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MASTER SPACES FOR STABLE PAIRS

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Introduction

In this paper we construct master spaces for certain coupled vector bundle problems over a fixed projective variety X.

From a technical point of view, master spaces classify oriented pairs $(\mathcal{E},\varepsilon,\varphi)$ consisting of a torsion free coherent sheaf $\mathcal{E}$ with fixed Hilbert polynomial, an orientation ε of the determinant of $\mathcal{E}$, and a framing $\varphi:\mathcal{E}\longrightarrow\mathcal{E}_0$ with values in a fixed reference sheaf $\mathcal{E}_0$, satisfying certain semistability conditions. The relevant stability concept is new and does not involve the choice of a parameter, but it can easily be compared to the older parameter-dependent stability concepts for (unoriented) pairs.

The corresponding moduli spaces $\mathcal{M}$ have the structure of polarized projective varieties endowed with a natural $\mathbb{C}^*$-action which can be exploited in two interesting ways:

1. The fixed point set $\mathcal{M}^{\mathbb{C}^*}$ of the $\mathbb{C}^*$-action is a union

$$\mathcal{M}^{\mathbb{C}^*}=\mathcal{M}_{source}\cup\mathcal{M}_{sink}\cup\mathcal{M}_R\ ,$$

where $\mathcal{M}_{source}$ is a Gieseker moduli space of semistable oriented sheaves, $\mathcal{M}_{sink}$ is a certain (possible empty) Grothendieck Quot-scheme, and the third term $\mathcal{M}_R:=\mathcal{M}^{\mathbb{C}^*}\setminus(\mathcal{M}_{source}\cup\mathcal{M}_{sink})$ is the so-called "variety of reductions", which consists essentially of lower rank objects. The structure as a $\mathbb{C}^*$-space can be used to relate "correlation functions" associated with the different parts of $\mathcal{M}^{\mathbb{C}^*}$ to each other [OT2].

2. Master spaces are also useful for the investigation of the birational geometry of the moduli spaces $\mathcal{M}_\delta$ of δ-semistable pairs in the sense of [HL2].

Indeed, each of the $\mathcal{M}_\delta$'s can be obtained as a suitable $\mathbb{C}^*$-quotient of the master space $\mathcal{M}$, and it can be shown that every two quotients $\mathcal{M}_\delta$, $\mathcal{M}_{\delta'}$ are related by a chain of generalized flips in the sense of [Th].

When X is a projective curve with trivial reference sheaf $\mathcal{E}_0=\mathcal{O}_X^{\oplus k}$, our master space can be considered as a natural compactification of the one described in [BDW]. Their space becomes an open subset of ours whose complement is the

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Quot-scheme $\mathcal{M}_{sink}$ alluded to above ($\mathcal{M}_{sink}$ is empty iff $k < \mathrm{rk}(\mathcal{E})$). Applying the ideas of 1. in this situation leads to formulas for volumina and characteristic numbers and to a new proof of the Verlinde formula when $k = 1$, and allows to relate Gromov-Witten invariants for Grassmannians to simpler vector bundle data when $k > \mathrm{rk}(\mathcal{E})$.

In the case of an algebraic surface X, master spaces can be viewed as algebraic analoga of certain gauge theoretic moduli spaces of monopoles which can be used to relate Seiberg-Witten invariants and Donaldson polynomials [OT1], [T1]. The latter application was actually our original motivation for the construction of master spaces.

The study of non-abelian monopoles on Kähler surfaces leads naturally to the investigation of a certain moment map on an infinite dimensional Kähler space.

The associated stability concept, which is expected to exist on general grounds [MFK], is precisely the one which gave rise to the stability definition for oriented pairs [OT2]. Since the moduli space of non-abelian monopoles admit an Uhlenbeck type compactification [T1], it was natural to look for a corresponding Gieseker type compactification of their algebro-geometric analoga. These compactifications, the master spaces for stable pairs, provide very useful models for understanding the ends of monopole moduli spaces in the more difficult gauge theoretical context [T2]. Understanding these ends is the essential final step in our program for relating Donaldson polynomials and Seiberg-Witten invariants [OT1], [T1]. Let us now briefly describe the main ideas and results of this paper.

The construction of master spaces requires the study of GIT-quotients for direct sums of representations, i.e. the construction of quotients $\mathbb{P}(A \oplus B)^{ss} /\!\!/ G$, where G is a reductive group acting linearly on vector spaces A and B. Since the Hilbert criterion is difficult to apply in this situation, we have chosen another approach instead. The idea is to use the natural $\mathbb{C}^*$-action $z \cdot \langle a, b\rangle := \langle a, zb\rangle$ on $\mathbb{P}(A \oplus B)$ which commutes with the given action of G. Our first main result characterizes G-semistable points in $\mathbb{P}(A \oplus B)$ in terms of G-semistability of their images in all possible $\mathbb{C}^*$-quotients of $\mathbb{P}(A \oplus B)$. The proof is based on a commuting principle for actions of products of groups.

These results, which we prove in the first section, explain in particular why chains of flips occur in GIT-problems for $G \times \mathbb{C}^*$-actions [DH], [Th].

In the second section of our paper, after defining stability for oriented pairs $(\mathcal{E}, \varepsilon, \varphi)$, we prove a crucial boundedness result and construct the corresponding parameter space $\mathfrak{B}$. This space admits a morphism $\iota : \mathfrak{B} \longrightarrow \mathbb{P}(\mathfrak{Z})$ into a certain Gieseker space $\mathbb{P}(\mathfrak{Z})$ which is equivariant w.r.t. a natural action of a product $\mathrm{SL} \times \mathbb{C}^*$ of $\mathbb{C}^*$ with a special linear group. The SL-action on $\mathbb{P}(\mathfrak{Z})$ possesses a linearization in a suitable line bundle, and the preimage of the subset $\mathbb{P}(\mathfrak{Z})^{ss}$ of SL-semistable points is precisely the open subspace $\mathfrak{B}^{ss} \subset \mathfrak{B}$ of points representing semistable oriented pairs. In order to prove this, we apply our GIT-Theorem from the first section to the $\mathrm{SL}\times\mathbb{C}^*$-action on $\mathbb{P}(\mathfrak{Z})$, and thereby reduce the proof to results in [G] and [HL1].

Then we show that the induced map $\iota|_{\mathfrak{B}^{ss}}:\mathfrak{B}^{ss}\longrightarrow\mathbb{P}(\mathfrak{Z})^{ss}$ is finite and hence descends to a finite map $\bar\iota:\mathfrak{B}^{ss}/\!\!/\mathrm{SL}\longrightarrow\mathbb{P}(\mathfrak{Z})^{ss}/\!\!/\mathrm{SL}$. The quotient $\mathfrak{B}^{ss}/\!\!/\mathrm{SL}$, which is therefore a projective variety, is our master space.

The ideas and techniques of this paper can also be applied to construct master spaces in other interesting situations, e.g. by coupling with sections in twisted endomorphism bundles. When X is a curve and the twisting line bundle is the canonical bundle, one obtains a natural compactification of the moduli spaces of Higgs bundles [H], [S].

Similar ideas should also apply to coupling with singular objects like parabolic structures. We refer to [OT2] for a general description of the underlying coupling principle and its application to computations of correlation functions.

Conventions. Our ground field is $\mathbb{C}$. A *polarization* on a quasi-projective variety X is an equivalence class $[L]$ of ample line bundles, where two line bundles L_1 and L_2 are *equivalent*, if there exist positive integers n_1 and n_2 such that $L_1^{\otimes n_1}\cong L_2^{\otimes n_2}$.

If W is a finite dimensional vector space, we denote by $\mathbb{P}(W)$ its projectivization in the sense of Grothendieck, i.e., the closed points of $\mathbb{P}(W)$ correspond to lines in the dual space $W^\vee$. We do not distinguish notationally between a vector space W and its associated scheme.

1. A theorem from Geometric Invariant Theory

1.1. **Background material from GIT.** Let G be a reductive algebraic group and let $\gamma\colon G\longrightarrow\mathrm{GL}(W)$ be a rational representation in the finite dimensional vector space W. The map γ defines an action of G on the dual space $W^\vee$ given by

$$g\cdot w:=w\circ\gamma(g^{-1})\qquad\forall g\in G; w\in W^\vee,$$

an action $\overline{\gamma}$ on the projective space $\mathbb{P}(W)$, and a linearization of this action in $\mathcal{O}_{\mathbb{P}(W)}(1)$. In the following we identify $H^0(\mathbb{P}(W),\mathcal{O}_{\mathbb{P}(W)}(k))$ with S^kW.

Recall that a point $x\in\mathbb{P}(W)$ is γ-*semistable* if and only if the orbit closure $\overline{G\cdot w}$ of any lift $w\in W^\vee\backslash\{0\}$ does not contain 0. Denote by $\mathbb{P}(W)^{ss}_\gamma\subset\mathbb{P}(W)$ the open set of semistable points and by $\mathbb{P}(W)^{ps}_\gamma$ the set of γ-*polystable* points, i.e. the semistable points whose orbit is closed in $\mathbb{P}(W)^{ss}_\gamma$. Equivalently, a point $x\in\mathbb{P}(W)$ is polystable if and only if the orbit $G\cdot w$ of any lift $w\in W^\vee\backslash\{0\}$ is closed in $W^\vee$. With this terminology, $x\in\mathbb{P}(W)$ is γ-*stable* if and only if it is polystable and its stabilizer G_x is finite. Let $\pi_\gamma\colon\mathbb{P}(W)^{ss}_\gamma\longrightarrow Q_\gamma:=\mathbb{P}(W)/\!\!/_\gamma G$ be the categorical quotient. For sufficiently large n, Q_γ admits a projective embedding

$j_n\colon Q_\gamma \hookrightarrow \mathbb{P}(S^n W^G)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{P}(W)^{ss}_\gamma \subset \mathbb{P}(W) \arrow[r, hook, "v_n"] \arrow[d, "\pi_\gamma"'] & \mathbb{P}(S^n W) \arrow[d, dashed, "p_G"] \\ Q_\gamma \arrow[r, hook, "j_n"] & \mathbb{P}(S^n W^G) \end{tikzcd} \tag{1}$$

In this diagram, v_n stands for the n-th Veronese embedding and p_G is the projection induced by the inclusion $S^n W^G \subset S^n W$. The space Q_γ comes with a natural polarization represented by $L_n := j_n^* \mathcal{O}_{\mathbb{P}(S^n W^G)}(1)$. Indeed, by (2) we have $\pi_\gamma^* L_n \cong \mathcal{O}_{\mathbb{P}(W)^{ss}_\gamma}(n)$, and from the commutative diagram

$$\begin{tikzcd} \mathbb{P}(W)^{ss}_\gamma \subset \mathbb{P}(W) \arrow[r, hook, "v_{n_1}"] \arrow[d, "\pi_\gamma"'] & \mathbb{P}(S^{n_1} W) \arrow[rr, hook, "v_{n_2}"] & & \mathbb{P}(S^{n_1 n_2} W) \arrow[d, dashed, "p_G"'] \\ Q_\gamma \arrow[r, hook, "j_{n_1}"] & \mathbb{P}(S^{n_1} W^G) \arrow[r, hook, "v_{n_2}"] & \mathbb{P}(S^{n_2}(S^{n_1} W^G)) \arrow[r, dashed] & \mathbb{P}(S^{n_1 n_2} W^G) \end{tikzcd} \tag{2}$$

we infer $L_{n_1}^{\otimes n_2} \cong L_{n_1 n_2}$, hence

$$L_{n_1}^{\otimes n_2} \cong L_{n_2}^{\otimes n_1}, \quad \forall n_1, n_2 \quad \text{large enough.} \tag{3}$$

Remark 1.1.1. In the following, we will mainly consider actions on projective spaces. However, if X is a quasi-projective variety with an action of an algebraic group G which is linearized in an ample line bundle L, then $L^{\otimes n}$ induces, for n large enough, a G-invariant embedding of X into $\mathbb{P} := \mathbb{P}(H^0(L^{\otimes n}))$ such that the semistable, polystable, and stable points of X are mapped to the semistable, polystable, and stable points of $\mathbb{P}$. Hence all the results which we will prove hold also in this more general setting, and will be used in this generality in Section 2.

1.2. **Polarized $\mathbb{C}^*$-quotients.** Let $\lambda\colon \mathbb{C}^* \to \mathrm{GL}(W)$ be a rational representation of $\mathbb{C}^*$ in the finite dimensional vector space W and let $\overline{\lambda}\colon \mathbb{C}^* \times \mathbb{P}(W) \longrightarrow \mathbb{P}(W)$ be the induced action. The space $W^\vee$ splits as a direct sum

$$W^\vee = \bigoplus_{i=1}^m W_i^\vee,$$

where $W_i^\vee$ is the eigenspace of the character $\chi_{d_i}\colon \mathbb{C}^* \longrightarrow \mathbb{C}^*, z \longmapsto z^{d_i}$. We assume $d_1 < d_2 < \cdots < d_m$. Let $x \in \mathbb{P}(W)$ and choose a lift $w \in W^\vee \backslash \{0\}$ of x. Define

$$\begin{aligned} d^\lambda_{\min}(x) &:= \min\Big\{ d_i \mid w \text{ has a non-trivial component in } W_i^\vee \Big\} \\ d^\lambda_{\max}(x) &:= \max\Big\{ d_i \mid w \text{ has a non-trivial component in } W_i^\vee \Big\}. \end{aligned}$$

Proposition 1.2.1.
i) *A point* $x \in \mathbb{P}(W)$ *is* λ*-semistable if and only if* $d^\lambda_{\min}(x) \le 0 \le d^\lambda_{\max}(x)$.
ii) *A point* $x \in \mathbb{P}(W)$ *is* λ*-polystable if and only if either* $d^\lambda_{\min}(x) = 0 = d^\lambda_{\max}(x)$ *or* $d^\lambda_{\min}(x) < 0 < d^\lambda_{\max}(x)$.

Proof. Let $w = (w_1, ..., w_n) \in W^\vee \backslash \{0\}$ be a lift of x, where we take coordinates with respect to a basis of eigenvectors. For $z \in \mathbb{C}^*$, we get

$$z \cdot w = (0, ..., 0, z^{d^\lambda_{\min}(x)} \cdot w_{i_0}, ..., z^{d^\lambda_{\max}(x)} \cdot w_{i_r}, 0, ..., 0).$$

Using this description, the assertion becomes obvious. □

As remarked above, we can view λ as a linearization of the action $\overline{\lambda}$. There are two natural ways of changing this linearization:

1. Multiplying λ by a character: Let d be an integer, and denote by λ_d the representation $z \longmapsto z^d \cdot \lambda(z)$ of $\mathbb{C}^*$ in $\mathrm{GL}(W)$. This means that we change the $\mathcal{O}_{\mathbb{P}(W)}(1)$-linearization of $\overline{\lambda}$ by multiplying it with the character $\chi_d\colon \mathbb{C}^* \longrightarrow \mathbb{C}^*, z \longmapsto z^d$.
2. Replacing λ by a symmetric power: Let $\lambda^k\colon \mathbb{C}^* \longrightarrow \mathrm{GL}(S^k W)$ be the k-th symmetric power of λ. This induces an $\mathcal{O}_{\mathbb{P}(W)}(k)$-linearization of $\overline{\lambda}$.

