Damped quantum harmonic oscillator

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ABSTRACT

In the framework of the Lindblad theory for open quantum systems the damping of the harmonic oscillator is studied. A generalization of the fundamental constraints on quantum mechanical diffusion coefficients which appear in the master equation for the damped quantum oscillator is presented; the Schrödinger and Heisenberg representations of the Lindblad equation are given explicitly. On the basis of these representations it is shown that various master equations for the damped quantum oscillator used in the literature are particular cases of the Lindblad equation and that the majority of these equations are not satisfying the constraints on quantum mechanical diffusion coefficients. Analytical expressions for the first two moments of coordinate and momentum are also obtained by using the characteristic function of the Lindblad master equation. The master equation is transformed into Fokker-Planck equations for quasiprobability distributions. A comparative study is made for the Glauber P representation, the antinormal ordering Q representation and the Wigner W representation. It is proven that the variances for the damped harmonic oscillator found with these representations are the same. By solving the Fokker-Planck equations in the steady state, it is shown that the quasiprobability distributions are two-dimensional Gaussians with widths determined by the diffusion coefficients. The density matrix is represented via a generating function, which is obtained by solving a time-dependent linear partial differential equation derived from the master equation. Illustrative examples for specific initial conditions of the density matrix are provided.

1. Introduction

In the last two decades, more and more interest arose about the problem of dissipation in quantum mechanics, i.e. the consistent description of open quantum systems [1-4]. The quantum description of dissipation is important in many different areas of physics. In quantum optics we mention the quantum theory of lasers and photon detection. There are some directions in the theory of atomic nucleus in which dissipative processes play a basic role. In this sense we mention the nuclear fission, giant resonances and deep inelastic collisions of heavy ions. Dissipative processes often occur also in many body or field-theoretical systems.

The irreversible, dissipative behaviour of the vast majority of physical phenomena comes into an evident contradiction with the reversible nature of our basic models. The very restrictive principles of conservative and isolated systems are unable to deal with more complicated situations which are determined by the features of open systems.

The fundamental quantum dynamical laws are of the reversible type. The dynamics of

a closed system is governed by the Hamiltonian that represents its total energy and which is a constant of motion. In this way the paradox of irreversibility arises: the reversibility of microscopic dynamics contrasting with the irreversibility of the macroscopic behaviour we are trying to deduce from it.

One way to solve this paradox of irreversibility is to use models to which Hamiltonian dynamics and Liouville's theorem do not apply, but the irreversible behaviour is clearly present even in the microscopic dynamical description. The reason for replacing Hamiltonian dynamics and Liouville's theorem is that no system is truly isolated, being subject to uncontrollable random influences from outside. For this reason these models are called open systems. There are two ways of treating quantitatively their interaction with the outside. One is to introduce specific stochastic assumptions to simulate this interaction, the other is to treat them according to the usual laws of dynamics, by regarding the open system as a subsystem of a larger system which is closed (i.e. which obeys the usual laws of dynamics, with a well-defined Hamiltonian). The dissipation arises in general from the subsystem interactions with this larger system, often referred to as the reservoir or bath. The first of these two approaches has been used for the study of steady-state transport processes, in systems obeying classical mechanics. The second of the two approaches has been mainly used in quantum mechanics. The main general result [1, 5-7] is that under certain conditions the time evolution of an open system can be described by a dynamical semigroup $\Phi_t(t > 0)$. For a closed finite system with Liouville operator the evolution operator is not restricted to nonnegative t. The importance of the dynamical semigroup concept is that it generalizes the evolution operator to open systems, for which there is no proper Liouville operator and no Φ_t for negative t. The mathematical theory of dynamical semigroups has been developed in [1, 8-12].

The quantum mechanics of the unidimensional damped harmonic oscillator represents a fundamental theoretical problem with applications in different domains of quantum optics, solid state physics, molecular and nuclear physics. In the present paper the quantum harmonic oscillator is treated in the Lindblad axiomatic formalism of quantum dynamical semigroups.

In Sect.2 the notion of the quantum dynamical semigroup is defined using the concept of a completely positive map [10]. The Lindblad formalism replaces the dynamical group uniquely determined by its generator, which is the Hamiltonian operator of the system, by the completely positive dynamical semigroup with bounded generators. Then the general form of Markovian quantum mechanical master equation is given.

In Sect.3 we give the fundamental constraints on quantum mechanical diffusion coefficients which appear in the corresponding master equations [17]. The Schrödinger and Heisenberg representations of the Lindblad equation are given explicitly. On the basis of these representations it is shown that various master equations for the damped quantum oscillator used in the literature for the description of the damped collective modes in deep inelastic collisions or in quantum optics are particular cases of the Lindblad equation and that the majority of these equations are not satisfying the constraints on quantum mechanical diffusion coefficients. Explicit expressions of the mean values and variances are also given [17,18].

In Sect.4 we solve the master equation with the characteristic function [19]. This

function is found as a solution of a corresponding partial differential equation. By this method one can derive explicit formulae for the centroids and variances and, in general, for moments of any order.

In Sect.5 we explore the applicability of quasiprobability distributions to the Lindblad theory [22]. The methods of quasiprobabilities have provided technical tools of great power for the statistical description of microscopic systems formulated in terms of the density operator [58]. The first quasiprobability distribution was the one introduced by Wigner [43] in a quantum-mechanical context. In quantum optics the P representation, introduced by Glauber [44,45,50], provided many practical applications of quasiprobabilities. The development of quantum-mechanical master equations was combined with the Glauber P representation to give a Fokker-Planck equation for the laser [47,48]. The master equation of the one-dimensional damped harmonic oscillator is transformed into Fokker-Planck equations for the Glauber P, antinormal Q and Wigner W quasiprobability distributions associated with the density operator. The resulting equations are solved by standard methods and observables directly calculated as correlations of these distribution functions. We solve also the Fokker-Planck equations for the steady state and show that variances found from the P, Q and W distributions are the same [22].

In Sect.6 we study the time evolution of the density matrix that follows from the master equation of the damped harmonic oscillator [20,21]. We calculate the physically relevant solutions of the master equation by applying the method of generating function. This means that we represent the density matrix with a generating function which is the solution of a time-dependent partial differential equation of second order, derived from the master equation of the damped harmonic oscillator. We discuss stationary solutions of the generating function and derive the Bose-Einstein density matrix as example. Then, formulas for the time development of the density matrix are presented and illustrative examples for specific initial conditions provided. The same method of generating function was already used by Jang [39] who studied the damping of a collective degree of freedom coupled to a Bosonic reservoir at finite temperature with a second order RPA master equation in the collective subspace.

The conclusions are given in Sect.7.

2. Lindblad theory for open quantum systems

The standard quantum mechanics is Hamiltonian. The time evolution of a closed physical system is given by a dynamical group U_t which is uniquely determined by its generator H, which is the Hamiltonian operator of the system. The action of the dynamical group U_t on any density matrix ρ from the set $\mathbf{D}(\mathbf{H})$ of all density matrices of the quantum system, whose corresponding Hilbert space is \mathbf{H} , is defined by

$$\rho(t) = U_t(\rho) = e^{-\frac{i}{\hbar}Ht}\rho e^{\frac{i}{\hbar}Ht}$$

for all $t \in (-\infty, \infty)$. We remind that, according to von Neumann, density operators $\rho \in \mathbf{D}(\mathbf{H})$ are trace class $(Tr\rho < \infty)$, self-adjoint $(\rho^+ = \rho)$, positive $(\rho > 0)$ operators with $Tr\rho = 1$. All these properties are conserved by the time evolution defined by U_t .

In the case of open quantum systems the main dificulty consists of finding such time evolutions Φ_t for density operators $\rho(t) = \Phi_t(\rho)$ which preserve these von Neumann conditions for all times. From this requirement it follows that Φ_t must have the following properties:

- (i) $\Phi_t(\lambda_1\rho_1 + \lambda_2\rho_2) = \lambda_1\Phi_t(\rho_1) + \lambda_2\Phi_t(\rho_2); \lambda_1, \lambda_2 \ge 0 \text{ with } \lambda_1 + \lambda_2 = 1,$
- $(ii) \Phi_t(\rho^+) = \Phi_t(\rho)^+,$
- (iii) $\Phi_t(\rho) > 0$,
- $(iv) Tr\Phi_t(\rho) = 1.$

But these conditions are not restrictive enough in order to give a complete description of the mappings Φ_t as in the case of the time evolutions U_t for closed systems. Even in the last case one has to impose other restrictions to U_t , namely, it must be a group $U_{t+s} = U_t U_s$. Also, it is evident that in this case $U_0(\rho) = \rho$ and $U_t(\rho) \to \rho$ in the trace norm when $t \to 0$. For the dual group \widetilde{U}_t acting on the observables $A \in \mathbf{B}(\mathbf{H})$, i.e. on the bounded operators on \mathbf{H} ,

$$\widetilde{U}_t(A) = e^{\frac{i}{\hbar}Ht}Ae^{-\frac{i}{\hbar}Ht}.$$

Then $\widetilde{U}_t(AB) = \widetilde{U}_t(A)\widetilde{U}_t(B)$ and $\widetilde{U}_t(I) = I$, where I denotes the identity operator on \mathbf{H} . Also, $\widetilde{U}_t(A) \to A$ ultraweakely when $t \to 0$ and \widetilde{U}_t is an ultraweakely continuous mapping [1,7,9,10,13]. These mappings have a strong positivity property called complete positivity:

$$\sum_{i,j} B_i^+ \widetilde{U}_t(A_i^+ A_j) B_j \ge 0, A_i, B_i \in \mathbf{B}(\mathbf{H}).$$

Because the detailed physically plausible conditions on the systems, which correspond to these properties are not known, it is much more convenient to adopt an axiomatic point of view which is based mainly on the simplicity and the succes of physical applications. Accordingly [1,7,9,10,13] it is convenient to suppose that the time evolutions Φ_t for open systems are not very different from the time evolutions for closed systems. The simplest dynamics Φ_t which introduces a preferred direction in time, which is characteristic for dissipative processes, is that in which the group condition is replaced by the semigroup condition [6,7,11,12,14]

$$\Phi_{t+s} = \Phi_t \Phi_s, \ t, s > 0.$$

The duality condition

$$Tr(\Phi_t(\rho)A) = Tr(\rho\widetilde{\Phi}_t(A))$$
 (2.1)

defines $\widetilde{\Phi}_t$, the dual of Φ_t acting on $\mathbf{B}(\mathbf{H})$. Then the conditions

$$Tr\Phi_t(\rho) = 1$$

and

$$\widetilde{\Phi}_t(I) = I \tag{2.2}$$

are equivalent. Also the conditions

$$\widetilde{\Phi}_t(A) \to A$$
 (2.3)

ultraweakely when $t \to 0$ and

$$\Phi_t(\rho) \to \rho$$

in the trace norm when $t \to 0$, are equivalent. For the semigroups with the properties (2.2), (2.3) and

$$A \ge 0 \to \widetilde{\Phi}_t(A) \ge 0$$
,

it is well known that there exists a (generally bounded) mapping \widetilde{L} -the generator of $\widetilde{\Phi}_t$. $\widetilde{\Phi}_t$ is uniquely determined by \widetilde{L} . The dual generator of the dual semigroup Φ_t is denoted by L:

$$Tr(L(\rho)A) = Tr(\rho \widetilde{L}(A)).$$

The evolution equations by which $L(\widetilde{L})$ determine uniquely $\Phi_t(\widetilde{\Phi}_t)$ are

$$\frac{d\Phi_t(\rho)}{dt} = L(\Phi_t(\rho)) \tag{2.4}$$

and

$$\frac{d\widetilde{\Phi}_t(A)]}{dt} = \widetilde{L}(\widetilde{\Phi}_t(A)), \tag{2.5}$$

respectively, in the Schrödinger and Heisenberg picture. These equations replace in the case of open systems the von-Neumann-Liouville equations

$$\frac{dU_t(\rho)}{dt} = -\frac{i}{\hbar}[H, U_t(\rho)]$$

and

$$\frac{d\widetilde{U}_t(A)}{dt} = \frac{i}{\hbar}[H, \widetilde{U}_t(A)],$$

respectively.

For any applications eqs.(2.4) and (2.5) are only useful if the detailed structure of the generator $L(\widetilde{L})$ is known and can be related to the concrete properties of the open systems, which are described by such equations.

Such a structural theorem was obtained by Lindblad [10] for the class of dynamical semigroups $\widetilde{\Phi}_t$ which are completely positive and norm continuous. For such semigroups the generator \widetilde{L} is bounded. In many applications the generator is unbounded.

