Assignment 11.

1. Find eigenvectors.

$$\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 6 \\ 6 & 7-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 50 = 0$$
$$\Rightarrow (\lambda+5)(\lambda-10) = 0$$

$\therefore \lambda_1 = -5, \ \lambda_2 = 10$

For $\lambda_1 = -5$. $A - \lambda I = \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}$

$$\left[\begin{array}{cc|c} 3 & 6 & 0 \\ 6 & 12 & 0 \end{array}\right] \xrightarrow{\times 1/3} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 6 & 12 & 0 \end{array}\right] \xrightarrow{\times(-6)} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

$\therefore x_1 + 2x_2 = 0 \quad \vec{v_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$

For $\lambda_2 = 10$. $A - \lambda I = \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -12 & 6 & 0 \\ 6 & -3 & 0 \end{array}\right] \xrightarrow{\times(-\frac{1}{12})} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 6 & -3 & 0 \end{array}\right] \xrightarrow{\times(-6)} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore x_1 + (-\frac{1}{2})x_2 = 0. \quad \vec{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\|\vec{v_1}\| = \sqrt{(-2)^2+1^2} = \sqrt{5} \qquad q_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\|\vec{v_2}\| = \sqrt{1^2+2^2} = \sqrt{5} \qquad q_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore Q = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 10 & 0 \\ 0 & -5 \end{pmatrix}$

2. $\det(A - \lambda I) = 0$.

$$\begin{bmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{bmatrix} = 0 \Rightarrow (9-\lambda)(16-\lambda) - 12 \times 12 = 0$$
$$\lambda(\lambda - 25) = 0$$
$$\lambda_1 = 0. \quad \lambda_2 = 25$$

For $\lambda_1 = 0$. $A - \lambda I = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$

$$\left[\begin{array}{cc|c} 9 & 12 & 0 \\ 12 & 16 & 0 \end{array}\right] \xrightarrow{\times \frac{1}{9}} \left[\begin{array}{cc|c} 1 & \frac{4}{3} & 0 \\ 12 & 16 & 0 \end{array}\right] \xrightarrow{\times(-12)} \left[\begin{array}{cc|c} 1 & \frac{4}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore x_1 + \frac{4}{3} x_2 = 0$. $\vec{v}_1 = \begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix}$

For $\lambda_2 = 25$. $A - \lambda I = \begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -16 & 12 & 0 \\ 12 & -9 & 0 \end{array}\right] \xrightarrow{\times(\frac{-1}{16})} \left[\begin{array}{cc|c} 1 & -\frac{3}{4} & 0 \\ 12 & -9 & 0 \end{array}\right) \xrightarrow{\times(-12)} \left[\begin{array}{cc|c} 1 & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 + \frac{3}{4} x_2 = 0$. $\vec{v}_2 = \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}$

$\|\vec{v}_1\| = \sqrt{(-\frac{4}{3})^2 + 1^2} = \frac{5}{3}$ $\quad q_1 = \frac{3}{5}\begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix}$ $\quad \therefore Q = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$

$\|\vec{v}_2\| = \sqrt{(\frac{3}{4})^2 + 1^2} = \frac{5}{4}$ $\quad q_2 = \frac{4}{5}\begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}$.

$$\Lambda = \begin{bmatrix} -\frac{4}{3} & \frac{3}{4} \\ 1 & 1 \end{bmatrix}$$

3. (a) $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ -6 & 4-\lambda & 4 \\ 3 & -1 & -\lambda \end{vmatrix} = (2-\lambda)(4-\lambda)(-\lambda) + (2-\lambda)\cdot 4 \cdot (-1) = 0.$$

$-\lambda^3 + 6\lambda^2 - 12\lambda + 8 = 0 \rightarrow (\lambda - 2)^3 = 0$

$\therefore \lambda = 2$. Algebric multiplicity $= 3$.

(b) for $\lambda = 2$. $A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ -6 & 2 & 4 \\ 3 & -1 & -2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -6 & 2 & 4 & 0 \\ 3 & -1 & -2 & 0 \end{array}\right] \xrightarrow{\text{swap}} \left[\begin{array}{ccc|c} -6 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & -1 & -2 & 0 \end{array}\right] \xrightarrow{\times(-\frac{1}{6})} \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 3 & -1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore x_1 + (-\frac{1}{3})x_2 + (-\frac{2}{3})x_3 = 0$

$$\vec{v_1} = \begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} \frac{2}{3} \\ 0 \\ 1 \end{pmatrix}$$

4. $\det(A - \lambda I) = \begin{pmatrix} 2-\lambda & 0 \\ -1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - (-1)\cdot 0 = 0 \quad (\lambda-2)^2 = 0.$

$\therefore \lambda = 2.$ algebric multiplicity $= 2$.

$$A - \lambda I = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = 0 \Rightarrow V = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(A - \lambda I)^2 x = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $(A - \lambda I)^{2-1} x_1 \neq 0$. $\quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

For $x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $(A - \lambda I)^{2-1} x_2 = 0$. not generalized eigenvector

$\therefore M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = P$. $\quad J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ (as there's only one λ).

$$P^{-1} = \frac{1}{0-(-1)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\therefore$ $A = PJP^{-1}$ is written as

$$\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$