HARMONIC HARDY SPACES ON SMOOTH DOMAINS

Tomasz Luks

Contents

1	Introduction	2
2	Preliminaries	2
3	Properties of smooth domains in \mathbb{R}^N	3
4	The spaces $h^{p}\left(D\right)$	19
5	The Fatou Theorem	32
6	The Local Fatou Theorem	36
7	The Area Theorem	54

1 Introduction

The harmonic Hardy spaces h^p are certain classes of harmonic functions, usually defined on the unit ball or the upper half-space. Many facts from h^p theory have their source in complex analysis and holomorphic Hardy spaces H^p . They are named in honor of the mathematician G. H. Hardy, who first studied them.

The objective of this paper is to characterize h^p classes and a boundary behavior of harmonic functions on a smooth domain in real Euclidean space. Most of the presented results come from the Elias M. Stein's book [5]. We will concentrate on supplementing the missing or incomplete proofs; the basis will be the well-known theory of h^p spaces and nontangential convergence on the ball or the upper half-space.

2 Preliminaries

Throughout this paper, we will deal with harmonic functions, defined on open subsets of real Euclidean space \mathbb{R}^N , where N will denote a fixed positive integer greater than 1. Let Ω be an open subset of \mathbb{R}^N ; a twice continuously differentiable, complex-valued function u is harmonic on Ω if

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_N^2} = 0$$

at every point of Ω . The operator Δ is called the *Laplacian*, and the equation $\Delta u = 0$ is called *Laplace's equation*. We will use some well-known properties of harmonic functions, as the maximum principle or the mean value property, without a comment; all this properties may be found in [1].

 $x=(x_1,...,x_N)$ denote a typical point in \mathbb{R}^N , and $|x|=(x_1^2+...+x_N^2)^{1/2}$ is the Euclidean norm of x. By $\langle \cdot, \cdot \rangle$ we denote the usual Euclidean inner product. Recall, that for every $x,y \in \mathbb{R}^N$

$$|x+y|^2 = |x|^2 + 2\langle x, y \rangle + |y|^2.$$

All functions in this paper are assumed to be complex valued unless stated otherwise. For fixed positive integer k, a function f is of class C^k on Ω , if f is k times continuously differentiable on Ω ; f is of class C^{∞} on Ω , if f is of class C^k for every k. We say, that f is of class $C^{1,1}$ on Ω , if f is of class C^1 , and ∇f satisfies the Lipschitz condition

$$|\nabla f(x) - \nabla f(y)| < A|x - y|, \quad \forall x, y \in \Omega,$$

where $A < \infty$ is a positive constant, and

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_N}\right),\,$$

is the gradient of f. For fixed positive integers n, k, a vector-valued function $F: \Omega \to \mathbb{R}^n$, $F = (F_1, ..., F_n)$, is of class C^k on Ω , if the component functions F_i , i = 1, ..., n, are of class C^k on Ω .

For $E \subset \mathbb{R}^N$, C(E) denote the space of continuous functions on E. $B(a,r) = \{x \in \mathbb{R}^N : |x-a| < r\}$ is the open ball centered at $a \in \mathbb{R}^N$ and radius r > 0. If the dimension is important, we write $B_N(a,r)$ in place of B(a,r). The unit sphere, the boundary of B(0,1), is denoted by S.

We will also deal with smooth domains in \mathbb{R}^N ; by a smooth domain, we mean a bounded domain with the boundary at least of class C^2 . More precisely, we say that a bounded domain $D \subset \mathbb{R}^N$ has a boundary of class C^2 , if there exists a real-valued function λ defined in a neighborhood of \overline{D} with the following properties:

- 1. λ is of class C^2 .
- 2. $\lambda(x) < 0$ if and only if $x \in D$.
- 3. $\{x:\lambda(x)=0\}=\partial D$.
- 4. $|\nabla \lambda(x)| > 0$ if $x \in \partial D$.

Throughout this paper, we will assume that D is a bounded domain in \mathbb{R}^N with the boundary of class C^2 , and a function λ of the above type will be called a *characterizing function* for D.

3 Properties of smooth domains in \mathbb{R}^N

In this section we will show some important properties of D. Let λ be a characterizing function for D. For $y \in \partial D$ denote

$$\nu_y = \frac{\nabla \lambda(y)}{|\nabla \lambda(y)|},$$

the outward unit normal vector field to ∂D . By the property 1 of λ (and mean value theorem), there exists a positive constant c, such that

$$|\nabla \lambda(x) - \nabla \lambda(y)| < c|x - y|$$

for every $x, y \in \partial D$; property 4 implies, that there exists positive constant c', such that for each $y \in \partial D$

$$|\nabla \lambda(y)| \ge c'$$
.

Denoting $c_0 = 2c/c'$, we conclude

$$\begin{aligned} |\nu_{x} - \nu_{y}| &= \left| \frac{\nabla \lambda(x)}{|\nabla \lambda(x)|} - \frac{\nabla \lambda(y)}{|\nabla \lambda(y)|} \right| = \left| \frac{\nabla \lambda(x)}{|\nabla \lambda(x)|} - \frac{\nabla \lambda(y)}{|\nabla \lambda(x)|} + \frac{\nabla \lambda(y)}{|\nabla \lambda(x)|} - \frac{\nabla \lambda(y)}{|\nabla \lambda(y)|} \right| \\ &\leq \frac{|\nabla \lambda(x) - \nabla \lambda(y)|}{|\nabla \lambda(x)|} + |\nabla \lambda(y)| \left| \frac{1}{|\nabla \lambda(x)|} - \frac{1}{|\nabla \lambda(y)|} \right| \\ &= \frac{|\nabla \lambda(x) - \nabla \lambda(y)|}{|\nabla \lambda(x)|} + |\nabla \lambda(y)| \left| \frac{|\nabla \lambda(y)| - |\nabla \lambda(x)|}{|\nabla \lambda(x)||\nabla \lambda(y)|} \right| \\ &\leq 2 \frac{|\nabla \lambda(x) - \nabla \lambda(y)|}{|\nabla \lambda(x)|} \leq c_0 |x - y|, \end{aligned}$$

for every $x, y \in \partial D$.

For $y \in \partial D$ and r > 0 let $K(y, r) = B(y, r) \cap (\partial D)$.

Lemma 3.1 There exist positive constants ρ , M_1 , M_2 , so that to each point $y \in \partial D$ there corresponds a local coordinate system (\overline{x}, z) , where $\overline{x} \in \mathbb{R}^{N-1}$, $z \in \mathbb{R}$ and every point $x \in K(y, \rho)$ is represented as $x = (\overline{x}, z)$, and a C^2 function $\varphi_y \colon B_{N-1}(\overline{y}, \rho) \to \mathbb{R}$, such that

$$\left| \frac{\partial \varphi_y}{\partial x_i} \right| \le M_1, \quad \left| \frac{\partial^2 \varphi_y}{\partial x_i \partial x_j} \right| \le M_2, \quad \forall i, j \in \{1, ..., N-1\},$$

and

$$K(y,\rho) = \{(\overline{x}, \varphi_y(\overline{x})) : \overline{x} \in B_{N-1}(\overline{y}, \rho)\} \cap B_N(y,\rho).$$

Proof. Since λ is of class C^2 in a neighborhood of \overline{D} , there exists $c_1, c_2 > 0$ such that for every $y \in \partial D$ and $i, j \in \{1, ..., N\}$

$$\left| \frac{\partial \lambda}{\partial x_i}(y) \right| \le c_1, \quad \left| \frac{\partial^2 \lambda}{\partial x_i \partial x_j}(y) \right| \le c_2.$$

Moreover, because ∂D is compact and $|\nabla \lambda(x)| \geq c$ on ∂D for some c > 0,

there exist r, c' > 0, such that for every $y \in \partial D$ we can choose $i \in \{1, ..., N\}$ so that for $x \in \partial D$ and |x - y| < r we have

$$\left| \frac{\partial \lambda}{\partial x_i}(x) \right| \ge c'.$$

So let $y \in \partial D$ and take $i \in \{1, ..., N\}$ like above; for $x \in K(y, r)$ let

$$\overline{x} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_N), \quad z = x_i,$$

and denote

$$(\overline{x}, z) = (x_1, ..., x_{i-1}, z, x_{i+1}, ..., x_N).$$

By implicit function theorem, there exist constants $\rho, \rho', 0 < \rho < \rho'$, and a C^1 function $\varphi_y \colon B_{N-1}(\overline{y}, \rho') \to \mathbb{R}$, such that:

- 1. $K(y,\rho) = \{(\overline{x}, \varphi_y(\overline{x})) : \overline{x} \in B_{N-1}(\overline{y}, \rho')\} \cap B_N(y,\rho).$
- 2. For every $\overline{x} \in B_{N-1}(\overline{y}, \rho'), \lambda(\overline{x}, \varphi_y(\overline{x})) = 0$ and

$$\frac{\partial \varphi_y}{\partial x_i}(\overline{x}) = -\frac{\frac{\partial \lambda}{\partial x_i}(\overline{x}, \varphi_y(\overline{x}))}{\frac{\partial \lambda}{\partial z}(\overline{x}, \varphi_y(\overline{x}))}, \quad i = 1, ..., N - 1.$$

Since ∂D is compact, we may choose ρ, ρ' independently on y, and take $\rho' = \rho$. Additionally we may assume, that $\rho < r$. Hence

$$\left| \frac{\partial \varphi_y}{\partial x_i}(\overline{x}) \right| \le c_1/c' = M_1.$$

Moreover, by 2. we conclude, that φ_y is of class C^2 and

$$\frac{\partial^2 \varphi_y}{\partial x_j \partial x_i}(\overline{x}) = \frac{\partial}{\partial x_j} \left(-\frac{\frac{\partial \lambda}{\partial x_i}(\overline{x}, \varphi_y(\overline{x}))}{\frac{\partial \lambda}{\partial z}(\overline{x}, \varphi_y(\overline{x}))} \right)$$

$$=\frac{\frac{\partial}{\partial x_{j}}\left(\frac{\partial \lambda}{\partial z}(\overline{x},\varphi_{y}(\overline{x}))\right)\frac{\partial \lambda}{\partial x_{i}}(\overline{x},\varphi_{y}(\overline{x}))-\frac{\partial}{\partial x_{j}}\left(\frac{\partial \lambda}{\partial x_{i}}(\overline{x},\varphi_{y}(\overline{x}))\right)\frac{\partial \lambda}{\partial z}(\overline{x},\varphi_{y}(\overline{x}))}{\left(\frac{\partial \lambda}{\partial z}(\overline{x},\varphi_{y}(\overline{x}))\right)^{2}}$$

$$=\frac{\left(\frac{\partial^2 \lambda}{\partial x_j \partial z}(\overline{x}, \varphi_y(\overline{x})) + \frac{\partial^2 \lambda}{\partial z^2}(\overline{x}, \varphi_y(\overline{x})) \frac{\partial \varphi_y}{\partial x_j}(\overline{x})\right) \frac{\partial \lambda}{\partial x_i}(\overline{x}, \varphi_y(\overline{x}))}{\left(\frac{\partial \lambda}{\partial z}(\overline{x}, \varphi_y(\overline{x}))\right)^2}$$

$$-\frac{\left(\frac{\partial^2 \lambda}{\partial x_j \partial x_i}(\overline{x}, \varphi_y(\overline{x})) + \frac{\partial^2 \lambda}{\partial z \partial x_i}(\overline{x}, \varphi_y(\overline{x})) \frac{\partial \varphi_y}{\partial x_j}(\overline{x})\right) \frac{\partial \lambda}{\partial z}(\overline{x}, \varphi_y(\overline{x}))}{\left(\frac{\partial \lambda}{\partial z}(\overline{x}, \varphi_y(\overline{x}))\right)^2}.$$

Therefore

$$\left| \frac{\partial^2 \varphi_y}{\partial x_j \partial x_i} (\overline{x}) \right| \le 2 \frac{c_1 c_2}{c'} \left(1 + \frac{c_1}{c'} \right) = M_2.$$

Lemma 3.2 (ball condition)

There exists r > 0, such that for each $y \in \partial D$ there are balls $B(c_y, r)$, $B(\tilde{c}_y, r)$ that satisfy

- 1. $\overline{B}(c_y, r) \cap \overline{D^c} = \{y\}.$
- 2. $\overline{B}(\tilde{c}_y, r) \cap \overline{D} = \{y\}.$

Proof. By Lemma 3.1, there exist positive constants ρ and M, such that to each point $y \in \partial D$ there corresponds a local coordinate system (\overline{x}, z) , where $\overline{x} \in \mathbb{R}^{N-1}$, $z \in \mathbb{R}$ and every point $x \in K(y, \rho)$ is represented as $x = (\overline{x}, z)$, and a C^2 function $\varphi_y \colon B_{N-1}(\overline{y}, \rho) \to \mathbb{R}$, such that

$$\left|\frac{\partial^{2}\varphi_{y}}{\partial x_{i}\partial x_{j}}\right|\leq M,\quad\forall i,j\in\left\{ 1,...,N-1\right\} ,$$

and

$$K(y,\rho) = \{(\overline{x}, \varphi_y(\overline{x})) : \overline{x} \in B_{N-1}(\overline{y}, \rho)\} \cap B_N(y,\rho).$$

By the mean value theorem, for some M'>0, every $y\in\partial D$ and every $x\in K(y,\rho)$ we have

$$|\nabla \varphi_y(\overline{x}) - \nabla \varphi_y(\overline{x}')| \le M'|\overline{x} - \overline{x}'|.$$

We show, that there exists positive constant $r = r(M', \rho)$, such that for every $y \in \partial D$, $B(y - r\nu_y, r) \subset D$ and $B(y + r\nu_y, r) \subset \mathbb{R}^N \backslash D$. Let $y \in \partial D$; without loss of generality we may assume that y = 0 and $z = x_N$ (in the local coordinate system near 0). Moreover, we may assume (by a rotation of the coordinate system if necessary), that $\nu_0 = e_N = (0, ..., 0, 1)$. This implies, that $\nabla \varphi_0(0) = 0$; therefore $|\nabla \varphi_0(\overline{x})| \leq M' |\overline{x}|$ if $|\overline{x}| < \rho$. Moreover, the Taylor expansion gives $\varphi_0(\overline{x}) = \langle \nabla \varphi_0(\theta \overline{x}), \overline{x} \rangle$ for some $\theta \in (0, 1)$, and hence

$$|\varphi_0(\overline{x})| \le |\nabla \varphi_0(\theta \overline{x})| \cdot |\overline{x}| \le M' \theta |\overline{x}|^2 \le M' |\overline{x}|^2.$$

Now observe, that $S(re_N, r)$, the sphere with center in (0, ..., 0, r) and radius r, touches the hyperplane $\{x_N = 0\}$ at the origin and the lower hemisphere is represented as

$$x_N = \psi_r(\overline{x}) = r - \sqrt{r^2 - |\overline{x}|^2} = r \left(1 - \sqrt{1 - \left(\frac{|\overline{x}|}{r}\right)^2} \right).$$

Since $\sqrt{1-t} \le 1-t/2$ for $0 \le t \le 1$, it follows that for $|\overline{x}| < r$

$$\psi_r(\overline{x}) \ge \frac{|\overline{x}|^2}{2r}.$$

So take $r < \min \{1/2M', \rho/2\}$; for $|\overline{x}| < r$ we have

$$-\psi_r(\overline{x}) \le \varphi_0(\overline{x}) \le \psi_r(\overline{x}),$$

where $-\psi_r$ represents the upper hemisphere of $S(-re_N, r)$. Because $r < \rho/2$, we conclude, that

$$B(-re_N, r) \subset D, \quad B(re_N, r) \subset \mathbb{R}^N \backslash D.$$

Obviously, in the lemma above we take $c_y = y - r\nu_y$ and $\tilde{c}_y = y + r\nu_y$.

Observe, that Lemma 3.2 may be proved without the assertion, that the functions φ_y from Lemma 3.1 are of class C^2 with uniformly bounded derivatives. In fact, it suffices to assume, that for every $y \in \partial D$, $\nabla \varphi_y$ satisfies the Lipschitz condition with a constant, which does not depend on y. This property characterize the domains with the boundary of class $C^{1,1}$. More precisely, we say that a bounded domain $\Omega \subset \mathbb{R}^N$ has a boundary of class $C^{1,1}$, if there exist positive constants ρ , A, such that to each point $y \in \partial \Omega$ there corresponds a local coordinate system (\overline{x}, z) , where $\overline{x} \in \mathbb{R}^{N-1}$, $z \in \mathbb{R}$ and every point $x \in (\partial \Omega) \cap B_N(y, \rho)$ is represented as $x = (\overline{x}, z)$, and a C^1 function $\varphi_y \colon B_{N-1}(\overline{y}, \rho) \to \mathbb{R}$, such that

$$|\nabla \varphi_y(\overline{x}) - \nabla \varphi_y(\overline{x}')| \le A|\overline{x} - \overline{x}'|$$

and

$$(\partial\Omega)\cap B_N(y,\rho)=\{(\overline{x},\varphi_y(\overline{x})):\overline{x}\in B_{N-1}(\overline{y},\rho)\}\cap B_N(y,\rho).$$

Moreover, it is possible to prove the reverse assertion to Lemma 3.2: if a bounded domain $\Omega \subset \mathbb{R}^N$ satisfies the ball condition, then $\partial\Omega$ is of class $C^{1,1}$ (for the proof, see [2]).

In the next part of this paper, by σ we will denote the area measure on ∂D . For $y \in \partial D$, let $\rho > 0$ and φ_y be as in Lemma 3.1. Then, by definition of σ , for an open set $E \subset K(y, \rho)$ we have

$$\sigma(E) = \int_{\{\overline{x}: x \in E\}} \sqrt{1 + |\nabla \varphi(\overline{x})|^2} d\overline{x},$$

where \overline{x} means the projection of x into $\mathbb{R}^{k-1} \times \{0\} \times \mathbb{R}^{N-k}$, for some $k \in \{1, ..., N\}$ (for more details, see [4]).

