# Quantum White Noises and The Master Equation for Gaussian Reference States

John Gough
Department of Computing and Mathematics,
Nottingham Trent University,
Nottingham
NG1 4BU
United Kingdom

#### Abstract

We show that a basic quantum white noise process formally reproduces quantum stochastic calculus when the appropriate normal / chronological orderings are prescribed. By normal ordering techniques for integral equations and a generalization of the Araki-Woods representation, we derive the master and random Heisenberg equations for an arbitrary Gaussian state: this includes thermal and squeezed states.

# 1 Quantum White Noises

It is possible to develop a formal theory of quantum white noises which nevertheless provides a powerful insight into quantum stochastic processes. We present the "bare bones" of the theory: the hope is that the structure will be more apparent without the mathematical gore and viscera.

Essentially, we need a Hilbert space  $\mathcal{H}_0$  to describe a system of interest. We postulate a family of pseudo-operators  $\{a_t^+:t\geq 0\}$  called creation noises; a formally adjoint family  $\{a_t^-:t\geq 0\}$  called annihilation noises and a vector  $\Psi$  called the vacuum vector. The noises describe the environment and are assumed to act trivially on the observables of  $\mathcal{H}_0$ .

The first structural relations we need is

$$a_t^- \Psi = 0 \tag{QWN1}$$

which implies that the annihilator noises annihilate the vacuum. The second

relations are given by the commutation relations

$$\left[a_t^-, a_s^+\right] = \kappa \delta_+ (t - s) + \kappa^* \delta_- (t - s)$$
 (QWN2)

Otherwise the creation noises all commute amongst themselves, as do the annihilation noises. Here  $\kappa = \frac{1}{2}\gamma + i\sigma$  is a complex number with  $\gamma > 0$ . We have introduced the functional kernels  $\delta_{\pm}$  having the action

$$\int_{-\infty}^{\infty} f(s) \,\delta_{\pm} (t - s) := f\left(t^{\pm}\right) \tag{1}$$

for any Riemann integrable function f.

If the right-hand side of (QWN2) was just  $\gamma\delta\left(t-s\right)$ , then we would have sufficient instructions to deal with integrals of the noises wrt. Schwartz functions; however introducing the  $\delta_{\pm}$ -functions we can formally consider integrals wrt. piecewise Schwartz functions as we now have a rule for what to do at discontinuities. In particular, we can consider integrals over simplices  $\{t>t_1>\cdots>t_n>0\}$ . The objective is to use the commutation rule (QWN2) to convert integral expressions involving the postulated noises  $a_t^{\pm}$  to equivalent normal ordered expressions, so that we only encounter the integrals of the  $\delta_{\pm}$ -functions against Riemann integrable functions.

We remark that the commutation relations (QWN2) arose from considerations of Markovian limits of field operators  $a_t^{\pm}(\lambda)$  in the Heisenberg picture satisfying relations of the type  $\left[a_t^{-}(\lambda), a_s^{+}(\lambda)\right] = K_{t-s}(\lambda) \theta(t-s) + K_{t-s}(\lambda)^{*} \theta(s-t)$  where  $\theta$  is the Heaviside function and the right-hand side is a Feynman propagator.

## 1.1 Quantum Stochastic Calculus

## 1.1.1 Fundamental Stochastic Processes

For real square-integrable functions f = f(t), we define the following four fields

$$A^{ij}(f) := \int_0^\infty \left[ a_s^+ \right]^i \left[ a_s^- \right]^j f(s) ds, \qquad i, j \in \{0, 1\}$$
 (2)

where on the right-hand side the superscript denotes a power, that is  $[a]^0 = 1$ ,  $[a]^1 = a$ .

With  $1_{[0,t]}$  denoting the characteristic function of the time interval [0,t], we define the four fundamental processes

$$A_t^{ij} := A^{ij} \left( 1_{[0,t]} \right) = \int_0^t \left[ a_s^+ \right]^i \left[ a_s^- \right]^j ds, \qquad i, j \in \{0, 1\};$$
 (3)

## 1.1.2 Testing Vectors

The Hilbert space spanned by the symmetrized vectors  $f_1 \hat{\otimes} \dots \hat{\otimes} f_n := A^{10}(f_1) \dots A^{10}(f_n) \Psi$  is denoted as  $\mathcal{F}^{(n)}$ . The Hilbert spaces  $\mathcal{F}^{(n)}$  are then naturally orthogonal for different n and their direct sum is the (Bose) Fock space  $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}$ . The exponential vector with test function  $f \in L^2(\mathbb{R}^+)$  is defined to be

$$\varepsilon(f) := \sum_{n=0}^{\infty} \frac{1}{n!} \hat{\otimes}^n f. \tag{4}$$

Note that  $\langle \varepsilon \left( f \right) | \varepsilon \left( g \right) \rangle = \exp \gamma \left\langle f | g \right\rangle$  and that  $\Psi \equiv \varepsilon \left( 0 \right)$ .

