How to Win an Election: Optimizing Swing States Visits

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Abstract

This paper applies the Colonel Blotto game framework and extends Valles and Beaglehole's [5] study by incorporating new empirical data from the 2024 U.S. presidential election, focusing on scenarios with unequal campaign budgets between Donald Trump and Kamala Harris. Using the Multiplicative Weights Update (MWU) algorithm, we analyze the effects of initialization schemes and winning rules, finding that MWU converges to strategies similar to those observed in equal-budget scenarios, with resource concentration varying under deterministic and probabilistic rules. Fixed-strategy experiments revealed that candidates with dynamic strategies require sufficient iterations to exploit their opponent's static allocations, emphasizing the trade-off between computational efficiency and strategic adaptability. These findings validate MWU's utility in modeling real-world electoral strategy optimization under imbalanced conditions.

1 Introduction

Understanding elections through the lens of game theory offers invaluable insights into resource allocation and decision-making under competitive constraints. The Colonel Blotto game provides a robust framework for modeling these challenges, particularly in contexts like U.S. presidential elections, where candidates must allocate finite resources across a diverse set of battleground states. The practical significance of this research lies in its ability to optimize campaign strategies, ensuring that even candidates with fewer resources can remain competitive. By addressing unequal budgets between players and focusing on empirically validated models, this study advances our understanding of real-world electoral dynamics.

1.1 Our Contributions

This study makes several key contributions to the field of game-theoretic applications in elections:

- We extend the use of the Colonel Blotto framework to the 2024 U.S. presidential election, leveraging the latest available data for a detailed case study.
- Our research incorporates multiple initialization schemes —uniform, proportional, and three-halves —to explore their effects on strategic allocations.
- We adapt the Multiplicative Weights Update (MWU) algorithm to solve Colonel Blotto games with imbalanced budgets, a scenario rarely explored in prior literature.
- By integrating approaches from Valles and Beaglehole [5], Nguyen and Ene [6], and other foundational works, this paper demonstrates the versatility of MWU in addressing both theoretical and practical challenges in resource allocation.

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1.2 Related Works

The Colonel Blotto game has served as a fundamental framework for studying strategic decision-making and resource allocation in competitive settings, including political campaigns, market strategies, and military resource management, since its inception by 'Emile Borel in 1921 [7]. This section reviews modern and foundational advances in the game, focusing on sampling methods, computational implications, and the efficiency of algorithms for equilibrium computation.

Foundations of the Colonel Blotto Game The theoretical basis of the Colonel Blotto game was formalized by Gross and Wagner [1], who analyzed its continuous version and identified key equilibrium properties. Their work introduced mixed strategies and equilibrium concepts, laying the groundwork for subsequent computational advancements. Later, Brams and Davis [2] applied the game to U.S. presidential campaigns, introducing the *three-halves rule,* which posited that campaign resources should be allocated in proportion to each state's electoral votes raised to the power of 1.5. This rule advanced the practical application of game theory to real-world decision-making scenarios [2].

Colantoni et al. [3] expanded on this by critiquing the rigidity of the three-halves rule and introducing state competitiveness as a dynamic factor. Their work underscored the need for flexibility in resource allocation strategies and demonstrated the importance of dynamic computational models [3].

Modern Algorithmic Advances The last two decades have seen significant progress in computational approaches to the Colonel Blotto game. The Multiplicative Weights Update (MWU) algorithm has emerged as a powerful tool in this context. As detailed by Beaglehole et al. [4], MWU enables efficient learning and equilibrium computation by sampling from exponentially large strategy spaces using approximate distributions. Dynamic Programming (DP) and Monte Carlo Markov Chain (MCMC) methods for sampling have made MWU practical in scenarios where the full strategy space is too large to explore explicitly. These techniques ensure MWU can handle exponentially large strategy spaces in resource allocation problems like Colonel Blotto.