Lemma 3.3 There exist positive constants c_1, c_2 , such that for each r > 0, $r \le \text{diam}(D)$, and each $y \in \partial D$ we have:

$$c_1 r^{N-1} \le \sigma \{K(y, r)\} \le c_2 r^{N-1}.$$

Proof. By Lemma 3.1, there exist positive constants ρ and M, such that to each point $y \in \partial D$ there corresponds a local coordinate system (\overline{x}, z) , where $\overline{x} \in \mathbb{R}^{N-1}$, $z \in \mathbb{R}$ and every point $x \in K(y, \rho)$ is represented as $x = (\overline{x}, z)$, and a C^2 function $\varphi_y \colon B_{N-1}(\overline{y}, \rho) \to \mathbb{R}$, such that

$$\left| \frac{\partial \varphi_y}{\partial x_i} \right| \le M, \quad i = 1, ..., N - 1,$$

and

$$K(y,\rho) = \{(\overline{x}, \varphi_y(\overline{x})) : \overline{x} \in B_{N-1}(\overline{y}, \rho)\} \cap B_N(y,\rho).$$

By first inequality,

$$|\varphi_y(\overline{x}) - \varphi_y(\overline{x}')| \le \sup |\nabla \varphi_y| \cdot |\overline{x} - \overline{x}'| \le \widetilde{M} |\overline{x} - \overline{x}'|.$$

Let
$$r \leq \rho$$
. If $x \in \left\{ (\overline{x}, \varphi_y(\overline{x})) : \overline{x} \in B_{N-1}\left(\overline{y}, r/\sqrt{1 + \widetilde{M}^2}\right) \right\}$, then

$$|x - y| = |(\overline{x} - \overline{y}, \varphi_y(\overline{x}) - \varphi_y(\overline{y}))| = \sqrt{|\overline{x} - \overline{y}|^2 + |\varphi_y(\overline{x}) - \varphi_y(\overline{y})|^2}$$

$$\leq \sqrt{1 + \widetilde{M}^2} |\overline{x} - \overline{y}| < r,$$

which means, that $x \in K(y, r)$, and hence

$$\sigma\left\{K(y,r)\right\} \ge \sigma\left(\left\{(\overline{x},\varphi_y(\overline{x})): \overline{x} \in B_{N-1}\left(\overline{y},r/\sqrt{1+\widetilde{M}^2}\right)\right\}\right)$$

$$= \int_{B_{N-1}\left(\overline{y},r/\sqrt{1+\widetilde{M}^2}\right)} \sqrt{1+|\nabla\varphi_y(\overline{x})|^2} d\overline{x}$$

$$\ge m_{N-1}\left\{B_{N-1}\left(\overline{y},r/\sqrt{1+\widetilde{M}^2}\right)\right\} = c \cdot r^{N-1},$$

where m_{N-1} denotes the N-1 dimensional Lebesque measure. Obviously $K(y,r) \subset \{(\overline{x}, \varphi_y(\overline{x})) : \overline{x} \in B_{N-1}(\overline{y},r)\}$. Therefore

$$\sigma\left\{K(y,r)\right\} \le \int_{B_{N-1}(\overline{y},r)} \sqrt{1 + |\nabla \varphi_y(\overline{x})|^2} d\overline{x}$$

$$\le \sqrt{1 + \widetilde{M}^2} m_{N-1}(B_{N-1}(\overline{y},r)) = c' \cdot r^{N-1}.$$

So the estimate is proved for $r \leq \rho$. Since ∂D is compact, we conclude that $\sigma(\partial D) < \infty$. Now suppose $r > \rho$. We have

$$\sigma\left\{K(y,r)\right\} \ge \sigma\left\{K(y,\rho)\right\} \ge c \cdot \rho^{N-1} = c \cdot \left(\frac{\rho}{r}\right)^{N-1} r^{N-1}$$

$$\ge c \cdot \left(\frac{\rho}{\operatorname{diam}(D)}\right)^{N-1} r^{N-1} = c_1 r^{N-1},$$

$$\sigma\left\{K(y,r)\right\} \le \sigma(\partial D) \le \frac{\sigma(\partial D)}{\rho^{N-1}} r^{N-1} \le c_2 r^{N-1},$$

where $c_2 = \max \{c', \sigma(\partial D)/\rho^{N-1}\}.$

Denote $\delta(x) = \operatorname{dist}(x, \partial D)$, the distance of x to the boundary of D. Observe, that for every $x, x' \in \mathbb{R}^N$ we have

$$\delta(x) \le |x - x'| + \delta(x').$$

By symmetry

$$\delta(x') \le |x - x'| + \delta(x),$$

and thus

$$\delta(x') - |x - x'| \le \delta(x) \le |x - x'| + \delta(x')$$
$$-|x - x'| \le \delta(x) - \delta(x') \le |x - x'|$$
$$|\delta(x) - \delta(x')| \le |x - x'|.$$

In particular, δ is continuous function on \mathbb{R}^N . For r>0, denote

$$D_r = \left\{ x \in \overline{D} : \delta(x) \le r \right\}.$$

Lemma 3.4 Let r be the constant from Lemma 3.2, and let $r_0 = r/4$. There exists a map $\pi \colon D_r \to \partial D$ such that for every $x \in D_r$ we have

$$|\pi(x) - x| = \delta(x)$$

and

$$|\pi(x) - \pi(y)| \le 4|x - y|$$

for every $x, y \in D_{r_0}$.

Proof. By Lemma 3.2, there exists a unique map π on D_r , that satisfies

$$|\pi(x) - x| = \delta(x).$$

In fact, since ∂D is compact, for every $x \in D_r$ there exists $x' \in \partial D$, such that $|x - x'| = \delta(x)$. Because $0 < \delta(x) \le r$, $B(x, \delta(x)) \cap \partial D = \{x'\}$ by Lemma 3.2, and we set $\pi(x) = x'$.

Now choose $x, y \in D_{r_0}$ and suppose $|x - y| < \delta(x)$. Observe, that $\pi(x) = \pi(z)$ for each z from the set

$$I = \left\{ \pi(x) + t \frac{x - \pi(x)}{\delta(x)} : t \in [0, 2\delta(x)] \right\}.$$

Because $I \subset \overline{B}(x, \delta(x))$, there exists $x' \in I$, such that $|y - x'| = \operatorname{dist}(y, I)$ and $\langle y - x', x' - \pi(x) \rangle = 0$. Obviously $\delta(x') = |x' - \pi(x)| < 2r_0$; by Lemma 3.2

$$B(\pi(y) - 2(\pi(y) - y), 2\delta(y)) \subset D$$

and

$$B(\pi(x) - 2(\pi(x) - x'), 2\delta(x')) \subset D.$$

Hence we have

1.
$$2|\pi(y) - y| \le |\pi(y) - 2(\pi(y) - y) - \pi(x)|$$

2.
$$2|\pi(x) - x'| - |y - \pi(x)| \le |\pi(y) - y|$$
.

From 1 we conclude

$$4|\pi(y) - y|^2 \le |y - \pi(y) + y - \pi(x)|^2$$

$$4|\pi(y) - y|^2 \le |y - \pi(y)|^2 + 2\langle y - \pi(y), y - \pi(x) \rangle + |y - \pi(x)|^2$$

$$|\pi(y) - y|^2 - 2\langle y - \pi(y), y - \pi(x) \rangle \le |y - \pi(x)|^2 - 2|y - \pi(y)|^2$$

$$|\pi(y) - y|^2 - 2\langle y - \pi(y), y - \pi(x) \rangle + |y - \pi(x)|^2 \le 2|y - \pi(x)|^2 - 2|y - \pi(y)|^2$$

$$|\pi(x) - \pi(y)|^2 \le 2|y - \pi(x)|^2 - 2|y - \pi(y)|^2.$$

2 gives

$$|\pi(x) - \pi(y)|^2 \le 2|y - \pi(x)|^2 - 2(2|\pi(x) - x'| - |y - \pi(x)|)^2$$

$$=8\delta(x')(|y-\pi(x)|-\delta(x'))=\frac{8\delta(x')(|y-\pi(x)|^2-\delta(x')^2)}{|y-\pi(x)|+\delta(x')}.$$

Because $\langle y-x',x'-\pi(x)\rangle=0$, we have $|y-\pi(x)|^2=|x'-y|^2+\delta(x')^2$, and hence

$$\frac{8\delta(x')(|y-\pi(x)|^2 - \delta(x')^2)}{|y-\pi(x)| + \delta(x')} \le \frac{8\delta(x')(|y-\pi(x)|^2 - \delta(x')^2)}{2\delta(x')}$$
$$= 4(|x'-y|^2 + \delta(x')^2 - \delta(x')^2) = 4|x'-y|^2.$$

Obviously $|x'-y| \le |x-y|$, and thus $|\pi(x)-\pi(y)| \le 2|x-y|$. Now if $|x-y| \ge \delta(x)$, then we have

$$|\pi(x) - \pi(y)| \le |\pi(x) - x| + |x - y| + |y - \pi(y)| = \delta(x) + |x - y| + \delta(y)$$

$$= 2\delta(x) + |x - y| + \delta(y) - \delta(x) \le 3|x - y| + |\delta(y) - \delta(x)|$$

$$\le 4|x - y|,$$

what gives the conclusion of the lemma.

The map π from Lemma 3.4 will be called an *orthogonal projection*. Observe, that if r is the constant from Lemma 3.2, then for $x \in D_r$ we have

$$x = \pi(x) - \delta(x)\nu_{\pi(x)}.$$

Lemma 3.5 Let r be the constant from Lemma 3.2, and let $r_0 = r/4$. Then the function δ is of class $C^{1,1}$ inside D_{r_0} . Moreover, for $a \in S$ we have

$$\lim_{h \to 0} \frac{\delta(x + ha) - \delta(x)}{h} = \langle a, -\nu_{\pi(x)} \rangle.$$

Proof. Choose $x \in \text{Int}(D_{r_0})$ and $a \in S$. First observe, that

$$2\delta(x) - |ha - (x - \pi(x))| \le \delta(x + ha) \le |ha + x - \pi(x)|$$

for $|h| < \delta(x)$. We have

$$\begin{split} &\delta(x+ha)-\delta(x) \geq \delta(x)-|ha-(x-\pi(x))|\\ &=\delta(x)-\sqrt{h^2-2h\langle a,x-\pi(x)\rangle+(\delta(x))^2}\\ &=h\frac{2\langle a,x-\pi(x)\rangle-h}{\delta(x)+\sqrt{h^2-2h\langle a,x-\pi(x)\rangle+(\delta(x))^2}}; \end{split}$$

$$\delta(x+ha) - \delta(x) \le |ha + x - \pi(x)| - \delta(x)$$

$$= \sqrt{h^2 + 2h\langle a, x - \pi(x)\rangle + (\delta(x))^2} - \delta(x)$$

$$= h \frac{2\langle a, x - \pi(x) \rangle + h}{\sqrt{h^2 + 2h\langle a, x - \pi(x) \rangle + (\delta(x))^2} + \delta(x)}.$$

If h > 0, then

$$\frac{\delta(x+ha)-\delta(x)}{h} \geq \frac{2\langle a,x-\pi(x)\rangle-h}{\delta(x)+\sqrt{h^2-2h\langle a,x-\pi(x)\rangle+(\delta(x))^2}} \xrightarrow{h\to 0} \frac{\langle a,x-\pi(x)\rangle}{\delta(x)},$$

similarly

$$\frac{\delta(x+ha)-\delta(x)}{h} \leq \frac{2\langle a,x-\pi(x)\rangle+h}{\delta(x)+\sqrt{h^2+2h\langle a,x-\pi(x)\rangle+(\delta(x))^2}} \xrightarrow{h\to 0} \frac{\langle a,x-\pi(x)\rangle}{\delta(x)};$$

if h < 0 the inequalities are reverse. Because $|x - \pi(x)| = \delta(x)$, we have

$$\frac{\pi(x) - x}{\delta(x)} = \nu_{\pi(x)}.$$

Now $|\pi(x) - \pi(y)| \le 4|x - y|$ on D_{r_0} by Lemma 3.4; recalling, that $|\nu_z - \nu_{z'}| \le c_0|z - z'|$ on ∂D , we conclude

$$\left| \frac{\partial \delta}{\partial x_i}(x) - \frac{\partial \delta}{\partial x_i}(y) \right| = \left| \langle e_i, \nu_{\pi(y)} - \nu_{\pi(x)} \rangle \right| \le |\nu_{\pi(y)} - \nu_{\pi(x)}|$$

$$\le c_0 |\pi(y) - \pi(x)| \le 4c_0 |x - y|$$

for every $x, y \in \text{Int}(D_{r_0})$.

Lemma 3.6 Let π be the orthogonal projection. There exists r > 0, such that π is of class C^1 inside D_r .

Proof. Choose $y \in \partial D$. It suffices to show, that there exists r > 0, such that π is of class C^1 on $D \cap B(y,r)$. Let ρ , M_1 , M_2 be the constants from Lemma 3.1. Assume additionally, that ρ satisfies the assertion of Lemma 3.2. Then there exists a local coordinate system (\overline{x}, z) , where $\overline{x} \in \mathbb{R}^{N-1}$, $z \in \mathbb{R}$

and every point $x \in K(y, \rho)$ is represented as $x = (\overline{x}, z)$, and a C^2 function $\varphi_y \colon B_{N-1}(\overline{y}, \rho) \to \mathbb{R}$, such that

$$\left| \frac{\partial \varphi_y}{\partial x_i} \right| \le M_1, \quad \left| \frac{\partial^2 \varphi_y}{\partial x_i \partial x_j} \right| \le M_2, \quad \forall i, j \in \{1, ..., N-1\},$$

and

$$K(y,\rho) = \{(\overline{x}, \varphi_y(\overline{x})) : \overline{x} \in B_{N-1}(\overline{y}, \rho)\} \cap B_N(y, \rho).$$

Without loss of generality we may assume, that $z = x_N$. Additionally we may assume, that

$$B_N(y,\rho) \cap D = \{(\overline{x},x_N) : \overline{x} \in B_{N-1}(\overline{y},\rho) \land x_N < \varphi_y(\overline{x})\} \cap B_N(y,\rho).$$

Then for $\overline{x} \in B_{N-1}(\overline{y}, \rho)$, $w(\overline{x}) = (-\nabla \varphi_y(\overline{x}), 1)$ is the outward orthogonal vector field to ∂D in $x = (\overline{x}, \varphi_y(\overline{x})) \in K(y, \rho)$. Since $|w(\overline{x})| > 0$, we have

$$\frac{w(\overline{x})}{|w(\overline{x})|} = \nu_x.$$

Let

$$F(\overline{x},t) = (\overline{x}, \varphi_y(\overline{x})) - t \cdot w(\overline{x}), \quad \overline{x} \in B_{N-1}(\overline{y}, \rho), t \in \mathbb{R}.$$

Because φ_y is of class C^2 , F is of class C^1 on $B_{N-1}(\overline{y},\rho) \times \mathbb{R}$.

Observe, that since ρ satisfies the condition of Lemma 3.2, for $\rho' \leq \rho/2$ and $x \in D \cap B_N(y, \rho')$ we have

$$|\pi(x) - y| \le |\pi(x) - x| + |x - y| = \delta(x) + |x - y| \le 2|x - y| < \rho,$$

so $\pi(x) \in K(y, \rho)$. Therefore, if we denote $\pi(x) = \left(\overline{\pi(x)}, \varphi_y\left(\overline{\pi(x)}\right)\right)$, then

$$F\left(\overline{\pi(x)}, \frac{\delta(x)}{|w(\overline{x})|}\right) = \left(\overline{\pi(x)}, \varphi_y\left(\overline{\pi(x)}\right)\right) - \delta(x) \frac{w(\overline{x})}{|w(\overline{x})|}$$
$$= \pi(x) - \delta(x)\nu_{\pi(x)} = x.$$

Since $|w(\overline{x})| \ge 1$, we have

$$D \cap B_N(y, \rho') \subset F(B_{N-1}(\overline{y}, \rho) \times (0, \rho')).$$

The Jacobian matrix of $F(\overline{x},t) = (\overline{x} + t\nabla\varphi_y(\overline{x}), \varphi_y(\overline{x}) - t)$ has a form

$$J(\overline{x},t) = \begin{bmatrix} 1 + t \frac{\partial^2 g}{\partial x_1^2}(\overline{x}) & t \frac{\partial^2 g}{\partial x_2 \partial x_1}(\overline{x}) & \dots & t \frac{\partial^2 g}{\partial x_{N-1} \partial x_1}(\overline{x}) & \frac{\partial g}{\partial x_1}(\overline{x}) \\ t \frac{\partial^2 g}{\partial x_1 \partial x_2}(\overline{x}) & 1 + t \frac{\partial^2 g}{\partial x_2^2}(\overline{x}) & \dots & t \frac{\partial^2 g}{\partial x_{N-1} \partial x_2}(\overline{x}) & \frac{\partial g}{\partial x_2}(\overline{x}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t \frac{\partial^2 g}{\partial x_1 \partial x_{N-1}}(\overline{x}) & t \frac{\partial^2 g}{\partial x_2 \partial x_{N-1}}(\overline{x}) & \dots & 1 + t \frac{\partial^2 g}{\partial x_{N-1}^2}(\overline{x}) & \frac{\partial g}{\partial x_{N-1}}(\overline{x}) \\ \frac{\partial g}{\partial x_1}(\overline{x}) & \frac{\partial g}{\partial x_2}(\overline{x}) & \dots & \frac{\partial g}{\partial x_{N-1}}(\overline{x}) & -1 \end{bmatrix}.$$

Therefore, if $t \to 0$, then J tends to the matrix

$$J_0(\overline{x}) = \begin{bmatrix} 1 & 0 & \dots & 0 & \frac{\partial g}{\partial x_1}(\overline{x}) \\ 0 & 1 & \dots & 0 & \frac{\partial g}{\partial x_2}(\overline{x}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \frac{\partial g}{\partial x_{N-1}}(\overline{x}) \\ \frac{\partial g}{\partial x_1}(\overline{x}) & \frac{\partial g}{\partial x_2}(\overline{x}) & \dots & \frac{\partial g}{\partial x_{N-1}}(\overline{x}) & -1 \end{bmatrix}.$$

Since φ_y is of class C^2 and

$$\left| \frac{\partial \varphi_y}{\partial x_i} \right| \le M_1, \quad \left| \frac{\partial^2 \varphi_y}{\partial x_i \partial x_j} \right| \le M_2, \quad \forall i, j \in \{1, ..., N-1\},$$

det $J \stackrel{t\to 0}{\longrightarrow}$ det J_0 uniformly on $B_{N-1}(\overline{y}, \rho)$. A simple calculation shows, that det $J_0(\overline{x}) = -|\nabla \varphi_y(\overline{x})|^2 - 1$. Hence, there exists r > 0, such that det $J(\overline{x}, t) \neq 0$ for every $x \in B_{N-1}(\overline{y}, \rho)$ and $t \in (0, r)$. We may assume, that $r \leq \rho/2$. By inverse function theorem, F is invertible in $B_{N-1}(\overline{y}, \rho) \times (0, r)$, and F^{-1} is of class C^1 in $F(B_{N-1}(\overline{y}, \rho) \times (0, r))$. In particular, F^{-1} is of class C^1 in $D \cap B_N(y, r)$.

Now if $x \in D \cap B_N(y,r)$, then $x = F(\overline{x}',t)$ for some $\overline{x}' \in B_{N-1}(\overline{y},\rho)$ and $t \in (0,r)$. Thus, if we denote $F^{-1}(x) = (F_1^{-1}(x),...,F_N^{-1}(x))$, then for $i \in \{1,...,N-1\}$ we have $F_i^{-1}(x) = x_i'$. On the other side,

$$x' = (\overline{x}', \varphi_y(\overline{x}')) = \pi(x) = (\pi_1(x), ..., \pi_N(x)),$$

since ρ satisfies the condition of Lemma 3.2, and $r < \rho$. Hence

$$\pi_i(x) = x_i' = F_i^{-1}(x), \ i = 1, ..., N - 1.$$

Moreover,

$$\pi_N(x) = x'_N = \varphi_y(\overline{x}') = \varphi_y(\pi_1(x), ..., \pi_{N-1}(x)),$$

and thus π is of class C^1 in $D \cap B(y,r)$, as desired.

Corollary 3.1 Let r_0 be as in Lemma 3.5. There exists r > 0, $r \le r_0$, such that δ is of class C^2 inside D_r .

Proof. By Lemma 3.5, for every $x \in D_{r_0}$ and i = 1, ..., N we have

$$\frac{\partial \delta}{\partial x_i}(x) = \langle e_i, -\nu_{\pi(x)} \rangle = \frac{\langle e_i, x - \pi(x) \rangle}{\delta(x)} = \frac{x_i - \pi_i(x)}{\delta(x)}.$$

By Lemma 3.6, there exists r > 0, such that $\pi = (\pi_1, ..., \pi_N)$ is of class C^1 inside D_r . If we assume additionally, that $r \leq r_0$, then δ is of class C^2 inside D_r .

Now we will introduce some facts from classical harmonic analysis on a smooth domain $D \subset \mathbb{R}^N$. Let $G_D(x,y)$ be the Green's function for D, defined in $(D \times \overline{D}) \setminus \{(x,x) : x \in D\}$. It is uniquely determined by the following properties:

- 1. G_D is of class C^2 on $(D \times D) \setminus \{(x, x) : x \in D\}$ and of class $C^{2-\varepsilon}$ up to $(D \times \overline{D}) \setminus \{(x, x) : x \in D\}$.
- 2. $\Delta_y G_D(x,y) = 0$ for every $y \in D$ and $y \neq x$.
- 3. $G_D(x,y) + \Gamma_N(x-y)$ is harmonic on D for each fixed $x \in D$, where Γ_N is the fundamental solution for the Laplacian on \mathbb{R}^N , given by

$$\Gamma_N(x) = \begin{cases} (2\pi)^{-1} \log |x|, & N = 2, \\ (2-N)^{-1} \omega_{N-1}^{-1} |x|^{2-N}, & N > 2, \end{cases}$$

 ω_{N-1} is the area measure of S in \mathbb{R}^N .

4. $G_D(x,y)|_{y\in\partial D}=0$ for each fixed $x\in D$.

