High energy limits of various actions

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Preface

When people ask me what I do, or what kind of subjects I study, I have sometimes ironically answered that I do abstract art. This thesis does not have practical applications, in the sense that it can tell you how to wash your clothes, make coffee, or help you to calculate numbers that can in turn be measured with some apparatus. The applications are theoretical – the thesis casts some light on mathematical relations within a theoretical framework. And the results that are obtained are purely analytical.

Some people demand from physics that it shall be concerned with observable things. With such a narrow view of the field this thesis can hardly be called a work of physics. But at the same time, I would guess that mathematicians would be horrified by the lack of stringency, so calling it a piece of mathematics is no less dangerous.

It is on this basis that I like to call it abstract art. Its value lies in the aesthetics. One kind of art has an immediate value because of its beauty to the eye or ear, or because of its direct associations. Another kind of art is more indirect. Its value may require some background information, and lies more in the meaning than in the sensation. For me the beauty in the abstract art of physics is of this kind, and lies in the understanding of certain mathematical relations – that in turn are somehow related to the world we live in.

This aesthetics is my prior motivation for the interest in theoretical physics. I cannot claim to do what I do because I think it will make the world a better place. It is beauty more than importance that attracts me.

For people not so fascinated by art and aesthetical values, I want to emphasize that this concerns only my subjectice motivation. What is actually done, the derivations and interpretations, shall of course satisfy the standards of reliable science.

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Chapter 1

Introduction

The goal of this thesis is twofold. First, it is meant to give a thorough presentation of two methods for deriving tensionless limits of strings, and the analogue in other models. Second, the applicability of these methods are investigated by explicitly going through the calculations for a variety of models.

We start by going through some background theory in this introductory chapter about constrained Hamiltonian systems, and different symmetry considerations. A general result concerning the Hamiltonian of diff invariant theories is derived in section 1.4.2. Then the methods are presented within the general picture, before the simplest examples are given in chapter 2. In the subsequent chapters we study D-branes, rigid strings, general relativity and take a brief look on Yang-Mills and Chern-Simons theory. A summary and concluding remarks are finally given in chapter 7.

Throughout this thesis we use the standard summation notation $a^i b_i \equiv \sum_i a^i b_i$, and natural units where c = 1.

1.1 Motivations

The question naturally arises of why we should bother with the kind of limits that we are going to study in this thesis. Do not particles have mass, and strings tension? The question is of course crucial, and worthy of an answer. If we were not able to give one, the present undertaking would seem like a meaningless activity beyond any interest apart from the purely academical one.

But those who are anxious can safely relax: There is a motivation. Actually, there are several different aspects of these limits that are of interest.

The limits represent physical situations If we start from the action of a massive particle and derive the massless limit, we will end up with a theory for massless particles describing e.g. photons (if we disregard spin and charge). And their existence at least is without doubt. Whether tensionless strings also have direct physical

applications is not as easy to tell since strings themselves are not yet sufficiently well understood.

But if we want to test the existence of tensionless strings (or analogous limits of other theories) we need to have a theory for them. And the methods we will use are ways to arrive at such theories.

High energy limit For high energies the mass of a particle becomes unimportant compared to its kinetic energy. Therefore the massless limit gives an approximation of the behaviour at high energies. The same is supposed to be true for strings: The tensionless limit may be viewed as a high energy limit. And high energy physics is important for several reasons. When we study systems at a very small scale we unavoidably (by the Heisenberg principle) have systems at high energy. Also, at the earliest stages of the evolution of the universe, the energy density was very high. An understanding of the "childhood" of the universe thus requires an understanding of high energy physics.

Conformal invariance Another characteristic of these limits is that they lead us to conformally invariant theories. Those are theories with a higher degree of symmetry and interesting in their own right.

And then of course we still have the purely academical reason to investigate just for the investigation in itself. For, who knows, we may find something interesting and important. Or we may at least get new insight into known theories.

1.2 Lagrange and Hamilton formalism

When it comes to solving simple problems in classical mechanics, the formulation by Newton is usually the most natural machinery to use. However, for analytical discussions the formalism developed by 18th century physicists like Lagrange and Hamilton has shown much more fruitful. Since this formalism will be used extensively throughout this thesis, we begin by giving a short review. A more thorough introduction is found in textbooks on field theory, e.g. [1–4].

1.2.1 Lagrange formalism

The starting point in this formalism is the Lagrangian density (in the following just called the Lagrangian or Lagrange function), which is a function of some fields $\phi^i(x^a)$ and their derivatives $\partial_a \phi^i = \frac{\partial \phi^i}{\partial x^a}$,

$$L = L(\phi^i, \partial_a \phi^i). \tag{1.1}$$

The index i is used, when necessary, to distinguish the different fields in the Lagrangian. a is an index running over all coordinates. a=0 denotes the time coordinate, and we write time derivatives as $\partial_0 \phi = \dot{\phi}$. For spatial (a>0) derivatives

we write $\nabla \phi$. This convention will be convenient when we work in the Hamilton formalism, since time has a special role there.

The above Lagrangian is written as a function of first order derivatives only, and hence we call it a *first order* Lagrangian. It is also possible to allow for higher order derivatives in the Lagrangian, but such Lagrangians can usually be reduced to first order ones by introduction of extra fields, as discussed in [5].

From the Lagrange function we construct the *action* as the integral over the configuration space,

$$S[\phi^i] = \int dx L(\phi^i(x), \partial_a \phi^i(x)), \tag{1.2}$$

where $x = \{x^a\}$ is usually a set of coordinates, and the integral is then manydimensional. The action integral is a functional of the fields $\phi^i(x)$, which means that it takes functions to numbers (in contrast to functions, which take numbers to numbers.) Suppose we make small variations in the arguments of the Lagrangian. In other words, consider the transformation $\phi^i \to \phi^i + \delta \phi^i$, with $\delta \phi^i = 0$ on the boundaries. The values of ϕ^i that represent the dynamical behaviour of the classical system are the ones that leave the action unchanged under such infinitesimal variations. This is the principle of extremal action, also called Hamilton's principle.

If we consider a variation $\delta \phi^i$ in the fields, and demand the action to be extremal (i.e. zero under the transformation), we arrive at some equations which we call the field equations, equations of motion or Euler-Lagrange equations. For Lagrangians that have only first order derivatives, they can be deduced quite simply as follows.

$$\delta\phi^{i} \Rightarrow \delta S = \int dx \left(\frac{\partial L}{\partial \phi^{i}} \delta\phi^{i} + \frac{\partial L}{\partial (\partial_{a}\phi^{i})} \delta(\partial_{a}\phi^{i}) \right). \tag{1.3}$$

We can change the order of variation and derivation and write $\delta(\partial_a \phi^i) = \partial_a (\delta \phi^i)$. Together with a partial integration this gives

$$\delta S = \int dx \left(\frac{\partial L}{\partial \phi^i} \delta \phi^i + \partial_a \left(\frac{\partial L}{\partial (\partial_a \phi^i)} \delta \phi^i \right) - \partial_a \left(\frac{\partial L}{\partial (\partial_a \phi^i)} \right) \delta \phi^i \right)$$

$$= \int dx \left(\frac{\partial L}{\partial \phi^i} - \partial_a \left(\frac{\partial L}{\partial (\partial_a \phi^i)} \right) \right) \delta \phi^i, \tag{1.4}$$

where we have disregarded the total derivative. This is allowed since it will only give rise to a boundary (surface) term that vanishes since $\delta \phi^i$ are then zero. Demanding $\delta S = 0$ for arbitrary infinitesimal variations $\delta \phi^i$ we find the field equations,

$$\psi_i \equiv \partial_a \left(\frac{\partial L}{\partial (\partial_a \phi^i)} \right) - \frac{\partial L}{\partial \phi^i} = 0. \tag{1.5}$$

The fields that satisfy the equations of motion are said to span the classical path.

As long as we are in the classical domain of physics, the action integral is just a convenient and compact notation that contains the field equations and the symmetries of the theory. The introduction of this formalism is, however, crucial when we want to do quantum mechanics. Classically, two actions that gives rise to the same equations

of motion are equivalent, but this is not true quantum mechanically. Classically equivalent actions normally lead to different quantum physics. This fact may serve as a justification of our eagerness to find classically equivalent actions.

One of the major advantages of this formalism is that many symmetries are manifest in the Lagrangian. And this is also of great help when we want to write down the Lagrange function in the first place: It must have a form that satisfies the symmetries of the theory under consideration.

There is generally no way to *deduce* the Lagrangian. But that is no weakness of the formalism. Any theory needs a starting point, and in this formalism we start with the Lagrangian, and take the extremum of the action as a first principle. The right attitude is not to try to derive a Lagrangian, but to argue from general principles (e.g. symmetry) and analogies with other theories that it should take some specific form.

Variational derivatives

Consider a function F constructed from some fields $\phi^i(x)$ and their first order derivatives, $F = F(\phi, \partial \phi)$. (The typical example is the Lagrangian.) Variations in $\phi^i(x)$ will then give a variation in F(x) that we can write

$$\delta\phi(x) \Rightarrow \delta F(x) = \int d\tilde{x} \left[\frac{\partial F(x)}{\partial \phi^i(\tilde{x})} - \partial_a \frac{\partial F(x)}{\partial (\partial_a \phi^i(\tilde{x}))} \right] \delta\phi^i(\tilde{x})$$
 (1.6)

$$\equiv \int d\tilde{x} \frac{\delta F(x)}{\delta \phi^i(\tilde{x})} \delta \phi^i(\tilde{x}). \tag{1.7}$$

The variational derivative $\frac{\delta F(x)}{\delta \phi^i(\tilde{x})}$ of F gives the contribution to the variation of F(x) from a variation of $\phi^i(\tilde{x})$. To get the total variation of F(x) we have to sum over the discrete index i, and integrate over the continuous parameter \tilde{x} .

By comparison with (1.4) we see that the Euler-Lagrange equations can be written by means of the variational derivative as

$$\frac{\delta L}{\delta \phi^i} = 0. ag{1.8}$$

1.2.2 Hamilton formalism

If we already have the Lagrangian, we define the canonical conjugate momentum (density) associated with ϕ^i as

$$\pi_i = \frac{\partial L}{\partial \dot{\phi}^i},\tag{1.9}$$

where, again, dot denotes differentiation with respect to the time parameter.

With the definition (1.9) we can perform a Legendre transformation from configuration space to phase space. First, define the quantity h by

$$h(\phi, \partial \phi, \pi) = \pi_i \dot{\phi}^i - L(\phi, \partial \phi). \tag{1.10}$$

A variation in h can then be written

$$\delta h = \pi_i \delta \dot{\phi}^i + \dot{\phi}^i \delta \pi_i - \frac{\partial L}{\partial \phi^i} \delta \phi^i - \frac{\partial L}{\partial \dot{\phi}^i} \delta \dot{\phi}^i - \frac{\partial L}{\partial \nabla \phi^i} \delta \nabla \phi^i.$$

Substitute for the definition of π and arrive at

$$\delta h = \dot{\phi}^i \delta \pi_i - \frac{\partial L}{\partial \phi^i} \delta \phi^i - \frac{\partial L}{\partial \nabla \phi^i} \delta \nabla \phi^i.$$

The variation δh can be written by means of variations in ϕ , π and $\nabla \phi$. This means that we can express $h = \pi \dot{\phi} - L$ as a function of these variables, omitting $\dot{\phi}$. This is true even though (1.9) may not always be possible to solve explicitly for $\dot{\phi}^i$. Written this way we call h the Hamiltonian, but now denoted with a capital H,

$$H = H(\phi, \pi, \nabla \phi) = \pi_i \dot{\phi}^i - L(\phi, \partial \phi) \tag{1.11}$$

The transformation from L to H is a Legendre transformation. We will call the Hamiltonian obtained in this way the *naive Hamiltonian*, H_{naive} , to distinguish it from the total Hamiltonian which we introduce in the next section.

Hamilton's modified principle states that the phase space action,

$$S^{PS} = \int dx (\pi_i \dot{\phi}^i - H(\phi, \pi, \nabla \phi)), \qquad (1.12)$$

is unchanged under variations of π and ϕ , which are now considered independent of each other. We find¹

$$\delta S^{PS} = \int dx \left[\int d\tilde{x} \delta(x - \tilde{x}) \left(\dot{\phi}^{i}(x) \delta \pi_{i}(\tilde{x}) - \dot{\pi}_{i}(x) \delta \phi(\tilde{x}) \right) - \int d\tilde{x} \left(\frac{\delta H(x)}{\delta \phi^{i}(\tilde{x})} \delta \phi^{i}(\tilde{x}) + \frac{\delta H(x)}{\delta \pi_{i}(\tilde{x})} \delta \pi_{i}(\tilde{x}) \right) \right]$$

$$= \int d\tilde{x} \left[\left(\dot{\phi}^{i}(x) - \int dx \frac{\delta H(x)}{\delta \pi_{i}(\tilde{x})} \right) \delta \pi_{i}(\tilde{x}) + \left(-\dot{\pi}_{i}(x) - \int dx \frac{\delta H(x)}{\delta \phi^{i}(\tilde{x})} \right) \delta \phi^{i}(\tilde{x}) \right]. \tag{1.13}$$

Using Hamilton's modified principle $\delta S^{PS} = 0$, this leads to Hamilton's equations:

$$\dot{\phi}^{i}(x) = \int dx' \frac{\delta H(x')}{\delta \pi_{i}(x)}, \qquad (1.14)$$

$$-\dot{\pi}_i(x) = \int dx' \frac{\delta H(x')}{\delta \phi^i(x)}.$$
 (1.15)

These equations are equivalent to the Euler-Lagrange equations. This is not trivial to prove, and it is a remarkable result that it is true.

¹ We use a trick and write $\dot{\phi}^i(x)\delta\pi_i(x) = \int d\tilde{x}\delta(x-\tilde{x})\dot{\phi}^i(x)\delta\pi_i(\tilde{x})$ (and the same for $\dot{\pi}_i(x)\delta\phi^i(x)$).

1.2.3 Constrained systems

References for the theory discussed in this section are [6–8].

Suppose that we for a given Lagrangian have found the momenta $\pi_i = \frac{\partial L}{\partial \dot{\phi}^i}$. The way to the Hamiltonian picture is easy if we can invert this procedure to find $\dot{\phi}^i$ as functions of the fields and momenta, i.e. $\dot{\phi}^i = \dot{\phi}^i(\phi, \pi, \nabla \phi)$. Then we can just substitute for $\dot{\phi}^i$ in h and immediately arrive at the naive Hamiltonian $H_{naive}(\phi, \pi, \nabla \phi)$. This is often possible, but not if there exists some gauge freedom in the theory. There are also non-gauge theories which fail to be invertible in this way [7].

It is therefore of interest to study this class of systems, which are called *constrained systems*. The non-invertibility means that the derived momenta will not be independent, and there exist some relations between them. These relations can be expressed as functions $\theta_m^I(\phi, \pi, \nabla \phi) = 0, m = 1, ..., M$, where M is the number of such functions. We call these the *primary constraints* since they follow directly from the definition of the momenta.

Suppose that i can take n different values (i.e. there are n field variables). Then define the $n \times n$ matrix $C_{ij} \equiv \frac{\partial^2 L}{\partial \dot{\phi}^i \partial \dot{\phi}^j}$. If r is the rank of this matrix, then the number of independent primary constraints is n-r.

For constrained systems the naive Hamiltonian is not unique, since we may add to it any linear combination of the constraint functions θ^I . This fact leads to a modification of Hamilton's equations. The modified versions are

$$\dot{\phi}^{i}(x) = \int dx' \frac{\delta H_{1}(x')}{\delta \pi_{i}(x)}$$
(1.16)

$$-\dot{\pi}_i(x) = \int dx' \frac{\delta H_1(x')}{\delta \phi^i(x)}, \qquad (1.17)$$

where $H_1 \equiv H_{naive} + \lambda_I^m \theta_m^I$ and λ_I^m are coefficients that do not depend on ϕ and π . They are called *Lagrange multipliers*.

Poisson brackets The notation of Poisson brackets is very convenient in this formalism. Consider two functions F and G that are constructed from ϕ and π . If we write F(x) as short for $F(\phi(x), \pi(x))$, their Poisson bracket is defined as

$$\{F(x), G(x')\} \equiv \int d\tilde{x} \left(\frac{\delta F(x)}{\delta \phi^{i}(\tilde{x})} \frac{\delta G(x')}{\delta \pi_{i}(\tilde{x})} - \frac{\delta F(x)}{\delta \pi_{i}(\tilde{x})} \frac{\delta G(x')}{\delta \phi^{i}(\tilde{x})} \right). \tag{1.18}$$

The brackets can easily be shown to satisfy the following relations:

- 1. Antisymmetry $\{F,G\} = -\{G,F\}$
- 2. Linearity $\{F + G, H\} = \{F, H\} + \{G, H\}$
- 3. Product law $\{FG, H\} = F\{G, H\} + \{F, H\}G$
- 4. The Jacobi identity $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$

It is also easy to show that the fundamental Poisson brackets are

$$\{\phi^i(x), \pi_i(x')\} = \delta^i_i \delta(x - x'), \tag{1.19}$$

$$\{\phi^i(x), \phi^j(x')\} = \{\pi_i(x), \pi_j(x')\} = 0.$$
 (1.20)

When we work with Poisson brackets in the present circumstance the following is important: Poisson brackets must be evaluated before we make use of the constraint equations. In other words, we should perform the calculations in phase space, and restrict to the constraint surface $(\theta = 0)$ at the end. To emphasize this point we use Dirac's notation [6] and say that the constraint equations are weakly zero, and write them with a new weak equality sign " \approx " as:

$$\theta_m^I(\phi, \pi) \approx 0. \tag{1.21}$$

This makes a difference, for even though $\theta(\phi, \pi)$ is dynamically zero (i.e. zero when ϕ and π satisfy Hamilton's equations) it is not zero throughout phase space.

Now, let us consider the time evolution of the function $F(\phi, \pi)$. By the chain rule we have

$$\dot{F}(x) = \int d\tilde{x} \left(\frac{\delta F(x)}{\delta \phi^{i}(\tilde{x})} \dot{\phi}^{i}(\tilde{x}) + \frac{\delta F(x)}{\delta \pi_{i}(\tilde{x})} \dot{\pi}_{i}(\tilde{x}) \right). \tag{1.22}$$

Using the (modified) Hamilton's equations (1.16) and (1.17) we can insert for $\dot{\phi}$ and $\dot{\pi}$ and get

$$\dot{F}(x) = \int dx' \int d\tilde{x} \left(\frac{\delta F(x)}{\delta \phi^{i}(\tilde{x})} \frac{\delta H_{1}(x')}{\delta \pi_{i}(\tilde{x})} - \frac{\delta F(x)}{\delta \pi_{i}(\tilde{x})} \frac{\delta H_{1}(x')}{\delta \phi^{i}(\tilde{x})} \right)
= \int dx' \left\{ F(x), H_{1}(x') \right\}.$$
(1.23)

The constraints must hold (i.e. be weakly equal to zero) for all times. This means that their time derivatives should vanish (weakly):

$$\dot{\theta}_m^I(\phi,\pi) \approx 0. \tag{1.24}$$

By putting $F = \theta_m^I$ in the equation above, we thus get the following consistency conditions:

$$\int dx' \{\theta_m^I(x), H_1(x')\} \approx 0, \qquad (1.25)$$

or more explicitly

$$\int dx' \left[\{ \theta_m^I(x), H_{naive}(x') \} + \lambda_I^n \{ \theta_m^I(x), \theta_n^I(x') \} \right] \approx 0.$$
 (1.26)

These equations may lead to three different situations. The first is that they do not give anything new at all, we merely end up with 0 = 0. In this case we have already found the full constraint structure.

Another possibility is that we arrive at equations not involving the λ_I 's. Then we get new constraints on ϕ and π on the form

$$\theta_p^{II}(\phi, \pi) \approx 0.$$
 (1.27)

These are called *secondary constraints*.

The third kind of equations we may end up with also depend on the Lagrange multipliers λ_I . The consistency conditions will then impose a condition on the λ_I 's.

If we get secondary constraints we must go on and check the consistency conditions on them, $\dot{\theta}^{II} \approx 0$ in just the same way as for the primary constraints. This "loop" should be continued until we get no more new conditions.

The distinction between primary and secondary (and tertiary etc.) constraints is just a matter of how they appear, and is not physically important. In fact, different Lagrangians that describe the same physical system will in general give rise to the constraints in a different order and different combinations.

When we have found all constraints and conditions on the λ 's (if any) we are ready to write down the *total Hamiltonian*:

$$H_T = H' + \lambda^a \theta_a. \tag{1.28}$$

If we had no conditions imposed on the λ 's, H' will be identical to the naive Hamiltonian, and $\lambda^a = \{\lambda_I^m, \lambda_{II}^p, \ldots\}$. If, on the other hand, such conditions were present, H' will be shifted by some factor, and the Lagrange multipliers can be redefined so that all the λ^a 's are independent. This will not be important for our considerations, and is explained in more detail in [6,7].

Several examples of calculations of the kind explained in this section are given later on in the thesis.

1.3 Symmetries

Let S be some symmetry, and let T_S be the transformation associated with this symmetry. We say that a theory has the symmetry S if the action $S = \int L dx$ describing the theory is left unchanged under transformations T_S . We then say that the action is S-invariant. In this section we will see a brief description of some symmetries that we will frequently encounter. Most of the examples presented are studied in more detail later on.

1.3.1 Diffeomorphism symmetry

Suppose we have an action integral parameterized by x (x may be one parameter or a whole set), then we write the action as

$$S = \int dx L(\phi), \tag{1.29}$$

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where ϕ symbolizes the field variables. (ϕ may be one or a whole set of fields.) A diffeomorphism is a general coordinate transformation of the form

$$x \to \tilde{x} = \tilde{x}(x). \tag{1.30}$$

The name comes from the fact that \tilde{x} is a differentiable function of x (usually $\in C^{\infty}$). In the context of particles, strings etc. where the parameters in the action are not the spacetime coordinates we usually refer to the transformation as a reparameterization. Now, for a diffeomorphism symmetry to be present in the model, the action has to be invariant under this transformation. Whether that is the case of course depends on how the fields transform, $\phi(x) \to \tilde{\phi}(\tilde{x})$, and of the form of the Lagrangian. For short, diffeomorphism is often written diff (e.g. diff invariance).

Example: The relativistic point particle. Here the action is (c.f. section 2.2)

$$S = m \int d\tau \sqrt{-\eta_{\mu\nu} \frac{dX^{\mu}}{d\tau} \frac{dX^{\nu}}{d\tau}}$$
 (1.31)

Consider the reparameterization $\tau \to \tilde{\tau}(\tau)$. The fields are scalars under this transformation, in the sense that $\tilde{X}^{\mu}(\tilde{\tau}) = X^{\mu}(\tau)$. (The position of the particle is obviously independent of the parameterization.) The transformed action (i.e. the action described by the transformed quantities) is then

$$\tilde{S} = \int d\tilde{\tau} \sqrt{-\eta_{\mu\nu}} \frac{d\tilde{X}^{\mu}(\tilde{\tau})}{d\tilde{\tau}} \frac{d\tilde{X}^{\nu}(\tilde{\tau})}{d\tilde{\tau}}
= \int d\tau \frac{d\tilde{\tau}}{d\tau} \sqrt{-\eta_{\mu\nu}} \frac{dX^{\mu}(\tau)}{d\tau} \frac{dX^{\nu}(\tau)}{d\tau} (\frac{d\tau}{d\tilde{\tau}})^{2}
= \int d\tau \sqrt{-\eta_{\mu\nu}} \frac{dX^{\mu}(\tau)}{d\tau} \frac{dX^{\nu}(\tau)}{d\tau} = S.$$
(1.32)

The symbolic form of the action is unchanged, which is precicely the symmetry criterion.

1.3.2 Weyl symmetry

When we talk about Weyl symmetry we study theories which involve some metric g_{ab} . A Weyl transformation is then a position-dependent rescaling of this metric. (We could also imagine rescalings of other fields, but they are not so interesting.) The transformation can be written as

$$g_{ab}(x) \to \tilde{g}_{ab}(x) = e^{\omega(x)} g_{ab}(x), \tag{1.33}$$

where $\omega(x)$ is any function. An action that is unchanged by this transformation is consequently called Weyl invariant.

Example: The Weyl-invariant string action [9, 10]:

$$S = \int d^2 \xi \sqrt{-g} g^{ab} \gamma_{ab}. \tag{1.34}$$

Here $\gamma_{ab} \equiv G_{\mu\nu}\partial_a X^{\mu}\partial_b X^{\nu}$, where $G_{\mu\nu}$ is the background (fixed) spacetime metric, and $\partial_a \equiv \frac{\partial}{\partial \xi^a}$. The field variables in the theory are the intrinsic worldsheet metric components $g_{ab}(\xi)$ and the position field $X^{\mu}(\xi)$. Consider then the Weyl transformation $g_{ab}(\xi) \to e^{\omega(\xi)} g_{ab}(\xi)$. Under this rescaling of the metric we will have

$$g \equiv \det g_{ab} \quad \to \quad e^{2\omega} g$$
$$g^{ab} \quad \to \quad e^{-\omega} g^{ab}.$$

And the action will transform as

$$S \to \int d^2 \xi \sqrt{-e^{2\omega}g} e^{-\omega} g^{ab} \gamma_{ab} = S. \tag{1.35}$$

So the action is worldsheet Weyl invariant (as its name correctly announces). Note also that the dimensionality D=2 enters crucially in this derivation.

1.3.3 Poincaré symmetry

A Poincaré transformation is a coordinate transformation that consists of the familiar Lorentz transformation plus translation. It is also called an *inhomogeneous* Lorentz transformation, a name which is obvious from the form:

$$X^{\mu} \to \tilde{X}\mu = \Lambda^{\mu}_{\nu}X^{\nu} + a^{\mu}. \tag{1.36}$$

 Λ^{μ}_{ν} are the Lorentz transformation coefficients and a^{μ} are constants describing the translational part.

Lorentz transformations can further be split into into *boosts* and *rotations* in addition to the discrete transformations of time and space inversion.

A general result worth noting is that Lorentz invariance is automatically achieved if the Lagrangian is written in covariant form as a Lorentz scalar.

One very interesting feature of the Poincaré transformation is that gauge theory based on *local* Poincaré invariance (i.e. the coefficients in the above transformation are position dependent), gives rise to a theory for gravitation.

Example: The relativistic point particle.

We start again from the action (1.31). The coefficients Λ^{ν}_{μ} and a^{ν} are constants, so the derivatives of X^{μ} are transformed as

$$\dot{X}^{\mu} \equiv \frac{dX^{\mu}}{d\tau} \to \Lambda^{\mu}_{\nu} \dot{X}^{\nu}.$$

1.3 Symmetries 11

This is the only quantity that is changed under the Poincaré transformation, so the action will transform as

$$S \rightarrow \int d\tau \sqrt{-\eta_{\mu\nu}\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}\dot{X}^{\rho}\dot{X}^{\sigma}}$$
$$= \int d\tau \sqrt{-\eta_{\rho\sigma}\dot{X}^{\rho}\dot{X}^{\sigma}} = S,$$

where we have used the general result $\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}$. Thus, we see that the relativistic point particle action has Poincaré symmetry.

1.3.4 Conformal symmetry

A conformal transformation is a mapping from flat space (Minkowski or Euclidian) onto itself, such that the flat metric η_{ab} is left invariant up to a rescaling. In other words, it has the same effect as the substitution $\eta_{ab} \to \Omega(x)\eta_{ab}$ for some positive definite function $\Omega(x)$. Because of this, the line element will transform as $ds^2 \to \Omega ds^2$, showing that the causal structure is conserved. In particular, light cones will transform into light cones.

Alternatively, the conformal transformation may be viewed as a composite diff and Weyl transformation that leaves the flat metric invariant. (The conformal symmetry group is a subgroup of the diff×Weyl symmetry group.)

In general relativity terminology, what is here called Weyl transformation is often named conformal transformation. But they are not to be considered as the same.

