Backpropagation – Mathematical Background

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Product and Quotient Rule

Product Rule

$$\frac{d}{dx}(f(x) * (g(x)) = \frac{d}{dx}(f(x)) * g(x) + f(x) * \frac{d}{dx}(g(x)) \qquad (1)$$

Quotient Rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{\frac{d}{dx}(f(x)) * g(x) - f(x) * \frac{d}{dx}(g(x))}{[g(x)]^2} \qquad (2)$$

Chain Rule

$$\frac{d}{dx}\left[f(g(x))\right] = f'\left[g(x)\right] * g'(x) \tag{3}$$

Remarks:

- $f(u)$ is is called *the outside function*
- $g(x)$ is is called *the inside function*
- The chain rule can be stated in words as follows: $\frac{df}{dx}$ equals the product of the derivatives of the outside function $\frac{df}{du}$ and of the inside function $\frac{du}{dx}$.
- The chain rule is often helpful when taking derivatives of functions such as: $\frac{d}{dx}(5x-4)^6$, $\frac{d}{dx}\sqrt{x^2-1}$, $\frac{d}{dx}\frac{1}{x^2-4x+5}$

Chain Rule – Example

$$\frac{d}{dx}\left[f(g(x))\right] = f'\left[g(x)\right] * g'(x) \tag{4}$$

Find the derivative of $f(x) = (x^2 + 1)^3$

$$\begin{aligned} f'(x) &= 3(x^2+1)^{3-1} * 2x^{2-1} \\ &= 3((x^2+1)^2(2x) \\ &= 6x((x^2+1)^2 \end{aligned} \tag{5}$$

Derivative of Sigmoid Function

$$
\begin{aligned}
\frac{d}{dx}\sigma(x) &= \frac{d}{dx}\left[\frac{1}{1+e^{-x}}\right] \\
&= \frac{(0)(1+e^{-x}) - (-e^{-x})(1)}{(1+e^{-x})^2} \\
&= \frac{e^{-x}}{(1+e^{-x})^2} \\
&= \frac{1}{1+e^{-x}}\frac{e^{-x}}{1+e^{-x}}
\end{aligned}
\tag{6}
$$

Derivative of Sigmoid Function (continued)

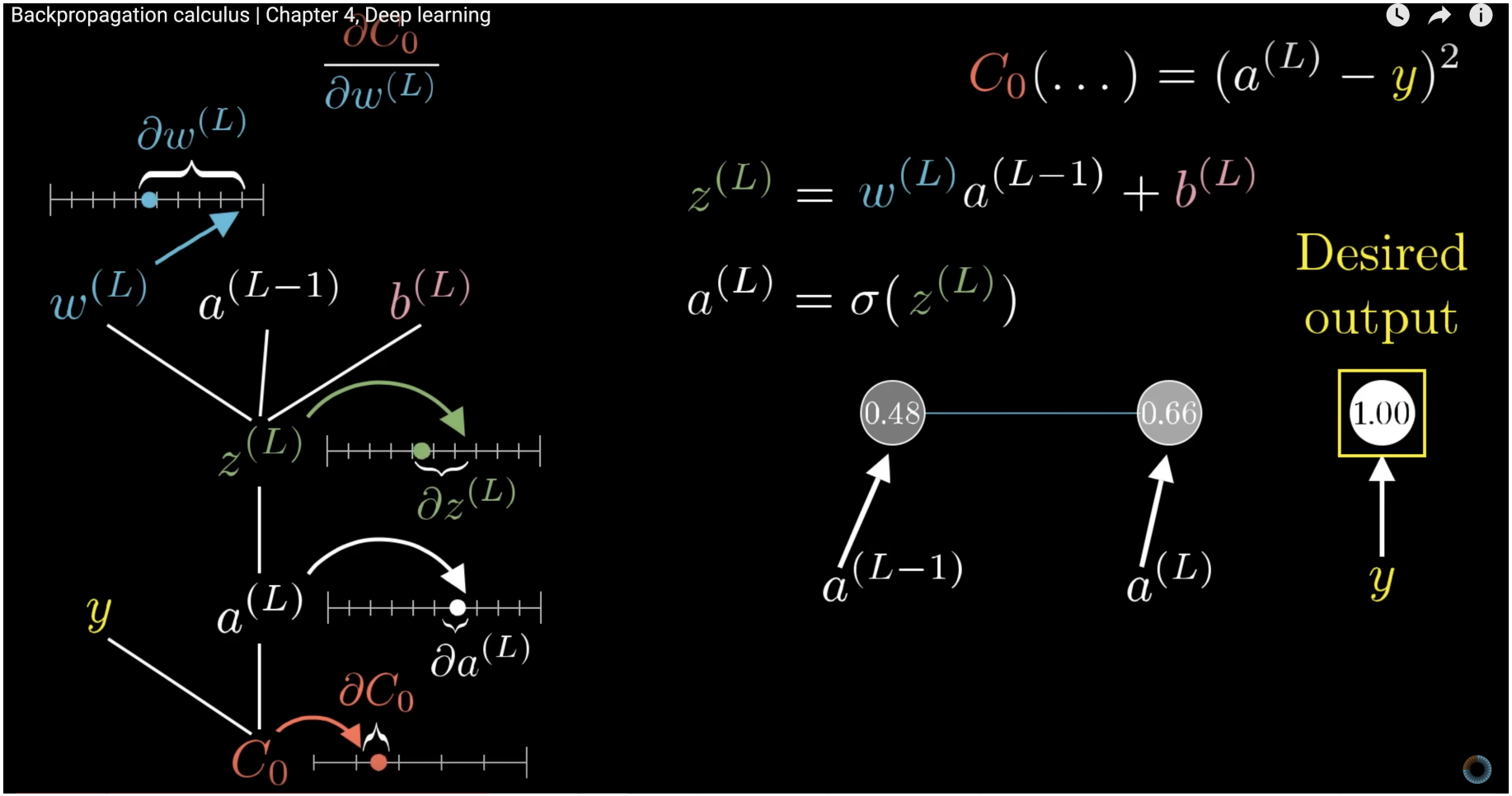
$$
\begin{aligned}
&= \frac{1}{1+e^{-x}} \frac{e^{-x}+(1-1)}{1+e^{-x}} \\
&= \frac{1}{1+e^{-x}} \frac{1+e^{-x}-1}{1+e^{-x}} \\
&= \frac{1}{1+e^{-x}} \left[\frac{1+e^{-x}}{1+e^{-x}} - \frac{1}{1+e^{-x}} \right] \\
&= \frac{1}{1+e^{-x}} \left[1 - \frac{1}{1+e^{-x}} \right] \\
&= \sigma(x)(1-\sigma(x))
\end{aligned}
\tag{7}
$$

