# THE SHAPE OF THURSTON'S MASTER TEAPOT

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ABSTRACT. We establish basic geometric and topological properties of Thurston's Master Teapot and the Thurston set for superattracting unimodal continuous self-maps of intervals. In particular, the Master Teapot is connected, contains the unit cylinder, and its intersection with a set $\mathbb{D} \times \{c\}$ grows monotonically with $c$. We show that the Thurston set described above is not equal to the Thurston set for postcritically finite tent maps, and we provide an arithmetic explanation for why certain gaps appear in plots of finite approximations of the Thurston set.

## 1. INTRODUCTION

When a continuous dynamical system on a compact space $(f, X)$ admits a Markov partition, the Perron-Frobenius theorem implies that the exponential of its topological entropy, $e^{h_{top}}(f)$, is a weak Perron number, i.e. an algebraic integer whose modulus is greater than or equal to those of its Galois conjugates. The *Thurston set* of a family $\mathcal{F}$ of such systems is the closure in $\mathbb{C}$ of the set of Galois conjugates of numbers of the form $e^{h_{top}(f)}$ for $f \in \mathcal{F}$. In this work, $\mathcal{F}$ is the family of superattracting real quadratic polynomials, and we investigate the geometry and topology of the associated Thurston set, $\Omega_2$:

$$\Omega_2 = \overline{\{z \in \mathbb{C} \mid z \text{ is a Galois conjugate of } e^{h_{top}(f)} \text{ for some } f \in \mathcal{F}\}}.$$

The *Master Teapot* for $\mathcal{F}$, defined by W. Thurston in [Thu14], is a three-dimensional set whose geometry encodes information about which maps in $\mathcal{F}$ correspond to which regions of the Thurston set:

$$\Upsilon_2 = \overline{\{(z, \lambda) \in \mathbb{C} \times \mathbb{R} \mid \lambda = e^{h_{top}(f)} \text{ for some } f \in \mathcal{F}, z \text{ is a Galois conjugate of } \lambda\}}.$$

In [Thu14], Thurston plotted the Galois conjugates of the *growth rates* (the numbers $e^{h_{top}(f)}$) of a selection of postcritically finite (PCF) quadratic real polynomials; Thurston's visually stunning image (see Figure 2) showed that the Thurston set has a rich geometric structure.

Our first main theorem is a geometric description of the part of the Master Teapot, $\Upsilon_2$, inside the unit cylinder:

**Theorem 1** (Persistence). *Fix $(z, \lambda) \in \Upsilon_2$ with $z \in \mathbb{D}$. Then $\{z\} \times [\lambda, 2] \subset \Upsilon_2$.*

In other words, $\Upsilon_2 \cap (\mathbb{D} \times \{c\})$ grows monotonically with $c$. The proof of Theorem 1 is at the end of §8.

In [Thu14, Figure 7.7], Thurston describes the part of the Master Teapot outside the unit cylinder as "a network of very frizzy hairs, . . . sometimes joining and splitting, but always transverse to the horizontal planes." As a counterpart to Thurston's "frizzy hairs," Theorem 1 suggests a description of the part of the Master Teapot inside the unit cylinder as a collection of "icicles" hanging down transverse to the horizontal planes.

Thurston was aware of this phenomenon, writing: "Roots in the closed unit disk do not depend continuously on $\lambda$, but they are confined to (and dense in) closed sets that include the unit circle and increases monotonically with $\lambda$, converging at $\lambda = 2$ to the inside portion of [the Thurston set]" [Thu14, caption of Figure 7.8]. However, [Thu14] gives no further explanation.

Theorem 2 describes the geometry of the Master Teapot in a neighborhood of the unit cylinder:

**Theorem 2.** *There exists $R > 0$ such that for any $n \in \mathbb{N}$,*

$$\left\{(z, \lambda) \in \mathbb{C} \times \mathbb{R} \mid \left(R^{-1}\right)^{\frac{1}{2^n}} \leq |z| \leq 1, 2^{\frac{1}{2^n}} \leq \lambda \leq 2\right\} \subset \Upsilon_2.$$

*In particular, the Master Teapot contains the unit cylinder, i.e.*

$$S^1 \times [1, 2] \subset \Upsilon_2.$$

Connectivity of the Master Teapot follows from Theorems 1 and 2 together with a proof by Tiozzo [Tio18, proof of Theorem 1.3] of connectivity of the region outside the unit cylinder:

**Theorem 3.** *The Master Teapot, $\Upsilon_2$, is connected. Furthermore $\Upsilon_2 \cap (\overline{\mathbb{D}} \times [1, 2])$ is path-connected.*

A heretofore mysterious feature of plots of finite approximations of the Thurston set, formed by bounding the length of the postcritical orbits, was the appearance of visible "gaps" or holes at fourth roots of unity, sixth roots of unity, and certain other algebraic numbers (see Figure 2). The gaps on the unit circle get filled in as the length of the postcritical orbits approaches infinity [Tio18, Proposition 6.1]. It is known, however, that $\Omega_2 \cap \mathbb{D}$ does have a hole other than the large central hold around the origin [CKW17]. Theorem 4 provides an arithmetic explanation for these visible gaps in finite approximations of $\Omega_2$.

**Theorem 4** (Gap theorem)**.** *For $n \in \mathbb{N}$, let $\omega_n$ denote the set of Galois conjugates of growth rates of superattracting tent maps with postcritical length at most $n$. Let $R$ be one of the rings $\mathbb{Z}[\sqrt{-D}]$ or $\mathbb{Z}[\frac{1+\sqrt{-D}}{2}]$ for $D = 1, 2, 3$ or $5$, and set $c = \inf\{|z| : z \in R, z \neq 0\}$. Then for any $x \in R$,*

$$B_{r(x)}(x) \cap \omega_n \subset \{x\},$$

*where*

$$r(x) = \begin{cases} \min\{\frac{c}{(2n^2+3n+1)|x|^n e}, \frac{1}{n+1}\} & \text{if } |x| \geq 1, \\ \min\{\frac{c}{(2n^2+3n+1)|x|e}, \frac{1}{n+1}\} & \text{if } |x| \leq 1. \end{cases}$$

Tiozzo proves there is a hole of radius 1/2 around the origin in the Thurston set, [Tio18, Lemma 2.4]. Our proof strategy is different: we use techniques resembling those of Solomyak for $\beta$-transformations with standard signature $E = (1, 1)$ [Sol94].

We define the preperiodic Thurston set $\Omega_2^{pre}$ as the Thurston set for the family of postcritically finite tent maps. That is, $\Omega_2^{pre}$ is the closure of the set of Galois conjugates of growth rates of postcritically finite tent maps. This includes tent maps that are both superattracting and strictly preperiodic.

**Theorem 5.** *The Thurston set $\Omega_2$ and the preperiodic Thurston set $\Omega_2^{pre}$ are not equal.*

The caption for Thurston's image [Thu14, Figure 1.1] states that the image shows the roots of the defining polynomials for "a sample of about $10^7$ postcritically finite quadratic maps of the interval with postcritical orbit of length $\leq 80$." We suspect that Thurston's image shows only roots of superattracting tent maps, i.e. shows $\Omega_2$ and not $\Omega_2^{pre}$ (c.f. Figures 2, 4).

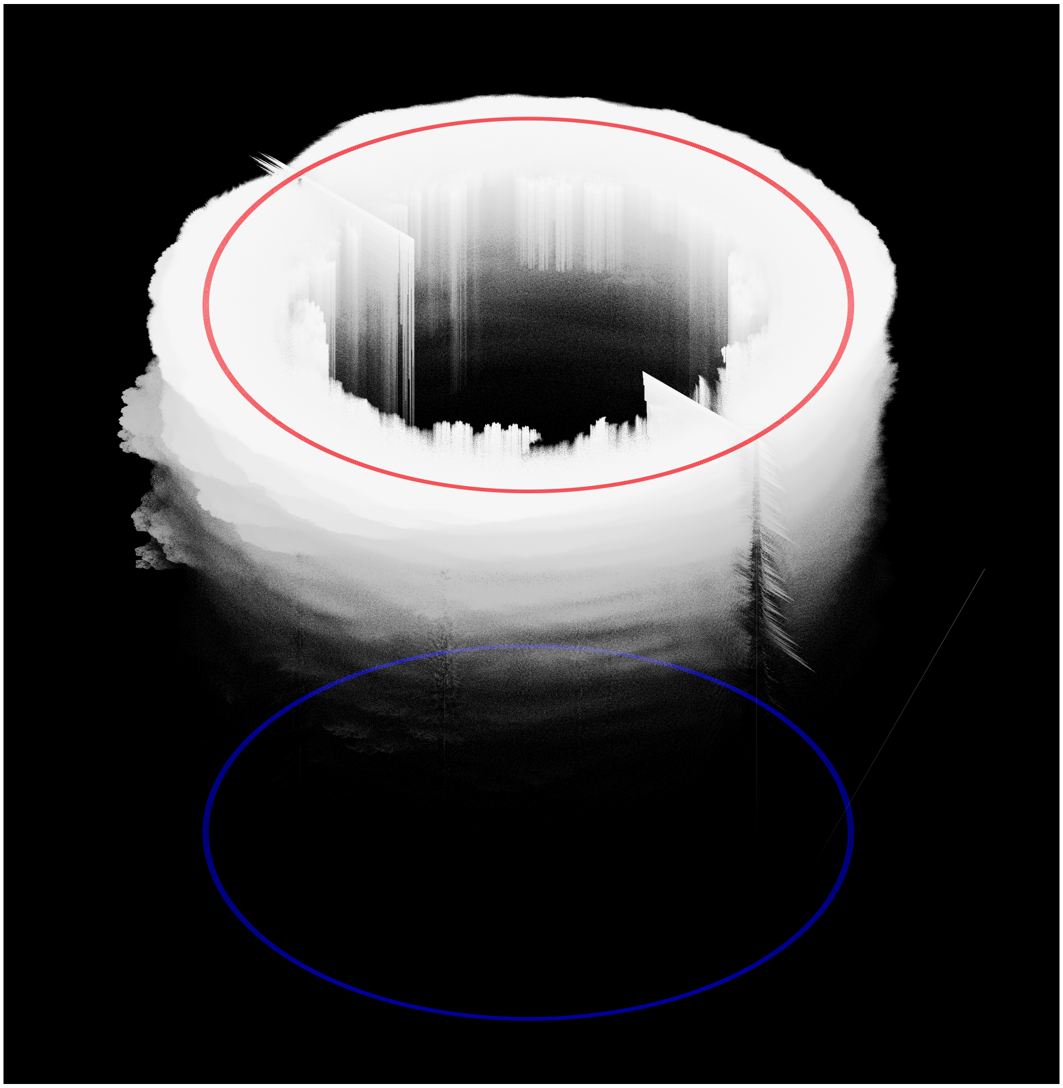

FIGURE 1. An approximation of Thurston's Master Teapot, $\Upsilon_2$. The horizontal plane is $\mathbb{C}$ and the vertical axis is $\mathbb{R}$. The projection onto $\mathbb{C}$ of the Master Teapot, $\Upsilon_2$, is the Thurston set, $\Omega_2$. The slice of the teapot at level $z = 1$ is the unit circle (blue); the unit circle is also shown at level $z = 2$ (red). The faint "spout" on the right consists of points the form $(\beta, 0, \beta) \in \mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$.

A uniform $\lambda$-expander is a continuous, piecewise linear self-map of an interval such that the slope of each piece is either $\lambda$ or $-\lambda$ (by convention, $\lambda > 0$). Thanks to a theorem of Thurston and Milnor, from the point of view of topological entropy, it suffices to consider uniform expanders:

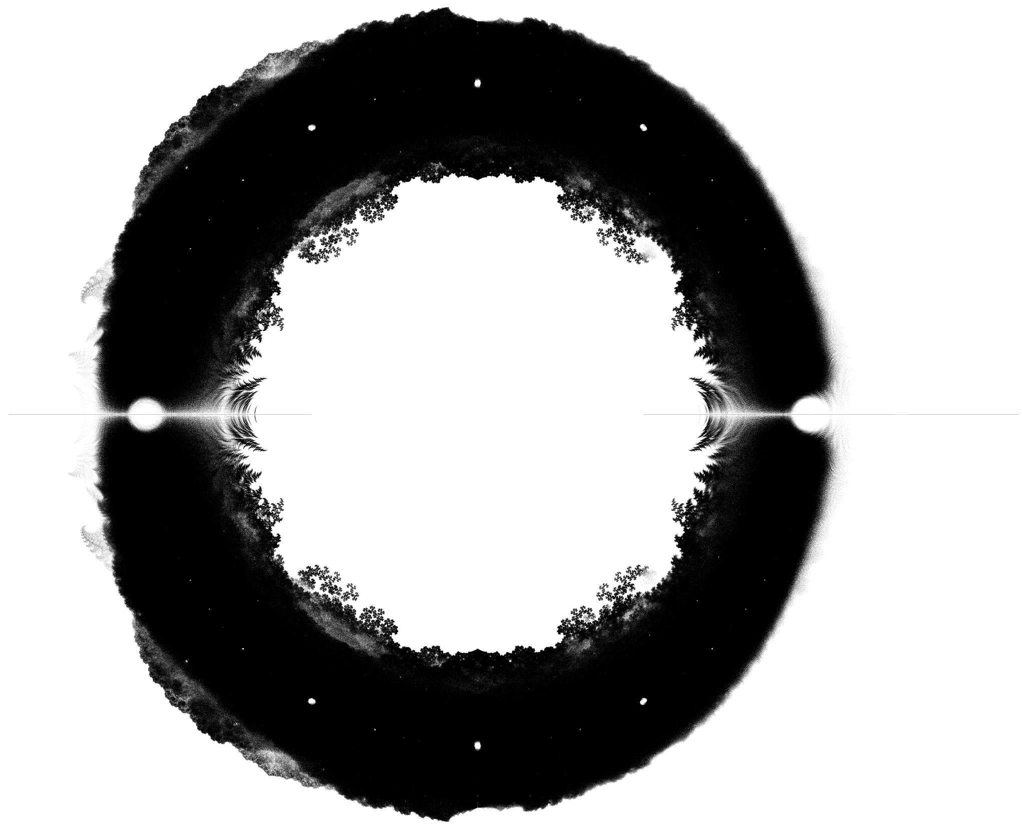

FIGURE 2. An approximation of the Thurston set, $\Omega_2$, containing the roots of the Parry polynomials for all of the (approximately $10^7$) postcritically finite quadratic superattracting tent maps of the interval with postcritical orbit of length $\leq 29$. Notice the "gaps" visible at the fourth and sixth roots of unity.

**Theorem 1.1.** [MT88, Theorem 7.4] *Every continuous self-map $g$ of an interval with finitely many turning points and with $h_{top}(g) > 0$ is semi-conjugate to a uniform $\lambda$-expander $PL(g)$ with the same topological entropy $h_{top}(g) = \log \lambda$. If $g$ is postcritically finite, so is $PL(g)$.*

A criterion for conjugacy to a uniform expander was also obtained in [Par66].

Uniform expanders may be thought of as one-dimensional analogues of pseudo-Anosov surface diffeomorphisms. For topological quadratic maps (i.e. maps with one turning point), this amounts to studying tent maps on the unit interval.

There are numerous characterizations of $\Omega_2$ arising from different points of view, and our results build (directly of indirectly) on a long history of research in each of these areas:

1. *Combinatorial.* The root of the combinatorial approach is the theory of $\beta$-expansions of real numbers and Parry polynomials. First introduced in [Par60] for maps of the form $x \mapsto \beta x \mod 1$ and later extended to larger classes of interval self-maps (e.g. [Ǵ07, IS09,

DMP11, Ste13, LSS16]), the Parry polynomial for a superattracting tent map is a monic polynomial with integer coefficients that is determined by combinatorial data about the critical orbit and and has the growth rate of the tent map as a root. Parry polynomials are not necessarily irreducible, but the collection of roots of Parry polynomials associated to a family of functions contains the Thurston set for that family. Parry polynomials were used to study the Thurston sets in [Sol94, Tho17]. We prove the relationship between Parry polynomials and kneading determinants for superattracting tent maps in § 4.

2. *Complex dynamics and kneading theory.* One may view a unimodal interval self-map as arising via the restriction to the real line of a quadratic polynomial with real coefficients on $\mathbb{C}$, and apply kneading theory (e.g. [Guc79, MT88]). The part of $\Omega_2$ that is outside the closed unit disk can be characterized as the set of points $z \in \mathbb{C} \setminus \mathbb{D}$ whose inverse is the root of a kneading determinant for a parameter in the real slice of the Mandelbrot set. The growth rate of a real PCF map can be viewed as a specific case of the core entropy of a complex polynomial [Tio15, Tio16, GT17].

3. *Iterated function systems.* A point $z \in \mathbb{D}$ is in $\Omega_2$ if and only if $0$ is in the limit set of the iterated function system generated by the two maps $x \mapsto zx + 1$ and $x \mapsto zx - 1$ [Tio18]. These IFS and their limit sets are the focus of numerous works, including [BH85, Bou88, Bou92, Ban02, SX03, Sol04, Sol05, CKW17].

4. *Power series with prescribed coefficients.* The set $\Omega_2 \cap \mathbb{D}$ equals the set of roots of all power series with coefficients $\pm 1$. There is a large body of literature that investigating the roots of polynomials and power series with all coefficients in a prescribed set (see, for example, [OP93, BBBP98, BEK99, Kon99, SS06, BEL08]). Different normalizations of the IFS give rise to power series with different coefficients. The polynomials most closely related to the Thurston set are perhaps Littlewood, Newman and Borwein polynomials, polynomials whose coefficients belong to the sets $\{\pm 1\}$, $\{0, 1\}$ and $\{-1, 0, +1\}$ respectively.

### 1.1. **Structure of the paper.**

**§2: Preliminaries** provides background on *tent maps*, the transformations we study in this paper. We define the *$\beta$-itinerary* of a point under such a transformation, the associated sequence of *digits*, the *cumulative sign* for sequences, the *$\beta$-tent map expansion*, the notion of being *postcritically finite*, and the *Parry polynomials*. We define *twisted lexicographic ordering* and give the *admissibility criterion* for itineraries, which are key tools. Finally, we give some background on Milnor-Thurston kneading theory, discuss the connection with quadratic maps, and give an *iterated function system* description.

**§3: Auxiliary sequences** defines the *auxiliary sequences* associated to sequences of digits, which we will use to characterize admissible sequences, and to define the important notion of *dominant* words that will be essential in § 5.

**§4: Relating kneading polynomials and Parry polynomials** shows how to convert between kneading polynomials and Parry polynomials.