The function

$$P_D(x,y) = -\frac{\partial G_D(x,y)}{\partial \nu_y},$$

defined in $D \times \partial D$ is the Poisson kernel for D. It is continuous and strictly positive on $D \times \partial D$. Moreover, by the use of Green's theorem, one can prove the following property: if u is harmonic on D and continuous on \overline{D} , then

$$u(x) = \int_{\partial D} P_D(x, y) u(y) d\sigma(y), \quad x \in D$$

(for more details and proofs, see [3]). Now, since D is a domain with the boundary of class C^2 , the Dirichlet Problem is solvable. More precisely, for each $f \in C(\partial D)$, there exists $F \in C(\overline{D})$, such that $F|_{\partial D} = f$ and F is harmonic on D (a good source for these results is [1]). Hence

$$F(x) = \int_{\partial D} P_D(x, y) f(y) d\sigma(y), \quad x \in D.$$

Moreover, the maximum principle for harmonic functions implies, that F is unique (since D is bounded). Using this facts, one can prove the following known properties of the Poisson kernel for D:

- 1. $P_D(x,\cdot) \ge c_x > 0$, for each fixed $x \in D$.
- 2. For every $x \in D$,

$$\int_{\partial D} P_D(x, y) d\sigma(y) = 1.$$

3. For any $\eta > 0$ and any fixed $y \in \partial D$,

$$\lim_{D\ni x\to y} \int_{\partial D\setminus K(y,p)} P_D(x,y) d\sigma(y) = 0.$$

4. $P_D(\cdot, y)$ is harmonic on D for each fixed $y \in \partial D$.

Unfortunately, P_D almost never can be computed explicitly. However, in the special case when D = B(a, r), the Poisson kernel is given by

$$P_{B(a,r)}(x,y) = \frac{1}{\omega_{N-1}r} \cdot \frac{r^2 - |x-a|^2}{|x-y|^N}$$

By the formula above and Lemma 3.2, we have the following inequality (more details also may be found in [3]).

Lemma 3.7 There exists a positive constant C, such that for every $x \in D$ and $y \in \partial D$ we have

$$P_D(x,y) \le \frac{C}{|x-y|^{N-1}}.$$

Proof. Fix $x \in D$, $y \in \partial D$. Choose, by Lemma 2, r > 0 (which does not depend on y), such that

$$\widetilde{B}_y = B(y + r\nu_y, r) \subset D^c.$$

Let $G_{\widetilde{B}_y^c}$ be the Green's function for \widetilde{B}_y^c . Observe that $G_D(x,\cdot)=0$ on ∂D , whereas $G_{\widetilde{B}_y^c}(x,\cdot)\geq 0$ on ∂D . Since $G_{\widetilde{B}_y^c}(x,\cdot)-G_D(x,\cdot)$ is harmonic on D, it follows that

$$G_D(x,t) \le G_{\widetilde{B}_y^c}(x,t), \quad t \in \overline{D}$$

$$G_D(x,y) = G_{\widetilde{B}_y^c}(x,y) = 0.$$

Therefore

$$P_D(x,y) = -\frac{\partial G_D(x,y)}{\partial \nu_y} \le -\frac{\partial G_{\widetilde{B}_y^c}(x,y)}{\partial \nu_y} = P_{\widetilde{B}_y^c}(x,y).$$

The Poisson kernel for \widetilde{B}_y has a form

$$P_{\widetilde{B}_y}(x,t) = \frac{1}{\omega_{N-1}r} \cdot \frac{r^2 - |x - \widetilde{c}_y|^2}{|x - t|^N}, \quad x \in \widetilde{B}_y, t \in \partial \widetilde{B}_y,$$

where $\tilde{c}_y = y + r\nu_y$. By Kelvin transform, the Poisson kernel for \widetilde{B}_y^c is

$$P_{\tilde{B}_{y}^{c}}(x,t) = \frac{1}{\omega_{N-1}r} \cdot \frac{|x - \tilde{c}_{y}|^{2} - r^{2}}{|x - t|^{N}}.$$

Hence we have

$$P_D(x,y) \le \frac{1}{\omega_{N-1}r} \cdot \frac{|x - \tilde{c}_y|^2 - r^2}{|x - t|^N} = \frac{1}{\omega_{N-1}r} \cdot \frac{(|x - \tilde{c}_y| - |\tilde{c}_y - y|)(|x - \tilde{c}_y| + r)}{|x - t|^N}$$

$$\leq \frac{1}{\omega_{N-1}r} \cdot \frac{(|x-y|)(|x-y|+2r)}{|x-t|^N} \leq \frac{\operatorname{diam}(D) + 2r}{\omega_{N-1}r} \cdot \frac{1}{|x-y|^{N-1}} = \frac{C}{|x-y|^{N-1}}.$$

In the next sections we will deal with the Banach spaces $L^p(\partial D)$, where $1 \leq p \leq \infty$. When $p \in [1, \infty)$, $L^p(\partial D)$ consists of the Borel measurable functions f on ∂D for which

$$||f||_p = \left(\int_{\partial D} |f|^p d\sigma\right)^{\frac{1}{p}} < \infty;$$

 $L^{\infty}(\partial D)$ consists of the Borel measurable functions f on ∂D for which $||f||_{\infty} < \infty$, where $||f||_{\infty}$ denotes the essential supremum norm on ∂D with respect to σ . The number $q \in [1, \infty]$ is said to be conjugate to p if 1/p + 1/q = 1. If $1 \leq p < \infty$ and q is conjugate to p, then $L^q(\partial D)$ is the dual space of $L^p(\partial D)$; we identify $g \in L^q(\partial D)$ with the linear functional Λ_g on $L^p(\partial D)$ defined by

$$\Lambda_g(f) = \int_{\partial D} f g \ d\sigma.$$

Let $1 \leq p < \infty$ and let q be conjugate to p; we say, that the sequence $\{g_n\} \subset L^q(\partial D)$ converges weak* to $g \in L^q(\partial D)$, if $\Lambda_{g_n}(f) \stackrel{n}{\to} \Lambda_g(f)$ for every $f \in L^p(\partial D)$. Note that because σ is a finite measure on ∂D , $L^p(\partial D) \subset L^1(\partial D)$ for all $p \in [1, \infty]$. Recall also that $C(\partial D)$ is dense in $L^p(\partial D)$ for 1 .

For $f \in L^1(\partial D)$ and $\mu \in M(\partial D)$ (the set of complex Borel measures on ∂D), we define the Poisson integrals of f and μ , respectively as

$$P_D[f](x) = \int_{\partial D} P_D(x, y) f(y) d\sigma(y),$$
$$P_D[\mu](x) = \int_{\partial D} P_D(x, y) d\mu(y).$$

Differentiating under the integral sign, we see that $P_D[f]$ and $P_D[\mu]$ are harmonic for every $f \in L^1(\partial D)$ and $\mu \in M(\partial D)$.

4 The spaces $h^p(D)$

As we said before, D is a bounded domain with the boundary of class C^2 , and λ is a characterizing function for D. Of course there are infinitely many such characterizing functions. Now we will show, that every characterizing function is comparable to δ near ∂D . Recall that $D_r = \{x \in \overline{D} : \delta(x) \leq r\}$.

Lemma 4.1 Let λ be a characterizing function for D. There exist positive constants C, r, such that for every $x \in D_r$

$$\frac{\delta(x)}{C} \le -\lambda(x) \le C\delta(x).$$

Proof. By the properties of characterizing function, there exist positive constants c_1, c_2, r_1 , such that $c_1 \leq |\nabla \lambda(x)| \leq c_2$ for every $x \in D_{r_1}$. Assume additionally, that r_1 satisfies the condition of Lemma 3.2. Moreover, we may assume, that for some $c_3 > 0$ and every $x, y \in D_{r_1}$ we have

$$|\nabla \lambda(x) - \nabla \lambda(y)| \le c_3|x - y|,$$

since λ is of class C^2 in a neighborhood of \overline{D} . Hence, similarly as in section 3 we conclude

$$\left| \frac{\nabla \lambda(x)}{|\nabla \lambda(x)|} - \frac{\nabla \lambda(y)}{|\nabla \lambda(y)|} \right| \le c_4 |x - y|, \quad x, y \in D_{r_1},$$

where $c_4 = 2c_3/c_1$. Take $r_2 = \min\{r_1, 1/c_4\}$, and fix $x \in D_{r_2}$. By Lemmas 3.2 and 3.4, $B(x, \delta(x)) \cap \partial D = \{\pi(x)\}$. Moreover, for every $t \in (0, 1)$,

$$tx + (1-t)\pi(x) \in D_{r_2}.$$

Let

$$f(t) = \lambda(tx + (1 - t)\pi(x)), \quad t \in [0, 1].$$

Then f is differentiable, and

$$f'(t) = \langle \nabla \lambda(tx + (1 - t)\pi(x)), x - \pi(x) \rangle.$$

Thus we have

$$-\lambda(x) = -(\lambda(x) - \lambda(\pi(x))) = -(f(1) - f(0)) = -f'(\theta)$$
$$= \langle \nabla \lambda(\theta x + (1 - \theta)\pi(x)), \pi(x) - x \rangle,$$

for some $\theta \in (0,1)$. Denote $x_{\theta} = \theta x + (1-\theta)\pi(x)$; therefore

$$-\lambda(x) = \langle \nabla \lambda(x_{\theta}), \pi(x) - x \rangle = \langle \frac{\nabla \lambda(x_{\theta})}{|\nabla \lambda(x_{\theta})|}, \frac{\pi(x) - x}{\delta(x)} \rangle |\nabla \lambda(x_{\theta})| \delta(x)$$

$$= \langle \frac{\nabla \lambda(x_{\theta})}{|\nabla \lambda(x_{\theta})|}, \frac{\nabla \lambda(\pi(x))}{|\nabla \lambda(\pi(x))|} \rangle |\nabla \lambda(x_{\theta})| \delta(x)$$

$$= \frac{2 - \left| \frac{\nabla \lambda(x_{\theta})}{|\nabla \lambda(x_{\theta})|} - \frac{\nabla \lambda(\pi(x))}{|\nabla \lambda(\pi(x))|} \right|^{2}}{2} |\nabla \lambda(x_{\theta})| \delta(x).$$

Now observe, that

$$\left| \frac{\nabla \lambda(x_{\theta})}{|\nabla \lambda(x_{\theta})|} - \frac{\nabla \lambda(\pi(x))}{|\nabla \lambda(\pi(x))|} \right| \le c_4 |x_{\theta} - \pi(x)| = c_4 \theta |x - \pi(x)| \le c_4 \delta(x) \le c_4 r_2 \le 1,$$

since $x_{\theta} \in D_{r_2}$. Hence

$$\frac{c_1}{2}\delta(x) \le -\lambda(x) \le c_2\delta(x),$$

what gives the conclusion of the lemma.

Each characterizing function λ determines a family of approximating subdomains $D_{\lambda}^{\varepsilon}$, for ε sufficiently small and positive. Clearly, by the properties of characterizing function, there exists c, r' > 0, such that for every $x \in D_{r'}$, $|\nabla \lambda(x)| \geq c$. Moreover, we can choose $\varepsilon_{\lambda} > 0$ with the property, that if $0 < \varepsilon < \varepsilon_{\lambda}$ and $\lambda(x) = -\varepsilon$, then $x \in D_{r'}$ (Lemma 4.1 may be helpful here). For ε as above let $D_{\lambda}^{\varepsilon} = \{x : \lambda(x) < -\varepsilon\}$. Then $\partial D_{\lambda}^{\varepsilon}$ is the level surface $\{x : \lambda(x) = -\varepsilon\}$, and $\lambda(x) + \varepsilon$ is a characterizing function for $D_{\lambda}^{\varepsilon}$ (thus $\partial D_{\lambda}^{\varepsilon}$ is of class C^2).

Let σ_{ε} be the area measure on $\partial D_{\lambda}^{\varepsilon}$. Let π be the orthogonal projection onto ∂D . Then we may choose ε_{λ} so small, so that for each $0 < \varepsilon < \varepsilon_{\lambda}$, σ_{ε} is locally a transform of the measure σ in the following sense. For $y \in \partial D_{\lambda}^{\varepsilon}$ let (\overline{x}, z) be the local coordinate system near y and $\varphi_{y}^{\varepsilon}$ a real-valued C^{2} function as in Lemma 3.1 (thus every point $x \in \partial D_{\lambda}^{\varepsilon}$ near y is represented as $x = (\overline{x}, z)$, where $z = \varphi_{y}^{\varepsilon}(\overline{x})$, and \overline{x} means the projection of x into $\mathbb{R}^{k-1} \times \{0\} \times \mathbb{R}^{N-k}$, for some $k \in \{1, ..., N\}$). We may assume, that the same local coordinate system (\overline{x}, z) corresponds to the neighborhood of $\pi(y)$ in ∂D (which means the same projection \overline{x} , but here $z = \varphi_{\pi(y)}(\overline{x})$). For \overline{x} near \overline{y} denote

$$\widetilde{\pi}_{\varepsilon}(\overline{x}) = \overline{\pi(\overline{x}, \varphi_y^{\varepsilon}(\overline{x}))}.$$

Then there exists $\rho_0 > 0$ (which does not depend on ε and y, $0 < \varepsilon < \varepsilon_{\lambda}$, $y \in \partial D_{\lambda}^{\varepsilon}$), such that

$$\det J_{\widetilde{\pi}_{\varepsilon}}(\overline{x}) \ge c_{\varepsilon} > 0, \quad \forall \overline{x} \in \{\overline{x} : x \in B(y, \rho_0) \cap \partial D_{\lambda}^{\varepsilon}\},$$

where $J_{\widetilde{\pi}_{\varepsilon}}$ is the Jacobian matrix of $\widetilde{\pi}_{\varepsilon}$. Moreover, $\det J_{\widetilde{\pi}_{\varepsilon}}(\overline{x})$ tends to 1 as $\varepsilon \to 0$ uniformly with respect to $\overline{x} \in \{\overline{x} : x \in B(y, \rho_0) \cap \partial D_{\lambda}^{\varepsilon}\}$ (what can be proved explicitly, by the use of some facts contained in the proof of Lemma 3.6). Thus $\widetilde{\pi}_{\varepsilon}$ is invertible in $\{\overline{x} : x \in B(y, \rho_0) \cap \partial D_{\lambda}^{\varepsilon}\}$, and we have

$$\sigma_{\varepsilon}(B(y,\rho_{0})\cap\partial D_{\lambda}^{\varepsilon}) = \int_{\left\{\overline{x}:x\in B(y,\rho_{0})\cap\partial D_{\lambda}^{\varepsilon}\right\}} \sqrt{1+|\nabla\varphi_{y}^{\varepsilon}(\overline{x})|^{2}} d\overline{x}$$

$$= \int_{\widetilde{\pi}_{\varepsilon}\left(\left\{\overline{x}:x\in B(y,\rho_{0})\cap\partial D_{\lambda}^{\varepsilon}\right\}\right)} \sqrt{1+|\nabla\varphi_{y}^{\varepsilon}(\widetilde{\pi}_{\varepsilon}^{-1}(\overline{w}))|^{2}} \det J_{\widetilde{\pi}_{\varepsilon}^{-1}}(\overline{w}) d\overline{w}$$

$$= \int_{K_{\varepsilon}} \frac{\sqrt{1+|\nabla\varphi_{y}^{\varepsilon}(\widetilde{\pi}_{\varepsilon}^{-1}(\overline{w}))|^{2}}}{\sqrt{1+|\nabla\varphi_{y}(\overline{w})|^{2}}} \det J_{\widetilde{\pi}_{\varepsilon}^{-1}}(\overline{w}) d\sigma(w),$$

where

$$K_{\varepsilon} = \left\{ (\overline{w}, \varphi_{\pi(y)}(\overline{w})) : \overline{w} \in \widetilde{\pi}_{\varepsilon} \left(\{ \overline{x} : x \in B(y, \rho_0) \cap \partial D_{\lambda}^{\varepsilon} \} \right) \right\}.$$

Therefore we conclude, that if $0 < \varepsilon < \varepsilon_{\lambda}$, then $\pi_{\varepsilon} := \pi|_{\partial D_{\lambda}^{\varepsilon}}$ is invertible, and for $f \in C(\partial D_{\lambda}^{\varepsilon})$ we have

$$\int_{\partial D_{\lambda}^{\varepsilon}} f(x) d\sigma_{\varepsilon}(x) = \int_{\partial D} f(\pi_{\varepsilon}^{-1}(y)) \phi_{\varepsilon}(y) d\sigma(y),$$

where ϕ_{ε} is locally well defined, and tends to 1 uniformly as $\varepsilon \to 0$ (thus we may assume, that $\phi_{\varepsilon} \leq 2$ for every $0 < \varepsilon < \varepsilon_{\lambda}$).

Now for fixed characterizing function λ , ε_{λ} as above and $1 \leq p \leq \infty$, we define a space $h^p(D)$ to be the class of function u harmonic on D, for which

$$||u||_{h^p}^{\lambda} = \sup_{0 < \varepsilon < \varepsilon_{\lambda}} \left(\int_{\partial D_{\lambda}^{\varepsilon}} |u(x)|^p d\sigma_{\varepsilon}(x) \right)^{\frac{1}{p}} < \infty.$$

The spaces $h^p(D)$ are called "harmonic Hardy spaces". Note that $h^{\infty}(D)$ is simply the collection of functions harmonic and bounded on D, and that

$$||u||_{h^{\infty}}^{\lambda} = ||u||_{h^{\infty}} = \sup_{x \in D} |u(x)|.$$

Moreover, $h^p(D) \subset h^q(D)$ for $1 \le q .$

Observe, that since $\sigma_{\varepsilon}(\partial D_{\lambda}^{\varepsilon}) \leq 2\sigma(\partial D)$ for every $0 < \varepsilon < \varepsilon_{\lambda}$, the harmonic function $u \in h^{p}(D)$ if and only if

$$\sup_{0<\varepsilon<\varepsilon'} \left(\int_{\partial D_{\lambda}^{\varepsilon}} |u(x)|^p d\sigma_{\varepsilon}(x) \right)^{\frac{1}{p}} < \infty$$

for every $0 < \varepsilon' < \varepsilon_{\lambda}$.

The next lemma shows, that the definition of $h^p(D)$ does not depend on characterizing function.

Lemma 4.2 (Stein)

Let λ_1 and λ_2 be two characterizing functions for D. Then for each p, $1 \le p \le \infty$, and each harmonic function u on D, the two conditions

$$\sup_{0<\varepsilon<\varepsilon_{\lambda_i}}\left(\int_{\partial D^\varepsilon_{\lambda_i}}|u(x)|^pd\sigma^i_\varepsilon(x)\right)^{\frac{1}{p}}<\infty,\quad i=1,2,$$

are equivalent $(\sigma^i_{\varepsilon}$ is the area measure on $\partial D^{\varepsilon}_{\lambda_i})$.

Proof. It suffices to show, that the last condition for i=1 implies the same condition for i=2. Because for $p=\infty$ the conclusion is trivial, we may assume that $1 \le p < \infty$. Denote

$$M = \sup_{0 < \varepsilon < \varepsilon_{\lambda_1}} \int_{\partial D^{\varepsilon}_{\lambda_1}} |u(x)|^p d\sigma^1_{\varepsilon}(x).$$

Let r > 0 be the constant from Lemma 4.1, such that λ_1 and λ_2 are comparable to δ in D_r . Clearly, there exists C > 0, such that

$$\frac{\delta(x)}{C} \le -\lambda_i(x) \le C\delta(x), \quad i = 1, 2, \quad x \in D_r.$$

We may assume, that $C \geq 1$. Take $\varepsilon_0 > 0$, so that if $0 < \varepsilon < \varepsilon_0$ and $\lambda_i(x) = -\varepsilon$, i = 1, 2, then $x \in D_r$. Assume additionally, that

$$\varepsilon_0 \le \frac{\min\left\{\varepsilon_{\lambda_1}, \varepsilon_{\lambda_2}, r\right\}}{2C^2}.$$

First we show, that there exist positive constants c, c_1, c_2 , so that if $0 < \varepsilon < \varepsilon_0$ and $x \in \partial D_{\lambda_2}^{\varepsilon}$, then

$$B(x, c\varepsilon) \subset \{y : -c_1\varepsilon < \lambda_1(y) < -c_2\varepsilon\} = L_{\varepsilon}.$$

Let $c < \frac{1}{2C}$, and choose $0 < \varepsilon < \varepsilon_0$, $x \in \partial D_{\lambda_2}^{\varepsilon}$. Thus, by Lemma 4.1

$$\frac{\delta(x)}{C} \le -\lambda_2(x) \le C\delta(x),$$

$$\frac{\delta(x)}{C} \le \varepsilon \le C\delta(x),$$

$$\frac{\varepsilon}{C} \le \delta(x) \le C\varepsilon,$$

since $x \in D_r$. If $y \in B(x, c\varepsilon)$, then we have

$$\delta(x) - |x - y| \le \delta(y) \le \delta(x) + |x - y|,$$

$$\varepsilon \left(\frac{1}{C} - c\right) < \delta(y) < (C + c)\varepsilon.$$

Since $(C+c)\varepsilon < 2C\varepsilon_0 \le r, y \in D_r$ and thus

$$\frac{\delta(y)}{C} < -\lambda_1(y) < C\delta(y),$$

$$\frac{\varepsilon}{C}\left(\frac{1}{C}-c\right) < -\lambda_1(y) < C(C+c)\varepsilon.$$

Denote $c_1 = C(C+c)$, $c_2 = \frac{1}{C} \left(\frac{1}{C} - c\right)$. Therefore we have

$$B(x, c\varepsilon) \subset L_{\varepsilon} = \{y : -c_1 \varepsilon < \lambda_1(y) < -c_2 \varepsilon\}.$$

for every $0 < \varepsilon < \varepsilon_0$ and $x \in \partial D_{\lambda_2}^{\varepsilon}$.

