Master Operators Govern Multifractality in Percolation

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Using renormalization group methods we study multifractality in percolation at the instance of noisy random resistor networks. We introduce the concept of master operators. The multifractal moments of the current distribution (which are proportional to the noise cumulants $C_R^{(l)}(x,x')$ of the resistance between two sites x and x' located on the same cluster) are related to such master operators. The scaling behavior of the multifractal moments is governed exclusively by the master operators, even though a myriad of servant operators is involved in the renormalization procedure. We calculate the family of multifractal exponents $\{\psi_l\}$ for the scaling behavior of the noise cumulants, $C_R^{(l)}(x,x') \sim |x-x'|^{\psi_l/\nu}$, where ν is the correlation length exponent for percolation, to two-loop order.

Percolation [1] is a leading paradigm for disorder. Though it represents the simplest model of a disordered system, it has many applications, e.g., polymerization, porous and amorphous materials, thin films, spreading of epidemics etc. In particular the transport properties of percolation clusters have gained a vast amount of interest over the last decades. Random resistor networks (RRN) are the most prominent model for transport on percolation clusters. Here we discuss RRN in the context of multifractality [2]. We propose the concept of master operators ('operator' in the sense of 'composite field'). Each moment of the multifractal measure (the square of the bond currents) has a master operator as field theoretic counterpart. These master operators are highly and dangerously irrelevant in the renormalization group sense. Therefore, each master operator needs in general a myriad of other irrelevant operators for renormalization. However, the renormalization of these servant operators does not induce their master. It follows, that the servant operators can be neglected in determining the scaling index of their master operator, i.e., one is spared the computation and diagonalization of gigantic renormalization matrices.

Systems where multifractality has been observed include, beside RRN, turbulence, diffusion near fractals, electrons in disordered media, polymers in disordered media, random ferromagnets, and chaotic dissipative systems. Multifractality in RRN is exhibited by the moments of the current distribution measured between two connected terminals x and x'. The multifractal moments of the current distribution are related to the noise cumulants $C_R^{(n)}(x,x')$ of the resistance by Cohn's theorem [3]. At criticality the noise cumulants scale as $C_R^{(n)}(x,x') \sim |x-x'|^{-x_n}$, where the x_n constitute an infinite set of exponents which are not related to each other in a linear way as commonly occurs in critical phenomena under the name of gap scaling. The multifractality manifests itself in the nonlinear dependence of x_n on n. The existence of the set $\{x_n\}$ was proposed by Rammal *et al* [4]. A set of exponents $\{\zeta_{2n}\}$ equivalent to $\{-x_n\nu\}$, where ν is the correlation length exponent for percolation, was also proposed by Arcangelis *et al.* [5].

The field theoretic description of multifractality in RRN was pioneered by Park, Harris and Lubensky (PHL) [6]. Based on an approach by Stephen [7] they formulated a $D{\times}E$-fold replicated Hamiltonian to study the effects of noise in RRN. The contributions to the Hamiltonian leading to multifractal behavior contain powers of replica space gradients analog to powers of real space gradients, which were accounted for as an origin of multifractality by Duplantier and Ludwig [8]. PHL introduced a set of exponents $\{\psi_n\}$ identical to the set $\{-x_n\nu\}$ and calculated it to first order in $\epsilon = 6 - d$, where d denotes the spatial dimension.

Consider a d-dimensional lattice, where bonds between nearest neighboring sites i and j are randomly occupied with probability p or empty with probability $1-p$. Each occupied bond $b = \langle i,j \rangle$ has a conductance σ_b. Unoccupied bonds have conductance zero. The bonds obey Ohm's law $\sigma_{i,j}(V_j - V_i) = I_{i,j}$, where $I_{i,j}$ is the current flowing through the bond from j to i and V_i is the potential at site i. The σ_b are equally and independently distributed random variables with distribution function f, mean $\overline{\sigma}$, and higher cumulants $\Delta^{(n \geq 2)}$ satisfying $\Delta^{(n)} \ll \overline{\sigma}^n$. The noise average is denoted by $\{\cdots\}_f = \int \prod_b d\sigma_b f(\sigma_b) \cdots$ and its nth cumulant by $\{\cdots^n\}_f^{(c)}$. Both kinds of disorder, the random dilution of the lattice and the noise, influence the statistical properties of the resistance $R(x,x')$ of the backbone between two sites x and x'. They are reflected by the noise cumulants

