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The Master Field for Rainbow Diagrams and Free Non-Commutative Random Variables

L. Accardi, I.Ya.Aref'eva * and I.V.Volovich †

Centro V. Volterra Università di Roma Tor Vergata
via di Tor Vergata, 00133 – Roma

Abstract

The master field for a subclass of planar diagrams, so called rainbow diagrams, for higher dimensional large N theories is considered. An explicit representation for the master field in terms of noncommutative random variables in the modified interaction representation in the Boltzmannian Fock space is given. A natural interaction in the Boltzmannian Fock space is formulated by means of a rational function of the interaction Lagrangian instead of the ordinary exponential function in the standard Fock space.

*Steklov Mathematical Institute, Vavilov 42, GSP-1, 117966, Moscow, e-mail: Arefeva@arevol.mian.su
†On leave absence from Steklov Mathematical Institute, Vavilov 42, GSP-1, 117966, Moscow

1 Introduction

The problem of summation of all planar diagrams in higher dimensional space-time is still out of reach. Its solution is closely related with problem of finding the leading asymptotics in matrix models for large N and may have important applications to the hadron dynamics [1, 2, 3]. Summation of planar diagrams has been performed only in low dimensional space-time [4].

One can write a closed system of equations for invariant correlation functions in the large N limit, so called the planar Schwinger-Dyson equations, for arbitrary dimension of space-time. In the early 80-s it was suggested [3] that there exists the master field $\Phi(x)$ such that the correlation functions for this field $\Phi(x)$ are equal to the large N limit of invariant correlation functions for matrix models,

$$\lim_{N\to\infty}\frac{1}{N^{1+n/2}} < \mathrm{tr}\,(\,M(x_n)...M(x_1)) > = < 0|\Phi(x_n)...\Phi(x_1)|0 > \tag{1.1}$$

It was suggested that $\Phi(x)$ satisfies the equation [5]

$$[i\frac{\delta S[\Phi]}{\delta\Phi(x)}+2\Pi(x)]|0>=0 \tag{1.2}$$

where S is an action and Π and Φ are the subjects of the relation [5, 6]

$$[\Pi(x),\Phi(y)]=i\delta^{(D)}(x-y)|0><0| \tag{1.3}$$

An operator realization of this algebra proposed in [5] has used the knowledge of all correlation functions. This was considered as an evident drawback of such approach. In that time it was also discussed the problem of finding a generating functional reproducing the planar Schwinger-Dyson equations or equations (1.2), (1.3). It was pointed out that the generating functional cannot depend on one commutative source [7]. It was proposed to use some auxiliary fermionic fields to reproduce the planar Schwinger-Dyson equations [8]. For gauge theories the suitable generating functional is nothing but a functional on paths, i.e. the Wilson loops, that satisfies the Makeenko-Migdal equation [9]. The stochastic equation for large N master fields was proposed in [10].

Recently it has been a reveal of an interest to the problem of constructing the master field for planar graphs. One of origins for this are mathematical works [11, 12, 13] devoted to non-commutative probability, for a review see [14, 15]. Singer has advocated that the essential difficulty of the large number of degrees of freedom in higher dimensional large N matrix models is dealt with finding master fields which live in "large" operator algebras such as the type II_1 factor associated with free group. In the recent paper by Gopakumar and Gross [16] the basic concepts of non-commutative probability have been reviewed and applied to the large N limit of matrix models. They stress that if one can solve a matrix model then one can write an explicit expression for the master field as an operator in a well defined Hilbert space. Douglas also proposed to use ideas of non-commutative probability to the large N stochastic approach [17]. The explicit construction of the master fields for several low-dimensional models including QCD_2 has been given [16]-[19].

