BETHE ALGEBRA OF THE $\mathfrak{gl}_{N+1}$ GAUDIN MODEL AND ALGEBRA OF FUNCTIONS ON THE CRITICAL SET OF THE MASTER FUNCTION

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ABSTRACT. Consider a tensor product of finite-dimensional irreducible $\mathfrak{gl}_{N+1}$-modules and its decomposition into irreducible modules. The $\mathfrak{gl}_{N+1}$ Gaudin model assigns to each multiplicity space of that decomposition a commutative (Bethe) algebra of linear operators acting on the multiplicity space. The Bethe ansatz method is a method to find eigenvectors and eigenvalues of the Bethe algebra. One starts with a critical point of a suitable (master) function and constructs an eigenvector of the Bethe algebra.

In this paper we consider the algebra of functions on the critical set of the associated master function and show that the action of this algebra on itself is isomorphic to the action of the Bethe algebra on a suitable subspace of the multiplicity space.

As a byproduct we prove that the Bethe vectors corresponding to different critical points of the master function are linearly independent and, in particular, nonzero.

1. INTRODUCTION

Let $L_{\boldsymbol{\lambda}}$ be the irreducible finite-dimensional $\mathfrak{gl}_{N+1}$-module of highest weight $\boldsymbol{\lambda}$. Let $L_{\boldsymbol{\Lambda}} = \otimes_{s=1}^{n} L_{\boldsymbol{\lambda}^{(s)}}$ be a tensor product of such modules, and $L_{\boldsymbol{\Lambda}} = \oplus_{\boldsymbol{\lambda}^{(\infty)}} L_{\boldsymbol{\lambda}^{(\infty)}} \otimes W_{\boldsymbol{\lambda}^{(\infty)}}$ the decomposition into irreducible representations. The multiplicity space $W_{\boldsymbol{\lambda}^{(\infty)}}$ of $L_{\boldsymbol{\lambda}^{(\infty)}}$ can be identified with $\operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}] \subset L_{\boldsymbol{\Lambda}}$, the subspace of singular vectors of weight $\boldsymbol{\lambda}^{(\infty)}$. To each multiplicity space $\operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$ and distinct complex numbers $z_1, \ldots, z_n$, the $\mathfrak{gl}_{N+1}$ Gaudin model assigns a commutative subalgebra of $\operatorname{End}(\operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}])$ called the Bethe algebra and denoted by $\mathcal{B}_{\boldsymbol{\Lambda}, \boldsymbol{\lambda}^{(\infty)}, \boldsymbol{z}}$.

The $\mathcal{B}_{\mathbf{\Lambda},\boldsymbol{\lambda}^{(\infty)},\boldsymbol{z}}$-module Sing $L_{\mathbf{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$ has an interesting geometric realization. In [MTV2] we constructed an isomorphism of the $\mathcal{B}_{\mathbf{\Lambda},\boldsymbol{\lambda}^{(\infty)},\boldsymbol{z}}$-module Sing $L_{\mathbf{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$ and the regular representation of the algebra of functions on the scheme-theoretical intersection of suitable Schubert cycles. This isomorphism can be viewed as the geometric Langlands correspondence in the $\mathfrak{gl}_{N+1}$ Gaudin model. In [MTV4] we argued that this geometric Langlands correspondence extends to the third, equally important, player — the algebra of functions on the critical set of the corresponding master function. In this paper we prove another result supporting that principle.

The master and weight functions are useful objects associated with each multiplicity space Sing $L_{\mathbf{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$, see [SV]. They are functions of some auxiliary variables $\boldsymbol{t} = (t_j^{(i)})$. The master function $\Phi(\boldsymbol{t})$ is a scalar function and the weight function $\omega(\boldsymbol{t})$ is an $L_{\mathbf{\Lambda}}$-valued function. They are used in the Bethe ansatz method to construct eigenvectors of the Bethe algebra $\mathcal{B}_{\mathbf{\Lambda},\boldsymbol{\lambda}^{(\infty)},\boldsymbol{z}}$. Namely, if $\boldsymbol{p}$ is a critical point of the master function then the vector $\omega(\boldsymbol{p})$ lies in Sing $L_{\mathbf{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$ and is an eigenvector of the Bethe algebra, see [MTV1].

In this paper we consider the algebra A_Φ of functions on the critical set of the master function. With the help of the weight function, we construct a linear embedding $\alpha : A_\Phi \to$ Sing $L_{\mathbf{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$ and show that $\alpha(A_\Phi)$ is a $\mathcal{B}_{\mathbf{\Lambda},\boldsymbol{\lambda}^{(\infty)},\boldsymbol{z}}$-submodule of Sing $L_{\mathbf{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$. We denote the image of $\mathcal{B}$ in $\mathrm{End}(\alpha(A_\Phi))$ by A_B. We construct an algebra isomorphism $\beta : A_\Phi \to A_B$ and show that the A_B-module $\alpha(A_\Phi)$ is isomorphic to the regular representation of A_Φ. That statement is our main result, see Theorems 5.5 and 7.1. As a byproduct we show that for any critical point $\boldsymbol{p}$ of the master function, the vector $\omega(\boldsymbol{p})$ is nonzero. For a nondegenerate critical point that fact was proved in [MV2] and [V].

The paper is organized as follows. In Section 2 we define the master function Φ and the algebra A_Φ of functions on the critical set C_Φ of the master function. The algebra A_Φ is the direct sum of local algebras $A_{\boldsymbol{p},\Phi}$ corresponding to points $\boldsymbol{p} \in C_\Phi$. In Theorem 2.1 we describe useful generators of the algebra $A_{\boldsymbol{p},\Phi}$. We prove Theorem 2.1 in Section 3. In Section 4 the Bethe algebra is introduced. We define the weight function in Section 5 and formulate our first main result Theorem 5.5. We prove Theorem 5.5 in Section 6. Our second main result, Theorem 7.1 is formulated and proved in Section 7.

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2. ALGEBRA A_Φ

2.1. **Lie algebra $\mathfrak{gl}_{N+1}$.** Let e_{ij}, $i, j = 1, \ldots, N + 1$, be the standard generators of the Lie algebra $\mathfrak{gl}_{N+1}$ satisfying the relations $[e_{ij}, e_{sk}] = \delta_{js}e_{ik} - \delta_{ik}e_{sj}$. Let $\mathfrak{h} \subset \mathfrak{gl}_{N+1}$ be the Cartan subalgebra generated by e_{ii}, $i = 1, \ldots, N + 1$. Let $\mathfrak{h}^*$ be the dual space. Let ϵ_i, $i = 1, \ldots, N + 1$, be the basis of $\mathfrak{h}^*$ dual to the basis e_{ii}, $i = 1, \ldots, N + 1$, of $\mathfrak{h}$. Let $\alpha_1, \ldots, \alpha_N \in \mathfrak{h}^*$ be simple roots, $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Let $(,)$ be the standard scalar product on $\mathfrak{h}^*$ such that the basis ϵ_i, $i = 1, \ldots, N + 1$, is orthonormal.

A sequence of integers $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{N+1})$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{N+1} \geqslant 0$ is called a *partition with at most $N + 1$ parts*. Denote $|\boldsymbol{\lambda}| = \sum_{i=1}^{N+1} \lambda_i$. We identify partitions $\boldsymbol{\lambda}$ with vectors $\lambda_1 \epsilon_1 + \cdots + \lambda_{N+1} \epsilon_{N+1}$ of $\mathfrak{h}^*$.

