~~MA204~~ Problem sheet – III

① ⓐ Prove that $\|A\|_1 = \sup_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$

Sol: $\|A\|_1 = \max_{x\neq\bar{0}} \dfrac{\|Ax\|_1}{\|x\|_1}$

Consider,

$$(Ax)(i) = \sum_{j=1}^{n} a_{ij}x_j, \quad i=1,2,3,\dots,n$$

$$\|Ax\|_1 = \sum_{i=1}^{n} |(Ax)(i)|$$

$$= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij}x_j \right|$$

$$\le \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}||x_j| \quad \text{(using triangl inequality)}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}||x_j| \quad \text{(Interchangi the order)}$$

$$= \left(\sum_{j=1}^{n} |x_j|\right)\left(\sum_{i=1}^{n} |a_{ij}|\right)$$

$$\le \alpha \|x\|_1, \quad \alpha = \max_{j} \sum_{i=1}^{n} |a_{ij}|$$

$\Rightarrow$ for $x\neq\bar{0}$, $\dfrac{\|Ax\|_1}{\|x\|_1} \le \alpha \Rightarrow \|A\|_1 = \max_{x\neq\bar{0}} \dfrac{\|Ax\|_1}{\|x\|}$
$\le \alpha$ --

Now $\|A\|_1 \le \alpha = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ij_0}|$ for some j_0.

Consider, $Ae_{j_0} = \begin{bmatrix} a_{1j_0} \\ a_{2j_0} \\ \vdots \\ a_{nj_0} \end{bmatrix}$, So, $\|Ae_{j_0}\|_1 = \alpha$ and $\|e_{j_0}\|$

Then $\alpha = \dfrac{\|Ae_{j_0}\|}{\|e_{j_0}\|} \le \|A_1\|$ -- ②

From ① & ②, the result follows.

ⓑ $\|Ax\|_\infty = \max_{1\le i\le n} |(Ax)(i)|$

$$= \max_{1\le i\le n} \left|\sum_{j=1}^{n} a_{ij} x_j\right|$$

$$\le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}||x_j| \quad \text{(Using triangle inequality)}$$

$$\le \|x\|_\infty \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

$$\le \beta \|x\|_\infty, \quad \beta = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

$$\Rightarrow \|A\|_\infty = \max_{x\ne \bar{0}} \frac{\|Ax\|_\infty}{\|x\|_\infty} \le \beta$$

$$\Rightarrow \|A\|_\infty \le \beta, \text{ where } \beta = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{i_0 j}|$$

Define,

$$y_j = \begin{cases} \dfrac{|a_{i_0 j}|}{a_{i_0 j}}, & a_{i_0 j} \ne 0 \\ 0, & a_{i_0 j} = 0 \end{cases} \Rightarrow \|y\|_\infty = 1$$

$$(Ay)(i_0) = \sum_{j=1}^{n} a_{i_0 j} y_j = \sum_{j=1}^{n} |a_{i_0 j}| = \beta \quad \text{since}$$

$$\therefore \beta = |Ay(i_0)| \le \|Ay\|_\infty \le \|A\|_\infty \|y\|_\infty = \|A\|_\infty$$

So, $\|A\|_\infty = \beta$

②ⓐ $\|A\|_2 = \max_{x\ne\bar{0}} \dfrac{\|Ax\|_2}{\|x\|_2}$; $\|x\|_2 = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$

Consider,

$$\|Ax\|_2^2 = \sum_{i=1}^{n} (Ax(i))^2$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j\right)^2$$

$$\le \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^2\right)\left(\sum_{j=1}^{n} x_j^2\right) \quad \text{(Using Cauchy-Schwartz inequality)}$$

$$= \|A\|_F^2 \, \|x\|_2^2 .$$

where $\|A\|_F^2 = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right).$

(b) Given that Q is orthogonal matrix.

$$QQ^T = Q^T Q = I .$$

$$\langle x, y \rangle = y^T x = \sum_{j=1}^{n} x_j y_j$$

$$\Rightarrow \|x\|_2 = (x^T x)^{2}$$

Consider,

$$\|Qx\|_2^2 = \langle Qx, Qx \rangle$$
$$= (Qx)^T (Qx)$$
$$= (x^T Q^T)(Qx) = x^T x = \|x\|_2^2$$

For $x \neq \bar{0}$,

$$\frac{\|Qx\|_2}{\|x\|_2} = \frac{\|x\|_2}{\|x\|_2} = 1 \Rightarrow \|Q\|_2 = \max_{x \neq \bar{0}} \frac{\|Qx\|}{\|x\|} = 1$$

Also, $(Q^T)^T Q^T = QQ^T = I \Rightarrow Q^T$ is orthogonal.

$$\therefore \|Q^T\|_2 = \|Q^{-1}\|_2 = 1 .$$

$$\Rightarrow \kappa(Q) = \|Q\|_2 \|Q^{-1}\|_2 = 1$$

If $\|C\| < 1$, then $I + C$ is invertiable.

