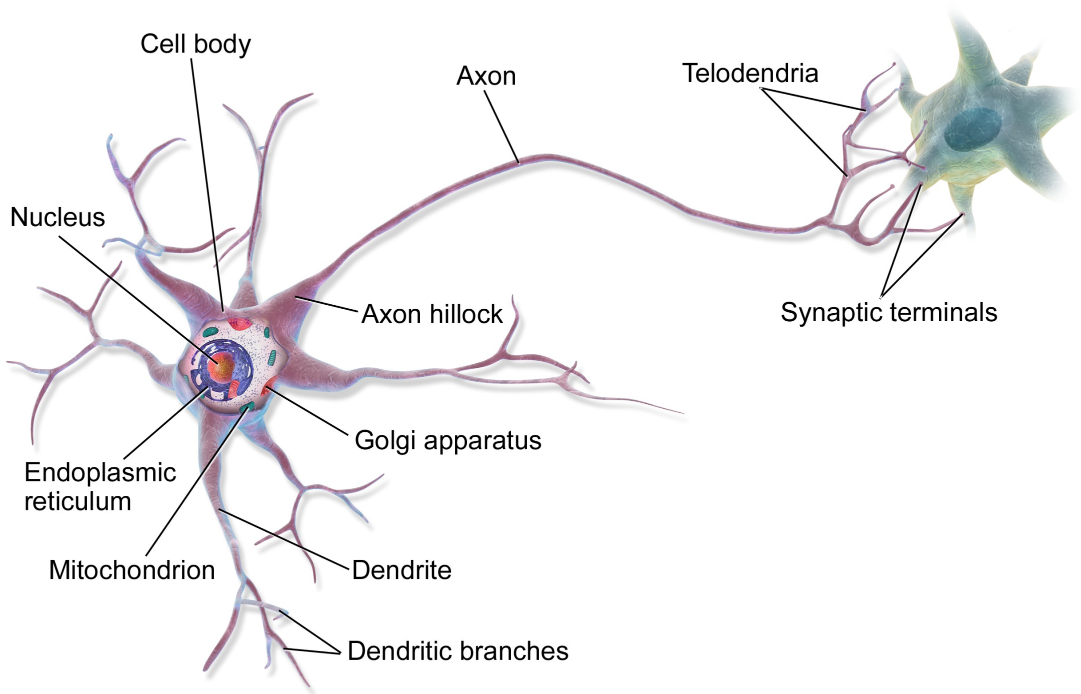
Simple Neural Networks and Neural Language Models

Units in Neural Networks

This is in your brain



Neural Network Unit

This is not in your brain

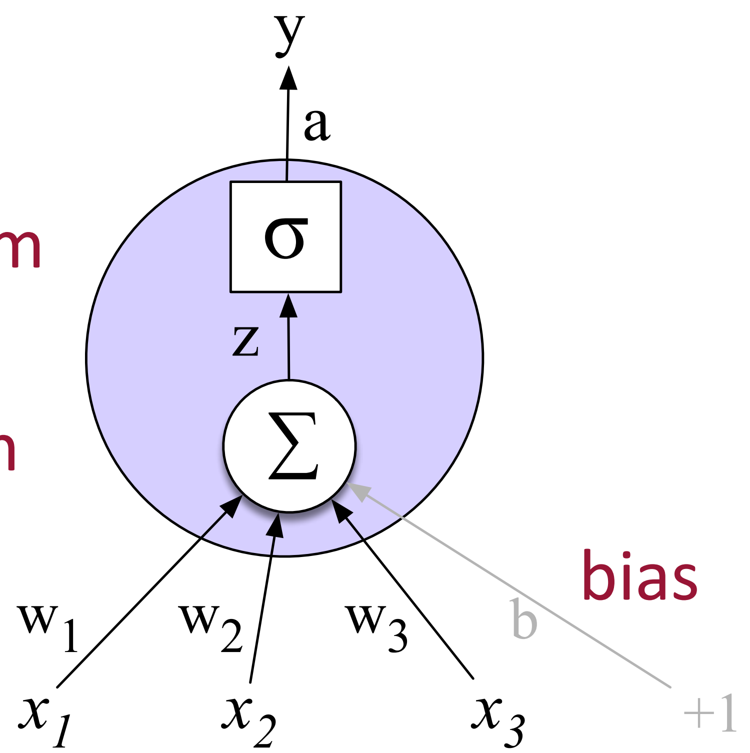
Output value

Non-linear transform

Weighted sum

Weights

Input layer



Neural unit

Take weighted sum of inputs, plus a bias

$$z = b + \sum_i w_i x_i$$

$$z = w \cdot x + b$$

Instead of just using z, we'll apply a nonlinear activation function f:

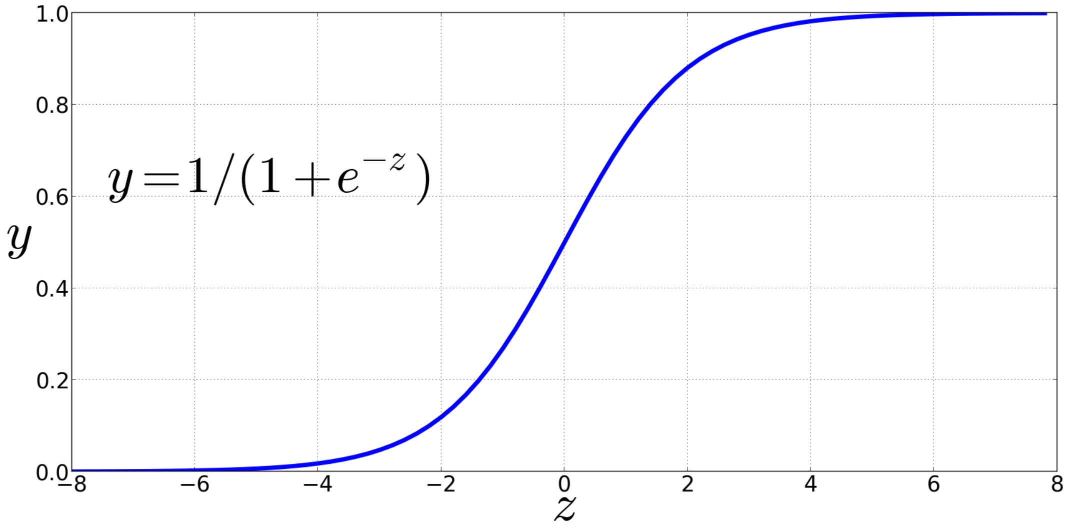
$$y = a = f(z)$$

Non-Linear Activation Functions

We've already seen the sigmoid for logistic regression:

Sigmoid

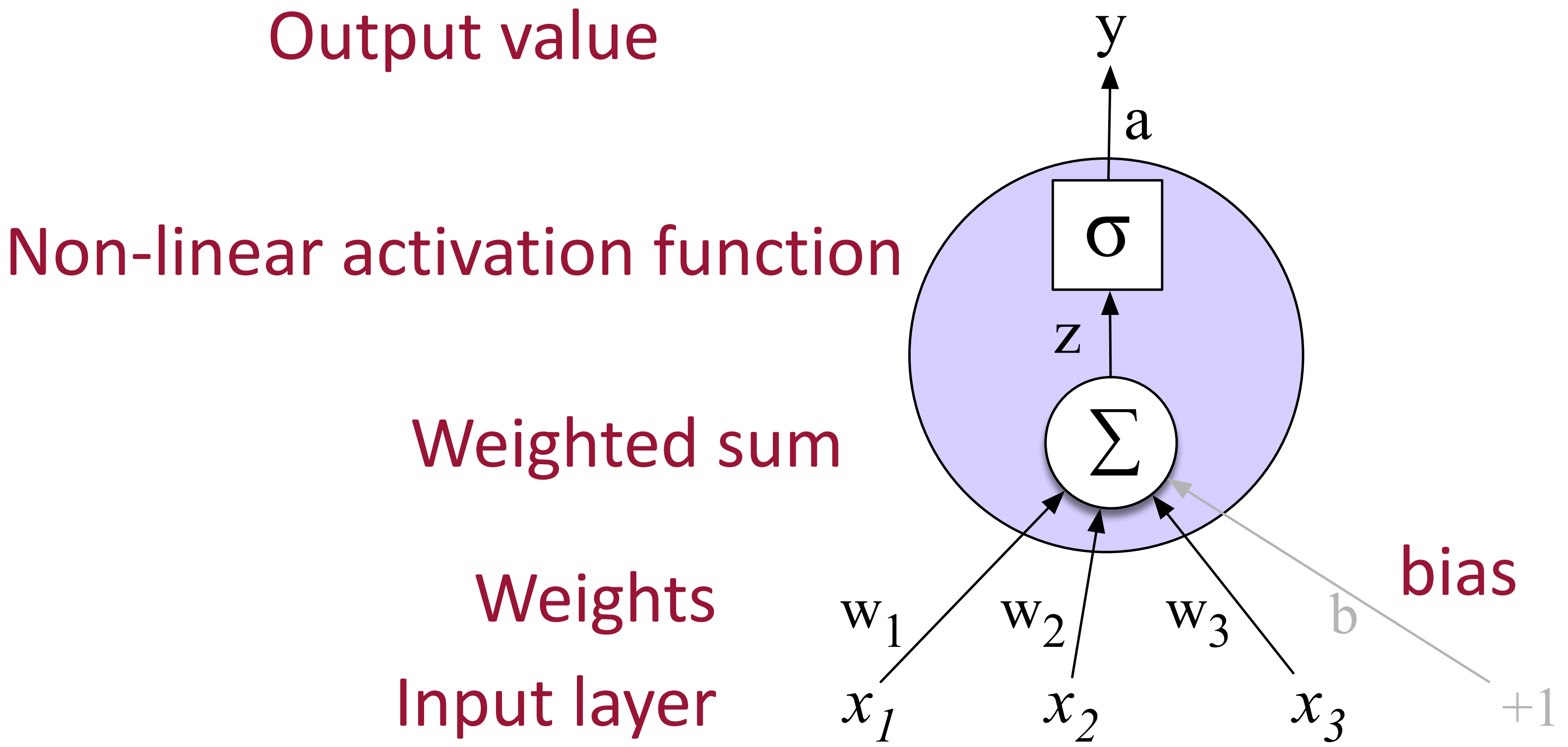
$$y = \sigma(z) = \frac{1}{1 + e^{-z}}$$



Final function the unit is computing

$$y = \sigma(w \cdot x + b) = \frac{1}{1 + \exp(-(w \cdot x + b))}$$

Final unit again



An example

Suppose a unit has:

$w = [0.2, 0.3, 0.9]$

$b = 0.5$

What happens with input x:

$x = [0.5, 0.6, 0.1]$

How is the output y computed?

$y = ?$

Neural unit

Take weighted sum of inputs, plus a bias

$$z = b + \sum_i w_i x_i$$

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$$y = \sigma(w \cdot x + b) =$$

An example

Suppose a unit has:

$w = [0.2, 0.3, 0.9]$

$b = 0.5$

What happens with the following input x?

$x = [0.5, 0.6, 0.1]$

$$y = \sigma(w \cdot x + b) = \frac{1}{1 + e^{-(w \cdot x + b)}} =$$

An example

Suppose a unit has:

$w = [0.2, 0.3, 0.9]$

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What happens with input x:

$x = [0.5, 0.6, 0.1]$

$$y = \sigma(w \cdot x + b) = \frac{1}{1 + e^{-(w \cdot x + b)}} =$$

$$\frac{1}{1 + e^{-(.5*.2+.6*.3+.1*.9+.5)}} =$$

An example

Suppose a unit has:

$w = [0.2, 0.3, 0.9]$

$b = 0.5$

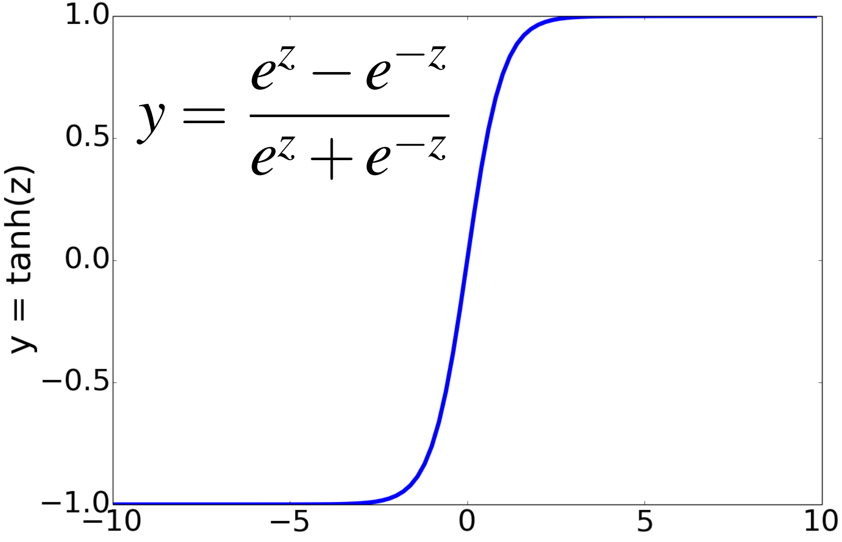
What happens with input x:

$x = [0.5, 0.6, 0.1]$

$$y = \sigma(w \cdot x + b) = \frac{1}{1 + e^{-(w \cdot x + b)}} =$$

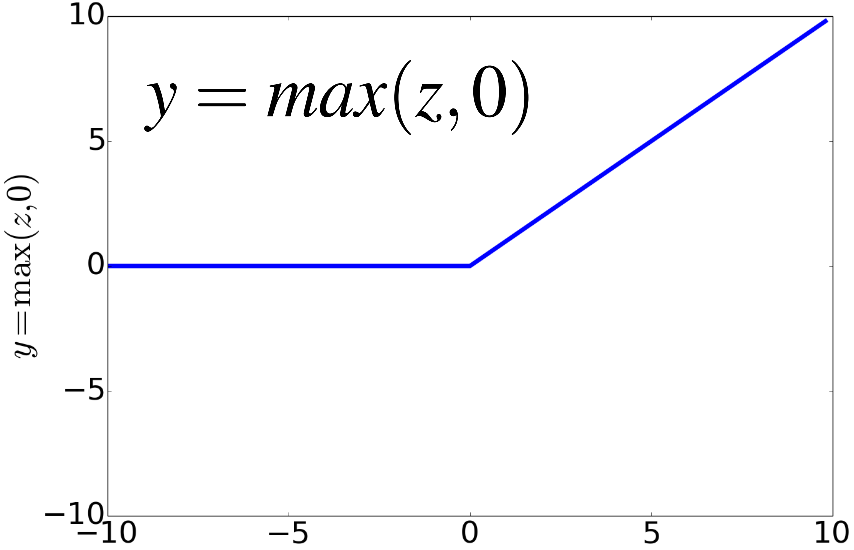
$$\frac{1}{1 + e^{-(.5 * .2 + .6 * .3 + .1 * .9 + .5)}} = \frac{1}{1 + e^{-0.87}} = .70$$

Non-Linear Activation Functions besides sigmoid



tanh

Most Common:



ReLU

Rectified Linear Unit

Simple Neural Networks and Neural Language Models

Units in Neural Networks

Simple Neural Networks and Neural Language Models

The XOR problem

The XOR problem

Minsky and Papert (1969)

Can neural units compute simple functions of input?

