Linear algebra

Session 01: Introduction

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October 26, 2023

Homework

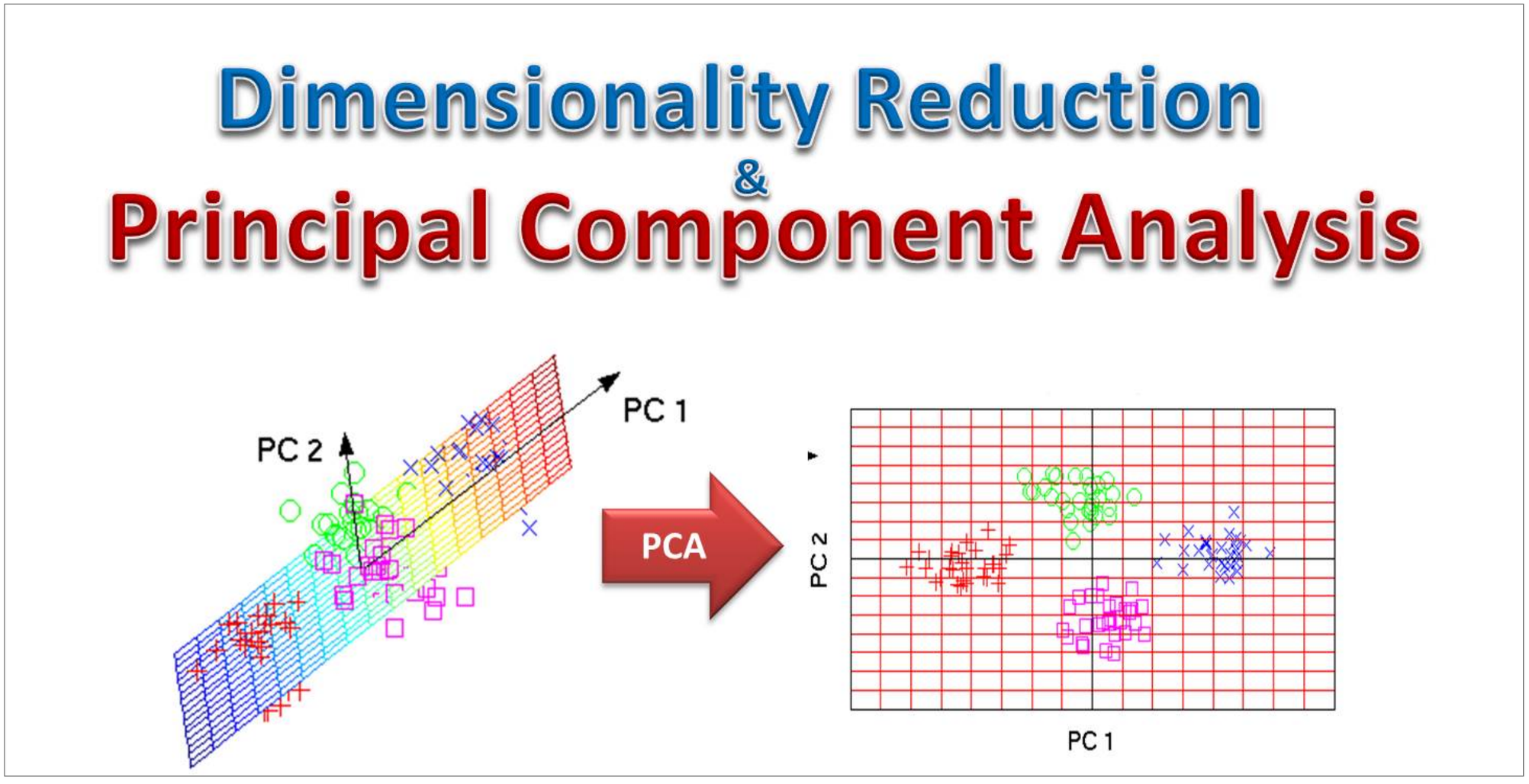
Homework assignments have to be submitted as pdf files via Moodle. You can write them on the computer, but you can also write them by hand to upload fotos.