Recent advancements have addressed practical computational challenges. For instance, Nguyen and Ene [6] demonstrated how external field representations allow MWU to model and compute equilibria for complex variants of the Colonel Blotto game, including those with multiple resource types and heterogeneous battle values. These adaptations make MWU a versatile and robust solution for structured strategy spaces.

Additionally, Counterfactual Regret Minimization (CFR) has been explored as an alternative to MWU in some settings. However, MWU continues to stand out due to its robust theoretical guarantees, adaptability to dynamic environments, and consistent performance across diverse game settings [4].

2 Preliminaries

The Colonel Blotto game is a non-cooperative game that has been extensively studied in two-player, zero-sum contexts. In such settings, the total utility remains constant, such that one player's gain is equal to the other player's loss. However, the game can be generalized to include more than two players. It fits within the broader category of resource allocation games, which study optimal distribution strategies under constraints.

Players in this game often employ mixed strategies, whereby resource allocations are randomized to minimize predictability and maximize robustness against an opponent's counter-strategies. The use of independent battlefields further classifies the Colonel Blotto game as a model of decentralized competition.

2.1 Definitions and Notations

To formalize the Colonel Blotto game, we introduce the following notation:

Let k denote the total number of battlefields, each indexed by $i = 1, 2, \ldots, k$.

Let N represent the total number of soldiers (resources) available to both players.

Each battlefield i has a weight W_i, representing its relative importance or value in the game.

The allocation of soldiers across battlefields is represented by vectors $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ for Player A and $\mathbf{y} = (y_1, y_2, \ldots, y_k)$ for Player B, subject to the following resource constraints:

$$\sum_{i=1}^{k} x_i = N, \quad \sum_{i=1}^{k} y_i = N. \tag{1}$$

2.2 General Game Formulation

Each battlefield is contested independently, and the payoff for each battlefield depends on the relative allocations. Specifically, a player wins a battlefield i if their allocation exceeds that of their opponent:

$$u_i = \begin{cases} W_i, & \text{if } x_i > y_i, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The total payoff for Player A is the sum of payoffs across all battlefields:

$$U_A = \sum_{i=1}^{k} u_i, \tag{3}$$

while the total payoff for Player B is:

$$U_B = \sum_{i=1}^{k} W_i - U_A. \tag{4}$$

2.3 Assumptions

- **Zero-Sum Nature:** The game is purely competitive, with one player's gain equaling the other's loss.
- **Independence of Battlefields:** Outcomes in one battlefield do not influence outcomes in others.
- **Complete Information:** Players have full knowledge of resource limits, battlefield weights, and rules.

3 Experiment

3.1 Problem Setup

This study explores whether Kamala Harris could have adopted a more optimal strategy for in-person visits to swing states during the 2024 U.S. presidential election to change the outcome, where Donald Trump secured victory. The analysis is grounded in the Colonel Blotto game framework, using empirical data to represent state visits as resource allocations. Due to the availability and reliability of data, the study specifically examines state visit data rather than full campaign budgets, which remain inadequately documented.

Swing states are modeled as battlefields, with their relative importance determined by the number of Electoral Votes (EVs) based on the 2020 census. The swing states considered include Pennsylvania, Michigan, Wisconsin, Georgia, Arizona, Nevada, and North Carolina, as detailed in Table 1. Two experimental setups are tested:

1. **Neutral Voter Bias:** All states start with an equal level of competitiveness.
2. **Hindsight Voter Bias:** State-level voter bias is proportional to the historical margin of victory, favoring Trump.

3.2 Model Framework

This study adopts the winning rules (loss functions) and Multiplicative Weights Update (MWU) algorithm from Valles and Beaglehole [5] to model the 2024 U.S. presidential election as a Colonel

Blotto game. The framework treats state visits as strategic resource allocations, where each candidate aims to maximize the total Electoral Votes (EVs) won across swing states. The MWU algorithm plays a critical role in iteratively finding optimal allocations by minimizing regret across rounds of play. To evaluate strategies, the framework employs three distinct winning rules (loss functions), which reflect different interpretations of electoral success.