AND			OR			XOR		
x1	x2	y	x1	x2	y	x1	x2	y
0	0	0	0	0	0	0	0	0
0	1	0	0	1	1	0	1	1
1	0	0	1	0	1	1	0	1
1	1	1	1	1	1	1	1	0

Perceptrons

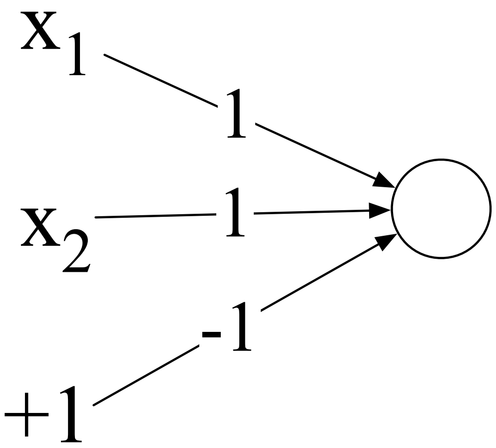
A very simple neural unit

- Binary output (0 or 1)
- No non-linear activation function

$$y = \begin{cases} 0, & \text{if } w \cdot x + b \leq 0 \\ 1, & \text{if } w \cdot x + b > 0 \end{cases}$$

Easy to build AND or OR with perceptrons

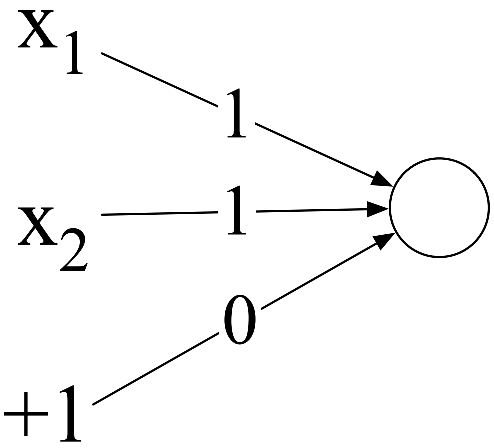
$$y = \begin{cases} 0, & \text{if } w \cdot x + b \leq 0 \\ 1, & \text{if } w \cdot x + b > 0 \end{cases}$$



AND

AND

x1	x2	y
0	0	0
0	1	0
1	0	0
1	1	1



OR

OR

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0	1	1
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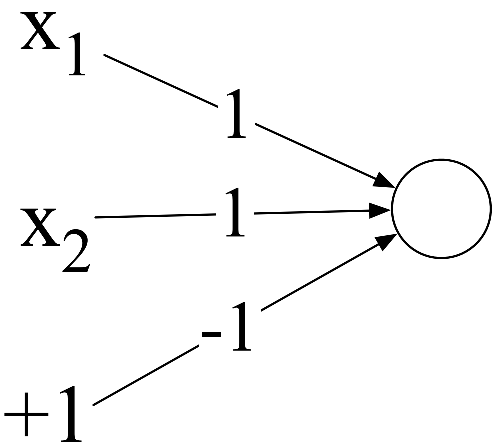
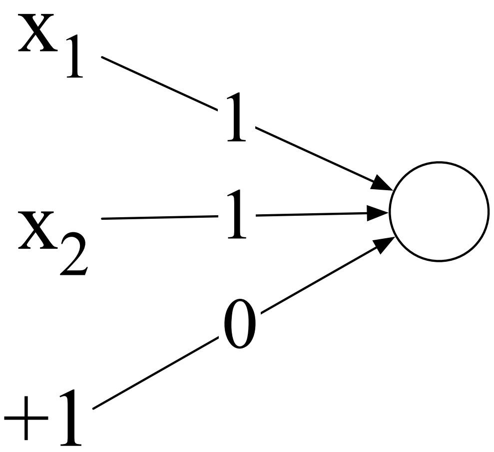


AND

AND		
x1	x2	y
0	0	0
0	1	0
1	0	0
1	1	1

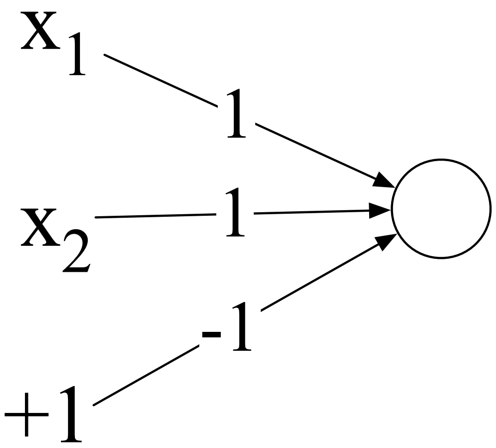


OR

OR		
x1	x2	y
0	0	0
0	1	1
1	0	1
1	1	1

Easy to build AND or OR with perceptrons

$$y = \begin{cases} 0, & \text{if } w \cdot x + b \leq 0 \\ 1, & \text{if } w \cdot x + b > 0 \end{cases}$$



AND

AND

x1	x2	y
0	0	0
0	1	0
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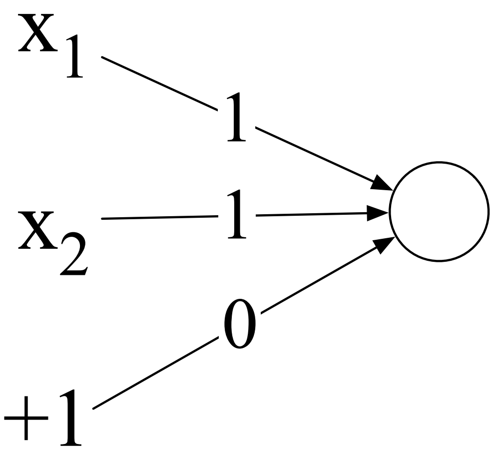


OR

OR

x1	x2	y
0	0	0
0	1	1
1	0	1
1	1	1

Not possible to capture XOR with perceptrons

Pause the lecture and try for yourself!

Why? Perceptrons are linear classifiers

Perceptron equation given x_1 and x_2, is the equation of a line

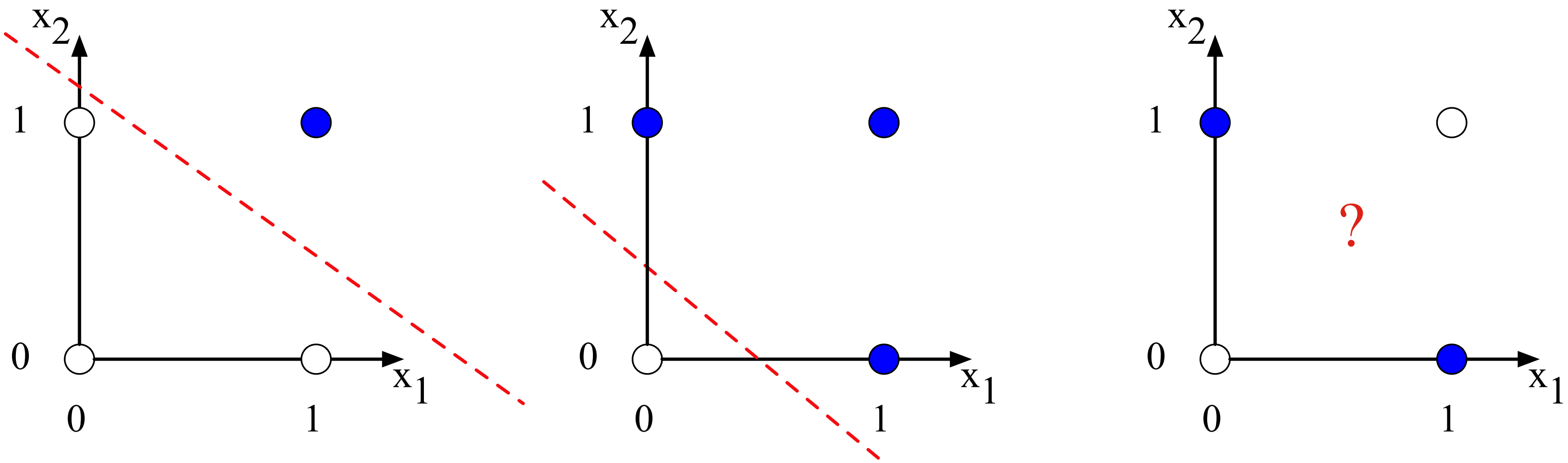
$$w_1x_1 + w_2x_2 + b = 0$$

(in standard linear format: $x_2 = (-w_1/w_2)x_1 + (-b/w_2)$)

This line acts as a **decision boundary**

- 0 if input is on one side of the line
- 1 if on the other side of the line

Decision boundaries



a) x_1 AND x_2 b) x_1 OR x_2 c) x_1 XOR x_2

XOR is not a **linearly separable** function!

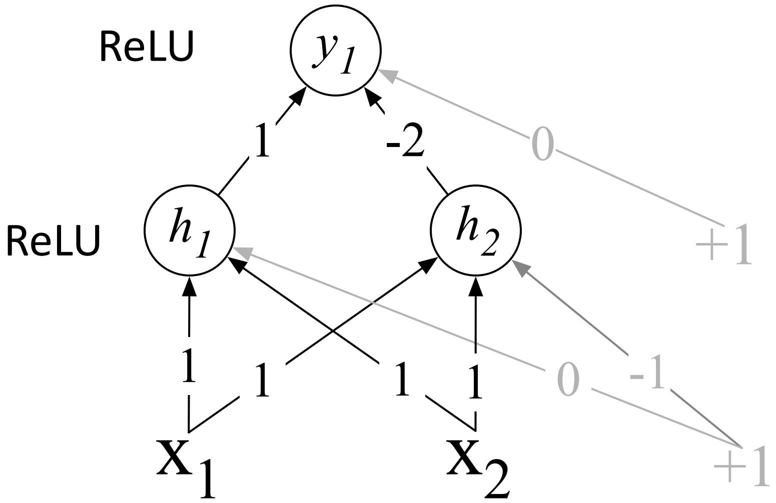
Solution to the XOR problem

XOR **can't** be calculated by a single perceptron

XOR **can** be calculated by a layered network of units.

XOR

x1	x2	y
0	0	0
0	1	1
1	0	1
1	1	0



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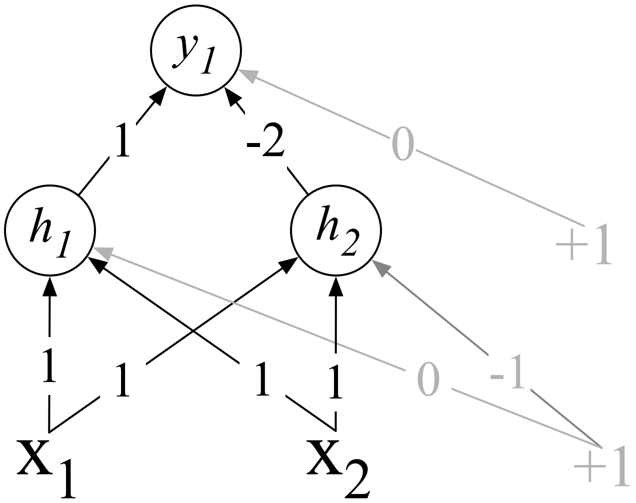
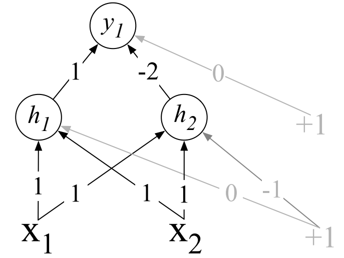
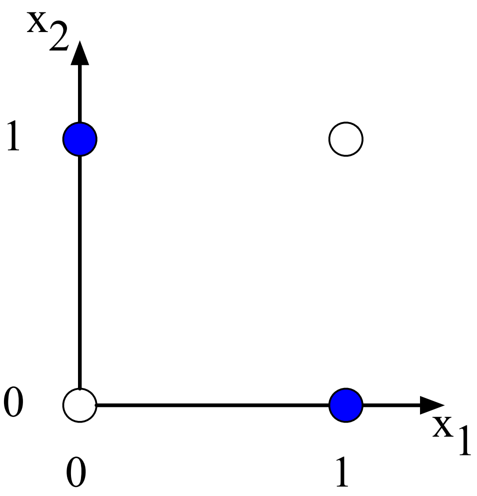
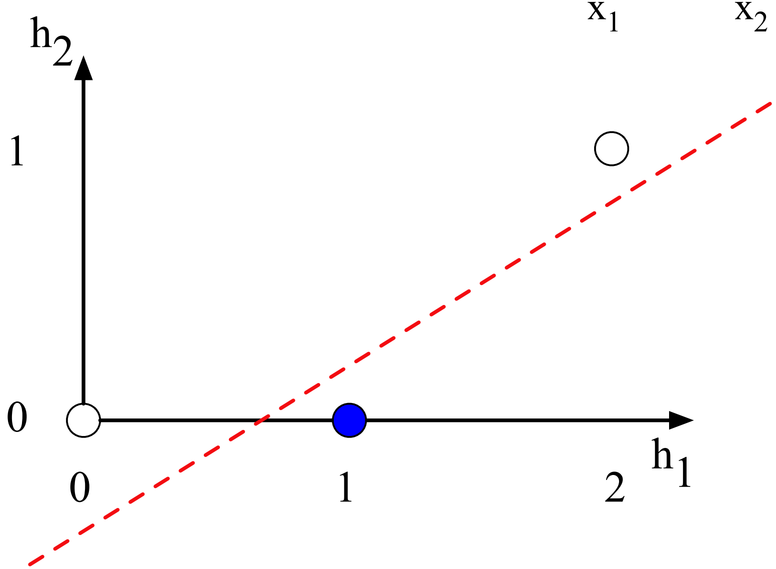