A bounded mapping, $\widetilde{L}: \mathbf{B}(\mathbf{H}) \to \mathbf{B}(\mathbf{H})$ which satisfies $\widetilde{L}(I) = 0$, $\widetilde{L}(A^+) = \widetilde{L}(A)^+$ and

$$\widetilde{L}(A^+A) - \widetilde{L}(A^+)A - A^+\widetilde{L}(A) \ge 0$$

is called dissipative. The 2-positivity property of the completely positive mapping $\widetilde{\Phi}_t$:

$$\widetilde{\Phi}_t(A^+A) \ge \widetilde{\Phi}_t(A^+)\widetilde{\Phi}_t(A),$$

with equality at t=0, implies that \widetilde{L} is dissipative. Lindblad [10] has shown that conversely, the dissipativity of \widetilde{L} implies that $\widetilde{\Phi}_t$ is 2-positive. \widetilde{L} is called completely dissipative

if all trivial extensions of \widetilde{L} are dissipative. Lindblad has also shown that there exists a one-to-one correspondence between the completely positive norm continuous semigroups $\widetilde{\Phi}_t$ and completely dissipative generators \widetilde{L} . The structural theorem of Lindblad gives the most general form of a completely dissipative mapping \tilde{L} [10,11]:

Theorem: \hat{L} is completely dissipative and ultraweakely continuous if and only if it is of the form

$$\widetilde{L}(A) = \frac{i}{\hbar}[H, A] + \frac{1}{2\hbar} \sum_{i} (V_j^+[A, V_j] + [V_j^+, A]V_j), \tag{2.6}$$

where $V_j, \sum_j V_j^+ V_j \in \mathbf{B}(\mathbf{H}), \ H \in \mathbf{B}(\mathbf{H})_{s.a.}$. The dual generator on the state space (Schrödinger picture) is of the form

$$L(\rho) = -\frac{i}{\hbar}[H, \rho] + \frac{1}{2\hbar} \sum_{j} ([V_{j}\rho, V_{j}^{+}] + [V_{j}, \rho V_{j}^{+}]). \tag{2.7}$$

Eqs. (2.4) and (2.7) give an explicit form for the most general time-homogeneous quantum mechanical Markovian master equation with a bounded Liouville operator.

Talkner [16] has shown that the assumption of a semigroup dynamics is only applicable in the limit of weak coupling of the subsystem with its environment, i.e. for long relaxation times.

We should like to mention that all Markovian master equations found in the literature are of this form after some rearrangement of terms, even for unbounded generators.

It is also an empirical fact that for many physically interesting situations the time evolutions Φ_t drive the system toward a unique final state $\rho(\infty) = \lim_{t\to\infty} \Phi_t(\rho(0))$ for all $\rho(0) \in \mathbf{D}(\mathbf{H}).$

The evolution equations of Lindblad, being operator equations, the problem of finding their solutions is, in general, rather difficult. In cases when the equations are exactly solvable, these solutions give complete informations about the studied problem - they permit the calculation of expectation values of the observables at any moment.

3. Master equations for damped quantum harmonic oscillator

In this Section the case of damped quantum harmonic oscillator is considered in the spirit of the ideas presented in the previous Section. The basic assumption is that the general form (2.7) of a bounded mapping L given by Lindblad theorem [10] is also valid for an unbounded completely dissipative mapping L:

$$L(\rho) = -\frac{i}{\hbar}[H, \rho] + \frac{1}{2\hbar} \sum_{j} ([V_{j}\rho, V_{j}^{+}] + [V_{j}, \rho V_{j}^{+}]). \tag{3.1}$$

This assumption gives one of the simplest way to construct an appropriate model for this quantum dissipative system. Another simple condition imposed to the operators H, V_i, V_i^+ is that they are functions of the basic observables of the one-dimensional quantum mechanical system q and p with $[q,p]=i\hbar I$, where I is the identity operator on **H** of such kind that the obtained model is exactly solvable. A precise version for this last condition is that linear spaces spanned by the first degree (respectively second degree) noncommutative polynomials in p and q are invariant to the action of the completely dissipative mapping L. This condition implies [15] that V_j are at most the first degree polynomials in p and q and H is at most a second degree polynomial in p and q.

Because in the linear space of the first degree polynomials in p and q the operators p and q give a basis, there exist only two C-linear independent operators V_1, V_2 which can be written in the form

$$V_i = a_i p + b_i q, \ i = 1, 2$$

with $a_i, b_i = 1, 2$ complex numbers [15]. The constant term is omitted because its contribution to the generator L is equivalent to terms in H linear in p and q which for simplicity are chosen to be zero. Then H is chosen of the form

$$H = H_0 + \frac{\mu}{2}(pq + qp), \quad H_0 = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2.$$
 (3.2)

With these choices the Markovian master equation can be written:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H_0, \rho] - \frac{i}{2\hbar} (\lambda + \mu) [q, \rho p + p \rho] + \frac{i}{2\hbar} (\lambda - \mu) [p, \rho q + q \rho] -
- \frac{D_{pp}}{\hbar^2} [q, [q, \rho]] - \frac{D_{qq}}{\hbar^2} [p, [p, \rho]] + \frac{D_{pq}}{\hbar^2} ([q, [p, \rho]] + [p, [q, \rho]]).$$
(3.3)

Here we used the notations:

$$D_{qq} = \frac{\hbar}{2} \sum_{j=1,2} |a_j|^2, D_{pp} = \frac{\hbar}{2} \sum_{j=1,2} |b_j|^2, D_{pq} = D_{qp} = -\frac{\hbar}{2} Re \sum_{j=1,2} a_j^* b_j, \lambda = -Im \sum_{j=1,2} a_j^* b_j, \lambda = -Im$$

where D_{pp} , D_{qq} and D_{pq} are the diffusion coefficients and λ the friction constant. They satisfy the following fundamental constraints as shown in [17]:

i)
$$D_{pp} > 0$$

ii) $D_{qq} > 0$
iii) $D_{pp}D_{qq} - D_{pq}^{2} \ge \lambda^{2}\hbar^{2}/4$. (3.4)

Introducing the annihilation and creation operators

$$a = \frac{1}{\sqrt{2\hbar}} (\sqrt{m\omega}q + \frac{i}{\sqrt{m\omega}}p),$$

$$a^{+} = \frac{1}{\sqrt{2\hbar}} (\sqrt{m\omega}q - \frac{i}{\sqrt{m\omega}}p),$$
(3.5)

obeying the commutation relation $[a, a^+] = 1$, we have

$$H_0 = \hbar\omega(a^+ a + \frac{1}{2}) \tag{3.6}$$

and the master equation has the form

$$\frac{d\rho}{dt} = \frac{1}{2}(D_1 - \mu)(\rho a^+ a^+ - a^+ \rho a^+) + \frac{1}{2}(D_1 + \mu)(a^+ a^+ \rho - a^+ \rho a^+) +$$

$$+\frac{1}{2}(D_2 - \lambda - i\omega)(a^+\rho a - \rho a a^+) + \frac{1}{2}(D_2 + \lambda + i\omega)(a\rho a^+ - a^+a\rho) + h.c.,$$
 (3.7)

where

$$D_{1} = \frac{1}{\hbar} (m\omega D_{qq} - \frac{D_{pp}}{m\omega} + 2iD_{pq}),$$

$$D_{2} = \frac{1}{\hbar} (m\omega D_{qq} + \frac{D_{pp}}{m\omega}).$$
(3.8)

In the literature, equations of this kind are encountered in concrete theoretical models for the description of different physical phenomena in quantum optics, the damping of collective modes in deep inelastic collisions of heavy ions or in the quantum mechanical description of the dissipation for the one-dimensional harmonic oscillator. In the following we show that all these master equations are particular cases of the Lindblad equation and that the majority of these equations are not satisfying the constraints on quantum mechanical diffusion coefficients, and therefore the uncertainty principle is violated.

- 1) The Dekker master equation for the damped quantum harmonic oscillator [4,23-26] supplemented with the fundamental constraints (3.4) obtained in [23] from the condition that the time evolution of this master equation does not violate the uncertainty principle at any time, is a particular case of the Lindblad master equation (3.7) when $\mu = \lambda$.
- 2) The quantum master equation considered in [27,28] by Hofmann et al. for treating the charge equilibration process as a collective high frequency mode is a particular case of the Lindblad master equation (3.3) if $\lambda = \gamma(\omega)/2m = \mu$, $D_{qq} = 0$, $D_{pp} = \gamma(\omega)T^*(\omega)$, $D_{pq} = 0$, but the fundamental constraints (3.4) are not satisfied.
- 3) For the quantum master equation considered in [29] for the description of heavy ion collisions we have $\lambda = \mu = \gamma/2$, $D_{pp} = D$, $D_{qq} = 0$, $D_{pq} = D_{qp} = -d/2$ and consequently the fundamental constraints are not fulfilled.
- 4) In [30], Spina and Weidenmüller considered two kinds of master equations I and II for describing the damping of collective modes in deep inelastic collisions of heavy ions. Eq.I can be obtained from eq.(3.3) by replacing H_0 with $H_0 \frac{1}{2}Am\omega q^2 + f(t)q$ and setting $\lambda = \mu = \Gamma/2$, $D_{pp} = D/2$, $D_{qq} = 0$ and $D_{pq} = D_{qp} = B/2$. Then the constraints (3.4) are not satisfied. Eq.II is obtained from (3.3) by putting $H_0 (1/2)A^{II}m\omega q^2 (1/2m\omega)A^{II}p^2 + f(t)q$ and $\Gamma_R^{II} = \Gamma_p^{II} = \lambda$, $\mu = 0$, $D_{pp} = D_p^{II}/2$, $D_{qq} = D_R^{II}/2$, $D_{pq} = 0$ and the last condition (3.4) is satisfied for all values of the parameters.
- 5) The master equation for the density operator of the electromagnetic field mode coupled to a squeezed bath [31,32] can be obtained from the master equation (3.7) if we set

$$\mu = 0, \lambda = \gamma, \frac{1}{2\lambda} \left(\lambda - \frac{m\omega D_{qq}}{\hbar} - \frac{D_{pp}}{\hbar m\omega}\right) = -N, \frac{1}{2\lambda} \left(\frac{m\omega D_{qq}}{\hbar} - \frac{D_{pp}}{\hbar m\omega} + 2i\frac{D_{pq}}{\hbar}\right) = M.$$

6) The master equation for the density operator of a harmonic oscillator coupled to an environment of harmonic oscillators considered in [33-36] is a particular case of the master equation (3.7) if we put

$$\lambda = \mu = \gamma, D_{qq} = 0, D_{pq} = 0, \frac{1}{2\lambda} (\lambda - \frac{D_{pp}}{\hbar m\omega}) = -\bar{n}$$

and the fundamental constraints (3.4) are not fulfilled.

7) The master equation written in [37] for different models of correlated- emission lasers can also be obtained from the master equation (3.7) by putting

$$\frac{1}{2}(D_1 + \mu) = \Lambda_4, \frac{1}{2}(D_1 - \mu) = \lambda_3, \frac{1}{2}(D_2 + \lambda + i\omega) = \Lambda_2, \frac{1}{2}(D_2 - \lambda - i\omega) = \Lambda_1.$$

8) Two master equations were introduced by Jang in [38,39], where the nuclear dissipative pocess is described as the damping of a collective degree of freedom coupled to a bosonic reservoir at finite temperature. The resulting RPA master equation within the observed collective subspace is derived in a purely dynamical way. The master equation written in [38] in the resonant approximation (rotating-wave approximation) can be obtained as a particular case of the Lindblad master equation (3.7). For this one has to set

$$D_{pp} = m^2 \omega^2 D_{qq}, D_{pq} = \mu = 0, \frac{4m\omega D_{qq}}{\hbar} = (2 < n > +1)\Gamma, \lambda = \frac{\Gamma}{2},$$

where $\langle n \rangle$ is the average number of the RPA collective phonons at thermal equilibrium and Γ is the width (friction parameter). The fundamental constraints (3.4) are fulfilled in this case.

The master equation derived recently [39] in order to extend the calculations carried out in [33] with the before-mentioned master equation in the resonant approximation, can also be obtained as a particular case of the master equation (3.7) by taking

$$D_{qq} = D_{pq} = 0, D_{pp} = \frac{\hbar m\omega}{2} (2 < n > +1)\Gamma, \mu = \lambda = \frac{\Gamma}{2}$$

or $D_2 = D_1 = (2 < n > +1)\Gamma/2$ and in this case the fundamental constraints (3.4) are not fulfilled.

