Quasithermodynamic Representation of the quantum master equations: its existence , advantages and applications

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We propose a new representation for several quantum master equations in so-called quasithermodynamic form. This representation (when it exists) let one to write down dynamical equations both for diagonal and nondiagonal elements of density matrix of the quantum system of interest in unified form by means of nonequilibrium potential ("entropy") that is a certain quadratic function depending on all variables describing the state. We prove that above representation exists for the general Pauli master equation and for the Lindblad master equation (at least in simple cases) as well. We discuss also advantages of the representation proposed in the study of kinetic properties of open quantum systems particularly of its relaxation to the stationary state.

I. INTRODUCTION

The dynamic equations method is the powerful mathematical tool in the study of a behavior of various complex systems and therefore widely used in physics, chemistry, population biology, and other sciences. This method may be successfully applied both to deterministic systems (as in the case of Newtonian mechanics where it initially arose) and for probabilistic description of physical and nonphysical systems both of classical and quantum nature. For example in quantum theory of open systems (QTOS) and in quantum optics extensively used the method of quantum master equations that describes the evolution through time of density matrix of a system we are interested in.It should be noted that as a rule two large classes of dynamical systems are considered more detail in literature:1) conservative systems and 2) dissipative ones. The first class is described by the equations of the Hamiltonian or Lagrangian type and the second one by equations of gradient type. In both cases there is a single function of a system state (the Hamiltonian or Lagrangian function in the first case and the Lyapunov(dissipative) function in the second case that completely determines its futher evolution. In the present paper we should like to draw attention to another important class of dynamical systems the evolution of which is determined by two independent functions of its state. It should be emphazised here that yet in 1865 R. Clausius one of the fathers of classical thermodynamics had formulated [1] its two basic laws in the following lapidar form:

I.Die Energie der Welt ist constant
II.Die Entropie der Welt strebt einem Maximum zu
(1)

It turns out that classical thermodynamics is although very important but not the only example of similar systems.

In the paper [2] it was proposed to call the systems that satisfy the two conditions Eq. (1) as quasithermodynamic (QT) systems .Note that for various dynamical systems including numerous systems of nonphysical nature the terms "energy" and "entropy" might be understood in fact only in Pickwick sense as certain labels for two functions that satisfy to the Clausius conditions Eq. (1). The main goal of present paper is to demonstrate that certain classes of well- known in physics Markov master equations describe substantially the systems of QT nature in which diagonal ρ_{ii} and nondiagonal ρ_{ik} elements of density matrix represent the set of dynamical variables. The pecularity of these systems mainly consists in the fact that the "energy" conservation in this case reduces merely to the standard normalization condition: $\sum_i \rho_{ii} = 1$. In such situation only one non-trivial task remains open namely to find the explicit form of entropy function that generates the required master equation. Of course the natural question arises here : what advantages such QT representation provides one compared with ordinary dynamical approach and surely we will discuss this question in this paper.

The paper organized as follows. In Section 1 we outline those facts and information about QT systems that are necessary for the understanding of the remainder part of the paper. In Section 2 (which is the central part of the paper) we consider the Pauli master equation and show initially at the simple examples and after that in general case this equation admits the requied QT representation. We discuss also the advantages of QT representation in the study of kinetic properties of quantum open systems particularly in the study of the character of their relaxation to stationary state.Also in this section we consider the case of the Lindblad master equation that widely applyed for the description of Markov quantum systems and show that in simple cases it admits QT representation as

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well.In the Section 3 we consider more special but curious issue closely connected with QT reperesentation namely : how under such representation "entropy" of composite Markov system can be expressed by means of entropies of its subsystems. We demonstrate that even in the case of noninterecting subsystems this link turns out to be nonadditive. It should be noted however that important issue affected in this part are needed more detail elaboration. Now let us pass to the presentation of the results .