The hidden representation h





a) The original *x* space



b) The new (linearly separable) *h* space

(With learning: hidden layers will learn to form useful representations)

Simple Neural Networks and Neural Language Models

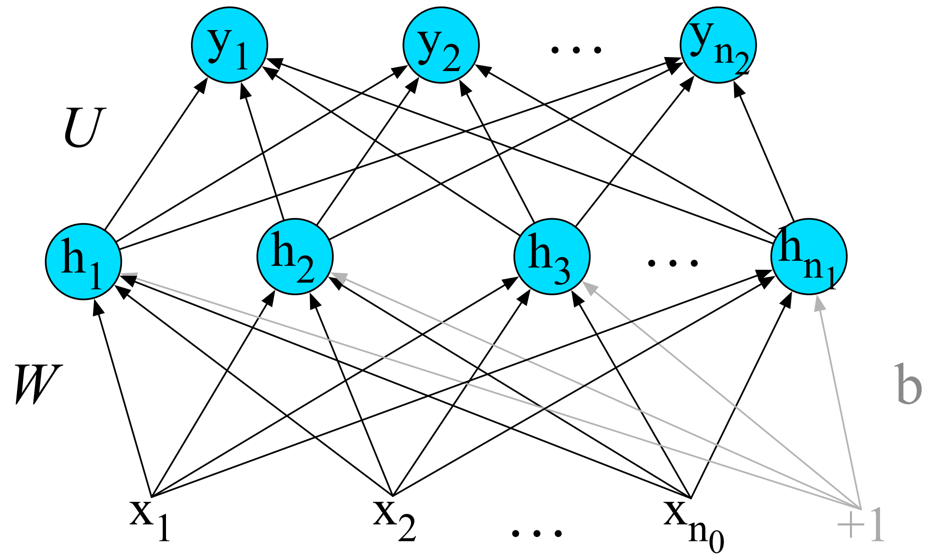
The XOR problem

Simple Neural Networks and Neural Language Models

Feedforward Neural Networks

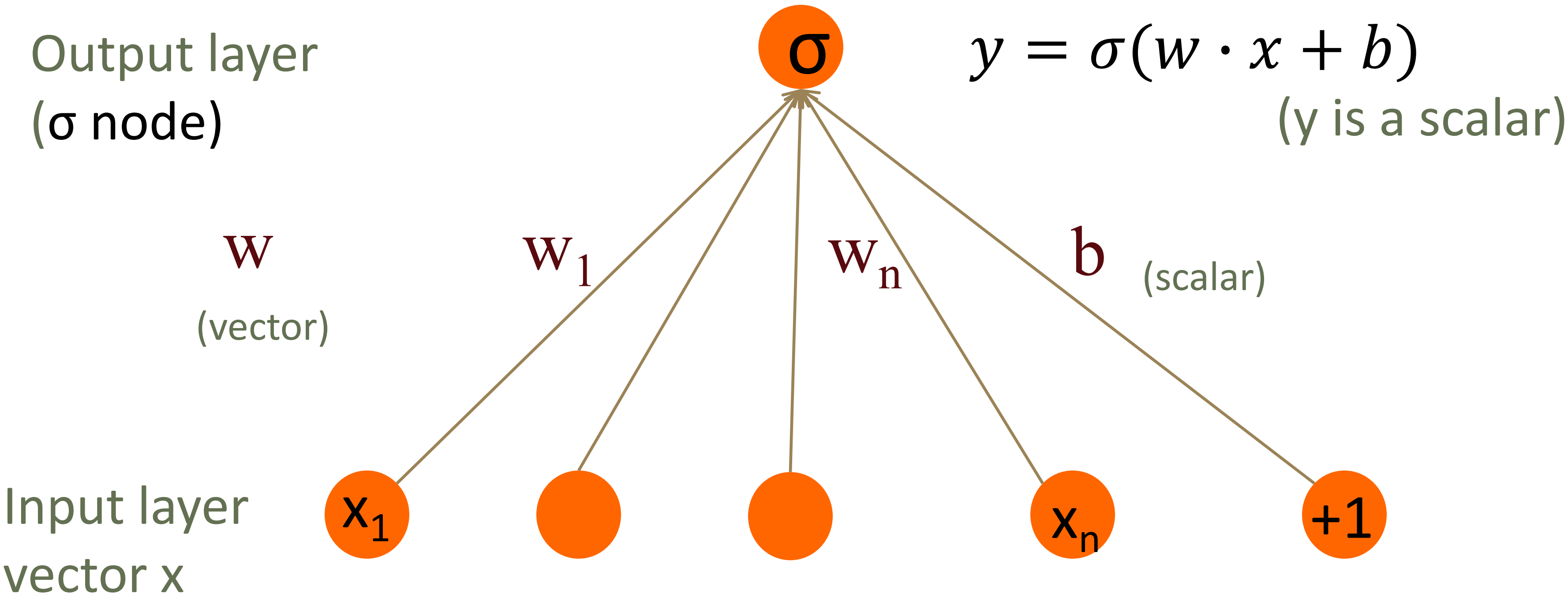
Feedforward Neural Networks

Can also be called **multi-layer perceptrons** (or **MLPs**) for historical reasons

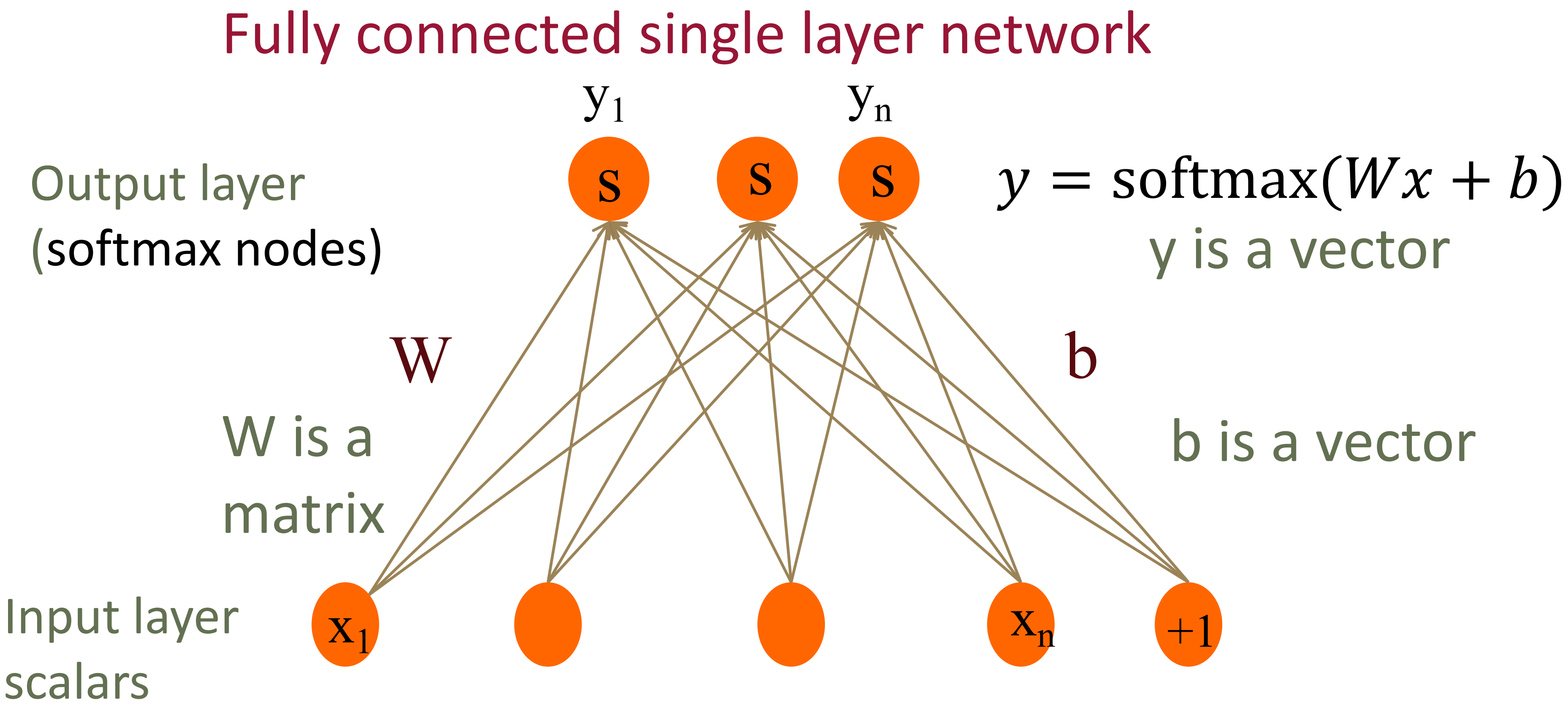


Binary Logistic Regression as a 1-layer Network

(we don't count the input layer in counting layers!)



Multinomial Logistic Regression as a 1-layer Network



Reminder: softmax: a generalization of sigmoid

For a vector *z* of dimensionality *k*, the softmax is:

$$\text{softmax}(z) = \left[\frac{\exp(z_1)}{\sum_{i=1}^{k}\exp(z_i)}, \frac{\exp(z_2)}{\sum_{i=1}^{k}\exp(z_i)}, \ldots, \frac{\exp(z_k)}{\sum_{i=1}^{k}\exp(z_i)}\right]$$

$$\text{softmax}(z_i) = \frac{\exp(z_i)}{\sum_{j=1}^{k}\exp(z_j)} \quad 1 \leq i \leq k$$

Example:

$$z = [0.6, 1.1, -1.5, 1.2, 3.2, -1.1]$$

$$\text{softmax}(z) = [0.055, 0.090, 0.006, 0.099, 0.74, 0.010]$$

Two-Layer Network with scalar output

Output layer
(σ node)

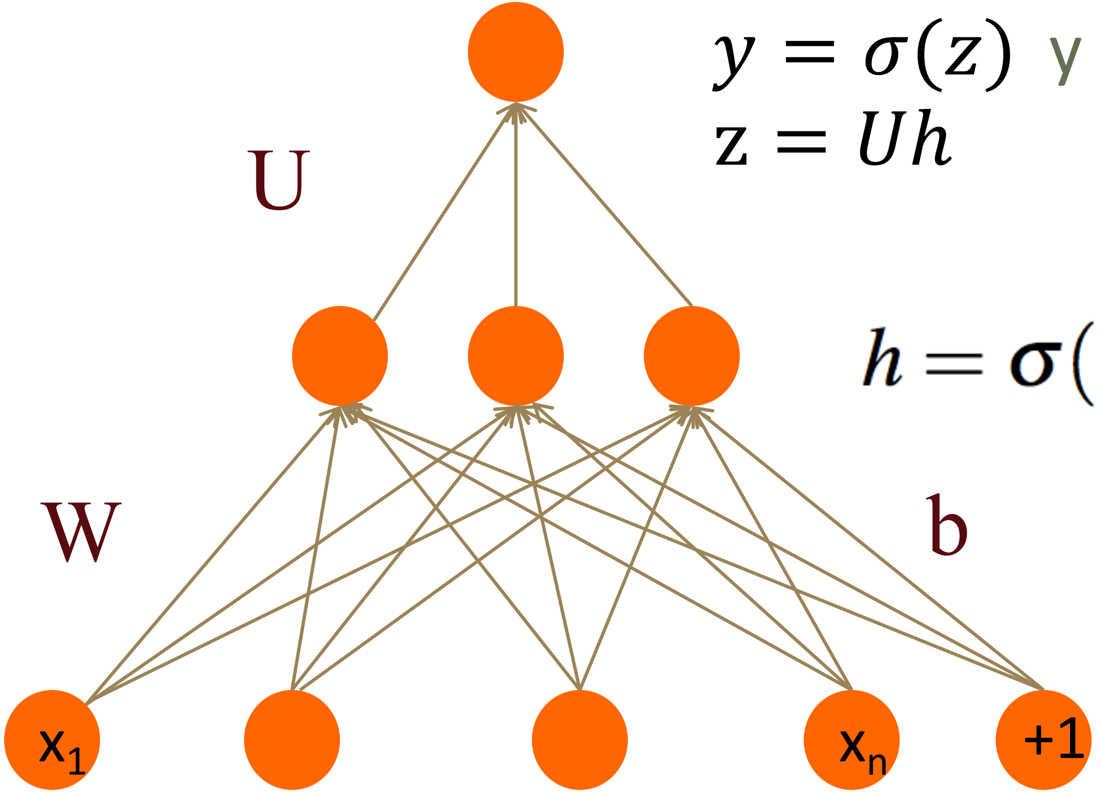
$y = \sigma(z)$ y is a scalar
$z = Uh$

hidden units
(σ node)

$h = \sigma(Wx + b)$

Could be ReLU
Or tanh

Input layer
(vector)



Two-Layer Network with scalar output

Output layer
(σ node)