Derivatives of ReLU and tanh

$$\frac{dReLU(z)}{dz} = \begin{cases} 0 & \text{for z} < 0 \\ 1 & \text{for z} \geq 0 \end{cases} \tag{8}$$

$$\frac{dtanh(z)}{dz} = 1 - tanh^2(z) \tag{9}$$

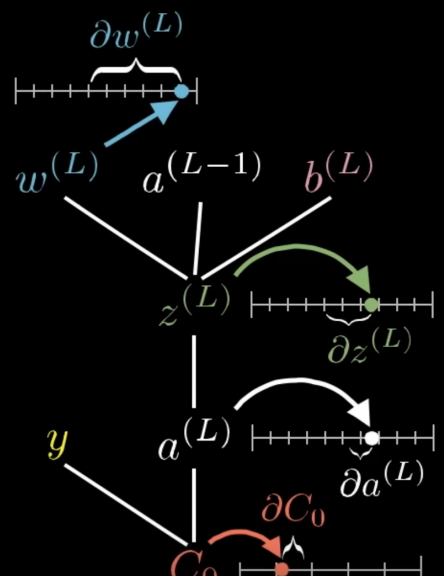
Backprop in a Simple FFN



Applying the Chain Rule

$$\frac{\partial C_0}{\partial w^{(L)}} = \frac{\partial z^{(L)}}{\partial w^{(L)}} \frac{\partial a^{(L)}}{\partial z^{(L)}} \frac{\partial C0}{\partial a^{(L)}}$$

Chain rule

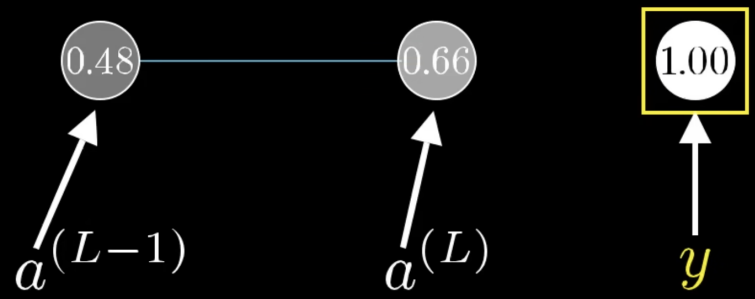


$$C_0(\dots) = (a^{(L)} - y)^2$$

$$z^{(L)} = w^{(L)} a^{(L-1)} + b^{(L)}$$

$$a^{(L)} = \sigma(z^{(L)})$$

Desired output



Computing Relevant Derivatives

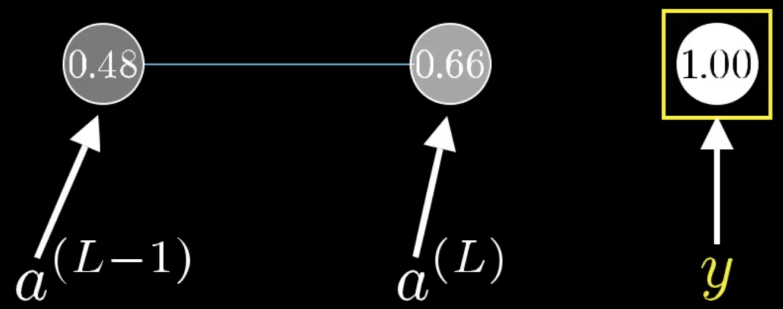
$$\frac{\partial C_0}{\partial w^{(L)}} = \frac{\partial z^{(L)}}{\partial w^{(L)}} \frac{\partial a^{(L)}}{\partial z^{(L)}} \frac{\partial C0}{\partial a^{(L)}}$$