**§5: Dominant Strings** shows that growth rates corresponding to dominant strings are dense in $\left[\sqrt{2}, 2\right]$, by proving the same result for the leading roots of Parry polynomials of dominant strings, and the fact that growth rates and leading roots are equivalent.

**§6: Compatibility of orderings** shows that the orderings on the sets of admissible words, kneading determinants, and growth rates are compatible.

**§7: Persistence on $[\sqrt{2}, 2]$** shows that roots of postcritically finite $\beta$-transformations *persist* inside the unit disk, for growth rates in the interval $\left[\sqrt{2}, 2\right]$. Using Thurston's terminology, this shows that this portion of the "Master Teapot" picture is connected. To do so, we first prove a technical fact: that certain words can be concatenated such that the concatenation is admissible. Dominant strings will be essential for this concatenation.

**§8: Period doubling** introduces the tool of *period doubling* to extend the persistence result to all growth rates in the interval $(1, 2]$, proving Theorem 1. Previous sections gave results for growth rates in $[\sqrt{2}, 2]$, and period doubling extends this to $[\sqrt[4]{2}, \sqrt{2}]$, then to $[\sqrt[8]{2}, \sqrt[4]{2}]$, and so on, which extends the results to all of $(1, 2]$.

**§9: The unit cylinder and connectivity** shows that the Master Teapot is connected inside the unit cylinder, and uses this structure to prove Theorems 2 and 3.

**§10: Gaps in the Thurston set** explains why there appear to be "holes" near primitive roots of unity in the Thurston set (Figure 2). We show that these holes are associated to discrete subgroups, proving Theorem 4.

**§11: $\Omega_2$ and $\Omega_2^{pre}$ are not equal** shows that the periodic and preperiodic Thurston sets are not equal, proving Theorem 5.

1.2. **Acknowledgements.** The authors gratefully acknowledge Giulio Tiozzo, Daniel Thompson, Sarah Koch, and Dylan Thurston for helpful conversations. This work began at the AMS Mathematics Research Communities program in June 2017. The authors are immensely grateful to the MRC program for stimulating this collaboration, and to Daniel Thompson introducing us to this subject while at the MRC. This material is based upon work supported by the National Science Foundation under Grant Number DMS 1641020. The first author was supported in part by NSF RTG grant 1045119.

## 2. Preliminaries

2.1. **Basic definitions.** Denote the unit interval $[0, 1]$ by $I$. Throughtout this work, a *tent map* will mean a map $f_\beta : I \to I$ of the following form. Fix a real number $\beta \in (1, 2]$, let $I_0^\beta = [0, \frac{1}{\beta}]$ and $I_1^\beta = (\frac{1}{\beta}, 1]$. The *$\beta$-tent map* is the map $f_\beta \colon I \to I$ defined by

$$f_\beta = \begin{cases} \beta x & \text{for } x \in [0, \frac{1}{\beta}], \\ -\beta x + 2 & \text{for } x \in [\frac{1}{\beta}, 1]. \end{cases}$$

The number $\beta$ is the *growth rate* of the map $f_\beta$; equivalently, $\beta = e^{h_{top}(f_\beta)}$. This equivalence follows from the fact that for a continuous self-map $f$ of an interval with finitely many turning points,

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log(\mathrm{Var}(f^n)), \tag{1}$$

where $\mathrm{Var}(f)$ denotes the total variation of $f$ [MS80].

The *$\beta$-itinerary sequence* of a point $x$ in $I$ is the sequence $\mathrm{It}_\beta(x, \cdot) \colon \mathbb{N} \to \{0, 1\}$ defined by

$$\mathrm{It}_\beta(x, j) = k$$

where $f_\beta^{j-1}(x) \in I_k^\beta$.

Equivalently, $\mathrm{It}_\beta(x, j) = \lfloor \beta \cdot f_\beta^{j-1}(x) \rfloor$, where $\lfloor \cdot \rfloor$ is the integer floor.

The sequence of *digits* associated to the $\beta$-itinerary sequence of a point $x$ is the sequence $d_\beta(x, \cdot) \colon \mathbb{N} \to \{0, 2\}$ defined by

$$d_\beta(x, j) = \begin{cases} 0 & \text{if } \mathrm{It}_\beta(x, j) = 0, \\ 2 & \text{if } \mathrm{It}_\beta(x, j) = 1. \end{cases}$$

For any point $x \in I$, $\beta \in (1, 2]$ and integer $j \geq 0$, define the *sign* $e_\beta(x, j)$ by

$$e_\beta(x,j) = d_\beta(x,j) = \begin{cases} +1 & \text{if } \mathrm{It}_\beta(x,j) = 0, \\ -1 & \text{if } \mathrm{It}_\beta(x,j) = 1. \end{cases}$$

The *sign vector* associated to any tent map is the function $E : \{0,1\} \to \{-1,+1\}$ defined by $E(0) = +1$ and $E(1) = -1$. The sign vector $E$ encodes the information that for any tent map $f_\beta$, the graph has positive slope on $I_0^\beta$ and negative slope on $I_1^\beta$.

The *cumulative sign* associated to a $\beta$-itinerary sequence of a point $x$ is the sequence $s_\beta(x,\cdot) : \mathbb{N} \to \{+1,-1\}$ defined inductively by $s_\beta(x,1) = 1$ and

$$s_\beta(x,j+1) = \prod_{k=1}^{j} e_\beta(x,k) \tag{2}$$

for $j \geq 1$. In fact, cumulative signs can be defined for any word in the alphabet $\{0,1\}$, not just those that arise as $\beta$-itineraries. For any sequence $w = w_1 w_2 w_3 \cdots \in \{0,1\}^{\mathbb{N}}$, define the sequence of cumulative signs $s_w : \mathbb{N} \to \{+1,-1\}$ inductively by $s_w(1) = +1$ and $s_w(i+1) = E(w_i)s(i)$ for $i \in \mathbb{N}$. For a finite string $w = w_1 \dots w_n$, define the cumulative sign of $w$ to be $s_w(n)$.

**Remark 2.1.** We will use the term *string* to refer to an ordered list of letters in some alphabet, and this list may be either finite or infinite. We adopt the convention that a *word* is always a finite string, and a *sequence* is always an infinite string. An itinerary is also assumed to be an infinite string.

The formula for the $\beta$-tent map expansion of $x$ is well known, but since Parry polynomials, which we will use extensively, come from $\beta$-expansions, we include an (original) proof below for completeness.

**Proposition 2.2** ($\beta$-tent map expansion of $x$)**.** *For any $\beta \in (1,2]$ and any $x \in I$,*

$$x = \sum_{j=1}^{\infty} \frac{s(x,j)d(x,j)}{\beta^j}. \tag{3}$$

*Proof.* Fix $1 < \beta \leq 2$ and let $f$ be the tent map of growth rate $\beta$. For any $x \in I$, $f(x) = d(x,1) + e(x,1)\beta x$. Then for any integer $n > 1$, $f^n(x) = d(x,n) + e(x,n)\beta f^{n-1}(x)$. By induction on $n$, one obtains that for any $n \in \mathbb{N}$ and $x \in [0,1]$,

$$\begin{aligned} f^n(x) = d(x,n) + \beta^1 d(x,n-1) \prod_{j=n}^{n} e(x,j) + \beta^2 d(x,n-2) \prod_{j=n-1}^{n} e(x,j) \\ + \cdots + \beta^{n-1} d(x,1) \prod_{j=2}^{n} e(x,j) + \beta^n x \prod_{j=1}^{n} e(x,j). \end{aligned}$$

Dividing through by $\beta^n \prod_{j=1}^n e(x,j)$ yields

$$\begin{aligned} \frac{f^n(x)}{\beta^n \prod_{j=1}^n e(x,j)} = \frac{d(x,n)}{\beta^n \prod_{j=1}^n e(x,j)} + \frac{d(x,n-1)}{\beta^{n-1} \prod_{j=1}^{n-1} e(x,j)} + \frac{d(x,n-2)}{\beta^{n-2} \prod_{j=1}^{n-2} e(x,j)} \\ + \cdots + \frac{d(x,1)}{\beta^1 \prod_{j=1}^{1} e(x,j)} + x. \end{aligned}$$

Taking the limit as $n \to \infty$ gives

$$0 = x + \sum_{i=1}^{\infty} \frac{d(x,i)}{\beta^i s(x,i+1)} = x + \sum_{i=1}^{\infty} \frac{d(x,i)s(x,i)e(x,i)}{\beta^i}. \tag{4}$$

Since for tent maps $d(x,i) \neq 0$ if and only if $e(x,i) = -1$, equation (4) implies

$$0 = x - \sum_{i=1}^{\infty} \frac{d(x,i)s(x,i)}{\beta^i}.$$

□

The *topological critical points* of the tent map $f_\beta$ are the points $0, 1/\beta$, and $1$. A tent map $f_\beta$ is said to be *postcritically finite* if the union of the forward orbits of the critical points of $f_\beta$ is a finite set. The definition of the tent map $f_\beta$ immediately implies that $f_\beta(0) = 0$ and $f_\beta(1/\beta) = 1$. Therefore, a tent map $f_\beta$ is postcritically finite if and only if the orbit of $1$ is finite. A postcritically finite orbit of $1$ may be (strictly) *periodic*, meaning that there exists $n \in \mathbb{N}$ such that $f^n(1) = 1$ or it may be (strictly) *preperiodic*, meaning that the orbit is not strictly periodic, but there exists $k, n \in \mathbb{N}$ such that $f^n(f^k(1)) = f^k(1)$. We call $f_\beta$ *superattracting* if the orbit of $1$ under $f_\beta$ is (strictly) periodic. The terminology "superattracting" is borrowed from complex dynamics (see §2.4).

If $f_\beta$ is superattracting, meaning that $1$ is (strictly) periodic under $f_\beta$, the $\beta$-tent map expansion of $1$ (equation 3) becomes a geometric series. Denoting the period of $1$ by $p$ and substituting the value of the geometric series, the $\beta$-tent map expansion of $1$ becomes

$$1 = \beta^p - \sum_{j=1}^{p} s(1,j)d(1,j)\beta^{p-j}. \tag{5}$$

**Definition 2.3.** The *Parry polynomial* for a superattracting tent map $f_\beta$ with critical period $p$ is the polynomial

$$P_\beta(z) := z^p - s(1,1)d(1,1)z^{p-1} - \cdots - s(1,p)d(1,p) - 1.$$

**Remark 2.4.** The Parry polynomial for a word $w$ in the alphabet $\{0,1\}$ is defined similarly; interpret the word $w$ as one period of the itinerary of $1$ under a tent map, compute the digits and cumulative signs, and form the Parry polynomial $P_w$ as above.

Thus, if $f_\beta$ is a superattracting tent map, it follows from equation (5) that $\beta$ is a root of the associated Parry polynomial. The minimal polynomial for $\beta$ is a factor of $P_\beta$. However, $P_\beta$ is never irreducible, as it always has a factor of $(z-1)$ (see Proposition 4.2), and may also have other factors.

In the case that $f_\beta$ is strictly preperiodic, a similar procedure using the sum of a power series produces a polynomial associated to a strictly preperiodic $f_\beta$.

2.2. **Irreducibility.** To establish irreducibility, we will use two lemmas from [Tio18] which are derived from Eisenstein's criterion.

**Lemma 2.5.** [Tio18, Lemma 4.1] *Let $d = 2^n - 1$ with $n \geq 1$, and choose a sequence $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ with each $\epsilon_k \in \{\pm 1\}$ such that $\sum_{k=0}^{d} \epsilon_k \equiv 2 \bmod 4$. Then the polynomial*

$$f(x) := \epsilon_0 + \epsilon_1 x + \cdots + \epsilon_d x^d$$

*is irreducible in $\mathbb{Z}[x]$.*

**Lemma 2.6.** [Tio18, Lemma 4.2] *Let $f(x) = 1 + \sum_{k=1}^{d} \epsilon_k x^k$ be a polynomial with $\epsilon_k \in \{\pm 1\}$ for all $1 \le k \le d$ and $\epsilon_k = -1$ for some $k$. If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then for all $n \ge 1$, the polynomial $f(x^{2^n})$ is irreducible in $\mathbb{Z}[x]$.*

Parry polynomials and kneading polynomials may not be irreducible; all Galois conjugates of $\beta$ are roots of $P_\beta$, but $P_\beta$ may have roots which are not Galois conjugates of $\beta$. The terms $\beta$-conjugates or generalized $\beta$-conjugates refer to the roots of a Parry polynomial associated to a $\beta$-map or generalized $\beta$-map. The distribution of $\beta$-conjugates was studied in [VG08a, VG08b].

2.3. **Ordering and admissibility of itineraries.**

**Definition 2.7** (Twisted lexicographic ordering)**.**

(1) Define the ordering $\le_E$ on the set of sequences in $\{0,1\}^{\mathbb{N}}$ as follows. Given two distinct words $w = w_1 w_2 \dots$ and $v = v_1 v_2 \dots$ in $\{0,1\}^{\mathbb{N}}$, define $w <_E v$ if and only if at the first integer $n$ such that $w_n \neq v_n$,
$$\begin{cases} w_n < v_n & \text{if } s_w(n) = +1, \\ w_n > v_n & \text{if } s_w(n) = -1. \end{cases}$$
(2) Define the ordering $\le_E$ on the set of words in the alphabet $\{0,1\}$ as follows. Given two words $w$ and $v$, write $w <_E v$ if and only if $w^\infty <_E v^\infty$.

Notice in Definition 2.7 that $s_w(n) = s_v(n)$ since $n$ is the first digit in which $w$ and $v$ differ.

**Definition 2.8** (Admissibility)**.**

(1) A sequence $w = (w_1 w_2 \dots)$ in the alphabet $\{0,1\}$ is *admissible* if there exists $\beta \in (1,2]$ such that $w$ is the itinerary of $1$ under the tent map $f_\beta$.
(2) A word $w = (w_1 \dots w_n)$ is *admissible* if the infinite string $(w_1 \dots w_n)^\infty$ is admissible.

Let $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the shift map, i.e. $\sigma(w_1 w_2 w_3 \dots) = w_2 w_3 \dots$.

**Theorem 2.9.** [MT88, Theorem 12.1] *A word $w \in \{0,1\}^{\mathbb{N}}$ is admissible if and only if $\sigma^j(w) \le_E w$ for all $j \in \mathbb{N}$.*

2.4. **The real slice of the Mandelbrot set & Milnor-Thurston kneading theory.** Every quadratic polynomial on $\mathbb{C}$ is conformally equivalent to a unique polynomial of the form $f_c(z) = z^2 + c$. The Mandelbrot set $\mathcal{M}$ is the set of parameters $c$ for which the filled Julia set for the map $f_c$ is connected. A parameter $c \in \mathcal{M}$ is said to be *hyperbolic* if the critical point for $f_c$ tends tends to the (necessarily unique) attracting cycle in $\mathbb{C}$. The hyperbolic parameters of $\mathcal{M}$ form an open set; connected components of this set are called hyperbolic components. Each hyperbolic component $H$ is conformally equivalent to $\mathbb{D}$ under the map $\lambda$ which assigns to each $c \in H$ the multiplier of its (unique) attracting cycle. The *center* and *root* of $H$ are $\lambda^{-1}(0)$ and $\lambda^{-1}(1)$, respectively. The set of all real hyperbolic parameters is dense in $\mathcal{M} \cap \mathbb{R} = [-2, 1/4]$; in particular, every component of the interior of $\mathcal{M}$ which meets the real line is hyperbolic [Lyu97]. The parameter $c$ and the map $f_c$ are said to be *superattracting* if the critical point $z = 0$ is strictly periodic under $f_c$. Each superattracting parameter $c$ is the center of a hyperbolic component of the Mandelbrot set.

For a parameter $c \in \partial\mathcal{M} \cap \mathbb{R}$, the *dynamic root* $r_c$ of $f_c$ is defined to be the critical value $c$ if $c$ belongs to the Julia set of $f_c$, and the smallest real value of $J(f_c)$ larger than $c$ if $c$ does not belong to the Julia set. For $c \in \partial\mathcal{M} \cap \mathbb{R}$, there exists a unique angle $\theta_c \in [0, 1/2]$ such that the dynamic rays $R_c(\pm\theta(c))$ land at the dynamic root $r_c$ of $f_c$; in the parameter plane, the two rays $R_{\mathcal{M}}(\pm\theta(c))$, and only these rays, contain $c$ in their impression [Zak03]. This angle angle $\theta_c$ is called the *characteristic angle* for the parameter $c \in \partial\mathcal{M} \cap \mathbb{R}$.

In the context of quadratic maps of the form $f_c(z) = z^2 + c$, define the *sign* of a real number $x \neq 0$ by $\epsilon(x) = -1$ if $x < 0$ and $\epsilon(x) = +1$ if $x > 0$. Define the sequence of cumulative signs by $\eta_n(x) = \prod_{i=0}^{n-1} \epsilon(f^i(x))$. (The use of $\epsilon$ and $\eta_n$ in this context are analogous to $e$ and $s_n$ in §2.1.) When the critical point 0 is not a periodic point for $f_c$, the *kneading series* of $x$, denoted by $K(x,t)$ is the formal series

$$K(x,t) = 1 + \sum_{n=1}^{\infty} \eta_n(x) t^n.$$

For each $c \in \mathbb{C}$, define the *kneading determinant* $K_c(t)$ of $f_c$ by

$$K_c(t) = \begin{cases} K(c,t) \text{ if the critical point is not periodic under } f_c \\ \lim_{C \to c^+} K(C,t) \text{ if the critical point is periodic under } f_c \end{cases}$$

where the limit as $C \to c^+$ is taken over the set of $C$'s such that the critical point is not periodic under $f_C$.

**Theorem 2.10.** [MT88, Theorem 6.3] *Let $s$ be the growth rate of $f_c$. Then the function $K_c(t)$ has no zeros on the interval $[0, 1/s)$, and if $s > 0$ we have $K_c(1/s) = 0$.*

A formal power series with coefficients $\pm 1$ is said to be *admissible* if it is the kneading determinant of some real quadratic polynomial. A formal power series $\phi(t)$ is said to be *positive* if its first non-zero coefficient is positive. Two formal power series satisfy $\phi_1(t) < \phi_2(t)$ if $\phi_2(t) - \phi_1(t)$ is positive. The absolute value $|\phi(t)|$ of a power series equals $\phi(t)$ if $\phi(t)$ is positive and equals $-\phi(t)$ otherwise.