However an *effective* (i.e. without the knowledge of correlation fonctions) operator realization for the master field for all planar diagrams in higher dimensional space-time is still unknown. Therefore it is worth to try to find an effective operator realization of

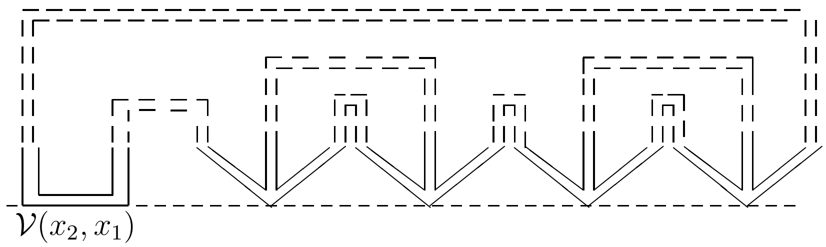

Figure 1: Wick theorem for rainbow diagrams

the master field for some subset of the planar diagrams. The goal of this letter is to construct an explicit operator realization of the master field for *rainbow* graphs. Rainbow graphs form a subset of planar graphs. It is turn out that rainbow correlation functions may be obtained by average of the fields with Boltzman statistics. The construction does not require as input correlation functions. More exactly we show that to get a closed set of equations for correlations functions for a model with an interaction in the Boltzmannian Fock space one has to deal with a modified interaction representation. This new interaction representation involves not the ordinary exponential function of the interaction but a rational function and will be given by the formula

$$< \phi(x_m)...\phi(x_1)\frac{1}{1-g\int dyV_{int}(\phi(y))} > \tag{1.4}$$

The paper is organized as follows. In Sect.2 we present the Schwinger-Dyson equations for rainbow diagrams. Sect.3 contains a necessary information about the Boltzmannian Fock space. We argue also that to get a closed set of the Schwinger-Dyson type of equation we have to deal with the interaction representation in the Boltzmannian Fock space in the form (1.4). We also show that the corresponding Schwinger-Dyson equations reproduce the Schwinger-Dyson equations for rainbow diagrams.

2 Rainbow Diagrams

Let us consider the correlation functions of the form

$$< \mathcal{V}(x_n,...x_1) >= \frac{1}{N^{1+n/2}} < \text{ tr } (M(x_n)...M(x_1)) > \tag{2.1}$$

$< \cdot >$ means

$$< \mathcal{O}(M) >= \frac{1}{Z}\int \mathcal{O}(M)\exp\{-S[M]\}dM, \tag{2.2}$$

where

$$S[M] = \int dx[\frac{1}{2}\text{tr}\,(M(-\triangle+m^2)M) + \frac{g}{4N}\text{tr}\,M^{\,4}(x)] \tag{2.3}$$

Here $M(x)$ is $N\times N$ matrix function. We assume all necessary regularizations. For our purpose it is essential that the regularization is such that the free propagator is