Now we combine both methods, i.e., we change λ^k to the representation λ^k_d of $\mathbb{C}^*$ in $\mathrm{GL}(S^k W)$. As above, this defines an $\mathcal{O}_{\mathbb{P}(W)}(k)$-linearization of $\overline{\lambda}$. Altogether, we have a family λ^k_d, $k \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}$, of linearizations of $\overline{\lambda}$. Since two $\mathcal{O}_{\mathbb{P}(W)}(k)$-linearizations of $\overline{\lambda}$ differ by a character of $\mathbb{C}^*$, these are indeed all possible linearizations.

Every linearization λ^k_d yields a polarized GIT-quotient $\left(Q^k_d := \mathbb{P}(W)/\!\!/_{\lambda^k_d}\mathbb{C}^*, [L^k_d]\right)$, and $(Q^k_d, [L^k_d])$ and $(Q^{k'}_{d'}, [L^{k'}_{d'}])$ are isomorphic as polarized varieties when the ratios d/k and d'/k' coincide. To see this, one just has to observe that, for any positve integer t, the linearization $\lambda^{t\cdot k}_{t\cdot d}$ is the t-th symmetric power of the linearization λ^k_d.

Since for a point $x \in \mathbb{P}(W)$ we have

$$d^{\lambda^k_d}_{\min}(x) = k \cdot d^\lambda_{\min} - d, \quad d^{\lambda^k_d}_{\max}(x) = k \cdot d^\lambda_{\max} - d,$$

we obtain the following corollary to Proposition 1.2.1:

Proposition 1.2.2.
i) *The point* x *is* λ^k_d*-semistable if and only if* $d^\lambda_{\min}(x) \le d/k \le d^\lambda_{\max}(x)$.
ii) *The point* x *is* λ^k_d*-polystable if and only if either* $d^\lambda_{\min}(x) = d/k = d^\lambda_{\max}(x)$ *or* $d^\lambda_{\min}(x) < d/k < d^\lambda_{\max}(x)$. *In particular, every point* $x \in \mathbb{P}(W)$ *is* λ^k_d*-polystable for suitable numbers* $k \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}$.

For integers i with $1 \leq i \leq 2m$ we define the following intervals in $\mathbb{P}^1_{\mathbb{Q}}$:

$$I_i := \begin{cases} \mathbb{P}^1_{\mathbb{Q}} \setminus [d_m, d_1] & \text{if } i = 2m \\ \{d_{\frac{i+1}{2}}\} & \text{if } i \text{ is odd} \\ (d_{\frac{i}{2}}, d_{\frac{i}{2}+1}) & \text{if } i \text{ is even.} \end{cases}$$

Corollary 1.2.3. $\mathbb{P}(W)^{ss}_{\lambda^k_d} = \mathbb{P}(W)^{ss}_{\lambda^{k'}_{d'}}$ *if and only if there is an* i *with* $1 \leq i \leq 2m$*, such that* I_i *contains both* d/k *and* d'/k'.

We see that for the given action $\overline{\lambda}$ there are exactly $2m$ notions of stability. Denote by Q_i, $i = 1, ..., 2m$, the corresponding unpolarized GIT-quotients, where $Q_{2m} = \emptyset$. Then, for any $i = 1, ..., 2m$, there is a k with $Q_i = Q^k_2$.

Remark 1.2.4. Białynicki-Birula and Sommese [BS] investigated $\mathbb{C}^*$-actions in a more general context. Specialized to our situation, their main result is the following: Let λ be a $\mathbb{C}^*$-action on W with a decomposition of the dual space $W^\vee = \bigoplus_{i=1}^m W_i^\vee$ as above. The fixed point set of the induced $\mathbb{C}^*$-action on $\mathbb{P}(W)$ is given by $\bigcup_{i=1}^m \mathbb{P}(W_i)$. Set $F_i := \mathbb{P}(W_i)$, and define for each index i:

$$\begin{aligned} X_i^+ &:= \Big\{ x \in \mathbb{P}(W) \mid \lim_{z \longrightarrow 0} z \cdot x \in F_i \Big\} = \mathbb{P}(W_i \oplus \cdots \oplus W_m) \\ X_i^- &:= \Big\{ x \in \mathbb{P}(W) \mid \lim_{z \longrightarrow \infty} z \cdot x \in F_i \Big\} = \mathbb{P}(W_1 \oplus \cdots \oplus W_i), \end{aligned}$$

and for $i \neq j$ set $C_{ij} := (X_i^+ \setminus F_i) \cap (X_j^- \setminus F_j)$. This means C_{ij} is empty for $i \geq j$ and equal to $\mathbb{P}(W_i \oplus \cdots \oplus W_j) \setminus (\mathbb{P}(W_i) \cup \mathbb{P}(W_j))$ for $i < j$. We write $F_i < F_j$ when $C_{ij} \neq \emptyset$, i.e.

$$F_1 < F_2 < \cdots < F_m.$$

In the terminology of [BS], F_1 is the *source* and F_m is the *sink*. For each i with $1 \leq i \leq m-1$, one has a partition of $A := \{1, ..., m\}$:

$$A = A_i^- \cup A_i^+, \quad \text{with } A_i^- := \{1, ..., i\} \text{ and } A_i^+ = \{i+1, ..., m\},$$

and an associated open set

$$U_i := \bigcup_{\mu \in A_i^-, \nu \in A_i^+} C_{\mu\nu}.$$

The main theorem of [BS] asserts that the U_i are the only Zariski-open $\mathbb{C}^*$-invariant subsets of $\mathbb{P}(W)$ not intersecting the fixed point set whose quotients by the $\mathbb{C}^*$-action are compact. One checks directly that U_i is the set of λ^k_d-semistable points for any pair k, d with $d/k \in (d_i, d_{i+1})$.

Example 1.2.5. Consider an action λ of $\mathbb{C}^*$ on a finite dimensional vector space W such that the dual space decomposes as $W^\vee = W_1^\vee \oplus W_2^\vee$ with weights $d_1 < d_2$. If $d \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$ are such that $d_1 < d/k < d_2$, then the set of λ^k_d-semistable points is $\mathbb{P}(W_1 \oplus W_2)\setminus(\mathbb{P}(W_1) \cup \mathbb{P}(W_2))$ and the quotient $Q_{\lambda_d^k}$ is naturally isomorphic to $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$. The quotient map

$$\pi\colon \mathbb{P}(W_1 \oplus W_2)\setminus(\mathbb{P}(W_1) \cup \mathbb{P}(W_2)) \subset \mathbb{P}(W_1 \oplus W_2) \dashrightarrow \mathbb{P}(W_1) \times \mathbb{P}(W_2)$$

is the obvious one.

Claim 1. *The polarization induced by λ_d^k on $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$ is the equivalence class of the bundle $\mathcal{O}_{\mathbb{P}(W_1)\times\mathbb{P}(W_2)}(kd_2 - d, -kd_1 + d)$. In particular, for every $m, n \in \mathbb{Z}_{>0}$, the class $[\mathcal{O}_{\mathbb{P}(W_1)\times\mathbb{P}(W_2)}(m,n)]$ occurs as an induced polarization.*

Proof. Let $L := \mathcal{O}_{\mathbb{P}(W_1)\times\mathbb{P}(W_2)}(m,n)$ represent the induced polarization. From the description of π it follows that $H^0(\pi^* L)^{\lambda_d^k} = \pi^* H^0(L) = S^m W_1 \otimes S^n W_2$ is the set of bihomogenous polynomials of bidegree (m,n), for some m,n. If $S^m W_1 \otimes S^n W_2$ occurs as an eigenspace of the induced $\mathbb{C}^*$-action on the space $H^0(\mathcal{O}_{\mathbb{P}(W_1\oplus W_2)}(m \cdot n))$, then it must obviously be an eigenspace for the character $\chi_{-(md_1+nd_2)+((m+n)/k)d}$. Now invariance implies $md_1 + nd_2 - ((m+n)/k)d = 0$, which can be written as $m(kd_1 - d) + n(kd_2 - d) = 0$. This yields the first assertion.

To prove the second part of the claim one has to find positive integers k, r and an integer d such that the following equations hold

$$\begin{aligned} kd_2 - d &= rm \\ -kd_1 + d &= rn; \end{aligned}$$

this results from a straightforward computation.

The other quotients are $\mathbb{P}(W_1)$, $\mathbb{P}(W_2)$ with the obvious polarizations, and $\emptyset$.

1.3. **Stability for actions of products of groups.** Consider now two reductive groups G, H and a rational representation $\rho\colon G \times H \longrightarrow \mathrm{GL}(W)$ in the finite dimensional space W. We denote by γ and λ the induced representations of G and H, respectively. Choose n large enough in order to obtain an embedding $j_n\colon Q_\gamma \hookrightarrow \mathbb{P}(S^n W^G)$. Since the actions of G and H commute, λ induces actions of H on Q_γ, on $S^n W^G$, and on $\mathbb{P}(S^n W^G)$; for these actions j_n is H-equivariant. The action of H on Q_γ possesses a natural linearization in $j_n^* \mathcal{O}_{\mathbb{P}(S^n W^G)}(1)$. By 1.1(3), the corresponding concept of stability does not depend on the choice of n. Let us denote the set of semistable points by Q_γ^{ss} and the set of polystable points by Q_γ^{ps}.

Proposition 1.3.1. *The set of ρ-semistable points in the projective space $\mathbb{P}(W)$ is given by $\mathbb{P}(W)^{ss}_\rho = \mathbb{P}(W)^{ss}_\gamma \cap \pi_\gamma^{-1}(Q_\gamma^{ss})$, and there exists a natural isomorphism $Q_\gamma /\!\!/_\lambda H \cong Q_\rho$.*

Proof. Suppose $x \in \mathbb{P}(W)$ is γ-semistable and its image $\pi_\gamma(x)$ is λ-semistable in Q_γ. If n is large, $j_n(\pi_\gamma(x))$ is semistable in $\mathbb{P}(S^n W^G)$, so that there exists an integer $k \geq 1$ and a section $\overline{s} \in H^0(\mathbb{P}(S^n W^G), \mathcal{O}_{\mathbb{P}(S^n W^G)}(k))^H$ not vanishing at $j_n(\pi_\gamma(x))$. Identifying $\overline{s} \in S^k(S^nW^G)^H$ with an element of $S^{kn}W^{G\times H}$, we obtain a $G\times H$-invariant section in $\mathcal{O}_{\mathbb{P}(W)}(kn)$ not vanishing at x, hence x is ρ-semistable.

Conversely, suppose $x \in \mathbb{P}(W)^{ss}_\rho$. Then there exists, for some $m \geq 1$, a section $s \in H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(m))^{G\times H}$ with $s(x) \neq 0$. Viewing $s \in S^mW^{G\times H}$ as an H-invariant element of S^mW^G, we see that $x \in \mathbb{P}(W)^{ss}_\gamma \cap \pi_\gamma^{-1}(Q^{ss}_\gamma)$. This proves the first assertion.

The second assertion follows immediately from the first one and the universal property of the categorical quotient. □

The corresponding result for the polystable points is

Proposition 1.3.2. *The set of ρ-polystable points is $\mathbb{P}(W)^{ps}_\rho = \mathbb{P}(W)^{ps}_\gamma \cap \pi_\gamma^{-1}(Q^{ps}_\gamma)$.*

Proof. Let $x \in \mathbb{P}(W)$ be a γ-polystable point with $\pi_\gamma(x) \in Q^{ps}_\gamma$. By 1.3.1, x is ρ-semistable. Choose a ρ-polystable point $y \in \overline{(G\times H)\cdot x} \cap \mathbb{P}(W)^{ss}_\rho$. Projecting onto Q_γ, it follows that $\pi_\gamma(y)$ is contained in $\overline{H\cdot \pi_\gamma(x)}$ and hence in $H\cdot\pi_\gamma(x)$, because $\pi_\gamma(x)$ is polystable by assumption. Therefore, there exists an $h \in H$ with $\pi_\gamma(x) = h\cdot\pi_\gamma(y) = \pi_\gamma(h\cdot y)$. But this means that the closures of the G-orbits of x and $h\cdot y$ intersect, so that $G\cdot x \subset \overline{G\cdot(h\cdot y)} \cap \mathbb{P}(W)^{ss}_\gamma$, since x is γ-polystable. In particular, $x \in \overline{(G\times H)\cdot y} \cap \mathbb{P}(W)^{ss}_\rho = (G\times H)\cdot y$. Hence x is also ρ-polystable.

To prove the converse, suppose x is a ρ-polystable point. We first show that x is γ-polystable, too. Let $y \in \overline{G\cdot x} \cap \mathbb{P}(W)^{ss}_\gamma$ be a γ-polystable point. Since $\pi_\gamma(y) = \pi_\gamma(x)$, it follows from 1.3.1 that $\pi_\gamma(y) \in Q^{ss}_\gamma$. Applying 1.3.1 again, we see that $y \in \mathbb{P}(W)^{ss}_\rho$. The orbit $(G\times H)\cdot x$ being closed in $\mathbb{P}(W)^{ss}_\rho$, there exist $g \in G$ and $h \in H$ with $y = g\cdot h\cdot x$, i.e. $x = h^{-1}\cdot g^{-1}\cdot y$. Now $g^{-1}\cdot y$ is γ-polystable, hence x is γ-polystable too, because γ and λ commute. Finally, we must show that $\pi_\gamma(x) \in Q^{ps}_\gamma$. Choose y such that $\pi_\gamma(y) \in \overline{H\cdot\pi_\gamma(x)} \cap Q^{ps}_\gamma$. We may assume that y is γ-polystable. By what we have already proved, y is ρ-polystable. Now $\pi_\gamma(y)$ and $\pi_\gamma(x)$ are mapped to the same point in $Q_\gamma/\!\!/_\lambda G = Q_\rho$. But the projection $\pi_\rho\colon \mathbb{P}(W)^{ss}_\rho \longrightarrow Q_\rho$ separates closed ρ-orbits, thus $(G\times H)\cdot x = (G\times H)\cdot y$, and therefore $H\cdot\pi_\gamma(x) = H\cdot\pi_\gamma(y)$ is closed in Q^{ss}_γ. □

1.4. **Applications to $G\times\mathbb{C}^*$-actions.** Let G be a reductive algebraic group possessing only the trivial character, so that for any action of G on a projective variety V and any line bundle L on V there is at most one L-linearization of the given action. Consider a rational representation ρ of $G\times\mathbb{C}^*$ in the finite dimensional vector space W. As above we denote by γ and λ the induced representations of G and $\mathbb{C}^*$, respectively, and by $\overline{\rho}$, $\overline{\gamma}$, and $\overline{\lambda}$ the induced action of $G\times\mathbb{C}^*$, G, and $\mathbb{C}^*$ on $\mathbb{P}(W)$. Let $\mathbb{P}(W)^s_i \subset \mathbb{P}(W)^{ps}_i \subset \mathbb{P}(W)^{ss}_i$ be the set stable, polystable, or semistable points w.r.t. the i-th stability concept for the action $\overline{\lambda}$, and let I_i, $i = 1, ..., 2m$, be the associated intervals of rational numbers. The representation ρ induces an action of G on Q^k_d which is equipped with a natural linearization in the ample line bundle L_d^k, and there is no natural way to alter this linearization, because G does not possess a non-trivial character. The corresponding concept of G-stability depends only on the rational parameter d/k.