II.   PRELIMINARY INFORMATION

Let us start our brief account of the theory of QT systems with the simplest example of the dynamical system that is described by two variables x_1, x_2 and assume it obeys the follows system of equations :

$$\frac{dx_i}{dt} = \varepsilon_{ik} \frac{\partial H}{\partial x_k} \{S, H\}. \qquad (2)$$

In Eq. (2) $H(x_1, x_2)$ and $S(x_1, x_2)$ are two fixed functions of a state (x_1, x_2) of the system , ε_{ik} is standard asymmetric tensor of the second rank and $\{f, g\}$ is the Poisson bracket for two functions $f(x_1, x_2)$ and $g(x_1, x_2)$ that is $\{f, g\} = \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_1}$. It is easy to verify that equations of motion Eq. (2) imply the required two relations: 1) $\frac{dH}{dt} = 0$ and 2) $\frac{dS}{dt} = \{S, H\}^2 \geqslant 0$. Thus the functions H and S satisfy to the Clausius conditions I) and II) and hence can be considered as "energy" and "entropy" of corresponding QT. Similarly one can consider the QT system with three variables x_1, x_2, x_3 the equations of motion of which have the form:

$$\frac{dx_i}{dt} = \varepsilon_{ikl} \frac{\partial H}{\partial x_k} A_l, \qquad (3)$$

where the vector $A_l \equiv \varepsilon_{lmn} \frac{\partial S}{\partial x_m} \frac{\partial H}{\partial x_n}$ and ε_{ikl} is completely antisymmetric tensor of the third rank. The equation Eq. (3) can be written down also in the equivalent form:

$$\frac{dx_i}{dt} = \frac{\partial S}{\partial x_i} \sum_k \left(\frac{\partial H}{\partial x_k}\right)^2 - \frac{\partial H}{\partial x_i} \sum_k \left(\frac{\partial H}{\partial x_k} \frac{\partial S}{\partial x_k}\right). \qquad (4)$$

It should be noted however that Eq. (3) and Eq. (4) are not the most general form of equations for QT systems with three variables. In fact we may add in r.h.s. of the equation Eq. (3) the hamilton- like term- $r\varepsilon_{ikl} \frac{\partial S}{\partial x_k} \frac{\partial H}{\partial x_l}$ (where r is an arbitrary multiplier) without changing its QT nature. So the general form of QS with three variables may be written as:

$$\frac{dx_i}{dt} = \epsilon_{ikl} \frac{\partial H}{\partial x_k} \left(A_l - r \frac{\partial S}{\partial x_l}\right). \qquad (5)$$

The construction of the explicit form of equations of motion for QT systems with more than three variables can be solved in principle by the similar way and we will

return to it in the next part where the general Paulu master equation would be considered as the example of QT system. Now we briefly explain: why for probabilistic systems the equations of motion in many cases admit QT representation. Indeed let us assume that probabilistic system of interest is described by dynamical equations that have the following schematic form:

$$\frac{dp_i}{dt} = F_i \{p_\alpha\}. \qquad (6)$$

In Eq. (6) p_i is a probability to detect the system in the state i $(i = 1, 2...N)$. The above formulation of the problem assumes that normalization condition $\sum_{i=1}^{N} p_i = 1$ and hence the restriction $\sum_{i=1}^{N} F_i = 0$ is satisfied. Thus we conclude that for probabilistic QT systems one of the two functions that determine such systems namely energy must be defined as $H = \sum_{i=1}^{N} p_i$. In respect of the entropy function its existence and concrete form should be established in every individual case separately.

III.   THE PAULI MASTER EQUATION AND ITS QT REPRESENTATION

Now we consider one extensive class of probabilistic systems namely those which behavior can be described by the Pauli-master equation (PME) [3].It is well known the PME describes the evolution through time only diagonal elements of density matrix of N-state open quantum system of interest and may be written down in the next standard form:

$$\frac{dP_i}{dt} = \sum_{k=1}^{N} (W_{ik} P_k - P_i W_{ki}). \qquad (7)$$

In Eq. (7) $P_i(t)$ is a probability to detect the system of interest in quantum state $|i\rangle$, W_{ik} is a probability (per unit time) of transition from state $|k\rangle$ to state $|i\rangle$. It is assumed the set of states $\{|i\rangle\}$ $(i = 1...N)$ forms some complete basis in N state vector space. Comparing Eq. (6) and Eq. (7) we conclude that in the case of the PME $F_i \{P_\alpha\} \equiv \sum_{k=1}^{N} (W_{ik} P_k - P_i W_{ki})$ and the necessary restriction $\sum F_i = 0$ is satisfied automatically. Note that kinetic properties of the system described by the Eq.(7) to a large extent depend on the restrictions imposed on the coefficients W_{ik}. In his prominent paper [4] J.S. Tomsen obtained some important relations connecting symmetry properties of coefficients W_{ik} with the character of relaxation process in corresponding quantum system obeying the PME. For example if these coefficients are symmetric that is $W_{ik} = W_{ki}$ then final probabilities $\{P_i^0\}$ to find the system in its stationary state are identical i.e. the ergodic hypothesis in this case is true. On the