$y = \sigma(z)$ y is a scalar

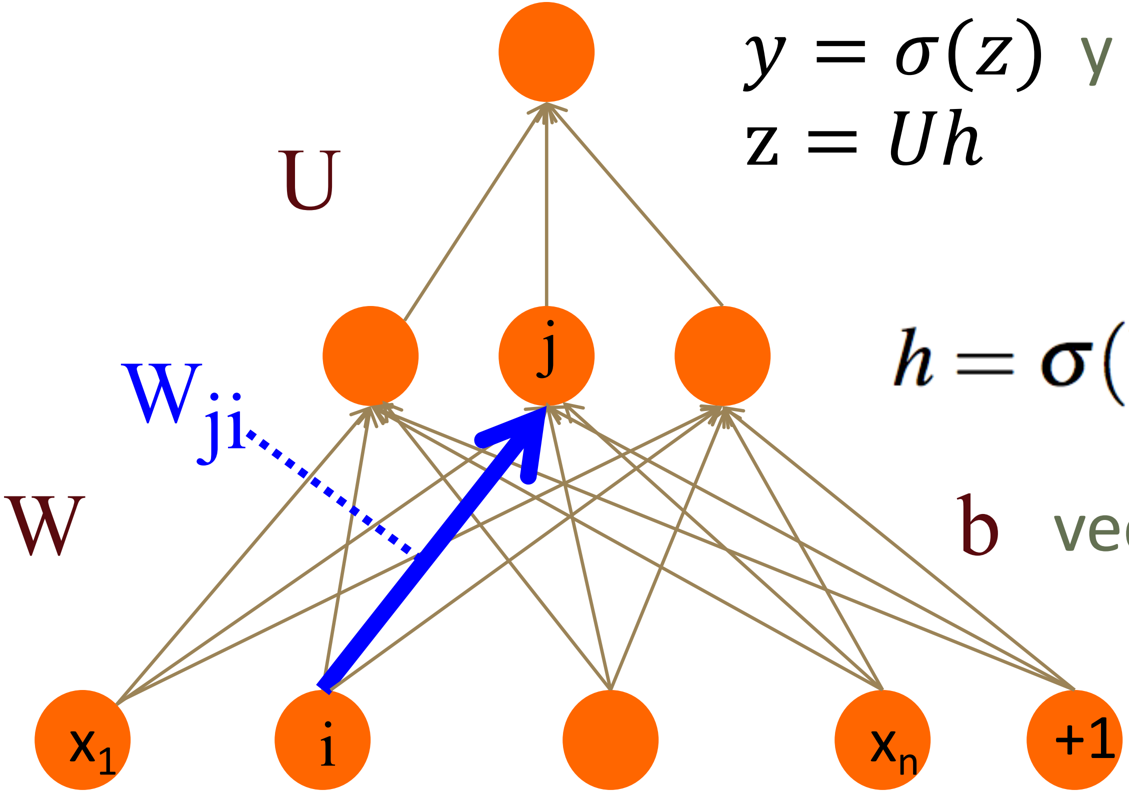
$z = Uh$

hidden units
(σ node)

$h = \boldsymbol{\sigma}(Wx + b)$

b vector

Input layer
(vector)

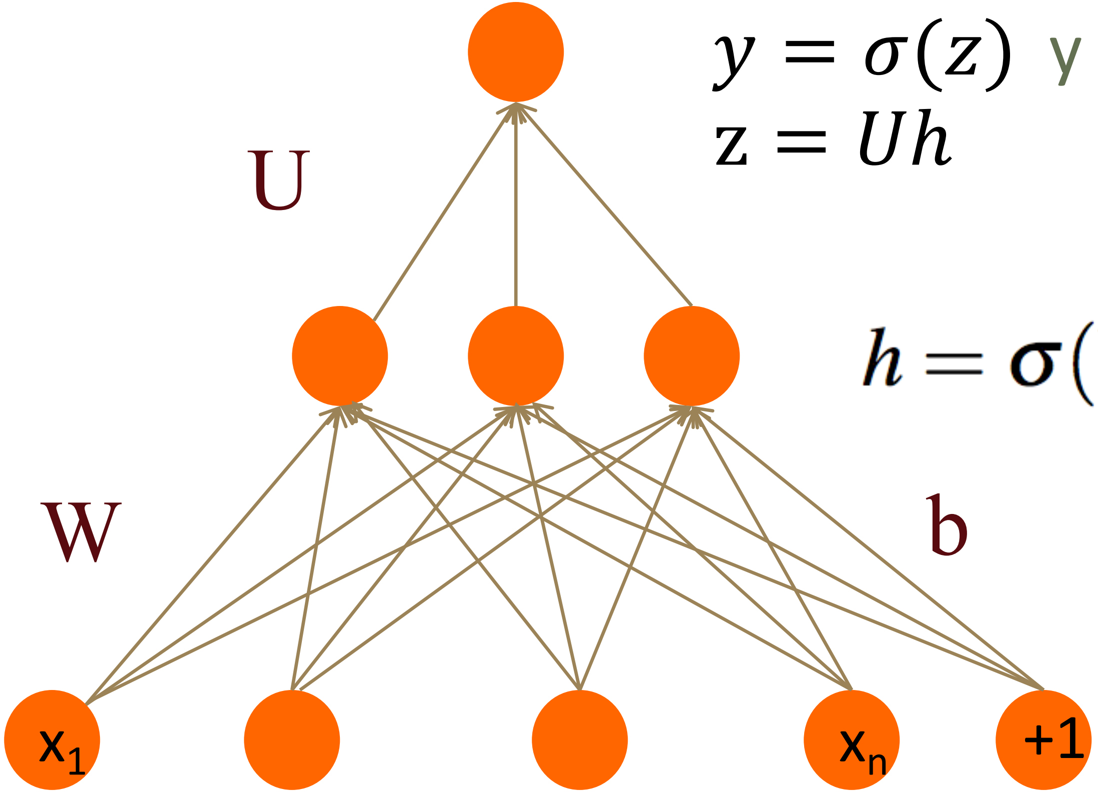


Two-Layer Network with scalar output

Output layer
(σ node)

hidden units
(σ node)

Input layer
(vector)



$y = \sigma(z)$ y is a scalar

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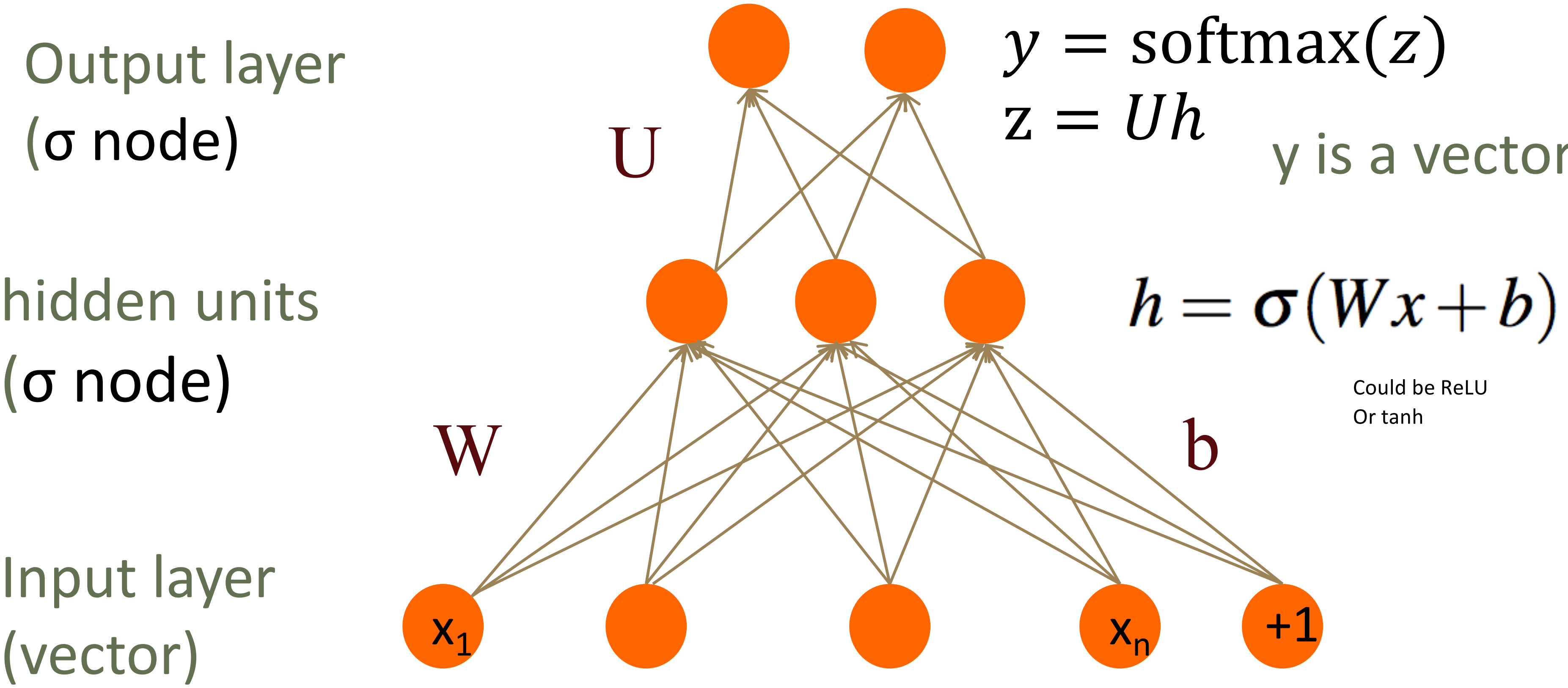
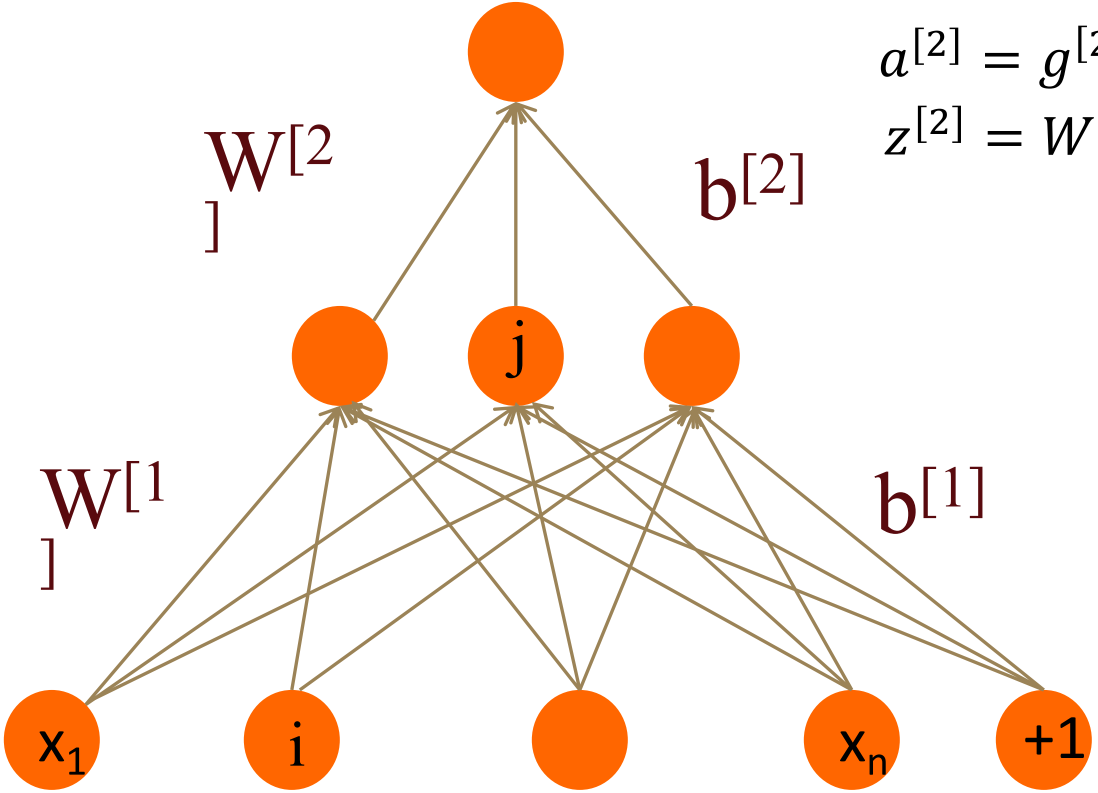
Could be ReLU
Or tanh

Two-Layer Network with softmax output



Multi-layer Notation



$y = a^{[2]}$

$a^{[2]} = g^{[2]}(z^{[2]})$ sigmoid or softmax

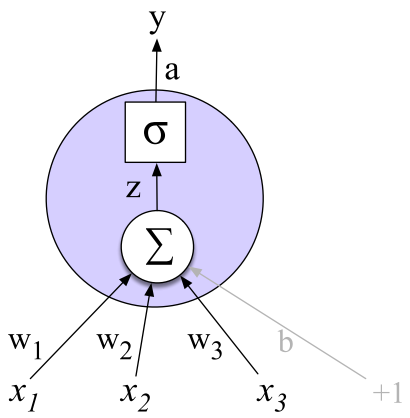
$z^{[2]} = W^{[2]}a^{[1]} + b^{[2]}$

$a^{[1]} = g^{[1]}(z^{[1]})$ ReLU

$z^{[1]} = W^{[1]}a^{[0]} + b^{[1]}$

$a^{[0]}$

Multi Layer Notation



$$
\begin{aligned}
z^{[1]} &= W^{[1]}a^{[0]} + b^{[1]} \\
a^{[1]} &= g^{[1]}(z^{[1]}) \\
z^{[2]} &= W^{[2]}a^{[1]} + b^{[2]} \\
a^{[2]} &= g^{[2]}(z^{[2]}) \\
\hat{y} &= a^{[2]}
\end{aligned}
$$

for i **in** 1..n

$$
\begin{aligned}
z^{[i]} &= W^{[i]}\, a^{[i-1]} + b^{[i]} \\
a^{[i]} &= g^{[i]}(z^{[i]})
\end{aligned}
$$

$$\hat{y} = a^{[n]}$$

Replacing the bias unit

Let's switch to a notation without the bias unit

Just a notational change

1. Add a dummy node $a_0=1$ to each layer
2. Its weight w_0 will be the bias
3. So input layer $a^{[0]}_0=1$,
   - And $a^{[1]}_0=1$, $a^{[2]}_0=1$,...

