A CHARACTERIZATION OF THURSTON'S MASTER TEAPOT

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ABSTRACT. We prove an explicit characterization of the points in Thurston's Master Teapot. This description can be implemented algorithmically to test whether a point in $\mathbb{C}\times\mathbb{R}$ belongs to the complement of the Master Teapot. As an application, we show that the intersection of the Master Teapot with the unit cylinder is not symmetrical under reflection through the plane that is the product of the imaginary axis of $\mathbb{C}$ and $\mathbb{R}$.

1. INTRODUCTION

When a continuous self-map f of an interval is postcritically finite, the exponential of its topological entropy, $e^{h_{top}(f)}$, also called its *growth rate*, is a *weak Perron number* – a real algebraic integer whose modulus is greater than or equal to that of all of its Galois conjugates. This is because cutting the interval at the critical and postcritical sets yields a Markov partition; each of the resulting subintervals is mapped to a finite union of subintervals. The leading eigenvalue of the associated incidence matrix is $e^{h_{top}(f)}$, which the Perron-Frobenius Theorem implies is a weak Perron number. We consider the family $\mathcal{F}$ of continuous, unimodal, interval self-maps that are not only postcritically finite, but *superattracting* – meaning that the orbit of the critical point is strictly periodic. The *Master Teapot* for $\mathcal{F}$, first described by W. Thurston in [Thu14], is the set

$$\Upsilon_2 := \overline{\{(z,\lambda)\in\mathbb{C}\times\mathbb{R} \mid \lambda = e^{h_{top}(f)} \text{ for some } f\in\mathcal{F}, z \text{ is a Galois conjugate of } \lambda\}}.$$

A finite approximation of Υ_2 is shown in Figure 1. The image of the projection of the Master Teapot Υ_2 to $\mathbb{C}$ via $(z,\lambda)\mapsto z$ is the *Thurston set*, Ω_2, for $\mathcal{F}$ (Proposition 6.2):

$$\Omega_2 := \overline{\{z\in\mathbb{C} \mid z \text{ is a Galois conjugate of } e^{h_{top}(f)} \text{ for some } f\in\mathcal{F}\}}.$$

The Master Teapot and Thurston set for $\mathcal{F}$ have rich geometrical and topological structures that have been investigated in several recent works, including [Tio18, Tio15, CKW17, Thu14, Tho17, BDLW19].

The main result of this paper is a set of necessary and sufficient conditions for points in $\mathbb{C}\times\mathbb{R}$ to belong to Υ_2. We treat the parts of Υ_2 inside and outside of the unit cylinder $\mathbb{D}\times[1,2]$ separately. Theorem 1 characterizes the points in $\Upsilon_2\cap(\mathbb{D}\times[\sqrt{2},2])$. For $z\in\mathbb{C}\setminus\{0\}$, define the functions $f_{0,z}^{-1}, f_{1,z}^{-1}:\mathbb{C}\to\mathbb{C}$ by

$$f_{0,z}^{-1}(x)=\frac{x}{z} \quad \text{and} \quad f_{1,z}^{-1}(x)=\frac{2-x}{z}.$$

Theorem 1. *Let $\lambda\in(\sqrt{2},2]$ be the growth rate of a superattracting tent map and fix a point $z\in\mathbb{D}\setminus\{0\}$. For each $m\in\mathbb{N}$, define*

$$C_m^\lambda := \{(a_1\ldots a_m)\in\{0,1\}^m \mid (a_k\ldots a_m) \text{ is } \lambda\text{-suitable } \forall 1\le k\le m\}.$$

Then $(z, \lambda) \in \Upsilon_2$ if and only if for every $M \in \mathbb{N}$ there exists a word $(a_1 \dots a_M) \in C_M^\lambda$ such that

$$\left| f_{a_1,z}^{-1} \circ \cdots \circ f_{a_M,z}^{-1}(1) \right| \leq \frac{2}{1-|z|}. \tag{1}$$

The notion of λ-suitability (see Definition 3.5) is a combinatorial condition on finite words that is related to the critical itinerary of tent map with slope λ. Theorem 2 characterizes the part of Υ_2 that is outside the unit cylinder.

Theorem 2. *A point (z, λ), with $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and $\lambda \in [1, 2]$, is in Υ_2 if and only if $K_{w_\lambda}(\frac{1}{z}) = 0$.*

Here, K_{w_λ} is the kneading series associated to the tent map with growth rate λ, as defined in §2.5. The characterization of Υ_2 is completed by the results that $\Upsilon_2 \cap (\mathbb{C} \times \{1\})$ equals $S^1 \times \{1\}$ and $S^1 \times [1, 2] \subset \Upsilon_2$ (see §2.7). Theorem 1 can be extended to a necessary and sufficient condition for points to be in $\Upsilon_2 \cap (\mathbb{D} \times (1, \sqrt{2})$ via the technique of Period Doubling (which is described in §2.7). This yields the following characterization of Υ_2:

Theorem 3. *A point $(z, \lambda) \in \mathbb{C} \times [1, 2]$ belongs to Υ_2 if and only if one of the following conditions holds:*

(1) $|z| = 1$, $\lambda \in [1, 2]$.
(2) (z, λ) satisfies the assumptions of Theorem 1 or Theorem 2.
(3) There is some $k \in \mathbb{N}$ such that (z^{2^k}, λ^{2^k}) satisfies the assumptions in Theorem 1

In contrast to constructive nature of the definition of Υ_2, Theorems 1, 2 and 3 are "if and only if," and thus can be used to prove that points do *not* belong to Υ_2. Theorem 1 can be implemented algorithmically by ruling out points (z, λ) for which a word $(a_1, \dots, a_M)$ that violates (1) can be found – that is, by representing Υ_2 as the intersection of a nested, decreasing sequence of sets indexed by M. Remark 4.3 describes how to implement Theorem 2 as an algorithm. Figures 4 and 5 are plots of approximations of "slices" of the unit cylinder part of Υ_2 (sets of the form $\Upsilon_2 \cap (\mathbb{D} \times \{c\})$ computed using Theorem 1.

As an application of Theorem 1, we prove:

Theorem 4. *The teapot Υ_2 is not invariant under the map $(z, \lambda) \mapsto (-z, \lambda)$.*

Since Galois conjugates occur in complex conjugate pairs, it is immediate that $(x + iy, \lambda) \in \Upsilon_2$ if and only if $(x - iy, \lambda) \in \Upsilon_2$. Consequently, Theorem 4 implies that Υ_2 is not symmetrical under reflection across the imaginary axis. Theorem 4 is suprising because the Thurston set, Ω_2, which is the projection to $\mathbb{C}$ of Υ_2, *is* symmetrical under the map $z \mapsto -z$ (Proposition 6.1). The proof of Theorem 4 consists of first exhibiting a point $(x + iy, \lambda) \in \Upsilon_2)$ and then using Theorem 3.2 to prove $(-x + iy, \lambda) \notin \Upsilon_2$. However, this asymmetry is confined to the slices of heights $\geq \sqrt{2}$; Lemma 6.3 proves that the unit cylinder part of slices of height $< \sqrt{2}$ are symmetrical under reflection across the imaginary axis.

The Thurston set Ω_2 is known to be path-connected and locally connected (Theorem 1.3 of [Tio18]). It follows from Theorem 2 that for many heights $\lambda \in (1, 2]$, the part of the slice of height λ that is outside the unit cylinder consists of more than one connected component.

Conjecture 1.1. *There exist values of c such that $\Upsilon_2 \cap (\mathbb{D} \times \{c\})$ consists of more than one connected component.*

Figure 2 shows a constructive plot (in black) of the slice $\Upsilon_2 \cap (\mathbb{D} \times \{1.8\}$, while Figure 3 shows (in white) points of $\mathbb{D} \times \{1.8\}$ that are in $(\mathbb{D} \times \{1.8\}) \setminus \Upsilon_2$. Comparison of these images suggests the existence of multiple small connected components in the region $\mathrm{Re}(z) < 0$ near the inner boundary of the "ring."

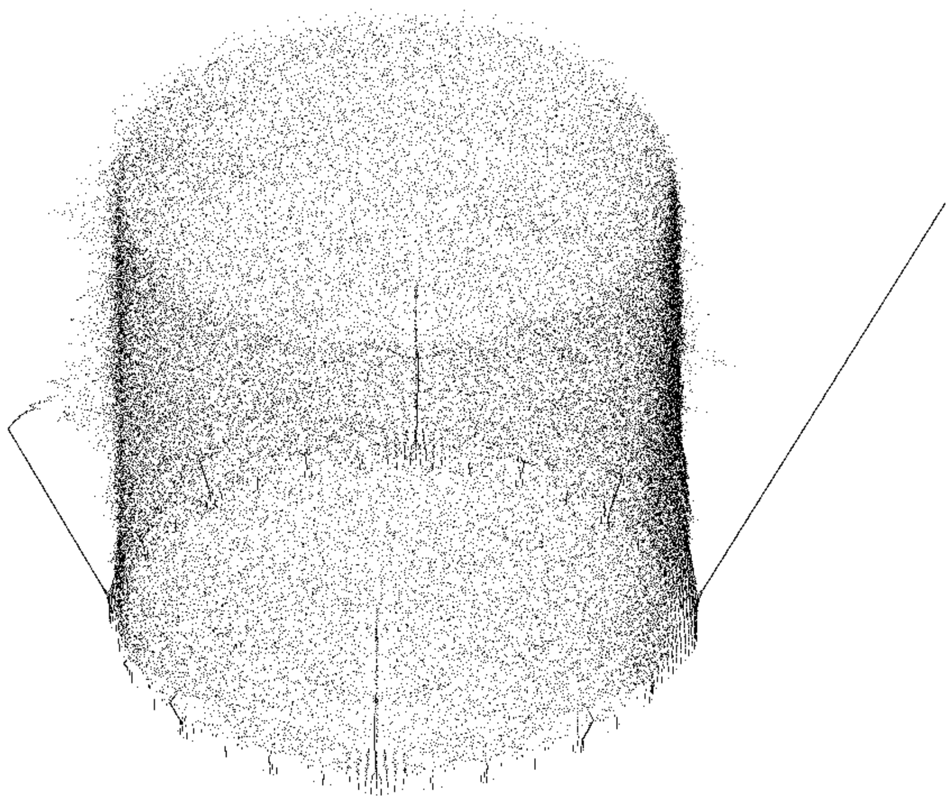

FIGURE 1. A constructive approximation of the part of Υ_2 outside the unit cylinder. This plot shows the 56737 points outside the cylinder $S^1 \times [1, 2]$ that are roots of the degree 100 partial sums of the kneading power series for 1000 different growth rates λ in $[1, 2]$. The "spout" on the right side of the image consists of points of the form (λ, λ).

Conjecture 1.1 could be potentially proven by computation via an effective version of Theorem 1 similar to Proposition 7.1. However, a tighter bound than that obtained in Proposition 7.1 would probably be needed for the computation to be feasible.