$$C_0 = (a^{(L)} - y)^2$$

$$\frac{\partial C0}{\partial a^{(L)}} = 2(a^{(L)} - y)$$

$$a^{(L)} = \sigma(z^{(L)})$$

$$\frac{\partial a^{(L)}}{\partial z^{(L)}} = \sigma'(z^{(L)})$$



$$\frac{\partial z^{(L)}}{\partial w^{(L)}} = a^{(L-1)}$$

$$z^{(L)} = w^{(L)} a^{(L-1)} + b^{(L)}$$

Average of All Training Examples

$$\frac{\partial C_0}{\partial w^{(L)}} = \frac{\partial z^{(L)}}{\partial w^{(L)}} \frac{\partial a^{(L)}}{\partial z^{(L)}} \frac{\partial C0}{\partial a^{(L)}} = a^{(L-1)} \sigma'(z^{(L)}) 2(a^{(L)} - y)$$

$$C_0 = (a^{(L)} - y)^2$$

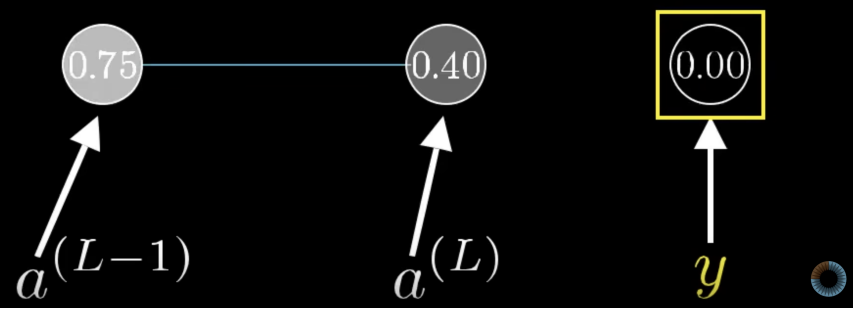
Average of all training examples

$$\underbrace{\frac{\partial C}{\partial w^{(L)}}} = \overbrace{\frac{1}{n} \sum_{k=0}^{n-1} \frac{\partial C_k}{\partial w^{(L)}}}$$

Derivative of full cost function

$$z^{(L)} = w^{(L)} a^{(L-1)} + b^{(L)}$$

$$a^{(L)} = \sigma(z^{(L)})$$



Gradient

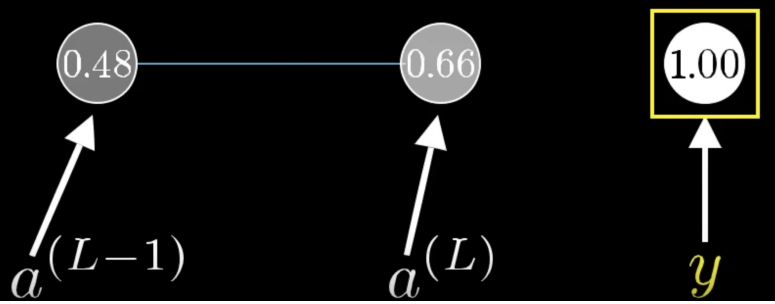
$$\frac{\partial C_0}{\partial w^{(L)}} = \frac{\partial z^{(L)}}{\partial w^{(L)}} \frac{\partial a^{(L)}}{\partial z^{(L)}} \frac{\partial C0}{\partial a^{(L)}} = a^{(L-1)} \sigma'(z^{(L)}) 2(a^{(L)} - y)$$

$$\nabla C = \begin{bmatrix} \frac{\partial C}{\partial w^{(1)}} \\ \frac{\partial C}{\partial b^{(1)}} \\ \vdots \\ \frac{\partial C}{\partial w^{(L)}} \\ \frac{\partial C}{\partial b^{(L)}} \end{bmatrix}$$

