G.C. Layek

An Introduction to Dynamical Systems and Chaos

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Dedicated to my father
Late Bijoychand Layek
for his great interest in my education.

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Preface

This book is the outcome of my teaching and research on dynamical systems, chaos, fractals, and fluid dynamics for the past two decades in the Department of Mathematics, University of Burdwan, India. There are a number of excellent books on dynamical systems that cover different aspects and approaches to nonlinear dynamical systems and chaos theory. However, there lies a gap among mathematical theories, intuitions, and perceptions of nonlinear science and chaos. There is a need for amalgamation among theories, intuitions, and perceptions of the subject and it is also necessary for systematic, sequential, and logical developments in the subject, which helps students at the undergraduate and postgraduate levels. Teachers and researchers in this discipline will be benefitted from this book. Readers are presumed to have a basic knowledge in linear algebra, mathematical analysis, topology, and differential equations.

Over the past few decades an unprecedented interest and progress in nonlinear systems, chaos theory, and fractals have been noticed, which are reflected in the undergraduate and postgraduate curriculum of science and engineering. The essence of writing this book is to provide the basic ideas and the recent developments in the field of nonlinear dynamical systems and chaos theory; their mathematical theories and physical examples. Nonlinearity is a driving mechanism in most physical and biological phenomena. Scientists are trying to understand the inherent laws underlying these phenomena over the centuries through mathematical modeling. We know nonlinear equations are harder to solve analytically, except for a few special equations. The superposition principle does not hold for nonlinear equations. Scientists are now convinced about the power of geometric and qualitative approaches in analyzing the dynamics of a system that governs nonlinearly. Using these techniques, some nonlinear intractable problems had been analyzed from an analytical point of view and the results were found to be quite interesting. Solutions of nonlinear system may have extremely complicated geometric structure. Historically, these types of solutions were known to both Henri Poincaré (1854–1912), father of nonlinear dynamics, and George David Brikhoff (1884–1944) in the late nineteenth and early twentieth centuries. In the year 1963,

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Edward Lorenz published a paper entitled “Deterministic Nonperiodic Flow” that described numerical results obtained by integrating third-order nonlinear ordinary differential equations, which was nothing but a simplified version of convection rolls in atmosphere. This work was most influential and the study of chaotic systems began. Throughout the book, emphasis has been given to understanding the subject mathematically and then explaining the dynamics of systems physically. Some mathematical theorems are given so that the reader can follow the logical steps easily and, also, for further developments in the subject. In this book, continuous and discrete time systems are presented separately, which will help beginners. Discrete-time systems and chaotic maps are given more emphasis. Conjugacy/semi-conjugacy relations among maps and their properties are also described. Mathematical theories for chaos are needed for proper understanding of chaotic motion. The concept and theories are lucidly explained with many worked-out examples, including exercises.

Bankura, India G.C. Layek
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About the Book

The materials of the book have been assembled from different articles and books published over the past 50 years along with my thinking and research experience. The book contains 13 chapters covering all aspects of nonlinear dynamical systems at the basic and advanced levels. The book is self-contained as the initial three chapters cover mainly ordinary differential equations and the concept of flows. The first chapter contains an introduction followed by a brief history of nonlinear science and discussions of one-dimensional continuous systems. Flows and their mathematical basis, qualitative approach, analysis of one-dimensional flows with examples, some important definitions, and conservative-dissipative systems are discussed in this chapter. Chapter 2 presents the solution technique of homogeneous linear systems using eigenvalue–eigenvector method and the fundamental matrix method. Discussions and theories on linear systems are presented. The solutions of a linear system form a vector space. The solution technique for higher dimensional systems and properties of exponential matrices are given in detail. The solution technique for nonhomogeneous linear equations using fundamental matrix is also given in this chapter. Flows in $\mathbb{R}^2$ that is, phase plane analysis, the equilibrium points and their stability characters, linearization of nonlinear systems, and its limitations are subject matters in Chap. 3. Mathematical pendulum problems and linear oscillators are also discussed in this chapter. Chapter 4 gives the theory of stability of linear and nonlinear systems. It also contains the notion of hyperbolicity, stable and unstable subspaces, Hartman–Grobman theorem, stable manifold theorem, and their applications. The most important contribution to the history of nonlinear dynamical systems is the theory of nonlinear oscillations. The problem of constructing mathematical tools for the study of nonlinear oscillations was first formulated by Mandelstham around 1920, in connection with the study of dynamical systems in radio-engineering. In 1927, Andronov, the most famous student of Mandelstham presented his thesis on a topic “Poincare’s limit cycles and the theory of oscillations.” Subsequently, van der Pol and Liénard made significant contributions with practical applications of nonlinear oscillations. Chapter 5 deals with linear and nonlinear oscillations with some important theorems and physical

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applications. Bifurcation is the study of possible changes in the structure of the orbit of a dynamical system depending on the changing values of the parameters. Chapter 6 presents the bifurcations in one-dimensional and two-dimensional systems. Lorenz system and its properties, for example in fluid system, are also given in this chapter. Hamiltonian systems are elegant and beautiful concepts in classical mechanics. Chapter 7 discusses the basics of Lagranergian and Hamiltonian systems and their derivations. Hamiltonian flows, their properties, and a number of worked-out examples are presented in this chapter. Symmetry is an inherent character in many physical phenomena. Symmetry analysis is one of the important discoveries of the ninetieth century. This is based on a continuous group of transformations discovered by the great Norwegian mathematician Sophus Lie (1842–1899). Symmetry groups or simply symmetries are invariant transformations that do not change the structural form of the equation under consideration. Knowledge of symmetries of a system definitely enhances our understanding of complex physical phenomena and their inherent laws. It has been presumed that students must be familiar with symmetry analysis of simple nonlinear systems for understanding natural phenomena in-depth. With this motivation we introduce the Lie symmetry under continuous group of transformations, invariance principle, and systematic calculation of symmetries for ordinary and partial differential equations in Chap. 8. Maps and their compositions have a vast dynamics with immense applications. Chapter 9 discusses maps, their iterates, fixed points and their stabilities, periodic cycles, and some important theorems. In Chap. 10 some important maps, namely tent map, logistic map, shift map, Hénon map, etc., are discussed elaborately. Chapter 11 deals with conjugacy/semi-conjugacy relations among maps, their properties, and proofs of some important theorems. In the twenty-first century, chaos and its mathematical foundation are crucially important. The chaos theory is an emergent area in twenty-first century science. The chaotic motion was first formulated by the French mathematician Henri Poincare in his paper on the stability of the solar system. What kinds of systems exhibit chaotic motion? Is there any universal quantifying feature of chaos? Chapter 12 contains a brief history of chaos and its mathematical theory. Emphasis has been given to establish mathematical theories on chaotic systems, quantifying chaos and universality. Routes of chaos, chaotic maps, Sharkovskii ordering, and theory are discussed in this chapter. The term 'fractal' was coined by Benoit Mandelbrot. It appeared as mathematical curiosities at the end of the twentieth century and its connection with chaotic orbit. Fractals are complex geometric shapes with fine structure at arbitrarily small scales. The self-similarity property is evidenced in most fractal objects. The dimension of a fractal object is not an integer. Chaotic orbit may be represented by fractals. Chapter 13 is devoted to the study of fractals, their self-similarities, scaling, and dimensions of fractal objects with many worked-out examples.

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Chapter 1
Continuous Dynamical Systems

Dynamics is a time-evolutionary process. It may be deterministic or stochastic. Long-term predictions of some systems often become impossible. Even their trajectories cannot be represented by usual geometry. In many natural and social phenomena there is unpredictability. Unpredictability is an intrinsic property which is present in the phenomenon itself. It has great impact on human civilization as well as scientific thoughts. There are numerous questions in human mind e.g., how can a deterministic trajectory be unpredictable? What are the causes in formation of symmetric crystals and snowflakes in Nature? How can one find chaotic trajectories? Can a deterministic trajectory be random? How can one define and explain turbulence in fluid motion? Is there any local symmetry in chaos? How can one relate chaotic dynamics with fractal object? For answering these questions we have no way but to study nonlinear dynamics.

Dynamical systems are generally described by differential or difference equations. Studies of differential equations in mathematics were devoted mainly of finding analytical solutions of equations for more than two centuries. But the dynamical behaviors of a system may not always be determined by analytical or closed-form solutions. Moreover, analytical solution of nonlinear equations is difficult to obtain except in a few special cases. The subject dynamical systems had evolved at the end of nineteenth century and made significant contributions to understanding some nonlinear phenomena. The dynamics of a system may be expressed either as a continuous-time or as a discrete-time-evolutionary process. The simplest mathematical models of continuous systems are those consisting of first-order differential equations. In first-order autonomous system (explicit in time), the dynamics is a very restrictive class of system since its motion is in the real line. In simple nonautonomous cases, on the other hand, the dynamics is very rich.

Nonlinear science and its dynamics have been a matter of great importance in the field of natural and social sciences. Examples include physical science (e.g., earth's atmosphere, laser, electronic circuit, superconductivity, fluid turbulence, etc.), chemistry (Belousov–Zhabotinsky reaction, Brusselator model, etc.), biology (neural and cardiac systems, biochemical processes), ecology and social sciences (spreading of fading, spreading diseases, price fluctuations of markets and stock markets, etc.), to mention a few. Nonlinear systems are harder (if not sometime impossible) to solve than linear systems, because the latter follow the superposition

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principle and can be divided into parts. Each part can be solved individually and adding them all provides the final result. However, solutions of linear systems are helpful for the analysis of nonlinear systems.

In this chapter we discuss some important definitions, concept of flows, their properties, examples, and analysis of one-dimensional flows for an easy way to understand the nonlinear dynamical systems.

1.1 Dynamics: A Brief History

The explicit time behaviors of a system and its dependency on initial conditions of solutions began after the 1880s. It is well known that analytical or closed-form solutions of nonlinear equations cannot be obtained except for very few special forms. Moreover, the solution behaviors at different initial conditions or their asymptotic characters are sometimes cumbersome to determine from closed-form solutions. In this situation scientists felt the necessity for developing a method that determines the qualitative features of a system rather than the quantitative analysis. The French mathematician *Henri Poincaré* (1854–1912) pioneered the qualitative approach, a combination of analysis and geometry which was proved to be a powerful approach for analyzing behaviors of a system and brought *Poincaré* recognition as the "father of nonlinear dynamics." The time-evolutionary process governed either by linear or nonlinear equations gives the dynamical system. Dynamics and its representations are inextricably tied with mathematics. The subject initiated informally from the different views of mathematicians and physicists. Studies began in the mid-1600s when *Newton* (1643–1727) invented calculus, differential equations, the laws of motion, and universal gravitation. With the help of Newton's discoveries the laws of planetary motions, already postulated by *Jonaesh Kepler*, a German astrologist (1609, 1619) were established mathematically and the study of dynamical systems commenced. In the qualitative approach, the local and asymptotic behaviors of an equation could be explained. Unfortunately, the qualitative study was restricted to mathematicians only. However, the power and necessity of the qualitative approach for analyzing the dynamical evolution of a system were subsequently enriched by *A.M. Lyapunov* (1857–1918), *G.D. Birkhoff* (1908–1944) and a group of mathematicians from the Russian schools, viz. *A.A. Andronov, V.I. Arnold,* and *co-workers* (1937, 1966, 1971, 1973).

In fact, Poincaré studied continuous systems in connection with an international competition held in honor of the 60th birthday of King Oscar II of Sweden and Norway. Of the four questions announced in the competition, he opted for the stability of the solar system. He won the prize. But the published memoir differed significantly from the original due to an error. In the study of dynamics he found it convenient to replace a continuous flow of time with a discrete analog. In celestial mechanics, Newton solved two-body problems: the motion of the Earth around the Sun. This is the famous inverse-square law: $F(\text{gravitational force}) \propto$

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(distance between two bodies)$^{-2}$. Many great mathematicians and physicists tried to extend Newton's analytical method to the three-body problem (Sun, Earth, and Moon), but three or more than three-body problems were found to be remarkably difficult. At this juncture the situation seemed completely hopeless. This means that instead of asking about the exact positions of the planets always, one may ask "Is the solar system stable forever?" Answering this question Poincaré devised a new way of analysis which emphasized the qualitative approach. This eventually gave birth to the subject of '*Dynamical Systems.*' The Russian Schools, viz. Nonlinear Mechanics and the Gorki (*Andronov* or *Mandelstham Andronov*) contributed immensely to the mathematical theories for dynamical systems. In the dynamic evolution stability of a system is an important property. The Russian academician *A.M. Lyapunov* made a significant contribution to the stability/instability of a system. The mathematical definition of stability, construction of Lyapunov function, and Lyapunov theorem are extensively used for analyzing the stability of a particular class of systems. Also, Lyapunov exponent, assuming the exponential growth/decay with time of nearby orbits are applied for quantifying in chaotic motions.