For a subset  $\mathcal{T}$  of  $L^2(\mathbb{R}^+)$ , the subset of Fock space generated by the elements of  $\mathcal{T}$  is  $\mathfrak{E}(\mathcal{T}) = \{\varepsilon(f) : f \in \mathcal{T}\}$ . In general, we shall denote the Fock space over a one-particle Hilbert space  $\mathfrak{h}$  by  $\Gamma(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} (\hat{\otimes}^n \mathfrak{h})$ ; thus  $\mathcal{F} \equiv \Gamma(L^2(\mathbb{R}^+))$ .

### 1.1.3 Quantum Stochastic Processes

Let  $\mathcal{H}$  be a fixed Hilbert space, we wish to consider operators on the tensor product  $\mathcal{H} \otimes \mathcal{F}$ . A family of operators  $(X_t)_{t\geq 0}$  defined on a common domain  $D \otimes \mathfrak{E}(\mathcal{T})$  can be understood as a mapping from :  $D \times D \times \mathcal{T} \times \mathcal{T} \times \mathbb{R}^+ \to \mathbb{C}$  :  $(\phi, \psi, f, g, t) \mapsto \langle \phi \otimes \varepsilon(f) | X_t \psi \otimes \varepsilon(g) \rangle$ .

## 1.1.4 Quantum Stochastic Integrals

Let  $\left(X_t^{ij}\right)_{t\geq 0}$  be four adapted processes. The quantum stochastic integral having these processes as integrands is

$$X_{t} = \int_{0}^{t} ds \, \left[a_{s}^{+}\right]^{i} X_{s}^{ij} \left[a_{s}^{-}\right]^{j} \tag{5}$$

where we use the Einstein convention that repeated indices are summed (over 0,1). We shall use the differential notation  $dX_t = \left[a_t^+\right]^i X_t^{ij} \left[a_t^-\right]^j dt$  or even  $\frac{dX_t}{dt} = \left[a_t^+\right]^i X_t^{ij} \left[a_t^-\right]^j$ .

 $\frac{dX_t}{dt} = \left[a_t^+\right]^i X_t^{ij} \left[a_t^-\right]^j.$  The key feature is that the noises appear in normal ordered form in the differentials. Suppose that the  $\left(X_t^{ij}\right)_{t\geq 0}$  are defined on domain  $D\otimes\mathfrak{E}(\mathcal{R})$ , then it follows that

$$\left\langle \phi \otimes \varepsilon \left( f \right) | \frac{dX_{t}}{dt} \psi \otimes \varepsilon \left( g \right) \right\rangle \equiv \left[ f^{*} \left( t \right) \right]^{i} \left\langle \phi \otimes \varepsilon \left( f \right) | X_{t}^{ij} \psi \otimes \varepsilon \left( g \right) \right\rangle \left[ g \left( t \right) \right]^{j}$$

for all  $\phi, \psi \in D$  and  $f, g \in \mathcal{R}$ . (The equivalence can be understood here as being almost everywhere.)

## Quantum Itô's Formula

Let  $X_t$  and  $Y_t$  be quantum stochastic integrals, then the product  $X_tY_t$  may be brought to normal order using (QWN2). In differential terms we may write this

$$d(X_{t}Y_{t}) = (dX_{t})Y_{t} + X_{t}(dY_{t})$$

$$= (\hat{d}X_{t})Y_{t} + X_{t}(\hat{d}Y_{t}) + (\hat{d}X_{t})(\hat{d}Y_{t})$$
(6)

where the Itô differentials are defined as  $\left(\hat{d}X_{t}\right)Y_{t}:=\left[a_{t}^{+}\right]^{i}X_{t}^{ij}\left(Y_{t}\right)\left[a_{t}^{-}\right]^{j}dt;$  $X_t\left(\hat{d}Y_t\right) := \left[a_t^+\right]^k(X_t) \ Y_t^{kl} \ \left[a_t^-\right]^l \ dt; \ \left(\hat{d}X_t\right) \left(\hat{d}Y_t\right) := \left[a_t^+\right]^i X_t^{i1} Y_t^{1l} \left[a_t^-\right]^l \ dt.$  The quantum Itô "table" corresponds to following relation for the funda-

mental processes:

$$\left(\hat{d}A_t^{i1}\right)\left(\hat{d}A_t^{1j}\right) = \gamma\left(\hat{d}A_t^{ij}\right). \tag{7}$$