$$C_R^{(n)}(x,x') = \left\langle \chi(x,x') \{R(x,x')^n\}_f^{(c)} \right\rangle_C / \langle \chi(x,x') \rangle_C \; , \tag{1}$$

where $\langle\cdots\rangle_C$ denotes the average over the configurations C of the randomly occupied bonds and $\chi(x,x')$ is an indicator function which is is unity if x and x' are connected and zero otherwise. $C_R^{(n)}$ is related to the $2n$th multifractal moment of the current distribution via Cohn's theorem [3] (cf. [6]),

$$C_R^{(n)}(x,x') = v_n \left\langle \chi(x,x') \sum_b (I_b/I)^{2n} \right\rangle_C \Big/ \langle \chi(x,x')\rangle_C \ , \tag{2}$$

where $v_n = \{(\delta\rho_b)^n\}_f^{(c)}$ is the nth cumulant of the deviation $\delta\rho_b = \rho_b - \overline{\rho}$ of the bond resistance $\rho_b = \sigma_b^{-1}$ from its average $\overline{\rho}$.

Our aim is to determine $C_R^{(n)}$. Hence, the task is to solve the set of Kirchhoff's equations and to perform the averages over the diluted lattice configurations and the noise. It can be achieved by employing the replica technique [7]. PHL introduced $D\times E$-fold replicated voltages $V_x \to \overset{\leftrightarrow}{V}_x = \left(V_x^{(\alpha,\beta)}\right)_{\alpha,\beta=1}^{D,E}$ and $\psi_{\overset{\leftrightarrow}{\lambda}}(x) = \exp\left(i\overset{\leftrightarrow}{\lambda}\cdot\overset{\leftrightarrow}{V}_x\right)$, where $\overset{\leftrightarrow}{\lambda}\cdot\overset{\leftrightarrow}{V}_x = \sum_{\alpha,\beta=1}^{D,E}\lambda^{(\alpha,\beta)}V_x^{(\alpha,\beta)}$ and $\overset{\leftrightarrow}{\lambda}\neq\overset{\leftrightarrow}{0}$. The corresponding correlation functions are defined as

$$G\left(x,x';\overset{\leftrightarrow}{\lambda}\right) = \lim_{D\to 0}\left\langle\left\{\frac{1}{\prod_{\beta=1}^E Z\left(\left\{\sigma_b^{(\beta)}\right\},C\right)^D}\int\prod_j\prod_{\alpha,\beta=1}^{D,E}dV_j^{(\alpha,\beta)}\right.\right.$$
$$\left.\left.\times\exp\left[-\frac{1}{2}P\left(\{\overset{\leftrightarrow}{V}\}\right)+i\overset{\leftrightarrow}{\lambda}\cdot\left(\overset{\leftrightarrow}{V}_x-\overset{\leftrightarrow}{V}_{x'}\right)\right]\right\}_f\right\rangle_C, \tag{3}$$

where $P\left(\{\overset{\leftrightarrow}{V}\}\right) = \sum_{\alpha,\beta=1}^{D,E}\sum_{\langle i,j\rangle}\sigma_{i,j}^{(\beta)}\left(V_i^{(\alpha,\beta)}-V_j^{(\alpha,\beta)}\right)^2$ is the power dissipated on the backbone and Z is the usual normalization.