$$< M_{ij}(x)M_{j'i'}(y) >^{(0)}= \delta_{ii'}\delta_{jj'}D(x-y), \tag{2.4}$$

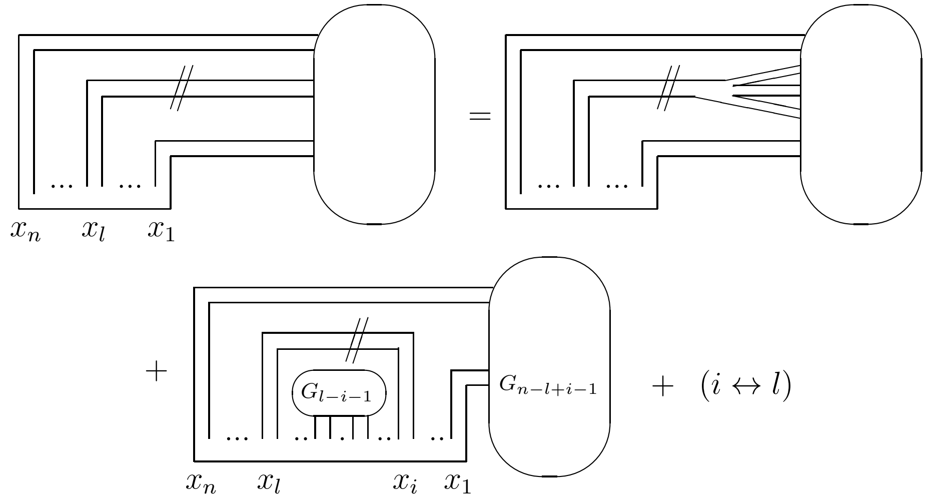

Figure 2: Schwinger-Dyson equation for planar graphs in the large N limit

$$(-\triangle + m^2)_x D(x-y) = \delta^{(D)}(x-y). \tag{2.5}$$

We shall consider the external lines corresponding to global invariant Green functions as the lines corresponding to generalized vertex. The rainbow diagrams for $<\mathcal{V}(x_n, ...x_1)>$ are defined as a part of planar non-vacuum diagrams which are topologically equivalent to the graphs with all vertexes lying on some stright line on the the right of generalized vertex and all propagators lying in the half plane. The rainbow diagramms are illustraited for the M^3- interaction on Fig.1, where vertexies are drawn by solid double lines and all contraction (propagators) by double dash lines; in that follows we will use also solid lines for propagators.

To write down the rainbow Schwinger-Dyson equations one has to select rainbow diagrams from the both hand sides of the planar Schwinger-Dyson equations. In the large N limit the planar Schwinger-Dyson equations have the form

$$\begin{aligned}(-\triangle + m^2)_{x_l} G_n(x_n, ...x_1) &= gG_{n+2}(x_n, ...x_{l+1}, x_l, x_l, x_l, x_{l-1}, ...x_1) \\ &+ \sum_{i<l} \delta(x_l - x_i) G_{l-i-1}(x_{i-1}, ...x_{l+1}) G_{n+i-l-1}(x_n, ...x_{l+1}, x_{i-1}, ...x_1) \\ &+ \sum_{l<i} \delta(x_l - x_i) G_{i-l-1}(x_{l-1}, ...x_{i+1}) G_{n+l-i-1}(x_n, ...x_{i+1}, x_{l-1}, ...x_1),\end{aligned} \tag{2.6}$$

where

$$G_n(x_n, ...x_1) = \lim_{N\to\infty} \frac{1}{N^{1+n/2}} < \mathrm{tr}\,(\,M(x_n)...M(x_1)) > \tag{2.7}$$

The planar Schwinger-Dyson equations (2.6) are written for the case of quartic interaction (2.3) and they are symbolically presented on Fig.2. Now let us consider a modification of the right hand side of (2.6) for correlation functions corresponding to the rainbow diagrams

$$\lim_{N\to\infty} \frac{1}{N^{1+n/2}} < \mathrm{tr}\,(\,M(x_n)...M(x_1)) >_{rb} = W_{n+2}(x_{n+2}, ...x_1) \tag{2.8}$$

