ASYMPTOTIC SOLUTIONS TO THE sl_2 KZ EQUATION AND THE INTERSECTION OF SCHUBERT CLASSES

I. SCHERBAK

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel
e-mail: scherbak@post.tau.ac.il

Abstract. The hypergeometric solutions to the KZ equation contain a certain symmetric "master function", [SV]. Asymptotics of the solutions correspond to critical points of the master function and give Bethe vectors of the inhomogeneous Gaudin model, [RV]. The general conjecture is that the number of orbits of critical points equals the dimension of the relevant vector space, and that the Bethe vectors form a basis. In [ScV], a proof of the conjecture for the sl_2 KZ equation was given. The difficult part of the proof was to count the number of orbits of critical points of the master function. Here we present another, "less technical", proof based on a relation between the master function and the map sending a set of polynomials into the Wronski determinant. Within these frameworks, the number of orbits becomes the intersection number of appropriate Schubert classes. Application of the Schubert calculus to the sl_p KZ equation is discussed.

1. Introduction

Denote $Z = \{ z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j ,\ 1 \le i < j \le n \}$. Let $M = (m_1, \dots, m_n)$ be a vector with positive integer coordinates, $|M| = m_1 + \dots + m_n$. For $z \in Z$, consider the following function in k variables, $k \le |M|/2$,

$$\Phi(t) = \Phi_{k,n}(t; z, M) = \prod_{i=1}^{k} \prod_{l=1}^{n} (t_i - z_l)^{-m_l} \prod_{1 \le i < j \le k} (t_i - t_j)^2 , \tag{1}$$

defined on

$$\begin{aligned} \mathcal{C}(t,z) &= \mathcal{C}_{k,n}(t,z) = \\ &= \left\{ t = (t_1, \dots, t_k) \in \mathbb{C}^k \mid t_i \neq z_j,\ t_i \neq t_l,\ 1 \le i < l \le k,\ 1 \le j \le n \right\} . \end{aligned} \tag{2}$$

The critical point system of the function $\Phi(t)$ provides the Bethe equations for the sl_2 Gaudin model of an inhomogeneous magnetic chain, [G]. The symmetric group acts on $\mathcal{C}(t,z)$ permuting coordinates $t_1, \dots, t_k$, and the action preserves the critical set of $\Phi(t)$. The orbits of nondegenerate critical points give common eigenvectors of the system of commuting Hamiltonians of the Gaudin model and define asymptotic solutions to the sl_2 KZ equation. For generic $z \in Z$, these eigenvectors generate the subspace Sing_k of singular vectors of weight $|M| - 2k$ in the tensor product of irreducible $sl_2(\mathbb{C})$ representations with highest weights $m_1, \ldots, m_n$, [RV].

One of the main results in [ScV] asserts that for generic z all critical points of the function $\Phi(t)$ are nondegenerate and the number of orbits equals the dimension of Sing_k. In other words, each common eigenvector of the Hamiltonians of the Gaudin model is represented by the solutions to the Bethe equations exactly once. In general, for other integrable models "parasite" solutions to the Bethe equations appear.

The difficult part of the proof given in [ScV] is to get an appropriate upper bound for the number of orbits of critical points of the function $\Phi(t)$. The aim of the present paper is to calculate this upper bound in another, "less technical", way using the Schubert calculus.

The Wronski determinant of two polynomials in one variable defines a map from the Grassmannian of two-dimensional planes of the linear space of polynomials to the space of monic polynomials (Sec. 3.1). It turns out that there is a one-to-one correspondence between the orbits of critical points of the function $\Phi(t)$ and the planes in the preimage under this map of the polynomial

$$W(x) = (x - z_1)^{m_1} \ldots (x - z_n)^{m_n}\,. \tag{3}$$

In fact, this is a classical result going back to Heine and Stieltjes (Sec. 3.3; cf.[S], Ch. 6.8).

To calculate the cardinality of the preimage is a problem of enumerative algebraic geometry, and an upper bound can be easily obtained in terms of the intersection number of special Schubert classes (Sec. 4). A well-known relation between representation theory and the Schubert calculus implies that the obtained upper bound coincides with the dimension of the space Sing_k.

The Schubert calculus seems to be a useful tool for the sl_p KZ equations as well. A function related in a very similar way to the Wronski map from the Grassmannian of p-dimensional planes was found by A. Gabrielov, [Ga]. This function turns to be the master function of the sl_p KZ equation associated with the tensor product of symmetric powers of the standard sl_p-module. The Schubert calculus provides an upper bound for the number of orbits of critical points as the intersection number of appropriate Schubert classes. Arguments similar to those of the sl_2 case prove then that the Bethe vectors form a basis of the corresponding space of singular vectors, Sec. 5.1.

The Wronski map corresponds to some special rational curve in the Grassmannian ([EG], see also Sec. 5.2). It would be interesting to find rational curves corresponding to the master functions associated with other tensor products.

Our presentation of the KZ equation theory in Sec. 2 is borrowed from [RV], [V]. The exposition of the Schubert calculus and its relation to the sl_2 representation theory in Sec. 3 and 4 follows [GH] and [F].

The author is thankful to A. Eremenko, A. Gabrielov, F. Sottile and A. Varchenko for useful suggestions, to G. Fainshtein and D. Karzovnik for stimulating comments.

2. The master function

2.1. sl_2 **modules.** The Lie algebra $sl_2 = sl_2(\mathbb{C})$ is generated by elements e, f, h such that $[e,f] = h,\ [h,e] = 2e,\ [h,f] = -2f$.

For a nonnegative integer m, denote L_m the irreducible sl_2-module with highest weight m and highest weight singular vector v_m. We have $hv_m = mv_m$, $ev_m = 0$.

Denote S_m the unique bilinear symmetric form on L_m such that $S_m(v_m, v_m) = 1$ and $S_m(hx, y) = S_m(x, hy)\,,\ S_m(ex, y) = S_m(x, fy)$ for all $x, y \in L_m$. The vectors $v_m, fv_m, \dots, f^m v_m$ are orthogonal with respect to S_m and form a basis of L_m.