The integrations in Eq. (3) can be carried out by employing the saddle point method, which is exact in this case. The maximum of the integrand is determined by the solution of Kirchhoff's equations and thus

$$G\left(x,x';\overset{\leftrightarrow}{\lambda}\right) = \left\langle\prod_{\beta=1}^E\left\{\exp\left[-\frac{\vec{\lambda}^{(\beta)2}}{2}R^{(\beta)}(x,x')\right]\right\}_f\right\rangle_C. \tag{4}$$

On defining $K_l\left(\overset{\leftrightarrow}{\lambda}\right) = \sum_{\beta=1}^E\left[\sum_{\alpha=1}^D\left(\lambda^{(\alpha,\beta)}\right)^2\right]^l$ one obtains

$$G\left(x,x';\overset{\leftrightarrow}{\lambda}\right) = \left\langle\exp\left[\sum_{l=1}^\infty\frac{(-1/2)^l}{l!}K_l\left(\overset{\leftrightarrow}{\lambda}\right)\left\{R(x,x')^l\right\}_f^{(c)}\right]\right\rangle_C, \tag{5}$$

i.e., G represents a generating function for $C_R^{(n)}$.

To guarantee that the limit $\lim_{D\to 0}Z^{DE}$ is well defined one switches to voltage variables $\overset{\leftrightarrow}{\theta}$ and current variables $\overset{\leftrightarrow}{\lambda}$ taking discrete values on a $D\times E$-dimensional torus. Upon Fourier transformation in replica space one introduces the Potts spins [9] $\Phi_{\overset{\leftrightarrow}{\theta}}(x) = (2M)^{-DE}\sum_{\overset{\leftrightarrow}{\lambda}\neq\overset{\leftrightarrow}{0}}\exp\left(i\overset{\leftrightarrow}{\lambda}\cdot\overset{\leftrightarrow}{\theta}\right)\psi_{\overset{\leftrightarrow}{\lambda}}(x) = \delta_{\overset{\leftrightarrow}{\theta},\overset{\leftrightarrow}{\theta}_x} - (2M)^{-DE}$ subject to the condition $\sum_{\overset{\leftrightarrow}{\theta}}\Phi_{\overset{\leftrightarrow}{\theta}}(x) = 0$. The effective Hamiltonian

$$H_{\rm rep} = -\ln\left\langle\left\{\exp\left[-\frac{1}{2}P\left(\{\overset{\leftrightarrow}{\theta}\}\right)\right]\right\}_f\right\rangle_C \tag{6}$$

can be expanded in terms of the Potts spins as

$$H_{\rm rep} = -\sum_{\langle i,j\rangle}\sum_{\overset{\leftrightarrow}{\theta},\overset{\leftrightarrow}{\theta}'}K\left(\overset{\leftrightarrow}{\theta}-\overset{\leftrightarrow}{\theta}'\right)\Phi_{\overset{\leftrightarrow}{\theta}}(i)\Phi_{\overset{\leftrightarrow}{\theta}'}(j) \tag{7}$$

where

$$K\left(\overset{\leftrightarrow}{\theta}\right) = \ln\left\{1+\frac{p}{1-p}\exp\left[\sum_{l=1}^\infty\frac{(-1/2)^l}{l!}\Delta^{(l)}K_l\left(\overset{\leftrightarrow}{\theta}\right)\right]\right\}. \tag{8}$$

The Fourier transform $\widetilde{K}\left(\overset{\leftrightarrow}{\lambda}\right)$ of $K\left(\overset{\leftrightarrow}{\theta}\right)$ in turn has the expansion

$$\widetilde{K}\left(\overset{\leftrightarrow}{\lambda}\right) = \tau + w\,\overset{\leftrightarrow}{\lambda}{}^2 + \sum_{l=2}^{\infty} v_l K_l\left(\overset{\leftrightarrow}{\lambda}\right) + \cdots , \tag{9}$$

with $\tau, w \sim \overline{\sigma}^{-1}$, and $v_l \sim \Delta^{(l)}/\overline{\sigma}^{2l}$ being expansion coefficients. We display only the leading parts of the expansion. The terms proportional to $K_l\left(\overset{\leftrightarrow}{\lambda}\right)$ are retained because they are required in extracting the multifractal moments from $G\left(x, x'; \overset{\leftrightarrow}{\lambda}\right)$.