other hand the more weak property of matrix W_{ik} namely its double stochasticity : $\sum_k W_{ik} = \sum_k W_{ki}$ for all indexes i implies the Boltzmann-Shannon entropy function $S_{BS} = - \sum_i P_i \ln P_i$ increases in time($\frac{dS}{dt} \geqslant 0$).Therefore we believe that in symmetric case the PME in fact describes the evolution of the quantum system of interest to its equilibrium state.Note that in present paper we do not assume (until contrary is not approved) any special symmetry properties of coefficients W_{ik}.

Let us begin our study with the simplest case of two state quantum system described by the PME.We write down the PME for the diagonal matrix elements of its density matrix $\widehat{\rho}$, namely $p_1 \equiv \rho_{11}$ and $p_2 \equiv \rho_{22}$ as:

$$\begin{aligned} \frac{dp_1}{dt} &= W_{12}p_2 - p_1 W_{21}, \\ \frac{dp_2}{dt} &= W_{21}p_1 - p_2 W_{12}. \end{aligned} \quad (8)$$

One can directly verify that the system Eq. (8) may be represented in required QT form : $\frac{dp_i}{dt} = \varepsilon_{ik} \frac{\partial H}{\partial p_k} \{S, H\}$ if one takes the "energy" function as $H = p_1 + p_2$ and "entropy" function as $S = - \frac{W_{12}p_1^2}{2} - \frac{W_{21}p_2^2}{2}$. It should be noted if the symmetry condition $W_{12} = W_{21}$ is realized the above entropy function in fact coincides with linear Boltzmann-Shannon entropy that provides the relaxation of the system to its equilibrium state with $p_1^0 = p_2^0 = \frac{1}{2}$ .But for general two state Markov system we obtain the stationary probabilities as : $p_1^0 = \frac{W_{12}}{W_{12}+W_{21}}$ and $p_2^0 = \frac{W_{21}}{W_{12}+W_{21}}$ and ergodic hypothesis does not hold.It is clear that two state case is too simple to shed light on behavior of general N state Markov system but in the next in complexity three-state case which admits the complete inquiry as well all key elements of required general construction can be recognized. So let us consider the case of three-state Markov system more detail. The dynamical equations for such system can be written in the following form

$$\begin{aligned} \frac{dp_1}{dt} &= -(a+b)p_1 + cp_2 + ep_3, \\ \frac{dp_2}{dt} &= ap_1 - (c+d)p_2 + fp_3, \\ \frac{dp_3}{dt} &= bp_1 + dp_2 - (e+f)p_3. \end{aligned} \quad (9)$$

It is clear that there is complete coincidance between the PME Eq. (7) in the case when $N = 3$ and the system Eq. (9). To this end enough to identify a with W_{21}, b with W_{31}, c with W_{12}, d with W_{32}, e with W_{13} and f with W_{23}. Note that in the situation of general N-state PME one has obviously $N(N-1)$ independent coefficients in it and hence in the three state case there are precisely six free parameters. Now let us seek the desired representation of the system Eq. (9) in the required QT form as

$$\frac{dp_i}{dt} = \varepsilon_{ikl} \frac{\partial H}{\partial p_k} \left(A_l - r \frac{\partial S}{\partial p_l} \right), \quad (10)$$

where all indices take values 1, 2, 3, and according to definition the vector $A_l = \varepsilon_{lmn} \frac{\partial S}{\partial p_m} \frac{\partial H}{\partial p_n}$, $H = p_1 + p_2 + p_3$,

and r is scalar factor. Note that in fact Eq. (10) coincides with Eq. (5) but with concrete energy function.