The tensor product

$$L^{\otimes M} = L_{m_1} \otimes \cdots \otimes L_{m_n}\,, \tag{4}$$

where $M = (m_1, \dots, m_n)$ and all m_i are nonnegative integers, becomes an sl_2-module if the action of $x \in sl_2$ is defined by

$$x \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes x \otimes \cdots \otimes 1 \cdots + 1 \otimes 1 \otimes \cdots \otimes x.$$

The bilinear symmetric form on $L^{\otimes M}$ given by

$$S = S_{m_1} \otimes \cdots \otimes S_{m_n} \tag{5}$$

is called *the Shapovalov form.*

Let $j_1, \dots, j_n$ be integers such that $0 \le j_i \le m_i$ for any $1 \le i \le n$. Denote

$$f^J v_M = f^{j_1} v_{m_1} \otimes \cdots \otimes f^{j_n} v_{m_n}\,,\ J = (j_1, \dots, j_n)\,. \tag{6}$$

The vectors $\left\{ f^J v_M \right\}$ are orthogonal with respect to the Shapovalov form and provide a basis of the space $L^{\otimes M}$.

For $A = (a_1, \dots, a_n)$, write $|A| = a_1 + \cdots + a_n$. We have

$$h(f^J v_M) \ = \ (|M| - 2|J|) f^J v_M\,, \quad e(f^J v_M) = 0\,,$$

i.e. the vector $f^J v_M$ is *a singular vector of weight* $|M| - 2|J|$.

2.2. **The** sl_2 **KZ equation.** For the Casimir element,

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h \in sl_2 \otimes sl_2,$$

and for $1 \le i < j \le n$, denote $\Omega_{ij} : L^{\otimes M} \to L^{\otimes M}$ the operator which acts as Ω on i-th and j-th factors and as the identity on all others. For $z \in Z$, define operators $H_1(z), \dots, H_n(z)$ on $L^{\otimes M}$ as follows,

$$H_i(z) = \sum_{j \ne i} \frac{\Omega_{ij}}{z_i - z_j}\,, \quad i = 1, \dots, n. \tag{7}$$

The Knizhnik–Zamolodchikov (KZ) equation on a function $u : Z \to L^{\otimes M}$ is the system of partial differential equations

$$\kappa \frac{\partial u}{\partial z_i} = H_i(z) u\,, \quad i = 1, \dots, n, \tag{8}$$

where κ is a parameter. This equation appeared first in Wess–Zumino models of conformal field theory, [KZ].

2.3. **Asymptotic solutions to the KZ equation** ([V]).

The series

$$\phi(z) = e^{I(z)/\kappa} \left(\phi_0(z) + \kappa \phi_1(z) + \kappa^2 \phi_2(z) + \dots \right), \tag{9}$$

where I and ϕ_i are functions on Z, provides *an asymptotic solution to the equation* (8) if the substitution of ϕ into (8) gives 0 in the expansion into a formal power series in κ.

If $\phi(z)$ is an asymptotic solution, then the substitution into (8) gives recurrence relations for the functions ϕ_i, and in general ϕ_i can be recovered from ϕ_0 for any $i > 0$. Moreover the substitution shows that

$$\phi_0(z) \;\; \textit{is an eigenvector of} \;\; H_i(z) \;\; \textit{with the eigenvalue} \; \frac{\partial I}{\partial z_i}(z) \tag{10}$$

for any $1 \le i \le n$.

2.4. **The Bethe Ansatz.**

For any $z \in Z$, the operators $H_i(z)$ commute, are symmetric with respect to the Shapovalov form, and therefore have a common basis of eigenvectors.

The Bethe Ansatz is a method for diagonalizing of a system of commuting operators H_i in a vector space L called *the space of states*. The idea is to consider some L-valued function $w : \mathbb{C}^k \to L$, and to determine its argument $t = (t_1, \dots, t_k)$ in such a way that the value of this function, $w(t)$, is an eigenvector. The equations which determine these special values of the argument are called *the Bethe equations* and the vector $w(t^0)$, where t^0 is a solution to the Bethe equations, is called *the Bethe vector*.

In all known examples the Bethe equations coincide with the critical point system of a suitable function called *the master function* of the model. The Bethe vectors correspond to the critical points of the master function.

The standard conjectures are that the (properly counted) number of critical points of the master function is equal to the dimension of the space of states, and that the Bethe vectors form a basis.

The operators $H_i(z)$ given by (7) appear in the sl_2 Gaudin model of inhomogeneous magnetic chain [G]. According to (10), the asymptotic solutions to the sl_2 KZ equation (8) are labeled by the Bethe vectors of the sl_2 Gaudin model.

2.5. **Subspaces of singular vectors.** Let k be a nonnegative integer, $k \leq |M|/2$, and $\mathrm{Sing}_k = \mathrm{Sing}_k(L^{\otimes M})$ the subspace of singular vectors of weight $|M| - 2k$ in $L^{\otimes M}$,

$$\mathrm{Sing}_k(L^{\otimes M}) = \left\{ w \in L^{\otimes M} \mid ew = 0\,,\ hw = (|M| - 2k)w \right\}. \tag{11}$$

This space is generated by the vectors $f^J v_M$ as in (6) with $|J| = k$.

Theorem 1. ([ScV], Theorem 5) *We have*

$$\dim \mathrm{Sing}_k(L^{\otimes M}) \;=\; \sum_{q=0}^{n} (-1)^q \sum_{1 \leq i_1 < \dots < i_q \leq n} \binom{k + n - 2 - m_{i_1} - \dots - m_{i_q} - q}{n-2}.$$

As usually we set $\binom{a}{b} = 0$ for $a < b$.

The number $\dim \mathrm{Sing}_k(L^{\otimes M})$ is clearly the multiplicity of $L_{|M|-2k}$ in the decomposition of the tensor product $L^{\otimes M}$ into a direct sum of irreducible sl_2-modules. Equivalently, this is the multiplicity of the trivial sl_2-module, L_0 , in the decomposition of the tensor product

$$L_{m_1} \otimes \ldots \otimes L_{m_n} \otimes L_{|M|-2k}\,.$$

It seems the explicit formula was unknown to experts in representation theory.

The KZ equation preserves Sing_k for any k, and the subspaces Sing_k generate the whole of $L^{\otimes M}$. Therefore it is enough to produce solutions with values in a given $\mathrm{Sing}_k = \mathrm{Sing}_k(L^{\otimes M})$.

