Assignment 4.

1. (1). Step ① $A-\lambda I = \begin{bmatrix}1&4\\2&3\end{bmatrix} - \begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix} = \begin{bmatrix}1-\lambda&4\\2&3-\lambda\end{bmatrix}$

Step ② $\text{Det}\begin{bmatrix}1-\lambda&4\\2&3-\lambda\end{bmatrix} = (1-\lambda)(3-\lambda) - (2)(4)$

$= \lambda^2 - 4\lambda - 5$

Step ③ $\Rightarrow (\lambda-5)(\lambda+1) = 0.$

$\therefore \lambda_1 = 5. \quad \lambda_2 = -1. \quad \Rightarrow$ Eigenvalues.

Step ④. for $\lambda_1 = 5$ $\begin{bmatrix}1-\lambda&4\\2&3-\lambda\end{bmatrix} \rightarrow \begin{bmatrix}1-5&4\\2&3-5\end{bmatrix} \rightarrow \begin{bmatrix}-4&4\\2&-2\end{bmatrix}$ Let it be B.

solve $B\bar{x} = \bar{0}$

i.e. $\begin{bmatrix}-4&4\\2&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow \left[\begin{array}{cc|c}-4&4&0\\2&-2&0\end{array}\right] \xrightarrow{\times(-\frac{1}{4})} \left[\begin{array}{cc|c}1&-1&0\\2&-2&0\end{array}\right] \xrightarrow{\times(-2)} \left[\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right]$

$\therefore x_1 + (-x_2) = 0.$ Let $x_1 = 1$, then $\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = v_1.$

for $\lambda_2 = -1.$ $\begin{bmatrix}1-\lambda&4\\2&3-\lambda\end{bmatrix} \Rightarrow \begin{bmatrix}2&4\\2&4\end{bmatrix}$ ~~let it be~~

solve $\begin{bmatrix}2&4\\2&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow \left[\begin{array}{cc|c}2&4&0\\2&4&0\end{array}\right] \xrightarrow{\times\frac{1}{2}} \left[\begin{array}{cc|c}1&2&0\\2&4&0\end{array}\right] \xrightarrow{\times(-2)} \left[\begin{array}{cc|c}1&2&0\\0&0&0\end{array}\right]$

$\therefore x_1 + 2x_2 = 0.$ Let $x_2 = 1$, $v_2 = \begin{pmatrix}-2\\1\end{pmatrix}$

(2). Let $A + I = \begin{bmatrix}2&4\\2&4\end{bmatrix} = C.$

Step ① $C - \lambda I = \begin{bmatrix}2&4\\2&4\end{bmatrix} - \begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix} = \begin{bmatrix}2-\lambda&4\\2&4-\lambda\end{bmatrix}$

Step ② $\text{Det}\begin{bmatrix}2-\lambda&4\\2&4-\lambda\end{bmatrix} = (2-\lambda)(4-\lambda) - (2)(4)$

$= \lambda^2 - 6\lambda$

Step ③ $\Rightarrow \lambda(\lambda-6) = 0.$

$\therefore \lambda_1 = 0. \quad \lambda_2 = 6.$

Step ④ Repeat the process as above. $\lambda_1 = 0, \; v_1 = \begin{pmatrix}-2\\1\end{pmatrix}$

$\lambda_2 = 6, \; v_2 = \begin{pmatrix}1\\1\end{pmatrix}$

2. (1) $\det(A-\lambda I) = \det\begin{bmatrix} 3-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3) = 0$.

$\therefore \lambda_1 = 1.\quad \lambda_2 = 3.$

For $\lambda_1 = 1$. $A - \lambda I = \begin{bmatrix} 3-1 & 0 \\ 1 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$

$\therefore \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \overset{\times 1/2}{=} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \overset{\times(-1)}{=} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore x_1 = 0.$ let $x_2 = 1.$ $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = 3$. $A - \lambda I = \begin{bmatrix} 3-3 & 0 \\ 1 & 1-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$.

$\therefore \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -2 & 0 \end{array}\right] \overset{\text{swap}}{=} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore x_1 + (-2)x_2 = 0.$ let $x_2 = 1$, $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(2) $\det(B-\lambda I) = \det\begin{bmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - 0 = \lambda^2 - 4\lambda + 3$
$= (\lambda-1)(\lambda-3) = 0.$

$\therefore \lambda_1 = 1,\quad \lambda_2 = 3.$

For $\lambda_1 = 1$. $A - \lambda I = \begin{bmatrix} 1-1 & 1 \\ 0 & 3-1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

$\therefore \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] \overset{\times(-2)}{=} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore x_2 = 0.$ let $x_1 = 1.$ $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 3$. $A - \lambda I = \begin{bmatrix} 1-3 & 1 \\ 0 & 3-3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$.

$\therefore x_1 \cdot (-2) + x_2 = 0.$ let $x_2 = 1$ $v_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$.

3. Rewrite the matrix $P = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $P - \lambda I = \begin{bmatrix} 1/5-\lambda & 2/5 & 0 \\ 2/5 & 4/5-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

for $\lambda_1 = 0$, $\left[\begin{array}{ccc|c} 1/5 & 2/5 & 0 & 0 \\ 2/5 & 4/5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\times 5} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2/5 & 4/5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\times(-\frac{2}{5})} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore x_1 + 2x_2 = 0.$ $x_3 = 0.$ $\left\{ x_2 \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$.

let $x_2 = 1.$ $v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 1$. $\begin{bmatrix} -4/5 & 2/5 & 0 \\ 2/5 & -1/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\times(-\frac{5}{4})} \begin{bmatrix} 1 & -1/2 & 0 \\ 2/5 & -1/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\times(-\frac{2}{5})} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore x_1 + (-\frac{1}{2})x_2 = 0.$

$x_2 = x_2.$

$x_3 = x_3.$

$X = \begin{bmatrix} 1/2\, x_2 \\ x_2 \\ x_3 \end{bmatrix} \quad x_2 \cdot \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

let $x_2 = 1$. $x_3 = 0$. $v_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

let $x_2 = 0$. $x_3 = 1$. $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$v_1 . v_2 . v_3 \qquad B = \begin{bmatrix} -2 & \frac{1}{2} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$\therefore$ There're linearly independent.

4. $\det(P - \lambda I) = \begin{bmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 1 & 0 & 0-\lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix}$

$= (-\lambda)(-1)^{1+1} \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} - 1(-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 1 & -\lambda \end{vmatrix} + 0(-1)^{1+3} \begin{vmatrix} 0 & -\lambda \\ 1 & 0 \end{vmatrix}$

$= (-\lambda)[(-\lambda)(-\lambda) - (0 \times 1)] - (-1)[0 \times (-\lambda) + 1 \times 1] + 0$

$= -\lambda(\lambda^2 + 0) - (-1) = -\lambda \cdot \lambda^2 + 1 = 0$

namely $-\lambda^3 + 1 = 0$.

$(\lambda - 1)(\lambda^2 + \lambda + 1) = 0$.

$\therefore \lambda_1 = 1$

$\lambda_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2ac} = \dfrac{-1 \pm \sqrt{3}}{2} \Rightarrow$ complex

$\therefore \lambda = 1$.