Superintegrable systems with position dependent mass: master symmetry and action-angle methods

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Abstract

We consider the issue of deriving superintegrable systems with position dependent mass (PDM) in two dimensions from certain known superintegrable systems using the recently introduced method of master symmetries and complex factorization by M. Ranada [29, 30, 31, 32]. We introduce a noncanonical transformation to map the Hamiltonian of the PDM systems to that of ordinary unit mass systems. We observe a duality between these systems. We also study Tsiganov's method [39, 40, 41, 15] to derive polynomial integrals of motion using addition theorems for the action-angle variables using famous Chebyshev's theorem on binomial differentials. We compare Tsiganov's method of generating an additional integral of motion with that of Ranada's master symmetry method.

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1 Introduction

It is well known that a Hamiltonian system with n degrees of freedom is integrable in the Liouville sense if it possesses n functionally independent constants of motion which are in involution. If H denotes the Hamiltonian then there must exists constants of motion I_a such that

$${H, I_a} = 0, \quad a = 1, 2, ..., n - 1$$

with

$${I_a, I_b} = 0, \ a, b = 1, 2, ..., n - 1.$$

It is generally true that integrable systems are exceptional as most dynamical systems governing physical phenomena rarely possess the requisite number of constants of motion to ensure their integrability. Having said this it is interesting to note that there are systems which possess even more integrals of motion then that required by integrability. Such systems are generally termed as superintegrable. The formal definition of such systems may be stated as follows.

An integrable system is superintegrable if it allows additional integrals of motion $J_b(q, p)$ such that $\{H, J_b\} = 0$ for all b = 1, 2, ..., k with the set $\{H, I_1, ... I_{n-1}, J_1, ... J_k\}$ being functionally independent so that

$$rank \frac{\partial(H, I_1, ..., I_{n-1}, J_1, ..., J_k)}{\partial(q_1, ..., q_n, p_1, ..., p_n)} = n + k$$

In [10] Fris et al studied systems admitting separability in two different coordinate systems in the Euclidean space and obtained four families of potentials possessing three functionally independent integrals of the motion. The first family of such potentials, is known as the Smorodinsky-Winternitz (SW) potential and is a two-dimenensional generalization of the isotonic oscillator. The superintegrability of the SW potential has also been investigated by Evans [5, 42] in the more general case of n degrees of freedom. In general integrable systems may be broadly divided into two classes depending on whether they are separable or non-separable. Systems for which the Hamilton-Jacobi equation is separable in a particular coordinate system are integrable with constants of motion which are typically quadratic in the respective momenta. On the other hand non-separable systems are typically characterized by the existence of higher-order constants of motion, i.e., when the momenta are of degree more than two [2, 14, 22, 20, 23, 36]. A recent example being the Tremblay-Turbiner-Winternitz (TTW) system [27, 30, 37, 38] which is directly related to the SW potential. The so-called TTW system and and Post-Winternitz (PW) models [26] which have recently attracted some interest and provide concrete examples of superintegrable systems with non-central potentials defined on Euclidean plane while spherical and pseudospherical generalizations of these models represent non-isotropic superintegrable systems on curved configuration spaces.

In the case of superintegrable systems one can identify three possible classes [1] namely: superseparable, separable and nonseparable. Most of the known superintegrable systems turn out to be superseparable, i.e., separable in more than one coordinate system; separable superintegrable systems are generally endowed with a mixture of quadratic and higher-order

constants of motion while for non-separable ones the constants of motion are all of higher-order (up to the Hamiltonian).

While mechanical energies are obvious constants of motion for the deduction of additional constants a variety of methods are usually employed. These additional integrals of motion are often polynomials in the momenta of order higher than two. In many cases Rañada and his coworkers obtained these by the method of complex factorization. Evans et al. [5] and Rodriguez et al. [33, 34] obtained them by making use of dimensional reduction. Fordy [9] used the Kaluza-Klein construction in reverse to construct lower dimensional superintegrable systems from the higher dimension one. It is noteworthy to say that the Kaluza-Klein reduction deals with (pseudo-)Riemannian metrics, where we consider Hamiltonians in natural form. After the reduction, the lower dimensional Hamiltonian will have electromagnetic terms, which could turn out to be trivial.

Master symmetries were introduced by Fokas and Fuchssteiner [7] and were also studied by Oevel [24] and Fuchssteiner [11]. It was first applied to nonlinear partial differential equations (PDEs) (infinite-dimensional Hamiltonian systems) and then to finite-dimensional systems [4, 6]. These symmetries are related to the existence of compatible Poisson structures and recursion operators.

Transformations mapping one integrable system to another have been put to good use in the literature. A particular type of transformation, known as coupling constant metamorphosis (CCM), was formulated by Hietarinta et al [17]. Using this technique Kress [19] mapped the (flat space) superintegrable system with Hamiltonian $H = p_x^2 + p_y^2 + \alpha x$ to a non-flat space superintegrable system. It is known that all nondegenerate two-dimensional superintegrable systems having constants quadratic in the momenta can be obtained by coupling constant metamorphosis from those on constant curvature space. The classification problem of classical second-order superintegrable systems is almost settled. Most of the results obtained rely on the use of separation of variables. Equivalence of superintegrable systems in two dimensions are usually studied via quadratic algebras. For a recent review addressing the classification of second-order superintegrable systems in two-dimensional Riemannian and pseudo-Riemannian spaces we may cite [20] and references therein. It is based on the study of the quadratic algebras of the integrals of motion and on the equivalence of different systems under coupling constant metamorphosis.