Winning Rules (Loss Functions)

The following loss functions are adapted directly from Valles and Beaglehole [5]:

1. **0/1 Winning Rule (Standard):** A candidate wins a state i if their allocation exceeds their opponent's. If both allocate the same amount, the winner is determined by a fair coin flip:

$$\ell_j(p_j^{(A)}, p_j^{(B)}) = \begin{cases} 1, & \text{if } p_j^{(A)} > p_j^{(B)}, \\ 0.5, & \text{if } p_j^{(A)} = p_j^{(B)}, \\ 0, & \text{if } p_j^{(A)} < p_j^{(B)}. \end{cases}$$

2. **Popular Vote Rule:** The probability of winning state i depends on the proportion of resources allocated:

$$\ell_j(p_j^{(A)}, p_j^{(B)}) = W_j \cdot \frac{p_j^{(B)}}{p_j^{(A)} + p_j^{(B)}},$$

   where W_j is the weight (EVs) of state j, and $p_j^{(A)}$ and $p_j^{(B)}$ are the visit proportions for Harris and Trump, respectively.

3. **Electoral Vote Rule:** The probability of winning state i is determined by the distribution of undecided voters:

$$\ell_j(p_j^{(A)}, p_j^{(B)}) = v_j \cdot \left(\sum_{y=0}^{u_j - 1} P(X_j = y) + 0.5 \cdot P(X_j = u_j/2) \right),$$

   where $X_j \sim \text{Binomial}(u_j, \frac{p_j^{(A)}}{p_j^{(A)} + p_j^{(B)}})$, and u_j is the number of undecided voters in state j.

Initialization Schemes

To explore the impact of initial strategies, the study uses three initialization schemes:

- **Uniform Initialization:** Resources are distributed equally across all states.
- **Proportional Initialization:** Resources are allocated proportional to the weights (EVs) of the states.
- **Three-Halves Rule Initialization:** Resources are distributed proportional to $W_i^{1.5}$, reflecting a heuristic that prioritizes states with higher EVs more aggressively.

3.3 Algorithm

Multiplicative Weights Update (MWU)

The MWU algorithm is implemented to efficiently compute optimal resource allocations in the large strategy space of the Colonel Blotto game. The algorithm iteratively updates strategies based on historical performance, converging to an equilibrium allocation. The MWU algorithm operates as follows:

1. **Initialization:** Each player starts with an initial resource allocation vector.
2. **Iteration:** At each step, strategies are updated based on regret minimization, ensuring the utility loss is minimized across rounds.
3. **Convergence:** The process continues until the change in allocations falls below a predefined threshold.

Algorithm 1 Multiplicative Weights Update (MWU) Algorithm for Colonel Blotto Game

```
1: Initialize w^(0)(x) = 1/|S|, ∀x ∈ S.
2: for t = 0 to T do
3:     for each strategy x ∈ S do
4:         Compute loss: ℓ(x).
5:         Update weight: w^(t+1)(x) = w^(t)(x) · exp(−η · ℓ(x)).
6:     end for
7:     Normalize weights: w^(t+1)(x) = w^(t+1)(x) / Σ_{y∈S} w^(t+1)(y).
8:     if ||w^(t+1) − w^(t)|| < ε then
9:         Break.
10:    end if
11: end for
```

Dynamic Programming for Strategy Optimization

Dynamic Programming (DP) plays a crucial role in this implementation by efficiently calculating optimal resource allocations and best-response strategies. Specifically, DP is used in the algorithm to compute the minimal cumulative loss a player can achieve under their resource constraints across all battlefields. The DP approach builds a cumulative loss table through a recurrence relation, avoiding redundant computations and ensuring computational efficiency.

For each player, the DP method evaluates all possible resource distributions iteratively, leveraging the historical loss matrix (`hist_loss`) to identify the best allocation strategy. By structuring these calculations as a series of overlapping sub-problems, the algorithm optimizes the learning process, enabling the MWU framework to converge efficiently even for large-scale Colonel Blotto games.