One of the most remarkable breakthroughs in the early nineteenth century was the discovery of solitary waves in shallow water. Solitary waves are disturbances occurring on the surface of a fluid. They are dispersive in nature and form a single hump above the surface by displacing an equal amount of fluid, creating a bore at the place. Furthermore, these waves spread while propagating without changing their shape and velocity. The speed of these waves is proportional to the fluid depth, which causes large amplitude of the wave. Consequently, the speed of the wave increases with increase in the height of the wave. When a high amplitude solitary wave is formed behind a low amplitude wave, the former overtakes the latter keeping its shape unchanged with only a shift in position. This preservation of shape and velocity after collision suggests a particle like character of these waves and therefore called as solitary wave or solition, coined by Zabusky and Kruskal relevant with photon, proton, etc. John Scott Russel, the Scottish naval engineer first observed solitary wave on the Edinburgh-Glasgow canal in 1834 and he called it the '*great wave of translation.*' Russel reported his observations to the British Association in 1844 as 'Report on waves.' The mathematical form of these waves was given by Boussinesque in 1871 and subsequently by Lord Rayleigh in 1876. The equation for solitary wave was later derived by Korteweg and de Vries in 1895 and was popularly known as the KdV equation. This is a nonlinear equation with a balance between the nonlinear advection term and dispersion resulting in the propagation of solitary waves in an inviscid fluid.

In the first half of the twentieth century nonlinear dynamics was mainly concerned with nonlinear oscillations and their applications in physics, electrical circuits, mechanical engineering, and biological science. Oscillations occur widely in nature and are exploited in many manmade devices. Many great scientists, viz. *van der Pol* (1889–1959), *Alfred-Marie Liénard* (1869–1958), *Georg Duffing* (1861–1944), *John Edensor Littlewood* (1885–1977), *A.A. Andronov* (1901-1952), *M.L. Cartwright* (1900–1998), *N. Levinson* (1912–1975), and others made

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mathematical formulations and analyzed different aspects of nonlinear oscillations. Balthasar van der Pol had made significant contributions to areas such as limit cycles (isolated closed trajectory but neighboring trajectories are not closed either as they spiral toward the closed trajectory or away from it), relaxation oscillations (limiting cycles exhibit an extremely slow buildup followed by a sudden discharge, and then followed by another slow buildup and sudden discharge, and so on) of nonlinear electrical circuits, forced oscillators hysteresis and bifurcation phenomena. The well-known van der Pol equation first appeared in his paper entitled "On relaxation oscillations" published in the Philosophical Magazine in the year 1926. The van der Pol oscillator in a triode circuit is a simple example of a system with a limit cycle. He and var der Mark used van der Pol nonlinear equation to describe the heartbeat and an electrical model of the heart. Limit cycles were found later in mechanical and biological systems. The existence of limit cycle of a system is important scientifically and stable limit cycle exhibits self-sustained oscillations.

Species live in harmony in Nature. The existence of one species depends on the other, otherwise, one of the species would become extinct. Coexistence and sometimes mutual exclusion occur in reality in which one of the species becomes extinct. *Alfred James Lotka* (1880–1949), *Vito Volterra* (1860–1940), *Ronald Fisher* (1890–1962) and *Nicols Rashevsky* (1899–1972), and many others had explored the area of mathematical biology. The interaction dynamics of species, its mathematical model, and their asymptotic behaviors are useful tools in population dynamics of interacting species. Interaction dynamics among species have a great impact on the ecology and environment. The two-species predator–prey model in which one species preys on another was formulated by Lotka in 1910 and later by Volterra in 1926. This is known as the Lotka–Volterra model. In reality, the predator–prey populations rise and fall periodically and the maximum and minimum values (amplitudes) are relatively constant. However, this is not true for the Lotka–Volterra model. Different initial conditions can have solutions with different amplitudes. Holling and Tanner (1975) constructed a mathematical model for predator–prey populations whose solutions have the same amplitudes in the long time irrespective of the initial populations. The mathematical ecologist *Robert May* (1972) and many other scientists formulated several realistic population models that are useful in analyzing the population dynamics.

The perception of unpredictability in natural and social phenomena has a great impact on human thoughts and also in scientific evolutions. The conflict between determinism and free-will has been a long-standing continuing debate in philosophy. Nature is our great teacher. In the nineteenth century, the French engineer *Joseph Fourier* (1770–1830) wrote "*The study of Nature is the most productive source of mathematical discoveries. By offering a specific objective, it provides the advantage of excluding vague problems and unwieldy calculations. It is also a means to formulate mathematical analysis, and to isolate the most important aspects to know and to conserve. These fundamental elements are those which appear in all natural effects.*"

Newtonian mechanics gives us a deterministic view of an object in which the future is determined from the laws of force and the initial conditions. There is no

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question of unpredictability or free-will in the Newtonian setup. In the beginning of the twentieth century experimental evidence, logical description, and also philosophical perception of physical phenomena, both in microscopic and macroscopic, made a breakthrough in science as a whole. The perception of infinity, how we approach the stage of infinitum, was a matter of great concern in the scientific community of the twentieth century. In the macroscopic world, studies particularly in oscillations in electrical, mechanical, and biological systems and the emergence of statistical mechanics either in fluid system or material body established the role and consequence of nonlinearity on their dynamics.

The existence of a chaotic orbit for a forced van der Pol equation (nonlinear equation) was proved mathematically by *M.L. Cartwright*, *J.E. Littlewood* about the 1950s. During this period mathematician *N. Levinson* showed that a physical model had a family of solutions that is unpredictable in nature. On the other hand, the turbulence in fluid flows is an unsolved and challenging problem in classical mechanics. The Soviet academician *A.N. Kolmogorov* (1903–1987), the greatest probabilist of the twentieth century and his co-workers made significant contributions to isotropic turbulence in fluids, the famous Kolmogorov-5/3 law (K41 theory) in the statistical equilibrium range. Kolmogorov's idea was based on the assumption of statistical equilibrium in an isotropic fluid turbulence. In turbulent motion large unstable eddies form and decay spontaneously into smaller unstable eddies, so that the energy-eddy cascade continues until the eddies reach a size so small that the cascade is damped effectively by fluid viscosity. G*eoffrey Ingram Taylor* (1886–1975), *von Karman* (1881–1963), and co-workers made significant contributions to the statistical description of turbulent motion. Yet, till today the nature of turbulent flow and universal law remain elusive. In nonlinear dynamics the well-known Kolmogorov–Arnold–Moser (KAM) theorem proves the existence of a positive measure set of quasi-periodic motions lying on invariant tori for Hamiltonian flows that are sufficiently close to completely integrable systems. This is the condition of weak chaotic motion in conservative systems. In chemistry, oscillation in chemical reaction such as the Belousov–Zhabotinsky reaction provided a wonderful example of relaxation oscillation. The experiment was conducted by the Russian biochemist Boris Belousov around the 1950s. However, he could not publish his discovery as in those days it was believed that chemical reagents must go monotonically to equilibrium solution, no oscillatory motion. Later, Zhabotinsky confirmed Belousov's results and brought this discovery to the notice of the scientific community at an international conference in Prague in the year 1968. For the progress of nonlinear science in the twentieth century both in theory and experiments such as hydrodynamic (water, helium, liquid mercury), electronic, biological (heart muscles), chemical, etc., scientists believed that simple looking systems can display highly complex seemingly random behavior. It was Henri Poincaré who first reported the notion of sensitivity to initial conditions in his work. The quotation from his essay on Science and Method is relevant here: "*It may happen that small differences in the initial produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the later prediction becomes impossible.*" Perhaps the most intriguing characteristic of

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a chaotic system is the extreme sensitivity to initial conditions. Naturally, there is a need to develop the science of the unpredictable. The real breakthrough came from the computational result of a simple nonlinear system. In the year 1963, *Edward Lorenz* (1917–2008) published a paper entitled "*Deterministic Non-periodic Flow.*" In this paper he derived equations for thermal convection in a simplified model of atmospheric flow and noticed a very strange thing that the solutions of the equations could be unpredictable and irregular despite being deterministic. The sensitive dependence of the evolution of a system for an infinitesimal change of initial conditions is called the butterfly effect. Deterministic systems may exhibit a regular behavior for some values of their control parameters and irregular behavior for other values. Deterministic systems can give rise to motions that are essentially random and the long-term prediction is impossible. Another paper from the discrete system "*Differential Dynamical Systems*" published by *Stephen Smale* proved mathematically the existence of chaotic solutions and gave a geometric description of the chaotic set, the Smale horseshoe map. Mathematicians/physicists such as *Lev. D. Laudau* (fluid dynamics and stability), *James Yorke* ("Period three implies chaos"), *Robert May* (mathematical biology), *Enrico Fermi* (ergodicity), *Stanislaw Ulam* (the growth of patterns in cellular automata, lattice dynamics), *J.G. Senai* (ergodic theory), *Sarkovskii* (ordering of infinitely many periodic points of a map), *Ruelle* and *Takens* (fluid turbulence and 'strange attractor'), *A. Libchaber,* and *J. Maurer* (intermittency as a route to fluid turbulence) and many others are the great contributors to the development of nonlinear science and chaos theory. In the mid-1970s a remarkable discovery was made by *Mitchell Feigenbaum*: the universality in chaotic regime for unimodal maps undergoing period doubling bifurcation.

The concept of fractal geometry or fractal objects is about 50 years old and was first introduced by the Polish–French–American mathematician *Benoit Mandelbrot* (1924–2010) in 1975. Fractals are structures that are irregular, erratic and self-similarity is intrinsic in most of these objects. Fractal objects consist of self-similarity between scales, that is, the patterns observed in larger scales repeat in ever decreasing smaller and smaller scales. In short, a fractal object is made up of parts similar to the whole in some way but lacks a characteristic smallest scale to measure. Fractal geometry is different from Euclidean geometry, and finds order in chaotic shapes and processes. Chaotic orbit can be expressed in terms of fractal object. Scaling and self-similarity are important features in most natural and manmade fractal objects. There exist numerous examples of fractals in natural and physical sciences. One can also find a number of examples of fractals in the human anatomy. For instance, lungs, heart, and many other anatomical structures are either fractal or fractal-like. Moreover, in recent years the idea of fractals is being exploited to find applications in medical science to curb fatal diseases. Mandelbrot and other researchers have shown how fractals could be explored in different areas and chaos in particular. The phenomenon of chaos is a realistic phenomenon and therefore one has to understand and realize chaos in usual incidents happening in our everyday life. The study of chaotic phenomena has begun in full length nowadays and is widely applied in different areas. The theory of chaos is now

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applied in computer security, digital watermarking, secure data aggregation and video surveillance successfully. Thus, chaotic phenomena are not only destructive as in tsunami, tornado, etc., but can also be effectively utilized for the welfare of human beings. Chaos has been considered as the third greatest discovery, after relativity and quantum mechanics in the twentieth century science and philosophy. In the past 20 years scientists and technologists have been realizing the potential use of chaos in natural and technological sciences.

1.2 Dynamical Systems

Dynamics is primarily the study of the time-evolutionary process and the corresponding system of equations is known as dynamical system. Generally, a system of n first-order differential equations in the space $\mathbb{R}^n$ is called a dynamical system of dimension n which determines the time behavior of evolutionary process. Evolutionary processes may possess the properties of determinacy/non-determinacy, finite/infinite dimensionality, and differentiability. A process is called **deterministic** if its entire future course and its entire past are *uniquely* determined by its state at the present time. Otherwise, the process is called **nondeterministic**. However, the process may be **semi-deterministic** (determined, but not uniquely). In classical mechanics the motion of a system whose future and past are uniquely determined by the initial positions and the initial velocities is an example of a deterministic dynamical system. The evolutionary process may describe, viz. (i) a continuous-time process and (ii) a discrete-time process. The continuous-time process is represented by differential equations, whereas the discrete-time process is by difference equations (or maps). The continuous-time dynamical systems may be described mathematically as follows:

Let $\underset{\sim}{x} = \underset{\sim}{x}(t) \in \mathbb{R}^n$, $t \in I \subseteq \mathbb{R}$ be the vector representing the dynamics of a continuous system (continuous-time system). The mathematical representation of the system may be written as

$$\frac{\mathrm{d}\underset{\sim}{x}}{\mathrm{d}t} = \dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}, t) \tag{1.1}$$

where $\underset{\sim}{f}(\underset{\sim}{x}, t)$ is a sufficiently smooth function defined on some subset $U \subset \mathbb{R}^n \times \mathbb{R}$. Schematically, this can be shown as

$$\underset{(\text{state space})}{\mathbb{R}^n} \times \underset{(\text{time})}{\mathbb{R}} = \underset{(\text{space of motions})}{\mathbb{R}^{n+1}}$$

The variable t is usually interpreted as time and the function $\underset{\sim}{f}(\underset{\sim}{x}, t)$ is generally nonlinear. The time interval may be finite, semi-finite or infinite. On the other hand, the discrete system is related to a discrete map (given only at equally spaced points of time) such that from a point x_0, one can obtain a point x_1 which in turn maps into

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x_2, and so on. In other words, $x_{n+1} = g(x_n) = g(g(x_{n-1}))$, etc. This is also written in the form $x_{n+1} = g(x_n) = g^2(x_{n-1}) = \cdots$. The discrete system will be discussed in the later chapters.

If the right-hand side of Eq. (1.1) is explicitly time independent then the system is called **autonomous**. The trajectories of such a system do not change in time. On the other hand, if the right-hand side of Eq. (1.1) has explicit dependence on time then the system is called **nonautonomous**. An n-dimensional nonautonomous system can be converted into autonomous form by introducing a new dependent variable x_{n+1} such that $x_{n+1} = t$. In general, the solution of Eq. (1.1) is difficult or sometimes impossible to obtain when the function $f(\underset{\sim}{x}, t)$ is nonlinear, except in some special cases. Examples of autonomous and nonautonomous systems are given below.