Tsiganov carried out a systematic study of superintegrable Hamiltonian systems separable in Cartesian coordinates using action-angle variables. In a series of papers he [39, 40, 41] constructed polynomial integrals of motion using addition theorems for the action-angle variables. For instance, by adding action variables I_1 and I_2 one gets Hamiltonian H which is in involution with the following integral of motion

$$X = F(I_1, I_2, \theta_2 \theta_1), \qquad \{H, F\} = 0$$

which is functionally independent from I_1, I_2 . Recently Grigoriev and Tsiganov [15] proposed the study of superintegrable systems of Thompson's type separable in Cartesian coordinates.

In 1984 Thompson [35] proved superintegrability of the Hamiltonian

$$H = p_1^2 + p_2^2 + a(x_1 - x_2)^{-\frac{2}{2n-1}}, \qquad n \in \mathbb{Z}_+$$

. In [15] Grigoriev et al have shown the existence of additional integrals of motion of such superintegrable systems which are related to the famous Chebyshev theorem [3] of binomial differentials. Recently Gonera and Kaszubska [13] obtained 2D superintegrable systems defined on 2D spaces of constant curvature using actiona-angle method. These systems are separable in the so called geodesic polar coordinates. In particular, Gonera [12] proved the superintegrability of the TTW model using action-angle methods.

Motivation and result The main purpose of this paper is to study superintegrable systems with position dependent mass both using the method of master symmetries due mainly to Rañada [29, 30, 31] and using addition theorems for action-angle variables proposed by Tsiganov [39, 40, 41, 15]. We make a comparative study explore the power of these two methods to probe superintegrable systems. We elucidate this study with various examples.

In particular, we obtain the first integrals of the Fokas-Lagerstrom [8] and Holt [16] or deformed 2:1 harmonic oscillator potentials using this method. To the best of our knowledge this is the first time Fokas-Lagerstrom first integrals are computed via complex factorization This factorization [28] is obtained as a deformation of the quadratic version of the factorization [18, 25] of the integrals of motion of the linear oscillator. We show how the idea of coupling constant metamorphosis (CCM) can be applied to position dependent mass systems by finding a transformation between a position dependent mass 2D oscillator and Smorodinsky-Winternitz systems and their corresponding unit mass systems. It is known that CCM in general does not preserve the structure of the symmetry algebras, however we can map all the conserved quantities.

The paper is **organized** as follows. In section 2 we recollect the method of complex factorization and apply it to the Fokas-Lagerstrom and Holt potentials. In section 3 we introduce the notion of a master symmetry to study superintegrable systems. Finally in section 4 we give the main result of the paper, the position dependent mass superintegrable systems.

2 Generalized oscillator systems

In two dimensions an oscillatory system is typically characterized by the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega_0^2(n_1^2q_1^2 + n_2^2q_2^2). \tag{2.1}$$

It is obvious that such a system admits two integrals of motion given by

$$I_1 = \frac{1}{2}(p_1^2 + \omega_0^2 n_1^2 q_1^2), \quad I_2 = \frac{1}{2}(p_2^2 + \omega_0^2 n_2^2 q_2^2)$$
 (2.2)

A particularly simple algorithm for finding the integrals of motion for oscillators was introduced in [25, 18] based on the product of powers of the complex functions

$$A_1 = p_1 + in_1\omega_0 q_1, \quad A_2 = p_2 + in_2\omega_0 q_2.$$
 (2.3)

involving factorization of the third integral. We illustrate below the procedure for finding additional integrals through factorization by considering the example of a Fokas-Lagerstrom potential.

Example 1: The Fokas-Lagerstrom potential

The Hamiltonian for the Fokas-Lagerstrom potential is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}q_1^2 + \frac{1}{18}q_2^2$$
 (2.4)

and corresponds to the choice $n_1 = 1$ and $n_2 = 1/9$ in (2.1) with $\omega_0 = 1$. The Hamiltons equations of motion are

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_1 = -q_1, \quad \dot{p}_2 = -\frac{1}{9}q_2.$$

The complex functions now have the appearance

$$A_1 = p_1 + iq_1, \quad A_2 = p_2 + \frac{i}{3}q_2,$$

and it follows that

$$\frac{dA_1}{dt} = iA_1, \quad \frac{dA_2}{dt} = \frac{i}{3}A_2.$$

Consequently it is easily verified that $C = A_1 A_2^{*3}$ is a complex constant of motion whose imaginary part

$$C_I = p_2^2(q_1p_2 - q_2p_1) + \frac{1}{27}q_2^3p_1 - \frac{1}{3}q_1q_2^2p_2,$$

yields a real constant of motion. Note the cubic dependance on the momenta. On the other hand the real part also gives us the constant of motion

$$C_R = p_2^2(p_1p_2 + q_1q_2) - \frac{q_2^2}{3}(p_1p_2 + \frac{1}{9}q_1q_2).$$

As the Hamiltonian in (2.4) is clearly separable two obvious integrals of motion are given by

$$I_1 = \frac{1}{2}(p_1^2 + q_1^2), \quad I_2 = \frac{1}{2}(p_2^2 + \frac{1}{9}q_2^2).$$

Relabelling the constants $C_I = I_3$ and $C_R = I_4$ one may show that they are not independent but satisfy the relation

$$I_3^2 + I_4^2 = 16I_1I_2^3.$$

In [30] the general case of a separable Hamiltonian of the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega_0^2(n_1^2q_1^2 + n_2^2q_2^2) + \frac{k_1}{2q_1^2} + \frac{k_2}{2q_2^2},$$
(2.5)

was tackled in the above spirit by defining additional complex functions

$$B_1 = A_1^2 + \frac{k_1}{q_1^2}, \quad B_2 = A_2^2 + \frac{k_2}{q_2^2},$$

which satisfy the equations

$$\frac{dB_1}{dt} = 2in_1\omega_0 B_1, \quad \frac{dB_2}{dt} = 2in_2\omega_0 B_2,$$
(2.6)

whence it follows that the functions

$$B_{ij} = (B_i)^{n_j} (B_j^*)^{n_i}, \quad i, j = 1, 2$$

are constants of motion.

