# Derivation of Lindblad master equation for the quantum Ising model interacting with a heat bath

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Starting from the Liouville-von Neumann equation, under a weak coupling limit we derive the Lindblad master equation for the one-dimensional quantum Ising model in a Markov approximation and a rotating wave approximation. We also prove that the steady solution of the Lindblad equation is the canonical distribution independent of the dissipation rate.

#### I. INTRODUCTION

The Lindblad master equation plays an important role in open quantum systems. As an example, it has been widely used in quantum optics<sup>1</sup>. It was originally derived by Lindblad using quantum dynamical semigroups<sup>2</sup>. Attal and Joye<sup>3</sup> obtained it through taking a continuous limit of repeated interactions with a sequence of baths in a given density matrix state. Recently, Brasil, Fanchini, and Napolitano<sup>4</sup> provided a relatively simple derivation of the Lindblad equation starting from a general Hamiltonian and the Liouville-von Neumann equation in a Markov approximation and a rotating wave approximation under weak-coupling limit.

For many-body systems, many studies focus on the Lindblad equation for open systems interacting with the environment at their boundaries<sup>5,6</sup>. Here we derive the Lindblad master equation for a one-dimensional quantum Ising system interacting with a heat bath at each site following the schema given in Ref. 4.

Our derivation and discussion of the Lindblad master equation will proceed as follows: In Sec. II we diagonalize the Hamiltonian for a composite system consisting of a quantum Ising chain and a heat bath and write it in the interaction picture. In Sec. III we start from the Liouville-von Neumann equation describing the evolution of the density matrix of the composite system to derive the Born-Markov equation under a weak system-environment interaction limit and a Markov approximation. These results are then combined to obtain the Lindblad equation for the quantum Ising model with the bosonic heat bath interacting at each site in Sec. IV. The steady solution of Lindblad equation is shown to be the canonical distribution independent of the dissipation rate in Sec. V, followed by a brief summary in Sec. VI.

### II. HAMILTONIAN AND ITS DIAGONALIZATION

The Hamiltonian for the quantum Ising chain interacting with a bosonic heat bath is

$$H = H_S + H_B + H_{SB},\tag{1}$$

with

$$H_S = -h_x \sum_{i=1}^{N} \sigma_i^x - J \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z,$$
 (2)

$$H_B = \sum_{\beta,i} \omega_\beta b_{\beta i}^\dagger b_{\beta i},\tag{3}$$

$$H_{SB} = \sum_{\beta,i} \lambda_{\beta} (b_{\beta i}^{\dagger} + b_{\beta i}) \sigma_{i}^{x}, \tag{4}$$

where  $H_S$ ,  $H_B$ , and  $H_{SB}$  are the Hamiltonians for the quantum Ising chain, the heat bath, and their interaction, respectively, N is the total number of spins,  $\sigma_i^j$  represents the Pauli matrix along j direction at site i,  $h_x$  is a traverse field,  $\lambda_{\beta}$  is the coupling strength,  $b_{\beta i}^{\dagger}$  ( $b_{\beta i}$ ) creates (annihilates) in mode  $\beta$  with an energy  $\omega_{\beta}$  a boson coupling to the spin at site i. Equations (1) to (4) have been studied in Ref. 7 using Green's functions.

In order to diagonalize  $H_S$ , we first apply a Jordan-Wigner transformation<sup>8</sup> to  $H_S$  and  $H_{SB}$ . After the transformation, we have

$$H_S = -h_x \sum_{j=1}^{N} (1 - C_j^{\dagger} C_j) - J \sum_{j=1}^{N} (C_j^{\dagger} C_{j+1}^{\dagger} + C_j^{\dagger} C_{j+1} - C_j C_{j+1}^{\dagger} - C_j C_{j+1}), \tag{5}$$

$$H_{SB} = \sum_{\beta,i} \lambda_{\beta} (b_{\beta i}^{\dagger} + b_{\beta i}) (1 - C_j^{\dagger} C_j), \tag{6}$$