We proceed with the usual coarse graining step and replace the Potts spins $\Phi_{\overset{\leftrightarrow}{\theta}}(x)$ by order parameter fields $\varphi\left(\mathbf{x}, \overset{\leftrightarrow}{\theta}\right)$ which inherit the constraint $\sum_{\overset{\leftrightarrow}{\theta}} \varphi\left(\mathbf{x}, \overset{\leftrightarrow}{\theta}\right) = 0$. We model the corresponding field theoretic Hamiltonian $\mathcal{H}$ in the spirit of Landau as a mesoscopic free energy from local monomials of the order parameter field and its gradients in real and replica space. The gradient expansion is justified since the interaction is short ranged in both spaces for large $\overline{\sigma}$. Purely local terms in replica space have to respect the full $S_{(2M)^{DE}}$ Potts symmetry. After these remarks we write down the Landau-Ginzburg-Wilson type Hamiltonian

$$\mathcal{H} = \int d^d x \sum_{\overset{\leftrightarrow}{\theta}} \left\{ \frac{1}{2}\varphi\, K\left(\nabla, \nabla_{\overset{\leftrightarrow}{\theta}}\right)\varphi + \frac{g}{6}\varphi^3 \right\} , \tag{10}$$

where

$$K\left(\nabla, \nabla_{\overset{\leftrightarrow}{\theta}}\right) = \tau + \nabla^2 + w \sum_{\alpha,\beta=1}^{D,E} \frac{-\partial^2}{\left(\partial \theta^{(\alpha,\beta)}\right)^2} + \sum_{l=2}^{\infty} v_l \sum_{\beta=1}^{E}\left[\sum_{\alpha=1}^{D} \frac{-\partial^2}{\left(\partial \theta^{(\alpha,\beta)}\right)^2}\right]^l . \tag{11}$$

In Eq. (10) we have neglected terms which are irrelevant in the renormalization group sense. τ, w and v_l are now coarse grained analogues of the original coefficients appearing in Eq. (9). Note that $\mathcal{H}$ reduces to the usual $(2M)^{DE}$ states Potts model Hamiltonian by setting $v_l = 0$ and $w = 0$ as one retrieves purely geometrical percolation in the limit of vanishing v_l and w.

Now we analyze the relevance of the v_l. A straightforward scaling analysis reveals that

$$C_R^{(l)}\left((\mathbf{x}, \mathbf{x}')\,; \tau, w, \{v_l\}\right) = w^l f_l\left((\mathbf{x}, \mathbf{x}')\,; \tau, \left\{\frac{v_k}{w^k}\right\}\right) , \tag{12}$$

where f_l is a scaling function. Note that the coupling constants v_k appear only as v_k/w^k. Dimensional analysis of the Hamiltonian shows that $w\,\overset{\leftrightarrow}{\lambda}{}^2 \sim \mu^2$ and $v_k K_k\left(\overset{\leftrightarrow}{\lambda}\right) \sim \mu^2$, where μ is an inverse length scale. Thus, $v_k/w^k \sim \mu^{2-2k}$ and hence the v_k/w^k have a negative naive dimension which decreases drastically with increasing k. This leads to the conclusion that the v_k are highly irrelevant couplings. Though irrelevant, one must not set $v_k = 0$ in calculating the noise exponents. We expand the scaling function f_l yielding

$$C_R^{(l)}\left((\mathbf{x}, \mathbf{x}')\,; \tau, w, \{v_l\}\right) = v_l \left\{ C_l^{(l)} + C_{l+1}^{(l)} \frac{v_{l+1}}{w v_l} + \cdots \right\} , \tag{13}$$

with $C_k^{(l)}$ being expansion coefficients depending on $\mathbf{x}$, $\mathbf{x}'$, and τ. It is important to recognize that $C_{k<l}^{(l)} = 0$ because the corresponding terms are not generated in the perturbation calculation. The first term on the right hand side of Eq. (13) gives the leading behavior. Thus, $C_R^{(l)}$ vanishes upon setting $v_l = 0$ and we cannot gain any further information about $C_R^{(l)}$. In other words, the v_l are dangerously irrelevant in investigating the critical properties of the $C_R^{(l\geq 2)}$.