As regards to the "entropy" function S we will seek it in the form of arbitrary quadratic function of basic variables p_i

$$S = \frac{Ap_1^2}{2} + \frac{Bp_2^2}{2} + \frac{Cp_3^2}{2} + \alpha p_1 p_2 + \beta p_1 p_3 + \gamma p_2 p_3. \quad (11)$$

It is easy to see that the transformation $S \rightarrow S + k(p_1 + p_2 + p_3)^2$ does not change equations of motion Eq. (10) and therefore without loss of generality one can put the value of γ to be equal zero. Substituting the expression Eq. (11) into Eq. (10) and comparing the coefficients of identical powers of p_1 , p_2 , p_3 with Eq. (9) one can find all unknown parameters A, B, C, α, β and reconstruct QT representation of the PME Eq. (9) in explicit form. We adduce here only the expression of the parameter r that does not depend on the concrete choice of the entropy function: $r \equiv \frac{1-\varkappa}{1+\varkappa}$, where $\varkappa = \frac{b+c+f}{a+d+e}$. It turns out and this fact is the instructive argument in behalf of QT representation that the condition $r = 0$ results in to monotonic relaxation of the system of interest to its stationary state. Let us prove this statement now. Indeed we can seek the solutions of linear PME Eq. (9) in standard form as $p_i(t) = C_i e^{\lambda t}$ and after a simple algebra we obtain the qubic secular equation for its three roots. One root is precisely equal to zero since the sum $\sum\limits_{i=1}^{i=3} p_i$ is conserved. The other two roots can be obtained from the following quadratic equation:

$$\lambda^2 + \xi\lambda + \eta(a+b+e) - (e-c)(f-a) = 0, \quad (12)$$

where, $\xi = a+b+c+d+e+f, \eta = c+d+f$. The condition that the determinant of this equation less than zero implies two roots of Eq. (12) will be real and negative. Thus the necessary and sufficient condition of monotonic relaxation of open Markov system Eq. (9) to its stationary state may be written as:

$$\xi^2 + 4(e-c)(f-a) - 4\eta(a+b=e) \leqslant 0. \quad (13)$$

Let us use the notation: $k = e-c, l = f-a, m = b-d$ and $\omega = (a+d+e) - (b+c+f)$. In this notation the condition Eq. (13) looks as $\omega^2 + 4\omega(l+m) + 4(l^2+m^2+lm) \leqslant 0$ or in more convinient form as $\left(\sqrt{3}u + \frac{2}{\sqrt{3}}\omega\right)^2 + v^2 - \frac{\omega^2}{3} \leqslant 0$ where $u \equiv l+m$ and $v \equiv l-m$. We see that the boundary of the region in parameter space of the PME Eq. (9), where the nonmonotonic relaxation of its solutions is possible may be represented by the ellipse: $\left(\sqrt{3}u + \frac{2}{\sqrt{3}}\omega\right)^2 + v^2 = \frac{\omega^2}{3}$. Obviously when $\omega = 0$, i.e. the condition $a+d+e = b+c+f$ or $r = 0$ holds, the ellipse degenerates into single point and all solutions of Eq. (9) monotonically decrese in time. On the other hand if $\omega \neq 0$ there is a finite region of parameters (the greater the more ω is) where nonmonotonic behavior of solutions of Eq. (9) is possible. So the result

stated above is proved. Now let us prove the existence of QT representation of the PME in general case of N state open quantum system. It should be noted that the construction of QT representation of the Eq. (9) may be realized with necessary changes in general case as well. We propose here only the outline of a complete proof. So let us consider the N state Markov system described by corresponding PME Eq. (7) with $N(N-1)$ independent coefficients.We claim that required QT representation of this PME may be represented in the next form:

$$\frac{dp_i}{dt} = \varepsilon_{i,i_1...i_{N-1}} \frac{\partial H}{\partial p_{i_1}} A_{i_2...i_{N-1}} + \sum_{\alpha=1}^{M} r_\alpha H_i^{(\alpha)}, \quad (14)$$