(i) **Autonomous systems**

(a) $\ddot{x} + \alpha\dot{x} + \beta x = 0, \alpha, \beta > 0$. This is a damped linear harmonic oscillator. The parameters α and β are, respectively, the strength of damping and the strength of linear restoring force.

(b) $\ddot{x} + \omega^2 \sin x = 0,\ \omega = \sqrt{g/L}$. g is the gravitational acceleration, L the string length. This is a simple undamped nonlinear oscillator (pendulum).

(c) $\left.\begin{array}{r}\dot{x} = \alpha x - \beta xy \\ \dot{y} = -\gamma y + \delta xy\end{array}\right\}$. This is the well-known Lotka–Volterra predator–prey model, where $\alpha, \beta, \gamma, \delta$ are all positive constants.

(d) $\ddot{x} - \mu(1 - x^2)\dot{x} + \beta x = 0, \mu > 0$. This is the well-known van der Pol oscillator.

(ii) **Nonautonomous systems**

(a) $\ddot{x} + \alpha\dot{x} + \beta x = f\ \cos\omega t, \alpha, \beta > 0$. This is an example of linear oscillator with external time-dependent force. f and ω are the amplitude and frequency of driving force, respectively.

(b) $\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \beta x^3 = f\ \sin\omega t$. This is a Duffing nonlinear oscillator with cubic restoring force. α is the strength of damping, ω_0 is the natural frequency and β is the strength of the nonlinear restoring force.

(c) $\ddot{x} - \mu(1 - x^2)\dot{x} + \beta x = f\ \cos\omega t, \mu > 0$. This is a van der Pol nonlinear forced oscillator.

(d) $\ddot{x} - \mu(1 - x^2)\dot{x} + \omega_0^2 x + \beta x^3 = f\ \cos\omega t$. This is a Duffing-van der Pol nonlinear forced oscillator.

Some examples of dynamical systems

(a) The most common example of a dynamical system is Newtonian systems governed by Newton's law of motion. This law states that the acceleration of a particle is determined by the force per unit mass. The force can be a function of the velocity $(\dot{x})$ and the position (x) and so the Newtonian systems take the form

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$$m\ddot{x} = F(x, \dot{x}), \quad (m = \text{mass}, F = \text{force}). \tag{1.2}$$

Equation (1.2) may be written as a system of two first-order differential equations as

$$\dot{x} = y, \text{ and } \dot{y} = F(x, y) \tag{1.3}$$

System (1.3) may be viewed as a dynamical system of dimension two in the xy-plane and the dynamics is a set of trajectories giving time evolution of motion.

(b) The simple exponential growth model for a single population is expressed mathematically as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx \text{ with } x = x_0 \text{ at } t = 0, \tag{1.4}$$

where $r > 0$ is the population growth parameter. The solution of (1.4) is $x(t) = x_0 e^{rt}$. This solution expresses the simplest model for population growth with time in unrestricted resources and the population $x(t) \to \infty$ as $t \to \infty$. Obviously, this model does not obey realistic population growth of any species.

The simple population growth model, considering effects like intraspecies competitions, depletion of resources with population growth is given as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (r - bx)x \tag{1.5}$$

with the condition $x = x_0$ at $t = 0$. The solution of (1.5) is given as

$$x(t) = \frac{\left(\frac{r}{b}\right)x_0}{x_0 + \left(\frac{r}{b} - x_0\right)e^{-rt}} \tag{1.6}$$

Clearly $x \to \frac{r}{b}$ as $t \to \infty$ for both the cases $x_0 > \frac{r}{b}$ and $x_0 < \frac{r}{b}$.

This growth model is known as the **Logistic growth model** of population. The graphical representation of the above solution is shown in Fig. 1.1.

This simple model shows that population $x(t)$ is of constant growth rate after some time t.

(c) Populations of two competing species (predator and prey populations) could be modeled mathematically. The predator–prey population model was first formulated by *Alfred J. Lotka* (1880–1949) in the year 1910 and later by *Vito Volterra* (1860–1940) in the year 1926. This is known as Lotka–Volterra predator–prey model. In this model the fox population preys on the rabbit population. The population density of rabbit affects the population density of fox, since the latter relies on the former for food. If the density of rabbit is high, the fox population decreases, while when the fox population increases,

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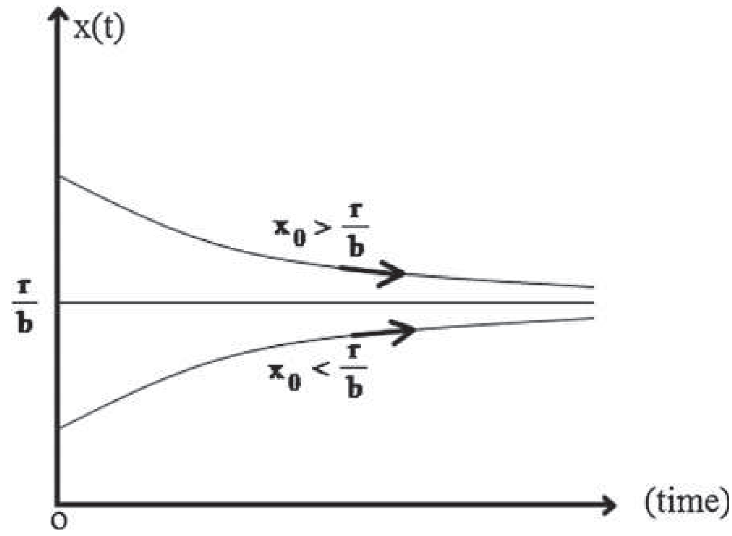
Fig. 1.1 Graphical representation of population growth model



the rabbit population decreases. When the rabbit population falls, the fox population also falls. When the fox population drops, the rabbits can multiply again and so on. The growth or decrease of two populations could be analyzed using dynamical system principles. The dynamical equations for predator–prey model are given as

$$\left.\begin{aligned} \dot{x} &= \alpha x - \beta xy \\ \dot{y} &= -\gamma y + \delta xy \end{aligned}\right\} \tag{1.7}$$

where x denotes the population density of the prey and y, the population density of the predator. The parameter α represents the growth rate of the prey in the absence of intersection with the predators whereas the parameter γ represents the death rate of the predators in the absence of interaction with the prey and β, δ are the interaction parameters and are all constants (for simple model). Using the dynamical principle one can obtain a necessary condition for coexistence of the two species. In this model the survival of the predators depends entirely on the population of the prey. If initially $x = 0$, then $\dot{y} = -\gamma y$, that is, $y(t) = y(0)e^{-\gamma t}$ and $y(t) \to 0$ as $t \to \infty$ (see the book Arrowsmith and Place [1]).

(d) Suppose we have an LCR circuit consisting of a resistor of resistance R, a capacitor of capacitance C, and an inductor of inductance L. In a simple electrical circuit the values of R, C, and L are always nonnegative and are independent of time t. Kirchhoff's current law (the sum of the currents flowing into a node is equal to the sum of the currents flowing out of it) is satisfied if we pass a current I to the closed loop as shown in Fig. 1.2.

According to Kirchhoff's voltage law of the circuit (the sum of the potential differences around any closed loop in a circuit is zero), we have the equation

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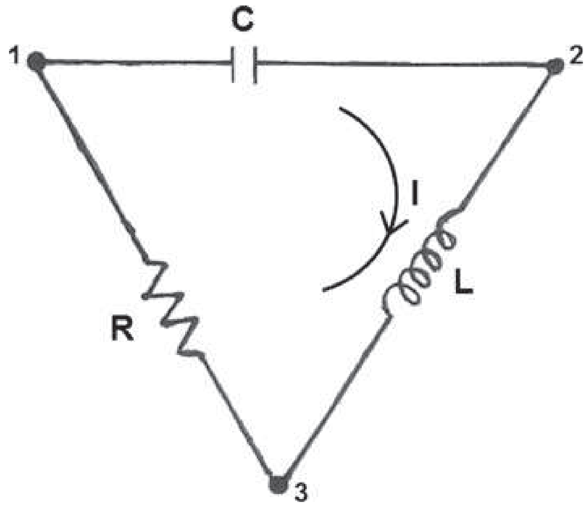

Fig. 1.2 Schematic of an LCR circuit

$$V_{12} + V_{23} + V_{31} = 0 \tag{1.8}$$

Here V_{ij} denotes the voltage difference between node i and node j. From Ohm's law, we get the relation

$$V_{31} = \mathrm{IR} \tag{1.9}$$

Also, from the definition of capacitance C, we have

$$C\frac{\mathrm{d}V_{12}}{\mathrm{d}t} = I \tag{1.10}$$

Again, Faraday's law of inductance gives

$$L\frac{\mathrm{d}I}{\mathrm{d}t} = V_{23} \tag{1.11}$$

Substituting (1.8) and (1.10) into (1.11) and writing $V_{12} = V$, we get

$$V + L\frac{\mathrm{d}I}{\mathrm{d}t} + \mathrm{IR} = 0$$

or,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{R}{L}I - \frac{V}{L} \tag{1.12}$$

Again, from (1.10)

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{I}{C}. \tag{1.13}$$

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Thus finally we obtain the following equations

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{I}{C} \text{ and } \frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{R}{L}I - \frac{V}{L}. \tag{1.14}$$

These equations represent a dynamical system of dimension two in the *VI* plane. This is a simple linear model in LCR circuit. The linear models have undoubtedly had good success, but they also have limitations. Linear models can only produce persistent oscillations of a harmonic (trigonometric) type.

A circuit is called nonlinear when it contains at least one nonlinear circuit element like a nonlinear resistor, a nonlinear capacitor or a nonlinear inductor. Chua's diode model equation is a simple example of nonlinear electric circuit (see Lakshmanan and Rajasekar [2] for nonlinear electrical circuit).

1.3 Flows

The time-evolutionary process may be described as a flow of a vector field. Generally, flow is frequently used for discussing the dynamics as a whole rather than the evolution of a system at a particular point. The solution $\underset{\sim}{x}(t)$ of a system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$ which satisfies $\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0$ gives the past $(t < t_0)$ and future $(t > t_0)$ evolutions of the system. Mathematically, the flow is defined by $\phi_t(\underset{\sim}{x}) : U \to \mathbb{R}^n$ where $\phi_t(\underset{\sim}{x}) = \phi(t, \underset{\sim}{x})$ is a smooth vector function of $\underset{\sim}{x} \in U \subseteq \mathbb{R}^n$ and $t \in I \subseteq \mathbb{R}$ satisfying the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(\underset{\sim}{x}) = \underset{\sim}{f}(\phi_t(\underset{\sim}{x}))$$

for all t such that the solution through $\underset{\sim}{x}$ exists and $\phi(0, \underset{\sim}{x}) = \underset{\sim}{x}$. The flow $\phi_t(\underset{\sim}{x})$ satisfies the following properties:

(a) $\phi_o = I_d$, (b) $\phi_{t+s} = \phi_t \circ \phi_s$.

Some flows may also satisfy the property

$$\phi(t+s,\ \underset{\sim}{x}) = \phi(t, \phi(s,\ \underset{\sim}{x})) = \phi(s, \phi(t,\ \underset{\sim}{x})) = \phi(s+t, \underset{\sim}{x}).$$

Flows in $\mathbb{R}$: Consider a one-dimensional autonomous system represented by $\dot{x} = f(x), x \in \mathbb{R}$. We can imagine that a fluid is flowing along the real line with local velocity $f(x)$. This imaginary fluid is called the **phase fluid** and the real line is called the **phase line**. For solution of the system $\dot{x} = f(x)$ starting from an arbitrary initial position x_0, we place an imaginary particle, called a **phase point**, at x_0 and watch how it moves along with the flow in phase line in varying time t. As time goes on, the phase point (x, t) in the one-dimensional system $\dot{x} = f(x)$ with $x(0) = x_0$ moves along the x-axis according to some function $\phi(t, x_0)$. The function $\phi(t, x_0)$ is called the **trajectory** for a given initial state x_0, and the set $\{\phi(t, x_0) | \, t \in I \subseteq \mathbb{R}\}$

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is the orbit of $x_0 \in \mathbb{R}$. The set of all qualitative trajectories of the system is called **phase portrait**.

Flows in $\mathbb{R}^2$: Consider a two-dimensional system represented by the following equations $\dot{x} = f(x, y),\ \dot{y} = g(x, y),\ (x, y) \in \mathbb{R}^2$. An imaginary fluid particle flows in the plane $\mathbb{R}^2$, known as phase plane of the system. The succession of states given parametrically by $x = x(t)$, $y = y(t)$ trace out a curve through some initial point $P(x(t_0), y(t_0))$ is called a **phase path**. The set $\{\phi(t, \underset{\sim}{x}_0) | t \in I \subseteq \mathbb{R}\}$ is the orbit of $\underset{\sim}{x}_0$ in $\mathbb{R}^2$. There are an infinite number of trajectories that would fill the phase plane when they are plotted. But the qualitative behavior can be determined by plotting a few trajectories with different initial conditions. The phase portrait displays how the qualitative behavior of a system is changing as x and y varies with time t. An orbit is called periodic if $x(t + p) = x(t)$ for some $p > 0$, for all t. The smallest integer p for which the relation is satisfied is called the prime period of the orbit. Flows in $\mathbb{R}$ cannot have oscillatory or closed path.