Now fix a rational parameter $\eta := d/k \in I_i$ for some index i. A point $y \in Q_i$ is called *η-stable (η-polystable, η-semistable)* if it is G-stable (G-polystable, G-semistable) w.r.t. the G-linearized line bundle L_d^k on $Q_i = Q_d^k$.

Recall that every point $x \in \mathbb{P}(W)$ lies in $\mathbb{P}(W)_i^{ps}$ for a suitable index i; let $\pi_i(x) \in Q_i$ be its image under $\pi_i\colon \mathbb{P}(W)_i^{ps} \longrightarrow Q_i$.

Theorem 1.4.1. *Fix a point $x \in \mathbb{P}(W)$. Then the following conditions are equivalent:*

i) *The point x is G-semistable (G-polystable).*

ii) *There exists an index i and a parameter $\eta \in I_i$ such that $x \in \mathbb{P}(W)_i^{ss}$ ($x \in \mathbb{P}(W)_i^{ps}$) and $\pi_i(x)$ is η-semistable (η-polystable).*

Proof. We explain the semistable case; the arguments in the polystable case are similar. Suppose first that $x \in \mathbb{P}(W)$ is G-semistable. Choose n large enough (cf. Section 1.1) in order to obtain a commutative diagram as in 1.1(2). Since γ and λ commute, the representation $\lambda^n\colon \mathbb{C}^* \longrightarrow \mathrm{GL}(S^nW)$ induces a representation $\lambda'\colon \mathbb{C}^* \longrightarrow \mathrm{GL}(S^nW^G)$. By 1.2.2, we find $k \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}$ such that $\pi_\gamma(x)$ is semistable w.r.t. the stability concept induced by $(\lambda')_d^k$ on Q_γ. Since $(\lambda')_d^k$ is induced by the representation

$$\lambda_d^{nk}\colon \mathbb{C}^* \longrightarrow \mathrm{GL}\left(S^k(S^nW)\right),$$

we may replace n by kn and, therefore, assume that $\pi_\gamma(x)$ is semistable w.r.t. the stability concept induced by $(\lambda')_d$ on Q_γ, for some integer d. We now apply Proposition 1.3.1 to the representation

$$(\gamma^n \times \lambda_d^n)\colon G \times \mathbb{C}^* \longrightarrow \mathrm{GL}(S^nW).$$

(Note that this representation induces the action $\overline{\rho}$ on $\mathbb{P}(W)$.) Since $x \in \mathbb{P}(W)$ is γ-semistable, it is also γ^n-semistable. By construction, $\pi_\gamma(x)$ is semistable w.r.t. the induced $\mathbb{C}^*$-action on Q_γ, and hence x is $\gamma^n \times \lambda_d^n$-semistable by 1.3.1. Applying 1.3.1 the other way round, setting $\eta := d/k$ and choosing i with $\eta \in I_i$, it follows that $x \in \mathbb{P}(W)_i^{ss}$ and that $\pi_i(x)$ is η-semistable. This settles the implication i)$\Rightarrow$ii).

To prove the other implication suppose $x \in \mathbb{P}(W)$ fulfills the assumptions of ii). By definition and by Proposition 1.3.1, this means that there are $k \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}$ with $\eta = d/k$ such that $x \in \mathbb{P}(W)$ is $\gamma^k \times \lambda_d^k$-semistable. This implies that x is γ^k- and hence γ-semistable. This concludes the proof. $\square$

Remark 1.4.2. At this point it becomes clear why chains of flips appear: Let G, ρ, γ, and λ be as above. We have constructed a family $(\gamma^k \times \lambda_d^k)$ of linearizations of the action $\overline{\rho}$ on $\mathbb{P}(W)$. Each of these linearizations yields a GIT-quotient of $\mathbb{P}(W)$ by the action $\overline{\rho}$. This family of quotients can be constructed in another manner: First take the G-quotient in order to obtain a polarized variety $(\widetilde{Q} := \mathbb{P}(W)/\!\!/_{\gamma}G, [L])$. The resulting $\mathbb{C}^*$-action on this variety yields a family of quotients Q_i, $i = 1, ..., 2n$, where $2n$ is usually (much) larger than $2m$, the number of *unpolarized* $\mathbb{C}^*$-quotients of $\mathbb{P}(W)$ (see 1.4.3). But the family Q_i, $i = 1, ..., 2n$, coincides with the family $\mathbb{P}(W)/\!\!/_{\gamma^k \times \lambda_d^k} G \times \mathbb{C}^*$, $k \in \mathbb{Z}_{>0}, d \in \mathbb{Z}$. This phenomenon is responsible for the occurence of chains of flips in these situations. It explains the question which was left open in [R], 2.4 Remark (2), 2.5.

Example 1.4.3. Let $W^\vee := S^3\mathbb{C}^{2^\vee} \oplus \mathbb{C}^{2^\vee}$ and let $\mathrm{SL}_2(\mathbb{C})$ act on $W^\vee$ in the following way: Given $(f, p) \in W^\vee$ and $m \in \mathrm{SL}_2(\mathbb{C})$, we interpret f and p as functions on $\mathbb{C}^2$ and set $(m \cdot f)(v) := f(m^t \cdot v)$ and $(m \cdot p)(v) := p(m^t \cdot v)$; then we define $m \cdot (f, p) := (m \cdot f, m \cdot p)$. Let $\mathbb{C}^*$ act on $W^\vee$ by multiplication with z^{d_1} on the first factor and by multiplication with z^{d_2} on the second one. The quotient $V := W^\vee/\!\!/\,\mathrm{SL}_2(\mathbb{C})$ is of the form $\mathrm{Spec}\,\mathbb{C}[I, J, D, R]$, where I, J, D, and R are certain bihomogenous polynomials of bidegrees $(2, 2)$, $(3, 3)$, $(4, 0)$, and $(1, 3)$ in the coordinates of $S^3\mathbb{C}^{2^\vee}$ and $\mathbb{C}^{2^\vee}$. Furthermore, I, D, and R are algebraically independent, and there is a relation

$$27J^2 = \frac{1}{256}DR^2 + I^3.$$

We examine the $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^*$-action on $\mathbb{P}(W)$. The quotient $Q := \mathbb{P}(W)/\!\!/\,\mathrm{SL}_2(\mathbb{C})$ is given by $\mathrm{Proj}\,\mathbb{C}[I, J, D, R]$ where I, J, D, and R have weights 4, 6, 4, and 4, respectively. The ring $\mathbb{C}[I, J, D, R]_{(12)}$ is generated by its elements in degree 1, i.e. by $I^3, I^2D, I^2R, ID^2, IR^2, IDR, J^2, D^3, D^2R, DR^2, R^3$; hence there is an embedding $Q \hookrightarrow \mathbb{P}(S^{12}W^{\mathrm{SL}_2(\mathbb{C})})$. The $\mathbb{C}^*$-action on Q can be extended to $\mathbb{P}(S^{12}W^{\mathrm{SL}_2(\mathbb{C})})$ such that the weights of the corresponding action on $S^{12}W^{\mathrm{SL}_2(\mathbb{C})^\vee}$ are

$$6d_1{+}6d_2, 8d_1{+}4d_2, 5d_1{+}7d_2, 10d_1{+}2d_2, 4d_1{+}8d_2, 7d_1{+}5d_2, 12d_1, 9d_1{+}3d_2, 3d_1{+}9d_2.$$

For a point in $p \in Q$, $d_{\min}(p)$ and $d_{\max}(p)$ can take the values $6d_1 + 6d_2$, $12d_1$, and $3d_1 + 9d_2$. Hence, for $d_1 \neq d_2$, there are 6 different notions of semistability on Q, hence 6 different notions of $\mathrm{SL}_2 \times \mathbb{C}^*$-semistability on $\mathbb{P}(W)$, whereas there are only 4 different notions of $\mathbb{C}^*$-semistability on $\mathbb{P}(W)$.

2. Oriented pairs and their moduli

Let X be a smooth projective variety over the field of complex numbers and fix an ample divisor H on X. All degrees will be taken with respect to H and the corresponding line bundle will be denoted by $\mathcal{O}_X(1)$. Fix a torsion free coherent sheaf $\mathcal{E}_0$ and a Hilbert polynomial P. Finally, let $\mathrm{Pic}(X)$ be the Picard scheme of X and choose a Poincaré line bundle $\mathcal{L}$ over $\mathrm{Pic}(X) \times X$. If S is a scheme and $\mathfrak{E}_S$ a flat family of coherent sheaves over $S \times X$, then there is a morphism $\det_S \colon S \longrightarrow \mathrm{Pic}(X)$ mapping a closed point s to $[\det(\mathfrak{E}_{S|\{s\} \times X})]$.

We set $\mathcal{L}[\mathfrak{E}_S] := (\det_S \times \mathrm{id})^*(\mathcal{L})$; this line bundle depends only on the isomorphism class of the family $\mathfrak{E}_S$. The Hilbert polynomial of a sheaf $\mathcal{F}$ will be denoted by $P_{\mathcal{F}}$. For any non-trivial torsion free coherent sheaf $\mathcal{F}$ there is a unique subsheaf $\mathcal{F}_{\max}$ for which $P_{\mathcal{F}}/\operatorname{rk}\mathcal{F}$ is maximal and whose rank is maximal among the subsheaves $\mathcal{F}'$ with $P_{\mathcal{F}'}/\operatorname{rk}\mathcal{F}'$ maximal. Set $\mu_{\max}(\mathcal{F}) := \mu(\mathcal{F}_{\max})$.

2.1. **Oriented pairs.** An *oriented pair of type* $(P, \mathcal{L}, \mathcal{E}_0)$ is a triple $(\mathcal{E}, \varepsilon, \varphi)$ consisting of a torsion free coherent sheaf $\mathcal{E}$ with Hilbert polynomial $P_{\mathcal{E}} = P$, a homomorphism $\varepsilon\colon \det\mathcal{E} \longrightarrow \mathcal{L}[\mathcal{E}]$, and a homomorphism $\varphi\colon \mathcal{E} \longrightarrow \mathcal{E}_0$. The homomorphisms ε and φ will be called the *orientation* and the *framing* of the pair $(\mathcal{E}, \varepsilon, \varphi)$. Two oriented pairs $(\mathcal{E}_1, \varepsilon_1, \varphi_1)$ and $(\mathcal{E}_2, \varepsilon_2, \varphi_2)$ are said to be *equivalent*, if there is an isomorphism $\Psi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ with $\varepsilon_1 = \varepsilon_2 \circ \det\Psi$ and $\varphi_1 = \varphi_2 \circ \Psi$. When $\ker(\varphi) \neq 0$, we set

$$\delta_{\mathcal{E},\varphi} := P_{\mathcal{E}} - \frac{\operatorname{rk}\mathcal{E}}{\operatorname{rk}\ker(\varphi)_{\max}} P_{\ker(\varphi)_{\max}}.$$

An oriented pair $(\mathcal{E}, \varepsilon, \varphi)$ of type $(P, \mathcal{L}, \mathcal{E}_0)$ is *semistable*, if either φ is injective, or ε is an isomorphism, $\ker(\varphi) \neq 0$, $\delta_{\mathcal{E},\varphi} \geq 0$, and for all non-trivial subsheaves $\mathcal{F} \subset \mathcal{E}$

$$\frac{P_{\mathcal{F}}}{\operatorname{rk}\mathcal{F}} - \frac{\delta_{\mathcal{E},\varphi}}{\operatorname{rk}\mathcal{F}} \quad \leq \quad \frac{P_{\mathcal{E}}}{\operatorname{rk}\mathcal{E}} - \frac{\delta_{\mathcal{E},\varphi}}{\operatorname{rk}\mathcal{E}}.$$

The corresponding stability concept is slightly more complicated: An oriented pair $(\mathcal{E}, \varepsilon, \varphi)$ of type $(P, \mathcal{L}, \mathcal{E}_0)$ is *stable*, if either φ is injective, or ε is an isomorphism, $\ker(\varphi) \neq 0$, $\delta_{\mathcal{E},\varphi} > 0$, and one of the following conditions holds:

1. For all non-trivial proper subsheaves $\mathcal{F} \subset \mathcal{E}$:

$$\frac{P_{\mathcal{F}}}{\operatorname{rk}\mathcal{F}} - \frac{\delta_{\mathcal{E},\varphi}}{\operatorname{rk}\mathcal{F}} \quad < \quad \frac{P_{\mathcal{E}}}{\operatorname{rk}\mathcal{E}} - \frac{\delta_{\mathcal{E},\varphi}}{\operatorname{rk}\mathcal{E}}.$$

2. $\varphi \neq 0$, $\ker(\varphi)_{\max}$ is stable, and $\mathcal{E} \cong \ker(\varphi)_{\max} \oplus \mathcal{E}'$, where the pair $(\mathcal{E}', \varphi)$ satisfies

$$\begin{aligned} \frac{P_{\mathcal{F}}}{\operatorname{rk}\mathcal{F}} - \frac{\delta_{\mathcal{E},\varphi}}{\operatorname{rk}\mathcal{F}} \quad &< \quad \frac{P_{\mathcal{E}'}}{\operatorname{rk}\mathcal{E}'} - \frac{\delta_{\mathcal{E},\varphi}}{\operatorname{rk}\mathcal{E}'} \qquad \forall \text{ proper subsheaves } 0 \neq \mathcal{F} \subset \mathcal{E}' \ , \\ \frac{P_{\mathcal{F}}}{\operatorname{rk}\mathcal{F}} \quad &< \quad \frac{P_{\mathcal{E}'}}{\operatorname{rk}\mathcal{E}'} - \frac{\delta_{\mathcal{E},\varphi}}{\operatorname{rk}\mathcal{E}'} \qquad \forall \text{ proper subsheaves } 0 \neq \mathcal{F} \subset \mathcal{E}' \cap \ker(\varphi). \end{aligned}$$

Our (semi)stability concept is related to the *parameter dependent* (semi)stability concept of [HL1] and [HL2] in the following way: Let δ be a polynomial over the rationals with positive leading coefficient. Recall that a pair $(\mathcal{E}, \varphi)$ consisting of a torsion free coherent sheaf $\mathcal{E}$ with $P_{\mathcal{E}} = P$ and a non-zero homomorphism $\varphi\colon \mathcal{E} \longrightarrow \mathcal{E}_0$ is called *(semi)stable w.r.t.* δ, if for any non-trivial proper subsheaf

$\mathcal{F} \subset \mathcal{E}$ the following conditions hold:

$$\begin{aligned} \frac{P_{\mathcal{F}}}{\operatorname{rk}\mathcal{F}} - \frac{\delta}{\operatorname{rk}\mathcal{F}} \quad &(\leq) \quad \frac{P_{\mathcal{E}}}{\operatorname{rk}\mathcal{E}} - \frac{\delta}{\operatorname{rk}\mathcal{E}} \ , \\ \frac{P_{\mathcal{F}}}{\operatorname{rk}\mathcal{F}} \quad &(\leq) \quad \frac{P_{\mathcal{E}}}{\operatorname{rk}\mathcal{E}} - \frac{\delta}{\operatorname{rk}\mathcal{E}}, \qquad \text{when } \mathcal{F} \subset \ker(\varphi). \end{aligned}$$