where $M = \frac{(N-1)(N-2)}{2}$, p_i has the same sense as in the Eq. (6), $H = p_1 + ... p_N$, $A_{i_2.....i_{N-1}} = \varepsilon_{i,i_1.....i_{N-1}} \frac{\partial S}{\partial p_i} \frac{\partial H}{\partial p_{i_1}}$ ($\varepsilon_{i_1...i_N}$ is completely antisymmetric tensor of N rank) and each of the $\frac{(N-1)(N-2)}{2}$ hamiltonian like terms $H_i^{(\alpha)}$ can be constructed by the following procedure. Firstly let us consider in N dimensional vector space representing all states of the system the subspace (hyperplane) consisting of all states that are orthogonal to the vector $\frac{\partial H}{\partial p_i} = (1, 1...1)$. Obviously this hyperplane has dimensionality $N-1$. Further we choose from the basis of this hyperplane arbitrarily $N-3$ vectors and construct by standard way on these vectors the antisymmetric tensor of $N-3$ rank. Each of this tensors (with accompanying coefficient r_α) enters in the sum in r.h.s. of Eq. (14). It is clear that we obtain in this way precisely $C_{N-1}^{N-3} = C_{N-1}^2$ distinct hamiltonian- like terms and respectively C_{N-1}^2 free parameters r_α. Let us calculate now the total number of free parameters being in our disposal. The entropy function as symmetric quadratic form of N variables gives us $\left[\frac{N(N+1)}{2} - 1\right]$ parameters (we take here into account that S is defined up to the term $k(p_1 + ... p_N)^2$). Besides due to various choice of hamiltonian- like terms we get additional C_{N-1}^2 parameters.Thus as the final result we have $\frac{N(N+1)}{2} - 1 + \frac{(N-1)(N-2)}{2} = N(N-1)$ free parameters that is as much as we need for the initial PME Eq. (7). This simple reasoning in our opinion proves our original statement. . Now we demonstrate that the PME is not the only quantum master equation which admits QT representation.In particular well-known Lindblad master equation (LME) that describes the evolution of arbitrary quantum Markov system admits the similar QT representation at least in the special case of two state systems as well. So let us consider this special case more thoroughly.We start with general Lindblad master equation which has the following form [3]

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_{j=1}^{N} \left[R_j \rho, R_j^+\right] + h.c, \quad (15)$$

(where H some hermitian operator,describing intrinsic dynamics of the open system and operators $\{R_j\}$ are the set of nonhermitian operators that describe the interaction of the system of interest with environment.In the case of two state open systems that we are only interested in here it is convenient to use the Bloch representation for its density matrix,namely $\rho = \frac{1 + \overrightarrow{P} \overrightarrow{\sigma}}{2}$, where $\overrightarrow{P}$ is polarization vector of the state and $\overrightarrow{\sigma} = \{\sigma_k\}$ $(k = 1, 2, 3)$ are standard Pauli matrices. Taking into account that any 2×2 hermitian matrix can be decomposed in Pauli matrices one can write down all operators entering in Eq. (15) in the following form: $H = 2\overrightarrow{h} \ \overrightarrow{\sigma}$, and $R_j = \overrightarrow{A_j} \ \overrightarrow{\sigma} + i\overrightarrow{B_j} \ \overrightarrow{\sigma}$ where $i \equiv \sqrt{-1}$.The set of vectors $\overrightarrow{h}, \overrightarrow{A_j}, \overrightarrow{B_j}$ are completely characterizes the evolution of two state open system within the Lindblad equation approach. Based on the Eq. (15) and using the Bloch representation of input operators we can write down the LME in two state case as equation for the state vector $\overrightarrow{P}$, namely:

$$\frac{d\overrightarrow{P}}{dt} = \left(\overrightarrow{h} \times \overrightarrow{P}\right) + \sum_{j=1}^{N} 2\left(\overrightarrow{A_j} \times \overrightarrow{B_j}\right) \quad (16)$$
$$-\overrightarrow{A_j} \times \left(\overrightarrow{P} \times \overrightarrow{A_j}\right) - \overrightarrow{B_j} \times \left(\overrightarrow{P_j} \times \overrightarrow{B_j}\right).$$