### **Quantum Stochastic Differential Equations**

The differential equation  $\frac{dX_t}{dt} = \left[a_t^+\right]^i X_t^{ij} \left[a_t^-\right]^j$  with initial condition  $X_0 = x_0$  (a bounded operator in  $\mathcal{H}_0$ ) corresponds to the differential equation system

$$\left\langle \phi \otimes \varepsilon \left( f \right) \left| \frac{dX_t}{dt} \, \psi \otimes \varepsilon \left( g \right) \right\rangle \equiv \left[ f^* \left( t \right) \right]^i \left\langle \phi \otimes \varepsilon \left( f \right) \left| X_t^{ij} \, \psi \otimes \varepsilon \left( g \right) \right\rangle \left[ g \left( t \right) \right]^j \right\rangle$$

and in general one can show the existence and uniqueness of solution as a family of operators on  $\mathcal{H}_0 \otimes \mathcal{F}$ . The solution can be written as  $X_t = x_0 +$  $\int_0^t ds \, \left[a_s^+\right]^i X_s^{ij} \left[a_s^-\right]^j$ . In Hudson-Parthasarathy notation this is written as  $X_t =$  $x_0 + \int_0^t X_s^{ij} \otimes \hat{d}A_t^{ij}$ , where the tensor product sign indicates the continuous tensor product decomposition  $\mathcal{F} = \Gamma \left( L^2(0,t] \right) \otimes \Gamma \left( L^2(t,\infty) \right)$ .

#### 1.2 Quantum Stochastic Evolutions

A quantum stochastic evolution is a family  $(J_t)_{t>0}$  mapping from the bounded observables on  $\mathcal{H}_0$  to the bounded observables on  $\mathcal{H}_0 \otimes \mathcal{F}$ . We are particularly interested in those taking the form

$$J_t\left(X\right) \equiv U_t^{\dagger} X U_t$$

where  $U_t$  is a unitary, adapted process satisfying some linear gsde (a stochastic Schrödinger equation). In the rest of this section we establish a Wick's theorem for working with such processes.

## 1.2.1 Normal-Ordered QSDE

Let  $V_t$  be the solution to the qsde

$$\frac{dV_t}{dt} = L_{ij} \left[ a_t^+ \right]^i V_t \left[ a_t^- \right]^j, \qquad V_0 = 1;$$
 (8)

where the  $L_{ij}$  are bounded operators in  $\mathcal{H}_0$ . The associated integral equation is  $V_t = 1 + \int_0^t dt_1 L_{ij} \left[a_{t_1}^+\right]^i V_{t_1} \left[a_{t_1}^-\right]^j$  which can be iterated to give the formal series

$$V_{t} = 1 + \sum_{n=1}^{\infty} \int_{t>t_{1}>...t_{n}>0} dt_{1}...dt_{n} L_{i_{1}j_{1}} \cdots L_{i_{n}j_{n}}$$

$$\times \left[a_{t_{1}}^{+}\right]^{i_{1}} \cdots \left[a_{t_{n}}^{+}\right]^{i_{n}} \left[a_{t_{n}}^{-}\right]^{j_{n}} \cdots \left[a_{t_{1}}^{-}\right]^{i_{1}}$$

$$= \tilde{\mathbf{N}} \exp \left\{ \int_{0}^{t} ds L_{ij} \left[a_{s}^{+}\right]^{i} \left[a_{s}^{-}\right]^{j} \right\},$$
(10)

where  $\tilde{\mathbf{N}}$  is the normal ordering operation for the noise symbols  $a_t^{\pm}$ . Necessary and sufficient conditions for unitary of  $V_t$  are that