In this terminology, (semi)stable oriented pairs can be characterized as follows:

Lemma 2.1.1. i) *An oriented pair $(\mathcal{E}, \varepsilon, \varphi)$ is semistable if and only if it satisfies one of the following three conditions:*

1. *φ is injective.*
2. *$\mathcal{E}$ is semistable and ε is an isomorphism.*
3. *$\varphi \neq 0$, ε is an isomorphism, and $(\mathcal{E}, \varphi)$ is semistable w.r.t. some $\delta > 0$.*

ii) *An oriented pair $(\mathcal{E}, \varepsilon, \varphi)$ is stable if and only if it satisfies one of the following four conditions:*

1. *φ is injective.*
2. *$\mathcal{E}$ is stable and ε is an isomorphism.*
3. *$\varphi \neq 0$, ε is an isomorphism, and $(\mathcal{E}, \varphi)$ is stable w.r.t. some $\delta > 0$.*
4. *$\varphi \neq 0$, $\delta_{\mathcal{E},\varphi} > 0$, ε is an isomorphism, and $\mathcal{E}$ splits as $\ker(\varphi)_{\max} \oplus \mathcal{E}'$, where $\ker(\varphi)_{\max}$ is stable and $(\mathcal{E}', \varphi)$ is stable w.r.t. $\delta_{\mathcal{E},\varphi}$.*

We note that the stable oriented pairs appearing in Lemma 2.1.1.ii)4. are precisely those pairs $(\mathcal{E}, \varepsilon, \varphi)$, for which ε is isomorphic, $\varphi \neq 0$, $\delta_{\mathcal{E},\varphi} > 0$, the pair $(\mathcal{E}, \varphi)$ is polystable w.r.t. $\delta_{\mathcal{E},\varphi}$, and which have *only finitely many automorphisms*. To see this, recall from [HL2] that for a given $\delta \in \mathbb{Q}[x]$, $\delta > 0$, the polystable pairs $(\mathcal{E}, \varphi)$ are those for which $\mathcal{E}$ splits in the form

$$\mathcal{E} \cong \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{s-1} \oplus \mathcal{E}_s \ ,$$

where the sheaves $\mathcal{E}_1, ..., \mathcal{E}_{s-1}$ are stable subsheaves of $\ker(\varphi)$, $(\mathcal{E}_s, \varphi)$ is a stable pair w.r.t. δ, and $P_{\mathcal{E}_1}/\operatorname{rk}\mathcal{E}_1 = \cdots = P_{\mathcal{E}_{s-1}}/\operatorname{rk}\mathcal{E}_{s-1} = P_{\mathcal{E}_s}/\operatorname{rk}\mathcal{E}_s - \delta/\operatorname{rk}\mathcal{E}_s$.

This makes our assertion obvious.

Remark 2.1.2. Let $(\mathcal{E}, \varepsilon, \varphi)$ be a stable oriented pair of type 4. (see 2.1.1.ii)). Then $\delta_{\mathcal{E},\varphi}$ is the only rational polynomial with positive leading coefficient w.r.t. which the pair $(\mathcal{E}, \varphi)$ is semistable. This follows from the equalities

$$\begin{aligned} \frac{P_{\mathcal{E}'}}{\operatorname{rk}\mathcal{E}'} - \frac{\delta_{\mathcal{E},\varphi}}{\operatorname{rk}\mathcal{E}'} \quad &= \quad \frac{P_{\mathcal{E}}}{\operatorname{rk}\mathcal{E}} - \frac{\delta_{\mathcal{E},\varphi}}{\operatorname{rk}\mathcal{E}} \ , \\ \frac{P_{\ker(\varphi)_{\max}}}{\operatorname{rk}\ker(\varphi)_{\max}} \quad &= \quad \frac{P_{\mathcal{E}}}{\operatorname{rk}\mathcal{E}} - \frac{\delta_{\mathcal{E},\varphi}}{\operatorname{rk}\mathcal{E}} \ . \end{aligned}$$

For all stability concepts introduced so far, there are analogous notions of *slope-(semi)stability*. As usual, slope-stability implies stability and semistability implies slope-semistability.

Let S be a noetherian scheme. *A family of oriented pairs parametrized by S* is a quadruple $(\mathfrak{E}_S, \varepsilon_S, \widehat{\varphi}_S, \mathfrak{M}_S)$ consisting of a flat family $\mathfrak{E}_S$ of torsion free coherent sheaves over the product $S\times X$, an invertible sheaf $\mathfrak{M}_S$ on S, a morphism $\varepsilon_S\colon \det\mathfrak{E}_S \to \mathcal{L}[\mathfrak{E}_S]\otimes\pi_S^*\mathfrak{M}_S$, and a morphism $\widehat{\varphi}_S\colon S^r\mathfrak{E}_S \to \pi_X^* S^r\mathcal{E}_0\otimes\pi_S^*\mathfrak{M}_S$ with $\widehat{\varphi}_{S|\{s\}\times X} = S^r\varphi_s$ for any closed point $s\in S$ and a suitable $\varphi_s\in \mathrm{Hom}(\mathfrak{E}_{S|\{s\}\times X},\mathcal{E}_0)$, so that the pair $(\varepsilon_{S|\{s\}\times X}, \widehat{\varphi}_{S|\{s\}\times X})$ is non-zero.

Two families $(\mathfrak{E}_S^i, \varepsilon_S^i, \widehat{\varphi}_S^i, \mathfrak{M}_S^i)$, $i=1,2$, are called *equivalent*, if there exist an isomorphism $\Psi_S\colon \mathfrak{E}_S^1 \longrightarrow \mathfrak{E}_S^2$ and an isomorphism $\mathfrak{m}\colon \mathfrak{M}_S^1 \longrightarrow \mathfrak{M}_S^2$ such that $(\mathrm{id}_{\mathcal{L}[\mathfrak{E}_S^1]}\otimes\pi_S^*\mathfrak{m})\circ\varepsilon_S^1 = \varepsilon_S^2\circ\det\Psi$ and $(\mathrm{id}_{\pi_X^* S^r\mathcal{E}_0}\otimes\pi_S^*\mathfrak{m})\circ\widehat{\varphi}_S^1 = \widehat{\varphi}_S^2\circ S^r\Psi$.

With these notions, we define the functors $M^{ss}_{(P,\mathcal{L},\mathcal{E}_0)}$ and $M^{s}_{(P,\mathcal{L},\mathcal{E}_0)}$ of equivalence classes of families of semistable and stable oriented pairs of type $(P,\mathcal{L},\mathcal{E}_0)$.

Remark 2.1.3. Though the definition of a family may appear a little odd at first sight, it will become clear that families must be defined in this way for technical reasons. Families of the above type are precisely those which are locally induced by the universal family on the parameter space which we will construct in Section 2.3.

The functors defined above do depend on the choice of the Poincaré bundle and there is no natural way to compare functors associated to different Poincaré bundles.

2.2. **A boundedness result.** Here we show that the family of isomorphism classes of torsion free coherent sheaves occuring in oriented slope-semistable pairs of type $(P,\mathcal{L},\mathcal{E}_0)$ is bounded. We use Maruyama's boundedness criterion:

Theorem 2.2.1. [M2] *Let C be some constant. The set of isomorphism classes of torsion free coherent sheaves with Hilbert polynomial P and $\mu_{\max}\le C$ is bounded.*

Proposition 2.2.2. *The set of isomorphism classes of torsion free sheaves occuring in a slope-semistable oriented pair of type $(P,\mathcal{L},\mathcal{E}_0)$ is bounded.*

Proof. Set $C := \max\{\,\mu_{\max}(\mathcal{E}_0), \mu(\mathcal{E})\,\}$. Let $(\mathcal{E},\varepsilon,\varphi)$ be a slope-semistable oriented pair of type $(P,\mathcal{L},\mathcal{E}_0)$. We claim that $\mu_{\max}(\mathcal{E})\le C$; in view of Maruyama's theorem, this assertion proves the proposition.

Write a given non-trivial subsheaf $\mathcal{F}$ of $\mathcal{E}$ as an extension

$$0\longrightarrow \mathcal{F}\cap\ker(\varphi)\longrightarrow\mathcal{F}\longrightarrow\varphi(\mathcal{F})\longrightarrow 0.$$

If $\mathcal{F}$ is entirely contained in the kernel of φ, the definition of slope-semistability implies $\mu(\mathcal{F})\le\mu(\mathcal{E})\le C$. If $\mathcal{F}$ is isomorphic to $\varphi(\mathcal{F})$, then obviously $\mu(\mathcal{F})\le \mu_{\max}(\mathcal{E}_0)\le C$. In the remaining cases

$$\begin{aligned}\mu(\mathcal{F}) &= \frac{\mu(\mathcal{F}\cap\ker(\varphi))\,\mathrm{rk}(\mathcal{F}\cap\ker(\varphi)) + \mu(\varphi(\mathcal{F}))\,\mathrm{rk}\,\varphi(\mathcal{F})}{\mathrm{rk}\,\mathcal{F}}\\ &\le \frac{\mathrm{rk}(\mathcal{F}\cap\ker(\varphi))}{\mathrm{rk}\,\mathcal{F}}\mu(\mathcal{E}) + \frac{\mathrm{rk}\,\varphi(\mathcal{F})}{\mathrm{rk}\,\mathcal{F}}\mu_{\max}(\mathcal{E}_0)\le C.\end{aligned}$$

□

2.3. **The parameter space for semistable oriented pairs.** By the boundedness result of the previous paragraph, there is a natural number m_0 such that for all torsion free coherent sheaves $\mathcal{E}$ occuring in a semistable oriented pair, and for all $m \geq m_0$ the following properties hold true: $\mathcal{E}(m)$ is globally generated and $H^i(X, \mathcal{E}(m)) = 0$ for $i > 0$. Let V be a complex vector space of dimension $p := P(m)$. There exists a quasi-projective scheme $\mathfrak{Q}$, the Quot-scheme of torsion free coherent quotient sheaves of $V \otimes \mathcal{O}_X(-m)$ with Hilbert polynomial P, and a universal quotient on $\mathfrak{Q} \times X$:

$$q_{\mathfrak{Q}} \colon V \otimes \pi_X^* \mathcal{O}_X(-m) \longrightarrow \mathfrak{E}_{\mathfrak{Q}}.$$

Let $\mathcal{N}$ be the sheaf $\pi_{\mathfrak{Q}*}(\det(\mathfrak{E}_{\mathfrak{Q}})^\vee \otimes \mathcal{L}[\mathfrak{E}_{\mathfrak{Q}}])$. By the universal property of the Picard scheme, there is a line bundle $\mathfrak{M}$ on $\mathfrak{Q}$ such that

$$\det(\mathfrak{E}_{\mathfrak{Q}})^\vee \otimes \mathcal{L}[\mathfrak{E}_{\mathfrak{Q}}] \cong \pi_{\mathfrak{Q}}^* \mathfrak{M}.$$

This implies that $\mathcal{N}$ is invertible and

$$\mathcal{N}\langle [q] \rangle \cong H^0(X, \det(\mathfrak{E}^\vee_{\mathfrak{Q}|\{[q]\}\times X}) \otimes \mathcal{L}[\mathfrak{E}_{\mathfrak{Q}|\{[q]\}\times X}]) \ .$$

Let $\mathfrak{N} \longrightarrow \mathfrak{Q}$ be the associated geometric line bundle. The space $\mathfrak{N}$ is a parameter space for equivalence classes $[q \colon V \otimes \mathcal{O}_X(-m) \longrightarrow \mathcal{E}, \varepsilon]$ consisting of a quotient $q \colon V \otimes \mathcal{O}_X(-m) \longrightarrow \mathcal{E}$ and an orientation $\varepsilon \colon \det(\mathcal{E}) \longrightarrow \mathcal{L}[\mathcal{E}]$. Here two objects $(q_i \colon V \otimes \mathcal{O}_X(-m) \longrightarrow \mathcal{E}_i, \varepsilon_i)$, $i = 1, 2$, are *equivalent*, if there is an isomorphism $\Psi \colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ with $\Psi \circ q_1 = q_2$ and $\varepsilon_1 = \varepsilon_2 \circ \det(\Psi)$.

Next we have to construct a parameter space for all oriented pairs. We choose $m \geq m_0$ so large that $\mathcal{E}_0(m)$ is also globally generated. Every oriented pair yields an element in $K := \mathrm{Hom}(V, H^0(\mathcal{E}_0(m)))$ and hence an element in $S^r K$.

On the projective bundle $\mathfrak{P} := \mathbb{P}((\mathfrak{N} \times S^r K)^\vee) \xrightarrow{\mathfrak{p}} \mathfrak{Q}$ there is a (nowhere vanishing) tautological section

$$\mathfrak{s} \colon \mathcal{O}_{\mathfrak{P}} \longrightarrow \mathfrak{p}^*(\mathcal{N} \oplus (S^r K \otimes \mathcal{O}_{\mathfrak{Q}})) \otimes \mathcal{O}_{\mathfrak{P}}(1).$$

Let

$$q_{\mathfrak{P}} \colon V \otimes \pi_X^* \mathcal{O}_X(-m) \longrightarrow \mathfrak{E}_{\mathfrak{P}}$$

be the pullback of the universal quotient on $\mathfrak{Q} \times X$ to $\mathfrak{P} \times X$. We view the pullback $\pi_{\mathfrak{P}}^* \mathfrak{s}$ of $\mathfrak{s}$ to $\mathfrak{P} \times X$ as a pair consisting of a homomorphism

$$\varepsilon_{\mathfrak{P}} \colon \det(\mathfrak{E}_{\mathfrak{P}}) \longrightarrow \mathcal{L}[\mathfrak{E}_{\mathfrak{P}}] \otimes \pi_{\mathfrak{P}}^* \mathcal{O}_{\mathfrak{P}}(1)$$

and a homomorphism

$$\kappa_{\mathfrak{P}} \colon S^r V \otimes \mathcal{O}_{\mathfrak{P}\times X} \longrightarrow S^r H^0(\mathcal{E}_0(m)) \otimes \pi_{\mathfrak{P}}^* \mathcal{O}_{\mathfrak{P}}(1).$$

