A Master Equation Approach to the '3 + 1' Dirac Equation

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Abstract

A derivation of the Dirac equation in '3 + 1' dimensions is presented based on a master equation approach originally developed for the '1 + 1' problem by McKeon and Ord. The method of derivation presented here suggests a mechanism by which the work of Knuth and Bahrenyi on causal sets may be extended to a derivation of the Dirac equation in the context of an inference problem.

1 Introduction

The Feynman Checkerboard (or Chessboard) problem[1, 2] is a model from which the Dirac Equation[3] in '1 + 1' dimensions may be derived. Feynman's version of the problem was first published in his textbook on path integral methods[1]. A combinatoric solution was published some years later[4]. That work, along with other combinatoric approaches, are critically reviewed and corrected elsewhere[5]. In addition to combinatoric methods of solution, Gersch[6] published a solution based on the correspondence between the Feynman Checkerboard model and the one-dimensional Ising model. This approach was developed by Ord and coworkers in a series of papers[7, 8, 9, 10]. In particular, the paper by McKeon and Ord[9] introduced a master equation approach for the solution of the '1 + 1' version of the problem which considered contributions from propagation events forwards and backwards in time. By imposing a causality constraint, the '1 + 1' Dirac equation emerged in a fairly straightforward fashion. The impressive variety of methods available for solving the Checkerboard problem is complemented by the range of systems to which the basic model may be applied. As an example, work by Kholodenko[11, 12] showed how the available approaches could be extended to, *e.g.*, polymer dynamics and heterotic strings, arenas seemingly far removed from the original provenance of a textbook exercise.

One shortcoming of the '1 + 1' Feynman Checkerboard model is that it does not account for spin. The history of spin within the context of a path integral approach is a checkered one, to coin a phrase. One possible approach was spearheaded by Schulman[13]. In some sense, the success of the '1 + 1 model is due to the observation that spin seems not to exist as an independent concept in one spatial dimension[2]. Extensions to three dimensions have hitherto all seemed to founder on the difficulty of accommodating the required $\boldsymbol{\sigma} \cdot \mathbf{p}$ operator. The work presented here indicates one possible method for accomplishing this.

An intriguing application of this approach may be found in an extension of the investigations of Knuth and Bahrenyi[14] on causal sets, or posets, to a derivation of special relativity. The duality property of posets[15] suggested to the author that the forward and backward master equation approach exploited by McKeon and Ord[9] might offer some insights into how to extend the poset approach of Knuth and Bahrenyi[14] to a derivation of the '3 + 1' Dirac Equation. This is work in progress. The advantage of the master equation approach is that explicit expressions for real-valued quantities may be obtained at each time step, facilitating comparison to the poset approach. Complex amplitudes appear naturally as a result of a discrete (invertible) Fourier transform from time and space variables to a momentum space representation. At each stage, invertible, unitary transformations allow one to 'follow the breadcrumbs' of a '3 + 1' Dirac equation back to the original, real-valued master equation. These assertions will be validated in the sequel.

2 Master Equation Approach

Define a transition rate a for changing direction on a line. It will be seen that this transition rate may be interpreted as a particle mass. The transition probability in time Δt will then depend on $a\Delta t$. McKeon and Ord[9] then write down a master equation for the probability amplitude of heading towards increasing values $+$ or decreasing values $-$ of x

$$P_\pm(x, t + \Delta t) = (1 - a\Delta t)P_\pm(x \mp \Delta x, t) + a\Delta t P_\mp(x \pm \Delta x, t).$$

As McKeon and Ord showed[9], the master equation for $P_\pm(x, t)$ may be iterated to obtain

$$P_+(x, t) = \sum_{\text{paths}} (1 - a\Delta t)^{n-R}(a\Delta t)^R$$

where n is the number of steps, and R is the number of reversals. Setting $a\Delta t = i\epsilon$ and $(1 - a\Delta t) \approx 1$ reproduces Feynman's expression[1] for the probability amplitude. It is not clear from the analysis, however, that the substitution $a\Delta t = i\epsilon$ is a reasonable thing to do, although it does give the right answer.