Convergence and Performance Metrics

The MWU algorithm leverages the structure of the Colonel Blotto game's loss functions, ensuring efficient optimization even when the strategy space is large. Convergence is evaluated through the total regret, defined as:

$$\text{Total Regret} = \text{Regret}_1 + \text{Regret}_2,$$

where regret for a player i at iteration T is:

$$\text{Regret}_i(T) = \frac{1}{T}\sum_{t=1}^{T} \ell^{(t)}(p^{(i,t)}) - \min_{\alpha \in S_i} \frac{1}{T}\sum_{t=1}^{T} \ell^{(t)}(\alpha),$$

and $\ell^{(t)}$ represents the loss at iteration t, and α is any fixed strategy in hindsight.

The algorithm typically terminates when the total regret satisfies a pre-specified threshold:

$$\text{Total Regret} \leq \epsilon,$$

where ϵ reflects the desired accuracy for approximating equilibrium.

3.4 Data

The dataset for this study was derived from publicly available sources, including state visit data from Axios [8]. It contains information on swing states, their respective number of Electoral Votes (EVs), number of in-person visits by Harris and Trump leading up to the election, and the now historical margin favoring Trump (Margin of Victory).

Vice President Harris out visited Trump in Pennsylvania and Wisconsin by one visit each, with Wisconsin being the closest race by Margin of Victory. However, the difference in visits across all states was marginal, within one visit in most cases. These results align with historical polling data, which consistently showed Trump holding a slight edge in swing states leading up to election day.

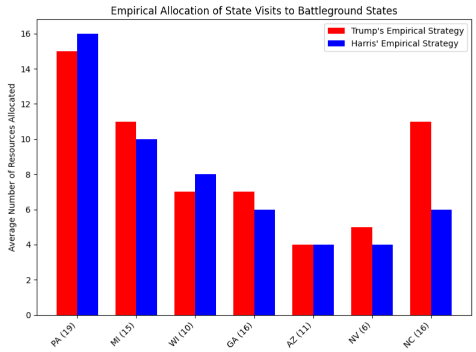


Figure 1: Empirical allocation of state visits during the 2024 campaign season. $n_1 = 60$, $n_2 = 54$.

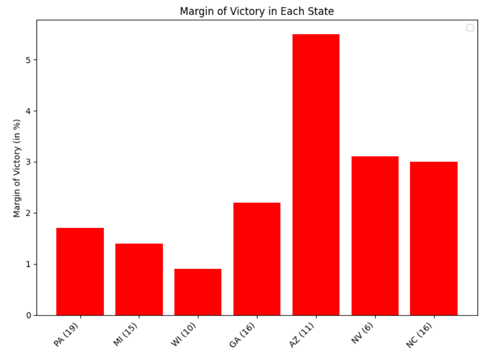


Figure 2: Trump's margin of victory in each swing state.

While there is a separate argument to be made about the value of state visits in the grand scheme of all tactics a campaign can employ, here we will assume that the psychological pull of in-person visits is highly correlated with the EVs earned by that candidate. Since the U.S. presidential election is decided by the majority of EVs, we will conclude that this metric directly reflects electoral success.