$$L_{ij} + L_{ji}^{\dagger} + \gamma L_{1i}^{\dagger} L_{1j} = 0. \tag{11}$$

(Necessity is immediate from the isometry condition  $d\left(U_t^{\dagger}U_t\right) = U_t^{\dagger}\hat{d}\left(U_t\right) + \hat{d}\left(U_t^{\dagger}\right)U_t + \hat{d}\left(U_t^{\dagger}\right)\hat{d}\left(U_t\right) = U_t^{\dagger}\left(L_{ij} + L_{ji}^{\dagger} + \gamma L_{1i}^{\dagger}L_{1j}\right)U_t \otimes \hat{d}A_t^{ij} = 0$ , but suffices to establish co-isometry  $d\left(U_tU_t^{\dagger}\right) = 0$ .) Equivalently, we can require that

$$\begin{split} L_{11} &= \frac{1}{\gamma} \left( W - 1 \right), \quad L_{10} = L, \\ L_{01} &= -L^{\dagger} W, \qquad L_{00} = -\frac{1}{2} \gamma L^{\dagger} L - i H; \end{split}$$

where W is unitary, H is self-adjoint and L is arbitrary.

## 1.2.2 Time-Ordered QSDE

Let  $U_t$  be the solution to the qsde

$$\frac{dU_t}{dt} = iE_{ij} \left[ a_t^+ \right]^i \left[ a_t^- \right]^j U_t, \qquad U_0 = 1; \tag{12}$$

where the  $E_{ij}$  are bounded operators in  $\mathcal{H}_0$ . Here we naturally interpret  $\Upsilon_t = E_{ij} \left[ a_t^+ \right]^i \left[ a_t^- \right]^j$  as a stochastic Hamiltonian. (For  $\Upsilon_t$  to be Hermitian we would need  $E_{11}$  and  $E_{00}$  to be self-adjoint while  $E_{10}^{\dagger} = E_{01}$ .)

Iterating the associated integral equation leads to

$$U_{t} = 1 + \sum_{n=1}^{\infty} (-i)^{n} \int_{t>t_{1}>\dots t_{n}>0} dt_{1} \dots dt_{n} E_{i_{1}j_{1}} \dots E_{i_{n}j_{n}}$$

$$\times \left[a_{t_{1}}^{+}\right]^{i_{1}} \left[a_{t_{1}}^{-}\right]^{i_{1}} \dots \left[a_{t_{n}}^{+}\right]^{i_{n}} \left[a_{t_{n}}^{-}\right]^{j_{n}}$$

$$= \tilde{\mathbf{T}} \exp \left\{-i \int_{0}^{t} ds E_{ij} \left[a_{s}^{+}\right]^{i} \left[a_{s}^{-}\right]^{j}\right\},$$
(13)

where  $\tilde{\mathbf{T}}$  is the time ordering operation for the noise symbols  $a_t^{\pm}$ .

## 1.2.3 Conversion From Time-Ordered to Normal-Ordered Forms

Using the commutation relations (QWN2) we can put the time-ordered expressions in (12) to normal order. The most efficient way of doing this is as follows:

$$[a_t^-, U_t] = \left[ a_t^-, 1 - i \int_0^t ds \, E_{ij} \left[ a_s^+ \right]^i \left[ a_s^- \right]^j U_s \right]$$

$$= -i \int_0^t ds \, E_{1j} \left[ a_t^{-\prime} a_s^+ \right] \left[ a_s^- \right]^j U_s$$

$$= -i \kappa E_{1j} \left[ a_t^- \right]^j U_t$$

or  $a_t^- U_t - U_t a_t^- = -i\kappa E_{11} a_t^- U_t - i\kappa E_{10} U_t$ . This implies the rewriting rule

$$a_t^- U_t = (1 + i\kappa E_{11})^{-1} \left\{ U_t a_t^- - i\kappa E_{10} U_t \right\}. \tag{15}$$

Thus  $iE_{ij} \left[ a_t^+ \right]^i \left[ a_t^- \right]^j U_t = iE_{i0} \left[ a_t^+ \right]^i U_t + iE_{i1} \left[ a_t^+ \right]^i (1 + i\kappa E_{11})^{-1} \left\{ U_t a_t^- - i\kappa E_{10} U_t \right\}$ . From this we deduce the following result:

Theorem 1 Time-ordered and normal-ordered forms are related as

$$\tilde{\mathbf{T}}\exp\left\{-i\int_{0}^{t}ds\,E_{ij}\left[a_{s}^{+}\right]^{i}\left[a_{s}^{-}\right]^{j}\right\} \equiv \tilde{\mathbf{N}}\exp\left\{\int_{0}^{t}ds\,L_{ij}\left[a_{s}^{+}\right]^{i}\left[a_{s}^{-}\right]^{j}\right\}$$
(16)

where

$$L_{11} = -iE_{11} (1 + i\kappa E_{11})^{-1}, \quad L_{10} = -i (1 + i\kappa E_{11})^{-1} E_{10},$$
  

$$L_{01} = -iE_{01} (1 + i\kappa E_{11})^{-1}, \quad L_{00} = -iE_{00} + \kappa E_{01} (1 + i\kappa E_{11})^{-1} E_{10}.$$
(17)

## 2 Gaussian States

The above construction requires the existence of a vacuum state  $\Psi$ ; by application of creation fields we should be able to reconstruct a Hilbert-Fock space for which  $\Psi$  is cyclic. But what about non-vacuum states? We describe now the trick we shall use in order to consider more general states for the simple case of one bosonic degree of freedom.

Let  $a, a^{\dagger}$  satisfy the commutation relations  $\left[a, a^{\dagger}\right] = 1$ . A state  $\langle \ \rangle$  is said to be Gaussian or quasi-free if we have

$$\langle \exp\{iz^*a + iza^{\dagger}\} \rangle = \exp\{\frac{1}{2}n(n+1)zz^* + m^*z^2 + mz^{*2} + iz^*\alpha + iz\alpha^*\};$$
(18)

in particular,  $\langle a \rangle = \alpha$ ,  $\langle aa^{\dagger} \rangle = n+1$ ,  $\langle aa \rangle = m$ . From the observation that  $\langle (a + \lambda a^{\dagger})^{\dagger} (a + \lambda a^{\dagger}) \rangle \geq 0$  it follows that the restriction  $|m|^2 \leq n (n+1)$  must apply.

Now suppose that  $a_1, a_1^{\dagger}$  and  $a_2, a_2^{\dagger}$  are commuting pairs of Bose variables and let

$$a = xa_1 + ya_2^{\dagger} + za_2 + \alpha \tag{19}$$

where  $x,y,z,\alpha$  are complex numbers. The commutation relations are maintained if  $|x|^2-C|y|^2+|z|^2=1$ . Taking the vacuum state for both variables  $a_i,a_i^{\dagger}$  (i=1,2) then we can reconstruct the state  $\langle \ \rangle$  if  $|x|^2+|z|^2=n+1$  and yz=m. That is

$$x = \sqrt{n+1 - \frac{|m|^2}{n}}, \ y = \sqrt{n}, \ z = \frac{m}{\sqrt{n}}.$$
 (20)

## 2.1 Generalized Araki-Woods Construction

Let2  $\mathfrak h$  be a fixed one-particle Hilbert space and let j be a anti-linear conjugation on  $\mathfrak h$ , that is  $\langle j\phi|j\psi\rangle_{\mathfrak h}=\langle\psi|\phi\rangle_{\mathfrak h}$  for all  $\phi,\psi\in\mathfrak h$ . Denote by  $A(\phi)$  the annihilator on  $\Gamma(\mathfrak h)$  with test function  $\phi$  (that is,  $A(\phi)\,\varepsilon\,(\psi)=\langle\phi|\psi\rangle_{\mathfrak h}\,\varepsilon\,(\psi)$ ). A state  $\langle\,\,\rangle$  on  $\Gamma(\mathfrak h)$  is said to be (mean-zero) Gaussian / quasi-free if there is a positive operator  $N\geq 0$ ; an operator M with  $|M|^2\leq N\,(N+1)$  and [N,M]=0; and a fixed anti-linear conjugation j such that

$$\left\langle \exp\left\{iA\left(\phi\right)+iA^{\dagger}\left(\phi\right)\right\}\right\rangle = \exp\left\{-\frac{1}{2}\left\langle\phi\right|\left(2N+1\right)\phi\right\rangle_{\mathfrak{h}} - \frac{1}{2}\left\langle M\phi|j\phi\right\rangle_{\mathfrak{h}} - \frac{1}{2}\left\langle j\phi|M\phi\right\rangle_{\mathfrak{h}}\right\} \tag{21}$$

for all  $\phi \in \mathfrak{h}$ . In particular, we have the expectations

The state is said to be gauge-invariant when M=0. The case where  $N=\left(1-e^{-\beta H}\right)^{-1}$  and M=0 yields the familiar thermal state at inverse temperature  $\beta$  for the non-interacting Bose gas with second quantization of H as Hamiltonian. The vacuum state is, of course N=0, M=0.