**Theorem 2.11.** [MT88, Theorem 12.1] *Let*

$$\phi(t) = 1 + \sum_{k=1}^{\infty} \epsilon_k t^k$$

*be a formal power series with $\epsilon_k \in \{\pm 1\}$. Then $\phi(t)$ is admissible if and only if*

$$\phi(t) \leq \left| \sum_{k=n}^{\infty} \epsilon_k t^{k-n} \right|$$

*for each $n \geq 1$.*

For a superattracting parameter $c$, denote the length of the critical orbit by $n$. Then the coefficients of the kneading determinant, $K_c(t)$ are periodic, and so there exists a polynomial $P_{c,knead}(t)$ of degree $n-1$ with coefficients in $\{+1, -1\}$ such that

$$K_c(t) = \frac{P_{c,knead}(t)}{1 - t^n}. \tag{6}$$

The polynomial $P_{c,knead}(t)$ is the *kneading polynomial* of $f_c$.

**Theorem 2.12.** [MT88, Theorem 13.1, Corollary 13.2] *The function $h_{top}(f_c|_{\mathbb{R}})$ is a continuous, nonincreasing function of $c$.*

**Theorem 2.13.** [Tio15, Theorem 1.1] *Let $c \in [-2, 1/4]$. Then*

$$\frac{h_{top}(f_c|_{\mathbb{R}})}{\log 2} = H.dim\{\theta \in S^1 \mid R_{\mathcal{M}}(\theta) \text{ lands on } \partial\mathcal{M} \cap [c, 1/4]\}.$$

An immediate consequence of Theorem 2.13 is that $h_{top}(f_c|_{\mathbb{R}})$ as a function of $c \in \mathbb{R}$ is constant on real hyperbolic components.

2.5. **Iterated function system description.** A point $z \in \mathbb{D} \setminus \{0\}$ defines a contracting iterated function system (IFS) generated by the two maps

$$f_z : x \mapsto zx + 1, \quad g_z : x \mapsto zx - 1.$$

The *attractor* or *limit set* $\Lambda_z$ of this IFS is defined to be the unique fixed, nonempty, compact set $S \subset \mathbb{C}$ such that $S = f_z(S) \cup g_z(S)$. The existence and uniqueness of $\Lambda_z$ is a consequence of the contraction mapping principle.

The image of a point $x \in \mathbb{C}$ under a word $w$ of length $n$ in the alphabet $\{f, g\}$ is

$$xz^n + \sum_{i=0}^{n-1} c_i z^i,$$

where $c_i \in \{-1, 1\}$ is determined according to whether the $i^{th}$ letter of $w$ is $f_z$ or $g_z$. Thus, the limit set $\Lambda_z$ of the IFS generated by $f_z$ and $g_z$ is the set of values of power series in $z$ with coefficients $\pm 1$.

Tiozzo showed, roughly speaking, that all finite strings occur as the suffixes of kneading sequences, thereby proving that $\Omega_2 \cap \mathbb{D}$ equals the closure of the set of roots in $\mathbb{D}$ of all power series with $\pm 1$ coefficients [Tio18, Proposition 5.2]. Therefore, a point $z \in \mathbb{D}$ is in $\Omega_2$ if and only if $0$ is in the limit set of the iterated function system generated by $f_z, g_z$

**Lemma 2.14.** [CKW17, Lemma 3.1.1]

$$\Lambda_z \subset B_{\frac{1}{1-|z|}}(0).$$

The statement in [CKW17] uses a different normalization on the maps. Lemma 2.14 above and its proof below are exact translations of the versions in [CKW17].

*Proof.* Let $D$ denote the ball of radius $R$ centered at $0$. Then $f(D)$ and $g(D)$ are balls of radius $|z|R$ centered at $1$ and $-1$, respectively. Hence, if $\frac{1}{1-|z|} < R$, we have $f(D), g(D) \subset D$. This implies $\Lambda_z \subset D$. □

## 3. Auxiliary sequences

Auxiliary strings will serve two purposes: first, admissible sequences can be characterized in terms of auxiliary strings, and second, auxiliary strings feature in the definition of dominant words, which we will use to obtain a set of tent maps whose growth rates are dense in $[1, 2]$. The definitions of auxiliary and dominant used here are translations from the complex dynamics setting of notions with the same names introduced in [Tio15].

**Lemma 3.1.** *Let $w$ be a word in the alphabet $\{0, 1\}$ such that $w^\infty$ is admissible. Then the first letter of $w$ is $1$ and the second letter is $0$.*

*Proof.* Let $w$ be a word in alphabet $\{0, 1\}$ such that $w^\infty$ is the itinerary of $1$ under the tent map of slope $\beta$, $f_\beta$. It suffices to prove that $f_\beta(1) \in [0, \frac{1}{\beta}]$. This holds if and only if $2-\beta \leq \frac{1}{\beta}$, which is equivalent to $-\beta^2 + 2\beta - 1 \leq 0$ since $\beta > 0$, and thus also to $(\beta - 1)^2 \geq 0$. Since $\beta > 1$ (by the definition of a tent map), the claim follows. □

**Lemma 3.2.** *Let $w$ be a word in the alphabet $\{0, 1\}$ that starts with $10$. Then $w^\infty$ is admissible if and only if for every nontrivial decomposition $w = xy$ such that $y$ starts with $10$, $yx \leq_E xy$.*

**Remark 3.3.** If $w$ is a word for which $w^\infty$ is admissible, then an immediate consequence of Lemma 3.2 is that any suffix of $w$ is smaller than or equal to the prefix of $w$ of the same length in the twisted lexicographical ordering.

*Proof of Lemma 3.2.* By Fact 2.9, a word $w$ is admissible if and only if for every nontrivial decomposition of $w$ as $w = xy$, we have

$$yx \leq_E xy. \tag{7}$$

If $w$ is admissible, then equation (7) holds for every nontrivial decomposition $w = xy$, including those for which $y$ starts with 10, proving one direction of the statement.

Now suppose $w$ starts with 10 and $yx \leq_E xy$ for every decomposition $w = xy$ such that $y$ starts with 10. Since $w = xy$ starts with 10, which is maximal in the ordering $\leq_E$, equation (7) automatically holds for any decomposition $w = x'y'$ such that $y'$ does not start with 10. Therefore equation (7) holds for every decomposition $w = xy$, and so $w$ is admissible. □

**Definition 3.4** (Auxiliary string)**.**

(1) Let $w = w_1 w_2 \dots$ be an infinite string in the alphabet $\{0, 1\}$ such that $w_1 = 1$. Let $i_1, i_2, \dots$ be the increasing sequence of indices $i$ such that $w_i = 1$. For each $j \in \mathbb{N}$, define $n_j = i_{n+1} - i_n - 1$. The *auxiliary string* $w_{aux}$ associated to $w$ is the sequence of nonnegative integers

$$w_{aux} = n_1 n_2 n_3 \dots.$$

(2) Let $w = w_1 \dots w_n$ be a word in the alphabet $\{0, 1\}$ such that $w_1 = 1$. Let $i_1, \dots, i_p$, $p \geq 1$, be the increasing string of indices $i$ such that $w_i = 1$. For each $j < p$, define $n_j = i_{n+1} - i_n - 1$, and define $n_p$ to be the number of 0's to the right of $w_p$ in $w$. The *auxiliary string* $w_{aux}$ associated to $w$ is the finite string of nonnegative integers

$$w_{aux} = n_1 \dots n_p.$$

**Remark 3.5.** Note that the auxiliary string is always defined for admissible sequences; since $f_\beta$ is uniformly expanding with slope $\beta > 1$ in the first interval $I_0$, the $f_\beta$-orbit of 1 must eventually leave the interval $I_0$ if it ever enters $I_0$.

The term $n_j$ in $w_{aux}$ represents the number of 0's after the $j^{\text{th}}$ occurance of 1 in the string $w$. If the last letter of a finite string $w$ is a 1, there are zero 0's to the right, so $n_j = 0$. Otherwise, the value of $n_j$ is zero if and only if the $j^{\text{th}}$ 1 and the $(j+1)^{\text{th}}$ 1 are adjacent. Notice that if $w$ is a finite string in the alphabet $\{0, 1\}$ that begins with 1, $(w^\infty)_{aux} = (w_{aux})^\infty$.

**Definition 3.6.** The *alternating lexicographical order* on a the set of length $n$ strings of nonnegative integers (where $n$ is either a finite positive integer or $\infty$) is defined as follows: $(a_i)_{i=1}^n <_{alt} (b_i)_{i=1}^n$ if, denoting by $k$ the index of the first digit in which the sequences differ,

$$\begin{cases} a_k <_{alt} b_k & \text{if } k \text{ is even,} \\ a_k >_{alt} b_k & \text{if } k \text{ is odd.} \end{cases}$$

If there is no such $k$, meaning that the two strings are the same, write $(a_i)_{i=1}^n \leq_{alt} (b_i)_{i=1}^n$ and $(b_i)_{i=1}^n \leq_{alt} (a_i)_{i=1}^n$.

For example, $21 <_{alt} 11 <_{alt} 12$.

**Definition 3.7.** Let $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_m)$ be two finite strings of positive integers (possibly of different lengths). Write

$$A \ll_{alt} B$$

if there exists a positive integer index $k \leq \min\{m, n\}$ such that $(a_1, \ldots, a_{k-1}) = (b_1, \ldots, b_{k-1})$ and $(a_1, \ldots, a_k) <_{alt} (b_1, \ldots, b_k)$.

**Definition 3.8.**

(1) A finite string of nonnegative integers $w$ is *extremal* if for any decomposition $w = xy$ where $x$ and $y$ are nontrivial, $xy \leq_{alt} yx$.
(2) An infinite string of nonnegative integers $S$ is *extremal* if for any decomposition $w = xy$ where $x$ has finite length,

$$xy \leq_{alt} y.$$

Recall (Definition 2.8) that a word $w$ is *admissible* if $w^\infty$ is the itinerary of 1 under a PCF tent map.

**Proposition 3.9.** *Let $w$ be a word in the alphabet $\{0, 1\}$ with first letters 10. Then $w$ is admissible if and only if $w_{aux}$ is extremal.*

*Proof.* Lemma 3.2 allows us to only consider decompositions in which $y$ starts with 10, meaning that the auxiliary sequence for $y$ is defined, and if $x_{aux} = (n_1, \ldots, n_\ell)$ and $y_{aux} = (n_{\ell+1}, \ldots, n_p)$, then $(xy)_{aux} = (n_1, \ldots, n_\ell, n_{\ell+1}, \ldots, n_p)$.

Let $w = xy$ be any such decomposition. Compare $w = xy$ to the shift $yx$. Note that equality of $w$ and its shift is trivial, so consider the case where they differ. Let $t$ be the last 1 in $w = xy$ at which $xy$ and $yx$ agree. More precisely, we are assuming that every $k^{\text{th}}$ term in $w$ up to (and including) this $t^{\text{th}}$ 1 agrees with the $k^{\text{th}}$ term of the shift $yx$. Then $w$ and its shift $yx$ differ in the number of consecutive zeros following the $t^{\text{th}}$ 1. Let us express this in terms of the auxiliary sequences. If we denote the auxiliary sequence of $w$ by

$$w_{aux} = (xy)_{aux} = (n_1 \ldots, n_t, \ldots, n_p)$$

then the auxiliary sequence of the shift $xy$ is given by

$$yx_{aux} = (n_\ell, \ldots, n_{t+\ell}, \ldots, n_p, \ldots, n_{\ell-1})$$

where $\ell$ is the length of the auxiliary sequence for $x$. Thus, $xy$ and $yx$ agree at least up to the $t^{\text{th}}$ one of $w = xy$ if and only if $n_{t+1}$, which is the $t + 1^{\text{st}}$ term of $xy_{aux}$, and $n_{t+\ell+1}$, the $t + 1^{\text{st}}$ term of $yx_{aux}$, are the first terms at which the sequences $xy_{aux}$ and $yx_{aux}$ differ. The direction of the inequality will be determined by the parity of $t$.

For this special case of the tent map, $E(0) = +1$ and $E(1) = -1$, so the cumulative sign at the $m^{\text{th}}$ term of a string is equal to $(-1)^{n(m)}$ where $n(m)$ is the number of 1's in the string before the $m^{\text{th}}$ term. Thus, $t$ even implies the cumulative sign at the point where $xy$ and $yx$ differ is positive. It follows that $xy >_E yx$ if and only if the $(t+1)^{\text{st}}$ 1 of $xy$ appears earlier in the sequence than the $(t+1)^{\text{st}}$ 1 of $yx$, which is equivalent to $n_t < n_{\ell+t}$. Since $t$ is even, $xy_{aux} <_{alt} yx_{aux}$.

Similarly, if $t$ is odd, then the cumulative sign at the point where $xy$ and $yx$ differ is negative. So $xy >_E yx$ if and only if $n_t > n_{\ell+t}$ if and only if $xy_{aux} <_{alt} yx_{aux}$. □

**Remark 3.10.** Proposition 3.9 is equivalent to Lemma 9.3 of [Tio15], which is developed from the point of complex dynamics (e.g. using external angles of the Mandelbrot set).

## 4. Relating kneading polynomials and Parry polynomials

For a characteristic angle $\theta_c$ of a real hyperbolic parameter, Tiozzo associates an auxiliary strings $w_c$ as follows: Write the binary expansion of $\theta_c$, and let $w_c$ be the sequence

$w_c = a_1 a_2 a_3 \dots$ whose entries counts how many digits in a row of the binary expansion of $\theta_c$ are the same:

$$\theta_c = 0.\underbrace{0\dots0}_{a_1}\underbrace{1\dots1}_{a_2}\underbrace{0\dots0}_{a_3}\dots \tag{8}$$

(Notice that the sequence $w_c = a_1 a_2 \dots$ is independent of whether one uses the binary expansion of $+\theta_c$ or $-\theta_c$.)

**Definition 4.1.** For a word $w$ in the alphabet $\{0,1\}$, let $w_{aux}^T$ be the sequence $a_1, \dots, a_n$ defined as above in equation (8) for the binary expansion $(0.w)$.

**Proposition 4.2.** *Let $w = (w_1 \dots w_p)$ be an admissible word in the alphabet $\{0,1\}$ such that $\sum_{i=1}^p w_i$ is even. Let $w_{aux} = (a_1, \dots, a_n)$ and $w_{aux}^T = (b_1, \dots, b_\ell)$. Then $n = \ell$ and $a_i = b_i - 1$ for all $i = 1 \dots, n$. Furthermore,*

$$(t-1)t^{p-1} P_{c,knead}(t^{-1}) = P_{Parry}(t),$$

*where $P_{Parry}$ is the Parry polynomial for the tent map associated to $w$ and $P_{c,knead}$ is the kneading polynomial associated to a real quadratic map $f_c$ whose auxiliary sequence is $w_{aux}^T$.*

*Proof.* Let $w$ be an admissible word of length $p$ and positive cumulative sign, and let $(b_1, \dots, b_n) = w_{aux}^T$ be the Tiozzo auxiliary string for $w$. By the Milnor-Thurston admissibility criterion (Theorem 2.11), there exists a parameter $c \in [1/4, 2]$ such that

$$P_{c,\text{knead}}(t) = 1 + \left( \sum_{k=1}^{n} (-1)^k \sum_{j=b_1+\dots+b_{k-1}+1}^{b_1+\dots+b_k} t^j \right) - t^p \tag{9}$$

and the smallest root of $P_{c,\text{knead}}$ is $1/\beta$ for some $\beta \in [1,2]$.

Since $n$ is even, the last term of $P_{c,\text{knead}}(t)$ in the summation at $k = n$ over $j$ is $t^{b_1+\dots+b_n} = t^p$, which cancels with $-t^p$. Thus,

$$P_{c,\text{knead}}(t) = \begin{cases} 1 + \left( \sum_{k=1}^{n-1} (-1)^k \sum_{j=b_1+\dots+b_{k-1}+1}^{b_1+\dots+b_k} t^j \right) + \sum_{j=b_1+\dots+b_{n-1}+1}^{b_1+\dots+b_n-1} t^j & \text{if } b_n > 1, \\ 1 + \sum_{k=1}^{n-1} (-1)^k \sum_{j=b_1+\dots+b_{k-1}+1}^{b_1+\dots+b_k} t^j & \text{if } b_n = 1. \end{cases}$$

Then we compute when $b_n > 1$ that

$$\begin{aligned}
P(t) &:= (t-1)t^{p-1} P_{c,\text{knead}}(t^{-1}) \\
&= (t^p - t^{p-1}) \left( 1 + \left( \sum_{k=1}^{n-1} (-1)^k \sum_{j=b_1+\dots+b_{k-1}+1}^{b_1+\dots+b_k} t^{-j} \right) + \sum_{j=b_1+\dots+b_{n-1}+1}^{b_1+\dots+b_n-1} t^{-j} \right) \\
&= t^p - t^{p-1} + \left( \sum_{k=1}^{n-1} (-1)^k \sum_{j=b_1+\dots+b_{k-1}+1}^{b_1+\dots+b_k} t^{p-j} - t^{p-(j+1)} \right) + \sum_{j=b_1+\dots+b_{n-1}+1}^{b_1+\dots+b_n-1} t^{p-j} - t^{p-(j+1)} \\
&= t^p - t^{p-1} + \left( \sum_{k=1}^{n-1} (-1)^k \left( t^{p-(b_1+\dots+b_{k-1}+1)} - t^{p-(b_1+\dots+b_k+1)} \right) \right) + t^{p-(b_1+\dots+b_{n-1}+1)} - 1 \\
&= t^p - 2t^{p-1} + 2t^{p-(b_1+1)} - 2t^{p-(b_1+b_2+1)} + \dots + 2t^{p-(b_1+b_2+\dots+b_{n-1}+1)} - 1 \\
&= t^p - 2t^{p-1} - 2 \left( \sum_{k=1}^{n-1} (-1)^k t^{p-(b_1+\dots+b_k+1)} \right) - 1.
\end{aligned}$$

When $b_n = 1$,

$$\begin{aligned}
P(t) &= (t^p - t^{p-1})\left(1 + \sum_{k=1}^{n-1}(-1)^k \sum_{j=b_1+\cdots+b_{k-1}+1}^{b_1+\cdots+b_k} t^{-j}\right) \\
&= t^p - t^{p-1} + \sum_{k=1}^{n-1}(-1)^k \sum_{j=b_1+\cdots+b_{k-1}+1}^{b_1+\cdots+b_k} t^{p-j} - t^{p-(j+1)} \\
&= t^p - 2t^{p-1} + 2t^{p-(b_1+1)} - \cdots + t^{p-(b_1+\cdots+b_{n-1}+1)} \\
&= t^p - 2t^{p-1} + 2\left(\sum_{k=1}^{n-1}(-1)^k t^{p-(b_1+\cdots+b_k+1)}\right) - 1
\end{aligned}$$

because $t^{p-(b_1+\cdots+b_{n-1}+1)} = t^{p-(b_1+\cdots+b_n)} = 1$ when $b_n = 1$. Therefore, we recover the same polynomial regardless of whether $b_n = 1$ or $b_n > 1$.