2.2. **Master function.** Let $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(n)})$ be a collection of partitions, where $\boldsymbol{\lambda}^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_{N+1}^{(i)})$ and $\lambda_{N+1}^{(i)} = 0$. Let $\boldsymbol{l} = (l_1, \ldots, l_N)$ be nonnegative integers such that

$$\boldsymbol{\lambda}^{(\infty)} = \sum_{i=1}^{n} \boldsymbol{\lambda}^{(i)} - \sum_{j=1}^{N} l_j \alpha_j$$

is a partition. Denote $l = l_1 + \cdots + l_N$,

$$\boldsymbol{t} = (t_1^{(1)}, \ldots, t_{l_1}^{(1)}, t_1^{(2)}, \ldots, t_{l_2}^{(2)}, \ldots, t_1^{(N)}, \ldots, t_{l_N}^{(N)}) .$$

Fix a collection of distinct complex numbers $\boldsymbol{z} = (z_1, \ldots, z_n)$. Let $\Phi(\boldsymbol{t})$ be the master function associated with this data,

$$\Phi(\boldsymbol{t}) = \prod_{i=1}^{N} \prod_{1 \leqslant j < j' \leqslant l_i} (t_j^{(i)} - t_{j'}^{(i)})^2 \prod_{i=1}^{N-1} \prod_{j=1}^{l_i} \prod_{j'=1}^{l_{i+1}} (t_j^{(i)} - t_{j'}^{(i+1)})^{-1} \prod_{i=1}^{N} \prod_{j=1}^{l_i} \prod_{s=1}^{n} (t_j^{(i)} - z_s)^{-(\boldsymbol{\lambda}^{(s)}, \alpha_i)}.$$

Denote

$$U = \{\boldsymbol{p} \in \mathbb{C}^l \mid \Phi \text{ is well-defined at } \boldsymbol{p} \text{ and } \Phi(\boldsymbol{p}) \neq 0\}. \tag{2.1}$$

The set U is the complement in $\mathbb{C}^l$ to a union of hyperplanes. The master function is a rational function regular on U.

Denote by $\mathbb{C}(\boldsymbol{t})_U$ the algebra of rational functions on $\mathbb{C}^l$ regular on U. The partial derivatives

$$\Psi_{ij} = \partial(\log \Phi)/\partial t_j^{(i)}, \qquad i = 1, \ldots, N, \quad j = 1, \ldots, l_i ,$$

are elements of $\mathbb{C}(\boldsymbol{t})_U$. Denote by $I_\Phi \subset \mathbb{C}(\boldsymbol{t})_U$ the ideal generated by Ψ_{ij}, $i = 1, \ldots, N$, $j = 1, \ldots, l_i$, and set

$$A_\Phi = \mathbb{C}(\boldsymbol{t})_U/I_\Phi. \tag{2.2}$$

Denote by C_Φ the zero set of the ideal. The zero set is finite, [MV1]. The algebra A_Φ is finite-dimensional and is the direct sum of local algebras,

$$A_\Phi = \oplus_{\boldsymbol{p} \in C_\Phi} A_{\boldsymbol{p}, \Phi}$$

corresponding to points $\boldsymbol{p} \in C_\Phi$. For $\boldsymbol{p} \in C_\Phi$, the local algebra $A_{\boldsymbol{p}, \Phi}$ may be defined as the quotient of the algebra of germs at $\boldsymbol{p}$ of holomorphic functions modulo the ideal $I_{\boldsymbol{p}, \Phi}$ generated by all the functions Ψ_{ij}. The algebra $A_{\boldsymbol{p}, \Phi}$ contains the maximal ideal $\mathfrak{m}_{\boldsymbol{p}}$ generated by the germs of functions equal to zero at $\boldsymbol{p}$.

2.3. **Generators of the local algebra of a critical point.** Let u be a variable. Define an N-tuple of polynomials $T_1, \ldots, T_N \in \mathbb{C}[u]$,

$$T_i(u) = \prod_{s=1}^{n} (u - z_s)^{(\boldsymbol{\lambda}^{(s)}, \alpha_i)},$$

an N-tuple of polynomials $y_1, \ldots, y_N \in \mathbb{C}[u, \boldsymbol{t}]$,

$$y_i(u, \boldsymbol{t}) = \prod_{j=1}^{l_i} (u - t_j^{(i)}),$$

and the differential operator

$$\mathcal{D}_\Phi = (\partial_u - \log'(\frac{T_1 \ldots T_N}{y_1}))$$

$$\times (\partial_u - \log'(\frac{y_1 T_2 \ldots T_N}{y_2})) \ldots (\partial_u - \log'(\frac{y_{N-1} T_N}{y_N}))(\partial_u - \log'(y_N)),$$

where $\partial_u = d/du$ and $\log' f$ denotes $(df/du)/f$. We have

$$\mathcal{D}_\Phi = \partial_u^{N+1} + \sum_{i=1}^{N+1} G_i \partial_u^{N+1-i}, \qquad G_i = \sum_{j=i}^{\infty} G_{ij} u^{-j}, \tag{2.3}$$

where $G_{ij} \in \mathbb{C}[\boldsymbol{t}]$.

For $\boldsymbol{p} \in C_\Phi$, $f \in \mathbb{C}(\boldsymbol{t})_U$ denote by $\bar{f}$ the image of f in $A_{\boldsymbol{p}, \Phi}$. Denote

$$\bar{\mathcal{D}}_\Phi = \partial_u^{N+1} + \sum_{i=1}^{N+1} \bar{G}_i \partial_u^{N+1-i},$$

where $\bar{G}_i = \sum_{j=i}^{\infty} \bar{G}_{ij} u^{-j}$.

Theorem 2.1. *For any $\boldsymbol{p} \in C_\Phi$, the elements $\bar{G}_{ij}$, $i = 1, \ldots, N$, $j \geqslant i$, generate $A_{\boldsymbol{p}, \Phi}$.*

Theorem 2.1 is proved in Section 3.4.

2.4. **Polynomials h_i.** Let A be a commutative algebra. For $g_1, \ldots, g_i \in A[u]$, denote by $\mathrm{Wr}(g_1(u), \ldots, g_i(u))$ the Wronskian,

$$\mathrm{Wr}(g_1(u), \ldots, g_i(u)) = \det \begin{pmatrix} g_1(u) & g_1'(u) & \ldots & g_1^{(i-1)}(u) \\ g_2(u) & g_2'(u) & \ldots & g_2^{(i-1)}(u) \\ \ldots & \ldots & \ldots & \ldots \\ g_i(u) & g_i'(u) & \ldots & g_i^{(i-1)}(u) \end{pmatrix},$$

where $g^{(j)}(u)$ denotes the j-th derivative of $g(u)$ with respect to u.