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2. PRELIMINARIES

2.1. **Words.** An *alphabet* $\mathcal{A}$ is a set. A finite *word* in the alphabet $\mathcal{A}$ is map

$$w : \{1, 2, \ldots, n\} \to \mathcal{A}$$

for some $n \in \mathbb{N}$. We will denote such a word by $w = w_1 w_2 \ldots w_n$, where $w_i = w(i)$, and write $|w| = n$ for the *length* of w. We will use $\cdot$ or adjacency to denote concatenation of

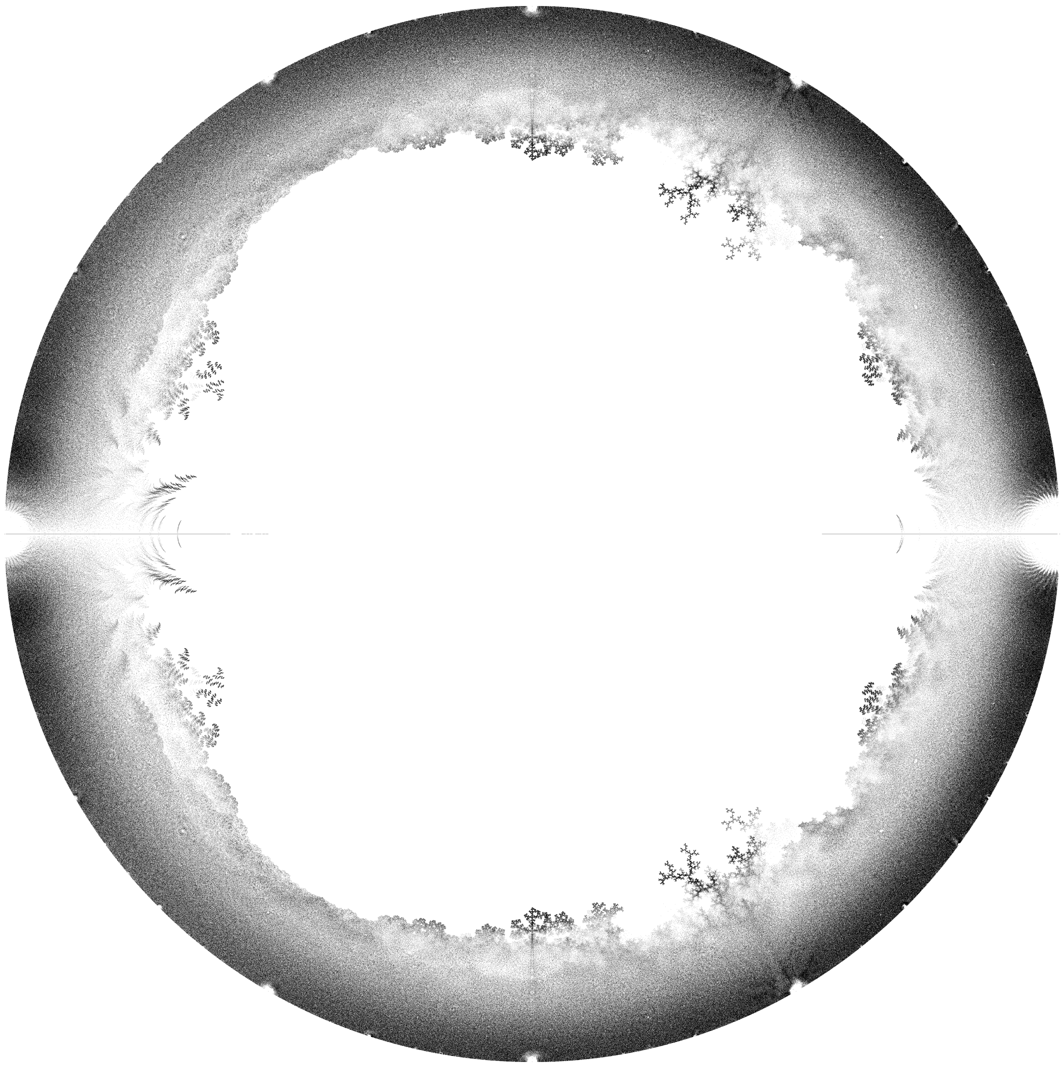


FIGURE 2. A constructive plot of the slice $\Upsilon_2 \cap (\mathbb{D} \times \{1.8\})$. The plotted black points are all the roots of modulus ≤ 1 of all Parry polynomials for superattracting tent maps with growth rate < 1.8 and critical length at most 29.

words. For example, given two finite words $w = w_1 \ldots w_n$ and $v = v_1 \ldots v_m$,

$$wv = w \cdot v = w_1 \ldots w_n v_1 \ldots v_m.$$

An *infinite word* or *sequence* in the alphabet $\mathcal{A}$ is a function $w : \mathbb{N} \to \mathcal{A}$, and we denote such a word by $w = w_1 w_2 \ldots$, where $w_i = w(i)$. Given any finite word w and $n \in \mathbb{N}$, we use w^n to denote the length $n|w|$ word formed by concatenating n copies of w, and w^∞ denotes the infinite word formed by concatenating infinitely many copies of w. For any finite word

FIGURE 3. The upper half of the slice $\Upsilon_2 \cap (\mathbb{D} \times \{1.8\})$ plotted using Theorem 1. Specifically, the plotted white points were shown to be in the complement of Υ_2 (by checking the condition of Theorem 1 for all $m \leq 18$).

$w = w_1 \dots w_{|w|}$ and integer $1 \leq k < |w|$, the *length k prefix* of w is

$$\mathrm{Prefix}_k(w) := w_1 \dots w_k$$

and the *length k suffix* of w is

$$\mathrm{Suffix}_k(w) := w_{|w|-k+1} \dots w_{|w|}.$$

A finite word w in an alphabet $\mathcal{A}$ is *irreducible* if there exists no shorter word v in the alphabet $\mathcal{A}$ such that $w^\infty = v^\infty$. For finite words w and v with $|w| > |v|$, we say w is an *extension* of v if $\mathrm{Prefix}_{|v|}(w) = v$. When $\mathcal{A}$ consists of self-maps of a set X, the action of a word w in alphabet $\mathcal{A}$ on X is from left to right, i.e. the action of $w = w_1 \dots w_n$ on a point $x \in X$ is

$$w(x) = w_1 \dots w_n(x) = w_n \circ \cdots \circ w_1(x).$$

We denote the *shift map* on a set of infinite words by σ; the action of σ is defined by

$$\sigma(w_1 w_2 \dots) = w_2 w_3 \dots$$

2.2. **Uniform expanders and tent maps.** A *uniform expander* is a continuous, piecewise (with finitely many pieces) linear self-map of an interval such that there exists a real number $\lambda > 0$ so that the slope of each piece is either λ or $-\lambda$. Theorem 7.4 of [MT88] asserts that every continuous self-map g of an interval with finitely many turning points and $h_{top}(g) > 0$ is semi-conjugate to a uniform expander $\tilde{g}$ with the same topological entropy and slope $\pm e^{h_{top}(g)}$.

Denote the unit interval $[0,1]$ by I. Throughout this work, a *tent map* will mean a map $g_\lambda : I \to I$ of the following form. Fix a real number $\lambda \in (1,2]$, let $I_0^\lambda = [0, \frac{1}{\lambda}]$ and $I_1^\lambda = (\frac{1}{\lambda}, 1]$. The *$\lambda$-tent map* or *tent map of growth rate* λ is the map $g_\lambda \colon I \to I$ defined by

$$g_\lambda(x) = \begin{cases} \lambda x & \text{for } x \in [0, \frac{1}{\lambda}], \\ -\lambda x + 2 & \text{for } x \in [\frac{1}{\lambda}, 1]. \end{cases}$$

The number λ is the *growth rate* of the map g_λ; equivalently, $\lambda = e^{h_{\text{top}}(g_\lambda)}$. This equivalence follows from the fact that for a continuous self-map g of an interval with finitely many turning points,

$$h_{\text{top}}(g) = \lim_{n\to\infty} \frac{1}{n} \log(\text{Var}(g^n)), \tag{2}$$

where $\text{Var}(g)$ denotes the total variation of f [MS80].

A tent map g is said to be *superattracting* if there exists $n \in \mathbb{N}$ such that $g^n(1) = 1$; the smallest such n is the *period* of 1 under g, also called the *critical period* of g. The set of growth rates of superattracting tent maps is dense in $[1, 2]$, which is the set of growth rates of all tent maps.

2.3. **Admissible finite words and Parry polynomials.** Let g be a superattracting tent map of growth rate λ and critical period p Then the *word associated to* g is the word $w_\lambda = w_1 \dots w_p \in \{0,1\}^p$ such that w_i is the index (0 or 1) of the interval I_0^λ or I_1^λ that contains $g_\lambda^{i-1}(1)$ for each i (using the convention that $1 = g_\lambda^0(1) \in I_1^\lambda$). (Note that the word associated to a superattracting tent map g is, by definition, irreducible, since its length is equal to the critical period.) A finite word $w \in \{0,1\}^p$, for $p \in \mathbb{N}$, is said to be *admissible* if there exists a superattracting tent map g such that $w^\infty = v^\infty$, where v is the word associated to g. In this case, the map g is called the *tent map associated to* w.

Consider a finite word $w = w_1 \dots w_p \in \{0,1\}^p$. The *cumulative sign vector* associated to w is the vector

$$(s_0, s_1, \dots, s_p) := (1, (-1)^{w_1}, (-1)^{w_1+w_2}, \dots, (-1)^{w_1+\cdots+w_p}) \in \{-1,+1\}^{p+1}$$

and the *digit vector* is the vector

$$(d_1, \dots, d_p) := (2w_1, \dots, 2w_p) \in \{0,2\}^p.$$

We will also call s_p the *cumulative sign* of w, and write

$$\text{sgn}(w) = (-1)^{w_1+\cdots+w_p}.$$

We will adopt the convention that the cumulative sign of the empty word is $+1$. The *Parry polynomial* for w is the polynomial $P_w : \mathbb{C} \to \mathbb{C}$ defined by

$$P_w(z) = z^p - s_0 d_1 z^{p-1} - \cdots - s_{p-1} d_p - 1.$$

If w is admissible, the growth rate of the tent map associated to w is a root of P_w. For any admissible w, $P_w(z)$ has a factor of $(z-1)$. Parry polynomials and kneading polynomials are closely related; see Proposition 4.2 of [BDLW19] for the precise relationship.

The *twisted lexicographic ordering* $\leq_E$ on $\{0,1\}^{\mathbb{N}}$ is defined as follows. Two sequences $w = w_1 w_2 \dots$ and $v = v_1 v_2 \dots$ in $\{0,1\}^{\mathbb{N}}$ are equal in the ordering $\leq_E$ if and only if $w = v$, i.e. $w_i = v_i$ for all i. For words $v \neq w$, define $w <_E v$ if, denoting by n the smallest natural number such that $w_n \neq v_n$, either $w_n < v_n$ and $\text{sgn}(w_1 \dots w_{n-1}) = 1$, or $w_n > v_n$ and $\text{sgn}(w_1 \dots w_{n-1}) = -1$.

Theorem 2.1 (Admissibility Condition, Theorem 12.1 of [MT88]). *A word $w \in \{0,1\}^{\mathbb{N}}$ is admissible if and only if $\sigma^j(w) \leq_E w$ for all $j \in \mathbb{N}$.*

It follows immediately from the definition that a finite word w is admissible if and only if w^∞ is admissible and w is irreducible.