It is also important not to confuse conformal symmetry with the *diff* invariance of general relativity. Conformal symmetry is a symmetry in flat space theory, with no independent metric fields to vary. Hence, conformal transformations in general actually change the distances between points. From this it follows that conformally symmetric theories have no length scale.

A full conformal transformation involves a Poincaré transformation (Lorentz + translation), a dilatation and a special conformal transformation. It acts on the spacetime coordinates, and the infinitesimal transformation can be written $x^{\mu} \rightarrow \tilde{x}^{\mu} = x^{\mu} + \delta x^{\mu}$, with

Lorentz:
$$\delta_{\omega} x^{\mu} = \omega^{\mu}_{\ \nu} x^{\nu},$$
 (1.37)

translation:
$$\delta_a x^\mu = a^\mu$$
, (1.38)

dilatation:
$$\delta_c x^\mu = c x^\mu$$
, (1.39)

special conformal:
$$\delta_b x^\mu = b^\nu x_\nu x^\mu - \frac{1}{2} x^\nu x_\nu b^\mu$$
. (1.40)

In 4 dimensions there are 15 independent parameters of this symmetry group: 6 Lorentz ($\omega_{\mu\nu} = -\omega_{\nu\mu}$); 4 translations (a^{μ}); 1 dilatation (c); 4 special (b^{μ}).

Example: The massless scalar field.

Consider the action

$$S = \int dx \partial_{\mu} \phi \partial_{\nu} \phi \eta^{\mu\nu}. \tag{1.41}$$

Since ϕ is a scalar we have

$$\tilde{\phi}(\tilde{x}) = \phi(x); \quad \tilde{\partial}_{\mu}\tilde{\phi}(\tilde{x}) = \frac{\partial x^{\nu}}{\partial \tilde{x}^{\mu}}\partial_{\nu}\phi(x) = (\delta^{\mu}_{\nu} - \partial_{\nu}\delta x^{\mu})\partial_{\mu}\phi(x), \tag{1.42}$$

where $\tilde{\partial}_{\mu} \equiv \frac{\partial}{\partial \tilde{x}^{\mu}}$. The transformed action can then be written as

$$\tilde{S} = \int d\tilde{x} \tilde{\partial}_{\mu} \tilde{\phi} \tilde{\partial}_{\nu} \tilde{\phi} \eta^{\mu\nu} \tag{1.43}$$

$$= \int dx \det(\frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}}) (\delta^{\rho}_{\mu} - \partial_{\mu} \delta x^{\rho}) (\delta^{\sigma}_{\nu} - \partial_{\nu} \delta x^{\sigma}) \partial_{\rho} \phi \partial_{\sigma} \phi \eta^{\mu\nu}. \tag{1.44}$$

The Jacobi determinant is to first order $\det(\frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}}) = 1 + \partial_{\alpha} \delta x^{\alpha}$. This gives

$$\begin{split} \tilde{S} &= S + \delta S; \qquad \delta S &= \int dx \partial_{\mu} \phi \partial_{\nu} \phi X^{\mu\nu}, \\ X^{\mu\nu} &\equiv -\eta^{\mu\nu} \partial_{\alpha} \delta x^{\alpha} + \eta^{\alpha\nu} \partial_{\alpha} \delta x^{\mu} + \eta^{\mu\alpha} \partial_{\alpha} \delta x^{\nu}. \end{split}$$

Let us now consider the conformal transformations one by one.

Lorentz Consider first the Lorentz transformation $\delta x^{\mu} = \omega^{\mu}_{\alpha} x^{\alpha}$. We find then $\partial_{\mu} \delta x^{\nu} = \omega^{\nu}_{\mu}$, which together with the antisymmetry of $\omega_{\mu\nu}$ gives $X^{\mu\nu} = 0$. So the action is Lorentz invariant. As mentioned earlier, this could be concluded solely from the fact that the Lagrangian is written covariantly as a Lorentz scalar.

Translation For translations we have $\delta x^{\mu} = a^{\mu}$, which gives $\partial_{\nu} \delta x^{\nu} = 0$ and immediately $X^{\mu\nu} = 0$. Thus the action is always Poincaré invariant.

Dilatation Turn then to the dilatation, $\delta x^{\mu} = cx^{\mu}$. We find $\partial_{\mu}\delta x^{\nu} = c\delta^{\nu}_{\mu}$, and $X^{\mu\nu} = \eta^{\mu\nu}c(\delta^{\alpha}_{\alpha} - 2)$ which is zero in *two* dimensions (D = 2). For general spacetime dimensions D, we get $\tilde{S} = \int dx \partial_{\mu}\phi \partial_{\nu}\phi \eta^{\mu\nu}\Omega$, with $\Omega = 1 + c(D - 2) \approx 1 > 0$. Thus we find that the effect of a dilatation is the same as a rescaling of the metric.

Special conformal Special conformal transformations have $\delta x^{\mu} = b^{\nu} x_{\nu} x^{\mu} - \frac{1}{2} x^{\nu} x_{\nu} b^{\mu}$, which gives $\partial_{\mu} \delta x^{\nu} = b_{\mu} x^{\nu} + b \cdot x \delta^{\nu}_{\mu} - x_{\mu} b^{\nu}$. This gives $X^{\mu\nu} = \eta^{\mu\nu} b \cdot x (\delta^{\alpha}_{\alpha} - 2)$, which is again zero for D = 2. In general we find $\tilde{S} = \int dx \partial_{\mu} \phi \partial_{\nu} \phi \eta^{\mu\nu} \Omega'$, with $\Omega' = 1 + b \cdot x (D - 2) \approx 1 > 0$.

We have now seen explicitly that a conformal transformation on the massless scalar field has the effect of a rescaling of the flat metric $(\eta^{\mu\nu} \to \eta^{\mu\nu}(\Omega + \Omega'))$, with Ω and Ω' as defined above). Furthermore, we found that it is conformal invariant in two dimensions.

1.4 More on coordinate transformations

In this section we will use symmetry principles to derive some important general results in field theory.

Consider a general action integral,

$$S = \int dx L(\phi^i, \partial \phi^i), \tag{1.45}$$

where ϕ^i are some general fields; scalars, vectors or whatever. We have assumed that there is no explicit coordinate dependence. In the following we will see what happens if we make the coordinate transformation,

$$x^a \to \tilde{x}^a = x^a + a^a, \tag{1.46}$$

where $a^a = a^a(x)$ is infinitesimal. We now define the *Jacobi matrix* J_b^a and the *Jacobi determinant* J as

$$J_b^a \equiv \frac{\partial x^a}{\partial \tilde{x}^b} = \delta_b^a - \partial_b a^a, \tag{1.47}$$

$$J \equiv \det(J_b^a) = \det(\delta_b^a - \partial_b a^a) \cong 1 - \partial_c a^c. \tag{1.48}$$

Whit this definition the integral measure transforms as $dx \to d\tilde{x} = dxJ^{-1}$.

Under the transformation (1.46) the action will transform to

$$S \to \tilde{S} = \int d\tilde{x} L(\tilde{\phi}(\tilde{x}), \tilde{\partial}\tilde{\phi}(\tilde{x})). \tag{1.49}$$

Let the transformation of the fields be written

$$\phi^{i}(x) \to \tilde{\phi}^{i}(\tilde{x}) = \phi^{i}(x) + \epsilon^{i}(x)$$
 (1.50)

This defines ϵ^i . Note that the fields are taken at different points on the left and right hand side. This is not the usual way to compare fields, but convenient for the moment. The notation, and the subsequent calculation, is inspired by Frøyland [1]. For scalars we have $\epsilon = 0$.

The derivatives of the fields will transform as

$$\tilde{\partial}_{b}\tilde{\phi}^{i}(\tilde{x}) = \frac{\partial x^{c}}{\partial \tilde{x}^{b}}\partial_{c}(\phi^{i}(x) + \epsilon^{i})$$

$$\cong \partial_{b}\phi^{i}(x) + \partial_{b}\epsilon^{i}(x) - \partial_{b}a^{c}\partial_{c}\phi(x). \tag{1.51}$$

The transformed action integral may now be Taylor expanded and rewritten in the following way:

$$\tilde{S} = \int dx J^{-1} L(\phi^i + \epsilon^i, \partial_a \phi^i + \partial_a \epsilon^i - \partial_a a^b \partial_b \phi^i)
= \int dx (1 + \partial_c a^c) \left(L(\phi, \partial \phi^i) + \frac{\partial L}{\partial \phi^i} \epsilon^i + \frac{\partial L}{\partial (\partial_a \phi^i)} (\partial_a \epsilon^i - \partial_a a^b \partial_b \phi^i) \right)$$

$$= \int dx \left(L + \partial_{c} a^{c} L + \frac{\partial L}{\partial \phi^{i}} \epsilon^{i} + \underbrace{\frac{\partial L}{\partial (\partial_{a} \phi^{i})} \partial_{a} \epsilon^{i}}_{-\partial_{a} (\frac{\partial L}{\partial (\partial_{a} \phi^{i})}) \epsilon^{i}} - \underbrace{\frac{\partial L}{\partial (\partial_{a} \phi^{i})} \partial_{b} \phi^{i} \partial_{a} a^{b}}_{-\partial_{a} (\frac{\partial L}{\partial (\partial_{a} \phi^{i})}) \epsilon^{i}} \right)$$

$$= \int dx \left(L - \underbrace{\left[\partial_{a} (\frac{\partial L}{\partial (\partial_{a} \phi^{i})}) - \frac{\partial L}{\partial \phi^{i}} \right]}_{=\psi_{i}} \epsilon^{i} + \partial_{a} a^{b} \underbrace{\left[\delta^{a}_{b} L - \frac{\partial L}{\partial (\partial_{a} \phi^{i})} \partial_{b} \phi^{i} \right]}_{\equiv T^{a}_{b}} \right)}_{\equiv T^{a}_{b}}$$

$$= S + \int dx \left(\partial_{a} a^{b} T^{a}_{b} - \psi_{i} \epsilon^{i} \right), \tag{1.52}$$

where we have performed a partial integration and assumed the fields to vanish at infinity. For fields that satisfy the Euler-Lagrange equations, $\psi^i = 0$, we find

$$\delta S = \int dx \partial_a a^b T^a_b. \tag{1.53}$$

Another partial integration gives

$$\delta S = -\int dx (\partial_a T^a_b) a^b. \tag{1.54}$$

In the calculations above we have assumed that a^a is position dependent. But consider now the *global* transformation (i.e. $a^a = const$), which is an infinitesimal translation. This is usually a symmetry of the action, in which case we have $\delta S = 0$. Equation (1.54) then gives the condition

$$\partial_a T^a_{\ b} = 0. \tag{1.55}$$

In other words, the (global) symmetry leads to a conserved translation current T^a_b . This is a special case of *Noether's theorem* which states that any symmetry implies a conserved current.

1.4.1 Energy-momentum tensor

The translation current is often called the *canonical energy-momentum tensor*, and we defined it as

$$\mathcal{T}^{a}_{b} = \delta^{a}_{b} L - \frac{\partial L}{\partial (\partial_{a} \phi^{i})} \partial_{b} \phi^{i}. \tag{1.56}$$

However, the tensor $\mathcal{T}^{ab} = \eta^{bd}\mathcal{T}^a_{d}$ is not symmetric and therefore cannot be used on the right hand side of Einstein's field equations for general relativity. Neither is it always possible to generalize to curved spacetime.

We will now present another way of defining the energy-momentum tensor, which avoids these problems. Consider the action for general relativity coupled to matter, which can be written (c.f. chapter 5 and [11])

$$S = \int d^4x \sqrt{-g} (\frac{1}{\kappa} R + \mathcal{L}_M), \qquad (1.57)$$

where κ is a constant, $g = \det(g_{ab})$ is the determinant of the metric, R is the Ricci curvature scalar and \mathcal{L}_M is the term describing the matter field. It is the same as the

Lagrangian expressed in flat spacetime with the flat metric exchanged by the general metric g_{ab} . (This is not valid for spinors.) If we define $L_M \equiv \sqrt{-g}\mathcal{L}_M$, a variation δg_{ab} leads to the field equations

$$-\frac{1}{\kappa}\sqrt{-g}(R^{ab} - \frac{1}{2}g^{ab}R) + \frac{\delta L_M}{\delta g_{ab}} = 0.$$
 (1.58)

We recover the Einstein field equations, $R^{ab} - \frac{1}{2}g^{ab}R = \frac{\kappa}{2}T^{ab}$, if we use

$$T^{ab} = \frac{2}{\sqrt{-g}} \frac{\delta L_M}{\delta q_{ab}}.$$
 (1.59)

This tensor is manifestly symmetric, and gives a convenient definition of the energy-momentum tensor. In the following we will go through the necessary calculations to prove that the two definitions of the energy-momentum tensor are equivalent in flat spacetime, provided that ϕ^i couples to gravity via g_{ab} .

Equivalence of (1.56) and (1.59)

Consider again the infinitesimal transformation $x^a \to \tilde{x}^a + a^a(x)$. The metric g_{ab} transforms as a second rank tensor, i.e.

$$g_{ab} \to \tilde{g}_{ab}(\tilde{x}) = \Lambda_a^c \Lambda_b^d g_{ab}(x),$$
 (1.60)

$$\Lambda_a^c \equiv \frac{\partial x^c}{\partial \tilde{x}^a} = J_a^c = \delta_a^c - \partial_a a^b. \tag{1.61}$$

This gives

$$\tilde{q}_{ab}(x) = q_{ab}(x) + \delta q_{ab}; \quad \delta q_{ab} = -(a^c \partial_c q_{ab} + q_{ac} \partial_b a^c + q_{cb} \partial_a a^c). \tag{1.62}$$

We recognize δg_{ab} as a Lie derivative. Using the results from appendix A.4 we find

$$\delta g_{ab} = -\mathcal{L}_{\vec{a}} g_{ab} = -\nabla_a a_b - \nabla_b a_a = -\nabla_{(a} a_{b)}. \tag{1.63}$$

Now, consider the flat space action $S = \int dx L(\phi)$. Coupling to gravity gives

$$S_G = \int dx L_G(\phi, g); \quad L_G = \sqrt{-g} L(\phi, g). \tag{1.64}$$

The infinitesimal diffeomorphism $x^a \to x^a + a^a$ gives a variation in the action, which we write

$$S_G \to S_G + \delta S_G; \quad \delta S_G = \delta_\phi S_G + \delta_\sigma S_G.$$
 (1.65)

The first term in δS_G is proportional to $\delta \phi$ and the second is proportional to δg_{ab} Since any gravity-coupled action is generally coordinate invariant, i.e. diff invariant, we must have

$$\delta S = \delta_{\phi} S_G + \delta_a S_G = 0, \tag{1.66}$$

and as a special result

$$(\delta_{\phi}S_G + \delta_q S_G)|_{q=\eta} = 0. \tag{1.67}$$

If we use equation (1.53) we get

$$\delta_{\phi} S_G|_{g=\eta} = \delta_{\phi} S = \int dx \partial_a a^b \mathcal{T}^a{}_b = \int dx \partial_a a_b \mathcal{T}^{ab}$$
 (1.68)

where \mathcal{T}^{ab} is the canonical energy-momentum tensor. Furthermore, by use of (1.63) we find

$$\delta_{g}S_{G}|_{g=\eta} = \int dx \frac{\delta L_{G}}{\delta g_{ab}} \delta g_{ab}|_{g=\eta} = -\int dx \frac{\delta L_{G}}{\delta g_{ab}} \nabla_{(a}a_{b)}|_{g=\eta}$$

$$= -2 \int dx \frac{\delta L_{G}}{\delta g_{ab}}|_{g=\eta} \partial_{a}a_{b}$$
(1.69)

Equation (1.67) gives then the following relation in flat space:

$$\mathcal{T}^{ab} = 2\frac{\delta L_G}{\delta g_{ab}}|_{g=\eta} = T^{ab}|_{g=\eta},\tag{1.70}$$

which is exactly what we wanted to show. It says that the two definitions (1.56) and (1.59) are the same in flat spacetime for models that couple to gravity according to (1.64). Since the two definitions are the same, we immediately find that even the canonical energy-momentum tensor \mathcal{T}^{ab} is symmetric. As noted earlier, this is not a general result, but comes here as a consequence of the assumpsion (1.64) that ϕ^i is a kind of field that couples to gravity via g_{ab} .

Energy-momentum tensor for spinors

A description of gravity models with spinors is most easily done in the vielbein formalism (see appendix A.3). Denote the vielbeins $e_a{}^A(x)$ and their determinants $\det(e_a{}^A) \equiv e$. We then have $\sqrt{-g} = e$, and a gravity-coupled model can be written $L_G = eL(\phi^i, e_a{}^A)$, where $L(\phi^i)$ is the non-coupled theory. ϕ^i may now be spinors, but also tensors.

We define again the energy-momentum tensor $T^a_A = \frac{\delta L_G}{\delta e_a^A}$ as the translation current:

$$\delta x^a \Rightarrow \delta S_G = \int dx \left[\frac{\delta L_G}{\delta e_a^A} \delta e_a^A + \frac{\delta L_G}{\delta \phi^i} \delta \phi^i \right].$$
 (1.71)

For Lorentz transformations Λ_B^A we have

$$\delta e_a^{\ A} = e_a^{\ B} \Lambda_B^{\ A}; \qquad \Lambda^{AB} = -\Lambda^{BA}, \tag{1.72}$$

which gives

$$\begin{split} \delta_{\Lambda} S_G &= \int dx \left[e_a{}^B \Lambda_B^{\ A} T^C_{\ A} e_C^{\ a} + \frac{\delta L_G}{\delta \phi^i} \delta_{\Lambda} \phi^i \right] \\ &= \int dx \left[\Lambda^{BA} T_{BA} + \frac{\delta L_G}{\delta \phi^i} \delta_{\Lambda} \phi^i \right]. \end{split}$$

The Lorentz transformation is a symmetry of the theory, so $\delta_{\Lambda}S = 0$. Furthermore, if the fields ϕ^i satisfy the equations of motion $\frac{\delta L_G}{\delta \phi^i} = 0$, we get $\int dx \Lambda^{AB} T_{AB} = 0$ which means that the antisymmetric part of the energy-momentum tensor is zero, i.e. $T_{[AB]} = 0$. In other words, the energy-momentum tensor is symmetric for fields that satisfy the equations of motion, but not generally.

Energy-momentum tensor for conformally invariant theories

As noted in section 1.3.4, a conformal transformation has the same effect as a rescaling of the metric. Thus we may consider the variation of the action S as a result of a variation in g_{ab} with $\delta g_{ab} = \Omega g_{ab}$. This means that we can write

$$\delta S = \int dx \frac{\delta L}{\delta q_{ab}} \delta g_{ab} = \int dx 2T^{ab} \Omega g_{ab} = 2 \int dx T^{ab} g_{ab} \Omega. \tag{1.73}$$

Invariance means $\delta S = 0$ so we have for conformally invariant theories that the energy-momentum tensor is traceless, i.e.

$$T^{a}_{a} = T^{ab}g_{ab} = 0. (1.74)$$

This is indeed a simple way to determine conformal invariance.

Example: We now return to the massless scalar field we considered in section 1.3.4, with the Lagrangian

$$L = \partial_{\mu}\phi \partial_{\nu}\phi \eta^{\mu\nu}. \tag{1.75}$$

The canonical energy-momentum tensor is found to be

$$\mathcal{T}^{\mu}_{\nu} \equiv \delta^{\mu}_{\nu} L - \frac{\partial L}{\partial \partial_{\mu} \phi} \partial_{\nu} \phi$$

$$= \partial_{\alpha} \phi \partial_{\beta} \phi (\delta^{\mu}_{\nu} \eta^{\alpha \beta} - 2 \delta^{\beta}_{\nu} \eta^{\mu \alpha}),$$

$$\mathcal{T}^{\mu \nu} = \partial_{\alpha} \phi \partial_{\beta} \phi (\eta^{\mu \nu} \eta^{\alpha \beta} - 2 \eta^{\mu \alpha} \eta^{\nu \beta}).$$
(1.76)

To use the other definition of the energy-momentum tensor, we couple the model to gravity, and get

$$L_G = \sqrt{-g}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}.$$
 (1.77)

Then we find

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta L_G}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}} \frac{\partial L_G}{\partial g_{\mu\nu}}$$
$$= \partial_{\alpha} \phi \partial_{\beta} \phi (g^{\mu\nu} g^{\alpha\beta} - 2g^{\mu\alpha} g^{\nu\beta}). \tag{1.78}$$

We see immediately that $\mathcal{T}^{\mu\nu} = T^{\mu\nu}|_{q=\eta}$. Furthermore, the trace is

$$T^{\mu}_{\ \mu} = g_{\mu\nu}T^{\mu\nu} = \partial_{\alpha}\phi\partial_{\beta}\phi(D-2)g^{\alpha\beta},\tag{1.79}$$

which says that the energy-momentum tensor is traceless in two dimension, D = 2. This is in complete agreement with the fact that the massless scalar field is conformal invariant in two dimensions.

1.4.2 Naive Hamiltonian for diff invariant theories

If we let a be position dependent, and arbitrary, the transformation (1.46) is identical to an infinitesimal diffeomorphism. If we demand the action to be diff invariant, but do not restrict to fields that satisfy the equations of motion, we get from (1.52) the condition

$$\int dx \left[T^a_b \partial_a a^b - \psi_i \epsilon^i \right] = 0. \tag{1.80}$$

We want to show how this can give us an expression for the Hamiltonian. To do so we need to know the form of ϵ . Let us consider scalar, vector and second rank tensor fields, $\phi^i = \{\phi, A_a, A_{ab}\}$, and define $\Lambda^b_a \equiv \frac{\partial x^b}{\partial \bar{x}^a} = \delta^b_a - \partial_a a^b$.

Scalars For scalar fields we have simply

$$\epsilon^{\phi} = 0. \tag{1.81}$$

Vectors For vector fields we have

$$\tilde{A}_a(\tilde{x}) = \Lambda_a^b A_b(x) = (\delta_a^b - \partial_a a^b) A_b(x), \tag{1.82}$$

which gives

$$\epsilon_b^A = -\partial_b a^a A_a. \tag{1.83}$$

Second rank tensors In this case we have

$$\tilde{F}_{ab}(\tilde{x}) = \Lambda_a^c \Lambda_b^d F_{cd}(x), \tag{1.84}$$

which gives

$$\epsilon_{ab}^{F} = -\partial_c a^d (\delta_a^c F_{db} + \delta_b^c F_{ad}) \tag{1.85}$$

If the action depends on both scalars, vectors and second rank tensors, their contributions will just add up. The diff symmetry criterion is then

$$\int dx \partial_a a^b \left[T^a_b + \psi^a A_b + \psi^{ac} F_{bc} + \psi^{da} F_{dc} \right] = 0, \tag{1.86}$$

where ψ^a are the Euler-Lagrange equations associated with A_a and ψ^{ab} are the Euler-Lagrange equations associated with F_{ab} . For the equation to be true for arbitrary a^a we must have

$$T^{a}_{b} = -\psi^{a} A_{b} - \psi^{ac} F_{bc} - \psi^{da} F_{dc}. \tag{1.87}$$

Furthermore, we recognize T_0^0 as the naive Hamiltonian with opposite sign. In other words we have

$$h = -T_0^0 = \psi^0 A_0 + \psi^{0c} F_{0c} + \psi^{d0} F_{d0}$$
 (1.88)

We get the Hamiltonian H_{naive} by elimination of time derivatives in favour of momenta in the expression for h. Immediately, we see that if the action depends only on scalar fields, the Hamiltonian will be zero. This is an important result that we can state as a theorem:

1.5 Methods

Theorem 1 Diffeomorphism invariant theories with Lagrangians that depend on scalar-transforming fields and their first derivatives have vanishing (naive) Hamiltonian.

The same result has been proved by von Unge in [12].

The significance of the more general result (1.88) is perhaps not obvious. For the case of scalar fields it is of course simple and easily applicable. If we also have vectors or higher rank tensors, (1.88) gives certainly not any simpler route for a calculation of the Hamiltonian than its definition itself. On the other hand, it allows us to make an interesting interpretation. Observe that the Hamiltonian is proportional to the ψ 's, which by the equations of motions (Euler-Lagrange equations) are zero. This means that at any point on the classical path the Hamiltonian will be zero, since the ψ 's are then zero. In this sense the we say that the Hamiltonian is dynamically zero.

Theorem 2 Diffeomorphism invariant theories with a Lagrangian that depends only on tensor fields (of any rank) and their derivatives have a Hamiltonian that is dynamically zero.

This result is not completely general, since we have still considered only tensor fields. It is not necessarily true if the Lagrangian depends on e.g. spinors. In this thesis, however, we will consider only fields of the first kind.

An example that validates the result (1.88) is given in section 2.5.3. There we will derive the Hamiltonian for the Polyakov string both directly from its definition, and using the result in this section.

1.5 Methods

The main purpose of this thesis is to describe and apply two methods for deriving high energy limits of various actions. This section is devoted to a general description of these methods. The first is the simplest. It can be applied to any model, although it does not always lead o any interesting field equations. The second requires more calculations, but makes it at the same time possible to derive several limits. The limit found by the first method is usually one of these.

The starting point is an action of the form

$$S = T \int dx \mathcal{L}(\phi, \partial \phi), \tag{1.89}$$

where T is some dimensionful constant, like mass or string tension. It is a basic assumption that we can write the Lagrangian as $L = T\mathcal{L}$. The quantity \mathcal{L} can be called the reduced Lagrangian, since we have taken out the constant T. This action is clearly not very suitable for studying the $T \to 0$ limit. The philosophy now is to search for an action that is classically equivalent to (1.89) as long as $T \neq 0$, but also well defined for T = 0. We will then treat this new action (with T = 0 inserted) as a $T \to 0$ limit of the original model. The methods described below are systematic ways for finding such actions.

1.5.1 Method I: Auxiliary field

This is the simplest approach, and involves the introduction of an auxiliary field χ . A reference for the this method is Karlhede and Lindström [13]. We use the \mathcal{L} from the original action and write

$$S_{\chi} = \frac{1}{2} \int dx (\chi \mathcal{L}^2 + \frac{T^2}{\chi}). \tag{1.90}$$

This action is equivalent to (1.90). To show this explicitly, we solve the equations of motion for χ :

$$\delta \chi \Rightarrow \delta S_{\chi} = \frac{1}{2} \int dx (\delta \chi \mathcal{L}^2 - \frac{T^2}{\chi^2} \delta \chi)$$

$$= \frac{1}{2} \int dx (\mathcal{L}^2 - \frac{T^2}{\chi^2}) \delta \chi.$$

Using Hamilton's principle and demanding $\delta S_{\chi}=0$ for arbitrary variations $\delta\chi$ gives

$$\mathcal{L}^2 - \frac{T^2}{\chi^2} = 0$$

$$\chi = \frac{T}{L}; \quad \text{when} \quad T \neq 0.$$
(1.91)

If we put this back into (1.90) we get

$$S_{\chi} = \frac{1}{2} \int dx (\frac{T}{\mathcal{L}} \mathcal{L}^2 + \frac{\mathcal{L}}{T} T^2) = T \int dx \mathcal{L} = S.$$

Thus the two actions S and S_{χ} are equivalent for $T \neq 0$. In addition S_{χ} allows us to take the $T \to 0$ limit simply by setting T = 0 in the action. This gives

$$S_{\chi}^{T=0} = \frac{1}{2} \int dx \chi \mathcal{L}^2. \tag{1.92}$$

You may ask what this new field χ really is. From the current point of view we cannot say anything more than we already have – that it helps us in our calculations. Hence the name *auxiliary* field.

Note however, that in the simplest case of a massless particle in section 2.2.2 we are lead to interpret χ as the einbein.