where  $C_j^{\dagger}$  and  $C_j$  are creation and annihilation operators at site j for the Jordan-Wigner fermions, respectively. Next, a Fourier transformation leads to

$$H_S = 2J \sum_{q>0} (C_q^{\dagger} C_{-q}) \begin{pmatrix} \frac{h_x}{J} - \cos q & i \sin q \\ -i \sin q & -\frac{h_x}{J} + \cos q \end{pmatrix} \begin{pmatrix} C_q \\ C_{-q}^{\dagger} \end{pmatrix}, \tag{7}$$

$$H_{SB} = \frac{-1}{\sqrt{N}} \sum_{k,q,\beta} \left[ b_{\beta q}^{\dagger} (C_k^{\dagger} C_{k+q} - C_{k+q} C_k^{\dagger}) + b_{\beta q} (C_{k+q}^{\dagger} C_k - C_k C_{k+q}^{\dagger}) \right], \tag{8}$$

$$H_B = \sum_{\beta,q} \omega_q b_{\beta q}^{\dagger} b_{\beta q}, \tag{9}$$

where  $C_q^{\dagger}$  and  $C_q$  are the corresponding operators in momentum space.

Further, a Bogoliubov transformation<sup>9</sup> of

$$\eta_q = u_q^* C_q + v_{-q} C_{-q}^{\dagger} \tag{10}$$

results in

$$H_S = \sum_q \Omega_q \eta_q^{\dagger} \eta_q, \tag{11}$$

$$H_{SB} = \sum_{j=1}^{3} H_{jSB},\tag{12}$$

with

$$H_{jSB} = \frac{-1}{\sqrt{N}} \sum_{k,q,\beta} \lambda_{\beta} (b_{\beta j}^{\dagger} a_{jkq} + b_{\beta j} a_{jkq}^{\dagger}), \tag{13}$$

and

$$a_{1kq} = (u_k^* u_{k+q} - v_{k+q} v_k^*) (\eta_k^{\dagger} \eta_{k+q} - \eta_{k+q} \eta_k^{\dagger}),$$

$$a_{2kq} = (v_{-k} u_{k+q} - v_{k+q} u_{-k}) \eta_{-k} \eta_{k+q},$$

$$a_{3kq} = (u_k^* v_{-k-q}^* - u_{-k-q}^* v_k^*) \eta_k^{\dagger} \eta_{-k-q}^{\dagger},$$
(14)

where  $\Omega_q = 2J\sqrt{(h_x/J)^2 + 1 - 2(h_x/J)\cos q}$ .

In the interaction picture defined by

$$A^{I}(t) = e^{i(H_S + H_B)t} A e^{-i(H_S + H_B)t}, (15)$$

for an operator A,

$$H_{SB}^{I}(t) = \frac{-1}{\sqrt{N}} \sum_{j=1}^{3} \sum_{k,q,\beta} \lambda_{\beta} \left( b_{\beta j}^{\dagger} a_{jkq} e^{-i\omega_{jkq\beta}t} + b_{\beta j} a_{jkq}^{\dagger} e^{i\omega_{jkq\beta}t} \right), \tag{16}$$

where the superscript I denotes the interaction picture and  $\omega_{jkq\beta} = \omega_{jkq} - \omega_{\beta}$  with  $\omega_{1kq} = \Omega_{k+q} - \Omega_k$ ,  $\omega_{2kq} = \Omega_{-k} + \Omega_{k+q}$ , and  $\omega_{3kq} = -\Omega_k - \Omega_{-k-q}$ . In Eq. (16), we have made use of the relations

$$\exp(iH_S t)a_{jkq}\exp(-iH_S t) = a_{jkq}\exp(-i\omega_{jkq}t), \qquad \exp(iH_S t)a_{jkq}^{\dagger}\exp(-iH_S t) = a_{jkq}^{\dagger}\exp(i\omega_{jkq}t), \tag{17}$$

derived from Eqs. (11) and (14).

