CS5003 Foundations of C.S. Spring, 2022

Quiz 0010

PRINT NAME: _____

$\mathcal{SIGN}$: _____

1. (2 pts) Let $X = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid n + m \text{ is odd}\}$.

Give a *recursive definition* of the set X.

Make sure that your definition is complete, and defines the set exactly.

♣ Before any definition is the plan. The pairs in question look like: $(2n, 2m + 1)$ or $(2n + 1, 2m)$, since one has to be even and one has to be odd. If you subtract one from each coordinate, you either get another element in the set with a smaller coordinate sum, or, you get a a negative coordinate because the even coordinate was zero. For those you get a pair with a smaller coordinate sum if you subtract instead two from the odd term, and that will work unless the odd term is 1. That reasoning gives me this construction:

BASIS: $(0, 1), (0, 1) \in L$. (Equivalently, $L_0 = \{(0, 1), (1, 0)\}$.)

RECURSIVE STEP: If $(n, m) \in L$ then $(n + 1, m + 1)$, $(n + 2, m)$ and $(n, m + 2)$ are all in L.

CLOSURE: All elements of the set can be obtained from the closure after a finite number of applications of the recursive step.

My plan above gives me most of the argument for half of the proof that this solution is correct.

Do you see which half?

Do you see why there are two halves?

The complete proof is one of the problems for presentation.

♣

2. (8 pts) Suppose A, B and C are sets. Suppose that there are two functions $f : A \to B$ and $g : C \to A$ both of which are one-to-one. Suppose also that A is countably infinite. Label each of the following as TRUE or FALSE or X, if there is not enough information given.

For each, just give a word or two of explanation.

__ $B \times A$ is countably infinite.

__ $B \cap A$ is countably infinite.

__ $C \times B$ is countably infinite.

__ $\mathcal{P}(B \cup C)$ is uncountable.

The given one-to-one functions tell us that $|A| \leq |B|$ and $|C| \leq |A|$. That tells us that C is countable, (so countably infinite or finite), and B is infinite, (perhaps countable and perhaps not).

Now it is easy to answer each in order:

First: X: This is only countable is B is. which is unknown.

Second: X: Every subset of a countably infinite set is countable, but not necessarily infinite.

Third: X: Same as the first, we don't know if B is countable.

Fourth T: C could be finite, but we know that C is definitely infinite, and the power set of any infinite set is uncountable.