Ma2201/CS2022 **Foundations of C.S.** Spring, 2021

Quiz 0011

PRINT NAME: ____________

$\mathcal{SIGN}$: ____________

1. (6 pts) Let $L \subseteq \{a, b, c\}^*$ be the language defined recursively by

BASIS: $\lambda \in L$

RECURSIVE STEP: If $w \in L$ then awb and $w^2 \in L$.

CLOSURE: Every element in L can be generated from the basis after a finite number of applications of the recursive step.

Prove *carefully* by induction that every element in L has even length.

♣ *The result we want to prove here is simple, and some might say obvious. The task at hand is to prove it carefully by induction.*

Proof: *We will prove it by induction on the number, k, of rules applied in generating an element of the recursively defined set.*

Base Case: $k = 0$. After no rule applications, the element of L is in the basis of the recursively defined set, so it must be λ, and the the length of λ is zero, which is even.

Inductive Step: Let $k \geq 0$ be given and suppose that every element constructed after k applications of the recursive step is of even length. Suppose u has been constructed by $k + 1$ rule applications. So u has been constructed by applying a rule to w and w has been constructed by k rule applications, and consequently w has even length by the inductive hypothesis. There are two cases. Either $u = awb$ or $u = w^2$. If $u = awb$, then length(u) = length$(w) + 2$, which is even. On the other hand, if $u = ww$, then length$(u) = 2$length(w), which is also even. So, in either case, u has even length.

Therefore, the result is true for all $k \geq 0$ by induction.

Here is another proof, equally valid.

Proof: *According to the recursive construction, $L = \bigcup_{N=0}^{\infty} L_N$. We will show that every element of L_N has even length by induction on N.*

For $N = 0$, $L_0 = \{\lambda\}$ and length$(\lambda) = 0$, which is even.

Suppose, for a given $N \geq 0$ that L_N consists only of even length elements. Every element u of L_{N+1} is either an element of L_N, hence has even length; or $u = awb$ or $u = ww$ for some $w \in L_N$. By the inductive hypothesis, length$(w) = 2m$, $m \in \mathbb{N}$. So length$(u) = 2m + 2$, or length$(u) = 4m$, both of which are even, so every element of L_{N+1} has even length.

Therefore, the result is true for all $k \geq 0$ by induction. ♣

2. (4 pts) Let p_k be statement for $k \in \mathbb{N}$. Suppose that for all k that $p_k \Rightarrow p_{2k}$.

Suppose that p_8 is true and p_{800} is false.

For each of the following, label it T if it *must* be true, F if it *must* be false, and X if it cannot be determined from the given information.

___ p_{1600}.

___ p_{25}.

___ p_{24}.

___ p_{16}.

♣ *Here is what we know for sure is true.* p_8, *which allows us to conclude sequentially*

$$p_{16}, p_{32}, p_{64}, p_{128}$$

and, by induction, the same for all larger powers of two. That is all we can conclude from p_8, *and we are left in the dark about* p_1, p_2 *and* p_4.

We are also given that p_{800} *is false, but that doesn't contradict what we already have since* 800 *is not a power of* 2. *We can't help but notice that* $800 = 25 \cdot 2^5$, *so none of*

$$p_{25}, p_{50}, p_{100}, p_{200}, p_{400}$$

could be true, since any would imply p_{800} *by induction.*

With that we can fill in the questions

$\underline{\ X\ }$ p_{1600}.

$\underline{\ F\ }$ p_{25}.

$\underline{\ X\ }$ p_{24}.

$\underline{\ T\ }$ p_{16}. ♣