In order to address this issue, McKeon and Ord[9] developed a more versatile approach based on coupled master equations, allowing for the possibility of forward and backward propagation subject to a causality constraint. In a notation adapted from Ord and McKeon[10] one may write down coupled master equations for propagation forwards and backwards along the z axis

$$
\begin{aligned}
Z_\pm(z, t) &= [1 - (\zeta_+ + \zeta_-)\Delta t]Z_\pm(z \mp \Delta z, t - \Delta t) \\
&\quad + \zeta_\mp \Delta t \bar{Z}_\pm(z \mp \Delta z, t + \Delta t) + \zeta_\pm \Delta t Z_\mp(z \pm \Delta z, t - \Delta t) \quad (1) \\
\bar{Z}_\mp(z \pm \Delta z, t + \Delta t) &= [1 - (\zeta_+ + \zeta_-)\Delta t]\bar{Z}_\mp(z, t) + \zeta_\mp \Delta t Z_\mp(z, t) + \zeta_\pm \Delta t \bar{Z}_\pm(z, t) \quad (2) \\
Z_\pm(z, t) &= \bar{Z}_\mp(z \pm \Delta z, t + \Delta t), \quad (3)
\end{aligned}
$$

where $Z_\pm$ is the forward time propagation probability towards larger z, i.e., Z_+, or smaller z, i.e., Z_-, $\bar{Z}$ is the backward time propagation probability, and Equation 3 is the causality constraint. In addition, $\zeta_\pm$ is the transition rate for propagation towards larger $(+)$ or smaller

$(-)$ values of z. McKeon and Ord[9] note that iteration of coupled equations can be difficult. Instead, they inferred a differential equation from a short time expansion of Equations 1 and 2 subject to the constraint Equation 3. Performing a Taylor series expansion in $\Delta z = v\Delta t$ and retaining terms only to order Δt, one finds the following differential equation for the difference of $Z_\pm$ and $\bar{Z}_\mp$

$$\pm v \left[\frac{\partial Z_\pm}{\partial z} - \frac{\partial \bar{Z}_\mp}{\partial z} \right] + \left[\frac{\partial Z_\pm}{\partial t} - \frac{\partial \bar{Z}_\mp}{\partial t} \right] + [\zeta_+ + \zeta_-] \left[Z_\pm - \bar{Z}_\mp \right] = [\zeta_\pm - \zeta_\mp] \left[Z_\mp - \bar{Z}_\pm \right] \quad (4)$$

Define $A_\pm(z,t) = \exp\left([\zeta_+ + \zeta_-]\,t\right) (Z_\pm(z,t) - \bar{Z}_\mp(z,t))$. One can interpret the integrating factor $\exp\left([\zeta_+ + \zeta_-]\,t\right)$ as a chemical activity, which controls the 'concentration' of 'up' and 'down' transitions. Substituting the definition of $A_\pm$ into Equation 4, one finds

$$\pm v \frac{\partial A_\pm}{\partial z} + \frac{\partial A_\pm}{\partial t} = (\zeta_\pm - \zeta_\mp)\, A_\pm \quad (5)$$

In order to make further progress, it is useful to Fourier transform Equation 5 to eliminate the z and t derivatives and work in the energy momentum representation. Set $v = c$ and define Fourier amplitudes as follows

$$A \equiv \sum_{p,E} \exp(-i(pz - Et)/\hbar) a_\pm(p, E) \quad (6)$$

$$\equiv \sum_{p,E} \exp(-i(pz + Et)/\hbar) \bar{a}_\pm(p, E) \quad (7)$$

Substituting Equations 6 and 7 into Equation 5 and noting that if the result is to hold for all times over all values of z, then the coefficients $a_\pm(p, E)$ and $\bar{a}_\pm(p, E)$ must satisfy the following constraints

$$\mp icp a_\pm + iE a_\pm = \hbar \left(\zeta_\pm - \zeta_\mp\right) a_\mp \quad (8)$$
$$\mp icp \bar{a}_\pm - iE \bar{a}_\pm = \hbar \left(\zeta_\pm - \zeta_\mp\right) \bar{a}_\mp \quad (9)$$

One may allow p and E to depend on z and t to account for space and time varying potentials, as will be shown. To be consistent in the order of Δt retained, it is sufficient to retain only the first term in derivatives of $\exp(-i(pz \pm Et))$ with respect to z and t, i.e., terms in $\partial p/\partial z$, $\partial p/\partial t$, $\partial E/\partial z$ and $\partial E/\partial t$ and higher are dropped in the limit $\Delta t \to 0$.