Example 2: Holt potential

Our next example concerns the Holt system for which the Hamiltonian is

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + 4q_2^2) + \frac{\delta}{q_1^2}.$$
 (2.7)

Defining the complex functions

$$A_1 = p_1 + iq_1, \quad A_2 = p_2 + i2q_2, \quad B_1 = A_1^2 + \frac{2\delta}{q_1^2}$$
 (2.8)

we find, using the relevant equations of motion,

$$\frac{dB_1}{dt} = 2iB_1, \quad \frac{dA_2}{dt} = 2iA_2.$$
 (2.9)

Consequently it follows that $C_{12} = B_1 A_2^*$ is a constant of motion with

$$Re(C_{12}) = p_1^2 p_2 - q_1^2 p_2 + \frac{2\delta}{q_1^2} p_2 + 4q_1 q_2 p_1$$
(2.10)

being a cubic integral of motion.

3 Superintegrability and Master symmetries

There exists a close relationship between superintegrability and the concept of master symmetries. Given a Hamiltonian H, we say that a function T(q, p) is a generator of the constants of motion of degree m for H if it satisfies the following conditions, namely

$$\frac{d^k T}{dt^k} \neq 0, \quad k = 1, ..., m, \qquad \frac{d^{m+1} T}{dt^{m+1}} = 0. \tag{3.1}$$

It is evident from these conditions that T is a function which generates an integral of motion by time derivation. Following [29, 30, 31] we note that for n=2 Hamiltonian system if T_1 and T_2 are two generators of degree m=1 such that I_1 and I_2 defined by $I_k=dT_k/dt$ for k=1,2 are constants of motion in involution then one can construct time-dependent constants of motion I_1^t and I_2^t by means of the following definition

$$I_k^t = T_k - I_k t. (3.2)$$

Note the linear dependence on the time t which allows us to define an additional time-independent constant of motion given by

$$I_{12} = T_1 I_2 - T_2 I_1, (3.3)$$

thereby ensuring that the system is superintegrable.

A similar procedure exists in case of higher degree generators as we illustrate below.

Example 3: A linear potential

We consider the Hamiltonian given by

$$H = p_x^2 + p_y^2 + \alpha x. (3.4)$$

The equations of motion are given by

$$\dot{x} = 2p_x, \ \dot{y} = 2p_y, \ \dot{p}_x = -\alpha, \ \dot{p}_y = 0$$

Two obvious first integrals are provided by

$$E_x = p_x^2 + \alpha x, \quad E_y = p_y$$

Let $T_1 = xp_y$ then it follows that $\dot{T}_1 = 2p_xp_y$, $\ddot{T}_1 = -2\alpha p_y$ and $\ddot{T}_1 = 0$ so that $I_1 = -2\alpha p_y$ is a first integral. Similarly if we take $T_2 = yp_x$ then we find that $\dot{T}_2 = 2p_xp_y - \alpha y$, $\ddot{T}_2 = -4\alpha p_y$ and $\ddot{T}_2 = 0$ so that $I_2 = -4\alpha p_y$ is a first integral. Consequently we may deduce the following two time-dependent first integrals using (3.2)

$$I_1^t := \dot{T}_1 - I_1 t = 2p_y(p_x + \alpha t), \quad I_2^t := \dot{T}_2 - I_2 t = 2p_x p_y - \alpha y + 4\alpha p_y t$$

To deduce a time independent first integral we note that the time derivative of

$$I_{12} = \dot{T}_1 I_2 - \dot{T}_2 I_1$$

is by construction zero. Hence the required time-independent first integral is

$$I_{12} = -4\alpha p_y (p_x p_y + \frac{\alpha}{2}y)$$

Since here p_y is itself conserved one may scale this first integral and take, $p_x p_y + y\alpha/2$, to be the additional (third) first integral thus ensuring superintegrability of the system under

consideration.

In fact there exists another first integral for this system in the literature which may be obtained as follows. Let us consider the following m = 1 generators namely

$$T_3 = 2p_x p_y - y p_x \quad T_4 = y,$$

whence it follows that

$$\dot{T}_3 = 2p_x p_y + \alpha y, \quad \ddot{T}_3 = 0,$$

 $\dot{T}_4 = 2p_y, \quad \ddot{T}_4 = 0.$

We immediately recognize $I_3 := 2p_xp_y + \alpha y$ as representing the integral I_{12} obtained earlier, so setting $I_4 := 2p_y$ we obtain using (3.3) the following first integral

$$I_{34} := T_3 I_4 - T_4 I_3 = 4 \left[(x p_y - y p_x) p_y - \frac{\alpha}{4} y^2 \right]$$

Relabelling these first integrals as

$$K = p_y$$
, $R_1 = (xp_y - yp_x)p_y - \frac{\alpha}{4}y^2$, $R_2 = p_x p_y + \frac{\alpha}{2}y$, $H = E_x + E_y$

where we have used the notation of [19] we note that these four integrals are not functionally independent for they are related by

$$R_2^2 + K^4 - HK^2 + \alpha R_1 = 0.$$

Although this example is well know the fact that all its first integrals can be recast in the language of master symmetries reveals an interesting aspect of the system described by a linear potential.

As our main interest is on position dependent mass systems we consider below the case when the mass function is not a constant and study the resulting impact on the methods outlined above.

