CS5003 Final Exam

Foundations of C.S.

Spring, 2022

PRINT NAME: ____________________

$\mathcal{SIGN}$: ____________________

This is an open book exam. Open notes. No internet. No collaboration. Do any six problems. Write your answers clearly and neatly. Use the back if necessary.

1. Let L be the language with recurse definition

BASIS: $a \in L$, and $b \in L$

RECURSIVE STEP: If $w \in L$ then wab, wba, waa, awa, bwb, and wbb are in L.

CLOSURE: Every element in L can be obtained from the basis after a finite number of applications of the recursive step.

a) Compute L_0 and L_1.

b) Prove that $L = L'$, where $L' = \{w \in \{a, b\}^* \mid \text{length}(w) = 2k + 1; k \in \mathbb{N}\}$.

2. Let L be the language of the regular expression $(a^* \cup b^*)^3$.
Let L' be the language of the regular expression $((a \cup b)^2)^*$.
a) Find any string in both languages.

b) Show that $L \not\subseteq L'$.

c) Show that $L' \not\subseteq L$.

d) Give a set theoretic description of both languages.

3. Design a regular expression for each of the following languages. Make sure that your principles of design are clear, that the expression matches every string required, and does not match any string not in the language.

a) L is the set of all strings on $\{a, b, c\}$ not containing the substring c^3.

b) L' is the set of all strings w on $\{a, b, c\}$ with $n_a(w) + n_b(w) = 3k$, $k \in \mathbb{N}$.

4. For each of the following, label the statement as TRUE or FALSE, and provide a short explanation.

__ The language of the regular expression $(((a \cup b^2)^* \cup c^2)^* \cup abc)^+$ is countably infinite.

__ The language of every regular grammar is countable.

__ The set of all regular languages is countably infinite.

__ The set of all languages which are not regular is countably infinite.

(TTTF) Every recursively defined set is countable, and the set of all subsets of a countable set is uncountable. That says the first three are countabe, first and third definitely infinite. And the last must be countable, since we are only removing a countable collection of languages.

5. Consider the context free grammar

$$\begin{aligned} G: S &\to AB \\ A &\to a^2Ab \mid a^2b \\ B &\to bBc^3 \mid bc^3 \end{aligned}$$

a) Write $L(G)$ in set notation. You do not have to prove that your set is correct.

b) Give a regular grammar for the same language, or show that none exists.

6. a) Use CHAIN to convert to an equivalent grammar with no chain rules.

$$
\begin{aligned}
G: S &\rightarrow aA \mid a \\
A &\rightarrow baB \mid B \mid E \mid ba \\
B &\rightarrow abaC \mid aba \\
C &\rightarrow ababD \mid D \mid ababa \\
D &\rightarrow A \mid babab E \mid babab \\
E &\rightarrow bababa
\end{aligned}
$$

b) Convert to an equivalent grammar with no left recursion.

$$
\begin{aligned}
G: S &\rightarrow SABC \mid a \mid b \\
A &\rightarrow ABC \mid AA \mid a \\
B &\rightarrow BBB \mid BB \mid b \\
C &\rightarrow CBA \mid CC \mid c
\end{aligned}
$$

T - Chomsky is a context free grammar. T - Greibach is a context free grammar. F Doesn't follow since the L(G3) could be an irregular grammar and L4 could be the set of all strings. F doesn't follow since L6 since it might be the union of an irregular language wih the empty set. T L8* is regular, every regular grammar is context free, every context free grammar has a chomsky normal form

7. Label each of the following as True or False. Read them carefully.

___ If G_1 is in Chomsky Normal Form then $L(G_1)$ must be is context free,

___ If G_2 is a grammar in Greibach normal form, then $L(G_2)$ must be is context free, regular grammar.

___ If G_3 is in Chomsky Normal Form and L_4 has a regular expression, then $L(G_3) \cap L_4$ has a regular expression.

___ If G_5 is in Chomsky Normal Form and L_6 has a regular expression, then $L(G_5) \cup L_6$ has a regular expression.

___ If L_8 has a regular expression, there is a grammar G_9 in Chomsky normal with $(L_8)^* = L(G_9)$

8. a) Suppose we have a grammar whose rules for the letter B are

$$B \to aA \mid Ba \mid ABBA \mid BBA$$

Give equivalent rules which satisfy the requirements of Chomsky Normal Form.

b) Suppose we have a grammar whose rules for the letter C are

$$C \to aB \mid bA \mid CbA \mid CaB \mid CCC$$

Give equivalent rules which satisfy the requirements of Greibach Normal Form.