2.6. **Hypergeometric solutions** ([SV])**.** For $z \in Z$, consider the function

$$\Psi_{k,n}(t,z) = \Psi_{k,n}(t,z;M) = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{m_i m_j/2} \prod_{i=1}^{k} \prod_{l=1}^{n} (t_i - z_l)^{-m_l} \prod_{1 \leq i < j \leq k} (t_i - t_j)^2\,,$$

which is defined on $\mathcal{C}(t,z)$ given by (2). This function is invariant with respect to the group S^k of permutations of $t_1, \dots, t_k$.

For $J = (j_1, \dots, j_n)$ with integer coordinates satisfying $0 \leq j_i \leq m_i$ and $|J| = k$, set

$$A_J(t,z) = \frac{1}{j_1! \dots j_n!} \mathrm{Sym}_t \left[\prod_{l=1}^{n} \prod_{i=1}^{j_l} \frac{1}{t_{j_1 + \dots + j_{l-1} + i} - z_l} \right],$$

where

$$\mathrm{Sym}_t F(t) = \sum_{\sigma \in S^k} F\left(t_{\sigma(1)}, \dots, t_{\sigma(k)}\right)$$

is the sum over all permutations of $t_1, \dots, t_k$.

Theorem 2. ([SV]) *Let $\kappa \neq 0$ be fixed. The function*

$$u^{\gamma(z)} = \sum_{J\,:\,|J|=k} \left(\int_{\gamma(z)} \Psi_{k,n}^{1/\kappa}(t,z) A_J(t,z)\, dt_1 \wedge \dots \wedge dt_k \right) \cdot f^J v_M\,,$$

where $\gamma(z)$ is an appropriate k-cycle lying in $\mathcal{C}_{k,n}(z)$, provides a nontrivial solution to the KZ equation (8) *and takes values in* Sing_k.

2.7. **The master function of the Gaudin model** ([RV])**.** Asymptotic solutions can be produced by taking the limit of $u^{\gamma(z)}$ as $\kappa \to 0$. According to the steepest descend method, the leading terms, ϕ_0, are defined by the critical points of $\Psi_{k,n}^{1/\kappa}(t,z)$ with respect to t. In order to study the critical points, one can clearly replace $\Psi_{k,n}^{1/\kappa}(t,z)$ with the function $\Phi(t)$ given by (1).

Theorem 3. ([RV]) *(i) If t^0 is a nondegenerate critical point of the function $\Phi(t)$, then the vector*

$$w(t^0, z) = \sum_{J\,:\,|J|=k} A_J(t^0, z) f^J v_M$$

is an eigenvector of the set of commuting operators $H_1(z), \dots, H_n(z)$.
(ii) For generic $z \in Z$, the eigenvectors $w(t^0, z)$ generate the space Sing_k .

Thus the function $\Phi(t)$ is the master function of the Gaudin model, and $w(t^0, z)$ are the Bethe vectors. The critical point system of the function $\Phi(t)$ is as follows,

$$\sum_{l=1}^{n} \frac{m_l}{t_i - z_l} = \sum_{j \neq i} \frac{2}{t_i - t_j}, \quad i = 1, \dots, n\,. \tag{12}$$

The set of critical points of the function $\Phi(t)$ is invariant with respect to the permutations of $t_1, \dots, t_k$. Critical points belonging to the same orbit clearly define the same vector. One of the main results of [ScV] is as follows.

Theorem 4. ([ScV]) *For generic $z \in Z$,*
(i) all critical points of the function $\Phi(t)$ given by (1) *are nondegenerate;*
(ii) the number of orbits of critical points equals the dimension of Sing_k .

Thus for generic $z \in Z$, the Bethe vectors form a basis of Sing_k .

The proof of the statement *(i)* of Theorem 4 given in [ScV] is short, see Theorem 6 in [ScV]. The statements (ii) of Theorem 3 and (i) of Theorem 4 imply that the number of orbits is *at least* $\dim \mathrm{Sing}_k$. The difficult part of [ScV] was to prove that the number of orbits is *at most* $\dim \mathrm{Sing}_k$, see Theorems 9–11 in [ScV].

As it was pointed out in [ScV], Sec. 1.4, the orbits of critical points of the function $\Phi(t)$ are labeled by certain two-dimensional planes in the linear space of complex polynomials. This observation suggests to apply the Schubert calculus to the problem under consideration.

3. The Bethe vectors and the Wronski map

3.1. **The Wronski map.** Let Poly_d be the vector space of complex polynomials of degree $\le d$ in one variable and let $G_2(\mathrm{Poly}_d)$ be the Grassmannian of two-dimensional planes in Poly_d. The complex dimension of $G_2(\mathrm{Poly}_d)$ is $2d-2$.

For any $V \in G_2(\mathrm{Poly}_d)$ define *the degree of* of V as the maximal degree of its polynomials and the *order* of V as the minimal degree of its non-zero polynomials. Let $V \in G_2(\mathrm{Poly}_d)$ be a plane of order a and of degree b. Clearly $a < b$. Choose in V two monic polynomials, $F(x)$ and $G(x)$, of degrees a and b respectively. They form a basis of V. *The Wronskian of* V is defined as the monic polynomial

$$W_V(x) = \frac{F'(x)G(x) - F(x)G'(x)}{a-b}\,.$$

The following lemma is evident.

Lemma 1. *(i) The degree of* $W_V(x)$ *is* $a+b-1 \le 2d-2$.
(ii) The polynomial $W_V(x)$ *does not depend on the choice of a monic basis.*
(iii) All polynomials of degree a *in* V *are proportional.* $\triangleleft$

Thus the mapping sending V to $W_V(x)$ is a well-defined map from $G_2(\mathrm{Poly}_d)$ to $\mathbb{CP}^{2d-2}$. We call it *the Wronski map.* This is a mapping between smooth complex algebraic varieties of the same dimension, and hence the preimage of any polynomial consists of a finite number of planes. On Wronski maps see [EG].