The following notations will be used:

$$\sigma_{q}(t) = Tr(\rho(t)q),$$

$$\sigma_{p}(t) = Tr(\rho(t)p),$$

$$\sigma_{qq} = Tr(\rho(t)q^{2}) - \sigma_{q}^{2}(t),$$

$$\sigma_{pp} = Tr(\rho(t)p^{2}) - \sigma_{p}^{2}(t),$$

$$\sigma_{pq}(t) = Tr(\rho(t)\frac{pq + qp}{2}) - \sigma_{p}(t)\sigma_{q}(t).$$
(3.9)

In the Heisenberg picture the master equation has the following symmetric form:

$$\begin{split} \frac{d\widetilde{\Phi}_t(A)}{dt} &= \widetilde{L}(\widetilde{\Phi}_t(A)) = \frac{i}{\hbar}[H_0, \widetilde{\Phi}_t(A)] - \frac{i}{2\hbar}(\lambda + \mu)([\widetilde{\Phi}_t(A), q]p + p[\widetilde{\Phi}_t(A), q]) + \\ &+ \frac{i}{2\hbar}(\lambda - \mu)(q[\widetilde{\Phi}_t(A), p] + [\widetilde{\Phi}_t(A), p]q) - \frac{D_{pp}}{\hbar^2}[q, [q, \widetilde{\Phi}_t(A)]] - \end{split}$$

$$-\frac{D_{qq}}{\hbar^{2}}[p,[p,\widetilde{\Phi}_{t}(A)]] + \frac{D_{pq}}{\hbar^{2}}([p,[q,\widetilde{\Phi}_{t}(A)]] + [q,[p,\widetilde{\Phi}_{t}(A)]]).$$

Denoting by A any selfadjoint operator we have

$$\sigma_A(t) = Tr(\rho(t)A), \sigma_{AA}(t) = Tr(\rho(t)A^2) - \sigma_A^2(t).$$

It follows that

$$\frac{d\sigma_A(t)}{dt} = TrL(\rho(t))A = Tr\rho(t)\widetilde{L}(A)$$
(3.10)

and

$$\frac{d\sigma_{AA}(t)}{dt} = TrL(\rho(t))A^2 - 2\frac{d\sigma_A(t)}{dt}\sigma_A(t) = Tr\rho(t)\widetilde{L}(A^2) - 2\sigma_A(t)Tr\rho(t)\widetilde{L}(A).$$
 (3.11)

An important consequence of the precise version of solvability condition formulated at the beginning of the present Section is the fact that when A is put equal to p or q in (3.10) and (3.11), then $d\sigma_p(t)/dt$ and $d\sigma_q(t)/dt$ are functions only of $\sigma_p(t)$ and $\sigma_q(t)$ and $d\sigma_{pp}(t)/dt$, $d\sigma_{qq}(t)/dt$ and $d\sigma_{pq}(t)/dt$ are functions only of $\sigma_{pp}(t)$, $\sigma_{qq}(t)$ and $\sigma_{pq}(t)$. This fact allows an immediate determination of the functions of time $\sigma_p(t)$, $\sigma_q(t)$, $\sigma_{pp}(t)$, $\sigma_{qq}(t)$, $\sigma_{pq}(t)$. The results are the following:

$$\frac{d\sigma_q(t)}{dt} = -(\lambda - \mu)\sigma_q(t) + \frac{1}{m}\sigma_p(t),$$

$$\frac{d\sigma_p(t)}{dt} = -m\omega^2\sigma_q(t) - (\lambda + \mu)\sigma_p(t) \tag{3.12}$$

and

$$\frac{d\sigma_{qq}(t)}{dt} = -2(\lambda - \mu)\sigma_{qq}(t) + \frac{2}{m}\sigma_{pq}(t) + 2D_{qq},$$

$$\frac{d\sigma_{pp}}{dt} = -2(\lambda + \mu)\sigma_{pp}(t) - 2m\omega^{2}\sigma_{pq}(t) + 2D_{pp},$$

$$\frac{d\sigma_{pq}(t)}{dt} = -m\omega^{2}\sigma_{qq}(t) + \frac{1}{m}\sigma_{pp}(t) - 2\lambda\sigma_{pq}(t) + 2D_{pq}.$$
(3.13)

All equations considered in various papers in connection with damping of collective modes in deep inelastic collisions are obtained as particular cases of Eqs. (3.13), as we already mentioned before.

The integration of Eqs.(3.12) is straightforward. There are two cases: $a) \mu > \omega$ (overdamped) and $b) \mu < \omega$ (underdamped). If S(t) denotes the vector

$$\begin{pmatrix} \sigma_q(t) \\ \sigma_p(t) \end{pmatrix}$$

and M the 2×2 matrix

$$M = \begin{pmatrix} -(\lambda - \mu) & 1/m \\ -m\omega^2 & -(\lambda + \mu) \end{pmatrix},$$

then (3.12) becomes

$$\frac{dS(t)}{dt} = MS(t). (3.14)$$

Now M can be written as $M = N^{-1}FN$ with F a diagonal matrix. It follows that the solution of (3.14) is

$$S(t) = N^{-1}e^{Ft}NS(0).$$

In the case a) with the notation $\nu^2 = \mu^2 - \omega^2$ the matrices N, N^{-1} and F are given by

$$N = \begin{pmatrix} m\omega^2 & \mu + \nu \\ m\omega^2 & \mu - \nu \end{pmatrix},$$

$$N^{-1} = \frac{1}{2m\omega^2\nu} \begin{pmatrix} -(\mu - \nu) & \mu + \nu \\ m\omega^2 & -m\omega^2 \end{pmatrix},$$

$$F = \begin{pmatrix} -(\lambda + \nu) & 0 \\ 0 & -(\lambda - \nu) \end{pmatrix}.$$

Then

$$N^{-1}e^{Ft}N = e^{-\lambda t} \begin{pmatrix} \cosh \nu t + \frac{\mu}{\nu} \sinh \nu t & \frac{1}{m\nu} \sinh \nu t \\ -\frac{m\omega^2}{\nu} \sinh \nu t & \cosh \nu t - \frac{\mu}{\nu} \sinh \nu t \end{pmatrix},$$

i.e.,

$$\sigma_q(t) = e^{-\lambda t} \left((\cosh \nu t + \frac{\mu}{\nu} \sinh \nu t) \sigma_q(0) + \frac{1}{m\nu} \sinh \nu t \sigma_p(0) \right),$$

$$\sigma_p(t) = e^{-\lambda t} \left(-\frac{m\omega^2}{\nu} \sinh \nu t \sigma_q(0) + (\cosh \nu t - \frac{\mu}{\nu} \sinh \nu t) \sigma_p(0) \right). \tag{3.15}$$

If $\lambda > \nu$, then $\sigma_q(\infty) = \sigma_p(\infty) = 0$. If $\lambda < \nu$, then $\sigma_q(\infty) = \sigma_p(\infty) = \infty$. In the case b) with the notation $\Omega^2 = \omega^2 - \mu^2$, the matrices N, N^{-1} and F are given by

$$N = \begin{pmatrix} m\omega^2 & \mu + i\Omega \\ m\omega^2 & \mu - i\Omega \end{pmatrix},$$

$$N^{-1} = \frac{1}{2im\omega^2\Omega} \begin{pmatrix} -(\mu - i\Omega) & \mu + i\Omega \\ m\omega^2 & -m\omega^2 \end{pmatrix},$$

$$F = \begin{pmatrix} -(\lambda + i\Omega) & 0 \\ 0 & -(\lambda - i\Omega) \end{pmatrix}.$$

Then

$$N^{-1}e^{Ft}N = e^{-\lambda t} \begin{pmatrix} \cos\Omega t + \frac{\mu}{\Omega}\Omega t & \frac{1}{m\Omega}\sin\Omega t \\ -\frac{m\omega^2}{\Omega}\sin\Omega t & \cos\Omega t - \frac{\mu}{\Omega}\sin\Omega t \end{pmatrix},$$

i.e.,

$$\sigma_q(t) = e^{-\lambda t} \left((\cos \Omega t + \frac{\mu}{\Omega} \sin \Omega t) \sigma_q(0) + \frac{1}{m\Omega} \sin \Omega t \sigma_p(0) \right),$$

$$\sigma_p(t) = e^{-\lambda t} \left(-\frac{m\omega^2}{\Omega} \sin \Omega t \sigma_q(0) + (\cos \Omega t - \frac{\mu}{\Omega} \sin \Omega t) \sigma_p(0) \right)$$
(3.16)

and $\sigma_q(\infty) = \sigma_p(\infty) = 0$.

In order to integrate Eqs. (3.13), it is convenient to consider the vector

$$X(t) = \begin{pmatrix} m\omega\sigma_{qq}(t) \\ \frac{1}{m\omega}\sigma_{pp}(t) \\ \sigma_{pq}(t) \end{pmatrix}.$$

Then the system of equations (3.13) can be written in the form

$$\frac{dX(t)}{dt} = RX(t) + D,$$

where R is the following 3×3 matrix

$$R = \begin{pmatrix} -2(\lambda - \mu) & 0 & 2\omega \\ 0 & -2(\lambda + \mu) & -2\omega \\ -\omega & \omega & -2\lambda \end{pmatrix}$$

and D is the following vector

$$D = \begin{pmatrix} 2m\omega D_{qq} \\ \frac{2}{m\omega} D_{pp} \\ 2D_{pq} \end{pmatrix}.$$

Then there exists a matrix T with property $T^2 = I$ where I is the identity matrix and a diagonal matrix K such that R = TKT. From this it follows that

$$X(t) = (Te^{Kt}T)X(0) + T(e^{Kt} - I)K^{-1}TD.$$
(3.17)

In the overdamped case $(\mu > \omega)$ the matrices T and K are given by

$$T = \frac{1}{2\nu} \begin{pmatrix} \mu + \nu & \mu - \nu & 2\omega \\ \mu - \nu & \mu + \nu & 2\omega \\ -\omega & -\omega & -2\mu \end{pmatrix}$$

and

$$K = \begin{pmatrix} -2(\lambda - \nu) & 0 & 0\\ 0 & -2(\lambda + \nu) & 0\\ 0 & 0 & -2\lambda \end{pmatrix}$$

with $\nu^2 = \mu^2 - \omega^2$.

In the underdamped case $(\mu < \omega)$ the matrices T and K are given by

$$T = \frac{1}{2i\Omega} \begin{pmatrix} \mu + i\Omega & \mu - i\Omega & 2\omega \\ \mu - i\Omega & \mu + i\Omega & 2\omega \\ -\omega & -\omega & -2\mu \end{pmatrix}$$

and

$$K = \begin{pmatrix} -2(\lambda - i\Omega) & 0 & 0\\ 0 & -2(\lambda + i\Omega) & 0\\ 0 & 0 & -2\lambda \end{pmatrix}$$

with $\Omega^2 = \omega^2 - \mu^2$.

From (3.17) it follows that

$$X(\infty) = -(TK^{-1}T)D = -R^{-1}D \tag{3.18}$$

(in the overdamped case the restriction $\lambda > \nu$ is necessary). Then Eq.(3.17) can be written in the form

$$X(t) = (Te^{Kt}T)(X(0) - X(\infty)) + X(\infty).$$
(3.19)

Also

$$\frac{dX(t)}{dt} = (TKe^{Kt}T)(X(0) - X(\infty)) = R(X(t) - X(\infty))$$

and

$$\frac{dX(t)}{dt}|_{t=0} = (TKT)(X(0) - X(\infty)) = R(X(0) - X(\infty)).$$

The formula (3.18) is remarcable because it gives a very simple connection between the asymptotic values of $\sigma_{qq}(t)$, $\sigma_{pp}(t)$, $\sigma_{pq}(t)$ and the diffusion coefficients D_{qq} , D_{pp} , D_{pq} . As an immediate consequence of (3.18) this connection is *the same* for both cases, underdamped and overdamped, and has the following explicit form:

$$\sigma_{qq}(\infty) = \frac{1}{2(m\omega)^{2}\lambda(\lambda^{2} + \omega^{2} - \mu^{2})} ((m\omega)^{2}(2\lambda(\lambda + \mu) + \omega^{2})D_{qq} + \omega^{2}D_{pp} + 2m\omega^{2}(\lambda + \mu)D_{pq}),$$

$$\sigma_{pp}(\infty) = \frac{1}{2\lambda(\lambda^{2} + \omega^{2} - \mu^{2})} ((m\omega)^{2}\omega^{2}D_{qq} + (2\lambda(\lambda - \mu) + \omega^{2})D_{pp} - 2m\omega^{2}(\lambda - \mu)D_{pq}), \quad (3.20)$$

$$\sigma_{pq}(\infty) = \frac{1}{2m\lambda(\lambda^{2} + \omega^{2} - \mu^{2})} (-(\lambda + \mu)(m\omega)^{2}D_{qq} + (\lambda - \mu)D_{pp} + 2m(\lambda^{2} - \mu^{2})D_{pq}).$$

These relations show that the asymptotic values $\sigma_{qq}(\infty)$, $\sigma_{pp}(\infty)$, $\sigma_{pq}(\infty)$ do not depend on the initial values $\sigma_{qq}(0)$, $\sigma_{pp}(0)$, $\sigma_{pq}(0)$. In other words,

$$R^{-1} = \frac{-1}{4\lambda(\lambda^2 + \omega^2 - \mu^2)} \begin{pmatrix} 2\lambda(\lambda + \mu) + \omega^2 & \omega^2 & 2\omega(\lambda + \mu) \\ \omega^2 & 2\lambda(\lambda - \mu) + \omega^2 & -2\omega(\lambda - \mu) \\ -(\lambda + \mu)\omega & (\lambda - \mu)\omega & 2(\lambda^2 - \mu^2) \end{pmatrix}.$$