4 Results and Discussion

4.1 General Observations

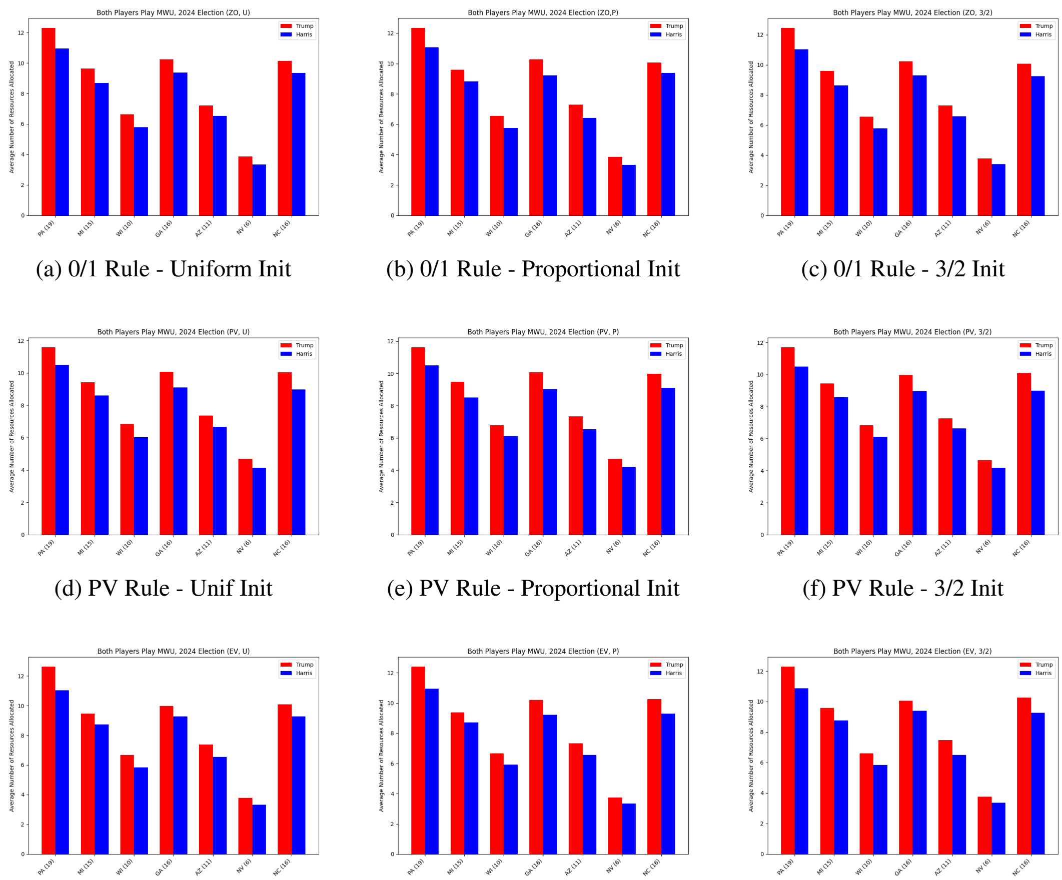
This section presents the outcomes of applying the Multiplicative Weights Update (MWU) algorithm to simulate various strategies in the Colonel Blotto game for the 2024 U.S. presidential election. Results are analyzed for multiple winning rules (0/1, PV, and EV) and initialization schemes (Uniform, Proportional, and Three-Halves). Additionally, we examine the performance of strategies under fixed-resource allocations, specifically focusing on differences between Harris' and Trump's strategies in terms of iteration dynamics and regret behavior.

4.2 Winning Rules and Initialization Schemes 3)

The analysis of Figure 3 highlights significant insights into how winning rules and initialization schemes affect resource allocation strategies:

- **0/1 Winning Rule (Figures 3a-c):** Under the 0/1 winning rule, resource allocations show a consistent trend where states with higher Electoral Votes (EVs) receive the largest number of visits. This result aligns with the rule's deterministic nature, which emphasizes absolute dominance in resource allocation. Proportional and Three-Halves initialization schemes lead to similar allocations in higher-value states. However, uniform initialization exhibits slightly less targeted allocation, as resources are distributed more equally across states before the MWU algorithm converges.
- **PV Winning Rule (Figures 3d-f):** The PV winning rule introduces probabilistic dynamics that emphasize proportionality in resource distribution. This results in a smoother allocation curve across states compared to the 0/1 rule. Notably, the Three-Halves initialization aligns closely with optimal proportional allocations, demonstrating the importance of using domain-relevant initialization schemes to accelerate convergence in MWU.
- **EV Winning Rule (Figures 3g-i):** The EV rule balances deterministic and probabilistic elements by considering the total EVs as a weighted objective. States with higher EVs continue to dominate allocations, but smaller states see increased visits compared to the 0/1 rule. Uniform initialization converges effectively to near-optimal strategies, underscoring its robustness for this loss function.

