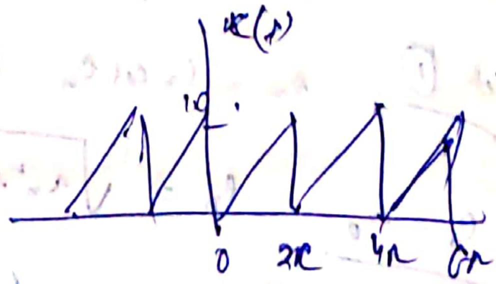
Q.

$(0,0) \quad (2\pi, 10)$



$$y - 0 = \frac{10}{2\pi}(x - 0).$$

$$y - 10 = \frac{10}{2\pi}(x - 2\pi)$$

$$y - 10 = \frac{10}{2\pi}x - 10$$

$$y = \frac{10}{2\pi}x$$

The waveform is periodic with period $T = 2\pi$.

$$\omega_0 = \frac{2\pi}{2\pi} = 1. \qquad \omega_0 = 2\pi f = 2\pi\frac{1}{T}$$

$$x(t) = \frac{10}{2\pi}t \quad \text{for } 0 < t < 2\pi$$

$\sin 2\theta = \ldots \cos^2\theta$

$1 - h$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right]$$

a_0

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt.$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{10}{2\pi}t\,dt.$$

$$= \frac{10}{(2\pi)^2}\left[\frac{t^2}{2}\right]_0^{2\pi}$$

$$= \frac{10}{(2\pi)^2}\left(\frac{4\pi^2}{2}\right) = 5.$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{T}\int_0 \frac{10}{2\pi}t\cos(n\omega_0 t)\,dt.$$

$\int \sin^2 n\omega t$

$\int_0^T 1 - \cos 2\theta$

$2\cos^2\theta - 1$

$\Big|_0^T$

$T = 0.$

Fourier Series

It is a mathematical tool that allows the representation of any periodic signal as the sum of harmonically related sinusoids.

Any periodic signal → $x(t) = x(t+T)$. can be expressed by a fourier series provided that.

1) If it is discontinuous, there are finite no. of discontinuities in the period T.
2. It has a finite average value over the period T.
3) It has a finite no. of +ve & -ve maxima in the period T.

Trigonometric Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right]$$

a_0, a_n, b_n are the trigonometric fourier series coefficients.

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt.$$

$$a_n = \frac{2}{T}\int_0^T x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T}\int_0^T x(t) \sin(n\omega_0 t)\, dt.$$

Complex exponentials is appropriate since they are periodic & relatively easy to manipulate mathematically.

Integration can be carried out from $-T/2$ to $T/2$, or over any other full period that might simplify the calculation.

The representation of a non-sinusoid periodic signal in terms of complex exponentials or equivalently in terms of sine & cosine waveforms, leads to the Fourier Series.

Page 2

Q.

$(0,0)$ $(2\pi, 10)$

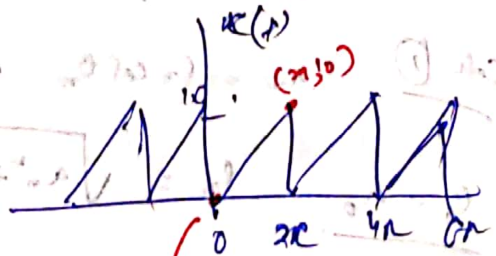
(x_1, y_1) (x_2, y_2)

$y - 0 = \frac{10}{2\pi}(x - 0)$

$y - 10 = \frac{10}{2\pi}(x - 2\pi)$

$y - 10 = \frac{10}{2\pi}x - 10$

$y = \frac{10}{2\pi}x$



(0,0)

$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$

slope

or

$\boxed{y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)}$

The waveform is periodic with period $T = 2\pi$.

$\omega_0 = \frac{2\pi}{2\pi} = 1.$ $\omega_0 = 2\pi f = 2\pi \frac{1}{T}$

$x(t) = \frac{10}{2\pi}t$ for $0 < t < 2\pi$

$\sin 2\theta = 1 - \cos^2\theta$

$1 - k$

$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right]$

a_0

$a_0 = \frac{1}{T}\int_0^T x(t)\, dt.$

$= \frac{1}{2\pi}\int_0^{2\pi} \frac{10}{2\pi} t\, dt.$

$= \frac{10}{(2\pi)^2}\left[\frac{t^2}{2}\right]_0^{2\pi}$

$= \frac{10}{(2\pi)^2}\left(\frac{4\pi^2}{2}\right) = 5.$

$\int \sin^2 n\omega_0 t$

$\int_0^T 1 - \cos 2\theta\, dt$

$\cos 2\theta$

$2\cos^2\theta - 1$

$t \Big]_0^T$

$T - 0.$

$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\, dt$

$= \frac{2}{T}\int_0^T \frac{10}{2\pi} t \cos(n\omega_0 t)\, dt.$

$$= \frac{2}{T} \times \frac{10}{2\pi} \int_0^T t \cos n\omega_0 t \, dt \qquad \left[\because T = 2\pi,\ \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1\right]$$