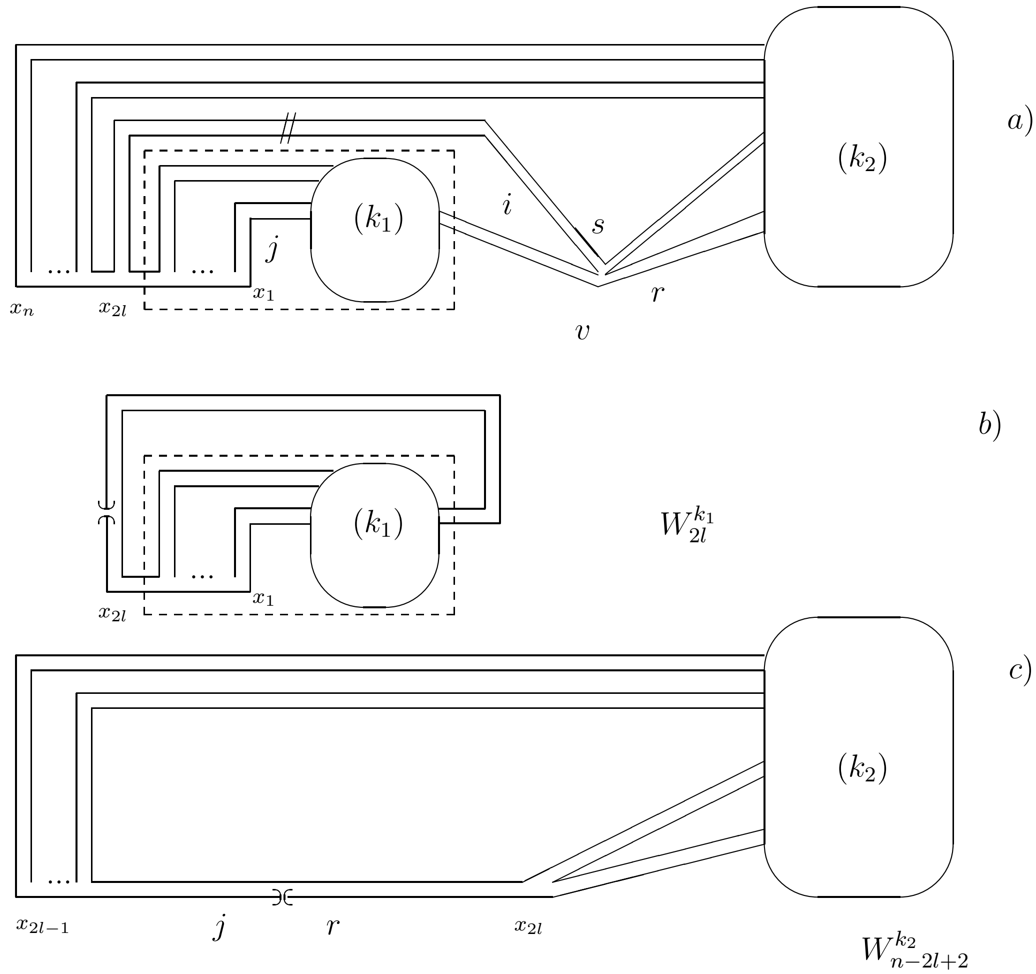

Figure 3: One term in the Schwinger-Dyson equation for rainbow graphs

There are modifications in the term representing the interaction and also in the Schwinger terms. Indeed, let us consider all possible contractions of a given point x_m with a vertex v of the rainbow diagrams. We have to distinguish the cases of odd and even m. For the even $m = 2l$ on the left of this vertex v (see Fig.3a) one has subgraphs corresponding to rainbow diagrams of the correlation function (Fig.3b)

$$< (M(x_{2l})M(x_{2l-1})...M(x_1)(V_{int})^{k_1})_{rj} > \tag{2.9}$$

or

$$< (M^3(x_{2l})M(x_{2l-1})...M(x_1)(V_{int})^{k_1})_{rj} > \tag{2.10}$$

By using

$$< (M(x_i)...M(x_k))_{jj'} >= \frac{\delta_{jj'}}{N} < \text{tr}\,(M(x_i)...M(x_k) > \tag{2.11}$$

these terms reproduce $W_{2l}^{k_1}(x_{2l}, x_{2l-1}...x_1)$ and $W_{2l+2}^{k_1}(x_{2l}, x_{2l}, x_{2l}, x_{2l-1}...x_1)$, respectively. The rest of the diagram Fig.3a corresponds to $W_{n-2l+2}^{k_2}(x_n, ...x_{2l+1}, x_{2l}, x_{2l})$, $k_2 = k-k_1-1$.

One has also to modify contributions from the Schwinger terms, since only one correlator (in our case the correlator corresponding to $G_{n-l+i-1}$) can contain the interaction. Finally we get the following system of equations

$$(-\triangle + m^2)_{x_{2l}} W_n(x_n, ...x_1) = g(W_{2l}(x_{2l}, x_{2l-1}, ...x_1)W_{n-2l+2}(x_n, ...x_{2l+1}, x_{2l}, x_{2l})$$

$$+W_{2l+2}(x_{2l},x_{2l},x_{2l},x_{2l-1},...x_1)W_{n-2l}(x_n,...x_{2l+1}))$$

$$+\sum_{i<2l}\delta(x_{2l}-x_i)W_{n-2l+i-3}(x_n,...x_{2l+1},x_{i-1},...x_1)W^{(0)}_{2l-i+1}(x_{2l-1},...x_{i+1})$$