Introduce a set

$$P = \{d_1, d_2, \ldots, d_{N+1}\}, \qquad d_i = \lambda_i^{(\infty)} + N + 1 - i. \tag{2.4}$$

Theorem 2.2. *There exist unique polynomials $h_1, \ldots, h_{N+1} \in A_{\boldsymbol{p}, \Phi}[u]$ of the form*

$$h_i = u^{d_i} + \sum_{j=1, \ d_i - j \notin P}^{d_i} h_{ij} u^{d_i - j} \tag{2.5}$$

such that $h_{N+1} = y_N$ and

$$\mathrm{Wr}(h_{N+1}, h_N \ldots, h_{N+1-j}) \ = \ y_{N-j} T_N^j T_{N-1}^{j-1} \ldots T_{N-j+1}^1 \prod_{N+1-j \leqslant i < i' \leqslant N+1} (d_i - d_{i'}) \tag{2.6}$$

for $j = 1, \ldots, N$, where $y_0 = 1$. Moreover, each of the polynomials $h_1, \ldots, h_{N+1}$ is a solution of the differential equation $\bar{\mathcal{D}}_\Phi h(u) = 0$. $\qquad\square$

Proof. The existence of unique polynomials h_i satisfying (2.6) is proved in [BMV] gener-alizing the corresponding result in Section 5 of [MV1]. The fact that the polynomials h_i satisfy the differential equation $\bar{\mathcal{D}}_\Phi h(u) = 0$ is proved like in Section 5 of [MV1]. $\qquad\square$

Lemma 2.3. *The subalgebra of $A_{\boldsymbol{p}, \Phi}$ generated by elements $\bar{G}_{ij}$, $i = 1, \ldots, N$, $j \geqslant i$, contains all the coefficients h_{ij}, $i = 1, \ldots, N+1$, $j = 1, \ldots, d_i$, $d_i - j \notin P$.*

The proof of the lemma is the same as the proof of Lemma 3.4 in [MTV2].

3. Algebra A_{Gr}

3.1. **Algebra $\mathcal{O}_{\boldsymbol{\lambda}^{(\infty)}}$.** Let $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(n)})$, $\boldsymbol{\lambda}^{(\infty)}$, $\boldsymbol{z} = (z_1, \ldots, z_n)$ be partitions and numbers as in Section 2.2. Let d be a natural number such that $d - N - 1 \geqslant \lambda_1^{(\infty)}$ and $d - N - 1 \geqslant \lambda_1^{(i)}$ for $i = 1, \ldots, n$.

Let $\mathbb{C}_d[u]$ be the space of polynomials in u of degree less than d. Let $\mathrm{Gr}(N+1, d)$ be the Grassmannian of all $N + 1$-dimensional subspaces of $\mathbb{C}_d[u]$.

For a complete flag $\mathcal{F} = \{0 \subset F_1 \subset F_2 \subset \cdots \subset F_d = \mathbb{C}_d[u]\}$ and a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{N+1})$ with $\lambda_1 \leqslant d - N - 1$, define a Schubert cell $\Omega_{\boldsymbol{\lambda}}(\mathcal{F}) \subset \mathrm{Gr}(N+1, d)$,

$$\Omega_{\boldsymbol{\lambda}}(\mathcal{F}) = \{\boldsymbol{q} \in \mathrm{Gr}(N+1, d) \mid \dim(\boldsymbol{q} \cap F_{d-j-\lambda_j}) = N + 1 - j \, , \ \dim(\boldsymbol{q} \cap F_{d-j-\lambda_j-1}) = N - j\} \, .$$

We have $\mathrm{codim} \, \Omega_{\boldsymbol{\lambda}}(\mathcal{F}) = |\boldsymbol{\lambda}|$.

Let $P = \{d_1, d_2, \ldots, d_{N+1}\}$ be defined in (2.4). Introduce a new partition

$$\boldsymbol{\lambda}^{(\vee)} = (d - N - 1 - \lambda_N^{(\infty)}, d - N - 1 - \lambda_{N-1}^{(\infty)}, \ldots, d - N - 1 - \lambda_1^{(\infty)}) \, . \tag{3.1}$$

Denote

$$\mathcal{F}(\infty) = \{0 \subset \mathbb{C}_1[u] \subset \mathbb{C}_2[u] \subset \cdots \subset \mathbb{C}_d[u]\} \, .$$

Consider the Schubert cell $\Omega_{\boldsymbol{\lambda}^{(\vee)}}(\mathcal{F}(\infty))$. We have $\dim \Omega_{\boldsymbol{\lambda}^{(\vee)}}(\mathcal{F}(\infty)) = |\boldsymbol{\lambda}^{(\infty)}|$.

The Schubert cell $\Omega_{\boldsymbol{\lambda}^{(\vee)}}(\mathcal{F}(\infty))$ consists of $N + 1$-dimensional subspaces $\boldsymbol{q} \subset \mathbb{C}_d[u]$ with a basis $\{f_1, \ldots, f_{N+1}\}$ of the form

$$f_i = u^{d_i} + \sum_{j=1, \ d_i - j \notin P}^{d_i} f_{ij} u^{d_i - j} \, . \tag{3.2}$$

Such a basis is unique.

Denote by $\mathcal{O}_{\boldsymbol{\lambda}(\infty)}$ the algebra of regular functions on $\Omega_{\boldsymbol{\lambda}^{(\vee)}}(\mathcal{F}(\infty))$. The cell $\Omega_{\boldsymbol{\lambda}^{(\vee)}}(\mathcal{F}(\infty))$ is an affine space with coordinate functions f_{ij}. The algebra $\mathcal{O}_{\boldsymbol{\lambda}(\infty)}$ is the polynomial algebra in variables f_{ij},

$$\mathcal{O}_{\boldsymbol{\lambda}(\infty)} = \mathbb{C}[f_{ij}, \ i = 1, \ldots, N+1, \ j = 1, \ldots, d_i, \ d_i - j \notin P]. \tag{3.3}$$

3.2. **Intersection of Schubert cells.** For $z \in \mathbb{C}$, consider the complete flag

$$\mathcal{F}(z) \ = \ \left\{ 0 \subset (u-z)^{d-1}\mathbb{C}_1[u] \subset (u-z)^{d-2}\mathbb{C}_2[u] \subset \cdots \subset \mathbb{C}_d[u] \right\}.$$

Denote by $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda}(\infty),\boldsymbol{z}}$ the set-theoretic intersection and by A_{Gr} the scheme-theoretic intersection of the $n+1$ Schubert cells $\Omega_{\boldsymbol{\lambda}^{(\vee)}}(\mathcal{F}(\infty))$, $\Omega_{\boldsymbol{\lambda}^{(s)}}(\mathcal{F}(z_s))$, $s = 1, \ldots, n$, see Section 4 of [MTV2]. The set-theoretic intersection $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda}(\infty),\boldsymbol{z}}$ is a finite set and the scheme-theoretic intersection A_{Gr} is a finite-dimensional algebra ("of functions on $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda}(\infty),\boldsymbol{z}}$"). The algebra of functions on $\Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda}(\infty),\boldsymbol{z}}$ is the direct sum of local algebras,

$$A_{\mathrm{Gr}} \ = \ \oplus_{\boldsymbol{q} \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda}(\infty),\boldsymbol{z}}} A_{\boldsymbol{q},\mathrm{Gr}} \ ,$$

corresponding to points $\boldsymbol{q} \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda}(\infty),\boldsymbol{z}}$. The algebra A_{Gr} is the quotient of the algebra $\mathcal{O}_{\boldsymbol{\lambda}(\infty)}$ of functions on $\Omega_{\boldsymbol{\lambda}^{(\vee)}}(\infty)$ by a suitable ideal. For $\boldsymbol{q} \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda}(\infty),\boldsymbol{z}}$ and $f \in \mathcal{O}_{\boldsymbol{\lambda}(\infty)}$ denote by $\bar{f}$ the image of f in $A_{\boldsymbol{q},\mathrm{Gr}}$.

