Gauge Symmetries of the Master Action

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We study the geometry of the Lagrangian Batalin–Vilkovisky theory on an antisymplectic manifold. We show that gauge symmetries of the BV-theory are essentially the symmetries of an *even symplectic* structure on the stationary surface of the master action.

1 Introduction

In this paper, we investigate gauge symmetries in the Lagrangian Batalin–Vilkovisky (BV) formalism [1, 2], which is the most universal approach to the quantization of general gauge theories. The version of the BV-quantization in which the coordinates are not explicitly separated into fields and antifields is known as the covariant approach [3, 4, 5, 6, 7, 8, 9]. The partition function is then given by a path integral of the exponential of the master action over the gauge-fixing surface $\mathcal{L}$, which is a Lagrangian submanifold of the odd-symplectic manifold $\mathcal{M}$. The gauge independence is realized as the independence from the choice of $\mathcal{L}$ and is ensured by the master equation imposed on the master action.

While the gauge symmetries of the original action are no longer explicitly present in the formalism, the covariant formulation itself has its own "gauge" transformations. Each observable (a BRST-closed function) determines a gauge symmetry. Studied in [10] were the gauge symmetries corresponding to the trivial observables (BRST-exact functions). It was shown there that the space of functions modulo the BRST-exact ones, called *the space of gauge parameters* [10], is endowed with the structure of a Lie algebra, which is induced by the Lie algebra structure on the space of BRST-trivial gauge symmetries.

In this paper, we study symmetries of the BV formalism using the geometrical setting provided by viewing the BV data as a QP-manifold [7, 11]. These are supermanifolds equipped with an antisymplectic structure (the P-structure) and an odd nilpotent Hamiltonian vector field (the Q-structure); in the BV setting, the latter is given by the antibracket with the master action. An important characteristics of the Q-structure is the *zero locus* $\mathcal{Z}_{\mathbf{Q}}$ of the odd vector field. The most interesting case in applications is where $\mathcal{Z}_{\mathbf{Q}}$ is a smooth $(n|N-n)$-dimensional submanifold (assuming the antisymplectic supermanifold $\mathcal{M}$ to be $(N|N)$-dimensional). We call such QP-manifolds the proper QP-manifolds; then the QP-structure induces a symplectic structure on the zero locus of $\mathbf{Q}$ (see (2.8)).[1]

It turns out that symmetries of the BV "master system" are to a considerable degree determined by Hamiltonian vector fields on the stationary surface of the master action (which are Hamiltonian with respect to the *Poisson bracket*). We will explicitly define a nondegenerate Poisson bracket on the quotient algebra of all smooth functions modulo the functions vanishing on $\mathcal{Z}_{\mathbf{Q}}$; this generalises the bilinear operation of [10], which was not a Poisson bracket since it failed to satisfy the Leibnitz rule (in fact, it was defined on the space that is not an algebra under the associative multiplication). We show

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[1]The existence of a symplectic structure on the zero locus can also be inferred from [11]; the Poisson bracket on (,)-Lagrangian submanifolds was described in [12]; see also [13].

that each symmetry of a proper QP-manifold—i.e., a vector field preserving the QP-structure—can be restricted to $\mathcal{Z}_{\mathbf{Q}}$ and, moreover, this restriction is a locally Hamiltonian vector field with respect to the Poisson bracket on $\mathcal{Z}_{\mathbf{Q}}$. Conversely, locally Hamiltonian vector fields on $\mathcal{Z}_{\mathbf{Q}}$ can be lifted to vector fields on $\mathcal{M}$, into symmetries of the proper QP-manifold. At the same time, the *globally* Hamiltonian vector fields on $\mathcal{Z}_{\mathbf{Q}}$ lift to BRST-trivial symmetries. In this way, we obtain a "translation table" between the objects pertaining to the antisymplectic geometry on $\mathcal{M}$ and to those of the symplectic geometry on $\mathcal{Z}_{\mathbf{Q}}$.

We further select the *on-shell symmetries*, i.e., the symmetries modulo those vanishing on the stationary surface. We show that the Lie algebra $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$ of on-shell gauge symmetries is isomorphic to the Lie algebra of locally Hamiltonian vector fields on the stationary surface $\mathcal{Z}_{\mathbf{Q}}$.

The Lie algebras of on-shell gauge symmetries ($\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$), of gauge parameters [10] ($\widetilde{\mathcal{O}}^c_{\text{triv}}$), and of gauge symmetries of the master action ($\mathcal{O}^c$), as well as their quantum counterparts, are related to each other as shown in (3.11) and to the cohomology, as shown in (3.16), (3.18), and (3.19).

We consider two examples of the general construction. We explicitly calculate the Lie algebras $\mathcal{O}^c$ and $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$ in the *abelianized* [18] gauge theory. It is not difficult to see then that the algebra of gauge symmetries of the original ("bare") classical theory is embedded into the Lie algebra $\mathcal{O}^c$ as a subalgebra in such a way that the algebra of on-shell gauge symmetries of the original theory is embedded into the Lie algebra $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$. As another example, we consider the theory with the vanishing action on a Lie group. Not surprisingly, the Poisson structure on the "stationary surface" is then related to the Kirillov bracket [16] on the coalgebra.

In section 2, we study the geometry of QP-manifolds. In section 3, we give a short reminder on the BV-quantization prescription and then study the quantum and classical gauge symmetries. In section 4, we demonstrate the main points of our construction in two characteristic examples.

2 Geometry of proper QP-manifolds

In this section, we study the geometry of QP-manifolds and define proper QP-manifolds, which are needed for applications to the BV-quantization. Then we show that the zero locus $\mathcal{Z}_{\mathbf{Q}}$ of the vector field $\mathbf{Q}$ on a proper QP-manifold is a symplectic manifold. Moreover, the vector fields that are *symmetries* of a proper QP-manifold correspond to locally Hamiltonian vector fields on $\mathcal{Z}_{\mathbf{Q}}$; the *BRST-trivial* symmetries then correspond to globally Hamiltonian vector fields.

2.1 A Poisson structure

Let $\mathcal{M}$ be an $(N|N)$-dimensional supermanifold, and let $\mathbb{C}_{\mathcal{M}}$ denote the algebra of smooth functions on $\mathcal{M}$. Let $(\cdot\,,\cdot) : \mathbb{C}_{\mathcal{M}} \times \mathbb{C}_{\mathcal{M}} \to \mathbb{C}_{\mathcal{M}}$ be an antibracket on $\mathcal{M}$. It satisfies

$$\epsilon((F\,,G)) \;=\; \epsilon(F) + \epsilon(G) + 1\,, \tag{2.1}$$

$$(F\,,G) \;=\; -(-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}(G\,,F)\,, \tag{2.2}$$

$$(F\,,GH) \;=\; (F\,,G)H + (-1)^{\epsilon(G)(\epsilon(F)+1)}G(F\,,H)\,, \tag{2.3}$$

$$0 \;=\; \underset{F,G,H}{\text{cycle}}(-1)^{(\epsilon(F)+1)(\epsilon(H)+1)}(F\,,(G\,,H))\,. \tag{2.4}$$

In a local coordinate system Γ^A, $A = 1,\ldots,2N$, we have the matrix $E^{AB} = (\Gamma^A\,,\Gamma^B)$ that defines a bivector E such that $(F\,,G) = E(dF,dG)$. We assume the antibracket to be nondegenerate.

Consider an odd vector field $\mathbf{Q} : \mathbb{C}_\mathcal{M} \to \mathbb{C}_\mathcal{M}$ on $\mathcal{M}$ satisfying the following conditions:

1. $\mathbf{Q}$ preserves the antibracket, i.e., $\mathcal{L}_\mathbf{Q} E = 0$, where $\mathcal{L}_\mathbf{Q}$ is the Lie derivative along $\mathbf{Q}$; equivalently, $\mathbf{Q}$ differentiates the antibracket:

$$\mathbf{Q}\,(F\,,\,G) - (\mathbf{Q}\,F\,,\,G) - (-1)^{\epsilon(F)+1}(F\,,\,\mathbf{Q}\,G) = 0\,, \quad F, G \in \mathbb{C}_\mathcal{M}\,; \tag{2.5}$$

2. $\mathbf{Q}$ is nilpotent: $\mathbf{Q}\,(\mathbf{Q}\,F) = 0\,,\ F \in \mathbb{C}_\mathcal{M}\,,\ \Longleftrightarrow\ [\mathbf{Q}\,,\,\mathbf{Q}\,] = 0$.

Definition 2.1 ([7, 11]) *A supermanifold $\mathcal{M}$ equipped with a nondegenerate antibracket $(\cdot\,,\,\cdot)$ and an odd nilpotent vector field $\mathbf{Q}$ that satisfies condition (2.5) is called the* QP*-manifold.*

The main object of our analysis is the set $\mathcal{Z}_\mathbf{Q}$ of zeroes of $\mathbf{Q}$, i.e. the set defined by equations $Q^A = 0$, where $\mathbf{Q} = Q^A\partial_A$ in a local coordinate system Γ^A. We assume $\mathcal{Z}_\mathbf{Q}$ to be a submanifold in $\mathcal{M}$. Denote by $\mathbb{I}_{\mathcal{Z}_\mathbf{Q}} \subset \mathbb{C}_\mathcal{M}$ the ideal of all smooth functions vanishing on $\mathcal{Z}_\mathbf{Q}$. We also assume the 'regularity condition', i.e. that $\mathbf{Q}$-exact functions generate the ideal $\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$, which means that any function $f \in \mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$ admits a representation $f = (\mathbf{Q}\,h)\,g$ with some $g, h \in \mathbb{C}_\mathcal{M}$. The quotient $\mathbb{C}_{\mathcal{Z}_\mathbf{Q}} = \mathbb{C}_\mathcal{M}/\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$ is the algebra of smooth functions on the submanifold $\mathcal{Z}_\mathbf{Q}$. Obviously, $\mathbf{Q}\,\mathbb{C}_\mathcal{M} \subset \mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$.