4 Position dependent mass and superintegrability

Consider the anisotropic two dimensional harmonic oscillator with a position dependent mass described by the following Hamiltonian:

$$H = \frac{p_1^2}{2m_1(q_1)} + \frac{p_2^2}{2m_2(q_2)} + \frac{1}{2}m_1(q_1)\omega_1^2 q_1^2 + \frac{1}{2}m_2(q_2)\omega_2^2 q_2^2. \tag{4.1}$$

We assume that (q_i, p_i) for i = 1, 2 represent a set of canonical variables and satisfy the standard Poisson algebra $\{q_i, p_j\} = \delta_{ij}$ and $\{q_i, q_j\} = \{p_i, p_j\} = 0$. The Hamilton's equation of motion are then given by

$$\dot{q}_i = \frac{p_i}{m_i(q_i)}, \quad \dot{p}_i = \frac{m_i'(q_i)}{2m_i^2} p_i^2 - \frac{1}{2}\omega_i^2 \frac{d}{dq_i}(m_i q_i^2), \quad i = 1, 2.$$
 (4.2)

4.1 Position dependent mass and complex factorization

By analogy with (2.3) we now define the complex functions

$$A_i = \left[\frac{p_i}{\sqrt{m_i}} + i\omega_i \sqrt{m_i} q_i \right], \quad i = 1, 2, \tag{4.3}$$

it being understood that the arguments of m_i are their respective coordinates. Then

$$\frac{dA_i}{dt} = i\omega_i \left[1 + \frac{q_i}{2} \frac{m_i'(q_i)}{m_i(q_i)} \right] A_i. \tag{4.4}$$

Defining $C_{ij} = A_i A_j^*$ it follows that

$$C_{ii} = \frac{p_i^2}{2m_i(q_i)} + \frac{1}{2}m_i(q_i)\omega_i^2 q_i^2, \quad i = 1, 2,$$
(4.5)

are constants of motion.

In (4.4) suppose $\omega_i = n_i \omega_0$ while the mass functions are such that

$$\left[1 + \frac{q_i}{2} \frac{m_i'(q_i)}{m_i(q_i)}\right] = \lambda_i \quad i = 1, 2, \tag{4.6}$$

where λ_i are constants. It turns out that

$$\frac{dA_i}{dt} = i\omega_0 n_i \lambda_i A_i,$$

and

$$C_{ij} = A_i^{n_j \lambda_j} (A_j^*)^{n_i \lambda_i}, \tag{4.7}$$

is a constant of motion. From (4.6) we recover the form of the mass function as

$$m_i(q_i) = m_{0i}q_i^{2(\lambda_i - 1)},$$
 (4.8)

where m_{0i} (i=1,2) is a constant. As an illustration consider the case of $n_1=1$ and $n_2=2$ while $\lambda_1=2$ and $\lambda_2=-1$ which corresponds to the Hamiltonian

$$H = \frac{p_1^2}{2q_1^2} + \frac{p_2^2}{2q_2^{-4}} + \frac{1}{2}\omega_0^2 q_1^4 + \frac{1}{2}\omega_0^2 \frac{4}{q_2^2},\tag{4.9}$$

taking $m_{0i} = 1$. These choices result in

$$\frac{dA_1}{dt} = 2i\omega_0 A_1, \quad \frac{dA_2}{dt} = -2i\omega_0 A_2,$$

leading to the following constants of motion, viz

$$I_3 = Re(A_1 A_2) = p_1 p_2 \frac{q_2^2}{q_1} - 2\omega_0^2 \frac{q_1^2}{q_2},$$

$$I_4 = Im(A_1A_2) = \left[q_1^2q_2^2p_2 + \frac{2p_1}{q_1q_2}\right].$$

Let us denote by H_1 and H_2 the two decoupled components of H as appearing in (4.9). Furthermore suppose I_1 and I_2 represent the two one-dimensional energies corresponding to Hamiltonians H_1 and H_2 , then an interesting property is that the Poisson bracket of I_1 with I_4 is just I_3 .

4.2 Smorodinsky-Winternitz system with position dependent mass

The n=2 Smorodinsky-Winternitz system has the following Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + k_0(q_1^2 + q_2^2) + \frac{k_1}{q_1^2} + \frac{k_2}{q_2^2}.$$
 (4.10)

The generators for the particular case of $k_0 = 0$ was considered in [29] and they are of the form

$$T_i = q_i p_i, \quad i = 1, 2.$$

We consider below a modification of the above Hamiltonian when the mass is position dependent (assuming $k_0 = 0$),

$$H = \frac{1}{2} \left(\frac{p_1^2}{m_1(q_1)} + \frac{p_2^2}{m_2(q_2)} \right) + \frac{k_1}{q_1^2} + \frac{k_2}{q_2^2}. \tag{4.11}$$

We assume the following form of the generator of master symmetries

$$T_i = f_i(q_i)p_i$$
, which satisfy the conditions $\frac{dT_i}{dt} \neq 0$, $\frac{d^2T_i}{dt^2} = 0$, $i = 1, 2$. (4.12)

It follows that

$$\frac{dT_i}{dt} = X_i(q_i)p_i^2 + 2k_i \frac{f_i}{q_i^3} \neq 0,$$
(4.13)

$$\frac{d^2T_i}{dt^2} = \left[X_i'(q_i) + \frac{m_i'}{m_i} X_i(q_i) \right] \frac{p_i^3}{m_i} + \left[X_i(q_i) \frac{4k_i}{q_i^3} + \frac{2k_i}{m_i} \left(\frac{f_i}{q_i^3} \right)' \right] p_i = 0 \tag{4.14}$$

where

$$X_i(q_i) = \left(\frac{f_i'}{m_i} + f_i \frac{m_i'}{2m_i^2}\right) \quad i = 1, 2.$$

and the 'denotes differentiation with respect to the appropriate argument. Equating the coefficients of the powers of p_i from (4.14) we obtain the following equations for determining the function f_i and the mass m_i ,

$$X_i'(q_i) + \frac{m_i'}{m_i} X_i(q_i) = 0 (4.15)$$

$$X_i(q_i)\frac{2}{q_i^3} + \frac{1}{m_i} \left(\frac{f_i}{q_i^3}\right)' = 0. {(4.16)}$$

