Assignment 9.

1. (1). Step ① $A-\lambda I = \begin{bmatrix}1&4\\2&3\end{bmatrix} - \begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix} = \begin{bmatrix}1-\lambda&4\\2&3-\lambda\end{bmatrix}$

Step ② $\mathrm{Det}\begin{bmatrix}1-\lambda&4\\2&3-\lambda\end{bmatrix} = (1-\lambda)(3-\lambda)-(2)(4)$

$= \lambda^2-4\lambda-5$

Step ③ $\Rightarrow (\lambda-5)(\lambda+1)=0.$

$\therefore \lambda_1=5. \quad \lambda_2=-1. \Rightarrow$ Eigenvalues.

Step ④. for $\lambda_1=5$ $\begin{bmatrix}1-\lambda&4\\2&3-\lambda\end{bmatrix} \to \begin{bmatrix}1-5&4\\2&3-5\end{bmatrix} \to \begin{bmatrix}-4&4\\2&-2\end{bmatrix}$ Let it be B.

solve $B\bar{x}=\bar{0}$

i.e. $\begin{bmatrix}-4&4\\2&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \to \left[\begin{array}{cc|c}-4&4&0\\2&-2&0\end{array}\right] \xrightarrow{\times(-\frac{1}{4})} \left[\begin{array}{cc|c}1&-1&0\\2&-2&0\end{array}\right] \xrightarrow{\times(-2)} \left[\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right]$

$\therefore x_1+(-x_2)=0.$ Let $x_1=1$, then $\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = v_1.$

for $\lambda_2=-1.$ $\begin{bmatrix}1-\lambda&4\\2&3-\lambda\end{bmatrix} \Rightarrow \begin{bmatrix}2&4\\2&4\end{bmatrix}$. ~~let it be~~

solve $\begin{bmatrix}2&4\\2&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \to \left[\begin{array}{cc|c}2&4&0\\2&4&0\end{array}\right] \xrightarrow{\times\frac{1}{2}} \left[\begin{array}{cc|c}1&2&0\\2&4&0\end{array}\right] \xrightarrow{\times(-2)} \left[\begin{array}{cc|c}1&2&0\\0&0&0\end{array}\right]$

$\therefore x_1+2x_2=0.$ let $x_2=1$, $v_2=\begin{pmatrix}-2\\1\end{pmatrix}$

(2). Let $A+I = \begin{bmatrix}2&4\\2&4\end{bmatrix} = C.$

Step ① $C-\lambda I = \begin{bmatrix}2&4\\2&4\end{bmatrix} - \begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix} = \begin{bmatrix}2-\lambda&4\\2&4-\lambda\end{bmatrix}$

Step ② $\mathrm{Det}\begin{bmatrix}2-\lambda&4\\2&4-\lambda\end{bmatrix} = (2-\lambda)(4-\lambda)-(2)(4)$

$= \lambda^2-6\lambda$

Step ③ $\Rightarrow \lambda(\lambda-6)=0.$

$\therefore \lambda_1=0. \quad \lambda_2=6.$

Step ④ Repeat the process as above, $\lambda_1=0, \quad v_1=\begin{pmatrix}-2\\1\end{pmatrix}$

$\lambda_2=6, \quad v_2=\begin{pmatrix}1\\1\end{pmatrix}$

2. (1) $\det(A-\lambda I) = \det\begin{bmatrix}3-\lambda & 0\\ 1 & 1-\lambda\end{bmatrix} = \lambda^2-4\lambda+3 = (\lambda-1)(\lambda-3)=0.$

$\therefore \lambda_1=1. \quad \lambda_2=3.$

For $\lambda_1=1$. $\quad A-\lambda I = \begin{bmatrix}3-1 & 0\\ 1 & 1-1\end{bmatrix} = \begin{bmatrix}2 & 0\\ 1 & 0\end{bmatrix}$

$\therefore \left[\begin{array}{cc|c}2&0&0\\1&0&0\end{array}\right] \overset{\times 1/2}{=} \left[\begin{array}{cc|c}1&0&0\\1&0&0\end{array}\right] \overset{\times(-1)}{=} \left[\begin{array}{cc|c}1&0&0\\0&0&0\end{array}\right]$

$\therefore x_1=0.$ $\quad$ let $x_2=1$. $\quad v_1=\begin{pmatrix}0\\1\end{pmatrix}$

For $\lambda_2=3$. $\quad A-\lambda I = \begin{bmatrix}3-3 & 0\\ 1 & 1-3\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & -2\end{bmatrix}.$

$\therefore \left[\begin{array}{cc|c}0&0&0\\1&-2&0\end{array}\right] \overset{\text{swap}}{=} \left[\begin{array}{cc|c}1&-2&0\\0&0&0\end{array}\right]$

