \hat{\psi} \right](y) \, \check{\mathcal{P}} \left[\hat{\psi}, t \right] \right\} \\ & = \int d^d y \left[\left\{ \sum_j^{1,n} \sum_i^{1,n} \frac{\partial}{\partial x_i^{(n)}} \left(a_i^{(n)} \left(\mathbf{x}^{(n)}, t \right) p^{(n)} \left(\mathbf{x}^{(n)}, t \right) \right) e_j(y) \, e_j(y) \right\}_{n \in \mathbb{N}} \right] \end{split}$$

$$= \left[\left\{ \sum_{j=1}^{1,n} \sum_{i=1}^{1,n} \frac{\partial}{\partial x_{i}^{(n)}} \left(a_{i}^{(n)} \left(\boldsymbol{x}^{(n)}, t \right) p^{(n)} \left(\boldsymbol{x}^{(n)}, t \right) \right) \langle e_{j}, e_{j} \rangle_{\mathcal{S}(\mathbb{R}^{d})} \right\}_{n \in \mathbb{N}} \right] \\
= \left[\left\{ \sum_{j=1}^{1,n} \sum_{i=1}^{1,n} \frac{\partial}{\partial x_{i}^{(n)}} \left(a_{i}^{(n)} \left(\boldsymbol{x}^{(n)}, t \right) p^{(n)} \left(\boldsymbol{x}^{(n)}, t \right) \right) \delta_{ij} \right\}_{n \in \mathbb{N}} \right] \\
= \left[\left\{ \sum_{i=1}^{1,n} \frac{\partial}{\partial x_{i}^{(n)}} \left(a_{i}^{(n)} \left(\boldsymbol{x}^{(n)}, t \right) p^{(n)} \left(\boldsymbol{x}^{(n)}, t \right) \right) \right\}_{n \in \mathbb{N}} \right] \tag{C.17}$$

and we also have

$$\frac{\delta \check{\mathcal{P}}\left[\hat{\boldsymbol{\psi}},t\right]}{\delta \hat{\boldsymbol{\psi}}(x)} = \left[\left\{ \sum_{i}^{1,n} \frac{\partial p^{(n)}\left(\boldsymbol{x}^{(n)},t\right)}{\partial x_{i}^{(n)}} e_{i}(x) \right\}_{n \in \mathbb{N}} \right]$$

and

$$\frac{\delta^2 \check{\mathcal{P}}\left[\hat{\boldsymbol{\psi}},t\right]}{\delta \hat{\boldsymbol{\psi}}(x)^2} = \left[\left\{ \sum_{j=1}^{1,n} \sum_{i=1}^{1,n} \frac{\partial^2 p^{(n)}\left(\boldsymbol{x}^{(n)},t\right)}{\partial x_j^{(n)} \partial x_i^{(n)}} e_j(x) e_i(x) \right\}_{n \in \mathbb{N}} \right] .$$

Thus, integrating over the x variable, we get

$$\int d^{d}x \frac{\delta^{2} \tilde{\mathcal{P}}\left[\hat{\psi}, t\right]}{\delta \hat{\psi}(x)^{2}} = \left[\left\{ \sum_{j}^{1,n} \sum_{i}^{1,n} \frac{\partial^{2} p^{(n)}(\boldsymbol{x}^{(n)}, t)}{\partial x_{j}^{(n)} \partial x_{i}^{(n)}} \left\langle e_{i}, e_{j} \right\rangle_{\mathcal{S}(\mathbb{R}^{d})} \right\}_{n \in \mathbb{N}} \right]$$

$$= \left[\left\{ \sum_{j}^{1,n} \sum_{i}^{1,n} \frac{\partial^{2} p^{(n)}(\boldsymbol{x}^{(n)}, t)}{\partial x_{j}^{(n)} \partial x_{i}^{(n)}} \delta_{ij} \right\}_{n \in \mathbb{N}} \right]$$

$$= \left[\left\{ \sum_{i}^{1,n} \frac{\partial^{2} p^{(n)}(\boldsymbol{x}^{(n)}, t)}{\partial x_{i}^{(n)} \partial x_{i}^{(n)}} \right\}_{n \in \mathbb{N}} \right] . \quad (C.18)$$