Eqn
$$(4.15)$$
 implies

$$m_i X_i(q_i) = \lambda_i, \quad i = 1, 2 \tag{4.17}$$

with λ_i being a constant. On the other hand from (4.16) upon using (4.17) we find that

$$f_i(q_i) = \lambda_i q_i + \mu_i q_i^3, \quad i = 1, 2$$
 (4.18)

where μ_i is also a constant. The explicit form of the mass function now follows from (4.17) and is given by

$$m_i(q_i) = \frac{1}{(\lambda_i + \mu_i q_i^2)^3}, \quad i = 1, 2.$$
 (4.19)

Thus it is evident that when $\mu_i = 0$ the mass function becomes a constant while $f_i = q_i$. This precisely corresponds to the situation considered in [29]. In our case the time-independent first integrals, given by dT_i/dt , are

$$I_i = (\lambda_i + \mu_i q_i^2) \left[\lambda_i (\lambda_i + \mu_i q_i^2)^2 p_i^2 + \frac{2k_i}{q_i^2} \right], \quad i = 1, 2.$$
 (4.20)

The corresponding time-dependent functions therefore have the form

$$I_i^t = (\lambda_i q_i + \mu_i q_i^3) p_i - \left[(\lambda_i + \mu_i q_i^2) \left(\lambda_i (\lambda_i + \mu_i q_i^2)^2 p_i^2 + \frac{2k_i}{q_i^2} \right) \right] t, \quad i = 1, 2$$
 (4.21)

Thus from (3.3) it at once follows that there exists a time -independent constant of motion given by

$$I_{12} = (\lambda_1 + \mu_1 q_1^2)(\lambda_2 + \mu_2 q_2^2) \left[\left((\lambda_2 + \mu_2 q_2^2)^2 \lambda_2 p_2^2 + \frac{2k_2}{q_2^2} \right) q_1 p_1 - \left((\lambda_1 + \mu_1 q_1^2)^2 \lambda_1 p_1^2 + \frac{2k_1}{q_1^2} \right) q_2 p_2 \right]. \tag{4.22}$$

In a similar manner it may be shown that for the Hamiltonian

$$H = \frac{p_x^2}{2m_x} + \frac{p_y^2}{2m_y} + k_2 x + \frac{k_3}{y^2}$$
 (4.23)

where m_x and m_y are functions of x and y respectively the generators of the first integrals are given by

$$T_1 = (B_1 - C_1 x)p_x, \quad m_x(x) = \frac{A_1}{(B_1 - C_1 x)^3}$$

$$T_2 = (B_2 y + C_2 y^3) p_y, \quad m_y(y) = \frac{A_2}{(B_2 + C_2 y^2)^3}$$

with a time-independent first integral

$$I_1 = \frac{C_1}{2A_1} (B_1 - C_1 x)^3 p_x^2 - k_2 (B_1 - C_1 x)$$

5 Metamorphosis and duality between position dependent mass systems and unit mass oscillators

In this section we show that a duality transformation exist between position dependent mass 2D oscillator and constant mass 2D oscillator. In the 1980's, a number of papers were devoted to the investigation of certain duality properties of pairs of Hamiltonians. The underlying idea was based on the work of Hietarinta *et al.* [17] and received a lot of attention as the result of one integrable system automatically implied the existence of another type of (or version) integrable system.

In order to illustrate this feature we consider the standard harmonic oscillator, whose Lagrangian and Hamiltonian are given by $L = \frac{1}{2}v_x^2 - \frac{1}{2}\omega^2x^2$ and $H = \frac{1}{2}p_x^2 + \frac{1}{2}\omega^2x^2$, respectively. Consider the following change of variables

$$(x, v_x) = (q, v_q);$$
 $x = q^{\lambda}, \quad v_x = \lambda q^{\lambda - 1} v_q$

The transformed Lagrangian and Hamiltonian in new coordinates are given by

$$\tilde{L} = \frac{1}{2}\lambda^2 q^{2(\lambda - 1)} v_q^2 - \frac{1}{2}\omega^2 q^{2\lambda}, \qquad \tilde{H} = \frac{1}{2}\frac{1}{\lambda^2} \frac{p_q^2}{q^{2(\lambda - 1)}} + \frac{1}{2}\omega^2 q^{2\lambda}.$$

Let us introduce the following notation

$$m(q) = q^{2(\lambda - 1)}, \qquad \omega = n\omega_0, \tag{5.1}$$

then \tilde{H} can be written as follows

$$\tilde{H} = \frac{1}{\lambda^2} \left[\frac{1}{2} \frac{p_q^2}{m(q)} + \frac{1}{2} (\lambda^2 n^2) \omega_0^2 m(q) q^2 \right], \tag{5.2}$$

which can be normalized $\tilde{H} = \frac{1}{\lambda^2} \tilde{H}_{\lambda}$. A nonlinear oscillator with a position dependent mass m(q) is the standard harmonic oscillator but just written in a new system of coordinates.