3.2. **Planes with a given Wronskian.**

Lemma 2. *Any plane with a given Wronskian is uniquely determined by any of its polynomial.*

Proof: Let $W(x)$ be the Wronskian of a plane V and $f(x) \in V$. Take any polynomial $g(x) \in V$ linearly independent with $f(x)$. The plane V is the solution space of the following second order linear differential equation with respect to unknown function $u(x)$,

$$\begin{vmatrix} u(x) & f(x) & g(x) \\ u'(x) & f'(x) & g'(x) \\ u''(x) & f''(x) & g''(x) \end{vmatrix} = 0.$$

The Wronskian of the polynomials $f(x)$ and $g(x)$ is clearly proportional to $W(x)$. Therefore this equation can be re-written in the form

$$W(x)u''(x) - W'(x)u'(x) + h(x)u(x) = 0\,,$$

where

$$h(x) \;=\; \frac{-W(x)f''(x) + W'(x)f'(x)}{f(x)}\,,$$

as $f(x)$ is clearly a solution to this equation. $\triangleleft$

We call a plane V *generic* if for any $x_0 \in \mathbb{C}$ there is a polynomial $P(x) \in V$ such that $P(x_0) \neq 0$. In a generic plane, the polynomials of any basis do not have common roots, and almost all polynomials of the bigger degree do not have multiple roots.

The following lemma is evident.

Lemma 3. *Let V be a generic plane.*
(i) If $P(x) = (x - x_1)\dots(x - x_l) \in V$ is a polynomial without multiple roots, then $W_V(x_i) \neq 0$ for all $1 \leq i \leq l$.
(ii) If x_0 is a root of multiplicity $\mu > 1$ of a polynomial $Q(x) \in V$, then x_0 is a root of $W_V(x)$ of multiplicity $\mu - 1$. ◁

A generic plane V is *nondegenerate* if the polynomials of the smaller degree in V do not have multiple roots.

Lemma 4. *Let V be a nondegenerate plane. If the Wronskian $W_V(x)$ has the form*

$$W_V(x) = x^m \tilde{W}(x)\,, \quad \tilde{W}(0) \neq 0\,,$$

then there exists a polynomial $F_0(x) \in V$ of the form

$$F_0(x) = x^{m+1} \tilde{F}(x)\,, \quad \tilde{F}(0) \neq 0\,.$$

Proof: Let $G(x) \in V$ be a polynomial of the smaller degree. We have $G(0) \neq 0$, due to Lemma 3. Let $F(x) \in V$ be a polynomial of the bigger degree. The polynomial

$$F_0(x) \;=\; F(x) - \frac{F(0)}{G(0)}\, G(x) \;\in V$$

satisfies $F_0(0) = 0$ and therefore has the form

$$F_0(x) = x^l \tilde{F}(x)\,, \quad \tilde{F}(0) \neq 0\,,$$

for some integer $l \geq 1$. The polynomials G and F_0 form a basis of V, therefore the polynomial $F_0'(x)G(x) - F_0(x)G'(x)$ is proportional to $W_V(x)$. The smallest degree term in this polynomial is $l \cdot a_l \cdot G(0)x^{l-1}$, where a_l is the coefficient of x^l in $F_0(x)$. Therefore $l = m + 1$. ◁

3.3. **The master function and the Wronski map.** In the XIX century, Heine and Stieltjes in their studies of second order linear differential equations with polynomial coefficients and a polynomial solution of a prescribed degree (cf. the proof of Lemma 2 in Sec. 3.2) arrived at the result, which can be formulated as follows.

Theorem 5. (Cf. [S], Ch. 6.8) *Let t^0 be a critical point of the function $\Phi(t)$ given by* (1). *Then $F(x) = (x - t_1^0)\dots(x - t_k^0)$ is a polynomial of the smaller degree in a nondegenerate plane with the Wronskian $W(x)$ given by* (3).

Conversely, if $F(x) = (x - t_1^0)\dots(x - t_k^0)$ is a polynomial of the smaller degree in a plane V such that $W_V(x) = W(x)$, then $t^0 = (t_1^0, \dots, t_k^0)$ is a critical point of $\Phi(t)$. ◁

Corollary 1. *There is a one-to-one correspondence between the orbits of critical points of the master function* $\Phi(t)$ *given by* (1) *and the nondegenerate planes of order* k *and of degree* $|M|+1-k$ *having Wronskian* $W(x)$ *given by* (3)*.*

Remark. The function $\Phi(t)$ given by (1) can be re-written in the form

$$\Phi(t) \,=\, \Phi(W\,,F) = \frac{\mathrm{Disc}^2(F)}{\mathrm{Res}(W,F)}\,,$$

where $\mathrm{Res}(W,F)$ is the resultant of the polynomials $W(x)$ and $F(x)$, $\mathrm{Disc}(F)$ is the discriminant of the polynomial $F(x)$, the polynomial $W(x)$ is given by (3) and $F(x) = (x-t_1)\dots(x-t_k)$. If $W = F'Q - FQ'$ for some polynomial $Q = Q(x)$, then $W' = F''Q - FQ''$, and we have

$$\frac{W'}{W} \,=\, \frac{F''Q - FQ''}{F'Q - FQ'}\,.$$

If polynomials F and Q do not have common roots, then at each root t_i of F we clearly have

$$\frac{W'(t_i)}{W(t_i)} \,=\, \frac{F''(t_i)}{F'(t_i)}\,,\quad i = 1,\dots,k.$$

This is exactly the critical point system of the function $\Phi(t)$, cf. (12). Thus the function Φ considered as a function of polynomials W and F is *the generating function* of the Wronski map – for a given polynomial W, the critical points of the generating function determine the nondegenerate planes in the preimage of W.

4. The preimage of a given Wronskian

The number of nondegenerate planes with a given Wronskian can be estimated from above by the intersection number of Schubert classes.