The final expression of $P(t)$ has the form of an admissible Parry polynomial. Note that by definition, since the smallest root of $P_{\text{knead},c}$ is $1/\beta$, the leading root of $P$ is $\beta$. The first term of the itinerary associated to $P$ is a 1 because of the coefficient $-2$ in front of the $t^{p-1}$ term. The next 1 appears at the $(p-1)-(p-b_1-1) = b_1$th term, so there are $b_1 - 1$ many 0's in between the first 1 and the second 1, and so on. (Note that $b_1$ should always be at least 2, so there is at least one 0 before the second 1, and then $b_i \geq 1$ for all $i = 2, \ldots, n$.) Thus, the $i$th term of the auxiliary sequence we extract from this polynomial is $b_i - 1$, where $b_i$ is the $i$th term of Tiozzo's auxiliary sequence $w^T_{aux}$. From this we recover the same itinerary $w$, and we see that $w_{aux} = (a_1, \ldots, a_n)$ and $w^T_{aux} = (b_1, \ldots, b_n)$ if and only if $a_i = b_i - 1$ for $i = 1, \ldots, n$. □

## 5. Dominant strings

The main goal of this section is to prove the following result:

**Proposition 5.1.** *The leading roots of Parry polynomials of dominant strings are dense in* $[\sqrt{2}, 2]$.

Proposition 5.1 is a reformulation of a result (Proposition 5.10 below) by Tiozzo [Tio18]; translating it into non-complex-dynamics language is somewhat delicate. Proposition 5.1 makes no guarantee that the leading roots of these polynomials correspond to growth rates (since the polynomials may have multiple factors). Proposition 7.5 will show how to add suffixes to these dominant strings so that the associated Parry polynomial is, after dividing by a factor of $(1-z)$, irreducible.

**Definition 5.2.** A finite string $S$ of positive integers is *dominant* if $XY \ll_{alt} Y$ for any nontrivial decomposition $S = XY$.

**Remark 5.3.** It is straightforward to verify that $S$ is dominant if and only if proper prefixes of $S$ are smaller than proper suffixes of $S$ of the same length. More precisely, for any proper prefix $X$ of $S$, if $Y$ is the suffix of $S$ with $|X| = |Y|$ then $X \ll_{alt} Y$.

Note that the last letter of $S$ must be 0, else the inequality would not be strict with the suffix 1 and prefix 1. In the alternating ordering, $1 <_{alt} 0$ is indeed true.

**Definition 5.4.** Define a word $w = w_1 \ldots w_n$ in the alphabet $\{0, 1\}$ such that $w_1 = 1$ to be *dominant* if and only if $w$ has positive cumulative sign and the auxiliary sequence $w_{aux}$ is dominant.

**Definition 5.5.** A word $w$ in the alphabet $\{0,1\}$ is *irreducible* if there exists no shorter word $w_0$ in the alphabet $\{0,1\}$ and integer $n \geq 2$ such that $w = (w_0)^n$.

The definition of dominant immediately implies that dominant words are irreducible.

**Corollary 5.6.** *Dominant strings are admissible.*

*Proof.* It is clear that dominant strings are extremal strings, so the statement follows immediately from Proposition 3.9. □

We prove an equivalent characterization of dominance of a word which is intrinsic to the word and the twisted lexicographical ordering:

**Lemma 5.7.** *Let $w$ be a word in the alphabet $\{0,1\}$ that starts with $10$ and has positive cumulative sign. Then $w$ is dominant if and only if for any proper suffix $b$ of $w$, the word $b1$ is (strictly) smaller than the prefix of $w$ of length $|b|+1$ in the twisted lexicographical ordering $<_E$.*

*Proof.* First assume $w$ is dominant. Let $b$ be any proper suffix of $w$. Any suffix $b$ with first term $0$ satisfies $b <_E w$ immediately, so we consider when the first letter of $b$ is 1. Then $b$ has a well-defined auxiliary string; if we denote $w_{aux} = (a_1, \ldots, a_n)$ and assume the first term of $b$ is the $k$th 1 of $w$, then $b_{aux} = (a_k, \ldots, a_n)$. Let $m \in \{1, \ldots, n-k\}$ be the index of the first term where $b_{aux}$ and $w_{aux}$ differ, which exists by dominance of $w$. For such an $m$, if $m$ is even, then

$$a_{k-1+m} > a_m \iff (a_k, \ldots, a_{k-1+m}) >_{alt} (a_1, \ldots, a_m) \tag{10}$$

$$\iff (a_k, \ldots, a_{k-1+m}, \ldots, a_n) >_{alt} (a_1, \ldots, a_m, \ldots, a_{n-k}), \tag{11}$$

and if $m$ is odd, then

$$a_{k-1+m} < a_m \iff (a_k, \ldots, a_{k-1+m}) >_{alt} (a_1, \ldots, a_m) \tag{12}$$

$$\iff (a_k, \ldots, a_{k-1+m}, \ldots, a_n) >_{alt} (a_1, \ldots, a_m, \ldots, a_{n-k}). \tag{13}$$

Note that in Equations (11) and (13), we compare a proper suffix of $w_{aux}$ to a proper prefix of $w_{aux}$ of the same length, where properness follows because $b$ was a proper suffix of $w$ by assumption. Since $w$ is dominant, these inqualities are true by definition. In the case where $m$ is even, $a_{k-1+m} > a_m$ is equivalent to more 0's appearing after the $m^{\text{th}}$ 1 in $b$ than after the $m^{\text{th}}$ 1 in $w$. Equivalently, the $(m+1)^{\text{st}}$ 1 of $(b1)$ appears later in the sequence than the $(m+1)^{\text{st}}$ 1 in $w$ (note that adding a 1 to $b$ allows for the case $m = n-k \leq n-1$). Since $m$ is even, at this point where $b$ and $w$ first differ, i.e. $w$ has a 1 but $b$ has a 0, there are an even number of 1's. Hence the strings have positive cumulative sign, and $(b1) <_E w$ as desired.

In the case where $m$ is odd, $a_{k-1+m} < a_m$ is equivalent to fewer 0's appear after the $m$th 1 in $b$ than the $m$ 1 in $w$. In other words, the $m+1$st 1 of $(b1)$ appears earlier than the $m+1$st 1 of $w$. Since $m$ is odd, the ordering is reserved at the first point where $(b1)$ and $w$ differ. Thus, $(b1) <_E w$ again.

Conversely, consider any proper suffix $(a_k, \ldots, a_n)$ of $w_{aux}$. Then there exists a proper suffix $b$ of $w$ with first letter is the $k$th 1 of $w$; in other words, $b$ admits an auxiliary string, and that string must be $b_{aux} = (a_k, \ldots, a_n)$ by design. By assumption, $(b1) <_E w$; define $m \in \{1, \ldots, n-k\}$ such that the initial difference between $(b1)$ and $w$ follows the $m$th 1 of $w$. Then indeed $a_{k-1+m} \neq a_m$. Again by the definition of the twisted lexicographical ordering, as in the previous arguments, if $m$ is even then $(b1) <_E w$ implies $a_{k-1+m} > a_m$; and if $m$ is odd then $(b1) <_E w$ implies $a_{k-1+m} > a_m$. In both cases, in Equations (11) and (13) we see that the proper suffix $(a_k, \ldots, a_n)$ of $w_{aux}$ is larger than the proper prefix of $w_{aux}$ of the same length in the alternating ordering. By definition, $w_{aux}$ is dominant and hence $w$ is dominant. □

Tiozzo defines a real parameter $c$ to be dominant if there exists a finite string $S$ of positive integers such that $w_c = \overline{S}$ and $S$ is dominant. To distinguish between dominant in the sense of Definition 5.4 (which uses $w_{aux}$) and dominant in the sense of Tiozzo (which uses $w_{aux}^T$), we will call a word $w$ for which $w_{aux}^T$ is dominant *T-dominant*. We will see in Proposition 4.2 that in fact these two notions of dominant are equivalent – that a word $w$ is dominant if and only if it is T-dominant.

**Definition 5.8.** Let $w = w_1 \dots w_p$ be a word in the alphabet $\{0, 1\}$ such that $w_1 = 1$ and $w$ has positive cumulative sign. The word $w$ is defined to be *T-dominant* if $w_{aux}^T$ is dominant.

**Lemma 5.9.** *A word $w$ is dominant if and only if it is $T$-dominant.*

*Proof.* As a consequence of Proposition 4.2, any $w$ that satisfies the assumptions of the proposition is $T$-dominant if and only if $w$ is dominant. Note that to be dominant, $w$ must satisfy the conditions of the proposition: it $w$ must start with $10$, and $w$ must has positive cumulative sign. □

**Proposition 5.10.** [Tio15, Proposition 9.6] *Let $\theta_c \in [0, 1/2]$ be the characteristic angle of a real, non-renormalizable parameter c, with $c \neq -1$. Then $\theta_c$ is the limit point from below of characteristic angles of T-dominant parameters.*

A non-renormalizable parameter $c \in \mathbb{C}$ is a parameter in the Mandelbrot set $\mathcal{M}$ that does not live inside a "baby Mandelbrot set." A hyperbolic component $W$ of the Mandelbrot set is a connected component of the interior of $\mathcal{M}$ such that for all $c \in W$, the orbit of the critical point is attracted to a periodic cycle under $f_c$. Associated to any hyperbolic component $W$ of $\mathcal{M}$ there is a tuning map $\iota_W : \mathcal{M} \to \mathcal{M}$ that sends the main cardioid of $\mathcal{M}$ to $W$ and all o $\mathcal{M}$ to a baby Mandelbrot set. Denote by $\tau_W$ the associated map on external angles, i.e. if $\theta$ is a characteristic for $c \in \partial\mathcal{M}$, then $\tau_W(\theta)$ is a characteristic angle of $\iota_W(c)$. Tiozzo proves

**Proposition 5.11.** [Tio15, Proposition 11.2] *Let $W$ be a hyperbolic component of period $p$ and let $c \in \mathcal{M}$. Then H.dim $\tau_W(H_c) = \frac{1}{p}$H.dim $H_c$.*

Here H.dim $H_c$ is equal to $h_{top}(f_c|T_c)/\log 2$ (by [Tio15, Theorem 7.1]), where $T_c$ is the Hubbard tree of $f_c$, in the case that $f_c$ is topologically finite (meaning that its Julia set is connected and locally connected and its Hubbard tree is homeomorphic to a finite tree.) The set of topologically finite parameters contains all postcriticially finite parameters [Tio15]. Since $2$ is the minimum possible value for $p$, Proposition 5.11 implies that if $c$ is renormalizable and PCF,

$$\frac{h_{top}(f_c|T_c)}{\log 2} = \text{H.dim } H_c \leq \frac{1}{2} \sup_{c \in \mathcal{M}} \{\text{H.dim } H_c\} = \frac{1}{2},$$

and hence

$$e^{h_{top}(f_c|T_c)} \leq \sqrt{2}. \tag{14}$$

Combining Theorem 2.12, Proposition 5.10 and equation 14, we have now proven the following:

**Proposition 5.12.** *If $\sqrt{2} < \lambda \leq 2$ is the growth rate of a PCF tent map, then $\lambda$ is the limit from below of a sequence of growth rates of maps corresponding to T-dominant parameters.*

In [Tio18], Tiozzo expresses the kneading polynomial for a parameter $c$ in terms of the associated auxiliary word $w_{aux}^T$. Namely, from [Tio18], if $c$ is a T-dominant (real) parameter with auxiliary string $S = (a_1, \ldots, a_n)$, then the associated kneading polynomial $P_{c,\text{knead}}$ can be written as

$$P_{c,\text{knead}}(t) = 1 + \left( \sum_{k=1}^{n} (-1)^k \sum_{j=a_1+\cdots+a_{k-1}+1}^{a_1+\cdots+a_k} t^j \right) - t^p. \tag{15}$$

Recall that if $s$ is the growth rate of a superattracting map $f_c$, then $1/s$ is a root of $P_{c,\text{knead}}$.

*Proof of Proposition 5.1.* By Lemma 5.9, for a word $w$ in the alphabet $\{0,1\}$, the auxiliary sequence $w_{aux}$ is dominant if and only if $w_{aux}^T$ is dominant. By Proposition 5.12, any $\lambda \in (\sqrt{2}, 2]$ is the limit from below of a sequence of growth rates of tent maps for which the associated word $w_{aux}^T$ is dominant. ☐

## 6. Compatibility of orderings

We will make use of the compatibility of corresponding orderings on three related sets: the set of admissible words (with the twisted lexicographic ordering), kneading determinants, and growth rates.

Recall that the ordering on the the additive group $\mathbb{Z}[[t]]$ of formal power series with integer coefficients is defined by setting $\alpha = a_0 + a_1 t + \cdots > 0$ whenever $a_0 = \cdots = a_{n-1} = 0$ but $a_n > 0$ for some $n \geq 0$.

**Lemma 6.1.** *For tent maps, the kneading determinant is a monotone decreasing function of the growth rate.*

*Proof.* For the real one-parameter family of maps $f_a(x) = (x^2 - a)/2$, [MT88, Theorem 13.1] asserts that the kneading determinant $D(f_a) \in \mathbb{Z}[[t]]$ is monotone decreasing as a function of the parameter $a$; and Corollary 13.2 asserts the growth rate is monotone increasing as a function of $a$. The family of maps $\{f_a\}$ takes on all possible growth rates; this can be seen from the fact that $f_a$ is conjugate to the map $q_{(-a/4)}(z) = z^2 + (-a/4)$ via the conjugation map $h(z) = z/2$, growth rate is a continuous function of $c$ (Theorem 2.12), and the Intermediate Value Theorem. ☐

**Lemma 6.2.** *Let $f$ be a tent map with kneading determinant $\alpha$ and denote the itinerary of $1$ under $f$ by $w_\alpha$; let $g$ be a tent map with kneading determinant $\beta$ and denote the itinerary of $1$ under $g$ by $w_\beta$. If $\alpha > \beta$, then $w_\alpha >_E w_\beta$.*

*Proof.* [MT88, Lemma 4.5] implies that if $f$ is a tent map and $\alpha = 1 + \sum a_i t^i$ is the kneading determinant associated to $f$, then

$$a_n = \text{sign}\left( \frac{d}{dx} f^{n-1}(x)\Big|_{x=1} \right).$$

By the definition of the cumulative sign (equation 2),

$$\text{sign}\left( \frac{d}{dx} f^{n-1}(x)\Big|_{x=1} \right) = s(1,n),$$

so $a_n = s(1,n)$.

Now suppose $\alpha$ is the kneading determinant $\alpha = 1 + \sum_{i=1}^{\infty} a_i t^i$, $\beta$ is the kneading determinant $\beta = 1 + \sum_{i=1}^{\infty} b_i t^i$, and $\alpha > \beta$. Let $n$ be the smallest natural number such that $a_n \neq b_n$. We must have $a_1 = b_1$, so we may assume $n \geq 2$.

Denoting the cumulative signs for the tent map with kneading determinant $\alpha$ by $s_\alpha(1,\cdot)$ and with kneading determinant $\beta$ by $s_\beta(1,\cdot)$, the statement $\alpha > \beta$ means $s_\alpha(1,j) = s_\beta(1,j)$ for all $1 \leq j \leq n-1$ and $s_\alpha(1,n) > s_\beta(1,n)$. Hence $\mathrm{It}_\alpha(1,j) = \mathrm{It}_\beta(1,j)$ for $1 \leq j \leq n-2$ and $\mathrm{It}_\alpha(1,n-1) \neq \mathrm{It}_\beta(1,n-1)$.

There are two possibilities:

$$s_\alpha(1,n-1) = s_\beta(1,n-1) = +1, \quad \mathrm{It}_\alpha(1,n-1) = 0, \quad \mathrm{It}_\beta(1,n-1) = 1, \text{ or}$$

$$s_\alpha(1,n-1) = s_\beta(1,n-1) = -1, \quad \mathrm{It}_\alpha(1,n-1) = 1, \quad \mathrm{It}_\beta(1,n-1) = 0.$$

In both cases,

$$\mathrm{It}_\alpha(1,1)\ldots\mathrm{It}_\alpha(1,n-1) <_E \mathrm{It}_\beta(1,1)\ldots\mathrm{It}_\beta(1,n-1).$$

□

**Corollary 6.3.** *Let $f$ be a tent map with growth rate $\lambda_f$ and denote the itinerary of $1$ under $f$ by $w_f$; let $g$ be a tent map with growth rate $\lambda_g$ and denote the itinerary of $1$ under $g$ by $w_g$. If $\lambda_f > \lambda_g$, then $w_f >_E w_g$.*

*Proof.* Suppose $\lambda_f > \lambda_g$. By Lemma 6.1, $D(f) < D(g)$, where $D(f)$ and $D(g)$ denote the kneading determinants of $f$ and $g$, respectively. Then by Lemma 6.2, $w_f >_E w_g$. □

## 7. Persistence on $[\sqrt{2}, 2]$

In this section, we prove a restriction of the persistence theorem for Galois conjugates inside the unit disk associated to growth rates in the interval $[\sqrt{2}, 2]$. This proof relies on the fact that growth rates of dominant strings are dense in $[\sqrt{2}, 2]$ (Proposition 5.1). To prove the full persistence theorem, we will need to apply the period doubling procedure, which is treated in the next section.

To motivate this approach to the persistence theorem, we prove in the following proposition that density of dominant strings in the interval $[\sqrt{2}, 2]$ is indeed optimal. The proof is well-understood and only included for completeness. Our proof is a combinatorial argument, but the result can also be obtained from the perspective of complex dynamics.

**Proposition 7.1.** *The set of growth rates of dominant words is contained in $[\sqrt{2}, 2]$.*

**Remark 7.2.** Recall from §2.1 that a word in the alphabet $\{0,1\}$ has *positive cumulative sign* if it contains an even number of 1's, and otherwise has *negative cumulative sign*.

It is straightforward to check by the definition of the twisted lexicographical ordering that if a word $a$ has positive cumulative sign, then for any words $v, w$, we have $w <_E v$ if and only if $aw <_E av$. Similarly, if $a$ has negative cumulative sign, then $w <_E v$ if and only if $aw >_E av$.