Next we study the inverse transformation. Under the transformation

$$Q_1 = \frac{1}{2}q_1^2, \quad P_1 = \frac{p_1}{q_1}, \quad Q_2 = -\frac{1}{q_2}, \quad P_2 = p_2q_2^2,$$
 (5.3)

the Hamiltonian (4.9) reduces to

$$\bar{H} = \frac{1}{2}P_1^2 + \frac{1}{2}(2\omega_0)^2 Q_1^2 + \frac{1}{2}P_2^2 + \frac{1}{2}(2\omega_0)^2 Q_2^2.$$
 (5.4)

This clearly corresponds to the constant mass scenario. The corresponding first integrals I_3 and I_4 now become just the Fradkin tensor and the angular momentum respectively, i.e.,

$$I_3 = P_1 P_2 + (2\omega_0)^2 Q_1 Q_2, \quad I_4 = 2(Q_1 P_2 - Q_2 P_1).$$

5.1 Duality of PDM Smorodinsky-Winternitz equation

In order to extend the above idea to the PDM scenario let us consider the Hamiltonian of the position dependent mass SW equation

$$H = \frac{1}{2}p_1^2(1+q_1^2)^3 + \frac{1}{2}p_2^2(1+q_2^2)^3 + \frac{k_1}{q_1^2} + \frac{k_2}{q_2^2}.$$
 (5.5)

We define $P_1 = p_1(1+q_1^2)^{3/2}$ and $P_2 = p_2(1+q_2^2)^{3/2}$, and fix the form of $Q_1 = Q_1(q_1, p_1)$ via the canonical requirement

$$\{Q_1, P_1\} = \frac{\partial Q_1}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_1}{\partial q_1} = 1,$$

from which we obtain

$$Q_{1q_1}(1+q_1^2) - 3p_pq_1(1+q_1^2)^{1/2}Q_{1p_1} = (1+q_1^2)^{-1/2}.$$

The corresponding Lagrange system of equations is

$$\frac{dq_1}{(1+q_1^2)} = \frac{dp_1}{-3p_1q_1} = \frac{dQ_1}{(1+q_1^{-1/2})}.$$

and yields the characteristics

$$C_1 = p_1(1+q_1^2)^{3/2}, \quad C_2 = Q_1 - \frac{q_1}{\sqrt{1+q_1^2}}$$

and hence the general solution $C_2 = F(C_1)$. Choosing the arbitrary function F to be the null function immediately yields a particularly simple solution for Q_1 namely $Q_1 = \frac{q_1}{\sqrt{1+q_1^2}}$ and similarly $Q_2 = \frac{q_2}{\sqrt{1+q_2^2}}$.

Proposition 5.1 Let $H = \frac{1}{2}p_1^2(1+q_1^2)^3 + \frac{1}{2}p_2^2(1+q_2^2)^3 + \frac{k_1}{q_1^2} + \frac{k_2}{q_2^2}$ be the position dependent Smorodinsky-Winternitz system, then under the transformation $Q_i = \frac{q_i}{\sqrt{1+q_i^2}}$ and $P_i = p_i(1+q_i^2)^{3/2}$ H becomes the Hamiltonian of the unit mass SW equation upto an additive constant viz

$$H \longrightarrow \tilde{H} = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + \frac{k_1}{Q_1^2} + \frac{k_2}{Q_2^2} + (k_1 + k_2)$$

and the transformation may be used to map the integrals of motion of the two systems.

6 Generalized oscillatory systems and master symmetries

Let us consider once again the Hamiltonian of (4.1) and the associated equations of motion as given by (4.2). Following Ranada [29] we define the variable

$$u_i = \frac{\omega_i f_i(q_i) p_i}{E_i}, \quad E_i = \frac{p_i^2}{2m_i(q_i)} + \frac{1}{2} m_i(q_i) \omega_i^2 q_i^2, \quad i = 1, 2$$
 (6.1)

where it is easy to verify, in view of (4.2) that $dE_i/dt = 0$. Suppose $T_i = \arcsin(u_i)$ so that

$$\frac{dT_i}{dt} = \frac{1}{\sqrt{1 - u_i^2}} \frac{du_i}{dt}.$$
(6.2)

Employing the definitions given in (6.1) it follows that

$$\frac{dT_i}{dt} = \frac{2\omega_i}{\sqrt{E_i^2 - \omega_i^2 f_i^2 p_i^2}} \left[f_i \left(\frac{m_i'(q_i)}{2m_i^2} p_i^2 - \frac{1}{2} \omega_i^2 \frac{d}{dq_i} (m_i q_i^2) \right) + \frac{f_i'}{m_i} p_i^2 \right].$$

Next let us demand that $dT_i/dt = 2\omega_i\lambda_i$ where λ_i is an arbitrary constant, which essentially means that

$$\left[f_i \left(\frac{m_i'(q_i)}{2m_i^2} p_i^2 - \frac{1}{2} \omega_i^2 \frac{d}{dq_i} (m_i q_i^2) \right) + \frac{f_i'}{m_i} p_i^2 \right] = \lambda_i \sqrt{E_i^2 - \omega_i^2 f_i^2 p_i^2}.$$
(6.3)

It follows that $d^2T_i/dt^2 = 0$ and hence T_i is a generator of a master symmetry for the PDM Hamiltonian (4.1). Upon squaring both sides of (6.3) and equating the coefficients of the different powers of p_i we obtain the following set of equations (i = 1, 2), namely

$$f_i \left(\frac{m_i'}{2m_i} + \frac{f_i'}{f_i} \right) = \pm \lambda_i, \tag{6.4}$$

$$q_i^2 f_i \left[f_i \left(\frac{m_i'}{2m_i} + \frac{f_i'}{f_i} \right) \right] \frac{(m_i q_i^2)'}{(m_i q_i^2)} = 2\lambda_i^2 (2f_i^2 - q_i^2), \tag{6.5}$$

$$\frac{(m_i q_i^2)'}{(m_i q_i^2)} = \pm \frac{2\lambda_i}{f_i}.$$
 (6.6)