Flows in $\mathbb{R}^n$: Let us now define an autonomous system representing n ordinary differential equations as

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n) \\ &\ \vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) \end{aligned}\right\}$$

which can also be written in symbolic notation as $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$, where $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ and $\underset{\sim}{f} = (f_1, f_2, \ldots, f_n)$. The solution of this system with the initial condition $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ can be thought as a continuous curve in the phase space $\mathbb{R}^n$ parameterized by time $t \in I \subseteq \mathbb{R}$. So the set of all states of the evolutionary process is represented by an n-valued vector field in $\mathbb{R}^n$. The solutions of the system with different initial conditions describe a family of phase curves in the phase space, called the phase portrait of the system. The vector field $\underset{\sim}{f}(\underset{\sim}{x})$ is everywhere tangent to these curves and their orientation is directed by the direction of the tangent vector of $\underset{\sim}{f}(\underset{\sim}{x})$.

1.4 Evolution

Consider a system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}), \underset{\sim}{x} \in \mathbb{R}^n$ with initial conditions $\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0$. Let $E \subset \mathbb{R}^n$ be an open set and $\underset{\sim}{f} \in C^1(E)$. For $\underset{\sim}{x}_0 \in E$, let $\phi(t, \underset{\sim}{x}_0)$ be a solution of the above system on the maximum interval of existence $I(\underset{\sim}{x}_0) \subset \mathbb{R}$. The mapping $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\phi_t(\underset{\sim}{x}_0) = \phi(t, \underset{\sim}{x}_0)$ is known as **evolution operator** of the system. The linear flow for the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ with $\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0$, is defined by $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ and $\phi_t = e^{At}$, the exponential matrix. The mappings ϕ_t for both linear and nonlinear systems satisfy the following properties:

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(i) $\phi_0(\underset{\sim}{x}) = \underset{\sim}{x}$
(ii) $\phi_s(\phi_t(\underset{\sim}{x})) = \phi_{s+t}(\underset{\sim}{x}), \forall s, t \in \mathbb{R}$
(iii) $\phi_t(\phi_{-t}(\underset{\sim}{x})) = \phi_{-t}(\phi_t(\underset{\sim}{x})) = \underset{\sim}{x}, \forall t \in \mathbb{R}$

In general a dynamical system may be viewed as group of nonlinear / linear operators evolving as $\{\phi_t(\underset{\sim}{x}), t \in \mathbb{R}, \underset{\sim}{x} \in \mathbb{R}^n\}$. The following dynamical group properties hold good:

(i) $\phi_t\phi_s \in \{\phi_t(\underset{\sim}{x}), t \in \mathbb{R}, \underset{\sim}{x} \in \mathbb{R}^n\}$ (Closure property)
(ii) $\phi_t(\phi_s\phi_r) = (\phi_t\phi_s)\phi_r$ (Associative property)
(iii) $\phi_0(\underset{\sim}{x}) = \underset{\sim}{x}$, ϕ_0 being the Identity operator.
(iv) $\phi_t\phi_{-t} = \phi_{-t}\phi_t = \phi_0$, where ϕ_{-t} is the Inverse of ϕ_t.

For some cases the flow satisfies the commutative property $\phi_t\phi_s = \phi_s\phi_t$.

1.5 Fixed Points of a System

The notion of fixed point is important in analyzing the local behavior of a system. The fixed point is nothing but a constant or equilibrium or invariant solution of a system. A point is a fixed point of the flow generated by an autonomous system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}),\ \underset{\sim}{x} \in \mathbb{R}^n$ if and only if $\phi(t, \underset{\sim}{x}) = \underset{\sim}{x}$ for all $t \in \mathbb{R}$. Consequently in continuous system, this gives $\dot{\underset{\sim}{x}} = \underset{\sim}{0} \Rightarrow \underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{0}$. For nonautonomous systems fixed point can be defined for a fixed time interval. A fixed point is also known as a **critical point** or an **equilibrium point** or a **stationary point**. This point is also called **stagnation point** with respect to the flow ϕ_t in $\mathbb{R}^n$. Flows on line may have no fixed points, only one fixed point, finite number of fixed points, and infinite number of fixed points. For example, the flow $\dot{x} = 5$ (no fixed points), $\dot{x} = x$ (only one fixed point), $\dot{x} = x^2 - 1$ (two fixed points), and $\dot{x} = \sin x$ (infinite number of fixed points).

1.6 Linear Stability Analysis

A fixed point, say $\underset{\sim}{x}_0$ is said to be stable if for a given $\varepsilon > 0$, there exists a $\delta > 0$ depending upon ε such that for all $t \geq t_0$, $\| \underset{\sim}{x}(t) - \underset{\sim}{x}_0(t) \| < \varepsilon$, whenever $\| \underset{\sim}{x}(t_0) - \underset{\sim}{x}_0(t_0) \| < \delta$, where $\|.\| : \mathbb{R}^n \to \mathbb{R}$ denotes the norm of a vector in $\mathbb{R}^n$. Otherwise, the fixed point is called unstable. In linear stability analysis the quadratic and higher order terms in the Taylor series expansion about a fixed point x^* of a system $\dot{x} = f(x),\ x \in \mathbb{R}$ are neglected due to the smallness of the terms. Consider a small perturbation quantity $\xi(t)$, away from the fixed point x^*, such that $x(t) = x^* + \xi(t)$. We see whether the perturbation grows or decays as time goes on. So we get the perturbation equation as

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$$\dot{\xi} = \dot{x} = f(x) = f(x^* + \xi).$$

Taylor series expansion of $f(x^* + \xi)$ gives

$$\dot{\xi} = f(x^*) + \xi f'(x^*) + \frac{\xi^2}{2} f''(x^*) + \cdots$$

According to linear stability analysis, we get

$$\dot{\xi} = \xi f'(x^*)[\because f(x^*) = 0]$$

Assuming $f'(x^*) \neq 0$, the perturbation $\xi(t)$ grows exponentially if $f'(x^*) > 0$ and decays exponentially if $f'(x^*) < 0$. Linear theory fails if $f'(x^*) = 0$ and then higher order derivatives must be considered in the neighborhood of fixed point for stability analysis of the system.

Example 1.1 Find the evolution operator ϕ_t for the one-dimensional flow $\dot{x} = -x^2$. Show that ϕ_t forms a dynamical group. Is it a commutative group?

Solution The solutions of the given system are obtained as below:

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = -x^2 \Rightarrow \frac{1}{x} = t + A \Rightarrow x(t) = \frac{1}{t + A}$$

in any interval of $\mathbb{R}$ that does not contain the point $x = 0$, where A is a constant.

If we take starting point $x(0) = x_0$, then $A = 1/x_0$ and so we get

$$x(t) = \frac{x_0}{1 + x_0 t}, \quad t \neq -1/x_0.$$

The point $x = 0$ is not included in this solution. But it is the fixed point of the given system, because $\dot{x} = 0 \Leftrightarrow x = 0$. Therefore, $\phi_t(0) = 0$ for all $t \in \mathbb{R}$. So the evolution operator of the system is given as $\phi_t(x) = \frac{x}{1 + xt}$ for all $x \in \mathbb{R}$.

The evolution operator ϕ_t is not defined for all $t \in \mathbb{R}$. For example, if $t = -1/x$, $x \neq 0$, then ϕ_t is undefined. Thus we see that the interval in which ϕ_t is defined is completely dependent on x.

We shall now examine the group properties of the evolution operator ϕ_t below:

(i) $\phi_r \phi_s \in \{\phi_t(x), t \in \mathbb{R}, x \in \mathbb{R}\} \, \forall r, s \in \mathbb{R}$ (Closure property)
Now,

$$\begin{aligned}
\phi_r(y) &= \frac{y}{1 + yr}. \text{ Take } y = \frac{x}{1 + xs} \\
&= \frac{x/1 + sx}{1 + \frac{xr}{1 + xs}} = \frac{x}{1 + xs + xr} = \frac{x}{1 + x(s + r)} \\
&= \phi_{s+r} \in \{\phi_t(x), t \in \mathbb{R}, x \in \mathbb{R}\}
\end{aligned}$$

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(ii) $\phi_t(\phi_s\phi_r) = (\phi_t\phi_s)\phi_r$ (Associative property)

$$\text{L.H.S.} = \phi_t((\phi_s\phi_r)(x)) = \phi_t(y) = \frac{y}{1+yt} = \frac{z}{1+zs} = \frac{x}{1+x(r+s)}, y = \phi_s(\phi_r(x))$$

$$\left(\text{where } y = \phi_s(z), z = \phi_r(x) = \frac{x}{1+rx}\right)$$

$$\therefore \text{L.H.S} = \frac{x}{1+x(t+r+s)} = \phi_{t+r+s}(x)$$

$$\text{R.H.S.} = ((\phi_t\phi_s)\phi_r(x))$$

Now,

$$\begin{aligned}
\phi_t(y) &= \frac{y}{1+yt}, y = \phi_s(x) = \frac{x}{1+sx} \\
&= \frac{x}{1+x(t+s)} = \phi_{t+s}(x) \\
\phi_{t+s}(\phi_r)(x) &= \phi_{t+s}(z) = \frac{z}{1+z(t+s)}, z = \phi_r(x) = \frac{x}{1+rx} \\
\phi_{t+s}(\phi_r)(x) &= \frac{x}{1+x(t+s+r)} = \phi_{t+s+r}(x)
\end{aligned}$$

Hence, $\phi_t(\phi_s\phi_r)(x) = (\phi_s\phi_r)\phi_t(x), \forall x \in \mathbb{R}$.

(iii) $\phi_0(x) = \frac{x}{1+x\cdot 0} = x$, ϕ_0 is the identity operator.

(iv)
$$\begin{aligned}
\phi_t\phi_{-t}(x) &= \phi_t(y) = \frac{y}{1+ty}, y = \phi_{-t}(x) = \frac{x}{1-tx} \\
&= \frac{x}{1-tx+tx} = x = \phi_0(x) \quad (\phi_{-t} \text{ is the inverse of } \phi_t)
\end{aligned}$$

Hence the flow evolution operator forms a dynamical group.

(v) $\phi_t\phi_s = \phi_s\phi_t$

Now,

$$\begin{aligned}
(\phi_t\phi_s)(x) &= \phi_t(y) = \frac{y}{1+ty}, \ y = \phi_s(x) = \frac{x}{1+xs} \\
&= \frac{x}{1+x(t+s)} = \phi_{t+s}(x) \\
\phi_s\phi_t(x) &= \phi_s(z) = \frac{z}{1+sz}, \ z = \phi_t(x) = \frac{x}{1+tx} \\
&= \frac{x}{1+tx+sx} = \frac{x}{1+(s+t)x} = \phi_{s+t}(x)
\end{aligned}$$

So, $\phi_t\phi_s = \phi_s\phi_t$ (Commutative property).

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Thus, the evolution operator ϕ_t forms a commutative group.

Example 1.2 Find the evolution operator ϕ_t for the system $\dot{x} = x^2 - 1$ and also verify that $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$ for all $s, t \in \mathbb{R}$. Show that the evolution operator forms a dynamical group. Examine whether it is commutative dynamical group.

Solution The solutions of the system satisfy the equation

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 - 1 \Rightarrow \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| &= t + A, \quad x \neq 1, -1 \\ \Rightarrow \frac{x-1}{x+1} &= Be^{2t}, \quad B = \pm\, e^{2A} \\ \Rightarrow x(t) &= \frac{Be^{2t}+1}{1-Be^{2t}}\end{aligned}$$

If we take $x(0) = x_0$, then from the above relation we get $B = (x_0 - 1)/(x_0 + 1)$ and so the solution can be written as

$$x(t) = \frac{(x_0 - 1)e^{2t} + x_0 + 1}{x_0 + 1 - (x_0 - 1)e^{2t}}.$$

This solution is defined for all $t \in \mathbb{R}$ and for $x_0 \neq -1, 1$. But the points $x = -1, 1$ are the fixed points of the system. Therefore, the evolution operator of the given one-dimensional system is

$$\phi_t(x) = \frac{(x-1)e^{2t} + x + 1}{x + 1 - (x-1)e^{2t}} \text{ for all } x, t \in \mathbb{R}.$$

Now for all $x, t, s \in \mathbb{R}$, we have

$$\phi_t(\phi_s(x)) = \phi_t(y) = \frac{(y-1)e^{2t} + y + 1}{y + 1 - (y-1)e^{2t}}$$

where $y = \phi_s(x) = \frac{(x-1)e^{2s} + x + 1}{x + 1 - (x-1)e^{2s}}$

Substituting this value of y into the above expression we get

$$\phi_t(\phi_s(x)) = \frac{(x\, -\, 1)e^{2(t+s)} + x + 1}{x + 1 - (x-1)e^{2(t+s)}} = \phi_{t+s}(x).$$