Lemma 3.1. *For any $\boldsymbol{q} \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda}(\infty),\boldsymbol{z}}$, the elements $\bar{f}_{ij}$, $i = 1, \ldots, N+1$, $j = 1, \ldots, d_i$, $d_i - j \notin P$, generate $A_{\boldsymbol{q},\mathrm{Gr}}$.* $\qquad\square$

3.3. **Isomorphism of algebras.**

Theorem 3.2 ([MV1]). *Let $\boldsymbol{p} \in C_\Phi$. Let $h_1, \ldots, h_{N+1} \in A_{\boldsymbol{p},\Phi}[u]$ be polynomials defined in Theorem 2.2. Denote by $\tilde{h}_1, \ldots, \tilde{h}_{N+1}$ the projection of the polynomials to $A_{\boldsymbol{p},\Phi}/\mathfrak{m}_{\boldsymbol{p}}[u] = \mathbb{C}[u]$. Then $\langle \tilde{h}_1, \ldots, \tilde{h}_{N+1} \rangle \in \Omega_{\boldsymbol{\Lambda},\boldsymbol{\lambda}(\infty),\boldsymbol{z}}$.* $\qquad\square$

Denote $\boldsymbol{q} = \langle \tilde{h}_1, \ldots, \tilde{h}_{N+1} \rangle$. Let $\bar{f}_{ij} \in A_{\boldsymbol{q},\mathrm{Gr}}$ be elements of Lemma 3.1. Let h_{ij} be coefficients of the polynomials $h_1, \ldots, h_{N+1}$ in Theorem 3.2.

Theorem 3.3 ([BMV]). *The map $\bar{f}_{ij} \mapsto h_{ij}$, $i = 1, \ldots, N+1$, $j = 1, \ldots, d_i$, $d_i - j \notin P$, extends uniquely to an algebra isomorphism $A_{\boldsymbol{q},\mathrm{Gr}} \to A_{\boldsymbol{p},\Phi}$.* $\qquad\square$

Corollary 3.4. *The elements h_{ij}, $i = 1, \ldots, N+1$, $j = 1, \ldots, d_i$, $d_i - j \notin P$, generate $A_{\boldsymbol{p},\Phi}$.* $\qquad\square$

3.4. **Proof of Theorem 2.1.** By Lemma 2.3 the subalgebra of $A_{\boldsymbol{p},\Phi}$ generated by all the elements $\bar{G}_{ij}$ contains all the coefficients h_{ij}. By Corollary 3.4 the coefficients h_{ij} generate $A_{\boldsymbol{p},\Phi}$. Theorem 2.1 is proved.

4. Bethe algebra

4.1. **Lie algebra $\mathfrak{gl}_{N+1}[t]$.** Let $\mathfrak{gl}_{N+1}[t] = \mathfrak{gl}_{N+1} \otimes \mathbb{C}[t]$ be the Lie algebra of $\mathfrak{gl}_{N+1}$-valued polynomials with the pointwise commutator. For $g \in \mathfrak{gl}_{N+1}$, we set $g(u) = \sum_{s=0}^{\infty}(g \otimes t^s)u^{-s-1}$.

We identify $\mathfrak{gl}_{N+1}$ with the subalgebra $\mathfrak{gl}_{N+1} \otimes 1$ of constant polynomials in $\mathfrak{gl}_{N+1}[t]$. Hence, any $\mathfrak{gl}_{N+1}[t]$-module has a canonical structure of a $\mathfrak{gl}_{N+1}$-module.

For each $a \in \mathbb{C}$, there exists an automorphism ρ_a of $\mathfrak{gl}_{N+1}[t]$, $\rho_a : g(u) \mapsto g(u-a)$. Given a $\mathfrak{gl}_{N+1}[t]$-module M, we denote by $M(a)$ the pull-back of M through the automorphism ρ_a. As $\mathfrak{gl}_{N+1}$-modules, M and $M(a)$ are isomorphic by the identity map.

We have the evaluation homomorphism, $\mathfrak{gl}_{N+1}[t] \to \mathfrak{gl}_{N+1}$, $g(u) \mapsto gu^{-1}$. Its restriction to the subalgebra $\mathfrak{gl}_{N+1} \subset \mathfrak{gl}_{N+1}[t]$ is the identity map. For any $\mathfrak{gl}_{N+1}$-module M, we denote by the same letter the $\mathfrak{gl}_{N+1}[t]$-module, obtained by pulling M back through the evaluation homomorphism.

4.2. **Definition of row determinant.** Given an algebra A and an $(N+1) \times (N+1)$-matrix $C = (c_{ij})$ with entries in A, we define its *row determinant* to be

$$\operatorname{rdet} C = \sum_{\sigma \in \Sigma_{N+1}} (-1)^{\sigma}\, c_{1\sigma(1)}c_{2\sigma(2)} \ldots c_{N+1\,\sigma(N+1)}\,.$$

4.3. **Definition of Bethe algebra.** Define the universal differential operator $\mathcal{D}_{\mathcal{B}}$ by the formula

$$\mathcal{D}_{\mathcal{B}} = \operatorname{rdet} \begin{pmatrix} \partial_u - e_{11}(u) & -e_{21}(u) & \ldots & -e_{N+1\,1}(u) \\ -e_{12}(u) & \partial_u - e_{22}(u) & \ldots & -e_{N+1\,2}(u) \\ \ldots & \ldots & \ldots & \ldots \\ -e_{1\,N+1}(u) & -e_{2\,N+1}(u) & \ldots & \partial_u - e_{N+1\,N+1}(u) \end{pmatrix}.$$

We have

$$\mathcal{D}_{\mathcal{B}} = \partial_u^{N+1} + \sum_{i=1}^{N+1} B_i\,\partial_u^{N+1-i}, \qquad B_i = \sum_{j=i}^{\infty} B_{ij}u^{-j}, \qquad B_{ij} \in U(\mathfrak{gl}_{N+1}[t])\,. \qquad (4.1)$$

The unital subalgebra of $U(\mathfrak{gl}_{N+1}[t])$ generated by B_{ij}, $i = 1, \ldots, N+1$, $j \geqslant i$, is called the *Bethe algebra* and denoted by $\mathcal{B}$.

By [T], cf. [MTV1], the algebra $\mathcal{B}$ is commutative, and $\mathcal{B}$ commutes with the subalgebra $U(\mathfrak{gl}_{N+1}) \subset U(\mathfrak{gl}_{N+1}[t])$.

As a subalgebra of $U(\mathfrak{gl}_{N+1}[t])$, the algebra $\mathcal{B}$ acts on any $\mathfrak{gl}_{N+1}[t]$-module M. Since $\mathcal{B}$ commutes with $U(\mathfrak{gl}_{N+1})$, it preserves the $\mathfrak{gl}_{N+1}$ weight subspaces of M and the subspace $\operatorname{Sing} M$ of $\mathfrak{gl}_{N+1}$-singular vectors.