9. Consider the three languages

$$\begin{aligned} L_1 &= \{w \in \{a,b\}^* \mid n_a(w) \geq 2;\; n_b(w) = 2k;\; k \geq 0\} \\ L_2 &= \{w \in \{a,b\}^* \mid a^3 w' b^3;\; w' \in L_1\} \\ L_3 &= \{w \in \{a,b\}^* \mid w \notin L_1\} \end{aligned}$$

Design one automaton for each Language. (Just draw in the state diagram.) At least one Machine must be deterministic. At most one machine can have λ-rules.

Nicole Conill

#1 a)

$L_0 = \{a, b\}$

$L_1 = L_0 \cup \{aab, aba, aaa, aaa, bab, abb$
$bab, bba, baa, aba, bbb, bbb\}$

b) Prove that $L = L'$, where $L' = \{w \in \{a,b\}^* \mid \text{length}(w) = 2k+1,\ k \in \mathbb{N}\}$

First, we prove $L \subseteq L'$.

Let $w \in L$. Show $w \in L'$ by induction on the number of rules applied.

BASE CASE: $k=0$. After no rule applications, the element w is in the basis of L, and must be either $w=a$ or $w=b$, bith of which are length 1 and thus odd.

INDUCTIVE STEP: Let $k \geq 0$ be given and suppose that every element constructed after k applications of the recursive step is of odd length. Suppose u has been constructed by $k+1$ applications. So u has been constructed by applying a rule to w, and w has been constructed by k rule applications, and w has odd length by the inductive hypothesis.

There are 6 possible elements after a rule application: $u = wab$, $u = wba$, $u = waa$, $u = awa$, $u = bwb$, and $u = wbb$. For every rule application, $u = \text{length}(w) + 2$, which is odd, and so u has odd length, and $u \in L'$ as required.

Second, we prove $L' \subseteq L$.

Let $w \in L'$, so $\text{length}(w) = 2k+1$. Show $w \in L$ by induction on the length of w.

BASE CASE: $\text{length}(w) = 1$, so $w = a$ or $w = b$, and $2k+1 = 2(0)+1 = 1$, and $w \in L$

INDUCTIVE STEP: Suppose $w \in L$ for $\text{length}(w) = 2k+1$. Let $u \in L$ with $\text{length}(u) = 2(k+1)+1$.

Since $u = wab, wba, waa, awa, bwb,$ or wbb, then $u = \text{length}(w) + 2$.

So $2(k+1)+1 = 2k+2+1 = 2k+3$, which is odd, so $u \in L$ as required.

(2.) Let L be the language of the regular expression $(a^* \cup b^*)^3$.
Let L' be the language of the regular expression $((a \cup b)^2)^*$.
a) Find any string in both languages.

$L = (a^* \cup b^*)^3$ contains a string $\boxed{aaaaaa}$

$(aa)^3$

L' also contains the string aaa aaa

$(a^2)^3$

b) Show that $L \not\subseteq L'$.

To show that $L \not\subseteq L'$, we need to find an element of L which is not an element of L'.

One such string is bbb.

You can take b from b^* then $b^3 = bbb$.

$((a \cup b)^2)^*$ is a string of even length, thus $b^3 \notin L'$.

Thus $L \not\subseteq L'$, since there is a string in L which is not in L'

c) Show that $L' \not\subseteq L$.

To show that $L' \not\subseteq L$ we need to find an element of L' which is not an element of L.

One such string is abab.

For L', $(a \cup b)^2$ can derive ab, and then $(ab)^*$ can derive abab.

Thus $abab \in L'$.

In L, a and b can alternate only if we take, for instance, a single a, then a single b, and then a single a, so aba. The alternation cannot expand beyond that because during each one of the 3 "pumps", one can only take a string of as or a string of bs. Here, since $abab \notin L$, but $abab \in L'$, $\boxed{L' \not\subseteq L}$.

d) Give a set theoretic description of both languages.