Conversely, if the relations $D = -RX(\infty)$ are considered, i.e.,

$$\begin{pmatrix} 2m\omega D_{qq} \\ \frac{2}{m\omega} D_{pp} \\ 2D_{pq} \end{pmatrix} = - \begin{pmatrix} -2(\lambda - \mu) & 0 & 2\omega \\ 0 & -2(\lambda + \mu) & -2\omega \\ -\omega & \omega & -2\lambda \end{pmatrix} \begin{pmatrix} m\omega \sigma_{qq}(\infty) \\ \frac{1}{m\omega} \sigma_{pp}(\infty) \\ \sigma_{pq}(\infty) \end{pmatrix},$$

then

$$D_{qq} = (\lambda - \mu)\sigma_{qq}(\infty) - \frac{1}{m}\sigma_{pq}(\infty),$$

$$D_{pp} = (\lambda + \mu)\sigma_{pp}(\infty) + m\omega^{2}\sigma_{pq}(\infty),$$

$$D_{pq} = \frac{1}{2}(m\omega^{2}\sigma_{qq}(\infty) - \frac{1}{m}\sigma_{pp}(\infty) + 2\lambda\sigma_{pq}(\infty)).$$
(3.21)

Hence, from (3.4) the fundamental constraints on $\sigma_{qq}(\infty)$, $\sigma_{pp}(\infty)$ and $\sigma_{pq}(\infty)$ follow:

$$D_{qq} = (\lambda - \mu)\sigma_{qq}(\infty) - \frac{1}{m}\sigma_{pq}(\infty) > 0,$$

$$D_{pp} = (\lambda + \mu)\sigma_{pp}(\infty) + m\omega^{2}\sigma_{pq}(\infty) > 0,$$

$$D_{qq}D_{pp} - D_{pq}^{2} = (\lambda^{2} - \mu^{2})\sigma_{qq}(\infty)\sigma_{pp}(\infty) - \omega^{2}\sigma_{pq}^{2}(\infty) +$$

$$+(\lambda - \mu)m\omega^{2}\sigma_{qq}(\infty)\sigma_{pq}(\infty) - \frac{(\lambda + \mu)}{m}\sigma_{pp}(\infty)\sigma_{pq}(\infty) -$$

$$-\frac{1}{4}(m\omega^{2})^{2}\sigma_{qq}^{2}(\infty) - \frac{1}{4m^{2}}\sigma_{pp}^{2}(\infty) - \lambda^{2}\sigma_{pq}^{2}(\infty) + \frac{1}{2}\omega^{2}\sigma_{qq}(\infty)\sigma_{pp}(\infty) -$$

$$-m\omega^{2}\lambda\sigma_{qq}(\infty)\sigma_{pq}(\infty) + \frac{\lambda}{m}\sigma_{pp}(\infty)\sigma_{pq}(\infty) \geq \frac{\lambda^{2}\hbar^{2}}{4}.$$
(3.22)

The constraint (3.22) can be put in a more clear form:

$$4(\lambda^2 + \omega^2 - \mu^2)(\sigma_{qq}(\infty)\sigma_{pp}(\infty) - \sigma_{pq}(\infty)^2) -$$

$$-(m\omega^2\sigma_{qq}(\infty) + \frac{1}{m}\sigma_{pp}(\infty) + 2\mu\sigma_{pq}(\infty))^2 \ge \hbar^2\lambda^2.$$
(3.23)

If $\mu < \omega$ (the underdamped case), then $\lambda^2 + \omega^2 - \mu^2 > \lambda^2$. If $\mu > \omega$ (the overdamped case), then $0 \le \lambda^2 + \omega^2 - \mu^2 < \lambda^2$ ($\lambda > \nu$) and the constraint (3.23) is more strong than the uncertainty inequality $\sigma_{qq}(\infty)\sigma_{pp}(\infty) - \sigma_{pq}^2(\infty) \ge \hbar^2/4$. By using the fact that the linear positive mapping $\mathbf{B}(\mathbf{H}) \to C$ defined by $A \to Tr(\rho A)$ is completely positive (hence 2-positive), in [17] the following inequality was obtained:

$$D_{qq}\sigma_{pp}(t) + D_{pp}\sigma_{qq}(t) - 2D_{pq}\sigma_{pq}(t) \ge \frac{\hbar^2 \lambda}{2}.$$

From this inequality which must be valid for all values of $t \in (0, \infty)$ it follows that

$$D_{qq}\sigma_{pp}(\infty) + D_{pp}\sigma_{qq}(\infty) - 2D_{pq}\sigma_{pq}(\infty) \ge \frac{\hbar^2 \lambda}{2}.$$

Using Eq. (3.21) this inequality is equivalent with the uncertainty inequality

$$\sigma_{qq}(\infty)\sigma_{pp}(\infty) - \sigma_{pq}(\infty)^2 \ge \frac{\hbar^2}{4}.$$

A restriction connecting the initial values $\sigma_{qq}(0), \sigma_{pp}(0), \sigma_{pq}(0)$ with the asymptotic values $\sigma_{pp}(\infty), \sigma_{qq}(\infty), \sigma_{pq}(\infty)$ is also obtained:

$$D_{qq}\sigma_{pp}(0) + D_{pp}\sigma_{qq}(0) - 2D_{pq}\sigma_{pq}(0) \ge \frac{\hbar^2 \lambda}{2}.$$

More explicitly

$$\lambda(\sigma_{qq}(\infty)\sigma_{pp}(0) + \sigma_{pp}(\infty)\sigma_{qq}(0) - 2\sigma_{pq}(\infty)\sigma_{pq}(0)) -$$

$$-\mu(\sigma_{qq}(\infty)\sigma_{pp}(0) - \sigma_{pp}(\infty)\sigma_{qq}(0)) - \frac{1}{m}(\sigma_{pq}(\infty)\sigma_{pp}(0) - \sigma_{pp}(\infty)\sigma_{pq}(0)) +$$

$$+m\omega^{2}(\sigma_{pq}(\infty)\sigma_{qq}(0) - \sigma_{qq}(\infty)\sigma_{pq}(0)) \ge \frac{\hbar^{2}\lambda}{2}.$$
(3.24)

If the asymptotic state is a Gibbs state

$$\rho_G(\infty) = e^{-\frac{H_0}{kT}} / Tr(e^{-\frac{H_0}{kT}}),$$

then

$$\sigma_{qq}(\infty) = \frac{\hbar}{2m\omega} \coth \frac{\hbar\omega}{2kT}, \sigma_{pp}(\infty) = \frac{\hbar m\omega}{2} \coth \frac{\hbar\omega}{2kT}, \sigma_{pq}(\infty) = 0$$
 (3.25)

and

$$D_{pp} = \frac{\lambda + \mu}{2} \hbar m\omega \coth \frac{\hbar\omega}{2kT}, D_{qq} = \frac{\lambda - \mu}{2} \frac{\hbar}{m\omega} \coth \frac{\hbar\omega}{2kT}, D_{pq} = 0$$
 ((3.26)

and the fundamental constraints (3.4) are satisfied only if $\lambda > \mu$ and [15]:

$$(\lambda^2 - \mu^2)(\coth \frac{\hbar\omega}{2kT})^2 \ge \lambda^2.$$

If the initial state is the ground state of the harmonic oscillator, then

$$\sigma_{qq}(0) = \frac{\hbar}{2m\omega}, \sigma_{pp}(0) = \frac{m\hbar\omega}{2}, \sigma_{pq}(0) = 0.$$

Then (3.24) becomes

$$\lambda(\sigma_{qq}(\infty)m\omega + \frac{\sigma_{pp}(\infty)}{m\omega}) - \mu(\sigma_{qq}(\infty)m\omega - \frac{\sigma_{pp}(\infty)}{m\omega}) \ge \hbar\lambda.$$

For example, in the case (3.25), this implies $\coth \hbar \omega / 2kT \ge 1$ which is always valid.

Now, the explicit time dependence of $\sigma_{qq}(t)$, $\sigma_{pp}(t)$, $\sigma_{pq}(t)$ will be given for both underand overdamped cases. From Eq. (3.19) it follows that in order to obtain this explicit time dependence it is necessary to obtain the matrix elements of $Te^{Kt}T$. In the overdamped case $(\mu > \omega)$, $\nu^2 = \mu^2 - \omega^2$ we have

$$Te^{Kt}T = \frac{e^{-2\lambda t}}{2\nu^2} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

with

$$a_{11} = (\mu^{2} + \nu^{2}) \cosh 2\nu t + 2\mu\nu \sinh 2\nu t - \omega^{2},$$

$$a_{12} = (\mu^{2} - \nu^{2}) \cosh 2\nu t - \omega^{2},$$

$$a_{13} = 2\omega(\mu \cosh 2\nu t + \nu \sinh 2\nu t - \mu),$$

$$a_{21} = (\mu^{2} - \nu^{2}) \cosh 2\nu t - \omega^{2},$$

$$a_{22} = (\mu^{2} + \nu^{2}) \cosh 2\nu t - 2\mu\nu \sinh 2\nu t - \omega^{2},$$

$$a_{23} = 2\omega(\mu \cosh 2\nu t - \nu \sinh 2\nu t - \mu),$$

$$a_{31} = -\omega(\mu \cosh 2\nu t + \nu \sinh 2\nu t - \mu),$$

$$a_{32} = -\omega(\mu \cosh 2\nu t - \nu \sinh 2\nu t - \mu),$$

$$a_{33} = -2(\omega^{2} \cosh 2\nu t - \nu \sinh 2\nu t - \mu),$$

$$a_{33} = -2(\omega^{2} \cosh 2\nu t - \mu^{2}).$$

In the underdamped case $(\mu < \omega)$, $\Omega^2 = \omega^2 - \mu^2$ we have

$$Te^{Kt}T = -\frac{e^{-2\lambda t}}{2\Omega^2} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

with

$$b_{11} = (\mu^{2} - \Omega^{2})\cos 2\Omega t - 2\mu\Omega\sin 2\Omega t - \omega^{2},$$

$$b_{12} = (\mu^{2} + \Omega^{2})\cos 2\Omega t - \omega^{2},$$

$$b_{13} = 2\omega(\mu\cos 2\Omega t - \Omega\sin 2\Omega t - \mu),$$

$$b_{21} = (\mu^{2} + \Omega^{2})\cos 2\Omega t - \omega^{2},$$

$$b_{22} = (\mu^{2} - \Omega^{2})\cos 2\Omega t + 2\mu\Omega\sin 2\Omega t - \omega^{2},$$

$$b_{23} = 2\omega(\mu\cos 2\Omega t + \Omega\sin 2\Omega t - \mu),$$

$$b_{31} = -\omega(\mu\cos 2\Omega t - \Omega\sin 2\Omega t - \mu),$$

$$b_{32} = -\omega(\mu\cos 2\Omega t + \Omega\sin 2\Omega t - \mu),$$

$$b_{33} = -2(\omega^{2}\cos 2\Omega t - \mu^{2}).$$
(3.28)

4. The method of the characteristic function

Instead of solving the master equation (3.7) directly, we first introduce the normally ordered quantum characteristic function $\chi(\Lambda, \Lambda^*, t)$ defined in terms of the density operator ρ by

$$\chi(\Lambda, \Lambda^*, t) = Tr[\rho(t) \exp(\Lambda a^+) \exp(-\Lambda^* a)], \tag{4.1}$$

where Λ is a complex variable and the trace is performed over the states of system. Substituting Eq.(4.1) into the master equation (3.7) and using the operator relations

$$a \exp(\Lambda a^{+}) = \exp(\Lambda a^{+})(a + \Lambda),$$

$$a^{+} \exp(-\Lambda^{*} a) = \exp(-\Lambda^{*} a)(a^{+} + \Lambda^{*}),$$

$$Tr[\rho(t) \exp(\Lambda a^{+}) \exp(-\Lambda^{*} a)(a^{+} + \Lambda^{*})] = \partial_{\Lambda} \chi,$$

$$Tr[\rho(t) \exp(\Lambda a^{+}) \exp(-\Lambda^{*} a)a] = -\partial_{\Lambda^{*}} \chi$$

or applying the rules:

$$\rho \leftrightarrow \chi$$

$$a\rho \leftrightarrow -\frac{\partial}{\partial \Lambda^*} \chi$$

$$a^+\rho \leftrightarrow (\frac{\partial}{\partial \Lambda} - \Lambda^*) \chi$$

$$\rho a \leftrightarrow (-\frac{\partial}{\partial \Lambda^*} + \Lambda) \chi$$

$$\rho a^+ \leftrightarrow \frac{\partial}{\partial \Lambda} \chi,$$

the following partial differential equation for χ is found:

$$\{\partial_t + [(\lambda - i\omega)\Lambda + \mu\Lambda^*]\partial_\Lambda + [(\lambda + i\omega)\Lambda^* + \mu\Lambda]\partial_{\Lambda^*}\}\chi(\Lambda, \Lambda^*, t) =$$

$$= \{L|\Lambda|^2 + C\Lambda^2 + C^*\Lambda^{*2}\}\chi(\Lambda, \Lambda^*, t), \tag{4.2}$$

where

$$L = \lambda - D_2, C = \frac{1}{2}(\mu + D_1^*).$$

We consider the state of the system initially to be a superposition of coherent states. The coherent states $|\alpha>$ of the harmonic oscillator are minimum uncertainty states having mean coordinate < q > and mean momentum given by

$$< q> = < \alpha |q|\alpha > = \sqrt{\frac{2\hbar}{m\omega}} Re\alpha, = < \alpha |p|\alpha > = \sqrt{2\hbar m\omega} Im\alpha.$$
 (4.3)