(ii) $\phi_0(x) = \frac{x-1+x+1}{x\,+\,1-x+1} = x$, the identity operator

$$(iii) \qquad \phi_t\phi_{-t}(x) = \phi_t(y) = \frac{(y-1)e^{2t}+y+1}{y+1-(y-1)e^{2t}}, y = \phi_{-t}(x) = \frac{(x-1)e^{-2t}+x+1}{x+1-(x-1)e^{-2t}}$$

$$= \frac{\left(\frac{(x-1)e^{-2t}+x+1}{(x+1)-(x-1)e^{-2t}}-1\right)e^{2t}+\left(\frac{(x-1)e^{-2t}+x+1}{(x+1)-(x-1)e^{-2t}}+1\right)}{\left(\frac{(x-1)e^{-2t}+x+1}{(x+1)-(x-1)e^{-2t}}+1\right)-\left(\frac{(x-1)e^{-2t}+x+1}{(x+1)-(x-1)e^{-2t}}-1\right)e^{2t}}$$

$$= \frac{\left(\frac{(x-1)e^{-2t}+x+1-(x+1)+(x-1)e^{-2t}}{(x+1)-(x-1)e^{-2t}}\right)e^{2t}+\left(\frac{(x-1)e^{-2t}+x+1+(x+1)-(x-1)e^{-2t}}{(x+1)-(x-1)e^{-2t}}\right)}{\left(\frac{(x-1)e^{-2t}+x+1+(x+1)-(x-1)e^{-2t}}{(x+1)-(x-1)e^{-2t}}\right)-\left(\frac{(x-1)e^{-2t}+x+1-(x+1)+(x-1)e^{-2t}}{(x+1)-(x-1)e^{-2t}}\right)e^{2t}}$$

$$= \frac{2(x-1)+2(x+1)}{2(x+1)-2(x-1)} = x$$

Again,

$$\phi_{-t}\phi_t(x) = \phi_{-t}(y) = \frac{(y-1)e^{-2t}+y+1}{y+1-(y-1)e^{-2t}},$$

$$y = \phi_t(x) = \frac{(x-1)e^{2t}+x+1}{x+1-(x-1)e^{2t}}$$

$$= \frac{\left(\frac{(x-1)e^{2t}+(x+1)}{(x+1)-(x-1)e^{2t}}-1\right)e^{-2t}+\left(\frac{(x-1)e^{2t}+(x+1)}{(x-1)-(x-1)e^{2t}}+1\right)}{\left(\frac{(x-1)e^{2t}+(x+1)}{(x+1)e^{2t}-(x-1)e^{2t}}+1\right)-\left(\frac{(x-1)e^{2t}+(x+1)}{(x+1)-(x-1)e^{2t}}-1\right)e^{-2t}}$$

$$= \frac{\left(\frac{(x-1)e^{2t}+(x+1)-(x+1)-(x-1)e^{2t}}{(x+1)-(x-1)e^{2t}}\right)e^{-2t}+\left(\frac{(x-1)e^{2t}+(x+1)+(x+1)-(x-1)e^{2t}}{(x-1)-(x-1)e^{2t}}\right)}{\left(\frac{(x-1)e^{2t}+(x+1)+(x+1)-(x-1)e^{2t}}{(x+1)e^{2t}-(x-1)e^{2t}}\right)-\left(\frac{(x-1)e^{2t}+(x+1)-(x+1)-(x-1)e^{2t}}{(x+1)-(x-1)e^{2t}}\right)e^{-2t}}$$

$$= \frac{2(x-1)+2(x+1)}{2(x+1)-2(x-1)} = x$$

Hence, the evolution operator forms a dynamical group.

(iv) $\phi_t(\phi_s\phi_r) = (\phi_t\phi_s)\phi_r$ (Associative property)

L.H.S. $\phi_t(y) = \frac{(y-1)e^{2t}+y+1}{y+1-(y-1)e^{2t}}$
where,

$$y = \phi_s(z), z = \phi_r(x) = \frac{(x-1)e^{2r}+x+1}{x+1-(x-1)e^{2r}}$$

$$= \frac{(z-1)e^{2s}+z+1}{z+1-(z-1)e^{2s}} = \frac{(x-1)e^{2(r+s)}+x+1}{x+1-(x-1)e^{2(r+s)}}$$

$$\text{L.H.S.} = \frac{(x-1)e^{2(r+s+t)}+x+1}{x+1-(x-1)e^{2(r+s+t)}} = \phi_{t+r+s}(x)$$

$$\text{R.H.S.} = (\phi_t\phi_s)\phi_r$$

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Now,

$$\phi_t(y) = \frac{(y-1)e^{2t}+y+1}{y+1-(y-1)e^{2t}}, y = \phi_s(x) = \frac{(x-1)e^{2s}+x+1}{x+1-(x-1)e^{2s}}$$
$$= \frac{(x-1)e^{2(t+s)}+x+1}{x+1-(x-1)e^{2(t+s)}} = \phi_{t+s}(x)$$
$$\phi_{t+s}(\phi_r)(x) = \phi_{t+s}(z) = \frac{(z-1)e^{2t}+z+1}{z+1-(z-1)e^{2t}}, z = \phi_r(x) = \frac{(x-1)e^{2r}+x+1}{x+1-(x-1)e^{2r}}$$
$$\phi_{t+s}(\phi_r)(x) = \frac{(x-1)e^{2(t+s+r)}+x+1}{x+1-(x-1)e^{2(t+s+r)}} = \phi_{t+s+r}(x)$$
$$\Rightarrow \phi_t(\phi_s\phi_r) = (\phi_s\phi_r)\phi_t$$

(v) $\phi_t\phi_s = \phi_s\phi_t$

Now,

$$\phi_t(y) = \frac{(y-1)e^{2t}+y+1}{y+1-(y-1)e^{2t}}, y = \phi_s(x) = \frac{(x-1)e^{2s}+x+1}{x+1-(x-1)e^{2s}}$$
$$= \frac{\left(\frac{(x-1)e^{2s}+x+1}{x+1-(x-1)e^{2s}} - 1\right)e^{2t} + \left(\frac{(x-1)e^{2s}+x+1}{x+1-(x-1)e^{2s}} + 1\right)}{\left(\frac{(x-1)e^{2s}+x+1}{x+1-(x-1)e^{2s}} + 1\right) - \left(\frac{(x-1)e^{2s}+x+1}{x+1-(x-1)e^{2s}} - 1\right)e^{2t}}$$
$$= \frac{2(x-1)e^{2(s+t)}+2(x+1)}{2(x+1)-2(x-1)e^{2(s+t)}}$$
$$= \frac{(x-1)e^{2(s+t)}+(x+1)}{(x+1)-(x-1)e^{2(s+t)}} = \phi_{s+t}(x)$$
$$\phi_s\phi_t(x) = \phi_s(z) = \frac{(z-1)e^{2s}+(z+1)}{(z+1)-(z-1)e^{2s}}, z = \phi_t(x) = \frac{(x-1)e^{2t}+(x+1)}{(x+1)-(x-1)e^{2t}}$$
$$= \frac{\left(\frac{(x-1)e^{2t}+x+1}{x+1-(x-1)e^{2t}} - 1\right)e^{2s} + \left(\frac{(x-1)e^{2t}+x+1}{x+1-(x-1)e^{2t}} + 1\right)}{\left(\frac{(x-1)e^{2t}+x+1}{x+1-(x-1)e^{2t}} + 1\right) - \left(\frac{(x-1)e^{2t}+x+1}{x+1-(x-1)e^{2t}} - 1\right)e^{2s}}$$
$$= \frac{2(x-1)e^{2(t+s)}+2(x+1)}{2(x+1)-2(x-1)e^{2(t+s)}} = \frac{(x+1)+(x-1)e^{2(t+s)}}{(x+1)-(x-1)e^{2(t+s)}} = \phi_{t+s}(x)$$

So, $\phi_t\phi_s = \phi_s\phi_t$ (commutative property).

Hence the flow evolution operator ϕ_t forms a commutative dynamical group.

Example 1.3 Find the maximal interval of existence for unique solution of the following systems

$$\text{(i)}\, \dot{x}(t) = x^2 + \cos^2 t, t > 0, x(0) = 0$$
$$\text{(ii)}\, \dot{x} = x^2, x(0) = 1$$

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Solution (i) By maximal interval of existence of solution we mean the largest interval for which the solution of the equation exists. The given system is nonautonomous and $f(t,x) = x^2 + \cos^2 t$. Consider the rectangle $R = \{(t,x) : 0 \le t \le a, |x| \le b, a > 0, b > 0\}$ containing the point $(0,0)$. Clearly, $f(t,x)$ is continuous and $\frac{\partial f}{\partial x} = 2x$ is bounded on R. The Lipschitz condition $|f(t,x_1) - f(t,x_2)| \le K|x_1 - x_2|, \forall (t,x_1),(t,x_2) \in R$, K being the Lipschitz constant, is satisfied on R. Since $|f(t,x)| = |x^2 + \cos^2 t| \le |x^2| + |\cos^2 t| \le |x^2| + 1$, and $M = \max|f(t,x)| = 1 + b^2$ in R. Therefore, from Picard's theorem (if $f(t,x)$ is a continuous function in a rectangle $R = \{(t,x) : |t - t_0| \le a, |x - x_0| \le b, a > 0, b > 0\}$ and satisfies Lipschitz condition therein, then the initial value problem $\dot{x} = f(t,x), x(t_0) = x_0$ has a unique solution in the rectangle $R' = \{(t,x) : |t - t_0| \le h, |x - x_0| \le b\}$, where $h = \min\{a, b/M\}$, $M = \max|f(t,x)|$ for all $(t,x) \in R$, see the books Coddington and Levinson [3], Arnold [4]). Now $h = \min\{a, \frac{b}{M}\} = \min\left\{a, \frac{b}{1+b^2}\right\}$. We now determine the maximum/minimum value(s) of $b/(1+b^2)$. Let $g(b) = \frac{b}{1+b^2}$. Then $g'(b) = \frac{1-b^2}{(1+b^2)^2}$ and $g''(b) = \frac{2b(b^2-3)}{(1+b^2)^3}$. For max or min value(s) of $g(b)$, $g'(b) = 0$. This gives $b = 1$. Since $g''(1) = -1/2 < 0$, $g(b)$ is maximum at $b = 1$ and the maximum value is given by $g(1) = \frac{1}{2}$. Now, if $a \ge 1/2$, then $h = \frac{b}{1+b^2} \le 1/2$ and if $a < 1/2$, then $h < 1/2$. Thus we must have $h \le 1/2$. Hence the maximum interval of existence of the solution of the given system is $0 \le t \le 1/2$.

(ii) Here $f(t,x) = x^2$. Consider the rectangle $R = \{(t,x) : |t| \le a, |x - 1| \le b, a > 0, b > 0\}$ containing the point $(0,1)$. Clearly, $f(t,x)$ is continuous and $\frac{\partial f}{\partial x} = 2x$ is bounded on R. Hence the Lipschitz condition is satisfied on R. Also in R, $M = \max|f(t,x)| = (1+b)^2$. Therefore, $h = \min\{a, \frac{b}{M}\} = \min\left\{a, \frac{b}{(1+b)^2}\right\}$. It can be shown, as earlier, that $g(b) = \frac{b}{(1+b)^2}$ is maximum at $b = 1$ and the maximum value is $g(1) = 1/4$. Now if $a \ge 1/4$, then $h = \frac{b}{(1+b)^2} \le 1/4$ and if $a < 1/4$, then $h < 1/4$. Thus we must have $h \le 1/4$. Hence the maximum interval of existence of solution of the given system is $|t| \le 1/4$, that is, $-1/4 \le t \le 1/4$. Note that the Picard's theorem gives the local region of existence of unique solution for a system.

Example 1.4 Using linear stability analysis determine the stability of the critical points for the following systems:

$$\text{(i)}\,\dot{x} = \sin x, \quad \text{(ii)}\,\dot{x} = x^2.$$

Solution (i) The given system has infinite numbers of critical points. The critical points are $x_n^* = n\pi, n = 0, \pm 1, \pm 2, \ldots$. When n is even, $f'(x_n^*) = \cos(x_n^*) = \cos(n\pi) = (-1)^n = 1 > 0$. So, these critical points are unstable. When n is odd, $f'(x_n^*) = -1 < 0$, and so these critical points are stable.

(ii) The critical point of the system is at $x^* = 0$. Now, $f'(x^*) = 0$ and $f''(x^*) = 2 > 0$. Hence, x^* is attracting when $x < 0$ and repelling when $x > 0$. Actually, the critical point is semi-stable in nature.

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1.7 Analysis of One-Dimensional Flows

As we know qualitative approach is the combination of analysis and geometry and is a powerful tool for analyzing solution behaviors of a system qualitatively. By drawing trajectories in phase line/plane/space, the behaviors of phase points may be found easily. In qualitative analysis we mainly look for the following solution behaviors:

(i) Local stabilities of fixed points for a system;
(ii) Analyzing the existence of periodic/quasi-periodic solutions, limit cycle, relaxation oscillation, hysteresis, etc.;
(iii) Local and asymptotic solution behaviors of a system;
(iv) Topological features of flows such as bifurcations, catastrophe, topological equivalence, transitiveness, etc.

We shall now analyze a simple one-dimensional system as follows.

Consider a one-dimensional system represented as $\dot{x}(t) = \sin x$ with the initial condition $x(t=0) = x(0) = x_0$. The characteristic features of the system are (i) it is a one-dimensional system, (ii) nonlinear system (iii) autonomous system, and (iv) its closed-form solution (analytical solution) exists. This is a one-dimensional flow and we analyze the system on the basis of flow. The analytical solution of the system is obtained easily

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin x \Rightarrow \mathrm{d}t = \operatorname{cosec}(x)\mathrm{d}x$$

Integrating, we get

$$\begin{aligned} t &= \int \operatorname{cosec}(x)\,\mathrm{d}x \\ &= -\log|\operatorname{cosec}(x) + \cot(x)| + c, \end{aligned}$$

where c is an integrating constant. Using the initial condition $x(0) = x_0$, we get the integrating constant c as

$$c = \log|\operatorname{cosec}(x_0) + \cot(x_0)|.$$

Thus the solution of the system is given as

$$t = \log\left|\frac{\operatorname{cosec}(x_0) + \cot(x_0)}{\operatorname{cosec}(x) + \cot(x)}\right|.$$

From this closed-form solution, the behaviors of solutions for any initial conditions are difficult to analyse. Moreover, the asymptotic values of the system are

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also difficult to obtain. The qualitative approach can give better dynamical behavior about this simple system.