The additional requirement imposed on $\mathbf{Q}$ is that its local cohomology (that is, the cohomology evaluated in a sufficiently small neighbourhood of a point) is trivial (constants only) at every point $p \in \mathcal{Z}_\mathbf{Q}$. In local coordinates, then, the nilpotent operator $\partial Q^A/\partial \Gamma^B\big|_{Q^A=0}$ has the vanishing cohomology on the tangent space to every $p \in \mathcal{Z}_\mathbf{Q}$ [7, 11]. This in turn implies the condition from [1]:

$$\mathsf{rank}\left(\frac{\partial Q^A}{\partial \Gamma^B}\right)\bigg|_{Q^A=0} = N\,. \tag{2.6}$$

In particular, it follows from (2.6) that $\mathcal{Z}_\mathbf{Q}$ is $(n|N-n)$-dimensional submanifold.

Definition 2.2 *A proper* QP*-manifold is a* QP*-manifold on which the local cohomology of $\mathbf{Q}$ is trivial at every point from $\mathcal{Z}_\mathbf{Q}$.*

Lemma 2.3 *The submanifold $\mathcal{Z}_\mathbf{Q}$ of a proper* QP*-manifold is Lagrangian with respect to the antibracket. In particular, the ideal $\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$ is closed under the antibracket:*

$$(\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}\,,\,\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}) \subset \mathbb{I}_{\mathcal{Z}_\mathbf{Q}}\,. \tag{2.7}$$

PROOF. Since $\mathbf{Q}$-exact functions *generate* the ideal $\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$, any function from $\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$ can be represented as the product $(\mathbf{Q}\,f)\,h$ with some $h \in \mathbb{C}_\mathcal{M}$. Thus, it suffices to check that the antibracket of $\mathbf{Q}$-exact functions is $\mathbf{Q}$-exact, which is obvious in view of $(\mathbf{Q}\,g\,,\,\mathbf{Q}\,h) = \mathbf{Q}\,(g\,,\,\mathbf{Q}\,h)$. □

In fact the submanifold $\mathcal{Z}_\mathbf{Q}$ is endowed with a natural Poisson structure. This is given by a construction of the type of those used, with some variations, in [10, 12, 13, 15], namely

$$\{F\,,\,G\} = (F\,,\,\mathbf{Q}\,G)\,, \quad F, G \in \mathbb{C}_\mathcal{M}\,. \tag{2.8}$$

We interpret this structure as a bilinear mapping on the quotient algebra $\mathbb{C}_{\mathcal{Z}_\mathbf{Q}}$. Functions from $\mathbb{C}_\mathcal{M}$ considered modulo $\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$ represent functions on $\mathcal{Z}_\mathbf{Q}$. We then have

Theorem 2.4 *For any proper* QP*-manifold,*

1. *Equation (2.8) defines a Poisson bracket $\{\cdot\,,\,\cdot\} : \mathbb{C}_{\mathcal{Z}_\mathbf{Q}} \times \mathbb{C}_{\mathcal{Z}_\mathbf{Q}} \to \mathbb{C}_{\mathcal{Z}_\mathbf{Q}}$ on the submanifold $\mathcal{Z}_\mathbf{Q}$;*

2. *moreover, the Poisson bracket $\{\cdot\,,\,\cdot\}$ is nondegenerate (thus, $\mathcal{Z}_\mathbf{Q}$ is symplectic).*

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PROOF. First of all, we must prove that definition (2.8) does not depend on the choice of representatives of the equivalence classes, i.e., $\{F + f, G + g\} - \{F, G\} \in \mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$ whenever $f, g \in \mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$. Since $\mathbf{Q}$ differentiates the antibracket, we can check that

$$\{F + f, G + g\} - \{F, G\} = (-1)^{\epsilon(F)}(\mathbf{Q}\,F, g) + (-1)^{\epsilon(F)+1}\mathbf{Q}\,(F, g) + (f, \mathbf{Q}\,(G + g))\,.$$

The first and the third terms belong to $\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$ by Lemma 2.3 and the second term is in $\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$ because it is $\mathbf{Q}$-exact. Thus (2.8) defines a mapping $\{\cdot\,,\,\cdot\} : \mathbb{C}_{\mathcal{Z}_\mathbf{Q}} \times \mathbb{C}_{\mathcal{Z}_\mathbf{Q}} \to \mathbb{C}_{\mathcal{Z}_\mathbf{Q}}$. It is antisymmetric because

$$\begin{aligned}
\{F, G\} + (-1)^{\epsilon(F)\epsilon(G)}\{G, F\} &= (F, \mathbf{Q}\,G) + (-1)^{\epsilon(F)\epsilon(G)}(G, \mathbf{Q}\,F) \\
&= (-1)^{\epsilon(F)+1}\mathbf{Q}\,(F, G) \in \mathbb{I}_{\mathcal{Z}_\mathbf{Q}}\,,
\end{aligned}$$

where we used (2.5) again. Next, to prove the Leibnitz rule, we evaluate

$$\begin{aligned}
\{F, GH\} - \{F, G\}H &- (-1)^{\epsilon(F)\epsilon(G)}G\{F, H\} \\
&= (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)}(\mathbf{Q}\,G)(F, H) + (-1)^{\epsilon(G)}(F, G)(\mathbf{Q}\,H)\,,
\end{aligned}$$

which evidently vanishes modulo terms from $\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$. Finally, to prove the Jacobi identity we have to show that

$$\operatorname*{cycle}_{F,G,H} (-1)^{\epsilon(F)\epsilon(H)}\{F, \{G, H\}\} = \operatorname*{cycle}_{F,G,H} (-1)^{\epsilon(F)\epsilon(H)}(F, \mathbf{Q}\,(G, \mathbf{Q}\,H)) \equiv 0 \bmod \mathbb{I}_\mathbf{Q}$$

In view of the Leibnitz rule and the nilpotency condition this rewrites, modulo terms from $\mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$, as

$$\operatorname*{cycle}_{\mathbf{Q}\,F, G, \mathbf{Q}\,H} (-1)^{(\epsilon(\mathbf{Q}\,F)+1)(\epsilon(\mathbf{Q}\,H)+1)}(\mathbf{Q}\,F, (G, \mathbf{Q}\,H))\,,$$

which vanishes by virtue of the Jacobi identity for the antibracket.

To prove that the Poisson bracket is nondegenerate on $\mathcal{Z}_\mathbf{Q}$, we recall a standard fact from symplectic geometry, namely that in some neighborhood of $\mathcal{Z}_\mathbf{Q}$ there exists a coordinate system x^i, ξ_i such that the antibracket takes the canonical form $(x^i, \xi_j) = \delta^i_j$ and $\mathcal{Z}_\mathbf{Q}$ is determined by $\xi_i = 0$. Locally, the vector field $\mathbf{Q}$ can be written in the form $\mathbf{Q} = (S, \cdot)$ with some function $S \in \mathbb{C}_\mathcal{M}$ (since a vector field preserving a nondegenerate (anti)bracket is locally Hamiltonian). Expanding S as

$$S = S_0(x) + \xi_i S^i(x) + \xi_i S^{ij}(x)\xi_j + \xi_i\xi_j\xi_k S^{ijk}(x) + \ldots\,,$$

we see that $S_0(x) = \mathrm{const}$ and $S^i(x) = 0$, because $\mathbf{Q} = (S, \cdot)$ vanishes as $\xi = 0$. Then condition (2.6) means that the matrix $S^{ij}(x)$ is non-degenerate at each point of $\mathcal{Z}_\mathbf{Q}$. On the other hand, $S^{ij}(x)|_{\xi=0} = -\frac{1}{2}\{x^i, x^j\}|_{\xi=0}$, which shows the theorem. $\square$

Note that the symbol $\{\cdot\,,\,\cdot\}$ is used for the formal operation (2.8) on the manifold $\mathcal{M}$ and also for the Poisson bracket on the submanifold $\mathcal{Z}_\mathbf{Q}$. We do not introduce two different symbols and hope that this will not lead to confusion.

2.2 Symmetries of QP-manifolds

For applications to the BV-quantization in the subsequent sections, we will need some facts about symmetries of QP-manifolds.

Definition 2.5 *A vector field* $\mathbf{X}$ *on a* QP-*manifold* $\mathcal{M}$ *is called a symmetry of* $\mathcal{M}$ *if*

1. $\mathbf{X}$ *preserves the antibracket* $(\cdot\,,\,\cdot)$:

$$\mathbf{X}\,(F, G) - (\mathbf{X}\,F, G) - (-1)^{(\epsilon(F)+1)\epsilon(\mathbf{X})}(F, \mathbf{X}\,G) = 0\,, \qquad F, G \in \mathbb{C}_\mathcal{M}\,, \qquad (2.9)$$

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2. **X** *preserves the odd vector field* **Q**: $[\mathbf{Q}\,,\mathbf{X}\,] = 0$.

Our aim is to demonstrate that symmetries of a proper QP-manifold restrict to the zero locus of **Q** and to study the properties of these restrictions. Let us therefore begin with characterising, in the standard way, those vector fields on $\mathcal{M}$ that restrict to $\mathcal{Z}_{\mathbf{Q}}$:

Lemma 2.6 *A vector field* **X** *on a proper* QP-*manifold* $\mathcal{M}$ *restricts to* $\mathcal{Z}_{\mathbf{Q}}$ *if and only if*

$$[\mathbf{X}\,,\mathbf{Q}\,]\big|_{\mathcal{Z}_{\mathbf{Q}}} = 0. \tag{2.10}$$

PROOF.   A vector field restricts to $\mathcal{Z}_{\mathbf{Q}}$ if and only if it preserves the ideal of functions vanishing on $\mathcal{Z}_{\mathbf{Q}}$: $\mathbf{X}\,\mathbb{I}_{\mathcal{Z}_{\mathbf{Q}}} \subset \mathbb{I}_{\mathcal{Z}_{\mathbf{Q}}}$. Now, $[\mathbf{X}\,,\mathbf{Q}\,]\big|_{\mathcal{Z}_{\mathbf{Q}}} = 0 \iff [\mathbf{X}\,,\mathbf{Q}]f \in \mathbb{I}_{\mathcal{Z}_{\mathbf{Q}}}\ \forall f \in \mathbb{C}_{\mathcal{M}}$, which rewrites as $\mathbf{X}\,\mathbf{Q}\,f \in \mathbb{I}_{\mathcal{Z}_{\mathbf{Q}}}$. Since $\mathbf{Q}$-exact functions generate the ideal, we conclude that $\mathbf{X}\,\mathbb{I}_{\mathcal{Z}_{\mathbf{Q}}} \subset \mathbb{I}_{\mathcal{Z}_{\mathbf{Q}}}$. The converse is now obvious. $\square$

It follows from this Lemma that any vector field **X** that is a symmetry of a proper QP-manifold can be restricted to $\mathcal{Z}_{\mathbf{Q}}$. We now recall that the zero locus of **Q** is endowed with a Poisson structure.