4.1. **Schubert calculus** ([GH], [F])**.** Let $G_2(d+1) = G_2(\mathbb{C}^{d+1})$ be the Grassmannian variety of two-dimensional subspaces $V \subset \mathbb{C}^{d+1}$. A chosen basis $e_1,\dots,e_{d+1}$ of $\mathbb{C}^{d+1}$ defines the flag of linear subspaces

$$E_\bullet\,:\quad E_1 \subset E_2 \subset \dots \subset E_d \subset E_{d+1} = \mathbb{C}^{d+1}\,,$$

where $E_i = \mathrm{Span}\{e_1,\dots,e_i\}$, $\dim E_i = i$. For any integers a_1 and a_2 such that $0 \le a_2 \le a_1 \le d-1$, the *Schubert variety* $\Omega_{a_1,a_2}(E_\bullet) \subset G_2(d+1)$ is defined as follows,

$$\Omega_{a_1,a_2} = \Omega_{a_1,a_2}(E_\bullet) = \{\, V \in G_2(d+1) \mid \dim\,(V \cap E_{d-a_1}) \ge 1\,,\ \dim\,(V \cap E_{d+1-a_2}) \ge 2\,\}\,.$$

The variety $\Omega_{a_1,a_2} = \Omega_{a_1,a_2}(E_\bullet)$ is an irreducible closed subvariety of $G_2(d+1)$ of the complex codimension $a_1 + a_2$.

The homology classes $[\Omega_{a_1,a_2}]$ of Schubert varieties Ω_{a_1,a_2} are independent of the choice of flag, and form a basis for the integral homology of $G_2(d+1)$. Define σ_{a_1,a_2} to be the cohomology class in $H^{2(a_1+a_2)}(G_2(d+1))$ whose cap product with the fundamental class of $G_2(d+1)$ is the homology class $[\Omega_{a_1,a_2}]$. The classes σ_{a_1,a_2} are called *Schubert classes.*

They give a basis over $\mathbb{Z}$ for the cohomology ring of the Grassmannian. The product or *intersection* of any two Schubert classes σ_{a_1,a_2} and σ_{b_1,b_2} has the form

$$\sigma_{a_1,a_2}\cdot\sigma_{b_1,b_2} = \sum_{c_1+c_2=a_1+a_2+b_1+b_2} C(a_1,a_2;b_1,b_2;c_1,c_2)\sigma_{c_1,c_2}\,,$$

where $C(a_1,a_2;b_1,b_2;c_1,c_2)$ are nonnegative integers called *the Littlewood–Richardson coefficients.*

If the sum of the codimensions of classes equals $\dim G_2(d+1) = 2d-2$, then their intersection is an integer (identifying the generator of the top cohomology group $\sigma_{d-1,d-1}\in H^{4d-4}(G_2(d+1))$ with $1\in\mathbb{Z}$) called *the intersection number.*

When $(a_1,a_2)=(q,0)$, $0\le q\le d-1$, the Schubert varieties $\Omega_{q,0}$ are called *special* and the corresponding cohomology classes $\sigma_q=\sigma_{q,0}$ are called *special Schubert classes.*

We will need the following fact connecting the Schubert calculus and representation theory.

Denote L_q the irreducible sl_2-module with highest weight q.

Proposition 1. ([F]) *Let $q_1,\dots,q_{n+1}$ be integers such that $0\le q_i\le d-1$ for all $1\le i\le n+1$ and $q_1+\dots+q_{n+1}=2d-2$. Then the intersection number of the corresponding special Schubert classes, $\sigma_{q_1}\cdot\ \dots\ \cdot\sigma_{q_{n+1}}$, coincides with the multiplicity of the trivial sl_2-module L_0 in the tensor product $L_{q_1}\otimes\dots\otimes L_{q_{n+1}}$.* ◁

This proposition and Theorem 1 imply the explicit formula for the intersection number of special Schubert classes which seems to be unknown to experts.

Corollary 2. *Let $q_1,\dots,q_{n+1}$ be integers such that $0\le q_i\le d-1$ for all $1\le i\le n+1$ and $q_1+\dots+q_{n+1}=2d-2$. Then we have*

$$\sigma_{q_1}\cdot\ \dots\ \cdot\sigma_{q_{n+1}} = \sum_{l=1}^{n}(-1)^{n-l}\sum_{1\le i_1<\dots<i_l\le n}\binom{q_{i_1}+\dots+q_{i_l}+l-d-1}{n-2}\,. \tag{13}$$

4.2. **The planes with a given Wronskian and Schubert classes.** Applying the Schubert calculus to the Wronski map, we arrive at the following result.

Theorem 6. *Let $m_1,\dots,m_n$ be positive integers such that $|M|=m_1+\dots+m_n\le 2d-2$. Then for generic $z\in Z$ and for any integer k such that $0<k<|M|+1-k\le d$, the preimage of the polynomial* (3) *under the Wronski map consists of at most*

$$\sigma_{m_1}\cdot\ \dots\ \cdot\sigma_{m_n}\cdot\sigma_{|M|-2k}$$

planes of order k and of degree $<|M|+1-k$.

Proof: Any plane of order k with the Wronskian of degree $|M|$ lies in $G_2(\mathrm{Poly}_{|M|+1-k})$, as Lemma 1 shows. Therefore it is enough to consider the Wronski map on $G_2(\mathrm{Poly}_{|M|+1-k})$.

For any z_i, define the flag $\mathcal{F}_{z_i}$ in $\mathrm{Poly}_{|M|+1-k}$,

$$\mathcal{F}_0(z_i) \subset \mathcal{F}_1(z_i) \subset \dots \mathcal{F}_{|M|+1-k}(z_i), = \mathrm{Poly}_{|M|+1-k} ,$$

where $\mathcal{F}_j(z_i)$ consists of the polynomials $P(x) \in \mathrm{Poly}_{|M|+1-k}$ of the form

$$P(x) = a_j(x-z_i)^{|M|+1-k-j} + \dots + a_0(x-z_i)^{|M|+1-k} .$$

We have $\dim \mathcal{F}_j(z_i) = j+1$. Lemma 4 implies that the nondegenerate planes with a Wronskian having at z_i a root of multiplicity m_i lie in the special Schubert variety

$$\Omega_{m_i,0}(\mathcal{F}_{z_i}) \subset G_2(\mathrm{Poly}_{|M|+1-k}) .$$

The maximal possible degree of the Wronskian $W_V(x)$ for $V \in \mathrm{Poly}_{|M|+1-k}$ is clearly $2|M|-2k$. Denote $m_\infty = |M|-2k$, the difference between $2|M|-2k$ and $|M|$. If m_∞ is positive, it is *the multiplicity of* $W(x)$ *at infinity.*