We consider t as time, x as the position of an imaginary particle moving along the flow in real line and $\dot{x}$ as the velocity of that particle. The differential equation $\dot{x} = \sin x$ represents a vector field on the line. It gives the velocity vector $\dot{x}$ at each position x. The arrows point to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. We shall draw the graph of $\sin x$ versus x in $x\dot{x}$- plane which gives the flow in the x-axis (see Fig. 1.3).

We may imagine that fluid is flowing steadily along the x-axis with a velocity $\dot{x}$ which varies from place to place, according to equation $\dot{x} = \sin x$. At points $\dot{x} = 0$, there is no flow and such points are called equilibrium points (fixed points). According to the definition of fixed point, the equilibrium points of this system are obtained as $\sin x = 0 \Rightarrow x = n\pi (n = 0, \pm 1, \pm 2, \ldots)$. This simple looking autonomous system has infinite numbers of equilibrium points in $\mathbb{R}$. We can see that there are two kinds of equilibrium points. The equilibrium point where the flow is toward the point is called **sink** or **attractor** (neighboring trajectories approach asymptotically to the point as $t \to \infty$). On the other hand, when the flow is away from the point, the point is called **source** or **repellor** (neighboring trajectories move away from the point as $t \to \infty$). From the above figure the solid circles represent the sinks that are stable equilibrium points and the open circles are the sources, which are unstable equilibrium points. The names are given because the sinks and sources are common in fluid flow problems. From the geometric approach one can get local stability behavior of the equilibrium points of the system easily and is valid for all time. We shall now re-look the analytical solution of the system. The analytical solution can be expressed as

$$t = \log|\tan(x/2)| + c \Rightarrow x(t) = 2\tan^{-1}(Ae^{t})$$

where A is an integrating constant.

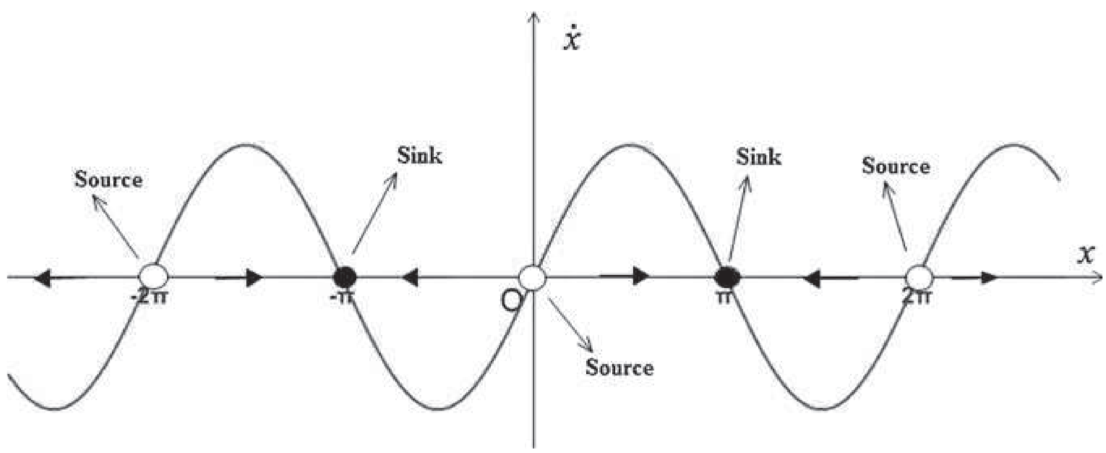

Fig. 1.3 Graphical representation of flow generated by $\sin(x)$

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Let the initial condition be $x_0 = x(0) = \pi/4$. Then from the above solution we obtain

$$A = \tan(\pi/8) = -1 + \sqrt{2} = 1/\left(1 + \sqrt{2}\right).$$

So the solution is expressed as

$$x(t) = 2\tan^{-1}\left(\frac{e^t}{1 + \sqrt{2}}\right).$$

We see that the solution $x(t) \to \pi$ and $t \to \infty$.

Without using analytical solution for this particular initial condition the same result can be found by drawing the graph of x versus t. So the solution's behavior at any initial condition can be obtained easily by geometric approach. This simple one-dimensional system also has an interesting application. For a slow motion of a spring immersed in a highly viscous fluid such as grease or viscoelastic fluid (the combined effects of fluid viscosity and elasticity for example, synovial fluid in the joints of human bones), the viscous damping force is very strong compared to the inertia of motion. So one can neglect acceleration term (that is, inertia) and the spring-mass system may be governed by the equation $\alpha\dot{x} = \sin x$, where $\alpha > 0$ (string constant) is a real number and the dynamics can be obtained using this approach for different values of α (see the book Strogatz [5] for more physical examples and explanations).

We shall discuss a few worked out examples presented below.

Example 1.5 With the help of flow concept discuss the local stability of the fixed points of $\dot{x} = f(x) = (x^2 - 1)$.

Solution The fixed points of the given autonomous system are given by setting $f(x) = 0$. This gives $x = \pm 1$. So the fixed points of the system are 1 and -1. For the local stability of the system about these fixed points we plot the graph of the function $f(x)$ and then sketch the vector field. The flow is to the right direction, indicated by the symbol '$\rightarrow$', where the velocity $\dot{x} > 0$, that is, where $(x^2 - 1) > 0$ and to the left direction, indicated by the symbol '$\leftarrow$', where $\dot{x} < 0$, that is, $(x^2 - 1) < 0$. We also use solid circles to represent stable fixed points and open circles for unstable fixed points.

In Fig. 1.4 the arrows indicate the flow of the system. From the figure, we see that the fixed point $x = 1$ is unstable, since it acts as a source point and the fixed point $x = -1$ is stable, since it acts as a sink point.

Example 1.6 Discuss the stability character of the fixed points for the system $\dot{x} = x(1 - x)$ using the concept of flow.

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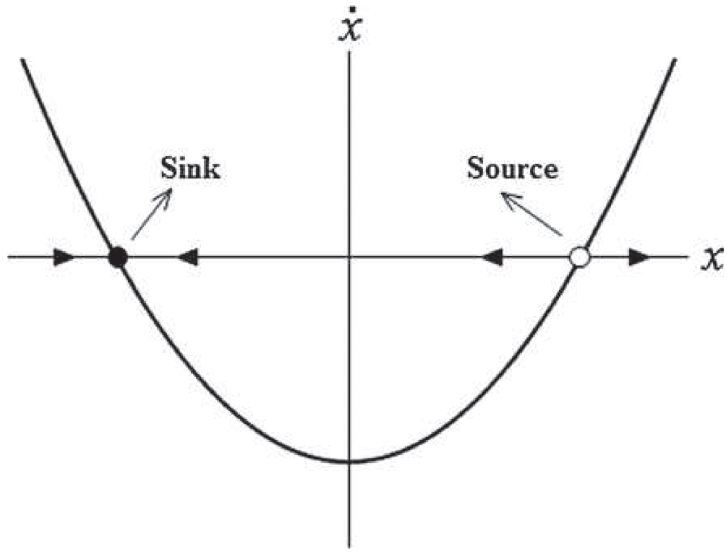

Fig. 1.4 Graphical representation of $f(x) = (x^2 - 1)$

Solution Here $f(x) = x(1 - x)$. Then for the fixed points, we have

$$f(x) = 0 \Rightarrow x(1 - x) = 0 \Rightarrow x = 0, 1.$$

Thus the fixed points are 0 and 1. To discuss the stability of these fixed points we plot the system (x versus $\dot{x}$) and then sketch the vector field. The flow is to the right direction, indicated by the symbol '$\longrightarrow$', when the velocity $\dot{x} > 0$, and to the left direction, indicated by the symbol '$\longleftarrow$', when $\dot{x} < 0$. We also use solid circle to represent stable fixed point and open circle to represent unstable fixed point.

From Fig. 1.5 we see that the fixed point $x = 1$ is stable whereas the fixed point $x = 0$ is unstable.

Example 1.7 Find the fixed points and analyze the local stability of the following systems (i) $\dot{x} = x + x^3$ (ii) $\dot{x} = x - x^3$(iii) $\dot{x} = -x - x^3$

Solution (i) Here $f(x) = x + x^3$. Then for fixed points $f(x) = 0 \Rightarrow x + x^3 = 0 \Rightarrow x = 0$, as $x \in \mathbb{R}$. So, 0 is the only fixed point of the system. We now see that when $x > 0$, $\dot{x} > 0$ and when $x < 0$, $\dot{x} < 0$. Hence the fixed point $x = 0$ is unstable. The graphical representation of the flow generated by the system is displayed in Fig. 1.6.

(ii) Here $f(x) = x - x^3$. Then $f(x) = 0 \Rightarrow x - x^3 = 0 \Rightarrow x = 0, 1, -1$. Therefore, the fixed points of the system are $0, 1, -1$. We now see that

(a) when $x < -1$, then $\dot{x} > 0$
(b) when $-1 < x < 0$, $\dot{x} < 0$
(c) when $0 < x < 1$, $\dot{x} > 0$
(d) when $x > 1$, then $\dot{x} < 0$.

This shows that the fixed points 1 and -1 are stable whereas the fixed point 0 is unstable (Fig. 1.7).

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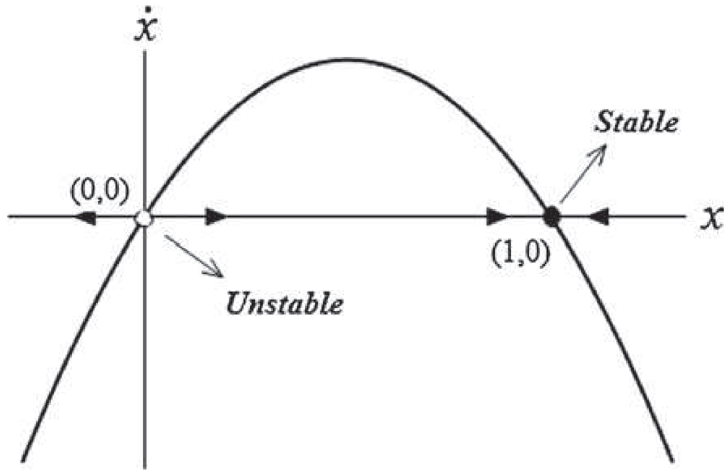

Fig. 1.5 Pictorial representation of $f(x) = x(1-x)$

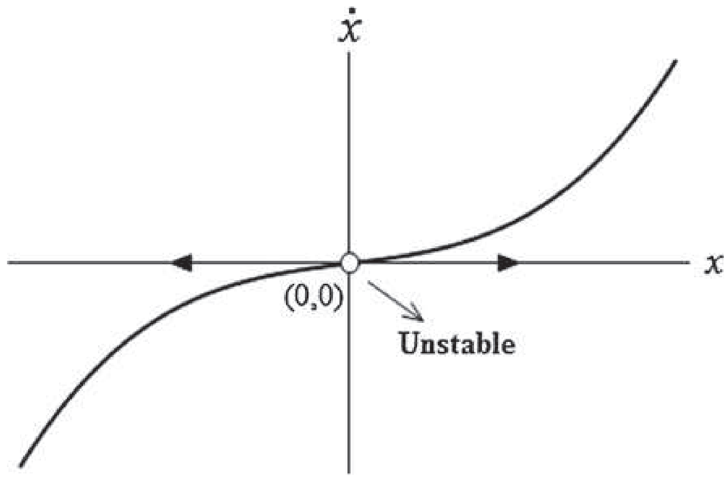

Fig. 1.6 Graphical representation of $f(x) = (x + x^3)$

(iii) Here $f(x) = -x - x^3$. Then $f(x) = 0 \Rightarrow -x - x^3 = 0 \Rightarrow x = 0$, as $x \in \mathbb{R}$. So $x = 0$ is the only fixed point of the system. We now see that $\dot{x} > 0$ when $x < 0$ and $\dot{x} < 0$ when $x > 0$. This shows that the fixed point $x = 0$ is stable. The graphical representation of the flow generated by the system is displayed in Fig. 1.8.

Example 1.8 Determine the equilibrium points and sketch the phase diagram in the neighborhood of the equilibrium points for the system represented as $\dot{x} + x\,\mathrm{sgn}(x) = 0$.