Consequently, we take as the initial density operator

$$\rho(0) = \int d\alpha d\beta N(\alpha, \beta) |\alpha\rangle \langle \beta|.$$

The quantum characteristic function corresponding to the operator $|\alpha \rangle \langle \beta|$ is given from Eq. (4.1) by

$$\chi = <\beta |\alpha> \exp(\lambda \beta^* - \lambda^* \alpha). \tag{4.4}$$

We look for a solution of (4.2) having the exponential form

$$\chi(\Lambda, \Lambda^*, t) = \int d\alpha d\beta N(\alpha, \beta) < \beta |\alpha| > \exp[A(t)\Lambda + B(t)\Lambda^* + f(t)\Lambda^2 + f^*(t)\Lambda^{*2} + h(t)|\Lambda|^2]. \tag{4.5}$$

The form of the solution (4.5) is suggested from the fact that the left-hand side of Eq. (4.2) contains first-order derivatives with respect to the time and variables Λ and Λ^* and is symmetric with respect to complex conjugation. The functions A(t), B(t), f(t) and h(t) depend only on time. Corresponding to the initial factor in Eq. (4.4), these functions have to satisfy the initial conditions

$$A(0) = \beta^*, B(0) = \alpha, f(0) = 0, h(0) = 0.$$
(4.6)

When we introduce the function (4.5) into Eq. (4.2) and equate the coefficients for equal powers of Λ and Λ^* , we get the following two systems of linear differential equations of first order with constant coefficients:

$$\frac{dA(t)}{dt} + (\lambda - i\omega)A(t) + \mu B(t) = 0$$

$$\frac{dB(t)}{dt} + \mu A(t) + (\lambda + i\omega)B(t) = 0$$

$$\frac{dR(t)}{dt} + 2\lambda R(t) + 2\omega I(t) + \mu h(t) = ReC$$

$$\frac{dI(t)}{dt} + 2\lambda I(t) - 2\omega R(t) = ImC$$

$$\frac{dh(t)}{dt} + 4\mu R(t) + 2\lambda h(t) = L,$$
(4.7a)

where R(t) = Ref(t), I(t) = Imf(t) with the initial conditions R(0) = I(0) = h(0) = 0. Subject to the initial conditions (4.6), the homogeneous system (4.7a) has the solution [19]:

$$A(t) = u(t)\beta^* - v(t)\alpha,$$

$$B(t) = -u^*(t)\alpha + v(t)\beta^*,$$
(4.8)

where

$$u(t) = \frac{1}{2} \left[\exp(-\mu_{-}t) + \exp(-\mu_{+}t) + \frac{2i\omega}{\mu_{-} - \mu_{+}} (\exp(-\mu_{+}t) - \exp(-\mu_{-}t)) \right],$$

$$v(t) = \frac{\mu}{\mu_{-} - \mu_{+}} (\exp(-\mu_{-}t) - \exp(-\mu_{+}t)). \tag{4.9}$$

The eigenvalues μ_{\pm} are given by

$$\mu_{\pm} = \lambda \pm \sqrt{\mu^2 - \omega^2}, \gamma \equiv \sqrt{\mu^2 - \omega^2}. \tag{4.10}$$

The system (4.7b) has the eigenvalues -2λ , $-2(\lambda \pm \sqrt{\mu^2 - \omega^2}) = -2\mu_{\pm}$ and in order to integrate it we apply the same method as for the system (3.13) in the preceding Section. We obtain:

$$f(t) = \frac{P}{2\mu} \exp(-2\mu + t)(\gamma - i\omega) - \frac{N}{2\mu} \exp(-2\mu - t)(\gamma + i\omega) - \frac{i\mu M}{2\omega} \exp(-2\lambda t) + f(\infty),$$

$$h(t) = M \exp(-2\lambda t) + N \exp(-2\mu_{-}t) + P \exp(-2\mu_{+}t) + h(\infty). \tag{4.11}$$

Here $M, N, P, f(\infty)$ and $h(\infty)$ are constants given by [19]:

$$M = \frac{\omega}{\lambda \gamma^2} (\mu ImC + \frac{1}{2}\omega L),$$

$$N = \frac{\mu}{2\gamma^2(\lambda - \gamma)} (\gamma ReC - \omega ImC - \frac{\mu L}{2}),$$

$$P = -\frac{\mu}{2\gamma^2(\lambda + \gamma)}(\gamma ReC + \omega ImC + \frac{\mu L}{2})$$

and the asymptotic values connected with the diffusion coefficients D_{qq} , D_{pp} and D_{pq} are:

$$R(\infty) = \frac{2(\lambda ReC - \omega ImC) - L\mu}{4(\lambda^2 - \gamma^2)},$$

$$I(\infty) = \frac{2\omega\lambda ReC + 2(\lambda^2 - \mu^2)ImC - L\mu\omega}{4\lambda(\lambda^2 - \gamma^2)},$$

$$h(\infty) = \frac{L(\lambda^2 + \omega^2) - 2\mu(\lambda ReC - \omega ImC)}{2\lambda(\lambda^2 - \gamma^2)}.$$

$$(4.12)$$

By knowing the characteristic function (4.5), (4.8)-(4.11) corresponding to the initial density operator which represents a superposition of coherent states, it is easy to obtain explicit formulae for the moments:

$$\langle a^{+m}(t)a^n(t) \rangle = Tr[a^{+m}(t)a^n(t)\rho(t)] = (-1)^n \frac{\partial^{n+m}}{\partial \Lambda^{*n}\partial \Lambda^m}\chi(\Lambda,\Lambda^*,t)|_{\Lambda=\Lambda^*=0}.$$

In the following, we take the density operator ρ in the coherent state representation

$$\rho(0) = \int P(\alpha)|\alpha\rangle \langle \alpha|d^2\alpha,$$

where $P(\alpha)$ is the diagonal or P Glauber distribution and $d^2\alpha = dRe\alpha dIm\alpha$. The integration covers the entire complex α plane. Then the characteristic function (4.5) becomes:

$$\chi(\Lambda, \Lambda^*, t) = \int d^2 \alpha P(\alpha) \exp[(u\alpha^* - v\alpha)\Lambda + (-u^*\alpha + v\alpha^*)\Lambda^*] \exp[f\Lambda^2 + f^*\Lambda^{*2} + h|\Lambda|^2].$$

Let us assume that the damped oscillator is at t=0 prepared in a pure coherent state, say $|\alpha_0>$, corresponding to $P(\alpha)=\delta(Re\alpha-Re\alpha_0)\delta(Im\alpha-Im\alpha_0)$. One has

$$\chi^{(0)}(\Lambda, \Lambda^*, t) = \exp[(u\alpha_0^* - v\alpha_0)\Lambda + (-u^*\alpha_0 + v\alpha_0^*)\Lambda^*] \exp[f\Lambda^2 + f^*\Lambda^{*2} + h|\Lambda|^2].$$

The first moments are given by

$$\langle a^+(t) \rangle = \frac{\partial \chi^{(0)}(t)}{\partial \Lambda} |_{\Lambda = \Lambda^* = 0} = u\alpha_0^* - v\alpha_0,$$

$$\langle a(t) \rangle = -\frac{\partial \chi^{(0)}(t)}{\partial \Lambda^*}|_{\Lambda = \Lambda^* = 0} = u^* \alpha_0 - v \alpha_0^*.$$

Then, with the notations (3.9), using (4.3) and the transformations

$$q(t) = \sqrt{\frac{\hbar}{2m\omega}}(a^{+}(t) + a(t)),$$

$$p(t) = i\sqrt{\frac{\hbar m\omega}{2}}(a^{+}(t) - a(t))$$

for the displacement operator q(t) and the momentum operator p(t) of the oscillator, we obtain the following mean values:

$$\sigma_q(t) = \sqrt{\frac{\hbar}{2m\omega}}((u-v)\alpha_0^* + (u^*-v)\alpha_0),$$

$$\sigma_p(t) = i\sqrt{\frac{\hbar m\omega}{2}}((u+v)\alpha_0^* - (u^*+v)\alpha_0),$$

with u, v given by (4.9), (4.10). There are two cases:

a) the overdamped case: $\mu > \omega, \nu^2 = \mu^2 - \omega^2, \gamma \equiv \nu$; then

$$u(t) = \exp(-\lambda t)(\cosh \nu t + \frac{i\omega}{\nu} \sinh \nu t),$$

$$v(t) = -\frac{\mu}{\nu} \exp(-\lambda t) \sinh \nu t$$
(4.13)

and $\sigma_q(t), \sigma_p(t)$ take the previous form (3.15); b) the underdamped case: $\mu < \omega, \Omega^2 = \omega^2 - \mu^2, \gamma \equiv i\Omega$; then

$$u(t) = \exp(-\lambda t)(\cosh \Omega t + \frac{i\omega}{\Omega} \sinh \Omega t),$$

$$v(t) = -\frac{\mu}{\Omega} \exp(-\lambda t) \sinh \Omega t$$
 (4.14)

and $\sigma_q(t), \sigma_p(t)$ take the previous form (3.16).

For the variances one finds:

$$\langle a^{2}(t) \rangle = \frac{\partial^{2} \chi^{(0)}(t)}{\partial \Lambda^{*2}} |_{\Lambda = \Lambda^{*} = 0} = (u^{*} \alpha_{0} - v \alpha_{0}^{*})^{2} + 2f^{*},$$

$$\langle a^{+2}(t) \rangle = \frac{\partial^{2} \chi^{(0)}(t)}{\partial \Lambda^{2}} |_{\Lambda = \Lambda^{*} = 0} = (u \alpha_{0}^{*} - v \alpha_{0})^{2} + 2f,$$

$$\langle a^{+}(t) a(t) \rangle = -\frac{\partial^{2} \chi^{(0)}(t)}{\partial \Lambda \partial \Lambda^{*}} |_{\Lambda = \Lambda^{*} = 0} = (u \alpha_{0}^{*} - v \alpha_{0})(u^{*} \alpha_{0} - v \alpha_{0}^{*}) - h.$$

$$(4.15)$$

Then the relations (3.9) will give us the explicit time dependence of the variances $\sigma_{qq}(t)$, $\sigma_{pp}(t)$, $\sigma_{pq}(t)$. The asymptotic values of these variances are given by the following expresions [19]:

$$\sigma_{qq}(\infty) = \frac{\hbar}{m\omega} (f + f^* - h + \frac{1}{2})|_{t \to \infty},$$

$$\sigma_{pp}(\infty) = -\hbar m\omega (f + f^* + h - \frac{1}{2})|_{t \to \infty},$$

$$\sigma_{pq}(\infty) = i\hbar (f - f^*)|_{t \to \infty}.$$

With $f(\infty) = R(\infty) + iI(\infty)$ and using the formulas (4.12) for $R(\infty), I(\infty), h(\infty)$, the asymptotic values of the variances take the same form (3.20) as in the preceding Section, as expected.

With the relations (4.15), the expectation value of the energy operator can be calculated:

$$E(t) = \hbar\omega(\langle a^+a \rangle + \frac{1}{2}) + \frac{i\hbar\mu}{2}(\langle a^{+2} \rangle - \langle a^2 \rangle).$$

The asymptotic mean value of the energy of the open harmonic oscillator is:

$$E(\infty) = \frac{1}{2m}\sigma_{pp}(\infty) + \frac{1}{2}m\omega^2\sigma_{qq}(\infty) + \mu\sigma_{pq}(\infty),$$

or, as a function of diffusion coefficients, by using (3.20):

$$E(\infty) = \frac{1}{\lambda} \left(\frac{1}{2m} D_{pp} + \frac{m\omega^2}{2} D_{qq} + \mu D_{pq} \right).$$

5. Quasiprobability distributions for damped harmonic oscillator

The methods of quasiprobabilities have provided technical tools of great power for the statistical description of microscopic systems formulated in terms of the density operator [40-42,58]. The first quasiprobability method was that introduced by Wigner [43] in a quantum-mechanical context. In quantum optics the P representation introduced by Glauber [44,45] and Sudarshan [46] provided many practical applications of quasiprobabilities. The development of quantum-mechanical master equations was combined with the Glauber P representation to give a Fokker-Planck equation for the laser [47,48]. One useful way to study the consequences of the master equation for the one-dimensional damped harmonic oscillator is to transform it into equations for the c-number quasiprobability distributions associated with the density operator. The resulting differential equations of the Fokker-Planck type for the distribution functions can be solved by standard methods and observables directly calculated as correlations of these distribution functions. However, the Fokker-Planck equations do not always have positive-definite diffusion coefficients. In this case one can treat the problem with the generalized P distribution [41].