Theorem 2.7 *If a vector field* **X** *is a symmetry of a proper* QP-*manifold* $\mathcal{M}$*, its restriction* $\mathbf{X}\big|_{\mathcal{Z}_{\mathbf{Q}}}$ *to the zero locus of* **Q** *preserves the Poisson bracket from Theorem 2.4:*

$$\mathbf{X}\big|_{\mathcal{Z}_{\mathbf{Q}}}\{F\,,G\} - \{\mathbf{X}\big|_{\mathcal{Z}_{\mathbf{Q}}}F\,,G\} - (-1)^{\epsilon(F)\epsilon(\mathbf{X})}\{F\,,\mathbf{X}\big|_{\mathcal{Z}_{\mathbf{Q}}}G\} = 0\,, \qquad F,G \in \mathbb{C}_{\mathcal{Z}_{\mathbf{Q}}}\,. \tag{2.11}$$

PROOF.   Let us choose two representatives $F,G \in \mathbb{C}_{\mathcal{M}}$ of the equivalence classes of functions on $\mathcal{Z}_{\mathbf{Q}}$. Using the properties stated in Definition 2.5, we have

$$\begin{aligned}
\mathbf{X}\{F\,,G\} &= \mathbf{X}\,(F\,,\mathbf{Q}\,G) = (\mathbf{X}\,F\,,\mathbf{Q}\,G) + (-1)^{(\epsilon(F)+1)\epsilon(\mathbf{X})}(F\,,\mathbf{X}\,\mathbf{Q}\,G) \\
&= \{\mathbf{X}\,F\,,G\} + (-1)^{\epsilon(F)\epsilon(\mathbf{X})}\{F\,,\mathbf{X}\,G\}.
\end{aligned}$$

$\square$

It follows from the nondegeneracy of the Poisson bracket (2.8) that any vector field $\mathbf{x}$ on $\mathcal{Z}_{\mathbf{Q}}$ that preserves the Poisson bracket can be written as $\mathbf{x} = \{H\,,\cdot\}$ with some (locally) defined function H. We will refer to this as a *locally Hamiltonian* vector field. Every $\mathbf{x} = \{H\,,\cdot\}$ with a globally defined H will be called *globally Hamiltonian*. It is well known that globally Hamiltonian vector fields form an ideal in the Lie algebra of locally Hamiltonian vector fields.

2.3   Lifts and restrictions of vector fields

In this section, we are interested in relations between Hamiltonian vector fields on the symplectic manifold $\mathcal{Z}_{\mathbf{Q}}$ and the Hamiltonian vector fields on $\mathcal{M}$,[2] in particular, symmetries of the proper QP-manifold $\mathcal{M}$.

To explain why there exists a correspondence between symmetries of proper QP-manifold $\mathcal{M}$ and symmetries of its symplectic submanifold $\mathcal{Z}_{\mathbf{Q}}$, we begin with an example [15]. Consider an

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[2]   Thus, whenever we speak about Hamiltonian vector fields on $\mathcal{M}$, these are Hamiltonian with respect to the *anti*bracket, while the Hamiltonian vector fields on $\mathcal{Z}_{\mathbf{Q}}$ are Hamiltonian with respect to the Poisson bracket.

N-dimensional symplectic manifold $\mathcal{K}$ with the symplectic form $\widehat{\omega}$ which defines the nondegenerate Poisson bracket $\{\cdot\,,\,\cdot\}$ (in general, $\mathcal{K}$ can be a supermanifold, but we assume for simplicity that it is an even manifold). In a local coordinate system x^i, we have the invertible matrix $\omega^{ij} = \{x^i\,,\,x^j\}$. Let $\Pi T^*\mathcal{K}$ be the cotangent bundle with the flipped parity. In the canonical coordinates x^i, ξ_i (with the antibracket $(x^i\,,\,\xi_j) = \delta^i_j$), the manifold $\mathcal{K}$ can be identified with the zero section $\xi_i = 0$ of $\Pi T^*\mathcal{K}$. The function $S = \frac{1}{2}\xi_i\omega^{ij}\xi_j$ satisfies $(S\,,\,S) = 0$ because ω^{ij} is the matrix of a Poisson bracket. Then the submanifold $\mathcal{K}$ is the zero locus of the vector field

$$\mathbf{Q} = (S\,,\,\cdot\,) = \tfrac{1}{2}\xi_i(\tfrac{\vec{\partial}}{\partial x^k}\omega^{ij})\xi_j\tfrac{\vec{\partial}}{\partial\xi_k} - \xi_i\omega^{ij}\tfrac{\vec{\partial}}{\partial x^j}\,. \tag{2.12}$$

It is easy to see that $\mathbf{Q}$ meets the conditions of Definition 2.1. In addition, condition (2.6) is satisfied because the dimension of $\mathcal{K}$ is N. Thus $\Pi T^*\mathcal{K}$ is a QP-manifold and in fact a proper QP-manifold (because ω^{ij} is nondegenerate). With the help of the symplectic form, we can identify $\Pi T^*\mathcal{K}$ with the tangent bundle $\Pi T\mathcal{K}$ (with $\xi^i = \omega^{ij}\xi_j$ being the coordinates on the fibers). Then we can rewrite $\mathbf{Q}$ as

$$\mathbf{Q} = \xi^i\tfrac{\vec{\partial}}{\partial x^i}\,, \tag{2.13}$$

Upon the identification of functions on $\Pi T\mathcal{K}$ with differential forms on $\mathcal{K}$, $\mathbf{Q}$ becomes the De Rham differential on $\mathcal{K}$ [11].

Further, every locally Hamiltonian vector field $\mathbf{x}$ on $\mathcal{K}$ can be lifted to a globally Hamiltonian vector field $\mathbf{X}$ on $\Pi T^*\mathcal{K}$. Namely, if $\mathbf{x} = \{H\,,\,\cdot\}$ (where we allow H to be multivalued), we take $F = (-1)^{\epsilon(H)}\mathbf{Q}\,(\pi^*H)$, where π^* is the pullback with respect to the canonical projection $\pi : \Pi T^*\mathcal{K} \to \mathcal{K}$. Then the vector field $\mathbf{X} = (F\,,\,\cdot\,)$ *is well-defined* (independent of the multivaluedness of H) and satisfies

$$\mathbf{X}\big|_{\mathcal{K}} = \mathbf{x} \tag{2.14}$$

and is a symmetry of $\Pi T^*\mathcal{K}$ in the sense of Definition 2.5. Conversely, any symmetry of $\Pi T^*\mathcal{K}$ determines a locally Hamiltonian vector field on $\mathcal{K}$ (see Theorem 2.7), which is obviously a symmetry of this symplectic manifold.

The above is a particular case of a more general construction. Namely, there exists a similar correspondence between symmetries of proper QP-manifold $\mathcal{M}$ and symmetries of its symplectic submanifold $\mathcal{Z}_\mathbf{Q}$, even though $\mathcal{M}$ can be an arbitrary proper QP-manifold.

We have seen in theorem 2.7 that every symmetry of a proper QP-manifold $\mathcal{M}$ restricts to $\mathcal{Z}_\mathbf{Q}$ as a locally Hamiltonian vector field. Consider now the converse problem. We say that a symmetry $\mathbf{X}$ of proper QP-manifold $\mathcal{M}$ is a lift from $\mathcal{Z}_\mathbf{Q}$ of a locally Hamiltonian vector field $\mathbf{x}$ if $\mathbf{X}$ restricts to $\mathcal{Z}_\mathbf{Q}$ and $\mathbf{X}\big|_{\mathcal{Z}_\mathbf{Q}} = \mathbf{x}\,.$

Theorem 2.8 *Every locally Hamiltonian vector field* $\mathbf{x}$ *on* $\mathcal{Z}_\mathbf{Q}$ *admits a lift to a symmetry of* $\mathcal{M}$ *that is a globally Hamiltonian vector field on* $\mathcal{M}$ *with a* $\mathbf{Q}$-*closed Hamiltonian. If* $\mathbf{x}$ *is globally Hamiltonian on* $\mathcal{Z}_\mathbf{Q}$*, it is lifted to a globally Hamiltonian vector field with a* $\mathbf{Q}$-*exact Hamiltonian.*

In what follows, symmetries of $\mathcal{M}$ of the form $\mathbf{X} = (F\,,\,\cdot\,)$ with a $\mathbf{Q}$-exact Hamiltonian F are called *BRST-trivial symmetries*.