$$+\sum_{2l<i}\delta(x_{2l}-x_i)W_{n+2l-i-3}(x_n,...x_{i+1},x_{2l-1},...x_1)W^{(0)}_{i-2l+1}(x_{i+1},...x_{2l+1}); \tag{2.12}$$

$$(-\triangle+m^2)_{x_{2l+1}}W_n(x_n,...x_1)=g(W_{2l}(x_{2l},x_{2l},...x_1)W_{n-2l+2}(x_n,...x_{2l++2},x_{2l+1},x_{2l+1},x_{2l+1})$$

$$+W_{2l+2}(x_{2l+1},x_{2l+1},x_{2l},...x_1)W_{n-2l}(x_n,...x_{2l+2}x_{2l-1}))$$

$$+\sum_{i<2l}\delta(x_{2l}-x_i)W_{n-2l+i-3}(x_n,...x_{2l+1},x_{i-1},...x_1)W^{(0)}_{2l-i+1}(x_{2l-1},...x_{i+1})$$

$$+\sum_{2l<i}\delta(x_{2l}-x_i)W_{n+2l-i-3}(x_n,...x_{i+1},x_{2l-1},x_1)W^{(0)}_{i-2l+1}(x_{i+1},...x_{2l+1}) \tag{2.13}$$

Note that for planar correlation functions it is enough to write down the equation differentiated only on x_1 since the function $G(x_n,...x_1)$ is invariant under cyclic permutation. The rainbow correlation functions loss this property and we have to write differential equations for all points x_i. In the the right hand side of (2.12) and (2.13) enters t he free rainbow correlation functions $W^0_n(x_n,...x_1)$. They satisfy the system of differential equations being the subject of equations (2.6) with $g=0$. In distinguish of the interaction case the free rainbow correlators posses the property of invariance under cyclic permutation. This is due to the fact that free planar correlators coincide with free rainbow correlators, i.e. $G^{(0)}_n=W^{(0)}_n$. To find $W^{(0)}_{2m}$ it is enough to apply the Wick theorem as it is shown on Fig.4a.

3 The Master Field for Rainbow Diagrams

3.1 Free Field Theory

Let us consider an algebra generated by operators $A(p)$ and $A^+(p)$ satisfying the relations

$$A(p)A^+(q)=\delta^{(D)}(p-q). \tag{3.1}$$

One can realized this algebra in a space which is an analogue of the usual Fock space [5, 16]. This space is generated by the vacuum vector $|0>$, $A(p)|0>=0$ and n-particle states of n non-identical particles,

$$|p_1,...p_n>=A^+(p_1)...A^+(p_n)|0> \tag{3.2}$$

There is no symmetization or antisymmetrization as in the Bose or Fermi cases. We shall call this Fock space the Boltzmannian Fock space (it is also called the free Fock space). One defines

$$\phi(x)=\phi^+(x)+\phi^-(x)=\frac{1}{(2\pi)^{D/2}}\int\frac{d^Dp}{\sqrt{p^2+m^2}}(A^+(p)e^{ipx}+A(p)e^{-ipx}) \tag{3.3}$$

and therefore

$$<0|\phi(x)\phi(y)|0>=D(x-y)=\frac{1}{(2\pi)^D}\int\frac{d^Dp}{p^2+m^2}e^{ip(x-y)} \tag{3.4}$$

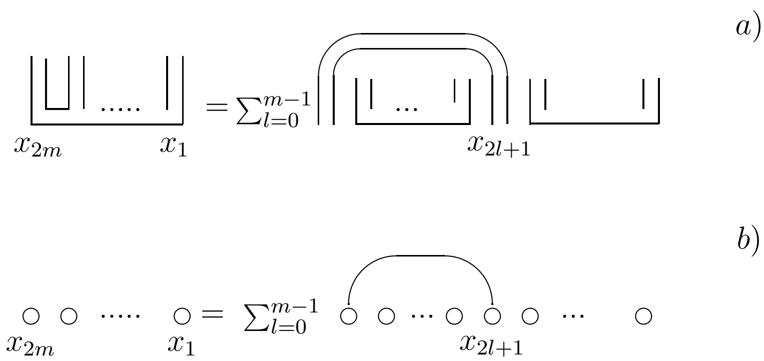

Figure 4: a) Wick theorem for rainbow graphs in free matrix theory; b) Wick theorem in the Bolzmannian Fock space