Replacing the bias unit

Instead of:

$$x = x_1, x_2, \ldots, x_{n0}$$

$$h = \sigma(Wx + b)$$

$$h_j = \sigma\left(\sum_{i=1}^{n_0} W_{ji}x_i + b_j\right)$$

We'll do this:

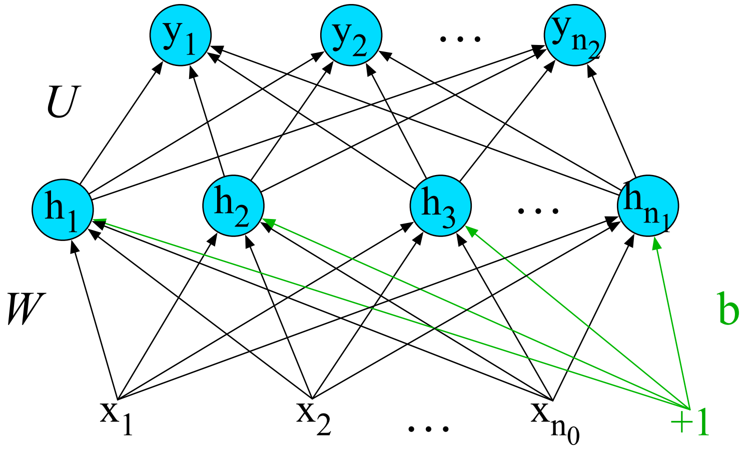
$$x = x_0, x_1, x_2, \ldots, x_{n0}$$

$$h = \sigma(Wx)$$

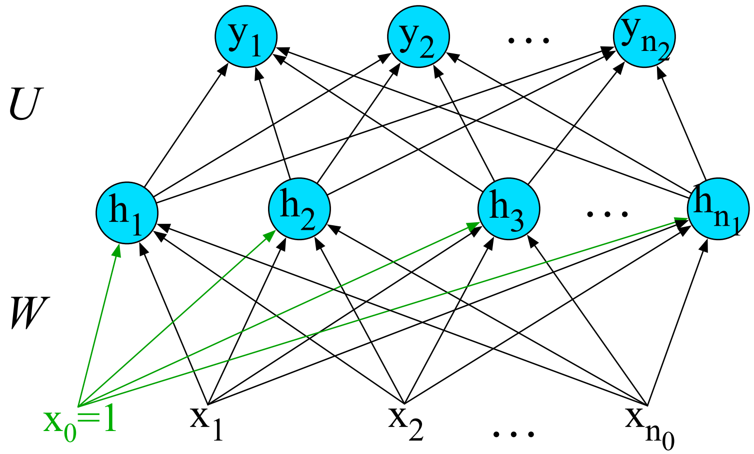
$$\sigma\left(\sum_{i=0}^{n_0} W_{ji}x_i\right)$$

Replacing the bias unit

Instead of:



We'll do this:



Simple Neural Networks and Neural Language Models

Feedforward Neural Networks

Simple Neural Networks and Neural Language Models

Applying feedforward networks to NLP tasks

Use cases for feedforward networks

Let's consider 2 (simplified) sample tasks:

1. Text classification
2. Language modeling

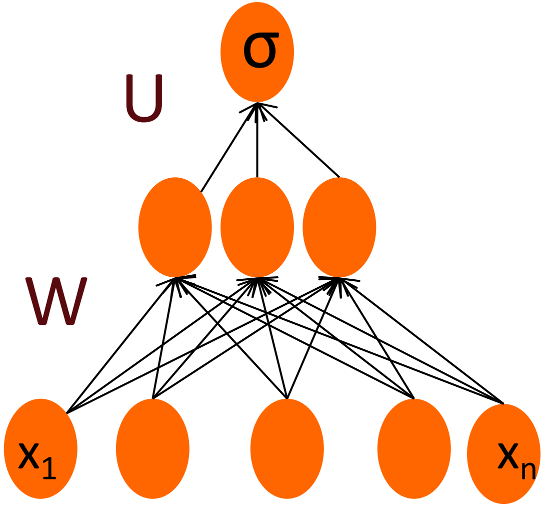
State of the art systems use more powerful neural architectures, but simple models are useful to consider!

Classification: Sentiment Analysis

We could do exactly what we did with logistic regression

Input layer are binary features as before

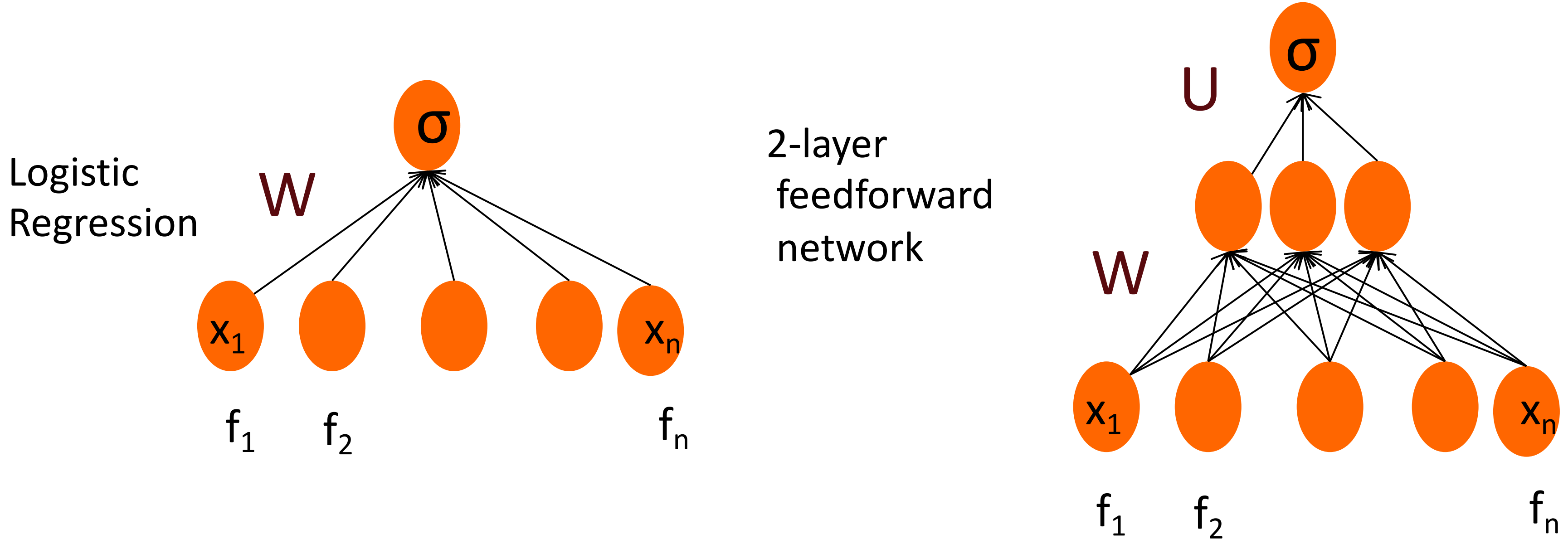
Output layer is 0 or 1



Sentiment Features

Var	Definition
x_1	$\text{count(positive lexicon)} \in \text{doc)}$
x_2	$\text{count(negative lexicon)} \in \text{doc)}$
x_3	$\begin{cases} 1 & \text{if ``no''} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$
x_4	$\text{count}(\text{1st and 2nd pronouns} \in \text{doc})$
x_5	$\begin{cases} 1 & \text{if ``!''} \in \text{doc} \\ 0 & \text{otherwise} \end{cases}$
x_6	$\log(\text{word count of doc})$

Feedforward nets for simple classification



Just adding a hidden layer to logistic regression

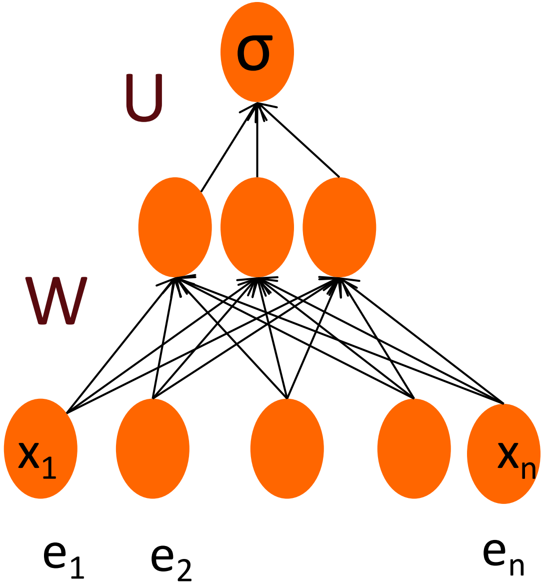
- allows the network to use non-linear interactions between features
- which may (or may not) improve performance.

Even better: representation learning

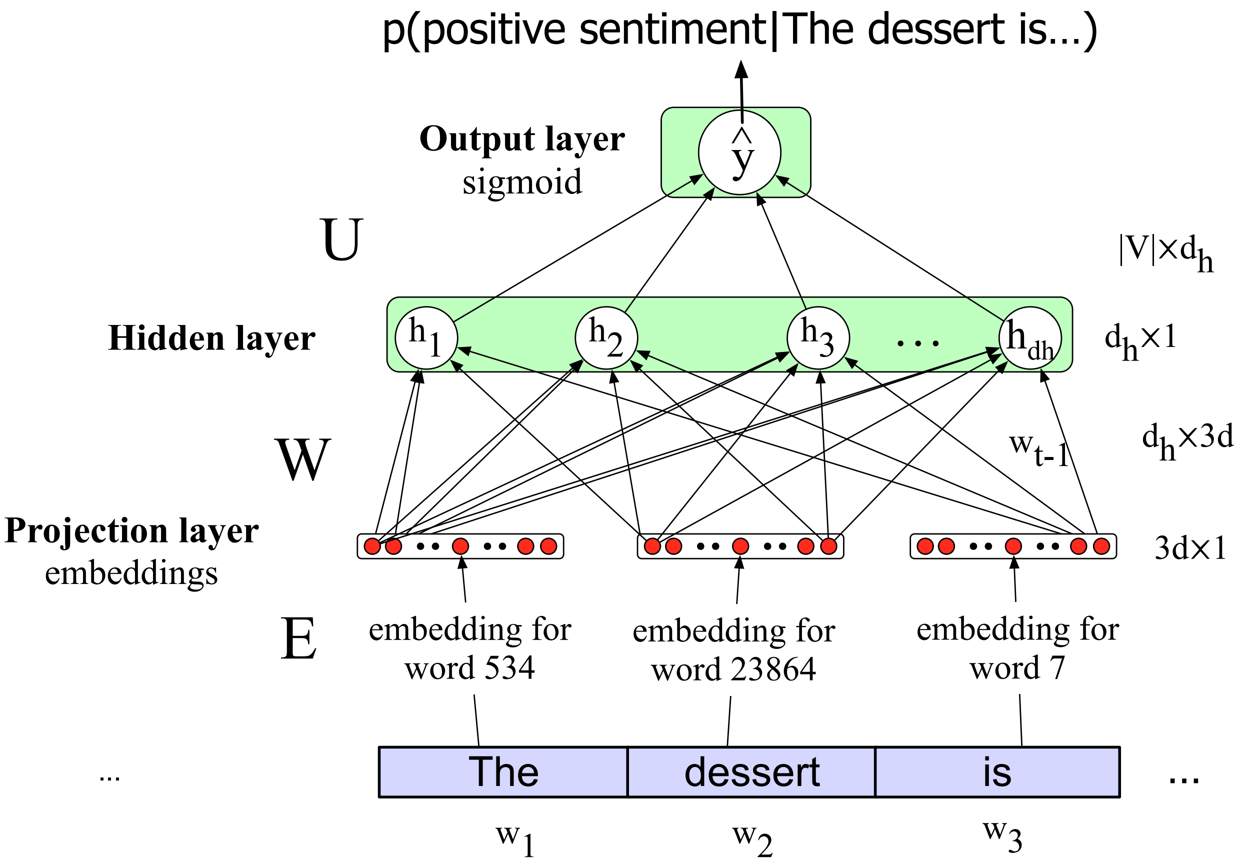
The real power of deep learning comes from the ability to **learn** features from the data

Instead of using hand-built human-engineered features for classification

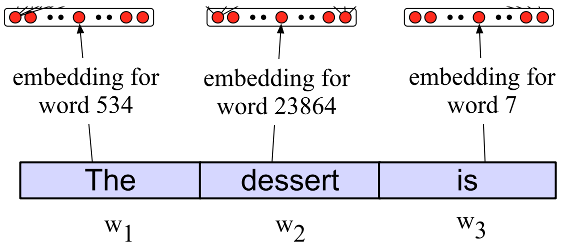
Use learned representations like embeddings!



Neural Net Classification with embeddings as input features!



Issue: texts come in different sizes



This assumes a fixed size length (3)!

Kind of unrealistic.