If L is a $\mathcal{B}$-module, then the image of $\mathcal{B}$ in $\operatorname{End}(L)$ is called the *Bethe algebra of L*.

4.4. **Bethe algebra of** $\operatorname{Sing} L_{\mathbf{\Lambda}}[\mathbf{\lambda}^{(\infty)}]$. For a partition $\mathbf{\lambda}$ with at most $N+1$ parts denote by $L_{\mathbf{\lambda}}$ the irreducible $\mathfrak{gl}_{N+1}$-module with highest weight $\mathbf{\lambda}$.

Let $\mathbf{\Lambda} = (\mathbf{\lambda}^{(1)}, \dots, \mathbf{\lambda}^{(n)})$, $\mathbf{\lambda}^{(\infty)}$, $\mathbf{z} = (z_1, \dots, z_n)$ be partitions and numbers as in Section 2.2. Denote $L_{\mathbf{\Lambda}} = L_{\mathbf{\lambda}^{(1)}} \otimes \cdots \otimes L_{\mathbf{\lambda}^{(n)}}$. Let

$$L_{\mathbf{\Lambda}}[\mathbf{\lambda}^{(\infty)}] = \{v \in L_{\mathbf{\Lambda}} \mid e_{ii}v = \lambda_i^{(\infty)}v \text{ for } i = 1, \dots, N+1\},$$

$$\operatorname{Sing} L_{\mathbf{\Lambda}}[\mathbf{\lambda}^{(\infty)}] = \{v \in L_{\mathbf{\Lambda}}[\mathbf{\lambda}^{(\infty)}] \mid e_{ij}v = 0 \text{ for } i < j\}$$

be the subspace of vectors of $\mathfrak{gl}_{N+1}$-weight $\mathbf{\lambda}^{(\infty)}$ and the subspace of $\mathfrak{gl}_{N+1}$-singular vectors of $\mathfrak{gl}_{N+1}$-weight $\mathbf{\lambda}^{(\infty)}$, respectively. Consider on $L_{\mathbf{\Lambda}}$ the $\mathfrak{gl}_{N+1}[t]$-module structure of the tensor product of evaluation modules, $L_{\mathbf{\Lambda}} = \otimes_{s=1}^n L_{\mathbf{\lambda}^{(s)}}(z_s)$. Then $\operatorname{Sing} L_{\mathbf{\Lambda}}[\mathbf{\lambda}^{(\infty)}]$ is a $\mathcal{B}$-submodule. We denote by $\mathcal{B}_{\mathbf{\Lambda}, \mathbf{\lambda}^{(\infty)}, \mathbf{z}}$ the Bethe algebra of $\operatorname{Sing} L_{\mathbf{\Lambda}}[\mathbf{\lambda}^{(\infty)}]$.

4.5. **Shapovalov Form.** Let $\tau : \mathfrak{gl}_{N+1} \to \mathfrak{gl}_{N+1}$ be the anti-involution sending e_{ij} to e_{ji} for all (i, j). Let M be a highest weight $\mathfrak{gl}_{N+1}$-module with a highest weight vector w. The Shapovalov form S on M is the unique symmetric bilinear form such that

$$S(w, w) = 1, \qquad S(xu, v) = S(u, \tau(x)v)$$

for all $u, v \in M$ and $x \in \mathfrak{gl}_{N+1}$.

Fix highest weight vectors $v_{\mathbf{\lambda}^{(s)}} \in L_{\mathbf{\lambda}^{(s)}}$, $s = 1, \dots, n$. Define a symmetric bilinear form on the tensor product $L_{\mathbf{\Lambda}} = L_{\mathbf{\lambda}^{(1)}} \otimes \cdots \otimes L_{\mathbf{\lambda}^{(n)}}$ by the formula

$$S_{\mathbf{\Lambda}} = S_1 \otimes \cdots \otimes S_n, \tag{4.2}$$

where S_s is the Shapovalov form on $L_{\mathbf{\lambda}^{(s)}}$. The form $S_{\mathbf{\lambda}}$ is called the tensor Shapovalov form.

Theorem 4.1 ([MTV1]). *Consider $L_{\mathbf{\Lambda}}$ as the $\mathfrak{gl}_{N+1}[t]$-module $\otimes_{s=1}^n L_{\mathbf{\lambda}^{(s)}}(z_s)$. Then any element $B \in \mathcal{B}$ acts on $L_{\mathbf{\Lambda}}$ as a symmetric operator with respect to the tensor Shapovalov form, $S_{\mathbf{\Lambda}}(Bu, v) = S_{\mathbf{\Lambda}}(u, Bv)$ for any $u, v \in L_{\mathbf{\Lambda}}$.* $\square$

5. Weight function

5.1. **Definition of the weight function.** Let $\mathbf{\Lambda} = (\mathbf{\lambda}^{(1)}, \dots, \mathbf{\lambda}^{(n)})$, $\mathbf{\lambda}^{(\infty)}$, $\mathbf{z} = (z_1, \dots, z_n)$ be partitions and numbers as in Section 2.2. Recall the construction of a rational map

$$\omega : \mathbb{C}^l \to L_{\mathbf{\Lambda}}[\mathbf{\lambda}^{(\infty)}]$$

called the weight function, see [SV], cf. [M], [RSV].

Denote by $P(\mathbf{l}, n)$ the set of sequences $C = (c_1^1, \dots, c_{b_1}^1; \ \dots; \ c_1^n, \dots, c_{b_n}^n)$ of integers from $\{1, \dots, N\}$ such that for every $i = 1, \dots, N$, the integer i appears in C precisely l_i times.

Denote by $\Sigma(C)$ the set of all bijections σ of the set $\{1, \dots, l\}$ onto the set of variables $\{t_1^{(1)}, \dots, t_{l_1}^{(1)}, t_1^{(2)}, \dots, t_{l_2}^{(2)}, \dots, t_1^{(N)}, \dots, t_{l_N}^{(N)}\}$ with the following property. For every $a = 1, \dots, l$ the a-th element of the sequence C equals i, if $\sigma(a) = t_j^{(i)}$.

To every $C \in P(\boldsymbol{l}, n)$ we assign a vector

$$e_C v \;=\; e_{c_1^1+1,c_1^1} \ldots e_{c_{b_1}^1+1,c_{b_1}^1} v_{\boldsymbol{\lambda}^{(1)}} \otimes \cdots \otimes e_{c_1^n+1,c_1^n} \ldots e_{c_{b_n}^n+1,c_{b_n}^n} v_{\boldsymbol{\lambda}^{(n)}} \quad \in \; L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}].$$

To every $C \in P(\boldsymbol{l}, n)$ and $\sigma \in \Sigma(C)$, we assign a rational function

$$\omega_{C,\sigma} \;=\; \omega_{\sigma;1,2,\ldots,b_1}(z_1) \;\cdots\; \omega_{\sigma;b_1+\cdots+b_{n-1}+1,b_1+\cdots+b_{n-1}+2,\ldots,b_1+\cdots+b_{n-1}+b_n}(z_n),$$

where

$$\omega_{\sigma;a,a+1,\ldots,a+j}(z) \;=\; \frac{1}{(\sigma(a)-\sigma(a+1))\ldots(\sigma(a+j-1)-\sigma(a+j))(\sigma(a+j)-z)}.$$

