CASE ASSIGNMENT 1

Group 9

Samadrita Roy Chowdhury, Anurag Reddy Katta, Dev Patel, Laxmi Lalitha Rachana Vepa

Pension Fund Optimization for Armco Incorporated

Objective:

To develop an LP model that minimizes the initial cash allocation required to meet future pension obligations using bond investments.

Context:

James Judson is tasked with funding Armco's pension fund using three bonds available for immediate purchase.

Payments are required over the next 15 years, with excess cash invested at a 4% annual return.

Base Case Model & Key Variables

```python
# Create a new model
m = gp.Model('base_case')

# Create variables
I = m.addVars(B, vtype=GRB.CONTINUOUS, name="I", lb=0.0) # Investment in each bond (shares to buy)
C = m.addVar(vtype=GRB.CONTINUOUS, name="C", lb=0.0) # Cash needed up front
A = m.addVars(range(T), vtype=GRB.CONTINUOUS, name="A", lb=0.0) # Cash Available in time T

#Objective
m.setObjective(C, GRB.MINIMIZE)
```

CONTENT:

1.Inputs

COSTS[b]: Initial cost to purchase each bond b.
D[b]: Duration for each bond b, defining when the bond matures.
BASE_INCOMES[b]: Coupon payment of each bond b (inflows for years prior to maturity).
PAYOUTS[b]: Payment received at bond maturity.
BASE_r: Fixed reinvestment interest rate.
Payments[t]: Required cash flow for each year t, specified by the pension fund obligations.
T: Number of years over which the pension fund needs to make payments.

2.Decision Variables:

I[i]: Investment in Bond i (for i=0,1,2)
C: Initial cash needed to cover future payments.
A[t]: Cash available at the start of year t+1.

3.Constraints: $A[0]=C-\sum(\text{COSTS}[b]\cdot Ib)$ $At=(1+r)\cdot At-1+\sum(\text{INCOMES}[b,t]\cdot Ib)-\text{payments}[t]$

$At.>=0$

Inferences based on Optimal Solution Values

Base Case (Optimal Values)

- **I[0]**: 73.69479810424208
- **I[1]**: 77.2083720154974
- **I[2]**: 28.837209302325583
- **C**: 186768.39680049528

Bad Case (Optimal Values)

- **I[0]**: 95.33811594860505
- **I[1]**: 78.14733915164764
- **I[2]**: 28.837209302325583
- **C**: 211818.5235016796

Insight on Quantity of bonds purchased

- Bond 1 matures sooner than Bonds 2 and 3 (maturing in year 6 with a payout of $1,060).
- By buying more of Bond 1, the model can reuse cash flows from its maturity payment to meet obligations in subsequent years without committing as much initial cash upfront.
- The maturity payouts of Bonds 2 and 3 are not available until years 12 and 15, respectively, which may be too late to meet early cash requirements especially in bad case where Interest rates drop after a few years, affecting the earnings from any cash on hand. Bond 1 provides predictable payouts and a maturity payout earlier on, which can mitigate the reduced returns from holding cash as rates decline.

Interpretation of Shadow Prices and Impact of Payment Reduction

```
Shadow Price Information:
    Initial_Cash_Allocation: -1.0, (RHS = 0.0), (LB = -inf), (UB = 186768.4)
    Year_2: -0.9615384615384615, (RHS = -12000.0), (LB = -inf), (UB = -2248.64)
    Year_3: -0.9245562130177513, (RHS = -14000.0), (LB = -inf), (UB = -4271.45)
    Year_4: -0.8889963586709146, (RHS = -15000.0), (LB = -inf), (UB = -7375.16)
    Year_5: -0.8548041910297255, (RHS = -16000.0), (LB = -inf), (UB = -11603.02)
    Year_6: -0.7190625344760272, (RHS = -18000.0), (LB = -63634.39), (UB = 60116.49)
    Year_7: -0.6914062831500261, (RHS = -20000.0), (LB = -67459.76), (UB = 49989.73)
    Year_8: -0.6648137337981019, (RHS = -21000.0), (LB = -70358.15), (UB = 38457.91)
    Year_9: -0.6392439748058671, (RHS = -22000.0), (LB = -73332.48), (UB = 25464.82)
    Year_10: -0.6146576680825645, (RHS = -24000.0), (LB = -77385.78), (UB = 9952.0)
    Year_11: -0.5910169885409273, (RHS = -25000.0), (LB = -80521.21), (UB = -7181.33)
    Year_12: -0.44999411599795885, (RHS = -30000.0), (LB = -91212.56), (UB = 52226.92)
    Year_13: -0.4326866499980373, (RHS = -31000.0), (LB = -94661.06), (UB = 25565.3)
    Year_14: -0.4160448557673435, (RHS = -31000.0), (LB = -97207.5), (UB = -2162.79)
    Year 15: -0.3593845096134623, (RHS = -31000.0), (LB = -95998.23), (UB = 0.0)
```