Now generalising the procedure used so far to define objects in ${}^*\mathcal{E}_{\{e_n\}}$ ($\mathbb{R}^d \to {}^*\mathbb{R}$), we see that the NonStandard Fokker-Planck Equation for the NonStandard Stochastic Process, which proceeding as in the previous definitions is

$$\frac{\partial \left[\left\{ p^{(n)}(\boldsymbol{x}^{(n)}, t) \right\}_{n \in \mathbb{N}} \right]}{\partial t} = \\
= -\left[\left\{ \sum_{i}^{1,n} \frac{\partial}{\partial x_{i}^{(n)}} \left(a_{i}^{(n)}(\boldsymbol{x}^{(n)}, t) p^{(n)}(\boldsymbol{x}^{(n)}, t) \right) \right\}_{n \in \mathbb{N}} \right] + \\
+ \beta \left[\left\{ \sum_{i}^{1,n} \frac{\partial^{2} p^{(n)}(\boldsymbol{x}^{(n)}, t)}{\partial x_{i}^{(n)}} \right\}_{n \in \mathbb{N}} \right] , \tag{C.19}$$

can be rewritten, using equations (C.17-C.18), as

$$\frac{\partial \check{\mathcal{P}}\left[\hat{\boldsymbol{\psi}},t\right]}{\partial t} = -\int d^{d}y \frac{\delta}{\delta \hat{\boldsymbol{\psi}}(y)} \left[\left(\boldsymbol{\mathcal{A}}_{t}^{+} \hat{\boldsymbol{\psi}}(y) \right) \check{\mathcal{P}}\left[\hat{\boldsymbol{\psi}},t\right] \right] + \beta \int d^{d}y \frac{\delta^{2} \check{\mathcal{P}}\left[\hat{\boldsymbol{\psi}},t\right]}{\delta \hat{\boldsymbol{\psi}}(y)^{2}} \quad . \tag{C.20}$$

This is the forward Fokker-Planck equation. The backward Fokker-Planck equation is also satisfied,

$$\frac{\partial \check{\mathcal{P}}\left[\hat{\boldsymbol{\psi}},t\right]}{\partial t} = -\int d^{d}y \frac{\delta}{\delta \hat{\boldsymbol{\psi}}(y)} \left[\left(\boldsymbol{\mathcal{A}}_{t}^{-} \hat{\boldsymbol{\psi}}(y) \right) \check{\mathcal{P}}\left[\hat{\boldsymbol{\psi}},t\right] \right] - \beta \int d^{d}y \frac{\delta^{2} \check{\mathcal{P}}\left[\hat{\boldsymbol{\psi}},t\right]}{\delta \hat{\boldsymbol{\psi}}(y)^{2}}$$
(C.21)

Consider now a stochastic process $\hat{\mathbf{\Psi}}(t)$ on ${}^*\mathcal{E}_{\{e_n\}}\left(\mathbb{R}^d \to {}^*\mathbb{R}\right)$ space, defined as in (C.9). Thus $\hat{\mathbf{\Psi}}$ is a non-standard random variable, defined in terms of the 1, 2, ..., i, \ldots, d dimensional random variables $\mathbf{X}^{(1)}, \mathbf{X}^{(n)}, \ldots, \mathbf{X}^{(i)}, \ldots$ A natural definition then arises:

Definition C.13 (Non Standard Expectation Value).

Let

$$\hat{\boldsymbol{\Psi}}(t) = \left[\left\{ \sum_{j=1}^{n} X_j^{(n)}(t) e_j \right\}_{n \in \mathbb{N}} \right]$$

be a NonStandard Stochastic Process and

$$\hat{\mathcal{R}}\left[\hat{\boldsymbol{\Psi}}(t),t\right] = \left[\left\{R^{(n)}\left(\boldsymbol{X}^{(n)}(t),t\right)\right\}_{n\in\mathbb{N}}\right]$$

a NonStandard Random Variable defined starting from $\hat{\psi}$.