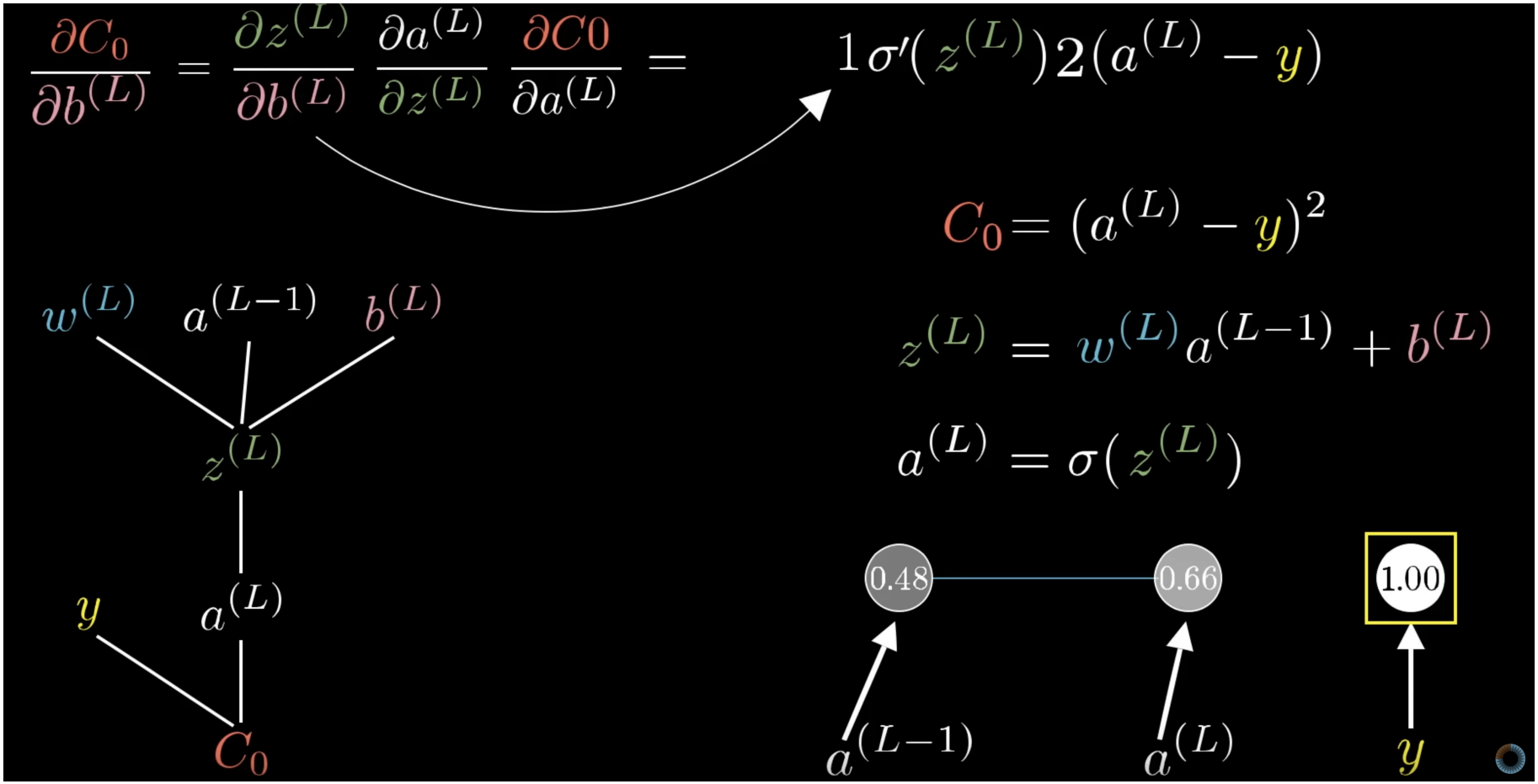
$$C_0 = (a^{(L)} - y)^2$$

$$z^{(L)} = w^{(L)} a^{(L-1)} + b^{(L)}$$

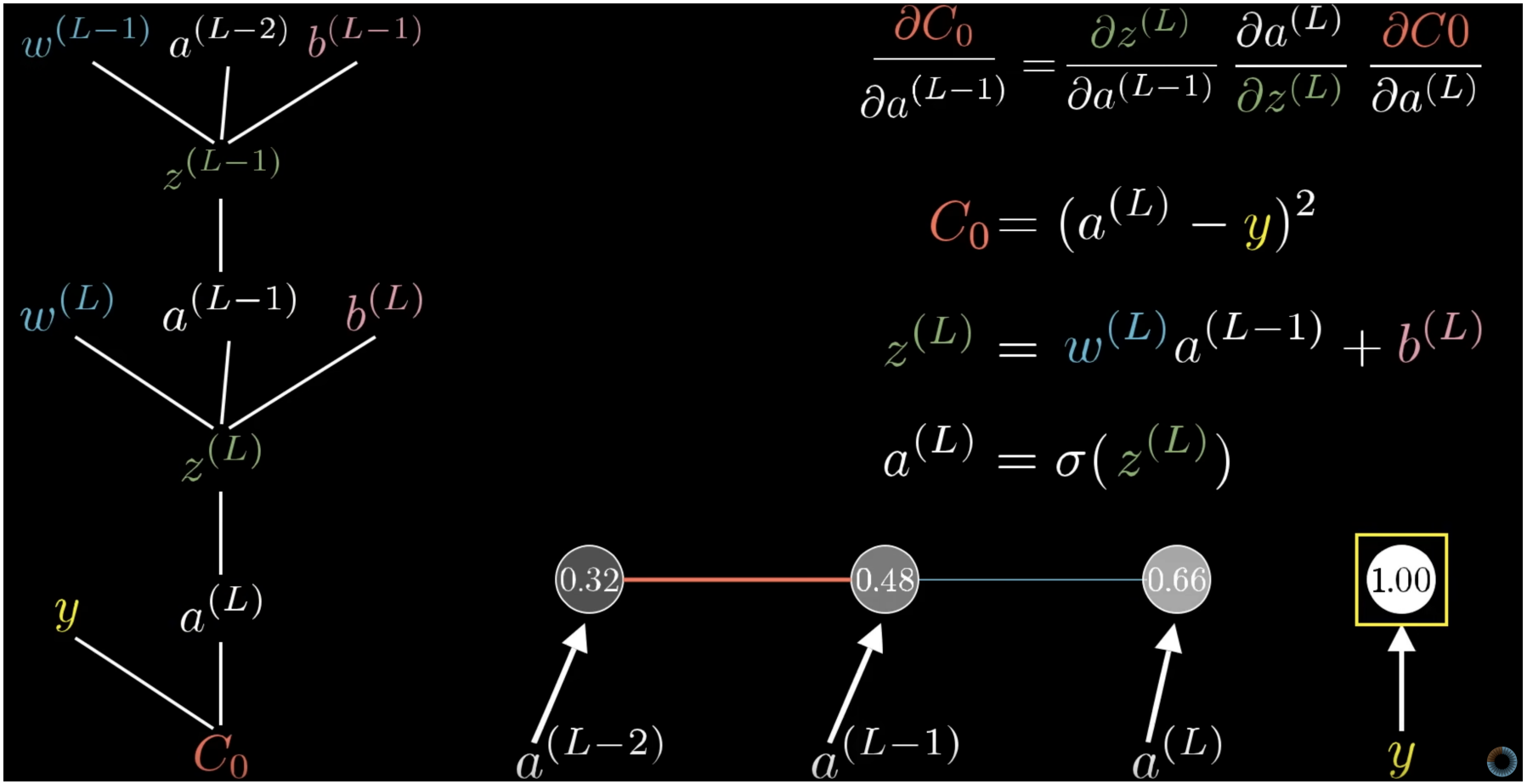
$$a^{(L)} = \sigma(z^{(L)})$$



Derivative of Bias



Adding Layers