Some simple solutions (more sophisticated solutions later)

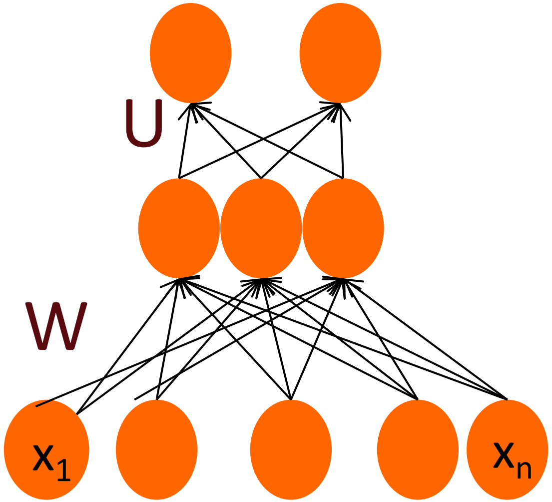
1. Make the input the length of the longest review
    - If shorter then pad with zero embeddings
    - Truncate if you get longer reviews at test time
2. Create a single "sentence embedding" (the same dimensionality as a word) to represent all the words
    - Take the mean of all the word embeddings
    - Take the element-wise max of all the word embeddings
        - For each dimension, pick the max value from all words

Reminder: Multiclass Outputs

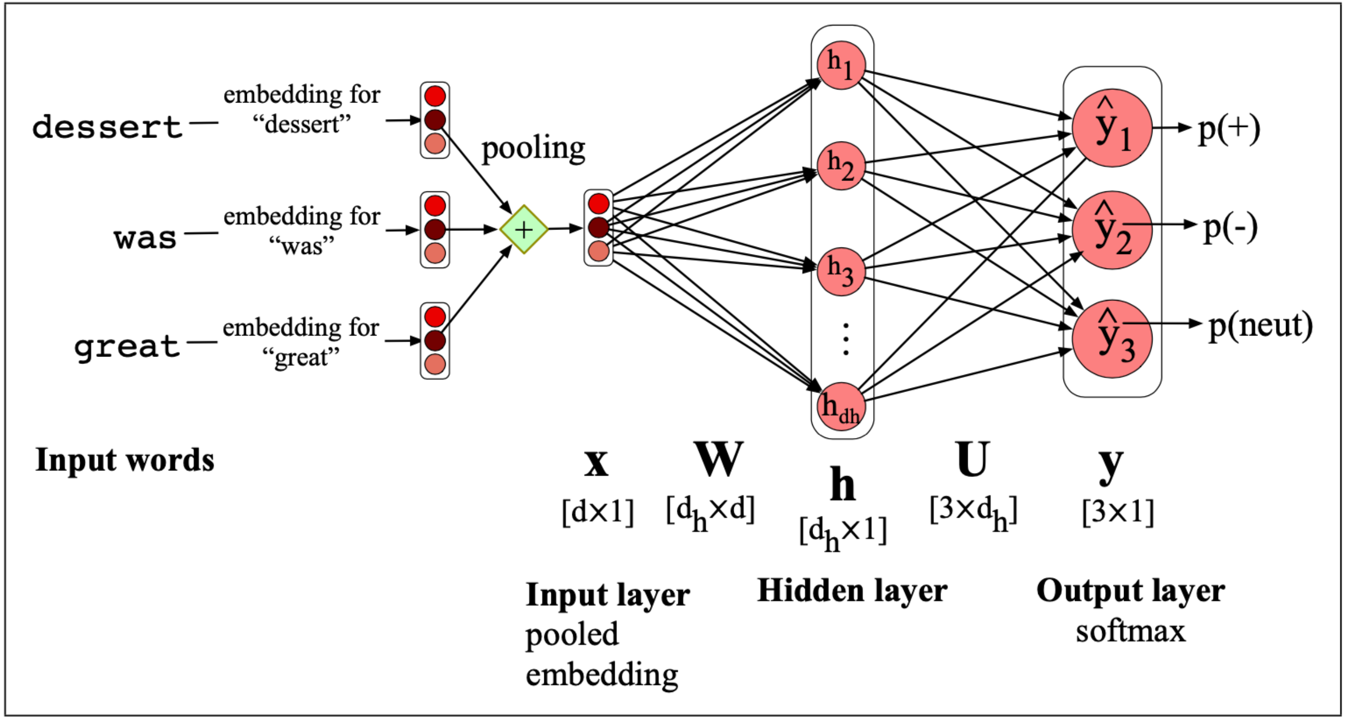
What if you have more than two output classes?

- Add more output units (one for each class)
- And use a "softmax layer"

$$\text{softmax}(z_i) = \frac{e^{z_i}}{\sum_{j=1}^{k} e^{z_j}} \quad 1 \le i \le D$$



Pooling



dessert
embedding for "dessert"
was
embedding for "was"
great
embedding for "great"
pooling
+
h1
h2
h3
hdh
ŷ1
ŷ2
ŷ3
p(+)
p(-)
p(neut)
Input words
x
[d×1]
W
[dh×d]
h
[dh×1]
U
[3×dh]
y
[3×1]
Input layer
pooled
embedding
Hidden layer
Output layer
softmax

Pooling: Vectors and matrices for a single example

$$
\begin{aligned}
\mathbf{x} &= \text{mean}(\mathbf{e}(w_1), \mathbf{e}(w_2), \ldots, \mathbf{e}(w_n)) \\
\mathbf{h} &= \sigma(\mathbf{Wx} + \mathbf{b}) \\
\mathbf{z} &= \mathbf{Uh} \\
\hat{y} &= \text{softmax}(\mathbf{z})
\end{aligned}
\tag{7.21}
$$

Generalizing to **m** test examples

- Pack all input feature vectors for each input **x** into a single matrix **X**, with each row *i* a row vector consisting of the pooled embedding for input example $x^{(i)}$ (i.e. the vector $x^{(i)}$).
- If the dimensionality of the pooled input embedding is *d*, **X** will be a matrix of shape [m x d].

$$\mathbf{H} = \sigma(\mathbf{X}\mathbf{W}^{\mathsf{T}} + \mathbf{b})$$
$$\mathbf{Z} = \mathbf{H}\mathbf{U}^{\mathsf{T}}$$
$$\hat{\mathbf{Y}} = \mathrm{softmax}(\mathbf{Z}) \qquad (7.22)$$

with the following shapes:

$\mathbf{X}$:[m x d],$\mathbf{W}$: [d_h x d],$\mathbf{H}$: [m x d_h] ,$\mathbf{U}$: 3 x d_h , $\mathbf{Z}$: [m x 3], $\hat{\mathbf{Y}}$: [m x 3]

Neural Language Models (LMs)

Language Modeling: Calculating the probability of the next word in a sequence given some history.

- We've seen N-gram based LMs
- But neural network LMs far outperform n-gram language models

State-of-the-art neural LMs are based on more powerful neural network technology like Transformers

But **simple feedforward LMs** can do almost as well!

Simple feedforward Neural Language Models

Task: predict next word w_t

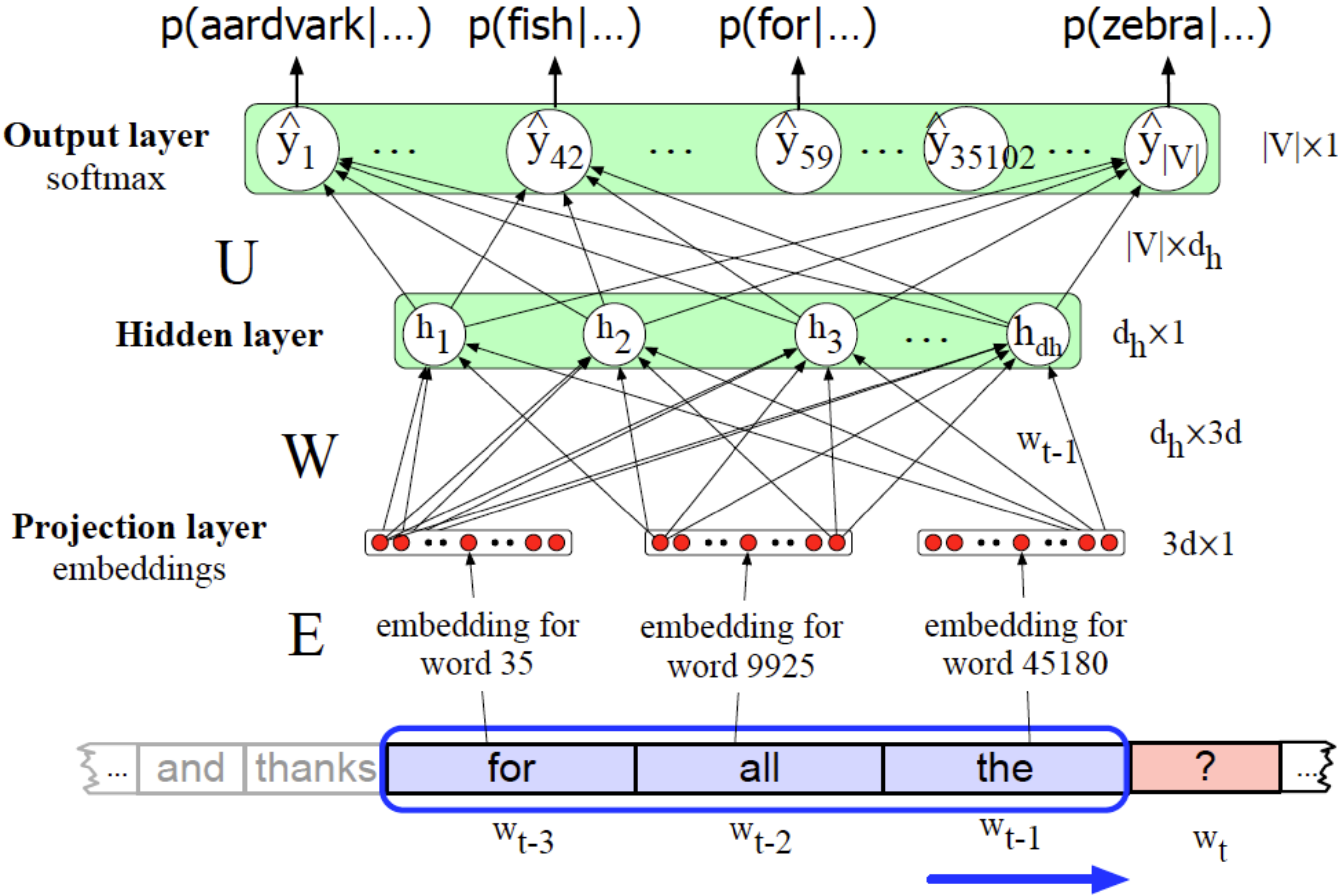
given prior words w_{t-1}, w_{t-2}, w_{t-3}, …

Problem: Now we're dealing with sequences of arbitrary length.

Solution: Sliding windows (of fixed length)

$$P(w_t | w_1^{t-1}) \approx P(w_t | w_{t-N+1}^{t-1})$$

Neural Language Model



Why Neural LMs work better than N-gram LMs

Training data:

We've seen: I have to make sure that the cat gets fed.

Never seen: dog gets fed

Test data:

I forgot to make sure that the dog gets ___

N-gram LM can't predict "fed"!

Neural LM can use similarity of "cat" and "dog" embeddings to generalize and predict “fed” after dog

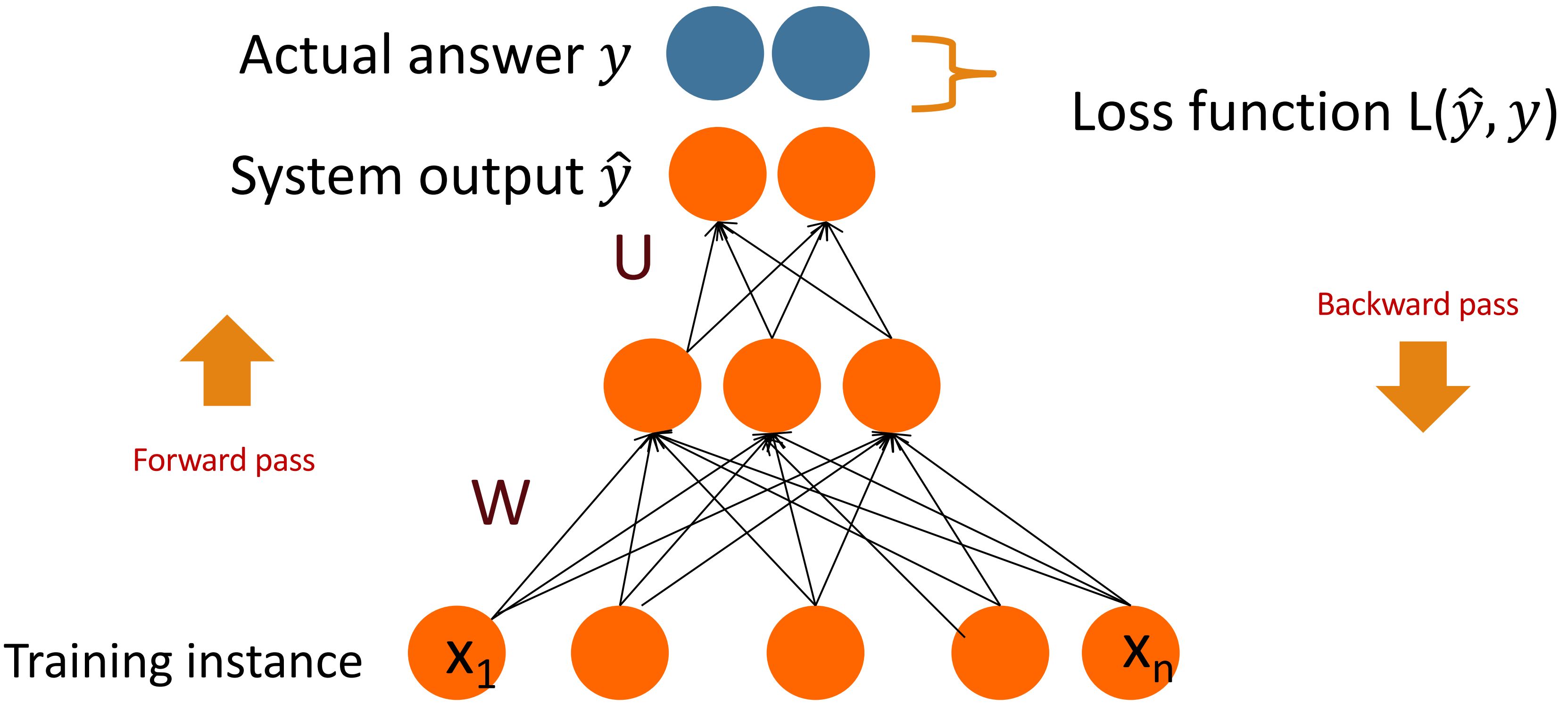
Simple Neural Networks and Neural Language Models