First we present a short summary of the theory of quasiprobability distributions. For the master equation (3.7) of the harmonic oscillator, physical observables can be obtained from the expectation values of polynomials of the annihilation and creation operators. The expectation values are determined by using the quantum density operator ρ . Usually one expands the density operator with the aid of coherent states, defined as eigenstates of the annihilation operator: $a|\alpha>=\alpha|\alpha>$. They are given in terms of the eigenstates of the harmonic oscillator as

$$|\alpha> = \exp(-\frac{|\alpha|^2}{2}) \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \alpha^n |n>,$$

with the normalization $|\langle \beta | \alpha \rangle|^2 = \exp(-|\alpha - \beta|^2)$. In order to solve the master equation (3.7) we represent the density operator ρ by a distribution function over a c-number phase space. The chosen distribution function, introduced in [49] is defined as follows:

$$\Phi(\alpha, s) = \frac{1}{\pi^2} \int \chi(\Lambda, s) \exp(\alpha \Lambda^* - \alpha^* \Lambda) d^2 \Lambda, \tag{5.1}$$

with the characteristic function

$$\chi(\Lambda, s) = Tr[\rho D(\Lambda, s)],$$

where $D(\Lambda, s)$ is the displacement operator

$$D(\Lambda, s) = \exp(\Lambda a^{+} - \Lambda^{*} a + \frac{1}{2} s |\Lambda|^{2}).$$

The interval of integration in Eq. (5.1) is the whole complex Λ plane. Because of

$$\delta^{2}(\alpha) = \frac{1}{\pi^{2}} \int \exp(\alpha \Lambda^{*} - \alpha^{*} \Lambda) d^{2} \Lambda,$$

the characteristic function $\Phi(\alpha, s)$ is the Fourier transform of the characteristic function. Since the density operator is normalized by $Tr\rho = 1$, one obtains the normalization of Φ :

$$\int \Phi(\alpha, s) d^2 \alpha = 1.$$

In this paper we restrict ourselves to distribution functions with the parameters s = 1, 0 and -1. These distribution functions can be used to calculate expectation values of products of annihilation and creation operators. For that purpose one first expands the displacements operator in a power series of the operators a and a^+ :

$$D(\Lambda, s) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Lambda^m (-\Lambda^*)^n}{m! n!} \{a^{+m} a^n\}_s.$$
 (5.2)

The braces with the index s indicate the special representations of the polynomials depending on s. For example for n = m = 1 we have the s-ordered operators:

$$\{a^{+}a\}_{s=1} = a^{+}a,$$

$$\{a^{+}a\}_{s=0} = \frac{1}{2}(a^{+}a + aa^{+}),$$

$$\{a^{+}a\}_{s=-1} = aa^{+}.$$
(5.3)

Expectation values of the s-ordered operators can be calculated as follows:

$$<\{a^{+m}a^n\}_s> = Tr[\rho\{a^{+m}a^n\}_s] = Tr[\rho(\frac{\partial}{\partial\Lambda})^m(-\frac{\partial}{\partial\Lambda^*})^nD(\Lambda,s)|_{\Lambda=0}] =$$

$$= \left(\frac{\partial}{\partial \Lambda}\right)^m \left(-\frac{\partial}{\partial \Lambda^*}\right)^n \chi(\Lambda, s)|_{\Lambda = 0} = \int (\alpha^*)^m \alpha^n \Phi(\alpha, s) d^2 \alpha.$$

For the last step we apply the inverse relation to Eq. (5.1):

$$\chi(\Lambda, s) = \int \Phi(\alpha, s) \exp(\Lambda \alpha^* - \Lambda^* \alpha) d^2 \alpha.$$

In the following we discuss the distribution functions for s = 1, 0 and -1 in more detail. For s = 1 we obtain the Glauber P function [44,45,50], for s = 0 the Wigner function [43] and for s = -1 the Q function [49]. For s = 1 we have

$$D(\Lambda, 1) = \exp(\Lambda a^+) \exp(-\Lambda^* a).$$

Then the s ordering in Eq. (5.2) corresponds to normal ordering. Since the Glauber P function is the Fourier transform of the characteristic function

$$\chi_N(\Lambda) = Tr[\rho \exp(\Lambda a^+) \exp(-\Lambda^* a)] = \chi(\Lambda, 1),$$

it follows from Eq. (5.1) that the distribution $\Phi(\alpha, 1)$ is identical to the P function. This function is used for an expansion of the density operator in diagonal coherent state projection operators [44-46,51]:

$$\rho = \int P(\alpha)d^2\alpha |\alpha \rangle \langle \alpha|.$$

Calculating the expectation value of normally ordered operator products we obtain the relation

$$\langle a^{+m}a^n \rangle = \int (\alpha^*)^m \alpha^n \Phi(\alpha, 1) d^2 \alpha = \int (\alpha^*)^m \alpha^n P(\alpha) d^2 \alpha,$$

from which we again derive $P(\alpha) = \Phi(\alpha, 1)$. Despite the formal similarity to averaging with a classical probability distribution, the function $P(\alpha)$ is not a true probability distribution. Because of the overcompleteness of the coherent states, the P function is not a unique, well-behaved positive function for all density operators.

Cahill [52] studied the P representation for density operators which represent pure states and found a narrow class of states for which the P representation exists. They can be generated from a particular coherent state $|\alpha\rangle$ by the application of a finite number of creation operators. For example, for the ground state of the harmonic oscillator it is easy to show that $\chi_N(\Lambda) = 1$ for all Λ . In that case the P function becomes $P(\alpha) = \delta^2(\alpha)$. The delta function and its derivatives are examples of a class of generalized functions known as tempered distributions [50]. Also Cahill [53] introduced a representation of the density operator of the electromagnetic field that is suitable for all density operators and that reduces to the coherent state P representation when the latter exists. The representation has no singularities.

Sudarshan [46] offered a singular formula for the P representation in terms of an infinite series of derivatives of the delta function. From the mathematical point of view, such a series is usually not considered to be a distribution function [50,51].

For s = -1 we have

$$D(\Lambda, -1) = \exp(-\Lambda^* a) \exp(\Lambda a^+).$$

The s ordering corresponds to antinormal ordering. Because the Q function is the Fourier transform of the characteristic function

$$\chi_A(\Lambda) = Tr[\rho \exp(-\Lambda^* a) \exp(\Lambda a^+)] = \chi(\Lambda, -1),$$

it follows from Eq. (5.1) that the distribution $\Phi(\alpha, -1)$ is the Q function. It is given by the diagonal matrix elements of the density operator in terms of coherent states:

$$Q(\alpha) = \frac{1}{\pi} < \alpha |\rho| \alpha > .$$

Though for all density operators the Q function is bounded, non-negative and infinitely differentiable, it has the disadvantage that not every positive Q function corresponds to a positive semidefinite Hermitian density operator. Evaluating moments is only simple in the Q representation for antinormally ordered operator products.

For s=0, the distribution $\Phi(\Lambda,0)$ becomes the Wigner function W. The latter function is defined as the Fourier transform of the characteristic function

$$\chi_S(\Lambda) = Tr[\rho \exp(\Lambda a^+ - \Lambda^* a)] = \chi(\Lambda, 0).$$

Because this characteristic function is identical to $\chi(\Lambda,0)$, we conclude that $\Phi(\alpha,0)$ is the Wigner function $W(\alpha)$. Therefore, the Wigner function can be used to calculate expectation values of symmetrically ordered operators:

$$\langle \{a^{+m}a^n\}_{s=0} \rangle = \int (\alpha^*)^m \alpha^n W(\alpha) d^2 \alpha.$$

The symmetrically ordered operators are the arithmetic average of (m+n)!/(m!n!) differently ordered products of m factors of a^+ and n factors of a. An example for m=n=1 is given in Eq. (5.3).

The Wigner function is a nonsingular, uniformly continuous function of α for all density operators and may in general assume negative values. It is related to the density operator as follows:

$$W(\alpha) = \frac{1}{\pi^2} \int d^2 \Lambda Tr[\exp(\Lambda(a^+ - \alpha^*) - \Lambda^*(a - \alpha))\rho].$$

Also it can be obtained from the P representation:

$$W(\alpha) = \frac{2}{\pi} \int P(\beta) \exp(-2|\alpha - \beta|^2) d^2\beta.$$

By using the standard transformations [54,55], the master equation (3.7) can be transformed into a differential equation for a corresponding c-number distribution. In this Section we apply these methods to derive Fokker-Planck equations for the before mentioned distributions: the Glauber P, the Q and Wigner distributions. Using the relations

$$\frac{\partial D(\Lambda, s)}{\partial \Lambda} = [(s-1)\frac{\Lambda^*}{2} + a^+]D(\Lambda, s) = D(\Lambda, s)[(s+1)\frac{\Lambda^*}{2} + a^+],$$
$$\frac{\partial D(\Lambda, s)}{\partial \Lambda^*} = [(s+1)\frac{\Lambda}{2} - a]D(\Lambda, s) = D(\Lambda, s)[(s-1)\frac{\Lambda}{2} - a],$$

one can derive the following rules for transforming the master equation (3.7) into Fokker-Planck equations in the Glauber P(s = 1), the Q(s = -1) and Wigner W(s = 0) representations:

$$a\rho \leftrightarrow (\alpha - \frac{s-1}{2} \frac{\partial}{\partial \alpha^*})\Phi$$

$$a^+\rho \leftrightarrow (\alpha^* - \frac{s+1}{2} \frac{\partial}{\partial \alpha})\Phi$$

$$\rho a \leftrightarrow (\alpha - \frac{s+1}{2} \frac{\partial}{\partial \alpha^*})\Phi$$

$$\rho a^+ \leftrightarrow (\alpha^* - \frac{s-1}{2} \frac{\partial}{\partial \alpha})\Phi.$$

Applying these operator correspondences (repeatedly, if necessary), we find the following Fokker-Planck equations for the distributions $\Phi(\alpha, s)$:

$$\frac{\partial \Phi(\alpha, s)}{\partial t} = -\left(\frac{\partial}{\partial \alpha} d_{\alpha} + \frac{\partial}{\partial \alpha^{*}} d_{\alpha}^{*}\right) \Phi(\alpha, s) +
+ \frac{1}{2} \left(\frac{\partial^{2}}{\partial \alpha^{2}} D_{\alpha \alpha} + \frac{\partial^{2}}{\partial \alpha^{*2}} D_{\alpha \alpha}^{*} + 2 \frac{\partial^{2}}{\partial \alpha \partial \alpha^{*}} D_{\alpha \alpha^{*}}\right) \Phi(\alpha, s).$$
(5.4)

Here, $\Phi(\alpha, s)$ is P(s = 1), Q(s = -1) or W(s = 0). While the drift coefficients are the same for the three distributions, the diffusion coefficients are different:

$$d_{\alpha} = -(\lambda + i\omega)\alpha + \mu\alpha^*, D_{\alpha\alpha} = D_1 + s\mu, D_{\alpha\alpha^*} = D_2 - s\lambda.$$

The Fokker-Planck equation (5.4) can also be written in terms of real coordinates x_1 and x_2 defined by

$$\alpha = x_1 + ix_2 \equiv \sqrt{\frac{m\omega}{2\hbar}} < q > +i\frac{1}{\sqrt{2\hbar m\omega}} ,$$

$$\alpha^* = x_1 - ix_2 \tag{5.5}$$

as follows:

$$\frac{\partial \Phi(x_1, x_2)}{\partial t} = -(\frac{\partial}{\partial x_1} d_1 + \frac{\partial}{\partial x_2} d_2) \Phi(x_1, x_2) +$$

$$+\frac{1}{2}\left(\frac{\partial^{2}}{\partial x_{1}^{2}}D_{11} + \frac{\partial^{2}}{\partial x_{2}^{2}}D_{22} + 2\frac{\partial^{2}}{\partial x_{1}\partial x_{2}}D_{12}\right)\Phi(x_{1}, x_{2}),\tag{5.6}$$

with the new drift and difusion coefficients given by

$$d_1 = -(\lambda - \mu)x_1 + \omega x_2, d_2 = -\omega x_1 - (\lambda + \mu)x_2,$$

$$D_{11} = \frac{1}{\hbar} m\omega D_{qq} - \frac{s}{2}(\lambda - \mu), D_{22} = \frac{1}{\hbar} \frac{D_{pp}}{m\omega} - \frac{s}{2}(\lambda + \mu), D_{12} = \frac{1}{\hbar} D_{pq}.$$

We note that the diffusion matrix

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{pmatrix}$$

for the P ddistribution (s = 1) needs not to be positive definite.