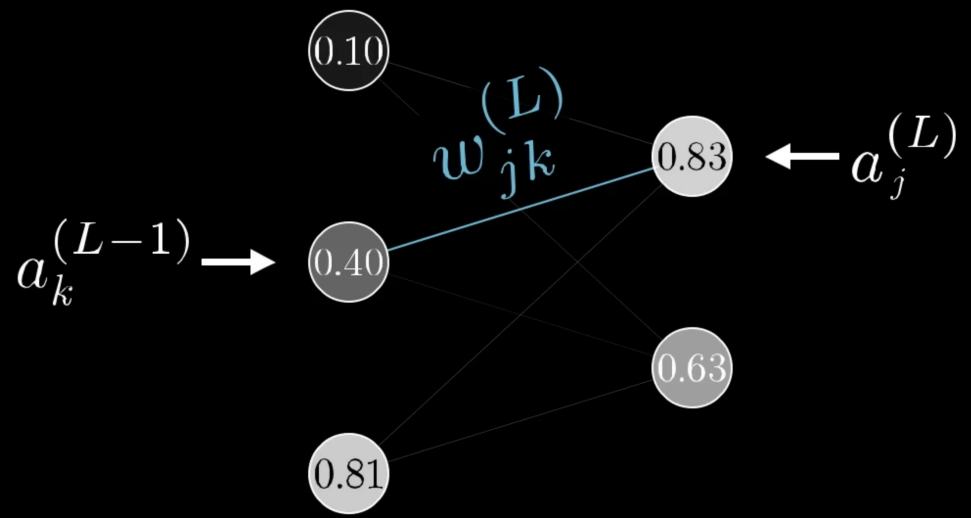
Adding Nodes Per Layer

$$\frac{\partial C_0}{\partial w_{jk}^{(L)}} = \frac{\partial z_j^{(L)}}{\partial w_{jk}^{(L)}} \frac{\partial a_j^{(L)}}{\partial z_j^{(L)}} \frac{\partial C_0}{\partial a_j^{(L)}}$$