$L = \{ w \in \{a,b\}^* \mid w = uvx$, where u is either a^k or b^k, v is either a^j or b^j, and x is either a^i or $b^i \mid i, j, k \in \mathbb{N} \}$

$L' = \{ w \in \{a,b\}^* \mid w = u^i$, length$(u) = 2 \mid i \in \mathbb{N} \}$

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#3 a)

$(c \cup c^2)((a \cup b)^+ (c \cup c^2))^*$ → words starting / ending with c's, but no more than 2 consecutive

$(a \cup b)^*$ → words with no c's

$$\boxed{[(a \cup b)^* (c \cup c^2)((a \cup b)^+ (c \cup c^2))^* (a \cup b)^*] \cup (a \cup b)^*}$$

b)

$(a \cup b)$ → restriction

$$((a \cup b)\ c^*\ (a \cup b)\ c^*\ (a \cup b)) \cup c)^*$$

$$\boxed{(((a \cup b)\ c^*\ (a \cup b)\ c^*\ (a \cup b)) \cup c)^*}$$

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#5 a)

$$\boxed{L(G) = \{ w \in \{a,b,c\}^* \mid a^{2i} b^{i+j} c^{3j};\ i,j > 0 \}}$$

b)

Let N be given.

Consider the string $w = a^{2N} b^{N+1} c^{3} \in L(G)$.

The pumping lemma would guarantee a pumpable substring among the first N characters. The pumpable string would have to be a^i for $1 \leq i \leq N$, but that is impossible since $a^{2N+i} b^{N+1} c^{3} \notin L(G)$.

So $L(G)$ violates the conclusion of the pumping lemma, and thus a regular grammar does not exist.

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#6 a)

CHAIN(S) = {S}
CHAIN (A) = {A, B, E}
CHAIN (B) = {B}
CHAIN (C) = {A, B, C, D, E}
CHAIN (D) = {A, B, D, E}
CHAIN (E) = {E}

G`: S → aA | a
A → baB | abaC | aba | bababa | ba
B → abaC | aba
C → ababD | baB | abaC | aba | bababa | ba | babab E | babab | ababa
D → baB | abaC | aba | bababa | ba | babab E | babab
E → bababa

b)

G`: S → a | b | aW | bW
A → a | aX
B → b | bY
C → c | cZ

W → ABC | ABCW
X → BC | A | BCX | AX
Y → BB | B | BBY | BY
Z → BA | C | BAZ | CZ

T8

a) $B \to A'A \mid BA' \mid AX_1 \mid BX_2$

$A' \to a$

$X_1 \to BX_2$

$X_2 \to BA$

b) eliminate left recursion

$C \to aB \mid bA \mid CbA \mid CaB \mid CCC$

$C \to aB \mid bA \mid aBX \mid bAX$

$X \to bA \mid bAX \mid aB \mid aBX \mid CC \mid CCX$

then X becomes $X \to bA \mid bAX \mid aB \mid aBX \mid aBC \mid bAC \mid aBXC \mid bAXC \mid$

$aBCX \mid bACX \mid aBXCX \mid bAXCX$

So, in conclusion, the Greibach Normal Form is

$C \to aB \mid bA \mid aBX \mid bAX$

$X \to bA \mid bAX \mid aB \mid aBX \mid aBC \mid bAC \mid aBXC \mid bAXC \mid$

$aBCX \mid bACX \mid aBXCX \mid bAXCX$

T9

1) $L_1 = \{ w \in \{a,b\}^* \mid n_a(w) \geq 2 ;\ n_b(w) = 2k ;\ k \geq 0 \}$

Role:

S ① prefix has no a & has even b $\to aB \mid bA$

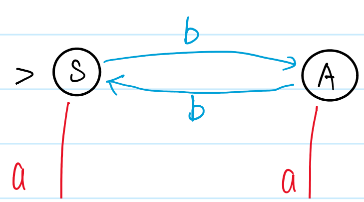
A ② prefix has no a & has odd b $\to aC \mid bS$