$$= \frac{10}{2\pi^2}\left\{\left[t \frac{\sin nt}{n}\right]_0^{2\pi} - \int_0^{2\pi} \left(\frac{dt}{dt} \int \cos nt \, dt\right)\right\}$$

$$= \frac{10}{2\pi^2}\left[t\frac{\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^{2\pi}$$

$$= \frac{10}{2\pi^2}\left[\frac{2\pi \cancel{\sin n 2\pi}^{0}}{n} + \frac{\cos n 2\pi}{n^2} - 0 - \frac{1}{n^2}\right] \quad \left[\because \sin(\text{integer multiple of } \pi) = 0\right]$$

$$= \frac{10}{2\pi^2}\left[\frac{1}{n^2} - \frac{1}{n^2}\right] \quad \left[\because \cos(\text{even multiple of } \pi) = 1\right]$$

$$= 0.$$

$$b_n = \frac{2}{T}\int_0^T x(t) \sin n\omega_0 t \, dt$$

$$= \frac{2}{2\pi}\int_0^T \frac{10}{2\pi} t \sin nt \, dt \qquad [\because T = 2\pi,\ \omega_0 = 1]$$

$$= \frac{10}{2\pi^2}\int_0^T t \sin nt \, dt$$

$$= \frac{10}{2\pi^2}\left\{\left[-t\frac{\cos nt}{n}\right]_0^{2\pi} - \int_0^{2\pi} (-)\frac{\cos nt}{n} \, dt\right\}$$

$$= \frac{10}{2\pi^2}\left[-\frac{t \cos nt}{n} + \frac{\sin nt}{n^2}\right]_0^{2\pi}$$

$$= \frac{10}{2\pi^2}\left[\frac{\cancel{\sin n 2\pi}^{0}}{n^2} - \frac{2\pi \cos n 2\pi}{n} - 0 - 0\right]$$

$$= \frac{-10}{2\pi^2} \times \frac{2\pi \cos n 2\pi}{n}$$

$$= -\frac{10}{2\pi^2} \times \frac{2\pi}{n} = -\frac{10}{\pi n}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

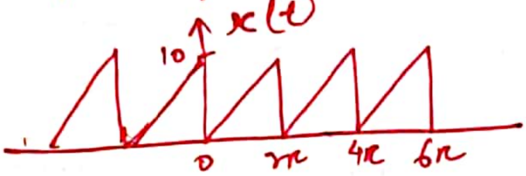
$$= 5 - \sum_{n=1}^{\infty} \frac{10}{\pi n} \sin nt = 5 - \frac{10}{\pi}\sin t - \frac{10}{\pi}\sin 2t - \frac{10}{\pi}\sin 3t + \cdots$$

Analysis of the Fourier Series expansion ~~Symmetry Conditions~~

As seen with the above problem, the fourier series expansion of the signal $x(t) = \frac{10}{2\pi} t$ can be expressed

as $x(t) = 5 - \frac{10}{\pi} \sin t - \frac{10}{2\pi} \sin 2t - \frac{10}{3\pi} \sin 3t + \dots$

The waveform of the signal was



Time period $(T) = 2\pi$ i.e. the signal is repeating itself after 2π amount of time, so fundamental frequency

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$ rad/sec

Now analyzing the result of fourier series expansion of the signal $x(t) \rightarrow x(t) = 5 - \frac{10}{5} \sin 1t - \frac{10}{2\pi} \sin(2\times 1)t - \frac{10}{3\pi} \sin(3\times 1)t$

$- \frac{10}{2\pi} \sin(4\times 1)t - \dots$

The signal has a d.c. component 5,

(i) The signal has sine waves of harmonics of fundamental frequency $\omega_0 = 1$ ($\sin 1t$), as $2\times\omega_0 = 2\times 1$ ($\sin 2t$), $3\times\omega_0 = 3\times 1$ ($\sin(3\times 1)t$), ... etc

(ii) Here it follows the definition of Fourier Series i.e it is representation of any periodic signal as the sum of harmonically related sinusoids.

(iii) $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$

a_0, a_n & b_n are called Fourier series Coefficients

$a_0 \rightarrow$ checks the d.c component

$a_n \rightarrow$ checks the harmonic of cosine waves

$b_n \rightarrow$ check the harmonics of sine waves.

Symmetry Conditions

The Fourier Series expansion may contain only sine terms or cosine terms, sometimes only odd/even harmonics whether the series contains sine, cosine or both type of terms.

Such conditions is the result of certain types of symmetry exhibited by signal waveform $x(t)$.

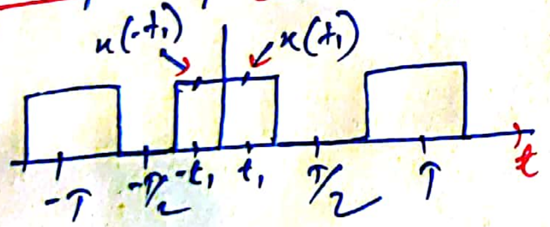
Knowledge of such symmetry conditions results in reduced calculations.