A general remark on symmetry properties can already be made. Consider diffeomorphism invariance. We know that the integral measure transforms as $dx \to dxJ^{-1}$, where J is the Jacobi determinant as defined by equation (1.48). If the original action is to be diff invariant, the Lagrangian must transform as a density, i.e. $\mathcal{L} \to J\mathcal{L}$. We see then that S_{χ} is also diff invariant if we demand χ to be an inverse density (i.e. scalar density of weight -1, c.f. appendix A.1). And since χ was introduced as an auxiliary field with no a priori physical interpretation, this transformation property is something we can impose on χ .

1.5 Methods 21

Dynamics

A variation of χ gives one equation of motion,

$$\delta \chi \Rightarrow \qquad \mathcal{L}^2 = 0 \quad \Rightarrow \quad \mathcal{L} = 0. \tag{1.93}$$

The effect of a variation in ϕ^i , on the other hand, depends on the form of \mathcal{L} :

$$\delta\phi^{i} \Rightarrow \delta S = \frac{1}{2} \int dx \chi 2\mathcal{L}\delta\mathcal{L}$$

$$= \int dx \chi \mathcal{L} \left[\frac{\partial \mathcal{L}}{\partial \phi^{i}} \delta\phi^{i} + \frac{\partial \mathcal{L}}{\partial (\partial_{a}\phi^{i})} \partial_{a}\delta\phi^{i} \right]$$

$$= \int dx \left[\chi \mathcal{L} \frac{\partial \mathcal{L}}{\partial \phi^{i}} - \partial_{a} \left(\chi \mathcal{L} \frac{\partial \mathcal{L}}{\partial (\partial_{a}\phi^{i})} \right) \right] \delta\phi^{i}.$$

The field equation is found from demanding $\delta S = 0$ for arbitrary $\delta \phi^i$. The result is

$$\delta \phi \Rightarrow \chi \mathcal{L} \frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_a \left(\chi \mathcal{L} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi^i)} \right) = 0.$$
 (1.94)

Often, this equation will reduce to an identity by use of equation (1.93), $\mathcal{L} = 0$. But it does give non-trivial equations in cases where $\frac{\partial \mathcal{L}}{\partial \phi^i} \sim \frac{1}{\mathcal{L}}$ or $\frac{\partial \mathcal{L}}{\partial (\partial_a \phi^i)} \sim \frac{1}{\mathcal{L}}$. (Then the factors \mathcal{L} are eliminated from equation (1.94).) However, this is not the general situation, so method I has limited applicability.

1.5.2 Method II: Phase space

This method of arriving at an action that admits taking the $T \to 0$ limit is designed for constrained systems. Demonstrations of the method can be found in [8,13–18], but also later in this thesis. Again we start from the action (1.89). We derive the canonical conjugate momenta

$$\pi_i = \frac{\partial L}{\partial \dot{\phi}^i},\tag{1.95}$$

and find the total Hamiltonian as in section 1.2.2,

$$H = H' + \lambda^m \theta_m. \tag{1.96}$$

The derivations of the total Hamiltonian H involves working out the constraint structure, which can be a cumbersome task. But since we are interested only in the limit T=0, these can be simplified by putting T=0 as early as possible.

Now, having found the total Hamiltonian, we write down the phase space action

$$S^{PS} = \int dx \left(\pi_i \dot{\phi}^i - H(\phi, \pi, \nabla \phi) \right). \tag{1.97}$$

The momenta π_i can then be eliminated by solving their equations of motion. (This is often called "integrating out the momenta", a notation which is natural in the

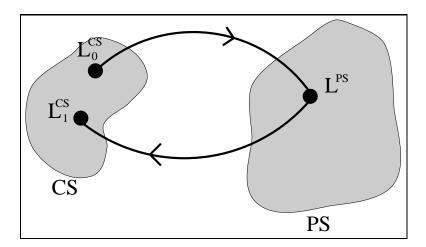


Figure 1.1: We start off with a Lagrangian L_0^{CS} in configuration space (CS), and perform a Legendre transform to the phase space (PS) Lagrangian L^{PS} . When we go back to configuration space we may, for constrained systems, end up with a new (but equivalent) Lagrangian $L_1^{CS} \neq L_0^{CS}$. And although L_0^{CS} is not defined in the limit T = 0, L_1^{CS} may be.

context of path integrals. In that case we go from phase space to configuration space by literary integrating out the momenta from the functional integral.) Substituting for the solutions of π_i , we arrive at the configuration space action

$$S^{CS} = \int dx \left[\pi(\phi, \partial \phi) \dot{\phi} - H(\phi, \pi(\phi, \partial \phi), \nabla \phi) \right]. \tag{1.98}$$

Unless the system under study is non-constrained (giving $H = H_{naive}$) this action will contain something new compared to the one we started with. In other words, it is different from the original configuration space action, but still equivalent (see figure 1.1) to it. Hopefully the new aspects make it possible to take the $T \to 0$ limit. What is new are the Lagrange multipliers, which are now independent fields. As discussed later, these may often be reinterpreted as components of some metric, or as (degenerated) vielbeins.

In cases where there are no constraints imposed from the definition of the momentum, the calculations above will give only a circle where we end up at the point we started. However, it will show possible (c.f. 2.5) via some redefinitions in phase space to allow for the $T \to 0$ limit in a sensible way. This situation is better discussed when it appears.

Chapter 2

Bosonic strings

2.1 Introduction to strings and branes

Short history of strings

Strings originated in the late sixties as a model describing strong interactions [19]. Quarks are known always to exist in bound states, and the string approach was a proposal for explaining this quark confinement. Very simplified, the picture was that of quarks attached to strings.

This theory was pushed aside by the successful QCD (quantum chromo-dynamics) theory. But in 1974, Scherk and Schwarz [20] made the remarkable suggestion that string theory was a correct mathematical theory of a different problem, the unification of elementary particle interactions with gravity.

After this the theory attained much attention, but had no real breakthrough. Many properties made it attractive, but the problems were too serious. However, with the introduction of supersymmetry (a symmetry between bosons and fermions) into the superstring theory [21–23] (in contrast to the old bosonic string theory), a lot of the problems disappeared. And this is the theory that has attracted enormous attention over the last fifteen years. In these years there have appeared different types of superstring theories, but they are now thought to be limits of one fundamental theory, which is called M-theory (or matrix theory), and is for the moment under constant investigation.

String theory

Strings are one-dimensional objects with a length of the order of the Planck length, 10^{-34} m. They are free to vibrate, much like ordinary guitar strings. The possible modes of vibration are determined by the string tension, which is the only fundamental parameter in string theory.

We know that at larger scales the strings must behave as particles with certain masses. The model is that the different vibration modes of the one fundamental string, give rise to the whole zoo of particles we know from elementary particle physics. Different vibration modes mean different frequencies or energies, and hence, different masses of the particles. Thus, in principle, string theory could be used to *derive* the mass spectrum of all particles. This is one of the ultimate goals, but is in practice very difficult.

As was mentioned above, this theory is a promising candidate for a unification of quantum field theory (elementary particle physics) and general relativity (gravitation). Actually, string theory is not consistent without gravity. This aspect of string theory is probably the most important, but there are also other sides that make such a large number of theoretical physicists talk warmly about it.

Another of its advantages is that it avoids altogether the divergences that unavoidably appear in quantum field theories. This is due to the fact that strings are not pointlike, but have extension in space. Also in contrast to quantum field theories, string theory involves no arbitrary choice of gauge symmetry group and choice of representation: string theory is essentially unique.

The last comment is an example of a general attractive feature of string theory, namely that there is very little freedom. Once the basic theory is formulated, important results follow directly or by consistency. The spacetime dimension is also fixed in this way. Superstring theory is consistent only with 10 (1 + 9) spacetime dimensions. (For bosonic strings there must be 26 dimensions.) At first this may sound like a catastrophe, since we know by everyday experience that the world is 4-dimensional (1 + 3). The way to handle this difficulty, is to say that the extra 6 dimensions are compactified, or curled up so that they play no role at large scales. This is very much an *ad hoc* assumption, but at least string theory gives a way to understand the spacetime dimensionality.

In this thesis we will not consider the supersymmetric string theory (superstrings), but only different models for *bosonic* strings. Bosonic string theory does not have fermions, and is not a realistic theory. However, it is a good starting point, and gives insight in crucial aspects of the more realistic models as well.

What is real?

To be or not to be, that is the question. Hamlet certainly had other things than elementary particle physics in mind, but his question is indeed fundamental for our discussion as well: Do strings really exist, or do they not? And are strings the fundamental building blocks of the universe?

The traditional view is that the world is built up of pointlike particles. These are "things" with certain properties like mass and spin, but without extension in space. In other words, they are thought to be zero-dimensional. String theory changes this view only insofar as the strings are not points, but have extension in one dimension, i.e. are one-dimensional.

With such views we are immediately faced with the question of model versus reality. Is our model only a mathematical construction that by coincidence happens to resemble the physical reality, or does the success of the model give a deeper understanding of *reality* itself?

To meet this question, we must know what we mean by reality. What does it actually mean that something is real? One answer to this is to say that the real world is the *observable* world. This description works quite well in everyday life. However, when it comes to elementary particles, the task of observing becomes very difficult. For instance, our understanding of *shape* is useless at such small scales. The quantum theory of physics, with its Schrödinger equation and Heisenberg relation, leads us to view the elementary particles as rather diffuse objects that are somehow smeared out in spacetime.

Returning then to the question of the being of the fundamental building blocks, one possible answer is to say that they *are* neither particles nor strings. But they *behave* at very small scales as strings, and at larger scales as points. And someone has also said that *you are what you do*.

Branes

As we have accepted the leap from points to strings, it is natural to go further and consider even higher-dimensional objects. Perhaps the fundamental objects are membranes, two-dimensional surfaces. Or why not p-dimensional p-branes?

p-Brane¹ theory is the obvious generalization from string theory. However, strings seem to be special among the branes with their success as a fundamental model. For instance, increasing the world surface dimensionality increases the probability of finding divergences (from integrations on the world surface) similar to those found in quantum field theories.

A non-technical introduction to string theory is found in [24–26]. Thorough textbooks on string theory are [23, 27].

2.2 The point particle

It is now time to do some real calculations to demonstrate how everything we have said so far applies. And we naturally start with the simplest possible case, the relativistic point particle.

2.2.1 The action

The action of a relativistic point particle can be written as its mass times the length of its world line:

$$S = m \int ds. (2.1)$$

This length is a distance in spacetime. We let τ parameterize the world line, and let $X^{\mu}(\tau)$ be the particle's position at some moment. Then we can write

$$ds^{2} = -G_{\mu\nu}dX^{\mu}dX^{\nu} = -dX^{\mu}dX_{\mu} = -\frac{dX^{\mu}}{d\tau}\frac{dX_{\mu}}{d\tau}d\tau^{2}.$$
 (2.2)

¹ The notation is such that a point particle is called a 0-brane, and a string is called a 1-brane.

 $G_{\mu\nu}$ denotes the spacetime metric. With the metric signature $(-,+,\ldots,+)$, a timelike vector v^{μ} has negative norm, $v^2 < 0$. And since $\frac{dX^{\mu}}{d\tau}$ is timelike, the overall minus sign above is a conventional choice to make ds^2 a positive quantity. We take the square root of (2.2) and plug back in (2.1) to get the more familiar expression for the relativistic point particle action:

$$S = m \int d\tau \sqrt{-\dot{X}^{\mu} \dot{X}_{\mu}} = m \int d\tau \sqrt{-\dot{X}^2}, \qquad (2.3)$$

where $\dot{X}^{\mu} \equiv \frac{\partial X^{\mu}}{\partial \tau} = \frac{dX^{\mu}}{d\tau}$. This action is reparameterization (diff) invariant by construction. Since it is covariantly written as a scalar, and only contains derivatives of X^{μ} , it easy to see that it is also Poincaré invariant.

To show that (2.3) is indeed an appropriate action for the point particle, and to give an example of how Hamilton's principle can be applied, we will now deduce the equations of motion from this action (in Minkowski space). Consider a small variation δX^{α} in X^{α} . This will give a small variation δS in the action, which we find as follows.

$$\delta X^{\alpha} \Rightarrow \delta S = m\delta \int d\tau \sqrt{-\dot{X}^2} = m \int d\tau \delta \sqrt{-\dot{X}^2}$$

$$= -m \int d\tau \frac{1}{2} (-\dot{X}^{\mu} \dot{X}^{\nu})^{-\frac{1}{2}} 2\dot{X}_{\alpha} \delta \dot{X}^{\alpha}$$
(2.4)

The order of variation and differentiation can be interchanged, so we have $\delta \dot{X}^{\alpha} = \frac{d}{d\tau}(\delta X^{\alpha})$. This, together with a partial integration gives

$$\delta S = -m \int d\tau \frac{-\dot{X}_{\alpha}}{\sqrt{\dot{X}^{2}}} \frac{d}{d\tau} (\delta X^{\alpha})$$

$$= -m \int d\tau \left(\underbrace{\frac{d}{d\tau} \left[\frac{-\dot{X}_{\alpha}}{\sqrt{\dot{X}^{2}}} \delta X^{\alpha} \right]}_{\to 0} - \frac{d}{d\tau} \left[\frac{\dot{X}_{\alpha}}{\sqrt{-\dot{X}^{2}}} \right] \delta X^{\alpha} \right)$$
(2.5)

The first part leads to a boundary term, which gives zero contribution since the fields are held fixed at the boundaries. Hamilton's principle states that the action must be extremal for the dynamically allowed fields. In other words, we must have $\delta S = 0$ for arbitrary δX^{α} . Thus we end up with the equation

$$\frac{d}{d\tau} \left[\frac{m\dot{X}_{\alpha}}{\sqrt{-\dot{X}^2}} \right] = 0. \tag{2.6}$$

We recognize the quantity within the brackets as the relativistic *momentum* for a point particle. (We will soon recover it by direct calculation.) The equation then says that the momentum is conserved, which is a well known consequence of translational invariance.

Furthermore, if we let τ be proper time, we have $\eta_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}=-1$. In this case the equations of motion read

$$\ddot{X}^{\mu} = 0, \tag{2.7}$$

i.e. the acceleration is zero. This is the familiar result for a free point particle.

We would deduce the same equation of motion if we started from the alternative action $S = m \int d\tau \dot{X}^2$. But this action has the disadvantage that the parameter τ has to be the proper time, i.e. it is not reparameterization invariant. For this reason we will in the following consider the action (2.3).

Massless limit From now on we will focus on the massless limit $(m \to 0)$ of the action (2.3) above. We read off the Lagrangian and find

$$L = m\mathcal{L}(X, \dot{X}) = m\sqrt{-\dot{X}^{\mu}\dot{X}_{\mu}}.$$
 (2.8)

This will now be our starting point for a discussion on the massless limit. We know that a particle's energy can be split into the rest energy, which is a constant $(E_0 = m)$ and a kinetic energy. If the total energy becomes very high, it is a good approximation to neglect the rest energy. Thus the massless limit represents a high energy limit of the point particle.

The derivations presented here for the point particle are also found in [15].

2.2.2 Method I

As explained before, the easiest way to find a massless limit is by introduction of an auxiliary field χ . We write, according to the general theory,

$$S_{\chi} = \frac{1}{2} \int d\tau (\chi \mathcal{L}^2 + \frac{m^2}{\chi})$$
$$= \frac{1}{2} \int d\tau (-\chi \dot{X}^{\mu} \dot{X}_{\mu} + \frac{m^2}{\chi}). \tag{2.9}$$

The massless limit is obtained directly by putting m = 0.

$$S_{\chi}^{m=0} = -\frac{1}{2} \int d\tau \chi \dot{X}^{\mu} \dot{X}_{\mu}. \tag{2.10}$$

It was mentioned as part of the motivation for studying the massless (or tensionless) limits that they lead to conformal invariant theories. We will now see an example of this. First, we note that the action (2.10) is obviously *Poincaré* invariant.

Under the dilatation $X^{\mu} \to (1+c)X^{\mu}$, we get $\dot{X}^2 \to (1+c)^2 \dot{X}^2 = (1+2c)\dot{X}^2$ to first order in c. The action will then be invariant under dilatations if χ transforms as

$$\delta_c X^{\mu}: \quad \chi \to (1 - 2c)\chi$$
 (2.11)

to first order.

The special conformal transformations $X^{\mu} \to X^{\mu} + b \cdot XX^{\mu} - \frac{1}{2}X^2b^{\mu}$ give to first order in b: $\dot{X}^2 \to (1+2b\cdot X)\dot{X}^2$. Thus the action will be invariant if χ transforms as

$$\delta_b X^\mu: \quad \chi \to (1 - 2b \cdot X)\chi.$$
 (2.12)

The field χ is an auxiliary field, so the transformation properties (2.11) and (2.12) are something we can *impose* on χ . And given these properties, we see that the action is conformally invariant. Thus, in the massless limit, we have that the original Poincaré symmetry is enlarged to full conformal symmetry.

The equations of motion derived from (2.10) are

$$\delta \chi \Rightarrow \qquad \dot{X}^2 = 0, \tag{2.13}$$

$$\delta \chi \Rightarrow \qquad \dot{X}^2 = 0,$$

$$\delta X_{\mu} \Rightarrow \qquad \frac{d}{d\tau} (\chi \dot{X}^{\mu}) = 0. \tag{2.13}$$

 \dot{X}^{μ} is a tangent to the world line, so the first equation says that the particle follows a lightlike (or null-) curve. With the conformal symmetry in mind, this is a natural result, since conformal transformations are transformations that preserve the light cone (c.f. section 1.3.4).

An interesting observation is that the action (2.9) leads us to identify χ as an inverse einbein field (c.f. appendix A.3). In [28] it is shown that the action for a point-particle coupled to one-dimensional gravity through the einbein field $e=e_1^1$ can be written

$$S = \frac{1}{2} \int d\tau \left(\frac{1}{e} \dot{X}^2 - em^2 \right), \tag{2.15}$$

if we disregard the spin. And by comparison with (2.9) we have the identification $\chi = -\frac{1}{e}$. Let us now see how this compares to the results of method II.

2.2.3 Method II

From the Lagrangian $L(X, \dot{X}) = m\sqrt{-\dot{X}^2}$ we find the canonical momenta

$$P_{\mu} = \frac{\partial L}{\partial \dot{X}^{\mu}} = -\frac{m\dot{X}_{\mu}}{\sqrt{-\dot{X}^2}}.$$
 (2.16)

This is the familiar relativistic expression for the momentum of a free particle. Notice also that this form sesults independently of which metric is used. This is true because a metric is a function of positions only, not of their derivatives; $G_{\mu\nu} = G_{\mu\nu}(X)$.

The Hamiltonian is defined as

$$H_{naive} = P_{\mu}\dot{X}^{\mu} - L(X, \dot{X}). \tag{2.17}$$

The fields X^{μ} are scalars under diffeomorphisms, so the naive Hamiltonian vanishes, as discussed in section 1.4.2. This can also easily be seen explicitly by noting that

$$P_{\mu}\dot{X}^{\mu} = \frac{-m\dot{X}_m}{\sqrt{-\dot{X}^2}}\dot{X}^{\mu} = m\sqrt{-\dot{X}^2} = L. \tag{2.18}$$

The expression for P_{μ} is not invertible so in accordance with what we said in section 1.2.3 there must exist some constraints. And indeed we find

$$P^{2} \equiv P^{\mu} P_{\mu} = \frac{m \dot{X}^{\mu}}{\sqrt{-\dot{X}^{2}}} \frac{m \dot{X}_{\mu}}{\sqrt{-\dot{X}^{2}}} = m^{2} \frac{\dot{X}^{2}}{-\dot{X}^{2}} = -m^{2}$$

$$\Rightarrow P^{2} + m^{2} = 0. \tag{2.19}$$

Since the naive Hamiltonian vanishes, the total Hamiltonian is made from the constraint as follows

$$H = \lambda(P^2 + m^2), \tag{2.20}$$

where the coefficient λ is a Lagrange multiplier. We write the phase space action as described earlier,

$$S^{PS} = \int d\tau \left(P_{\mu} \dot{X}^{\mu} - \lambda (P^{\mu} P_{\mu} + m^2) \right). \tag{2.21}$$

Now it is time to start simplifying and return to configuration space. To do so we need to eliminate the momenta. A variations in P_{μ} gives

$$\delta P_{\mu} \Rightarrow \delta S^{PS} = \int d\tau (\dot{X}^{\mu} - 2\lambda P^{\mu}) \delta P_{\mu}.$$
 (2.22)

For δS^{PS} to be zero for arbitrary (though infinitesimal) variations δP_{μ} , we need to have

$$\dot{X}^{\mu} - 2\lambda P^{\mu} = 0$$

$$P^{\mu} = \frac{\dot{X}^{\mu}}{2\lambda}.$$
(2.23)

Plugging this back into (2.21) we end up with a configuration space action,

$$S^{CS} = \int d\tau \left(\frac{\dot{X}_{\mu}}{2\lambda} \dot{X}^{\mu} - \lambda \left(\frac{\dot{X}^{\mu} \dot{X}_{\mu}}{4\lambda^2} + m^2 \right) \right)$$
$$= \frac{1}{2} \int d\tau \left(\frac{1}{2\lambda} \dot{X}^{\mu} \dot{X}_{\mu} - 2\lambda m^2 \right). \tag{2.24}$$

Comparison with the action (2.9) we found by using method I, reveals that the two methods give exactly the same result. We just have to identify the auxiliary field χ with the Lagrange multiplier as $\chi = (-2\lambda)^{-1}$.

A discussion of the massless limit m=0 was given in the previous section.

2.3 The Nambu-Goto string

2.3.1 The action

The action for the point particle is proportional to the length of its world line. This suggests that we can generalize to a string which sweeps out a world *surface* and say

that its action is proportional to the area of this surface. In mathematical terms, we have

 $S = T \int dA. \tag{2.25}$

T will have have the dimension of energy/length, or $(length)^{-2}$ in natural units. We thus call it the string tension. It plays a role analogous to the particle mass.

Let us denote the spacetime metric by $G_{\mu\nu}$. If ξ^a ; a=0,1 are world sheet coordinates that parameterize the world surface, we can write the spacetime points of the surface as $X^{\mu} = X^{\mu}(\xi)$. The *induced metric* γ_{ab} is then given by (see appendix A.5)

$$\gamma_{ab} = G_{\mu\nu} \frac{\partial X^{\mu}}{\partial \xi^{a}} \frac{\partial X^{\nu}}{\partial \xi^{b}} = G_{\mu\nu} \partial_{a} X^{\mu} \partial_{b} X^{\nu}. \tag{2.26}$$

We write the inverse of this matrix as γ^{ab} , i.e. $\gamma^{ab}\gamma_{bc} = \delta^a_c$, and its determinant simply as $\gamma \equiv \det(\gamma_{ab})$.

With the introduction of the induced metric, an infinitesimal area element of a 2-dimensional surface embedded in spacetime can be written as (see e.g. [29])

$$dA = \sqrt{-\det(\gamma_{ab})} d\xi^0 d\xi^1. \tag{2.27}$$

Then we can write the string action as

$$S = T \int d^2 \xi \sqrt{-\gamma}. \tag{2.28}$$

This is the famous Nambu-Goto form [30,31] of the action for a (bosonic) string.

Tensionless limit The high-energy limit of the strings has been studied with different approaches. For a short review, and a list of references, the reader may consult [14].

In analogy with the point particle, we expect the tensionless limit $T \to 0$ of strings to give insight into the high-energy behaviour, just as the massless limit of particles does. Schild [32] was the first to study this limit. Later, the tensionless limit of strings (not just the Nambu-Goto string) has been studied by several authors [13–17,33–36].

We will in the following go through the derivation of different tensionless limits, starting from the Nambu-Goto string (2.28).

2.3.2 Method I

The (reduced) Lagrangian is $\mathcal{L}(X, \partial X) = \sqrt{-\gamma}$, and following the general recipe, we write

$$S_{\chi} = \frac{1}{2} \int d^{2}\xi (\chi \mathcal{L}^{2} + \frac{T^{2}}{\chi})$$

$$= -\frac{1}{2} \int d^{2}\xi (\chi \gamma - \frac{T^{2}}{\chi}), \qquad (2.29)$$

where χ is the auxiliary field. This action is equivalent to (2.28), and allows us to take the T=0 limit. We find simply

$$S_{\chi}^{T=0} = -\frac{1}{2} \int d^2 \xi \ \chi \gamma \tag{2.30}$$

Variations in χ and X^{μ} give the equations of motion:

$$\delta \chi \Rightarrow \qquad \gamma \equiv \det(\gamma_{ab}) = 0,$$
 (2.31)

$$\delta \chi \Rightarrow \qquad \gamma \equiv \det(\gamma_{ab}) = 0, \tag{2.31}$$

$$\delta X_{\mu} \Rightarrow \qquad \partial_{b} \left[\chi \epsilon^{ac} \epsilon^{bd} \gamma_{cd} \partial_{a} X^{\mu} \right] = 0. \tag{2.32}$$

The induced metric is degenerate, which means that the surface is a null surface. It means that the world surface has tangent vectors v^a that are null (lightlike), i.e. $v^2 = 0$. Tensionless strings are for this reason often referred to as null-strings.

A difference from the point particle is that we cannot give χ a geometric interpretation.

Conformal invariance As was the case for the point particle, this action is conformally invariant given that χ transforms in a special way. If we now for a moment generalize to D-dimensional surfaces, the induced metric (which is then D-dimensional) transforms under Poincaré, dilatation and special conformal transformations as

$$\delta_{\omega,a}X^{\mu}: \qquad \gamma_{ab} \to \gamma_{ab},
\delta_{c}X^{\mu}: \qquad \gamma_{ab} \to (1+2c)\gamma_{ab},
\delta_{b}X^{\mu}: \qquad \gamma_{ab} \to (1+2b^{\nu}X_{\nu})\gamma_{ab}.$$
(2.33)

Thus, the combination $\chi\gamma$ in (2.30) is conformally invariant provided that χ transforms as

$$\delta_{\omega,a} X^{\mu}: \qquad \chi \to \chi,
\delta_{c} X^{\mu}: \qquad \chi \to (1 - 2Dc)\chi,
\delta_{b} X^{\mu}: \qquad \chi \to (1 - 2Db^{\nu} X_{\nu})\chi. \tag{2.34}$$

Putting D=2 gives the string result, which we are interested in here.