Side remark

I will illustrate some concepts computationally, using *Python* and the packages *numpy* and *sympy*.
You can use these tools if you find them helpful, but this is not a required part of the course. It is possible to do all assignments purely with pencil and paper.

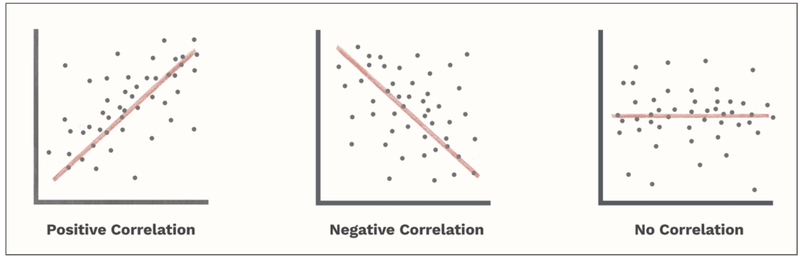
Applications of linear algebra

Exploratory data analysis, e.g.



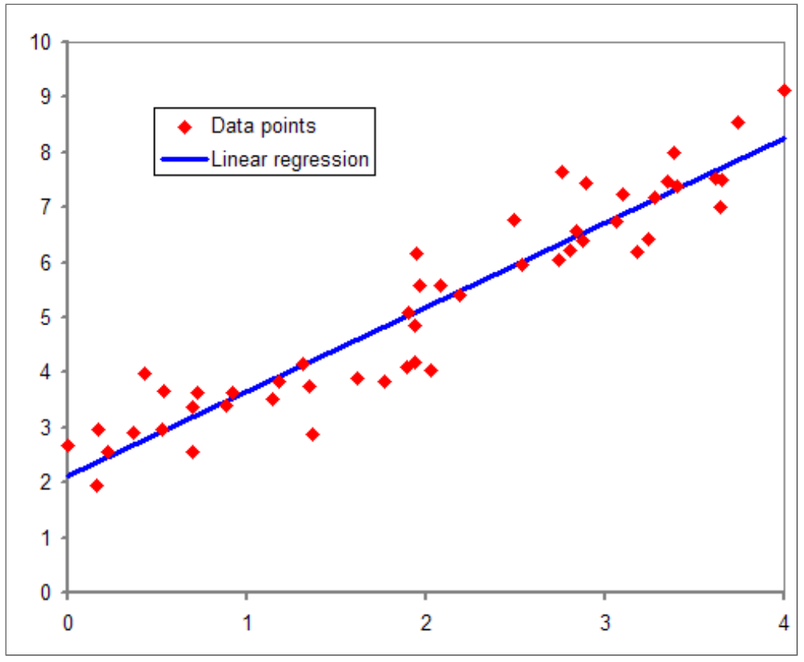
Applications of linear algebra

Descriptive statistics, e.g correlation



Applications of linear algebra

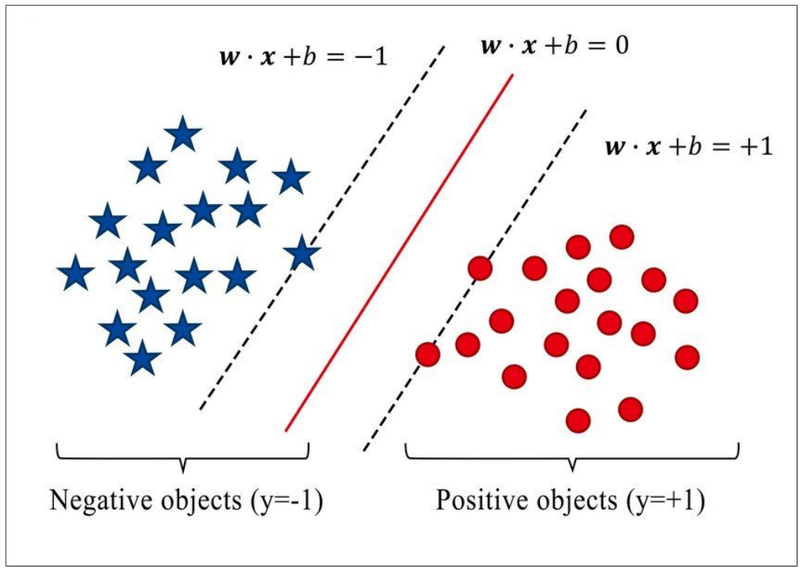
Inferential statistics, e.g. linear regression



