CS5003 Quiz 0011

Foundations of C.S.

Spring, 2022

PRINT NAME: ___________________

$\mathcal{SIGN}$: ___________________

1. (7 pts) Let X be defined recursively via
BASIS: $(0,1),(1,0) \in X$.
RECURSIVE STEP: If $(n,m) \in X$ then $(n+1,m+1)$, $(n+2,m)$ and $(n,m+2)$ are all in X.
CLOSURE: All elements of the set can be obtained from the closure after a finite number of applications of the recursive step.

On the back of this page, prove carefully by induction that every element (p,q) in the set X satisfies $p+q+1$ is positive and divisible by 2

♣ Proof: By the recursive construction, $X = \bigcup_{k=0}^{\infty} X_k$, so we will show the result by induction on k.

Base Case: The property holds for every $x \in X_0$. So we have to check the property for $(1,0)$ and $(0,1)$. Both of them satisfy $p+q+1=2$ which is both positive and divisible by 2.

Inductive Step: Suppose the property holds for all elements in X_k. Let $(n,m) \in X_{k+1}$. Then there is an element $(n',m') \in X_k$ such that applying a rule to (n',m') gives (n,m). Also, by the induction hypothesis, the sum of its coordinates is positive and divisible by 2, so $n'+m'+1 \ = 2j > 0$. There are three rules, so there are three possible forms for (n,m): So

$(n,m) = (n'+1, m'+1)$; in which case $n+m+1 = (n'+1)+(m'+1)+1 = (n'+m'+1)+2 = 2j+2$, which is positive, since j is, and divisible by 2, or

$(n,m) = (n', m'+2)$; in which case $n+m+1 = n'+(m'+2)+1 = (n'+m'+1)+2 = 2j+2$, again positive and divisible by 2, or

$(n,m) = (n'+2, m')$; in which case $n+m+1 = (n'+2)+m'+1 = (n'+m'+1)+2 = 2j+2$, again positive and divisible by 2.

So regardless which rule was applied, (n,m) has coordinate sum positive and divisible by 2, completing the induction step.

Therefore, every for all $k \in \mathbb{N}$, every element of X_k satisfies the property, and so every element of their union.

(This is a long version. It is possible to shorten the argument considerably.)

Short Version:

Proof: We will show the result by induction on the number of rules applied.

For the base case, if no rules have been applied, either we have $(0,1)$ or $(1,0)$, and in either case $p+q+1=2$ which is both even and positive.

Suppose after k rules have been applied we have (n,m) with $n+m+1$ both even and positive. Applying one more rule gives either $(n+1,m+1)$, $(n+2,m)$ or $(n,m+2)$, each which has coordinate sum plus one equal to $(n+m+1)+2$, which is even and positive, as required.

Therefore the statement is true after any infinite number of rules is applied, by induction, and so is true for all elements of X. ♣

2. (3 pts) Let L_1 be the language of all strings on $\Sigma = \{a, b, c\}$ containing exactly five b's.

Let L_2 be the language of all strings on $\Sigma = \{a, b, c\}$ matching
$(a \cup b \cup c^2)^*(a \cup b^2 \cup c)^*(a^2 \cup b \cup c)^*$.

Let $L = L_1 \cup L_2$.

Label each of the following as TRUE or FALSE or X, if there is not enough information given.

For each, just give a word or two of explanation.

__ $L_1 \cup L_2$ is countably infinite.

__ $(L_1 \cup L_2)^*$ is countably infinite.

__ $L_1 \cup L_2$ is a regular language.

♣ It is nice to notice right away that L_1 is regular, matching $[(a \cup c)^* b (a \cup c)^*]^5$

L_2 is regular, since it has a regular expression, so their union is regular, which is the third one.

All languages are countable sets, because the set of all strings on a finite alphabet is countably infinite, the only issue remaining is if they are infinite, which they are.

So all three are true. ♣