#### III. DERIVATION OF BORN-MARKOV EQUATION

The composite system evolves according to the Liouville-von Neumann equation

$$\frac{\partial \rho}{\partial t} = -i[H, \rho],\tag{18}$$

where  $\rho$  is the density matrix of the composite system. In the interaction picture, the Liouville-von Neumann equation becomes

$$\frac{\partial \rho^I(t)}{\partial t} = -i[H_{SB}^I(t), \rho^I(t)]. \tag{19}$$

Integrating Eq. (19) from 0 to t and substituting the result back into it gives

$$\frac{\partial \rho^{I}(t)}{\partial t} = -i[H_{SB}^{I}(t), \rho^{I}(0)] - [H_{SB}^{I}(t), \int_{0}^{t} dt' [H_{SB}^{I}(t'), \rho^{I}(t')]], \tag{20}$$

where  $\rho^I(0) = \rho^I_S(0) \otimes \rho^I_B(0)$ .

Our goal is the evolution of  $\rho_S^I(t) = \text{Tr}_B[\rho^I(t)]$ , the density matrix operator of the system itself. So,

$$\frac{\partial \rho_S^I(t)}{\partial t} = -i \operatorname{Tr}_B[H_{SB}^I(t), \rho^I(0)] - \operatorname{Tr}_B[H_{SB}^I(t), \int_0^t dt' [H_{SB}^I(t'), \rho^I(t')]]. \tag{21}$$

Assuming

$$\rho_B^I(0) = \frac{\prod_{\beta q} \exp\left(\frac{-i\omega_{\beta} b_{\beta q}^{\dagger} b_{\beta q}}{k_B T}\right)}{\operatorname{Tr}_B\left[\prod_{\beta q} \exp\left(\frac{-i\omega_{\beta} b_{\beta q}^{\dagger} b_{\beta q}}{k_B T}\right)\right]},\tag{22}$$

for an equilibrium bath at the temperature T ( $k_B$  is Boltzmann's constant), one sees that  $-i\text{Tr}_B[H^I_{SB}(t), \rho^I(0)] = 0$  because  $\text{Tr}_B(b^{\dagger}_{\beta q}\rho^I_B(0)) = \text{Tr}_B(b_{\beta q}\rho^I_B(0)) = 0$ . Also, in Born approximation of weak system-environment interaction limit<sup>10</sup>, we can write  $\rho^I(t) = \rho^I_S(t) \otimes \rho^I_B(0)$  as the reduced density matrix for the bath changes little with time because of its huge size. So, Eq. (21) becomes

$$\frac{\partial \rho_S^I(t)}{\partial t} = -\text{Tr}_B[H_{SB}^I(t), \int_0^t dt' [H_{SB}^I(t'), \rho_S^I(t') \otimes \rho_B^I]], \tag{23}$$

or

$$\rho_S^I(t') - \rho_S^I(t) = -\int_t^{t'} dt'' \operatorname{Tr}_B[H_{SB}^I(t''), \int_0^{t''} dt''' [H_{SB}^I(t'''), \rho_S^I(t''') \otimes \rho_B^I]], \tag{24}$$

after integration. We see that the difference between  $\rho_S^I(t')$  and  $\rho_S^I(t)$  is of second order in  $H_{SB}^I$ . As a result, in the weak system-environment interaction limit, we can replace  $\rho^I(t')$  in Eq. (23) with  $\rho^I(t)$  and arrive at

$$\frac{\partial \rho_S^I(t)}{\partial t} = -\text{Tr}_B[H_{SB}^I(t), \int_0^t dt' [H_{SB}^I(t'), \rho_S^I(t) \otimes \rho_B^I]]. \tag{25}$$

In the Markov approximation, the system is memoryless. This can be obtained by changing the integrated variable to t - t' and sending the upper limit of the integral to  $+\infty^{10}$ . The result is a Born-Markov equation,

$$\frac{\partial \rho_S^I(t)}{\partial t} = -\text{Tr}_B[H_{SB}^I(t), \int_0^{+\infty} dt' [H_{SB}^I(t - t'), \rho_S^I(t) \otimes \rho_B^I]]. \tag{26}$$

## IV. DERIVATION OF LINDBLAD EQUATION

In this section, we shall derive the Lindblad equation from the Born-Markov equation (26) in the rotating wave approximation 10.