PROOF. For a (locally) Hamiltonian vector field $\mathbf{x}$ on $\mathcal{Z}_\mathbf{Q}$, the equation $\mathbf{x} = \{H\,,\,\cdot\}$ can be solved for H in a sufficiently small neighbourhood of every point $p \in \mathcal{Z}_\mathbf{Q}$. Different such solutions can be considered as a multivalued Hamiltonian H. This can be extended to a multivalued function $\widetilde{H}$ on $\mathcal{M}$

that restricts to H on $\mathcal{Z}_{\mathbf{Q}}$ (for example, consider a neighbourhood U of $\mathcal{Z}_{\mathbf{Q}}$ in $\mathcal{M}$ and identify it with a neighbourhood of the zero section of a vector bundle over $\mathcal{Z}_{\mathbf{Q}}$; if $\widetilde{h}$ is the pullback of the multivalued function H to the bundle, we can choose a function $\alpha \in \mathbb{C}_{\mathcal{M}}$ such that $\alpha|_{\mathcal{Z}_{\mathbf{Q}}} = 1$ and $\alpha = 0$ outside U, which yields the lifting $\widetilde{H} = \alpha \widetilde{h}$ of the multivalued Hamiltonian H).

Then consider the function $F = (-1)^{\epsilon(H)} \mathbf{Q} \widetilde{H}$ on $\mathcal{M}$; F is a *single-valued* function that is $\mathbf{Q}$-closed by construction but in general is not $\mathbf{Q}$-exact (because its $\mathbf{Q}$-primitive is not necessarily single-valued). Now, let $\mathbf{X} = (F, \cdot)$. For any function $\widetilde{G} \in \mathbb{C}_{\mathcal{M}}$, we have

$$(F, \widetilde{G})\Big|_{\mathcal{Z}_{\mathbf{Q}}} = (\widetilde{H}, \mathbf{Q}\widetilde{G})\Big|_{\mathcal{Z}_{\mathbf{Q}}},$$

which coincides with $\{\widetilde{H}\big|_{\mathcal{Z}_{\mathbf{Q}}}, \widetilde{G}\big|_{\mathcal{Z}_{\mathbf{Q}}}\} = \mathbf{x}\widetilde{G}\big|_{\mathcal{Z}_{\mathbf{Q}}}$ (see (2.8)). Thus $\mathbf{X}$ is a lift of $\mathbf{x}$ to a symmetry of $\mathcal{M}$.

Whenever the Hamiltonian H of $\mathbf{x}$ on $\mathcal{Z}_{\mathbf{Q}}$ is globally defined on $\mathcal{Z}_{\mathbf{Q}}$, the function F is obviously $\mathbf{Q}$-exact and thus $\mathbf{X} = (F, \cdot)$ is a BRST trivial symmetry of $\mathcal{M}$. $\square$

This theorem can also be seen by noticing that the cohomology of $\mathbf{Q}$ evaluated on an appropriately chosen neighbourhood U of $\mathcal{Z}_{\mathbf{Q}}$ in $\mathcal{M}$ coincides with the De Rham cohomology of $\mathcal{Z}_{\mathbf{Q}}$ [11]. Namely, one can identify the neighbourhood U of $\mathcal{Z}_{\mathbf{Q}}$ with some neighbourhood of the zero section of $\Pi T \mathcal{Z}_{\mathbf{Q}}$. Then vector field $\mathbf{Q}$ can be written as $\mathbf{Q} = \xi^i \frac{\overrightarrow{\partial}}{\partial x^i}$, where x^i and ξ^i are coordinates on $\mathcal{Z}_{\mathbf{Q}}$ and on the fibers, respectively. Thus $\mathbf{Q}$ coincides with the De Rham differential of $\mathcal{Z}_{\mathbf{Q}}$ if one identifies functions on U that are homogeneous in ξ with the differential forms on $\mathcal{Z}_{\mathbf{Q}}$. In particular, every closed but not exact 1-form $f = dx^i f_i$ leads to the function $F = \xi^i f_i$ on U that is obviously in the cohomology of $\mathbf{Q}$. At the same time, the 1-form $f = dx^i f_i$ gives rise to the locally Hamiltonian vector field $\mathbf{x} = (-1)^{\epsilon(x^i)\epsilon(f_i)} f_i \omega^{ij} \frac{\overrightarrow{\partial}}{\partial x^j}$ on $\mathcal{Z}_{\mathbf{Q}}$ (where $\omega^{ij} = \{x^i, x^j\}$). Therefore $\mathbf{x}$ lifts to the vector field $(-1)^{\epsilon(F)+1}(F, \cdot)$ on U whose Hamiltonian is the same function $F = \xi^i f_i$.

We thus see, in particular, that a locally Hamiltonian vector field representing the first cohomology of $\mathcal{Z}_{\mathbf{Q}}$ corresponds to an element of the $\mathbf{Q}$-cohomology on $\mathcal{M}$. To complete the section let us single out those $(\cdot, \cdot)$-Hamiltonian vector fields on $\mathcal{M}$ that restrict to $\{\cdot, \cdot\}$-Hamiltonian vector fields on $\mathcal{Z}_{\mathbf{Q}}$, and describe the full arbitrariness of the lifts of Hamiltonian vector fields on $\mathcal{Z}_{\mathbf{Q}}$ to Hamiltonian vector fields on $\mathcal{M}$. The following is proved by directly generalising the proof of Theorem 2.8.

Theorem 2.9

1. *Let $\mathbf{X} = (F, \cdot)$ be a globally Hamiltonian vector field on a proper QP-manifold $\mathcal{M}$ with the Hamiltonian satisfying $\mathbf{Q}F \in \mathbb{I}^3_{\mathcal{Z}_{\mathbf{Q}}}$ (i.e., $\mathbf{Q}F = Q^A Q^B Q^C Y_{ABC}$ with some $Y_{ABC} \in \mathbb{C}_{\mathcal{M}}$). Then $\mathbf{X}$ restricts to a locally Hamiltonian vector field on $\mathcal{Z}_{\mathbf{Q}}$.*

2. *Every locally Hamiltonian vector field $\mathbf{x}$ on $\mathcal{Z}_{\mathbf{Q}}$ admits a lift to a globally Hamiltonian vector field on $\mathcal{M}$ with the Hamiltonian F satisfying $\mathbf{Q}F = 0$. If $\mathbf{x}$ is globally Hamiltonian on $\mathcal{Z}_{\mathbf{Q}}$ with the Hamiltonian H, the Hamiltonians of all its lifts to $\mathcal{M}$ are of the form*

$$F = (-1)^{\epsilon(\widetilde{H})} \mathbf{Q}\widetilde{H} + K + \mathrm{const}, \quad K \in \mathbb{I}^2_{\mathcal{Z}_{\mathbf{Q}}}, \tag{2.15}$$

where $\widetilde{H}$ is any function on $\mathcal{M}$ such that $\widetilde{H}\big|_{\mathcal{Z}_{\mathbf{Q}}} = H$.

3 Gauge symmetries of the master-action

We now interpret classical gauge symmetries in the covariant BV formalism as symmetries of the corresponding proper QP-manifold. Using the results of the previous section, we then show that the Lie algebra of locally Hamiltonian vector fields on the stationary surface of the master action coincides with the algebra of *on-shell gauge symmetries*. Section 3.1 contains a brief reminder on the BV formalism, so the reader may wish to go directly to Sec. 3.2.

3.1 Batalin–Vilkovisky quantization

The geometrical background of the covariant formulation of the BV quantization is an $(N|N)$-dimensional supermanifold $\mathcal{M}$ equipped with a nondegenerate antibracket $(\cdot\,,\,\cdot)$ and a volume form $d\mu = \rho\,d\Gamma$, where $\rho = \rho(\Gamma)$ is a density (and Γ^A, $A = 1,\dots 2N$, are some local coordinates). The density should be compatible with the antibracket in such a way that the BV Δ operator

$$\Delta_\rho H = \tfrac{1}{2}\operatorname{div}_\rho(\mathbf{V}_H) \tag{3.1}$$

be nilpotent, $\Delta_\rho^2 = [\Delta_\rho,\,\Delta_\rho] = 0$. Here, div_ρ denotes the divergence of a vector field with respect to the density ρ and $\mathbf{V}_H = (H\,,\,\cdot\,)$ is the globally Hamiltonian vector field with the Hamiltonian H.

The physics is determined by the quantum master-action $W \in \mathbb{C}_{\mathcal{M}}[[\hbar]]$ (a formal power series in $\hbar$ with coefficients in $\mathbb{C}_{\mathcal{M}}$) that satisfies the quantum master equation

$$\Delta_\rho e^{\frac{i}{\hbar}W} = 0 \quad\Longleftrightarrow\quad \tfrac{1}{2}(W\,,\,W) = i\hbar\Delta_\rho W\,. \tag{3.2}$$

Writing $W = S + i\hbar W_1 + (i\hbar)^2 W_2 + \dots$, we rewrite (3.2) as

$$(S\,,\,S) = 0\,, \tag{3.3}$$
$$(S\,,\,W_1) = \Delta_\rho S\,, \tag{3.4}$$

and so on. Equation (3.3) is the classical master equation, and the function $S = W|_{\hbar=0}$ is called the classical master action.

In addition to the master equation, one should impose boundary conditions on W. This requires fixing a Lagrangian submanifold $\mathcal{L}_0$ in $\mathcal{M}$ (in the canonical coordinates, the manifold of fields, with the antifields set to zero) and a function $\mathcal{S}$ on $\mathcal{L}_0$, which is the *original ("bare") action* of the classical theory that is being quantised. Then one requires $W(\Gamma,\hbar)$ to be such that $W(\cdot,0)|_{\mathcal{L}_0} = \mathcal{S}$.

By definition, a quantum observable is a function $A \in \mathbb{C}_{\mathcal{M}}[[\hbar]]$ that satisfies $\delta_W A = 0$, where

$$\delta_W A = (W, A) - i\hbar\Delta_\rho A\,. \tag{3.5}$$

It follows from (3.2) that $\delta_W^2 = 0$ and therefore any function A of the form $A = \delta_W B$ is an observable; these are called *trivial* observables. Expanding $A = A_0 + i\hbar A_1 + (i\hbar)^2 A_2 + \dots$, we rewrite equation $\delta_W A = 0$ as

$$(S\,,\,A_0) = 0\,, \tag{3.6}$$
$$(S\,,\,A_1) + (W_1\,,\,A_0) = \Delta_\rho A_0\,, \tag{3.7}$$

and so on. An $\hbar$-independent function A_0 satisfying (3.6) is called the *classical observable*. It is easy to see that if $A = \delta_W B$ then $A_0 = (S, B_0)$, where $B_0 = B|_{\hbar=0}$. Any classical observable A_0 of the form $A_0 = (S, B_0)$ with some $\hbar$-independent function B_0 is called the *trivial classical observable*.