By eliminating the terms involving the mass m_i it readily follows that the only acceptable form of the unknown function f_i is given by

$$f_i(q_i) = q_i, \quad i = 1, 2.$$
 (6.7)

This is precisely the form with which the author of [29] began. However the interesting feature here is that in presence of a position dependent mass term with this form of the function f_i one finds from the above set of equations that the mass function has either of the following two forms depending on the \pm sign, viz

$$m_i(q_i) = q_i^{2(\lambda_i - 1)}, \quad \text{for } + \text{sign}$$
 (6.8)

$$m_i(q_i) = q_i^{-2(\lambda_i+1)}, \quad \text{for - sign}$$
 (6.9)

The specific choice $\lambda_i = 1$ leads to the case where each $m_i = 1$ and was considered in [29]. Even with this choice of λ_i there exists a second possibility (corresponding to the negative sign) wherein $m_i = q_i^{-4}$ for which the Hamiltonian of (4.1) assumes the following form

$$H = \sum_{i=1}^{2} \left[\frac{1}{2} p_i^2 q_i^4 + \frac{1}{2} \frac{\omega_i^2}{q_i^2} \right]$$
 (6.10)

The equation of motion for q_i following from the above Hamiltonian represent second-order ordinary differential equations of the Liénard -II type namely

$$\ddot{q}_i + \frac{2}{q_i}\dot{q}_i^2 - \omega_i^2 q_i = 0, \quad i = 1, 2.$$
(6.11)

7 Action-angle method: Tsiganov's approach

Recently the concept of action-angle variables has been employed to derive an additional first integral typically for the systems considered in this paper. The method has proved to be complimentary to that of Master symmetries and is in some sense closer in spirit to the very notion of integrability itself being dependent on action-angle variables. To this end we consider example 3 again where

$$H = p_1^2 + p_2^2 + \alpha q_1$$

for which two first integrals are obvious, namely

$$I_1 = p_1^2 + \alpha q_1, \quad I_2 = p_2^2$$

We define the angle variables

$$\phi_1 = \frac{\partial}{\partial I_1} \int p_1 dq_1 = \frac{\partial}{\partial I_1} \int^{q_1} \sqrt{I_1 - \alpha x} dx = \frac{1}{2} \int^{q_1} \frac{dx}{\sqrt{I_1 - \alpha x}}$$

and similarly

$$\phi_2 = \frac{1}{2} \int^{q_2} \frac{dx}{\sqrt{I_2}}$$

One can easily verify that

$$\{\phi_1, I_1\} = \{\phi_2, I_2\} = 1, \ \{\phi_1, \phi_2\} = \{I_1, I_2\} = 0$$

Following [15] et al we note that any function $F = F(I_1, I_2, \phi_1 - \phi_2)$ can be regarded as an additional functionally independent first integral. We verify this by taking $X = \phi_1 - \phi_2$. It is possible to evaluate X explicitly to obtain

$$X = \phi_1 - \phi_2 = \frac{1}{2} \left[\int_{-1}^{q_1} \frac{dx}{\sqrt{I_1 - \alpha x}} - \int_{-1}^{q_2} \frac{dx}{\sqrt{I_2}} \right] = -\frac{1}{\alpha} \sqrt{I_1 - \alpha q_1} - \frac{q_2}{2\sqrt{I_2}}$$

The equations of motion following from the above Hamiltonian are

$$\dot{q_1} = p_1, \ \dot{p_1} = -\alpha, \ \dot{q_2} = p_2, \ \dot{p_2} = 0$$

and one can verify that

$$\frac{dX}{dt} = \dot{\phi}_1 - \dot{\phi}_2 = 0$$

where

$$X = -\frac{1}{\alpha}p_2^{-1}(p_1p_2 + \frac{\alpha}{2}q_2) = -\frac{1}{\alpha\sqrt{I_2}}(p_1p_2 + \frac{\alpha}{2}q_2),$$

which is clearly consistent with the earlier result.

As a second illustration we consider the following example taken from Ranada [32]

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega_0^2(n_1q_1^2 + n_2^2q_2^2) + \frac{k_1}{2q_1^2} + k_2q_2.$$
 (7.1)

Once again it is obvious that

$$I_1 = \frac{1}{2}p_1^2 + \frac{1}{2}\omega_0^2 n_1^2 q_1^2 + \frac{k_1}{2q_1^2}, \quad I_2 = \frac{1}{2}p_2^2 + \frac{1}{2}\omega_0^2 n_2^2 q_2^2 + k_2 q_2, \tag{7.2}$$

are first integrals. Proceeding as in the previous example we define the coresponding angle variables

$$\phi_1 = \frac{\partial}{\partial I_1} \int^x p_1 dx' = \frac{1}{\sqrt{2}} \int^x \frac{dx'}{\sqrt{I_1 - \frac{1}{2}\omega_0^2 n_1^2 x'^2 - \frac{k_1}{2x'^2}}},\tag{7.3}$$

$$\phi_2 = \frac{\partial}{\partial I_2} \int^y p_2 dy' = \frac{1}{\sqrt{2}} \int^y \frac{dy'}{\sqrt{I_1 - \frac{1}{2}\omega_0^2 n_2^2 y'^2 - k_2 y'}}.$$
 (7.4)

According to the Chebyshev theorem [3] on integrals of differential binomials of the form

$$\int x^m (a+bx^n)^p \, dx,$$

such integrals can be evaluated in terms of elementary functions if and only if

- (a) p is an integer, then we expand $(a + bx^n)^p$ by the binomial formula in order to rewrite the integrand as a rational function of simple radicals $x^{j/k}$. By a simple substitution $x = t^r$ we remove the radicals entirely and obtain integral on rational function.
- (b)m+1/n is an integer, then setting $t=a+bx^n$ we convert the integral to $\int t^p(t-a)^{m-1/n-1}dt$.
- (c)m+1/n+p is an integer, then we transform the integral by factoring out x^n and resultant new integral of the differential binomial belongs to case (b).