We first consider the contribution from  $H_{1SB}$ . Denoting the corresponding density matrix as  $\rho_{1S}^{I}$ , substituting Eq. (16) into Eq. (26), and using<sup>1</sup>

$$\operatorname{Tr}_{B}(b_{\beta q}b_{\beta'q'}^{\dagger}\rho_{B}^{I}) = (\langle n(\omega_{\beta})\rangle + 1)\delta_{\beta\beta'}\delta_{qq'},$$

$$\operatorname{Tr}_{B}(b_{\beta'q'}^{\dagger}b_{\beta q}\rho_{B}^{I}) = \langle n(\omega_{\beta})\rangle\delta_{\beta\beta'}\delta_{qq'},$$

$$\operatorname{Tr}_{B}(b_{\beta q}b_{\beta'q'}\rho_{B}^{I}) = \operatorname{Tr}_{B}(b_{\beta q}^{\dagger}b_{\beta'q'}^{\dagger}\rho_{B}^{I}) = 0,$$

$$(27)$$

with  $\langle n(\omega_{\beta}) \rangle = \langle b_{\beta q}^{\dagger} b_{\beta q} \rangle = 1/[\exp(\omega_{\beta}/k_B T) - 1]$ , we have

$$\frac{\partial \rho_{1S}^{I}}{\partial t} = \frac{-1}{N} \int_{0}^{+\infty} dt' \sum_{kk'q\beta} \lambda_{\beta}^{2}$$

$$\times \left\{ \left[ a_{1kq}^{\dagger} a_{1k'q} \rho_{1S}^{I} (\langle n(\omega_{\beta}) \rangle + 1) e^{i\omega_{1kq\beta}t - i\omega_{1k'q\beta}(t - t')} + a_{1kq} a_{1k'q}^{\dagger} \rho_{1S}^{I} \langle n(\omega_{\beta}) \rangle e^{-i\omega_{1kq\beta}t + i\omega_{1k'q\beta}(t - t')} \right] \right.$$

$$- \left[ a_{1kq}^{\dagger} \rho_{1S}^{I} a_{1k'q} \langle n(\omega_{\beta}) \rangle e^{i\omega_{1kq\beta}t - i\omega_{1k'q\beta}(t - t')} + a_{1kq} \rho_{1S}^{I} a_{1k'q}^{\dagger} (\langle n(\omega_{\beta}) \rangle + 1) e^{-i\omega_{1kq\beta}t + i\omega_{1k'q\beta}(t - t')} \right]$$

$$- \left[ a_{1k'q}^{\dagger} \rho_{1S}^{I} a_{1kq} \langle n(\omega_{\beta}) \rangle e^{-i\omega_{1kq\beta}t + i\omega_{1k'q\beta}(t - t')} + a_{1k'q} \rho_{1S}^{I} a_{1k'q}^{\dagger} (\langle n(\omega_{\beta}) \rangle + 1) e^{i\omega_{1kq\beta}t - i\omega_{1k'q\beta}(t - t')} \right]$$

$$+ \left[ \rho_{1S}^{I} a_{1k'q}^{\dagger} a_{1kq} (\langle n(\omega_{\beta}) \rangle + 1) e^{-i\omega_{1kq\beta}t + i\omega_{1k'q\beta}(t - t')} + \rho_{1S}^{I} a_{1k'q} a_{1kq}^{\dagger} \langle n(\omega_{\beta}) \rangle e^{i\omega_{1kq\beta}t - i\omega_{1k'q\beta}(t - t')} \right] \right\}. (28)$$