The quantum expectation of an observable is defined via the path-integral over a Lagrangian submanifold $\mathcal{L}$,

$$\langle A \rangle = \int_{\mathcal{L}} d\lambda_\rho A e^{\frac{i}{\hbar}W}\,, \qquad (3.8)$$

where $d\lambda_\rho$ is the volume form on $\mathcal{L}$ determined by the volume form $d\mu = \rho d\Gamma$ on $\mathcal{M}$ and by the antisymplectic structure as follows [4, 3]:

$$d\lambda_\rho(e^1, \ldots, e^N) = (d\mu(e^1, \ldots, e^N, f_1, \ldots, f_N))^{\frac{1}{2}}\,, \qquad (3.9)$$

where $e^i \in T\mathcal{L}$ and $f_j \in T\mathcal{M}$ are any vectors that satisfy $\widehat{E}(e^i, f_j) = \delta^i_j$ and $\widehat{E}$ is the antisymplectic two-form on $\mathcal{M}$. It follows from (3.9) that the volume form $d\lambda_{\rho'}$ corresponding to the density function $\rho' = \rho e^H$ is related to $d\lambda_\rho$ as $d\lambda_{\rho'} = d\lambda_\rho e^{\frac{1}{2}H}$ (this is the origin of the exponent in Definition 3.1). If the submanifold $\mathcal{L}$ is determined by equations $G_\alpha = 0$, $\alpha = 1, \ldots, N$, it would be Lagrangian whenever $(G_\alpha, G_\beta) = U^\gamma_{\alpha\beta}G_\gamma$.

An important part of the BV axioms is the nondegeneracy conditions. Submanifold $\mathcal{L}$ in (3.8) must be such that the restriction of $S = W|_{\hbar=0}$ to $\mathcal{L}$ be nondegenerate. In terms of the equations $G_\alpha = 0$, the matrix $\partial_A G_\alpha$ and the Hessian matrix $\partial_A \partial_B S$ should have no common null vectors at the points where $\partial_A S = 0$ and $G_\alpha = 0$ [1, 2, 3]. Whenever the set $\mathcal{Z}_{(S,\cdot)}$ defined by equations $\partial_A S = 0$ is a submanifold, this requirement means that $\mathcal{Z}_{(S,\cdot)}$ intersects $\mathcal{L}$ transversely. It also follows that the rank of the Hessian matrix $H_{AB} = (\partial_A \partial_B S)|_{\partial_A S = 0}$ satisfies $\mathsf{rank}\,(H_{AB}) \geq N$. At the same time, the classical master equation (3.3) implies that $\mathsf{rank}\,(H_{AB}) \leq N$, whence [1, 2, 3]

$$\mathsf{rank}\left(\frac{\partial^2 S}{\partial \Gamma^A \partial \Gamma^B}\right)\bigg|_{\partial_A S = 0} = N\,. \qquad (3.10)$$

The solution of classical master-equation (3.3) that satisfies (3.10) is called a *proper solution*.

The key statement of the BV formalism is that the path integral constructed as in (3.8) is invariant under infinitesimal deformations of the Lagrangian submanifold $\mathcal{L}$ [1, 2, 4, 3] for every quantum observable A. In the case where $A = 1$, this is often called the gauge independence of the partition function.

3.2 Lie algebras of gauge symmetries

We now study Lie algebras of gauge symmetries in the BV quantization scheme. These are Lie algebras $\mathcal{O}^q$ and $\mathcal{O}^c$ of *quantum and classical gauge symmetries*, respectively. In addition to these two basic algebras, it is useful to consider several more Lie algebras, which we define in what follows and which can be arranged into the following commutative diagram of homomorphisms of Lie algebras:

$$
\begin{array}{ccccc}
\widetilde{\mathcal{O}}^q_{\text{triv}} & \longrightarrow & \mathcal{O}^q_{\text{triv}} & \longrightarrow & \mathcal{O}^q \\
\big\downarrow {\scriptstyle \hbar=0} & & \big\downarrow {\scriptstyle \hbar=0} & & \big\downarrow {\scriptstyle \hbar=0} \\
\widetilde{\mathcal{O}}^c_{\text{triv}} & \longrightarrow & \mathcal{O}^c_{\text{triv}} & \longrightarrow & \mathcal{O}^c
\end{array}
\qquad (3.11)
$$
$$
\searrow \qquad \big\downarrow \qquad \swarrow
$$
$$
\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}
$$

9

In addition, we have homomorphisms (3.16), (3.18), and (3.19), whose constructions will also be explained in what follows. Here $\mathbb{H}_{\mathcal{Z}_\mathbf{Q}}$ is the Lie algebra of *on-shell gauge symmetries* (which, as we show, is the algebra of locally Hamiltonian vector fields on $\mathcal{Z}_\mathbf{Q}$), $\mathcal{O}_{\mathrm{triv}}^q$ is the Lie algebra *of BRST-trivial quantum gauge symmetries*, and $\widetilde{\mathcal{O}}_{\mathrm{triv}}^q$ is the Lie algebra *of quantum gauge parameters* [10, 6]. Lie algebras $\mathcal{O}^c$, $\mathcal{O}_{\mathrm{triv}}^c$ and $\widetilde{\mathcal{O}}_{\mathrm{triv}}^c$ are the classical counterparts of $\mathcal{O}^q$, $\mathcal{O}_{\mathrm{triv}}^q$, and $\widetilde{\mathcal{O}}_{\mathrm{triv}}^q$ respectively. We now proceed to the exact definitions.

Definition 3.1 ([10, 14]) *A vector field* $\mathbf{X}(\hbar)$ *is called the quantum gauge symmetry if it preserves the antibracket and the measure* $\rho e^{\frac{2i}{\hbar}W} d\Gamma$ *(viewed as formal power series in $\hbar$).*

The Lie algebra $\mathcal{O}^q$ of these vector fields is called the Lie algebra of quantum gauge symmetries.

It follows from this definition that a quantum gauge symmetry $\mathbf{X}(\hbar)$ satisfies

$$\mathrm{div}_\rho(\mathbf{X}(\hbar)) + \tfrac{2i}{\hbar}\mathbf{X}(\hbar)W = 0, \tag{3.12}$$

$$\mathbf{X}(F,G) - (\mathbf{X}F,G) - (-1)^{(\epsilon(F)+1)\epsilon(\mathbf{X})}(F,\mathbf{X}G) = 0, \qquad F,G \in \mathbb{C}_{\mathcal{M}}. \tag{3.13}$$

Equation (3.13) implies that there exists, at least locally, a function $A(\hbar)$ such that $\mathbf{X}(\hbar) = (A(\hbar),\,\cdot\,)$. Then Eq. (3.12) implies that $(W, A(\hbar)) - i\hbar\Delta_\rho A(\hbar) = 0$. Whenever $A(\hbar)$ is globally defined, it is a quantum observable. We explicitly indicate the $\hbar$-dependence of $\mathbf{X}(\hbar)$ because $\rho e^{\frac{2i}{\hbar}W}$ should be preserved for any value of $\hbar$; we assume $\mathbf{X}(\hbar)$ to be a formal power series in $\hbar$ with coefficients in the vector fields on $\mathcal{M}$.

Although $\mathbf{X}(\hbar)$ is not a symmetry of any classical system (in particular, it preserves neither the quantum master action nor the measure $d\mu$), we call it a quantum gauge symmetry because its classical counterpart, obtained by taking the limit as $\hbar \to 0$, does preserve the classical master action S. (The latter can be considered as the action of some classical system defined on $\mathcal{M}$. Then the classical master equation can be viewed as an additional constraint imposed on the system with the action S. One can naturally identify gauge symmetries of this system with the transformations preserving both S and the master equation imposed on S.)

To make contact with the literature, we consider the Lie algebra $\mathcal{O}_{\mathrm{triv}}^q$ of quantum *BRST-trivial gauge symmetries* studied in [10]. These are quantum gauge symmetries $\mathbf{X}_B(\hbar) = (\delta_W B(\hbar),\,\cdot\,)$ whose Hamiltonians are trivial observables (see (3.5)), which span an ideal in $\mathcal{O}^q$.

Now, the (Hamiltonian) mapping $\mathbb{C}_{\mathcal{M}}[[\hbar]] \to \mathcal{O}_{\mathrm{triv}}^q$ allows us to pullback the Lie bracket from $\mathcal{O}_{\mathrm{triv}}^q$ to the space of $\hbar$-dependent functions. Namely,

$$\left[B^1(\hbar), B^2(\hbar)\right]^q = (B^1(\hbar),\, \delta_W B^2(\hbar)), \tag{3.14}$$

which implies

$$[\mathbf{X}_{B^1}(\hbar), \mathbf{X}_{B^2}(\hbar)] = (\delta_W(B^1(\hbar),\, \delta_W B^2(\hbar)),\,\cdot\,) = \mathbf{X}_{[B^1(\hbar), B^2(\hbar)]^q}. \tag{3.15}$$

The bracket (3.14) was shown in [10, 6] to determine a Lie algebra structure on the quotient space

$$\widetilde{\mathcal{O}}_{\mathrm{triv}}^q = \mathbb{C}_{\mathcal{M}}[[\hbar]]\Big/\delta_W \mathbb{C}_{\mathcal{M}}[[\hbar]]$$

of all $\hbar$-dependent functions modulo the δ_W-exact ones. $\widetilde{\mathcal{O}}_{\mathrm{triv}}^q$ was called the *Lie algebra of quantum gauge parameters* in [10, 6].[3]

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[3]This is not an algebra under the *associative* multiplication because the multiplication does not preserve the equivalence classes $B(\hbar) \sim B(\hbar) + \delta_W C(\hbar)$; in particular, (3.14) is not a Poisson bracket.