Remark 2.3.1. For a scheme S, giving a morphism $f \colon S \longrightarrow \mathfrak{P}$ is equivalent to giving a map $\overline{f} \colon S \longrightarrow \mathfrak{Q}$ - which yields the family $\mathfrak{E}_S := (\overline{f} \times \mathrm{id}_X)^* \mathfrak{E}_{\mathfrak{Q}}$ - , a line bundle $\mathfrak{M}_S$ on S, and homomorphisms

$$\varepsilon_S \colon \det(\mathfrak{E}_S) \longrightarrow \mathcal{L}[\mathfrak{E}_S] \otimes \pi_S^* \mathfrak{M}_S \ ,$$

$$\kappa_S \colon S^r V \otimes \mathcal{O}_{S\times X} \longrightarrow S^r H^0(\mathcal{E}_0(m)) \otimes \pi_S^* \mathfrak{M}_S$$

on $S \times X$ such that the pair $(\varepsilon_{S|\{s\}\times X}, \kappa_{S|\{s\}\times X})$ is non-zero for every closed point $s \in S$. Of course, for the morphism f determined by $\overline{f}$ and $(\varepsilon_S, \kappa_S, \mathfrak{M}_S)$, we have $\overline{f} = \mathfrak{p} \circ f$, and there is an isomorphism $\mathfrak{m}\colon \mathfrak{M}_S \longrightarrow \overline{f}^*\mathcal{O}_{\mathfrak{P}}(1)$ such that

$$(\mathrm{id}_{\mathcal{L}[\mathfrak{E}_S]} \otimes \pi_S^*\mathfrak{m}) \circ \varepsilon_S = (f \times \mathrm{id}_X)^*(\varepsilon_{\mathfrak{P}}) \ ,$$

$$(\mathrm{id}_{\pi_X^* S^r H^0(\mathcal{E}_0(m))} \otimes \pi_S^*\mathfrak{m}) \circ \kappa_S = (f \times \mathrm{id}_X)^*(\kappa_{\mathfrak{P}}) \ .$$

Our parameter space $\mathfrak{B}$ will be a closed subscheme of $\mathfrak{P}$ whose closed points are of the form $[[q\colon V \otimes \mathcal{O}_X(-m) \longrightarrow \mathcal{E}, \varepsilon], S^r k]$, with $[q, \varepsilon] \in \mathfrak{N}$ and $k \in K$, such that there is a map $\varphi\colon \mathcal{E} \longrightarrow \mathcal{E}_0$ making the following diagramm commutative:

$$\begin{tikzcd} V \otimes \mathcal{O}_X(-m) \arrow[r, "q"] \arrow[d, "k"'] & \mathcal{E} \arrow[d, "\varphi"] \\ H^0(\mathcal{E}_0(m)) \otimes \mathcal{O}_X(-m) \arrow[r, "ev"] & \mathcal{E}_0 \end{tikzcd}$$

Scheme-theoretically, $\mathfrak{B}$ is constructed as follows: On $\mathfrak{P} \times X$, there is a homomorphism

$$\overline{\varphi}_{\mathfrak{P}}\colon S^r V \otimes \pi_X^*\mathcal{O}_X(-rm) \longrightarrow \pi_X^* S^r \mathcal{E}_0 \otimes \pi_{\mathfrak{P}}^*\mathcal{O}_{\mathfrak{P}}(1).$$

Set $\widehat{\mathcal{G}} := \ker(S^r q_{\mathfrak{P}})$, choose $n \geq m$ large enough so that $\widehat{\mathcal{G}}_{|\{b\}\times X}(n)$ is globally generated and without higher cohomology for any closed point $b \in \mathfrak{P}$, and let

$$\widehat{\gamma}\colon \mathcal{G} := \widehat{\mathcal{G}} \otimes \pi_X^*\mathcal{O}_X(n) \longrightarrow \pi_X^* S^r \mathcal{E}_0(n) \otimes \pi_{\mathfrak{P}}^*\mathcal{O}_{\mathfrak{P}}(1)$$

be the induced homomorphism. We first define a scheme $\widehat{\mathfrak{B}}$ whose closed points are those elements $b \in \mathfrak{P}$ for which $\widehat{\gamma}_{|\{b\}\times X}$ is the zero map. Since $\mathcal{G}_{|\{b\}\times X}$ and $S^r\mathcal{E}_0(n)$ are globally generated for any closed point $b \in \mathfrak{P}$, the scheme $\widehat{\mathfrak{B}}$ is the zero locus of the following homomorphism between locally free sheaves:

$$\gamma := \pi_{\mathfrak{P}*}(\widehat{\gamma})\colon \pi_{\mathfrak{P}*}\mathcal{G} \longrightarrow \pi_{\mathfrak{P}*}(\pi_X^* S^r \mathcal{E}_0(n) \otimes \pi_{\mathfrak{P}}^*\mathcal{O}_{\mathfrak{P}}(1)) = H^0(S^r\mathcal{E}_0(n)) \otimes \mathcal{O}_{\mathfrak{P}}(1).$$

The scheme $\mathfrak{B}$ we are looking for is the scheme-theoretic intersection of $\widehat{\mathfrak{B}}$ with the image in $\mathfrak{P}$ of the weighted projective bundle associated with the vector bundle $\mathfrak{N} \times K$ over $\mathfrak{Q}$. There exists a universal family $(\mathfrak{E}_{\mathfrak{B}}, \varepsilon_{\mathfrak{B}}, \widehat{\varphi}_{\mathfrak{B}}, \mathfrak{M}_{\mathfrak{B}})$: $\mathfrak{M}_{\mathfrak{B}}$ is the restriction of $\mathcal{O}_{\mathfrak{P}}(1)$ to $\mathfrak{B}$, $q_{\mathfrak{B}}$ and $\varepsilon_{\mathfrak{B}}$ are the restrictions of $q_{\mathfrak{P}}$ and $\varepsilon_{\mathfrak{P}}$, and $\widehat{\varphi}_{\mathfrak{B}}$ is induced by the restriction of $\widehat{\varphi}_{\mathfrak{P}}$ which factorizes through $S^r\mathfrak{E}_{\mathfrak{B}}$ by definition. In the following, a closed point $b = [[q\colon V\otimes \longrightarrow \mathcal{O}_X(-m), \varepsilon], S^r k] \in \mathfrak{B}$ will be denoted by $[q, \varepsilon, \varphi]$; here φ is the unique framing on $\mathcal{E}$ induced by k.

Remark 2.3.2. By construction, a morphism $\widehat{f}\colon S \longrightarrow \mathfrak{P}$ factorizes through $\mathfrak{B}$ if and only if it factorizes through the image of the associated weighted projective bundle of $\mathfrak{N} \times K$, and $(\widehat{f} \times \mathrm{id}_X)^*(\widehat{\varphi}_{\mathfrak{P}})$ is identically zero on the kernel of the map $(\widehat{f} \times \mathrm{id}_X)^*(S^r q_{\mathfrak{P}})$.

On the parameter space $\mathfrak{B}$, there is a natural action (from the right) of the group $\mathrm{SL}(V)$. To define this action, it suffices to construct a $\mathrm{SL}(V)$-action on $\mathfrak{P}$ which leaves the scheme $\mathfrak{B}$ invariant. The standard representation of $\mathrm{SL}(V)$ on V gives us the homomorphism

$$\Gamma\colon V\otimes \mathcal{O}_{\mathfrak{Q}\times \mathrm{SL}(V)\times X}\longrightarrow V\otimes \mathcal{O}_{\mathfrak{Q}\times \mathrm{SL}(V)\times X}.$$

Moreover, on $\mathfrak{Q}\times \mathrm{SL}(V)\times X$ there is the pullback of the universal quotient

$$\pi^*_{\mathfrak{Q}\times X}(q_{\mathfrak{Q}})\colon V\otimes \pi_X^*\mathcal{O}_X(-m)\longrightarrow \pi^*_{\mathfrak{Q}\times X}\mathfrak{E}_{\mathfrak{Q}}.$$

By the universal property of the Quot-scheme, $\pi^*_{\mathfrak{Q}\times X}(q_{\mathfrak{Q}})\circ\Big(\Gamma\otimes \mathrm{id}_{\pi_X^*\mathcal{O}_X(-m)}\Big)$ yields a morphism $\overline{f}\colon \mathfrak{Q}\times\mathrm{SL}(V)\longrightarrow \mathfrak{Q}$ such that there is a well-defined isomorphism

$$\Psi_{\mathfrak{Q}\times\mathrm{SL}(V)}\colon (\overline{f}\times \mathrm{id}_X)^*\mathfrak{E}_{\mathfrak{Q}}\longrightarrow \pi^*_{\mathfrak{Q}\times X}\mathfrak{E}_{\mathfrak{Q}}$$

with $\Psi_{\mathfrak{Q}\times\mathrm{SL}(V)}\circ(\overline{f}\times\mathrm{id}_X)^*(q_{\mathfrak{Q}})=\pi^*_{\mathfrak{Q}\times X}(q_{\mathfrak{Q}})\circ\Big(\Gamma\otimes\mathrm{id}_{\pi_X^*\mathcal{O}_X(-m)}\Big)$. Let $\Psi_{\mathfrak{P}\times\mathrm{SL}(V)}$ be the pullback of $\Psi_{\mathfrak{Q}\times\mathrm{SL}(V)}$ to $\mathfrak{P}\times\mathrm{SL}(V)\times X$, and set $\mathfrak{M}_{\mathfrak{P}\times\mathrm{SL}(V)}:=\pi_{\mathfrak{P}}^*\mathcal{O}_{\mathfrak{P}}(1)$,

$$\begin{aligned}\varepsilon_{\mathfrak{P}\times\mathrm{SL}(V)} &:= \pi^*_{\mathfrak{P}\times X}(\varepsilon_{\mathfrak{P}})\circ\det\Psi_{\mathfrak{P}\times\mathrm{SL}(V)}\ ,\\ \kappa_{\mathfrak{P}\times\mathrm{SL}(V)} &:= \pi^*_{\mathfrak{P}\times X}(\kappa_{\mathfrak{P}})\circ S^r\Big((\mathfrak{p}\times\mathrm{id}_{\mathrm{SL}(V)\times X})^*\Gamma\Big)\ .\end{aligned}$$

By Remark 2.3.1, the data $\overline{f}$ and $(\varepsilon_{\mathfrak{P}\times\mathrm{SL}(V)},\kappa_{\mathfrak{P}\times\mathrm{SL}(V)},\mathfrak{M}_{\mathfrak{P}\times\mathrm{SL}(V)})$ define an action

$$f\colon \mathfrak{P}\times\mathrm{SL}(V)\longrightarrow\mathfrak{P}.$$

Proposition 2.3.3. *Let S be a noetherian scheme and let $(\mathfrak{E}_S,\varepsilon_S,\widehat{\varphi}_S,\mathfrak{M}_S)$ be a family of semistable oriented pairs parametrized by S. Then S can be covered by open subschemes S_i for which there exist morphisms $\beta_i\colon S_i\longrightarrow\mathfrak{B}$ such that the restricted families $(\mathfrak{E}_{S|S_i},\varepsilon_{S|S_i},\widehat{\varphi}_{S|S_i},\mathfrak{M}_{S|S_i})$ are equivalent to the pullbacks of $(\mathfrak{E}_{\mathfrak{B}},\varepsilon_{\mathfrak{B}},\widehat{\varphi}_{\mathfrak{B}},\mathfrak{M}_{\mathfrak{B}})$ via the maps $\beta_i\times\mathrm{id}_X$.*

Proof. The scheme S can be covered by open subschemes S_i such that the family $\mathfrak{E}_{S|S_i}$ over $S_i\times X$ can be written as a family of quotients:

$$q_{S_i}\colon V\otimes\pi_X^*\mathcal{O}_X(-m)\longrightarrow\mathfrak{E}_{S|S_i}.$$

Each q_{S_i} defines a morphism $\overline{f}_i\colon S_i\longrightarrow\mathfrak{Q}$ such that there is a well defined isomorphism $\Psi_{S_i}\colon\mathfrak{E}_{S_i}:=(\overline{f}_i\times\mathrm{id}_X)^*\mathfrak{E}_{\mathfrak{Q}}\longrightarrow\mathfrak{E}_{S|S_i}$. Define $\mathfrak{M}_{S_i}:=\mathfrak{M}_{S|S_i}$,

$$\begin{aligned}\varepsilon_{S_i}\colon&\quad \det(\mathfrak{E}_{S_i})\xrightarrow{\det\Psi_{S_i}}\det\mathfrak{E}_{S|S_i}\xrightarrow{\varepsilon_{S|S_i}}\mathcal{L}[\mathfrak{E}_{S|S_i}]\otimes\pi_{S_i}^*\mathfrak{M}_{S_i}\ ,\\ \widehat{\varphi}_{S_i}\colon&\quad S^r\mathfrak{E}_{S_i}\xrightarrow{S^r\Psi_{S_i}}S^r\mathfrak{E}_{S|S_i}\xrightarrow{\widehat{\varphi}_{S|S_i}}\pi_X^*S^r\mathcal{E}_0\otimes\pi_{S_i}^*\mathfrak{M}_{S_i}.\end{aligned}$$

The homomorphism $\widehat{\varphi}_{S_i}$ yields a homomorphism

$$\overline{\kappa}_{S_i}\colon S^rV\otimes\mathcal{O}_{S_i\times X}\longrightarrow\pi_X^*S^r\mathcal{E}_0(m)\otimes\pi_{S_i}^*\mathfrak{M}_{S_i}$$

and hence a homomorphism

$$\kappa_{S_i}:=\pi_{S_i}^*\pi_{S_i*}(\overline{\kappa}_{S_i})\colon S^rV\otimes\mathcal{O}_{S_i\times X}\longrightarrow S^rH^0(\mathcal{E}_0(m))\otimes\pi_{S_i}^*\mathfrak{M}_{S_i};$$

here we have used the fact that our definition of a family implies that the map

$$\pi_{S_i*}(\overline{\kappa}_{S_i})\colon S^rV\otimes\mathcal{O}_{S_i}\longrightarrow H^0(S^r\mathcal{E}_0(m))\otimes\mathfrak{M}_{S_i}$$

factorizes through $S^rH^0(\mathcal{E}_0(m))\otimes\mathfrak{M}_{S_i}$.

By Remark 2.3.1, the quadruple $(\overline{f}_i,\varepsilon_{S_i},\kappa_{S_i},\mathfrak{M}_{S_i})$ determines a morphism $\beta_i\colon S_i\longrightarrow\mathfrak{P}$. It is clear that the morphism β_i factorizes through $\mathfrak{B}$ and that the family $(\mathfrak{E}_{S_i},\varepsilon_{S_i},\widehat{\varphi}_{S_i},\mathfrak{M}_{S_i})$ is the pullback of the universal family by $\beta_i\times\mathrm{id}_X$. The family $(\mathfrak{E}_{S_i},\varepsilon_{S_i},\widehat{\varphi}_{S_i},\mathfrak{M}_{S_i})$ is equivalent to $(\mathfrak{E}_{S|S_i},\varepsilon_{S|S_i},\widehat{\varphi}_{S|S_i},\mathfrak{M}_{S|S_i})$ by construction. □

Let $\mathfrak{B}^{\mathrm{iso}}$ be the open subscheme of oriented pairs $[q,\varepsilon,\varphi]$ for which $H^0(q(m))$ is an isomorphism. The maps constructed in the above proof factorize through $\mathfrak{B}^{\mathrm{iso}}$.