To proceed, note first that the integral

$$\int_{0}^{\infty} dt' e^{\pm i\omega_{1k'q\beta}t'} = \pi \delta(\omega_{1k'q\beta}) \pm i\mathcal{P} \frac{1}{\omega_{1k'q\beta}}$$
(29)

indicates that only the values of k', q, and  $\beta$  satisfying  $\omega_{1k'q\beta} = 0$  contribute to the real part of the righthand side of (28) due to the  $\delta$  function. Further, after the integration, one is left with exponentials of  $e^{i(\omega_{1k'q\beta}-\omega_{1kq\beta})t}$ , which, in the rotating wave approximation<sup>10</sup>, vanishes unless  $\omega_{1k'q\beta} = \omega_{1kq\beta}$  or k = k'. So, Eq. (28) becomes

$$\frac{\partial \rho_{1S}^{I}}{\partial t} = -i[H_{1LS}, \rho_{1S}^{I}] - c \sum_{kq} \left[ \gamma_{1kq} (\langle n(\omega_{1kq}) \rangle + 1) (\rho_{S}^{I} a_{1kq}^{\dagger} a_{1kq} + a_{1kq}^{\dagger} a_{1kq} \rho_{S}^{I} - 2a_{1kq} \rho_{S}^{I} a_{1kq}^{\dagger}) \right] 
- c \sum_{k,q} \left[ \gamma_{1kq} \langle n(\omega_{1kq}) \rangle (\rho_{S}^{I} a_{1kq} a_{1kq}^{\dagger} + a_{1kq} a_{1kq}^{\dagger} \rho_{S}^{I} - 2a_{1kq}^{\dagger} \rho_{S} a_{1kq}) \right],$$
(30)

where  $H_{1LS}=(1/N)\sum_{kq}(D_{1kq}[a^{\dagger}_{1kq},a_{1kq}]+\triangle\omega_{1kq}a^{\dagger}_{1kq}a_{1kq})$  with  $D_{1kq}=\sum_{\omega}\mathcal{P}c_{\omega}\langle n(\omega)\rangle/(\omega_{1kq}-\omega)$ ,  $\triangle\omega_{1kq}=\sum_{\omega}\mathcal{P}c_{\omega}/(\omega_{1kq}-\omega)$ , and  $c_{\omega}=\sum_{\beta}\lambda_{\beta}^{2}\delta_{\omega,\omega_{\beta}}$ ,  $\gamma_{1kq}=\pi c_{\omega_{1kq}}/(cN)$ , and  $c=\sum_{\omega}c_{\omega}/N'$  (N' is the total number of boson states), which is a dissipation rate describing how fast the system dissipates to equilibrium.

Similarly, the term containing only  $H_{2SB}^I$  contributes an equation similar to (30) because of the modes satisfying  $\omega_{2kq\beta} = \omega_{2k'q\beta}$  and again k = k'. The cross terms from  $H_{1SB}^I$  and  $H_{2SB}^I$  should vanish in the same approximation because the combination of k', k and q to meet  $\omega_{1kq\beta} = \omega_{2k'q\beta}$  or  $\omega_{2kq\beta} = \omega_{1k'q\beta}$ , if any, is much less than that satisfying either  $\omega_{1kq\beta} = \omega_{1k'q\beta}$  or  $\omega_{2kq\beta} = \omega_{2k'q\beta}$ . Terms containing  $H_{3SB}^I$  do not contribute to the real part of the righthand side of Eq. (28) because  $\omega_{3kq\beta}$  is always negative.

Collecting the relevant terms, we then obtain the Lindblad master equation in the interaction picture

$$\frac{\partial \rho_S^I}{\partial t} = -c \sum_{j=1}^2 \sum_{k,q} \left[ \gamma_{jkq} (\langle n(\omega_{jkq}) \rangle + 1) (\rho_S^I a_{jkq}^\dagger a_{jkq} + a_{jkq}^\dagger a_{jkq} \rho_S^I - 2 a_{jkq} \rho_S^I a_{jkq}^\dagger) \right] 
-c \sum_{j=1}^2 \sum_{k,q} \left[ \gamma_{jkq} (\langle n(\omega_{jkq}) \rangle (\rho_S^I a_{jkq} a_{jkq}^\dagger + a_{jkq} a_{jkq}^\dagger \rho_S^I - 2 a_{jkq}^\dagger \rho_S a_{jkq}) \right],$$
(31)