The nondegenerate planes with a Wronskian having given multiplicity m_∞ at infinity lie in the special Schubert variety $\Omega_{m_\infty,0}(\mathcal{F}_\infty)$, where $\mathcal{F}_\infty$ is the flag

$$\mathrm{Poly}_0 \subset \mathrm{Poly}_1 \subset \dots \subset \mathrm{Poly}_{|M|+1-k} .$$

We conclude that the nondegenerate planes which have the Wronskian (3) lie in the intersection of special Schubert varieties

$$\Omega_{m_1,0}(\mathcal{F}_{z_1}) \cap \Omega_{m_2,0}(\mathcal{F}_{z_2}) \cap \dots \cap \Omega_{m_n,0}(\mathcal{F}_{z_n}) \cap \Omega_{m_\infty,0}(\mathcal{F}_\infty) .$$

The dimension of $G_2(\mathrm{Poly}_{|M|+1-k})$ is exactly $m_1 + \dots + m_n + m_\infty$, therefore this intersection consists of a finite number planes, and the intersection number of the special Schubert classes

$$\sigma_{m_1} \cdot \ \dots \ \cdot \sigma_{m_n} \cdot \sigma_{|M|-2k}$$

provides an upper bound. ◁

The statement (ii) of Theorem 3, Corollary 1, Proposition 1 and Theorem 6 prove the statement *(ii)* of Theorem 4.

Corollary 3. *For generic* $z \in Z$*, all planes in the preimage under the Wronski map of the polynomial* $W(x)$ *given by* (3) *are nondegenerate.*

5. Comments: sl_p case

5.1. **The Schubert calculus and the sl_p KZ equations.** Any irreducible sl_2-module is a symmetric power of the standard one. It turns out that our arguments work for the sl_p KZ equations associated with the tensor product of symmetric powers of the standard sl_p-module. These arguments show that

the number of orbits of critical points of the master function
equals the dimension of the relevant subspace of singular vectors

and hence prove the conjecture on Bethe vectors for the corresponding Gaudin model. Here we describe briefly the steps following the scheme of sl_2 case (Sec. 2–4).

(I) *The master function.* For the Lie algebra $sl_p = sl_p(\mathbb{C})$, denote E the standard sl_p-module and $\alpha_1, \dots, \alpha_{p-1}$ the simple roots. Let $M = (m_1, \dots, m_n)$ be a fixed vector with positive integer coordinates. Set $L_j = \mathrm{Sym}^{m_j} E$ and $L = L_1 \otimes \dots \otimes L_n$. Every L_j is the irreducible sl_p-module with highest weight $\Lambda_j = (m_j, 0, \dots, 0)$. Write $\Lambda = \Lambda_1 + \dots + \Lambda_n$. Let $k_1, \dots, k_{p-1}$ be integers, $k_1 \geq \dots \geq k_{p-1} \geq 0$, $\mathbf{k} = (k_1, \dots, k_{p-1})$. Denote $\mathrm{Sing}_{\mathbf{k}} L$ the subspace of singular vectors in L of weight $\Lambda - k_1\alpha_1 - \dots - -k_{p-1}\alpha_{p-1}$.

For the KZ equation associated with the subspace $\mathrm{Sing}_{\mathbf{k}} L$, the master function $\tilde\Phi(t) = \tilde\Phi_{K,n}(t; z; M)$ is a function in $K = k_1 + \dots + k_{p-1}$ complex variables t,

$$t = \left(t^{(1)}, \dots, t^{(p-1)}\right), \quad t^{(l)} = \left(t_1^{(l)}, \dots, t_{k_l}^{(l)}\right), \quad l = 1, \dots, p-1,$$

defined on $\mathcal{C}_{K,n}(t; z)$, see (2), and given by

$$\begin{aligned} \tilde\Phi(t) = \tilde\Phi_{K,n}(t; z; M) &= \prod_{l=1}^{p-1} \prod_{1 \leq i < j \leq k_l} (t_i^{(l)} - t_j^{(l)})^2 \\ &\times \prod_{l=2}^{p-1} \prod_{i=1}^{k_{l-1}} \prod_{j=1}^{k_l} (t_i^{(l-1)} - t_j^{(l)})^{-1} \\ &\times \prod_{j=1}^{n} \prod_{i=1}^{k_1} (t_i^{(1)} - z_j)^{-m_j}, \end{aligned} \tag{14}$$

according to [RV]. The set of critical points of the function (14) is invariant with respect to the group $S = S^{k_1} \times \dots \times S^{k_{p-1}}$, where S^{k_l} is the group of permutations of $t_1^{(l)}, \dots, t_{k_l}^{(l)}$. If $k_l = 0$ for $l_0 \leq l \leq p-1$, then corresponding terms in (14) are missing.

(II) *The Wronski map.* The Wronskian of l polynomials $f_1(x), \dots, f_l(x)$ is the determinant

$$\mathrm{Wr}[f_1, \dots, f_l](x) = \det \begin{pmatrix} f_1(x) & \dots & f_l(x) \\ f_1'(x) & \dots & f_l'(x) \\ \dots & \dots & \dots \\ f_1^{(l-1)}(x) & \dots & f_l^{(l-1)}(x) \end{pmatrix}.$$

Denote $G_p(\mathrm{Poly}_d)$ the Grassmannian of p-dimensional planes in Poly_d. The Wronskian of any $V \in G_p(\mathrm{Poly}_d)$ is defined as a monic polynomial which is proportional to the Wronskian of some (and hence, any) basis of V. *The Wronski map* $\mathcal{W}: G_p(\mathrm{Poly}_d) \to \mathbb{CP}^{p(d+1-p)}$ sends $V \in G_p(\mathrm{Poly}_d)$ into its Wronskian, [EG].