1) Even signal, $x(t) = x(-t)$

Important points: a) Sum, product of two or more even signals is an even signal.

b) Addition of a constant to an even signal is still an even signal.

Examples of even signal:



$x(t_1) = x(-t_1)$

Generalised fourier series coefficients for an even signal a_0, a_n & b_n

1) $a_0 = \frac{2}{T}\int_0^T x(t)\,dt$

$= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\,dt$ [0 to T) or (-T/2 to T/2) covering a range of T, -T/2 to T/2 is considered to consider -ve time interval for applying symmetry conditions]

$= \frac{2}{T}\left[\int_{-T/2}^{0} x(t)\,dt + \int_0^{T/2} x(t)\,dt\right]$

$= \frac{2}{T}\left[\int_0^{T/2} x(-t)\,dt + \int_0^{T/2} x(t)\,dt\right]$ [-T/2 to 0 → gives -ve (t) so changing -T/2 to 0 to 0 to T/2, → -t

$$= \frac{2}{T}\left[\int_0^{T/2} x(-t)\,dt + \int_0^{T/2} x(t)\,dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} x(t)\,dt + \int_0^{T/2} x(t)\,dt\right] \quad \left[\because x(t) = x(-t) \text{ for even symmetry}\right]$$

$$= \frac{4}{T}\int_0^{T/2} x(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} x(t)\cos n\omega_0 t\,dt + \int_0^{T/2} x(t)\cos n\omega_0 t\,dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} x(-t)\underbrace{\cos n\omega_0 t}_{\Updownarrow\ \cos(-n\omega_0 t)}\,dt + \int_0^{T/2} x(t)\cos n\omega_0 t\,dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} x(t)\cos n\omega_0 t\,dt + \int_0^{T/2} x(t)\cos n\omega_0 t\,dt\right] \quad \left[x(t) = x(-t)\right]$$

$$= \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} x(t)\sin n\omega_0 t\,dt + \int_0^{T/2} x(t)\sin n\omega_0 t\,dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} x(-t)\sin n\omega_0(-t)\,dt + \int_0^{T/2} x(t)\sin n\omega_0 t\,dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} x(t)(-)\sin n\omega_0 t\,dt + \int_0^{T/2} x(t)\sin n\omega_0 t\,dt\right]$$

$$= \frac{2}{T}\left[-\int_0^{T/2} x(t)\sin n\omega_0 t\,dt + \int_0^{T/2} x(t)\sin n\omega_0 t\,dt\right]$$

$$= 0$$

Thus for an even symmetry signal $[x(t) = x(-t)]$

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \quad , \quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt \quad , \quad b_n = 0$$

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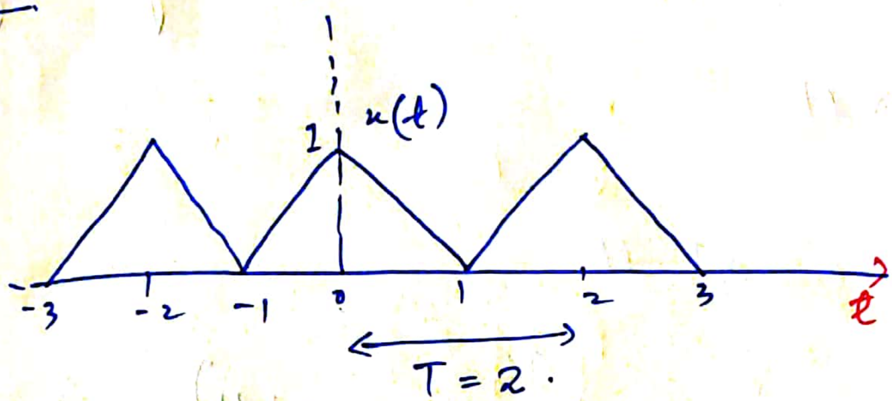
Problem



$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \qquad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt \qquad b_n = 0$$

To find $x(t)$,

For interval $0 \le t \le t = 1$, Coordinates $(0,1)$ & $(1,0)$
(x_1, y_1) (x_2, y_2)

$$y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

considering $(1,0) = (x_2, y_2)$

$$y - 0 = \frac{0-1}{1-0}(x-1)$$

$$\Rightarrow y = -(x-1) = -x+1$$

$$\Rightarrow x(t) = -t+1$$

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt = \frac{4}{2}\int_0^1 (1-t)\,dt \qquad T = 2 , \ T/2 = 1$$

$$= 2\int_0^1 1 - t\,dt = 2\left[t - \frac{t^2}{2}\right]_0^1 = 2\left[1 - \frac{1}{2}\right] = 1$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\, dt$$

$$= \frac{4}{2}\int_0^{1} (1-t)\cos n\pi t\, dt$$

$$= 2\left\{\left[(1-t)\frac{\sin n\pi t}{n\pi}\right]_0^1 - \left[(-1)\left(\frac{-\cos n\pi t}{n^2\pi^2}\right)\right]_0^1\right\}$$

$$= 2\left[(1-t)\frac{\cancel{\sin n\pi t}^{\,0}}{n\pi} - \frac{\cos n\pi t}{n^2\pi^2}\right]_0^1 \quad [\sin(\text{multiples of } \pi) = 0]$$

$$= \frac{2}{n^2\pi^2}\left[-\cos n\pi + 1\right] = \frac{2}{n^2\pi^2}\left[1 - \cos n\pi\right]$$

$$a_n = \frac{2}{n^2\pi^2}\left[1-(-1)\right] \quad \text{for } n \to \text{odd},\ n = 1, 3, 5 \ldots$$