2.3.3 Method II

As for a point particle we start by deriving the momenta. Defining $\dot{X}^{\mu} \equiv \partial_0 X^{\mu} = \frac{\partial X^{\mu}}{\partial \xi^0}$ and $\hat{X}^{\mu} \equiv \partial_1 X^{\mu} = \frac{\partial X^{\mu}}{\partial \xi^1}$, we get

$$P_{\mu} \equiv \frac{\partial L}{\partial \dot{X}^{\mu}} = T \frac{\partial}{\partial \dot{X}^{\mu}} \sqrt{-\gamma} = \frac{T}{2\sqrt{-\gamma}} \frac{\partial (-\gamma)}{\partial \dot{X}^{\mu}}$$

$$= \frac{T}{2\sqrt{-\gamma}} \frac{\partial}{\partial \dot{X}^{\mu}} \left((G_{\alpha\beta} \dot{X}^{\alpha} \dot{X}^{\beta})^{2} - (G_{\alpha\beta} \dot{X}^{\alpha} \dot{X}^{\beta}) (G_{\alpha\beta} \dot{X}^{\alpha} \dot{X}^{\beta}) \right)$$

$$= \frac{T}{\sqrt{-\gamma}} \left((\dot{X}^{\alpha} \dot{X}_{\alpha}) \dot{X}_{\mu} - (\dot{X}^{\alpha} \dot{X}_{\alpha}) \dot{X}_{\mu} \right). \tag{2.35}$$

As was the case for the point particle, we cannot use this expression for P_{μ} to solve for \dot{X}^{μ} , i.e. it is not invertible. Therefore we look for constraints. The constraints should not be functions of time derivatives (\dot{X}^{μ}) , and this limits the number of possible candidates. Anyway, we find

$$P^{2} = \frac{T^{2}}{-\gamma} ((\dot{X} \cdot \acute{X}) \acute{X}^{\mu} - (\acute{X}^{\mu})^{2} \dot{X}^{\mu}) ((\dot{X} \cdot \acute{X}) \acute{X}_{\mu} - \acute{X}_{\mu}^{2} \dot{X}_{\mu})$$

$$= \frac{T^{2}}{-\gamma} \underbrace{(\dot{X}^{2} \acute{X}^{2} - (\dot{X} \cdot \acute{X})^{2})}_{\gamma} = -T^{2} \acute{X}^{2}$$

$$P_{\mu} \acute{X}^{\mu} = \frac{T}{\sqrt{-\gamma}} ((\dot{X} \cdot \acute{X})^{2} \acute{X}^{2} - \acute{X}^{2} \dot{X} \cdot \acute{X}) = 0. \tag{2.36}$$

Thus, we have these primary constraints

$$\theta_0 \equiv P^2 + T^2 \acute{X}^2 \approx 0; \qquad \theta_1 \equiv P^\mu \acute{X}^\mu \approx 0.$$
 (2.37)

Also, we notice that

$$P_{\mu}\dot{X}^{\mu} = \frac{T}{\sqrt{-\gamma}} \left((\dot{X} \cdot \acute{X})^2 - \acute{X}^2 \dot{X}^2 \right) = T\sqrt{-\gamma} = L.$$
 (2.38)

So the naive Hamiltonian vanishes, in accordance with the general discussion in section 1.4.2. The primary constraints do not give rise to secondary constraints. Thus, the total Hamiltonian can be written as a sum of θ_0 and θ_1 ,

$$H = \lambda (P^2 + T^2 \acute{X}^2) + \rho P_{\mu} \acute{X}^{\mu}, \tag{2.39}$$

where λ and ρ are Lagrange multipliers. This gives the phase space action

$$S^{PS} = \int d^2 \xi \left[P_{\mu} \dot{X}^{\mu} - \lambda (P_{\mu} P^{\mu} + T^2 \dot{X}^{\mu} \dot{X}_{\mu}) - \rho P_{\mu} \dot{X}^{\mu} \right]. \tag{2.40}$$

The equations of motion for P_{μ} gives

$$\dot{X}^{\mu} - 2\lambda P^{\mu} - \rho \dot{X}^{\mu} = 0$$

$$P^{\mu} = \frac{\dot{X}^{\mu} - \rho \dot{X}^{\mu}}{2\lambda}.$$
(2.41)

Substituted back into (2.40) we find after a few rearrangements

$$S^{CS} = \int d^2 \xi \frac{1}{4\lambda} \left[\dot{X}^2 - 2\rho \dot{X} \cdot \dot{X} + (\rho^2 - 4T^2\lambda^2) \dot{X}^2 \right]. \tag{2.42}$$

A Weyl-invariant action We can now identify λ and ρ as components of a metric field in this way:

$$g^{ab} = \Omega \begin{pmatrix} 1 & -\rho \\ -\rho & \rho^2 - 4T^2\lambda^2 \end{pmatrix}, \tag{2.43}$$

where Ω is any scaling function. We have then $g = \det(g_{ab}) = \frac{1}{4T^2\lambda^2\Omega^2}$, and we can write (2.42) as

$$S = \frac{T}{2} \int d^2 \xi \sqrt{-g} g^{ab} \gamma_{ab}. \tag{2.44}$$

This is the Weyl-invariant string action, which is further discussed in section 2.5.

However, by this rewriting we are in no better position to study the limit $T \to 0$. To do so, we go back to (2.42), and make another interpretation of the Lagrange multipliers.

Limit one

Following [15] we introduce an auxiliary vector density (of weight $-\frac{1}{2}$ – see appendix A.1 for an explanation of what is meant by density),

$$V^a = \frac{1}{2\sqrt{\lambda}}(1, -\rho). \tag{2.45}$$

Noting that we can write

$$\lambda^2 \acute{X}^2 = \frac{\gamma_{11}}{4V^0 V^0} = \frac{\gamma \gamma^{00}}{4V^0 V^0}$$

we see that we can write the action (2.42) in the simpler form:

$$S^{CS} = \int d^2 \xi \left(V^a V^b \gamma_{ab} - \frac{T^2 \gamma \gamma^{00}}{V^0 V^0} \right). \tag{2.46}$$

The tensionless limit is then easily found if we set T=0. We end up with

$$S^{T=0} = \int d^2\xi \ V^a V^b \gamma_{ab}. \tag{2.47}$$

The equations of motion follow from variations in V^a and X^{μ} ,

$$\delta V^a \Rightarrow V^b \gamma_{ab} = 0, \qquad (2.48)$$

$$\delta X^\mu \Rightarrow \partial_b (V^a V^b \partial_a X^\mu) = 0. \qquad (2.49)$$

$$\delta X^{\mu} \Rightarrow \partial_b (V^a V^b \partial_a X^{\mu}) = 0.$$
 (2.49)

The first equation means that γ_{ab} has eigenvectors with zero eigenvalue, which again means that the determinant of the induced metric is zero, i.e. $\det(\gamma_{ab}) = 0$. We have thus the same situation as we found using method I. We may say that the two limits represent the same physical situation, but are differently formulated. And accepting that there must be a formal difference between the two results is not difficult, as we have in the present case one extra variable. (V^a are two variables, while χ is one.)

Owing to the diff invariance, we can choose a gauge (the transverse gauge) where $V^a = (v, 0)$, where v is a constant. The equations of motion then reduce to

$$\ddot{X}^{\mu} = 0; \qquad \dot{X}^2 = \dot{X}^{\mu} \acute{X}_{\mu} = 0.$$
 (2.50)

We conclude that the tensionless string behaves as a collection of massless particles moving transversally to the direction of the string.

We said above that V^a are densities, which is necessary to preserve worldsheet diff invariance. We can interpret the V^a fields further by comparing with the Weylinvariant form (2.44), which by introduction of inverse zweibeins e_A^a (c.f. appendix A.3) can be written

$$S = \frac{T}{2} \int d^2 \xi \ e \eta^{AB} e_A{}^a e_B{}^b \gamma_{ab} = \frac{1}{2} \int d^2 \xi \eta^{AB} \tilde{e}_A{}^a \tilde{e}_B{}^b \gamma_{ab}, \tag{2.51}$$

where $\tilde{e}_A{}^a \equiv \sqrt{Tee}_A{}^a$ are densities, and a,b,A,B=0,1. Clearly the limit $T\to 0$ (2.47) corresponds to the case when the zweibeins has become parallel, and we can make the substitution $\tilde{e}_A{}^a \to V^a$.

From the transformation properties (2.33) of γ_{ab} , it is easy to check that (2.47) is conformally invariant provided V^a transform under dilatations and special conformal transformations as

$$\delta_c X^{\mu}: \qquad V^a \to (1-c)V^a, \tag{2.52}$$

$$\delta_c X^{\mu}: V^a \to (1-c)V^a,$$
(2.52)
$$\delta_b X^{\mu}: V^a \to (1-b^{\nu}X_{\nu})V^a.$$
(2.53)

Another important symmetry observation is that, with this interpretation of the Lagrange multipliers, the manifest covariance is broken in (2.46), but recovered in the T = 0 limit, (2.47).

Limit two

To find a limit that resembles the result of method I, it is clear that we must at least eliminate one degree of freedom. This will now be done, and we start from (2.42) and eliminate ρ by solving its equation of motion. The result is immediate:

$$\rho = \frac{\dot{X} \cdot \dot{X}}{\dot{X}^2}.\tag{2.54}$$

Substituted back into the action, this gives

$$S = \int d^{\xi} \frac{1}{4\lambda} \left(\frac{\dot{X}^{2} \dot{X}^{2} - (\dot{X} \cdot \dot{X})^{2}}{\dot{X}^{2}} - 4T^{2} \lambda^{2} \dot{X}^{2} \right)$$
$$= \int d^{2} \xi \frac{1}{4\lambda} \left(\frac{1}{\gamma^{00}} - 4T^{2} \lambda^{2} \gamma \gamma^{00} \right), \tag{2.55}$$

where we use that $\acute{X}^2 = \gamma_{11} = \gamma \gamma^{00}$. If we define $V \equiv 2\lambda \gamma \gamma^{00}$, the action can be written in the form

$$S^{CS} = \frac{1}{2} \int d^2 \xi (\frac{\gamma}{V} - T^2 V). \tag{2.56}$$

This is identical to what we obtained using method I, by the identification $\chi = -V^{-1}$, and was discussed in the previous section. It is interesting to note that we can arrive at the same formulation of the limit with both methods. We also note that method II gives an additional possible limit, and is thus more general.

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2.4 p-Branes

The discussion in the previous section, can quite straight forwardly be generalized to p-branes. This is what we will do in this section. Most of what is here written is also found in [18] and [14].

2.4.1 The action

The p-brane action is a direct generalization of the Nambu-Goto string action (2.28). The action is now taken to be proportional to the "area" of the (p + 1)-dimensional world "surface". (We use the words area, surface or volume even though they may refer to higher dimensional objects.) We get

$$S = T \int d^{p+1}\xi \sqrt{-\gamma}, \tag{2.57}$$

where $\gamma_{ab} = \partial_a X^{\mu} \partial_b X_{\mu}$ is the induced metric as in the previous section. The indices a, b now take values $0, \ldots, p$. The action is invariant under the reparametrization $\xi^a \to \sigma^a(\xi)$, which must be true from the geometrical interpretation of the action. It is also easy to demonstrate, by noting the transformation properties of the integral measure and the induced metric. The Jacobi determinant (1.48) is $J = \det(\frac{\partial \xi^a}{\partial \sigma^b})$, and we get

$$d^{p+1}\xi \rightarrow J^{-1}d^{p+1}\xi,$$
 (2.58)

$$\gamma_{ab} \equiv \frac{\partial X^{\mu}}{\partial \xi^{a}} \frac{\partial X_{\mu}}{\partial \xi^{b}} \rightarrow (\frac{\partial \xi^{a}}{\partial \sigma^{c}}) (\frac{\partial \xi^{b}}{\partial \sigma^{d}}) \gamma_{cd}$$
 (2.59)

$$\sqrt{-\gamma} \rightarrow J\sqrt{-\gamma}.$$
 (2.60)

Thus the action transforms as

$$S \to T \int d^{p+1}\xi J^{-1} \sqrt{-\gamma} J = S, \tag{2.61}$$

which proves the invariance.

In the p-brane action, the constant T has natural dimensions of $(length)^{-(p+1)}$, which means mass for p = 0 (points) and tension for p = 1 (strings). Although the dimensionality varies with p, we generally refer to T as the p-brane tension. And we will now turn to the problem of finding models where this constant is allowed to be zero.

2.4.2 Method I

Formally this first method will give exactly the same as it did for strings, since the action (2.57) looks the same. The only difference is that $\gamma_{ab} = \partial_a X^{\mu} \partial_b X_{\mu}$ is no longer a 2×2 matrix, but a $(p+1) \times (p+1)$ matrix. The result is

$$S_{\chi} = -\frac{1}{2} \int d^{p+1\xi} (\chi \gamma - \frac{T^2}{\chi}),$$
 (2.62)

and the tensionless limit:

$$S_{\chi}^{T=0} = -\frac{1}{2} \int d^{p+1}\xi \ \chi \gamma. \tag{2.63}$$

It is found to be conformally invariant in the same way as the string, but now with D=p+1 in the χ transformations (2.34). As before, the χ equation of motion, $\det(\gamma_{ab})=0$, says that the world surface is degenerate.

2.4.3 Method II

As the attentive reader may already have guessed, we start by deriving the momenta. The Lagrangian is read off from (2.57) to give $L = T\sqrt{-\gamma}$. Then we find

$$P_{\mu} \equiv \frac{\partial L}{\partial(\partial_0 X^{\mu})} = -\frac{1}{2} T \sqrt{-\gamma} \gamma^{ab} \frac{\partial \gamma_{ab}}{\partial(\partial_0 X^{\mu})}, \tag{2.64}$$

where we have used the result in appendix A.2. Furthermore we have

$$\gamma_{ab} = G_{\alpha\beta}\partial_{a}X^{\alpha}\partial_{b}X^{\beta}
\frac{\partial}{\partial(\partial_{0}X^{\mu})}\gamma_{ab} = \delta_{a}^{0}\partial_{b}X_{\mu} + \delta_{b}^{0}\partial_{a}X_{\mu}
\gamma^{ab}\frac{\partial\gamma_{ab}}{\partial(\partial_{0}X^{\mu})} = 2\gamma^{a0}\partial_{a}X_{\mu}.$$
(2.65)

So the momenta can finally be written

$$P_{\mu} = T\sqrt{-\gamma}\gamma^{a0}\partial_{a}X_{\mu},\tag{2.66}$$

and are not invertible. Following the usual route, we go on by searching for constraints, and find

$$P^2 = -T^2 \gamma \gamma^{a0} \partial_a X_\mu \gamma^{b0} \partial_b X^\mu = -T^2 \gamma \gamma^{00}, \qquad (2.67)$$

$$P_{\mu}\partial_{k}X^{\mu} = T\sqrt{-\gamma}\gamma^{a0}\partial_{a}X_{\mu}\partial_{k}X^{\mu}$$

= $T\sqrt{-\gamma}\delta_{k}^{0} = 0$ when $k > 0$. (2.68)

We know that γ and γ^{00} contain time derivatives, so the first of these equations does not look like a proper constraint at first sight. But the time derivatives cancel in this special combination, as we easily see:

$$\gamma \gamma^{00} = \gamma \frac{\det \gamma_{ik}}{\det \gamma_{ab}} = \det \gamma_{ik}, \qquad (2.69)$$

where det γ_{ik} is the determinant of the spatial part of the γ -matrix. It contains no time derivatives. So we have found these constraints:

$$\theta_0 \equiv P^2 + T^2 \gamma \gamma^{00} \approx 0; \qquad \theta_k \equiv P_\mu \partial_k X^\mu \approx 0 \quad \text{for} \quad k > 0.$$
 (2.70)

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Since the action is diff invariant, and the theory contains only the scalar (under world volume diffeomorphisms) field X^{μ} , the naive Hamiltonian vanishes (c.f. theorem 1 in section 1.4.2). The total Hamiltonian is thus a sum of the constraints:

$$H = \lambda (P^2 + T^2 \gamma \gamma^{00}) + \rho^k P_\mu \partial_k X^\mu. \tag{2.71}$$

Now we are ready to write the phase space action:

$$S^{PS} = \int d^{p+1}\xi \left[P_{\mu} \partial_0 X^{\mu} - \lambda (P^2 + T^2 \gamma \gamma^{00}) - \rho^k P_{\mu} \partial_k X^{\mu} \right]. \tag{2.72}$$

The equations of motion for P_{μ} reads

$$P^{\mu} = \frac{1}{2\lambda} (\partial_0 X^{\mu} - \rho^k \partial_k X^{\mu}). \tag{2.73}$$

Inserted into the phase space action this gives a new configuration space action

$$S^{CS} = \int d^{p+1}\xi \frac{1}{4\lambda} \left[\gamma_{00} - 2\rho^k \gamma_{k0} + \rho^i \rho^k \gamma_{ik} - 4\lambda^2 T^2 \gamma \gamma^{00} \right]. \tag{2.74}$$

Limit one

In analogy to the string case we introduce the vector density

$$V^a \equiv \frac{1}{2\sqrt{\lambda}}(1, -\rho^k),\tag{2.75}$$

which is now (p+1)-dimensional. Using this variable we can write the action as

$$S_1 = \int d^{p+1}\xi \left[V^a V^b \gamma_{ab} - T^2 \frac{\gamma \gamma^{00}}{V^0 V^0} \right]. \tag{2.76}$$

The $T \to 0$ limit may now be taken and we end up with

$$S_1^{T=0} = \int d^{p+1}\xi V^a V^b \gamma_{ab}. \tag{2.77}$$

We see that, in this sense, the tensionless p-brane is a simple generalization from the string.

Limit two

Another way to a tensionless limit is to eliminate ρ^k in the action (2.74). The equations of motion for ρ^k gives

$$\rho^i \gamma_{ik} = \gamma_{k0}. \tag{2.78}$$

If we define G_{ij} to be the spatial part of γ_{ab} , i.e. $G_{ij} = \gamma_{ij}$, $i, j = 1, 2 \dots, p$, and G^{ij} as its inverse, we can write

$$\rho^i = G^{ik} \gamma_{0k}. \tag{2.79}$$

Inserted back into the action, this gives

$$S_2 = \int d^{p+1}\xi \frac{1}{4\lambda} \left[\gamma_{00} - G^{kj} \gamma_{0j} \gamma_{0k} - 4\lambda^2 T^2 \gamma \gamma^{00} \right]. \tag{2.80}$$

Furthermore, we find $\gamma^{00}(\gamma_{00} - G^{kj}\gamma_{0j}\gamma_{0k}) = 1$, which means that the action can be written

$$S_2 = \int d^{p+1}\xi \frac{1}{4\lambda} \left(\frac{1}{\gamma^{00}} - 4\lambda^2 T^2 \gamma \gamma^{00} \right). \tag{2.81}$$

Defining the scalar $V \equiv 2\lambda\gamma\gamma^{00}$ we end up with

$$S_2 = \frac{1}{2} \int d^{p+1}\xi \left(\frac{\gamma}{V} - T^2 V \right). \tag{2.82}$$

Again, this is exactly the same action as we found by using method I (2.62), if we let $\chi = -V^{-1}$.

It should be noted that as long as we are interested only in the tensionless limit, we could set T=0 already in the Hamiltonian (2.71). The subsequent calculations would then be a little simpler while giving the same result.

2.5 The string in the Polyakov form

2.5.1 The action

A Weyl-invariant form of the p-brane action can be written [5]

$$S = d^{-\frac{d}{2}}T \int d^d \xi \sqrt{-g} (g \cdot \gamma)^{\frac{d}{2}}, \qquad (2.83)$$

where d = p + 1 is the dimension of the world volume of the membrane. The fields to be taken as the independent variables are X^{μ} and g_{ab} .

The action is also world volume diff invariant, which is easy to check. Consider the reparametrization $\xi \to \sigma(\xi)$. Then the fields will transform as

$$g_{ab}(\xi) \rightarrow \tilde{g}_{ab}(\sigma) = \frac{\partial \xi^a}{\partial \sigma^c} \frac{\partial \xi^b}{\partial \sigma^d} g_{cd}(\xi),$$
 (2.84)

$$X^{\mu}(\xi) \rightarrow \tilde{X}^{\mu}(\sigma) = X^{\mu}(\xi).$$
 (2.85)

The Jacobi matrix is $J_b^a = \frac{\partial \xi^a}{\partial \sigma^b}$, and if we let its determinant be written J, we have

$$d^d \xi \quad \to \quad d^d \sigma = \frac{1}{I} d^d \xi, \tag{2.86}$$

$$\sqrt{-g} \rightarrow \sqrt{-\tilde{g}} = J\sqrt{-g},$$
 (2.87)

$$g \cdot \gamma \rightarrow \tilde{g} \cdot \tilde{\gamma} = g \cdot \gamma.$$
 (2.88)

The action will transform as

$$S \to \tilde{S} = \int d^d \sigma \sqrt{-\tilde{g}} \tilde{g} \cdot \tilde{\gamma}$$
 (2.89)

$$= \int d^d \xi \sqrt{-g} g \cdot \gamma = S, \qquad (2.90)$$

which proves the statement of diffeomorphism invariance.

The classical equivalence between the Weyl-invariant and Nambu-Goto type actions is shown by elimination of the metric fields g_{ab} . Start from the Weyl-invariant action (2.83), and demand it to extremal with respect to g^{ab} ,

$$\delta g^{ab} \Rightarrow -\frac{1}{2}\sqrt{-g}g_{ab}(g\cdot\gamma)^{\frac{d}{2}} + \frac{d}{2}\sqrt{-g}(g\cdot\gamma)^{\frac{d}{2}-1}\gamma_{ab} = 0$$

which gives

$$d^{-1}g_{ab}(g \cdot \gamma) = \gamma_{ab}$$

$$\det \Rightarrow d^{-d}g(g \cdot \gamma)^{d} = \gamma$$

$$d^{-\frac{d}{2}}\sqrt{-g}(g \cdot \gamma)^{\frac{d}{2}} = \sqrt{-\gamma}.$$
(2.91)

Substituted into the action, this gives

$$S = T \int d^d \xi \sqrt{-\gamma}, \tag{2.92}$$

which is exactly the Nambu-Goto action for a p-brane.

In the string case when d=2, the Weyl-invariant action is particularly neat. Since we have in mind to derive momenta, we are interested in the \dot{X}^{μ} -dependence of this action. The only place we find this dependence is in γ_{ab} ; γ_{00} is quadratic and γ_{0i} are linear in \dot{X} . This means that the derivative of $\gamma \cdot g$ will depend linearly on \dot{X} . Thus, for d=2 we will find a linear relationship between the momenta and \dot{X} , whereas $d\neq 2$ gives something more complicated because of the exponent $\frac{d}{2}$. These complications will make the calculations considerably more difficult. For this reason we will in the following consider only the string case (d=2):

$$S = \frac{T}{2} \int d^2 \xi \sqrt{-g} g \cdot \gamma. \tag{2.93}$$

This action was first described by Brink–Di Vecchia–Howe and by Deser–Zumino [9,10], but is often named after Polyakov, who has given important contributions to its investigation. A generalization of this action to p-branes (which is, in contrast to (2.83), not Weyl-invariant) is the Howe-Tucker action [37]

$$\frac{T}{2} \int d^{p+1} \xi \sqrt{-g} \left(g \cdot \gamma - (p-1) \right). \tag{2.94}$$

In the following we will consider the two-dimensional case, the *Polyakov action* (2.93), and try to derive tensionless limits of it. Since it is classically equivalent to the Nambu-Goto action, we expect the results here also to be equivalent to those we found in section 2.3.

2. Bosonic strings

2.5.2Method I

Following the usual logic we get an equivalent action,

$$S_{\chi} = \frac{1}{2} \int d^2 \xi \left(-g(g \cdot \gamma)^2 \chi + \frac{T^2}{4\chi} \right), \qquad (2.95)$$

where again χ is an auxiliary field. In the tensionless limit we have:

$$S_{\chi}^{T=0} = -\frac{1}{2} \int d^2 \xi \ \chi g(g \cdot \gamma)^2. \tag{2.96}$$

Variations of the fields give the equations of motion

$$\delta \chi \quad \Rightarrow \qquad g(g^{ab}\gamma_{ab})^2 = 0, \tag{2.97}$$

$$\delta g^{ab} \Rightarrow \chi \left[-gg_{ab}(g \cdot \gamma)^2 + 2g(g \cdot \gamma)\gamma_{ab} \right] = 0,$$
 (2.98)

$$\delta X_{\mu} \Rightarrow \partial_b \left[\chi g(g \cdot \gamma) g^{ab} \partial_a X^{\mu} \right] = 0.$$
 (2.99)

To see exactly what this means, we use the totally antisymmetric symbol² ϵ^{ab} to write the inverse of g_{ab} as $g^{ab} = g^{-1} \epsilon^{ac} \epsilon^{bd} g_{cd}$, which gives

$$g^{-1}(\epsilon^{ac}\epsilon^{bd}g_{cd}\gamma_{ab})^2 = 0, (2.100)$$

$$\chi \left[-g^{-1}g_{ab}(\epsilon^{ec}\epsilon^{fd}g_{cd}\gamma_{ef})^2 + 2(\epsilon^{ec}\epsilon^{fd}g_{cd}\gamma_{ef})\gamma_{ab} \right] = 0, \qquad (2.101)$$

$$\partial_b \left[\chi g^{-1} (\epsilon^{eg} \epsilon^{fh} g_{ef} \gamma_{gh}) \epsilon^{ac} \epsilon^{bd} g_{cd} \partial_a X^{\mu} \right] = 0.$$
 (2.102)

If $g^{-1} = 0$ we get from (2.101) the solution $\chi((\epsilon^{ec}\epsilon^{fd}g_{cd}\gamma_{ef})\gamma_{ab} = 0$, i.e. $\epsilon^{ec}\epsilon^{fd}g_{cd}\gamma_{ef} = 0$ 0 or $\chi = 0$. However, $g^{-1} = 0 \Leftrightarrow \det(g_{ab}) = g \to \infty$, which is a situation we are not interested in. So we assume that $g^{-1} \neq 0$, in which case we find

$$\epsilon^{ac}\epsilon^{bd}g_{cd}\gamma_{ab} = 0, (2.103)$$

which satisfies all the equations above. The fact that the other equations of motion reduce to identities by use of (2.103) stems from the form of the action (2.93), which is not as it should be for method I to work well (c.f. section 1.5.1).

By two-dimensional diff invariance we can choose a parameterization that fixes g_{ab} to $g_{ab} = \Omega \gamma_{ab}$, with Ω as a positive definite function. Then, by Weyl-invariance, we may rescale the metric to get $g_{ab} = \gamma_{ab}$. (This represents a gauge choice.) Then (2.103) gives $\det(\gamma_{ab}) = 0$. And since the induced metric is degenerate in one gauge, it is degenerate in all systems. So the tensionless Polyakov action has generally the solution

$$\det(\gamma_{ab}) = 0. \tag{2.104}$$

In the above gauge the X^{μ} equation (2.102) becomes

$$\frac{\partial_b \left[\chi \epsilon^{ac} \epsilon^{bd} \gamma_{cd} \partial_a X^{\mu} \right] = 0.}{\text{2 In two dimensions we have } \epsilon^{00} = \epsilon^{11} = 0, \ \epsilon^{01} = -1, \ \epsilon^{10} = 1.}$$

which is what we found for the $T \to 0$ limit of the Nambu-Goto string. Hence we conclude that method I gives exactly the same tensionless limits for the Nambu-Goto and Polyakov strings. In the next section we will see that this is true also for the second method.

2.5.3 Method II

We start again by deriving the momenta:

$$P_{\mu} \equiv \frac{\partial L}{\partial \dot{X}^{\mu}}$$

$$= \frac{1}{2} T \sqrt{-g} g^{ab} \frac{\partial \gamma_{ab}}{\partial \dot{X}^{\mu}}$$

$$= T \sqrt{-g} (g^{00} \dot{X}_{\mu} + g^{10} \dot{X}_{\mu}), \qquad (2.106)$$

$$\Pi^{\mu\nu} \equiv \frac{\partial L}{\partial \dot{g}_{\mu\nu}} = 0. \qquad (2.107)$$

Since $\Pi^{\mu\nu}$ vanishes everywhere, we know that $g_{\mu\nu}$ are not really dynamical variables. We will later see explicitly that they play (with a suitable identification) the role of Lagrange multipliers.

From the equation for P_{μ} we see easily that the transformation from configuration space to phase space is invertible. This means that we can obtain \dot{X}^{μ} from P^{μ} . Simple rearrangement indeed gives

$$\dot{X}_{\mu} = \frac{1}{g^{00}} \left(\frac{P_{\mu}}{T\sqrt{-g}} - g^{10} \acute{X}_{\mu} \right). \tag{2.108}$$

The fact that this is possible further means that the momenta are independent functions of \dot{X}^{μ} . Thus we will have no functions connecting them by $\theta_m(P,X,\dot{X})=0$, and hence there will be no constraints in the theory. For higher dimensional Weylinvariant p-branes we would not find the momenta to be invertible. In this sense the string case is special.