*Proof of Proposition 7.1.* By contrapositive, assume $w$ is an admissible word and that the growth rate of $w$ is at most $\sqrt{2}$. By monotonicity (Corollary 6.3) and that the itinerary of $\sqrt{2}$ is strictly preperiodic, we conclude

$$w^\infty <_E \mathrm{It}_{\sqrt{2}}(1) = 10 \cdot 1^\infty,$$

which implies

$$w \leq_E 10 \cdot 1^{|w|-2}. \tag{16}$$

In the case of equality, there are two possibilities. If $w$ has an even number of 1's, then $(w1)$ has an odd number of 1's. Then equation (16) and Remark 7.2 imply

$$w \cdot 10 >_E 10 \cdot 1^{|w|-2} \cdot 11,$$

which implies $w^\infty >_E 10 \cdot 1^\infty$ because admissible words start with $10$ (Lemma 3.1). This violates our assumption. On the other hand, if $w$ has an odd number of 1's then $w$ cannot be dominant by the definition (Definition 5.4), as desired.

Now consider the case where $w <_E 10 \cdot 1^{|w|-2}$. Then $w$ has at least two 0's. Moreover, there is at least one other term of $10$ in $w$ besides the first two letters in $w$, since $101 <_E 100$ implies $w$ must start with $101$. Let $b$ be a proper suffix of $w$ which begins with a term of $10$, and assume that $b$ is the shortest possible such choice. Then $b \cdot 1 = 10 \cdot 1^{|b|-1}$, which by the assumption (equation (16)) is greater than or equal to the prefix of $w$ of length $|b|+1$ in the twisted lexicographical ordering. By Lemma 5.7, $w$ is not dominant. □

7.1. **Constructing dominant extensions.** The development of persistence on $[\sqrt{2}, 2]$ hinges on a series of technical combinatorial lemmas.

**Proposition 7.3.** *Assume $w_1$ is dominant, $w_2$ is admissible and irreducible, $n$ is a positive integer such that*

$$2n|w_2| > |w_1| > n|w_2|,$$

*$w_1^\infty >_E w_2^\infty$, and $w_2^n$ has positive cumulative sign. Then $(w_1 w_2^n)^\infty$ is admissible.*

*Proof.* It suffices to show that

$$\sigma^k (w_1 w_2^n)^\infty \leq_E (w_1 w_2^n)^\infty$$

for all $k < |w_1| + n|w_2|$. If $1 < k < |w_1|$, denote by $b$ the proper suffix of $w_1$ of length $|w_1| - k$. Then $(b1)$ is a prefix of $\sigma^k(w_1 w_2^n)$ because the first letter of $w_2$ is 1 by admissibility and Lemma 3.1. By dominance of $w_1$ and Lemma 5.7, $(b1)$ is smaller than the prefix of $w_1$ of length $|b|+1$ in the twisted lexicographical ordering, which proves

$$\sigma^k(w_1 w_2^n) = b w_2^n <_E w_1$$

and provides the desired inequality.

If $k = |w_1|$, for contradiction, see that existence of $n$ such that $w_2^n \geq_E w_1$ implies

$$w_2^\infty <_E w_1^\infty \leq_E (w_2^n)^\infty = w_2^\infty,$$

which is impossible given the assumption that $w_2^\infty$ is smaller than $w_1^\infty$ in the twisted lexicographical ordering. Thus,

$$\sigma^{|w_1|}(w_1 w_2^n)^\infty = w_2^n (w_1 w_2^n)^\infty <_E (w_1 w_2^n)^\infty.$$

Lastly, we consider the shift by $k$ where $|w_1| < k < |w_1| + n|w_2|$. Let $r = k - |w_1|$, so that $1 < r < n|w_2|$. See that $\sigma^r w_2^n >_E w_1$ is impossible, because $\sigma^r w_2^n >_E w_1$ and admissibility of $w_2$ implies

$$w_1^\infty <_E \sigma^r(w_2)^\infty \leq_E w_2^\infty,$$

a contradiction. We conclude that $\sigma^r w_2^n \leq_E w_1$. If this inequality is strict, we are done: we would have

$$\sigma^k(w_1 w_2^n) = \sigma^{|w_1|+r}(w_1 w_2^n) = \sigma^r w_2^n <_E w_1$$

as desired.

We must now consider when this inequality is not strict; in other words, $\sigma^r w_2^n$ is a prefix of $w_1$. We will need to prove that such a string must always have cumulative negative sign. If it does, then $|w_1| - r < |w_1|$ implies

$$\sigma^{|w_1|-r}(w_1 w_2^n)^\infty \leq_E (w_1 w_2^n)^\infty$$

by dominance of $w_1$ discussed above. Then by Remark 7.2 and that $\sigma^r w_2^n$ has negative cumulative sign,

$$\begin{aligned}(w_1 w_2)^\infty = \sigma^r w_2^n \sigma^{|w_1|-r} w_1 w_2^n (w_1 w_2^n)^\infty &\geq_E \sigma^r w_2^n (w_1 w_2^n)^\infty \\ &= \sigma^{k-|w_1|} w_2^n (w_1 w_2^n)^\infty = \sigma^k (w_1 w_2^n)^\infty.\end{aligned}$$

It remains to prove that if $\sigma^r w_2^n$ is a prefix of $w_1$, then it cannot have cumulative positive sign. Consider the suffix $b = \sigma^r w_2^n$ of $w_2^n$. Since $w_2^n$ is admissible, $b \leq_E a$ where $a$ is the prefix of $w_2^n$ of the same length (see Remark 3.3). Since $w_2^\infty < w_1^\infty$, moreover $a$ is smaller than or equal to the prefix of $w_1$ of the same length, which is assumed to be equal to $b$. Then $b \leq_E a \leq_E b$ implies equality, and we conclude $w_2^n = ac = db = da$.

Now

$$w_2^\infty = (ac)^\infty = (da)^\infty \geq_E a \cdot (da)^\infty$$

implying

$$(ca)^\infty \geq_E (da)^\infty = w_2^\infty \geq_E (ca)^\infty$$

because we assumed $a$ has positive cumulative sign (see Remark 7.2) and $w_2$ is admissible, hence $(ca)^\infty = (ac)^\infty$. Then,

$$w_2^\infty = (ac)^\infty = a \cdot (ca)^\infty = a \cdot (ac)^\infty = a^2 (ca)^\infty = \cdots = a^\infty$$

implies $a = w_2^m$ for some $m$ because $w_2$ is irreducible.

Then $w_1 = af = w_2^m f$ for some suffix $f$, and again by dominance of $w_1$ and Lemma 5.7,

$$w_1^\infty = (w_2^m f)^\infty = w_2^m (f w_2^m)^\infty \leq_E (w_2^m w_1)^\infty = w_2^{2m} (f w_2^m)^\infty \leq_E \cdots \leq_E w_2^\infty$$

which contradicts the assumption that $w_1^\infty >_E w_2^\infty$. □

**Definition 7.4.** We say that a string $v$ is an *extension* of a word $w$ if $w$ is a proper prefix of $v$. If $v$ is finite then such a $v$ is a *finite extension* of $w$.

If the kneading determinant of $(w_1 w_2^n)$ was irreducible then we would be able to proceed immediately to the proof of persistence on $[\sqrt{2}, 2]$. However, there is no such guarantee.

We next prove that we can extend $w_1$ to a dominant word $w_1'$ which guarantees that the kneading determinant of the concatenation is irreducible via Lemma 2.5. We will exploit word monotonicity in the core entropy (Corollary 6.3) and that we are currently only studying strings with core entropy larger than $\sqrt{2}$. This allows us to append truncations of the itinerary of $\sqrt{2}$ to $w_1$ without compromising dominance.

For the next Proposition, we recall or advise the reader to verify that the itinerary of $\sqrt{2}$ is $10 \cdot 1^\infty$.

**Proposition 7.5.** *Let $w_1$ and $w_2$ be words in the alphabet $\{0, 1\}$ such that $w_1$ is dominant, $w_2$ is admissible and irreducible, and $w_1^\infty > w_2^\infty$ and there exists an $m$ such that*

$$2m|w_2| > |w_1| > m|w_2|.$$

*Then there exists a finite extension $w_1'$ of $w_1$ and an integer $m' \geq m$ such that $(w_1' w_2^{m'})^\infty$ is admissible, $|w_1'| > m'|w_2|$, and $P(z)/(z-1)$ is an irreducible polynomial, where $P$ is the Parry polynomial of $(w_1' w_2^{m'})$.*

The following Lemma will give us a recipe for extending $w_1$.

**Lemma 7.6.** *Let $w$ be a dominant string. Then the words*

$$w \cdot 10 \cdot 1^{\kappa} \cdot 10 \cdot 1^{|w|} \cdot \mathbf{01} \cdot 1^{|w|} \textit{ and } w \cdot 10 \cdot 1^{\kappa} \cdot 10 \cdot 1^{|w|} \cdot \mathbf{10} \cdot 1^{|w|}$$

*for any odd natural number $\kappa > |w|$, and*

$$w \cdot 1^{\kappa} \cdot 10 \cdot 1^{|w|} \cdot \mathbf{01} \cdot 1^{|w|} \textit{ and } w \cdot 1^{\kappa} \cdot 10 \cdot 1^{|w|} \cdot \mathbf{10} \cdot 1^{|w|}$$

*for any even natural number $\kappa > |w|$, are all dominant extensions of $w$.*

*Moreover, for each $\kappa$, the sums of the coefficients of the kneading polynomials for the two extensions differ by 2.*

*Proof of Lemma 7.6.* The parity condition on $\kappa$ is to guarantee that the new word has an even number of 1's, which is part of the definition of dominance.

We apply the alternate definition of dominance from Lemma 5.7. Let $w'$ be one of the possible extensions in the statement of the Lemma. Let $b$ be any suffix of $w'$. If a prefix of $b$ is a suffix of $w$, then $(b1)$ is smaller than the prefix of $w'$ of the same length in the twisted lexicographical ordering by dominance of $w$ and the construction of $w'$. If not, then if $b$ starts with 0 or 11,and the desired inequality is immediate, so the interesting case is if $b$ starts with 10 and no prefix of $b$ is a suffix of $w$. By construction, including our choice of $\kappa > |w|$ in the $\kappa$ odd case, we are comparing a prefix of $\mathrm{It}_{\sqrt{2}}(1)$ with length at least $|w| + 1$ to $w$, which must be smaller by monotonicity (Corollary 6.3).

For any natural number $\kappa$, odd or even, there are now two choices to extend $w$ to a dominant word. The two choices only differ by an exchange of 01 with 10 in one position. This exchange will change the sum of the coefficients of the kneading polynomials by a factor of 2. ☐

*Proof of Proposition 7.5.* We need to choose for $w_1'$ one of the extensions of $w_1$ from Lemma 7.6, and select $n$, $\kappa$, and $m'$ so that $|w_1'|$ has length $2^n - 1 - m'|w_2|$ and

$$2m'|w_2| > |w_1'| > m'|w_2|.$$

To do so, first define constants $C_1 = 1 + |w_1| + m|w_2|$ and $C_2 = |w_2|$. Then choose $n$ for which

$$2^n > \max\{C_2(10m + 3) + C_1, 18C_2 + C_1\}$$

and define

$$k_n = \left\lceil \frac{2^n - C_1}{2C_2} \right\rceil - 2, \qquad k_n' = \left\lceil \frac{2^n - C_1}{2C_2} \right\rceil - 3. \tag{17}$$

The two options $k_n$ and $k_n'$ are needed for parity reasons. Choosing $2^n > C_2(10m+3)+C_1$ ensures that

$$k_n > k_n' > 10m, \tag{18}$$

which becomes useful later in the proof when we define the length of the extension. The choice of $2^n > 18C_2 + C_1$ and the definition of $k_n, k_n'$ ensures (respectively) that

$$3k_n > 3k_n' > \frac{2^n - C_1}{C_2} > 2k_n > 2k_n'. \tag{19}$$

Let $m' = k_n + m$ if this is even, and else, replace $k_n$ with $k_n'$. We will proceed with the notational choice $m' = k_n + m$ and assume $m'$ is even, but note that the needed inequalities hold for both $k_n$ and $k_n'$.

Now, replacing $C_1, C_2$ with their definitions, applying Equation (19), and invoking the assumed relationship between $|w_1|$ and $|w_2|$, we see that

$$3m'|w_2| > 3k_n|w_2| + m|w_2| + |w_1| > 2^n - 1 > 2k_n|w_2| + m|w_2| + |w_1| > 2m'|w_2|$$

which implies

$$2m'|w_2| > 2^n - 1 - m'|w_2| > m'|w_2|. \tag{20}$$

We now adjust the extension $w_1'$ of $w_1$ to have length $|w_1'| = 2^n - 1 - m'|w_2|$, so that $(w_1' w_2^{m'})$ has total length $2^n - 1$.

If $|w_1|$ is odd, then $\kappa = (2^n - 1 - m'|w_2|) - 6 - 3|w_1|$ is even, as needed for

$$w \cdot 1^\kappa \cdot 10 \cdot 1^{|w|} \cdot \mathbf{01} \cdot 1^{|w|} \text{ and } w \cdot 1^\kappa \cdot 10 \cdot 1^{|w|} \cdot \mathbf{10} \cdot 1^{|w|}$$

to both be dominant extensions of $w_1$ by Lemma 7.6, each of length $2^n - 1 - m'|w_2|$.

If $|w_1|$ is even, then $\kappa = (2^n - 1 - m'|w_2|) - 4 - 3|w_1|$ is odd, as needed for

$$w_1 \cdot 1^\kappa \cdot 10 \cdot 1^{|w_1|} \cdot \mathbf{01} \cdot 1^{|w_1|} \text{ and } w_1 \cdot 1^\kappa \cdot 10 \cdot 1^{|w_1|} \cdot \mathbf{10} \cdot 1^{|w_1|}$$

to both be dominant extensions of $w_1$ by Lemma 7.6, each of length $2^n - 1 - m'|w_2|$. In all the above cases, $\kappa > |w_1|$ follows from Equation (18).

For each choice, $w_1^\infty >_E w_2^\infty$ implies $w_1'^\infty >_E w_2^\infty$, and $w_2^{m'}$ has positive cumulative sign because we ensured that $m'$ is even. Combined with Equation (20), we have all the necessary hypotheses to apply Proposition 7.3 and conclude that $(w_1' w_2^{m'})^\infty$ is admissible. We also designed $w_1'$ so that $|w_1'| > m'|w_2|$.

The sum of the coefficients of the kneading polynomial of $w_1' w_2^{m'}$ is even, because it has $2^n$ coefficients, each of which is either $-1$ or $+1$. By the final observation in Lemma 7.6, we can choose the extension so that the sum of the coefficients of the kneading polynomial for $w_1' w_2^{m'}$ is equal to $2 \mod 4$. Since the kneading polynomial has degree $2^n - 1$, we apply Lemma 2.5 to conclude irreducibility. □

### 7.2. **Controlling Galois conjugates and core entropies of concatenations.**

**Lemma 7.7.** *Let $w_2$ be a word whose Parry polynomial has a root at $z_0 \in \mathbb{D}$. Then for any $\epsilon > 0$, there exists an integer $N = N(\epsilon, w_2) \in \mathbb{N}$ such that $n > N$ implies that for every word $w_1$ for which $w_1 w_2^n$ is admissible, the Parry polynomial associated to $(w_1 w_2^n)$ has a root within distance $\epsilon$ of $z_0$.*

*Proof.* First, for any word $w$, denote the Parry polynomial associated to $w$ by $P_w$. Let $D$ be the closed disk radius $\epsilon$ centered at $z_0$, and let $C$ be the boundary of $D$. Without loss of generality, assume $\epsilon$ is small enough that $D \subset \mathbb{D}$, and that $D$ contains no root of $P_{w_2}$ except $z_0$.

For any $n \in \mathbb{N}$, it is straightforward to see that

$$P_{w_1 w_2^n}(z) = z^{n|w_2|} P_{w_1}(z) + \left(z^{(n-1)|w_2|} + z^{(n-2)|w_2|} + \cdots + 1\right) P_{w_2}(z)$$

Set $\alpha = \min_{z \in C} |P_{w_2}(z)|$, which exists and is positive by compactness and the assumption that $D$ contains no root of $P_{w_2}$ except $z_0$. Set

$$0 < \beta := \min_{z \in C} \left(1 - |z|^{|w_2|}\right) / \left(1 + |z|^{|w_2|}\right).$$

Then for all $z \in C$, we have

$$\left|\left(z^{(n-1)|w_2|} + z^{(n-2)|w_2|} + \cdots + 1\right) P_{w_2}(z)\right| \geq \left|\frac{1 - (z^{|w_2|})^n}{1 - z^{|w_2|}}\right| \alpha \geq \frac{1 - |z^{|w_2|}|}{1 + |z^{|w_2|}|} \alpha \geq \beta\alpha > 0$$

where the middle nonstrict inequality follows the triangle inequality and that $\left|z^{|w_2|}\right| < 1$.

Set $1 > m := \max_{z\in D} |z|$. Also for all $z \in C$, since all coefficients of $P_{w_1}$ have absolute value at most 3,

$$\left|z^{n|w_2|}P_{w_1}(z)\right| \leq |z|^{n|w_2|}\left(1+3\sum_{i=0}^{\infty}|z|^i\right) \leq m^{n|w_2|}\left(1+3\sum_{i=0}^{\infty}m^i\right).$$

Therefore, for sufficiently large $n \in \mathbb{N}$ depending only on $w_2$, we have

$$\left|z^{(n-1)|w_2|}P_{w_1}(z)\right| < \frac{\beta\alpha}{2}.$$

Consequently, the winding number around 0 of the image of $C$ under $P_{w_1w_2^n}$ equals the winding number around 0 of the image of $C$ under the map

$$z \mapsto \left(z^{(n-1)|w_2|} + z^{(n-2)|w_2|} + \cdots + 1\right)P_{w_2}(z).$$

The winding number of the image around 0 is related to number of zeros via the Argument Principle; for a holomorphic function $f$ and a simple closed contour $\Gamma$, the number $N$ of zeros of $f$ inside $\Gamma$ is given by

$$N = \frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{f(\Gamma)}\frac{dw}{w} \tag{21}$$

where $w = f(z)$. Since $P_{w_2}$ has a root in $D$, this implies $P_{w_1w_2^n}$ also has a root in $D$ for sufficiently large $n$. □

**Lemma 7.8.** *Let $w_1$ be an admissible word whose Parry polynomial $P_{w_1}$ has leading root $z_0 > 1$. For any $\epsilon > 0$, there exists an integer $N = N(\epsilon, w_1)$ such $n > N$ implies that for every word $w_2$ for which $w_1^n w_2$ is admissible, the leading root of the Parry polynomial $P_{w_1^n w_2}$ associated to $(w_1^n w_2)$ is within distance $\epsilon$ of $z_0$.*

*Proof.* The proof consists of three main steps. Step 1: Compute the Parry and kneading polynomials associated to $(w_1^n w_2)$. Step 2: Show that there exists $N$ such that $n > N$ implies that for every word $w_2$ for which $w_1^n w_2$ is admissible, the Parry polynomial $P_{w_1^n w_2}$ has a root within distance $\epsilon$ of $z_0$. Step 3: Show that no root of $P_{w_1^n w_2}$ is greater in modulus than $|z_0| + \epsilon$.