[cos (odd multiples of π) = -1]

$$= \frac{4}{n^2\pi^2}$$

$$a_n = 0 \quad \text{for } n \to \text{even},\ n = 0, 2, 4 \ldots$$

[cos (even multiples of π) = 1]

$b_n = 0$ for even symmetry

$$\therefore\ x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$a_0 = 1 \qquad a_n = \frac{4}{n^2\pi^2} \text{ for } n \to \text{odd}$$

$$= 0 \text{ for } n \to \text{even}$$

$\omega_0 = \pi$

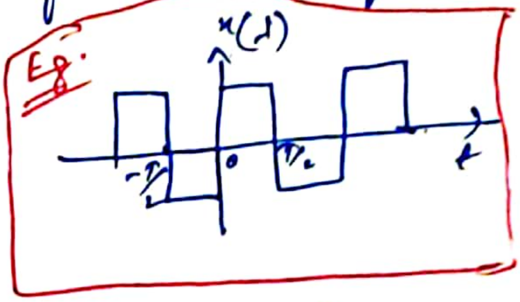
$$x(t) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi t + 0$$

$$= 1 + \frac{4}{\pi^2}\cos \pi t + \frac{4}{3^2\pi^2}\cos 3\pi t + \frac{4}{5^2\pi^2}\cos 5\pi t + \ldots$$

(*) Thus the even symmetry signal consists of only cosine terms of odd harmonics

2) Odd signal $x(t) = -x(-t)$

Derivation of Fourier Series Coefficients for an odd signal.



$$a_0 = \frac{2}{T}\int_0^T x(t)\,dt = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} x(t)\,dt + \int_0^{T/2} x(t)\,dt\right]$$

[(0 to T/2) or (-T/2 to T/2)]

$$= \frac{2}{T}\left[\int_0^{T/2} x(-t)\,dt + \int_0^{T/2} x(t)\,dt\right]$$

$$= \frac{2}{T}\left[-\int_0^{T/2} x(t)\,dt + \int_0^{T/2} x(t)\,dt\right] \qquad [\because x(t) = -x(-t)]$$

$$= 0$$

$$a_n = \frac{2}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt = \frac{2}{T}\left[\int_{-T/2}^{0} x(t)\cos n\omega_0 t\,dt + \int_0^{T/2} x(t)\cos n\omega_0 t\,dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} x(-t)\cos n\omega_0 (-t)\,dt + \int_0^{T/2} x(t)\cos n\omega_0 t\,dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} -x(t)\cos n\omega_0 t\,dt + \int_0^{T/2} x(t)\cos n\omega_0 t\,dt\right]$$

$$= 0$$

$$b_n = \frac{2}{T}\int_0^{T/2} x(t)\sin n\omega_0 t\,dt = \frac{2}{T}\int_{-T/2} x(t)\sin n\omega_0 t\,dt$$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} x(t)\sin n\omega_0 t\,dt + \int_0^{T/2} x(t)\sin n\omega_0 t\,dt\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} x(-t)\sin n\omega_0 (-t)\,dt + \int_0^{T/2} x(t)\sin n\omega_0 t\,dt\right]$$

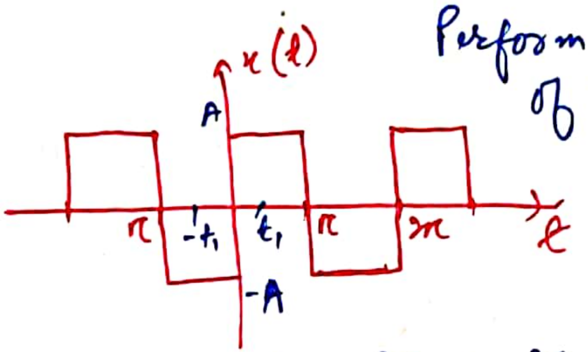
$$= \frac{2}{T}\left[\int_0^{T/2} (-)x(t)(-1)\sin n\omega_0 t\,dt + \int_0^{T/2} x(t)\sin n\omega_0 t\,dt\right]$$

$$= \frac{2}{T} \times 2 \int_0^{T/2} x(t) \sin n\omega_0 t \, dt$$

$$= \frac{4}{T} \int_0^{T/2} x(t) \sin n\omega_0 t \, dt \, .$$

$\therefore$ for an odd signal, $a_0 = 0$, $a_n = 0$ & $b_n = \frac{4}{T} \int_0^{T/2} x(t) \sin n\omega_0 t \, dt$

Problem 1) Perform Fourier Series expansion of the given signal $x(t)$



$$\boxed{x(t_1) = -x(-t_1)}$$

Soln

$T = 2\pi$, $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

$$\boxed{T/2 = \pi}$$

$x(t) = A$ for $0 \le t \le \pi$

$= -A$ for $\pi < t \le 2\pi$

$x(t) = -x(-t)$ so the given signal is an odd symmetric signal.