Our renormalization group improved perturbation calculation comprises the diagrams A to L given in Ref. [10]. Since they are irrelevant, the couplings proportional to v_l have to be treated by inserting [11]

$$\mathcal{O}^{(l)} = -\frac{1}{2}v_l \int d^d p \sum_{\left\{\overset{\leftrightarrow}{\lambda}\right\}} K_l\left(\overset{\leftrightarrow}{\lambda}\right) \phi\left(\mathbf{p}, \overset{\leftrightarrow}{\lambda}\right) \phi\left(-\mathbf{p}, -\overset{\leftrightarrow}{\lambda}\right) \tag{14}$$

into these diagrams. Here $\phi\left(\mathbf{p}, \overset{\leftrightarrow}{\lambda}\right)$ denotes the Fourier transform of $\varphi\left(\mathbf{x}, \overset{\leftrightarrow}{\theta}\right)$. In Schwinger parametrization the computation of the diagrams (including insertions) involves summations of the structure

$$- s_i v_l K_l \left(\overset{\leftrightarrow}{\lambda}_i \right) \exp\left[-w \sum_j s_j \overset{\leftrightarrow}{\lambda}_j^2 \right] . \tag{15}$$

We replace these summations by integrations which can be carried out by completing the squares in the exponential. In the limit $D \to 0$ we obtain

$$- s_i c_i \left(\{s\} \right)^{2l} v_l K_l \left(\overset{\leftrightarrow}{\lambda} \right) + \cdots , \tag{16}$$

where the $c_i \left(\{s\} \right)$ are homogeneous functions of the Schwinger paramters s_j of degree zero. After all, the total contribution of a diagram can be written as

$$I \left(\mathbf{p}^2, \overset{\leftrightarrow}{\lambda} \right) = - v_l K_l \left(\overset{\leftrightarrow}{\lambda} \right) \int_0^\infty \prod_i ds_i \sum_j s_j \, c_j \left(\{s\} \right)^{2l} D \left(\mathbf{p}^2, \{s\} \right) + \cdots . \tag{17}$$

Here $D \left(\mathbf{p}^2, \{s\} \right)$ stands for the integrand one obtains upon Schwinger parametrization of the corresponding diagram in the usual ϕ^3 theory.

The ellipsis in Eq. (17) stands for primitive divergences corresponding to all operators $\mathcal{O}_i^{(l)}$ of the generic form $\overset{\leftrightarrow}{\lambda}{}^{2a} \mathbf{p}^{2b} \phi^n$ having the same or a lower naive dimension than $\mathcal{O}^{(l)}$ $(a + b + n \leq l + 2)$. This myriad of newly generated operators is required as counterterms in the Hamiltonian. Inserting either of the $\mathcal{O}_i^{(l)}$, however, does not generate $\mathcal{O}^{(l)}$. Thus, the renormalization scheme is given by

$$\mathcal{O}^{(l)}_{\text{bare}} = Z^{(l)} \mathcal{O}^{(l)}_{\text{ren}} + \sum_i Z_i^{(l)} \mathcal{O}^{(l)}_{i, \text{ren}} \tag{18}$$

$$\mathcal{O}^{(l)}_{i, \text{bare}} = \sum_i Z_{i,j}^{(l)} \mathcal{O}^{(l)}_{j, \text{ren}} , \tag{19}$$

and one solely needs $Z^{(l)}$ in calculating the scaling index of $\mathcal{O}^{(l)}_{\text{ren}}$. Therefore, we refer to $\mathcal{O}^{(l)}_{\text{ren}}$ as master and to the $\mathcal{O}^{(l)}_{i, \text{ren}}$ as his servants.