$$z_j^{(L)} = \cdots + w_{jk}^{(L)} a_k^{(L-1)} + \cdots$$

$$a_j^{(L)} = \sigma(z_j^{(L)})$$

$$C_0 = \sum_{j=0}^{n_L - 1} (a_j^{(L)} - y_j)^2$$



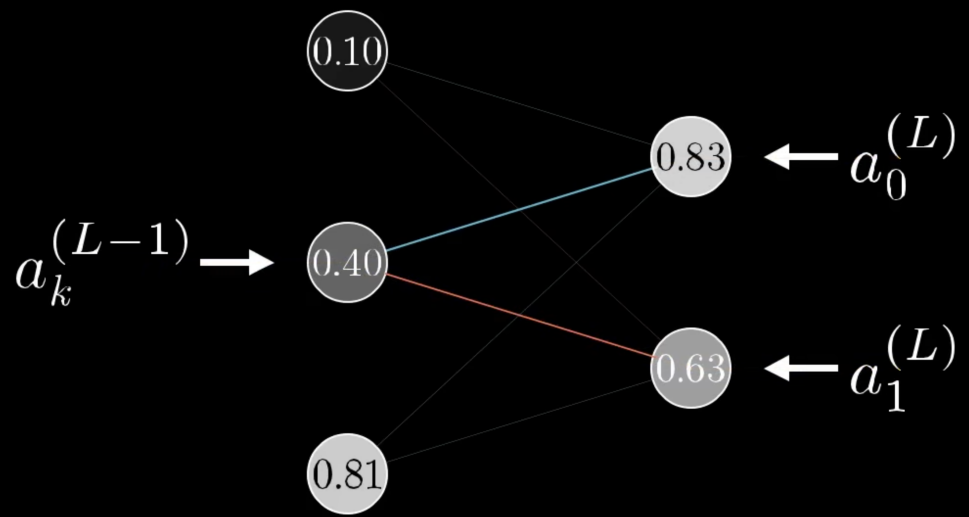
Summing Over Nodes Per Layer

$$\frac{\partial C_0}{\partial a_k^{(L-1)}} = \underbrace{\sum_{j=0}^{n_L-1} \frac{\partial z_j^{(L)}}{\partial a_k^{(L-1)}} \frac{\partial a_j^{(L)}}{\partial z_j^{(L)}} \frac{\partial C_0}{\partial a_j^{(L)}}}_{\text{Sum over layer L}}$$

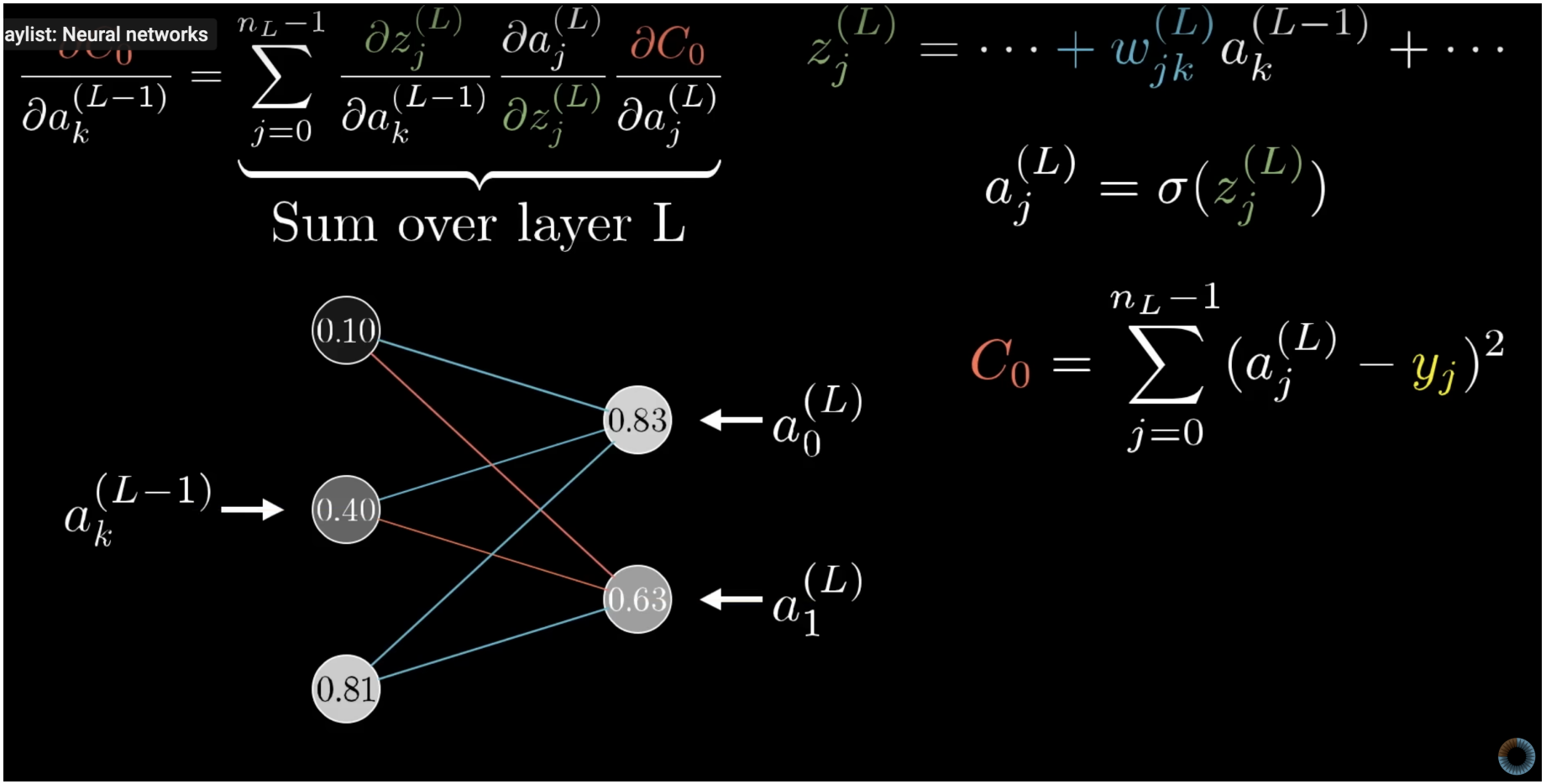
$$z_j^{(L)} = \cdots + w_{jk}^{(L)} a_k^{(L-1)} + \cdots$$