$\therefore$ $a_0 = 0$, $a_n = 0$ & $b_n = \frac{4}{T} \int_0^{T/2} x(t) \sin n\omega_0 t \, dt$.

$$b_n = \frac{4}{2\pi} \int_0^{\pi} A \sin nt \, dt$$

$$= \frac{4A}{2\pi} \left[-\frac{\cos nt}{n} \right]_0^{\pi} = \frac{4A}{2\pi} \left[-\frac{\cos n\pi}{n} + 1 \right]$$

$$= \frac{2A}{\pi} \left[\frac{1}{n} - \frac{\cos n\pi}{n} \right]$$

$$= \frac{2A}{n\pi} \left[1 - (-1) \right] = \frac{4A}{\pi n} = \frac{4A}{n\pi} \text{ for } n = 1, 3, 5 \ldots$$

$[\because \cos n\pi = -1, n \to \text{odd}]$

$b_n = 0$ for $n = 2, 4, 6 \ldots$ $[\because \cos n\pi = +1, n \to \text{even}]$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$a_0 = 0 \quad a_n = 0 \quad b_n = \frac{4A}{n\pi}$ for $n = 1, 3, 5 \ldots$

$$x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi} \sin nt \qquad [\because \omega_0 = 1]$$

$$= \frac{4A}{\pi} \sin 1t + \frac{4A}{3\pi} \sin 3t + \frac{4A}{5\pi} \sin 5t + \ldots$$

$\therefore$ The series contains only odd harmonics of sine terms since the given signal $x(t)$ is odd symmetric.

* i,e odd signal contains only ~~odd~~ harmonics of sine terms

sine wave is an odd function.

Exponential Fourier Series.

Consider a signal $x(t)$ periodic in nature with period T. The exponential Fourier Series of the given signal is expressed as

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

where $\Omega_0 = \frac{2\pi}{T} = 2\pi F_0$ = Fundamental frequency in rad/sec.

$[F_0 = \frac{1}{T}]$

F_0 = Fundamental frequency in cycles/sec or Hz

$\pm n\Omega_0$ = Harmonic frequencies

c_n = Fourier Series Coefficients of Exponential Fourier Series

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\Omega_0 t} dt \quad (\text{or}) \quad c_n = \frac{1}{T}\int_{0}^{T} x(t) e^{-jn\Omega_0 t} dt$$

* In Trigonometric Fourier Series, we have used ω_0 instead of Ω_0. It still represents the same meaning (fundamental frequency in rad/sec).

Limits of integration may be from $-T/2$ to $T/2$ or 0 to T or t_0 to $t_0 + T$

covering a range of the time period T.

Negative Frequency.

As seen in the Exponential Fourier Series expansion of a signal $x(t)$ periodic with time period T, there exists complex exponential harmonic components ($e^{\pm jn\Omega_0}$) of both positive and negative frequencies ($\pm n\Omega_0$).

When the positive and negative complex exponential components of same harmonic are added, it results to real sine or cosine signals, which is the essential requirement of expressing an unknown signal in terms of known sine & cosine terms.

Or in the other way, when the real sine or cosine signal has to be represented in terms of complex exponential, then a signal with negative frequency is required.

∴ It can be concluded that though a -ve frequency is physically not realizable, it helps in the mathematical representation of real signals in terms of complex exponentialsignals.

Relation between Fourier coefficients of Trigonometric & Exponential Form.

Trigonometric Fourier Series expansion of a periodic signal $x(t)$ is given as

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left[\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left[\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right] + \sum_{n=1}^{\infty} b_n j \left[\frac{-e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right]$$

[multiplying by j both in numerator & denominator]

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n e^{jn\omega_0 t}}{2} + \frac{a_n e^{-jn\omega_0 t}}{2} - \frac{j b_n}{2} e^{jn\omega_0 t} + \frac{j b_n}{2} e^{-jn\omega_0 t}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{\left[\frac{a_n - jb_n}{2}\right] e^{jn\omega_0 t} + \left[\frac{a_n + jb_n}{2}\right] e^{-jn\omega_0 t}\right\}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \left(\frac{a_n + jb_n}{2}\right) e^{-jn\omega_0 t}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} \frac{a_{-n} + jb_{-n}}{2} e^{jn\omega_0 t}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} \left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t}$$

$$\left[\because a_n = \frac{2}{T}\int_0^T x(t) \cos n\omega_0 t \, dt\right.$$

$$a_{-n} = \frac{2}{T}\int_0^T x(t) \cos n(-)\omega_0 t \, dt$$

$$= \frac{2}{T}\int_0^T x(t) \cos n\omega_0 t \, dt$$

$$a_{-n} = +a_n$$

$$\& \quad b_{-n} = \frac{2}{T}\int_0^T x(t) \sin(-)n\omega_0 t$$

$$\left. = -b_n \right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\alpha}\left(\frac{a_n - j b_n}{2}\right)e^{jn\omega_0 t} + \sum_{n=-\alpha}^{-1}\frac{(a_n - j b_n)}{2}e^{jn\omega_0 t}$$