Lemma 2.2 (Lemma 3.1 of [BDLW19]). *If w is a finite admissible word, then the first letter of w is 1 and the second letter is 0.*

2.4. **Dominant words.**

Definition 2.3. Let w be a word in the alphabet $\{0,1\}$ that starts with 10 and has positive cumulative sign. The word w is said to be *dominant* if for every integer $1 \leq k < |w|$,

$$\text{Suffix}_k(w) \cdot 1 <_E \text{Prefix}_{k+1}(w).$$

Proposition 2.4 (Proposition 5.1 of [BDLW19])**.** *The leading roots of Parry polynomials of dominant words are dense in* $[\sqrt{2}, 2]$.

Conversely, no dominant word has a growth rate less than $\sqrt{2}$. This definition of dominant words is closely related to and inspired by the notion of dominant words as defined in [Tio18]. See [BDLW19] for more information about dominant words.

2.5. **Kneading theory.** In the context of quadratic maps of the form $f_c(z) = z^2 + c$, define the *sign* of a real number $x \neq 0$ by $\epsilon(x) = -1$ if $x < 0$ and $\epsilon(x) = +1$ if $x > 0$. Define the sequence of cumulative signs by $\eta_n(x) = \prod_{i=0}^{n-1} \epsilon(f^i(x))$. When the critical point 0 is not a periodic or preperiodic point for f_c, the *kneading series* of x, denoted by $K(x,t)$ is the formal series

$$K(x,t) := 1 + \sum_{n=1}^{\infty} \eta_n(x) t^n. \tag{3}$$

For each $c \in \mathbb{C}$, define the *kneading determinant* $K_c(t)$ of f_c by

$$K_c(t) = \begin{cases} K(c,t) \text{ if the critical point is not periodic or preperiodic under } f_c \\ \lim_{C \to c^+} K(C,t) \text{ if the critical point is periodic or preperiodic under } f_c \end{cases}$$

where the limit as $C \to c^+$ is taken over the set of C's such that the critical point is not periodic or preperiodic under f_C.

Theorem 2.5. [MT88, Theorem 6.3] *Let s be the growth rate of f_c. Then the function $K_c(t)$ has no zeros on the interval $[0, 1/s)$, and if $s > 0$ we have $K_c(1/s) = 0$.*

For superattracting real $c \in [-2, 1/4]$ with critical period p, there exists a degree $p-1$ polynomial $P_{c,\text{knead}}$, called the *kneading polynomial*, such that the kneading determinant K_c can be written as

$$K_c(t) = \frac{P(t)}{1-t^p}.$$

Note that, in this case, $\lim_{C \to c^+} K(C,t) = \frac{P(t)}{1-t^p}$, while $\lim_{C \to c^-} K(C,t) = \frac{P(t)}{1+t^p}$, so the two limit functions have the same roots in $\mathbb{D}$ ([Tio18, Proposition 3.3]).

The analogous definitions work in the context of tent maps. Let $w = w_1 w_2, \ldots$ be the infinite word associated to a tent map f for which 1 is not a periodic or preperiodic point. Then the kneading series associated to w (or to f) is given by

$$K_w(t) := 1 + \sum_{n=1}^{\infty} s_n t^n \tag{4}$$

where $s = s_0 s_1 \ldots$ is the infinite sequence of cumulative signs associated to w.

For a superattracting tent map with associated word w_λ of period p, define

$$K_{w_\lambda}(t) = \lim_{\lambda_i \nearrow \lambda} K_{w_{\lambda_i}}(t)$$

where the limit is taken over the growth rates λ_i of non-periodic, non-preperiodic tent maps.

We now briefly sketch a proof that definitions (2.5) and (4) are compatible, meaning that if the growth rate of $z \mapsto z^2+c$ is λ, and w_λ is the word associated to the tent map of growth rate λ, then $K_c(t)$ and $K_{w_\lambda}(t)$ are the same power series in the nonperiodic case, and $K_c(t)$ and $K_{w_\lambda}(t)$ have the same roots in $\mathbb{D}$ in the superattracting case.

The map $\tau : x \mapsto 1-x$ conjugates a tent map to an "inverted tent map" which, by kneading theory (cf. [MT88]), is semiconjugate (via a nonndecreasing function h) to a quadratic map of the form $z \mapsto z^2 + c$. Thus, the tent map is semiconjugate to a map $z \mapsto z^2 + c$ via the map $j := h \circ \tau$. The function j sends the part of the interval in the tent map below the critical point to the positive numbers, while j sends the part of the interval that is above the critical point to negative numbers. Hence, the cumulative sign η_n, which is -1 raised to the parity of the number of times the trajectory of the critical value passes through the set of negative numbers for the map $z \mapsto z^2 + c$, equals s_n, which is -1 raised to the parity of the number of times the trajectory of the critical value passes through the set of numbers above the critical value for the tent map. Compatibility of definitions in the superattracting case then follows from the observation that the two directional limits have the same roots in $\mathbb{D}$.

Proposition 2.6 (Proposition 4.2 of [BDLW19]). *For a superattracting real parameter $c \in [-2, 1/4]$,*

$$(t-1)t^{p-1}P_{c,knead}(t^{-1}) = P_{c,Parry}(t).$$

Here, $P_{c,\text{knead}}$ is the kneading polynomial associated to the map $z \mapsto z^2 + c$, and $P_{c,\text{Parry}}$ is the Parry polynomial of the tent map of the same growth rate.

2.6. **Irreducibility.** To establish irreducibility, we will use two lemmas from [Tio18] which are derived from Eisenstein's criterion.

Lemma 2.7. [Tio18, Lemma 4.1] *Let $d = 2^n - 1$ with $n \geq 1$, and choose a sequence $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ with each $\epsilon_k \in \{\pm 1\}$ such that $\sum_{k=0}^{d} \epsilon_k \equiv 2 \bmod 4$. Then the polynomial*

$$f(x) := \epsilon_0 + \epsilon_1 x + \cdots + \epsilon_d x^d$$

is irreducible in $\mathbb{Z}[x]$.

Lemma 2.8. [Tio18, Lemma 4.2] *Let $f(x) = 1 + \sum_{k=1}^{d} \epsilon_k x^k$ be a polynomial with $\epsilon_k \in \{\pm 1\}$ for all $1 \leq k \leq d$ and $\epsilon_k = -1$ for some k. If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then for all $n \geq 1$, the polynomial $f(x^{2^n})$ is irreducible in $\mathbb{Z}[x]$.*

2.7. **Geometry of Υ_2.** We cite results that we will use later in the paper. First, it is evident from the definition of the set that $\Upsilon_2 \subset \Omega_2 \times [1, 2] \subset \{(z, \lambda) : |z| \leq 2, \lambda \in [1, 2]\}$ (cf. [Tio18]).

Theorem 2.9 ([Tio18]). *Fix a point $z \in \mathbb{D}$. Then the following are equivalent:*

(1) $z \in \Omega_2$.
(2) 0 *is in the limit set of the iterated function system generated by the maps* $x \mapsto zx + 1$ *and* $x \mapsto zx - 1$.
(3) z *is the root of a power series whose coefficients are all in* $\{\pm 1\}$.

We define a map $\wp$ to that takes a finite word of length n, for any $n \in \mathbb{N}$, and returns a word of length $2n$. The map $\wp$ interchanges 0s and 1s, and then adds a 1 between each pair of letters and at the beginning of the word. That is,

$$\wp(w_1 \cdot \ldots \cdot w_n) = 1 \cdot \flat(w_1) \cdot 1 \cdot \flat(w_2) \cdot \ldots \cdot 1 \cdot \flat(w_n).$$

where $\flat$ is defined by $\flat(1) = 0$ and $\flat(0) = 1$.

Proposition 2.10 (Period doubling, Lemma 8.1 and Proposition 8.2 of [BDLW19]).

(1) If $1 \leq \lambda \leq 2$ is the growth rate of a superattracting tent map, then so is $\sqrt{\lambda}$. Conversely, if $1 \leq \lambda \leq \sqrt{2}$ is the growth rate of a superattracting tent map, then so is λ^2.

(2) Let w_λ and w_{λ^2} be the words associated to superattracting tent maps with growth rates λ and λ^2, respectively. Then $w_\lambda = \wp(w_{\lambda^2})$.

Theorem 2.11 (Persistence Theorem, Theorem 1 of [BDLW19]). *Fix $(z, \lambda) \in \Upsilon_2$ with $z \in \mathbb{D}$. Then $\{z\} \times [\lambda, 2] \subset \Upsilon_2$.*

Lemma 2.12 (Corollary 6.3 of [BDLW19]). *Let f be a tent map with growth rate λ_f and denote the itinerary of 1 under f by w_f. Let g be a tent map with growth rate λ_g and denote the itinerary of 1 under g by w_g. If $\lambda_f > \lambda_g$, then $w_f >_E w_g$.*

Theorem 2.13 (Theorem 2 of [BDLW19]). *The Master Teapot contains the unit cylinder, i.e. $S^1 \times [1, 2] \subset \Upsilon_2$.*

3. Characterization inside the unit cylinder

The goal of this section is to prove Theorem 1, which characterizes points in $\Upsilon_2 \cap (\mathbb{D} \times [\sqrt{2}, 2])$. §3.1 proves that for any point $(z, \lambda) \in \Upsilon_2 \cap (\mathbb{D} \times [1, 2])$, the condition of Theorem 1 holds. §3.2 proves that if $(z, \lambda) \in \mathbb{D} \times [\sqrt{2}, 2]$ satisfies the condition of Theorem 1, then $(z, \lambda) \in \Upsilon_2$. The reason the constructive direction is only proven here for $\lambda \in [\sqrt{2}, 2]$ is that the proof uses dominant words, and the set of growth rates of dominant words is contained in and dense in $[\sqrt{2}, 2]$.

3.1. **The condition is necessary.** For any fixed $z \in \mathbb{C}$, define the two functions $f_{0,z}, f_{1,z} : \mathbb{C} \to \mathbb{C}$ by

$$f_{0,z}(x) = zx \quad \text{and} \quad f_{1,z}(x) = 2 - zx.$$

For a growth rate $1 < \lambda \leq 2$, the two branches of the tent map g_λ of growth rate λ are $f_{0,\lambda}$ and $f_{1,\lambda}$. For $z \neq 0$, the inverse maps are given by $f_{0,z}^{-1}(x) = \frac{x}{z}$ and $f_{1,z}^{-1}(x) = \frac{2-x}{z}$.