$$a_j^{(L)} = \sigma(z_j^{(L)})$$

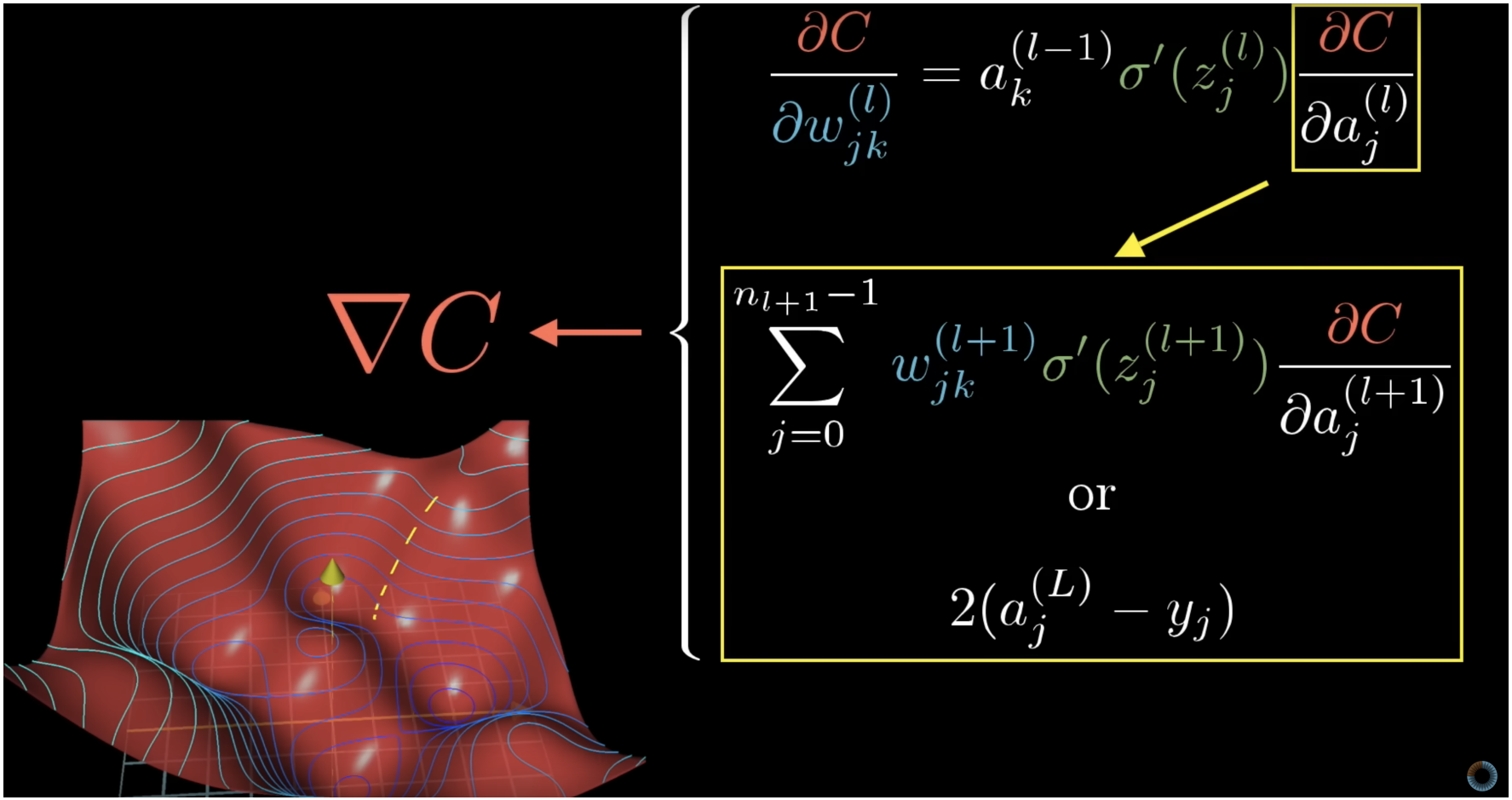
$$C_0 = \sum_{j=0}^{n_L-1} (a_j^{(L)} - y_j)^2$$



Summing Over Layers



Summary


$\nabla C \leftarrow$
$\frac{\partial C}{\partial w_{jk}^{(l)}} = a_k^{(l-1)} \sigma'(z_j^{(l)}) \frac{\partial C}{\partial a_j^{(l)}}$
$\sum_{j=0}^{n_{l+1}-1} w_{jk}^{(l+1)} \sigma'(z_j^{(l+1)}) \frac{\partial C}{\partial a_j^{(l+1)}}$
or
$2(a_j^{(L)} - y_j)$