Now, by the mean value property and Jensen inequality,

$$|u(x)|^p = \left| c_3 \varepsilon^{-N} \int_{B(x,c\varepsilon)} u(y) dy \right|^p \le c_3 \varepsilon^{-N} \int_{B(x,c\varepsilon)} |u(y)|^p dy.$$

Therefore, for $0 < \varepsilon < \varepsilon_0$ we have

$$\int_{\partial D_{\lambda_2}^{\varepsilon}} |u(x)|^p d\sigma_{\varepsilon}^2(x) \le c_3 \varepsilon^{-N} \int_{\partial D_{\lambda_2}^{\varepsilon}} \left(\int_{B(x,c\varepsilon)} |u(y)|^p dy \right) d\sigma_{\varepsilon}^2(x)$$

$$= c_3 \varepsilon^{-N} \int_{\mathbb{R}^N} \left(\int_{\partial D_{\lambda_2}^{\varepsilon}} \chi_{\varepsilon}(x,y) d\sigma_{\varepsilon}^2(x) \right) |u(y)|^p dy,$$

where $\chi_{\varepsilon}(x,y)$ is the characteristic function of the ball $B(x,c\varepsilon)$. Observe, that

$$\int_{\partial D_{\lambda_2}^{\varepsilon}} \chi_{\varepsilon}(x, y) d\sigma_{\varepsilon}^{2}(x) = 0$$

for $y \notin L_{\varepsilon} = \{y : -c_1 \varepsilon < \lambda_1(y) < -c_2 \varepsilon\}$. Moreover, there exists positive constant c_4 , such that for every $0 < \varepsilon < \varepsilon_0$ and $y \in L_{\varepsilon}$ we have

$$\int_{\partial D_{\lambda_2}^{\varepsilon}} \chi_{\varepsilon}(x, y) d\sigma_{\varepsilon}^{2}(x) \le c_4 \varepsilon^{N-1}$$

(by Lemma 3.3 and the properties of the transform ϕ_{ε}). Hence

$$\int_{\partial D_{\lambda_2}^{\varepsilon}} |u(x)|^p d\sigma_{\varepsilon}^2(x) \le c_3 c_4 \varepsilon^{-1} \int_{L_{\varepsilon}} |u(y)|^p dy$$
$$= c_5 \varepsilon^{-1} \int_{c_2 \varepsilon}^{c_1 \varepsilon} \left(\int_{\partial D_{\lambda_1}^{\eta}} |u(y)|^p d\sigma_{\eta}^1(y) \right) d\eta \le c_5 (c_1 - c_2) M,$$

since $c_1 \varepsilon < 2C^2 \varepsilon_0 \le \varepsilon_{\lambda_1}$. Thus the lemma is proved.

By definition, if u is harmonic and bounded on D, then $u \in h^{\infty}(D)$. The next lemma shows a similar result for $p < \infty$.

Lemma 4.3 Let $1 \le p < \infty$. Suppose u is harmonic on D, and there exists a positive harmonic function h on D, such that $|u(x)|^p \le h(x)$, for every $x \in D$. Then $u \in h^p(D)$.

Proof. Fix $x_0 \in D$ and let $\lambda_0(x) = -G_D(x_0, x)$. Since

$$|\nabla \lambda_0(y)| = |\nabla_y G_D(x_0, y)| = -\frac{\partial G_D(x_0, y)}{\partial \nu_y} = P_D(x_0, y) \ge c_{x_0} > 0,$$

for every $y \in \partial D$ and $\lambda_0|_{\partial D} \equiv 0$, λ_0 is a characterizing function for D (near the boundary). Let $\varepsilon_0 = \varepsilon_{\lambda_0} > 0$ be so small, so that $D_0^{\varepsilon} = D_{\lambda_0}^{\varepsilon}$ are well-defined approximating subdomains for $0 < \varepsilon < \varepsilon_0$. Then D_0^{ε} are of class C^2 and have their Green's functions $G_{D_0^{\varepsilon}}$ and Poisson kernels $P_{D_0^{\varepsilon}}$.

Now observe, that by the properties of the Green's function,

$$G_{D_0^{\varepsilon}}(x_0, x) = G_D(x_0, x) - \varepsilon,$$

for every $x \in D_0^{\varepsilon}$, $x \neq x_0$, since $G_D(x_0, \cdot)|_{\partial D_0^{\varepsilon}} \equiv \varepsilon$. Hence

$$P_{D_0^{\varepsilon}}(x_0, x) = |\nabla_x G_{D_0^{\varepsilon}}(x_0, x)| = |\nabla_x G_D(x_0, x)|, \quad x \in \partial D_0^{\varepsilon},$$

and

$$h(x_0) = \int_{\partial D_0^{\varepsilon}} P_{D_0^{\varepsilon}}(x_0, x) h(x) d\sigma_{\varepsilon}(x) = \int_{\partial D_0^{\varepsilon}} |\nabla_x G_D(x_0, x)| h(x) d\sigma_{\varepsilon}(x).$$

Now, because $|\nabla_y G_D(x_0, \pi(x))| = P_D(x_0, \pi(x)) \ge c_{x_0} > 0$ for every $x \in D$, where π the orthogonal projection, there exists $\varepsilon_1 > 0$, such that

$$|\nabla_y G_D(x_0, x)| \ge \frac{c_{x_0}}{2},$$

if $\delta(x) < \varepsilon_1$. Since $\delta|_{\partial D_0^{\varepsilon}}$ tends to 0 uniformly as $\varepsilon \to 0$, we may choose ε_2 , $0 < \varepsilon_2 < \varepsilon_0$, such that the last inequality holds for $x \in \partial D_0^{\varepsilon}$, whenever $0 < \varepsilon < \varepsilon_2$. Thus

$$h(x_0) = \int_{\partial D_0^{\varepsilon}} |\nabla_x G_D(x_0, x)| h(x) d\sigma_{\varepsilon}(x) \ge \frac{c_{x_0}}{2} \int_{\partial D_0^{\varepsilon}} h(x) d\sigma_{\varepsilon}(x),$$

for $0 < \varepsilon < \varepsilon_2$, and we have

$$\sup_{0<\varepsilon<\varepsilon_2} \int_{\partial D_{\delta}^{\varepsilon}} |u(x)|^p d\sigma_{\varepsilon}(x) \le \sup_{0<\varepsilon<\varepsilon_2} \int_{\partial D_{\delta}^{\varepsilon}} h(x) d\sigma_{\varepsilon}(x) \le \frac{2h(x_0)}{c_{x_0}}.$$

By Lemma 4.2 we conclude, that $u \in h^p(D)$.

A simple corollary of Lemma 4.3 is, that every positive harmonic function on D belongs to $h^1(D)$. In particular, for fixed $y \in \partial D$, $P_D(\cdot, y) \in h^1(D)$. Moreover we have

1. If
$$\mu \in M(\partial D)$$
, then $P_D[\mu] \in h^1(D)$.

2. If $1 \le p \le \infty$ and $f \in L^p(\partial D)$, then $P_D[f] \in h^p(D)$.

To see 1, choose $\mu \in M(\partial D)$ and let $u = P_D[\mu]$. Then for every $x \in D$ we have

$$|u(x)| = \left| \int_{\partial D} P_D(x, y) d\mu(y) \right| \le \int_{\partial D} P_D(x, y) d|\mu|(y).$$

Since $|\mu|$ is positive and finite measure on ∂D , $P_D[|\mu|]$ is positive and harmonic in D, and by Lemma 4.3, $u \in h^1(D)$. Proof of 2 is similar. For fixed $1 \leq p < \infty$, $f \in L^p(\partial D)$, let $u = P_D[f]$. Then by Jensen inequality we have

$$|u(x)|^p = \left| \int_{\partial D} P_D(x, y) f(y) d\sigma(y) \right|^p \le \int_{\partial D} P_D(x, y) |f(y)|^p d\sigma(y) = P_D\left[|f|^p\right](x),$$

for every $x \in D$. Since $P_D[|f|^p]$ is positive, $u \in h^p(D)$. The case $p = \infty$ is the easiest. For $f \in L^{\infty}(\partial D)$ and $u = P_D[f]$ we have

$$|u(x)| \le \int_{\partial D} P_D(x,y)|f(y)|d\sigma(y) \le ||f||_{\infty} \int_{\partial D} P_D(x,y)d\sigma(y) = ||f||_{\infty},$$

and thus $u \in h^{\infty}(D)$.

In the next part of h^p theory, we will need some stronger assertion about the functions $\{P_D(\cdot,y)\}_{y\in\partial D}$.

Lemma 4.4 Let λ be a characterizing function for D and choose $\varepsilon_{\lambda} > 0$ as before. There exists a positive constant C_{λ} , such that for every $y \in \partial D$

$$||P_D(\cdot,y)||_{h^1}^{\lambda} \leq C_{\lambda}.$$

Proof. Fix $x_0 \in D$ and let $\lambda_0(x) = -G_D(x_0, x)$. Then, as in the proof of Lemma 4.3 we conclude, that λ_0 is a characterizing function for D. Moreover, since $P_D(\cdot, y)$ are positive and harmonic on D for every $y \in \partial D$, there exists $\varepsilon_1 > 0$ and $M_{x_0} > 0$ (which does not depend on y), such that

$$\sup_{0<\varepsilon<\varepsilon_1} \int_{\partial D_{\lambda_0}^{\varepsilon}} P_D(x,y) d\sigma_{\varepsilon}(x) \le M_{x_0} P_D(x_0,y).$$

By Lemma 3.7, for some C > 0 and every $x \in D$, $y \in \partial D$,

$$P_D(x,y) \le \frac{C}{|x-y|^{N-1}} \le \frac{C}{\delta(x)^{N-1}}.$$

Therefore

$$\begin{aligned} \|P_D(\cdot,y)\|_{h^1}^{\lambda_0} &= \sup_{0 < \varepsilon < \varepsilon_{\lambda_0}} \int_{\partial D_{\lambda_0}^{\varepsilon}} P_D(x,y) d\sigma_{\varepsilon}(x) \\ &\leq \sup_{0 < \varepsilon < \varepsilon_1} \int_{\partial D_{\lambda_0}^{\varepsilon}} P_D(x,y) d\sigma_{\varepsilon}(x) + \sup_{\varepsilon_1 \le \varepsilon < \varepsilon_{\lambda_0}} \int_{\partial D_{\lambda_0}^{\varepsilon}} P_D(x,y) d\sigma_{\varepsilon}(x) \\ &\leq \frac{CM_{x_0}}{\delta(x_0)^{N-1}} + \frac{2C\sigma(\partial D)}{\operatorname{dist}(D_{\lambda_0}^{\varepsilon_1}, \partial D)} = C_{\lambda_0}. \end{aligned}$$

Now for any characterizing function λ , ε_{λ} as before, we may choose, as in the proof of Lemma 4.2, $\varepsilon_2 > 0$, $\varepsilon_2 < \min \{ \varepsilon_{\lambda_0}, \varepsilon_{\lambda} \}$, such that

$$\sup_{0<\varepsilon<\varepsilon_2} \int_{\partial D_{\lambda}^{\varepsilon}} P_D(x,y) d\sigma_{\varepsilon}(x) \le M_1 \|P_D(\cdot,y)\|_{h^1}^{\lambda_0},$$

for some $M_1 = M_1(\lambda, \lambda_0) > 0$. Because the functions $P_D(\cdot, y)$ are uniformly bounded by some $M_2 = M_2(\lambda) > 0$ on the sets $\partial D_{\lambda}^{\varepsilon}$ for $\varepsilon_2 \leq \varepsilon < \varepsilon_{\lambda}$, we have

$$||P_D(\cdot,y)||_{h^1}^{\lambda} \le M_1 ||P_D(\cdot,y)||_{h^1}^{\lambda_0} + M_2 2\sigma(\partial D) \le M_1 C_{\lambda_0} + M_2 2\sigma(\partial D) = C_{\lambda} < \infty,$$

what gives the conclusion of the lemma.

Recall, that for fixed characterizing function λ , $\varepsilon_{\lambda}>0$ as usually, the map

$$\pi_{\varepsilon} := \pi|_{\partial D_{\lambda}^{\varepsilon}}, \quad 0 < \varepsilon < \varepsilon_{\lambda},$$

is invertible. For u harmonic on D denote $u_{\varepsilon}(y) = u(\pi_{\varepsilon}^{-1}(y)), y \in \partial D$. If $f \in C(\partial D)$ and $u = P_D[f]$, then $u_{\varepsilon} \to f$ in $C(\partial D)$. Because $C(\partial D)$ is dense in $L^p(\partial D)$ for $1 \le p < \infty$, we have the following result on L^p -convergence.

Lemma 4.5 Suppose $1 \le p < \infty$. If $f \in L^p(\partial D)$ and $u = P_D[f]$, then

$$||u_{\varepsilon} - f||_p \to 0$$
 as $\varepsilon \to 0$.

Proof. Choose $1 \leq p < \infty$, $f \in L^p(\partial D)$ and let $u = P_D[f]$. Fix $\varepsilon > 0$. Let $C_{\lambda} \geq 1$ satisfy the assertion of Lemma 4.4, and choose $g \in C(\partial D)$, such that $\|f - g\|_p < \varepsilon/C_{\lambda}$. Let $v = P_D[g]$, and choose $\varepsilon_1 > 0$, such that $\|v_{\varepsilon'} - g\|_p < \varepsilon$ for every $0 < \varepsilon' < \varepsilon_1$. Then we have

$$||u_{\varepsilon'} - f||_p \le ||u_{\varepsilon'} - v_{\varepsilon'}||_p + ||v_{\varepsilon'} - g||_p + ||f - g||_p \le ||u_{\varepsilon'} - v_{\varepsilon'}||_p + 2\varepsilon.$$

However

$$||u_{\varepsilon'} - v_{\varepsilon'}||_p^p = \int_{\partial D} \left| \int_{\partial D} P_D(\pi_{\varepsilon'}^{-1}(y), z) (f(z) - g(z)) d\sigma(z) \right|^p d\sigma(y)$$

$$\leq \int_{\partial D} \int_{\partial D} P_D(\pi_{\varepsilon'}^{-1}(y), z) |f(z) - g(z)|^p d\sigma(z) d\sigma(y),$$

$$= \int_{\partial D} \int_{\partial D} P_D(\pi_{\varepsilon'}^{-1}(y), z) d\sigma(y) |f(z) - g(z)|^p d\sigma(z),$$

by Jensen inequality and Fubini's theorem. Now we may choose $\varepsilon_2 > 0$, $\varepsilon_2 < \varepsilon_1$, such that for every $0 < \varepsilon' < \varepsilon_2$, and every $z \in \partial D$ we have

$$\int_{\partial D} P_D(\pi_{\varepsilon'}^{-1}(y), z) d\sigma(y) \le 2 \int_{\partial D} P_D(\pi_{\varepsilon'}^{-1}(y), z) \phi_{\varepsilon'}(y) d\sigma(y)$$
$$= 2 \int_{\partial D_{\lambda}^{\varepsilon'}} P_D(x, z) d\sigma_{\varepsilon'}(x) \le 2 \|P_D(\cdot, z)\|_{h^1}^{\lambda} \le 2C_{\lambda}.$$

Therefore

$$||u_{\varepsilon'} - v_{\varepsilon'}||_p = \left(\int_{\partial D} \int_{\partial D} P_D(\pi_{\varepsilon'}^{-1}(y), z) d\sigma(y) |f(z) - g(z)|^p d\sigma(z)\right)^{\frac{1}{p}}$$

$$\leq (2C_{\lambda})^{\frac{1}{p}} \|f - g\|_{p} < 2\varepsilon,$$

and hence $||u_{\varepsilon'} - f||_p < 4\varepsilon$, for every $0 < \varepsilon' < \varepsilon_2$. Since ε is arbitrary, we have

$$||u_{\varepsilon} - f||_{p} \stackrel{\varepsilon \to 0}{\longrightarrow} 0,$$

as desired.

As we have seen before, if $1 \leq p \leq \infty$ and $f \in L^p(\partial D)$, then $P_D[f] \in h^p(D)$. To the end of this section we show, that for p > 1 each harmonic function from the space $h^p(D)$ can be characterized in terms of the Poisson kernel for D.

Theorem 4.1 Suppose $1 and let <math>u \in h^p(D)$. Then there exists $f \in L^p(\partial D)$, such that $u = P_D[f]$. Moreover, if λ is a characterizing function for D, then for some positive constant $\widetilde{C} = \widetilde{C}(\lambda, p)$ we have

$$||f||_{p} \le ||u||_{h^{p}}^{\lambda} \le \widetilde{C} ||f||_{p}.$$

Proof. Let $\{D_j\}$ be a finite cover of D with the following properties:

- 1. $D = \bigcup D_j$.
- 2. For every j, D_j is a domain with the boundary of class C^2 .
- 3. For every j, $\partial D_i \cap \partial D$ is a N-1 dimensional manifold with boundary.
- 4. There exists $\varepsilon_0 > 0$ and a vector $\nu_j = \nu_{y_j}$ for some $y_j \in \partial D_j \cap \partial D$, so that $\overline{D}_j \varepsilon \nu_j \subset D$ for every $0 < \varepsilon < \varepsilon_0$.

Let $P_j(x,y)$ be the Poisson kernel for D_j . Because for every $0 < \varepsilon < \varepsilon_0$, the functions $u_{\varepsilon}^j(x) = u(x - \varepsilon \nu_j)$ are harmonic on D_j and continuous on \overline{D}_j , we have

$$u_{\varepsilon}^{j}(x) = \int_{\partial D_{j}} P_{j}(x, y) u_{\varepsilon}^{j}(y) d\sigma_{j}(y), \quad x \in D_{j}.$$

Moreover, in view of Lemma 4.2,

$$\sup_{0<\varepsilon<\varepsilon_0}\int_{\partial D_i}|u_\varepsilon^j(y)|^pd\sigma_j(y)<\infty.$$

By Banach-Alaoglu theorem, there exists a subsequence $u_{\varepsilon_k}^j$, that converges weak* to some $f_j \in L^p(\partial D_j)$. Hence, if q is conjugate to p, then obviously $P_j(x,\cdot) \in L^q(\partial D_j)$, and we have

$$u(x) = \int_{\partial D_j} P_j(x, y) f_j(y) d\sigma_j(y), \quad x \in D_j.$$

Now observe, that if λ_j, λ_k are the characterizing functions for the domains D_j, D_k respectively, and $\partial D_j \cap \partial D_k$ contains some open subset of ∂D , then λ_j, λ_k are comparable on the set

$$\{y - t\nu_y : y \in \partial D_j \cap \partial D_k \cap \partial D, 0 < t < t_0\},\$$

for t_0 sufficiently small. Hence, by (a small modification of) Lemma 4.5, $f_j = f_k$ a.e. (with respect to σ) in $\partial D_j \cap \partial D_k \cap \partial D$. Thus $f \equiv f_j$ on $\partial D_j \cap \partial D$ is well defined; obviously, $f \in L^p(\partial D)$.