Lemma 3.1. *Fix $z \in \mathbb{D} \setminus \{0\}$ and let $C = \{x \in \mathbb{C} : |x| \geq \frac{2}{1-|z|}\}$. Then for every finite word w in the alphabet $\{f_{0,z}^{-1}, f_{1,z}^{-1}\}$, we have $w(C) \subset C$.*

Proof. Let D be the open ball $B_R(0)$. The map $f_{0,z}^{-1}$ increases distance to the origin, so if $z \notin B_R(0)$, then $f_{0,z}^{-1}(z) \notin B_R(0)$. We have $f_{1,z}^{-1}(\mathbb{C} \setminus D) = \mathbb{C} \setminus B_{R/|z|}(2/z)$. To guarantee that $\mathbb{C} \setminus B_{R/|z|}(2/z)$ has empty intersection with D, it suffices that $\frac{2}{|z|} + R \leq \frac{R}{|z|}$, which is equivalent to $\frac{2}{1-|z|} \leq R$. □

Lemma 3.2. *Let w be an irreducible finite word in an alphabet $\mathcal{A}$ such that $w = aba$ where a, b are nonempty finite words in the alphabet $\mathcal{A}$. Then $ab \neq ba$.*

Proof. We will show that if $ab = ba$, then w is not irreducible. So suppose $ab = ba$. Let $n = |w|$. Then $\sigma^n(w^\infty) = w^\infty$. Let $m = |ab|$, and note that $m \neq n$ and $m \neq 0$ since a, b are nonempty. Since $ab = ba$, $aba = baa$, so $\sigma^m(w^\infty) = w^\infty$. Since m and n are not equal and are not 0, they must have some common divisor $k < m, n$ with $\sigma^k(w^\infty) = w^\infty$. Thus w consists of $k > 1$ repeats of some shorter word, so w is not irreducible. □

Lemma 3.3 (Sandwich Lemma). *If w is an irreducible finite admissible word in the alphabet $\{0, 1\}$ that can be written as $w = aba$, where a, b are nonempty finite words in the alphabet $\{0, 1\}$, then sgn$(a) = -1$.*

Proof. By the Admissibility Condition (Theorem 2.1), and result of Lemma 3.2 that irreducibility implies $ab \neq ba$,

$$w^\infty = (aba)^\infty >_E a \cdot (aba)^\infty. \tag{5}$$

Denoting the length of a by $|a|$, we have

$$\sigma^{|a|}((aba)^\infty) = (baa)^\infty <_E (aba)^\infty = \sigma^{|a|}(a \cdot (aba)^\infty, \tag{6}$$

where the central inequality is again the Admissibility Condition (Theorem 2.1) and the fact that $ab \neq ba$. Inequality 6 shows that applying $\sigma^{|a|}$ to both words in inequality (5), words that both begin with a, flips the direction of the $>_E$ inequality. Therefore, $\mathrm{sgn}(a) = -1$. □

Lemma 3.4. *Suppose a and b are finite words in the alphabet $\{0,1\}$ such that $ab = ba$. Denote by m the greatest common divisor of $|a|$ and $|b|$. Then there exists a length m word w such that $a = w^{|a|/m}$ and $b = w^{|b|/m}$.*

Proof. Consider the $\mathbb{Z}$ action on finite set $\{\sigma^n(ab) : n \in \mathbb{Z}\}$. It is evident, by construction, that this action has only one orbit. Let m be the period of this orbit, then $\sigma^n(ab) = ab$ iff $p|n$, and $ab = w_0^{|ab|/p}$ for some irreducible word w_0 of length p. By assumption of the lemma, $\sigma^{|a|}(ab) = ba = ab$, $\sigma^{|ab|}(ab) = ab$, hence $p||a|$ and $p||ab|$, which implies p divides the greatest common divisor of $|a|$ and $|b|$, and we can let $w = w_0^{m/p}$. □

Definition 3.5. Let λ be the growth rate of a superattracting tent map and let w_λ be its associated word in the alphabet $\{0,1\}$. A finite word v in the alphabet $\{0,1\}$ is λ-*suitable* if the following conditions hold for all $1 \leq k \leq |w_\lambda|$:

(1) $\mathrm{Suffix}_k(v) \leq_E \mathrm{Prefix}_k(w_\lambda^\infty)$,
(2) If $\mathrm{Suffix}_k(v) = \mathrm{Prefix}_k(w_\lambda^\infty)$, then $\mathrm{sgn}(\mathrm{Suffix}_k(v)) = -1$.

Here, $\mathrm{Suffix}_k(w)$ and $\mathrm{Prefix}_k(w)$ denote the length k Suffix and Prefix, respectively, of the word w, and sgn denotes cumulative sign (see §2 for definitions of these terms).

Lemma 3.6. *Let λ' and λ be growth rates of superattracting tent maps with $\lambda' < \lambda$, and denote by $w_{\lambda'}$ the finite word in $\{0,1\}$ associated to λ'. Then $w_{\lambda'}$ is λ-suitable.*

Proof. Denote by w_λ the finite word in the alphabet $\{0,1\}$ associated to w. By Lemma 2.12, $(w_{\lambda'})^\infty <_E (w_\lambda)^\infty$. Then, by the Admissibility Condition (Theorem 2.1),

$$\sigma^k((w_{\lambda'})^\infty) \leq_E (w_{\lambda'})^\infty <_E (w_\lambda)^\infty \tag{7}$$

for every nonnegative integer k. Hence, for any integer $1 \leq n \leq |w_{\lambda'}|$,

$$\mathrm{Suffix}_n(w_{\lambda'}) \leq_E \mathrm{Prefix}_n(w_{\lambda'}) \leq_E \mathrm{Prefix}_n(w_\lambda^\infty),$$

which is condition (1) of Definition 3.5.

For condition (2) of Definition 3.5, suppose

$$\mathrm{Suffix}_k(w_{\lambda'}) = \mathrm{Prefix}_k(w_\lambda^\infty).$$

For convenience, set $a = \mathrm{Suffix}_k(w_{\lambda'})$. Then, from equation (7), we must also have $a = \mathrm{Prefix}_n(w_{\lambda'}^\infty)$. Set $\tilde{w}_{\lambda'} = w_{\lambda'} \cdot w_{\lambda'}$. Then $\tilde{w}_{\lambda'} = a \cdot b \cdot a$ for some finite nonempty word b.

Suppose $ab = ba$. Then Lemma 3.4 implies that $w_{\lambda'}$ is not irreducible, which contradicts the definition of the word associated to a superattracting tent map. Therefore $ab \neq ba$. Therefore the assumptions of the Sandwich Lemma (Lemma 3.3) apply, guaranteeing $\mathrm{sgn}(a) = -1$. □

Lemma 3.7. *Let λ_0 and λ be growth rates of superattracting tent maps with $\lambda < \lambda_0$. Define, for $m \in \mathbb{N}$,*

$$C_m^{\lambda} = \{(a_1 \dots a_m) \in \{0,1\}^m \mid (a_k \dots a_m) \text{ is } \lambda\text{-suitable } \forall 1 \le k \le m\},$$
$$C_m^{\lambda_0} = \{(a_1 \dots a_m) \in \{0,1\}^m \mid (a_k \dots a_m) \text{ is } \lambda_0\text{-suitable } \forall 1 \le k \le m\}.$$

Then $C_m^{\lambda} \subseteq C_m^{\lambda_0}$.

Proof. This is an immediate consequence of Lemma 2.12. □

Proposition 3.8. *Let λ be the growth rate of a superattracting tent map and fix a point $z \in \mathbb{D} \setminus \{0\}$. For each $m \in \mathbb{N}$, define*

$$C_m^{\lambda} = \{(a_1 \dots a_m) \in \{0,1\}^m \mid (a_k \dots a_m) \text{ is } \lambda\text{-suitable } \forall\, 1 \le k \le m\}.$$

If there exists $M \in \mathbb{N}$ such that C_M^{λ} is nonempty and for every word $(a_1 \dots a_M) \in C_M^{\lambda}$,

$$\left|f_{a_1,z}^{-1} \circ \dots \circ f_{a_M,z}^{-1}(1)\right| > \frac{2}{1-|z|}, \tag{8}$$

then for every $\lambda' < \lambda$,

$$(z, \lambda') \notin \Upsilon_2.$$

Proof. Fix λ and z as above. We will proceed by contradiction; suppose there exists $M \in \mathbb{N}$ such that equation (8) holds but there exists $\lambda' < \lambda$ such that $(z, \lambda') \in \Upsilon_2$. Since C_M^{λ} is finite, we may fix a real number $\epsilon > 0$ such that

$$\left|f_{a_1,z}^{-1} \circ \dots \circ f_{a_M,z}^{-1}(1)\right| > \frac{2}{1-|z|} + \epsilon,$$

for all $(a_1 \dots a_M) \in C_M^{\lambda}$ Let w_λ be the word in the alphabet $\{0,1\}$ associated to λ.

By the definition of Υ_2, there exists a sequence of points $(\tilde{z}_n)$ and a sequence of growth rates $(\tilde{\lambda}_n)$ such that $\tilde{z}_n \to z$, $\tilde{\lambda}_n \to \lambda'$, each $\tilde{\lambda}_n$ is the growth rate of a superattracting tent map, and $\tilde{z}_n$ is a Galois conjugate of $\tilde{\lambda}_n$. The Persistence Theorem (Theorem 2.11) then implies that there exists a sequence of points (z_n) and a sequence of growth rates (λ_n) such that $z_n \to z$, $\lambda_n < \lambda$ for all n, $\lambda_n \to \lambda$, each λ_n is the growth rate of a superattracting tent map, and z_n is a Galois conjugate of λ_n. For each n, let $w_n = a_1^n \dots a_{\ell_n}^n$ be the word in $\{0,1\}$ associated to λ_n. Also without loss of generality, we may assume $|w_n| = \ell_n \ge M$ for all n, since there are only finitely many words of any fixed length.

Thus, for each n,

$$f_{a_{\ell_n}^n,\lambda_n} \circ \dots \circ f_{a_1^n,\lambda_n}(1) = 1.$$

This is a polynomial in $\mathbb{Z}[\lambda_n]$. Hence, since z_n is a Galois conjugate of λ_n,

$$f_{a_{\ell_n}^n,z_n} \circ \dots \circ f_{a_1^n,z_n}(1) = 1,$$

and so

$$1 = f_{a_1^n,z_n}^{-1} \circ \dots \circ f_{a_{\ell_n}^n,z_n}^{-1}(1). \tag{9}$$

Since w_n is λ-suitable by Lemma 3.6, its length M suffix is also λ-suitable, so

$$a_{\ell_n-M}^n \dots a_{\ell_n}^n \in C_M^{\lambda}.$$

Then, by assumption,

$$\left|f_{a_{\ell_n-M}^n,z}^{-1} \circ \dots f_{a_{\ell_n}^n,z}^{-1}(1)\right| > \frac{2}{1-|z|} + \epsilon.$$

Since $z_n \to z$, by restricting to sufficiently large n, we can guarantee that

$$\left| f^{-1}_{a^n_{\ell_n - M},z} \circ \dots f^{-1}_{a^n_{\ell_n},z}(1) - f^{-1}_{a^n_{\ell_n - M},z_n} \circ \dots f^{-1}_{a^n_{\ell_n},z_n}(1) \right| < \frac{\epsilon}{2}$$

and

$$\left| \frac{2}{1-|z|} - \frac{2}{1-|z_n|} \right| < \frac{\epsilon}{2},$$

which together imply

$$f^{-1}_{a^n_{\ell_n - M},z_n} \circ \dots f^{-1}_{a^n_{\ell_n},z_n}(1) > \frac{2}{1-|z_n|}.$$

But by Lemma 3.1 and the fact that $\frac{2}{1-|z_n|} > 1$, this contradicts equation 9. ☐

3.2. The condition is sufficient for $\lambda \in [\sqrt{2}, 2]$.

Lemma 3.9. *Fix $z \in \mathbb{D} \setminus \{0\}$, λ the growth rate of a superattracting tent map, and $x \in \mathbb{C}$. Suppose that for every $n \in \mathbb{N}$, there exists a word $w_n = (\epsilon^n_1, \dots, \epsilon^n_n) \in \{0,1\}^n$ such that w_n is λ-suitable and*

$$|f^{-1}_{\epsilon^n_n,z} \circ \dots \circ f^{-1}_{\epsilon^n_1,z}(x)| \leq \frac{2}{1-|z|}.$$

Then there exists a sequence $\epsilon^*_1, \epsilon^*_2, \dots \in \{0,1\}^{\mathbb{N}}$ such that for every $k \in \mathbb{N}$, $(\epsilon^*_k, \dots, \epsilon^*_1)$ is λ-suitable and

$$|f^{-1}_{\epsilon^*_k,z} \circ \dots \circ f^{-1}_{\epsilon^*_1,z}(x)| \leq \frac{2}{1-|z|}. \tag{10}$$

Furthermore, there are infinitely many i such that $\epsilon^*_i = 1$ and infinitely many i such that $\epsilon^*_i = 0$.