*Step 1:* First, for any word $v$, denote the kneading polynomial associated to $v$ by $K_v$. It suffices to show that $K_{w_1^n w_2}$ can be made to have a root arbitrarily close to $1/z_0$ by choosing $n$ sufficiently big, with the choice of $n$ not depending on $w_2$.

For any $n \in \mathbb{N}$, the Parry polynomial $P_{w_1^n w_2}$ is given by

$$P_{w_1^n w_2}(z) = \left(z^{(n-1)|w_1|} + \cdots + z^{2|w_1|} + z^{|w_1|} + 1\right)\left(z^{|w_2|}\right)P_{w_1}(z) + P_{w_2}(z).$$

By Proposition 4.2,

$$(z-1)z^{|w_1^n w_2|}K_{w_1^n w_2}(z^{-1}) = P_{w_1^n w_2}(z).$$

Hence, for $z \neq 1$,

$$K_{w_1^n w_2}(z) = \frac{z}{1-z}z^{|w_1^n w_2|}P_{w_1^n w_2}(z^{-1}).$$

So

$$K_{w_1^n w_2}(z) = \frac{z}{1-z}\left(z^{n|w_1|+|w_2|}P_{w_2}(1/z) + (z^{|w_1|} + \cdots + z^{n|w_1|})P_{w_1}(1/z)\right)$$

Denote by $Q_{w_2}$ the reciprocal polynomial for $P_{w_2}$, i.e. $Q_{w_2} = z^{|w_2|}P_{w_2}(1/z)$. Notice $Q_{w_2}$ is a polynomial whose coefficients are all at most 3 is absolute value. Then

$$K_{w_1^n w_2}(z) = \frac{z}{1-z}\left(z^{n|w_1|}Q_{w_2}(z) + (z^{|w_1|} + \cdots + z^{n|w_1|})P_{w_1}(1/z)\right) \tag{22}$$

$$= \frac{z^{|w_1|+1}}{1-z}\left(z^{(n-1)|w_1|}Q_{w_2}(z) + (1 + \cdots + z^{(n-1)|w_1|})P_{w_1}(1/z)\right) \tag{23}$$

$$= \frac{z^{|w_1|+1}}{1-z}\left(z^{(n-1)|w_1|}Q_{w_2}(z) + \frac{1-(z^{|w_1|})^n}{1-z^{|w_1|}}P_{w_1}(1/z)\right) \tag{24}$$

*Step 2:* For any fixed $\epsilon_0$, let $D$ be the closed disk of radius $\epsilon_0 > 0$ centered at $1/z_0$, and let $C$ be the boundary of $D$. Without loss of generality, assume $\epsilon_0$ is small enough that $D \subset \mathbb{D}$ and that $D$ contains no root of $P_{w_1}(1/z)$ except $1/z_0$ and $D$ does not contain $0$.

We will show that on $C$, we can make the size of $z^{n|w_1|}Q_{w_2}(z)$ small enough relative to the size of $(1 + \cdots + z^{(n-1)|w_1|})P_{w_1}(1/z)$ that the winding number around $0$ of the image of $C$ under $K$ equals the winding number around $0$ of the image of $C$ under of $z \mapsto P_{w_1}(1/z)$.

Set $\alpha = \min_{z\in C}|P_{w_1}(1/z)|$, which exists and is positive by compactness and the assumption that $D$ contains no root of $P_{w_1}$ except $1/z_0$. Set

$$0 < \beta := \min_{z\in C}\left\{\frac{1-|z|^{|w_1|}}{1+|z|^{|w_1|}}\right\}.$$

Then for all $z \in C$, we have

$$\left|\frac{1-(z^{|w_1|})^n}{1-z^{|w_1|}}P_{w_1}(1/z)\right| \geq \left(\frac{1-|z|^{|w_1|n}}{1+|z|^{|w_1|}}\right)\alpha \geq \left(\frac{1-|z|^{|w_1|}}{1+|z|^{|w_1|}}\right)\alpha \geq \beta\alpha \tag{25}$$

Set $1 > m := \max_{z\in D}\{|z|\}$. Also for all $z \in C$,

$$\left|z^{(n-1)|w_1|}Q_{w_2}(z)\right| \leq |z|^{(n-1)|w_1|}\left(1 + 3\sum_{i=0}^{\infty}|z|^i\right) \leq m^{(n-1)|w_2|}\left(1 + 3\sum_{i=0}^{\infty}m^i\right). \tag{26}$$

Therefore, for sufficiently large $n$, $\left|z^{n|w_1|}Q_{w_2}(z)\right| \leq \frac{\alpha\beta}{2}$. Consequently, the winding number around $0$ of the image of $C$ under the map

$$k_{w_1^n w_2} : z \mapsto z^{(n-1)|w_1|}Q_{w_2}(z) + \frac{1-(z^{|w_1|})^n}{1-z^{|w_1|}}P_{w_1}(1/z)$$

equals the winding number $W$ of the image of $C$ around $0$ under the map

$$g_{w_1^n w_2} : z \mapsto \frac{1-(z^{|w_1|})^n}{1-z^{|w_1|}}P_{w_1}(1/z).$$

Since $P_{w_1}$ has a root at $1/z_0 \in D$, the argument principle (equation 21) implies the winding number $W$ is nonzero. Hence, the winding number around $0$ of the image of $C$ under $k_{w_1^n w_2}$ is nonzero. Therefore, $k_{w_1^n w_2}$ has a root in $D$, and thus $K_{w_1^n w_2}$ has a root in $D$. This implies $P_{w_1^n w_2}$ has a root in the set $\{z : 1/z \in D\}$. The diameter of this set decreases to $0$ as $\epsilon_0$ decreases to $0$, and $\epsilon_0$ was arbitrary.

*Step 3:* Set $r = |1/z_0| - \epsilon_0$. Without loss of generality, assume $\epsilon_0$ is small enough that $r > 0$ and $|1/z_0| + \epsilon_0 < 1$. Let $E$ be the closed disk of radius $r$ centered at $0$. Let $F$ be the boundary of $E$. Since $z_0$ is the leading root of $P_{w_1}$, the map $z \mapsto P_{w_1}(1/z)$ has no roots in $E$. Hence the map $g_{w_1^n w_2}$ has no roots in $E$, as $|z| < 1$ for all $z \in E$.

Set $\tilde{\alpha} = \min_{z\in F} |P_{w_1}(1/z)|$, which exists and is positive by compactness. Set

$$0 < \tilde{\beta} := \min_{z\in F} \left\{ \frac{(1-|z|^{|w_1|})}{1+|z|^{|w_1|}} \right\}.$$

By equation (25), for any $n$ and for any $z \in F$,

$$|g_{w_1^n w_2}(z)| = \left| \frac{1-(z^{|w_1|})^n}{1-z^{|w_1|}} P_{w_1}(1/z) \right| \geq \tilde{\beta}\tilde{\alpha}.$$

Thus, for any $n$, the image of the circle $F$ under $g_{w_1^n w_2}$ is a closed curve that has winding number $0$ about the origin and is contained in the set of points with absolute value at least $\tilde{\beta}\tilde{\alpha}$. By equation (26), for any $n$ and any $z \in F$,

$$\left| z^{(n-1)|w_1|} Q_{w_2}(z) \right| \leq (|1/z_0| + \epsilon_0)^{(n-1)|w_2|} \left( 1 + 3 \sum_{i=0}^{\infty} (|1/z_0| + \epsilon_0)^i \right). \tag{27}$$

Since $|1/z_0| + \epsilon_0 < 1$ by assumption, equation (27) implies that for sufficiently large $n$, $\left|z^{(n-1)|w_1|} Q_{w_2}(z)\right| < \tilde{\beta}\tilde{\alpha}/2$ for all $z \in F$. In order to perturb the image of $F$ under $g_{w_1^n w_2}$ so that it has nonzero winding number around the origin, some point in the image would have to move by at least $\tilde{\beta}\tilde{\alpha}/2$. Therefore, for sufficiently large $n$, the image of $F$ under $k_{w_1^n w_2}$ has zero winding number around $0$. The argument principle then implies $k_{w_1^n w_2}$ has no roots in $E$. By equation (24), the only root of $K_{w_1^n w_2}$ in $E$ is $z = 0$. Therefore $P_{w_1^n w_2}$ has no roots in $\mathbb{C}$ of modulus greater than $1/(|1/z_0| - \epsilon_0)$. $\square$

**Lemma 7.9.** *Let $v$ be a dominant word with growth rate $\beta$. Then the string $v^n \cdot 1^\infty$ is admissible for all $n$, and the growth rate of $v^n \cdot 1^\infty$ converges to $\beta$ as $n \to \infty$.*

*Proof.* Denote the growth rate of $v^n \cdot 1^\infty$ by $\zeta_n$. First, we show that $v^n \cdot 1^\infty$ is admissible. It is evident that $v^n \cdot 1^\infty \geq_E 1^\infty$, so one needs only to show that

$$v^n \cdot 1^\infty \geq_E \sigma^k(v^n \cdot 1^\infty) \tag{28}$$

for all $0 < k < n|v|$. If $k$ is a multiple of $|v|$, then $\sigma^k(v^n \cdot 1^\infty)$ is of the form $v^m \cdot 1^\infty$ for some natural number $m < n$. In this case, equation (28) then follows from the fact that $v^m$ has positive cumulative sign and $v^{n-m} \cdot 1^\infty \geq_E 1^\infty$. If $k$ is not a multiple of $|v|$, then $\sigma^k(v^n \cdot 1^\infty)$ starts with a word of the form $b \cdot 1$ where $b$ is a proper suffix of $v$. Hence by dominance of $v$ and Lemma 5.7, equation (28) holds with strict inequality in this case.

Proposition 2.2 gives us:

$$1 = \sum_{j=1}^{\infty} \frac{s(1,j)d(1,j)}{\beta^j} = \frac{1}{1-\beta^{-|v|}} \sum_{j=1}^{|v|} \frac{s(1,j)d(1,j)}{\beta^j},$$
$$1 = \sum_{j=1}^{\infty} \frac{s_{\zeta^n}(1,j)d_{\zeta^n}(1,j)}{\zeta_n^j} = \frac{1-\zeta_n^{-n|v|}}{1-\zeta_n^{-|v|}} \sum_{j=1}^{|v|} \frac{s(1,j)d(1,j)}{\zeta_n^j} + \frac{2\zeta_n^{-n|v|-1}}{1+\zeta_n^{-1}},$$

where $d(1,j)$ and $s(1,j)$ are the digits and cumulative signs associated to the string $v^\infty$, and $d_{\zeta^n}(1,j)$ and $s_{\zeta^n}(1,j)$ are the digits and cumulative signs associated to the string $v^n \cdot 1^\infty$.

Hence, the corresponding Parry polynomials are:

$$\beta^{|v|} - \left( \sum_{j=1}^{|v|} s(1,j)d(1,j)\beta^{|v|-j} \right) - 1 = 0,$$

$$(\zeta_n + 1)\zeta^{n|v|} - (\zeta_n + 1)\frac{\zeta_n^{n|v|} - 1}{\zeta_n^{|v|} - 1} \left( \sum_{j=1}^{|v|} s(1,j)d(1,j)\zeta_n^{|v|-j} \right) - 2 = 0.$$

It follows from kneading theory (Theorem 2.10) that $\beta$ and $\zeta_n$, respectively, are the leading roots of these Parry polynomials. Hence, $1/\beta$ and $1/\zeta_n$ are the smallest zeroes of the following analytic functions:

$$Q_\beta(z) = 1 - \left( \sum_{j=1}^{|v|} s(1,j)d(1,j)z^j \right) - z^{|v|},$$

$$Q_{\zeta_n}(z) = Q_\beta(z) - z^{n|v|}(Q_\beta(z) - 1) + z^{n|v|}(1 - z^{|v|}) - \frac{2z^{n|v|+1}(z^{|v|} - 1)}{z+1}.$$

Now it is evident that $Q_{\zeta_n} - Q_\beta$ converges uniformly to 0 as $n \to \infty$ in any compact subset of the open unit disc, hence by the same winding number argument used in the proof of Lemma 7.8, the smallest zeroes of $Q_{\zeta_n}$ converge to the smallest zero of $Q_\beta$.

□

**Proposition 7.10.** *For all $y \in (\sqrt{2}, 2)$ and all $\epsilon > 0$, there exists a sequence of dominant words $(w_n)_{i=1}^\infty$ such that for any admissible extension $w'_n$ of $w_n$, including the empty extension, the growth rate of $(w'_n)^\infty$ is within $\epsilon$ of $y$.*

*Proof.* By Proposition 5.1, there exists a dominant word $v$ with growth rate $\beta$ within $\epsilon/2$ of $y$. For each $n \in \mathbb{N}$, consider the admissible string $v^n \cdot 1^\infty$; denote the growth rate of the tent map associated to $v^n \cdot 1^\infty$ by $\zeta_n$.

Denote by $I_j^\eta$ the subinterval of $[0,1]$, with the partition into subintervals depending on the growth rate $\eta$ (as in §2.1), that contains the point $f_\eta^j(1)$. For each pair $k, n \in \mathbb{N}$, define the set of growth rates

$$U_k^n = \left\{ \eta \in [\sqrt{2}, 2] \mid f_\eta^j(1) \in \operatorname{int}(I_j^{\zeta_n}) \text{ for all } j = 1, \ldots, k \right\}.$$

Note that $\zeta_n \in U_k^n$ for all $k$ and $n$, since if at any point the $f_{\zeta_n}$-orbit of 1 landed on the boundary of $I_0$ or $I_1$, then either the tail of the itinerary would be $0^\infty$ or 1 would be periodic under $f_{\zeta_n}$ which contradicts the construction of the itinerary $v^n \cdot 1^\infty$. Evidently, $U_k^n$ is open for all $k, n$ by design.

By Lemma 7.9, there exists $N_1 \in \mathbb{N}$ such that whenever $n \geq N_1$, the growth rate $\zeta_n$ is within $\epsilon/2$ of $\beta$, and hence within $\epsilon$ of $y$. Therefore, for all $n, k \in \mathbb{N}$ with $n \geq N_1$, the set $U_k^n$ has nontrivial intersection with $(y - \epsilon, y + \epsilon)$. For integer $n \geq N_1$, fix an integer $k_n > |v^n|$. Since $U_{k_n}^n \cap (y - \epsilon, y + \epsilon)$ is open and nonempty, Proposition 5.1 implies there exists a dominant word $w_n$ with growth rate

$$\beta_n \in U_{k_n}^n \cap (y - \epsilon, y + \epsilon).$$

By the definition of the set $U_{k_n}^n$, the word $w_n$ agrees with $v^n \cdot 1^\infty$ for more than $|v^n|$ letters. Therefore, any extension $w'_n$ of $w_n$, including the empty extension, is also an extension of $v^n$. Let $N_2 = N_2(v, \epsilon)$ be the integer whose existence is guaranteed by Lemma 7.8, and let

$N = \max\{N_1, N_2\}$. Then whenever $n > N$, for any admissible extension $w'_n$ of $w_n$, the leading root of the Parry polynomial $P_{w'_n}$ is within $\epsilon$ of $\beta$. $\square$

### 7.3. Proof of main theorem of section.

**Theorem 7.11.** *Let $\alpha \in \mathbb{D}$ be a Galois conjugate of a superattracting $\beta \in (\sqrt{2}, 2)$. Then for any $y \in [\beta, 2]$ and any $\epsilon > 0$, there exists a superattracting $\beta'$ within $\epsilon$ of $y$ which has some Galois conjugate within $\epsilon$ of $\alpha$.*

*Proof.* Let $w$ be an irreducible, admissible word with growth rate $\beta \in [\sqrt{2}, 2]$ and fix $y \in [\beta, 2]$. If $y = \beta$ the statement is trivial, so assume $y > \beta$. Fix

$$0 < \epsilon < \frac{y - \beta}{2}.$$

Construct the sequence of dominant words $(w_n)$ as in Proposition 7.10; the words $w_n$ satisfy that for any admissible extension $w'_n$ of $w_n$, the growth rate of $w'_n$ is within $\epsilon$ of $y$. Denote the growth rate of $w_n$ by $\beta_n$. The inequality $\beta_n > \beta$, for all $n$, follows from $\epsilon < \frac{y-\beta}{2}$.

Because $\beta_n > \beta$, monotonicity (Corollary 6.3) implies $w_n^\infty >_E w_2^\infty$. Passing to subsequences as needed, we may assume that $|w_n| \to \infty$ as $n \to \infty$, since there are only finitely many words of bounded length.

For each $n$, let $M_n = \left\lceil \frac{|w_n|}{|w|} \right\rceil - 2$. Then

$$2M_n|w| \geq 2\left(\frac{|w_n|}{|w|} - 2\right)|w| = 2|w_n| - 4|w|. \tag{29}$$

Since $2|w_n| - 4|w| > |w_n|$ if and only if $|w_n| > 4|w|$, we have from equation (29) that

$$2M_n|w| > |w_n| \iff |w_n| > 4|w|.$$

Observe that

$$|w_n| = \frac{|w_n|}{|w|}|w| > \left(\left\lceil \frac{|w_n|}{|w|} \right\rceil - 2\right)|w| = M_n|w| \quad \text{for all } n$$

and $|w_n| \to \infty$. Therefore, for all $n$ large enough that $|w_n| > 4|w|$, there exists a positive integer $M_n$ such that

$$2M_n|w| > |w_n| > M_n|w|.$$

Note also that $M_n \to \infty$ as $n \to \infty$.

Thus, for sufficiently large $n$, the hypotheses of Proposition 7.5 hold, using $w_n$ in place of $w_1$ and $w$ in place of $w_2$. Then by Proposition 7.5, there exists an integer $m'_n > M_n$ and a dominant extension $w'_n$ of $w_n$ so that $(w'_n w^{m'_n})^\infty$ is admissible and the polynomial

$$\frac{P_{w'_n w^{m'_n}}(z)}{1 - z}$$

is irreducible, where $P_{w'_n w^{m'_n}}$ is the Parry polynomial of the admissible word $w'_n w^{m'_n}$.