Applying feedforward networks to NLP tasks

Simple Neural Networks and Neural Language Models

Training Neural Nets: Overview

Intuition: training a 2-layer Network



Intuition: Training a 2-layer network

For every training tuple (x, y)

- Run *forward* computation to find our estimate $\hat{y}$
- Run *backward* computation to update weights:
  - For every output node
    - Compute loss L between true y and the estimated $\hat{y}$
    - For every weight w from hidden layer to the output layer
      - Update the weight
  - For every hidden node
    - Assess how much blame it deserves for the current answer
    - For every weight w from input layer to the hidden layer
      - Update the weight

Reminder: Loss Function for binary logistic regression

A measure for how far off the current answer is to the right answer

Cross entropy loss for logistic regression:

$$L_{\mathrm{CE}}(\hat{y}, y) = -\log p(y|x) \;=\; -[y\log\hat{y} + (1-y)\log(1-\hat{y})]$$

$$= \; -[y\log\sigma(w\cdot x+b) + (1-y)\log(1-\sigma(w\cdot x+b))]$$

Reminder: gradient descent for weight updates

Use the derivative of the loss function with respect to weights $\frac{d}{dw}L(f(x;w),y)$

To tell us how to adjust weights for each training item

- Move them in the opposite direction of the gradient

$$w^{t+1} = w^t - \eta \frac{d}{dw}L(f(x;w),y)$$

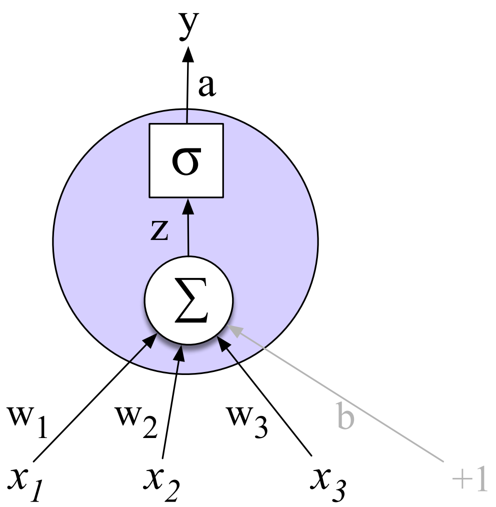
- For logistic regression

$$\frac{\partial L_{\mathrm{CE}}(\hat{y},y)}{\partial w_j} = [\sigma(w \cdot x + b) - y]x_j$$

Where did that derivative come from?

Using the chain rule! $f(x) = u(v(x))$ $\dfrac{df}{dx} = \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$

Intuition (see the text for details)



Derivative of the weighted sum

Derivative of the Activation

Derivative of the Loss

$$\frac{\partial L}{\partial w_i} = \frac{\partial L}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial w_i}$$

How can I find that gradient for every weight in the network?

These derivatives on the prior slide only give the updates for one weight layer: the last one!

What about deeper networks?

- Lots of layers, different activation functions?

Solution in the next lecture:

- Even more use of the chain rule!!
- Computation graphs and backward differentiation!

Simple Neural Networks and Neural Language Models

Training Neural Nets: Overview

Simple Neural Networks and Neural Language Models

Computation Graphs and Backward Differentiation

Why Computation Graphs

For training, we need the derivative of the loss with respect to each weight in every layer of the network

- But the loss is computed only at the very end of the network!

Solution: **error backpropagation** (Rumelhart, Hinton, Williams, 1986)

- **Backprop** is a special case of **backward differentiation**
- Which relies on **computation graphs**.

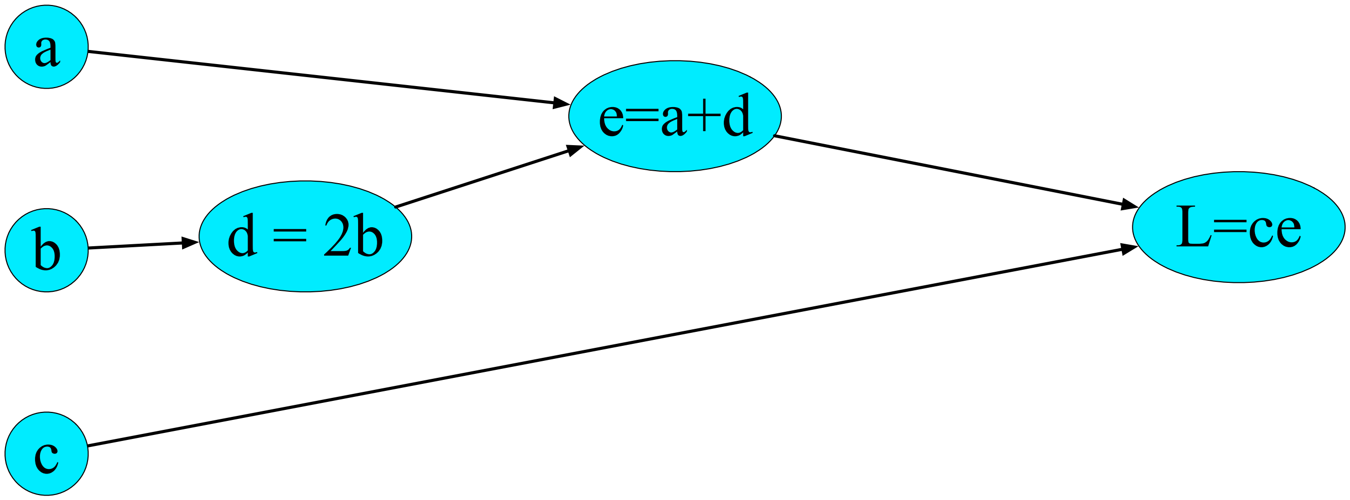
Computation Graphs

A computation graph represents the process of computing a mathematical expression

Example: $L(a,b,c) = c(a+2b)$

Computations:

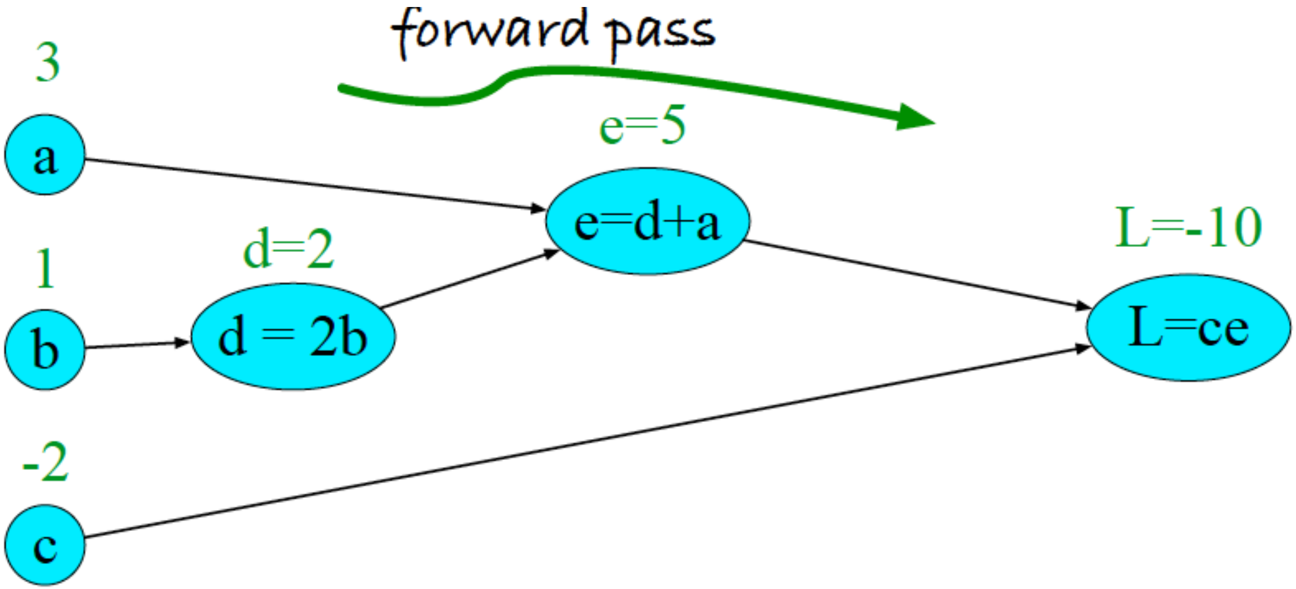
$$\begin{aligned} d &= 2*b \\ e &= a+d \\ L &= c*e \end{aligned}$$



Example: $L(a,b,c) = c(a+2b)$

Computations:

$$\begin{aligned} d &= 2*b \\ e &= a+d \\ L &= c*e \end{aligned}$$



Backwards differentiation in computation graphs

The importance of the computation graph comes from the backward pass

This is used to compute the derivatives that we'll need for the weight update.

Example $L(a,b,c) = c(a+2b)$

$$\begin{aligned} d &= 2*b \\ e &= a+d \\ L &= c*e \end{aligned}$$

We want: $\frac{\partial L}{\partial a}, \frac{\partial L}{\partial b}, \text{ and } \frac{\partial L}{\partial c}$

The derivative $\frac{\partial L}{\partial a}$, tells us how much a small change in *a* affects *L*.

The chain rule

Computing the derivative of a composite function:

$$f(x) = u(v(x)) \qquad \frac{df}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$f(x) = u(v(w(x))) \qquad \frac{df}{dx} = \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

Example $L(a,b,c) = c(a+2b)$

$$\begin{aligned} d &= 2*b \\ e &= a+d \\ L &= c*e \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial c} &= e \\ \frac{\partial L}{\partial a} &= \frac{\partial L}{\partial e}\frac{\partial e}{\partial a} \\ \frac{\partial L}{\partial b} &= \frac{\partial L}{\partial e}\frac{\partial e}{\partial d}\frac{\partial d}{\partial b} \end{aligned}$$

Example

$$L(a,b,c) = c(a+2b)$$

$$\begin{aligned} d &= 2*b \\ e &= a+d \\ L &= c*e \end{aligned}$$

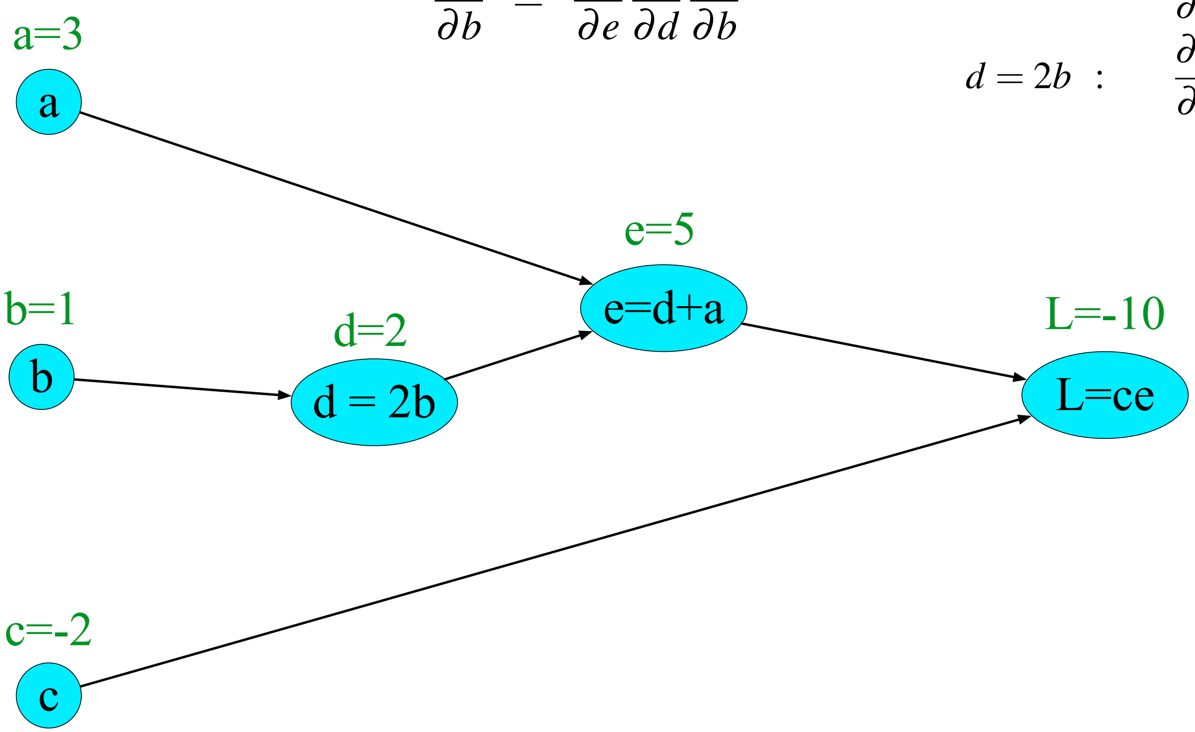
$$\begin{aligned} \frac{\partial L}{\partial a} &= \frac{\partial L}{\partial e}\frac{\partial e}{\partial a} \\ \frac{\partial L}{\partial b} &= \frac{\partial L}{\partial e}\frac{\partial e}{\partial d}\frac{\partial d}{\partial b} \end{aligned}$$

$$\begin{aligned} L = ce &: \quad \frac{\partial L}{\partial e} = c, \frac{\partial L}{\partial c} = e \\ e = a+d &: \quad \frac{\partial e}{\partial a} = 1, \frac{\partial e}{\partial d} = 1 \\ d = 2b &: \quad \frac{\partial d}{\partial b} = 2 \end{aligned}$$