Now, as we do not have to think about constraints, we go on by explicitly deriving the Hamiltonian. The total Hamiltonian will in this case equal the naive Hamiltonian. The way to do this is first to define the quantity

$$h = h(X, \partial X, P) \equiv P_{\mu} \dot{X}^{\mu} - L(X, \partial X). \tag{2.109}$$

If we manage to eliminate time derivatives of X (i.e. \dot{X}^{μ}), we arrive at the Hamiltonian H which is a function of X, P and *spatial* derivatives of X (i.e. \dot{X}^{μ}). Thus, if we start from h and substitute for \dot{X}^{μ} we find:

$$\begin{split} h &= P_{\mu} \dot{X}^{\mu} - L \\ &= P_{\mu} \dot{X}^{\mu} - \frac{1}{2} T \sqrt{-g} (g^{00} \dot{X}^{\mu} \dot{X}_{\mu} + 2 g^{10} \dot{X}^{\mu} \dot{X}_{\mu} + g^{11} \dot{X}^{\mu} \dot{X}_{\mu}) \\ &= \frac{1}{2T g^{00} \sqrt{-g}} P^2 - \frac{g^{01}}{g^{00}} P \cdot \dot{X} + \frac{1}{2} T \sqrt{-g} (\frac{(g^{10})^2}{g^{00}} - g^{11}) \dot{X}^2 \\ &\equiv H(P, X, \dot{X}). \end{split}$$

Rearranging the terms we can write this as

$$H = \frac{1}{2Tg^{00}\sqrt{-g}}(P^2 + T^2\dot{X}^2) - \frac{g^{01}}{g^{00}}P \cdot \dot{X}.$$
 (2.110)

Here we have T in the denominator of the first term. If we let $T \to 0$ that term will blow up, and the limit will not be well defined. To overcome this problem we can make the redefinitions

$$\lambda \equiv \frac{1}{2Tg^{00}\sqrt{-g}},\tag{2.111}$$

$$\rho \equiv -\frac{g^{10}}{g^{00}}. (2.112)$$

This is the same identification as we did in (2.43). Interpreting λ and ρ as Lagrange multipliers, this gives exactly the Hamiltonian for the Nambu-Goto string (2.39).

Alternative calculation of the Hamiltonian In section 1.4.2 (theorem 2) we saw that diffeomorphism invariant models made from tensor fields g_{ab} will have a naive Hamiltonian

$$H = \psi^{0a} g_{0a} + \psi^{a0} g_{a0}, \tag{2.113}$$

where $\psi^{ab} = 0$ are the Euler-Lagrange equations associated with g_{ab} . To show that this gives indeed the same result, we will now present an explicit calculation. We first have to find the ψ 's.

$$\psi^{ab} \equiv \partial_c \frac{\partial L}{\partial (\partial_c g_{ab})} - \frac{\partial L}{\partial g_{ab}}$$

$$= -\frac{\partial L}{\partial g_{ab}} = -\frac{1}{2} \sqrt{-g} (g^{ab} g^{cd} \gamma_{cd} - g^{ca} g^{db} \gamma_{cd}). \tag{2.114}$$

This gives

$$h = 2\psi^{0b}g_{0b}$$

$$= -T\sqrt{-g}(\frac{1}{2}g^{cd}\gamma_{cd} - g^{c0}\gamma_{c0})$$

$$= -T\sqrt{-g}(-\frac{1}{2}g^{00}\gamma_{00} + \frac{1}{2}g^{11}\gamma_{11}). \qquad (2.115)$$

Remember that $\gamma_{00} = \dot{X}^2$ and $\gamma_{11} = \dot{X}^2$, and use equation (2.108) to eliminate \dot{X} in favour of the momenta. Then we end up with

$$H = \frac{1}{2\sqrt{-g}g^{00}T}(P^2 + T^2\dot{X}^2) - \frac{g^{10}}{g^{00}},$$
(2.116)

which is just what was found above.

We find the same Hamiltonian as in the Nambu-Goto case, and hence the two models lead to the same phase space action, and of course the same limits. This result should

not come as a chock, as the two actions are classically equivalent. We also remember that in the phase space formulation we were able to deduce the Weyl-invariant action from the Nambu-Goto action (c.f. section 2.3.3).

One interesting observation concerns the degrees of freedoms. Naively, g_{ab} has three degrees of freedom, while λ and ρ are only two degrees of freedom. But owing to the symmetries, there is no real freedom in g_{ab} (remember that it is an auxiliary field). The 2D diff invariance "kills" two degrees of freedom, and Weyl-invariance "kills" the last. Similarly, in the Nambu-Goto case λ and ρ represent no real degrees of freedom, due to the two-parametric diff invariance. The superficial difference in degrees of freedom is then understood from the fact that the Nambu-Goto action has no Weyl-symmetry.

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Chapter 3

D-branes

D-branes¹ [38] are soliton-like, extended objects (so-called topological defects) that appear naturally in string theory. They are defined by the property that strings can end on them, but have their own dynamics.

Actions 3.1

In [27] the action for a Dp-brane is given as

$$S = T_p \int d^{p+1}\xi \ e^{-\phi} \sqrt{-\det(\gamma_{ab} + B_{ab} + F_{ab})}, \tag{3.1}$$

where T_p is a constant, ϕ is the dilation field, and $F_{ab} = 2\pi\alpha'\partial_{[a}A_{b]}$, A_a being a gauge field and $2\pi\alpha'$ the inverse of the fundamental string tension. Furthermore,

$$\gamma_{ab}(\xi) \equiv \partial_a X^{\mu} \partial_b X^{\nu} G_{\mu\nu}(X); \qquad B_{ab}(\xi) \equiv \partial_a X^{\mu} \partial_b X^{\nu} \mathcal{B}_{\mu\nu}(X)$$
 (3.2)

are the induced metric and antisymmetric tensor on the brane. $G_{\mu\nu}$ is the background (symmetric) metric, and $\mathcal{B}_{\mu\nu}$ is the background (antisymmetric) Kalb-Ramond field. The indices take values $\mu = 0, \dots, D-1$; $a = 0, \dots, p$.

The Born-Infeld action The original Born-Infeld action [39] was (unsuccessfully) invented to describe nonlinear electrodynamics, and has the form

$$S = \int d^4x \sqrt{-\det(\eta_{\mu\nu} + f_{\mu\nu})}, \tag{3.3}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, and $f_{\mu\nu}=\partial_{[\mu}A_{\nu]}$ is the electromagnetic field strength.

If we exchange $\eta_{\mu\nu}$ for a general metric $g_{\mu\nu}$ we will have a gravity-coupled model. Furthermore, if we consider the two-dimensional case and use the induced metric γ_{ab}

The D is an abbreviation for Dirichlet.
Remember that $\partial_a = \frac{\partial}{\partial \xi^a}$ and $\partial_{[a} A_{b]} = \partial_a A_b - \partial_b A_a$.

3. D-branes

we will get a kind of string. For higher dimensions we may interpret the action as describing some kind of p-brane:

$$S = T \int d^{p+1}\xi \sqrt{-\det(\gamma_{ab} + F_{ab})}, \tag{3.4}$$

where T is some constant. This resembles very much the D-brane action (3.1). Thus, we will call (3.1) the Born-Infeld action for D-branes.

The dilation $e^{-\phi}$ makes no difference to what concerns the dynamics, and can for our purposes be disregarded (or taken together with the constant T_p). This will be done in the following. The independent field variables are the embedding $X^{\mu}(\xi)$ and the gauge fields $A_a(\xi)$.

A Weyl-invariant action It is shown in [40] that the Born-Infeld action can be written in the two classically equivalent forms

$$S = T_p \int d^{p+1}\xi \sqrt{-s} (s^{ab}(\gamma_{ab} + B_{ab} + F_{ab}))^{\frac{p+1}{2}}, \tag{3.5}$$

$$S = \frac{T}{2} \int d^{p+1} \xi \sqrt{-s} \left[s^{ab} (\gamma_{ab} + B_{ab} + F_{ab}) - (p-1) \right], \tag{3.6}$$

where s_{ab} is an auxiliary tensor field with no symmetry assumed. In the usual way we have defined s as the determinant, $s \equiv \det(s_{ab})$ and s^{ab} as the inverse, $s^{ab}s_{bc} = \delta^a_c$. Elimination of s_{ab} gives back the original form (3.1). The first of these (3.5) is Weylinvariant (under rescalings of s_{ab}), while the second (3.6) is simpler when it comes to calculations.

In two dimensions (p = 1) the two actions are the same. For the same reason as we investigated only the Weyl-invariant *string* in section 2.5, we will for the moment consider only the two-dimensional case of the Weyl-invariant D-brane action.

A reference where the second of the alternative formulations of the D-brane action has been used is [41].

 $T_p \to 0$ **limit** The Dp-brane tension is given by $T_p = \frac{1}{g(2\pi)^p \sqrt{\alpha'}^{p+1}}$, where g is the string coupling. The $T_p \to 0$ limit can thus be viewed as a *strong coupling* limit where $g \to \infty$ and α' held fixed. We will focus on this limit in what follows.

3.2 The Born-Infeld action

Defining $M_{ab} \equiv \gamma_{ab} + B_{ab} + F_{ab}$ and $M \equiv \det(M_{ab})$ we write the Born-Infeld Dp-brane action as

$$S = T \int d^{p+1}\xi \sqrt{-M},\tag{3.7}$$

What is new compared to the usual p-brane action is the addition of the antisymmetric terms B_{ab} and F_{ab} . We will get the old p-brane in the limit $A_a = 0$ and $\mathcal{B}_{\mu\nu} = 0$.

3.2.1 Method I

Introduction of an auxiliary field χ , and taking the T=0 limit gives, in the usual way,

$$S_{\chi}^{T=0} = -\frac{1}{2} \int d^{p+1}\xi \ \chi \det(M_{ab}). \tag{3.8}$$

The equation of motion for χ is found from a variation $\delta \chi$:

$$\delta \chi \Rightarrow \det(M_{ab}) = 0.$$
 (3.9)

This is similar to what we found for the strings and p-branes. But in the present case the degeneracy does not imply that the world volume is a null surface (c.f. section 2.3.2). It only gives a relation between the X^{μ} and A_a fields.

3.2.2 Method II

The calculations presented here are given in more detail by Lindström and von Unge in [40]. In the following we have for simplicity set $\mathcal{B}_{\mu\nu} = 0$, which means $M_{ab} = \gamma_{ab} + F_{ab}$. Calculations with the antisymmetric $\mathcal{B}_{\mu\nu}$ field included are given in [42]. The Lagrangian is given from (3.7) as $L = T\sqrt{-M}$.

The field variables are $X^{\mu}(\xi)$ and $A_a(\xi)$, and the canonical conjugate momenta associated with them are

$$\Pi_{\mu} \equiv \frac{\partial L}{\partial \dot{X}^{\mu}} = \frac{T}{2} \sqrt{-M} M^{(a0)} \partial_a X_{\mu}, \tag{3.10}$$

$$P^{a} \equiv \frac{\partial L}{\partial \dot{A}_{a}} = \frac{T}{2} \sqrt{-M} M^{[a0]} 2\pi \alpha', \tag{3.11}$$

where M^{ab} is the inverse of M_{ab} . Round parenthesis and brackets around the indices denote symmetrization and antisymmetrization respectively, i.e. $M^{(ab)} = M^{ab} + M^{ba}$; $M^{[ab]} = M^{ab} - M^{ba}$.

The equations (3.10, 3.11) are not invertible, and give rise to the following primary constraints:

$$\Theta_i \equiv \Pi_\mu \partial_i X^\mu + \frac{P^j}{2\pi\alpha'} F_{ij} \approx 0, \tag{3.12}$$

$$\Theta_A \equiv \Pi_\mu \Pi^\mu + \frac{P^i \gamma_{ij} P^j}{(2\pi\alpha')^2} + T^2 \det(M_{ij}) \approx 0, \tag{3.13}$$

$$\Theta_B \equiv P^0 \approx 0, \tag{3.14}$$

where i takes spatial values $i = 1, \ldots, p$.

The naive Hamiltonian can be calculated straight forwardly, and reads

$$H_{naive} = P^a \partial_a A_0. (3.15)$$

(Equivalently, we could use theorem 2 of section 1.4.2 and write $h = \psi_A^0 A_0$, where ψ_A^0 is the Euler-Lagrange equation associated with A_0 . This would, up to a total

3. D-branes

derivative, $\partial_a X^a$, yield the same result.) Consistency conditions on the primary constraints give us one secondary constraint

$$\Theta_C \equiv \partial_c P^c \approx 0, \tag{3.16}$$

but there are no tertiary constraints. The total Hamiltonian will then be

$$H = P^{a} \partial_{a} A_{0} + \lambda \Theta_{A} + \rho^{i} \Theta_{i} + \sigma \Theta_{B} + \tau \Theta_{C}. \tag{3.17}$$

The phase space action is

$$L^{PS} = \Pi_{\mu} \partial_{0} X^{\mu} + P^{a} \partial_{0} A_{a} - P^{a} \partial_{a} A_{0} - \lambda \Theta_{A} - \rho^{i} \Theta_{i} - \sigma P^{0} - \tau \partial_{a} P^{a}$$
$$= \Pi_{\mu} \partial_{0} X^{\mu} + P^{i} \partial_{0} A_{i} - P^{i} \partial_{i} (A_{0} - \tau) - \lambda \Theta_{A} - \rho^{i} \Theta_{i} - (\sigma - \partial_{0} \tau) P^{0}(3.18)$$

We can redefine³ $A_0 - \tau \to A_0$ and $\sigma - \partial_0 \tau \to \sigma$. Thus, we get

$$L^{PS} = \Pi_{\mu} \partial_0 X^{\mu} + \frac{P^i F_{ij}}{2\pi\alpha'} - \lambda \Theta_A - \rho^i \Theta_i - \sigma P^0.$$
 (3.19)

Elimination of σ is trivial, and elimination of the momenta gives a configuration space action

$$S^{CS} = \int d^{p+1}\xi \frac{1}{4\lambda} \Big[\gamma_{00} - 2\rho^i \gamma_{0i} + \rho^i \rho^j \gamma_{ij} + \hat{\gamma}^{ij} (F_{0i} - \rho^k F_{ki}) (F_{0j} - \rho^l F_{lj}) - 4\lambda^2 T^2 \det(M_{ij}) \Big], \qquad (3.20)$$

where $\hat{\gamma}^{ij}$ is the inverse of the spatial part of γ_{ab} , i.e. $\hat{\gamma}^{ij}\gamma_{jk} = \delta_k^i$. The $T \to 0$ limit is now easily taken by dropping the last term. And as shown in [40] this gives rise to two different tensionless limits of the D-brane:

$$S_1^{T=0} = \frac{1}{4} \int d^{p+1} \xi V \det M, \tag{3.21}$$

$$S_2^{T=0} = \frac{1}{4} \int d^{p+1} \xi V^a W^b M_{ab}, \qquad (3.22)$$

V, V^a and W^a are scalar and vector fields that are defined by means of the Lagrange multipliers, but may be treated as independent fields. The result would be the same if we included the background field $\mathcal{B}_{\mu\nu}$ (with the proper modification of M_{ab}). The first action (3.21) is identical to what we found using method I.

If we define $V^a=e_0{}^a+e_1{}^a$ and $W^a=-e_0{}^a+e_1{}^a$, equation (3.22) can be rewritten as

$$S_3^{T=0} = \frac{1}{4} \int d^{p+1}\xi \left(\eta^{AB} e_A^{\ a} e_B^{\ b} - \epsilon^{AB} e_A^{\ a} e_B^{\ b} \right) M_{ab}, \tag{3.23}$$

³ Note that in the path integral picture, a shift in the fields does not make any difference, as $\int \mathcal{D}A_0F[A_0-\tau]=\int \mathcal{D}A_0F[A_0]$, in analogy with the simple result $\int dx f(x-a)=\int dx f(x)$.

⁴ In conventions where $\eta^{AB}=diag(-1,+1)$ and $\epsilon^{10}=+1$.

where A, B = 0, 1. We may identify e_A^a as zweibeins, and the form above then shows that the dynamics of the tensionless D-brane is governed by an action that involves a degenerate metric, $g^{ab} = \eta^{AB} e_A^a e_B^b$, of rank 2.

The equations of motion derived from this action can be shown [40] to imply that the world volume of the brane generally splits into a collection of tensile strings or, in special cases, massless particles. Thus, it leads to a parton picture of D-branes in this limit.

3.3 Weyl-invariant form

In the two-dimensional case we found the Weyl-invariant D-brane to be

$$S = \frac{T}{2} \int d^2 \xi \sqrt{-s} s^{ab} M_{ab}. \tag{3.24}$$

Throughout this section we will set $\mathcal{B}_{\mu\nu} = 0$, giving $M_{ab} = \gamma_{ab} + 2\pi\alpha'\partial_{[a}A_{b]}$. The subsequent calculations of tensionless limits resembles much those in the previous section and those for the Polyakov string.

Method I As discussed in the introduction (section 1.5.1) the action (3.24) is not on the form that we need for method I to give dynamical equations for the tensionless limit. This was also true for the Polyakov string action in section 2.5, but in that case we could nonetheless use method I to make interesting interpretations concerning the tensionless limit. An important property that made that possible was the symmetries that allowed us to choose a gauge where the auxiliary metric was equal to the induced metric, $g_{ab} = \gamma_{ab}$ (proportionality would be enough). In the present case, however, the auxiliary tensor s_{ab} is not symmetric and hence is 4-parametric. And since the Weyl+diff symmetry is still only 3-parametric we cannot by a gauge choice set $s_{ab} = M_{ab}$ similar to what we did for the Polyakov string.

3.3.1 Method II

The fields to be considered as independent variables in the action (3.24) are X^{μ} , A_a and s_{ab} . Let us first derive the canonical conjugate momenta associated with these fields.

$$\Pi_{\mu} \equiv \frac{\partial L}{\partial \dot{X}^{\mu}} = \frac{T}{2} \sqrt{-s} s^{cd} \frac{\partial \gamma_{dc}}{\partial \dot{X}^{\mu}} = T \sqrt{-s} \left(s^{00} \dot{X} + \frac{1}{2} (s^{01} + s^{10}) \partial_1 X_{\mu} \right)$$
(3.25)

$$P^{a} \equiv \frac{\partial L}{\partial \dot{A}_{a}} = \frac{T}{2} \sqrt{-s} s^{cd} \frac{\partial F_{dc}}{\partial (\partial_{0} A_{a})} = -T \sqrt{-s} \frac{1}{2} (s^{0a} - s^{a0}) 2\pi \alpha'$$
 (3.26)

$$\Sigma^{ab} \equiv \frac{\partial L}{\partial \dot{s}_{ab}} = 0 \tag{3.27}$$

The first equation (3.25) is invertible which means that we can find an explicit expression for \dot{X}^{μ} :

$$\dot{X}_{\mu} = \frac{1}{s^{00}} \left(\frac{\Pi_{\mu}}{T\sqrt{-s}} - \frac{1}{2} (s^{01} + s^{10}) \partial_1 X_{\mu} \right)$$
(3.28)

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The second equation (3.26) is obviously not invertible. We have actually found a momentum P that is completely independent of the fields A. Its definition gives then immediately rise to the constraints

$$\Theta_0 \equiv P^0 \approx 0, \tag{3.29}$$

$$\Theta_1 \equiv P^1 + \frac{T}{2}\sqrt{-s}(s^{01} - s^{10})2\pi\alpha' \approx 0.$$
 (3.30)

The last equation (3.27) says that the conjugate momenta to s_{ab} are identically zero. This follows the pattern of previous results, and s_{ab} are non-dynamical variables to be treated on the same footing as Lagrange multipliers.

We are now ready to derive the naive Hamiltonian. Disregarding Σ^{ab} , we have:

$$H_{naive} = \Pi_{\mu}\dot{X}^{\mu} + P^{a}\dot{A}_{a} - \frac{T}{2}\sqrt{-s}s^{ab}(\gamma_{ab} + 2\pi\alpha'(\partial_{a}A_{b} - \partial_{b}A_{a}))$$
$$= f(\Pi, \dot{X}, \partial_{1}X) + g(P, \dot{A}, \partial_{1}A)$$

To arrive at a proper Hamiltonian we have to eliminate all time derivatives. Consider first g:

$$g = P^{a}\partial_{0}A_{a} - T\sqrt{-s}\frac{1}{2}s^{ab}(\partial_{a}A_{b} - \partial_{b}A_{a})2\pi\alpha'$$

$$= P^{a}\partial_{0}A_{a} - T\sqrt{-s}\frac{1}{2}[s^{01}(\partial_{0}A_{1} - \partial_{1}A_{0}) + s^{10}(\partial_{1}A_{0} - \partial_{0}A_{1})]2\pi\alpha'$$

$$= P^{a}\partial_{a}A_{0}.$$

Now remains only to rewrite f. If we insert the expression for \dot{X}^{μ} and simplify we find:

$$f = \Pi_{\mu} \dot{X}^{\mu} - T \sqrt{-s} \frac{1}{2} s^{ab} \gamma_{ab}$$

$$= \frac{1}{2T s^{00} \sqrt{-s}} \Pi^{2} - \frac{s^{01} - s^{10}}{2s^{00}} \partial_{i} X^{\mu} \Pi_{\mu}$$

$$+ \frac{T \sqrt{-s}}{2s^{00}} \left(\frac{1}{4} (s^{01} + s^{10}) (s^{01} + s^{10}) - s^{00} s^{11} \right) \gamma_{11}$$

If we calculate P^1P^1 we see that we can still simplify the expression within the large brackets of the last term:

$$\frac{1}{4}(s^{01} + s^{10})(s^{01} + s^{10}) - s^{00}s^{11} = \frac{P^1P^1}{-T^2s(2\pi\alpha')^2} - \frac{1}{s}.$$
 (3.31)

If we put together terms with the same coefficients we can now write the naive Hamiltonian in its simplest form:

$$H_{naive} = \frac{1}{2Ts^{00}\sqrt{-s}} \left(\Pi_{\mu}\Pi^{\mu} + \frac{P^{1}\gamma_{11}P^{1}}{(2\pi\alpha')^{2}} + T^{2}\gamma_{11} \right) - \frac{s^{01} + s^{10}}{2s^{00}} \Pi_{\mu}\partial_{1}X^{\mu} + P^{a}\partial_{a}A_{0}.$$
(3.32)

The consistency condition on the primary constraint Θ_0 gives a secondary "Gauss law" constraint

$$\Theta_2 \equiv \partial_a P^a \approx 0, \tag{3.33}$$

while Θ_1 gives nothing new. There are no tertiary constraints. The four component fields of s_{ab} are Lagrange multipliers which can be redefined as

$$\lambda \equiv \frac{1}{2Ts^{00}\sqrt{-s}},\tag{3.34}$$

$$\rho \equiv -\frac{s^{01} + s^{10}}{2s^{00}},\tag{3.35}$$

$$\varphi \equiv \frac{T}{2}\sqrt{-s}(s^{01} - s^{10}). \tag{3.36}$$

Including the constraints, we can then write the total Hamiltonian as

$$H = \lambda(\Pi^{2} + \frac{P^{1}\gamma_{11}P^{1}}{(2\pi\alpha')^{2}} + T^{2}\gamma_{11}) + \rho\Pi_{\mu}\partial_{1}X^{\mu} + P^{a}\partial_{a}A_{0} + \sigma_{0}P^{0} + \sigma_{1}(P^{1} + \varphi) + \tau\partial_{a}P^{a}$$
(3.37)

The phase space Lagrangian is $L^{PS} = \Pi_{\mu} \partial_0 X^{\mu} + P^a \partial_0 A_a - H$, and a variation of φ gives $\sigma_1 = 0$, which means that the constraint $\Theta_1 = P^1 + \varphi$ in fact makes no difference. Then we see that we have exactly the same Hamiltonian and phase space Lagrangian as we derived from the Born-Infeld action for the two-dimensional D-brane (D-string) (3.17). Hence, we get the same tensionless limits (3.21) and (3.22)/(3.23).

We have thus the same situation as we found for the Polyakov string versus the Nambu-Goto string. This should not really come as a surprise, since the string action can be seen as a special case of the D-string action when we let $A_a = \mathcal{B}_{\mu\nu} = 0$, and s_{ab} be symmetric.

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Chapter 4

Rigid strings

Rigid strings are strings with an extra curvature term that depends on the spacetime embedding. They are also referred to as *smooth* strings, because this extra term (provided that it has the right sign) makes it energetically favourable for them to be less creased.

The rigid string was introduced in 1986 as an attempt to find a string that corresponds to QCD (quantum chromo-dynamics) [43], and, independently, to study the string near a phase transition [44]. Different aspects of rigid strings have later been investigated in [45–56].

4.1 The action

The intrinsic curvature of a surface Σ embedded in a flat background space S can be written by means of the extrinsic curvatures K_a^{ib} generally (up to a total derivative) as (c.f. appendix A.5)

$$R = K_a^{ia} K_{ib}^{\ \ b} - K_a^{ib} K_{ib}^{\ \ a}, \tag{4.1}$$

where a, b = 0, 1 are worldsheet indices and i = 2, 3, ..., D - 1 refer to directions normal to the surface. D is the spacetime dimension. Indices are raised and lowered by the induced metric γ_{ab} and its inverse γ^{ab} . The intrinsic curvature is a total derivative in two dimensions, but the separate terms in (4.1) are not. A generalization of the Nambu-Goto action to include curvature can therefore be written [43]

$$S = T \int d^2 \xi \sqrt{-\gamma} + \frac{1}{2\alpha} \int d^2 \xi \sqrt{-\gamma} K_a^{ia} K_{ib}{}^b, \tag{4.2}$$

where, again, $\gamma = \det(\gamma_{ab})$ and T is the string tension. The coupling constant α is referred to as the *rigidity parameter*.

The term describing the extrinsic curvature contains double derivatives. To get an action with only first derivatives, we introduce an extra field B^{μ} , and write the action as in [56],

$$S = T \int d^2 \xi \sqrt{-\gamma} \left(1 - \frac{\alpha T}{2} B^{\mu} B_{\mu} - \gamma^{ab} \partial_a X^{\mu} \partial_b B_{\mu} \right). \tag{4.3}$$

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This action is seen to be equivalent to the original one by elimination of B:

$$\delta B_{\mu} \Rightarrow \delta S = T \int d^{2}\xi \left(-\sqrt{-\gamma}\alpha T B^{\mu}\delta B_{\mu} - \sqrt{-\gamma}\gamma^{ab}\partial_{a}X^{\mu}\partial_{b}(\delta B_{\mu}) \right)$$

$$= T \int d^{2}\xi \left(-\sqrt{-\gamma}\alpha T B^{\mu} + \partial_{b}(\sqrt{-\gamma}\gamma^{ab}\partial_{a}X^{\mu}) \right) \delta B_{\mu}.$$
(4.4)

The covariant d'Alembertian operator is $\Box \equiv \gamma^{ab} \nabla_a \partial_b = \frac{1}{\sqrt{-\gamma}} \partial_a \sqrt{-\gamma} \gamma^{ab} \partial_b$, where ∇ is the covariant derivative with respect to the metric γ^{ab} . Demanding the action to be extremal, we find immediately

$$B^{\mu} = \frac{1}{\alpha T} \Box X^{\mu} \tag{4.5}$$

Inserting this solution for B in the action (4.3) we get

$$S = T \int d^2 \xi \sqrt{-\gamma} + \frac{1}{2\alpha} \int d^2 \sqrt{-\gamma} \xi \Box X^{\mu} \Box X_{\mu}. \tag{4.6}$$

Furthermore, we have the identity $\partial_a \partial_b X^\mu = \{^c_{ab}\} \partial_c X^\mu + K^i_{ab} n^\mu_i$, where $\{^c_{ab}\}$ is the Christoffel symbol associated with γ_{ab} , whereas K^i_{ab} is the extrinsic curvature and n^μ_i are normal vectors to the worldsheet. Using this together with the definition of $\Box X^\mu$ we find $\Box X^\mu = \gamma^{ab} K^i_{ab} n^\mu_i$, which gives $\Box X^\mu \Box X_\mu = K^{ia}_a K_{ib}^{\ b}$. Hence we recover the "second order" action (4.2).

The form (4.3) of the action may further be derived from a membrane action [57]. In the following we will focus on the tensionless limit of the rigid string.

4.2 Method I

The action (4.3) is not appropriate for method I (see section 1.5.1). For this reason we will here instead consider the action [53]

$$S = T \int d^2 \xi \sqrt{-\det(\gamma_{ab} + H_{ab})}; \qquad H_{ab} \equiv \alpha^{-1} (\nabla_a \partial_c X^{\mu} \nabla_b \partial_d X_{\mu}) \gamma^{cd}, \qquad (4.7)$$

which is equivalent to (4.2) to first order in α^{-1} . The tensionless limit is

$$S_{\chi}^{T=0} = -\frac{1}{2} \int d^2 \xi \ \chi \det(\gamma_{ab} + H_{ab}), \tag{4.8}$$

and a variation in χ gives

$$\delta \chi \Rightarrow \det(\gamma_{ab} + H_{ab}) = 0.$$
 (4.9)

This equation together with the equation we find from a variation in X^{μ} are the field equations for this model.