It remains to be shown, that $u = P_D[f]$. So fix $x_0 \in D$, and let $\lambda(x) = G_D(x_0, x)$. As in the proof of Lemma 4.3 we conclude, that

$$P_{D_{\lambda}^{\varepsilon}}(x_0, x) = |\nabla_x G_{D_{\lambda}^{\varepsilon}}(x_0, x)| = |\nabla_x G_D(x_0, x)|, \quad x \in \partial D_{\lambda}^{\varepsilon},$$

and

$$u(x_0) = \int_{\partial D_{\lambda}^{\varepsilon}} P_{D_{\lambda}^{\varepsilon}}(x_0, x) u(x) d\sigma_{\varepsilon}(x) = \int_{\partial D_{\lambda}^{\varepsilon}} |\nabla_x G_D(x_0, x)| u(x) d\sigma_{\varepsilon}(x)$$
$$= \int_{\partial D_{\lambda}^{\varepsilon}} |\nabla \lambda(x)| u(x) d\sigma_{\varepsilon}(x).$$

Moreover,

$$\int_{\partial D_{\lambda}^{\varepsilon}} |\nabla \lambda(x)| u(x) d\sigma_{\varepsilon}(x) = \int_{\partial D} |\nabla \lambda(\pi_{\varepsilon}^{-1}(y))| u(\pi_{\varepsilon}^{-1}(y)) \phi_{\varepsilon}(y) d\sigma(y).$$

Hence we have

$$u(x_0) = \int_{\partial D} |\nabla \lambda(\pi_{\varepsilon}^{-1}(y))| u(\pi_{\varepsilon}^{-1}(y)) \phi_{\varepsilon}(y) d\sigma(y)$$

$$= \int_{\partial D} (|\nabla \lambda(\pi_{\varepsilon}^{-1}(y))| \phi_{\varepsilon}(y) - |\nabla_y G_D(x_0, y)) u(\pi_{\varepsilon}^{-1}(y)) d\sigma(y)$$

$$+ \int_{\partial D} P_D(x_0, y) u(\pi_{\varepsilon}^{-1}(y)) d\sigma(y) = I_1 + I_2.$$

Now since $|\nabla \lambda(\pi_{\varepsilon}^{-1}(y))| \to |\nabla_y G_D(x_0, y)|$, $\phi_{\varepsilon} \to 1$ uniformly on ∂D as $\varepsilon \to 0$, and $u \in h^p(D) \subset h^1(D)$, we have $I_1 \stackrel{\varepsilon}{\to} 0$, and it suffices to show that $I_2 \stackrel{\varepsilon}{\to} P_D[f](x_0)$. For fixed j we have

$$\left| \int_{\partial D \cap \partial D_j} P_D(x_0, y) u(\pi_{\varepsilon}^{-1}(y)) d\sigma(y) - \int_{\partial D \cap \partial D_j} P_D(x_0, y) f(y) d\sigma(y) \right|$$

$$\leq \frac{C}{\delta(x_0)^{N-1}} \int_{\partial D \cap \partial D_j} |u(\pi_{\varepsilon}^{-1}(y)) - f(y)| d\sigma(y),$$

by Lemma 3.7. Thus if λ_j is a characterizing function for D_j , then λ and λ_j

are comparable on the set $\pi_{\varepsilon}^{-1}(\partial D \cap \partial D_j) \subset D_j$ (independently on ε), and by the use of Lemma 4.5 we obtain

$$\int_{\partial D \cap \partial D_i} |u(\pi_{\varepsilon}^{-1}(y)) - f(y)| d\sigma(y) \xrightarrow{\varepsilon \to 0} 0.$$

Because $\partial D = \bigcup \partial D_j \cap \partial D$ (and the sum is finite), we have $I_2 \stackrel{\varepsilon}{\to} P_D[f](x_0)$. Therefore $u = P_D[f]$.

Now let λ' be a characterizing function for D (which is not necessarily λ). Suppose first, that $p < \infty$. Then

$$\int_{\partial D_{\lambda'}^{\varepsilon}} |u(x)|^p d\sigma_{\varepsilon}(x) = \int_{\partial D} |u(\pi_{\varepsilon}^{-1}(y))|^p \phi_{\varepsilon}(y) d\sigma(y) \xrightarrow{\varepsilon \to 0} ||f||_p^p,$$

by Lemma 4.5. Therefore $||u||_{h^p}^{\lambda} \geq ||f||_p$. On the other hand,

$$\int_{\partial D_{\lambda'}^{\varepsilon}} |u(x)|^{p} d\sigma_{\varepsilon}(x) = \int_{\partial D_{\lambda'}^{\varepsilon}} \left| \int_{\partial D} P_{D}(x, y) f(y) d\sigma(y) \right|^{p} d\sigma_{\varepsilon}(x)
\leq \int_{\partial D} \int_{\partial D_{\lambda'}^{\varepsilon}} P_{D}(x, y) d\sigma_{\varepsilon}(x) |f(y)|^{p} d\sigma(y) \leq C_{\lambda'} \|f\|_{p}^{p},$$

by Lemma 4.4. Thus $||u||_{h^p}^{\lambda'} \leq (C_{\lambda'})^{1/p} ||f||_p$ (obviously, the estimation occurs even if $f \in L^1(\partial D)$).

If $p = \infty$, then $\|u\|_{h^{\infty}} = \sup |u| = \sup |P_D[f]| \le \|f\|_{\infty}$. By previous computations, $\|u\|_{h^q}^{\lambda'} \ge \|f\|_q$ for $1 \le q < \infty$. Since $\|u\|_{h^q}^{\lambda'} \to \|u\|_{h^{\infty}}$ as $q \to \infty$, we then get $\|u\|_{h^{\infty}} = \|f\|_{\infty}$. That finishes the proof of the theorem.

5 The Fatou Theorem

In this section, by the use of [3] and [6], we will prove the known property of the Poisson integrals on D, a so called "nontangential" convergence.

For each $y \in \partial D$ and $\alpha > 0$ we define the "cone" of aperture α and vertex y as

$$\Gamma_{\alpha}(y) = \{x \in D : |x - y| < (1 + \alpha)\delta(x)\}.$$

Obviously $\Gamma_{\alpha}(y) \subset \Gamma_{\beta}(y)$ for $\alpha < \beta$, and $\bigcup_{\alpha>0} \Gamma_{\alpha}(y) = D$. We say that a function u on D has a nontangential limit L at $y \in \partial D$ if, for every $\alpha > 0$, $u(x) \to L$ as $x \to y$ within $\Gamma_{\alpha}(y)$.

Theorem 5.1 Suppose $u = P_D[f]$, where $f \in L^1(\partial D)$. Then u has a non-tangential limit at almost every point of ∂D and

$$\lim_{\Gamma_{\alpha}(y)\ni x\to y}u(x)=f(y)\quad for\quad a.e.\quad y\in\partial D.$$

A key tool in the proof of Theorem 5.1 is the use of the Hardy-Littlewood maximal functions. For any $f \in L^1(\partial D)$ we define

$$M[f](y) = \sup_{r>0} \frac{1}{\sigma(K(y,r))} \int_{K(y,r)} |f(z)| d\sigma(z),$$

the maximal function of f.

Lemma 5.1 (Wiener)

For positive integer k, let $F \subset \mathbb{R}^k$ be a compact set that is covered by the open balls $\{B_{\alpha}\}_{{\alpha}\in A}$, $B_{\alpha}=B(c_{\alpha},r_{\alpha})$. There is a disjoint, finite subcover B_{α_1} , $B_{\alpha_2},...$, such that

$$F \subset \bigcup_i B(c_{\alpha_i}, 3r_{\alpha_i}).$$

Proof. Since F is compact, we may assume that $\{B_{\alpha}\}_{\alpha\in A}$ is finite. Let B_{α_1} be the ball in this collection, that has the largest radius. Let B_{α_2} be the ball that is disjoint from B_{α_1} and has the greatest radius, and so on. The process ends in finitely many steps. We claim that the B_{α_i} chosen above satisfy the conclusion of the lemma.

It is enough to show that $B_{\alpha} \subset \bigcup_{i} B(c_{\alpha_{i}}, 3r_{\alpha_{i}})$ for every α . Fix an α . If $\alpha = \alpha_{i}$ for some i, then we are done. If $\alpha \notin \{\alpha_{i}\}$, let $\alpha_{i_{0}}$ be the first index with $B_{\alpha} \cap B_{\alpha_{i}} \neq \emptyset$ (there must be one, or else the process would not have stopped). Hence $r_{\alpha} \leq r_{\alpha_{i_{0}}}$; otherwise, we selected $B_{\alpha_{i_{0}}}$ incorrectly. But then clearly $B(c_{\alpha}, r_{\alpha}) \subset B(c_{\alpha_{i_{0}}}, 3r_{\alpha_{i_{0}}})$, as desired.

Lemma 5.2 Suppose $f \in L^1(\partial D)$. There exists a positive constant C = C(N, D), such that

$$\sigma\left\{y\in\partial D:M[f](y)>t\right\}\leq\frac{C\left\|f\right\|_{L^{1}(\partial D)}}{t},\quad\forall t>0.$$

Proof. Choose t > 0 and let F be a compact subset of $\{y \in \partial D : M[f](y) > t\}$. Because σ is regular, it suffices to estimate $\sigma(F)$. Now for each $y \in F$ there exists $r_y > 0$, such that

$$\frac{1}{\sigma(K(y,r_y))} \int_{K(y,r_y)} |f(z)| d\sigma(z) > t.$$

The balls $\{B(y,r_y)\}_{y\in F}$ cover F. Choose, by Lemma 5.1, finite family of disjoint balls $\{B(y_i,r_{y_i})\}$ so that $\{B(y_i,3r_{y_i})\}$ cover F. Then

$$\sigma(F) \le \sum_{i} \sigma(K(y_i, 3r_{y_i})) \le 3^{N-1} \frac{c_2}{c_1} \sum_{i} \sigma(K(y_i, r_{y_i})),$$

where c_1, c_2 are the constants from Lemma 3.3. Denoting $C = 3^{N-1}c_2/c_1$ we conclude, that

$$\sigma(F) \le \frac{C}{t} \sum_{i} \int_{K(y_i, r_{y_i})} |f(z)| d\sigma(z) \le \frac{C \|f\|_{L^1(\partial D)}}{t}.$$

Lemma 5.3 Suppose $u = P_D[f]$, where $f \in L^1(\partial D)$, and let $\alpha > 0$. Then there exists $C_{\alpha} > 0$, such that for every $y \in \partial D$

$$\sup_{x \in \Gamma_{\alpha}(y)} |u(x)| \le C_{\alpha} M[f](y).$$

Proof. Choose $y \in \partial D$ and let $x \in \Gamma_{\alpha}(y)$; denote $\eta = |x - y|$. We have

$$|u(x)| \le \int_{\partial D} P_D(x, z) |f(z)| d\sigma(z) = \int_{|z-y| < 2\eta} P_D(x, z) |f(z)| d\sigma(z)$$

$$+ \sum_{k=2}^{\infty} \int_{2^{k-1}\eta < |z-y| < 2^k \eta} P_D(x, z) |f(z)| d\sigma(z).$$

By Lemma 3.7,

$$P_D(x,z) \le \frac{C}{|x-z|^{N-1}}$$

for some constant C > 0. Because $\delta(x) \leq |x-z|$, we then get that $P_D(x,z) \leq C(\delta(x))^{1-N}$. The cone condition, $|x-y| < (1+\alpha)\delta(x)$, shows, that $\delta(x) > \eta/(1+\alpha)$. Thus

$$\int_{|z-y|<2\eta} P_D(x,z)|f(z)|d\sigma(z) \le C(1+\alpha)^{N-1}\eta^{1-N} \int_{K(y,2\eta)} |f(z)|d\sigma(z)$$

34

$$\leq \frac{CC'2^{N}(1+\alpha)^{N-1}}{2} \cdot \frac{1}{\sigma(K(y,2\eta))} \int_{K(y,2\eta)} |f(z)| d\sigma(z),$$

where C' is the constant from the upper estimation of Lemma 3.3. Similarly, if

$$2^{k-1}\eta < |z - y| < 2^k \eta,$$

where $k \geq 2$, then

$$|x - z| \ge |z - y| - |y - x| \ge 2^{k-1}\eta - \eta \ge 2^{k-2}\eta$$

and

$$P_D(x,z) \le \frac{C}{|x-z|^{N-1}} \le C2^{2N}2^{-kN}\eta^{1-N}.$$

By Lemma 3.3,

$$\eta^{1-N} \le \frac{C'2^{kN}2^{-k}}{\sigma(K(y, 2^k\eta))};$$

therefore

$$\int_{2^{k-1}\eta < |z-y| < 2^k \eta} P_D(x,z) |f(z)| d\sigma(z) \le C 2^{2N} 2^{-kN} \eta^{1-N} \int_{K(y,2^k \eta)} |f(z)| d\sigma(z)
\le \frac{C C' 2^{2N}}{2^k} \cdot \frac{1}{\sigma(K(y,2^k \eta))} \int_{K(y,2^k \eta)} |f(z)| d\sigma(z).$$

Now, denoting $C_{\alpha} = \max \left\{ CC'2^{N}(1+\alpha)^{N-1}, CC'2^{2N} \right\}$, we conclude, that

$$|u(x)| \le C_{\alpha} \sum_{k=1}^{\infty} \frac{1}{2^k \sigma(K(y, 2^k \eta))} \int_{K(y, 2^k \eta)} |f(z)| d\sigma(z) \le C_{\alpha} M[f](y).$$

Proof of Theorem 5.1. For $f \in L^1(\partial D)$ and $\alpha > 0$, define the function $T_{\alpha}[f]$ on ∂D by

$$T_{\alpha}[f](y) = \lim_{\Gamma_{\alpha}(y) \ni x \to y} |P_{D}[f](x) - f(y)|.$$

We first show, that $T_{\alpha}[f] = 0$ almost everywhere on ∂D .

35

Note that

$$T_{\alpha}[f](y) \le \sup_{x \in \Gamma_{\alpha}(y)} |P_{D}[f](x)| + |f(y)| \le C_{\alpha}M[f](y) + |f(y)|$$

by Lemma 5.3, and that $T_{\alpha}[f_1+f_2] \leq T_{\alpha}[f_1]+T_{\alpha}[f_2]$. Note also that $T_{\alpha}[f] \equiv 0$ for every $f \in C(\partial D)$.

Now fix $f \in L^1(\partial D)$ and $\alpha > 0$. Also fixing $t \in (0, \infty)$, we wish to show that $\sigma(\{T_{\alpha}[f] > 2t\}) = 0$.

Given $\varepsilon > 0$, we may choose $g \in C(\partial D)$ such that $||f - g||_{L^1(\partial D)} < \varepsilon$. Then we have

$$T_{\alpha}[f] \le T_{\alpha}[f-g] + T_{\alpha}[g] = T_{\alpha}[f-g] \le C_{\alpha}M[f-g] + |f-g|.$$

Therefore

$$\{T_{\alpha}[f] > 2t\} \subset \{C_{\alpha}M[f-g] > t\} \cup \{|f-g| > t\}.$$

By Lemma 5.2 we conclude, that

$$\sigma(\left\{T_{\alpha}[f]>2t\right\}) \leq \frac{CC_{\alpha} \|f-g\|_{L^{1}(\partial D)}}{t} + \frac{\|f-g\|_{L^{1}(\partial D)}}{t} < \varepsilon \frac{CC_{\alpha}+1}{t}.$$

Since ε is arbitrary, we have shown that the set $\{T_{\alpha}[f] > 2t\}$ is contained in sets of arbitrarily small measure, and thus $\sigma(\{T_{\alpha}[f] > 2t\}) = 0$. Because this is true for every $t \in (0, \infty)$, we have proved, that $T_{\alpha}[f] = 0$ almost everywhere on ∂D .

Now for $k \in \mathbb{N}$ let $E_k = \{T_k[f] = 0\}$. We have shown that E_k is set of full measure on ∂D for each k, and thus $\bigcap_k E_k$ is a set of full measure. At each $y \in \bigcap_k E_k$, P[f] has nontangential limit f(y), which is what we set out to prove.

6 The Local Fatou Theorem

Theorem 5.1 has a local version. We require a definition. A function u on D is said to be nontangentially bounded at $y \in \partial D$ if u is bounded in $\Gamma_{\alpha}(y)$ for some $\alpha > 0$.

Theorem 6.1 Suppose u is harmonic on D and $E \subset \partial D$ is the set of points at which u is nontangentially bounded. Then u has a nontangential limit at almost every point of E.

This theorem was originally obtained by Privalov, Plessner, Marcinkiewicz and Zygmund, and Spencer in the classical case N=2 by the use of complex-variable techniques. Methods which are effective for the upper half-space in \mathbb{R}^N , $N \geq 2$, have been served by Calderon and Stein. The proof for this case may be found in [1].

Using similar methods as in [1], we will serve a detailed proof of Theorem 6.1 for the present case, when D is a bounded domain in \mathbb{R}^N , $N \geq 2$, with the boundary of class C^2 . In order to do this, we shall use a few important technical lemmas. We begin with some stronger assertion about the behavior of the Poisson kernel inside the cone.

Lemma 6.1 Let $\alpha > 0$. There exists positive constant A_{α} , such that for every $y \in \partial D$ and $x \in \Gamma_{\alpha}(y)$ we have

$$P_D(x,y) \ge \frac{A_{\alpha}}{\delta(x)^{N-1}}.$$

Proof. Notice that, if K is compact set in D, then the estimate we seek is trivial for $x \in K$ and $y \in \partial D$, since $P_D(x, y)$ is positive and continuous on $D \times \partial D$, and $\delta(x)$ is bounded away from zero on K.

Let r be the constant from Lemma 3.2, and let $r_0 = r/4$. Hence, for $y \in \partial D$ we have

$$\overline{B}(y + 4r_0\nu_y, 4r_0) \cap \overline{D} = \{y\},$$

$$\overline{B}(y - 4r_0\nu_y, 4r_0) \cap \overline{D^c} = \{y\}.$$

Suppose $x \in B(y - r_0\nu_y, r_0)$. This means that

$$|x - y + r_0 \nu_y| < r_0$$

 $|x - y|^2 - 2r_0 \langle x - y, -\nu_y \rangle + r_0^2 < r_0^2$
 $|x - y|^2 < 2r_0 \langle x - y, -\nu_y \rangle.$

Moreover

$$dist(x, \partial B(y + 4r_0\nu_y, 4r_0)) = |x - y - 4r_0\nu_y| - 4r_0,$$

$$dist(x, \partial B(y - 4r_0\nu_u, 4r_0)) = 4r_0 - |x - y + 4r_0\nu_u|,$$

and thus

$$|x - y - 4r_0\nu_y| - 4r_0 = \frac{|x - y - 4r_0\nu_y|^2 - 16r_0^2}{|x - y - 4r_0\nu_y| + 4r_0} = \frac{|x - y|^2 + 8r_0\langle x - y, -\nu_y\rangle}{|x - y - 4r_0\nu_y| + 4r_0}$$

$$< \frac{10r_0\langle x - y, -\nu_y\rangle}{8r_0} = \frac{5}{4}\langle x - y, -\nu_y\rangle,$$

$$4r_0 - |x - y + 4r_0 \nu_y| = \frac{16r_0^2 - |x - y + 4r_0 \nu_y|^2}{4r_0 + |x - y + 4r_0 \nu_y|} = \frac{-|x - y|^2 + 8r_0 \langle x - y, -\nu_y \rangle}{4r_0 + |x - y + 4r_0 \nu_y|}$$
$$> \frac{6r_0 \langle x - y, -\nu_y \rangle}{8r_0} = \frac{3}{4} \langle x - y, -\nu_y \rangle.$$

Therefore

$$|x - y - 4r_0\nu_u| - 4r_0 \le 2(4r_0 - |x - y + 4r_0\nu_u|).$$

Obviously

$$\delta(x) \le \operatorname{dist}(x, \partial B(y + 4r_0\nu_y, r_0)) = |x - y - 4r_0\nu_y| - 4r_0,$$

since $B(y + 4r_0\nu_u, 4r_0) \subset D^c$, and hence

$$\delta(x) \le 2(4r_0 - |x - y + 4r_0\nu_y|).$$

Now choose $y \in \partial D$ and let $x \in B(y - r_0\nu_y, r_0)$. Denote $c_y = y - 4r_0\nu_y$ and $B_y = B(c_y, 4r_0)$. Let G_D , G_{B_y} be the Green's functions for D and B_y , respectively. Observe, that $G_{B_y}(x,\cdot)$ is 0 on ∂B_y , whereas $G_D(x,\cdot) \geq 0$ on ∂B_y . Since $G_D(x,\cdot) - G_{B_y}(x,\cdot)$ is harmonic on B_y , it follows that

$$G_{B_y}(x,t) \le G_D(x,t), \quad t \in \overline{B_y}.$$

$$G_{B_y}(x,y) = G_D(x,y) = 0.$$

Therefore

$$P_{B_y}(x,y) = -\frac{\partial G_{B_y}(x,y)}{\partial \nu_y} \le -\frac{\partial G_D(x,y)}{\partial \nu_y} = P_D(x,y).$$

The Poisson kernel for B_y has a form

$$P_{B_y}(x,t) = \frac{1}{\omega_{N-1} 4r_0} \cdot \frac{16r_0^2 - |x - c_y|^2}{|x - t|^N}, \quad x \in B_y, t \in \partial B_y.$$

Hence we have

$$P_D(x,y) \ge \frac{1}{\omega_{N-1} 4r_0} \cdot \frac{16r_0^2 - |x - c_y|^2}{|x - y|^N}$$

$$= \frac{1}{\omega_{N-1} 4r_0} \cdot \frac{(4r_0 - |x - c_y|)(4r_0 + |x - c_y|)}{|x - y|^N}$$

$$\ge \frac{1}{\omega_{N-1}} \cdot \frac{(4r_0 - |x - c_y|)}{|x - y|^N} = \frac{1}{\omega_{N-1}} \cdot \frac{(4r_0 - |x - y + 4r_0\nu_y|)}{|x - y|^N} \ge \frac{1}{2\omega_{N-1}} \cdot \frac{\delta(x)}{|x - y|^N}.$$

Thus we have shown, that

$$P_D(x,y) \ge C \frac{\delta(x)}{|x-y|^N}.$$

for every $y \in \partial D$ and $x \in B(y - r\nu_y, r_0)$.