- **All shadow prices are negative** (e.g., -0.9615 for Year 1), meaning that **decreasing** the pension requirement in any of these years would **reduce the initial cash allocation** required.
- The value of shadow price (absolute) decreases linearly with time with significant shifts happening around the time of bond maturation.
- **Shadow Price (Year 1)**: -0.9615
- **Interpretation**: If we reduce the cash requirement in Year 1 by $1,000, the initial cash allocation CCC would decrease by approximately $961.50.
- **Bound Information**:
- **Upper Bound**: 797.39. This is the maximum allowable increase in the Year 1 payment that would still keep the current solution optimal.

Key Insight: The **relationship between shadow price and bond maturation** becomes significant because a bond's **maturity payment directly contributes to the cash flow in the year it matures and consequently reduces the impact of pension payment requirement which is something we can notice by looking at the shadow prices**

Reduced Cost Analysis

```
A[0]: 9376.304035863655
A[1]: 9354.378962234478
A[2]: 7331.576885660132
A[3]: 4227.862726022812
A[4]: 0.0
A[5]: 67297.82086917837
A[6]: 57171.06858262725
A[7]: 45639.24620461409
A[8]: 32646.15093148042
A[9]: 17133.331847421392
A[10]: 0.0
A[11]: 54389.70689417916
A[12]: 27728.08586762075
A[13]: 0.0
A[14]: 0.0
```

```
A[0]: 0.0, (Coef LB = -0.1), (Coef UB = 2.49)
A[1]: 0.0, (Coef LB = -0.1), (Coef UB = 3.26)
A[2]: 0.0, (Coef LB = -0.1), (Coef UB = 4.79)
A[3]: 0.0, (Coef LB = -0.11), (Coef UB = 9.4)
A[4]: 0.10697915517465717, (Coef LB = -0.11), (Coef UB = inf)
A[5]: 0.0, (Coef LB = -0.11), (Coef UB = 1.4)
A[6]: 0.0, (Coef LB = -0.11), (Coef UB = 1.72)
A[7]: 0.0, (Coef LB = -0.12), (Coef UB = 2.25)
A[8]: 0.0, (Coef LB = -0.12), (Coef UB = 3.32)
A[9]: 0.0, (Coef LB = -0.12), (Coef UB = 6.5)
A[10]: 0.1230231079030501, (Coef LB = -0.12), (Coef UB = inf)
A[11]: 0.0, (Coef LB = -0.04), (Coef UB = 2.73)
A[12]: 0.0, (Coef LB = -0.04), (Coef UB = 5.36)
A[13]: 0.04228496576934271, (Coef LB = -0.04), (Coef UB = inf)
A[14]: 0.3593845096134623, (Coef LB = -0.36), (Coef UB = inf)
```

Definition: The **reduced cost** of a variable indicates how much the objective function would change if that variable were to enter the solution at a positive (non-zero) level

For example:

A[4]=0 and has a reduced cost of 0.10698. This means that if we were to increase A[4] to a positive value, the initial cash allocation C would increase by approximately $0.10698 for every $1 added to A[4]. Thus, the model keeps A[4]=0 (we can use up all the amount for year 5 since the bond matures at year 6)to avoid this extra cost and maintain the most efficient solution.**non-zero reduced cost for A[t]=0** signifies

- Introducing any positive cash availability in that year would increase the initial cash requirement.
- This occurs because the current solution has optimized the cash flows without requiring surplus cash in that year, and adding any would only worsen the objective.

Bad Case Comparison & Key Insights

Goal: Compare the cash requirements and bond investments between the base and bad cases.

Inferences:

- **Additional Cash Requirement in Bad Case:** C(bad)−C(base) = 25050.127
- **Most Invested Bond in Bad Case:** Bond 1, as it offers higher returns during periods of uncertainty.
- **Insight on Payment Reduction:** The years valuable for payment reduction remain the same, underscoring the importance of early-year payments.
- The shadow prices and reduced cost remain largely consistent across both models, with the same general trends observed