$\therefore x_1+(-2)x_2=0.$ $\quad$ let $x_2=1$, $v_2=\begin{pmatrix}2\\1\end{pmatrix}.$

(2) $\det(B-\lambda I) = \det\begin{bmatrix}1-\lambda & 1\\ 0 & 3-\lambda\end{bmatrix} = (1-\lambda)(3-\lambda)-0 = \lambda^2-4\lambda+3$
$=(\lambda-1)(\lambda-3)=0.$

$\therefore \lambda_1=1, \quad \lambda_2=3.$

For $\lambda_1=1$. $\quad A-\lambda I = \begin{bmatrix}1-1 & 1\\ 0 & 3-1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}$

$\therefore \left[\begin{array}{cc|c}0&1&0\\0&2&0\end{array}\right] \overset{\times(-2)}{=} \left[\begin{array}{cc|c}0&1&0\\0&0&0\end{array}\right].$

$\therefore x_2=0.$ $\quad$ let $x_1=1$. $\quad v_1=\begin{pmatrix}1\\0\end{pmatrix}.$

For $\lambda_2=3$. $\quad A-\lambda I = \begin{bmatrix}1-3 & 1\\ 0 & 3-3\end{bmatrix} = \begin{bmatrix}-2 & 1\\ 0 & 0\end{bmatrix}.$

$\therefore x_1\cdot(-2)+x_2=0.$ $\quad$ let $x_2=1$ $\quad v_2=\begin{pmatrix}1/2\\1\end{pmatrix}.$

3. Rewrite the matrix $P=\begin{bmatrix}1/5 & 2/5 & 0\\ 2/5 & 4/5 & 0\\ 0 & 0 & 1\end{bmatrix}$ $\quad P-\lambda I=\begin{bmatrix}1/5-\lambda & 2/5 & 0\\ 2/5 & 4/5-\lambda & 0\\ 0 & 0 & 1-\lambda\end{bmatrix}$

For $\lambda_1=0$, $\left[\begin{array}{ccc|c}1/5&2/5&0&0\\2/5&4/5&0&0\\0&0&1&0\end{array}\right] \xrightarrow{\times 5} \left[\begin{array}{ccc|c}1&2&0&0\\2/5&4/5&0&0\\0&0&1&0\end{array}\right] \xrightarrow{\times(-\frac{2}{5})} \left[\begin{array}{ccc|c}1&2&0&0\\0&0&0&0\\0&0&1&0\end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c}1&2&0&0\\0&0&1&0\\0&0&0&0\end{array}\right]$

$\therefore x_1+2x_2=0.$ $\quad x_3=0.$ $\quad \left\{x_2\cdot\begin{pmatrix}-2\\1\\0\end{pmatrix}\right\}.$

let $x_2=1$. $\quad V_1=\begin{pmatrix}-2\\1\\0\end{pmatrix}$

For $\lambda_2 = 1$. $\begin{bmatrix} -4/5 & 2/5 & 0 \\ 2/5 & -1/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\times(-\frac{5}{4})} \begin{bmatrix} 1 & -1/2 & 0 \\ 2/5 & -1/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\times(-\frac{2}{5})} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore x_1 + (-\frac{1}{2}) x_2 = 0.$

$x_2 = x_2.$

$x_3 = x_3.$

$X = \begin{bmatrix} 1/2\, x_2 \\ x_2 \\ x_3 \end{bmatrix} \quad x_2 \cdot \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

let $x_2 = 1$. $x_3 = 0$. $v_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$ ✓

let $x_2 = 0$. $x_3 = 1$. $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ✓

3

$v_1, v_2, v_3 \quad B = \begin{bmatrix} -2 & \frac{1}{2} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

$\therefore$ There're linearly independent.

4. $\det(P - \lambda I) = \begin{bmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 1 & 0 & 0-\lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix}$

$= (-\lambda)(-1)^{1+1} \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} - 1(-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 1 & -\lambda \end{vmatrix} + 0(-1)^{1+3} \begin{vmatrix} 0 & -\lambda \\ 1 & 0 \end{vmatrix}$

$= (-\lambda)[(-\lambda)(-\lambda) - (0 \times 1)] - (-1)[0 \times (-\lambda) + 1 \times 1] + 0$

$= -\lambda(\lambda^2 + 0) - (-1) = -\lambda \cdot \lambda^2 + 1 = 0$

namely $-\lambda^3 + 1 = 0.$

$(\lambda - 1)(\lambda^2 + \lambda + 1) = 0.$

$\therefore \lambda_1 = 1$

$\lambda_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2ac} = \dfrac{-1 \pm \sqrt{3}}{2} \Rightarrow$ complex −

$\therefore \lambda = 1.$ ✓

7