Proof. For each $n \in \mathbb{N}$, define U_n to be the subset of $\{0,1\}^{\mathbb{N}}$ consisting of all sequences $(\epsilon_1, \epsilon_2, \dots) \in \{0,1\}^{\mathbb{N}}$ such that

$$|f^{-1}_{\epsilon_n,z} \circ \dots \circ f^{-1}_{\epsilon_1,z}(x)| \leq \frac{2}{1-|z|}$$

and $(\epsilon_n, \dots, \epsilon_1)$ is λ-suitable.

For any $\epsilon \in \{0,1\}$,

$$f^{-1}_{\epsilon} \left(\left\{ x \in \mathbb{C} : |x| > \frac{2}{1-|z|} \right\} \right) \subset \left\{ x \in \mathbb{C} : |x| > \frac{2}{1-|z|} \right\}$$

by Lemma 3.1, implying that for any sequence $(\epsilon_1, \epsilon_2, \dots) \in U_n$,

$$|f^{-1}_{\epsilon_k,z} \circ \dots \circ f^{-1}_{\epsilon_1,z}(x)| \leq \frac{2}{1-|z|}$$

for all $1 \leq k \leq n$. Hence $U_{n+1} \subset U_n$ for all n. Each U_n is nonempty by the assumption, since it contains w_n. Since each U_n is a finite union of clopen cylinder sets, U_n is a compact subset of $\{\pm 1\}^{\mathbb{N}}$ with the 2-adic metric topology. Thus $\{U_n\}$ is a nonincreasing sequence of nested, compact, nonempty sets. By Cantor's Intersection Theorem, $\bigcap_{n=1}^{\infty} U_n$ is nonempty. By construction, any sequence in this intersection satisfies 10 for all n.

Now suppose $\epsilon^*_1, \epsilon^*_2, \dots$ is a sequence in $\bigcap_{n=1}^{\infty} U_n$ but $\epsilon^*_i = 0$ for all but finitely many i. Thus, the tail of the sequence $\epsilon^*_1, \epsilon^*_2, \dots$ is all 1s. The map $f^{-1}_{1,z}$ is a linear expansion centered at the fixed point $2/(1+z)$. Hence, given any point, its image under $(f^{-1}_{1,z})^n$ for sufficiently large n will have modulus greater than $\frac{2}{1-|z|}$, contradicting (10). Therefore $\epsilon^*_i = 0$ for infinitely many i. The same argument shows $\epsilon^*_i = 1$ for infinitely many i. ☐

Lemma 3.10. *Fix a real number $\epsilon > 0$, let λ be the growth rate of a superattracting tent map, and fix $z \in \mathbb{D} \setminus \{0\}$. Suppose that for arbitrarily large $M \in \mathbb{N}$, there exists a λ-suitable word $w = (a_1, \ldots, a_M) \in \{0,1\}^N$ such that*

$$|f_{a_1,z}^{-1} \cdot \cdots \cdot f_{a_M,z}^{-1}(1)| \leq \frac{2}{1-|z|}.$$

Then, there exists a sequence $\epsilon_1^*, \epsilon_2^*, \cdots \in \{0,1\}^{\mathbb{N}}$ such that for arbitrarily large $N_0 \in \mathbb{N}$, there is some $N > N_0$ such that the word $w_N = (\epsilon_N^*, \ldots, \epsilon_1^*)$ is λ-suitable, $\epsilon_N^* = 1$, and the Parry polynomial P_{w_N} has a root within distance ϵ of z,

Proof. Let $\epsilon_1^*, \epsilon_2^*, \cdots \in \{0,1\}^{\mathbb{N}}$ be the sequence guaranteed by Lemma 3.9. Thus, for all $n \in \mathbb{N}$, $(\epsilon_n^*, \ldots, \epsilon_1^*)$ is λ-suitable and

$$|f_{\epsilon_n^*,z}^{-1} \circ \cdots \circ f_{\epsilon_1^*,z}^{-1}(1)| \leq \frac{2}{1-|z|}.$$

Therefore

$$|1 - f_{\epsilon_1^*,z} \circ \cdots \circ f_{\epsilon_n^*,z}(0)| \leq \frac{2|z|^n}{1-|z|},$$

since each of the maps $f_{0,z}, f_{1,z}$ contracts distances by a factor of $|z|$. So

$$\left|1 - \lim_{n\to\infty} f_{\epsilon_1^*,z} \cdot \cdots \cdot f_{\epsilon_n^*,z}(0)\right| = 0. \tag{11}$$

Set $d_i^* = 2\epsilon_i^*$. Set $e_i^* = (-1)^{\epsilon_i^*}$. An induction argument shows that for any $x \in \mathbb{D} \setminus \{0\}$,

$$f_{\epsilon_1^*,x} \circ \cdots \circ f_{\epsilon_n^*,x}(0) = d_1^* + x d_2^* \prod_{j=1}^{1} e_j^* + x^2 d_3^* \prod_{j=1}^{2} e_j^* + \cdots + x^{n-1} d_n^* \prod_{j=1}^{n-1} e_j^* \tag{12}$$

Now define the power series

$$S(x) := 1 - d_1^* - \sum_{k=1}^{\infty} x^k d_{k+1}^* \prod_{j=1}^{k} e_j^*.$$

By equation (12),

$$S(x) = 1 - \lim_{n\to\infty} f_{\epsilon_1^*,x} \cdot \cdots \cdot f_{\epsilon_n^*,x}(0)$$

and this limit converges for all $x \in \mathbb{D}$ by Lemma 3.11. By equation (11), z is a root of S. Note that because S is analytic, and hence holomorphic, all roots of S are isolated.

For integers $n \in \mathbb{N}$ such that $\epsilon_n^* = 1$ and $\prod_{j=1}^n e_j^* = 1$ (there are infinitely many of them due to Lemma 3.9), let S_n denote the n^{th} partial sum of S, i.e.

$$S_n(x) = 1 - f_{\epsilon_1^*,x} \circ \cdots \circ f_{\epsilon_{n-1}^*,x}(0) = 1 - d_1^* - x d_2^* (\prod_{j=1}^{1} e_j^*) - x^2 d_3^* (\prod_{j=1}^{2} e_j^*) - \cdots - x^{n-1} d_n^* (\prod_{j=1}^{n-1} e_j^*).$$

Then, since dividing and multiplying by -1 and 1 are equivalent,

$$S_n(x) = \frac{S_n(x)}{\prod_{j=1}^n e_j^*} = 1 - d_1^* x^0 (\prod_{j=1}^{n} e_j^*) - x^1 d_2^* (\prod_{j=2}^{n} e_j^*) - x^2 d_3^* (\prod_{j=3}^{n} e_j^*) - \cdots - x^{n-1} d_n^* (\prod_{j=n}^{n} e_j *).$$

Using the fact that d_i^* is nonzero precisely when $e_i^* = -1$, the above equation becomes

$$S_n(x) = 1 + d_1^* x^0 (\prod_{j=2}^{n} e_j^*) + x^1 d_2^* (\prod_{j=3}^{n} e_j^*) + x^2 d_3^* (\prod_{j=4}^{n} e_j^*) + \cdots + x^{n-2} d_{n-1} \prod_{j=n}^{n} e_j^* + x^{n-1} d_n^*. \tag{13}$$

Meanwhile, for n such that $\epsilon_n^* = 1$, the Parry polynomial associated to the word $(\epsilon_n^*, \ldots, \epsilon_1^*)$ is

$$P_{(\epsilon_n^*,\ldots,\epsilon_1^*)}(x) = -1 - d_1^* x^0 (\prod_{j=2}^{n} e_j^*) - d_2^* x^1 (\prod_{j=3}^{n} e_j^*) - \cdots - d_{n-1}^* x^{n-2} (\prod_{j=n}^{n} e_j^*) - d_n^* x^{n-1} + x^n \tag{14}$$

Combining equations (14) and (14), we have that for n such that $\epsilon_n^* = 1$,

$$P_{(\epsilon_n^*,\ldots,\epsilon_1^*)}(x) = x^n - S_n(x).$$

We wish to show that, for sufficiently large n, assuming $\epsilon_n^* = 1$, $P_{(\epsilon_n^*,\ldots,\epsilon_1^*)}$ has a root within ϵ of z. Let C be the circle of radius ϵ centered at z, and let D be the closed disk C bounds. Without loss of generality, assume ϵ is small enough that D is contained in $\mathbb{D} \setminus \{0\}$. Since S has isolated roots, we may furthermore assume without loss of generality that D contains no root of S except for z.

We will apply Rouché's Theorem to $-S$ and $P_{(\epsilon_n^*,\ldots,\epsilon_1^*)}$ on C. For any n,

$$|-S(x) - P_{(\epsilon_n^*,\ldots,\epsilon_1^*)}| = |-S(x) + S_n(x) + x^n| \geq \big||S(x) - S_n(x)| - |x^n|\big|. \tag{15}$$

By compactness and the fact that S has isolated roots, there exists $\delta > 0$ such that $|S| > \delta$ on C.

Notice

$$|S(x) - S_n(x)| \leq 2 \sum_{j=n+1}^{\infty} |x|^j = \frac{2|x|^{n+1}}{1-|x|}.$$

Therefore, for sufficiently large N, $|S(x) - S_N(x)| < \delta/4$ for all $x \in C$. Also for sufficiently large N, $|x^N| < \delta/4$. Hence, for sufficiently large N, from equation (15), we have $|-S(x) - P_{(\epsilon_N^*,\ldots,\epsilon_1^*)}(x)| > \delta/2$.

Thus, for sufficiently large N, the winding number around 0 of the image of $-S$ on C equals the winding number around 0 of the image of $P_{(\epsilon_N^*,\ldots,\epsilon_1^*)}$. Therefore, $P_{(\epsilon_N^*,\ldots,\epsilon_1^*)}$ has a root in D.