Now we briefly illustrate our perturbation calculation at one-loop level. Upon inserting $\mathcal{O}^{(2)}$ into the conducting propagators of diagrams A and B one finds in ϵ-expansion

$$A_{\mathcal{O}^{(2)}} - 2 B_{\mathcal{O}^{(2)}} = g^2 \frac{G_\epsilon}{\epsilon} \tau^{-\epsilon/2} \left\{ \frac{14}{15} v_2 K_2 \left(\overset{\leftrightarrow}{\lambda} \right) + \frac{1}{15} v_2 \left(\overset{\leftrightarrow}{\lambda}{}^2 \right)^2 + \frac{1}{15} \frac{v_2}{w} \overset{\leftrightarrow}{\lambda}{}^2 \mathbf{p}^2 \right\} , \tag{20}$$

where $G_\epsilon = (4\pi)^{(-d/2)} \Gamma \left(1 + \epsilon/2 \right)$. We learn, that not only primitive divergencies proportional to $K_2 \left(\overset{\leftrightarrow}{\lambda} \right)$, but also proportional to $\mathbf{p}^2 \overset{\leftrightarrow}{\lambda}{}^2$ and $\left(\overset{\leftrightarrow}{\lambda}{}^2 \right)^2$ are generated. Thus, one has, at least in principle, to consider $\mathcal{O}^{(2)}$ in liaison with

$$\mathcal{O}^{(2)}_1 = -\frac{1}{2} v_2 \int d^d p \sum_{\{\overset{\leftrightarrow}{\kappa}\}} \left(\overset{\leftrightarrow}{\lambda}{}^2 \right)^2 \phi \left(\mathbf{p}, \overset{\leftrightarrow}{\lambda} \right) \phi \left(-\mathbf{p}, -\overset{\leftrightarrow}{\lambda} \right) ,$$

$$\mathcal{O}^{(2)}_2 = -\frac{1}{2} \frac{v_2}{w} \int d^d p \sum_{\{\overset{\leftrightarrow}{\kappa}\}} \overset{\leftrightarrow}{\lambda}{}^2 \mathbf{p}^2 \phi \left(\mathbf{p}, \overset{\leftrightarrow}{\lambda} \right) \phi \left(-\mathbf{p}, -\overset{\leftrightarrow}{\lambda} \right) ,$$

$$\mathcal{O}^{(2)}_3 = -\frac{1}{18} \frac{v_2}{w} \int d^d p_1 d^d p_2 d^d p_3 \sum_{\{\overset{\leftrightarrow}{\lambda}_1, \overset{\leftrightarrow}{\lambda}_2, \overset{\leftrightarrow}{\lambda}_3\}} \sum_i \overset{\leftrightarrow}{\lambda}_i^2 \phi \left(\mathbf{p}_1, \overset{\leftrightarrow}{\lambda}_1 \right) \phi \left(\mathbf{p}_2, \overset{\leftrightarrow}{\lambda}_2 \right) \phi \left(\mathbf{p}_3, \overset{\leftrightarrow}{\lambda}_3 \right) , \tag{21}$$

where $\sum_i \overset{\leftrightarrow}{\lambda}_i = \overset{\leftrightarrow}{0}$ and $\sum_i \mathbf{p}_i = \mathbf{0}$. The diagrams one obtains upon insertion of these operators are depicted in Fig. 1. A straightforward calculation reveals that neither of the diagrams in Fig. 1 contains primitive divergencies proportional to $K_2 \left(\overset{\leftrightarrow}{\lambda} \right)$. Hence, these diagrams can be neglected in calculating the scaling index of $\mathcal{O}^{(2)}$.
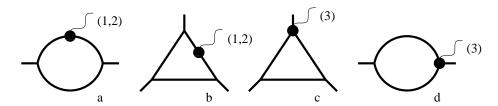
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FIG. 1. One-loop diagrams obtained by inserting (1) $\mathcal{O}_1^{(2)}$, (2) $\mathcal{O}_2^{(2)}$ and (3) $\mathcal{O}_3^{(2)}$ respectively.

The situation is different if one wishes to compute corrections to scaling associated with $\mathcal{O}_1^{(2)}$ [12]. All the diagrams in Fig. 1 enter the correction exponent and thus one has to compute and diagonalize a 3×3 renormalization matrix. This was overlooked by Harris and Lubensky [13] who erroneously neglected diagrams b, c, d and diagram a with the $\mathcal{O}_2^{(2)}$ insertion .