Because $w'_n w^{m'_n}$ is an admissible extension of $w_n$, which was constructed via Proposition 7.10, the growth rate of $w'_n w^{m'_n}$ is within $\epsilon$ of $y$. Since $M_n \to \infty$ as $n \to \infty$ and $m'_n > M_n$, we have $m'_n \to \infty$ as $n \to \infty$. Then by Lemma 7.7, for sufficiently large $n \in \mathbb{N}$, $P_{w'_n w^{M'_n}}$ has a root within $\epsilon$ of $\alpha$. $\square$

## 8. PERIOD DOUBLING

This section shows that if $1 < \lambda \leq 2$ is the growth rate of a superattracting tent map, then so is $\sqrt{\lambda}$, and relates the itineraries of these two maps; we refer to this mechanism as Period Doubling. We then use Period Doubling to extend Theorem 7.11, which holds for $\beta \in [\sqrt{2}, 2]$, to work for $\beta \in [1, 2]$ (Proposition 8.3), and then use this to prove Theorem 1. Period doubling is related to the process of "tuning" the Mandelbrot set in complex dynamics; see e.g. [Tio18, § 7.2].

Define $s : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the map that interchanges 0s and 1s, i.e.

$$s(b_1, b_2, b_3, \dots) = (b_1 + 1 \bmod 2, b_2 + 1 \bmod 2, b_3 + 1 \bmod 2, \dots).$$

**Lemma 8.1.** *Let $f$ be a tent map on $[0,1]$ with growth rate $1 < \lambda < \sqrt{2}$, and denote the itinerary of $1$ under $f$ by*

$$It_f(1) = a_1, a_2, a_3, \dots.$$

*Then*

(1) $a_{2k+1} = 1$ *for all nonnegative integers $k$, and*
(2) *there exists a tent map $g$ of growth rate $\lambda^2$ such that*

$$a_2, a_4, a_6, \dots = s(It_g(1)).$$

*Furthermore, $g$ is conjugate to the restriction of $f^2$ to the interval $[2 - \lambda, \frac{2}{1+\lambda}]$ via an affine scaling and flipping of the interval.*

*Proof.* Fix $1 < \lambda < \sqrt{2}$ and let $f$ be the tent map with growth rate $\lambda$. Let

$$J_1 = \left[2 - \lambda, \frac{2}{1+\lambda}\right], \qquad J_2 = \left[\frac{2}{1+\lambda}, 1\right]. \tag{30}$$

Notice that $2 - \lambda = f(1) \leq 1/\lambda$, and that $\frac{2}{1+\lambda}$ is the non-zero fixed point of $f$. Since $f(1) \in I_0$, $f^2(1) = \lambda \cdot f(1) = 2\lambda - \lambda^2$. The inequality $2x - x^2 \geq \frac{2}{1+\lambda}$ is true when $1 \leq x \leq \sqrt{2}$. Hence, $f^2(1) \in J_2$. Thus, $f(J_1) \subset J_2$ and $f(J_2) \subset J_1$.

The inequality $1/\lambda < 2/(1+\lambda)$ holds for $\lambda > 1$, so the critical point $1/\lambda$ of $f$ is in the interior of interval $J_1$. It follows that the restrictions $f^2 : J_1 \to J_1$ and $f^2 : J_2 \to J_2$ are piecewise linear, continuous, have one turning point, and have growth rate $\lambda^2$. The map $f^2|_{J_2}$ is a tent map on $J_2$. The map $f^2|_{J_1}$ is an "inverted tent map" on $J_1$. It is conjugate (via scaling and then flipping the interval so as to exchange the endpoints) to a tent map $g$ on $[0,1]$ of growth rate $\lambda^2$.

Denote the itinerary of $1$ under $f$ by $a_1, a_2, a_3, \dots$. Since $f^2(J_2) \subset J_2 \subset I_1$, all odd terms $a_{2k+1}$ are equal to 1. What about the even terms? By definition, the term $a_{2k} = 1$ if and only if $f^{2k-1}(1) \in I_1$. This happens if and only if

$$f^{2k-1}(1) \in J_1 \cap [1/\lambda, 2/(1+\lambda)],$$

which is equivalent to

$$(f^2)^{k-1}(f(1)) \in J_1 \cap [1/\lambda, 2/(1+\lambda)].$$

Because the map that conjugates $f^2|_{J_1}$ and $g$ involves an isometric flip that exchanges the endpoints of the interval, we have that

$$(f^2)^{k-1}(f(1)) \in J_1 \cap [1/\lambda, 2/(1+\lambda)]$$

if and only if the point $g^{k-1}(1)$ lies to the left of the critical point for $g$. Thus, $a_{2k} = 1$ if and only if the $(k-1)^{\text{th}}$ digit of the itinerary of 1 under $g$ equals 0. Hence

$$(a_2, a_4, a_6, \dots) = s(\mathrm{It}_g(1))$$

□

**Proposition 8.2** (Period Doubling)**.** *Let $g$ be a superattracting tent map with growth rate $1 < \lambda \leq 2$, and denote the itinerary of 1 under $g$ by*

$$It_g(1) = b_1, b_2, b_3, \dots.$$

*Then the sequence*

$$a_1, a_2, a_3, \dots$$

*defined by*

$$\begin{cases} a_{2k+1} = 1 & \textit{for nonnegative integers } k \\ a_{2k} = b_k + 1 \pmod 2 & \textit{for nonnegative integers } k \end{cases}$$

*is the itinerary of 1 under the superattracting tent map of growth rate $\sqrt{\lambda}$.*

*Proof.* Denote by $g$ the superattracting tent map of growth rate $1 < \lambda \leq 2$, and denote by $f$ the tent map of growth rate $\sqrt{\lambda}$. Let $J_1$ and $J_2$ be the intervals defined as in equation (30) for $f$ (with growth rate $\sqrt{\lambda}$). By Lemma 8.1, $g$ is conjugate to $f^2|_{J_1}$ via an affine map that scales and flips $J_1$ (exchanging the endpoints). Since $g$ is superattracting, the left endpoint of $J_1$, $2-\sqrt{\lambda}$, is a (strictly) periodic point for $f^2$. Since $f(J_1) \subset J_2$ and $f(J_2 \subset J_1)$ and $f$ is injective on $J_2$, this implies 1 is a strictly periodic point for $f$. Hence $f$ is superattracting.

The statement about the itineraries is a restatement of Lemma 8.1. □

**Proposition 8.3.** *Let $\alpha \in \mathbb{D}$ be a Galois conjugate of a superattracting $\beta \in [1, 2]$. Then for any $y \in [\beta, 2]$ and any $\epsilon > 0$, there exists a superattracting $\beta'$ within $\epsilon$ of $y$ which has some Galois conjugate within $\epsilon$ of $\alpha$.*

*Proof.* We will use Period Doubling (Proposition 8.2) to extend the conclusion of Theorem 7.11, which gives the desired result for $\beta \in (\sqrt{2}, 2]$, to all $\beta$ in the interval $(1, 2]$.

Let $\alpha, \beta, y, \epsilon$ be as in the statement of the theorem. Assume $\beta > 1$. The case $y \in [\sqrt{2}, 2]$ is covered by Theorem 7.11, so assume $y \in (1, \sqrt{2})$. Define $k \in \mathbb{N}$ so that $y^{2^k} \in [\sqrt{2}, 2]$. Set $\tilde{y} = y^{2^k}$ and $\tilde{\alpha} = \alpha^{2^k}$. By Theorem 7.11, there exists a superattracting $\tilde{\beta}'$ within $\epsilon$ of $\tilde{y}$ which has a Galois conjugate $\tilde{z}$ within $2^{-2^k}\epsilon$ of $\tilde{\alpha}$. Without loss of generality, we may assume $\tilde{\beta}' \in [\sqrt{2}, 2]$. Set $\beta'$ to be the real root $(\tilde{\beta}')^{-2^k}$, and pick $z$ to be a root of $(\tilde{z})^{-2^k}$ that minimizes distance to $\alpha$.

Let $f$ be the minimal polynomial for $\tilde{\beta}' \in [\sqrt{2}, 2]$. The polynomial $f$ is, by definition, irreducible in $\mathbb{Z}[z]$, and so satisfies the assumptions of Lemma 2.6. Thus, for all $n \geq 1$, the polynomial $f(z^{2^n})$ is irreducible in $\mathbb{Z}[z]$. By Period Doubling (Proposition 8.2), if a growth rate $1 < \lambda < 2$ is admissible, then the growth rate $\sqrt{\lambda}$ is also admissible. Consequently, $\beta'$ is an admissible slope and $z$ is a Galois conjugate of $\beta'$.

Taking positive square roots of positive numbers does not increase distance, so $|\beta' - y| < \epsilon$. Since the supremum over $\mathbb{D}$ of the absolute value of the derivative of the map $z \mapsto z^2$ is 2, the distance between $\tilde{z} = z^{2^k}$ and $\tilde{\alpha} = \alpha^{2^k}$ is at most $2^{2^k}|z - \alpha|$. Hence

$$|z - \alpha| \leq 2^{2^k}|\tilde{z} - \tilde{\alpha}| < 2^{2^k}2^{-2^k}\epsilon = \epsilon.$$

For $\beta = 1$, since $1$ has no nontrivial Galois conjugates, $(z, 1)$ must the the limit of a sequence of points $(z_n, \lambda_n) \in \Upsilon_2$ with $z_n \in \mathbb{D}$ and $\lambda_n > 1$. By the previous argument, we can approximate each $(z_n, \lambda_n)$ by a point $(c_n, \beta'_n)$ where $\beta'_n$ is a superattracting growth rate with a Galois conjugate $c_n$ so that $c_n$ is within $\epsilon/2$ of $z_n$. The claim follows. □

*Theorem 1.* For $(z, \lambda) \in \Upsilon_2$ with $z \in \mathbb{D}$ and $\lambda > 1$, the statement $\{z\} \times [\lambda, 2] \subset \Upsilon_2$ follows immediately from Proposition 8.3 and the fact that the Master Teapot $\Upsilon_2$ is closed. Thus, it suffices to deal with the case $(z, 1) \in \Upsilon_2$ with $z \in \mathbb{D}$. Since $1$ has no nontrivial Galois conjugates, $(z, 1)$ must the the limit of a sequence of points $(z_n, \lambda_n) \in \Upsilon_2$ with $z_n \in \mathbb{D}$ and $\lambda_n > 1$. Then the interval $\{z\} \times [1, 2] \subset \Upsilon_2$ by the first part and that $\Upsilon_2$ is closed. □

In fact, the case $(z, 1) \subset \Upsilon_2$ with $z \in \mathbb{D}$ discussed in the proofs of Theorem 1 and Proposition 8.3 cannot occur; Proposition 9.1 will show that the bottom level of the Master Teapot is the unit circle.

## 9. THE UNIT CYLINDER AND CONNECTIVITY

**Proposition 9.1.** *The unit circle is equal to the bottom level of the Master Teapot, i.e.*

$$S^1 \times \{1\} = \Upsilon_2 \cap (\mathbb{C} \times \{1\}).$$

*Proof.* We will first show $S^1 \times \{1\} \subset \Upsilon_2$. By Proposition 8.2, if the tent map of growth rate $1 < \lambda \leq 2$ is superattracting, then the tent map of growth rate $\sqrt{\lambda}$ is also superattracting. Fix $1 < \lambda \leq 2$ such that the tent map of growth rate $\lambda$ is superattracting and such that the kneading polynomial for that map satisfies the assumptions of Lemma 2.6. Thus, for any Galois conjugate $\alpha$ of $\lambda$ and for any $n \in \mathbb{N}$, each of the $2^n$ complex points $\alpha^{\frac{1}{2^n}}$ is a Galois conjugate of the positive real root $\lambda^{\frac{1}{2^n}}$. So each of the $2^n$ points $(\alpha^{\frac{1}{2^n}}, \lambda^{\frac{1}{2^n}}) \subset \Upsilon_2$. Taking the closure over all $n$, we have that $S^1 \times \{1\} \subset \Upsilon_2$.

To show $\Upsilon_2 \cap (\mathbb{C} \times \{1\}) \subset S^1 \times \{1\}$, suppose there exists a point $(y, 1) \in \Upsilon_2$ such that $|y| \neq 1$. Since $1$ has no nontrivial Galois conjugates, $(y, 1) \in \mathbb{C} \times \mathbb{R}$ must be the the limit of a sequence of points $(\alpha_n, \beta_n) \in \mathbb{C} \times \mathbb{R}$ such that $\beta_n$ is the growth rate of a superattracting tent map and $\alpha_n$ is a Galois conjugate of $\beta$. Thus, reindexing the sequence as necessary, we have that for any $k > 0$, there exists $\beta_k$ with $1 < \beta_k < 1 + \frac{1}{k}$ with Galois conjugate $\alpha_k$, so that $|\alpha_k - y| < \epsilon$. Now by Lemma 8.1, $\beta_k^{2^{n_k}} \leq 2$ is admissible, where $n_k$ is the maximal value of $n$ for which $\beta_k^{2^{n_k}} \leq 2$. The fact that $\alpha_k^{2^{n_k}}$ is a Galois conjugate of $\beta_k^{2^{n_k}}$ follows immediately from the definition of a Galois automorphism. Thus $(\alpha_k^{2^{n_k}}, \beta_k^{2^{n_k}}) \subset \Upsilon_2$.

Now, $|\alpha_k|$ is bounded away from 1 for for $k$ sufficiently large (because $\alpha_k \to y$), and $n_k \to \infty$ as $k \to \infty$, since $\beta_k \to 1$ as $k \to \infty$. Consequently, either $\alpha_k^{2^{n_k}} \to 0$ or $\alpha_k^{2^{n_k}} \to \infty$ as $k \to \infty$. This is a contradiction because

$$\Omega \subset \{z \in \mathbb{C} : 1/2 \leq z \leq 2\}$$

by [Tio18, Lemma 2.4], and the projection of $\Upsilon_2$ onto the first coordinate is $\Omega_2$. □

**Proposition 9.2.** *Fix $z \in \mathbb{D} \cap \Omega$ and $\epsilon > 0$. Then there exists $(y, \beta) \in \mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$ such that*

*(1) $d((z, 2), (y, \beta)) < \epsilon$,*
*(2) $y$ is a Galois conjugate of $\beta$, and*
*(3) the minimal polynomial for $\beta$ has coefficients in $\{\pm 1\}$, and not all its coefficients are equal.*

*Proof.* Fix any sequence $\{\epsilon_i\}_{i \in \mathbb{N}}$, $\epsilon_i \in \{\pm 1\}$. In the proof of [Tio18, Corollary 5.3], Tiozzo shows that for any $n \in \mathbb{N}$, there exist arbitrarily large $N \in \mathbb{N}$ and $\eta = \eta(N, n) \in \{\pm 1\}$ such that

$$P_{N,n}(t) = 1 - \left(\sum_{k=1}^{N} t^k\right) + \eta t^{N+1} + \left(\sum_{k=0}^{n} \epsilon_{n-k} t^{N+2+k}\right)$$

is an admissible kneading determinant for a superattracting tent map and the polynomial

$$\begin{aligned} Q_{N,n}(t) &= t^{N+n+2} P_{N,n}\left(\frac{1}{t}\right) \\ &= t^{N+n+2} - \left(\sum_{k=1}^{N} t^{N+n+2-k}\right) + \eta t^{n+1} + \left(\sum_{k=0}^{n} \epsilon_{n-k} t^{n-k}\right) \end{aligned}$$

is irreducible. The leading (real) root of $Q_{N,n}$ is the inverse growth rate of the associated superattracting tent map, and its Galois conjugates are the other roots of $Q_{N,n}$. By Rouché's Theorem, for any sequence $\{N_i\}_{i\in\mathbb{N}}$, each root of $\sum_{k=0}^{\infty} \epsilon_k x^k$ is the limit of roots of $Q_{N_i,i}(x)$ as $i \to \infty$.

We claim that for any fixed $n$, the limit as $N \to \infty$ of the leading root of $Q_{N,n}$ equals 2. Suppose $\{\lambda_N\}_{n\in\mathbb{N}}$ is a sequence of nonzero complex numbers with $3/2 < |\lambda_N| \leq 2$ such that $0 = Q_{N,n}(\lambda_N)$. Then

$$0 = P_{N,n}\left(\frac{1}{\lambda_N}\right) = 1 - \sum_{k=1}^{N} \left(\frac{1}{\lambda_N}\right)^k + \frac{1}{\lambda_N^{N+n+2}} \left(\eta \lambda_N^{n+1} + \sum_{k=0}^{n} \epsilon_{n-k} \lambda_N^{n-k}\right).$$

Now

$$\begin{aligned} (31) \qquad \left|\frac{1}{\lambda_N^{N+n+2}} \left(\eta \lambda_n^{n+1} + \sum_{k=0}^{n} \epsilon_{n-k} \lambda_N^{n-k}\right)\right| &\leq \frac{1}{|\lambda_N^{N+n+2}|} \left(\sum_{k=0}^{n+1} |\lambda_N|^k\right) \\ &\leq \frac{1}{|\lambda_N^{N+n+2}|} \left(\sum_{k=0}^{\infty} |\lambda_N|^k\right) \leq \frac{1}{(1-|\lambda_N|)|\lambda_N^{N+n+2}|} \leq \frac{2}{(3/2)^{N+n+2}}. \end{aligned}$$

Hence,

$$0 = \lim_{N\to\infty} P_{N,n}(\frac{1}{\lambda_N}) = 1 - \lim_{N\to\infty} \sum_{k=1}^{N} \left(\frac{1}{\lambda_N}\right)^k + \lim_{N\to\infty} \frac{1}{\lambda_N^{N+n+2}} \left(\eta \lambda_n^{n+1} + \sum_{k=0}^{n} \epsilon_{n-k} \lambda_N^{n-k}\right).$$

Thus, since the limit of the right hand term is 0 by (31),

$$1 = \lim_{N\to\infty} \sum_{k=1}^{N} \left(\frac{1}{\lambda_N}\right)^k.$$

Since $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$, this implies $\lim_{N\to\infty} \lambda_N = 2$.

□

*Proof of Theorem 2.* By [Tio18, Proposition 6.1], there exists $R > 1$ such that the inclusion

$$\{z \mid R^{-1} < |z| < R\} \subset \Omega_2$$

holds. Therefore, by the Persistence Theorem 1, we have that there exists $R > 0$ such that the annulus

$$A := \{(z, 2) \in \mathbb{C} \times \mathbb{R} \mid R^{-1} < |z| < 1\} \subset \Upsilon_2.$$

By Proposition 9.2, each point in $A$ is the limit of a sequence of points of the form $(y, \beta) \in \mathbb{C} \times \mathbb{R}$ such that $y$ is a Galois conjugate of $\beta < 2$, $\beta$ is the growth rate of a superattracting tent map, and the minimal polynomial for $\beta$ has all coefficients in $\{\pm 1\}$, with not all coefficients are equal.

