CSE3500-001 Homework #4

Exercise 4:

1, 2, 3, 5, 8, 9, 21.

Exercises

1. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

Let G be an arbitrary connected, undirected graph with a distinct cost $c(e)$ on every edge e. Suppose e^* is the cheapest edge in G; that is, $c(e^*) < c(e)$ for every edge $e \neq e^*$. Then there is a minimum spanning tree T of G that contains the edge e^*.

2. For each of the following two statements, decide whether it is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

(a) Suppose we are given an instance of the Minimum Spanning Tree Problem on a graph G, with edge costs that are all positive and distinct. Let T be a minimum spanning tree for this instance. Now suppose we replace each edge cost c_e by its square, c_e^2, thereby creating a new instance of the problem with the same graph but different costs.

True or false? T must still be a minimum spanning tree for this new instance.

(b) Suppose we are given an instance of the Shortest s-t Path Problem on a directed graph G. We assume that all edge costs are positive and distinct. Let P be a minimum-cost s-t path for this instance. Now suppose we replace each edge cost c_e by its square, c_e^2, thereby creating a new instance of the problem with the same graph but different costs.

True or false? P must still be a minimum-cost s-t path for this new instance.

3. You are consulting for a trucking company that does a large amount of business shipping packages between New York and Boston. The volume is high enough that they have to send a number of trucks each day between the two locations. Trucks have a fixed limit W on the maximum amount of weight they are allowed to carry. Boxes arrive at the New York station one by one, and each package i has a weight w_i. The trucking station is quite small, so at most one truck can be at the station at any time. Company policy requires that boxes are shipped in the order they arrive; otherwise, a customer might get upset upon seeing a box that arrived after his make it to Boston faster. At the moment, the company is using a simple greedy algorithm for packing: they pack boxes in the order they arrive, and whenever the next box does not fit, they send the truck on its way.

   But they wonder if they might be using too many trucks, and they want your opinion on whether the situation can be improved. Here is how they are thinking. Maybe one could decrease the number of trucks needed by sometimes sending off a truck that was less full, and in this way allow the next few trucks to be better packed.

   Prove that, for a given set of boxes with specified weights, the greedy algorithm currently in use actually minimizes the number of trucks that are needed. Your proof should follow the type of analysis we used for the Interval Scheduling Problem: it should establish the optimality of this greedy packing algorithm by identifying a measure under which it "stays ahead" of all other solutions.

5. Let's consider a long, quiet country road with houses scattered very sparsely along it. (We can picture the road as a long line segment, with an eastern endpoint and a western endpoint.) Further, let's suppose that despite the bucolic setting, the residents of all these houses are avid cell phone users. You want to place cell phone base stations at certain points along the road, so that every house is within four miles of one of the base stations.

   Give an efficient algorithm that achieves this goal, using as few base stations as possible.

8. Suppose you are given a connected graph G, with edge costs that are all distinct. Prove that G has a unique minimum spanning tree.

9. One of the basic motivations behind the Minimum Spanning Tree Problem is the goal of designing a spanning network for a set of nodes with minimum *total* cost. Here we explore another type of objective: designing a spanning network for which the *most expensive* edge is as cheap as possible.

Specifically, let $G = (V, E)$ be a connected graph with n vertices, m edges, and positive edge costs that you may assume are all distinct. Let $T = (V, E')$ be a spanning tree of G; we define the *bottleneck edge* of T to be the edge of T with the greatest cost.

A spanning tree T of G is a *minimum-bottleneck spanning tree* if there is no spanning tree T' of G with a cheaper bottleneck edge.

(a) Is every minimum-bottleneck tree of G a minimum spanning tree of G? Prove or give a counterexample.

(b) Is every minimum spanning tree of G a minimum-bottleneck tree of G? Prove or give a counterexample.

21. Let us say that a graph $G = (V, E)$ is a *near-tree* if it is connected and has at most $n + 8$ edges, where $n = |V|$. Give an algorithm with running time $O(n)$ that takes a near-tree G with costs on its edges, and returns a minimum spanning tree of G. You may assume that all the edge costs are distinct.