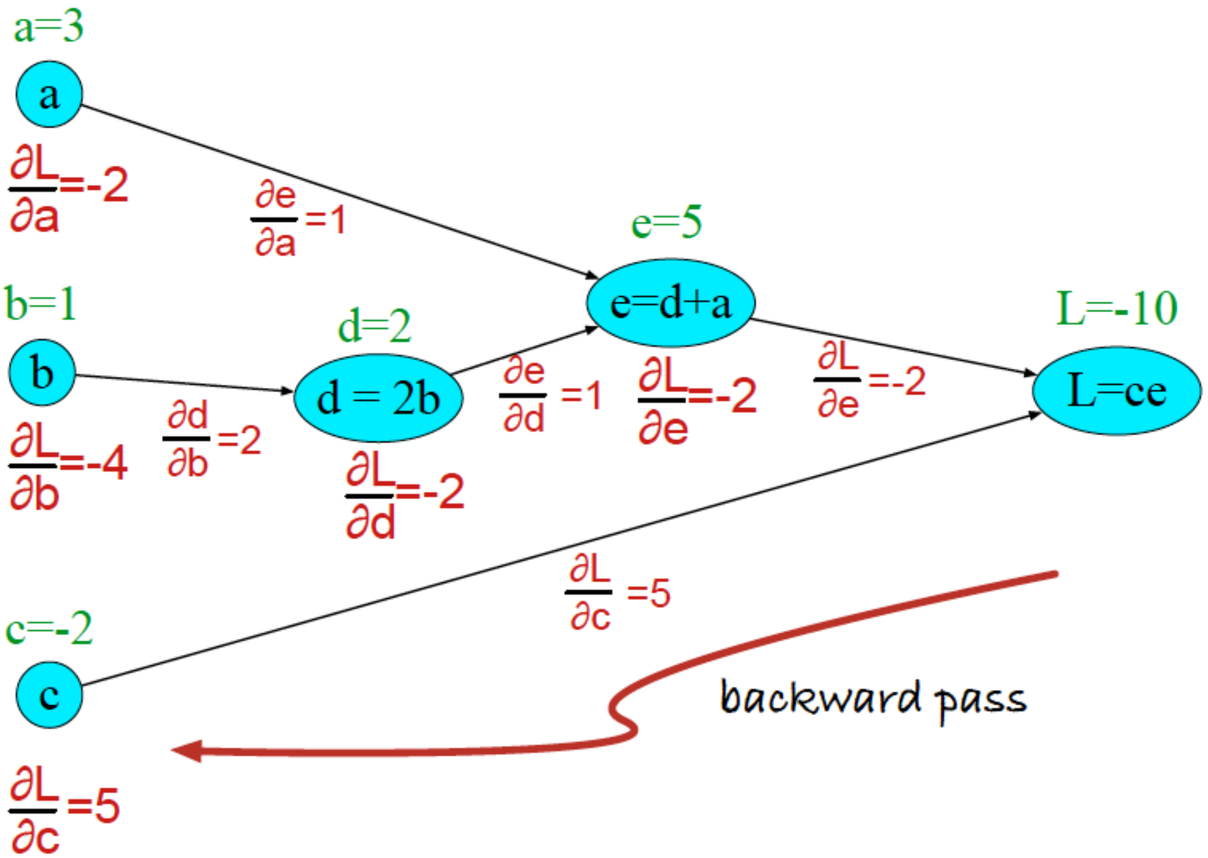
Example

$$\frac{\partial L}{\partial a} = \frac{\partial L}{\partial e}\frac{\partial e}{\partial a}$$
$$\frac{\partial L}{\partial b} = \frac{\partial L}{\partial e}\frac{\partial e}{\partial d}\frac{\partial d}{\partial b}$$

$$L = ce \ : \quad \frac{\partial L}{\partial e} = c, \frac{\partial L}{\partial c} = e$$
$$e = a + d \ : \quad \frac{\partial e}{\partial a} = 1, \frac{\partial e}{\partial d} = 1$$
$$d = 2b \ : \quad \frac{\partial d}{\partial b} = 2$$

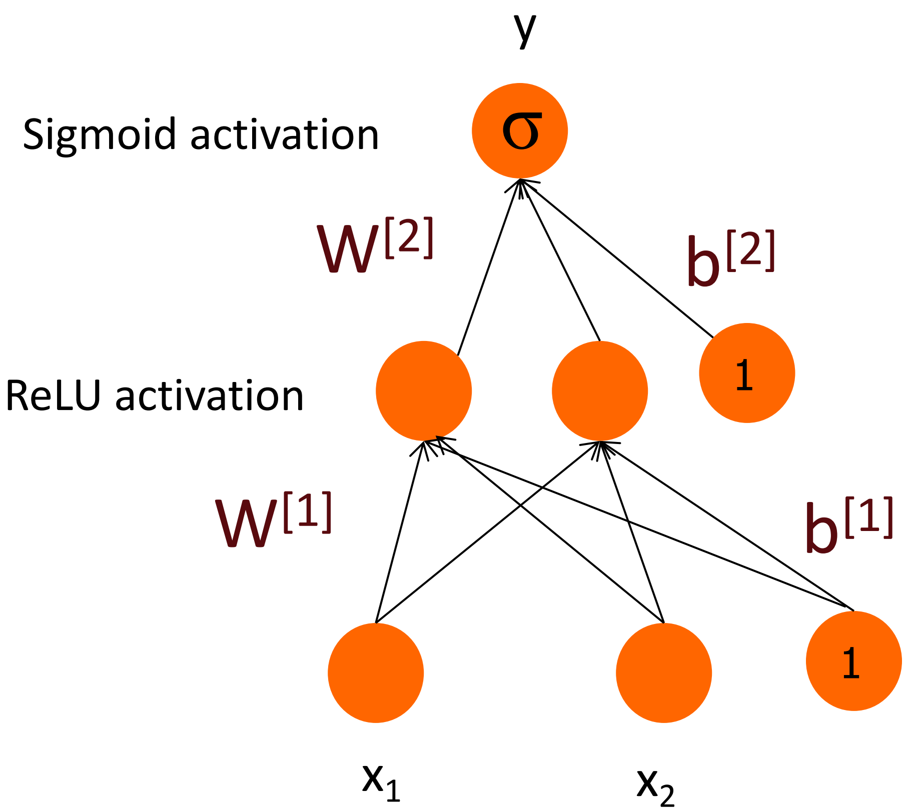


Example



a=3
a
$\frac{\partial L}{\partial a}$=-2
$\frac{\partial e}{\partial a}$=1
e=5
e=d+a
b=1
b
d=2
d = 2b
$\frac{\partial e}{\partial d}$=1
$\frac{\partial L}{\partial e}$=-2
$\frac{\partial L}{\partial e}$=-2
L=-10
L=ce
$\frac{\partial L}{\partial b}$=-4
$\frac{\partial d}{\partial b}$=2
$\frac{\partial L}{\partial d}$=-2
$\frac{\partial L}{\partial c}$=5
c=-2
c
backward pass
$\frac{\partial L}{\partial c}$=5

Backward differentiation on a two layer network



$$
\begin{aligned}
z^{[1]} &= W^{[1]}\mathbf{x} + b^{[1]} \\
a^{[1]} &= \mathrm{ReLU}(z^{[1]}) \\
z^{[2]} &= W^{[2]}a^{[1]} + b^{[2]} \\
a^{[2]} &= \sigma(z^{[2]}) \\
\hat{y} &= a^{[2]}
\end{aligned}
$$

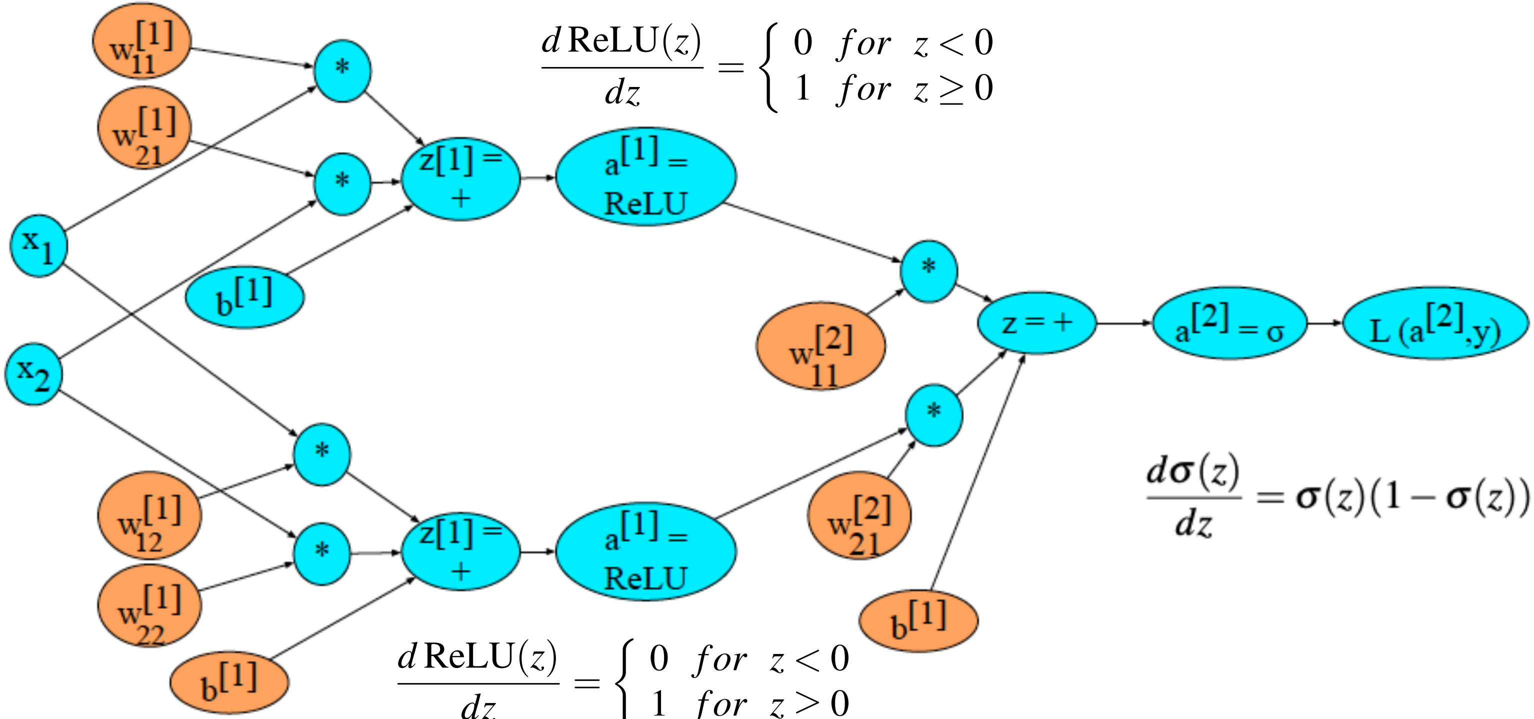
Backward differentiation on a two layer network

$$
\begin{aligned}
z^{[1]} &= W^{[1]}\mathbf{x} + b^{[1]} \\
a^{[1]} &= \mathrm{ReLU}(z^{[1]}) \\
z^{[2]} &= W^{[2]}a^{[1]} + b^{[2]} \\
a^{[2]} &= \sigma(z^{[2]}) \\
\hat{y} &= a^{[2]}
\end{aligned}
$$

$$
\frac{d\,\mathrm{ReLU}(z)}{dz} = \begin{cases} 0 & for \quad z < 0 \\ 1 & for \quad z \geq 0 \end{cases}
$$

$$
\frac{d\sigma(z)}{dz} = \sigma(z)(1 - \sigma(z))
$$

Backward differentiation on a 2-layer network



$$\frac{d\,\mathrm{ReLU}(z)}{dz} = \begin{cases} 0 & for \quad z < 0 \\ 1 & for \quad z \geq 0 \end{cases}$$

$$\frac{d\sigma(z)}{dz} = \sigma(z)(1-\sigma(z))$$

$$\frac{d\,\mathrm{ReLU}(z)}{dz} = \begin{cases} 0 & for \quad z < 0 \\ 1 & for \quad z \geq 0 \end{cases}$$

Starting off the backward pass: $\frac{\partial L}{\partial z}$

(I'll write a for $a^{[2]}$ and z for $z^{[2]}$)

$$\begin{aligned} z^{[1]} &= W^{[1]}\mathbf{x} + b^{[1]} \\ a^{[1]} &= \mathrm{ReLU}(z^{[1]}) \\ z^{[2]} &= W^{[2]}a^{[1]} + b^{[2]} \\ a^{[2]} &= \sigma(z^{[2]}) \\ \hat{y} &= a^{[2]} \end{aligned}$$

$$L(\hat{y}, y) = -(\mathrm{y}\log(\hat{y}) + (1-y)\log(1-\hat{y}))$$

$$L(a, y) = -(\mathrm{y}\log a + (1-y)\log(1-a))$$

$$\frac{\partial L}{\partial z} = \frac{\partial L}{\partial a}\frac{\partial a}{\partial z}$$

$$\begin{aligned} \frac{\partial L}{\partial a} &= -\left(\left(y\frac{\partial \log(a)}{\partial a}\right) + (1-\mathrm{y})\ \frac{\partial \log(1-a)}{\partial a}\right) \\ &= -\left(\left(y\frac{1}{a}\right) + (1-\mathrm{y})\ \frac{1}{1-a}(-1)\right) = -\left(\frac{y}{a} + \frac{y-1}{1-a}\right) \end{aligned}$$

$$\frac{\partial a}{\partial z} = a(1-a) \qquad \frac{\partial L}{\partial z} = -\left(\frac{y}{a} + \frac{y-1}{1-a}\right)a(1-a) = \mathrm{a} - \mathrm{y}$$

Summary

For training, we need the derivative of the loss with respect to weights in early layers of the network

- But loss is computed only at the very end of the network!

Solution: **backward differentiation**

Given a computation graph and the derivatives of all the functions in it we can automatically compute the derivative of the loss with respect to these early weights.

Simple Neural Networks and Neural Language Models

Computation Graphs and Backward Differentiation