The <u>NonStandard Expectation Value</u> of $\hat{\mathcal{R}}$ is

$$\hat{\mathbf{E}} \left\langle \hat{\mathbf{R}} \left[\hat{\mathbf{\Psi}}(t), t \right] \right\rangle \stackrel{\text{def.}}{=} \left[\left\{ \mathbf{E} \left\langle R^{(n)} \left(\mathbf{X}^{(n)}(t), t \right) \right\rangle \right\}_{n \in \mathbb{N}} \right] \\
= \int_{*\mathcal{E}_{\{e_n\}}(\mathbb{R}^d \to *\mathbb{R})} R \left[\hat{\boldsymbol{\psi}}(t), t \right] \check{\mathcal{P}} \left[\hat{\boldsymbol{\psi}}(t), t \right] \delta \hat{\boldsymbol{\psi}} \quad .$$
(C.22)

After this definition we can at the end define the most important quantities we are looking for.

Definition C.14 (Mean Non Standard Forward/Backward Derivative).

Let

$$\hat{\mathbf{\Psi}}(t) = \left[\left\{ \sum_{j=1}^{n} X_j^{(n)}(t) e_j \right\}_{n \in \mathbb{N}} \right]$$

be a non-standard stochastic process. The $\underline{\textit{Mean NonStandard Forward Derivative}}$ of the NonStandard Stochastic Variable

$$\hat{\mathcal{R}}\left[\hat{\boldsymbol{\Psi}}(t),t\right] = \left[\left\{R^{(n)}\left(X^{(n)}(t),t\right)\right\}_{n\in\mathbb{N}}\right]$$

is given by

$$\mathcal{D}^{+}\hat{\mathcal{R}}\left[\hat{\boldsymbol{\Psi}}(t),t\right] \stackrel{\text{def.}}{=} \lim_{\Delta t \to 0^{+}} \frac{\hat{\mathbf{E}}\left\langle\hat{\mathcal{R}}\left[\hat{\boldsymbol{\Psi}}(t+\Delta t),t+\Delta t\right] - \hat{\mathcal{R}}\left[\hat{\boldsymbol{\Psi}}(t),t\right] \middle| \hat{\boldsymbol{\Psi}}(t)\right\rangle}{\Delta t}$$

Similarly the Mean NonStandard Backward Derivative is

$$\mathcal{D}^{-}\hat{\mathcal{R}}\left[\hat{\mathbf{\Psi}}(t),t\right] \stackrel{\text{def.}}{=} \lim_{\Delta t \to 0^{+}} \frac{\hat{\mathbf{E}}\left\langle\hat{\mathcal{R}}\left[\hat{\mathbf{\Psi}}(t),t\right] - \hat{\mathcal{R}}\left[\hat{\mathbf{\Psi}}(t-\Delta t),t-\Delta t\right] \middle| \hat{\Psi}(t)\right\rangle}{\Delta t}$$

We can now prove the following.

Proposition C.5 (Mean NonStandard Forward/Backward Derivative).

Let

$$\hat{\mathcal{R}}\left[\hat{\boldsymbol{\Psi}}(t),t\right] = \left[\left\{R^{(n)}\left(X^{(n)}(t),t\right)\right\}_{n\in\mathbb{N}}\right]$$

be a NonStandard Stochastic Variable of the NonStandard Stochastic Process

$$\hat{\mathbf{\Psi}}(t) = \left[\left\{ \sum_{j=1}^{n} X_j^{(n)}(t) e_j \right\}_{n \in \mathbb{N}} \right]$$

and

$$\mathcal{D}^{+}\hat{\mathcal{R}}\left[\hat{\mathbf{\Psi}}(t),t
ight]$$

its Mean NonStandard Forward Derivative. This last expression can be rewritten as

$$\mathcal{D}^{+}\hat{\mathcal{R}}\left[\hat{\boldsymbol{\Psi}}(t),t\right] = \left(\frac{\partial}{\partial t} + \int d^{d}x \boldsymbol{\mathcal{A}}_{t}^{+}\hat{\boldsymbol{\psi}}(x)\,\frac{\delta}{\delta\psi(x)} + \beta \int d^{d}x \frac{\delta^{2}}{\delta\hat{\boldsymbol{\psi}}(x)^{2}}\right)\hat{\boldsymbol{\mathcal{R}}}\left[\hat{\boldsymbol{\Psi}}(t),t\right]$$