$$= \cancel{\frac{a_0}{2}+} \sum_{n=-\alpha}^{\alpha}\frac{(a_n - j b_n)}{2}e^{jn\omega_0 t}$$

$$= \cancel{c_0 +} \sum_{n=-\alpha}^{\alpha} c_n e^{jn\omega_0 t}$$

$$c_0 = \frac{a_0 - j b_0}{2} = \frac{a_0}{2}$$

$$b_0 = \frac{2}{T}\int_0^T x(t)\sin 0\times\omega_0 t\, dt = 0$$

$$a_0 = \frac{2}{T}\int_0^T x(t)\cos 0\, dt \to 1$$

where $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - j b_n}{2}$, $c_{-n} = \frac{a_n + j b_n}{2}$

$[\because a_{-n} = a_n \;\&\; b_{-n} = -b_n]$

$$= c_0 + \dots c_{-3}e^{-j3\omega_0 t} + c_{-2}e^{-j2\omega_0 t} + c_{-1}e^{-j\omega_0 t} + c_0 + c_1 e^{j\omega_0 t} + c_2 e^{j2\omega_0 t} + c_3 e^{j3\omega_0 t} + \dots$$

Thus consider the exponential Fourier series of $x(t)$

$$x(t) = \sum_{n=-\alpha}^{\alpha} c_n e^{jn\Omega_0 t}$$

c_n is the Fourier coefficient of n^{th} harmonic component

c_n is a complex quantity & it can be expressed in polar form as $[\Omega_0 = \frac{2\pi}{T}$, T is the time period of $x(t)]$

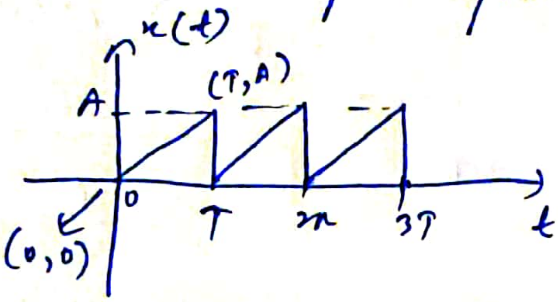
$c_n = |c_n|\angle c_n$, where $|c_n|$ = Magnitude of c_n;

$\angle c_n$ = phase of c_n

$\therefore$ The term $|c_n|$ represents the magnitude of n^{th} harmonic component & $\angle c_n$ represents the phase of the n^{th} harmonic component.

* The plot of harmonic magnitude / phase of a signal versus harmonic frequency $n\Omega_0$ is called frequency spectrum

Lets consider the ramp waveform as shown below



$$\boxed{\omega_0 = \frac{2\pi}{T}}$$

Fourier Series Coefficient c_n is given by

$$c_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt \qquad [\,\omega_0 \text{ or } \Omega_0\,] \text{ } \underline{\text{Same meaning}}$$

Lets get $x(t)$ equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \begin{array}{l}(x_1, y_1) \rightarrow (0, 0) \\ (x_2, y_2) \rightarrow (T, A)\end{array}$$

$$\Rightarrow y - 0 = \frac{A}{T}(x - 0)$$

$$\Rightarrow y = \frac{A}{T}x \quad \Rightarrow x(t) = \frac{A}{T}t$$

$$\therefore \; c_n = \frac{1}{T}\int_0^T \frac{A}{T}\,t\,e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T^2}\int_0^T t e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T^2}\left[\frac{t\,e^{-jn\omega_0 t}}{-jn\omega_0}\right]_0^T - \int_0^T 1 \times \frac{e^{-jn\omega_0 t}}{-jn\omega_0}\,dt$$

$$= \frac{A}{T^2}\left[\frac{t\,e^{-jn\omega_0 t}}{-jn\omega_0} - \frac{e^{-jn\omega_0 t}}{(-jn\omega_0)^2}\right]_0^T$$

$$= \frac{A}{T^2}\left[\frac{t\,e^{-jn\omega_0 t}}{-jn\omega_0} + \frac{e^{-jn\omega_0 t}}{n^2\omega_0^2}\right]_0^T$$

$$= \frac{A}{T^2}\left[-\frac{T e^{-jn\omega_0 T}}{jn\omega_0} + \frac{e^{-jn\omega_0 T}}{n^2\omega_0^2} - 0 - \frac{1}{n^2\omega_0^2}\right]$$

$$= \frac{A}{T^2}\left[-\frac{T e^{-jn\omega_0 T}}{jn\omega_0} + \frac{e^{-jn\omega_0 T}}{n^2\omega_0^2} - \frac{1}{n^2\omega_0^2} \right]$$

$$\boxed{\omega_0 = 2\pi/T}$$

$$= A/T^2\left[-\frac{T e^{-jn 2\pi/T \times T}}{jn \times 2\pi/T} + \frac{e^{-jn\frac{2\pi}{T}\times T}}{n^2\left(\frac{2\pi}{T}\right)^2} - \frac{1}{n^2\left(\frac{2\pi}{T}\right)^2} \right]$$