4.3 Method II 55

4.3 Method II

More interesting than method I is the calculations and results we obtain from the phase space method. It is a more general method, and usually makes it easier to see what is going on as we take the limit $T \to 0$.

We define $N \equiv \frac{\alpha T}{2}$ and allow for the possibility of N to remain finite as $T \to 0$. Our starting point is the Lagrangian

$$L = T\sqrt{-\gamma}(1 - NB^2 - \gamma^{ab}\partial_a X^{\mu}\partial_b B_{\mu}). \tag{4.10}$$

As usual, we start by deriving the canonical conjugate momenta ($\dot{\theta} = \frac{\partial}{\partial \xi^0}, \dot{\theta} = \frac{\partial}{\partial \xi^0}$),

$$\Pi_{\mu} \equiv \frac{\partial L}{\partial \dot{B}^{\mu}} = -T\sqrt{-\gamma}\gamma^{a0}\partial_{a}X_{\mu}, \tag{4.11}$$

$$P_{\mu} \equiv \frac{\partial L}{\partial \dot{X}^{\mu}} = T \sqrt{-\gamma} \Big[\gamma^{d0} \left((1 - NB^2) \partial_d X_{\mu} - \partial_d B_{\mu} \right) + (\gamma^{a0} \gamma^{bd} + \gamma^{b0} \gamma^{ad} - \gamma^{ab} \gamma^{d0}) (\partial_a X^{\nu} \partial_b B_{\nu}) \partial_d X_{\mu} \Big]. \tag{4.12}$$

Neither of these equations are invertible, and we find the following primary constraints:

$$\Theta_1 \equiv \Pi^2 + T^2 \acute{X}^2 \approx 0, \tag{4.13}$$

$$\Theta_2 \equiv \Pi_\mu \acute{X}^\mu \approx 0, \tag{4.14}$$

$$\Theta_3 \equiv P_\mu \acute{X}^\mu + \Pi_\mu \acute{B}^\mu \approx 0, \tag{4.15}$$

$$\Theta_4 \equiv P_\mu \Pi^\mu + T^2 \acute{X}_\mu \acute{B}^\mu + (1 - NB^2) \Pi^2 \approx 0. \tag{4.16}$$

The diffeomorphism invariance ensures that the naive Hamiltonian is zero, which is also easy to check by direct calculation. This means that the consistency conditions (1.26) on the primary constraints take the form

$$\int d^2 \xi' \ \lambda^n \{ \Theta_m(\xi), \Theta_n(\xi') \} \approx 0, \tag{4.17}$$

where λ^n is the Lagrange multiplier associated with the constraint Θ_n . Working out these conditions will give us the secondary constraints.

Secondary constraints Since we have so far never really done any thorough calculations to find secondary constraints, we will now do this in great detail. Refer back to section 1.2.3 for the general theory. We first have to find the variational derivatives of the fields. This is easily done, with the results:

$$\begin{split} \frac{\delta\Theta_{1}(\xi)}{\delta X^{\mu}(\tilde{\xi})} &= 2T^{2}\acute{X}_{\mu}\partial_{1}\delta(\xi-\tilde{\xi}), & \frac{\delta\Theta_{1}(\xi)}{\delta B^{\mu}(\tilde{\xi})} &= 0, \\ \frac{\delta\Theta_{1}(\xi)}{\delta P_{\mu}(\tilde{\xi})} &= 0, & \frac{\delta\Theta_{1}(\xi)}{\delta\Pi_{\mu}(\tilde{\xi})} &= 2\Pi^{\mu}(\xi)\delta(\xi-\tilde{\xi}); \end{split}$$

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$$\begin{split} &\frac{\delta\Theta_{2}(\xi)}{\delta X^{\mu}(\tilde{\xi})} = \Pi_{\mu}(\xi)\partial_{1}\delta(\xi - \tilde{\xi}), & \frac{\delta\Theta_{2}(\xi)}{\delta B^{\mu}(\tilde{\xi})} = 0, \\ &\frac{\delta\Theta_{2}(\xi)}{\delta P_{\mu}(\tilde{\xi})} = 0, & \frac{\delta\Theta_{2}(\xi)}{\delta\Pi_{\mu}(\tilde{\xi})} = \dot{X}^{\mu}(\xi)\delta(\xi - \tilde{\xi}); \\ &\frac{\delta\Theta_{3}(\xi)}{\delta X^{\mu}(\tilde{\xi})} = P_{\mu}(\xi)\partial_{1}\delta(\xi - \tilde{\xi}), & \frac{\delta\Theta_{3}(\xi)}{\delta B^{\mu}(\tilde{\xi})} = \Pi_{\mu}(\xi)\partial_{1}\delta(\xi - \tilde{\xi}), \\ &\frac{\delta\Theta_{3}(\xi)}{\delta P_{\mu}(\tilde{\xi})} = \dot{X}^{\mu}(\xi)\delta(\xi - \tilde{\xi}), & \frac{\delta\Theta_{3}(\xi)}{\delta\Pi_{\mu}(\tilde{\xi})} = \dot{B}^{\mu}(\xi)\delta(\xi - \tilde{\xi}); \\ &\frac{\delta\Theta_{4}(\xi)}{\delta X^{\mu}(\tilde{\xi})} = T^{2}\dot{B}(\xi)\partial_{1}\delta(\xi - \tilde{\xi}), & \frac{\delta\Theta_{4}(\xi)}{\delta B^{\mu}(\tilde{\xi})} = T^{2}\dot{X}_{\mu}(\xi)\partial_{1}\delta(\xi - \tilde{\xi}), \\ &\frac{\delta\Theta_{4}(\xi)}{\delta P_{\mu}(\tilde{\xi})} = \Pi^{\mu}(\xi)\delta(\xi - \tilde{\xi}), & \frac{\delta\Theta_{4}(\xi)}{\delta\Pi_{\mu}(\tilde{\xi})} = [P^{\mu}(\xi) + 2(1 - NB^{2}(\xi)) \\ &\frac{\delta\Theta_{4}(\xi)}{\delta\Pi_{\mu}(\tilde{\xi})} = [P^{\mu}(\xi) + 2(1 - NB^{2}(\xi))] \\ &\frac{\delta\Theta_{4}(\xi)}{\delta\Pi_{\mu}(\tilde{\xi})} =$$

Next, we must calculate the Poisson brackets between the constraints, which are in general given by

$$\{\Theta_{m}(\xi), \Theta_{n}(\xi')\} = \int d^{2}\tilde{\xi} \left[\frac{\partial \Theta_{m}(\xi)}{\partial X^{\mu}(\tilde{\xi})} \frac{\partial \Theta_{n}(\xi')}{\partial P_{\mu}(\tilde{\xi})} + \frac{\partial \Theta_{m}(\xi)}{\partial B^{\mu}(\tilde{\xi})} \frac{\partial \Theta_{n}(\xi')}{\partial \Pi_{\mu}(\tilde{\xi})} - \frac{\partial \Theta_{m}(\xi)}{\partial P_{\mu}(\tilde{\xi})} \frac{\partial \Theta_{n}(\xi')}{\partial X^{\mu}(\tilde{\xi})} - \frac{\partial \Theta_{m}(\xi)}{\partial \Pi_{\mu}(\tilde{\xi})} \frac{\partial \Theta_{n}(\xi')}{\partial B^{\mu}(\tilde{\xi})} \right]. \tag{4.18}$$

Performing the required calculations, we find¹

$$\{\Theta_1(\xi), \Theta_2(\xi')\} = 0,$$
 (4.19)

$$\{\Theta_1(\xi), \Theta_3(\xi')\} = 2 \left[\Pi^{\mu}(\xi) \Pi_{\mu}(\xi') + T^2 \acute{X}^{\mu}(\xi) \acute{X}_{\mu}(\xi') \right] \partial_1 \delta(\xi - \xi'), \quad (4.20)$$

$$\{\Theta_{1}(\xi), \Theta_{4}(\xi')\} = 2T^{2} \left[\Pi_{\mu}(\xi) \dot{X}^{\mu}(\xi') + \Pi_{\mu}(\xi') \dot{X}^{\mu}(\xi) \right] \partial_{1} \delta(\xi - \xi') + 4N\Pi^{2} \Pi_{\mu} B^{\mu} \delta(\xi - \xi'), \tag{4.21}$$

$$\{\Theta_2(\xi), \Theta_3(\xi')\} = \left[\Pi_{\mu}(\xi) \acute{X}^{\mu}(\xi') + \Pi_{\mu}(\xi') \acute{X}^{\mu}(\xi)\right] \partial_1 \delta(\xi - \xi'), \tag{4.22}$$

$$\{\Theta_{2}(\xi), \Theta_{4}(\xi')\} = \left[\Pi_{\mu}(\xi)\Pi^{\mu}(\xi') + T^{2}\dot{X}^{\mu}(\xi)\dot{X}_{\mu}(\xi')\right]\partial_{1}\delta(\xi - \xi') + 2N\Pi^{2}\dot{X}^{\mu}B_{\mu}\delta(\xi - \xi'), \tag{4.23}$$

$$\{\Theta_{3}(\xi), \Theta_{4}(\xi')\} = 2(1 - NB^{2}(\xi'))\Pi_{\mu}(\xi')\Pi^{\mu}(\xi)\partial_{1}\delta(\xi - \xi') + 2N\Pi^{2}\dot{B}^{\mu}B_{\mu}\delta(\xi - \xi'). \tag{4.24}$$

When put inside an integration, we can integrate by parts to get rid of the derivatives of the delta function. This will give rise to a vanishing surface term, and a

¹ We use the notation $\xi = (\xi^0, \xi^1)$ and $\partial_1 = \frac{\partial}{\partial \xi^1}$.

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term that vanishes (weakly) by use of the primary constraints. Let us show this for $\{\Theta_2(\xi), \Theta_3(\xi')\}$. The expression has meaning only within an integral, and we find

$$\int d^{2}\sigma\varphi(\sigma)\{\Theta_{2}(\xi),\Theta_{3}(\sigma)\}
= \int d^{2}\sigma\varphi(\sigma) \Big[\Pi_{\mu}(\xi)\dot{X}^{\mu}(\sigma) + \Pi_{\mu}(\sigma)\dot{X}^{\mu}(\xi)\Big] \partial_{1}\delta(\xi - \sigma)
= \int d^{2}\sigma\partial_{1} \Big(\varphi(\sigma) \Big[\Pi_{\mu}(\xi)\dot{X}^{\mu}(\sigma) + \Pi_{\mu}(\sigma)\dot{X}^{\mu}(\xi)\Big] \delta(\xi - \sigma)\Big)
- \int d^{2}\sigma\partial_{1} \Big(\varphi(\sigma) \Big[\Pi_{\mu}(\xi)\dot{X}^{\mu}(\sigma) + \Pi_{\mu}(\sigma)\dot{X}^{\mu}(\xi)\Big]\Big) \delta(\xi - \sigma)
= \partial_{1} \int d^{2}\sigma\varphi(\sigma) \Big[\Pi_{\mu}(\xi)\dot{X}^{\mu}(\sigma) + \Pi_{\mu}(\sigma)\dot{X}^{\mu}(\xi)\Big] \delta(\xi - \sigma)
- \int d^{2}\sigma\varphi(\sigma) \Big[\dot{\Pi}_{\mu}(\xi)\dot{X}^{\mu}(\sigma) + \Pi_{\mu}(\sigma)\partial_{1}\partial_{1}X^{\mu}(\xi)\Big] \delta(\xi - \sigma)
= \partial_{1} \Big(\varphi(\xi)2\Pi\dot{X}\Big) - \varphi(\xi)\partial_{1}(\Pi\dot{X})
\stackrel{\Theta_{2}}{\approx} 0.$$
(4.25)

The same can be shown to hold for all the other terms including the factor $\partial_1 \delta(\xi - \xi')$. Thus, the consistency conditions (4.17) yield in general three secondary constraints²,

$$4N\Pi^2\Pi^{\mu}B_{\mu} \approx 0, \tag{4.26}$$

$$2N\Pi^2 \acute{X}^{\mu} B_{\mu} \approx 0, \tag{4.27}$$

$$2N\Pi^2 \dot{B}^{\mu} B_{\mu} \approx 0. \tag{4.28}$$

The fields are here to be evaluated at the same world-sheet points ξ .

The limit B=0 From the original action integral (4.3) we see that in this limit we recover the Nambu-Goto action for a string. It is instructive to check that this will be true also in the phase space picture. If we let B=0 in the expressions for the momenta we find that $P^{\mu}=-\Pi^{\mu}=T\sqrt{-\gamma}\gamma^{a0}\partial_{a}X^{\mu}$, which is of course the same as for the Nambu-Goto string. Furthermore, $\Theta_{4}\approx 0$ will be reduced to an identity, and Θ_{3} will be identical to $-\Theta_{2}$. The remaining two constraints will be the same as in the Nambu-Goto case,

$$\Theta_1 = P^2 + T\dot{X}^2, (4.29)$$

$$\Theta_2 = P \cdot \acute{X}. \tag{4.30}$$

Since the naive Hamiltonian is zero, this immediately tells us that the phase space action will also be the same.

² This is different from what is found in [56]. In this article the authors use, instead of (4.16), $\Theta_4 = P\Pi + T^2 \dot{X} \dot{B} + (1 - NB^2)(\Pi^2 - T^2 \dot{X}^2)$ and find $\{\Theta_3, \Theta_4\} = 0$. This is obviously not equivalent with our results.

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The limit T = 0 This is the tensionless case that we are really interested in. The primary constraints are now reduced to

$$\Theta_1 = \Pi^2, \tag{4.31}$$

$$\Theta_2 = \Pi^{\mu} \acute{X}_{\mu}, \tag{4.32}$$

$$\Theta_3 = P^{\mu} \acute{X}_{\mu} + \Pi^{\mu} \acute{B}_{\mu},$$
 (4.33)

$$\Theta_4 = P^{\mu} \Pi_{\mu}. \tag{4.34}$$

Since $\Pi^2 \approx 0$ the consistency conditions (4.26–4.28) reduce to identities, so we have no secondary constraints in the tensionless limit.

The Hamiltonian is then just the sum of primary constraints,

$$H = a\Pi^2 + b\Pi \cdot \acute{X} + c[P \cdot \acute{X} + \Pi \cdot \acute{B}] + dP \cdot \Pi, \tag{4.35}$$

where a, b, c and d are Lagrange multipliers. Note that the parameter N did also vanish as we put T=0. What we now want to do is to write the phase space action, and eliminate the momenta to arrive at a new configuration space action. The phase space action is

$$S = \int d^2\xi \left[P\dot{X} + \Pi \dot{B} - H \right]. \tag{4.36}$$

This action is only linear in P, and a variation δP gives

$$\delta P_{\mu} \Rightarrow \dot{X}^{\mu} - c\dot{X}^{\mu} - d\Pi^{\mu} = 0, \tag{4.37}$$

$$\Rightarrow \Pi^{\mu} = \frac{1}{d}(\dot{X}^{\mu} - c\dot{X}^{\mu}). \tag{4.38}$$

Inserted back into the action, and defining $\rho = \frac{a}{d}$, this gives the configuration space action

$$S = \int d^2 \xi \frac{1}{d} \Big[-\rho \gamma_{00} + (2c\rho - b)\gamma_{01} - (c^2\rho - cb)\gamma_{11} + \beta_{00} - c\beta_{10} - c\beta_{01} + c^2\beta_{11} \Big].$$
 (4.39)

(Remember that $\gamma_{ab} = \partial_a X^{\mu} \partial_b X_{\mu} = \gamma_{ba}$ and $\beta_{ab} = \partial_a X^{\mu} \partial_b B_{\mu} \neq \beta_{ba}$.) Let us define the vector density

$$V^a \equiv \frac{1}{\sqrt{d}} \begin{pmatrix} 1 \\ -c \end{pmatrix}, \tag{4.40}$$

which means $V^0V^0 = \frac{1}{d}$, $V^0V^1 = -\frac{1}{d}c$ and $V^1V^1 = \frac{1}{d}c^2$. Redefining $\frac{1}{\sqrt{d}}b \to b$, we can write the Lagrangian in (4.39) as

$$L = -\rho V^a V^b \gamma_{ab} - b V^a \gamma_{a1} + V^a V^b \beta_{ab}. \tag{4.41}$$

If we also define

$$W^a \equiv \begin{pmatrix} \rho V^0 \\ \rho V^1 + b \end{pmatrix}, \tag{4.42}$$

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we can write the action for the tensionless rigid string in a manifestly covariant form,

$$S = \int d^2 \xi \left[V^a \partial_a X^\mu (V^b \partial_b B_\mu - W^b \partial_b X_\mu) \right]. \tag{4.43}$$

The four degrees of freedom of the Lagrange multipliers are now replaced by the four degrees of freedom in the two vector densities.

Variations in the fields give the equations of motion,

$$\delta B_{\mu} \Rightarrow \qquad \partial_a (V^a V^b \partial_b X^{\mu}) = 0, \tag{4.44}$$

$$\delta W^b \Rightarrow V^a \gamma_{ab} = 0, \tag{4.45}$$

$$\delta V^b \Rightarrow V^a \partial_{(a} X_\mu \partial_{b)} B^\mu = W^a \gamma_{ab}, \tag{4.46}$$

$$\delta B_{\mu} \Rightarrow \qquad \partial_{a} (V^{a} V^{b} \partial_{b} X^{\mu}) = 0, \qquad (4.44)$$

$$\delta W^{b} \Rightarrow \qquad V^{a} \gamma_{ab} = 0, \qquad (4.45)$$

$$\delta V^{b} \Rightarrow \qquad V^{a} \partial_{(a} X_{\mu} \partial_{b)} B^{\mu} = W^{a} \gamma_{ab}, \qquad (4.46)$$

$$\delta X_{\mu} \Rightarrow \qquad \partial_{a} \left[V^{(a} W^{b)} \partial_{b} X^{\mu} - V^{a} V^{b} \partial_{b} B^{\mu} \right] = 0. \qquad (4.47)$$

The first two equations are the same as we found for the tensionless Nabu-Goto string. What is special for the rigid string must then be found in the remaining two equations.

Equation (4.47) gives

$$V^{(a}W^{b)}\partial_b X^{\mu} - V^a V^b \partial_b B^{\mu} = \epsilon^{ab}\partial_b C^{\mu}, \tag{4.48}$$

where C^{μ} is some spacetime vector. Contracting this equation with $\partial_a X_{\mu}$ gives

$$2V^{a}W^{b}\gamma_{ab} - V^{a}V^{b}\beta_{ab} = \epsilon^{ab}\partial_{b}C^{\mu}\partial_{a}X_{\mu}$$
$$0 = \epsilon^{ab}\partial_{b}C^{\mu}\partial_{a}X_{\mu}, \tag{4.49}$$

which says that $\partial_b C^\mu \partial_a X_\mu$, is symmetric, i.e. $\partial_a C^\mu = c \partial_a X^\mu$, where c is some constant. Put into equation (4.47) this gives

$$V^{(a}W^{b)}\partial_{b}X^{\mu} - V^{a}V^{b}\partial_{b}B^{\mu} = c\epsilon^{ab}\partial_{b}X^{\mu}$$

$$\epsilon_{ca} \Rightarrow \epsilon_{ca} \left(V^{(a}W^{b)}\partial_{b}X^{\mu} - V^{a}V^{b}\partial_{b}B^{\mu}\right) = c\partial_{c}X^{\mu}$$

$$V^{c} \Rightarrow \epsilon_{ca}V^{c}V^{b}W^{a}\partial_{b}X^{\mu} = c\partial_{c}X^{\mu}V^{c}$$

$$\Rightarrow c = \epsilon_{ab}V^{a}W^{b}. \tag{4.50}$$

Thus we have determined the constant c. Using the relation $\epsilon^{ab}\epsilon_{cd} = \delta^a_d\epsilon^b_c - \epsilon^a_c\epsilon^b_d$ we can now write equuation (4.48)

$$V^a W^b \partial_b X^\mu + V^b W^a \partial_b X^\mu - V^a V^b \partial_b B^\mu = \partial_c X^\mu V^c W^a - \partial_d X^\mu V^a W^d,$$

which gives

$$V^b \partial_b B^\mu = 2W^b \partial_b X^\mu. \tag{4.51}$$

This equation put into (4.46) yields

$$V^b \partial_b X^\mu \partial_a B^\mu = -W^b \gamma_{ab}. \tag{4.52}$$

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A contraction of equation (4.51) with $\partial_a X_\mu$, and using (4.52) gives us

$$V^b \partial_b B^\mu \partial_a X_\mu = -2V^b \partial_b X^\mu \partial_a B^\mu, \tag{4.53}$$

which says that $det(\beta_{ab} + 2\beta_{ba}) = 0$. This relation can (in 2D) be written

$$\det(3\beta_{(ab)}) = -\hat{\beta}^2; \qquad \beta_{[ab]} = \hat{\beta}\epsilon_{ab}, \tag{4.54}$$

where $\hat{\beta} = \beta_{01} - \beta_{10}$.

We have seen that method II does indeed lead to a sensible theory for the tensionless limit of the rigid string, represented by the field equations (4.44; 4.45; 4.51; 4.52). The first two equations describe a tensionless Nambu-Goto string, which we derived in section 2.3. One special solution of the last two equations is the case where W^a is parallel to V^a , and $\partial_a B^\mu$ is parallel to $\partial_a X^\mu$, in which case we end up with exactly the same equations of motion as for the tensionless Nambu-Goto string.

An interesting question that we do not go into in any more detail, is what role the B field actually plays in these equations.

Note also that the action (4.43) can be made conformally invariant in the same way as the tensionless Nambu-Goto string, if we in addition demand that B^{μ} transforms like X^{μ} .

Chapter 5

General relativity

Consider a given manifold S with metric $g_{\mu\nu}$. Suppose furthermore that there exist a coordinate basis $\{x^{\mu}\}$ on the manifold, and let comma denote partial derivative with respect to the coordinates, i.e. $F_{,\mu} \equiv \frac{\partial F}{\partial x^{\mu}}$. Then we may define the the Christoffel symbol $\{^{\alpha}_{\mu\nu}\}$, the Riemann curvature tensor $R^{\alpha}_{\mu\beta\nu}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R as follows:

$$\{^{\alpha}_{\mu\nu}\} \equiv \frac{1}{2}g^{\alpha\beta}(g_{\beta\nu,\mu} + g_{\mu\beta,\nu} - g_{\mu\nu,\beta}), \tag{5.1}$$

$$R^{\alpha}_{\ \mu\beta\nu} \equiv {\alpha \atop \{\mu\nu\},\beta} - {\alpha \atop \mu\beta\},\nu} + {\alpha \atop \sigma\beta\},\nu} - {\alpha \atop \sigma\beta}, \qquad (5.2)$$

$$R_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu}, \tag{5.3}$$

$$R \equiv g^{\mu\nu}R_{\mu\nu}. \tag{5.4}$$

The Christoffel symbol is very often written $\Gamma^{\alpha}_{\mu\nu}$, and is an example of a connection. However, it is not a tensor, so this notation can be very confusing. We write it in the form above to emphasize this fact, but also because we will later use the symbol $\Gamma^{\alpha}_{\mu\nu}$ with a different meaning.

Although $\{^{\alpha}_{\mu\nu}\}$ is not a tensor, it can be shown that $R^{\alpha}_{\ \mu\beta\nu}$ is, and thus has earned the right to its name.

5.1 The action

Einstein's field equations (in vacuum) describing the dynamics of spacetime are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{5.5}$$

and may be found as the equations of motion derived from the *Hilbert action* (see standard textbooks on gravitation, e.g. [58, 59])

$$S[g] = \frac{1}{\kappa} \int d^4x \sqrt{-g} (R(g) - 2\Lambda). \tag{5.6}$$

Einstein's constant κ is defined as $\kappa \equiv \frac{8\pi}{c^3}G_N$, where G_N is Newton's gravitational constant. Λ is the cosmological constant, which may be thought of as being related to the energy of vacuum.

This action contains metric fields $g_{\mu\nu}$, their derivatives $\partial g_{\mu\nu}$, and also their second derivatives $\partial^2 g_{\mu\nu}$, and was used by Hilbert to derive Einstein's field equations only days after Einstein had published his results.

It has later become clear that we may treat the connections $\Gamma^{\alpha}_{\mu\nu}$ as independent field variables, and define the Riemann tensor by means of $\Gamma^{\alpha}_{\mu\nu}$ instead of $\{^{\alpha}_{\mu\nu}\}$. Equation (5.2) is then replaced by

$$R^{\alpha}_{\ \mu\beta\nu} \equiv \Gamma^{\alpha}_{\mu\nu,\beta} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\alpha}_{\sigma\beta}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\beta}. \tag{5.7}$$

This gives the so-called *Palatini action* [60]

$$S[g,\Gamma] = \frac{1}{\kappa} \int d^4x \sqrt{-g} (g^{\mu\nu} R_{\mu\nu}(\Gamma) - 2\Lambda). \tag{5.8}$$

Variation of Γ and g respectively give us

$$\delta\Gamma^{\alpha}_{\mu\nu} \Rightarrow \Gamma^{\alpha}_{\mu\nu} = {\alpha \atop \mu\nu},$$
 (5.9)

$$\delta g^{\mu\nu} \Rightarrow R_{\mu\nu}(\Gamma) - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} R_{\rho\sigma}(\Gamma) + \Lambda g_{\mu\nu} = 0.$$
 (5.10)

We see that the relation between the connection and the metric, $\Gamma^{\alpha}_{\mu\nu} = \{^{\alpha}_{\mu\nu}\}$, comes out as a solution of the field equations. With the substitution $\Gamma \to \{\}$, equation (5.10) is exactly the Einstein field equations for vacuum, with a cosmological constant included. Details on these calculations are found e.g. in [59].

If we eliminate Γ in $S[g,\Gamma]$ by solving its equation of motion and substituting back, we just have to replace Γ by $\{\}$, thus recover the Hilbert action S[g]. In this sense we call $S[g,\Gamma]$ a parent action for S[g].

Alternatively, we may eliminate the metric fields $g_{\mu\nu}$. Contraction with $g^{\mu\nu}$ in (5.10) gives

$$g^{\mu\nu}R_{\mu\nu}(g) = 4\Lambda. \tag{5.11}$$

(If we worked with another spacetime dimension than 4 we would get $\frac{2D}{D-2}\Lambda$ on the right hand side.) Inserted back into equation (5.10) we find

$$\Lambda g_{\mu\nu} = R_{\mu\nu}(\Gamma). \tag{5.12}$$

As long as $\Lambda \neq 0$ we can eliminate $g_{\mu\nu}$ in $S_{\Lambda}[g,\Gamma]$ and arrive at the *Eddington-Schrödinger action* [61,62]

$$S[\Gamma] = \frac{2}{\kappa \Lambda} \int d^4x \sqrt{-\det(R_{\mu\nu}(\Gamma))}.$$
 (5.13)

The actions S[g] and $S[\Gamma]$ are dual, i.e. they are both derivable from the same parent action $S[g,\Gamma]$. All three actions will of course give us equivalent equations of motion, so they are classically equivalent.

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If we compare with the p-brane actions, we see that $S[\Gamma]$ resembles the Nambu-Goto form (2.28), while $S[g,\Gamma]$ is very similar to the Howe-Tucker action (2.94), and partly the Polyakov action (c.f. section 2.5). But there are important differences as well.

First, the strings were supposed to "live" in a background space, i.e. a spacetime with some fixed metric. In the gravitational case, on the other hand, this spacetime is precisely what is under study. This aspect is in fact a major obstacle in the search for a quantum field theory of gravitation.