To calculate the n-point correlation function one has to apply a Boltzmannian Fock space analog of the ordinary Wick theorem. The specific feature of the Wick theorem in this case is that for a given diagram one has not additional symmetry factors related with that an annihilation operator can be contracted with any creation operator on the right. In the Boltzmannian Fock space an annihilation operator can been contracted only with a nearest creation operator on the right. Therefore one sees immediately from the Fig 4b that the correlation function

$$< 0|\phi(x_{2m})...\phi(x_1)|0 >= F^{(0)}_{2m}(x_{2m}, ...x_1) \tag{3.5}$$

satisfies to the same equations as W^0_{2m}, and therefore $F^{(0)}_{2m}(x_{2m}, ...x_1) = W^{(0)}_{2m}(x_{2m}, ...x_1)$, i.e.

$$\lim_{N\to\infty} \frac{1}{N^{1+m}} < \text{ tr}\,(\, M(x_{2m})...M(x_1)) >^{(0)}=< 0|\phi(x_{2m})...\phi(x_1)|0 > \tag{3.6}$$

Let us make a few comments about an operator realization of the algebra (1.3) with π satisfying the requirement

$$\pi(x)|0 >= \frac{i}{2}(- \triangle +m^2)_x\phi(x)|0 > . \tag{3.7}$$

First of all note that the operator algebra (1.3) is not an unique algebra which follows from the free planar Schwinger-Dyson equation. Indeed, one gets the same equation from the operator relation

$$(- \triangle +m^2)_x\phi(x) = \pi(x), \quad [\pi(x), \phi(y)] = -i\delta^D(x - y)|0 >< 0| + K(x, y) \tag{3.8}$$

with $K(x, y)$ being the subject of relations

$$< 0|K(y, x_1)\phi(x_2)...\phi(x_n)|0 > + < 0|\phi(x_1)K(y, x_2)...\phi(x_n)|0 > +... \tag{3.9}$$

$$+ < 0|\phi(x_1)...\phi(x_{n-1})K(y, x_n)|0 >= 0$$

The simplest solution of (3.8) and (3.9) is given by

$$\pi(y) = \frac{i}{2}(- \triangle +m^2)_y[\phi^+(y)|0 >< 0| - |0 >< 0|\phi^-(y)], \tag{3.10}$$

$$K(y,x)=\frac{i}{2}(-\triangle+m^2)_y[-\phi^+(x)\phi^+(y)|0><0|-|0><0|\phi^-(y)\phi^-(x) \tag{3.11}$$
$$+\phi^+(y)|0><0|\phi^-(x)+\phi^+(x)|0><0|\phi^-(y)]$$

This gives a hint to write a following operator realization of the commutation relations (1.3) with π satisfying the requirement (3.7)

$$\pi(y)=\frac{i}{2}(-\triangle+m^2)_y\{\phi^+(y)|0><0|-|0><0|\phi^-(y)+ \tag{3.12}$$
$$\sum_{n=1}^{\infty}\int dz_1(-\triangle+m^2)_{z_1}...\int dz_n(-\triangle+m^2)_{z_n}[\phi^+(z_1)...\phi^+(z_n)\phi^+(y)|0><0|\phi^-(z_n)...\phi^-(z_1$$
$$-\phi^+(z_1)...\phi^+(z_n)|0><0|\phi^-(y)\phi^+(z_n)...\phi^+(z_1)]\}$$