Similarly, for the Mean NonStandard Backward Derivative we get

$$\mathcal{D}^{-}\hat{\mathcal{R}}\left[\hat{\boldsymbol{\Psi}}(t),t\right] = \left(\frac{\partial}{\partial t} + \int d^{d}x \boldsymbol{\mathcal{A}}_{t}^{-}\hat{\boldsymbol{\psi}}(x)\frac{\delta}{\delta\hat{\boldsymbol{\psi}}(x)} - \beta \int d^{d}x \frac{\delta^{2}}{\delta\hat{\boldsymbol{\psi}}(x)^{2}}\right)\hat{\boldsymbol{\mathcal{R}}}\left[\hat{\boldsymbol{\Psi}}(t),t\right]$$

Proof:

The result is a direct consequence of the definitions of NonStandard Expectation Value of a Non-Standard Random Variable, and of the form of the NonStandard Fokker-Planck equations. Indeed we have

$$\mathcal{D}^{+}\hat{\mathcal{R}}\left[\hat{\mathbf{\Psi}}(t),t\right] = \lim_{\Delta t \to 0^{+}} \frac{\hat{\mathbf{E}}\left\langle\hat{\mathbf{R}}\left[\hat{\mathbf{\Psi}}(t+\Delta t),t+\Delta t\right] - \hat{\mathbf{R}}\left[\hat{\mathbf{\Psi}}(t),t\right]\left|\hat{\Psi}(t)\right\rangle\right.}{\Delta t}$$

$$= \lim_{\Delta t \to 0^{+}} \frac{\left[\left\{ \mathbb{E} \left\langle \hat{\mathcal{R}} \left[\boldsymbol{X}^{(n)}(t + \Delta t), t + \Delta t \right] - \hat{\mathcal{R}} \left[\boldsymbol{X}^{(n)}(t), t \right] \middle| \boldsymbol{X}^{(n)}(t) \right\rangle \right\}_{n \in \mathbb{N}} \right]}{\Delta t}$$

$$= \left[\left\{ \left(\frac{\partial}{\partial t} + \sum_{j}^{1,n} a_{j}^{(n)} \frac{\partial}{\partial x_{j}^{(n)}} + \beta \sum_{j}^{1,n} \frac{\partial^{2}}{\partial x_{j}^{(n)2}} \right) R^{(n)} \left(\boldsymbol{X}^{(n)}(t), t \right) \right\}_{n \in \mathbb{N}} \right]$$

$$= \left(\frac{\partial}{\partial t} + \int d^{d}x \mathcal{A}_{t}^{+} \hat{\psi}(x) \frac{\delta}{\delta \hat{\psi}(x)} + \beta \int d^{d}x \frac{\delta^{2}}{\delta \hat{\psi}(x)^{2}} \right) \hat{\mathcal{R}} \left[\hat{\boldsymbol{\Psi}}(t), t \right] .$$

Of course the same procedure gives the result for the mean non-standard backward derivative.

Appendix D

Loop Derivatives.

D.1 Overview

It may be useful to give a brief review about *Holographic Functional Derivatives*, because they are often confused with *(Ordinary) Functional Derivatives* in view of the formal relation

$$\frac{\delta}{\delta Y^{\mu}(s)} = Y^{\prime\nu}(s) \frac{\delta}{\delta Y^{\mu\nu}(s)} \quad . \tag{D.1}$$

Notwithstanding, there is a basic difference between these two types of operations.

To begin with, an infinitesimal shape variation corresponds to "cutting" the loop C at a particular point, say y, and then joining the two end-points to an infinitesimal loop δC_y . Accordingly,

$$\delta Y^{\mu\nu} [C; y] = \oint_{\delta C_{\mu}} Y^{\mu} dY^{\nu} \approx dY^{\mu} \wedge dY^{\nu} \rangle_{y}$$
 (D.2)

where $dY^{\mu} \wedge dY^{\nu}|_{y}$ is the elementary oriented area subtended by δC_{y} . A suggestive description of this procedure, due to Migdal [6], is that of adding a "petal" to the original loop. Then, we can introduce an "intrinsic distance" between the deformed and initial *Strings*, as the infinitesimal, oriented area variation,

$$\delta\sigma^{\mu\nu}\left[C;y\right] \equiv \, dY^\mu \wedge dY^\nu \big]_y \quad . \label{eq:delta-sigma}$$

"Intrinsic" means that the (spacelike) distance is invariant under reparametrization and/or embedding transformations. Evidently, there is no counterpart of this operation in the

case of point–particles, because of the lack of spatial extension. Note that, while $\delta Y^{\mu}(s)$ represents a *smooth* deformation of the loop $Y^{\mu} = Y^{\mu}(s)$, the addition of a petal introduces a singular "cusp" at the contact point. Moreover, cusps produce infinities in the ordinary variational derivatives but not in the *Holographic Functional Derivatives* [6].