Second, the $\Gamma^{\alpha}_{\mu\nu}$ fields in gravitation have a more complicated nature than the position fields X^{μ} in the string models. For instance, X^{μ} are scalars under diffeomorphisms, while the Γ 's are not even tensors.

The limit $\kappa \to \infty$ From the definition of κ we see that the limit may be viewed as a $c \to 0$ limit, or a $G_N \to \infty$ limit. This is the opposite of the Newtonian limit, which can be thought of as $c \to \infty$. As the speed of light approaches zero, lightcones will collapse into spacetime lines, and points in space will be disconnected. So this limit leads to an ultralocal field theory. These limits have been investigated e.g. in [63] and [64] in the search for a quantum description of gravity.

It is possible to make yet another interpretation of the $\kappa \to \infty$ limit by observing (which is possible in the Hamiltonian formulation) that it is equal to the so-called zero signature limit. This limit represents some intermediate stage between Euclidian space (signature +1) and Minkowski space (signature -1). This viewpoint is taken by Teitelboim [65]. His paper also includes an instructive discussion on what physical significance this limit has.

In the following we will see if we can use our well-developed methods as a successful approach to this limit. General relativity as a theory is quite different from the string models we have investigated so far, and it is difficult to say on beforehand whether any of the two methods will give anything of physical interest. But it is certainly worth a try.

5.2 Method I

Of the actions $S[g, \Gamma]$, S[g] and $S[\Gamma]$ only the latter has the form that makes method I applicable. We will henceforth consider this action. Introducing the auxiliary field χ we can write a classically equivalent action as

$$S[\chi, \Gamma] = \int d^4x \left[-\frac{4}{\Lambda^2} \det(R_{\mu\nu}(\Gamma))\chi + \frac{1}{\kappa^2 \chi} \right].$$
 (5.14)

Elimination of χ will as usual give back the original action, i.e. $\delta \chi \Rightarrow S[\Gamma]$.

However, we are interested in the $\kappa \to \infty$ limit, and the action above then gives

$$S^{0}[\chi,\Gamma] = -\frac{4}{\Lambda^{2}} \int d^{4}x \det(R_{\mu\nu}(\Gamma))\chi. \tag{5.15}$$

We could redefine the auxiliary field to get rid of the factor $\frac{4}{\Lambda^2}$. But the form above makes it manifestly clear that the excession is valid only for $\Lambda \neq 0$, so we keep it as it is.

The equations of motion are found by variations $\delta \chi$ and $\delta \Gamma^{\alpha}_{\mu\nu}$:

$$\delta \chi \Rightarrow \frac{1}{\Lambda^2} \det(R_{\mu\nu}(\Gamma)) = 0, \qquad (5.16)$$

$$\delta \Gamma^{\alpha}_{\nu\mu} \Rightarrow \frac{1}{\Lambda^2} \left[-\nabla_{\alpha} (\chi \mathcal{R} \mathcal{R}^{\mu\nu}) + \nabla_{\lambda} (\chi \mathcal{R} \mathcal{R}^{\lambda\nu}) \delta^{\mu}_{\alpha} \right] = 0,$$

where $\mathcal{R} \equiv \det(R(\Gamma))$, $\mathcal{R}^{\rho\sigma} = \frac{1}{3!} \frac{1}{\mathcal{R}} \epsilon^{\rho\alpha\beta\gamma} \epsilon^{\sigma\mu\nu\lambda} R_{\mu\alpha} R_{\nu\beta} R_{\lambda\gamma}$ is the inverse of $R_{\rho\sigma}$, and ∇_{α} is the covariant derivative with Γ as the connection. The determinant \mathcal{R} cancels in the combination $\mathcal{R}\mathcal{R}^{\mu\nu}$, so the second equation is non-trivial even when $\det(R) = 0$, which is imposed by the first equation.

The relation (5.12) connecting Γ to the metric, $R_{\mu\nu}(\Gamma) = \Lambda g_{\mu\nu}$, cannot be used as the equivalence between $S[g,\Gamma]$ and $S[\chi,\Gamma]$ breaks when we take the limit $\kappa \to \infty$. So even though we could solve the above equations for $\Gamma^{\alpha}_{\mu\nu}$, the interpretations of what *physics* this limit represents would not be immediate.

We do not elaborate more on this here, but turn instead to method II. This method employes a much more general and powerful formalism, and we have seen earlier that it may give rise to several different limits. Our hope is that within this possible class of limits we will find a physically significant one.

5.3 Method II

In the string case, the most straight forward way to arrive at tensionless limits was found by starting from the Nambu-Goto action. And because of its similarities to the Nambu-Goto action, we might therefore expect the action $S[\Gamma]$ to be the best starting point for deriving $\kappa \to \infty$ limits in the gravity case. However, we will see that the similarities are only formal.

The Lagrangian in $S[\Gamma]$ is $L = \frac{2}{\kappa\Lambda}\sqrt{-\det(R_{\mu\nu}(\Gamma))}$. With the definition of the Riemann tensor in terms of Γ instead of $\{\}$ we find the canonical conjugate momenta straight forwardly to be

$$\Pi_{\alpha}^{\mu\nu} \equiv \frac{\partial L}{\partial \Gamma_{\mu\nu}^{\alpha}} = \frac{2}{\kappa\Lambda} \sqrt{-\det(R)} \mathcal{R}^{\sigma\rho} \frac{\partial R_{\rho\sigma}}{\partial \Gamma_{\mu\nu,0}^{\alpha}}
= \frac{1}{\kappa\Lambda} \sqrt{-\det(R)} (\delta_{\alpha}^{0} \mathcal{R}^{(\mu\nu)} - \mathcal{R}^{0(\mu} \delta_{\alpha}^{\nu)}),$$
(5.17)

where again $\mathcal{R}^{\mu\nu}$ is the *inverse* of $R_{\mu\nu}$.

The naive Hamiltonian is found from

$$h = \Pi_{\alpha}^{\mu\nu} \Gamma_{\mu\nu,0}^{\alpha} - \frac{2}{\kappa\Lambda} \sqrt{-\det(R)}$$
$$= \frac{2}{\kappa\Lambda} \sqrt{-\det(R)} (\mathcal{R}^{\mu\nu} \Gamma_{\mu\nu,0}^{0} - \mathcal{R}^{0\mu} \Gamma_{\mu\nu,0}^{\nu} - 1). \tag{5.18}$$

This expression is not zero, and contains time derivatives both in $\det(R)$ and $\mathcal{R}^{\mu\nu}$. Still, the Hamiltonian is known to be a function independent of time derivatives, so it is always possible to eliminate time derivatives in favour of momenta. However, the relation (5.18) for h is clearly unwieldly and makes the subsequent analysis very hard. It is thus gratifying that there is another set of variables, the ADM variables, in which the analysis becomes tractable.

5.4 ADM approach (method II)

We will now follow the fruitful approach originally made by Arnowitt, Deser and Misner (ADM) [66].

The starting point is the Hilbert action with zero cosmological constant (for convenience),

$$S = \frac{1}{\kappa} \int d^4x \sqrt{-g} R(g). \tag{5.19}$$

The crucial point is the introduction of a new set of variables to replace the ten metric components of $g_{\mu\nu}$. In a Hamilton description we perform a space and time split. And because of the rather intricate role of time in general relativity, this is not as straight forwardly done as before. The way it is done, is by slicing the (4-dimensional) spacetime S into spacelike (3-dimensional) hypersurfaces Σ . Spacetime may then be parameterized by means of a continuous parameter, such that each value of this parameter corresponds to one of these hypersurfaces. This parameter is what we will call time, although it is not necessarily directly related to time as measured by clocks.

Let h_{ab} be the 3-dimensional induced metric on Σ , and let h^{ab} be its inverse. Indices referring to Σ (i.e. Latin indices, a, b) are raised and lowered with this metric. Now, introduce the *lapse function* N and *shift functions* N_a defined through

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_b \\ N_a^T & h_{ab} \end{pmatrix}; \qquad \begin{array}{c} \mu, \nu = 0, 1, 2, 3; \\ a, b = 1, 2, 3. \end{array}$$
 (5.20)

The metric components $\{g_{\mu\nu}\}$ are now replaced by the set of variables $\{h_{ab}, N, N_a\}$. If we let \vec{n} be a unit vector normal to the hypersurface Σ , we find the extrinsic curvature K_{ab} of Σ as a Lie derivative (c.f. appendix A.4) of the metric in the direction of \vec{n} . This can again be shown to depend on the 3-metric h_{ab} and the shift functions in the following manner [58]:

$$K_{ab} = \frac{1}{2} \mathcal{L}_n h_{ab} = \frac{1}{2N} (\dot{h}_{ab} - D_a N_b - D_b N_a), \tag{5.21}$$

where D_a denotes the covariant derivative associated with h_{ab} , and the dot means differentiation with respect to the "time" parameter.

The Gauss-Codazzi equation (c.f. appendix A.5) gives the relation between intrinsic curvatures (the Ricci scalars) R_S on S and R_Σ on S:

$$R_S = R_{\Sigma} + K_{ab}K^{ab} - K^2, (5.22)$$

where $K \equiv h^{ab}K_{ab}$ and $K^{ab} = h^{ac}h^{bd}K_{cd}$. From equation (5.20) we find

$$\sqrt{-g} = N\sqrt{h}. (5.23)$$

Put together this means that we can rewrite the action (5.19) to

$$S = \frac{1}{\kappa} \int dx^0 d^3x N \sqrt{h} (R + K_{ab} K^{ab} - K^2), \tag{5.24}$$

where R from now on refers to the intrinsic curvature on the hypersurface, i.e. $R = R_{\Sigma}$.

We may now derive the canonical conjugate momenta:

$$P \equiv \frac{\partial L}{\partial \dot{N}} = 0, \tag{5.25}$$

$$P^a \equiv \frac{\partial L}{\partial \dot{N}_a} = 0, \tag{5.26}$$

$$\pi^{ab} \equiv \frac{\partial L}{\partial \dot{h}_{ab}} = \frac{1}{\kappa} \sqrt{h} (K^{ab} - h^{ab} K). \tag{5.27}$$

P and P^a are identically zero, which means that N and N_a are non-dynamical variables. They play the role of Lagrange multipliers, as we will see explicitly later.

Equation (5.27) can be inverted to give an expression for K_{ab} and h_{ab} (by use of (5.21)),

$$K_{ab} = \frac{\kappa}{\sqrt{h}} (\pi_{ab} - \frac{1}{2} h_{ab} \pi), \qquad (5.28)$$

$$\dot{h}_{ab} = \frac{2N\kappa}{\sqrt{h}}(\pi_{ab} - \frac{1}{2}h_{ab}\pi) + D_aN_b + D_bD_a; \quad \pi \equiv h^{ab}\pi_{ab},$$
 (5.29)

which is easy to verify. The fact that we can obtain \dot{h}_{ab} from π_{ab} means that we do not have to search for constraints. This situation is very similar to what was found for the Polyakov form of the string in section 2.5.

The Hamiltonian is now straight forward to calculate:

$$H \equiv \pi^{ab}\dot{h}_{ab} - L$$

$$= \tilde{N} \left[\frac{1}{\sqrt{h}} (\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2) - \frac{1}{\kappa^2} \sqrt{h}R \right] + 2\pi^{ab}D_aN_b, \tag{5.30}$$

where we have redefined $\tilde{N} \equiv \kappa N$. This is done to allow for the $\kappa \to \infty$ limit. A similar argument was made in section 2.5 for the string case.

The last term in the Hamiltonian can be rewritten;

$$2\pi^{ab}D_a N_b = 2\underbrace{D_a(\pi^{ab}N_b)}_{\to 0} -2N_b D_a \pi^{ab}. \tag{5.31}$$

Again, the first term on the right hand side can be disregarded since it is only a total derivative. If we consider \tilde{N} and N_a as Lagrange multipliers we see that H is just a sum of these constraints:

$$\Theta = \frac{1}{\sqrt{h}} (\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2) - \frac{1}{\kappa^2} \sqrt{h} R, \qquad (5.32)$$

$$\Theta^b = -2D_a \pi^{ab}. (5.33)$$

Thus, we can write the Hamiltonian in this simple form

$$H = \tilde{N}\Theta + N_b\Theta^b. \tag{5.34}$$

The limit $\kappa \to \infty$

We see at once that we may take this limit simply by dropping the last term in Θ . And as mentioned in the beginning, this has the same effect as taking the zero signature limit $\varepsilon \to 0$. The signature $\varepsilon = \vec{n} \cdot \vec{n}$ of the spacetime metric only influences on this term, and enters in such a way that taking $\varepsilon \to 0$ removes the term proportional to $\sqrt{h}R$ [65].

This limit has in turn been shown by Henneaux [67] to correspond to the fourdimensional action

$$S = \int d^4x \Omega(x) (\mathcal{K}^{\alpha\beta} \mathcal{K}_{\alpha\beta} - \mathcal{K}^2); \quad \alpha, \beta = 0, 1, 2, 3.$$
 (5.35)

He does so by shoving that this action gives the same Hamilton formulation (5.34) as the $\varepsilon \to 0$ limit of the general relativity action.

The independent fields in the action (5.35) are the positive scalar density $\Omega(x)$ and the components of a symmetric covariant tensor $g_{\alpha\beta}(x)$. This "metric" $g_{\alpha\beta}$ is degenerate, i.e. $\det(g_{\alpha\beta}) = 0$, which means that it has only 9 independent components. Together with Ω this gives 10 independent fields, which is the same number as in the original action.

 $\mathcal{K}_{\alpha\beta}$ is the second fundamental tensor, defined as the Lie derivative in a unique direction \vec{e} ,

$$\mathcal{K}_{\alpha\beta} = \frac{1}{2} \mathcal{L}_e g_{\alpha\beta} = \frac{1}{2} (e^{\gamma} g_{\alpha\beta,\gamma} + e^{\gamma}_{,\alpha} g_{\gamma\beta} + e^{\gamma}_{,\beta} g_{\alpha\gamma}). \tag{5.36}$$

And the vector field \vec{e} is defined through

$$\mathcal{G}^{\mu\nu} = \Omega^2 e^{\mu} e^{\nu}; \qquad \mathcal{G}^{\alpha\beta} \equiv \frac{1}{3!} \varepsilon^{\alpha\lambda\mu\nu} \varepsilon^{\beta\rho\sigma\tau} g_{\lambda\rho} g_{\mu\sigma} g_{\nu\tau}, \qquad (5.37)$$

where $\mathcal{G}^{\mu\nu}$ is called the *minor* of $g_{\mu\nu}$. The vector \vec{e} is completely determined from $g_{\alpha\beta}$ and Ω . It satisfies $g_{\alpha\beta}e^{\beta}=0$ and is thus orthogonal to any other vector v^{α} , i.e. $g_{\alpha\beta}e^{\alpha}v^{\beta}=0$.

Since the metric is degenerate (i.e. the determinant vanishes), there is no inverse $g^{\alpha\beta}$ satisfying $g^{\alpha\beta}g_{\beta\gamma}=\delta^{\alpha}_{\gamma}$. However, the class of symmetric tensors $G^{\alpha\beta}$ defined by

$$G^{\alpha\beta}g_{\beta\gamma} = \delta^{\alpha}_{\gamma} - \theta_{\gamma}e^{\alpha}, \tag{5.38}$$

where θ_{γ} is an arbitrary vector satisfying $\theta_{\gamma}e^{\gamma}=1$, can instead serve as an "index raiser" which makes $\mathcal{K}=G^{\alpha\beta}\mathcal{K}_{\alpha\beta}$ and $\mathcal{K}^{\alpha\beta}\mathcal{K}_{\alpha\beta}$ well defined.

For an elaboration of the ideas presented here, the reader should consult [67].

The conclusion of this section is that we *are* able to find an interesting limit by use of the phase space method. This is not as straight forward as in the string cases, but by introduction of the more suitable ADM coordinates, it can be done.

The most troublesome part has turned out to be the way back from phase space to configuration space. This could not be done easily by elimination of momenta and Lagrange multipliers as before. Instead, we adopted the idea of first proposing a configuration space action, and then showing that it gives the right Hamiltonian.

Similar to the string models, we found the limit to correspond to a degenerate geometry. In the present case, this means a non-Riemannian space halfway between Euclidean and Minkowski space, which corresponds to a theory of gravity based on local *Caroll invariance*. (Normal gravitation is based on local Poincaré invariance.) The 10-parametric Caroll group was introduced by Levy-Lebond [68].

Chapter 6

Other models

We will now, very briefly, take a look on two models which are quite different from those we have studied thus far. The results we find here will tell us something about the generality of the methods.

6.1 Yang-Mills theory

In this section we will consider the simplest of all Yang-Mills theories, namely Abelian U(1) theory, which is the familiar Maxwell theory of electrodynamics.

6.1.1 The action

The gauge field of electrodynamics is the electromagnetic potential, denoted A^{μ} . The electromagnetic field strength $F_{\mu\nu}$ can then defined as $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. And the action for free electrodynamics (i.e. without sources) is

$$S = \frac{1}{g^2} \int d^4x F^{\mu\nu} F_{\mu\nu} = \frac{1}{g^2} \int d^4x \eta^{\mu\alpha} \eta^{\nu\beta} F_{\alpha\beta} F_{\mu\nu}, \tag{6.1}$$

where g is the coupling constant, equal to the electric charge in our case. Note that this action is not diff invariant. (It is not coupled to gravity.) The definition of $F_{\mu\nu}$ and a variation δA^{μ} together give the Maxwell equations.

From now on we will focus on the limit $g \to \infty$, which is a strong coupling limit relevant at high energies. We want to find out whether our methods can give any insight to this limit.

Method I We can immediatly say that this method will not lead to anything interesting, as the Lagrangian $\mathcal{L} = F^2$ in the action (6.1) does not have the proper form.

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6.1.2 Method II

To see the time dependence of the Lagrangian more explicitly, we rewrite it (keep in mind that we use the Minkowski metric $\eta^{\mu\nu} = (-, +, +, +)$),

$$L = \frac{1}{g^2} \eta^{\mu\alpha} \eta^{\nu\beta} F_{\alpha\beta} F_{\mu\nu}$$

= $\frac{1}{g^2} (-2F_{0i}F_{0i} + F_{ij}F_{ij}); \quad i, j = 1, 2, 3,$ (6.2)

where repeated indices are to be summed over. The momenta are now found to be

$$P^0 \equiv \frac{\partial L}{\partial(\partial_0 A_0)} = 0, \tag{6.3}$$

$$P^{i} \equiv \frac{\partial L}{\partial(\partial_{0}A_{i})} = -\frac{4}{g^{2}}F_{0i}. \tag{6.4}$$

The first equation says that A_0 is a non-dynamical variable to be treated as a Lagrange multiplier (which enforces Gauss' law). The dynamical variables are A_i , and equation (6.4) can be inverted to give an expression for $\partial_0 A_i$:

$$\partial_0 A_i = -\frac{g^2}{4} P^i + \partial_i A_0. \tag{6.5}$$

The naive Hamiltonian is

$$H \equiv P^{i}\partial_{0}A_{i} - \frac{1}{g^{2}}F^{\mu\nu}F_{\mu\nu}$$

$$= P^{i}\partial_{i}A_{0} - \frac{g^{2}}{8}P_{i}P^{i} - \frac{1}{g^{2}}F_{ij}F_{ij}.$$
(6.6)

This gives the phase space action

$$S^{PS} = \int d^4x \left[P^i F_{0i} + \frac{g^2}{8} P^i P^i + \frac{1}{g^2} F_{ij} F_{ij} \right]. \tag{6.7}$$

Elimination of the momenta P^i will of course give us the original action, since we have no constraints.

The coupling constant g appears both in the numerator and in the denominator, which means that the $g \to \infty$ limit is not well defined unless we can use a "trick" like in section 2.5 for eliminating g in the numerator. However, in the present case there is no Lagrange multiplier that can "swallow" the constant. The conclusion is that we cannot use the same method to define a $g \to \infty$ limit as we have applied earlier.

The assumption that the YM theory is Abelian is not essential. For non-Abelian YM theories things will be more complicated, but the problem of defining a $g \to \infty$ limit will certainly remain.

6.2 Chern-Simons theory

The calculations in this section are mainly based on [69]

6.2.1 The action

The action for free Abelian Chern-Simons theory is

$$S = \frac{\kappa}{4} \int d^3x \epsilon_{\mu\nu\rho} A^{\mu} F^{\nu\rho} = \frac{\kappa}{2} \int d^3x \epsilon_{\mu\nu\rho} A^{\mu} \partial^{\nu} A^{\rho}; \quad \mu, \nu, \rho = 0, 1, 2.$$
 (6.8)

Under diffeomorphisms $x^{\mu} \to \tilde{x}^{\mu}(x)$ we have the transformations

$$d^3x \rightarrow d^3J^{-1}; \qquad J \equiv \det(\Lambda_b^a); \quad \Lambda_b^a \equiv \frac{\partial x^a}{\partial \tilde{x}^b},$$
 (6.9)

$$\epsilon_{\mu\nu\rho}A^{\mu}\partial^{\nu}A^{\rho} \rightarrow \epsilon_{\mu\nu\rho}\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\Lambda^{\rho}_{\gamma}A^{\alpha}\partial^{\beta}A^{\gamma} = J\epsilon_{\alpha\beta\gamma}A^{\alpha}\partial^{\beta}A^{\gamma},$$
 (6.10)

which means that the action is diff invariant. We may view the Chern-Simons action as representing some kind of 3D gravitation. The equations of motion found from a variation δA^{α} are

$$\delta A^{\alpha} \Rightarrow \epsilon_{\alpha\mu\nu} \partial^{\mu} A^{\nu} = 0. \tag{6.11}$$

We now turn to the problem of finding a reasonable action describing the $\kappa \to 0$ limit of this model. As for the Yang-Mills theory, method I will fail to give anything interesting due to the form of the action.

6.2.2 Method II

The canonical conjugate momenta to the fields A^{μ} are

$$\Pi_{\rho} \equiv \frac{\partial L}{\partial (\partial^{0} A^{\rho})} = \frac{\kappa}{2} \epsilon_{\mu 0 \rho} A^{\mu}, \tag{6.12}$$

which gives $\Pi_0 = 0$ and $\Pi_i = \frac{\kappa}{2} \epsilon_{ij} A^j$, where i, j are spatial indices. This Legendre transformation from time derivatives to momenta is not invertible, and we get the primary constraints:

$$\Theta_0 \equiv \Pi_0 \approx 0, \tag{6.13}$$

$$\Theta_i \equiv \Pi_i - \frac{\kappa}{2} \epsilon_{ij} A^j \approx 0; \quad i, j = 1, 2.$$
 (6.14)

The naive Hamiltonian is found to be

$$H_{naive} \equiv \Pi_{\rho} \partial^{0} A^{\rho} - \frac{\kappa}{2} \epsilon_{\mu\nu\rho} A^{\mu} \partial^{\nu} A^{\rho} = -\frac{\kappa}{2} \epsilon_{\mu i \rho} A^{\mu} \partial^{i} A^{\rho}$$
$$= -\kappa A^{0} \epsilon_{ij} \partial^{i} A^{j}, \tag{6.15}$$

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where we have disregarded a total derivative. (Again, this expression is in complete agreement with what we would get by using theorem 2 of section 1.4.2.) The consistency conditions on the primary constraints are

$$\{\Theta_{\nu}, H_{naive}\} + \lambda^{\rho} \{\Theta_{\nu}, \Theta_{\rho}\} \approx 0,$$
 (6.16)

and lead to

$$\kappa \epsilon_{ij} \partial^i A^j \approx 0,$$
(6.17)

$$\kappa \epsilon_{1j} \partial^j A^0 - \lambda^2 \kappa \epsilon_{12} \approx 0, \tag{6.18}$$

$$\kappa \epsilon_{2j} \partial^j A^0 + \lambda^1 \kappa \epsilon_{12} \approx 0. \tag{6.19}$$

We thus get one secondary constraint,

$$\Theta_3 = \kappa \epsilon_{ij} \partial^i A^j, \tag{6.20}$$

and the conditions on two of the Lagrange multipliers, $\lambda^i = \partial^i A^0$. The consistency condition on the secondary constraint Θ_3 does not give any new constraints.

The total Hamiltonian can now be written

$$H = -\kappa A^{0} \epsilon_{ij} \partial^{i} A^{j} + \lambda^{0} \Pi_{0} + \partial^{i} A^{0} (\Pi_{i} - \frac{\kappa}{2} \epsilon_{ij} A^{j}) + \lambda^{3} \kappa \epsilon_{ij} \partial^{i} A^{j},$$
 (6.21)

and the phase space action is found to be

$$S^{PS} = \int d^3x \left[\Pi_i F^{0i} + \frac{\kappa}{2} \epsilon_{ij} A^0 \partial^i A^j + \Pi_0 \partial^0 A^0 - \Pi_0 \lambda^0 - \lambda^3 \kappa \epsilon_{ij} \partial^i A^j \right]. \tag{6.22}$$

Taking the $\kappa \to 0$ limit of this action gives

$$S^{\kappa=0} = \int d^3x \left[\Pi_i F^{0i} + \Pi_0 \partial^0 A^0 - \Pi_0 \lambda^0 \right]. \tag{6.23}$$

This action is linear in the momenta Π_{μ} . Hence we cannot eliminate the momenta and arrive at a sensible configuration space action. We must therefore conclude that this approach for studying the $\kappa \to 0$ limit does not work.

Chapter 7

Conclusion

One aim of this thesis was to present two methods for deriving tensionless limits of strings and the analogue in other models. This has been done thoroughly by first presenting the most important background theory in chapter 1, and by going through a series of examples in the subsequent chapters. The reader will hopefully understand and be able to apply the ideas on basis of this presentation.

One main result of the introductory chapter was the derivation of the naive Hamiltonian for *diff* invariant theories with only tensor fields. It was demonstrated that for such models the Hamiltonian is constrained to be zero when the fields satisfy the field equations.

A second important goal of the thesis was to investigate how widely applicable the methods are. It was known from before that they work well for a number of string models. The derivations in section 2.5 and 3.3 revealed that the methods work perfectly also if we start from the Weyl-invariant form of the bosonic string and D-string actions. These are models that already at the very beginning contain Lagrange multipliers (an auxiliary metric). And it is a noteworthy result that the methods apply in such cases as well, which emphasizes their generality.

The calculations for the rigid string also lead us to reasonable actions for the tensionless limit. But in contrast the other string models, we could not in this case use the two methods to derive the *same* form of the action.

Applying the methods to general relativity turned out to be a much more cumbersome task. Method I applied to the Eddington-Schrödinger action, while we had to change to ADM coordinates for method II to give expressions that we could handle in a reasonable way.

Finally the examples of chapter 6 demonstrated some of the limitations of the models. It was noted already in the introduction that method I requires a specific form of the Lagrangian. And when it comes to method II, it is of course not a priori given that it should work: We start with an action that is not defined for the specific limit, and hope that through some mathematical manipulations we can write an equivalent action that is also defined in the limit. The most surprising is rather the fact that it does indeed work so well for many theories.

7. Conclusion

As general remarks we note that in the tensionless limit, the string models naturally provide conformally invariant theories, and that the geometries turn out to be degenerate.

It would be interesting to know if there exist some crucial aspect that determine whether method II works, and if so, *what* this is. It would also be of interest to know if it is possible to say something general concerning the equality of method I and method II. These are proposals for further investigation.

Another natural extension of this work, would be to look at *supersymmetric* models. This has been done in some extent [15], but still not very thoroughly.

Appendix

A.1 Tensor densities

In this thesis we have not been very precise when talking about tensors. We have often referred to quantities as being e.g. scalars when they were really scalar *densities*. The Lagrangian is one example of this. The distinction has not been an important feature in our calculations, but we will now give a brief description of the difference between tensors and tensor densities.

For coordinate transformations $x^a \to \tilde{x}^a = \tilde{x}^a(x)$ we define the Jacobi matrix $J^a_b \equiv \frac{\partial x^a}{\partial \tilde{x}^b}$ and the Jacobi determinant $J \equiv \det(J^a_b)$. The Jacobi matrix is the same as what we often call the transformation matrix, denoted $\Lambda^a_b = J^a_b$.