For completeness let us present a known solution of equation (2.6) for $D=0$ and $g=0$, i.e equations

$$\langle\Phi^{2m}\rangle=\sum_{l=0}^{m-1}\langle\Phi^{2l}\rangle\langle\Phi^{2m-2l-2}\rangle \tag{3.13}$$

where $\Phi=a^++a$, $aa^+=1$. Let us denote $c_n=\langle\Phi^{2n}\rangle,\quad c_0=1$. Consider the generating function

$$Z(g)=c_0+c_1g+...+c_ng^n+...=\langle\frac{1}{1-g\Phi^2}\rangle \tag{3.14}$$

One has

$$Z(g)^2=c_0^2+(c_0c_1+c_1c_0)g+...+(c_0c_n+...+c_nc_0)g^n+...$$

From (3.13) one gets

$$Z(g)^2=c_1+c_2g+...+c_{n+1}g^n+...$$

Therefore $Z(g)$ satisfies the equation

$$gZ(g)^2=Z(g)-1.$$

One has to take the following solution of this equation

$$Z(g)=\frac{1-\sqrt{1-4g}}{2g}=1+C_2^1g+...+\frac{1}{n+1}C_{2n}^ng^n+...,$$

from which we get

$$c_n=<\Phi^{2n}>=\frac{1}{n+1}C_{2n}^n=\frac{2n!}{n!(n+1)!}$$

3.2 Interacting Theory

We want to derive the Schwinger-Dyson equations for theory with interaction in the Boltzmannian Fock space. To find the form of interaction let us consider the following correlation functions

$$F_m^{(k)}(x_m,...x_1)=<0|\phi(x_m)...\phi(x_1)(\int dy_1:(\phi(y_1))^4:)...(\int dy_k:(\phi(y_k))^4:)|0>' \tag{3.15}$$

where $\phi(x)$ is the free field (3.3) and $'$ means that we do not take into account the diagrams with vacuum subgraphs. We draw all operators ϕ on the staight line. The operators $\phi(x_i)$

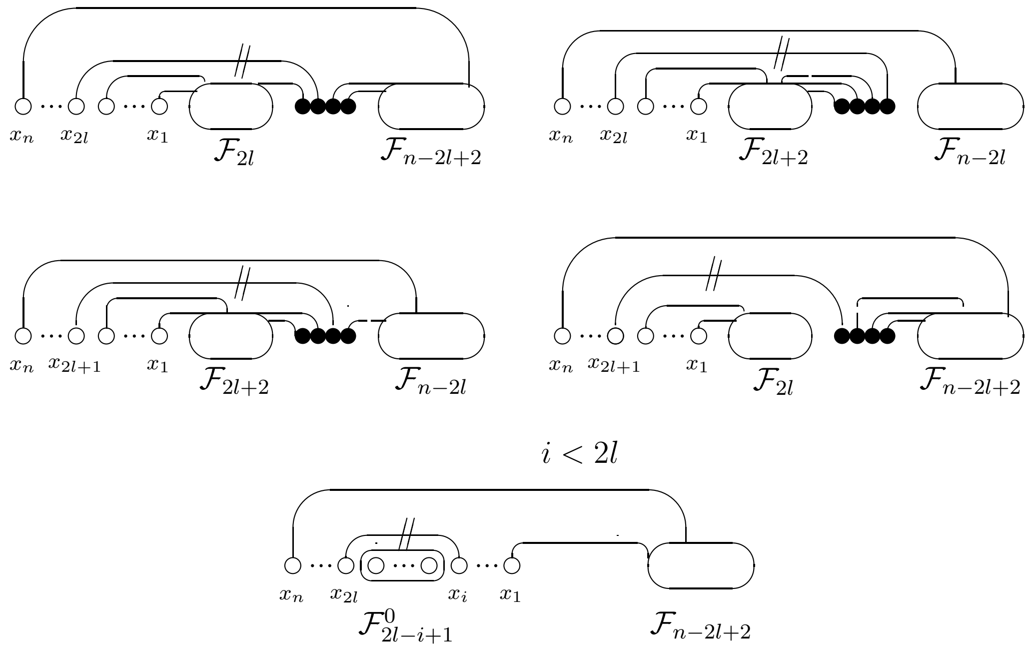

Figure 5: Some diagrams contributed to the Schwinger- Dyson equations in the Boltzmannian Fock space

corresponding to the external lines are represented by the circles and the operators $\phi(y_i)$ corresponding to the interaction vertices are represented by the filled circles on Fig.5.