Proposition 2.3.4. *Let S be a noetherian scheme and let $\beta_i\colon S\longrightarrow\mathfrak{B}^{\mathrm{iso}}$, $i=1,2$, be two morphisms such that the pullbacks of $(\mathfrak{E}_{\mathfrak{B}},\varepsilon_{\mathfrak{B}},\widehat{\varphi}_{\mathfrak{B}},\mathfrak{M}_{\mathfrak{B}})$ via the maps $(\beta_i\times\mathrm{id}_X)$ are equivalent families. Then there exists an étale cover $\eta\colon T\longrightarrow S$ and a morphism $g\colon T\longrightarrow\mathrm{SL}(V)$ such that $\beta_1\circ\eta=(\beta_2\circ\eta)\cdot g$.*

Proof. Denote the two families by $(\mathfrak{E}^i_S,\varepsilon^i_S,\widehat{\varphi}^i_S,\mathfrak{M}^i_S)$, and let $\Psi_S\colon\mathfrak{E}^1_S\longrightarrow\mathfrak{E}^2_S$ be the corresponding isomorphism. The bundles $\mathfrak{E}^i_S$ can be written as quotients $q^i_S\colon V\otimes\pi_X^*\mathcal{O}_X(-m)\longrightarrow\mathfrak{E}^i_S$, and there is a morphism $g_S\colon S\longrightarrow\mathrm{GL}(V)$ making the following diagramm commutative:

$$\begin{tikzcd}
V\otimes\pi_X^*\mathcal{O}_X(-m) \arrow[r, "g_S\otimes\mathrm{id}"] \arrow[d, "q^1_S"'] & V\otimes\pi_X^*\mathcal{O}_X(-m) \arrow[d, "q^2_S"] \\
\mathfrak{E}^1_S \arrow[r, "\Psi_S"] & \mathfrak{E}^2_S
\end{tikzcd}$$

As in the proof of [HL1], Lemma 1.15, one constructs an étale cover $\eta\colon T\longrightarrow S$ such that there is a morphism $\mathfrak{d}\colon T\longrightarrow\mathbb{C}^*$ with $(\mathfrak{d}(t))^p=\det(g_S(\eta(t)))$ for any closed point $t\in T$. Now define $g:=\mathfrak{d}\cdot(g_S\circ\eta)$. In view of the description of the $\mathrm{SL}(V)$-action at the beginning of this section, the assertion is obvious. □

2.4. **The GIT-construction.** Let $\mathfrak{A}$ be the union of the finitely many components of $\mathrm{Pic}(X)$ meeting the image of $\det_{\mathfrak{B}}\colon\mathfrak{B}\longrightarrow\mathrm{Pic}(X)$. We may choose m so large that the restriction of the line bundle $\mathcal{L}_{|\mathfrak{A}\times X}\otimes\pi_X^*\mathcal{O}_X(rm)$ to $\{a\}\times X$ is globally generated and without higher cohomology for any closed point $a\in\mathfrak{A}$. The direct image sheaf $\pi_{\mathfrak{A}_*}(\mathcal{L}_{|\mathfrak{A}\times X}\otimes\pi_X^*\mathcal{O}_X(rm))$ is then locally free and commutes with base change. The same holds for $\mathcal{H}om(\bigwedge^r V\otimes\mathcal{O}_{\mathfrak{A}},\pi_{\mathfrak{A}*}(\mathcal{L}_{|\mathfrak{A}\times X}\otimes\pi_X^*\mathcal{O}_X(rm)))$; let $\mathfrak{H}$ be the geometric vector bundle associated to this locally free sheaf. Consider the homomorphism

$$\sigma_{\mathfrak{N}}\colon\bigwedge^r V\otimes\mathcal{O}_{\mathfrak{N}\times X}\longrightarrow\det\mathfrak{E}_{\mathfrak{N}}\otimes\pi_X^*\mathcal{O}_X(rm)\xrightarrow{\varepsilon_{\mathfrak{N}}}\mathcal{L}[\mathfrak{E}_{\mathfrak{N}}]\otimes\pi_X^*\mathcal{O}_X(rm).$$

By the universal property of the scheme $\mathfrak{H}$, the pushforward $\pi_{\mathfrak{N}*}(\sigma_{\mathfrak{N}})$ determines a morphism of schemes $\mathfrak{N} \longrightarrow \mathfrak{H}$ and hence a morphism $\mathfrak{N} \times S^r K \longrightarrow \mathfrak{H} \times S^r K$. Let $\mathfrak{Z}$ be the vector bundle $(\mathfrak{H} \times S^r K)^\vee$ over $\mathfrak{A}$, and denote by $\mathbb{P}(\mathfrak{Z})$ the associated projective bundle. $\mathbb{P}(\mathfrak{Z})$ can be polarized by tensorizing $\mathcal{O}_{\mathbb{P}(\mathfrak{Z})}(1)$ with the pull back of a very ample line bundle from $\mathfrak{A}$.

On $\mathbb{P}(\mathfrak{Z})$ there is a natural action of the group $\mathrm{SL}(V)$ from the right, which is trivial on the base $\mathfrak{A}$ and admits a canonical linearization in the polarizing line bundle. We have a natural morphism

$$\iota\colon \mathfrak{B} \hookrightarrow \mathfrak{P} \longrightarrow \mathbb{P}(\mathfrak{Z})$$

which is equivariant w.r.t. the given actions.

Let us describe the effect of ι on closed points: Given $b \in \mathfrak{B}$, let $(\mathcal{E}_b, \varepsilon_b, \varphi_b)$ be the oriented pair induced by the restriction of $(\mathfrak{E}_{\mathfrak{B}}, \varepsilon_{\mathfrak{B}}, \widehat{\varphi}_{\mathfrak{B}})$ to $\{b\} \times X$, i.e., $\mathcal{E}_b$ and ε_b are the restrictions of $\mathfrak{E}_{\mathfrak{B}}$ and $\varepsilon_{\mathfrak{B}}$ and φ_b is a framing with $S^r\varphi_b = \widehat{\varphi}_{\mathfrak{B}|\{b\}\times X}$ (φ_b is unique up to an r-th root of unity). The point b is mapped to $[\mathcal{L}[\mathcal{E}_b], h, S^r k]$ with

$$h\colon \bigwedge^r V \longrightarrow H^0(\det(\mathcal{E}_b)(rm)) \xrightarrow{H^0(\varepsilon_b(rm))} H^0(\mathcal{L}[\mathcal{E}_b](rm))$$

and $k = H^0((\varphi_b \circ q)(m))$. A point in $\mathbb{P}(\mathfrak{Z})$ is $\mathrm{SL}(V)$-*(semi)stable* if it is semistable in the projective space $\mathbb{P}((\mathrm{Hom}(\bigwedge^r V, H^0(\mathcal{L}_{|\{a\}\times X}(rm))) \oplus S^r K)^\vee)$, where a is its image in $\mathfrak{A}$.

Let $\mathfrak{B}^{ss}$ ($\mathfrak{B}^s$) be the open subscheme of points $[q, \varepsilon, \varphi]$ such that the triple $(\mathcal{E}, \varepsilon, \varphi)$ is a semistable (stable) oriented pair and such that the homomorphism $H^0(q(m))\colon V \longrightarrow H^0(\mathcal{E}(m))$ is an isomorphism.

Theorem 2.4.1. *For m large enough, $\mathfrak{B}^{ss} = \iota^{-1}(\mathbb{P}(\mathfrak{Z})^{ss})$, and $\mathfrak{B}^s = \iota^{-1}(\mathbb{P}(\mathfrak{Z})^s)$.*

Before we can start with the proof, we have to recall some definitions and results from [HL1] and [HL2]. Let $(\mathcal{E}, \varphi)$ be a pair consisting of a torsion free coherent sheaf $\mathcal{E}$ with $P_{\mathcal{E}} = P$ and a non-trivial framing φ.

Let $\overline{\delta}$ be any positive rational number. The pair $(\mathcal{E}, \varphi)$ is called *sectional (semi)stable w.r.t. $\overline{\delta}$*, if there is a subspace $V \subset H^0(\mathcal{E})$ of dimension $\chi(\mathcal{E}) = P(0)$ such that the following conditions are satisfied:

1. For all non-trivial submodules $\mathcal{F}$ of $\ker(\varphi)$:
$$(\mathrm{rk}\,\mathcal{E}) \dim\left(H^0(\mathcal{F}) \cap V\right) (\leq) \,\mathrm{rk}\,\mathcal{F}(\chi(\mathcal{E}) - \overline{\delta}).$$
2. For all non-trivial submodules $\mathcal{F} \neq \mathcal{E}$:
$$(\mathrm{rk}\,\mathcal{E}) \dim\left(H^0(\mathcal{F}) \cap V\right) (\leq) \,\mathrm{rk}\,\mathcal{F}(\chi(\mathcal{E}) - \overline{\delta}) + (\mathrm{rk}\,\mathcal{E})\overline{\delta}.$$

Then we have the following result [HL2], Th. 2.1:

Theorem 2.4.2. *For any polynomial δ, there exists a natural number m_1 such that for all $m \geq m_1$ the following conditions are equivalent for a pair $(\mathcal{E}, \varphi)$:*
i) *$(\mathcal{E}, \varphi)$ is (semi)stable w.r.t. the polynomial δ.*
ii) *$(\mathcal{E}, \varphi)(m)$ is sectional (semi)stable w.r.t. $\delta(m)$.*

Let $(q\colon V\otimes\mathcal{O}_X(-m)\longrightarrow\mathcal{E},\varphi)$ be a pair consisting of a *generically* surjective map q of $V\otimes\mathcal{O}_X(-m)$ to a torsion free sheaf $\mathcal{E}$ with $P_{\mathcal{E}}=P$ and a non-zero homomorphism $\varphi\colon\mathcal{E}\longrightarrow\mathcal{E}_0$. We can associate to this pair an element $([h],[k])\in\mathbb{P}(H^\vee)\times\mathbb{P}(K^\vee)$, where $H:=\mathrm{Hom}(\bigwedge^r V, H^0(\mathcal{L}[\mathcal{E}](rm)))$. There is a natural $\mathrm{SL}(V)$-action on $\mathbb{P}(H^\vee)\times\mathbb{P}(K^\vee)$ which can be linearized in every sheaf $\mathcal{O}(a_1,a_2)$, where a_1 and a_2 are positive integers. Define $\nu:=a_2/a_1$ and $\overline{\delta}:=p\nu/(\mathrm{rk}\,\mathcal{E}+\nu)$. The proof of [HL1], Proposition 1.18 is valid in any dimension and yields the following

Theorem 2.4.3. *Let $(q\colon V\otimes\mathcal{O}_X(-m)\longrightarrow\mathcal{E},\varphi)$ be as above. The associated element $([h],[k])$ is (semi)stable w.r.t. the linearization in $\mathcal{O}(a_1,a_2)$ if and only if the following two conditions are satisfied:*
i) *The homomorphism $H^0(q(m))$ is injective.*
ii) *The pair $(\mathcal{E},\varphi)(m)$ is sectional (semi)stable w.r.t. $\overline{\delta}$.*

We also need the following obvious observation:

Lemma 2.4.4. *Let $(q\colon V\otimes\mathcal{O}_X(-m)\longrightarrow\mathcal{E},\varphi)$ be as above. The following conditions are equivalent:*
i) *The homomorphism $k=H^0((\varphi\circ q)(m))$ is injective.*
ii) *The associated element $[k]\in\mathbb{P}(K^\vee)$ is stable.*

After these preparations, we return to our situation. Let $\mathfrak{B}_0\subset\mathfrak{B}$ be the open set of all oriented pairs $[q,\varepsilon,\varphi]$ for which $\mathcal{E}$ is semistable, and define for each polynomial δ the set $\mathfrak{B}_\delta$ as the open set of oriented pairs $(\mathcal{E},\varepsilon,\varphi)$ with $\varphi\neq 0$ such that $(\mathcal{E},\varphi)$ is semistable w.r.t. δ. The union $\mathfrak{B}':=\mathfrak{B}_0\cup\bigcup\mathfrak{B}_\delta$ is quasi-projective, hence quasi-compact, so that there exist finitely many polynomials, say, δ_1,...,δ_s with $\mathfrak{B}'=\mathfrak{B}_0\cup\mathfrak{B}_{\delta_1}\cup\cdots\cup\mathfrak{B}_{\delta_s}$. Let M be some constant. By [M2], Theorem 1.7, the set of points $b\in\mathfrak{B}$ such that $\mu_{\max}(\mathfrak{E}_{\mathfrak{B}|\{b\}\times X})\le M$ is open. Since $\mathfrak{B}$ is quasi-compact, there is a constant μ_0 such that $\mu_{\max}(\mathfrak{E}_{\mathfrak{B}|\{b\}\times X})\le\mu_0$ for all $b\in\mathfrak{B}$. We also know that the family $\mathfrak{Ker}$ of kernels of framings of semistable oriented pairs is bounded. It follows that $\mu_{\max}(\ker(\varphi))$, for $\ker(\varphi)\in\mathfrak{Ker}$, can only take finitely many values. As in [HL2], Lemma 2.7, this implies that there are only finitely many polynomials of the form $P_{\ker(\varphi)_{\max}}$. In particular, there are only finitely many polynomials of the form

$$P_{\mathcal{E}}-(\mathrm{rk}\,\mathcal{E}/\,\mathrm{rk}\ker(\varphi)_{\max})P_{\ker(\varphi)_{\max}}.$$

We assume in the following that these polynomials are among $\delta_1,...,\delta_s$, and that the chosen m is large enough, so that Theorem 2.4.2 holds for all δ_i and set $\overline{\delta}_i:=\delta_i(m)$.

Theorem 2.4.5. *Suppose m is sufficiently large. Let $[q,\varepsilon,\varphi]\in\mathfrak{B}$ be a pair with $\varphi\neq 0$ which is not (semi)stable. Then there is no positive rational number $\overline{\delta}$ such that $(\mathcal{E},\varphi)(m)$ is sectional (semi)stable w.r.t. $\overline{\delta}$.*

Proof. Denote by $\mathfrak{S}$ the bounded set of equivalence classes of pairs $(\mathcal{E},\varphi)$ for which there is an element $[q,\varepsilon,\varphi]\in\mathfrak{B}$.