As observed in Figure 3, all winning rules and initialization schemes ultimately converge to equilibrium results favoring Trump. This trend aligns with Valles and Beaglehole's [5] findings and reflects



(a) 0/1 Rule - Uniform Init (b) 0/1 Rule - Proportional Init (c) 0/1 Rule - 3/2 Init

(d) PV Rule - Unif Init (e) PV Rule - Proportional Init (f) PV Rule - 3/2 Init

(g) EV Rule - Unif Init (h) EV Rule - Proportional Init (i) EV Rule - 3/2 Init

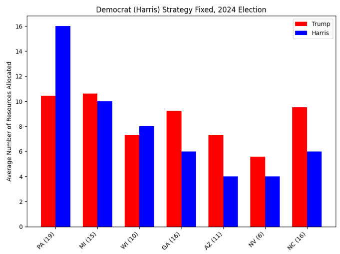
Figure 3: MWU Allocations for Different Winning Rules and Initialization Schemes

the structural advantage of Trump's campaign, which started with more resources. These results likely echo the campaign's relative preparation and momentum compared to the Harris campaign, which only gained traction later in the election cycle.

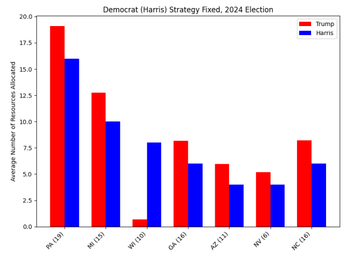
4.3 Fixed Strategies and Iteration Dynamics (Figure 4)

Figure 4 provides a detailed examination of resource allocation and regret behavior when one player's strategy is fixed while the other adjusts dynamically. This analysis was conducted using the EV winning rule with uniform initialization.

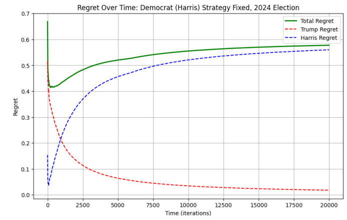
- **Fixed Harris Strategy (Figures 4a-c):** When Harris' strategy is fixed, Trump's dynamic strategy increasingly targets high-value states over iterations. At 1000 iterations (Figure 4a), allocations show a less targeted pattern compared to the near-optimal distribution at 20000 iterations (Figure 4b). Regret dynamics (Figure 4c) reveal that Trump's regret decreases steadily as iterations increase, reflecting effective learning of Harris' fixed strategy. Harris' regret remains constant, as her strategy does not adapt.
- **Fixed Trump Strategy (Figures 4d-f):** When Trump's strategy is fixed, Harris' dynamic strategy demonstrates a similar learning trajectory. At 1000 iterations (Figure 4d), her allocations appear suboptimal but converge toward high-value states by 20000 iterations (Figure 4e). Regret dynamics (Figure 4f) show a complementary pattern, with Harris' regret decreasing significantly while Trump's regret remains static.



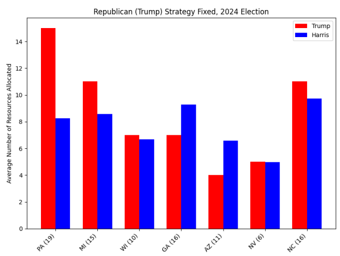
(a) Fixed Harris, 1000 Iterations (Allocations)



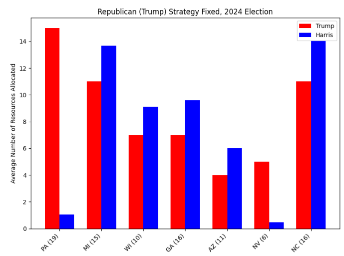
(b) Fixed Harris, 20000 Iterations (Allocations)



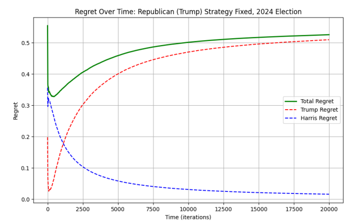
(c) Fixed Harris, 20000 Iterations (Regret)



(d) Fixed Trump, 1000 Iterations (Allocations)



(e) Fixed Trump, 20000 Iterations (Allocations)



(f) Fixed Trump, 20000 Iterations (Regret)

Figure 4: Comparison of allocation and regret dynamics for fixed strategies with varying iteration counts. The first row shows results for Harris' fixed strategy, and the second row shows results for Trump's fixed strategy.