On Fig.5 we draw all possible contractions of a given external line with a given vertex. As we have mentioned above there is no here additional factors related with symmetry of graphs. Therefore one has not here the standard factor $1/k!$ in the k-th order of perturbation theory. This remark leads to an important observation that to get a set of equations for correlation functions in the Boltzmannian Fock space we have to consider instead of the usual exponential factor $\exp\{V_{int}\}$ the rational function

$$\{1-V_{int}\}^{-1}, \tag{3.16}$$

(compare with $Z(g)$ (3.14)). Therefore we introduce the following correlation functions

$$F_m(x_m,...x_1)=\sum_{k=0} g^k F^{(k)}_m(x_m,...x_1) =<0|\phi(x_m)...\phi(x_1)\frac{1}{1-V_{int}}|0>' \tag{3.17}$$

On Fig.5 $V_{int}=g\int dy_1(\phi(y_1))^4$.

Examining all possible contractions of the point x_{2l} we see that the first two graphs on Fig.5 reproduce the first two sums on the right hand side of (2.12). In the similar way one sees that for odd point the graphs reproduce the first two sum in the the right hand side of (2.13).

Performing the normal ordering in the expression $\phi(x_m)...\phi(x_1)$ we get contributions corresponding to the Schwingers terms in the correlation function $(-\triangle+m^2)_{x_{2l}}F_m(x_m,...x_1)$. This consideration proves that F_m satisfies to the following equations

$$(-\triangle+m^2)_{x_{2l}}F_n(x_n,...x_1)=g(F_{2l}(x_{2l},x_{2l-1},...x_1)F_{n-2l+2}(x_n,...x_{2l+1},x_{2l},x_{2l})$$

$$+F_{2l+2}(x_{2l},x_{2l},x_{2l},x_{2l-1},...x_1)F_{n-2l}(x_n,...x_{2l+1}))$$

$$+\sum_{i<2l}\delta(x_{2l}-x_i)F_{n-2l+i-3}(x_n,...x_{2l+1},x_{i-1},...x_1)F^{(0)}_{2l-i+1}(x_{2l-1},...x_{i+1})$$

$$+\sum_{2l<i}\delta(x_{2l}-x_i)F_{n+2l-i-3}(x_n,...x_{i+1},x_{2l-1},...x_1)F^{(0)}_{i-2l+1}(x_{i+1},...x_{2l+1}); \qquad (3.18)$$

and the similar equations for the x_{2l+1}.

Comparing (3.18) with (2.12) we see that F_m satisfies to the equations for the rainbow diagrams. Therefore we get

$$\lim_{N\to\infty}\frac{1}{N^{1+m/2}} < \text{ tr }(\,M(x_m)...M(x_1))\exp\{\frac{g}{4N}\int dy\,\text{tr }(\,M(y))^4\} >^{\mathbf{F}}_{rb}= \qquad (3.19)$$

$$< 0|\phi(x_m)...\phi(x_1)\frac{1}{1-g\int dy\phi(y)^4}|0 >^{\mathbf{BF}}$$

Symbol $< \, . \, >^{\mathbf{F}}$ denotes the vacuum expectation value in the ordinary Euclidean bosonic Fock space and $< \, . \, >^{\mathbf{BF}}$ denotes nonvacuum diagramms in the Boltzmannian Fock space.

In conclusion, a model of quantum field theory with interaction in the Boltzmannian Fock space has been considered. We have used the new interaction representation with a rational function of the interaction Lagrangian instead of the exponential function in the standard interaction representation. The Schwinger-Dyson equations were derived and it was shown that the perturbation expansion for the model corresponds to the summation of the rainbow diagramms. The quantum field with this interaction can be interpreted as the master field for the rainbow diagramms in the largs N limit matrix model. The construction of the master field is effective in the sense that it is purely algebraic and doesn't require the knowledge of correlation functions of the theory with the interaction. Another aspects of quantum field theory in the Boltzmannian Fock space including the Minkowskian formulation are considered in [20].

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