where k and q take values satisfying  $\omega_{jkq\beta}=0$ . In Eq. (31), we have neglected a Lamb shift term  $-i[H_{LS},\rho]$  with  $H_{LS}=H_{1LS}+H_{2LS}+H_{3LS}$  because it is of higher order<sup>10</sup>. Transforming back to the Schrödinger picture, we find

$$\frac{\partial \rho_S}{\partial t} = -i[H_S, \rho_S] - c \sum_{j=1}^2 \sum_{k,q} \left[ \gamma_{jkq} (\langle n(\omega_{jkq}) \rangle + 1) (\rho_S a_{jkq}^{\dagger} a_{jkq} + a_{jkq}^{\dagger} a_{jkq} \rho_S - 2 a_{jkq} \rho_S a_{jkq}^{\dagger}) \right] 
- c \sum_{j=1}^2 \sum_{k,q} \left[ \gamma_{jkq} (\langle n(\omega_{jkq}) \rangle (\rho_S a_{jkq} a_{jkq}^{\dagger} + a_{jkq} a_{jkq}^{\dagger} \rho_S - 2 a_{jkq}^{\dagger} \rho_S a_{jkq}) \right].$$
(32)

In order to write Eq. (32) in the familiar Lindblad form, for two energy levels  $E_l$  and  $E_m$  with  $E_m - E_l = \omega_{jkq}$ , we may let  $V_{m\to l} = a_{jkq}$  and  $V_{m\to l}^{\dagger} = a_{jkq}^{\dagger}$ , which are respectively thermal jump operators representing emitting and absorbing a particle with energy  $\omega_{jkq}$  and jumping to a lower and higher energy state. Also, let  $W_{l\to m} = \gamma_{jkq} \langle n(\omega_{jkq}) \rangle$  and  $W_{m\to l} = \gamma_{jkq} \langle n(\omega_{jkq}) \rangle + 1$ , which are transition probabilities from lth state to the mth and vice versa, so that l1

$$\frac{W_{l\to m}}{W_{m\to l}} = \frac{\gamma_{jkq}(\langle n(\omega_{jkq})\rangle)}{\gamma_{jkq}(\langle n(\omega_{jkq})\rangle + 1)} = \exp\left(\frac{-\omega_{jkq}}{k_B T}\right) \equiv \exp\left(\frac{E_l - E_m}{k_B T}\right). \tag{33}$$

Equation (32) then becomes

$$\frac{\partial \rho_S}{\partial t} = -i[H_S, \rho_S] - c \sum_{m>l} W_{l\to m} (V_{l\to m}^{\dagger} V_{l\to m} \rho_S + \rho_S V_{l\to m}^{\dagger} V_{l\to m} - 2V_{l\to m} \rho_S V_{l\to m}^{\dagger})$$

$$- c \sum_{m>l} W_{m\to l} (V_{l\to m} V_{l\to m}^{\dagger} \rho_S + \rho_S V_{l\to m} V_{l\to m}^{\dagger} - 2V_{l\to m}^{\dagger} \rho_S V_{l\to m})$$

$$= -i[H_S, \rho_S] - c \sum_{l,m,m\neq l} W_{l\to m} (V_{l\to m}^{\dagger} V_{l\to m} \rho_S + \rho_S V_{l\to m}^{\dagger} V_{l\to m} - 2V_{l\to m} \rho_S V_{l\to m}^{\dagger}),$$
(34)

the usual Lindblad form. Although the form of the transition probability  $W_{l\to m}$  depends on the environment and determines the details of the process, universal properties only rely on  $W_{l\to m}/W_{m\to l}^{-11}$ . For example, in Ref. (12),  $W_{l\to m}=\beta_m$  with  $\beta_m$  the probability for the system to stay in the *m*th state in equilibrium. This completes our derivation of the Lindblad equation for the quantum Ising model.