In what follows for the simplicity we will consider the special case when $N = 1$ that is only one operator R in the r.h.s. of Eq. (15) is nonzero. In addition we assume that there is no hamiltonian- like term $\left(\overrightarrow{h} \times \overrightarrow{P}\right)$ in Eq. (16). After these assumptions the simplified version of Eq. (16) takes the form

$$\frac{d\overrightarrow{P}}{dt} = 2\left(\overrightarrow{A} \times \overrightarrow{B}\right) - \overrightarrow{A} \times \left(\overrightarrow{P} \times \overrightarrow{A}\right) \quad (17)$$
$$-\overrightarrow{B} \times \left(\overrightarrow{P} \times \overrightarrow{A}\right).$$

Note that Eq. (17) implies that the Bloch vector of the stationary state is equal to $\overrightarrow{P}_{st} = \frac{2\left(\overrightarrow{A} \times \overrightarrow{B}\right)}{A^2 + B^2}$ and hence if one takes the operator $\widehat{R} = \overrightarrow{A}\overrightarrow{\sigma} + i\overrightarrow{B}\overrightarrow{\sigma}$ so that $\left|\overrightarrow{A}\right| = \left|\overrightarrow{B}\right|$ and $\overrightarrow{A} \cdot \overrightarrow{B} = 0$ the final stationary state would be pure one irrespective of initial state of the system. Now it is easy to verify directly that the LME in vector form Eq. (17) can be represented also in gradient form as:

$$P_i = \frac{\partial S}{\partial P_i}. \quad (18)$$

To this end one need to choose the entropy function as

$$S\left(\overrightarrow{P}\right) = 2\left(\overrightarrow{A} \times \overrightarrow{B}\right) \cdot \overrightarrow{P} - \frac{P^2}{2}\left(A^2 + B^2\right) \quad (19)$$
$$+ \frac{\left(\overrightarrow{A}\overrightarrow{P}\right)^2}{2} + \frac{\left(\overrightarrow{B}\overrightarrow{P}\right)^2}{2}.$$

Thus we have seen that the LME Eq. (17) admits the representation in simple gradient form. Now we show

that specified gradient system with three variables P_i Eq. (18) can be in natural way represented as QT system with 6 variables. So let P_x, P_y, P_z are three components of the Bloch vector satisfying to Eq.(18) . Then by means of this components we introduce six new variables $\{p_i\}$ ($i = 1...6$) according to the rule: $p_1 = \frac{1+P_x}{2}$, $p_2 = \frac{1-P_x}{2}$, $p_3 = \frac{1+P_y}{2}$, $p_4 = \frac{1-P_y}{2}$, $p_5 = \frac{1+P_z}{2}$, $p_6 = \frac{1-P_z}{2}$. Now let us write the following QT system of equations for variables p_i :

$$\frac{dp_i}{dt} = N\varepsilon_{iklmnp}\frac{\partial H_1}{\partial p_k}\frac{\partial H_2}{\partial p_l}\frac{\partial H_3}{\partial p_m}A_{np}, \qquad (20)$$

where $H_1 = p_1 + p_2, H_2 = p_3 + p_4$, $H_3 = p_5 + p_6$ are three integrals of motion for the Eq. (20), ε_{iklmnp} is the antisymmetric tensor of the 6 rank, and tensor A_{np} according to definition is : $A_{np} = N\varepsilon_{nprstu}\frac{\partial S}{\partial p_r}\frac{\partial H_1}{\partial p_s}\frac{\partial H_2}{\partial p_t}\frac{\partial H_3}{\partial p_u}$ where entropy function of three variables $S = S(p_1 - p_2, p_3 - p_4, p_5 - p_6)$ up to notation should be coincide with entropy function Eq. (19), N is normalizing factor. Let us prove now that under appropriate choice of coefficient N the equations Eq. (20) are in fact entirely equivalent to equations Eq. (18). We test this statement only for the first pair of variables of Eq. (20) namely p_1 and p_2. All other equations may be obtain in a similar way as well. Indeed the first equation Eq. (20) in expended form looks as follows:

$$\frac{dp_1}{dt} = 2N\left(A_{64} + A_{45} + A_{36} + A_{53}\right). \qquad (21)$$