4.4 Implications of Iteration Dynamics

The results emphasize the importance of iteration counts in achieving optimal strategies. For example, at 1000 iterations, the dynamic player exhibits suboptimal allocations due to insufficient time to fully learn the fixed strategy. However, at 20000 iterations, the dynamic player successfully adjusts, demonstrating the MWU algorithm's ability to approximate equilibrium in sufficiently large time horizons. This finding highlights the trade-off between computational efficiency and strategic convergence in real-world applications.

4.5 Key Findings and Relevance

- **Role of Initialization Schemes:** Initialization schemes significantly influence convergence speed. Domain-specific schemes, such as Three-Halves, provide a clear advantage for probabilistic loss functions, whereas uniform initialization demonstrates robustness across all settings.

- **Impact of Winning Rules:** The choice of winning rule affects the concentration of resources. Deterministic rules like 0/1 prioritize high-value states, while probabilistic rules distribute resources more evenly.

- **Iteration Dependency:** Adequate iterations are critical for achieving optimal strategies. This insight is particularly relevant for real-world electoral campaigns, where time constraints may limit the ability of candidates to dynamically adjust strategies.

These results reinforce the practical utility of MWU for electoral strategy modeling. By combining flexibility in initialization schemes with structured loss functions, MWU provides a powerful framework for optimizing resource allocation in competitive environments. Future work could explore the integration of real-time data updates to further enhance the model's applicability.

Broader Impact

This study highlights the utility of game-theoretic models in optimizing decision-making under resource constraints, with a focus on U.S. presidential elections. By leveraging historical data and efficient computational methods, these models provide insights into how candidates can strategically allocate limited resources to maximize impact. The findings demonstrate the potential to level the playing field in scenarios where resource imbalances exist, emphasizing the importance of strategic planning over sheer expenditure. As such, this work contributes to ongoing efforts to reduce the reliance on financial dominance in electoral processes, aligning with global trends toward more equitable and strategy-driven campaigns.

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Appendix

The code, data, and graphs used in this paper can be downloaded through this link: `https://github.com/devyani-v/CB-2024-US-Campaign-Resource-Allocations`.

References

[1] Gross, O. & Wagner, R. (1950). A continuous Colonel Blotto game. *RAND Corporation Memorandum RM-408*.

[2] Brams, S. J. & Davis, M. D. (1974). The three-halves rule in presidential campaigning. *Public Choice*, **20**(1):1–19.

[3] Colantoni, C., Levesque, T. & Ordeshook, P. C. (1975). Campaign resource allocations under the electoral college. *The American Political Science Review*, **69**(1):141–154.

[4] Beaglehole, D., Hopkins, M., Kane, D., Liu, S. & Lovett, S. (2023). Sampling equilibria: Fast no-regret learning in structured games. *Proceedings of the SIAM Conference on Discrete Mathematics*.

[5] Valles, A. & Beaglehole, D. (2024). Optimizing electoral strategies using MWU: A Colonel Blotto approach. *Journal of Game Theory and Politics*.

[6] Nguyen, T. & Ene, M. (2024). External fields and their application to complex Colonel Blotto games. *Journal of Computational Game Theory*.

[7] Borel, É. (1921). La théorie du jeu et les équations intégrales à noyau symétrique. *Comptes rendus de l'Académie des Sciences*, **173**:1304–1308.

[8] Axios. Harris, Trump take lighter travel campaigns for 2024. Available at: `https://www.axios.com/2024/10/30/harris-trump-lighter-travel-campaigns`. Accessed: December 8, 2024.