There is a nice way to 'measure' how $\mathcal{O}_{\text{triv}}^q$ differs from $\widetilde{\mathcal{O}}_{\text{triv}}^q$. The (Hamiltonian) mapping $\mathbb{C}_{\mathcal{M}}[[\hbar]] \to \mathcal{O}_{\text{triv}}^q$ induces a homomorphism $\widetilde{\mathcal{O}}_{\text{triv}}^q \to \mathcal{O}_{\text{triv}}^q$ (see diagram (3.11)), whose kernel consists of functions (modulo δ_W-exact ones) satisfying $\delta_W B(\hbar) = \text{const}(\hbar)$. However, the fact that a function F satisfies $\delta_W F(\hbar) = \text{const}(\hbar)$ implies $\delta_W F(\hbar) = 0$.[4] We thus conclude that the homomorphism $\widetilde{\mathcal{O}}_{\text{triv}}^q \to \mathcal{O}_{\text{triv}}^q$ is included into the exact sequence that involves the cohomology of δ_W:

$$0 \to \mathcal{H}^q \to \widetilde{\mathcal{O}}_{\text{triv}}^q \to \mathcal{O}_{\text{triv}}^q \to 0\,, \qquad \mathcal{H}^q = \text{Ker}\,\delta_W\,/\,\text{Im}\,\delta_W\,. \tag{3.16}$$

The classical versions of these constructions are as follows.

Definition 3.2 *A vector field* $\mathbf{X}_0$ *is called a classical gauge symmetry if* $\mathbf{X}_0 S = 0$ *and* $\mathbf{X}_0$ *preserves the antibracket.*

The Lie algebra $\mathcal{O}^c$ *of these vector fields is called the Lie algebra of classical gauge symmetries.*

The classical *BRST-trivial gauge symmetries* are the vector fields $\mathbf{X}_0 = ((S\,,\,B_0)\,,\,\cdot\,)$ whose Hamiltonians are trivial classical observables (see (3.6)). These vector fields span the ideal $\mathcal{O}_{\text{triv}}^c \subset \mathcal{O}^c$, which is called the *classical BRST-trivial gauge symmetries*.

We have the obvious homomorphism $\mathcal{O}^q \xrightarrow{\hbar \to 0} \mathcal{O}^c$. This induces a homomorphism from the ideal $\mathcal{O}_{\text{triv}}^q \subset \mathcal{O}^q$ into the ideal $\mathcal{O}_{\text{triv}}^c \subset \mathcal{O}^c$ (which are shown in (3.11)).

The classical counterpart of $\widetilde{\mathcal{O}}_{\text{triv}}^q$ is the space $\widetilde{\mathcal{O}}_{\text{triv}}^c$ of all functions on $\mathcal{M}$ modulo the functions of the form $(S\,,\,C)$, where S is the classical master action satisfying $(S\,,\,S) = 0$. One can see that the space $\widetilde{\mathcal{O}}_{\text{triv}}^c$ is endowed with a Lie algebra structure with respect to the 'classical' bracket

$$\left[B_0^1,\,B_0^2\right]^c = (B_0^1\,,\,(S\,,\,B_0^2))\,, \tag{3.17}$$

Thus we have the Lie algebra homomorphism $\widetilde{\mathcal{O}}_{\text{triv}}^c \to \mathcal{O}_{\text{triv}}^c$ shown in (3.11). The kernel of the homomorphism coincides with the cohomology of $\mathbf{Q}$, therefore we have the following exact sequence involving the cohomology of $\mathbf{Q}$:

$$0 \to \mathcal{H}^c \to \widetilde{\mathcal{O}}_{\text{triv}}^c \to \mathcal{O}_{\text{triv}}^c \to 0\,, \qquad \mathcal{H}^c = \text{Ker}\,\mathbf{Q}\,/\,\text{Im}\,\mathbf{Q}\,. \tag{3.18}$$

We also observe that the relation between the quantum and the classical bracket is given by $\left[B_0^1,\,B_0^2\right]^c = \left[B^1(\hbar),\,B^2(\hbar)\right]^q\big|_{\hbar=0}$, where $B_0^i = B^i|_{\hbar=0}$. Therefore there exists a Lie algebra homomorphism $\widetilde{\mathcal{O}}_{\text{triv}}^q \xrightarrow{\hbar \to 0} \widetilde{\mathcal{O}}_{\text{triv}}^c$. Following [10, 6] we call $\widetilde{\mathcal{O}}_{\text{triv}}^c$ the *Lie algebra of classical gauge parameters*. We thus see how it is related to the other algebras in (3.11).

Of the algebras entering (3.11), it only remains to construct $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$, which we now do in the BV-setting.

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[4]In order to see this, consider first a function F_0 satisfying $\mathbf{Q} F_0 = (S\,,\,F_0) = \text{const}$. Since (as we see in the next subsection) $\mathcal{M}$ is a proper QP-manifold, the function $\mathbf{Q} F_0$ vanishes on the zero locus of $\mathbf{Q}$ and therefore $\mathbf{Q} F_0 = 0$. Now, to see that equation $\delta_W F(\hbar) = \text{const}(\hbar)$ leads to $\delta_W F(\hbar) = 0$, we rewrite δ_W and $F(\hbar)$ as power series in $\hbar$: $\delta_W = \delta_W^0 + i\hbar\delta_W^1 + (i\hbar)^2\delta_W^2 + \ldots$, where in particular $\delta_W^0 = \mathbf{Q}$, and $F = F_0 + i\hbar F_1 + (i\hbar)^2 F_2 + \ldots$. Since $\mathcal{M}$ is a proper QP-manifold, equation $\mathbf{Q} F_0 = (S\,,\,F_0) = 0$ implies that in some neighbourhood $F_0 = \mathbf{Q}\phi_0 + \text{const}$ with some function ϕ_0. Then in the first order in $\hbar$, the equation $\delta_W^1 F_0 + \delta_W^0 F_1 = \text{const}$ implies $\delta_W^1 F_0 + \delta_W^0 F_1 = 0$ because $\delta_W^1 F_0 = -\delta_W^0 \delta_W^1 \phi_0$. In higher orders in $\hbar$, a similar argument applies. Thus the kernel of the homomorphism $\widetilde{\mathcal{O}}_{\text{triv}}^q \to \mathcal{O}_{\text{triv}}^q$ coincides with the cohomology of δ_W evaluated on the space of formal power series in $\hbar$ with the coefficients in smooth function on $\mathcal{M}$.

3.3   The Hamiltonian algebra of on-shell gauge symmetries

The BV field-antifield manifold $\mathcal{M}$ and the classical master action S satisfying the BV quantization axioms are such that $\mathcal{M}$ is a proper QP-manifold. Indeed, the odd vector field $\mathbf{Q} = (S, \cdot)$ on the $(N|N)$-dimensional antisymplectic manifold $\mathcal{M}$ preserves the antibracket and therefore satisfies condition (2.5), the master equation imposed on S implies that $\mathbf{Q}$ is nilpotent, and, finally, the fact that S is a proper solution of the master equation implies the rank condition (2.6). The zero locus $\mathcal{Z}_\mathbf{Q}$ of $\mathbf{Q}$ determined by the equations $\partial_A S = 0$ will be referred to as the *stationary surface* of the action S. As before, we assume $\mathcal{Z}_\mathbf{Q}$ to be a smooth submanifold [5]. Then according to Theorem 2.4, $\mathcal{Z}_\mathbf{Q}$ has a natural symplectic structure. Further, the classical gauge symmetries (see Definition 3.2) are in fact symmetries of the proper QP-manifold $\mathcal{M}$.

Theorem 3.3 *Every classical gauge symmetry $\mathbf{X}_0$ determines a vector field $\mathbf{x} = \mathbf{X}_0\big|_{\mathcal{Z}_\mathbf{Q}}$ on $\mathcal{Z}_\mathbf{Q}$ that preserves the Poisson bracket on $\mathcal{Z}_\mathbf{Q}$.*

PROOF.   Indeed, any vector field $\mathbf{X}_0$ preserving the master action S and the antibracket commutes with $\mathbf{Q} = (S, \cdot)$ and is therefore a symmetry of $\mathcal{M}$ (Definition 2.5). As we saw in section 2.2, any vector field $\mathbf{X}_0$ that is a symmetry of $\mathcal{M}$ restricts to $\mathcal{Z}_\mathbf{Q}$ and $\mathbf{X}_0\big|_{\mathcal{Z}_\mathbf{Q}}$ is locally Hamiltonian on $\mathcal{Z}_\mathbf{Q}$. $\square$

We denote the Lie algebra of locally Hamiltonian vector fields on $\mathcal{Z}_\mathbf{Q}$ by $\mathbb{H}_{\mathcal{Z}_\mathbf{Q}}$. As we are going to see, this is the algebra of on-shell gauge symmetries.

Definition 3.4 *A classical gauge symmetry $\mathbf{X}_0$ is called on-shell trivial if it vanishes on the stationary surface $\mathcal{Z}_\mathbf{Q}$.*

We now show that the Lie algebra of locally Hamiltonian vector fields on $\mathcal{Z}_\mathbf{Q}$ is isomorphic to the algebra of on-shell gauge symmetries.