In view of the above one may evaluate these integrals in (7.3) and (7.4) explicitly and obtain their difference as

$$\phi_1 - \phi_2 = \frac{1}{2\omega_0 n_1} \arcsin\left(\frac{\frac{\omega_0 n_1}{\sqrt{2}} \left(x^2 - \frac{I_1}{\omega_0^2 n_1^2}\right)}{\sqrt{\frac{I_1^2}{2\omega_0^2 n_1^2} - \frac{k_1}{2}}}\right) - \frac{1}{\omega_0 n_2} \arcsin\left(\frac{\frac{\omega_0 n_2}{\sqrt{2}} \left(y + \frac{k_2}{\omega_0^2 n_2^2}\right)}{\sqrt{I_2 + \frac{k_2^2}{2\omega_0^2 n_2^2}}}\right). \tag{7.5}$$

It is possible to verify, using the Hamiltons equations of motion, that $X := \phi_1 - \phi_2$ is a constant of motion. Moreover, using the complex logarithmic version of the $\arcsin(z)$ function, namely

$$\arcsin(z) = -i\log(iz + \sqrt{1 - z^2})$$

we may exponentiate the constant of motion to obtain

$$e^{i2n_1n_2\omega_0X} = (iz_1 + \sqrt{1 - z_1^2})^{n_2}(iz_2 + \sqrt{1 - z_2^2})^{-2n_1}$$
(7.6)

where

$$z_1 = \left(\frac{\frac{\omega_0 n_1}{\sqrt{2}} \left(x^2 - \frac{I_1}{\omega_0^2 n_1^2}\right)}{\sqrt{\frac{I_1^2}{2\omega_0^2 n_1^2} - \frac{k_1}{2}}}\right), \quad z_2 = \left(\frac{\frac{\omega_0 n_2}{\sqrt{2}} \left(y + \frac{k_2}{\omega_0^2 n_2^2}\right)}{\sqrt{I_2 + \frac{k_2^2}{2\omega_0^2 n_2^2}}}\right).$$

Next we look at the Smorodinsky-Winternitz system for which the Hamiltonian is given by

$$H = \frac{1}{2}p_1^2 + k_0q_1^2 + \frac{k_1}{q_1^2} + \frac{1}{2}p_2^2 + k_0'q_1^2 + \frac{k_1'}{q_1^2}$$

As usual two obvious first integrals are

$$I_1 = \frac{1}{2}p_1^2 + k_0q_1^2 + \frac{k_1}{q_1^2}, \quad I_2 = \frac{1}{2}p_2^2 + k_0'q_2^2 + \frac{k_1'}{q_2^2}$$

A similar calculation as in the previous case gives the additional first integral

$$X_{SW} = \frac{1}{2\sqrt{2}} \left[\frac{1}{\sqrt{k_0}} \arcsin\left(\frac{q_1^2 - \frac{I_1}{2k_0}}{\frac{C_1}{\sqrt{k_0}}}\right) - \frac{1}{\sqrt{k'_0}} \arcsin\left(\frac{q_2^2 - \frac{I_2}{2k'_0}}{\frac{C_2}{\sqrt{k'_0}}}\right) \right], \tag{7.7}$$

where

$$C_1^2 = \frac{I_1^2}{4k_0} - k_1, \quad C_2^2 = \frac{I_2^2}{4k_0'} - k_1'$$

Similar manipulations, using the complex logarithmic form of the inverse sine function, now lead to the integral (complex)

$$I_{SW} = \left[\frac{\sqrt{2k_0}p_1q_1 - i\left(\frac{p_1^2}{2} - q_1^2 + \frac{k_1}{q_1^2}\right)}{\sqrt{I_1^2k_0 - 4k_0k_1}} \right]^{\sqrt{k_0'}} \left[\frac{\sqrt{2k_0'}p_2q_2 - i\left(\frac{p_2^2}{2} - q_2^2 + \frac{k_1'}{q_2^2}\right)}{\sqrt{I_2^2k_0' - 4k_0'k_1'}} \right]^{-\sqrt{k_0}}.$$
 (7.8)

8 Final comments

It is well known that there exist a variety of methods for proving the superintegrability of various potentials each having its own merits as far as computational flexibility is concerned. In this paper we have considered two approaches namely the master symmetry procedure and the method based on action-angle variables. The latter requires the explicit evaluation of certain integrals which has been aided by the Chebyshev theorem on binomial differentials. On the other hand the former relies more on the existence of a series of constructs leading

ultimately to an integral of motion by time derivation. In this context we have also looked at position dependent mass versions of some of the existing superintegrable systems.

We have illustrated and compared the two methods considered in this paper with several examples such as the Fokas-Lagerstorm, Holt type potential, Smorodinsky-Winternitz type equation and have demonstrated how in many cases the action-angle method, which plays a fundamental role in classical and quantum mechanics, captures the phenomena of the master symmetry method. It is known that the classical action-angle variables are defined only in some domain of the phase space which in many cases overlaps with domain of complex factorization, which is the heart of master symmetry method. In a sense the two almost equivalent methods have the advantage of possessing a great degree of elegance and simplicity which reflects their inherent robustness.

Finally it will obviously be interesting to study the quantum counterpart of the superintegrable Hamiltonian using quantum analogs of the action-angle variables, which play an important role in semi-classical quantization and also to explore quantum analogs using master symmetry or complex factorization method.

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