By employing dimensional regularization and minimal subtraction we proceed with standard techniques of renormalized field theory [11]. We calculate $Z^{(l)}$ to two-loop order. From the renormalization group equation and dimensional analysis we deduce that the correlation function G scales at criticality as

$$G\left(\mathbf{x}, \mathbf{x}'; \overset{\leftrightarrow}{\lambda}\right) = |\mathbf{x} - \mathbf{x}'|^{2-d-\eta}\left\{1 + w \overset{\leftrightarrow}{\lambda}^2 |\mathbf{x} - \mathbf{x}'|^{\phi/\nu} + v_l K_l \left(\overset{\leftrightarrow}{\lambda}\right) |\mathbf{x} - \mathbf{x}'|^{\psi_l/\nu} + \cdots\right\} . \tag{22}$$

ν and η are the well known critical exponents for percolation [14]. ϕ is the resistance exponent [15,16], $\phi = 1 + \epsilon/42 + 4\epsilon^2/3087 + O\left(\epsilon^3\right)$. For the noise exponents ψ_l, $l \geq 2$, we obtain here

$$\begin{aligned}
\psi_l = 1 &+ \frac{\epsilon}{7\left(1+l\right)\left(1+2l\right)} + \frac{\epsilon^2}{12348\left(1+l\right)^3\left(1+2l\right)^3} \\
&\times \left\{313 - 672\gamma + l\left\{3327 - 4032\gamma - 8l\left\{4\left(-389 + 273\gamma\right)\right.\right.\right. \\
&+ l\left[-2076 + 1008\gamma + l\left(-881 + 336\gamma\right)\right]\left.\right\}\left.\right\} \\
&- 672\left(1+l\right)^2\left(1+2l\right)^2 \Psi(1+2l)\left.\right\} + O\left(\epsilon^3\right) .
\end{aligned} \tag{23}$$

$\gamma = 0.5772...$ denotes Euler's constant and Ψ stands for the Digamma function. With Eq. (5) the desired scaling behavior of $C_R^{(n)}$ is now readily derived yielding

$$C_R^{(n)} \sim |\mathbf{x} - \mathbf{x}'|^{\psi_n/\nu} . \tag{24}$$

Our result for the noise exponents is in agreement to first order in ϵ with the one-loop calculation by PHL. We point out that Eq. (23) can be analytically continued to $l = 1$ and is in conformity with the result for ϕ cited above. Analytic continuation of ψ_l to $l = 0$ and comparison with the available ϵ-expansion results for D_B [17,18,10] shows that $\psi_0 = \nu D_B$ [5] up to order $O\left(\epsilon^3\right)$. Blumenfeld et $al.$ [19] proved that ψ_l is a convex monotonically decreasing function of l. Note that our result for ψ_l captures this feature for reasonable values of ϵ. It reduces to unity in the limit $l \to \infty$ as one expects from the relation of ψ_∞ to the fractal dimension of the singly connected (red) bonds [5], $\psi_\infty = d_{\text{red}}\nu$, and Coniglio's proof [20] of $d_{\text{red}} = 1/\nu$.

In conclusion, we introduced the concept of master and servant operators and showed that it works consistently as a tool to describe the multifractal properties of RRN by renormalized field theory. We presented the premier two-loop calculation of a family of multifractal exponents. Our result (23) is for dimensions near the upper critical dimension 6 the most accurate analytic estimate of $\{\psi_l\}$ that we know of. It fulfills several consistency checks.

We showed a one-to-one correspondence of the multifractal moments and the master operators. Though a myriad of servant operators is involved in the renormalization of the masters $\mathcal{O}^{(l)}$ the scaling behavior of the lth multifractal moment is governed by $\mathcal{O}^{(l)}$ only. The situation is different for operators which are irrelevant without being masters. The scaling behavior of the related quantities is influenced by the whole bunch of operators generated in the perturbation calculation.

The concept of master and servant operators should have many more applications. Indeed, its applicability might be a general feature of multifractal systems. Hence, our concept could prove to be a key in understanding multifractality, at least from the standpoint of renormalized field theory.

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