$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

Applications of linear algebra

Machine learning



Applications of linear algebra

Game theory

		PRISONER 2	
		Confess	Lie
PRISONER 1	Confess	-8 , -8	0 , -10
	Lie	-10 , 0	-1 , -1

$$E(u) = \tau' SRa$$

Vectors and linear equations

- basis of linear algebra: solving **linear equations**
- Examples:

$$\begin{aligned} 3x + 4y - z &= 0 \\ y + z &= 4 \\ x - y - z &= -5 \end{aligned}$$

or

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

Let's focus on the second example. We can write this in **matrix notation**.

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Schematically:

$$A\mathbf{x} = \mathbf{b}$$

The goal is to find x and y, (in general: $\mathbf{b}$) which solve the equation.
Two fundamental ways to approach this problem geometrically:

- **row picture**
- **column picture**

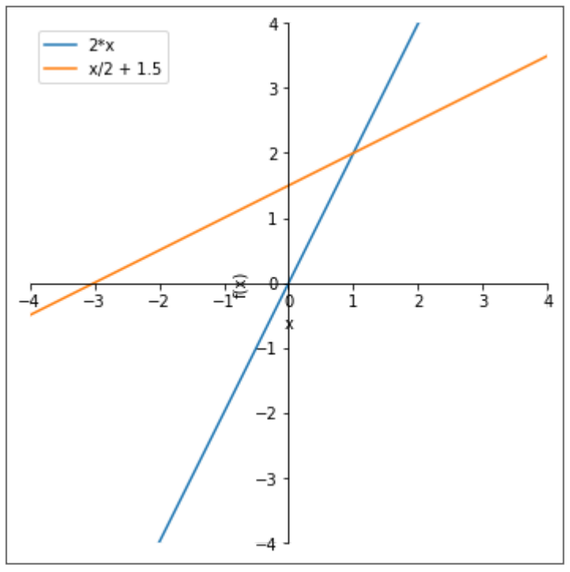
Row picture

$$2x - y = 0$$

describes a line in a Cartesian plane. So does

$$-x + 2y = 3$$

The solution is where the lines intersect.



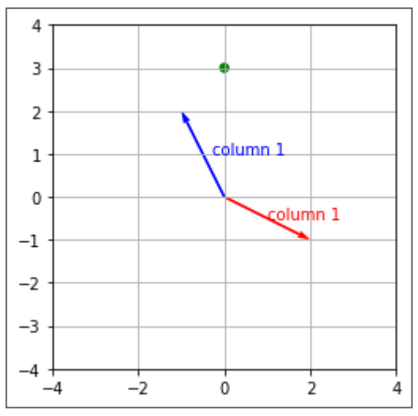
Column picture

recall

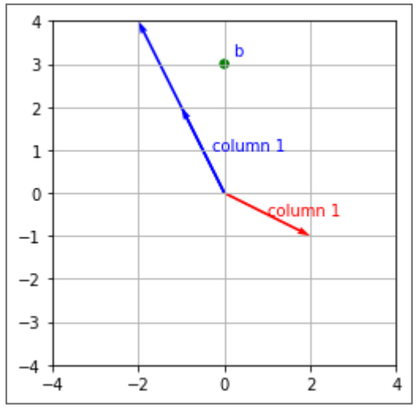
$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