Consider any such fixed $(\beta, y)$. By Period Doubling (Proposition 8.2), for any $n \in \mathbb{N}$, we have that $\beta^{\frac{1}{2^n}}$ is the growth rate of a superattracting tent map. By Lemma 2.6 [Tio18, Lemma 4.2], if $f(x)$ is the minimal polynomial for $\beta$, then $f(x^{2^n})$ is irreducible for all $n \in \mathbb{N}$. Hence, if $\gamma$ is any $(\frac{1}{2^n})^{\text{th}}$ root of $y$, then $\gamma$ is a Galois conjugate of $\beta^{\frac{1}{2^n}}$.

Consequently, for any $n \in \mathbb{N}$, the set

$$\left\{\left(z, 2^{\frac{1}{2^n}}\right) \in \mathbb{C} \times \mathbb{R} \mid \left(R^{-1}\right)^{\frac{1}{2^n}} < |z| < 1\right\} \subset \Upsilon_2.$$

Therefore, by the Persistence Theorem 1, for each $n \in \mathbb{N}$, we have the inclusion

$$\left\{(z, \lambda) \in \mathbb{C} \times \mathbb{R} \mid \left(R^{-1}\right)^{\frac{1}{2^n}} < |z| < 1, 2^{\frac{1}{2^n}} \leq \lambda \leq 2\right\} \subset \Upsilon_2.$$

Since $\Upsilon_2$ is closed, in fact we have the stronger inclusion

$$\left\{(z, \lambda) \in \mathbb{C} \times \mathbb{R} \mid \left(R^{-1}\right)^{\frac{1}{2^n}} \leq |z| \leq 1, 2^{\frac{1}{2^n}} \leq \lambda \leq 2\right\} \subset \Upsilon_2.$$

□

*Proof of Theorem 3.* Connnectivity of the part of the Master Teapot outside of the unit cylinder is due to Tiozzo [Tio18]. Namely, by [Tio18, Lemma 7.3], for any point $(z, \beta) \subset \mathbb{C} \times \mathbb{R}$ such that $\beta$ is the growth rate of a superattracting tent map, $z$ is a Galois of $\beta$, and $|z| > 1$, there exists a continuous path $(\gamma(x), x)$ in $\Upsilon_2$ connecting $(z, \beta)$ to a point $(w, 1)$. Consequently, since the unit cylinder is in $\Upsilon_2$ by Theorem 2, and since $\Upsilon_2$ is closed, this implies $\Upsilon_2 \cap (\{z : |z| \geq 1\} \times \mathbb{R})$ is connected. By the Persistence Theorem 1, the part of the Master Teapot inside the unit circle is connected. Thus, the entire Master Teapot, $\Upsilon_2$, is connected. □

## 10. Gaps in the Thurston set

Plots of finite approximations of the Thurston set consisting of the roots of all defining polynomials associated to superattracting tent maps of critical orbit length at most $n$, for fixed $n \in \mathbb{N}$, have "gaps" at certain algebraic integers, some of which are on the unit circle. The Thurston set contains a neighborhood of the unit circle [Tio18], but these gaps get filled in more slowly with $n$ than some other regions. See Figure 2 for a picture of the entire Thurston set, and Figure 3 for a closeup of one such gap. In this section, we prove an arithmetic justification for gaps:

**Theorem 10.1.** *Let $\alpha$ be an algebraic integer such that $\mathbb{Z}[\alpha]$ is a discrete subgroup of $\mathbb{C}$ and let $x \in \mathbb{Z}[\alpha]$. Set $c = \min\{|z| : z \in \mathbb{Z}[\alpha], z \neq 0\}$. Suppose there exists a superattracting tent map with postcritical length $n$ whose growth rate has a Galois conjugate of the form $x + \epsilon$ for some $\epsilon \in \mathbb{C}$ with $|\epsilon| \leq \frac{1}{n+1}$. Then*

(1) *if $|x| \geq 1$, then* $\dfrac{c}{(2n^2 + 3n + 1)|x|^n e} \leq \epsilon$.
(2) *if $|x| \leq 1$, then* $\dfrac{c}{(2n^2 + 3n + 1)|x| e} \leq \epsilon$.

*Proof.* Fix $x \in \mathbb{Z}[\alpha]$ and suppose there exists a real number $\beta$ associated to a generalized PCF $\beta$-map with $m$ intervals and postcritical length $n$ that has a Galois conjugate of the form $x + \epsilon$ for some $\epsilon \in \mathbb{C}$ with $|\epsilon| \leq 1$.

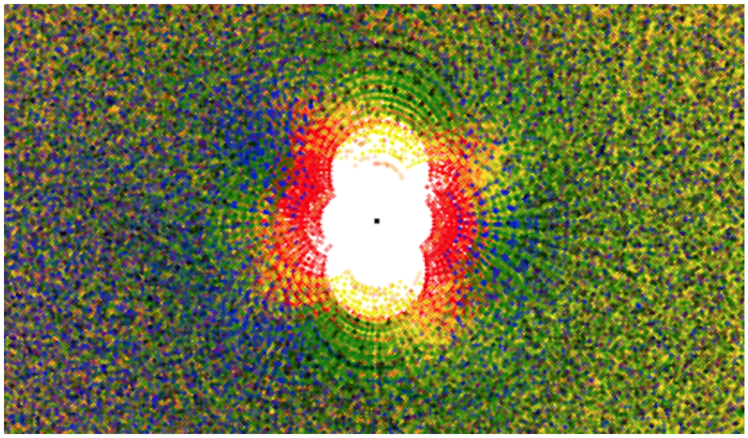

FIGURE 3. A closeup of the how the "gap" around the point $i$ fills in as postcritical length increases, for an approximation of the Thurston set. The points are color-coded by the length of the associated post-critical orbit. Blue is the shortest, followed by green, yellow, orange, and finally red with the longest orbit, of length 23.

Then $\beta$ is the root of the associated Parry polynomial $P_{\beta,E}$;

$$0 = z^{n+1} - (a_0 z^n + a_1 z^{n-1} + \cdots + a_n) - 1,$$

where $a_i \in \{-2, 0, 2\}$. Hence $(x+\epsilon)$ is also a root of $P_{\beta,E}$:

$$0 = (x+\epsilon)^{n+1} - (a_0(x+\epsilon)^n + a_1(x+\epsilon)^{n-1} + \cdots + a_n) - 1.$$

Therefore

$$\begin{aligned} 1 - x^{n+1} + a_0 x^n + \cdots + a_n = (x+\epsilon)^{n+1} - x^{n+1} - \big(a_0((x+\epsilon)^n - x^n) \\ + a_1((x+\epsilon)^{n-1} - x^{n-1}) + \cdots + a_{n-1}((x+\epsilon) - x)\big). \end{aligned}$$

We have $1 - x^{n+1} + a_0 x^n + \cdots + a_n \in Z[\alpha]$, so $c \leq |1 - x^{n+1} + a_0 x^n + \cdots + a_n|$. Then by the triangle inequality,

(32)

$$\begin{aligned} &c \leq |1 - x^{n+1} + a_0 x^n + \cdots + a_n| \\ &\leq |(x+\epsilon)^{n+1} - x^{n+1}| + |a_0||(x+\epsilon)^n - x^n| + |a_1||(x+\epsilon)^{n-1} - x^{n-1}| \ldots |a_{n-1}||(x+\epsilon) - x|. \end{aligned}$$

We now restrict to the case $|x| \geq 1$. For any $k \leq n+1$, by the binomial theorem, the triangle inequality, and $|\epsilon| \leq \frac{1}{n+1}$,

(33)
$$\begin{aligned}
|(x+\epsilon)^k - x^k| = \quad & \left|\sum_{i=1}^{k}\binom{k}{i}x^{k-i}\epsilon^i\right| \quad \le \quad \sum_{i=1}^{k}\left|\binom{k}{i}x^{k-i}\epsilon^i\right| \\
\le \quad & \sum_{i=1}^{k}\left|\frac{k^i}{(k-i)!}x^{k-i}\frac{1}{(n+1)^{i-1}}\epsilon\right| \quad = \quad \sum_{i=1}^{k}\left|\left(\frac{k}{n+1}\right)^{i-1}\frac{k}{(k-i)!}\,\epsilon\, x^{k-i}\right| \\
\le \quad & \epsilon k|x|^{k-1}\sum_{i=1}^{k}\frac{1}{(k-i)!} \quad = \quad \epsilon k|x|^{k-1}\sum_{i=0}^{k-1}\frac{1}{i!} \\
\le \quad & \epsilon k|x|^{k-1}\sum_{i=0}^{\infty}\frac{1}{i!} \quad = \quad \epsilon k|x|^{k-1}e.
\end{aligned}$$

Combining equations (32) and (33) yields

$$\begin{aligned}
c &\le \epsilon(n+1)e|x|^n + |a_0|\epsilon n e|x|^{n-1} + \cdots + |a_{n-1}|\epsilon 1|x|^0 e| \\
&\le \epsilon(n+1)e|x|^n\left(1+|a_0|+\cdots+|a_{n-1}|\right) \\
&\le \epsilon(n+1)e|x|^n\left(1+2n\right).
\end{aligned}$$

Thus
$$\frac{c}{e(1+2n)(n+1)|x|^n} \le \epsilon.$$

We now restrict to the case $|x| \le 1$. In this case, the estimate (33) becomes

$$\left|(x+\epsilon)^k - x^k\right| \le \epsilon k|x|e. \tag{34}$$

Combining equations (32) and (34) yields

$$c \le \epsilon(n+1)e|x|(1+|a_0|+|a_1|+\cdots+|a_{n-1}|) \le \epsilon(n+1)e|x|(1+2n).$$

Hence, for $|x| \ge 1$,
$$\frac{c}{(n+1)(1+2n)|x|e} \le \epsilon.$$

□

*Proof of Theorem 4.* In view of Theorem 10.1, it suffices to classify the discrete subgroups of $\mathbb{C}$. The classification of discrete subrings of $\mathbb{C}$ is well-known, and we include it for completeness: firstly, because it is a discrete additive subgroup, it is either $\mathbb{Z}$ or a lattice of rank 2. If it is the latter case, let $\{1, a\}$ be a basis of the lattice, then $a$ must be an algebraic integer of degree 2, so it can be chosen as something of either the form $\sqrt{-D}$ or $\frac{1+\sqrt{-D}}{2}$ (the latter only when $D = 4n+1$), where $D$ is some positive integer. Requiring that there is an element not on the real line and has absolute value less than 2 means that $D = 1, 2, 3, 5$. □

## 11. $\Omega_2$ AND $\Omega_2^{pre}$ ARE NOT EQUAL

In this section we prove Theorem 5, that $\Omega_2$ and $\Omega_2^{pre}$ are not equal. $\Omega_2$ is shown in Figure 2, and $\Omega_2^{pre}$ is shown in Figure 4.

As outlined in section §2.5, a point $z \in \mathbb{D}$ is in $\Omega_2$ if and only if $0$ is in the limit set of the iterated function system generated by $f_z, g_z$, where

$$f_z : x \mapsto zx+1, \quad g_z : x \mapsto zx-1.$$

Denote the alphabet $\{f_z, g_z\}$ by $\mathcal{F}_z$ and denote the alphabet of inverses $\{f_z^{-1}, g_z^{-1}\}$ by $\mathcal{F}_z^{-1}$. For a word $w = w_1, \ldots, w_n$ in the alphabet $\mathcal{F}_z$ or in the alphabet $\mathcal{F}_z^{-1}$, define the action of $w$ on $\mathbb{C}$ by

$$w(x) = w_n \circ \cdots \circ w_1(x).$$

**Lemma 11.1.** *Fix $z \in \mathbb{D} \setminus \{0\}$. If there exists $n \in \mathbb{N}$ such that*

$$\min\left\{|v(0)| : v \in (\mathcal{F}_z^{-1})^n\right\} > \frac{1}{1 - |z|},$$

*then $z \notin \Omega_2$.*

*Proof.* Suppose $z \in \mathbb{D} \cap \Omega_2$. Then $0$ is in the limit set $\Lambda_z$. Since $\Lambda_z = f_z(\Lambda_z) \cup g_z(\Lambda_z)$, it follows that $\Lambda_z$ is fixed by taking the union of the images of $\Lambda_z$ under all words of length $n$, for any $n \in \mathbb{N}$:

$$\Lambda_z = \bigcup_{w \in (\mathcal{F}_z)^n} w(\Lambda_z).$$

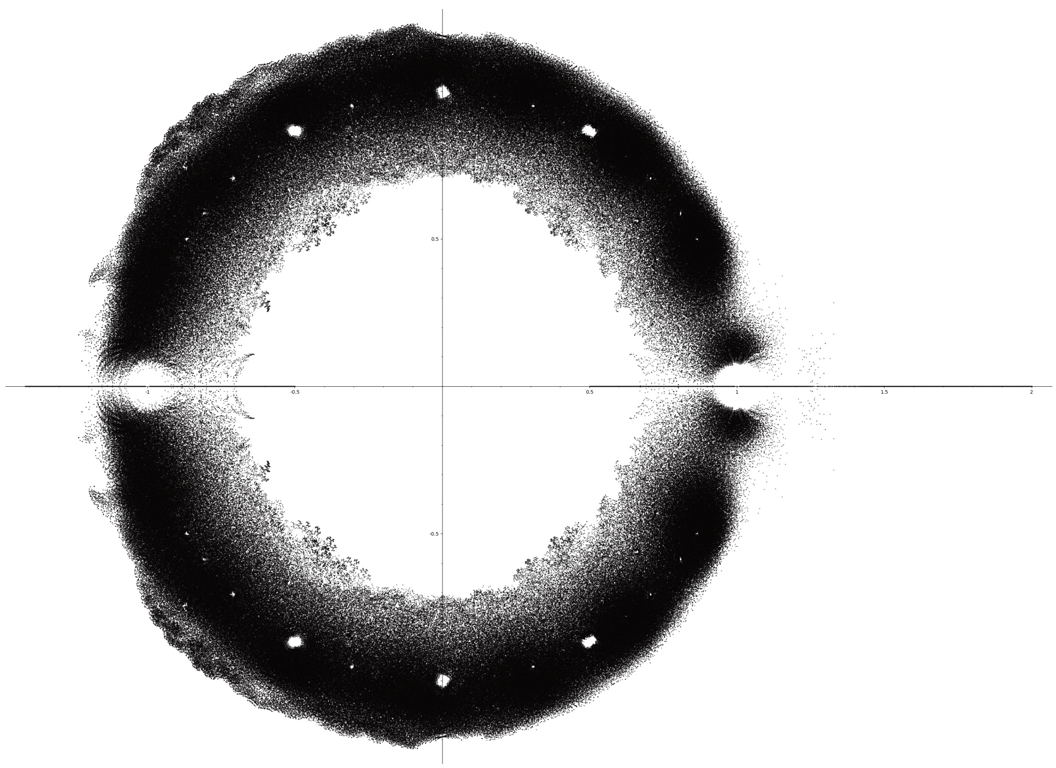

FIGURE 4. An approximation of the preperiodic Thurston set, $\Omega_2^{\text{pre}}$, consisting of the roots of all minimal polynomials associated to postcritically finite tent maps for which the sum of the pre-critical length and the period is at most 19. Compare this with the Thurston set $\Omega_2$ in Figure 2, and note in particular the difference in a large neighborhood of the point 1.

Hence, for any $n \in \mathbb{N}$, each point in $\Lambda_z$ is the image of a point $\Lambda_z$ under some word in $\mathcal{F}_z$ of length $n$. In particular, $0$ is the the image of a point in $\Lambda_z$ under some word in $\mathcal{F}$ of length $n$. Since $\Lambda_z \subset B_{\frac{1}{1-|z|}}(0)$ by Lemma 2.14, this implies that for any $n \in \mathbb{N}$,

$$\left( \bigcup_{v \in (\mathcal{F}_z^{-1})^n} v(0) \right) \cap B_{\frac{1}{1-|z|}}(0) \neq \emptyset.$$

□

*Proof of Theorem 5.* We will exhibit a point that is in $\Omega_2^{pre}$ but not in $\Omega_2$. Let $w$ be the preperiodic itinerary

$$w = 1000011100(101000)^\infty.$$

One may verify that $\sigma^j(w) \leq_E w$ for every integer $j \geq 0$. Hence, by the Admissibility Criterion (Fact 2.9), $w$ is the itinerary of $1$ under a preperiodic tent map. One may then calculate from $w$ the sequence of digits:

$$2000022200(202000)^\infty$$

and the sequence of cumulative signs:

$$+ - - - - - + - + + (+ - - + ++)^\infty.$$

Then the $\beta$-expansion of $1$, where $\beta$ is the slope of the associated tent map (Fact 2.2), is given by

$$1 = \frac{2}{\beta} - \frac{2}{\beta^6} + \frac{2}{\beta^7} - \frac{2}{\beta^8} + \frac{1}{\beta^{10}} \left( \frac{2}{\beta^1} - \frac{2}{\beta^3} \right) \sum_{n=0}^{\infty} \left( \frac{1}{\beta^6} \right)^n. \tag{35}$$

Substituting in the sum of the geometric series and clearing denominators, the equation (35) becomes

$$0 = 2 - 4\beta + 2\beta^2 + 2\beta^3 - 2\beta^6 - \beta^8 + 2\beta^{13} - \beta^{14},$$

which factors as

$$0 = \big(-1+\beta\big)\big(1+\beta\big)\big(2-4\beta+4\beta^2-2\beta^3+4\beta^4-2\beta^5+2\beta^6-2\beta^7+\beta^8-2\beta^9+\beta^{10}-2\beta^{11}+\beta^{12}\big).$$

Let $P$ be the irreducible polynomial

$$P(x) = x^{12} - 2x^{11} + x^{10} - x^9 + x^8 - 2x^7 + 2x^6 - 2x^5 + 4x^4 - 2x^3 + 4x^2 - 4x + 2.$$

By construction, the roots of $P$ are in $\Omega_2^{per}$. Let $p$ be the root of $P$ with approximate value

$$p \approx 0.5393738531461442 + 0.4050155839374199i.$$

Since $|p|$ is approximately $0.674509$, $p \in \mathbb{D} \cap \Omega_2^{pre}$.

Let $\mathcal{F}_p^{-1}$ be the alphabet consisting of the two maps $f_p^{-1}$ and $g_p^{-1}$, where

$$f_p^{-1} : x \mapsto \frac{x-1}{p}, \quad g_p^{-1} : x \mapsto \frac{x+1}{p}.$$

Computation shows that

$$\min \left\{ |v(0)| : v \in (\mathcal{F}_p^{-1})^5 \right\} \approx 4.3792,$$

which is much bigger than $\frac{1}{1-|p|} \approx 3.07228$. Consequently, Lemma 11.1 implies that $p \notin \Omega_2$.

□

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