The standard procedure for treating thermal states of the interacting Bose gas is to represent the canonical commutations (CCR) algebra on the tensor product  $\Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$  and realize state as a double Fock vacuum state. This approach generalizes to the problem at hand. For clarity we label the Fock spaces with subscripts 1 and 2 and consider the morphism from the CCR algebra over  $\Gamma(\mathfrak{h})$  to that over  $\Gamma(\mathfrak{h})_1 \otimes \Gamma(\mathfrak{h})_2$  induced by

$$A(\phi) \mapsto A_1(X\phi) \otimes 1_2 + 1_1 \otimes A_2^{\dagger}(jY\phi) + 1_1 \otimes A_2(Z\psi)$$
 (22)

where

$$X = \sqrt{N + 1 - \frac{|M|^2}{N}}, \ Y = \sqrt{N}, \ Z = \frac{M}{\sqrt{N}}.$$
 (23)

Here we write  $A_i(\psi)$  for annihilators on  $\Gamma(\mathfrak{h})_i$  (i=1,2).

# 3 Gaussian Noise

Now let  $a_1^{\pm}(t)$  and  $a_2^{\pm}(t)$  be independent (commuting) copies of quantum white noises with respective vacua  $\Psi_1$  and  $\Psi_2$ . Let x, y, x be as in (18) and let  $\alpha$  be arbitrary complex. We consider the quantum white noise(s) defined by

$$a_t^- := xa_1^-(t) + ya_2^+(t) + za_2^-(t) + \alpha. \tag{24}$$

We consider the stochastic dynamics generated by the formal Hamiltonian

$$\Upsilon_t = Ca_t^+ + C^\dagger a_t^- + F \tag{25}$$

where C and self-adjoint F are operators on  $\mathcal{H}_0$ . We are required to "normal order" the unitary process  $U_t = \tilde{\mathbf{T}} \exp\left\{-i\int_0^t ds \, \Upsilon_s\right\}$  for the given state and this means normal order the  $a_1^\pm\left(t\right)$  and the  $a_2^\pm\left(t\right)$ . By using the same technique as in (13) we find that, for instance,  $\left[a_1^-\left(t\right), U_t\right] = -i\int_0^t ds \, \left[a_1^-\left(t\right), \Upsilon_s\right] U_s = -i\kappa x C U_t$ . We can deduce the rewriting rules

$$a_{1}^{-}(t) U_{t} = U_{t} (a_{1}^{-}(t) - i\kappa x C);$$
  
 $a_{2}^{-}(t) U_{t} = U_{t} (a_{2}^{-}(t) - i\kappa z^{*}C - i\kappa y C^{\dagger}).$ 

The qsde  $\frac{dU_t}{dt} = -i\Upsilon_t U_t$  can be then be rewritten as

$$\frac{dU_t}{dt} = -i : \Upsilon_t U_t : -iCyU_t \left( -i\kappa z^*C - i\kappa yC^{\dagger} \right)$$
$$-iCxU_t \left( -i\kappa xC \right) - iC^{\dagger}zU_t \left( -i\kappa z^*C - i\kappa yC^{\dagger} \right)$$

where :  $\Upsilon_t U_t$ : is the reordering of  $\Upsilon_t U_t$  placing all creators  $a_1^+(t)$  and  $a_2^+(t)$  to the left and all annihilators  $a_1^-(t)$  and  $a_2^-(t)$  to the right. Rearranging gives

$$\frac{dU_{t}}{dt} = -iC\left(xa_{1}^{+}(t)U_{t} + yU_{t}a_{2}^{-}(t) + z^{*}a_{2}^{+}(t)U_{t} + \alpha^{*}U_{t}\right) 
-iC^{\dagger}\left(xU_{t}a_{1}^{-}(t) + ya_{2}^{+}(t)U_{t} + zU_{t}a_{2}^{-}(t) + \alpha U_{t}\right) 
-\left[iF + \kappa\left((n+1)C^{\dagger}C + nCC^{\dagger} + m^{*}CC + mC^{\dagger}C^{\dagger}\right)\right]U_{t}. (26)$$