This can be decomposed into

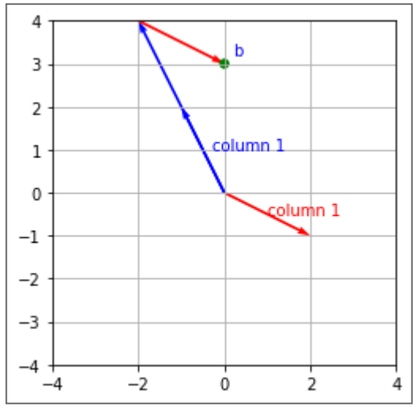
$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} x + \begin{pmatrix} -1 \\ 2 \end{pmatrix} y = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$



Vectors can be multiplied with real numbers.
Let us multiply the second column with 2.



Vectors can also be added. Let's add column one to 2*(column 1).



- so our solution is

$$x = 1$$
$$y = 2$$

3 equations with three unknowns

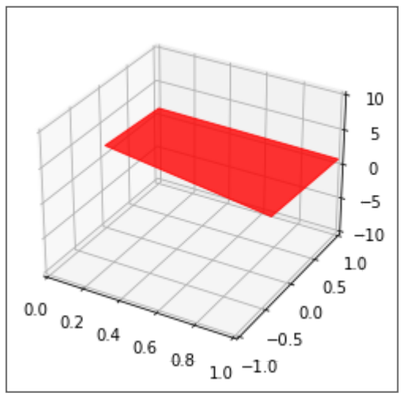
$$
\begin{aligned}
x + 2y + 3z &= 6 \\
2x + 5y + 2z &= 4 \\
6x - 3y + z &= 2
\end{aligned}
$$

row picture

A linear equation in three unknowns describes a **plane** in a 3-dimensional space.

