CS 5084 - Introduction to Algorithms: Design and Analysis

Section 3.1 Graphs - Basic Definitions and Applications

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Definition - A *graph* is a data structure represented as an ordered pair, $G = (V, E)$ where V is a set of nodes and E is a set of edges that pairwise connect them, i.e., we represent an edge $e \in E$ as a two-element subset of V: $e = \{u, v\}$ for some $u, v \in V$ where we call u and v the *ends* of e.

The definition given above is for an *undirected graph*, (the default assumption for a graph).

A *directed graph* is a graph, $G = (V, E)$ where E is a set of edges that are represented by ordered pairs, $e = (u, v)$ for some $u, v \in V$ where u is the tail of the edge, v is the head of the edge, and e is said to leave u and end at v.

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Example of Graph - Transportation Network

Transportation Network:

- Nodes - airports
- Edges - an edge from node u to v implies there exists a non-stop flight from u to v.
- Even though this structure implies a directed graph, there is almost always a return flight in the opposite direction.
- Typically, there are a small number of hubs with a large number of incident edges.
- It is possible to travel between any two nodes in a small number of hops.

Example of Graph - Communication Network

Communication Network:

- Nodes - computers
- Edges - an edge from node u to v implies there exists a *physical link* from u to v.

Example of Graph - Information Network

Information Network:

- Nodes - web pages
- Edges - hyperlinks - an edge from node u to v implies there exists a *hyperlink* (directional) from u to v.
- The structure of these hyperlink graphs *used to* imply the most important pages on a topic, and was the basis of Google's Page-Rank Algorithm.

Example of Graph - Social Network

Social Network:

- Nodes - people
- Edges - could mean a number of different things:
  1. u and v are friends with each other.
  2. a romantic relationship (undirected)
  3. u seeks advice from v.
  4. u lists v in his/her email address book.
  5. affiliation, such as in LinkedIn.
- Studied by sociologists extensively to understand the nature of dynamics between people, the concept of an "influencer" in a population, etc.

Example of Graph - Dependency Network

Dependency Network:

- Nodes - object
- Edges - $u \rightarrow v \implies u$ is a pre-requisite for v.
- Edges - $u \rightarrow v \implies u$ *calls* v as in a software system.

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Path

Definition: A *path*, p, is a sequence of nodes $v_1, v_2, v_3, \ldots, v_{k-1}, v_k$ such that each consecutive pair, v_i, v_{i+1} is joined by edge, $e \in E$, where $G = (V, E)$.

p is often called the path from v_1 to v_k, or we refer to the $v_1 - v_k$ path.

A path p is called *simple* if all its vertices are distinct from each other.

Cycle

Definition: A *cycle*, is a path, $p = v_1, v_2, v_3, \ldots, v_{k-1}, v_k$ such that we have the following:

- $k > 2$
- the first $k - 1$ nodes are all distinct
- $v_1 = v_k$

In other words, the sequence of nodes "cycles back" to the node where it began.

Cycle

All of these definitions carry over naturally to directed graphs, with the following change: in a directed path or cycle, each pair of consecutive nodes has the property that (v_i, v_{i+1}) is an edge.

Connectivity

Definition: An undirected graph is *connected* if for every pair of nodes, u and v, there is a path from u to v.

Definition: A directed graph is *strongly connected* if for every two nodes, u and v there is a path from u to v and a path from v to u.

Definition: The *distance* between u and v is the *minimum* number of edges between u and v.

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Definition - Tree

Definition: An undirected graph is a *tree* if it is connected and does not contain a cycle.

In a strong sense, trees are the simplest kind of connected graph - deleting any edge from a tree will disconnect it.

Tree - Other Definitions

- For thinking aobut the structure of tree T, it is useful to *root* it at a particular node, r. Physically, this is the operation of grabbing T at the node r and letting the rest of it hang downward under the force of gravity, like a mobile.
- We "orient" each edge of T away from r.
- **Definition:** For each other node, v, we declare the *parent* of v to be the node u that directly precedes v on its path *away* from r.
- **Definition:** We declare node w to be a *child* of v if v is the parent of w.

Tree - Two Drawings

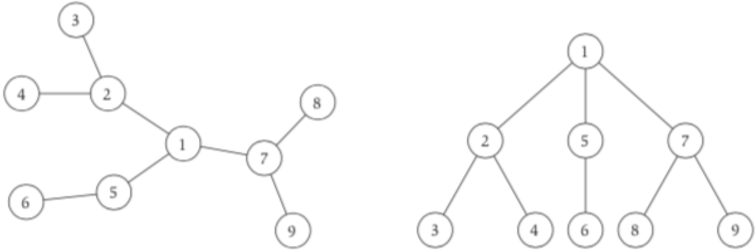

Figure 3.1 Two drawings of the same tree. On the right, the tree is rooted at node 1.

- The idea of *rooting* a tree encodes the notion of a *hierarchy*.
- Rooting a tree, T also makes certain concepts easier to grasp. For example:
  - For a given tree of n nodes, how many edges does it have?
  - Each node other than the root has a single edge leading "upward" to its parent, and conversely, each edge leads upward from precisely one non-root node.
  - Thus, every n-node tree has exactly $n - 1$ edges.

(3.1) Every n-node tree has exactly $n - 1$ edges.

(3.2) Let G be an undirected graph on n nodes. Any two of the following statements implies the third:

1. G is connected.
2. G does not contain a cycle.
3. G has exactly $n - 1$ edges.