D.2 Functional and Holographic Derivatives

Let us discuss the relation between Functional and Holographic Derivatives. A functional is a "function of functions", which we shall write as F[f]. We adopt the square brackets notation, in order to avoid confusion with the composition law of two functions F(f(x)).

In ordinary calculus, the derivative of a function f(x) is a map, which associates to each point the differential of the function at the given point. The first derivative of a functional is an application which associates to each "point", represented in this case by a function f, a linear functional F'[f], which acts on functions v(x). Thus, we obtain

$$F'[f](v) = \int dx \frac{\delta F[f]}{\delta f(x)} v(x) \quad , \tag{D.3}$$

where

$$\frac{\delta F[f]}{\delta f(x)} = \int dy \frac{\delta F(f(y))}{\delta f(x)} \quad . \tag{D.4}$$

On the other side, the *Holographic Derivative* of a loop, is given by a different, more geometrical procedure. It amounts to evaluate the linear increment of a functional over a closed loop C, when we deform C by adding a small loop (petal) δC in the way described above. To wash away any dependence from the attachment point, i.e. to implement reparametrization invariance, we "average" such a "petal addition" over the whole loop. Formally,

$$\delta F[C;s] = F[C + \delta C(s)] - F[C] = \int_{\delta C} \frac{\delta F[C]}{\delta Y^{\mu\nu}(s)} dY^{\mu} \wedge dY^{\nu}$$
 (D.5)

and

$$\frac{\delta F}{\delta C^{\mu\nu}} \equiv \frac{\delta F}{\delta Y^{\mu\nu}} \equiv \left(\oint_C dl(s)\right)^{-1} \oint_C dl(s) \frac{\delta F[C]}{\delta Y^{\mu\nu}(s)} \quad , \tag{D.6}$$

where, one can use the following equivalent notations F[C], or $F[\sigma^{\mu\nu}]$, and

$$F\left[\sigma^{\mu\nu}\right] \equiv F\left[\sigma^{\mu\nu}\left(s_0, s_1\right)\right] \quad . \tag{D.7}$$

In what follows, we would like to clarify the relation between Functional and Holographic Derivative.

Proposition D.1 (Functional and Holographic Derivatives).

The Functional and Holohgraphic Derivative are related in the following way:

$$\frac{\delta}{\delta Y^{\alpha}(s)} = Y'^{\beta}(s) \frac{\delta}{\delta Y^{\alpha\beta}(s)} \tag{D.8}$$

Proof:

To establish this relationship we recall the definition

$$Y_{\mu\nu} = \oint Y^{\mu} dY^{\nu} = \int_{\Gamma} ds Y^{\mu}(s) Y^{\prime\nu}(s)$$

and derive the following properties of the derivative [4]. The first Functional Derivative of the Holographic Coordinates is

$$\begin{split} \frac{\delta Y^{\mu\nu}\left[C\right]}{\delta Y^{\alpha}(t)} &= \oint_{\Gamma} ds \frac{\delta Y^{\mu\nu}\left[x(s)\right]}{\delta Y^{\alpha}(t)} \\ &= \oint_{\Gamma} ds \left[\frac{\delta Y^{\mu}(s)}{\delta Y^{\alpha}(t)} Y^{\prime\nu}(s) + Y^{\mu}(s) \frac{d}{ds} \left(\frac{\delta Y^{\nu}(s)}{\delta Y^{\alpha}(t)} \right) \right] \\ &= \oint_{\Gamma} ds \left[\delta^{\mu}_{\alpha} \delta(s-t) Y^{\prime\nu}(s) + Y^{\mu}(s) \frac{d}{ds} \left(\delta^{\nu}_{\alpha} \delta(s-t) \right) \right] \\ &= \delta^{\mu}_{\alpha} Y^{\prime\nu}(t) + \left[\delta^{\nu}_{\alpha} Y^{\mu}(s) \delta(s-t) \right]^{\partial \Gamma = \partial \partial \Sigma = \emptyset} + \\ &- \oint_{\Gamma} ds \delta^{\nu}_{\alpha} Y^{\prime\mu}(s) \delta(s-t) \\ &= \delta^{\mu}_{\alpha} Y^{\prime\nu}(t) - \delta^{\nu}_{\alpha} Y^{\prime\mu}(t) \\ &= \delta^{\mu\nu}_{\alpha\beta} Y^{\prime\beta} \quad ; \end{split} \tag{D.9}$$