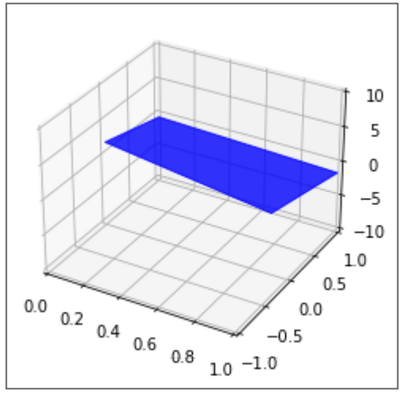
first row:

```
(-10.0, 10.0)
```



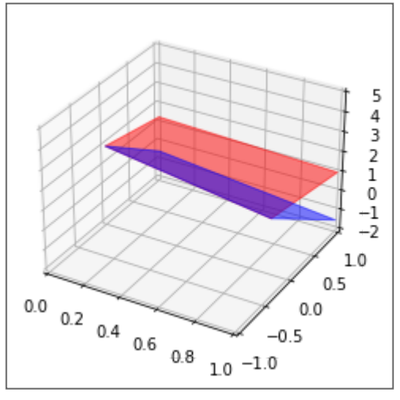
second row

```
(-10.0, 10.0)
```



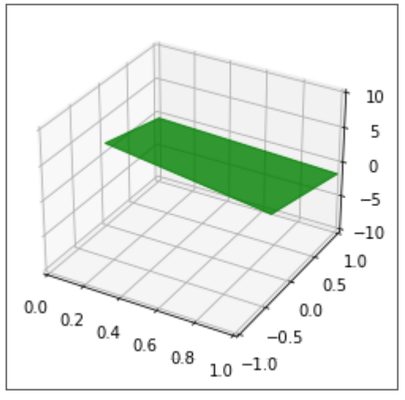
both rows together

```
(-2.0, 5.0)
```



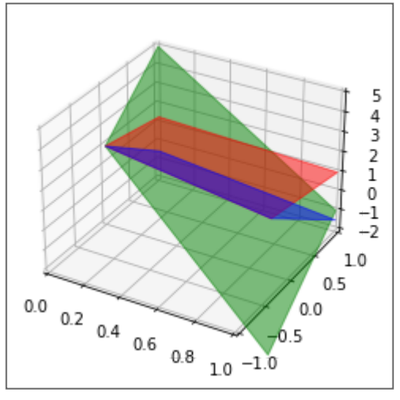
third row

```
(-10.0, 10.0)
```



all three rows together

```
(-2.0, 5.0)
```



- the solution set for each row is a **plane**
- the intersection of two plane (if they are not parallel) is a **line**
- the intersection of three planes is a **point**

This point is the solution of the system of equations.

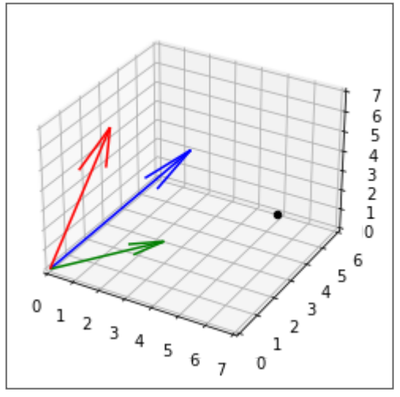
column picture

recall the system to be solved:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned}$$

- can be rewritten as

$$\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} x + \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} y + \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} z = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$$


0 1 2 3 4 5 6 7
0 1 2 3 4 5 6 7
0 1 2 3 4 5 6 7

- We can see with the bare eye that **b** is a multiple of the third column vector. So the solutions for x and y are 0. The solution for z happens to be 2.

We have

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & 3 & 1 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

The solution is

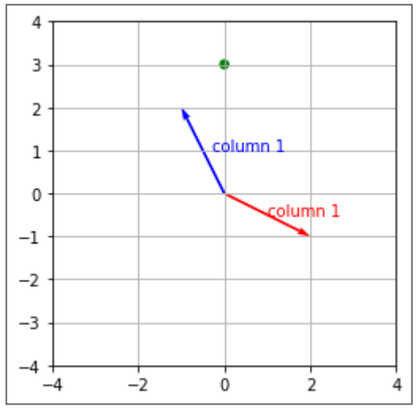
$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Big question?

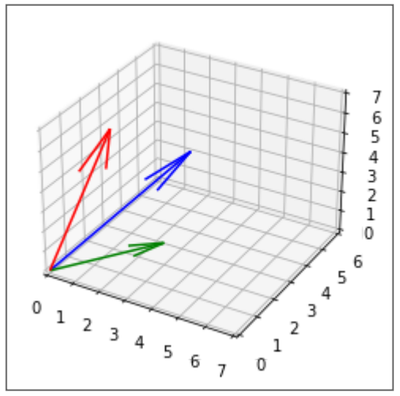
- Is there a solution of this equation for every $\mathbf{b}$?
- What would A have to look like to get a different answer?

Back to the basics

- a **vector** is an ordered sequence of n real numbers
- geometrically, a vector can be interpreted as a point in the n-dimensional space
- 2 dimensions equals plane



- 3 dimension equals space



- higher dimensions are hard to visualize, but beyond that, there is nothing special about them
- vectors are usually written as bold-faced lowercase letter, like $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}$
- when the individual cells are spelled out, a vector is written als a *column*

- individual components of a vector are indicated by subscript. If

$$\mathbf{x} = \begin{pmatrix} 8 \\ 6 \\ 1 \end{pmatrix},$$

then

$$\begin{aligned} x_1 &= 8 \\ x_2 &= 6 \\ x_3 &= 1 \end{aligned}$$

vector operations

- vectors can be **added**, provided they have the same length/dimensionality

$$\mathbf{x} + \mathbf{y} \doteq \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