Theorem 3.5 *The algebra $\mathcal{I}_0$ of on-shell trivial symmetries is an ideal in the Lie algebra $\mathcal{O}^c$ of gauge symmetries and the quotient algebra $\mathcal{O}^c/\mathcal{I}_0$ is isomorphic to the Lie algebra $\mathbb{H}_{\mathcal{Z}_\mathbf{Q}}$ of locally Hamiltonian vector fields on $\mathcal{Z}_\mathbf{Q}$.*

PROOF.   Let $\mathbf{Y}_0 \in \mathcal{I}_0$ and $\mathbf{X}_0 \in \mathcal{O}^c$. For any function $F \in \mathbb{C}_\mathcal{M}$, we have $\mathbf{Y}_0 F \in \mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$. Since $\mathbf{X}_0 \mathbb{I}_{\mathcal{Z}_\mathbf{Q}} \subset \mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$ and $\mathbf{Y}_0 F \in \mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$, we have $[\mathbf{X}_0, \mathbf{Y}_0]F = \mathbf{X}_0 \mathbf{Y}_0 F - (-1)^{\epsilon(\mathbf{X}_0)\epsilon(\mathbf{Y}_0)}\mathbf{Y}_0 \mathbf{X}_0 F \in \mathbb{I}_{\mathcal{Z}_\mathbf{Q}}$, therefore $\mathcal{I}_0$ is an ideal in the Lie algebra $\mathcal{O}^c$. Further, we have seen in Theorem 2.8 that any locally Hamiltonian vector field $\mathbf{x}$ on $\mathcal{Z}_\mathbf{Q}$ is a restriction of some vector field $\mathbf{X} \in \mathcal{O}^c$. Thus one can identify the quotient algebra $\mathcal{O}^c/\mathcal{I}_0$ with the Lie algebra of locally Hamiltonian vector fields on $\mathcal{Z}_\mathbf{Q}$. $\square$

It follows from the theorem that the homomorphism $\mathcal{O}^c \to \mathbb{H}_{\mathcal{Z}_\mathbf{Q}}$ is included into the exact sequence

$$0 \to \mathcal{I}_0 \to \mathcal{O}^c \to \mathbb{H}_{\mathcal{Z}_\mathbf{Q}} \to 0. \tag{3.19}$$

Note that we cannot replace $\mathcal{O}^c$ with $\mathcal{O}^c_{\text{triv}}$ here, because the homomorphism $\mathcal{O}^c_{\text{triv}} \to \mathbb{H}_{\mathcal{Z}_\mathbf{Q}}$ is *not* surjective whenever there exists the first cohomology of $\mathcal{Z}_\mathbf{Q}$. Indeed, a nonvanishing first cohomology implies that there exist locally Hamiltonian vector fields that are not globally Hamiltonian on $\mathcal{Z}_\mathbf{Q}$,

---

[5] Although in realistic examples the structure of the zero locus of $\mathbf{Q}$ can be very involved, we treat $\mathcal{Z}_\mathbf{Q}$ as a submanifold. Note in passing that the finite-dimensional models of gauge systems should be considered with some caution also in view of the results of [17].

which we have seen in theorem 2.8 to correspond to *BRST-nontrivial* gauge symmetries. Due to the existence of the latter, the mapping $\mathcal{O}^c_{\mathrm{triv}} \to \mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$ is not surjective in general.

Looking at diagram (3.11), it is natural to ask the following question: What is the analogue of $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$ for the upper line of the diagram, i.e. what is the *quantum* analogue of the on-shell gauge symmetries? We propose one possible answer to this question.

Note that Poisson bracket (2.8) on $\mathcal{Z}_{\mathbf{Q}}$ and Lie bracket (3.17) on the space of gauge parameters are defined by the same bilinear operation $(\,\cdot\,,\mathbf{Q}\,\cdot\,)$ on $\mathbb{C}_{\mathcal{M}}$. The difference between these two brackets is that (3.17) is defined on the quotient space $\mathbb{C}_{\mathcal{M}}/\mathrm{Im}\,\mathbf{Q}$, while the Poisson bracket is defined on $\mathbb{C}_{\mathcal{M}}/\mathbb{I}_{\mathcal{Z}_{\mathbf{Q}}}$. In the case of a proper QP-manifold, $\mathbb{I}_{\mathcal{Z}_{\mathbf{Q}}}$ is the ideal generated by $\mathrm{Im}\,\mathbf{Q}$, i.e. $\mathbb{I}_{\mathcal{Z}_{\mathbf{Q}}} = \mathbb{C}_{\mathcal{M}} \cdot \mathrm{Im}\,\mathbf{Q}$. At the same time, (3.17) is the limit as $\hbar \to 0$ of the quantum construction (3.14) defined on $\mathbb{C}_{\mathcal{M}}[[\hbar]]$. Therefore, in the quantum case one can construct a Poisson bracket as a direct generalisation of (2.8), as $\{\cdot\,,\cdot\}^q = (\cdot\,,\delta_W\,\cdot)$. The bracket $\{\cdot\,,\cdot\}^q$ would be well defined only on the quotient algebra of $\mathbb{C}_{\mathcal{M}}[[\hbar]]$ modulo the ideal $\mathbb{I}_{\delta_W}$ generated by $\mathrm{Im}\,\delta_W$. Obviously, $\mathrm{Im}\,\delta_W$ is generated by all series of the form $\delta_W f(\hbar)$, where $f(\hbar) = f_0 + f_1\hbar + f_2\hbar^2 + \dots$ with the coefficients f_i taking independently each value 1, Γ^A and $\Gamma^A\Gamma^B$. Since the matrix $E^{AB} = (\Gamma^A,\Gamma^B)$ is invertible, we thus see that $\mathbb{I}_{\delta_W}$ consists of the series of the form $\mathbb{I}_{\mathcal{Z}_{\mathbf{Q}}} + \mathbb{C}_{\mathcal{M}}\hbar + \mathbb{C}_{\mathcal{M}}\hbar^2 + \dots$. Thus, the quotient algebra $\mathbb{C}_{\mathcal{M}}[[\hbar]]/\mathbb{I}_{\delta_W}$ coincides with the algebra $\mathbb{C}_{\mathcal{Z}_{\mathbf{Q}}}$ of functions on the zero locus of $\mathbf{Q}$. This means that the algebra $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$ is in a certain sense the most general algebra of the on-shell symmetries not only of the classical but also of the quantum, master action.

4 Examples

4.1 Abelianized gauge theory

We now consider the field-antifield space and the master action corresponding to the simplest gauge theory, the abelianized gauge theory, which we choose as an instructive example that is free of additional complications because gauge symmetries are explicitly separated from the physical ones. We then explicitly construct the Poisson bracket and the Lie algebras $\mathcal{O}^c$ and $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$. Moreover, this example shows that the classical gauge symmetries $\mathcal{O}^c$ contain the Lie algebra of gauge symmetries of the *original* theory as a subalgebra and that, similarly, the on-shell gauge symmetries $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$ contain the on-shell symmetries of the abelianized gauge theory.

Let $S_0(X,x)$ be a polynomial action such that

$$\partial_\alpha S_0 = 0, \quad \det\left(\partial_i\partial_j S_0\right)\big|_{\partial_i S_0 = 0} \neq 0, \tag{4.1}$$

where we denote $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ and $\partial_i = \frac{\partial}{\partial X^i}$ and assume X^i and x^α to be bosonic for simplicity. Due to rank condition (4.1), the equations $\partial_i S_0 = 0$ admit only a finite set of solutions $\mathfrak{M}$. Thus, the stationary surface of this theory is the direct product of $\mathfrak{M}$ with the space parametrised by x^α. The gauge transformations preserving the action S_0 are of the form

$$\mathbf{Y}_0 = Y_0^\alpha(X,x)\partial_\alpha + \mu^{ij}(X,x)\partial_i S_0 \partial_j, \tag{4.2}$$

where $\mu^{ij}(X,x)$ is an antisymmetric matrix. These vector fields span a Lie algebra $\mathcal{A}$ with respect to the commutator of vector fields; those vanishing on the stationary surface span the ideal $\mathcal{A}_{\mathrm{triv}}$ in $\mathcal{A}$.

Then $\widetilde{\mathcal{A}} = \mathcal{A}/\mathcal{A}_{\mathrm{triv}}$ is the algebra of on-shell gauge symmetries, which can be identified with the Lie algebra of vector fields on the stationary surface.

To carry out the BV scheme, we choose the gauge generators in the form $R_\beta^\alpha = \delta_\beta^\alpha$ and introduce ghosts c^α and the antifields x_α^*, X_i^*, and c_α^*. The canonical antibracket is $(\phi^A, \phi_B^*) = \delta_B^A$, where $\phi^A = (X^i, x^\alpha, c^\alpha)$ and $\phi_A^* = (X_i^*, x_\alpha^*, c_\alpha^*)$. Then the master action

$$S = S_0 + x_\alpha^* c^\alpha \tag{4.3}$$

is a proper solution of the master equation $(S, S) = 0$. This action defines the vector field

$$\mathbf{Q} = (S, \cdot) = \partial_i S_0 \frac{\overrightarrow{\partial}}{\partial X_i^*} + c^\alpha \frac{\overrightarrow{\partial}}{\partial x^\alpha} + x_\alpha^* \frac{\overrightarrow{\partial}}{\partial c_\alpha^*}, \tag{4.4}$$

whose stationary surface $\mathcal{Z}_{\mathbf{Q}}$ is determined by $\partial_i S_0 = 0$, $x_\alpha^* = 0$, $c^\alpha = 0$. Thus $\mathcal{Z}_{\mathbf{Q}}$ is the direct product of $\mathfrak{M}$ (the set of solutions to the system of equations $\partial_i S_0 = 0$) with the space parametrised by $Y^A = \{X_i^*, x^\alpha, c_\alpha^*\}$. The Poisson bracket (2.8) on $\mathcal{Z}_{\mathbf{Q}}$ is then represented by the matrix

$$\Omega^{AB} = \{Y^A, Y^B\} = \begin{pmatrix} \partial_i \partial_j S_0 & 0 & 0 \\ 0 & 0 & \delta_\alpha^\beta \\ 0 & -\delta_\gamma^\nu & 0 \end{pmatrix}. \tag{4.5}$$