Now for h > 0 let $\Gamma_{\alpha}^{h}(y) = \Gamma_{\alpha}(y) \cap \{x : \delta(x) < h\}$. We will show, that there exists $h_{\alpha} > 0$, such that for every $y \in \partial D$, $\Gamma_{\alpha}^{h_{\alpha}}(y) \subset B(y - r_{0}\nu_{y}, r_{0})$. For $y \in \partial D$ denote $\widetilde{B}_{y} = B(y + 4r_{0}\nu_{y}, 4r_{0})$, and recall that $\overline{\widetilde{B}}_{y} \cap \overline{D} = \{y\}$. Let $\widetilde{\Gamma}_{\alpha}(y)$ be the cone in \widetilde{B}_{y}^{c} . That is,

$$\widetilde{\Gamma}_{\alpha}(y) = \left\{ x \in \widetilde{B}_{y}^{c} : |x - y| < (1 + \alpha) \operatorname{dist}(x, \partial \widetilde{B}_{y}) \right\};$$

let

$$\widetilde{\Gamma}_{\alpha}^{h}(y) = \widetilde{\Gamma}_{\alpha}(y) \cap \left\{ x : \operatorname{dist}(x, \partial \widetilde{B}_{y}) < h \right\}.$$

Since $\delta(x) \leq \operatorname{dist}(x, \partial \widetilde{B}_y)$, we have $\Gamma_{\alpha}(y) \subset \widetilde{\Gamma}_{\alpha}(y)$. Moreover, observe that

$$\Gamma^h_\alpha(y) = \{x \in D: |x-y| < (1+\alpha)\delta(x) \wedge \delta(x) < h\}$$

$$= \{ x \in D : |x - y| < (1 + \alpha)\delta(x) \land \delta(x) < h \land |x - y| < (1 + \alpha)h \}$$

$$\subset \Gamma_{\alpha}(y) \cap \{x \in D : |x - y| < (1 + \alpha)h\}$$

$$\subset \widetilde{\Gamma}_{\alpha}(y) \cap \{x : \operatorname{dist}(x, \partial \widetilde{B}_{y}) < (1 + \alpha)h\} = \widetilde{\Gamma}_{\alpha}^{h'}(y),$$

where $h' = (1 + \alpha)h$. Thus it suffices to show, that there exists $h'_{\alpha} > 0$, such that $\widetilde{\Gamma}_{\alpha}^{h'_{\alpha}}(y) \subset B(y - r_0\nu_y, r_0)$ for every $y \in \partial D$.

Suppose $h'_{\alpha} < 8r_0/5(1+\alpha)^2$, and fix $y \in \partial D$, $x \in \widetilde{\Gamma}_{\alpha}^{h'_{\alpha}}(y)$. We have

$$|x - y| < (1 + \alpha) \operatorname{dist}(x, \partial \widetilde{B}_y) = (1 + \alpha)(|x - y - 4r_0\nu_y| - 4r_0)$$

$$= (1+\alpha) \frac{|x-y-4r_0\nu_y|^2 - 16r_0^2}{|x-y-4r_0\nu_y| + 4r_0} = (1+\alpha) \frac{|x-y|^2 - 8r_0\langle x-y,\nu_y\rangle}{|x-y-4r_0\nu_y| + 4r_0}$$
$$\leq (1+\alpha) \frac{|x-y|^2 - 8r_0\langle x-y,\nu_y\rangle}{8r_0}.$$

Hence

$$\langle x - y, \nu_y \rangle < \frac{|x - y|^2}{8r_0} - \frac{|x - y|}{1 + \alpha} = \frac{5|x - y|^2}{8r_0} - \frac{|x - y|}{1 + \alpha} - \frac{|x - y|^2}{2r_0}$$
$$= |x - y| \left(\frac{5|x - y|}{8r_0} - \frac{1}{1 + \alpha}\right) - \frac{|x - y|^2}{2r_0} < -\frac{|x - y|^2}{2r_0},$$

since $|x-y| < (1+\alpha)h'_{\alpha}$. Thus we conclude

$$\langle x - y, \nu_y \rangle < -\frac{|x - y|^2}{2r_0}$$

$$|x - y|^2 + 2r_0 \langle x - y, \nu_y \rangle + r_0^2 < r_0^2$$

$$|x - y + r_0 \nu_y| < r_0,$$

what means, that $x \in B(y - r_0\nu_y, r_0)$, so $\widetilde{\Gamma}_{\alpha}^{h'_{\alpha}}(y) \subset B(y - r_0\nu_y, r_0)$. Denoting $h_{\alpha} = h'_{\alpha}/(1 + \alpha)$, we have $\Gamma_{\alpha}^{h_{\alpha}}(y) \subset B(y - r_0\nu_y, r_0)$.

Therefore, if $x \in \Gamma_{\alpha}^{h_{\alpha}}(y)$, then

$$P_D(x,y) \ge C \frac{\delta(x)}{|x-y|^N} > C \frac{\delta(x)}{(1+\alpha)^N \delta(x)^N} = \frac{C}{(1+\alpha)^N} \cdot \frac{1}{\delta(x)^{N-1}}.$$

Now, since $K_{\alpha} = \{x \in D : \delta(x) \geq h_{\alpha}\}$ is compact, we can find a positive constant $A_{\alpha} \leq C/(1+\alpha)^{N}$, such that for every $y \in \partial D$ and $x \in K_{\alpha}$

$$P_D(x,y) \ge \frac{A_{\alpha}}{\delta(x)^{N-1}}.$$

Thus the inequality is proved.

Lemma 6.2 Suppose $E \subset \partial D$ is Borel measurable, $\alpha > 0$, and

$$\Omega = \bigcup_{y \in E} \Gamma_{\alpha}(y).$$

Then there exists a positive harmonic function v on D such that $v \geq 1$ on $(\partial\Omega) \cap D$, and such that v has nontangential limit 0 almost everywhere on E.

Proof. Define a positive harmonic function w on D by

$$w(x) = \int_{\partial D} P_D(x, y) \chi_{E^c}(y) d\sigma(y),$$

where χ_{E^c} denotes the characteristic function of E^c , the complement of E in ∂D . By Theorem 2, w has nontangential limit 0 almost everywhere on E. We wish to show that w is bounded away from 0 on $(\partial\Omega) \cap D$. Choose $x \in (\partial\Omega) \cap D$; this means, that

$$|x - y| \ge (1 + \alpha)\delta(x) \quad \forall y \in E.$$

Because ∂D is compact, there exists $x' \in \partial D$, such that $|x - x'| = \delta(x)$. Hence for every $y \in E$ we have

$$|x - y| \ge (1 + \alpha)|x - x'|$$
$$|y - x'| \ge |y - x| - |x - x'| \ge \alpha|x - x'|$$
$$|y - x'| \ge \alpha\delta(x),$$

so $K(x', \alpha \delta(x)) \subset E^c$. Therefore

$$\int_{\partial D} P_D(x,y) \chi_{E^c}(y) d\sigma(y) = \int_{E^c} P_D(x,y) d\sigma(y) \geq \int_{K(x',\alpha\delta(x))} P_D(x,y) d\sigma(y).$$

On the other hand, if $y \in K(x', \alpha\delta(x))$, then

$$|x - y| \le \delta(x) + |x' - y| < (1 + \alpha)\delta(x),$$

so $x \in \Gamma_{\alpha}(y)$. By Lemmas 3.3 and 6.1, there exist positive constants c, A_{α} , such that

$$\sigma \{K(x', \alpha \delta(x))\} \ge c(\alpha \delta(x))^{N-1}$$

and

$$P_D(x,y) \ge \frac{A_{\alpha}}{\delta(x)^{N-1}} \quad \forall y \in K(x', \alpha \delta(x)).$$

Hence

$$\int_{K(x',\alpha\delta(x))} P_D(x,y) d\sigma(y) \ge \frac{A_\alpha}{\delta(x)^{N-1}} \sigma\left\{K(x',\alpha\delta(x))\right\} \ge cA_\alpha \alpha^{N-1}.$$

Denoting $c_{\alpha} = cA_{\alpha}\alpha^{N-1}$ (a constant greater than 0 that depends only on α and N), we see that if $v = w/c_{\alpha}$, then v satisfies the conclusion of the lemma.

Lemma 6.3 Choose $\alpha, r > 0$. Suppose $\nu_1, \nu_2 \in S$ and $|\nu_1 - \nu_2| < \frac{\alpha}{1+\alpha}$. Then for each $t \in (0, r]$ we have the following inequality

$$t < (1+\alpha)(r-|t\nu_1 - r\nu_2|).$$

Proof. Let $t \in (0, r]$; because $|\nu_1| = |\nu_2| = 1$, we have the following sequence of equivalent inequalities

$$t < (1+\alpha)(r-|t\nu_1 - r\nu_2|)$$
$$|t\nu_1 - r\nu_2| < r - \frac{t}{1+\alpha}$$
$$t^2 - 2rt\langle\nu_1, \nu_2\rangle + r^2 < \left(r - \frac{t}{1+\alpha}\right)^2$$

$$t^{2} - rt(2 - |\nu_{1} - \nu_{2}|^{2}) < \left(\frac{t}{1+\alpha}\right)^{2} - \frac{2rt}{1+\alpha}$$
$$|\nu_{1} - \nu_{2}|^{2} < 2 - \frac{2}{1+\alpha} - \frac{t}{r}\left(1 - \frac{1}{(1+\alpha)^{2}}\right).$$

Because $t \in (0, r]$, we have

$$\left(\frac{\alpha}{1+\alpha}\right)^2 \le 2 - \frac{2}{1+\alpha} - \frac{t}{r} \left(1 - \frac{1}{(1+\alpha)^2}\right),\,$$

and this gives the conclusion of the lemma.

Now let r be the constant from Lemma 3.2 and let $r_0 = r/4$. We may assume, that $r_0 \leq 1$. For $y \in \partial D$ denote

$$B_y = B(y - r_0 \nu_y, r_0), \quad B'_y = B(y + r_0 \nu_y, r_0)$$

and

$$\Gamma'_{\alpha}(y) = \{x \in B_y : |x - y| < (1 + \alpha)\operatorname{dist}(x, \partial B_y)\}.$$

Obviously

$$dist(x, \partial B_y) = r_0 - |y - r_0 \nu_y - x|,$$

and because $B_y \subset D$, we have

$$\Gamma'_{\alpha}(y) \subset \Gamma_{\alpha}(y).$$

Lemma 6.4 Let $\alpha > 0$. There exists a positive constant $d = d(\alpha)$ such that for every $x, y \in \partial D$ and $|x - y| \leq d$ we have

$$1. \ \forall t \in (0, r_0]$$

$$y - t\nu_x \in \Gamma'_{\alpha}(y).$$

2.
$$\forall t, s \in (0, r_0], t < s$$

$$\delta(y - s\nu_x) - \delta(y - t\nu_x) \ge \frac{s - t}{1 + \alpha}.$$

Proof.

Recall that

$$|\nu_x - \nu_y| \le c_0 |x - y| \quad \forall x, y \in \partial D.$$

We may assume, that $c_0 \geq 1$. So let $x, y \in \partial D$ and suppose

$$|x - y| < \frac{\alpha}{c_0(1 + \alpha)};$$

then

$$|\nu_x - \nu_y| < \frac{\alpha}{1 + \alpha},$$

and because $\nu_x, \nu_y \in S$, by Lemma 6.3 we have

$$|y - t\nu_x - y| = t < (1 + \alpha)(r_0 - |t\nu_x - r_0\nu_y|) = (1 + \alpha)(r_0 - |y - r_0\nu_y - (y - t\nu_x)|)$$

$$= (1 + \alpha) \operatorname{dist}(y - t\nu_x, \partial B_y) \quad \forall t \in (0, r_0].$$

This gives 1.

Now let π be the orthogonal projection; then for every $x, y \in D_{r_0}$ we have

$$|\pi(x) - \pi(y)| < 4|x - y|,$$

by Lemma 3.4. So let $x, y \in \partial D$ and assume

$$|x-y| < \frac{\alpha}{8c_0^2(1+\alpha)}.$$

Then we have

$$\langle \nu_x, \nu_y \rangle = 1 - \frac{|\nu_x - \nu_y|^2}{2} \ge 1 - \frac{c_0^2 |x - y|^2}{2} > 0,$$

and choosing $t \in (0, r_0)$ we conclude

$$|y - t\nu_x - (y - t\langle\nu_x, \nu_y\rangle\nu_y)| = t|\langle\nu_x, \nu_y\rangle\nu_y - \nu_x| = t\left(1 - \langle\nu_x, \nu_y\rangle^2\right)^{\frac{1}{2}}$$

$$= t \left[1 - \left(1 - \frac{|\nu_x - \nu_y|^2}{2} \right)^2 \right]^{\frac{1}{2}} = t \left(|\nu_x - \nu_y|^2 - \frac{|\nu_x - \nu_y|^4}{4} \right)^{\frac{1}{2}}$$

$$= t|\nu_x - \nu_y| \left(1 - \frac{|\nu_x - \nu_y|^2}{4}\right)^{\frac{1}{2}} \le r_0|\nu_x - \nu_y| \le c_0|x - y| < \frac{\alpha}{8c_0(1 + \alpha)}.$$

In particular, since $t\langle \nu_x, \nu_y \rangle \in (0, r_0)$, we have

$$|\pi(y - t\nu_x) - \pi(y - t\langle \nu_x, \nu_y \rangle \nu_y)| = |\pi(y - t\nu_x) - y| < \frac{\alpha}{2c_0(1 + \alpha)}.$$

Therefore

$$|\nu_{\pi(y-t\nu_x)} - \nu_y| \le c_0 |\pi(y - t\nu_x) - y| < \frac{\alpha}{2(1+\alpha)}.$$

Let $s \in (t, r_0]$; because

$$y - s\nu_x = \pi(y - t\nu_x) - \delta(y - t\nu_x)\nu_{\pi(y - t\nu_x)} - (s - t)\nu_x$$

and

$$|\nu_{\pi(y-t\nu_x)} - \nu_x| \le |\nu_{\pi(y-t\nu_x)} - \nu_y| + |\nu_y - \nu_x| < \frac{\alpha}{1+\alpha}$$

where $\nu_{\pi(y-t\nu_x)}, \nu_x \in S$, by Lemma 6.3 we have

$$y - s\nu_x \in B(\pi(y - t\nu_x) - [\delta(y - t\nu_x) + r_0]\nu_{\pi(y - t\nu_x)}, r_0).$$

Because

$$B(\pi(y_t) - (\delta_t + r_0)\nu_{\pi(y_t)}, r_0) \subset B(\pi(y_t) - (\delta_t + r_0)\nu_{\pi(y_t)}, \delta_t + r_0)$$

$$\subset B(\pi(y_t) - 2r_0\nu_{\pi(y_t)}, 2r_0) \subset D, \quad \delta_t := \delta(y - t\nu_x), \quad y_t := y - t\nu_x,$$

we have

$$\delta(y - s\nu_x) - \delta(y - t\nu_x) \ge r_0 - |y - s\nu_x - [\pi(y - t\nu_x) - (\delta(y - t\nu_x) + r_0)\nu_{\pi(y - t\nu_x)}]|$$

$$= r_0 - |r_0 \nu_{\pi(y-t\nu_x)} - (s-t)\nu_x|;$$

one more time by Lemma 6.3 we conclude

$$r_0 - |r_0 \nu_{\pi(y-t\nu_x)} - (s-t)\nu_x| > \frac{s-t}{1+\alpha},$$

what gives 2.

For $\alpha > 0$ let d_{α} be the constant from Lemma 6.4; assume additionally, that $d_{\alpha} < \rho$, where ρ is the constant from Lemma 3.1. Because ∂D is compact, there exists a finite family of balls $\{B(y_i, d_{\alpha}) : i = 1, ..., m\}$, where $y_i \in \partial D$ for each i = 1, ..., m, such that

$$\partial D \subset \bigcup_{i=1}^{m} B(y_i, d_{\alpha});$$

hence

$$\partial D = \bigcup_{i=1}^{m} K(y_i, d_{\alpha}).$$

For $i \in \{1,...,m\}$ denote $\nu_i := \nu_{y_i}$ and for r_0 as before, let

$$D_i^{\alpha} = \left\{ y - t\nu_i : y \in K\left(y_i, d_{\alpha}\right), t \in \left(0, \frac{r_0}{2}\right) \right\}.$$

Lemma 6.5 For every $\alpha > 0$ and $i \in \{1, ..., m\}$ we have

- 1. D_i^{α} is an open subset of D and $(\partial D_i^{\alpha}) \cap (\partial D) = \overline{K}(y_i, d_{\alpha})$.
- 2. Let E be the closed subset of ∂D and suppose $E \subset K(y_i, d_\alpha)$. Let

$$\Omega = \bigcup_{y \in E} \Gamma_{\alpha}(y);$$

then there exists $\varepsilon_0 > 0$, such that

$$\Omega \cap \{x \in D : \delta(x) < \varepsilon_0\} \subset D_i^{\alpha}$$
.

Proof. Choose $\alpha > 0$ and $i \in \{1, ..., m\}$. Without loss of generality we may assume, that $y_i = 0$ and $\nu_i = e_N = (0, ..., 0, 1)$. Obviously

$$\overline{D_{i}^{\alpha}} = \left\{ y - te_{N} : y \in \overline{K}(0, d_{\alpha}), t \in \left[0, \frac{r_{0}}{2}\right] \right\}.$$

By Lemma 6.4,

$$\forall y \in \overline{K}(0, d_{\alpha}) \, \forall t \in (0, r_0] \quad y - te_N \in \Gamma'_{\alpha}(y),$$

in particular $y - te_N \in B_y$. Hence $D_i^{\alpha} \subset D$. Moreover, $\overline{D_i^{\alpha}} \setminus \overline{K}(0, d_{\alpha}) \subset D$, and hence $(\partial D_i^{\alpha}) \cap (\partial D) = \overline{K}(0, d_{\alpha})$.

Now let $x \in D_i^{\alpha}$. Then $x = y - te_N$, where $y \in K(0, d_{\alpha})$ and $t \in (0, \frac{r_0}{2})$. By Lemma 3.1, since $d_{\alpha} < \rho$, $y = (\overline{y}, g(\overline{y}))$, where $\overline{y} \in \mathbb{R}^{N-1}$ is the projection of y into $\mathbb{R}^{N-1} \times \{0\}$ (since $\nu_0 = e_N$), and g is a real-valued, C^2 function defined on $B_{N-1}(0, \rho)$. Strictly, $\overline{K}(0, d_{\alpha})$ is a part of the graph of function g. Moreover, since $\nu_0 = e_N$, we may assume that

$$B_N(0,\rho) \cap D = B_N(0,\rho) \cap \{(\overline{x},x_N) : \overline{x} \in B_{N-1}(\overline{0},\rho) \land x_N < g(\overline{x})\}.$$

Therefore $x=(\overline{y},g(\overline{y})-t)$, and $\overline{x}=\overline{y}$. Then there exists $\varepsilon_1>0$, such that if $|x-x'|<\varepsilon_1$, then $y'=(\overline{x}',g(\overline{x}'))\in K(0,d_\alpha)$. Moreover, $x'=y'-(g(\overline{x}')-x_N')e_N$, and there exists $\varepsilon_2>0$, such that $t'=g(\overline{x}')-x_N'\in \left(0,\frac{r_0}{2}\right)$ whenever $|x-x'|<\varepsilon_2$. Thus if $\varepsilon_3=\min\left\{\varepsilon_1,\varepsilon_2\right\}$, and $|x-x'|<\varepsilon_3$, then $x'=y'-t'e_N$, where $y'\in K(0,d_\alpha)$ and $t'\in \left(0,\frac{r_0}{2}\right)$, so $B(x,\varepsilon_3)\subset D_i^\alpha$. This gives 1.