Calculating coefficients A_{np} entering in Eq. (21) in explicit form (using the above expressions for them) we obtain : $A_{64} = A_{45} = A_{36} = A_{53} = \frac{\partial S}{\partial p_1} - \frac{\partial S}{\partial p_2}$ and hence $\frac{dp_1}{dt} = 8N^2\left(\frac{\partial S}{\partial p_1} - \frac{\partial S}{\partial p_2}\right)$. By selfsame way we obtain that $\frac{dp_2}{dt} = 8N^2\left(\frac{\partial S}{\partial p_2} - \frac{\partial S}{\partial p_1}\right)$. Hence as a result for component $P_x = p_1 - p_2$ we obtain the equation $\frac{dP_x}{dt} = 16N^2\left(\frac{\partial S}{\partial p_1} - \frac{\partial S}{\partial p_2}\right) = 64N^2\frac{\partial S}{\partial P_x}$. Thus we conclude that if one choose the coefficient N as $\frac{1}{8}$ the equations Eq. (20) and Eq. (18) would be entirely equivalent.Thus we have proved that in two state case the LME admits the required QT representation. Relating to the possibility to represent the LME in more general situation note that this issue is rather delicate. We believe that this possibility is in fact closely connected with the problem of existence of hidden variables for system under consideration and therefore goes far beyond the scope of this paper.

IV.   THE SIMPLE COMPOSITE MARKOV SYSTEM CONSISTING OF TWO INDEPENDENT SUBSYSTEMS AND SUBEXTENSIVITY OF ITS ENTROPY FUNCTION.

In this part we consider one important issue closely connected with QT representation of quantum master equations namely: how the "entropy" function of composite quantum system satisfying to the PME may be expressed by means of entropies of its subsystems.In present paper we study only the simplest example of this problem namely we study the four state composite Markov system described by the PME that consists of a pair of two state independent subsystems . We show that even in this case required connection turns out to be subextensive that is to a certain extent is the same as in the nonextensive statistical thermodynamics [6]. We start with two statistical independent quantum systems A and B both of which may be described by the PME ,namely, for system A

$$\begin{aligned}\frac{dp_1}{dt} &= -ap_1 + bp_2, \\ \frac{dp_2}{dt} &= ap_1 - bp_2\end{aligned} \qquad (22)$$

and for system B

$$\begin{aligned}\frac{dq_1}{dt} &= -cq_1 + dp_2, \\ \frac{dq_2}{dt} &= cq_1 + dq_2.\end{aligned} \qquad (23)$$

In what follows we will consider only the situation when $a = b$ and $c = d$ because as we will see in this case QT representation of the PME for the composite system C consisting of subsystems A and B does not contain hamiltonian like terms. Note that in this case the entropies of subsystems A and B can be chosen in the next form:$S_A = -\frac{a(p_1-p_2)^2}{4}$ and $S_B = -\frac{c(q_1-q_2)^2}{4}$. Let us introduce now the probabilities of populations W_α ($\alpha = 1,..4$) for the four states of composite system C. In view of the statistical independence A and B one can express these probabilities W_α by means of probabilities p_i and q_k as follows : $W_1 = p_1q_1, W_2 = p_1q_2, W_3 = p_2q_1, W_4 = p_2q_2$. The conditions: $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$ imply that $\sum\limits_{\alpha=1}^{\alpha=4} W_\alpha = 1$. In addition dynamical equations Eq. (22) and Eq. (23) immediately imply the following equations of motion for probabilities W_α

$$\begin{aligned}\frac{dW_1}{dt} &= -(a+c)W_1 + cW_2 + aW_3, \\ \frac{dW_2}{dt} &= cW_1 - (a+c)W_2 + aW_4, \\ \frac{dW_3}{dt} &= aW_1 - (a+c)W_3 + cW_4, \\ \frac{dW_4}{dt} &= aW_2 + cW_3 - (a+c)W_4.\end{aligned} \qquad (24)$$

It is obvious that Eq. (24) is the PME for the composite system C and therefore as it was proved above admit QT representation with some entropy function S. Our task is to find explicit form of this function and to state its connection with subsystems entropies S_A and S_B. Let us seek the required QT representation of Eq. (24) in the form

$$\frac{dW_i}{dt} = N\varepsilon_{iklm}\frac{\partial H}{\partial W_k}A_{lm}. \qquad (25)$$