Solution Given system is $\dot{x} + x\,\mathrm{sgn}\,(x) = 0$, that is, $\dot{x} = -x\,\mathrm{sgn}\,(x)$, where the function $\mathrm{sgn}(x)$ is defined as

$$\mathrm{sgn}\,(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

For equilibrium points, we have

$$\dot{x} = 0 \Rightarrow x\,\mathrm{sgn}\,x = 0 \Rightarrow x = 0.$$

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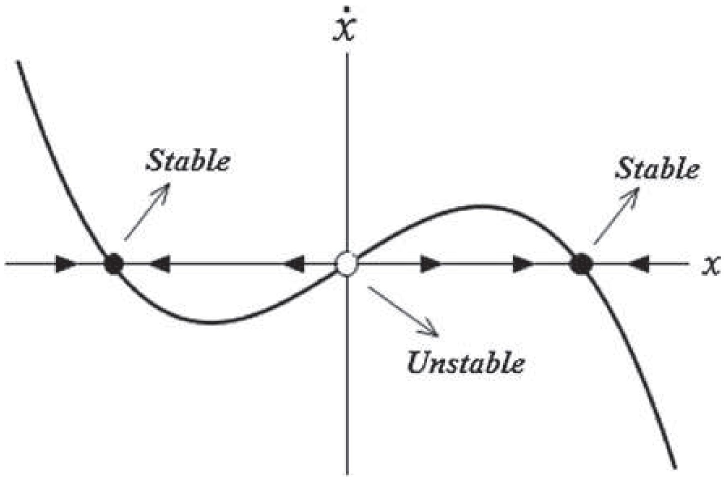

Fig. 1.7 Graphical representation of the flow generated by $(x - x^3)$

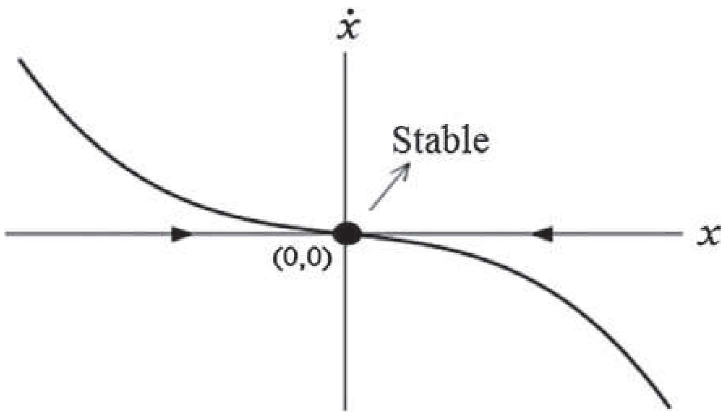

Fig. 1.8 Graphical representation of $f(x) = (-x - x^3)$ versus x

This shows that the system has only one equilibrium point at $x = 0$. In flow analysis we see that the velocity $\dot{x} < 0$ for all $x \neq 0$. The flow is to the right direction, when $\dot{x} > 0$, in the negative x-axis and to the left direction, when $\dot{x} < 0$, in the positive x-axis. This is shown in the phase diagram depicted in Fig. 1.9, which shows that the fixed point origin is semi-stable.

1.8 Conservative and Dissipative Dynamical Systems

The dichotomy of dynamical systems in conservative versus dissipative is very important. They have some fundamental properties. Particularly, conservative systems are the integral part of Hamiltonian mechanics. We give here only the formal definitions of conservative and dissipative systems. Consider an autonomous system represented as

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}), \underset{\sim}{x} \in \mathbb{R}^n. \tag{1.15}$$

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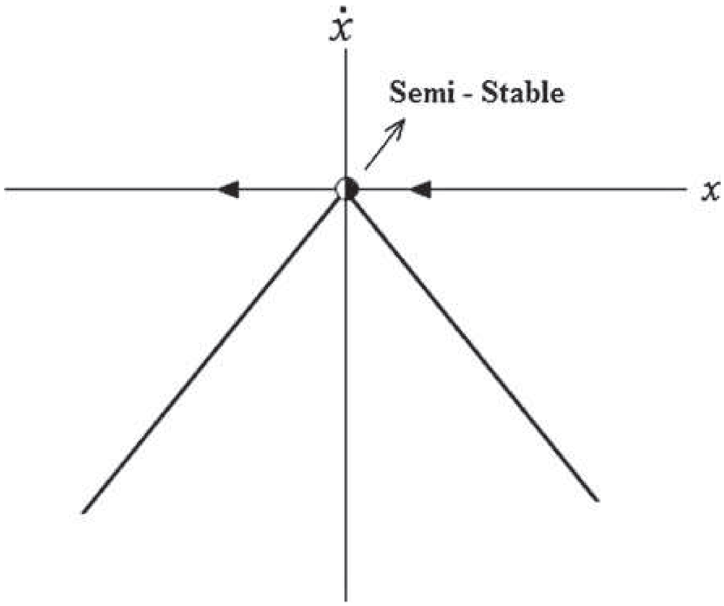

Fig. 1.9 Graphical representation of the flow $\dot{x} = -x\,\mathrm{sgn}\,x$

The conservative and dissipative systems are defined with respect to the divergence of the corresponding vector field, which in turn refers to the conservation of volume or area in their state space or phase plane, respectively as follows:

A system is said to be **conservative** if the divergence of its vector field is zero. On the other hand, it is said to be **dissipative** if its vector field has negative divergence. The phase volume in a conservative system is constant under the flow while for a dissipative system the phase volume occupied by the system is gradually decreased as the time t increases and shrinks to zero as $t \to \infty$. When divergence of vector field is positive, the phase volume is gradually expanding. We shall discuss it in a later chapter. We state a lemma below which gives the change of volume in a phase space for an autonomous system.

Sometimes, it is useful to find the evolution of volume in the phase space of a system $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}), \underset{\sim}{x} \in \mathbb{R}^n$. The system generates a flow $\phi(t, \underset{\sim}{x})$. We give Liouville's theorem which describes the time evolution of volume under the flow $\phi(t, \underset{\sim}{x})$. Before this we now give the following lemma.

Lemma 1.1 *Consider an autonomous vector field* $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}), \underset{\sim}{x} \in \mathbb{R}^n$ *and generates a flow* $\phi_t(\underset{\sim}{x})$. *Let* D_0 *be a domain in* $\mathbb{R}^n$ *and* $\phi_t(D_0)$ *be its evolution under the flow. If* $V(t)$ *is the volume of* D_t, *then the time rate of change of volume is given as* $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{t=0} = \int\limits_{D_0} \nabla \cdot f\, d\underset{\sim}{x}$.

Proof The volume $V(t)$ can be expressed in the following form using the definition of the Jacobian of a transformation as

$$V(t) = \int\limits_{D_0} \left| \frac{\partial \phi(t, \underset{\sim}{x})}{\partial x} \right| d\underset{\sim}{x}$$

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Expanding Taylor series of $\phi(t, \underset{\sim}{x})$ in the neighborhood of $t = 0$, we get

$$\phi(t, \underset{\sim}{x}) = \underset{\sim}{x} + f(\underset{\sim}{x})t + O(t^2)$$
$$\Rightarrow \frac{\partial \phi}{\partial \underset{\sim}{x}} = I + \frac{\partial f}{\partial \underset{\sim}{x}} t + O(t^2)$$

Here I is the $n \times n$ identity matrix and

$$\left|\frac{\partial \phi}{\partial \underset{\sim}{x}}\right| = \left|I + \frac{\partial f}{\partial \underset{\sim}{x}} t\right| + O(t^2)$$
$$= 1 + \text{trace}\left(\frac{\partial f}{\partial \underset{\sim}{x}}\right) t + O(t^2) \text{ [Using expansion of the determinant]}$$

Now, $\text{trace}\left(\frac{\partial f}{\partial \underset{\sim}{x}}\right) = \nabla \cdot f$, so we have

$$V(t) = V(0) + \int_{D_0} t\nabla \cdot f \, d\underset{\sim}{x} + O(t^2).$$

This gives $\left.\frac{dV}{dt}\right|_{t=0} = \int_{D_0} \nabla \cdot f \, d\underset{\sim}{x}$.

Theorem 1.1 (Liouville's Theorem) *Suppose* $\nabla \cdot f = 0$ *for a vector field f. Then for any region* $D_0 \subseteq \mathbb{R}^n$, *the volume V(t) generated by the flow* $\phi(t, \underset{\sim}{x})$ *is* $V(t) = V(0)$, $V(0)$ *being the volume of* D_0.

Proof Suppose the divergence of the vector field f is everywhere constant, that is, $\nabla \cdot f = c$. For arbitrary time t_0 the evolution equation for the volume is given as $\dot{V} = cV$. This gives $V(t) = V(0)e^{ct}$. When the vector field is divergence free, that is, $c = 0$, we get the result $\dot{V} = 0 \Rightarrow V(t) = V(0) =$ constant. Thus we can say that the flow generated by a time independent system is volume preserving.

Examples of conservative and dissipative systems are presented below.

(a) Consider a linear and undamped pendulum represented as $\ddot{x} + x = 0$. This is an example of a conservative system. Setting $\dot{x} = y$, we can write it as a system of equations

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \end{aligned}\right\}$$

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The system may also be written in the compact form $\underset{\sim}{\dot{x}} = \underset{\sim}{f}(\underset{\sim}{x})$, where $\underset{\sim}{f}(\underset{\sim}{x}) = \begin{pmatrix} y \\ -x \end{pmatrix}$. The divergence of the vector field $\underset{\sim}{f}$ is given by $\vec{\nabla} \cdot \underset{\sim}{f} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) = 0$. According to the definition, the system is conservative and the area occupied in the xy-phase plane is constant.

(b) The damped pendulum governed by $\ddot{x} + \alpha\dot{x} + \beta x = 0, \alpha, \beta > 0$ is an example of a dissipative system. Setting $\dot{x} = y$, we can write the system as

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\alpha y - \beta x \end{aligned}\right\}$$

The vector field is then expressed as $\underset{\sim}{f}(\underset{\sim}{x}) = \begin{pmatrix} y \\ -\alpha y - \beta x \end{pmatrix}$.

Now, $\vec{\nabla} \cdot \underset{\sim}{f} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-\alpha y - \beta x) = -\alpha < 0, \text{ since } \alpha > 0.$

This shows that the divergence of the vector field is negative. So the system is dissipative in nature and the area in the phase plane is decreasing as time goes on. This is the simplest linear oscillator with linear damping. It describes a spring-mass system with a damper in parallel. The spring force is proportional to the extension x of the spring and the damping or frictional force is proportional to the velocity $\dot{x}$. The two constants α and β are related to the stiffness of the spring and the degrees of friction in the damper, respectively. According to the above lemma, the change in phase area is given by

$$A(t) = cA(0)e^{-\alpha t}, \alpha > 0 \text{ as } t \to \infty, c \text{ being a constant.}$$

Example 1.9 Find the phase volume element for the systems (i) $\dot{x} = -x$, (ii) $\dot{x} = ax - bxy, \dot{y} = bxy - cy$ where $x, y \geq 0$ and a, b, c are positive constants.

Solution (i) The flow of the system $\dot{x} = -x$ is attracted toward the point $x = 0$. The time rate of change of volume element $V(t)$ under the flow is given as

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{t=0} = -\int\limits_{D(0)} \mathrm{d}x = -V(0)$$

$$\text{or,} \quad V(t) = V(0)e^{-t} \to 0 \text{ as } t \to \infty.$$

Hence the phase volume element $V(t)$ shrinks exponentially.

(ii) The given system is a Lotka-Volterra predator-prey population model. The rate of change in phase area $A(t)$ is given as

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$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\int \vec{\nabla} \cdot \underset{\sim}{f}\, \mathrm{d}x\mathrm{d}y$$
$$= -\int (a - c - by + bx)\mathrm{d}x\mathrm{d}y$$

This shows that a phase area periodically shrinks and expands.

1.9 Some Definitions

In this section we give some important preliminary definitions relating to flow of a system. The definitions given here are elaborately discussed in the later chapters for higher dimensional systems.

Invariant set A set $D \subset \mathbb{R}^n$ is said to be an invariant set under the flow ϕ_t if for any point $p \in D$, $\phi_t(p) \in D$ for all $t \in \mathbb{R}$. The set D is said to be positively invariant if $\phi_t(p) \in D$ for $t \geq 0$. Trajectories starting in an invariant set remain in the set for all times. An interval is called trapping if it is mapped into itself and is said to be invariant if it is mapped exactly onto itself. Moreover, if a bounded interval is trapping, then all of its trajectories are trapped inside and must converge to a closed, invariant, and bounded limit set. Basically these limit sets are the attractors of a system. So the periodic orbits are examples of invariant sets. We now define two limiting topological concepts which are relevant to the orbits of dynamical systems.

Limit points (ω- and α-limit points)

The asymptotic behavior of a trajectory may be related with limit points/sets or cycles and are termed as ω- and α-limit points/sets or cycles. We now give the definitions.