We set

$$\omega(\boldsymbol{t}) = \sum_{C \in P(\boldsymbol{l},n)} \sum_{\sigma \in \Sigma(C)} \omega_{C,\sigma} e_C v . \tag{5.1}$$

Examples. If $n = 2$ and $(l_1, l_2, \ldots, l_N) = (1, 1, 0, \ldots, 0)$, then

$$\omega(\boldsymbol{t}) = \frac{1}{(t_1^{(1)} - t_1^{(2)})(t_1^{(2)} - z_1)} \, e_{21}e_{32}v_{\boldsymbol{\lambda}^{(1)}} \otimes v_{\boldsymbol{\lambda}^{(2)}} + \frac{1}{(t_1^{(2)} - t_1^{(1)})(t_1^{(1)} - z_1)} \, e_{32}e_{21}v_{\boldsymbol{\lambda}^{(1)}} \otimes v_{\boldsymbol{\lambda}^{(2)}}$$

$$+ \frac{1}{(t_1^{(1)} - z_1)(t_1^{(2)} - z_2)} \, e_{21}v_{\boldsymbol{\lambda}^{(1)}} \otimes e_{32}v_{\boldsymbol{\lambda}^{(2)}} + \frac{1}{(t_1^{(2)} - z_1)(t_1^{(1)} - z_2)} \, e_{32}v_{\boldsymbol{\lambda}^{(1)}} \otimes e_{21}v_{\boldsymbol{\lambda}^{(2)}}$$

$$+ \frac{1}{(t_1^{(1)} - t_1^{(2)})(t_1^{(2)} - z_2)} \, v_{\boldsymbol{\lambda}^{(1)}} \otimes e_{21}e_{32}v_{\boldsymbol{\lambda}^{(2)}} + \frac{1}{(t_1^{(2)} - t_1^{(1)})(t_1^{(1)} - z_2)} \, v_{\boldsymbol{\lambda}^{(1)}} \otimes e_{32}e_{21}v_{\boldsymbol{\lambda}^{(2)}} .$$

If $n = 2$ and $(l_1, l_2, \ldots, l_N) = (2, 0, \ldots, 0)$, then

$$\omega(\boldsymbol{t}) = \Big(\frac{1}{(t_1^{(1)} - t_2^{(1)})(t_2^{(1)} - z_1)} + \frac{1}{(t_2^{(1)} - t_1^{(1)})(t_1^{(1)} - z_1)} \Big) e_{21}^2 v_{\boldsymbol{\lambda}^{(1)}} \otimes v_{\boldsymbol{\lambda}^{(2)}}$$

$$+ \Big(\frac{1}{(t_1^{(1)} - z_1)(t_2^{(1)} - z_2)} + \frac{1}{(t_2^{(1)} - z_1)(t_1^{(1)} - z_2)} \Big) e_{21}v_{\boldsymbol{\lambda}^{(1)}} \otimes e_{21}v_{\boldsymbol{\lambda}^{(2)}}$$

$$+ \Big(\frac{1}{(t_1^{(1)} - t_2^{(1)})(t_2^{(1)} - z_2)} + \frac{1}{(t_2^{(1)} - t_1^{(1)})(t_1^{(1)} - z_2)} \Big) v_{\boldsymbol{\lambda}^{(1)}} \otimes e_{21}^2 v_{\boldsymbol{\lambda}^{(2)}} .$$

Lemma 5.1 (Lemma 2.1 in [MV2]). *The weight function is regular on U.* $\square$

5.2. **Grothendieck residue and Hessian.** Let

$$\text{Hess}\, \log \Phi \;=\; \det\left(\frac{\partial^2}{\partial t_j^{(i)} \partial t_{j'}^{(i')}} \log \Phi \right)$$

be the Hessian of $\log \Phi$. Let $\boldsymbol{p} \in U$ be a critical point of Φ. Denote by $H_{\boldsymbol{p}}$ the image of the Hessian in the local algebra $A_{\boldsymbol{p},\Phi}$. It is known that $H_{\boldsymbol{p}}$ is nonzero and the one-dimensional subspace $\mathbb{C}H_{\boldsymbol{p}} \subset A_{\boldsymbol{p},\Phi}$ is the annihilating ideal of the maximal ideal $\mathfrak{m}_{\boldsymbol{p}} \subset A_{\boldsymbol{p},\Phi}$.

Let $\rho_{\boldsymbol{p}} : A_{\boldsymbol{p},\Phi} \to \mathbb{C}$, be the Grothendieck residue,

$$f \;\mapsto\; \frac{1}{(2\pi i)^l} \, \text{Res}_{\boldsymbol{p}} \, \frac{f}{\prod_{ij} \Psi_{ij}} \; .$$

It is known that $\rho_{\boldsymbol{p}}(H_{\boldsymbol{p}}) = \mu_{\boldsymbol{p}}$, where $\mu_{\boldsymbol{p}} = \dim A_{\boldsymbol{p},\Phi}$ is the Milnor number of the critical point $\boldsymbol{p}$. Let $(\,,\,)_{\boldsymbol{p}}$ be the Grothendieck residue bilinear form on $A_{\boldsymbol{p},\Phi}$,

$$(f,g)_{\boldsymbol{p}} \;=\; \rho_{\boldsymbol{p}}(fg) \;.$$

It is known that $(\,,\,)_{\boldsymbol{p}}$ is nondegenerate. These facts see for example in Section 5 of [AGV].

5.3. **Projection of the weight function.** Let $\boldsymbol{p} \in C_\Phi$ be a critical point of Φ. Let

$$\omega_{\boldsymbol{p}} \in L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}] \otimes A_{\boldsymbol{p},\Phi}$$

be the element induced by the weight function. Let $S_{\boldsymbol{\Lambda}}$ be the tensor Shapovalov form on $L_{\boldsymbol{\Lambda}}$.

Theorem 5.2 ([MV2], [V]). *We have*

$$S_{\boldsymbol{\Lambda}}(\omega_{\boldsymbol{p}}, \omega_{\boldsymbol{p}}) = H_{\boldsymbol{p}} \;. \tag{5.2}$$

$\square$

Theorem 5.3 ([SV]). *The element $\omega_{\boldsymbol{p}}$ belongs to $\operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}] \otimes A_{\boldsymbol{p},\Phi}$ where $\operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}] \subset L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$ is the subspace of singular vectors.* $\square$

Theorem 5.3 is a direct corollary of Theorem 6.16.2 in [SV], see also [RV] and [B].

5.4. **Bethe ansatz.** Let $\boldsymbol{p} \in C_\Phi$ be a critical point of Φ. Consider the differential operator

$$\mathcal{D}_\Phi \;=\; \partial_u^{N+1} + \sum_{i=1}^{N+1} G_i\,\partial_u^{N+1-i}, \qquad G_i = \sum_{j=i}^{\infty} G_{ij}u^{-j}, \qquad G_{ij} \in \mathbb{C}[\boldsymbol{t}],$$

described by (2.3), and projections $\bar{G}_{ij}$ of its coefficients to $A_{\boldsymbol{p},\Phi}$. Consider the differential operator

$$\mathcal{D}_{\mathcal{B}} = \partial_u^{N+1} + \sum_{i=1}^{N+1} B_i\,\partial_u^{N+1-i}, \qquad B_i \;=\; \sum_{j=i}^{\infty} B_{ij}u^{-j}\,, \qquad B_{ij} \in U(\mathfrak{gl}_{N+1}[t])\,,$$

described by (4.1).

