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CS 35 Lab 6

1 Inductive Proofs

Prove each of the following claims by induction

Claim 1. *The sum of the first n odd numbers is n^2. That is,* $\sum_{i=1}^{n}(2i-1) = n^2$

$P(n) : \sum_{i=1}^{n}(2i-1) = n^2$
Let us examine the claim for n=1 or P(1):
$\sum_{i=1}^{1}(2i-1) = 1^2$
$(2*1-1) = 1^2$
$1 = 1$
LHS = RHS
The claim is true for n=1, P(1) is true
Assume true for the case of n=k, or P(k) $\sum_{i=1}^{k}(2i-1) = k^2$
Now let us examine if the proposition holds true for k+1
Prove P(k+1) is true:
$\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$
LHS = $\sum_{i=1}^{k}(2i-1) + 2(k+1) - 1$
$= k^2 + 2k + 2 - 1$
$= k^2 + 2k + 1$
$= (k+1)^2 =$ RHS
LHS = RHS
P(k+1) is true when P(k) is true
Therefore P(n) is true for all $n \geq 1$ by principle of mathematical induction

Claim 2. $\sum_{i=1}^{n}\frac{1}{2^i} = 1 - \frac{1}{2^n}$

$P(n) : \sum_{i=1}^{n}\frac{1}{2^i} = 1 - \frac{1}{2^n}$
Let us examine the claim for $n = 1 or P(1) :$
$\sum_{i=1}^{1}\frac{1}{2^i} = 1 - \frac{1}{2^1}$
$= \frac{1}{2} = 1 - \frac{1}{2}$
The claim is true for $n = 1, P(1)$ is true
Assume true for the case of $n = k, or P(k)$

$\sum\limits_{i=1}^{k} \frac{1}{2^i} = 1 - \frac{1}{2^k}$
Now let us examine if the proposition holds true for k+1
Prove P(k+1) is true:
LHS $= \sum\limits_{i=1}^{k+1} \frac{1}{2^i}$
$= \sum\limits_{i=1}^{k} \frac{1}{2^i} + \frac{1}{2^{k+1}}$
$= 1 - \frac{1}{2^k} + \frac{1}{2 * 2^k}$
$= 1 - \frac{1}{2^k} + \frac{1}{2}\frac{1}{2^k}$
$= 1 - \frac{1}{2^k}(1 - \frac{1}{2})$
$= 1 - \frac{1}{2^k} * \frac{1}{2}$
$= 1 - \frac{1}{2^{k+1}} =$ RHS
LHS = RHS
P(k+1) is true when P(k) is true
Therefore P(n) is true for all $n \geq 1$ by principle of mathematical induction

Claim 3. $\sum\limits_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$P(n) : \sum\limits_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
Let us examine the claim for n=1 or P(1):
$\sum\limits_{i=1}^{1} i^2 = \frac{1(1+1)(2*1+1)}{6}$
$LHS = 1$
$RHS = 1 = LHS$
The claim is true for n=1, P(1) is true
Assume true for the case of n=k, or P(k)
$\sum\limits_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$
Now let us examine if the proposition holds true for k+1
Prove P(k+1) is true:
$P(k+1) : \sum\limits_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$
$LHS = \sum\limits_{i=1}^{k+1} i^2$
$= (k+1)^2 + \sum\limits_{i=1}^{k} i^2$
$= (k+1)^2 + \frac{k(k+1)(2k+1)}{6}$
Expanding this, we get:

$\dfrac{2k^3 + 8k^2 + 13k + 6}{6}$

$RHS = \dfrac{(k+1)(k+2)(2(k+1)+1)}{6}$

$= \dfrac{(k+1)(k+2)(2k+3)}{6}$

Expanding this, we get:

$\dfrac{2k^3 + 8k^2 + 13k + 6}{6} = LHS$

LHS = RHS

P(k+1) is true when P(k) is true

Therefore P(n) is true for all $n \geq 1$ by principle of mathematical induction

2 Recursive Invariants

The function `minEven`, given below in pseudocode, takes as input an array A of size n of numbers. It returns the smallest *even* number in the array. If no even numbers appear in the array, it returns positive infinity ($+\infty$). Using induction, prove that the `minEven` function works correctly. Clearly state your recursive invariant at the beginning of your proof.

```
Function minEven(A,n)
  If n = 0 Then
    Return +
  Else
    Set best To minEven(A,n-1)
    If A[n-1] < best And A[n-1] is even Then
      Set best To A[n-1]
    EndIf
    Return best
  EndIf
EndFunction
```

Let us examine the function `minEven` for an array A of size 0

In this case `minEven(A,0)` will return $+\infty$

Hence the function `minEven` works correctly for $n = 0$

This is the recursive invariant $P(0)$

Let us assume that the `minEven` function works correctly for an array `A` of size `k` and returns `m`

This is $P(k)$ and is assumed to be true.

We have to prove that $P(K+1)$ is true given $P(K)$ is true.

Assume that one more element `E` is added to the array `A` increasing its size to `k+1`

This element `E` will be added in the position `A[n-1]`
`E`'s value can have three cases:
Case 1: `E` is odd
Case 2: `E` is Even and `E` $\geq$ `m`
Case 3: `E` is Even and `E` $<$ `m`
Let us now run `minEven(A,K+1)`:
$k > 0$, so we go to the `Else` branch

```
Set best To minEven(A,n-1): best = m

If A[n-1] < best And A[n-1] is even Then - Case 3
  Set best To A[n-1]: best = E
EndIf
Return best : return m for Case 1 and Case 2, return k for Case 3
```

So `minEven` works correctly for array of size k+1 if it works for correctly for size k

for all cases of the added element /texttt E
Hence $P(K + 1)$ is true given $P(K)$ is true.
Hence this function works correctly for all arrays of integers
using the principle of mathematical induction.