Now suppose $E \subset K(0, d_{\alpha}), E = \overline{E}$ and let

$$\Omega = \bigcup_{y \in E} \Gamma_{\alpha}(y).$$

By Lemma 3.1, there exists M > 0, such that $|g(\overline{x}) - g(\overline{y})| \leq M|\overline{x} - \overline{y}|$ for every $\overline{x}, \overline{y} \in B_{N-1}(\overline{0}, \rho)$. Let $x \in \Omega$ and suppose

$$\delta(x) < \varepsilon_0 = \min \left\{ \frac{\operatorname{dist} \left[E, \partial B \left(0, d_{\alpha} \right) \right]}{\sqrt{1 + M^2} (1 + \alpha)}, \frac{r_0}{2(1 + M)(1 + \alpha)} \right\}.$$

Then for some $y \in E$ we have

$$|x-y| < (1+\alpha)\delta(x) < \operatorname{dist} [E, \partial B(0, d_{\alpha})].$$

Thus

$$|x| \le |x - y| + |y| < \operatorname{dist} [E, \partial B(0, d_{\alpha})] + |y| \le d_{\alpha} < \rho,$$

in particular $|\overline{x}| < \rho$. Hence $x = (\overline{x}, g(\overline{x})) - (g(\overline{x}) - x_N)e_N$, and it suffices to show that $(\overline{x}, g(\overline{x})) \in K(0, d_{\alpha})$ and $g(\overline{x}) - x_N \in (0, \frac{r_0}{2})$. We have

$$|(\overline{x},g(\overline{x}))| \leq |(\overline{x},g(\overline{x})) - (\overline{y},g(\overline{y}))| + |y| = \sqrt{|\overline{x} - \overline{y}|^2 + |g(\overline{x})) - g(\overline{y})|^2} + |y|$$

$$\leq \sqrt{1+M^2}|\overline{x}-\overline{y}|+|y|\leq \sqrt{1+M^2}|x-y|+|y|$$

$$<\sqrt{1+M^2}(1+\alpha)\delta(x)+|y|<\operatorname{dist}\left[E,\partial B\left(0,d_{\alpha}\right)\right]+|y|\leq d_{\alpha}.$$

Moreover, $g(\overline{x}) - x_N > 0$ since $x \in B(0, \rho) \cap D$, and

$$g(\overline{x}) - x_N \le |g(\overline{x}) - g(\overline{y})| + |y_N - x_N| \le M|\overline{x} - \overline{y}| + |x - y| \le (M + 1)|x - y| < \frac{r_0}{2}.$$

Therefore $x \in D_i^{\alpha}$, what gives 2.

The next lemma is the crux of the proof of Theorem 6.1.

Lemma 6.6 Let $E \subset \partial D$ be Borel measurable, let $\alpha > 0$, and let

$$\Omega = \bigcup_{y \in E} \Gamma_{\alpha}(y).$$

Suppose u is harmonic on D and bounded on Ω . Then for almost every $y \in E$, $\lim u(x)$ exists as $x \to y$ within $\Gamma_{\alpha}(y)$.

Proof. Because

$$\partial D = \bigcup_{i=1}^{m} K(y_i, d_{\alpha}),$$

it suffices to show, that the conclusion of the lemma holds for the set

$$E_i = K(y_i, d_\alpha) \cap E,$$

where $i \in \{1, ..., m\}$ is fixed. We may also assume that u is real valued and $|u| \le 1$ on Ω . Suppose first, that E_i is closed, and let

$$\Omega_i = \bigcup_{y \in E_i} \Gamma_{\alpha}(y).$$

For $k \in \mathbb{N}$, let

$$E_i^k = \left\{ \left[K(y_i, d_\alpha) - \frac{r_0}{k} \nu_i \right] \cap \Omega_i \right\} + \frac{r_0}{k} \nu_i.$$

Because Ω_i is an open set in \mathbb{R}^N , and $K(y_i, d_\alpha)$ is an open subset of ∂D , E_i^k is an open subset of ∂D too. Obviously $E_i^k \subseteq K(y_i, d_\alpha)$. By Lemma 6.4, for each $y \in E_i$ $y - \frac{r_0}{k}\nu_i \in \Gamma_\alpha(y)$; hence $E_i \subseteq E_i^k$ for every k. Let

$$f_i^k(y) = \chi_{E_i^k}(y)u\left(y - \frac{r_0}{k}\nu_i\right), \quad y \in \partial D.$$

Because $y - \frac{r_0}{k}\nu_i \in \Omega$ if $y \in E_i^k$, $|f_i^k| \leq 1$ on ∂D for every k. By Banach-Alaoglu theorem, there exists a subsequence, which we still denote by (f_i^k) , that converges weak* to some $f_i \in L^{\infty}(\partial D)$. Because E_i^k is open, each f_i^k is continuous on E_i^k , and thus, by the properties of the Poisson kernel listed in section 3, $P_D[f_i^k]$ extends continuously to $D \cup E_i^k$.

By Lemma 6.5, D_i^{α} is an open subset of D, additionally $(\partial D_i^{\alpha}) \cap (\partial D) = \overline{K}(y_i, d_{\alpha})$. Now if $x \in \overline{D_i^{\alpha}} \cap D$, then

$$x - \frac{r_0}{k}\nu_i = y - \left(t + \frac{r_0}{k}\right)\nu_i,$$

where $y \in \overline{K}(y_i d_{\alpha})$ and $t \in (0, \frac{r_0}{2}]; t + \frac{r_0}{k} \le r_0$ for $k \ge 2$, thus by Lemma 6.4, $x - \frac{r_0}{k} \nu_i \in D$. Therefore the function u_i^k given by

$$u_i^k(x) = P_D[f_i^k](x) - u\left(x - \frac{r_0}{k}\nu_i\right), \quad k \ge 2,$$

is harmonic on D_i^{α} and continuous on $\overline{D_i^{\alpha}} \cap D$. Moreover, u_i^k extends continuously to $(\overline{D_i^{\alpha}} \cap D) \cup E_i^k$, with $u_i^k = 0$ on E_i^k . We have assumed, that E_i is closed, and thus $\overline{\Omega}_i \cap (\partial D) = E_i$; by Lemma 6.5, there exists $\varepsilon_0 > 0$ such that

$$\Omega_i|_{\varepsilon_0} := \Omega_i \cap \{x \in D : \delta(x) < \varepsilon_0\} \subset D_i^{\alpha}.$$

Therefore we conclude, that in particular u_i^k is harmonic on $\Omega_i|_{\varepsilon_0}$ and continuous on $\overline{\Omega_i|_{\varepsilon_0}}$ with $u_i^k=0$ on E_i .

Let $x \in \Omega_i|_{\varepsilon_0}$; then $x = y - t\nu_i$, where $y \in K(y_i, d_\alpha)$, $t \in (0, \frac{r_0}{2})$ and there exists $a \in E_i$, such that $x \in \Gamma_\alpha(a)$. For $k \ge 2$ we have

$$\left| x - \frac{r_0}{k} \nu_i - a \right| = \left| y - t \nu_i - a - \frac{r_0}{k} \nu_i \right| < (1 + \alpha) \delta(y - t \nu_i) + \frac{r_0}{k}$$

$$\leq (1 + \alpha) \delta\left(y - \left(t + \frac{r_0}{k} \right) \nu_i \right),$$

where the last inequality is the result of Lemma 6.4. This means, that $\frac{x-\frac{r_0}{k}\nu_i}{\Omega_i|_{\varepsilon_0}}$. This means, that $\frac{x}{\Omega_i|_{\varepsilon_0}}$.

Now let v be the function of Lemma 6.2 with respect to Ω_i . Because $v \geq 1$ on $(\partial \Omega_i) \cap (\partial (\Omega_i|_{\varepsilon_0})) \cap D$, and there exists constant c > 0, such that $v \geq c$ on $\{x \in D : \delta(x) = \varepsilon_0\}$, thus the function

$$\tilde{v}(x) = \frac{v(x)}{\min\{c, 1\}}$$

has the same properties as v, but with respect to $\Omega_i|_{\varepsilon_0}$. Then

$$\lim_{x \to \partial(\Omega_i|_{\varepsilon_0})} (2\tilde{v} - u_i^k)(x) \ge 0.$$

By the minimum principle, $2\tilde{v} - u_i^k \geq 0$ in $\Omega_i|_{\varepsilon_0}$. Letting $k \to \infty$, we then see that $2\tilde{v} - (P_D[f_i] - u) \geq 0$ in $\Omega_i|_{\varepsilon_0}$. Because this argument applies as well to $2\tilde{v} + u_i^k$, we conclude that $|P_D[f_i] - u| \leq 2\tilde{v}$ in $\Omega_i|_{\varepsilon_0}$.

By Theorem 5.1, $P_D[f_i]$ has nontangential limits almost everywhere on ∂D , while Lemma 6.2 asserts \tilde{v} has nontangential limits 0 almost everywhere on E_i . From this and the last inequality, the desired limits for u follow.

Now if E_i is any Borel measurable and bounded set, then for each $k \in \mathbb{N}$ there exists a closed set $F_k \subseteq E_i$, such that $\sigma(E_i \backslash F_k) < \frac{1}{k}$. Let A_k be the set of points $y \in F_k$, such that $\lim u(x)$ doesn't exists as $x \to y$ within $\Gamma_{\alpha}(y)$; then

$$\sigma\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \sigma(A_k) = 0,$$

and because $\sigma(E_i \setminus \bigcup_{k=1}^{\infty} F_k) = 0$, the conclusion of the lemma holds with respect to E_i .

Let $E \subset \partial D$ be Borel measurable; a point $y \in E$ is said to be a point of density of E provided

$$\lim_{r \to 0} \frac{\sigma(K(y,r) \cap E)}{\sigma(K(y,r))} = 1.$$

By the Lebesque Differentiation Theorem, almost every point of E is a point of density of E.

Lemma 6.7 Suppose $E \subset \partial D$ is Borel measurable, $\alpha > 0$, and

$$\Omega = \bigcup_{y \in E} \Gamma_{\alpha}(y).$$

Suppose u is continuous on D and bounded on Ω . If y is a point of density of E, then u is bounded in $\Gamma_{\beta}(y)$ for every $\beta > 0$.

Proof. Let y be a point of density of E, and let $\beta > 0$. For h > 0 denote

$$\Gamma_{\beta}^{h}(y) = \{ x \in D : |x - y| < (1 + \beta)\delta(x) \land \delta(x) < h \}.$$

It suffices to show that $\Gamma^h_{\beta}(y) \subset \Omega$ for some h > 0.

Let c_1, c_2 be the constants from Lemma 3.3; there exists $r_1 > 0$ such that for each $r < r_1$ we have

$$\frac{\sigma(K(y,r)\cap E)}{\sigma(K(y,r))} > 1 - \frac{c_1}{c_2} \left(\frac{\alpha}{2+\alpha+\beta}\right)^{N-1}.$$

So let $h_1 = \frac{r_1}{2+\alpha+\beta}$ and fix $x \in \Gamma_{\beta}^{h_1}(y)$. There exists $x' \in \partial D$, such that

$$|x - x'| = \delta(x).$$

Suppose $y' \in K(x', \alpha \delta(x))$; we have

$$|y' - x'| < \alpha |x - x'|$$

$$|y' - x| \le |y' - x'| + |x' - x| < (1 + \alpha)|x - x'|,$$

what means, that $x \in \Gamma_{\alpha}(y')$. Hence it suffices to show, that $K(x', \alpha\delta(x)) \cap E$ is non-empty.

Because $x \in \Gamma^{h_1}_{\beta}(y)$, we have

$$|y - x'| - |x' - x| \le |y - x| < (1 + \beta)|x - x'|,$$

and thus $x' \in K(y, (2+\beta)\delta(x))$, what implies, that

$$K(x', \alpha\delta(x)) \subset K(y, (2 + \alpha + \beta)\delta(x)).$$

Moreover, because $\delta(x) < h_1$, we conclude

$$\frac{\sigma\left\{K(y,(2+\alpha+\beta)\delta(x))\cap E\right\}}{\sigma\left\{K(y,(2+\alpha+\beta)\delta(x))\right\}} > 1 - \frac{c_1}{c_2} \left(\frac{\alpha}{2+\alpha+\beta}\right)^{N-1}$$

$$\geq 1 - \frac{\sigma\left\{K(x', \alpha\delta(x))\right\}}{\sigma\left\{K(y, (2 + \alpha + \beta)\delta(x))\right\}} \geq \frac{\sigma\left\{K(y, (2 + \alpha + \beta)\delta(x))\setminus K(x', \alpha\delta(x))\right\}}{\sigma\left\{K(y, (2 + \alpha + \beta)\delta(x))\right\}},$$

and hence

$$\sigma \left\{ K(y, (2+\alpha+\beta)\delta(x)) \cap E \right\} > \sigma \left\{ K(y, (2+\alpha+\beta)\delta(x)) \setminus K(x', \alpha\delta(x)) \right\}.$$

Therefore $K(x', \alpha\delta(x)) \cap E$ is non-empty, and thus $\Gamma_{\beta}^{h_1}(y) \subset \Omega$, as desired.

Lemma 6.8 Let u be continuous in D and suppose $E \subset \partial D$ is the set of points at which u is nontangentially bounded. Then E is Borel measurable, and for any $\varepsilon > 0$, there exists a closed set $E_0 \subset E$, such that $\sigma(E \setminus E_0) < \varepsilon$ and u is bounded in

$$\Omega = \bigcup_{y \in E_0} \Gamma_{\alpha}(y)$$

for every $\alpha > 0$.

Proof.

For k = 1, 2, ..., set $E_k = \{y \in \partial D : |u| \le k \text{ on } \Gamma_{\frac{1}{k}}(y)\}$. Then each E_k is closed, and

$$E = \bigcup_{k=1}^{\infty} E_k$$

what means, that the set E is Borel measurable. Fix $k \in \mathbb{N}$; applying Lemma 6.7 to E_k , and recalling that the points of density of E_k form a set of full

measure in E_k , we see that there is a set $F_k \subset E_k$, $\sigma(E_k \setminus F_k) = 0$, such that u is bounded in $\Gamma_{\alpha}(y)$ for every $y \in F_k$ and every $\alpha > 0$. Let

$$F = \bigcup_{k=1}^{\infty} F_k;$$

obviously $\sigma(E \setminus F) = 0$. For fixed positive integer k, we can write F as $F = \bigcup_j F_j^k$, where $F_j^k = \{y \in F : |u| \leq j \text{ on } \Gamma_k(y)\}$. Now fix $\varepsilon > 0$. Because $F_j^k \subset F_{j+1}^k$ for each $j \in \mathbb{N}$, there exists j_k , such that $\sigma(E \setminus F_{j_k}^k) = \sigma(F \setminus F_{j_k}^k) < \varepsilon/2^k$. Hence for each k, u is bounded on the set

$$\Omega_k = \bigcup_{y \in F_{j_k}^k} \Gamma_k(y).$$

Let

$$E_0 = \bigcap_{k=1}^{\infty} F_{j_k}^k.$$

Then

$$\sigma(E \backslash E_0) = \sigma\left(E \cap \left(\bigcap_{k=1}^{\infty} F_{j_k}^k\right)^C\right) = \sigma\left(\bigcup_{k=1}^{\infty} (E \backslash F_{j_k}^k)\right) \le \sum_{k=1}^{\infty} \sigma(E \backslash F_{j_k}^k)$$
$$< \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon,$$

and it is easily seen, that u is bounded in

$$\Omega = \bigcup_{y \in E_0} \Gamma_{\alpha}(y)$$

for every $\alpha > 0$.

Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. By Lemma 6.8, E is Borel measurable, and for each $k \in \mathbb{N}$ we can choose $E_k \subset E$, such that $\sigma(E \setminus E_k) < \frac{1}{k}$ and u is bounded on the set

$$\Omega_k = \bigcup_{y \in E_k} \Gamma_j(y)$$

for every $j \in \mathbb{N}$. Fix j, k; by Lemma 6.6, for almost every $y \in E_k \lim u(x)$ exists as $x \to y$ within $\Gamma_j(y)$. Letting $k \to \infty$, we then get the same result for almost every $y \in E$. Now u has nontangential limit L at y if and only if $\lim u(x) = L$ as $x \to y$ within $\Gamma_j(y)$ for every $j \in \mathbb{N}$, and thus u has nontangential limit almost everywhere on E, as desired.

7 The Area Theorem

The question whether u has nontangential limits can also be answered in terms of the "area integral". As before we assume, that D is a bounded domain with the boundary of class C^2 . For any harmonic function u on D, $\alpha > 0$ and a point $y \in \partial D$, we consider $S_{\alpha}u(y)$ by

$$S_{\alpha}u(y) = \int_{\Gamma_{\alpha}(y)} |\nabla u(x)|^2 (\delta(x))^{2-N} dx.$$

A necessary and sufficient condition for the existence of nontangential limits, can be formulated as follows.

Theorem 7.1 Let u be harmonic on D.

- 1. Suppose $E \subset \partial D$ is the set of points at which u is nontangentially bounded. Then for a.e. $y \in E$, $S_{\alpha}u(y)$ is finite for every $\alpha > 0$.
- 2. Conversely, suppose E is the set of points $y \in \partial D$, such that $S_{\alpha}u(y)$ is finite, where α can depend on y. Then u has a nontangential limit at almost every point of E.

This theorem has been proved by Stein in [6], in the case, when D is a half-space in \mathbb{R}^N . By the use of very similar methods, we will prove the necessity part (1) of it in the present case. Because of some technical difficulties, we omit the sufficiency (2). The detailed proof of this part (by the use of other techniques) may be found in [7], together with various generalizations.

We begin with some technical lemmas. Let α and β be given positive quantities with $\alpha < \beta$.

Lemma 7.1 Let u be harmonic in the cone $\Gamma_{\beta}(y)$ and suppose that $|u| \leq 1$ there. Then

$$\delta(x)|\nabla u(x)| \le A \quad \forall x \in \Gamma_{\alpha}(y),$$

where $A = A(\alpha, \beta)$ depends only on the indicated parameters but not on y or u.

Proof. We shall need the following fact: if u is harmonic in B(0,r) and $|u| \leq 1$ there, then $|\nabla u(0)| \leq A/r$, where A does not depend on u (see [1]).

Let x be any point in $\Gamma_{\alpha}(y)$. Notice that since $\alpha < \beta$, there exists a fixed constant c > 0, which does not depend on x and y, so that

$$B(x, c\delta(x)) \subset \Gamma_{\beta}(y);$$

it suffices to take

$$c < \frac{\beta - \alpha}{1 + \beta}.$$

We now apply the previous fact to u and the ball $B(x, c\delta(x))$, and obtain

$$|\nabla u(x)| \le \frac{A}{c\delta(x)}.$$

Hence

$$\delta(x)|\nabla u(x)| \le A/c \quad \forall x \in \Gamma_{\alpha}(y).$$

For positive constants α , h and $y \in \partial D$ recall the notation

$$\Gamma_{\alpha}^{h}(y) = \{x \in D : |x - y| < (1 + \alpha)\delta(x), \delta(x) < h\}.$$

Let $E \subset \partial D$ be Borel measurable. For $\rho > 0$ and $a \in \partial D$ let $E(a, \rho) = E \cap \overline{B}(a, \rho)$. For positive α, h denote

$$\Omega(a, h, \rho, \alpha) = \bigcup_{y \in E(a, \rho)} \Gamma_{\alpha}^{h}(y).$$

Lemma 7.2 Suppose E is closed subset of ∂D and let $\alpha > 0$. There exist positive constants h, ρ , such that for each $a \in \partial D$ we can choose a family of regions $\{\Omega_n\}_{n=1}^{\infty}$ with the following properties:

- 1. $\overline{\Omega}_n \subset \Omega(a, h, \rho, \alpha) \quad \forall n$
- 2. $\Omega_n \subset \Omega_{n+1} \quad \forall n$
- 3. $\Omega(a, h, \rho, \alpha) = \bigcup \Omega_n$
- 4. The boundary B_n of Ω_n is piecewise of class C^2 , at a positive distance from ∂D , and there exists a constant A > 0 such that $\sigma_n(B_n) < A \ \forall n$.