B ③ prefix has 1 a & has even b $\to aD \mid bC \mid a$

C ④ prefix has 1 a & has odd b $\to aE \mid bB$

D ⑤ prefix has 2 or more a & has even b $\to aD \mid bE \mid a$

E ⑥ prefix has 2 or more a & has odd b $\to aE \mid bD \mid b$



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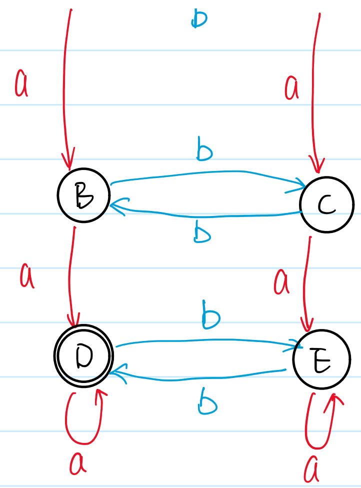
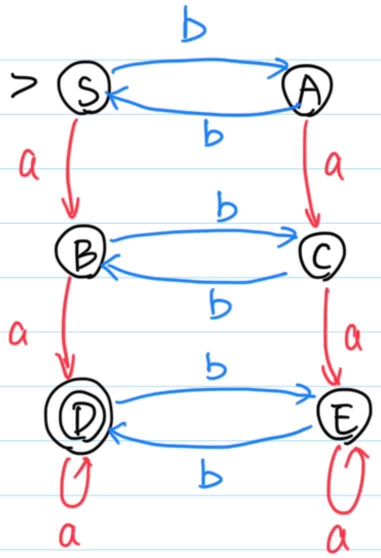
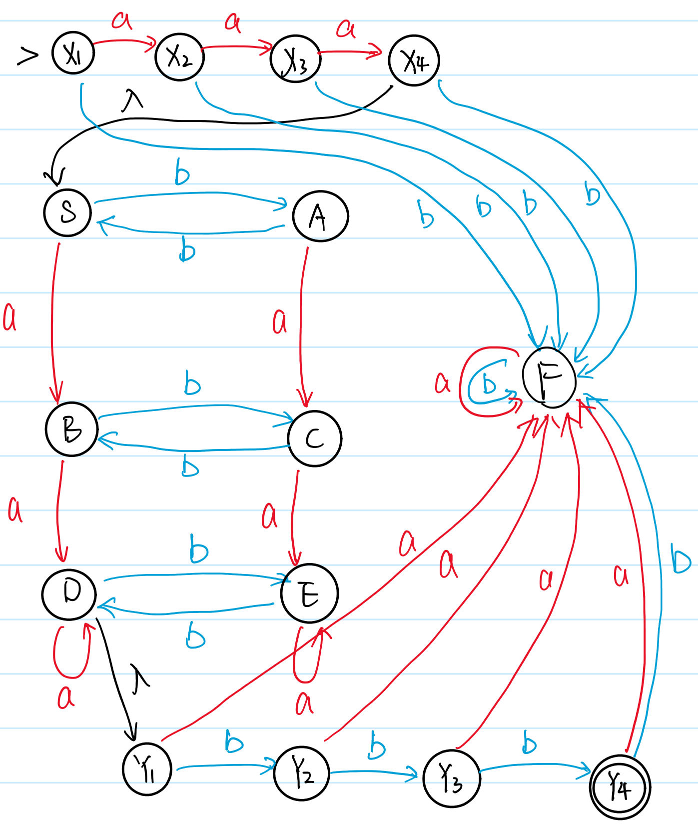


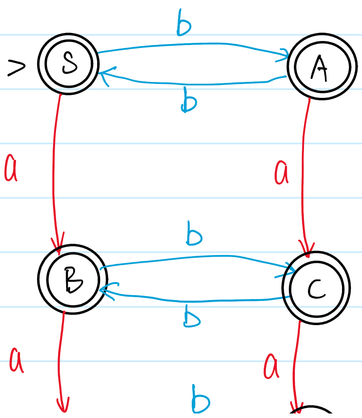


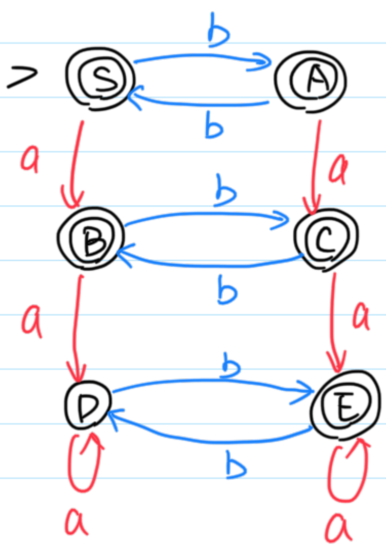
2) $\{ w \in \{a,b\}^* \mid a^3 w' b^3 \,;\, w' \in L_1 \}$

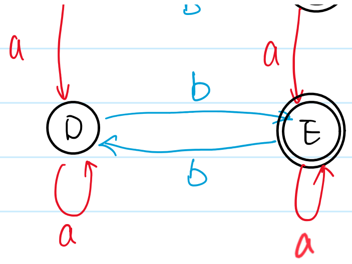
$X_1: \lambda$ $X_2: a$ $X_3: a^2$ $X_4: a^3$ $Y_1: a^3 w'$ $Y_2: a^3 w' b$ $Y_3: a^3 w' b^2$ $Y_4: a^3 w' b^3$ F: fail



3) $L_3 = \{ w \in \{a,b\}^* \mid w \notin L_1 \}$







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