In Eq. (24) according to definition the antisymmetric tensor $A_{lm} = N\varepsilon_{lmnp}\frac{\partial S}{\partial W_n}\frac{\partial H}{\partial W_p}$, ε_{iklm} is completely antisymmetric tensor of fourth rank, $H =$

$\sum_{i=1}^{i=4} W_i$, N - some normalizing factor and S is the entropy function of the composite system C which leads to the equations coinciding with Eq. (24). Let us take the entropy function S in the following subextensive form : $S = S_A + S_B - \lambda S_A S_B$ where the factor λ would be specified. Taking into account that $S_A = -\frac{a}{4}(p_1 - p_2)^2 = -\frac{a}{4}(W_1 + W_2 - W_3 - W_4)^2$, $S_B = -\frac{c}{4}(q_1 - q_2)^2 = -\frac{c}{4}(W_1 + W_3 - W_2 - W_4)^2$ and $S_A S_B = \frac{ac}{16}(W_1 + W_4 - W_2 - W_3)^2$ one can obtain the expression for the entropy function S as:

$$
\begin{aligned}
S = & -\frac{a}{4}(W_1 + W_2 - W_3 - W_4)^2 \\
& -\frac{c}{4}(W_1 + W_3 - W_2 - W_4)^2 \\
& -\lambda \frac{ac}{16}(W_1 + W_4 - W_2 - W_3)^2 .
\end{aligned} \tag{26}
$$

We prove that by an appropriate choice of factor λ QT representation of composite systems that is Eq. (25) coincides with the PME Eq. (24). We give here the proof only for the first equation of Eq. (24), all others equations can be obtained in a similar way. So, the first of Eq. (24) in extended form reads as:

$$
\frac{dW_1}{dt} = 2N(A_{23} + A_{34} + A_{42}). \tag{27}
$$

According to definition all tensors A_{lm} entering in Eq. (27) can be easily calculated and are equal to: $A_{23} = N\left(\frac{\partial S}{\partial W_1} - \frac{\partial S}{\partial W_4}\right)$, $A_{34} = N\left(\frac{\partial S}{\partial W_1} - \frac{\partial S}{\partial W_2}\right)$ and $A_{42} = N\left(\frac{\partial S}{\partial W_1} - \frac{\partial S}{\partial W_3}\right)$. Substituting these expressions in Eq. (27) we obtain that

$$
\frac{dW_1}{dt} = 2N^2\left(4\frac{\partial S}{\partial W_1} - \sum_{i=1}^{i=4}\frac{\partial S}{\partial W_i}\right). \tag{28}
$$

It is easy to see that expression Eq. (26) implies that $\sum_{i=1}^{i=4}\frac{\partial S}{\partial W_i} = 0$ and if one chooses the factor N so that $8N^2 = 1$ the equation for probability W_1 takes standard dissipative form:

$$
\frac{dW_1}{dt} = \frac{\partial S}{\partial W_1}. \tag{29}
$$

Using the expression Eq. (26) for the entropy function of the composite system it is not difficult to verify that if one choose the factor λ so that $\frac{ac\lambda}{8} = \frac{a+c}{2}$ the first equation of Eq. (24) coincides with Eq. (29) and the required result is proved. It should be noted here that in nonextensive thermodynamics (see i.e [6]) the following general expression for the nonextensive entropy takes place: $S_{AB} = S_A + S_B + \frac{1-q}{k}S_A S_B$ where k is the Boltzmann constant and q is the factor of nonextensivity. Note that superextensivity, extensivity and subextensivity occurs when $q \leqslant 1$, $q = 1$, $q \geqslant 1$ respectively. In our case $\left(q = 1 + \frac{k(a+c)}{4}\right)$ and hence in this model we are dealing with subextensive situation.

In conclusion let us express another general reason on behalf of possible advantage of QT representation of quantum master equations. It is well-known that the formulation of principles of mechanics and field theory in Hamiltonian or Lagrangian form enables one to describe dynamics of the complex conservative systems on the basis of knowledge their parts behavior and the symmetry of interaction between all their parts. Similarly we believe that the deeper understanding of QT representation of master equations let one construct more and more complex probabilistic models based on the collection of certain elementary constituents that admit detail analysis.

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