then the second one turns out to be

$$\begin{split} \frac{\delta^2 \sigma^{\mu\nu} \left[C\right]}{\delta x^\alpha(t) \delta Y^\beta(u)} &= \oint_{\Gamma} ds \frac{\delta^2 Y^{\mu\nu} \left[Y(s)\right]}{\delta Y^\alpha(t) \delta Y^\beta(u)} \\ &= \oint ds \frac{\delta}{\delta Y^\alpha(t)} \left[\delta^\mu_\beta \delta(s-u) Y'^\nu(s) + Y^\mu(s) \delta^\nu_\beta \frac{d}{ds} \left(\delta(s-u)\right) \right] \\ &= \oint ds \left[\delta^\mu_\beta \delta^\nu_\alpha \delta(s-u) \frac{d}{ds} \left(\delta(s-t)\right) + \right. \\ &\left. + \delta^\mu_\alpha \delta^\nu_\beta \delta(s-t) \frac{d}{ds} \left(\delta(s-u)\right) \right] \end{split}$$

$$= \delta^{\mu}_{\beta} \delta^{\nu}_{\alpha} \frac{d}{du} \left(\delta(u - t) \right) + \delta^{\nu}_{\beta} \delta^{\mu}_{\alpha} \frac{d}{dt} \left(\delta(t - u) \right)$$

$$= \delta^{\mu}_{\beta} \delta^{\nu}_{\alpha} \frac{d}{du} \left(\delta(u - t) \right) - \delta^{\nu}_{\beta} \delta^{\mu}_{\alpha} \frac{d}{du} \left(\delta(t - u) \right)$$

$$= \delta^{\mu}_{\beta} \delta^{\nu}_{\alpha} \frac{d}{du} \left(\delta(u - t) \right) - \delta^{\nu}_{\beta} \delta^{\mu}_{\alpha} \frac{d}{du} \left(\delta(u - t) \right)$$

$$= \delta^{\mu\nu}_{\alpha\beta} \frac{d}{du} \left(\delta(u - t) \right) . \tag{D.10}$$

This two result are in agreement with the more general case computed in section A.11, proposition A.13. We can now calculate the relation between *Holographic* and *Functional Derivatives* using a generalization of the *chain rule* to the functional case. We then get

$$\begin{split} \frac{\delta}{\delta Y^{\alpha}(s)} \text{1.} &= \frac{1}{2} \frac{\delta Y^{\mu\nu} \left[C\right]}{\delta Y^{\alpha}(s)} \frac{\delta}{\delta Y^{\mu\nu}(s)} \\ \text{2.} &= \frac{1}{2} \delta^{\mu\nu}_{\alpha\beta} Y'^{\beta}(s) \frac{\delta}{\delta Y^{\mu\nu}(s)} \\ \text{3.} &= Y'^{\beta}(s) \frac{\delta}{\delta Y^{\alpha\beta}(s)} \quad , \end{split}$$

which is the desired result.