$$= \frac{A}{T^2}\left[-\frac{T^2 e^{-jn2\pi}}{jn2\pi} + \frac{e^{-jn2\pi}}{\frac{n^2 4\pi^2}{T^2}} - \frac{1}{\frac{n^2 4\pi^2}{T^2}} \right]$$

$$= -\frac{A e^{-jn2\pi}}{jn2\pi} + \frac{A e^{-jn\pi}}{n^2 4\pi^2} - \frac{A}{n^2 4\pi^2} \qquad \left[e^{-jn\pi} = (\cos n2\pi - j\sin n2\pi) = 1 \right]$$

$$= -\frac{A}{jn\pi} + \frac{A}{n^2 4\pi^2} - \frac{A}{n^2 4\pi^2}$$

$$= -\frac{A}{jn\pi} = \frac{jA}{n2\pi} \; // \qquad \therefore C_n = j\frac{A}{n2\pi} \text{ for } n \neq 0$$

$$C_0 = \frac{a_0}{2} \qquad a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$= \frac{1}{T}\int_0^T \frac{A}{T} t\,dt$$

$$= \frac{A}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{A}{T^2} \times T^2/2 = A/2$$

Let $A = 20$

n	C_n	n	C_n
-3	$c_{-3} = -j\,10/3\pi = -j1.061 = 1.061\angle -90^\circ$	-1	$C_{-1} = -j\frac{10}{\pi} = -j3.183 = 3.183\angle 90^\circ$
$+3$	$C_3 = j\,10/3\pi = j1.061 = 1.061\angle 90^\circ$	1	$C_1 = j\,10/\pi = j3.183 = 3.183\angle +90^\circ$
-2	$C_{-2} = -j\,10/2\pi = -j1.592 = 1.592\angle -90^\circ$	0	$C_0 = \frac{20}{2} = 10 = 10\angle 0^\circ$
2	$C_2 = j\,10/2\pi = j1.592 = 1.592\angle 90^\circ$		

Lets rewrite the expansion by substituting the values of c_0 & c_n $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$

$$x(t) = \cancel{c_0} + \cdots\cdots c_{-3} e^{-j3\omega_0 t} + c_{-2} e^{-j2\omega_0 t} + c_{-1} e^{-j\omega_0 t} + c_0 e^0 + c_1 e^{j\omega_0 t} + c_2 e^{j2\omega_0 t} + c_3 e^{j3\omega_0 t} + \cdots\cdots$$

$$= 10 \cancel{+ -j\frac{10}{3\pi} e^{-j3\omega_0 t}}.$$

$$= 10 + \cdots\cdots -j\frac{10}{3\pi} e^{-j3\omega_0 t} - j\frac{10}{2\pi} e^{-j2\omega_0 t} - j\frac{10}{\pi} e^{-j\omega_0 t} + j\frac{10}{\pi} e^{j\omega_0 t} + j\frac{10}{2\pi} e^{j2\omega_0 t} + j\frac{10}{3\pi} e^{j3\omega_0 t}. \cancel{+ \cdots\cdots}$$

$$= 10 + \cdots\cdots + j\frac{10}{3\pi}\left(e^{+j3\omega_0 t} - e^{-j3\omega_0 t}\right) + j\frac{10}{2\pi}\left(e^{j2\omega_0 t} - e^{-j2\omega_0 t}\right) + j\frac{10}{\pi}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right). \cancel{+ \cdots\cdots}$$

$$= 10 + \cdots\cdots + j\frac{10}{3\pi} \times 2j \sin 3\omega_0 t + j\frac{10}{2\pi} \times 2j \sin 2\omega_0 t + j\frac{10}{\pi} \times 2j \sin \omega_0 t. \cancel{+ \cdots\cdots}$$

$$= 10 + \cdots\cdots - \frac{20}{3\pi} \sin 3\omega_0 t - \frac{20}{2\pi} \sin 2\omega_0 t - \frac{20}{\pi} \sin \omega_0 t$$

This result supports the concept mentioned in Page 13 (regarding negative frequency) that when the + & -ve complex exponential components of same harmonic are added, it results to real sine or cosine signals.

eg. $\left[c_{-3} e^{-j3\omega_0 t} + c_3 e^{j3\omega_0 t} = -\frac{20}{3\pi} \sin 3\omega_0 t\right]$.

Properties of Continuous Time Fourier Series

→ Helps in developing conceptual insights
→ Reduce complexity in evaluation of Fourier Series related problems.

Considering $x(t)/y(t)$ is periodic signal with period T & fundamental frequency $\omega_0 = 2\pi/T$
& X_n/Y_n is the Exponential Fourier Series Coefficient.
(E.F.S.C)

1) Linearity.