By the above, any pair $(\mathcal{E},\varphi)\in\mathfrak{S}$ satisfies $\mu_{\max}(\mathcal{E})\le\mu_0$. Let $\widetilde{\delta}$ be a rational polynomial of degree $\dim X-1$ whose leading coefficient $\widetilde{\delta}_0$ satisfies $\mu(\mathcal{E})+\widetilde{\delta}_0\ge\max\{\,0,\mu_0\,\}$. One can now copy the proof of [HL2], page 305, to show that there is a constant C such that for any submodule $(\widetilde{\mathcal{E}},\widetilde{\varphi})$ of a pair $(\mathcal{E},\varphi)\in\mathfrak{S}$ either $|\deg(\widetilde{\mathcal{E}})-\operatorname{rk}\widetilde{\mathcal{E}}\mu(\mathcal{E})|<C$, or for all m large enough

$$\frac{h^0(\widetilde{\mathcal{E}}(m))}{\operatorname{rk}\widetilde{\mathcal{E}}}-\frac{\widetilde{\delta}(m)}{\operatorname{rk}\widetilde{\mathcal{E}}} < \frac{P_{\mathcal{E}}(m)}{\operatorname{rk}\mathcal{E}}-\frac{\widetilde{\delta}(m)}{\operatorname{rk}\mathcal{E}} \qquad \text{if } \widetilde{\mathcal{E}}\not\subset\ker(\varphi)\ ,$$

$$\frac{h^0(\widetilde{\mathcal{E}}(m))}{\operatorname{rk}\widetilde{\mathcal{E}}} < \frac{P_{\mathcal{E}}(m)}{\operatorname{rk}\mathcal{E}}-\frac{\widetilde{\delta}(m)}{\operatorname{rk}\mathcal{E}} \qquad \text{otherwise.}$$

Recall that a submodule $\widetilde{\mathcal{E}}\subset\mathcal{E}$ is called *saturated*, if the quotient $\mathcal{E}/\widetilde{\mathcal{E}}$ is torsion free. The family of saturated submodules $\widetilde{\mathcal{E}}$ of modules $\mathcal{E}$ with $(\mathcal{E},\varphi)\in\mathfrak{S}$ satisfying $|\deg(\widetilde{\mathcal{E}})-\operatorname{rk}\widetilde{\mathcal{E}}\mu(\mathcal{E})|<C$ is bounded ([HL2], Lemma 2.7). Denote this family by $\widetilde{\mathfrak{S}}$. There are only finitely many possibilities for the Hilbert polynomials of those submodules. Let δ'_j be the finite family of polynomials of the form $P_{\mathcal{E}}-(\operatorname{rk}\mathcal{E}/\operatorname{rk}\mathcal{E}')P_{\mathcal{E}'}$ where $\mathcal{E}'$ is a saturated submodule of $\ker(\varphi)$ for some $(\mathcal{E},\varphi)\in\widetilde{\mathfrak{S}}$, and δ''_k be the finite family of polynomials of the form $(\operatorname{rk}\mathcal{E}''P_{\mathcal{E}}-\operatorname{rk}\mathcal{E}P_{\mathcal{E}''})/(\operatorname{rk}\mathcal{E}-\operatorname{rk}\mathcal{E}'')$ where $\mathcal{E}''$ is a saturated submodule of a pair $(\mathcal{E},\varphi)\in\widetilde{\mathfrak{S}}$ not contained in the kernel of φ. We may assume that $\widetilde{\delta}$, the δ'_j's and the δ''_k's with positive leading coefficients are among $\delta_1,\dots,\delta_s$. Next, we choose m large enough, so that $\widetilde{\mathcal{E}}(m)$ is globally generated and has no higher cohomology for all $\widetilde{\mathcal{E}}\in\widetilde{\mathfrak{S}}$. Let $(\mathcal{E},\varphi)$ be a pair which is not semistable w.r.t. any of the polynomials $\delta_1,\dots,\delta_s$. This is equivalent to $(\mathcal{E},\varphi)(m)$ not being sectional semistable w.r.t. any of the numbers $\overline{\delta}_1,\dots,\overline{\delta}_s$. Since $(\mathcal{E},\varphi)$ is not semistable w.r.t. $\widetilde{\delta}$, there is either a saturated submodule $\mathcal{E}'_0\subset\ker(\varphi)$ with $\delta_{\mathcal{E}'_0}:=P_{\mathcal{E}}-(\operatorname{rk}\mathcal{E}/\operatorname{rk}\mathcal{E}'_0)P_{\mathcal{E}'_0}<\widetilde{\delta}$, or there exists a saturated submodule $\mathcal{E}''_0\not\subset\ker(\varphi)$ such that

$$\delta_{\mathcal{E}''_0}:=(\operatorname{rk}\mathcal{E}''_0P_{\mathcal{E}}-\operatorname{rk}\mathcal{E}P_{\mathcal{E}''_0})/(\operatorname{rk}\mathcal{E}-\operatorname{rk}\mathcal{E}''_0)>\widetilde{\delta}\ .$$

In the first case suppose that $\delta_{\mathcal{E}'_0}$ is minimal and in the second that $\delta_{\mathcal{E}''_0}$ is maximal. We consider only the first case, since the second can be treated similarly. If $\delta_{\mathcal{E}'_0}\le 0$, then we are done. Otherwise, set $\delta'_{i_0}:=\delta_{\mathcal{E}'_0}$. By the minimality of δ'_{i_0}, any submodule $\mathcal{E}'$ of $\ker(\varphi)$ satisfies

$$(\operatorname{rk}\mathcal{E})\dim H^0(\mathcal{E}'(m))\le\operatorname{rk}\mathcal{E}'(p-\overline{\delta'}_{i_0}),$$

and for $\mathcal{E}'=\mathcal{E}'_0$ we have equality. Since $\mathcal{E}$ is not sectional semistable w.r.t. $\overline{\delta'}_{i_0}$, there must exist a submodule $\mathcal{E}''\not\subset\ker(\varphi)$ with

$$(\operatorname{rk}\mathcal{E})\dim H^0(\mathcal{E}''(m))>\operatorname{rk}\mathcal{E}''(p-\overline{\delta'}_{i_0})+(\operatorname{rk}\mathcal{E})\overline{\delta'}_{i_0}.$$

This makes it obvious that $(\mathcal{E},\varphi)$ cannot be sectional semistable w.r.t. to any positive rational number.

We still have to prove the "stable" version of the proposition. For this we enlarge the constant C such that $-C \le -\delta_i^0$, $i=1,...,s$, where δ_i^0 is the leading coefficient of δ_i. If $(\mathcal{E},\varphi)$ is a pair which is semistable w.r.t. the polynomial, say, δ_{i_0} but not stable w.r.t. any other polynomial δ, then there must exist submodules $\mathcal{E}' \subset \ker(\varphi)$ and $\mathcal{E}''$ belonging to $\widetilde{\mathfrak{S}}$ with

$$\frac{P_{\mathcal{E}'}}{\operatorname{rk}\mathcal{E}'} = \frac{P_{\mathcal{E}}-\delta_{i_0}}{\operatorname{rk}\mathcal{E}} \quad \text{and} \quad \frac{P_{\mathcal{E}''}-\delta_{i_0}}{\operatorname{rk}\mathcal{E}''} = \frac{P_{\mathcal{E}'}-\delta_{i_0}}{\operatorname{rk}\mathcal{E}}.$$

Since m was so large that all modules in $\widetilde{\mathfrak{S}}$ are globally generated and without higher cohomology, this gives

$$\begin{aligned}(\operatorname{rk}\mathcal{E})\dim H^0(\mathcal{E}'(m)) &= (\operatorname{rk}\mathcal{E}')(p-\overline{\delta}_{i_0})\\ (\operatorname{rk}\mathcal{E})\dim H^0(\mathcal{E}''(m)) &= (\operatorname{rk}\mathcal{E}'')(p-\overline{\delta}_{i_0}) + (\operatorname{rk}\mathcal{E})\overline{\delta}_{i_0},\end{aligned}$$

and hence the assertion. □

2.5. **Proof of Theorem 2.4.1.** For $b \in \mathfrak{B}$, put $H_b := \operatorname{Hom}(\bigwedge^r V, H^0(\mathcal{L}[\mathcal{E}_b](rm)))$ and $\mathbb{P}_b := \mathbb{P}((H_b \oplus S^rK)^\vee)$. The space $\mathbb{P}_b$ admits the following natural $\mathbb{C}^*$-action:

$$z \cdot [h,\widehat{k}] := [h, z\widehat{k}] = [z^{-1}h, \widehat{k}].$$

By 1.2.5, this $\mathbb{C}^*$-action can be linearized in such a way that the quotient is either $\mathbb{P}(H_b^\vee)$, $\mathbb{P}((S^rK)^\vee)$, or $\mathbb{P}(H_b^\vee)\times\mathbb{P}((S^rK)^\vee)$ equipped with the polarization $[\mathcal{O}(a_1,a_2)]$ for any prescribed ratio a_2/a_1. We are now able to apply our GIT-Theorem 1.4.1 to reduce Theorem 2.4.1 to Theorem 2.4.3.

First we explain the assertion about semistability: Suppose that $b=[q,\varepsilon,\varphi]$ lies in $\mathfrak{B}^{ss}$. Then either φ is injective, or $\mathcal{E}$ is semistable, or $\varphi \ne 0$ and the pair $(\mathcal{E},\varphi)$ is semistable w.r.t. some δ_i. If φ is injective, we linearize in such a way that we obtain $\mathbb{P}((S^rK)^\vee)$ as the quotient. By 2.4.4, the point $[k]$ is semistable in $\mathbb{P}(K^\vee)$ and hence $[S^rk]$ is semistable in $\mathbb{P}((S^rK)^\vee)$. This implies by 1.4.1 that $[h,S^rk]$ is semistable in $\mathbb{P}_b$. If $\mathcal{E}$ is semistable, we linearize the $\mathbb{C}^*$-action in such a way that the quotient $\mathbb{P}_b/\!\!/\mathbb{C}^*$ is given by $\mathbb{P}(H_b^\vee)$. By [G], Theorem 0.7 (which does not depend on dimension 2), the point $[h]$ is then semistable in $\mathbb{P}(H_b^\vee)$, and hence $[h,S^rk]$ is $\mathrm{SL}(V)$-semistable in $\mathbb{P}_b$ by 1.4.1. If $\varphi\ne 0$, $\varepsilon \ne 0$ and $(\mathcal{E},\varphi)$ is semistable w.r.t. δ_i, we choose the linearization of the $\mathbb{C}^*$-action in such a way that the quotient is $\mathbb{P}(H_b^\vee)\times\mathbb{P}((S^rK)^\vee)$, equipped with a polarization $[\mathcal{O}(ra_1,a_2)]$ satisfying $(a_2/a_1)=\operatorname{rk}\mathcal{E}\overline{\delta}_i/(p-\overline{\delta}_i)$. By Theorem 2.4.3, $([h],[S^rk])$ is semistable and thus $[h,S^rk]$ is semistable.

Conversely, suppose $[h,S^rk]$ is $\mathrm{SL}(V)$-semistable. By 1.4.1 there is a linearization of the $\mathbb{C}^*$-action such that the image of $[h,S^rk]$ is $\mathrm{SL}(V)$-semistable in the quotient $\mathbb{P}_b/\!\!/\mathbb{C}^*$. There are three possible quotients: If the quotient is $\mathbb{P}((S^rK)^\vee)$, then semistability implies that $[k]$ is semistable in $\mathbb{P}(K^\vee)$ and hence that k is injective. It follows that $\mathcal{E}$ is a subsheaf of $\mathcal{E}_0$, since we may assume that m is so large that $\ker(\varphi(m))$ is globally generated. If the quotient is $\mathbb{P}(H_b^\vee)$, then $\mathcal{E}$ is semistable by [G], loc. cit.. If the quotient is $\mathbb{P}(H_b^\vee)\times\mathbb{P}((S^rK)^\vee)$ with polarization $[\mathcal{O}(a_1,a_2)]$, then $(\mathcal{E},\varphi)$ is sectional semistable w.r.t.

$$\overline{\delta} := p(ra_2/a_1)/(\operatorname{rk}\mathcal{E} + (ra_2/a_1)) \ .$$

In view of 2.4.2 and 2.4.5, $(\mathcal{E},\varphi)$ is semistable w.r.t. some δ, hence $[q,\varepsilon,\varphi]$ lies in $\mathfrak{B}^{ss}$.

We still have to identify the stable points. As the proof of 2.3.4 shows, the oriented pair $(\mathcal{E},\varepsilon,\varphi)$ given by a point $b=[q,\varepsilon,\varphi]\in\mathfrak{B}$ has only finitely many automorphisms if and only if the associated point $[h,S^rk]\in\mathbb{P}_b$ has a finite $\mathrm{SL}(V)$-stabilizer. Let $b=[q,\varepsilon,\varphi]$ be a point whose associated element $[h,S^rk]$ in $\mathbb{P}_b$ is stable. If $h=0$ or $k=0$, then it is easy to see that the corresponding element $[S^rk]\in\mathbb{P}((S^rK)^\vee)$ or $[h]\in\mathbb{P}(H_b^\vee)$ is stable. Hence $H^0(q(m))$ is an isomorphism and either φ is injective or $\mathcal{E}$ is a stable sheaf. In both cases, the oriented pair $(\mathcal{E},\varepsilon,\varphi)$ is stable and $H^0(q(m))$ is an isomorphism, in other words $b\in\mathfrak{B}^s$. If both $h\neq 0$ and $k\neq 0$, then by 1.4.1 $([h],[S^rk])\in\mathbb{P}(H_b^\vee)\times\mathbb{P}((S^rK)^\vee)$ is a polystable point w.r.t. the polarization, say, $\mathcal{O}(a_1,a_2)$. By what we have already proved, $(\mathcal{E},\varphi)$ is a semistable pair. Remark 2.1.2 shows that either $(\mathcal{E},\varphi)$ is a stable pair or there is an $i\in\{\,1,...,s\,\}$ such that $(\mathcal{E},\varphi)$ is polystable w.r.t. δ_i. In the first case, we are done. In the second case, the finiteness of the stabilizer of $[h,S^rk]$ implies that the oriented pair $(\mathcal{E},\varepsilon,\varphi)$ has only finitely many automorphisms, hence it is a stable oriented pair.

Suppose now that $b\in\mathfrak{B}^s$. If $\varphi=0$, then $\mathcal{E}$ must be a stable coherent sheaf and thus $[h]\in\mathbb{P}(H_b^\vee)$ is a stable point. It follows that $[h,0]$ is a polystable point. But as $[h,0]$ is a fixed point of the $\mathbb{C}^*$-action, the $\mathrm{SL}(V)$-stabilizer of $[h,0]\in\mathbb{P}_b$ can be identified with the $\mathrm{SL}(V)$-stabilizer of $[h]\in\mathbb{P}(H_b^\vee)$, so that $[h,0]$ is indeed a stable point. If $\varepsilon=0$, then φ must be injective and we may argue in the same manner. If both $\varepsilon\neq 0$ and $\varphi\neq 0$, it suffices to show that $[h,S^rk]$ is a polystable point, since its stabilizer is finite by definition. By the stability of $(\mathcal{E},\varphi)$, by the "stable" version of 2.4.5, and by the choice of the δ_i, there exists an index $i\in\{\,1,...,s\,\}$ such that $(\mathcal{E},\varphi)$ is polystable w.r.t. δ_i. This in turn shows that $([h],[S^rk])\in\mathbb{P}(H_b^\vee)\times\mathbb{P}((S^rK)^\vee)$ is polystable w.r.t. the linearization in $\mathcal{O}(ra_1,a_2)$ satisfying $\overline{\delta}_i = p(a_2/a_1)/(\operatorname{rk}\mathcal{E}+(a_2/a_1))$.