□

Lemma 3.11. *Fix $z \in \mathbb{D} \setminus \{0\}$ and $x \in \mathbb{C}$. For a sequence $(\epsilon_1, \epsilon_2, \ldots) \in \{0,1\}^{\mathbb{N}}$, the limit*

$$\lim_{n\to\infty} f_{\epsilon_1,z} \circ f_{\epsilon_2,z} \circ \cdots \circ f_{\epsilon_n,z}(x)$$

is well-defined and does not depend on x.

Proof sketch. This is Definition 3.1.7 and Lemma 3.1.2 from [CKW17]. The idea is that all the maps are (uniform) contractions, and if you precompose with enough of such contractions, the effect of changing the initial point is very small. □

Proposition 3.12. *Let w_1 be a dominant, finite word in the alphabet $\{0,1\}$ and let λ be the growth rate of the associated tent map. Let w_2 be a finite λ-suitable word in the alphabet $\{0,1\}$ with sgn$(w_2) = +1$. Then for any $n \in \mathbb{N}$ such that $|w_1| > n|w_2|$ and any even integer $m \geq |w_1|$, the infinite word $(w_1 1^m w_2^n)^\infty$ is admissible.*

In regards to the condition $|w_1| > n|w_2|$, note that we will apply Proposition 3.12 to pairs w_1, w_2 for which $|w_1|$ is very big relative to $|w_2|$.

Proof. By the Admissibility Condition (Theorem 2.1), it suffices to show that

$$\sigma^k (w_1 1^m w_2^n)^\infty \leq_E (w_1 1^m w_2^n)^\infty$$

for all $k < |w_1| + m + n|w_2|$. Here σ denotes the shift map (defined in §2.1).

- Case 1: If $k < |w_1|$, due to the dominance of w_1, a suffix of w_1 with 1 added to the end is no larger than a prefix of w_1 of the same length, hence
$$\sigma^k(w_1 1^m w_2^n)^\infty \leq_E (w_1 1^m w_2^n)^\infty.$$
- Case 2: If $m + |w_1| - 1 > k \geq |w_1|$, then
$$\sigma^k(w_1 1^m w_2^n)^\infty \leq_E (w_1 1^m w_2^n)^\infty,$$
because the second letter of w_1 is 0.
- Case 3: $k = m + |w_1| - 1$, due to condition (1) of Definition 3.5 (λ-suitability), the number of 0s in the beginning of w_2 can not be more than the number of 0s following the first 1 in w_1. Suppose the number of 0s in the beginning of w_2 is smaller,
$$\sigma^k(w_1 1^m w_2^n)^\infty \leq_E (w_1 1^m w_2^n)^\infty$$
is true from the definition of $\leq_E$. If otherwise, then the last letter of w_1 must be 1, and this is true due to condition (1) of Definition 3.5 (λ-suitability).
- Case 4: If $k \geq |w_1| + m$, if $\sigma^k(w_1 1^m w_2^n)^\infty$ and $(w_1 1^m w_2^n)^\infty$ differ at the first $m + |w_1| + n|w_2| - k$ letters,
$$\sigma^k(w_1 1^m w_2^n)^\infty \leq_E (w_1 1^m w_2^n)^\infty$$
due to condition (1) of λ-suitability. If otherwise, after deleting the common prefix of length $m + |w_1| + n|w_2| - k$ on both itineraries, the second one is smaller due to the discussions in Case 1, 2, and 3, due to the second condition in λ-suitability we have
$$\sigma^k(w_1 1^m w_2^n)^\infty \leq_E (w_1 1^m w_2^n)^\infty.$$

□

Proposition 3.13. *Let words w_1 and w_2, and integer n satisfy the same assumptions as in Proposition 3.12. Then there exists a finite extension w_1' of w_1 and an even integer $m \geq |w_1'|$, such that $(w_1' 1^m w_2^n)^\infty$ is admissible, and $P(z)/(z-1)$ is an irreducible polynomial, where P is the Parry polynomial of $(w_1' 1^m w_2^n)$.*

Proof. The proof is essentially identical to the proof of Proposition 7.5 in [BDLW19]. Let j be an integer such that
$$2^j > 10|w_1| + 10.$$
By Lemma 7.6 of [BDLW19], if $|w_1|$ is odd, let
$$\kappa = 2^j - 1 - n|w_2| - 6 - 3|w_1|,$$
then w_1' is chosen as either
$$w_1 1^\kappa 1 0 1^{|w_1|} 0 1 1^{|w_1|}$$
or
$$w_1 1^\kappa 1 0 1^{|w_1|} 1 0 1^{|w_1|},$$
depending on which makes the sum of the coefficients of the kneading polynomial for $w_1' 1^{2^j} w_2^n$ equal to 2 mod 4. Similarly, if $|w_1|$ is even, let
$$\kappa = 2^j - 1 - n|w_2| - 4 - 3|w_1|$$
and repeat the process above. □

We will need the following two technical lemmas.

Lemma 3.14 (Lemma 7.7 of [BDLW19]). *Let w_2 be a finite word whose Parry polynomial has a root at $z_0 \in \mathbb{D}$. Then for any $\epsilon > 0$, there exists an integer $N = N(\epsilon, w_2) \in \mathbb{N}$ such that $n > N$ implies that for every finite word w_1 for which $w_1 w_2^n$ is admissible, the Parry polynomial associated to $(w_1 w_2^n)$ has a root within distance ϵ of z_0.*

Proposition 3.15 (Proposition 7.10 of [BDLW19]). *For all $y \in (\sqrt{2}, 2)$ and all $\epsilon > 0$, there exists a sequence of finite dominant words $(w_n)_{i=1}^{\infty}$ such that for any admissible extension w_n' of w_n, including the empty extension, the growth rate of $(w_n')^\infty$ is within distance ϵ of y.*

Proposition 3.16 proves the constructive direction of Theorem for growth rates in $[\sqrt{2}, 2)$.

Proposition 3.16. *Let $\sqrt{2} \leq \lambda < 2$ be the growth rate of a superattracting tent map and fix a point $z \in \mathbb{D} \setminus \{0\}$. For each $m \in \mathbb{N}$, define*

$$C_m^\lambda = \{(a_1 \dots a_m) \in \{0,1\}^m \mid (a_k \dots a_m) \text{ is } \lambda\text{-suitable } \forall\, 1 \leq k \leq m\}.$$

Suppose that for every $M \in \mathbb{N}$ there exists a word $(a_1 \dots a_M) \in C_M^\lambda$ such that

$$\left| f_{a_1,z}^{-1} \circ \cdots \circ f_{a_M,z}^{-1}(1) \right| \leq \frac{2}{1 - |z|}. \tag{16}$$

Then $(z, \lambda) \in \Upsilon_2$.

Proof. Fix $\epsilon > 0$. We will show that there exists a point $(z', \lambda') \in \Upsilon_2$ that is distance at most ϵ to (z, λ). Since Υ_2 is closed, this will imply the result.

By Lemma 3.10, there exists a finite, λ-suitable word $w_{N_1} = (\epsilon_{N_1}^*, \dots, \epsilon_1^*)$ such that $\epsilon_{N_1}^* = 1$ and the Parry polynomial $P_{w_{N_1}}$ has a root within distance $\epsilon/3$ of z.

By Lemma 3.14, there exists an even integer $N_2 \in \mathbb{N}$ such that for every finite word w_1 for which $w_1 w_{N_1}^{N_2}$ is admissible, the Parry polynomial associated to $w_1 w_{N_1}^{N_2}$ has a root within distance $\epsilon/3$ of the root of $P_{w_{N_1}}$ (and hence within distance $2\epsilon/3$ of z). Since N_2 is even, $\operatorname{sgn}(w_{N_1}^{N_2}) = +1$.

As a consequence of Proposition 3.15, there are infinitely many distinct finite dominant words $(v_i)_{i=1}^{\infty}$ such that for any admissible extension v_n' of v_n, including the empty extension, the growth rate of $(v_n')^\infty$ is in the interval $[\lambda, \lambda + \epsilon/3]$. Since there are only finitely many words of any bounded length, we may pick a v_i so that $|v_i| > N_2 |w_{N_1}|$. Denote this word v_i by w_{λ_0}, and let λ_0 be the associated growth rate. Since $\lambda_0 \geq \lambda$ and w_{N_1} is λ-suitable, w_{N_1} is also λ_0-suitable by Lemma 3.7.

Applying Proposition 3.13 (using w_{λ_0} in place of w_1, w_{N_1} in place of w_2, and N_2 in place of n) guarantees that there exists a finite extension w_{λ_0}' of w_{λ_0} and an even integer $m > |w_{\lambda_0}'|$ such that

$$(w_{\lambda_0}' \cdot 1^m \cdot w_{N_1}^{N_2})^\infty$$

is admissible and $P(z)/(z-1)$ is an irreducible polynomial, where P is the Parry polynomial for $(w_{\lambda_0}' \cdot 1^m \cdot w_N^n)^\infty$. By construction P has a root z' within distance $2\epsilon/3$ of z and a growth rate λ' within distance $\epsilon/3$ of λ. □

Proof of Theorem 1. Propositions 3.8 and 3.16 prove that the condition of Theorem 1 is both necessary and sufficient for a point $(z, \lambda) \in \mathbb{D} \times [\sqrt{2}, 2]$ to be in Υ_2.

□

4. CHARACTERIZATION OUTSIDE THE UNIT CYLINDER

The goal of this section is to prove Theorem 2, a characterization of the part of the Master Teapot that is outside the unit cylinder.

Lemma 4.1. *The set of growth rates of superattracting tent maps whose kneading polynomials are irreducible in $\mathbb{Z}[x]$ is dense in $[1,2]$.*

Proof. For any $\epsilon > 0$ and any $\beta \in (\sqrt{2}, 2)$, the proof of Theorem 7.11 of [BDLW19] constructs an admissible superattracting word w such that the kneading polynomial $K_w(x)$ is irreducible in $\mathbb{Z}[x]$ (irreducibility is because it is of degree $2^n - 1$ and the sum of its coefficients is equal to $2 \mod 4$ (cf. Lemma 2.7)), and the growth rate of the associated tent map is in $(\beta-\epsilon, \beta+\epsilon) \cap [1,2]$. Hence, the set of growth rates of superattracting tent maps whose kneading polynomials are irreducible in $\mathbb{Z}[x]$ is dense in $[\sqrt{2}, 2]$. Period Doubling (Proposition 2.10) together with Lemma 2.8 then implies density of this set on the interval $[1,2]$. $\square$

We will use the following result by Tiozzo:

Proposition 4.2 (Proposition 3.3 of [Tio18])**.** *The map Tr* : $[-2, 1/4] \to Com\, \widetilde{\mathbb{D}}$ *give by*

$$Tr(c) := \{z \in \mathbb{D} : K_c(z) = 0\} \cup \{\infty\}$$

is continuous in the Hausdorff topology.

Here, $\widetilde{\mathbb{D}} = \mathbb{D} \cup \{\infty\}$ is the one point compactification of the unit disk, and $Com\ V$ denotes the space of compact subsets of a compact metric space V.