Since the drift coefficients are linear in the variables x_1 and $x_2(i = 1, 2)$:

$$d_i = -\sum_{j=1}^{2} A_{ij} x_j, A_{ij} = -\frac{\partial d_i}{\partial x_j},$$

with

$$A = \begin{pmatrix} \lambda - \mu & -\omega \\ \omega & \lambda + \mu \end{pmatrix} \tag{5.7}$$

and the diffusion coefficients are constant with respect to x_1 and x_2 , Eq. (5.6) describes an Ornstein-Uhlenbeck process [56,57].

The solution of the Fokker-Planck equation (5.6) can immediately be written down provided that the diffusion matrix D is positive definite. However, the diffusion matrix in the Glauber P representation is not, in general, positive definite. For example, if

$$D_{11}^P D_{22}^P - (D_{12})^2 < 0,$$

the P distribution does not exist as a well-behaved function. In this situation, the so-called generalized P distributions can be taken that are well-behaved, normal ordering functions [41]. The Q and W distributions always exist; they are Gaussian functions if they are initially of Gaussian type.

From Eq. (5.6) one can directly derive the equations of motion for the expectation values of the variables x_1 and $x_2(i = 1, 2)$:

$$\frac{d < x_i >}{dt} = -\sum_{j=1}^{2} A_{ij} < x_j > . {(5.8)}$$

By using Eqs. (3.5), (5.5) and (5.8) we obtain the equations of motion for the expectation values $\sigma_q(t), \sigma_p(t)$ of coordinate and momentum of the harmonic oscillator which are identical with those derived in the preceding two Sections by using the Heisenberg representation and the method of characteristic function, respectively (see Eqs. (3.12)).

The variances of the variables x_1 and x_2 are defined by the expectation values

$$\sigma_{ij} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle, ij = 1, 2.$$

They are connected with the variances and covariance of the coordinate q and momentum p by

$$\sigma_{qq} = (2\hbar/m\omega)\sigma_{11}, \sigma_{pp} = 2\hbar m\omega\sigma_{22},$$

$$\sigma_{pq} = <\frac{1}{2}(pq + qp) > - < q > = 2\hbar\sigma_{12}.$$

They can be calculated with the help of the variances of the quasiprobability distributions (i, j = 1, 2):

$$\sigma_{ij}^{(s)} = \int x_i x_j \Phi(x_1, x_2, s) dx_1 dx_2 - \int x_i \Phi(x_1, x_2, s) dx_1 dx_2 \int x_j \Phi(x_1, x_2, s) dx_1 dx_2.$$

The following relations exist between the various variances:

$$\sigma_{ii} = \sigma_{ii}^P + \frac{1}{4} = \sigma_{ii}^Q - \frac{1}{4} = \sigma_{ii}^W, i = 1, 2,$$

$$\sigma_{12} = \sigma_{12}^P = \sigma_{12}^Q = \sigma_{12}^W.$$

The variances $\sigma_{ij}^{(s)}$ fulfill the following equations of motion:

$$\frac{d\sigma_{ij}^{(s)}}{dt} = -\sum_{l=1}^{2} (A_{il}\sigma_{lj}^{(s)} + \sigma_{il}^{(s)}A_{lj}^{T}) + D_{ij}^{(s)}.$$
 (5.9)

They can be written explicitly in the form:

$$\frac{d\sigma_{11}^{(s)}}{dt} = -2A_{11}\sigma_{11}^{(s)} - 2A_{12}\sigma_{12}^{(s)} + D_{11}^{(s)},$$

$$\frac{d\sigma_{22}^{(s)}}{dt} = -2A_{21}\sigma_{12}^{(s)} - 2A_{22}\sigma_{22}^{(s)} + D_{22}^{(s)},$$

$$\frac{d\sigma_{12}^{(s)}}{dt} = -(A_{11} + A_{22})\sigma_{12}^{(s)} - A_{21}\sigma_{11}^{(s)} - A_{12}\sigma_{22}^{(s)} + D_{12}^{(s)},$$

where the matrix elements A_{ij} are defined in Eq. (5.7). These relations are sufficient to prove that the equations of motion for the variances σ_{11} and σ_{22} and the covariance σ_{12} are the same irrespective of the choice of the representation as expected. The corresponding equations of motion of the variances and covariance of the coordinate and momentum coincide with those obtained in the preceding two Sections by using the Heisenberg representation and the method of characteristic function, respectively (see Eqs. (3.13)).

In order that the system approaches a steady state, the condition $\lambda > \nu$ must be met. Thus the steady-state solutions are

$$\Phi(x_1, x_2, s) = \frac{1}{2\pi\sqrt{\det(\sigma(\infty))}} \exp\left[-\frac{1}{2} \sum_{i, j=1, 2} (\sigma^{-1})_{ij}(\infty) x_i x_j\right], \tag{5.10}$$

where the stationary covariance matrix

$$\sigma(\infty) = \sigma^{(s)}(\infty) = \begin{pmatrix} \sigma_{11}^{(s)}(\infty) & \sigma_{12}^{(s)}(\infty) \\ \sigma_{12}^{(s)}(\infty) & \sigma_{22}^{(s)}(\infty) \end{pmatrix}$$

can be determined from the algebraic equation (see Eq. (5.9)):

$$\sum_{l=1}^{2} (A_{il}\sigma_{lj}^{(s)}(\infty) + \sigma_{il}^{(s)}(\infty)A_{lj}^{T}) = D_{ij}^{(s)}.$$

With the matrix elements A_{ij} given by (5.7), we obtain

$$\sigma_{11}^{(s)}(\infty) = \frac{(2\lambda(\lambda+\mu)+\omega^2)D_{11}^{(s)}+\omega^2D_{22}^{(s)}+2\omega(\lambda+\mu)D_{12}^{(s)}}{4\lambda(\lambda^2+\omega^2-\mu^2)},$$

$$\sigma_{22}^{(s)}(\infty) = \frac{\omega^2D_{11}^{(s)}+(2\lambda(\lambda-\mu)+\omega^2)D_{22}^{(s)}-2\omega(\lambda-\mu)D_{12}^{(s)}}{4\lambda(\lambda^2+\omega^2-\mu^2)},$$

$$\sigma_{12}^{(s)}(\infty) = \frac{-\omega(\lambda+\mu)D_{11}^{(s)}+\omega(\lambda-\mu)D_{22}^{(s)}+2(\lambda^2-\mu^2)D_{12}^{(s)}}{4\lambda(\lambda^2+\omega^2-\mu^2)}.$$

The explicit matrix elements $\sigma_{ij}^{(s)}$ for the three representations P, Q and W can be obtained by inserting the corresponding diffusion coefficients. The distribution functions (5.10) can be used to calculate the expectation values of the coordinate and momentum and the variances by direct integration. The following relations are noticed [37]:

$$\sigma^W_{ij}(\infty) = \frac{1}{2}(\sigma^P_{ij}(\infty) + \sigma^Q_{ij}(\infty)), i, j = 1, 2.$$

The uncertainty principle $\sigma_{11}\sigma_{22} \geq 1/16$ gives rise to the conditions $4\sigma_{11}^Q\sigma_{22}^Q \geq \sigma_{11}^Q + \sigma_{22}^Q$ and $\sigma_{11}^W\sigma_{22}^W \geq 1/16$ for the Q and W distributions, respectively.

6. Density matrix of the damped harmonic oscillator

In this section we explore the general results that follow from the master equation of the one-dimensional damped harmonic oscillator. Namely, we discuss the physically relevant solutions of the master equation, by using the method of the generating function. In particular, we provide extended solutions (including both diagonal and off-diagonal matrix elements) for different initial conditions.

The method used in this section follows closely the procedure of Jang [39]. Let us first rewrite the master equation (3.7) for the density matrix by means of the number representation. Specifically, we take the matrix elements of each term between different number states denoted by |n>, and using $a^+|n>=\sqrt{n+1}|n+1>$ and $a|n>=\sqrt{n}|n-1>$ we get

$$\frac{d\rho_{mn}}{dt} = -i\omega(m-n)\rho_{mn} + \lambda\rho_{mn} - (m+n+1)D_2\rho_{mn} +
+ \frac{1}{2}\sqrt{m(m-1)}(D_1 + \mu)\rho_{m-2,n} - \sqrt{m(n+1)}D_1\rho_{m-1,n+1} +
+ \frac{1}{2}\sqrt{(n+1)(n+2)}(D_1 - \mu)\rho_{m,n+2} + \frac{1}{2}\sqrt{(m+1)(m+2)}(D_1^* - \mu)\rho_{m+2,n} -
- \sqrt{m+1}nD_1^*\rho_{m+1,n-1} + \frac{1}{2}\sqrt{(n-1)n}(D_1^* + \mu)\rho_{m,n-2} +
+ \sqrt{(m+1)(n+1)}(D_2 + \lambda)\rho_{m+1,n+1} + \sqrt{mn}(D_2 - \lambda)\rho_{m-1,n-1}.$$
(6.1)

Here, we have used the abbreviated notation

$$\rho_{mn} = \langle m|\rho(t)|n \rangle$$
.

This master equation is complicated in form and in indices involved. It comprises not only the density matrix in symmetrical forms, such as $\rho_{m\pm 1,n\pm 1}$, but also those matrix elements in asymmetrical forms like $\rho_{m\pm 2,n}, \rho_{m,n\pm 2}$ and $\rho_{m\mp 1,n\pm 1}$. In order to solve Eq. (6.1) we use the method of a generating function which allows us to eliminate the variety of indices m and n implicated in the equation. When we define the double-fold generating function by

$$G(x, y, t) = \sum_{m,n} \frac{1}{\sqrt{m!n!}} x^m y^n \rho_{mn}(t),$$
 (6.2)

the density matrix can be evaluated from the inverse relation of Eq. (6.2):

$$\rho_{mn}(t) = \frac{1}{\sqrt{m!n!}} \left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n G(x, y, t)|_{x=y=0},\tag{6.3}$$

provided that the generating function is calculated beforehand. When we multiply both sides of Eq. (6.1) by $x^m y^n / \sqrt{m!n!}$ and sum over the result, we get a linear second order partial differential equation for G(x, y, t), namely

$$\frac{\partial}{\partial t}G(x,y,t) = \{ [-(i\omega + D_2)x - D_1^*y] \frac{\partial}{\partial x} + [-D_1x + (i\omega - D_2)y] \frac{\partial}{\partial y} + (D_2 + \lambda) \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} [(D_1^* - \mu) \frac{\partial^2}{\partial x^2} + (D_1 - \mu) \frac{\partial^2}{\partial y^2}] + (D_1 - \mu) \frac{\partial^2}{\partial y^2} + (D_2 - \lambda)(xy - 1)] \} G(x,y,t).$$
(6.4)

A special solution of Eq. (6.4) can be taken as

$$G(x,y,t) = \frac{1}{A} \exp\{xy - [B(x-C)^2 + D(y-E)^2 + F(x-C)(y-E)]/H\},$$
 (6.5)

where A, B, C, D, E, F and H are unknown functions of time which are to be determined. When we first substitute the expression (6.5) for G(x, y, t) into Eq. (6.4) and equate the coefficients of equal powers of x, y and xy on both sides of the equation, we get the following differential equations for the functions A, B, D, F and H:

$$-\frac{1}{A}\frac{dA}{dt} = -(D_1^* - \mu)\frac{B}{H} - (D_1 - \mu)\frac{D}{H} - (D_2 + \lambda)\frac{F}{H} + 2\lambda,\tag{6.6}$$

$$\frac{d}{dt}(\frac{B}{H}) = 2(\lambda - i\omega)\frac{B}{H} - \mu\frac{F}{H} - \frac{1}{2}(D_1 - \mu)\frac{F^2}{H^2} - 2(D_2 + \lambda)\frac{FB}{H^2} - 2(D_1^* - \mu)\frac{B^2}{H^2}, \quad (6.7)$$

$$\frac{d}{dt}(\frac{D}{H}) = 2(\lambda + i\omega)\frac{D}{H} - \mu\frac{F}{H} - \frac{1}{2}(D_1^* - \mu)\frac{F^2}{H^2} - 2(D_2 + \lambda)\frac{DF}{H^2} - 2(D_1 - \mu)\frac{D^2}{H^2}, \quad (6.8)$$

$$\frac{d}{dt}(\frac{F}{H}) = 2\lambda \frac{F}{H} - (D_2 + \lambda) \frac{F^2}{H^2} - 2\mu(\frac{B}{H} + \frac{D}{H}) - 4(D_2 + \lambda) \frac{DB}{H^2} - 2(D_1^* - \mu) \frac{BF}{H^2} - 2(D_1 - \mu) \frac{DF}{H^2}.$$
(6.9)