Scalars, vectors and second rank tensors are quantities that transform as

scalar
$$\phi(x) \to \tilde{\phi}(\tilde{x}) = \phi(x),$$
 (A.1)

vector
$$A_a(x) \to \tilde{A}_a(\tilde{x}) = \Lambda_a^b A_b(x),$$
 (A.2)

2nd rank tensor
$$F_{ab}(x) \to \tilde{F}_{ab}(\tilde{x}) = \Lambda_a^c \Lambda_b^d F_{cd}(x)$$
. (A.3)

On basis of this, we define tensor densities to be quantities that transform as tensors, but with an extra factor of the Jacobi determinant. Thus, a tensor density of weight n transforms as

scalar density
$$\varphi(x) \to \tilde{\varphi}(\tilde{x}) = J^n \varphi(x),$$
 (A.4)

vector density
$$\mathcal{A}_a(x) \to \tilde{\mathcal{A}}_a(\tilde{x}) = J^n \Lambda_a^b \mathcal{A}_b(x),$$
 (A.5)

2nd rank tensor density
$$\mathcal{F}_{ab}(x) \to \tilde{\mathcal{F}}_{ab}(\tilde{x}) = J^n \Lambda_a^c \Lambda_b^d \mathcal{F}_{cd}(x)$$
. (A.6)

Suppose we have the action $S = \int dx L(\phi^i)$. With a transformation of the parameter x the integral measure transforms as $dx \to dx J^{-1}$. For the action to be invariant, the Lagrangian L must then be a scalar density of weight 1 (i.e. $L \to JL$).

Other examples of scalar densities (of weight 1) are the square root of the spacetime metric $\sqrt{-g}$ in general relativity, and the square root of the induced metric $\sqrt{-\gamma}$, $\gamma_{ab} = \partial_a X^{\mu} \partial_b X_{\mu}$ (under world sheet reparameterizations, c.f. section 2.4).

A.2 Derivatives of determinants

Consider an $n \times n$ matrix A_{ab} . Write its inverse with upper indices, i.e. $(A^{-1})_{ab} \equiv A^{ab}$ and $A^{ab}A_{bc} = \delta_c^a$. Introduce the set of matrices (\tilde{A}_{ab}) , which are $(n-1) \times (n-1)$

matrices identical to A except that row a and column b are eliminated. (There are n^2 such matrices.) The cofactor matrix C_{ab} is an $n \times n$ matrix which has as its elements the determinant of the (\tilde{A}_{ab}) matrices, i.e. $C_{ab} = \det(\tilde{A}_{ab})$. Let us write, for short, $\det(A_{ab}) = A$. Then we can write

$$A^{ab} = (-1)^{a+b} \frac{C_{ba}}{A}, (A.7)$$

$$A = \sum (\pm 1) \prod A_{ab} = (-1)^{a+c} \sum_{c=1}^{n} A_{ac} C_{ac} \quad \text{independent of } a. \quad (A.8)$$

Then we easily see that $\frac{\partial A}{\partial A_{ab}} = (-1)^{a+b} C_{ab} = AA^{ba}$, which gives the very useful results

$$\frac{d}{dx}A = AA^{ab}\frac{dA_{ba}}{dx} = -AA_{ab}\frac{dA^{ba}}{dx},\tag{A.9}$$

$$\frac{d}{dx}\sqrt{-A} = \frac{1}{2}\sqrt{-A}A^{ab}\frac{dA_{ba}}{dx} = -\frac{1}{2}\sqrt{-A}A_{ab}\frac{dA^{ba}}{dx}.$$
 (A.10)

A.3 On the vielbein formalism

Consider a D-dimensional manifold S, and suppose that there exist a coordinate system $\{x^a\}$. Then we can introduce the coordinate basis forms $\{\tilde{d}x^a\}$. Any one-form on S can then be described by means of these basis forms, as $\tilde{V} = V_a \tilde{d}x^a$. We may also define coordinate basis vectors, $\vec{u}_a \equiv \frac{\partial}{\partial x^a}$, which then satisfy $\tilde{d}x^a(\vec{u}_b) = \delta_b^a$.

The metric tensor can then be written

$$\mathbf{g} = g_{ab}\tilde{d}x^a \otimes \tilde{d}x^b; \qquad a, b = 0, \dots, D - 1.$$
 (A.11)

and we find

$$\vec{u}_a \cdot \vec{u}_b = \mathbf{g}(\vec{u}_a, \vec{u}_b) = g_{cd}\tilde{d}x^c(\vec{u}_a)\tilde{d}x^d(\vec{u}_b) = g_{ab}. \tag{A.12}$$

We know that, locally, a spacetime manifold behaves as flat space. So it must be possible to introduce a new (position dependent) flat basis $\{\tilde{e}^A\}$, such that

$$\mathbf{g} = \eta_{AB}\tilde{e}^A \otimes \tilde{e}^B; \qquad A, B = 0, \dots, D - 1,$$
 (A.13)

where $\eta_{AB} = diag(-,+,...,+)$ is the Minkowski metric. Such a basis is often calleda tetrad or an orthonormal basis. The new basis forms can be written as a linear combinations of the coordinate basis forms,

$$\tilde{e}^A(x) = e_a{}^A(x)\tilde{d}x^a, \tag{A.14}$$

which defines the *vielbein* $e_a^A(x)$. We see that the vielbein can be viewed as a transformation matrix between the curved and flat bases. It can, however, not generally be expressed as a Jacobi matrix. Together with the equations (A.11) and (A.13), this gives

$$g_{ab} = \eta_{AB} e_a^{\ A} e_b^{\ B}. \tag{A.15}$$

A.4 On Lie derivatives 77

Writing $\det(g_{ab}) \equiv g$, $\det(e_a^A) \equiv e$ and remembering that $\det(\eta_{AB}) = -1$ this gives immediately

$$\sqrt{-g} = e. (A.16)$$

If we introduce the inverse vielbein $e_A{}^a$ satisfying $e_A{}^a e_a{}^B = \delta_A^B$ and $e_a{}^A e_A{}^b = \delta_a^b$, we can write $\tilde{d}x^a = e_A{}^a \tilde{e}^A$. A one-form (or covariant tensors in general) can now be written either in the coordinate or flat basis as $\tilde{V} = V_a \tilde{d}x^a = V_A \tilde{e}^A$, and the components are related through the vielbeins:

$$V_A = e_A{}^b V_b; \qquad V_a = e_a{}^B V_B.$$
 (A.17)

Indices referring to the coordinate basis are often called *Einstein indices* (denoted here with small letters), while indices referring to the flat basis are called *Lorentz indices* (denoted with capital letters).

Vielbeins are often called *vierbeins*, also in cases where they are not 4-dimensional. This formalism is particularly useful when one wants to consider particles with spin in curved spaces.

A.4 On Lie derivatives

The Lie derivative is in this thesis denoted $\mathcal{L}_{\vec{n}}$, and represents a kind of generalized directional derivative. For a more thorough presentation, the reader may consult [58].

The Lie derivative of a $scalar \ f$ is defined as

$$\mathcal{L}_{\vec{n}}f \equiv \vec{n}[f] = f_{,\mu}n^{\mu},\tag{A.18}$$

and gives the change in f along \vec{n} . The Lie derivative of a vector \vec{v} is

$$\mathcal{L}_{\vec{n}}\vec{v} \equiv [\vec{n}, \vec{v}] = \nabla_{\vec{n}}\vec{v} - \nabla_{\vec{v}}\vec{n},\tag{A.19}$$

and says something about how \vec{v} is changed under a parallel transport along \vec{n} . Here ∇ may be any derivative operator.

In a coordinate basis $\{x^{\alpha}\}$ we may write the Lie derivative of a 1-form field $\tilde{\sigma}$ and a second rank tensor T as

$$\mathcal{L}_{\vec{n}}\tilde{\sigma} = (\sigma_{\alpha,\beta}n^{\beta} + \sigma_{\beta}n^{\beta}_{,\alpha})\tilde{d}x^{\alpha}, \tag{A.20}$$

$$\mathcal{L}_{\vec{n}} T = (T_{\alpha\beta,\mu} n^{\mu} + T_{\mu\beta} n^{\mu}_{,\alpha} + T_{\alpha\mu} n^{\mu}_{,\beta}) \tilde{d}x^{\alpha} \otimes \tilde{d}x^{\beta}. \tag{A.21}$$

It can be shown that this holds also if we exchange the partial derivative "," with some covariant derivative ∇ . Thus, we can write in coordinate form

$$\mathcal{L}_{\vec{n}} T_{\alpha\beta} = (\nabla_{\mu} T_{\alpha\beta}) n^{\mu} + T_{\mu\beta} (\nabla_{\alpha} n^{\mu}) + T_{\alpha\mu} (\nabla_{\beta} n^{\mu}). \tag{A.22}$$

If we let ∇ be the covariant derivative associated with some metric $g_{\alpha\beta}$ (i.e. $\nabla_{\mu}n^{\alpha} = n^{\alpha}_{,\mu} + \{^{\alpha}_{\nu\mu}\}n^{\nu}$), then the following result is not very difficult to prove:

$$\pounds_{\vec{n}}g_{\alpha\beta} = \nabla_{\alpha}n_{\beta} + \nabla_{\beta}n_{\alpha}. \tag{A.23}$$

A.5 The Gauss-Codazzi equation

The discussion here is mainly due to Wald [58]. Consider a *D*-dimensional manifold S with a vector basis $\{\vec{e}_{\mu}; \mu = 0, \dots, D-1\}$ and a corresponding form basis $\{\tilde{\omega}^{\mu}\}$, such that $\tilde{\omega}^{\mu}(\vec{e}_{\nu}) = \delta^{\mu}_{\nu}$. The metric tensor \boldsymbol{g} on S can then be written

$$\mathbf{g} = g_{\mu\nu}\tilde{\omega}^{\mu} \otimes \tilde{\omega}^{\nu}; \qquad g_{\mu\nu} = \mathbf{g}(\vec{e}_{\mu}, \vec{e}_{\nu}) \equiv \vec{e}_{\mu} \cdot \vec{e}_{\nu}.$$
 (A.24)

Induced metric

Let Σ be a d-dimensional hypersurface embedded in this space. The basis vectors on this hypersurface are denoted $\{\vec{u}_a; a=0,\ldots,d-1\}$, and there are d such vectors. The corresponding basis forms are denoted $\{\tilde{w}^a\}$, and satisfy $\tilde{w}^a(\vec{u}_b) = \delta^a_b$. The induced metric γ on Σ is then

$$\gamma = \gamma_{ab}\tilde{w}^a \otimes \tilde{w}^b; \qquad \gamma_{ab} = \gamma(\vec{u}_a, \vec{u}_b) \equiv \vec{u}_a \cdot \vec{u}_b.$$
(A.25)

Define $\{\vec{n}_i; i=d,\ldots,D-1\}$ to be as set of D-d orthonormal vectors perpendicular to Σ . This means $\vec{n}_i \cdot \vec{u}_a = 0$ and $\vec{n}_i \cdot \vec{n}_j = \eta_{ij}$, where η_{ij} is a diagonal matrix with elements ± 1 , depending on whether the normal vectors are timelike (-1) or spacelike (+1).

It is now evident that we can write any vector in S as a linear combination of \vec{u}_a and \vec{n}_i . Especially, we have $\vec{e}_{\mu} = \sigma_{\mu}^{\ a} \vec{u}_a + \tau_{\mu}^{\ i} \vec{n}_i$, which gives

$$g_{\mu\nu} = \vec{e}_{\mu} \cdot \vec{e}_{\nu} = \sigma_{\mu}^{\ a} \sigma_{\nu}^{\ b} \gamma_{ab} + \tau_{\mu}^{\ i} \tau_{\nu}^{\ j} \eta_{ij}$$
 (A.26)

and

$$\vec{n}_i \cdot \vec{e}_{\mu} = \tau_{\mu}^{\ j} \eta_{ji}$$

$$\vec{u}_a \cdot \vec{e}_{\mu} = \sigma_{\mu}^{\ b} \gamma_{ab}. \tag{A.27}$$

But we can also decompose by means of \vec{e}_{μ} , and write $\vec{u}_a = u_a^{\ \mu} \vec{e}_{\mu}$, $\vec{n}_i = n_i^{\ \mu} \vec{e}_{\mu}$. With this decomposition we find directly

$$\vec{n}_i \cdot \vec{e}_{\mu} = n_i^{\ \nu} g_{\nu\mu} = n_{i\mu}$$

$$\vec{u}_a \cdot \vec{e}_{\mu} = u_a^{\ \nu} g_{\nu\mu} = u_{a\mu}. \tag{A.28}$$

Equations (A.27) and (A.28) together give $n_{i\mu} = \tau_{\mu}{}^{j}\eta_{ji}$, $u_{a\mu} = \sigma_{\mu}{}^{b}\gamma_{ba}$. If we introduce η^{ij} and γ^{ab} as the *inverse* of η_{ij} and γ_{ab} respectively, we find

$$\tau_{\mu}^{\ i} = \eta^{ij} n_{j\mu} \equiv n_{\mu}^{i}; \qquad \sigma_{\mu}^{\ a} = \gamma^{ab} u_{b\mu}. \tag{A.29}$$

Put into the expression (A.26) for $g_{\mu\nu}$, this gives

$$g_{\mu\nu} = h_{\mu\nu} + \eta_{ij} n^i_{\mu} n^j_{\nu}; \qquad h_{\mu\nu} \equiv \gamma^{ab} u_{a\mu} u_{b\nu}.$$
 (A.30)

We will soon demonstrate that $h_{\mu\nu}$ is the induced metric.

We raise and lower Greek and Latin (middle alphabet) indices with $g_{\mu\nu}$ and η_{ij} , and their inverses $g^{\mu\nu}$ and η^{ij} respectively. Thus we have $h_{\mu\nu} = g_{\mu\nu} - n^i_{\mu}n_{i\nu}$, $h^{\alpha}_{\nu} = \delta^{\alpha}_{\nu} - n^{\alpha}_{i}n^i_{\nu}$, and $h^{\alpha\beta} = g^{\alpha\beta} - n^{\alpha}_{i}n^{i\beta}$. Note that $h^{\mu\nu}$ is not the inverse of $h_{\mu\nu}$.

We find immediately

$$h_{\mu\nu}n_i^{\nu} = n_{i\mu} - n_{\mu}^j n_{j\nu} n_i^{\nu} = 0, \tag{A.31}$$

which states that h is a tensor tangent to Σ .

Consider now an arbitrary tensor T, which lies in Σ . This means that it can be decomposed in two ways:

$$T = T_{\mu\nu}\tilde{\omega}^{\mu} \otimes \tilde{\omega}^{\nu} = T_{ab}\tilde{w}^{a} \otimes \tilde{w}^{b}. \tag{A.32}$$

The relations between the components are

$$T_{ab} = T(\vec{u}_a, \vec{u}_b) = T_{\mu\nu}\tilde{\omega}^{\mu}(\vec{u}_a)\tilde{\omega}^{\nu}(\vec{u}_b) = T_{\mu\nu}u_a^{\mu}u_b^{\nu}. \tag{A.33}$$

The tensor h is of this kind (i.e. tangent to Σ), so by use of (A.33) and (A.30) we can write

$$h_{ab} = h_{\mu\nu} u_a^{\ \mu} u_b^{\ \nu} = \gamma^{cd} u_{c\mu} u_{d\nu} u_a^{\ \mu} u_b^{\ \nu} = \gamma^{cd} \gamma_{ac} \gamma_{bd} = \gamma_{ab}. \tag{A.34}$$

This shows that h is indeed the induced metric, i.e. $h = \gamma$.

Consider an arbitrary vector $\vec{v} = v^{\mu}\vec{e}_{\mu}$ in S. Define $\vec{u} \equiv h^{\mu}_{\nu}v^{\nu}\vec{e}_{\mu}$, or in component form $u^{\mu} = h^{\mu}_{\nu}v^{\nu} = v^{\mu} - n^{\mu}_{i}n^{i}_{\nu}v^{\nu}$. Then we have

$$\vec{u} \cdot \vec{n}_i = (v^{\mu} - n^j_{\nu} v^{\nu} n^{\mu}_i) \vec{e}_{\mu} \cdot \vec{e}_{\rho} n^{\rho}_i = 0.$$
(A.35)

In words, this means that the vector \vec{u} is tangent to Σ . Thus we may consider h^{μ}_{ν} as a projection operator from S to Σ .

If we have coordinate bases $\{\vec{e}_{\mu} = \frac{\partial}{\partial x^{\mu}}\}$ on S and $\{\vec{u}_{a} = \frac{\partial}{\partial \xi^{a}}\}$ on Σ , we can write

$$\vec{u}_a = \frac{\partial}{\partial \xi^a} = \frac{\partial x^{\mu}}{\partial \xi^a} \frac{\partial}{\partial x^{\mu}} = \frac{\partial x^{\mu}}{\partial \xi^a} \vec{e}_{\mu} \equiv u_a^{\ \mu} \vec{e}_{\mu}, \tag{A.36}$$

which gives

$$\gamma_{ab} = \vec{u}_a \cdot \vec{u}_b = u_a^{\ \mu} u_b^{\ \nu} g_{\mu\nu} = \frac{\partial x^{\mu}}{\partial \xi^a} \frac{\partial x^{\nu}}{\partial \xi^b} g_{\mu\nu}. \tag{A.37}$$

This is the expression we use for the induced metric on the string worldsheet.

Extrinsic curvature

We may define the *extrinsic curvature* of the hypersurface Σ as the Lie derivative of the metric in directions normal to Σ , i.e. along \vec{n}_i . In mathematical terms,

$$K_{\mu\nu}^{i} \equiv \frac{1}{2} \mathcal{L}_{\vec{n}_{i}} g_{\mu\nu} = \frac{1}{2} \mathcal{L}_{\vec{n}_{i}} h_{\mu\nu}$$

$$= \frac{1}{2} (\nabla_{\mu} n_{\nu}^{i} + \nabla_{\nu} n_{\mu}^{i})$$

$$= h_{\mu}^{\rho} \nabla_{\rho} n_{\nu}^{i}, \qquad (A.38)$$

where ∇ is the covariant derivative associated with $g_{\mu\nu}$. From the above, we see that the extrinsic curvature is symmetric, i.e. $K^i_{\mu\nu} = K^i_{\nu\mu}$, and, owing to the projection factor h^{ρ}_{μ} , tangent to Σ , i.e. $K^i_{\mu\nu}n^{\nu}_j = 0$.

Riemann curvature

We now restrict to cases where d = D - 1, i.e. there is only one normal direction to the hypersurface Σ . The "Minkowski metric" η_{ij} is then replaced by η , which is still +1 if the normal vector is spacelike, and -1 if it is timelike.

The Riemann curvature tensor of (S, g) is by definition given by

$$R^{\mu}_{\ \nu\alpha\beta}\omega_{\mu} = \nabla_{\alpha}\nabla_{\beta}\omega_{\nu} - \nabla_{\beta}\nabla_{\alpha}\omega_{\nu},\tag{A.39}$$

where ω_{ν} is any 1-form field on S. The corresponding result for vectors t^{μ} is

$$R^{\mu}_{\nu\alpha\beta}t^{\nu} = \nabla_{\alpha}\nabla_{\beta}t^{\mu} - \nabla_{\beta}\nabla_{\alpha}t^{\mu}. \tag{A.40}$$

Similarly, if we let $\bar{\nabla}$ be the covariant derivative associated with $h_{\mu\nu}$, we can write the Riemann tensor of (Σ, h) as

$$\bar{R}^{\mu}_{\ \nu\alpha\beta}\bar{\omega}_{\mu} = \bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\bar{\omega}_{\nu} - \bar{\nabla}_{\beta}\bar{\nabla}_{\alpha}\bar{\omega}_{\nu}, \tag{A.41}$$

 $\bar{\omega}_{\nu}$ now being a 1-form field on Σ . (Quantities with a bar are all referring to Σ .) The operator $\bar{\nabla}$ can be shown to be related to ∇ in the following simple way:

$$\bar{\nabla}_{\alpha}\bar{T}^{\mu\dots}_{\nu\dots} = h^{\mu}_{\rho}\cdots h^{\sigma}_{\nu}\cdots h^{\sigma}_{\alpha}\nabla_{\beta}\bar{T}^{\rho\dots}_{\sigma\dots}, \tag{A.42}$$

where \bar{T} is some tensor on Σ . We define the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R as

$$R_{\mu\nu} \equiv R^{\alpha}_{\ \mu\alpha\nu}; \qquad R \equiv g^{\mu\nu}R_{\mu\nu},$$
 (A.43)

$$\bar{R}_{\mu\nu} \equiv \bar{R}^{\alpha}_{\ \mu\alpha\nu}; \qquad \bar{R} \equiv g^{\mu\nu}\bar{R}_{\mu\nu}.$$
 (A.44)

Let us now derive four useful relations.

$$h_{\alpha}^{\beta}h_{\mu}^{\nu}\nabla_{\beta}h_{\nu}^{\rho} = h_{\alpha}^{\beta}h_{\mu}^{\nu}\nabla_{\beta}(\delta_{\nu}^{\rho} - \eta n_{\nu}n^{\rho})$$

$$= -\eta h_{\mu}^{\nu}n^{\rho}\underbrace{h_{\alpha}^{\beta}\nabla_{\beta}n_{\nu}}_{=K^{\alpha\nu}} - \eta h_{\alpha}^{\beta}\underbrace{h_{\mu}^{\nu}n_{\nu}}_{=0}\nabla_{\beta}n^{\rho}$$

$$= -\eta n^{\rho}h_{\mu}^{\nu}K_{\alpha\nu} = -\eta n^{\rho}K_{\alpha\mu}, \qquad (A.45)$$

$$h_{\beta}^{\nu}n^{\rho}\nabla_{\nu}\bar{\omega}_{\rho} = h_{\beta}^{\nu}\nabla_{\nu}\underbrace{(n^{\rho}\bar{\omega}_{\rho})}_{=0} - h_{\beta}^{\nu}\bar{\omega}_{\rho}\nabla_{\nu}n^{\rho} = -\bar{\omega}_{\rho}g^{\rho\mu}\underbrace{h_{\beta}^{\nu}\nabla_{\nu}n_{\mu}}_{K_{\beta\mu}}$$

$$= -g^{\rho\mu}K_{\beta\mu}\bar{\omega}_{\rho} = -K_{\beta}^{\rho}\bar{\omega}_{\rho}, \qquad (A.46)$$

$$h^{\alpha\mu}h^{\beta\nu}R_{\alpha\beta\mu\nu} = (g^{\alpha\mu} - \eta n^{\alpha}n^{\mu})(g^{\beta\nu} - \eta n^{\beta}n^{\nu})R_{\alpha\beta\mu\nu}$$

$$= R - 2\eta R_{\alpha\beta} n^{\alpha} n^{\beta}, \qquad (A.47)$$

$$R_{\alpha\beta} n^{\alpha} n^{\beta} = R^{\alpha}_{\beta\alpha\nu} n^{\beta} n^{\nu}$$

$$= n^{\nu} (\nabla_{\alpha} \nabla_{\nu} - \nabla_{\nu} \nabla_{\alpha}) n_{\alpha}$$

$$= n^{\nu} \nabla_{\alpha} \nabla_{\nu} n^{\alpha} - n^{\nu} \nabla_{\nu} \nabla_{\alpha} n^{\alpha}$$

$$= \nabla_{\alpha} (n^{\nu} \nabla_{\nu} n^{\alpha}) - (\nabla_{\alpha} n^{\nu}) (\nabla_{\nu} n^{\alpha})$$

$$- \nabla_{\nu} (n^{\nu} \nabla_{\alpha} n^{\alpha}) + (\nabla_{\nu} n^{\nu}) (\nabla_{\alpha} n^{\alpha})$$

$$= (K^{\nu}_{\nu})^{2} - K_{\alpha\nu} K^{\alpha\nu} \underbrace{-\nabla_{\nu} (n^{\nu} \nabla_{\alpha} n^{\alpha}) + \nabla_{\alpha} (n^{\nu} \nabla_{\nu} n^{\alpha})}_{\text{I}}, \qquad (A.48)$$

where t.d. is an abbreviation for total divergence. It will vanish under integration. Using the relation (A.42) between $\bar{\nabla}$ and ∇ and equation (A.45), we find

$$\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\bar{\omega}_{\mu} = h_{\alpha}^{\sigma}h_{\beta}^{\nu}h_{\mu}^{\rho}\nabla_{\sigma}\nabla_{\nu}\bar{\omega}_{\rho} - \eta h_{\mu}^{\rho}K_{\alpha\beta}n^{\nu}\nabla_{\nu}\bar{\omega}_{\rho} - \eta h_{\beta}^{\nu}K_{\alpha\mu}n^{\rho}\nabla_{\nu}\bar{\omega}_{\rho}. \tag{A.49}$$

Together with (A.46) this gives

$$\bar{R}^{\alpha}_{\beta\mu\nu} = h^{\alpha}_{\sigma}h^{\tau}_{\beta}h^{\lambda}_{\mu}h^{\rho}_{\nu}R^{\sigma}_{\tau\lambda\rho} - \eta K_{\mu\beta}K_{\nu}^{\ \alpha} + \eta K_{\nu\beta}K_{\mu}^{\ \alpha}. \tag{A.50}$$

This equation is known as the Gauss-Codazzi equation. Using the equations (A.47) and (A.48) we end up with

$$\bar{R} = R - \eta (K_{\alpha}^{\alpha})^2 + \eta K_{\alpha\beta} K^{\alpha\beta} + t.d., \tag{A.51}$$

which gives the general relationship between the intrinsic curvature \bar{R} on Σ and the extrinsic curvature $K_{\mu\nu}$.

We said that the extrinsic curvatures are tensors tangent to the space Σ , which means that they can be expressed by means of basis forms \tilde{w}^a on Σ instead of basis forms $\tilde{\omega}^{\mu}$ on S. In mathematical terms,

$$\mathbf{K} = K_{\mu\nu}\tilde{\omega}^{\mu} \otimes \tilde{\omega}^{\nu} = K_{ab}\tilde{w}^{a} \otimes \tilde{w}^{b}; \qquad \mu, \nu = 0, \dots, D-1; a, b = 0, \dots, d-1 = D-2. \quad (A.52)$$

On Σ we use the *induced* metric $\gamma_{ab}=h_{ab}$ and its inverse γ^{ab} to raise and lower indices. Thus we have

$$Tr(\mathbf{K}) = K_{\mu}^{\ \mu} = g^{\mu\nu} K_{\mu\nu} = \gamma^{ab} K_{ab},$$

$$Tr(\mathbf{K}^2) = \eta K_{\mu\nu} K^{\mu\nu} = \eta g^{\mu\alpha} g^{\nu\beta} K_{\mu\nu} K_{\alpha\beta} = \eta \gamma^{ac} \gamma^{bd} K_{ab} K_{cd}.$$

This means altogether that equation (A.51) takes the form

$$R = \bar{R} + \eta (K_a^a)^2 - \eta K_{ab} K^{ab} + t.d., \tag{A.53}$$

connecting quantities that refer to S on the left hand side, and quantities that refer to Σ on the right hand side.

In general relativity, a natural split is to let Σ be a spacelike 3-dimensional hypersurface. Then the normal direction \vec{n} is timelike, i.e. $\eta = -1$, which gives

$$R = \bar{R} - K^2 + K_{ab}K^{ab} + t.d. \tag{A.54}$$

This result is used in chapter 5 while doing the space-time split necessary for a Hamilton description of general relativity.

The results above can be generalized to situations where the dimension difference between S and Σ is more than one, so that there are several normal directions \vec{n}_i . In those cases equation (A.53) takes the more general form

$$R = \bar{R} + \eta_{ij} K_a^{i\ a} K_b^{j\ b} - \eta_{ij} K_{ab}^{i} K^{jab} + t.d., \tag{A.55}$$

which is a result that we apply in chapter 4 when investigating the rigid string.

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