Any element of $G_p(\mathrm{Poly}_d)$ has a basis of polynomials of pairwise distinct degrees, and these degrees are uniquely defined by the element. For $V \in G_p(\mathrm{Poly}_d)$, let $P_1(x), \dots, P_p(x)$ be a basis such that $\deg P_l = d_l$ for all $1 \leq l \leq p$ and $d \geq d_1 >$

$d_2 > \cdots > d_p \geq 0$. Denote $\mathrm{Wr}_l(x) = \mathrm{Wr}[P_{l+1}, \ldots, P_p](x)$ for any $0 \leq l \leq p-1$. In particular, $\mathrm{Wr}_0(x)$ is proportional to the Wronskian of V, and $\mathrm{Wr}_{p-1}(x) = P_p(x)$. We have $\deg \mathrm{Wr}_l = k_l$, where

$$k_l = d_p + \cdots + d_{l+1} - \frac{(p-l)(p-l-1)}{2}, \quad 0 \leq l \leq p-1. \tag{15}$$

If polynomials $\mathrm{Wr}_l(x)$ and $\mathrm{Wr}_{l+1}(x)$ do not have common roots for all $0 \leq l \leq p-2$, we call $V = \mathrm{Span}\{P_1(x), \ldots, P_p(x)\} \in G_p(\mathrm{Poly}_d)$ *a nondegenerate p-plane of type* $D = (d_1, \ldots, d_p)$.

The question of enumerative algebraic geometry we are interested in is as follows:

Given monic polynomial $W(x)$ and integers $d_1 > \cdots > d_p \geq 0$,
find the number of nondegenerate p-planes
of type $D = (d_1, \ldots, d_p)$ with the Wronskian $W(x)$.

Such p-planes belong to $G_p(\mathrm{Poly}_{d_1})$, therefore we write $d = d_1$.

(III) *Critical points of the master function and nondegenerate p-planes.* Fix integers $d_1 > d_2 > \cdots > d_p \geq 0$, $d_1 = d$. Let $k_1, \ldots, k_{p-1}$ be as in (15). One can re-write the function (14) in the form (cf. Remark in Sec. 3.3)

$$\tilde{\Phi}(t) \;=\; \frac{\mathrm{Disc}^2(W_1)\cdots \mathrm{Disc}^2(W_{p-1})}{\mathrm{Res}(W, W_1)\mathrm{Res}(W_1, W_2)\cdots \mathrm{Res}(W_{p-2}, W_{p-1})}\,,$$

where $W_l = W_l(x; t^{(l)}) = \prod_{j=1}^{k_l}(x - t_j^{(l)})$ is a polynomial in x of degree k_l for $1 \leq l \leq p-1$ and $W = W(x) = \prod_{j=1}^{n}(x - z_j)^{m_j}$ is the same as in (3). In this form the function $\tilde{\Phi}(t)$ was found by A. Gabrielov as the generating function of the Wronski map, [Ga]. If degrees of polynomials W_l vanish for $l_0 \leq l \leq p-1$, then the corresponding terms in $\tilde{\Phi}(t)$ are missing. The following statement is a generalization of the result of Heine and Stieltjes (cf. Theorem 5 in Sec. 3.3).

Theorem 7. *Let* (3) *be the Wronskian of some p-dimensional plane in the space of polynomials. Assume that there exists a basis $\{P_1(x), \ldots, P_p(x)\}$ of this plane such that the Wronskian of the polynomials $P_{l+1}(x), \ldots, P_p(x)$ is $W_l(x; s^{(l)}) = \prod_{j=1}^{k_l}(x - s_j^{(l)})$ for any $1 \leq l \leq p-1$ and $s = (s^{(1)}, \ldots, s^{(p-1)}) \in \mathcal{C}_{K,n}(t; z)$. Then the point s is a critical point of the master function* (14).

Conversely, any orbit of critical points of the master function (14) *defines a unique p-dimensional plane with these properties.*

Corollary 4. *The number of orbits of critical points of the master function* (14) *coincide with the number of nondegenerate p-planes of type $D = (d_1, \ldots, d_p)$ in the preimage of* (3) *under the Wronski map. Here $d_1 = d,\ d_l = k_{l-1} - k_l + p - l\ (2 \leq l \leq p-1),\ d_p = k_{p-1}$.*

(IV) *An upper bound: the Schubert calculus.* The nondegenerate p-planes of Corollary 4 lie in the intersection of Schubert varieties

$$\Omega_{(m_1)}(\mathcal{F}_{z_1}) \cap \Omega_{(m_2)}(\mathcal{F}_{z_2}) \cap \dots \cap \Omega_{(m_n)}(\mathcal{F}_{z_n}) \cap \Omega_w(\mathcal{F}_\infty)\,,$$

where $\Omega_{(m_j)} = \Omega_{m_j,0,\dots,0}$ are special Schubert varieties and $\Omega_w = \Omega_{w_1,\dots,w_p}$ is a Schubert variety with $w_l = d + l - p - d_{p+1-l}$, $1 \le l \le p$ or, in terms of $\mathbf{k}$,

$$w_1 = d + 1 - p - k_{p-1}\,, \quad w_l = d + 1 - p - k_{p-l} + k_{p-l+1} \text{ for } 2 \le l \le p-1, \quad w_p = 0.$$

Thus the number of orbits of critical points of the master function (14) is bounded from above by the intersection number of the corresponding Schubert classes. This number is the dimension of $\mathrm{Sing}_{\mathbf{k}} L$, [F].

(V) *A lower bound: Fuchsian equations.* In the sl_2 case this step was done in [RV] (see Sec. 2.7). In the sl_p case, results of N. Reshetikhin and A. Varchenko (see Theorem 10.4 of [RV]) say that the bound from below by the same number, $\dim \mathrm{Sing}_{\mathbf{k}} L$, holds for generic $\mathbf{z}$ if this is the case for $n = 2, z_1 = 0, z_2 = 1$. This is really the case, as we will see using the theory of Fuchsian differential equations ([R]).

Denote $\tilde\Phi^0(t) = \tilde\Phi_{K,2}(t; \{0,1\}; \{m_1, m_2\})$. This is the master function of the sl_p KZ equation associated with the subspace of singular vectors of the weight $\Lambda_1 + \Lambda_2 - k_1\alpha_1 - \dots - k_{p-1}\alpha_{p-1}$ in the tensor product of two symmetric powers of the standard sl_p-module.

Assume $m_1 \ge m_2$. The Pieri formula ([F]) says that the dimension of this subspace is 1 if $0 \le k_1 \le m_2$ and $k_2 = \dots = k_{p-1} = 0$ and 0 otherwise.