Proof. Recall, that for every $x, y \in \partial D$

$$|\nu_x - \nu_y| \le c_0 |x - y|.$$

Let E be closed subset of ∂D , $\alpha > 0$. Choose $a \in \partial D$; without loss of generality we may assume that $-\nu_a = e_N = (0, ..., 0, 1)$. Let d_{α} be the constant from Lemma 6.4, and let ρ , h be positive constants, such that

$$(2+\alpha)h + \rho \le \min\left\{d_{\alpha}, \frac{1}{c_0}\sqrt{\frac{\alpha}{1+\alpha}}\right\}.$$

Assume additionally, that h satisfies the condition of Corollary 3.1. Now observe, that

$$\Omega(a,h,\rho,\alpha) = \bigcup_{y \in E(a,\rho)} \Gamma_{\alpha}^{h}(y) = \left\{ x \in D : \widetilde{\delta}(x) < (1+\alpha)\delta(x), \quad \delta(x) < h \right\},\,$$

where $\widetilde{\delta}(x) = \operatorname{dist}(x, E(a, \rho))$. Let φ_t be a C^{∞} "approximation to the identity". It may be constructed as follows. Take a function φ of class C^{∞} on \mathbb{R}^N , such that $\varphi \geq 0$, $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B(0, 1)$ and

$$\int_{\mathbb{R}^N} \varphi(x) dx = 1.$$

For t > 0 let $\varphi_t(x) = t^{-N}\varphi(x/t)$, and denote

$$f_t(x) = \int_{\mathbb{R}^N} \widetilde{\delta}(x - y) \varphi_t(y) dy.$$

Then f_t is of class C^{∞} on \mathbb{R}^N , nonnegative and $f_t \to \widetilde{\delta}$ uniformly on each compact subset of \mathbb{R}^N as $t \to 0$. Take a sequence $\varepsilon_n = 1/2^n$ and choose $t_n = t(\varepsilon_n)$, such that

$$|f_{t_n}(x) - \widetilde{\delta}(x)| < \varepsilon_n \quad \forall x \in \overline{D}.$$

Let

$$\widetilde{\delta}_n(x) = f_{t_n}(x) + 2\varepsilon_n$$

and denote

$$\Omega_n = \left\{ x \in D : \widetilde{\delta}_n(x) < (1+\alpha)\delta(x), \quad \delta(x) < h - \varepsilon_n \right\}.$$

Because $\widetilde{\delta}_n(x) > \widetilde{\delta}(x) + \varepsilon_n$ on \overline{D} , we have $\overline{\Omega}_n \subset \Omega(a, h, \rho, \alpha)$. Moreover,

$$\widetilde{\delta}_n(x) < \widetilde{\delta}(x) + 3\varepsilon_n < \widetilde{\delta}_{n-1}(x) - \varepsilon_{n-1} + \frac{3}{2}\varepsilon_{n-1} < \widetilde{\delta}_n(x), \quad x \in \overline{D},$$

hence $\Omega_{n-1} \subset \Omega_n$. Since $\widetilde{\delta}_n \stackrel{n}{\to} \widetilde{\delta}$ on \overline{D} , we have

$$\Omega(a, h, \rho, \alpha) = \bigcup_{n=1}^{\infty} \Omega_n,$$

and thus the properties 1-3 are proved.

Now the boundary B_n of Ω_n consists of two pieces B_n^1, B_n^2 , such that

$$B_n^1 = \left\{ x \in D : (1+\alpha)\delta(x) - \widetilde{\delta}_n(x) = 0, \quad \delta(x) \le h - \varepsilon_n \right\},$$

$$B_n^2 = \left\{ x \in D : (1+\alpha)\delta(x) - \widetilde{\delta}_n(x) \ge 0, \quad \delta(x) = h - \varepsilon_n \right\}.$$

It is easily seen, similarly as in the case of $\delta(x)$, that the function $\widetilde{\delta}(x)$ satisfies the Lipschitz condition

$$|\widetilde{\delta}(x) - \widetilde{\delta}(y)| \le |x - y|,$$

since $E(a, \rho)$ is compact. Therefore

$$|\widetilde{\delta}_n(x) - \widetilde{\delta}_n(y)| = \left| \int_{\mathbb{R}^N} (\widetilde{\delta}(x-z) - \widetilde{\delta}(y-z)) \varphi_{t_n}(z) dz \right|$$

$$\leq \int_{\mathbb{R}^N} \left| \widetilde{\delta}(x-z) - \widetilde{\delta}(y-z) \right| \varphi_{t_n}(z) dz \leq \int_{\mathbb{R}^N} |x-y| \varphi_{t_n}(z) dz = |x-y|,$$

and thus

$$\left|\frac{\partial \widetilde{\delta}_n}{\partial x_i}(x)\right| \le 1, \quad i = 1, ..., N.$$

Because h satisfies the condition of Corollary 3.1, for $x \in \Omega(a, h, \rho, \alpha)$ we have

$$\left| \frac{\partial \delta}{\partial x_i}(x) \right| \le 1, \quad i = 1, ..., N.$$

Moreover, by Lemma 3.5 we conclude

$$\frac{\partial \delta}{\partial x_N}(x) = \langle e_N, -\nu_{\pi(x)} \rangle = \langle -\nu_a, -\nu_{\pi(x)} \rangle = \frac{2 - |\nu_a - \nu_{\pi(x)}|^2}{2}$$

$$\geq 1 - \frac{c_0^2}{2}|a - \pi(x)|^2 \geq 1 - \frac{c_0^2}{2}(|a - x| + \delta(x))^2 \geq 1 - \frac{c_0^2}{2}(\widetilde{\delta}(x)(x) + \rho + h)^2$$

$$\geq 1 - \frac{c_0^2}{2}((2+\alpha)h + \rho)^2 \geq 1 - \frac{\alpha}{2(1+\alpha)}.$$

Therefore

$$\frac{\partial}{\partial x_N} \left((1+\alpha)\delta(x) - \widetilde{\delta}_n(x) \right) = (1+\alpha)\frac{\partial \delta}{\partial x_N}(x) - \frac{\partial \widetilde{\delta}_n}{\partial x_N}(x)$$

$$\geq (1+\alpha) \left(1 - \frac{\alpha}{2(1+\alpha)} \right) - 1 = \frac{\alpha}{2}, \quad \forall x \in \Omega(a, h, \rho, \alpha).$$

Hence, if we denote $F_n(x) = (1 + \alpha)\delta(x) - \widetilde{\delta}_n(x)$, then for every $x \in B_n^1$, $F_n(x) = 0$ and

$$\left. \frac{\partial F_n}{\partial x_N}(x) \geq \frac{\alpha}{2}, \quad \left| \frac{\partial F_n}{\partial x_i}(x) \right| \leq \alpha + 2, \quad \forall x \in \Omega(a,h,\rho,\alpha), i = 1,...,N. \right.$$

Denote $x = (\overline{x}, x_N)$, where $\overline{x} \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. By implicit function theorem, for every $x \in B_N^1$ there exist $r_x > 0$, balls $B_N(x, r_x) \subset \mathbb{R}^N$, $B_{N-1}(\overline{x}, r_x) \subset \mathbb{R}^{N-1}$ and a function $g_x \colon B_{N-1}(\overline{x}, r_x) \to \mathbb{R}$ of class C^1 , such that:

- 1. $B_N(x, r_x) \subset \Omega(a, h, \rho, \alpha)$
- 2. $\{(\overline{y}, g_x(\overline{y})) : \overline{y} \in B_{N-1}(\overline{x}, r_x)\} \subset \Omega(a, h, \rho, \alpha)$
- 3. $B_n^1 \cap B_N(x, r_x) \subset \{(\overline{y}, g_x(\overline{y})) : \overline{y} \in B_{N-1}(\overline{x}, r_x)\}$

4. for every $\overline{y} \in B_{N-1}(\overline{x}, r_x), F_n(\overline{y}, g_x(\overline{y})) = 0$ and

$$\frac{\partial g_x}{\partial x_i}(\overline{y}) = -\frac{\frac{\partial F_n}{\partial x_i}(\overline{y}, g_x(\overline{y}))}{\frac{\partial F_n}{\partial x_N}(\overline{y}, g_x(\overline{y}))}.$$

Since B_n^1 is compact, we may choose r > 0, such that for every $x \in B_n^1$, the conditions 1-4 holds with $r_x = r$.

Denote $V_n = \{\overline{x} : x \in B_n^1\}$, $V = \{\overline{x} : x \in \Omega(a, h, \rho, \alpha)\}$. Then V_n is compact, and because of 2, we conclude

$$V_n \subset \bigcup_{x \in V_n} B_{N-1}(\overline{x}, r/3) \subset \Omega(a, h, \rho, \alpha).$$

By Lemma 5.1, we can choose a finite family of disjoint balls $\{B_{N-1}(\overline{x}_j, r/3)\}$, such that

$$V_n \subset \bigcup_j B_{N-1}(\overline{x}_j, r)$$
.

Now observe, that if $x, x' \in \Omega(a, h, \rho, \alpha)$ and $\overline{x} = \overline{x}'$, then, by (small modification of) Lemmas 6.4 and 6.5, for every $\theta \in (0, 1)$, $\theta x + (1 - \theta)x' \in \Omega(a, h, \rho, \alpha)$. Since

$$\frac{\partial F_n}{\partial x_N}(x) \ge \frac{\alpha}{2}, \quad \forall x \in \Omega(a, h, \rho, \alpha),$$

the projection $\Omega(a, h, \rho, \alpha) \cap \{x : F_n(x) = 0\} \ni x \mapsto \overline{x} \in V$ is "one to one", and we have

$$B_n^1 \subset \bigcup_{j \in J} \left\{ (\overline{y}, g_{x_j}(\overline{y})) : \overline{y} \in B_{N-1}(\overline{x}_j, r) \right\}.$$

Therefore

$$\sigma_{n}(B_{n}^{1}) \leq \sum_{j \in J} \int_{B_{N-1}(\overline{x}_{j},r)} \sqrt{1 + \sum_{i=1}^{N-1} \left(\frac{\partial g_{x_{j}}}{\partial x_{i}}(\overline{y})\right)^{2}} d\overline{y}$$

$$\leq \sqrt{1 + (N-1) \left(\frac{2(\alpha+2)}{\alpha}\right)^{2}} \sum_{j \in J} m_{N-1}(B_{N-1}(\overline{x}_{j},r))$$

$$= c_{\alpha} 3^{N-1} \sum_{j \in J} m_{N-1}(B_{N-1}(\overline{x}_{j},r/3)) = c_{\alpha} 3^{N-1} m_{N-1} \left(\bigcup_{j \in J} B_{N-1}(\overline{x}_{j},r/3)\right)$$

$$\leq c_{\alpha} 3^{N-1} m_{N-1}(V) \leq c_{\alpha} 3^{N-1} m_{N-1}(\{\overline{x}: x \in D\}) < \infty,$$

where m_{N-1} denotes N-1 dimensional Lebesque measure. Hence $\sigma_n(B_n^1)$ is bounded by a positive constant, which does not depend on n.

Now B_n^2 is a portion of the set $\{x \in \mathbb{R}^N : \delta(x) = h - \varepsilon_n\}$, moreover

$$|\nabla \delta(x)| \le \sqrt{N}$$

and

$$\left| \frac{\partial \delta}{\partial x_N}(x) \right| \ge 1 - \frac{\alpha}{2(1+\alpha)}$$

for every $x \in B_N^2$. Hence, by very similar arguments, $\sigma_n(B_n^2)$ is uniformly bounded with respect to n.

Proof of Theorem 7.1 part 1. By the use of Lemma 6.8, we may assume, without loss of generality (neglecting a set of arbitrarily small measure), that E is closed and u is bounded in the region

$$\Omega = \bigcup_{y \in E} \Gamma_{\alpha}(y)$$

for every $\alpha > 0$. We may also assume, that u is real valued.

Choose $\alpha > 0$ and let h, ρ be the constants from Lemma 7.1 with respect to α ; assume additionally, that h satisfies the condition of Corollary 3.1. Fix $a \in E$; we shall prove that

$$S^h_{\alpha}u(y) = \int_{\Gamma^h_{\alpha}(y)} |\nabla u(x)|^2 (\delta(x))^{2-N} dx$$

is finite for almost every $y \in E(a, \rho) = E \cap \overline{B}(a, \rho)$. It suffices to show that

$$\int_{E(a,\rho)} S^h_\alpha u(y) d\sigma(y) < \infty.$$

Let $\chi(x, y, \alpha)$ be the characteristic function of $\Gamma_{\alpha}^{h}(y)$. That is, $\chi(x, y, \alpha) = 1$ if $|x - y| < (1 + \alpha)\delta(x)$ and $\delta(x) < h$, otherwise $\chi(x, y, \alpha) = 0$. We have

$$\int_{E(a,\rho)} S_{\alpha}^{h} u(y) d\sigma(y) = \int_{E(a,\rho)} \left(\int_{\Gamma_{\alpha}^{h}(y)} |\nabla u(x)|^{2} (\delta(x))^{2-N} dx \right) d\sigma(y)$$

$$= \int_{E(a,\rho)} \left(\int_{\Omega(a,h,\rho,\alpha)} \chi(x,y,\alpha) |\nabla u(x)|^{2} (\delta(x))^{2-N} dx \right) d\sigma(y)$$

$$= \int_{\Omega(a,h,\rho,\alpha)} \left(\int_{E(a,\rho)} \chi(x,y,\alpha) d\sigma(y) \right) |\nabla u(x)|^2 (\delta(x))^{2-N} dx,$$

where

$$\Omega(a, h, \rho, \alpha) = \bigcup_{y \in E(a, \rho)} \Gamma_{\alpha}^{h}(y).$$

However

$$\int_{E(a,\rho)} \chi(x,y,\alpha) d\sigma(y) \leq \int_{K(\pi(x),(2+\alpha)\delta(x))} d\sigma(y) = \sigma \left\{ K(\pi(x),(2+\alpha)\delta(x)) \right\},$$

and by Lemma 3.3, there exists a positive constant c_{α} , such that

$$\sigma\left\{K(\pi(x), (2+\alpha)\delta(x))\right\} \le c_{\alpha}\delta(x)^{N-1}.$$

Thus it suffices to show that

$$\int_{\Omega(a,h,\rho,\alpha)} \delta(x) |\nabla u(x)|^2 dx < \infty.$$

We shall transform the last integral by Green's theorem. In order to do this we shall use the approximating smooth regions Ω_n discussed in Lemma 7.2. Therefore, by the properties of Ω_n , the last inequality is equivalent with

$$\int_{\Omega_n} \delta(x) |\nabla u(x)|^2 dx \le c < \infty,$$

where the constant c is independent of n. Since the region Ω_n has a sufficiently smooth boundary B_n , we apply to it Green's theorem in the form

$$\int_{B_n} \left(G \frac{\partial F}{\partial \nu_n} - F \frac{\partial G}{\partial \nu_n} \right) d\sigma_n = \int_{\Omega_n} \left(G \Delta F - F \Delta G \right) dx.$$

Here $\partial/\partial\nu_n$ indicates the directional derivative along the outward normal to B_n .

In the above formula we take $F = u^2$, and $G = \delta$. A simple calculation shows that $\Delta(u^2) = 2|\nabla u|^2$, since u is real valued and harmonic. Therefore we obtain

$$2\int_{\Omega_n} \delta(x) |\nabla u(x)|^2 dx = \int_{\Omega_n} u^2(x) \Delta \delta(x) dx$$
$$+ \int_{B_n} \left(\delta(x) \frac{\partial u^2}{\partial \nu_n}(x) - u^2(x) \frac{\partial \delta}{\partial \nu_n}(x) \right) d\sigma_n(x).$$

Now because $|\delta(x) - \delta(y)| \leq |x - y|$, we conclude, as in the proof of Lemma 7.2, that for $x \in \Omega(a, h, \rho, \alpha)$

$$\left| \frac{\partial \delta}{\partial x_k}(x) \right| \le 1, \quad k = 1, ..., N,$$

since h satisfies the condition of Corollary 3.1. Because $\overline{\Omega}_n \subset \Omega(a, h, \rho, \alpha)$, the inequality holds for $x \in \overline{\Omega}_n$. Moreover, by Lemma 3.5, there exists a constant M > 0 such that for every $x, y \in \Omega(a, h, \rho, \alpha)$ we have

$$\left| \frac{\partial \delta}{\partial x_k}(x) - \frac{\partial \delta}{\partial x_k}(y) \right| \le M|x - y|, \quad k = 1, ..., N.$$

Therefore

$$\left| \frac{\partial^2 \delta}{\partial x_k^2}(x) \right| \le M \quad \forall x \in \overline{\Omega}_n, \quad k = 1, ..., N.$$

For $\beta > \alpha$ we have

$$\overline{\Omega}_n \subset \Omega(a, h, \rho, \alpha) \subset \bigcup_{y \in E} \Gamma_{\beta}(y).$$

Hence, there exists $c_1 = c_1(\beta) > 0$, such that $|u| \leq c_1$ on $\overline{\Omega}_n$. Notice also that $\partial u^2/\partial \nu_n = 2u \cdot \partial u/\partial \nu_n$. Thus

$$\left| \delta(x) \frac{\partial u^2}{\partial \nu_n}(x) \right| \le 2|u(x)| \cdot \delta(x) \cdot \left| \frac{\partial u}{\partial \nu_n}(x) \right| \le 2|u(x)| \cdot \delta(x) \cdot |\nabla u(x)|$$

for $x \in \overline{\Omega}_n$. By Lemma 7.1, there exists $c_2 = c_2(\alpha, \beta)$, such that $\delta(x)|\nabla u(x)| \le c_2$ on $\Omega(a, h, \rho, \alpha)$. Therefore we have

$$\left| \int_{\Omega_n} u^2(x) \Delta \delta(x) dx + \int_{B_n} \left(\delta(x) \frac{\partial u^2}{\partial \nu_n}(x) - u^2(x) \frac{\partial \delta}{\partial \nu_n}(x) \right) d\sigma_n(x) \right|$$

References 63

$$\leq \int_{\Omega_n} |u(x)|^2 |\Delta \delta(x)| dx + \int_{B_n} \left(\left| \delta(x) \frac{\partial u^2}{\partial \nu_n}(x) \right| + |u(x)|^2 \left| \frac{\partial \delta}{\partial \nu_n}(x) \right| \right) d\sigma_n(x)
\leq c_1^2 \cdot NM \cdot m_N(\Omega_n) + 2c_1c_2\sigma_n(B_n) + c_1^2 \int_{B_n} |\nabla \delta(x)| d\sigma_n(x)
\leq c_1^2 \cdot NM \cdot m_N(D) + \left(2c_1c_2 + Nc_1^2 \right) \sigma_n(B_n),$$

where m_N is N dimensional Lebesque measure. By Lemma 7.2, $\sigma_n(B_n)$ is uniformly bounded, so the proof is complete.

References

[1] S. AXLER, P. BOURDON, AND W. RAMEY, *Harmonic Function Theory*, Springer-Verlag, New York, Berlin, **1992**

- [2] H. AIKAWA, T. KILPELÄINEN, N. SHANMUGALINGAM, AND X. ZHONG, Boundary Harnack Principle for p-harmonic Functions in Smooth Euclidean Domains, Potential Analysis 26 (2006), 281-301.
- [3] S.G. Krantz, Function theory of several complex variables, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, Singapore 1982
- [4] M. Spivak, Calculus on Manifolds, Benjamin, New York, 1965.
- [5] E.M. Stein, Boundary behavior of holomorphic functions of several complex variables, Princeton University Press and University of Tokyo Press, Princeton, New Jersey, 1972.
- [6] E.M. STEIN, On the theory of harmonic functions of several variables. II.Behavior near the boundary, Acta Math. 106 (1961), 137-174.
- [7] K.O. WIDMAN, On the boundary behavior of solutions to a class of elliptic partial differential equations, Ark. Mat. 6 (1966), 485-533.