3 Interpreting the Equations

The forward time amplitudes $a_\pm(p, E)$ and the time reversed amplitudes $\bar{a}_\mp$ encode the symmetry of the poset used in Knuth and Bahrenyi[14]. Note that $(\zeta_\pm - \zeta_\mp)$ has the same sign as v in Equation 5. One may therefore rewrite $(\zeta_\pm - \zeta_\mp) \equiv \pm\omega$ where ω is a positive definite constant. One may obtain a dimensionally consistent equation by defining $\hbar\omega = mc^2$, the Compton energy of the particle. Jacobson and Schulman[4] have an insightful physical interpretation of mass as being proportional to a rate of making path reversals. That concept seems to be extended here to a notion of inertia, where the probability of making a particular class of path reversal ζ_+ or ζ_- depends on which direction the particle is traveling in.

Multiplying Equations 8 and 9 through by i and setting $c = 1$, one may write down matrix equations for $a_\pm$ and $\bar{a}_\pm$ as follows

$$E \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = [p\sigma_z + m\sigma_y] \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \tag{10}$$

$$E \begin{pmatrix} \bar{a}_+ \\ \bar{a}_- \end{pmatrix} = -[p\sigma_z + m\sigma_y] \begin{pmatrix} \bar{a}_+ \\ \bar{a}_- \end{pmatrix} \tag{11}$$

It is a straightforward exercise to show that the Equations 10 and 11 satisfy the relativistic dispersion relation $E^2 = p^2 + m^2$ in a system of units where $c = 1$.

Using the transformation

$$\Phi = \bar{\Phi}^{-1} = \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}$$

it is possible to rewrite the equations for $\alpha_\pm = a_\pm e^{\mp i3\pi/4}$ and $\bar{\alpha}_\pm = \bar{a}_\pm e^{\pm i3\pi/4}$ in a convenient matrix form

$$\begin{bmatrix} p & -m & 0 & 0 \\ -m & -p & 0 & 0 \\ 0 & 0 & -p & -m \\ 0 & 0 & -m & p \end{bmatrix} \begin{bmatrix} \alpha_+ \\ \alpha_- \\ \bar{\alpha}_+ \\ \bar{\alpha}_- \end{bmatrix} = E \begin{bmatrix} \alpha_+ \\ \alpha_- \\ \bar{\alpha}_+ \\ \bar{\alpha}_- \end{bmatrix} \tag{12}$$

subject to the constraint $E^2 = p^2 + m^2$. Equation 12 is an eigenvalue equation in the form $H\Psi = E\Psi$. Focusing on H, one may apply the following symmetry transformation

$$\Sigma = \Sigma^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to put H into the following form

$$H = \begin{bmatrix} p & 0 & -m & 0 \\ 0 & -p & 0 & -m \\ -m & 0 & -p & 0 \\ 0 & -m & 0 & p \end{bmatrix} \tag{13}$$

Note that Equation 13 is in the form

$$H = \begin{bmatrix} P & -M \\ -M & -P \end{bmatrix}$$

Observe that M is proportional to the unit matrix and is invariant under unitary transformations. Define

$$P \to P'' = \begin{bmatrix} p''_z & 0 \\ 0 & -p''_z \end{bmatrix}$$

and consider a rotation about the y'' axis of the form

$$U_\theta = U_\theta^{-1} = \begin{bmatrix} \cos\theta/2 & \sin\theta/2 \\ \sin\theta/2 & -\cos\theta/2 \end{bmatrix}$$

such that $\tan\theta = p_x'/p_z'$. In this new coordinate system

$$P'' \xrightarrow{\theta} P' = \begin{bmatrix} p_z' & p_x' \\ p_x' & -p_z' \end{bmatrix}$$