According to Theorem 7, a critical point of the function $\tilde\Phi^0(t)$ corresponds to the solution space of the linear differential equation of order p with respect to unknown function $u(x)$,

$$(16) \qquad \det \begin{pmatrix} u(x) & P_1(x) & \dots & P_p(x) \\ u'(x) & P_1'(x) & \dots & P_p'(x) \\ \dots & \dots & \dots & \dots \\ u^{(p)}(x) & P_1^{(p}(x) & \dots & P_p^{(p)}(x) \end{pmatrix} = 0,$$

where $\{P_1(x), \dots, P_p(x)\}$ is a basis of this p-plane. This is a Fuchsian differential equation with regular singular points at 0, 1 and ∞. The exponents at 0 and 1 are $0, 1, \dots, p-2, m_1 + p - 1$ and $0, 1, \dots, p-2, m_2 + p - 1$, respectively. The Wronskian of this equation is the Wronskian of the p-plane,

$$\mathrm{Wr}[P_1, \dots, P_p](x) = x^{m_1}(x-1)^{m_2}.$$

Proposition 2. *The equation* (16) *has the form*

$$x(x-1)u^{(p)}(x) + (Ax + B)u^{(p-1)}(x) + Cu^{(p-2)}(x) = 0.$$

Hence we can assume that polynomials $P_3(x), \dots, P_{p-1}(x), P_p(x)$ are $x^{p-3}, \dots, x, 1$, respectively. Then the Wronskians $W_l(x) = \mathrm{Wr}[P_{l+1}, \dots, P_p](x) = 1$ for $2 \le l \le p-1$, i.e. $k_2 = \dots = k_{p-1} = 0$.

Corollary 5. *The function $\tilde\Phi^0(t)$ may have critical points only if $k_2 = \cdots = k_{p-1} = 0$.*

One way to calculate the number of critical points of the function $\tilde\Phi^0(t)$ is to note that for $k_2 = \cdots = k_{p-1} = 0$ it becomes the function $\Phi_{k_1,2}(t;\{0,1\};\{m_1,m_2\})$, see (1). Another way is to observe that the equation of Proposition 2 is the hypergeometric equation with respect to the function $u^{(p-2)}(x)$, [R]. In any way we get that the number of orbits of critical points of the function $\tilde\Phi^0(t)$ is 1 when the corresponding highest weight enters the tensor product, and 0 otherwise.

Thus the upper and lower bounds for the number of orbits of critical points of the function $\tilde\Phi(t)$ coincide, and this number is the dimension of $\mathrm{Sing}_{\mathbf{k}} L$.

5.2. **Master functions and rational functions in the Grassmannian.** As it is explained in [EG], any rational curve of degree A in the Grassmannian $G_p(\mathrm{Poly}_d)$ defines a projection from $G_p(\mathrm{Poly}_d)$ to the projective space of complex polynomials of degree at most A, considered up to proportionality. The construction is as follows.

Let $V \in G_p(\mathrm{Poly}_d)$. One can identify polynomials of degree at most d with vectors in $\mathbb{C}^{d+1}$ using coefficients as coordinates. Then V can be given by a $p \times (d+1)$-matrix of coordinates of p linearly independent polynomials of V. On the other hand, one can also identify these polynomials with linear forms on $\mathbb{C}^{d+1}$, and p linearly independent forms define a subspace $\tilde V$ of dimension $d+1-p$ in $\mathbb{C}^{d+1}$. This subspace can be represented by a $(d+1-p)\times(d+1)$-matrix. We denote this matrix K_V.

Now let a rational curve be given by a $p\times(d+1)$-matrix $P(\xi)$ of polynomials in ξ, and let K_V be a $(d+1-p)\times(d+1)$-matrix corresponding in the described way to any $V \in G_p(\mathrm{Poly}_d)$. Then $P(\xi)$ defines a projection by the formula

$$V \mapsto \det \begin{pmatrix} P(\xi) \\ K_V \end{pmatrix} . \tag{17}$$

As it was shown in [EG], the Wronski map corresponds to a rational curve given by the matrix

$$\begin{pmatrix} F(\xi) \\ F'(\xi) \\ \cdots \\ F^{(p-1)}(\xi) \end{pmatrix} ,$$

where $F(\xi)$ is a row of powers of ξ,

$$F(\xi) = (\xi^d \quad \xi^{d-1} \quad \dots \quad \xi \quad 1) .$$

Hence this rational curve corresponds to the master function (14). It would be interesting to understand what rational curves correspond to the master functions for sl_p KZ equations associated with other tensor products.

References

[EG] A. Eremenko and A. Gabrielov, Degrees of real Wronski maps, to appear in *Discrete and Computational Geometry*.

[F] W. Fulton, "Young Tableaux", Cambridge University Press, 1997.

[Ga] A. Gabrielov, Generating functions of Wronski maps, *private communication.*

[G] M. Gaudin, Diagonalization d'une class hamiltoniens de spin. *Journ. de Physique* **37**, no. 10 (1976), 1087 – 1098.

[GH] Ph. Griffits and J. Harris, "Principles of Algebraic Geometry", Wiley, New York, 1978.

[KZ] V. Knizhnik and A. Zamolodchikov, Current algebra and Wess–Zumino models in two dimensions, *Nucl. Phys.* **B247** (1984), 83–103.

[R] E. Rainville, Intermediate differential equations, The Macmillan Company, 1964.

[RV] N. Reshetikhin and A. Varchenko, Quasiclassical Asymptotics of Solutions to the KZ Equations. In: Geometry, Topology, and Physics for Raoul Bott, International Press, 1994, 293–322.

[SV] V. Schechtman and A. Varchenko, Arrangements of hyperplanes and Lie algebra homology. *Invent. Math.* **106** (1991), 139 – 194.

[ScV] I. Scherbak and A. Varchenko, Critical points of functions, sl_2 representations, and Fuchsian differential equations with only univalued solutions, *preprint* (2001), math.QA/0112269.

[S] G. Szego, "Orthogonal polynomials", AMS, 1939.

[V] A. Varchenko, Special functions, KZ type equations and Representation theory. *Notes of a course given at MIT during the spring of 2002*, math.QA/0205313.