- vector addition is
  - commutative

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

  - associative

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$

vector operations

- vectors can be **multiplied with real numbers**

$$a \cdot \mathbf{x} \doteq \begin{pmatrix} a \cdot x_1 \\ a \cdot x_2 \\ \vdots \\ a \cdot x_n \end{pmatrix}$$

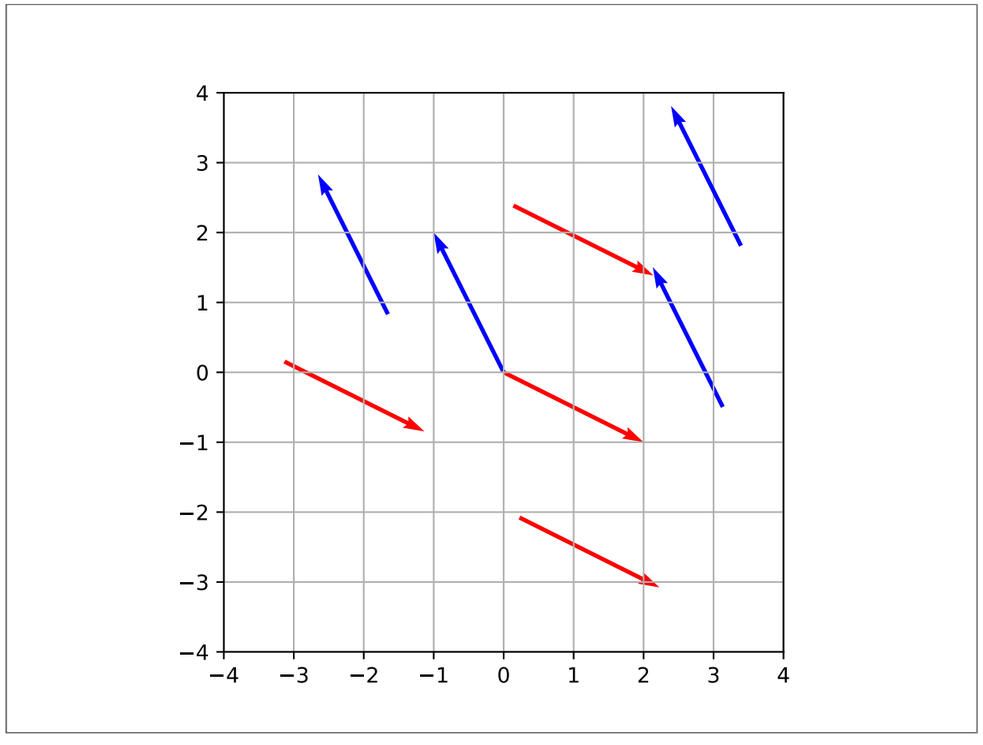
Real numbers are often called **scalars**, to distinguish them from vectors. Multiplication of a vector with a scalar is called **scalar multiplication**.

Scalar multiplication and vector addition obey the **distributive law**:

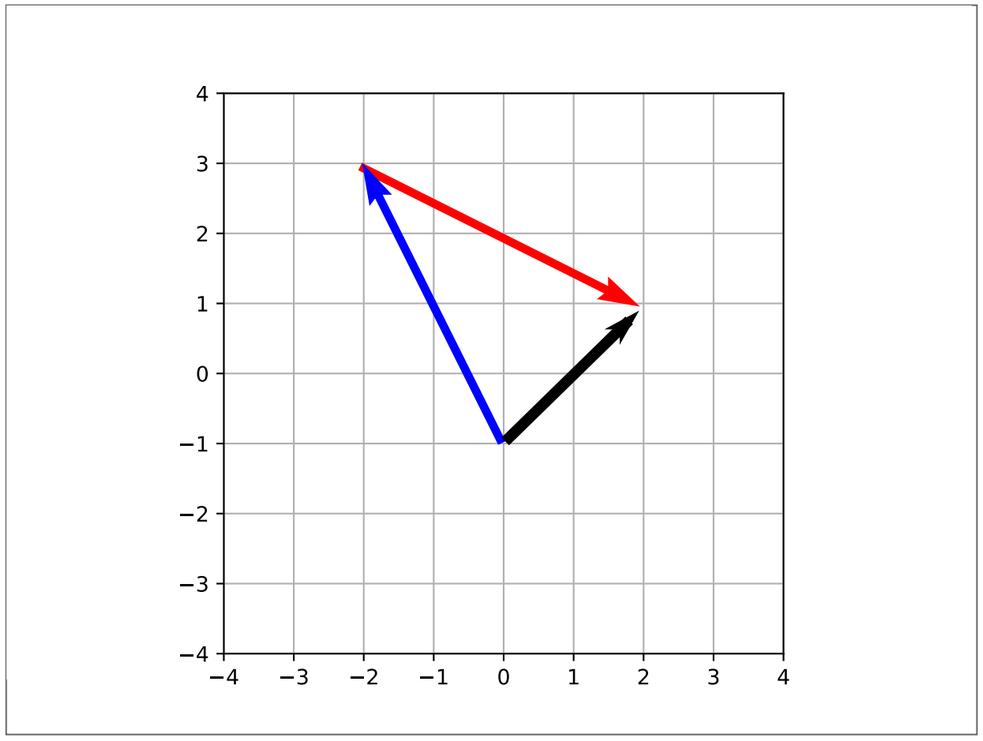
$$a \cdot (\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$$

geometric interpretation

- vectors can also be seen as *equivalence classes of pairs of points* in the n-dimensional space
- often drawn as an arrow
- two arrows represent the same vector if they have the same length and direction



- vector addition $\mathbf{x} + \mathbf{y}$ amounts to moving the start point of $\mathbf{y}$ to the end point of $\mathbf{x}$ and connecting the start point of $\mathbf{x}$ to the end point of $\mathbf{y}$.



important questions:

- Let $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. What is the set of vectors $\{a\mathbf{x} | a \in \mathbb{R}\}$?
- Let $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. What is the set of vectors $\{a\mathbf{x} + b\mathbf{y} | a, b \in \mathbb{R}\}$?

A **linear combination** of vectors is the result of applying scalar multiplication and vector addition to them.
So the last question amounts to: *What is the set of linear combinations of $\mathbf{x}$ and $\mathbf{y}$?*

inner product

- the **inner product** of two vectors is a scalar

$$\begin{aligned}\mathbf{x} \cdot \mathbf{y} &\doteq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \\ &= \sum_i x_i y_i\end{aligned}$$

(Sometimes the inner proudct is written $\langle \mathbf{x}, \mathbf{y} \rangle$.)

- the inner product is commutative

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$$

- Furthermore, the inner product is **linear** in both arguments

$$\begin{aligned}(a\mathbf{x}) \cdot (b\mathbf{y}) &= ab(\mathbf{x} \cdot \mathbf{y}) \\ \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) &= \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z} \\ (\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} &= \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}\end{aligned}$$

norm of a vector

The **norm** (=length) of a vector is defined as

$$\begin{aligned}\|\mathbf{x}\| &\doteq \sqrt{\mathbf{x}\cdot\mathbf{x}} \\ &= \sqrt{\sum_i x_i^2}\end{aligned}$$

properties of the norm

- for all vectors $\mathbf{x}, \mathbf{y}$ and scalars a:

$$\begin{aligned}\|\mathbf{x}\| &\geq 0 \\ \|\mathbf{x}\| &= 0 \text{ if and only if } \mathbf{x} = \mathbf{0}\ (\forall i.\, x_i = 0) \\ \|a\mathbf{x}\| &= a\|\mathbf{x}\| \\ \|\mathbf{x}+\mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\|\end{aligned}$$