To obtain an equivalent Hudson-Parthasarathy qsde, we introduce quantum Brownian motions  $A_t = \int_0^t \left(xa_1^-(s) + ya_2^+(s) + za_2^-(s)\right) ds$  with the under standing that, for  $R_t$  adapted,  $R_t \otimes dA_t = \left(xR_ta_1^-(t) + ya_2^+(t)R_t + zR_ta_2^-(t)\right)$  and  $R_t \otimes dA_t^{\dagger} = \left(xa_1^+(t)R_t + yR_ta_2^-(t) + z^*a_2^+(t)R_t\right)$ . Then

$$dU_t \equiv -iCU_t \otimes dA_t^{\dagger} - iC^{\dagger}U_t \otimes dA_t - GU_t \otimes dt$$
 (27)

where  $G = i \left( F + \alpha^* C + \alpha C^{\dagger} \right) + \kappa \left( (n+1) C^{\dagger} C + n C C^{\dagger} + m^* C C + m C^{\dagger} C^{\dagger} \right)$ . Note that the quantum Itô table will be

$$dA_t dA_t^{\dagger} = \gamma (n+1) dt; \qquad dA_t^{\dagger} dA_t = \gamma n dt; dA_t^{\dagger} dA_t^{\dagger} = \gamma m^* dt; \qquad dA_t dA_t = \gamma m dt.$$
 (28)

It is readily shown, either by normal ordering or by means of the quantum stochastic calculus, that the stochastic Heisenberg equation is then

$$dJ_t(X) = -iJ_t([X, C^{\dagger}]) \otimes dA_t^{\dagger} - iJ_t([X, C]) \otimes dA_t + J_t(L(X)) \otimes dt \quad (29)$$

where

$$L\left(X\right) = \gamma \left\{ \left(n+1\right)C^{\dagger}XC + nCXC^{\dagger} + mCXC + m^{*}C^{\dagger}XC^{\dagger} \right\} - XG - G^{\dagger}X. \tag{30}$$

Finally the master equation is obtained by duality:  $\frac{d}{dt}\varrho = L'(\varrho)$  where  $tr\{\varrho L(X)\} \equiv tr\{L'(\varrho)X\}$ .

## References

- [1] R.L. Hudson and K.R. Parthasarathy Quantum Itô's formula and stochastic evolutions. Commun.Math.Phys. 93, 301-323 (1984)
- [2] K. Itô Lectures on Stochastic Processes. Tata Inst. Fund. Research, Bombay (1961)
- [3] R. Stratonovich A New Representation For Stochastic Integrals And Equations SIAM J.Cont. 4, 362-371 (1966)
- [4] W. von Waldenfels Itô solution of the linear quantum stochastic differential equation describing light emission and absorption. Quantum Probability I, 384-411, Springer LNM 1055 (1986)
- [5] C.W. Gardiner Quantum Noise, Springer-Verlag
- [6] J. Gough A new approach to non-commutative white noise analysis, C.R. Acad. Sci. Paris, t.326, série I, 981-985 (1998)
- [7] L. Accardi, A. Frigerio, Y.G. Lu Weak coupling limit as a quantum functional central limit theorem Commun. Math. Phys. 131, 537-570 (1990)
- [8] L. Accardi, J. Gough, Y.G. Lu On the stochastic limit of quantum field theory Rep. Math. Phys. 36, No. 2/3, 155-187 (1995)
- [9] J. Gough Asymptotic stochastic transformations for non-linear quantum dynamical systems Reports Math. Phys. 44, No. 3, 313-338 (1999)
- $[10]\,$  L.Accardi, Y.G. Lu  $Low\ density\ limit$  J. Phys. A, Math. and Gen. 24: (15) 3483-3512 (1991)
- [11] J. Gough Causal structure of quantum stochastic integrators. Theoretical and Mathematical Physics 111, 2, 218-233 (1997)
- [12] J. Gough A new approach to non-commutative white noise analysis, C.R. Acad. Sci. Paris, t.326, série I, 981-985 (1998)

- [13] A.N. Chebotarev Symmetric form of the Hudson-Parthasarathy equation Mat. Zametki, **60**, 5, 725-750 (1996)
- [14] G. Lindblad On the generators of completely positive semi-groups Commun. Math. Phys 48, 119-130 (1976)