Theorem 5.4 ([MTV1]). *For any $i = 1, \ldots, N+1$, $j \geqslant i$, we have*

$$(B_{ij} \otimes 1)\,\omega_{\boldsymbol{p}} \;=\; (1 \otimes \bar{G}_{ij})\,\omega_{\boldsymbol{p}} \tag{5.3}$$

in $\operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}] \otimes A_{\boldsymbol{p},\Phi}$. $\square$

This statement is the Bethe ansatz method to construct eigenvectors of the Bethe algebra in the $\mathfrak{gl}_{N+1}$ Gaudin model starting with a critical point of the master function.

5.5. **Main result.** Let $g_1, \ldots, g_{\mu_{\boldsymbol{p}}}$ be a basis of $A_{\boldsymbol{p},\Phi}$ considered as a $\mathbb{C}$-vector space. Write $\omega_{\boldsymbol{p}} = \sum_i v_i \otimes g_i$, with $v_i \in \operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$. Denote by $M_{\boldsymbol{p}} \subset \operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$ the vector subspace spanned by $v_1, \ldots, v_{\mu_{\boldsymbol{p}}}$. Define a linear map

$$\alpha \; : \; A_{\boldsymbol{p},\Phi} \; \to \; M_{\boldsymbol{p}} \,, \qquad f \; \mapsto \; (f, \omega_{\boldsymbol{p}})_{\boldsymbol{p}} = \sum_{i=1}^{\mu_{\boldsymbol{p}}} (f, g_i)_{\boldsymbol{p}} \, v_i \,. \tag{5.4}$$

Theorem 5.5. *Let $\boldsymbol{p} \in C_\Phi$. Then the following statements hold:*
   (i) *The subspace $M_{\boldsymbol{p}} \subset \operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$ is a $\mathcal{B}$-submodule. Let $A_{\boldsymbol{p},B} \subset \operatorname{End}(M_{\boldsymbol{p}})$ be the Bethe algebra of $M_{\boldsymbol{p}}$. Denote by $\bar{B}_{ij}$ the image in $A_{\boldsymbol{p},B}$ of generators $B_{ij} \in \mathcal{B}$.*
  (ii) *The map $\alpha : A_{\boldsymbol{p},\Phi} \to M_{\boldsymbol{p}}$ is an isomorphism of vector spaces.*
 (iii) *The map $\bar{G}_{ij} \mapsto \bar{B}_{ij}$ extends uniquely to an algebra isomorphism $\beta : A_{\boldsymbol{p},\Phi} \to A_{\boldsymbol{p},B}$.*
 (iv) *The isomorphisms α and β identify the regular representation of $A_{\boldsymbol{p},\Phi}$ and the $\mathcal{B}$-module $M_{\boldsymbol{p}}$, that is, for any $f, g \in A_{\boldsymbol{p},\Phi}$ we have $\alpha(fg) = \beta(f)\alpha(g)$.*

Corollary 5.6. *Let $\boldsymbol{p} \in C_\Phi$. Then the value $\omega(\boldsymbol{p})$ of the weight function at $\boldsymbol{p}$ is a nonzero vector of $\operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$.*

Theorem 5.5 and Corollary 5.6 are proved in Section 6.4.

6. Proof of Theorem 5.5

6.1. **Proof of part (i) of Theorem 5.5.** It is enough to show that for any $f \in A_{\boldsymbol{p},\Phi}$ and any (i,j) we have $B_{ij}\alpha(f) \in M_{\boldsymbol{p}}$. Indeed, we have

$$B_{ij}\alpha(f) = \sum_{l=1}^{\mu_{\boldsymbol{p}}} (f, g_l)_{\boldsymbol{p}} B_{ij} v_l = \sum_{l=1}^{\mu_{\boldsymbol{p}}} (f, \bar{G}_{ij}g_l)_{\boldsymbol{p}} \bar{v}_l = \sum_{l=1}^{\mu_{\boldsymbol{p}}} (\bar{G}_{ij}f, g_l)_{\boldsymbol{p}} \bar{v}_l = \alpha(\bar{G}_{ij}f). \tag{6.1}$$

Here the second equality follows from Theorem 5.4 and the third equality follows from properties of the Grothendieck residue form.

6.2. **Bilinear form $(\,,\,)_S$.** Define a symmetric bilinear form $(\,,\,)_S$ on $A_{\boldsymbol{p},\Phi}$,

$$(f, g)_S = S_{\boldsymbol{\Lambda}}(\alpha(f), \alpha(g)) = \sum_{i,j=1}^{\mu_{\boldsymbol{p}}} S_{\boldsymbol{\Lambda}}(v_i, v_j) \, (f, g_i)_{\boldsymbol{p}} \, (g, g_j)_{\boldsymbol{p}}$$

for all $f, g \in A_{\boldsymbol{p},\Phi}$.

Lemma 6.1. *For all $f, g, h \in A_{\boldsymbol{p},\Phi}$ we have $(fg, h)_S = (f, gh)_S$.*

Proof. By Theorem 2.1 the elements $\bar{G}_{ij}$ generate $A_{\boldsymbol{p},\Phi}$. We have $(\bar{G}_{ij}f, h)_S = S_{\boldsymbol{\Lambda}}(\alpha(\bar{G}_{ij}f), \alpha(h)) = S_{\boldsymbol{\Lambda}}(\bar{B}_{ij}\alpha(f), \alpha(h)) = S_{\boldsymbol{\Lambda}}(\alpha(f), \bar{B}_{ij}\alpha(h)) = S_{\boldsymbol{\Lambda}}(\alpha(f), \alpha(\bar{G}_{ij}h)) = (f, \bar{G}_{ij}h)_S$. Here the third equality follows from Theorem 4.1. $\square$

Lemma 6.2. *There exists $F \in A_{\boldsymbol{p},\Phi}$ such that $(f, h)_S = (Ff, h)_{\boldsymbol{p}}$ for all $f, h \in A_{\boldsymbol{p},\Phi}$.*

Proof. Consider the linear function $A_{\boldsymbol{p},\Phi} \to \mathbb{C}$, $h \mapsto (1, h)_S$. The form $(\,,\,)_{\boldsymbol{p}}$ is nondegenerate. Hence there exits $F \in A_{\boldsymbol{p},\Phi}$ such that $(1, h)_S = (F, h)_{\boldsymbol{p}}$ for all $h \in A_{\boldsymbol{p},\Phi}$. Now the lemma follows from Lemma 6.1. $\square$

6.3. **Auxiliary lemmas.**

Lemma 6.3. *For any $f \in A_{\boldsymbol{p},\Phi}$, we have*

$$f H_{\boldsymbol{p}} = \frac{1}{\mu_{\boldsymbol{p}}}(f, H_{\boldsymbol{p}})_{\boldsymbol{p}} H_{\boldsymbol{p}}. \tag{6.2}$$

Proof. The lemma follows from the fact that formula (6.2) evidently holds for $1 \in A_{\boldsymbol{p},\Phi}$ and for any element of the maximal ideal. $\square$

For $f \in A_{\boldsymbol{p},\Phi}$, denote by L_f the linear operator $A_{\boldsymbol{p},\Phi} \to A_{\boldsymbol{p},\Phi}$, $h \mapsto fh$.