In addition to these equations, we get for the functions C and E

$$2B\frac{dC}{dt} + F\frac{dE}{dt} = (-2(\lambda - i\omega)B + \mu F)C + (2\mu B - (\lambda + i\omega)F)E, \tag{6.10}$$

$$2D\frac{dE}{dt} + F\frac{dC}{dt} = (-2(\lambda + i\omega)D + \mu F)E + (2\mu D - (\lambda - i\omega)F)C. \tag{6.11}$$

The equations (6.10) and (6.11) can be reformulated in order to eliminate the functions B, D and F, provided $BD - F^2/4 \neq 0$. We obtain

$$\frac{dC}{dt} = -(\lambda - i\omega)C + \mu E, \qquad (6.12)$$

$$\frac{dE}{dt} = -(\lambda + i\omega)E + \mu C. \tag{6.13}$$

The functions A, B, D, F and H are connected by the auxiliary condition that $Tr\rho$ is independent of time. The trace of ρ can be evaluated by summing the diagonal matrix elements ρ_{nn} given in Eq. (6.3) or directly by using the integral expression

$$Tr\rho = \sum_{n=0}^{\infty} \rho_{nn} = \frac{1}{(2\pi)^2} \int \exp(-k_1 k_2) \exp(ik_1 x + ik_2 y) G(x, y, t) dk_1 dk_2 dx dy.$$

We obtain with the generating function (6.5)

$$Tr(\rho) = \left(\frac{4A^2}{H^2}\left(\frac{F^2}{4} - BD\right)\right)^{-1/2}.$$
 (6.14)

This quantity is time-independent which can be verified by constructing an equation satisfied by the quantity $(F^2/4 - BD)/H^2$. Combining Eqs. (6.7)-(6.9) we get

$$\frac{d}{dt}(\frac{F^2/4 - BD}{H^2}) = 2[2\lambda - (D_1^* - \mu)\frac{B}{H} - (D_1 - \mu)\frac{D}{H} - (D_2 + \lambda)\frac{F}{H}](\frac{F^2/4 - BD}{H^2}).$$

We see immediately that the first factor on the right-hand side of this equation is identical with the right-hand side of Eq. (6.6). Accordingly, we find

$$\frac{d}{dt}((\frac{F^2}{4} - BD)\frac{A^2}{H^2}) = 0.$$

Since the scaling function H is arbitrary, we simplify the following equations by the choice

$$\frac{F^2}{4} - BD = -H. ag{6.15}$$

Setting $Tr\rho = 1$, we obtain from Eqs. (6.14) and (6.15) the normalization constant $A^2 = -H/4$. As a consequence, we can simplify Eqs. (6.7)-(6.9) by eliminating the function H from these equations. The resulting three equations are

$$\frac{dB}{dt} = -2(\lambda + i\omega)B - \mu F + 2(D_1 - \mu),$$

$$\frac{dD}{dt} = -2(\lambda - i\omega)D - \mu F + 2(D_1^* - \mu),$$

$$\frac{dF}{dt} = -2\mu(B + D) - 2\lambda F - 4(D_2 + \lambda).$$
(6.16)

These equations imply that the function D is complex conjugate to B, provided that the function F is real.

In order to integrate the equations for the time-dependent functions B, C, D, E and F we start with Eqs. (6.12) and (6.13). These equations imply that the function E is complex conjugate to the function C. By solving the coupled equations we find:

$$C(t) = E^*(t) = u(t)C(0) - v(t)C^*(0), (6.17)$$

where u(t) and v(t) are given by (4.13) and (4.14) for the two considered cases: overdamped and underdamped, respectively. For integrating the system (6.16) we proceed in the same way as for integrating the system (3.13). With the assumption that F is real and

$$D(t) = B^*(t) = R(t) + iI(t),$$

we obtain explicitly:

$$R(t) = \frac{1}{2} (e^{-2\mu_+ t} + e^{-2\mu_- t}) \widetilde{R} + \frac{1}{2} (e^{-2\mu_+ t} - e^{-2\mu_- t}) (\frac{\omega}{\gamma} \widetilde{I} + \frac{\mu}{2\gamma} \widetilde{F}) + R(\infty),$$

$$\begin{split} I(t) &= e^{-2\lambda t} (\frac{\mu^2}{\gamma^2} \widetilde{I} + \frac{\omega \mu}{2\gamma^2} \widetilde{F}) - \\ &- \frac{1}{2} (e^{-2\mu + t} + e^{-2\mu - t}) (\frac{\omega^2}{\gamma^2} \widetilde{I} + \frac{\omega \mu}{2\gamma^2} \widetilde{F}) - \frac{\omega}{2\gamma} (e^{-2\mu + t} - e^{-2\mu - t}) \widetilde{R} + I(\infty), \\ F(t) &= -e^{-2\lambda t} (\frac{2\omega \mu}{\gamma^2} \widetilde{I} + \frac{\omega^2}{\gamma^2} \widetilde{F}) + \\ &+ (e^{-2\mu + t} + e^{-2\mu - t}) (\frac{\omega \mu}{\gamma^2} \widetilde{I} + \frac{\mu^2}{2\gamma^2} \widetilde{F}) + \frac{\mu}{\gamma} (e^{-2\mu + t} - e^{-2\mu - t}) \widetilde{R} + F(\infty), \end{split}$$

where we used the notations:

$$\mu_{\pm} = \lambda \pm \gamma, \gamma \equiv \sqrt{\mu^2 - \omega^2},$$

$$\widetilde{R} = R(0) - R(\infty), \widetilde{I} = I(0) - I(\infty), \widetilde{F} = F(0) - F(\infty).$$

We can also obtain the connection between the asymptotic values of B(t), D(t), F(t) and the coefficients D_1 , D_2 , μ and λ :

$$R(\infty) = ReD(\infty) = \frac{\lambda(ReD_1 - \mu) + \omega ImD_1 + \mu(D_2 + \lambda)}{\lambda^2 - \gamma^2},$$

$$I(\infty) = ImD(\infty) = \frac{\omega\lambda(ReD_1 - \mu) + (\mu^2 - \lambda^2)ImD_1 + \omega\mu(D_2 + \lambda)}{\lambda(\lambda^2 - \gamma^2)},$$

$$F(\infty) = -2\frac{\mu[\lambda(ReD_1 - \mu) + \omega ImD_1] + (\lambda^2 + \omega^2)(D_2 + \lambda)}{\lambda(\lambda^2 - \gamma^2)}.$$

When all explicit expressions for A, B, C, D, E, F and H are introduced into Eq. (6.5), we obtain an analytical form of the generating function G(x, y, t) which allows us to evaluate the density matrix.

If the constants involved in the generating function satisfy the relations

$$C(0) = 0, R(0) = R(\infty), I(0) = I(\infty), F(0) = F(\infty),$$

we obtain the stationary solution

$$C(t) = E(t) = 0, R(t) = R(0), I(t) = I(0), F(t) = F(0),$$

so that

$$D(t) = B^*(t) = R(0) + iI(0),$$

$$H(t) = -4A^2(t) = R^2(0) + I^2(0) - F^2(0)/4.$$

Then the stationary solution of Eq. (6.4) is

$$G(x,y,t) = \frac{1}{A} \exp\{(1 - \frac{F}{H})xy - (Bx^2 + B^*y^2)/H\}.$$
 (6.18)

In addition, for a thermal bath [17] with

$$\frac{m\omega D_{qq}}{\hbar} = \frac{D_{pp}}{\hbar m\omega}, D_{pq} = 0, \mu = 0,$$

the stationary generating function is simply given by

$$G(x,y) = \frac{2\lambda}{D_2 + \lambda} \exp\left[\frac{D_2 - \lambda}{D_2 + \lambda}xy\right].$$

The same generating function can be found for large times, if the asymptotic state is a Gibbs state with $\mu = 0$. In this case we obtain with Eq. (3.26) and $\mu = 0$

$$D_2 = \lambda \coth \frac{\hbar \omega}{2kT}$$

and

$$G(x,y) = (1 - \exp(-\frac{\hbar\omega}{kT})) \exp(\exp(-\frac{\hbar\omega}{kT})xy).$$

The density matrix can be calculated with Eq. (6.3) and yields the Bose-Einstein distribution

$$\rho = (1 - \exp(-\frac{\hbar\omega}{kT})) \exp(-\frac{n\hbar\omega}{kT}) \delta_{nm}.$$

A formula for the density matrix can be written down by applying the relation (6.3) to the generating function (6.5). We get

$$\langle m|\rho(t)|n\rangle = \frac{\sqrt{m!n!}}{A} \exp\left[-(BC^2 + DE^2 + FCE)/H\right] \times$$

$$\sum_{n_1, n_2, n_3 = 0} \frac{(1 - \frac{F}{H})^{n_3} (-\frac{B}{H})^{n_1} (-\frac{D}{H})^{n_2} (2\frac{BC}{H} + \frac{FE}{H})^{m-2n_1 - n_3} (2\frac{DE}{H} + \frac{FC}{H})^{n-2n_2 - n_3}}{n_1! n_2! n_3! (m - 2n_1 - n_3)! (n - 2n_2 - n_3)!}.$$
 (6.19)

In the case that the functions C(t) and E(t) vanish, the generating function has the form of Eq. (6.18). Then the elements of the density matrix with an odd sum m+n are zero: $\rho_{mn}=0$ for m+n=2k+1 with k=0,1,2,... The lowest non-vanishing elements are given with $\rho_{mn}=\rho_{nm}$ as

$$\rho_{00} = \frac{1}{A}, \rho_{20} = -\frac{\sqrt{2}B}{AH}, \rho_{11} = \frac{1}{A}(1 - \frac{F}{H}),$$

$$\rho_{22} = \frac{2BB^*}{AH^2} + \frac{1}{A}(-\frac{F}{H})^2, \rho_{31} = -(1 - \frac{F}{H})\frac{\sqrt{6}B}{AH}, \rho_{40} = \frac{\sqrt{6}B^2}{AH^2}.$$

It is also possible to choose the constants in such a way that the functions B and D vanish at time t = 0 and F(0) = H(0). Then the density matrix (6.19) becomes at t = 0 ($E = C^*$):

$$< m|\rho(0)|n> = \frac{1}{\sqrt{m!n!}} (C^*(0))^m (C(0))^n \exp(-|C(0)|^2).$$
 (6.20)

This is the initial Glauber packet. The diagonal matrix elements of Eq. (6.20) represent a Poisson distribution used also in the study of multi-phonon excitations in nuclear physics. In the particular case when we assume

$$D_1 = \mu = 0, D_2 = \lambda,$$

$$B(0) = D(0) = 0, F(0) = H(0) = -4,$$

the differential equations (6.16) yield B(t) = D(t) = 0 and F(t) = H(t) = -4. Then the density matrix subject to the initial Glauber packet is (see also [39])

$$< m|\rho(t)|n> = \frac{1}{\sqrt{m!n!}} (C^*(t))^m (C(t))^n \exp(-|C(t)|^2),$$

where C(t) is given by Eq. (6.17).

4. Conclusions

The Lindblad theory provides a selfconsistent treatment of damping as a possible extension of quantum mechanics to open systems. In the present paper first we studied the damped quantum oscillator by using the Schrödinger and Heisenberg representations. According to this theory we have calculated the damping of the expectation values of coordinate and momentum and the variances as functions of time. The resulting time dependence of the expectation values yields an exponential damping. Second we have also shown how the quasiprobability distributions can be used to solve the problem of dissipation for the harmonic oscillator. From the master equation of the damped quantum oscillator we have derived the corresponding Fokker-Planck equations in the Glauber P, the antinormal ordering Q and the Wigner W representations and have made a comparative study of these quasiprobability distributions. We have proven that the variances found from the Fokker-Planck equations in these representations are the same. We have solved these equations in the steady state and showed that the Glauber P function (when it exists), the Q and the Wigner W functions are two-dimensional Gaussians with different widths. Finally, we have calculated the time evolution of the density matrix. For this purpose we applied the method of the generating function of the density matrix. In this case the density matrix can be obtained by taking partial derivatives of the generating function. The generating function depends on a set of time-dependent coefficients which can be calculated as solutions of linear differential equations of first order. Depending on the initial conditions for these coefficients, the density matrix evolves differently in time. For a thermal bath, when the asymptotic state is a Gibbs state, a Bose-Einstein distribution results as density matrix. Also for the case that the initial density matrix is chosen as a Glauber packet, a simple analytical expression for the density matrix has been derived. The density matrix can be used in various physical applications where a Bosonic degree of freedom moving in a harmonic oscillator potential is damped. For example, one needs to determine nondiagonal transition elements of the density matrix, if an oscillator is perturbed by a weak electromagnetic field in addition to its coupling to a heat bath. The density matrix can also be derived from the solution of the Fokker-Planck equation for the coherent state representation.

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