Proof: Consider, $(I + C)x = \bar{0}$

$\Rightarrow x = -Cx$

$\Rightarrow \|x\| \leq \|C\| \|x\|$

$\Rightarrow \|x\| = \bar{0}$, $I + C$ is 1-1. Hence proved.

(3) **Note:** If $\|\delta A\| \|A^{-1}\| < 1$, then $A + \delta A$ is invertiable.

Reason:

$$A + \delta A = A + \delta A I$$
$$= A + \delta A (A^{-1} A)$$
$$= (I + \delta A A^{-1}) A = (I + C) A$$

where $C = \delta A A^{-1}$

So, $\|C\| \leq \|\delta A\| \|A^{-1}\| < 1$ (Given).

$\therefore (I + C)$ and A are invertiable

$$\therefore (A + \delta A)^{-1} = A^{-1} (I + C)^{-1} .$$

see Jain, Iyengar, Jain pp137

Question 4

- $\rho(A) \leq \|A\|$ (Proved in class)

- $\lambda_{max} \leq \|A\|$

$$\frac{1}{\lambda_{min}} \leq \|A^{-1}\|$$

$$\chi(A) = \|A\| \, \|A^{-1}\| \geq \frac{\lambda_{max}}{\lambda_{min}}$$

- $A = \begin{bmatrix} 100 & -200 \\ -200 & 401 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$(100-\lambda)(401-\lambda) - 40000 = 0$$

$$\lambda^2 - 501\lambda + 100 = 0$$

$$\lambda = \frac{501 \pm \sqrt{501^2 - 400}}{2}$$

$$= 500.8003, \; 0.1997$$

$$K(A) \geqslant \frac{500.8003}{0.1997}$$

Question 5

Please see Theorem 3.1, pp 154 of Jain, Iyenger, Jain. sixth edi.

Question 6

Refer Jain, Iyenger, Jain pp. 154 or. Kincaid & Cheney (1991) pp 172 (or 198 for other version)

Question 7

Refer Kincaid & Cheney (1991) pp 174 (or 200)

Question 8

Ref. Therem 3.5 of Jain, Iyenger, Jain sixth edi.

Question 9

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 1 & 0 & 2 \\ 5 & 2 & -1 \end{bmatrix}$$

$$x_0 = [1, 1, 1]^T$$

$$\tilde{x}_0 = [1, 1, 1]^T$$

$$x_1 = A\tilde{x}_0 = \begin{bmatrix} 3 & 1 & 5 \\ 1 & 0 & 2 \\ 5 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix}$$

$$\lambda_{max} = \left(\frac{9}{1}, \frac{3}{1}, \frac{6}{1}\right) = (9, 3, 6)$$

$$\tilde{x}_1 = \frac{1}{\|x_1\|_\infty} x_1 = \frac{1}{9}\begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.3333 \\ 0.6667 \end{bmatrix}$$

$$x_2 = A\tilde{x}_1 = \begin{pmatrix} 6.6667 \\ 2.3333 \\ 5 \end{pmatrix}$$

$$\lambda_{max} = \left(\frac{6.6667}{1}, \frac{2.3333}{0.3333}, \frac{5}{0.6667}\right)$$

$$= (6.6667, 7.0000, 7.5000)$$

$$\tilde{x}_2 = \begin{pmatrix} 1 \\ 0.3500 \\ 0.7500 \end{pmatrix}$$

$$x_3 = A\tilde{x}_2 = \begin{pmatrix} 7.1000 \\ 2.5000 \\ 4.9500 \end{pmatrix}$$

$$\lambda_{max} = \left(\frac{7.1000}{1}, \frac{2.5000}{0.35}, \frac{4.9500}{0.75} \right)$$

$$= (7.1, 7.1429, 6.6)$$

$$\tilde{x}_3 = \begin{pmatrix} 1 \\ 0.3521 \\ 0.6972 \end{pmatrix}$$

$$x_4 = A\tilde{x}_3 = \begin{pmatrix} 6.8380 \\ 2.3944 \\ 5.0070 \end{pmatrix}$$

$$\lambda_{max} = \left(\frac{6.8380}{1}, \frac{2.3944}{0.3521}, \frac{5.0070}{0.6972} \right)$$

$$= (6.838, 6.8, 7.1818)$$

$\vdots$

Answer is 6.9911, $\begin{pmatrix} 1 \\ 0.3510 \\ 0.7180 \end{pmatrix}$