Proof of Theorem 2. For convenience of notation, set

$$\breve{\Upsilon}_2 := \Upsilon_2 \cap \big((\mathbb{C} \setminus \overline{\mathbb{D}}) \times [1,2]\big)$$
$$\breve{\mathfrak{U}}_2 := \{(z,\lambda) \in \big((\mathbb{C} \setminus \overline{\mathbb{D}}) \times [1,2]\big) : K_{w_\lambda}(\tfrac{1}{z}) = 0\}$$

We will prove $\breve{\Upsilon}_2 \subset \breve{\mathfrak{U}}_2 \cup (S^1 \times [1,2])$ and $\breve{\mathfrak{U}}_2 \subset \breve{\Upsilon}_2 \cup (S^1 \times [1,2])$.

$\breve{\Upsilon}_2 \subset \breve{\mathfrak{U}}_2 \cup (S^1 \times [1,2])$: First, observe that if λ is the growth rate of a superattracting tent map f and z, with $|z| > 1$, is a Galois conjugate of λ, then $\frac{1}{z}$ and $\frac{1}{\lambda}$ are roots of the kneading polynomial for f, and thus also of K_{w_λ}. Hence $(z,\lambda) \in \breve{\mathfrak{U}}_2$.

We now show that $\breve{\mathfrak{U}}_2 \cup (S^1 \times [1,2])$ is sequentially compact, and hence closed, and thus contains $\breve{\Upsilon}_2$. Let (z_i, λ_i), $i \in \mathbb{N}$, be a sequence of points in $\breve{\mathfrak{U}}_2$. Since $\breve{\mathfrak{U}}_2$ is bounded, we may assume without loss of generality that (z_i, λ_i) converges – i.e. that there exists $z_* \in \mathbb{C}$ and $\lambda_* \in [1,2]$ such that $z_i \to z_*$ and $\lambda_i \to \lambda_*$. If $(z_*, \lambda_*) \in (S^1 \times [1,2])$, there is nothing to prove. So suppose $(z_*, \lambda_*) \notin (S^1 \times [1,2])$. Set $\epsilon = (|z_*| - 1)/2$. The set

$$\mathfrak{U}_2 \cap (\{z \in \mathbb{C} : |z| \geq 1 + \epsilon\} \times \{\lambda_*\})$$

is finite because roots of analytic functions are isolated. . Hence

$$d(z_*, \mathfrak{U}_2 \cap (\{z \in \mathbb{C} : |z| \geq 1 + \epsilon\} \times \{\lambda_*\}) > 0.$$

This contradicts Proposition 4.2. Therefore $(z_*, \lambda_*) \in \breve{\mathfrak{U}}_2$.

$\breve{\mathfrak{U}}_2 \subset \breve{\Upsilon}_2 \cup (S^1 \times [1,2])$: By Lemma 4.1, the set of growth rates of superattracting tent maps whose kneading polynomials are irreducible in $\mathbb{Z}[x]$ is dense in $[1,2]$. By Proposition 4.2, roots in $\mathbb{D}$ of kneading polynomials vary continuously in the Hausdorff topology. Thus, $(z,\lambda) \in \breve{\mathfrak{U}}_2$ implies (z,λ) is in the closure of the set of points (z_0, λ_0) for which λ_0 is the growth rate of a superattracting tent map for which the associated kneading polynomial is irreducible in $\mathbb{Z}[x]$, and z_0 is a Galois conjugate of λ_0. Hence $(z, \lambda) \in \check{\Upsilon}_2 \cup (S^1 \times [1, 2])$. $\square$

Remark 4.3. This characterization gives us a way to plot the part of the teapot outside the unit cylinder, because whether or not an analytic function has a zero in a specific region can be determined via Rouché's theorem.

5. Period doubling and the proof of Theorem 3

Proof of Theorem 3. It follows from Period Doubling (Proposition 2.10) that a point $(z, \lambda) \in \mathbb{C} \times (1, \sqrt{2})$ is in Γ_2 if and only if $(z^{2k}, \lambda^{2k}) \in \Upsilon_2$ for $k \in \mathbb{N}$ such that $\lambda^{2k} \in [\sqrt{2}, 2]$. Theorem 1 characterizes the points in $\Upsilon_2 \cap (\mathbb{D} \times [\sqrt{2}, 2])$. Theorem 2.13 states that Υ_2 contains the unit cylinder $S^1 \times [1, 2]$. Theorem 2 characterizes the points in $\Upsilon \cap (\{z \in \mathbb{C} : |z| > 1\} \times [1, 2])$. $\square$

6. Symmetries, slices, and the Thurston Set

The following proposition is likely well-known to experts; we include the proof for completeness.

Proposition 6.1. *$\Omega_2 \cap \mathbb{D}$ is invariant under reflection across the real axis and across the imaginary axis.*

Proof. The set $\Omega_2 \cap \mathbb{D}$ is invariant under reflection across the real axis because Galois conjugates come in complex conjugate pairs. By Theorem 2.9, $\Omega_2 \cap \mathbb{D}$ is the set of all the roots in $\mathbb{D}$ of all power series with all coefficients in $\{\pm 1\}$. So if $z \in \mathbb{D}$ is a root of a power series S with coefficients in $\{\pm 1\}$, then $-z$ is a root of the power series formed from S by flipping the sign of the coefficients on all terms of odd degree. Therefore the complex conjugate, $\overline{-z}$, is in Ω_2. $\square$

Proposition 6.2. *The Thurston set Ω_2 is the image of the projection to $\mathbb{C}$ of the Master Teapot Υ_2.*

The Master Teapot Υ_2 is a subset of $\mathbb{C} \times \mathbb{R}$. By the projection to $\mathbb{C}$, we mean that map $\mathbb{C} \times \mathbb{R} \ni (z, \lambda) \mapsto z$.

Proof. Let π denote the projection map to $\mathbb{C}$. It follows immediately from the definitions of the sets that $\pi(\Upsilon_2)$ contains all Galois conjugates of all growth rates of superattracting tent maps, and also that $\Upsilon_2 \subset \pi^{-1}(\Omega_2)$. Furthermore, $\pi(\Upsilon_2)$ is compact because Υ_2 is compact. As a closet set that contains all Galois conjugates of all growth rates of superattracting tent maps, $\pi(\Upsilon_2)$ cannot be smaller than Ω_2. Hence $\pi(\Upsilon_2) = \Omega_2$.

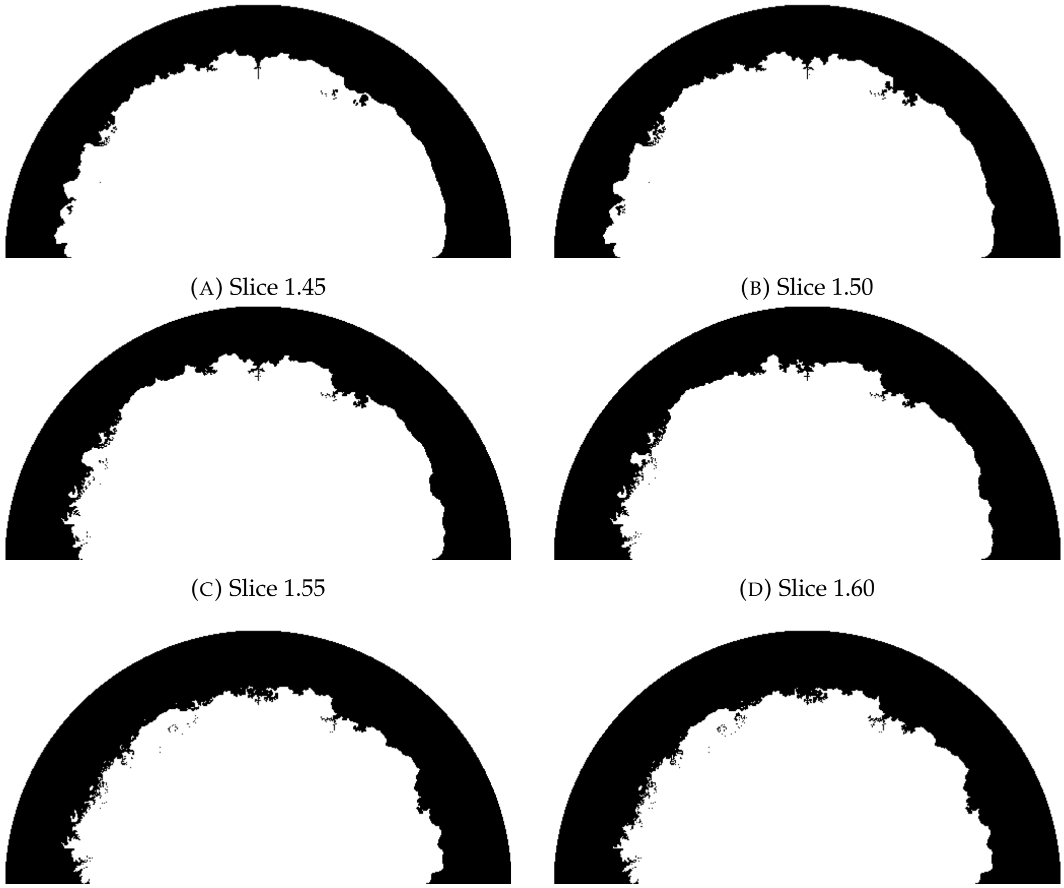
$\square$

Lemma 6.3. *For $1 < \lambda < \sqrt{2}$, the horizontal slice of height λ,*

$$\Upsilon_2 \cap (\mathbb{D} \times \{\lambda\}),$$

has left-right symmetry (i.e. is invariant under the map $(x + iy, \lambda) \mapsto (-x + iy, \lambda)$).

Proof. Since all slices are invariant under complex conjugation, it suffices to prove that $\Upsilon_2 \cap (\mathbb{D} \times \{\lambda\})$ is invariant under $(z, \lambda) \mapsto (-z, \lambda)$. Furthermore, it suffices to establish that if λ is an admissible growth rate and $z \in \mathbb{D}$ is a Galois conjugate of λ, then $(-z, \lambda) \in \Upsilon_2$. But for any growth rate λ in $[1, \sqrt{2})$, there exists an integer $n \geq 2$ such that $\lambda^{2^n} \in (\sqrt{2}, 2]$; then all 2^{-n} roots of z^{2^n} are Galois conjugates of λ. In particular, $-z$ is a Galois conjugate of λ. $\square$



(A) Slice 1.45

(B) Slice 1.50

(C) Slice 1.55

(D) Slice 1.60

(E) Slice 1.65

(F) Slice 1.70

FIGURE 4. Approximations of the "slices" $\Upsilon_2 \cap (\mathbb{D} \times \{c\})$ for various values of $c \in (\sqrt{2}, 2)$ based on Theorem 1. Since slices are invariant under taking complex conjugates, only the top half of each slice (points z with $\mathrm{Re}(z) \geq 0$) is shown. The images are formed by dividing the rectangle $\{x + iy : x \in [-1, 1], y \in [0, 1]\}$ into a 500×250 grid of squares and testing one point in each square; we color the square white if there exists an integer $1 \leq M \leq 20$ such that no word in C_M^λ satisfies equation (1), and otherwise we color the square black. Thus, a white square indicates that the associated point is definitely not in the slice, while a black square is inconclusive.