Now perform a rotation around the z' axis of the form

$$U_\varphi = \left(U_\varphi^{-1}\right)^* = \begin{bmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{bmatrix}$$

such that $\tan\varphi \equiv p_y/p_x$. In this new coordinate system $p_z' = p_z$. Thus

$$P'' \xrightarrow{\theta} P' \xrightarrow{\varphi} P = \begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix}$$

which properly encodes the operator $\boldsymbol{\sigma} \cdot \mathbf{p}$. The following orthogonal transformation (R^T indicates the matrix transpose of R)

$$R = \left(R^{-1}\right)^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

allows one to write

$$RHR^T = \begin{bmatrix} M & \boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -M \end{bmatrix}$$

which is equivalent to Dirac's time-independent equation in momentum space

$$[\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m]\psi = E\psi \tag{14}$$

where the α and β matrices are in the Dirac representation[3]. Note that the components of ψ are linear combinations of the Fourier amplitudes defined in Equations 10 and 11. The Fourier amplitudes may be traced back, ultimately, to the master equations for the forward and backward transition probability amplitudes defined in Equations 1 and 2 as all of the transformations are invertible. Given that the various α quantities are constantly being rephased and formed into new linear combinations, it is clear that they must be probability amplitudes and not probabilities *per se*. The picture that emerges is that a particle follows a stochastic trajectory in time and position, subject to the causality constraint introduced above. Manipulations of the relevant equations are facilitated by working in the momentum-energy representation.

4   Incorporating a Potential

Given the assumption that p and E may be functions of position and time, one may reinterpret the Fourier coefficients as canonical momenta. In this way, one would incorporate an electromagnetic four-vector as follows

$$p^r \to p^r - \frac{eA^r}{c}, \quad E \to E - eA^0/c,$$

where $r \in \{1, 2, 3\}$.

Under the assumption that

$$p = \sqrt{(p_x - eA_x/c)^2 + (p_y - eA_y/c)^2 + (p_z - eA_z/c)^2}$$

one may use the same series of steps used to derive Equation 14 to show that

$$RHR^T = \begin{bmatrix} M & \boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}) \\ \boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}) & -M \end{bmatrix}$$

where $c = 1$. Note that, expressed as a four-gradient, the four-momentum transforms as a covariant (lowered index) four-vector. With this observation the Dirac equation becomes

$$[\boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A}) + \beta m]\,\psi = (E - eA_0)\psi \tag{15}$$

One can recover the standard Dirac equation in the space time representation by substituting

$$\psi = \sum_{\mathbf{p},t} \exp(i(\mathbf{p} \cdot \mathbf{x} - Et))\Psi(\mathbf{p}, t)$$

in Equation 15. Note the argument of the exponent is a scalar, so if it is valid in one frame of reference, then it is valid in all reference frames. Then for $\psi(\mathbf{x}, t)$ to be valid over all space and time, one requires

$$[\boldsymbol{\alpha} \cdot (-i\boldsymbol{\nabla} - e\mathbf{A}) + \beta m]\,\psi(x, t) = \left(i\frac{\partial}{\partial t} - eA_0\right)\psi(x, t) \tag{16}$$

Equation 16 completes the derivation of the time-dependent Dirac equation in the presence of a potential from the master equation approach of McKeon and Ord[9]. In Feynman slash notation, one has

$$(\not{p} - \not{A} + m)\psi = 0.$$

The derivation given here is fairly simple and straightforward compared to other approaches to the Dirac equation in '3 + 1' dimensions based on the Feynman checkerboard problem[2, and references therein]. This may be due to the observation that the only essential property of spinors used here is that the quadratic expression $p_x^2 + p_y^2 + p_z^2 = p^2$ can be 'bilinearized' into the form

$$\begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix}\begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix} = \left(p_x^2 + p_y^2 + p_z^2\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

a result which has much more to do with analytic geometry than any notions of 'quantum strangeness'. In order to complete the program sketched in the abstract, it is necessary to show how the master equation may be inferred from the poset approach of Knuth and Bahrenyi[14]. As noted, this is work in progress.

Acknowledgements

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