#### V. CANONICAL DISTRIBUTION IS THE STEADY SOLUTION OF THE LINDBLAD EQUATION

In this section, we discuss the steady solution of Lindblad equation.

The meaning of the Lindblad equation (34) is clear. If the second term in the right hand side is neglected, it is the quantum Liouville equation determining the quantum fluctuations of the evolution of density operator  $\rho_S$ ; while if the first term on the right hand side is neglected, the diagonal part is

$$\frac{\partial \rho_{ii}}{\partial t} = c \sum_{j \neq i} (W_{j \to i} \rho_{jj} - W_{i \to j} \rho_{ii}), \tag{35}$$

which is the classical master equation, and the off-diagonal part is

$$\frac{\partial \rho_{ij}}{\partial t} = -\frac{c}{2} \left( \sum_{k \neq i} W_{i \to k} + \sum_{l \neq j} W_{j \to l} \right) \rho_{ij}, \tag{36}$$

which decays exponentially. Thus the Lindblad equation (34) naturally integrates the quantum and thermal fluctuations together.

It can be readily checked that for a time-independent Hamiltonian, the equilibrium density matrix of the canonical distribution  $\rho_E = \exp(-H_S/k_BT)/\text{Tr}[\exp(-H_S/k_BT)]$  is the steady solution of the Lindblad equation<sup>3</sup>. To this end, we note first that there is a detailed balance condition,

$$W_{l\to m}\rho_E V_{m\to l} = \gamma_{jkq} \langle n(\omega_{jkq}) \rangle \frac{\exp(-H_S/k_B T)}{\operatorname{Tr} \left[ \exp(-H_S/k_B T) \right]} a_{jkq} \exp\left(\frac{H_S}{k_B T}\right) \exp\left(-\frac{H_S}{k_B T}\right)$$

$$= \gamma_{jkq} \langle n(\omega_{jkq}) \rangle \exp\left(\frac{\omega_{jkq}}{k_B T}\right) a_{jkq} \frac{\exp(-H_S/k_B T)}{\operatorname{Tr} \left[ \exp(-H_S/k_B T) \right]}$$

$$= W_{m\to l} V_{m\to l} \rho_E,$$
(37)

where uses have been made of Eqs. (17) and (33). Then, on the righthand side of Eq. (34), the first term  $[H_S, \rho_E] = 0$ . For the second term, using the property of the thermal jump matrices  $V_{l\to m} = V_{m\to l}^{\dagger}$  from their definitions and substituting Eq. (37) into the righthand side of the Lindblad equation, the second term can be explicitly written as<sup>3</sup>,

$$W_{l\to m}(V_{l\to m}^{\dagger}V_{l\to m}\rho_E + \rho_E V_{l\to m}^{\dagger}V_{l\to m} - 2V_{l\to m}\rho_E V_{l\to m}^{\dagger}) = W_{l\to m}V_{l\to m}^{\dagger}V_{l\to m}\rho_E - W_{l\to m}V_{l\to m}V_{l\to m}^{\dagger}\rho_E, \tag{38}$$

and

$$W_{m\to l}(V_{m\to l}^{\dagger}V_{m\to l}\rho_E + \rho_E V_{m\to l}^{\dagger}V_{m\to l} - 2V_{m\to l}\rho_E V_{m\to l}^{\dagger}) = W_{m\to l}V_{m\to l}^{\dagger}V_{m\to l}\rho_E - W_{m\to l}V_{m\to l}^{\dagger}V_{m\to l}\rho_E.$$
(39)

These two terms cancel with each other and thus the second term of the right hand side of the Lindblad equation (34) equals zero too. This derivation is similar to that from Ref. (3), but we take the thermal jump involving all energy levels into account. Thus for the weak coupling situation, the Lindblad equation reduces to the canonical distribution solution in long time. Note that this steady solution does not depends on c.

#### VI. SUMMARY

We have derived the Lindblad equation for the quantum Ising chain weakly interacting with a heat bath. Further we have confirmed that the steady solution of this equation is the equilibrium canonical distribution independent of the dissipation rate.

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