A point $p \in \mathbb{R}^n$ is called an ω-(resp. a α-) limit point if there exists a sequence $\{t_i\}$ with $t_i \to \infty$ (resp. $t_i \to -\infty$) such that $\phi(t_i, x) \to p$ as $i \to \infty$. The ω-limit set(cycle) is denoted by $\Lambda(\underset{\sim}{x})$ and is defined as

$$\Lambda(\underset{\sim}{x}) = \left\{ \underset{\sim}{x} \in \mathbb{R}^n | \exists \{t_i\} \text{ with } t_i \to \infty \text{ and } \phi(t_i, \underset{\sim}{x}) \to p \text{ as } i \to \infty \right\}.$$

Similarly, the α-limit set (cycle), $\mu(\underset{\sim}{x})$, is defined as

$$\mu(\underset{\sim}{x}) = \left\{ \underset{\sim}{x} \in \mathbb{R}^n | \exists \{t_i\} \text{with } t_i \to -\infty \text{ and} \phi(t_i, \underset{\sim}{x}) \to p \text{ as } i \to \infty \right\}.$$

For example, consider a flow $\phi(t,x)$ on $\mathbb{R}^2$ generated by the system $\dot{r} = cr(1-r)$, $\dot{\theta} = 1$, c being a positive constant. For $x \neq 0$, let p be any point of the closed orbit C and take $\{t_i\}_{i=1}^{\infty}$ to be the sequence of $t > 0$. The trajectory through x crosses the radial line through p. So, $t_i \to \infty$ as $i \to \infty$ and

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$\phi(t_i, x) \to p$ as $i \to \infty$. If $\underset{\sim}{x}$ lies in the closed orbit C, then $\phi(t_i, x) = p$ for each i. Hence every point of C is a ω-limit point of $\underset{\sim}{x}$ and so $\Lambda(x) = C$ for every $x \neq 0$. When $|x| \leq 1$, the sequence $\{t_i\}_{i=1}^{\infty}$ with $t<0$ gives the α-limit set $\mu(x) = \begin{cases} \{0\} & \text{for } |x| < 1 \\ \text{closed orbit} & \text{for } |x| = 1 \end{cases}$.

When $|x| > 1$, there is no sequence $\{t_i\}_{i=1}^{\infty}$, with $t_i \to \infty$ as $i \to \infty$, such that $\phi(t_i, x)$ exists as $i \to \infty$. So, $\mu(x)$ is empty when $|x| > 1$. The closed orbit C is called a limit cycle of the system.

The trajectory of a system through a point $\underset{\sim}{x}$ is the set $\gamma(\underset{\sim}{x}) = \bigcup_{t \in \mathbb{R}} \phi(t, \underset{\sim}{x})$ and the corresponding positive semi-trajectory $\gamma^+(\underset{\sim}{x})$ and negative semi-trajectory $\gamma^-(\underset{\sim}{x})$ are defined as follows:

$$\gamma^+(\underset{\sim}{x}) = \bigcup_{t \geq 0} \phi(t, \underset{\sim}{x}) \text{ and } \gamma^-(\underset{\sim}{x}) = \bigcup_{t \leq 0} \phi(t, \underset{\sim}{x}).$$

We now state two lemmas below. Interested readers can try for proofs (see the book Glendinning [6]).

Lemma 1.2

(a) *The set D is invariant if and only if $\gamma(\underset{\sim}{x}) \subset D$ for all $\underset{\sim}{x} \in D$.*
(b) *D is invariant if and only if $\mathbb{R}^n \backslash D$ is invariant.*
(c) *Let (D_i) be a countable collection of invariant subsets of $\mathbb{R}^n$. Then $\bigcup_i D_i$ and $\bigcap_i D_i$ are also invariant subsets of $\mathbb{R}^n$.*

Lemma 1.3 *The set $\Lambda(\underset{\sim}{x}) = \bigcap_{\underset{\sim}{x} \in \gamma(\underset{\sim}{x})} \mathrm{cl}\left(\gamma^+(\underset{\sim}{x})\right)$, where cl denotes the closure of the set, is the ω-limit set.*

Non-wandering point A point p is called a non-wandering point if for any neighborhood U of p and for any $T > 0$, there exists some $|t| > T$ such that $\phi(t, U) \cap U \neq \varphi$. The nonwandering set, denoted by Ω, contains all such points $p \in U$ and it is closed. Non-wandring points give asymptotic behavior of the orbit. In the above definition, if $\phi(t, U) \cap U = \varphi$, then the point p is called a wandering point.

The examples of non-wandering points are fixed points and periodic orbits of a system. For the undamped oscillator $(\ddot{x} + x = 0)$, all points are non-wandering in $x\dot{x}$ phase plane while for the damped oscillator $(\ddot{x} + \alpha\dot{x} + x = 0)$, origin is the only non-wandering point.

Attracting set A closed invariant set $D \subset \mathbb{R}^n$ for a flow ϕ_t is said to be an attracting set if there exists some neighborhood U in D such that $\forall t \geq 0$, $\phi(t, U) \subset U$ and $\bigcap_{t>0} \phi(t, U) = D$.

Absorbing set A positive invariant compact subset $B \subseteq \mathbb{R}^n$ is said to be an absorbing set if there exists a bounded subset C of $\mathbb{R}^n$ with $C \supset B$ such that $t_C > 0 \Rightarrow \phi(t, C) \subset B \, \forall t \geq t_C$ (see the book by Wiggins [7] for details).

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Trapping zone An open set U in an invariant set $D \subset \mathbb{R}^n$ in an attracting set for a flow generated by a system is called a trapping zone. Let a set A be closed and invariant. The set A is said to be stable if and only if every neighborhood of A contains a neighborhood U of A which is trapping.

Basin of attraction The domain (called as basin of attraction) of an attracting set D is defined as $\bigcup_{t \le 0} \phi(t, U)$ where U is any open set in $D \subset \mathbb{R}^n$.

We now give an example from Ruelle [8]. This example is also discussed in the book by Guckenheimer and Holmes [9]. Consider the one-dimensional system $\dot{x} = -x^4 \sin(\pi/x)$. It has countably infinite set of fixed points at $x^* = 0,\ \pm\frac{1}{n}, n = 1, 2, 3, \ldots$. Now,

$$\begin{aligned} f(x) = -x^4 \sin(\pi/x) &\Rightarrow f'(x) = -4x^3 \sin(\pi/x) + \pi x^2 \cos(\pi/x) \\ &\Rightarrow f'(x^*)|_{x^* = \pm\frac{1}{n}} = \frac{\pi}{n^2}\cos(n\pi) = \frac{\pi}{n^2}(-1)^n. \end{aligned}$$

The fixed point $x^* = 0$ is neither attracting nor repelling. The interval $[-1, 1]$ is an attracting set of the given system. The fixed points $x^* = \pm\frac{1}{2n}$, $n = 1, 2, \ldots$ are repelling while the fixed points $x^* = \pm\frac{1}{(2n-1)}, n = 1, 2, \ldots$ are attracting.

1.10 Exercises

1.(a) What is a dynamical system? Write its importance.
(b) Discuss continuous and discrete dynamical systems with examples.
(c) Explain deterministic, semi-deterministic and nondeterministic dynamical processes with examples.
(d) What do you mean by ‘qualitative study’ of a nonlinear system? Write it importance in nonlinear dynamics.

2.(a) Give the mathematical definition of flow. Discuss the concept related to ‘a flow and its orbit’. Also indicate its implication on uniqueness theorem of differential equation.
(b) Show that the initial value problem $\dot{x} = x^{\frac{1}{3}}, x(0) = 0$ has an infinite number of solutions. How would you explain it in the context of flow?
(c) Consider a system $\dot{x} = |x|^{p/q}$ with $x(0) = 0$ where p and q are prime to each other. Show that the system has an infinite number of solutions if $p < q$, and it has a unique solution if $p > q$.

3. Find the maximum interval of existence for the solutions of the following equations
(a) $\dot{x} = x^2, x(0) = x_0$, (b) $\dot{x} = tx^2, x(0) = x_0$, (c) $\dot{x} = x^2 + t^2, x(0) = x_0$, (d) $\dot{x} = x(x-2), x(0) = 3$

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4. For what values of t_0 and x_0 does the initial value problem $\dot{x} = \sqrt{x}, x(t_0) = x_0$ have a unique solution?
5. Show that the initial value problem $\dot{x} = 3x^{2/3}$ with $x(0) = 0$ has two solutions passing through the point (0, 0). How do you explain the context of flow in this?
6. Show that the initial value problem $\dot{x} = |x|^{1/2}, x(0) = 0$ has four different solutions through the point $(0, 0)$. Sketch these solutions in the $t{-}x$ plane. Explain this from Picard's theorem.
7. Prove that the system $\dot{x} = x^3$ with the initial condition $x(0) = 2$ has a solution on an interval $(-\infty, c), c \in \mathbb{R}$. Sketch the solution $x(t)$ in the t – x plane and find the limiting behavior of solution $x(t)$ as $t \to c$-.
8. Prove that the solutions of the initial value problem $\dot{x} = \begin{cases} 0 & \text{when } x \leq 0 \\ x^{1/n} & \text{when } x > 0 \end{cases}$ with $x(0) = 0$ are not unique for $n = 2, 3, 4, \ldots$.
9. What do you mean by fixed point of a system? Determine the fixed points of the system $\dot{x} = x^2 - x, x \in \mathbb{R}$. Show that solutions exist for all time and become unbounded in finite time.
10. Give mathematical definitions of 'flow evolution operator' of a system. Write the basic properties of an evolution operator of a flow.
11. Show that the dynamical system (or evolution) forms a dynamical group. What can you say about commutative/non-commutative group of a system? Give reasons in support of your answer.
12. Find the evolution operators for the following systems: (i) $\dot{x} = x - x^2$, (ii) $\dot{x} = x^2$, (iii) $\dot{x} = x \ln x, x > 0$, (iv) $\dot{x} = \tanh(x)$, (v) $\dot{x} = x - x^3$, (vi) $\dot{x} = f \cos \omega t$, (vii) $\dot{x} = f \sin \omega t$.
    Verify that $\phi_t(\phi_s(x)) = \phi_{t+s}(x) \ \forall x, t, s \in \mathbb{R}$ for all cases. Also, show that the evolution operator ϕ_t for the system $\dot{x} = x^2$ forms a dynamical group.
13. Define fixed point of a system in the context of flow. Give its geometrical interpretation. How do you relate this concept with the usual notion of fixed point in a continuous dynamical system?
14.(a) Define source and sink for a one-dimensional flow. Illustrate them with examples.
    (b) Locate the source and sink for the system $\dot{x} = (x^2 - 1), x \in \mathbb{R}$.
15. Consider the one-dimensional system represented as $\dot{x} = ax + b$, where b is a constant and a is a nonzero parameter. Find all fixed points of the system and discuss their stability for different values of a.
16.(a) Sketch the region of the flow generated by the one-dimensional system $\dot{x} = 1/x, x > 0$ with the starting condition $x(0) = x_0$.
    (b) What do you mean by oscillating solution of a system? Explain with examples. Show that one-dimensional flow cannot oscillate.
17.(a) Find the critical points of the following systems: (i) $\dot{x} = e^x - 1$, (ii) $\dot{x} = x^2 - x - 1$, (iii) $\dot{x} = \sin(\pi x)$, (iv) $\dot{x} = \cos x$, (v) $\dot{x} = \sinh x$, (vi) $\dot{x} = rx - x^2$ for $r < 0,\ r = 0,\ r > 0$, (vii) $\dot{x} = x - \ln(1 + x) + r,\ r > 0$, (viii) $\dot{x} = x - x^3/6$.

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(b) Using linear stability analysis determine the stabilities/instabilities of the following systems about their critical points:
(i) $\dot{x} = x(x-1)(x-2)$, (ii) $\dot{x} = (x-2)(x-3)$, (iii) $\dot{x} = \log x$, (iv) $\dot{x} = \cos x$, (v) $\dot{x} = \tan x$, (vi) $\dot{x} = 2 + \sin x$, (vii) $\dot{x} = x - x^3/6$, (viii) $\dot{x} = 1 - x^2/2 + x^4/24$.

18. Sketch the family of solutions of the differential equation $\dot{x} = ax - bx^2,\ x > 0$ and $a, b > 0$. How does the velocity vector $\dot{x}$ behave when $(a/b) < x < \infty$?
19. Find the critical points and analyze the local stability about the critical points of each of the following systems: (i)$\dot{x} = x^4 - x^3 - 2x^2$, (ii) $\dot{x} = \sinh(x^2)$ (iii) $\dot{x} = \cos x - 1$, (iv) $\dot{x} = (x-a)^2$, (v) $\dot{x} = (x+1)(x+2)$, (vi) $\dot{x} = \tan x$, (vii) $\dot{x} = \log x$,(viii) $\dot{x} = e^x - x - 1$.
20. Classify all possible flows in $\mathbb{R}$ of the system $\dot{x} = a_0 + a_1 x + a_2 x^2 + x^3$, where $a_0, a_1, a_2 \in \mathbb{R}$.
21. Consider the one-dimensional system $\dot{x} = \mu x + x^3,\ \mu \geq 0$. Using geometric approach find the solution behavior for any initial condition $x_0 (\neq 0)$.
22. When is a flow called conservative? Give an example of conservative flow.
23. Prove that the phase volume of a conservative system is constant. Is the converse true? Give reasons in support of your answer.
24. What can you say about time rate of change of phase volume element in a dissipative dynamical system? Explain it geometrically. Give an example of a dissipative system.
25. Prove that the α- and ω-limit sets of a flow $\phi_t(x)$ are contained in the non-wandering set of the flow $\phi_t(x)$.
26. Define absorbing set of a flow. Write down the relation between trapping zones and absorbing sets. Prove that for an absorbing set A, $\bigcap_{t \geq 0} \phi(t, A)$ forms an attracting set.
27. Give the definition of invariant set of a flow. Write its importance in dynamical evolution of a system. Prove that the ω-limit set, $\Lambda(\underset{\sim}{x})$, is invariant and it is nonempty and compact if the positive orbit $\gamma^+(\underset{\sim}{x})$ of $\underset{\sim}{x}$ is bounded.
28. If two orbits $\gamma(x)$ and $\gamma(y)$ of autonomous systems satisfy $\gamma(x) \cap \gamma(y) \neq \varphi$, prove that both the orbits are coinciding.

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