Assignment 10.

1. (a). $\det(A-\lambda I) = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - 2\times 0$
$= (1-\lambda)(3-\lambda).$

$\therefore \lambda_1 = 1. \quad \lambda_2 = 3.$

When $\lambda_1 = 1.$ $\begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$

$\Rightarrow \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 2 & 0 \end{array}\right] \xrightarrow{\times 1/2} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] \xrightarrow{\times(-2)} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] = 0.$

$\therefore x_1 \times 0 + 1 \times x_2 = 0 \Rightarrow x_2 = 0.$

$\begin{cases} x_1 = x_1 \\ x_2 = 0. \end{cases} \quad x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \Rightarrow x_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Let $x_1 = 1.$ $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

When $\lambda_2 = 3.$ $\begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \overset{\times(-\frac{1}{2})}{=} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] = 0.$

$\therefore x_1 - x_2 = 0.$ $\Rightarrow$ Let $x_2 = 1.$ $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$
$x_2 = x_2$

$\therefore X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$

For $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ $\det(A-\lambda I) = \begin{bmatrix} 1-\lambda & 1 \\ 3 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - 3$
$= \lambda(\lambda - 4)$

$\therefore \lambda_1 = 0. \quad \lambda_2 = 4.$

For $\lambda_1 = 0.$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 3 & 3 & 0 \end{array}\right] \xrightarrow{\times(-3)} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\therefore \begin{cases} x_1 + x_2 = 0 \\ x_2 = x_2 \end{cases}$

Let $x_2 = 1.$ $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

For $\lambda_2 = 4.$ $\left[\begin{array}{cc|c} -3 & 1 & 0 \\ 3 & -1 & 0 \end{array}\right] \xrightarrow{\times(\frac{-1}{3})} \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 3 & -1 & 0 \end{array}\right] \xrightarrow{\times(-3)} \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore \begin{cases} x_1 + (-\frac{1}{3})x_2 = 0 \\ x_2 = x_2 \end{cases}$ Let $x = .$ $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\therefore X = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \quad X^{-1} = 1/4 \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

(b). $\Lambda^3 = X\Lambda^3 X^{-1}$.

For $\Lambda_1^3 = \begin{bmatrix}1&0\\0&3\end{bmatrix}^3$.

$\Lambda_1^2 = \begin{bmatrix}1&0\\0&3\end{bmatrix}\begin{bmatrix}1&0\\0&3\end{bmatrix} = \begin{bmatrix}1\times1+0\times0 & 1\times0+0\times3\\ 0\times1+3\times0 & 0\times0+3\times3\end{bmatrix} = \begin{bmatrix}1&0\\0&9\end{bmatrix}$

$\therefore \Lambda_1^3 = \begin{bmatrix}1&0\\0&9\end{bmatrix}\begin{bmatrix}1&0\\0&3\end{bmatrix} = \begin{bmatrix}1\times1+0\times0 & 1\times0+0\times3\\ 0\times1+9\times0 & 0\times0+9\times3\end{bmatrix} = \begin{bmatrix}1&0\\0&27\end{bmatrix}$.

$\therefore A_1^3 = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&27\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}$

For $\Lambda_2^3 = \begin{bmatrix}0&0\\0&4\end{bmatrix}^3$. Follow the same steps. $\Lambda_2^3 = \begin{bmatrix}0&0\\0&64\end{bmatrix}$

$\therefore A_2^3 = \begin{bmatrix}1&1\\-1&3\end{bmatrix}\begin{bmatrix}0&0\\0&4\end{bmatrix}\begin{bmatrix}\frac{3}{4}&-\frac{1}{4}\\\frac{1}{4}&\frac{1}{4}\end{bmatrix}$

For $\Lambda_1 = \begin{bmatrix}1&0\\0&3\end{bmatrix}$ $\Lambda_1^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

$= \frac{1}{1\times3-0\times0}\begin{bmatrix}3&-0\\-0&1\end{bmatrix} = \begin{bmatrix}1&0\\0&\frac{1}{3}\end{bmatrix}$

For $\Lambda_2 = \begin{bmatrix}0&0\\0&4\end{bmatrix}$ $\Lambda_2^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

$\because ad-bc = 0\times4-0=0$

$\therefore$ determinant$=0$.

Matrix is not inverteible.

So $A_1 = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}$ 2

A_2^{-1} could not be calculated.

2. $X = X_1, X_2 = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I$.

$\therefore X\Lambda X^{-1} = \Lambda = \begin{bmatrix}2&0\\0&5\end{bmatrix} = A$.

$\therefore A = \begin{bmatrix}2&0\\0&5\end{bmatrix}$. 2

3. (1) F. (2) T. (3) T. (4) F. 2

4. $\det(A-\lambda I) = \begin{bmatrix}2-\lambda&-1\\-1&2-\lambda\end{bmatrix} = (2-\lambda)^2-(-1)(-1) = (2-\lambda)^2-1$

$= \lambda^2-4\lambda+3 = (\lambda-1)(\lambda-3)$.

$\therefore \lambda_1 = 1$. $\lambda_2 = 3$.

For $\lambda_1 = 1$. $(A-\lambda_1 I)x = 0$

$\Rightarrow \left[\begin{array}{cc|c}2-1&-1&0\\-1&2-1&0\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}1&-1&0\\-1&1&0\end{array}\right] \to \left[\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right]$.

$\therefore x_1 + (-x_2) = 0$.

Let $x_1 = 1$. $v_1 = \begin{pmatrix}1\\1\end{pmatrix}$

For $\lambda_2 = 3$ $(A-\lambda I)x=0$

$$\Rightarrow \left[\begin{array}{cc|c} 2-3 & -1 & 0 \\ -1 & 2-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \xrightarrow{\times(-1)} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore x_1 + x_2 = 0$. let $x_2 = 1$ $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$\therefore D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \Lambda$. $X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $X^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

$$\therefore A^k = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^k \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1\times1+(-1)\times0 & 1\times0+(-1)\times3^k \\ 1\times1+1\times0 & 1\times0+1\times3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -3^k \\ 1 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1\times1+(-3^k)\times(-1) & 1\times1+(-3^k)\times1 \\ 1\times1+3^k(-1) & 1\times1+3^k\times1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$$

Proven.

2

5. $\det(B-\lambda I) = \begin{bmatrix} 5-\lambda & 1 \\ 0 & 4-\lambda \end{bmatrix} = (5-\lambda)(4-\lambda) - 0\times1 = (5-\lambda)(4-\lambda)$

$\therefore \lambda = 5$. $\lambda = 4$.

For $\lambda_1 = 5$. $(B-\lambda I)x=0$

$$\Rightarrow \left[\begin{array}{cc|c} 5-5 & 1 & 0 \\ 0 & 4-5 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore x_1\times0 + x_2 = 0 \Rightarrow x_2 = 0$. $\therefore v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 4$. $(B-\lambda I)x=0$.

$$\Rightarrow \left[\begin{array}{cc|c} 5-4 & 1 & 0 \\ 0 & 4-4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$ $\therefore x_1 + x_2 = 0$. let $x_2 = 1$. $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$$\therefore B^k = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times5^k+(-1)\times0 & 1\times0+(-1)\times4^k \\ 0\times5^k+1\times0 & 0\times0+1\times4^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5^k & -4^k \\ 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5^k\times1+(-4)^k\times0 & 5^k\times1+(-4^k)\times1 \\ 0\times1+4^k\times0 & 0\times1+4^k\times1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k-4^k \\ 0 & 4^k \end{bmatrix}.$$

Proven. 2