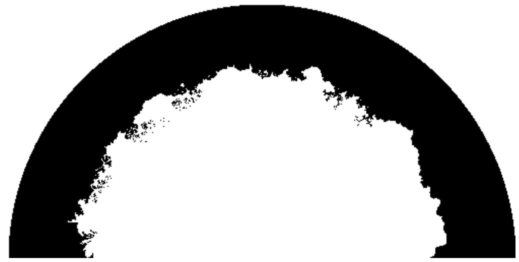
Proof of Theorem 4. Consider the length 16 itinerary

$$L = (1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1).$$

The reader may verify that L is admissible, the associated sequence of cumulative signs is

$$(+1, -1, -1, -1, +1, -1, -1, +1, -1, +1, -1, -1, +1, +1, -1, -1),$$

(A) Slice 1.75



(B) Slice 1.80

(C) Slice 1.85

(D) Slice 1.90

(E) Slice 1.95

FIGURE 5. A continuation of Figure 4

and the associated Parry polynomial is

$$P(z) = z^{16} - 2z^{15} + 2z^{12} - 2z^{11} + 2z^{9} - 2z^{8} + 2z^{7} - 2z^{6} + 2z^{4} - 2z^{2} + 1,$$

which factors as

$$P(z) = (-1+z)(1+z)(-1+z^2-z^4+z^6-2z^7+3z^8-4z^9+3z^{10}-2z^{11}+z^{12}-2z^{13}+z^{14})$$

Let λ_0 be the leading root of P; computation shows

$$\lambda_0 \approx 1.8149185987640513.$$

Let z_0 be the root of P with approximate coordinates

$$z_0 = -0.5840341196392905 + 0.4820600149798202i.$$

By construction, $(z_0, \lambda_0) \in \Upsilon_2$. A computer computation shows that for $M = 20$ every word $(a_1 \dots a_M) \in C_M^{\lambda_0}$ satisfies

$$\left| f^{-1}_{a_1,-z_0} \circ \cdots \circ f^{-1}_{a_M,-z_0}(1) \right| > \frac{2}{1 - |-z_0|}.$$

Therefore, $(-z_0, \lambda_0) \notin \Upsilon_2$ by Theorem 1.

□

7. Effective version of Proposition 3.8

Proposition 3.8 can be used to show that a single point $(z, \lambda) \notin \Upsilon_2$, for $z \in \mathbb{D} \setminus \{0\}$. However, this proposition can be bumped up to a sufficient condition for guaranteeing that a small open neighborhood U of z satisfies $(u, \lambda) \notin \Upsilon_2$ for all $u \in U$. This is the content of Proposition 7.1, and Lemma 7.2 is a coarse characterization of the small neighborhood $G_{z,\epsilon}$ to which Proposition 7.1 can be applied. One could, in theory, use these results to construct a chain of small overlapping disks in the complement of Υ_2 in a slice, and thereby prove results about the connected components of a slice (such as Conjecture 1.1). However, the open sets in the complement of Υ_2 constructed in Proposition 7.1 and Lemma 7.2 are likely too small for such a proof to be computationally feasible.

Proposition 7.1. *Let λ be the growth rate of a superattracting tent map and fix a point $z \in \mathbb{D} \setminus \{0\}$. For each $m \in \mathbb{N}$, define the set*

$$C_m := \{(a_1 \dots a_m) \mid a_i \in \{0,1\} \text{ and } (a_k \dots a_m) \leq_E \mathit{Prefix}_{m-k+1}(It_\lambda(1)) \; \forall 1 \leq k \leq m\}.$$

For each $\epsilon > 0$, define the set $G_{z,\epsilon}$ to be the set of points $x \in \mathbb{D} \setminus \{0\}$ such the following conditions hold:

(1) for every length M word $b_1 \dots b_M$,

$$\left| f^{-1}_{b_M,z} \circ \dots f^{-1}_{b_1,z}(1) - f^{-1}_{b_M,x} \circ \dots f^{-1}_{b_1,x}(1) < \frac{\epsilon}{2} \right|.$$

(2)

$$\left| \frac{2}{1-|z|} - \frac{2}{1-|x|} \right| < \frac{\epsilon}{2}$$

If there exists $M \in \mathbb{N}$ and $\epsilon > 0$ such that C_M is nonempty and for every word $(a_1 \dots a_M) \in C_M$,

$$\left| f^{-1}_{a_1,z} \circ \dots \circ f^{-1}_{a_M,z}(1) \right| > \frac{2}{1-|z|} + \epsilon, \tag{17}$$

then for every $\lambda' < \lambda$ and $y \in G_{z,\epsilon}$, the point $(y, \lambda') \notin \Upsilon_2$.

Proof. Fix λ and z as above, and assume there exists $M \in \mathbb{N}$ and $\epsilon > 0$ for which equation (17) holds. Let w_λ be the word in $\{0,1\}$ associated to λ. Since C_M is a finite set, we may pick a real number $\delta > 0$ such that

$$\left| f^{-1}_{a_1,z} \circ \dots \circ f^{-1}_{a_M,z}(1) \right| > \frac{2}{1-|z|} + \epsilon + \delta$$

for every word $(a_1 \dots a_M) \in C_M$.

Fix $y \in G_{z,\epsilon}$. Suppose $(y, \lambda') \in \Upsilon_2$ for some $\lambda' < \lambda$. By the definition of Υ_2, there exists a sequence of points $(\tilde{y}_n)$ and a sequence of growth rates $(\tilde{\lambda}_n)$ such that $\tilde{y}_n \to z$, $\tilde{\lambda}_n \to \lambda'$, each $\tilde{\lambda}_n$ is the growth rate of a superattracting tent map, and $\tilde{y}_n$ is a Galois conjugate of $\tilde{\lambda}_n$. The Persistence Theorem then implies that there exists a sequence of points (y_n) and a sequence of growth rates (λ_n) such that $y_n \to y$, $\lambda_n < \lambda$ for all n, $\lambda_n \to \lambda$, each λ_n is the growth rate of a superattracting tent map, and y_n is a Galois conjugate of λ_n. For each n, let $w_n = a_1^n \dots a_{\ell_n}^n$ be the word in $\{0,1\}$ associated to λ_n. Also without loss of generality, we may assume $|w_n| = a_{\ell_n} \geq M$ for all n because the growth rates of superattracting tent maps are dense. Thus, for each n,

$$f_{a^n_{\ell_n},\lambda_n} \circ \dots f_{a^n_1,\lambda_n}(1) = 1.$$

This is a polynomial in $\mathbb{Z}[\lambda_n]$. Hence, since y_n is a Galois conjugate of λ_n,

$$f_{a^n_{\ell_n},y_n} \circ \dots f_{a^n_1,y_n}(1) = 1,$$

and so

$$1 = f^{-1}_{a^n_1,y_n} \cdots \circ f^{-1}_{a^n_{\ell_n},y_n}(1). \tag{18}$$

The length M suffix of w_n, which is $a^n_{\ell_n - M} \dots a^n_{\ell_n}$, is $\leq_E$ the length n prefix of $(w_\lambda)^\infty$ by Lemma 3.6, so $a^n_{\ell_n - M} \dots a^n_{\ell_n} \in C_M$. Therefore, by assumption,

$$\left| f^{-1}_{a^n_{\ell_n - M},z} \circ \dots f^{-1}_{a^n_{\ell_n},z}(1) \right| > \frac{2}{1-|z|} + \epsilon + \delta. \tag{19}$$

Combing equation (19) with the defining equations for the set $G_{z,\epsilon}$ yields

$$\left| f^{-1}_{a^n_{\ell_n - M},y} \circ \dots f^{-1}_{a^n_{\ell_n},y}(1) \right| > \frac{2}{1-|z|} + \frac{\epsilon}{2} + \delta > \frac{2}{1-|y|} + \delta. \tag{20}$$

Since $y_n \to y$, by restricting to sufficiently large n, we can guarantee that

$$\left| f^{-1}_{a^n_{\ell_n - M},y} \circ \dots f^{-1}_{a^n_{\ell_n},y}(1) - f^{-1}_{a^n_{\ell_n - M},y_n} \circ \dots f^{-1}_{a^n_{\ell_n},y_n}(1) \right| < \frac{\delta}{2}$$

and

$$\left| \frac{2}{1-|y|} - \frac{2}{1-|y_n|} \right| < \frac{\delta}{2},$$

which, together with equation (20), yield

$$\left| f^{-1}_{a^n_{\ell_n - M},y_n} \circ \dots f^{-1}_{a^n_{\ell_n},y_n}(1) \right| > \frac{2}{1-|y_n|}$$

By Lemma 3.1 and the fact that $\frac{2}{1-|y_n|} > 1$, this contradicts equation 18. □

Lemma 7.2. *Fix $z \in \mathbb{C}$ with $\frac{1}{3} < |z| \leq 1$, $M \in \mathbb{N}$, and $\epsilon > 0$. Then for any $y \in \mathbb{C}$ with $\frac{1}{3} < |y| \leq 1$ such that*

$$|Arg(y) - Arg(z)| + |y - z| < \frac{\epsilon}{12\pi \cdot M \cdot 9^M},$$

we have that for every length M word $b_1 \dots b_M$,

$$\left| f^{-1}_{b_M,z} \circ \dots f^{-1}_{b_1,z}(1) - f^{-1}_{b_M,y} \circ \dots f^{-1}_{b_1,y}(1) < \frac{\epsilon}{2} \right|.$$

Proof. For any word $b_1 \dots b_M$, there exist $c_1, \dots, c_M \in \{-2, 0, 2\}$ and $s \in \{-1, +1\}$ such that

$$f^{-1}_{b_M,z} \circ \dots \circ f^{-1}_{b_1,z}(x) = \frac{c_1}{z} + \frac{c_2}{z^2} + \dots + \frac{c_M}{z^m} + \frac{s\,x}{z^m}$$

for all $x \in \mathbb{C}$, and $z \in \mathbb{D} \setminus \{0\}$. Hence, for any $z, y \in \mathbb{D}$ with $\frac{1}{3} \leq |x|, |y|$,

$$\begin{aligned}
&\left| f^{-1}_{b_M,z} \circ \dots \circ f^{-1}_{b_1,z}(1) - f^{-1}_{b_M,y} \circ \dots \circ f^{-1}_{b_1,y}(1) \right| = \\
&\qquad \left| c_1 \left(\frac{1}{z} - \frac{1}{y} \right) + c_2 \left(\frac{1}{z^2} - \frac{1}{y^2} \right) + \dots + (c_M + s) \left(\frac{1}{z^M} - \frac{1}{y^M} \right) \right| \leq \\
&\qquad 3 \left(\left| \frac{y-z}{zy} \right| + \left| \frac{y^2 - z^2}{z^2 y^2} \right| + \dots + \left| \frac{y^M - z^M}{z^M y^M} \right| \right) \leq \\
&\qquad 3 \left(9|(y-z)| + 9^2|y^2 - z^2| + \dots + 9^M|y^M - z^M| \right) \leq \\
&\qquad\qquad 3M \cdot 9^M \left(2\pi|\mathrm{Arg}(y) - \mathrm{Arg}(z)| + |y - z| \right).
\end{aligned}$$

□

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