Derivatives of log functions

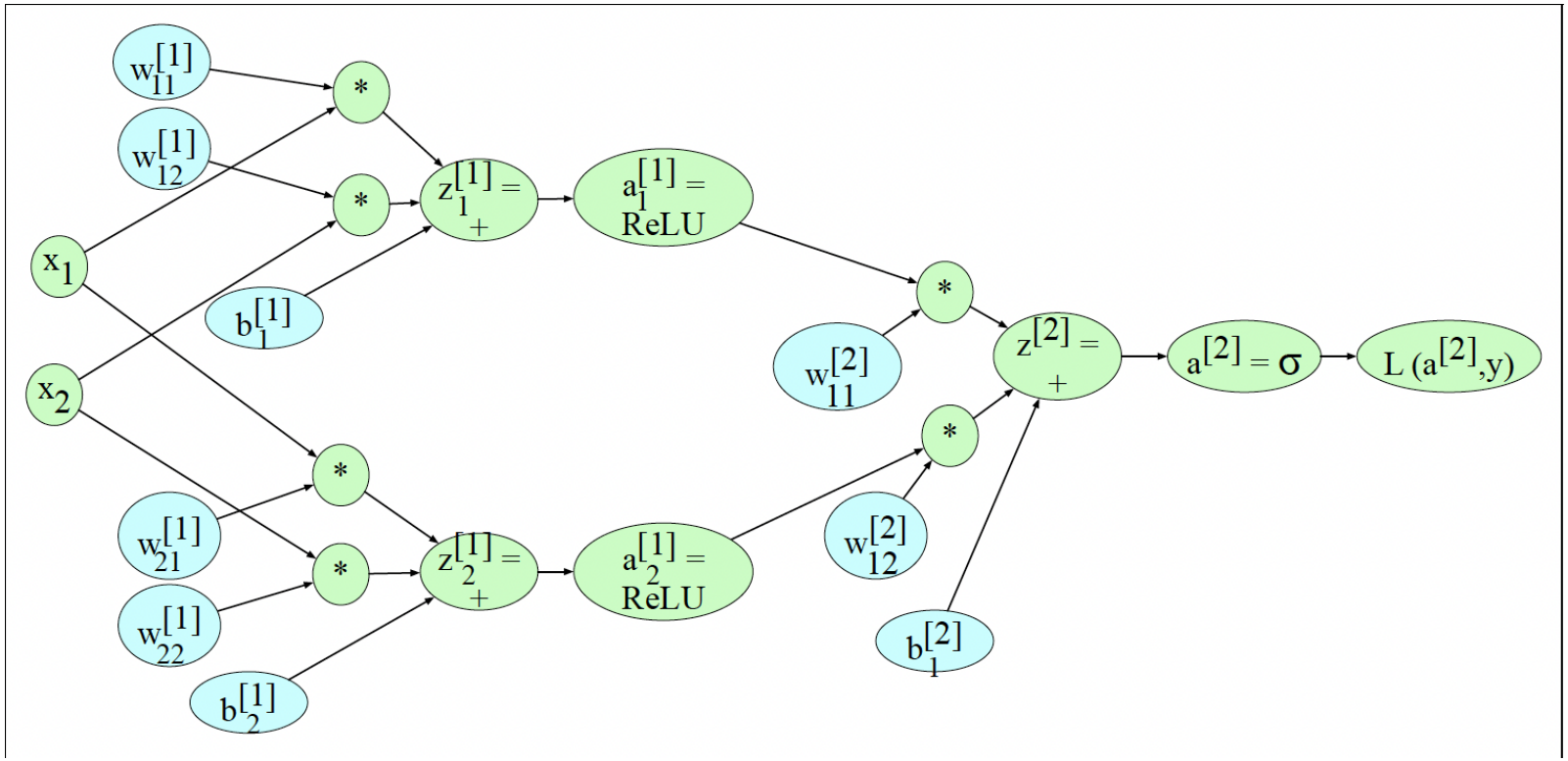
Derivative of Common Logarithm

$$\frac{d}{dx} log_a x = \frac{1}{x * ln(a)} \tag{10}$$

Derivative of Natural Logarithm

$$\frac{d}{dx} ln\ x = \frac{1}{x} \tag{11}$$

Sample Computation Graph for a simple 2-layer NN



Derivative of the CE Loss Function with Respect to z

$$L_{CE}(a^{[2]}, y) = -\left[y \ log \ a^{[2]} + (1-y) log(1-a^{[2]})\right] \tag{12}$$

$$\begin{aligned}
\frac{\partial L}{\partial a^{[2]}} &= -\left(\left(y\frac{\partial log(a^{[2]})}{\partial a^{[2]}}\right) + (1-y)\frac{\partial log(1-a^{[2]})}{\partial a^{[2]}}\right) \\
&= -\left(\left(y\frac{1}{a^{[2]}}\right) + (1-y)\frac{1}{1-a^{[2]}}(-1)\right) \qquad (13) \\
&= -\left(\frac{y}{a^{[2]}} + \frac{y-1}{1-a^{[2]}}\right)
\end{aligned}$$

Derivative of the Sigmoid and the Chain Rule

$$\frac{\partial a^{[2]}}{\partial z} = a^{[2]}(1 - a^{[2]}) \tag{14}$$

$$\begin{aligned}
\frac{\partial L}{\partial z} &= \frac{\partial L}{\partial a^{[2]}} \frac{\partial a^{[2]}}{\partial z} \\
&= -\left(\frac{y}{a^{[2]}} + \frac{y-1}{1-a^{[2]}} \right) a^{[2]}(1 - a^{[2]}) \qquad (15) \\
&= a^{[2]} - y
\end{aligned}$$