Bibliography

- B.Hatfield, Quantum Field Theory of Point Particles and Strings, Addison Wesley, (1992)
- [2] E.Witten, Physics Today, 24, April 1996
- [3] A.Aurilia, E.Spallucci, I.Vanzetta, Phys. Rev. **D50**, 6490, (1994)
- [4] S.Ansoldi, A.Aurilia, E.Spallucci, Phys. Rev. **D53**, 870, (1996)
- [5] T.Eguchi, Phys. Rev. Lett. 44, 126, (1980)
- [6] A.M.Polyakov, Nucl. Phys. **B164**, 171, (1980)
- [7] A.Migdal, Phys. Rep. **102**, 199, (1983)
- [8] H.Kawai, Progr. Theor. Phys. 65, 351, (1981)
- [9] K.Seo and A.Sugamoto, Phys. Rev. **D24**, 1630, (1981)
- [10] R.P.Feynman, A.R.Hibbs, Quantum Mechanics and Path Integrals, Mc Graw-Hill, N.Y., (1965)
- [11] L.F.Abbott, M.B.Wise, Am. J. Phys. 49, 37, (1981)
- [12] G.N.Ord, J. Phys. A 16, 1869 (1983)
 F.Cannata, L.Ferrari, Am. J. Phys. 56, 721 (1988)
- [13] L.Nottale, Int. J. Mod. Phys. A 4, 5047 (1989)
 L. Nottale, Fractal Spacetime and Microphysics, World Scientific, (1992)

- [14] C.Marshall, P.Ramond, Nucl. Phys. **B85**, 375, (1975)
- [15] A. Kyprianidis, Phys. Rep. 155, 1, (1987)
 J.R.Franchi, Found. of Phys. 23, 487, (1993)
- [16] L.Carson, Y.Hosotani, Phys. Rev. **D37**, 1492, (1988)
 C-Lin Ho, L.Carson, Y.Hosotani, Phys. Rev. **D37**, 1519, (1988)
- [17] M.Green, J.Schwartz, Nucl. Phys. B198, 252, 441, (1982)
 D.Gross, J.Harvey, E.Martinec, R.Rhom, Phys. Rev. Lett. 54, 502, (1985)
- [18] K.S.Narain, Phys. Lett. **169B**, 41, (1986)
- [19] A.Aurilia, E.Spallucci, I.Vanzetta, Phys. Rev. **D50**, 6490, (1994)
- [20] S.Ansoldi, A.Aurilia, E.Spallucci. Phys. Rev. **D53**, 870, (1996)
- [21] A.Schild, Phys. Rev. **D16**, 1722, (1977)
- [22] H.A.Kastrup, Phys. Lett. 82B, 237, (1979)
- [23] H.B.Nielsen, P.Olesen, Nucl. Phys. **B57**, 367, 441 (1973)
- [24] A.Aurilia, E.Spallucci, Class. Quantum Grav. 10, 1217, (1993)
 B. Harms, Y. Leblanc, Phys. Lett. B347, 230, (1995)
 Y.Ne'eman, E.Eizenberg, Membranes & Other Extendons, World Scientific Lecture
 Notes in Physics, Vol.39 (1995)
- [25] T.Banks, W.Fischler, S.H.Shenker, L.Susskind, Phys. Rev. **D55**, 5112, (1997)
- [26] N.Ishibashi, H.Kawai, Y.Kitazawa, A.Tsuchiya, Nucl. Phys. B498, 467, (1997)
- [27] C.Castro, Found. Phys. Lett. 10, 3 (1997)
- [28] P.A.M.Dirac Phys. Rev. **74**, 817 (1948)
- [29] A.M. Polyakov Phys.Lett. **B 103**, 207 (1981)
- [30] A.Aurilia, E.Spallucci Phys.Lett. **B282**, 50 (1992)

- [31] R.Gambini, J. Pullin, A.Ashtekar "Loops, knots, gauge theories and quantum gravity", Cambridge Univ. Press 1996
- [32] A.Aurilia, A.Smailagic, E.Spallucci Phys. Rev. **D47**, 2536 (1993)
- [33] S.Ansoldi, A.Aurilia, E.Spallucci Phys. Rev. **D56**, 2352 (1997)
 S.Ansoldi, A.Aurilia, E.Spallucci Chaos Sol. & Fract. **10**, n.1, 1 (1998)
- [34] Kastrup, Phys. Rep.
- [35] Ogielski, Phys. Rev. **D22**, 2407, (1980)
- [36] S.Ansoldi, A.Aurilia, E.Spallucci, Particle Propagator in Elementary Quantum Mechanics, submitted to Am. J. Phys.
- [37] Gantmacher, Differential Equations and Variational Calculus