2.6. **Moduli spaces of stable oriented pairs.** We need the following proposition

Proposition 2.6.1. *The map* $\iota_{|\mathfrak{B}^{ss}}\colon \mathfrak{B}^{ss}\longrightarrow\mathbb{P}(\mathfrak{Z})^{ss}$ *is finite.*

Proof. We claim that $\iota_{|\mathfrak{B}^{ss}}$ is proper and injective. Injectivity follows by standard arguments. For the proof of properness, we will make use of the discrete valuative criterion. Let $C=\operatorname{Spec}R$ be the spectrum of a discrete valuation ring, $c_0\in C$ the closed point, and $C_0 := C\setminus\{c_0\}$. Suppose there is a commutative diagram:

$$\begin{tikzcd} C_0 \arrow[r, "u"] \arrow[d] & \mathfrak{B}^{ss} \arrow[d, "\iota|_{\mathfrak{B}^{ss}}"] \\ C \arrow[r, "\bar{u}"] \arrow[ur, dashed, "\widetilde{u}"] & \mathbb{P}(\mathfrak{Z})^{ss} \end{tikzcd}$$

We have to construct a lifting $\widetilde{u}$ of the map $\overline{u}$. By assumption, we are given a family $(\mathfrak{E}_{C_0}, \varepsilon_{C_0}, \widehat{\varphi}_{C_0}, \mathcal{O}_{C_0})$ of semistable oriented pairs over $C_0 \times X$. Note that $\mathfrak{E}_{C_0}$ is torsion free. We claim that we can extend the quotient map

$$q_{C_0} \colon V \otimes \pi_X^* \mathcal{O}_X(-m) \longrightarrow \mathfrak{E}_{C_0}$$

to a homomorphism $q_C \colon V \otimes \pi_X^* \mathcal{O}_X(-m) \longrightarrow \mathfrak{E}_C$ over $C \times X$, where $\mathfrak{E}_C$ is a flat family of torsion free coherent sheaves extending $\mathfrak{E}_{C_0}$, q_C extends q_{C_0}, and its restriction to $\{c_0\} \times X$ is generically surjective. In order to prove this claim, we first extend the family $\mathfrak{E}_{C_0}$ to a flat family of quotients

$$\widetilde{q}_C \colon V \otimes \pi_X^* \mathcal{O}_X(-m) \longrightarrow \widetilde{\mathfrak{E}}_C.$$

There is a locally free sheaf $\mathcal{H}$ on X and an epimorphism $\pi_X^* \mathcal{H} \longrightarrow \widetilde{\mathfrak{E}}_C^\vee$. This yields a homomorphism

$$\lambda \colon V \otimes \pi_X^* \mathcal{O}_X(-m) \longrightarrow \widetilde{\mathfrak{E}}_C \longrightarrow \widetilde{\mathfrak{E}}_C^{\vee\vee} \longrightarrow \pi_X^* \mathcal{H}^\vee.$$

Let $\mathfrak{E}_C$ be the maximal subsheaf of $\pi_X^* \mathcal{H}^\vee$ with the following properties

$$\mathfrak{E}_{C|C_0 \times X} = \mathfrak{E}_{C_0}; \qquad \operatorname{Im} \lambda \subset \mathfrak{E}_C; \qquad \dim(\operatorname{supp}(\mathfrak{E}_C / \operatorname{Im} \lambda)) < \dim X.$$

Note that the set of subsheaves of $\pi_X^* \mathcal{H}^\vee$ having the above properties contains $\operatorname{Im} \lambda$. One checks that $\mathfrak{E}_{C|\{c_0\} \times X}$ is torsion free, using arguments as in [HL1], p.85. Let $t \in R$ be a generator of the maximal ideal. There is a well defined integer α such that $(t^\alpha \varepsilon_{C_0}, t^\alpha \widehat{\varphi}_{C_0})$ extends to the family $\mathfrak{E}_C$.

The classifying map to $\mathbb{P}(\mathfrak{Z})^{ss}$ induced by the resulting family

$$(q_C \colon V \otimes \pi_X^* \mathcal{O}_X(-m) \longrightarrow \mathfrak{E}_C, \widetilde{\varepsilon}_C, \widetilde{\widehat{\varphi}}_C, \mathcal{O}_C)$$

is the same as the one induced by $\overline{u}$. By the various stability criteria we have encountered so far, it follows that $H^0(q_{C|\{c_0\} \times X}(m))$ is injective and that the triple $(\mathfrak{E}_{C|\{c_0\} \times X}, \widetilde{\varepsilon}_{C|\{c_0\} \times X}, \widetilde{\varphi}_{c_0})$, where φ_{c_0} is a framing induced by $\widetilde{\widehat{\varphi}}_{C|\{c_0\} \times X}$, is a semistable oriented pair.

Thus, $\mathfrak{E}_{C|\{c_0\} \times X}(m)$ is globally generated and without higher cohomology, the map $q_{C|\{c_0\} \times X}$ is surjective, and hence $q_C \colon V \otimes \pi_X^* \mathcal{O}_X(-m) \longrightarrow \mathfrak{E}_C$ is a flat family of torsion free quotients. The family $(q_C \colon V \otimes \pi_X^* \mathcal{O}_X(-m) \longrightarrow \mathfrak{E}_C, \widetilde{\varepsilon}_C, \widetilde{\widehat{\varphi}}_C, \mathcal{O}_C)$ defines by 2.3.3 a morphism

$$\widetilde{u} \colon C \longrightarrow \mathfrak{B}^{ss}$$

which extends u by construction. □

By Proposition 2.6.1. and [G], Lemma 4.6, the quotient $\mathfrak{B}^{ss}/\!\!/\,\mathrm{SL}(V)$ exists as a projective scheme. We set

$$\mathcal{M}^{ss}_{(P,\mathcal{L},\mathcal{E}_0)} := \mathfrak{B}^{ss}/\!\!/\,\mathrm{SL}(V)\ ,$$
$$\mathcal{M}^{s}_{(P,\mathcal{L},\mathcal{E}_0)} := \mathfrak{B}^{s}/\!\!/\,\mathrm{SL}(V)\ .$$

Theorem 2.6.2. i) *There is a natural transformation of functors*

$$\tau\colon M^{ss}_{(P,\mathcal{L},\mathcal{E}_0)} \longrightarrow h_{\mathcal{M}^{ss}_{(P,\mathcal{L},\mathcal{E}_0)}},$$

such that for any scheme $\widetilde{\mathcal{M}}$ and any natural transformation of functors

$$\tau'\colon M^{ss}_{(P,\mathcal{L},\mathcal{E}_0)} \longrightarrow h_{\widetilde{\mathcal{M}}}$$

there is a unique morphism $\vartheta\colon \mathcal{M}^{ss}_{(P,\mathcal{L},\mathcal{E}_0)} \longrightarrow \widetilde{\mathcal{M}}$ such that $\tau' = h(\vartheta)\circ\tau$.

ii) *The space $\mathcal{M}^{s}_{(P,\mathcal{L},\mathcal{E}_0)}$ is a coarse moduli space for stable oriented pairs.*

Proof. The existence of the natural transformation is a direct consequence of Proposition 2.3.3 and 2.3.4. The minimality property of $\mathcal{M}^{ss}_{(P,\mathcal{L},\mathcal{E}_0)}$ follows from the universal property of the categorical quotient.

Since $\mathfrak{B}^{s}$ is contained in the set of $\mathrm{SL}(V)$-stable points, the set of closed points of $\mathcal{M}^{s}_{(P,\mathcal{L},\mathcal{E}_0)}$ is the set of equivalence classes of stable oriented pairs which means that $\mathcal{M}^{s}_{(P,\mathcal{L},\mathcal{E}_0)}$ is a coarse moduli space. □

In our applications [OT2] we shall also need a slightly modified version of the constructions and results above. We fix a line bundle $\mathcal{L}_0 \in \mathrm{Pic}(X)$ and consider only torsion free sheaves of determinant isomorphic to $\mathcal{L}_0$.

More precisely, an $\mathcal{L}_0$-*oriented pair of type* $(P,\mathcal{E}_0)$ is a triple $(\mathcal{E},\varepsilon,\varphi)$ consisting of a torsion free coherent sheaf $\mathcal{E}$ with Hilbert polynomial P and with $\det\mathcal{E}$ isomorphic to $\mathcal{L}_0$, a homomorphism $\varepsilon:\det\mathcal{E}\longrightarrow\mathcal{L}_0$, and a homomorphism $\varphi:\mathcal{E}\longrightarrow\mathcal{E}_0$.

Equivalence classes of such $\mathcal{L}_0$-oriented pairs, families, equivalence classes of families, (semi)stability and the corresponding functors $M^{ss}_{(P,\mathcal{L}_0,\mathcal{E}_0)}$ are defined as in 2.1. The same methods as above yield the following result:

Theorem 2.6.3. *There exist moduli spaces $\mathcal{M}^{ss}_{(P,\mathcal{L}_0,\mathcal{E}_0)}$ and $\mathcal{M}^{s}_{(P,\mathcal{L}_0,\mathcal{E}_0)}$ with the following properties:*

i) *There is a natural transformation of functors*

$$\tau\colon M^{ss}_{(P,\mathcal{L}_0,\mathcal{E}_0)} \longrightarrow h_{\mathcal{M}^{ss}_{(P,\mathcal{L}_0,\mathcal{E}_0)}},$$

such that for any scheme $\widetilde{\mathcal{M}}$ and any natural transformation of functors

$$\tau'\colon M^{ss}_{(P,\mathcal{L}_0,\mathcal{E}_0)} \longrightarrow h_{\widetilde{\mathcal{M}}}$$

there is a unique morphism $\vartheta\colon \mathcal{M}^{ss}_{(P,\mathcal{L}_0,\mathcal{E}_0)} \longrightarrow \widetilde{\mathcal{M}}$ such that $\tau' = h(\vartheta)\circ\tau$.

ii) *The space $\mathcal{M}^{s}_{(P,\mathcal{L}_0,\mathcal{E}_0)}$ is a coarse moduli space for stable $\mathcal{L}_0$-oriented pairs.*

2.7. **The closed points of $\mathcal{M}^{ss}_{(P,\mathcal{L},\mathcal{E}_0)}$.** Let $(\mathcal{E},\varepsilon,\varphi)$ be a semistable oriented pair of type $(P,\mathcal{L},\mathcal{E}_0)$. If $(\mathcal{E},\varepsilon,\varphi)$ is stable, then it defines a closed point in $\mathcal{M}^{ss}_{(P,\mathcal{L},\mathcal{E}_0)}$. If $(\mathcal{E},\varepsilon,\varphi)$ is not stable, then either $\mathcal{E}$ is a semistable but not stable coherent sheaf, or $\varphi\neq 0$ and there exists a $\delta\in\mathbb{Q}[x]$, $\delta>0$, such that $(\mathcal{E},\varphi)$ is semistable but not stable w.r.t. δ. In both cases, there is a Harder-Narasimhan filtration

$$0=\mathcal{E}_0\subset\mathcal{E}_1\subset\cdots\subset\mathcal{E}_s=\mathcal{E}$$

of $\mathcal{E}$, whose associated graded sheaf $\mathrm{gr}(\mathcal{E}):=\bigoplus_{i=1}^s\mathcal{E}_i/\mathcal{E}_{i-1}$ inherits a well-defined orientation $\varepsilon_{\mathrm{gr}}$ and a well-defined framing φ_{gr} from $(\mathcal{E},\varepsilon,\varphi)$. As usual, the resulting object $(\mathrm{gr}(\mathcal{E}),\varepsilon_{\mathrm{gr}},\varphi_{\mathrm{gr}})$ is determined up to equivalence. We call it the *graded object associated to* $(\mathcal{E},\varepsilon,\varphi)$. Using the techniques of Section 2.5, i.e., applying 1.4.1 in the "polystable" version, we reduce the polystability of $(\mathrm{gr}(\mathcal{E}),\varepsilon_{\mathrm{gr}},\varphi_{\mathrm{gr}})$ to the respective results of [HL1], [HL2], [G], and [M1]. Finally, one easily adapts the proof in [HL2], p.312, to show that a semistable oriented pair $(\mathcal{E},\varepsilon,\varphi)$ can be deformed into its graded object. If we call two semistable oriented pairs $(\mathcal{E}_i,\varepsilon_i,\varphi_i)$, $i=1,2$, *gr-equivalent* if their associated graded objects are equivalent, then we see that the closed points of $\mathcal{M}^{ss}_{(P,\mathcal{L},\mathcal{E}_0)}$ correspond to gr-equivalence classes of semistable oriented pairs of type $(P,\mathcal{L},\mathcal{E}_0)$.

2.8. **The $\mathbb{C}^*$-action on $\mathcal{M}^{ss}_{(P,\mathcal{L},\mathcal{E}_0)}$.** The moduli space possesses a natural $\mathbb{C}^*$-action, given by

$$z\cdot[\mathcal{E},\varepsilon,\varphi]:=[\mathcal{E},\varepsilon,z\varphi]=[\mathcal{E},z^{-r}\varepsilon,\varphi].$$

The set of fixed points of this action can easily be described: It consists of classes $[\mathcal{E},0,\varphi]$, $[\mathcal{E},\varepsilon,0]$, and of classes $[\ker(\varphi)_{\max}\oplus\mathcal{E}',\varepsilon,\varphi]$ with $0\neq\ker(\varphi)_{\max}$.

The $\mathbb{C}^*$-action is naturally linearized in an ample line bundle coming from the description of $\mathcal{M}^{ss}_{(P,\mathcal{L},\mathcal{E}_0)}$ as GIT-quotient. This line bundle and the polarization which it represents may, however, depend on an integer m chosen in the course of the construction. Nevertheless, we can state the following result which clarifies the birational geometry of the moduli spaces $\mathcal{M}^{ss}_\delta(X;\mathcal{E}_0,P)$ constructed in [HL2]:

Theorem 2.8.1. *Let $\delta_i\in\mathbb{Q}[x]$, $i=1,2$, be polynomials with positive leading coefficients. For a suitable choice of the polarization on $\mathcal{M}^{ss}_{(P,\mathcal{L},\mathcal{E}_0)}$ the following properties hold true:*

i) *$\mathcal{M}^{ss}_{\delta_i}(X;\mathcal{E}_0,P)$, $i=1,2$, are $\mathbb{C}^*$-quotients of the master space $\mathcal{M}^{ss}_{(P,\mathcal{L},\mathcal{E}_0)}$.*

ii) *$\mathcal{M}^{ss}_{\delta_1}(X;\mathcal{E}_0,P)$ and $\mathcal{M}^{ss}_{\delta_2}(X;\mathcal{E}_0,P)$ are related by a chain of generalized flips.*

Proof. Let m be so large that a pair $(\mathcal{E},\varphi)$ is semistable w.r.t. δ_i if and only the pair $(\mathcal{E}(m),\varphi(m))$ is sectional semistable w.r.t. $\delta_i(m)$, $i=1,2$, and that all the other requirements needed in the constructions are met. Then our proof of Theorem 2.4.1 together with the results of Section 1 easily yields the assertions of the theorem. □

We note that the δ_i for which the corresponding set of $\mathbb{C}^*$-stable points meets the fixed point set of the $\mathbb{C}^*$-action, i.e., for which the corresponding set of $\mathbb{C}^*$-stable points contains stable oriented pairs of the type $[\ker(\varphi)_{\max} \oplus \mathcal{E}', \varepsilon, \varphi]$ with $0 \neq \ker(\varphi)_{\max}$ are uniquely determined. The corresponding polynomial is $\mathrm{rk}\,\mathcal{E}'(P_{\mathcal{E}'} - P_{\ker(\varphi)_{\max}}/\mathrm{rk}\ker(\varphi)_{\max})$. The associated moduli spaces $\mathcal{M}_{\delta_i}$ are those which show up "at the top" of the flips.

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