We now want to show that the Lie algebras $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ of the original theory are subalgebras in the Lie algebras $\mathcal{O}^c$ and $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$, respectively. To do so, we calculate $\mathcal{O}^c$ in the master theory with the master action (4.3). Note that in the case of the abelianized gauge theory, the first cohomology group of the field-antifield space vanishes, therefore each Hamiltonian vector field has a globally defined Hamiltonian. Thus in order to find $\mathcal{O}^c$, it suffices to find the kernel of $\mathbf{Q}$ evaluated on the space of globally defined functions. Any element $A \in \mathsf{Ker}\,\mathbf{Q}$ can be written in the form $A = \mathbf{Q}\,F + G$, where F is an arbitrary smooth function on the field-antifield space and G is a representative of the cohomology class of $\mathbf{Q}$. In order to calculate the cohomology of $\mathbf{Q}$ [6], we write $\mathbf{Q} = \mathbf{Q}_1 + \mathbf{Q}_2$, where

$$\mathbf{Q}_1 = \partial_i S_0 \frac{\overrightarrow{\partial}}{\partial X_i^*}, \qquad \mathbf{Q}_2 = c^\alpha \frac{\overrightarrow{\partial}}{\partial x^\alpha} + x_\alpha^* \frac{\overrightarrow{\partial}}{\partial c_\alpha^*} \quad \text{and} \quad \mathbf{Q}_1^2 = \mathbf{Q}_2^2 = [\mathbf{Q}_1, \mathbf{Q}_2] = 0. \tag{4.6}$$

By the Poincaré lemma, the cohomology of $\mathbf{Q}_2$ consists of constants only. Thus the cohomology of $\mathbf{Q}$ is determined by the cohomology of $\mathbf{Q}_1$ on the space of functions $F(X, X^*)$. A function $F(X, X^*)$ belongs to the image of $\mathbf{Q}_1$ whenever $F(X, X^*) = \partial_i S_0 f^i(X, X^*)$, i.e. $F(X, X^*)$ vanishes at each point where $\partial_i S_0 = 0$. Thus, any element A from $\mathsf{Ker}\,\mathbf{Q}$ is of the form

$$A(X, X^*) = \mathbf{Q}\,F(X, X^*) + G(X), \tag{4.7}$$

where F is an arbitrary function and $G(X)$ is a function that does not vanish at least at one point from $\mathfrak{M}$. Whenever $\mathfrak{M}$ is an n-point set, the cohomology of $\mathbf{Q}$ is an n-dimensional vector space[7].

We thus see that $\mathcal{O}^c$ and $\mathcal{O}^c_{\mathrm{triv}}$ are spanned by the vector fields of the form $(\mathbf{Q}\,F + G(X), \cdot)$ and $(\mathbf{Q}\,F, \cdot)$ respectively. The algebra $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$ of on-shell symmetries consists of Hamiltonian vector

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[6] Note that in the case where $S_0 = \frac{1}{2}\delta_{ij}X^i X^j$, the vector field $\mathbf{Q}$ is nothing but the de Rham differential of $\mathcal{Z}_{\mathbf{Q}}$. In this case the cohomology of $\mathbf{Q}$ consists of constants only and $\mathcal{O}^c$ coincides with $\mathcal{O}^c_{\mathrm{triv}}$.

[7] In this example, the group of 'physical' symmetries is the group of the permutations of these n points. This group obviously acts on the cohomology of $\mathbf{Q}$.

fields $\{H(\mathfrak{m}, X^*, x, c^*)\,,\,\cdot\,\}$ on $\mathcal{Z}_{\mathbf{Q}}$ (where we label the Hamiltonian by $\mathfrak{m} \in \mathfrak{M}$ enumerating the different components of $\mathfrak{M}$).

To see that the algebra $\mathcal{A}$ of gauge transformations of the original theory is embedded into the Lie algebra $\mathcal{O}^c$ of the classical gauge symmetries, we note that vector fields of the form

$$\mathbf{Y} = (Y_0^\alpha(X, x)x_\alpha^* + \mu^{ij}(X, x)\partial_i S_0 X_j^*\,,\,\cdot\,) \tag{4.8}$$

form the subalgebra in $\mathcal{O}^c$. Moreover, these fields restrict to the subspace $c^\alpha = c_\alpha^* = x_\alpha^* = X_i^* = 0$ as elements of $\mathcal{A}$ (see (4.2)). Thus we have an embedding of $\mathcal{A}$ into $\mathcal{O}^c$ (obviously, the embedding is not unique).

As regards the on-shell gauge symmetries, observe that vector fields on $\mathcal{Z}_{\mathbf{Q}}$ of the form

$$\mathbf{y} = \{y^\alpha(\mathfrak{m}, x)c_\alpha^*\,,\,\cdot\,\} \tag{4.9}$$

(which define a subalgebra in $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$) restrict to the stationary surface of the original theory (which is a submanifold of $\mathcal{Z}_{\mathbf{Q}}$ determined by the equations $c_\alpha^* = 0$ and $X_i^* = 0$) and span $\widetilde{\mathcal{A}}$. Thus the algebra of on-shell gauge symmetries $\widetilde{\mathcal{A}}$ of the original theory is embedded into the Lie algebra $\mathbb{H}_{\mathcal{Z}_{\mathbf{Q}}}$ of the on-shell gauge symmetries.

4.2 A 'topological' field theory

We now apply Theorem 2.4 to the 'topological' theory with the vanishing action on a Lie group $\mathcal{G}$. We show that in this case bracket (2.8) is related to the Kirillov bracket on the coalgebra. Denote by x^i a coordinate system in the neighbourhood of $1 \in \mathcal{G}$. Let $\mathcal{R}_\alpha = R_\alpha^i \partial_i$ (where the Greek indices have the same cardinalities as the Latin ones) be the basis of the left invariant vector fields on $\mathcal{G}$. We have $[\mathcal{R}_\alpha, \mathcal{R}_\beta] = \mathcal{F}_{\alpha\beta}^\gamma \mathcal{R}_\gamma$, where $\mathcal{F}_{\alpha\beta}^\gamma$ are the structure constants.

In accordance with the BV prescription, we introduce the ghosts c^α and the antifields x_i^* and c_α^* such that $(x^i, x_j^*) = \delta_j^i$, $(c^\alpha, c_\beta^*) = \delta_\beta^\alpha$. The master action $S = x_i^* R_\alpha^i c^\alpha - \frac{1}{2}c_\gamma^* \mathcal{F}_{\alpha\beta}^\gamma c^\beta c^\alpha$ is a proper solution of $(S, S) = 0$. Then

$$\mathbf{Q} = (S\,,\,\cdot\,) = c^\alpha R_\alpha^i \frac{\overrightarrow{\partial}}{\partial x^i} + \frac{1}{2}\mathcal{F}_{\alpha\beta}^\gamma c^\beta c^\alpha \frac{\overrightarrow{\partial}}{\partial c^\gamma} + (x_i^* R_\alpha^i - c_\gamma^* \mathcal{F}_{\alpha\beta}^\gamma c^\beta)\frac{\overrightarrow{\partial}}{\partial c_\alpha^*}\,, \tag{4.10}$$

therefore the zero locus $\mathcal{Z}_{\mathbf{Q}}$ of $\mathbf{Q}$ is determined by the equations $c^\alpha = 0$, $x_i^* = 0$ and is coordinatized by $Y^A = \{x^i, c_\alpha^*\}$. In this case $\mathcal{Z}_{\mathbf{Q}} = T^*\mathcal{G} = \mathcal{G} \times \mathfrak{g}^*$ is the cotangent bundle to $\mathcal{G}$, where $\mathfrak{g}^*$ is the coalgebra. The matrix of Poisson bracket (2.8) takes the form

$$\Omega^{AB} = \{Y^A, Y^B\} = \begin{pmatrix} \{x^i, x^j\} & \{x^i, c_\beta^*\} \\ \{c_\alpha^*, x^j\} & \{c_\alpha^*, c_\beta^*\} \end{pmatrix} = \begin{pmatrix} 0 & R_\beta^i \\ -R_\alpha^j & -c_\gamma^* \mathcal{F}_{\alpha\beta}^\gamma \end{pmatrix}\,. \tag{4.11}$$

It is nondegenerate because R_α^i are nondegenerate everywhere on $\mathcal{Z}_{\mathbf{Q}}$ since R_α^i are the coefficients of the left invariant vector fields on the Lie group. The cotangent bundle to a Lie group is trivial, therefore we have the embedding $\mathfrak{g}^* \to T^*\mathcal{G}$, which induces a Poisson bracket on $\mathfrak{g}^*$. This gives us the Kirillov bracket [16] on the coalgebra $\mathfrak{g}^*$ parametrised by the coordinates c^*.

5 Conclusions

We have seen that a number of objects of the antisymplectic BV geometry are essentially determined by objects of the symplectic geometry on the stationary surface of the master action, where the

nondegenerate Poisson bracket is given by (2.8). In particular, every observable determines a symmetry of the master-action, which in turn restricts to a locally Hamiltonian vector field on $\mathcal{Z}_{\mathbf{Q}}$; at the same time, every trivial observable determines a symmetry of the master action such that the corresponding vector field on $\mathcal{Z}_{\mathbf{Q}}$ is globally Hamiltonian. Those Hamiltonian vector fields on $\mathcal{Z}_{\mathbf{Q}}$ that are not globally Hamiltonian correspond then to the BRST-nontrivial observables.

Recalling how the master theory is constructed in terms of the bare classical action $\mathcal{S}$, we were able to explicitly see, in the abelianized setting, that the gauge symmetries of $\mathcal{S}$ are dressed into symmetries of the master theory, i.e., into (,)-Hamiltonian vector fields; at the same time, on-shell gauge symmetries of $\mathcal{S}$ are dressed into { , }-Hamiltonian vector fields on the symplectic manifold $\mathcal{Z}_{\mathbf{Q}}$. This would be interesting to extend the setting of the general gauge theory.

Our analysis was performed in the framework of the finite-dimensional model; such models should be viewed with caution precisely for the reasons related to the existence of the BRST cohomology [17]. It would be interesting to see how our results can be reformulated in local field theory, where the gauge symmetries have been discussed in [19], and, possibly, also in application to string field theory [5, 6], which has been one of the motivations behind the geometrically covariant reformulation of the BV quantization.

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