Lemma 6.4. *We have* $\operatorname{tr} L_f = (f, H_{\boldsymbol{p}})_{\boldsymbol{p}}$.

Proof. The linear function $A_{\boldsymbol{p},\Phi} \to \mathbb{C}$, $f \mapsto \operatorname{tr} L_f$, is such that $1 \mapsto \mu_{\boldsymbol{p}}$ and $f \mapsto 0$ for all $f \in \mathfrak{m}_{\boldsymbol{p}}$. Hence this function equals the linear function $f \mapsto (f, H_{\boldsymbol{p}})_{\boldsymbol{p}}$. $\square$

Let $g_1^*, \ldots, g_{\mu_{\boldsymbol{p}}}^*$ be the basis of $A_{\boldsymbol{p},\Phi}$ dual to the basis $g_1, \ldots, g_{\mu_{\boldsymbol{p}}}$ with respect to the form $(\,,\,)_{\boldsymbol{p}}$. Then $H_{\boldsymbol{p}} = \sum_{i=1}^{\mu_{\boldsymbol{p}}}(H_{\boldsymbol{p}}, g_i^*)_{\boldsymbol{p}} g_i$. Indeed for any $f \in A_{\boldsymbol{p},\Phi}$, we have $f = \sum_i (f, g_i^*)_{\boldsymbol{p}} g_i$.

Lemma 6.5. *We have* $\sum_{i=1}^{\mu_{\boldsymbol{p}}} g_i^* g_i = H_{\boldsymbol{p}}$.

Proof. For $f \in A_{\boldsymbol{p},\Phi}$, we have $\operatorname{tr} L_f = \sum_i (g_i^*, fg_i)_{\boldsymbol{p}} = (\sum_i g_i^* g_i, f)_{\boldsymbol{p}}$. By Lemma 6.4, we get $(\sum_i g_i^* g_i, f)_{\boldsymbol{p}} = (H_{\boldsymbol{p}}, f)_{\boldsymbol{p}}$. Hence $\sum_i g_i^* g_i = H_{\boldsymbol{p}}$, since the form $(\,,\,)_{\boldsymbol{p}}$ is nondegenerate. $\square$

Lemma 6.6. *Let $F \in A_{\boldsymbol{p},\Phi}$ be the element defined in Lemma 6.2. Then F is invertible, $F H_{\boldsymbol{p}} = H_{\boldsymbol{p}}$, and the form $(\,,\,)_S$ is nondegenerate.*

Proof. By definitions we have

$$(f, h)_S = \sum_{ij} S_{\boldsymbol{\Lambda}}(v_i, v_j)(g_i, f)_{\boldsymbol{p}}(g_j, h)_{\boldsymbol{p}}$$

and

$$(f, h)_S = (Ff, h)_{\boldsymbol{p}} = \sum_i (g_i, Ff)_{\boldsymbol{p}}(g_i^*, h)_{\boldsymbol{p}} = \sum_i (Fg_i, f)_{\boldsymbol{p}}(g_i^*, h)_{\boldsymbol{p}}.$$

Hence $\sum_{ij} S_{\boldsymbol{\Lambda}}(v_i, v_j) g_i \otimes g_j = \sum_i F g_i \otimes g_i^*$ and therefore by Lemma 6.5 we get

$$\sum_{ij} S_{\boldsymbol{\Lambda}}(v_i, v_j) g_i g_j = \sum_i F g_i g_i^* = F H_{\boldsymbol{p}}.$$

By Theorem 5.2, $\sum_{ij} S_{\boldsymbol{\Lambda}}(v_i, v_j) g_i g_j = H_{\boldsymbol{p}}$. Hence $F H_{\boldsymbol{p}} = H_{\boldsymbol{p}}$, the element F is invertible, and the form $(\,,\,)_S$ is nondegenerate. $\square$

6.4. **Proof of Theorem 5.5 and Corollary 5.6.** Part (i) of Theorem 5.5 is proved in Section 6.1.

Assume that $\sum_{i=1}^{\mu_{\boldsymbol{p}}} \lambda_i v_i = 0$. Denote $h = \sum_i \lambda_i g_i^*$. Then $\alpha(h) = 0$ and $(f, h)_S = S_{\boldsymbol{\Lambda}}(\alpha(f), \alpha(h)) = 0$ for all $f \in A_{\boldsymbol{p},\Phi}$. Hence $h = 0$ since $(,)_S$ is nondegenerate. Therefore, $\lambda_i = 0$ for all i and the vectors $v_1, \ldots, v_{\mu_{\boldsymbol{p}}}$ are linearly independent. We have $\alpha(g_i^*) = v_i$ for all i. That proves part (ii) of Theorem 5.5.

Parts (iii-iv) easily follow from part (ii) and formula (6.1).

We have

$$\mu_{\boldsymbol{p}}\,\omega(\boldsymbol{p}) = (H_{\boldsymbol{p}}, \omega_{\boldsymbol{p}})_{\boldsymbol{p}} = \alpha(H_{\boldsymbol{p}}). \tag{6.3}$$

That implies that $\omega(\boldsymbol{p})$ is a nonzero vector.

7. Concluding remarks

Theorem 7.1. *Let $C_\Phi = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k\}$, be the critical set of Φ in U. Let $M_{\boldsymbol{p}_s} \subset$ $\operatorname{Sing} L_{\boldsymbol{\Lambda}}[\boldsymbol{\lambda}^{(\infty)}]$, $s = 1, \ldots, k$, be the corresponding subspaces defined in Section 5.5. Then the sum of these subspaces is direct.*

Proof. It follows from Theorem 5.5 that for any s and any (i, j) the operator $\bar{B}_{ij} - G_{ij}(\boldsymbol{p}_s)$ restricted to $M_{\boldsymbol{p}_s}$ is nilpotent. Moreover, the differential operators $\mathcal{D}_\Phi|_{\boldsymbol{t}=\boldsymbol{p}_s}$, $s = 1, \ldots, k$, which contain eigenvalues of the operators B_{ij}, are distinct. These observations imply Theorem 7.1. $\square$

Let $\alpha(A_\Phi) = \oplus_{s=1}^k M_{p_s}$. Denote by A_B the image of $\mathcal{B}$ in $\operatorname{End}(\alpha(A_\Phi))$. Consider the isomorphisms

$$\alpha = \oplus_{s=1}^k \alpha_s \; : \; \oplus_{s=1}^k A_{\boldsymbol{p}_s,\Phi} \; \to \; \oplus_{s=1}^k M_{\boldsymbol{p}_s}, \qquad \beta = \oplus_{s=1}^k \beta_s \; : \; \oplus_{s=1}^k A_{\boldsymbol{p}_s,\Phi} \; \to \; \oplus_{s=1}^k A_{\boldsymbol{p}_s,B}$$

of Theorem 5.5.

Corollary 7.2. *We have*

(i) $A_B = \oplus_{s=1}^k A_{p_s,B}$;
(ii) *The isomorphisms α, β identify the regular representation of the algebra A_Φ and the A_B-module $\alpha(A_\Phi)$.*

The corollary follows from Theorems 5.5 and 7.1.

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