$x(t)$ & $y(t)$ denote two periodic signals with period T

$$x(t) \xrightarrow{E.F.S.C} X_n \qquad z(t) \longrightarrow Z_n$$

similarly

$$y(t) \longrightarrow Y_n$$

$$z(t) = a(x(t)) + b(y(t)) \longleftrightarrow Z_n = aX_n + bY_n$$

2) Time shifting

$x(t)$ a periodic signal with period T

$$x(t) \longrightarrow X_n$$

$$x(t-t_0) \longrightarrow X_n e^{-jn\omega_0 t_0}$$

eg. a signal periodic with time period T.

$$x(t) \rightarrow X_n = \frac{jA}{n2\pi}$$

then $x(t-2) \Rightarrow X_n e^{-jn\omega_0 t_0}$ $\qquad \omega_0 = 2\pi/T \quad t_0 = 2.$

$$= \frac{jA}{n2\pi} e^{-j\frac{2\pi}{T}\times 2}$$

$$= e^{-j4\pi/T} \frac{jA}{n2\pi} //.$$

3 Frequency shifting. $x(t)$ periodic with period T

$x(t) \longrightarrow X_n$

$\cancel{e^{jm\omega_0 t}}$ $e^{jm\omega_0 t} x(t) \longrightarrow X_{n-m}$

eg. $x(t) \longrightarrow \frac{jA}{n 2\pi}$

$e^{j3\omega_0 t} x(t) \longrightarrow \frac{jA}{(n-3)2\pi}$ $[\omega_0 = 2\pi/T]$

4) Time Reversal

$x(t)$ periodic with period T.

$x(t) \longrightarrow X_n$

$x(-t) \longrightarrow X_{-n}$

if $x(t)$ is an even signal.

$x(t) = x(-t)$

$X_n = X_{-n}$

if $x(t)$ is an odd signal

$X_n = -X_{-n}$

5) Time scaling.

$x(t)$ periodic with time period T, $\omega_0 = 2\pi/T$

$x(t) \longrightarrow X_n$ [then $x(at)$ is periodic with time period T/a]

$x(at) \longrightarrow X_n$

Proof: $X_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$ [let $y(t) = x(at)$ so $y(t) \longrightarrow Y_n$]

$Y_n = \frac{1}{T/a} \int_0^{T/a} x(at) e^{-jn\omega_0 at} dt$ $[t \rightarrow at]$

$$Y_n = \frac{1}{T/a}\int_0^{T/a} x(at)\, e^{-jn\omega_0 at}\, dt$$

let $at = \tau$

Differentiating w.r.t t

$$\frac{d\tau}{dt} = a$$

$$\Rightarrow dt = \frac{d\tau}{a}$$

$$\boxed{\begin{array}{l} \text{at } t = 0 \\ \quad \tau = 0 \\ \text{at } t = T/a \\ \quad \tau = T \end{array}}$$

$$Y_n = \frac{1}{T/a}\int_0^{T} x(\tau)\, e^{-jn\omega_0 \tau}\, d\tau/a$$

$$= \frac{1}{T}\int_0^{T} x(\tau)\, e^{-jn\omega_0 \tau}\, d\tau$$

$$= \frac{1}{T}\int_0^{T} x(t)\, e^{-jn\omega_0 t}\, dt \quad [\tau = t, \text{ variable}]$$

$$= X_n$$

$\therefore \; x(at) \rightarrow X_n$

6/ Periodic Convolution

$x(t) \rightarrow X_n$ $\qquad z(t) = x(t) * y(t)$ [Linear convolution]

$y(t) \rightarrow Y_n$

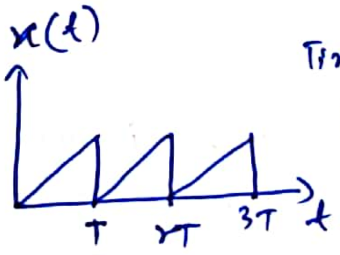
$Z_n = X_n Y_n$

7. Differentiation

$x(t) \rightarrow X_n$

$\frac{dx(t)}{dt} \rightarrow jn\omega_0 X_n$

Eg:



Time period $\rightarrow T$

$\omega_0 = 2\pi/T$

$x(t) \rightarrow X_n = \frac{jA}{n2\pi}$

$\frac{dx(t)}{dt} \rightarrow jn\frac{2\pi}{T} \times \frac{jA}{n2\pi}$

8. Integration.

$x(t) \longrightarrow X_n$

$\int_{-\infty}^{t} x(t)\,dt \longrightarrow \frac{1}{jn\omega_0} X_n$

9. Conjugation & Conjugate symmetry.

$x(t) \longrightarrow X_n$

$x^*(t) \longrightarrow X^*_{-n}$ [X^*_{-n} → Conjugate of time reversal of Fourier series coefficient X_{-n}]

Case ① if $x(t)$ is real

$x^*(t) = x(t)$

$x^*(t) \longrightarrow X^*_{-n}$ So $X^*_{-n} = X_n$

$x(t) \longrightarrow X_n$

$\Rightarrow X_{-n} = X^*_n$

But if $x(t)$ is real & even

$X_n = X^*_n = X_{-n}$ [$\because$ $x(t) = x(-t)$, $X_n = X_{-n}$].

$\therefore$ we can conclude that if $x(t)$ is real & even, then its Fourier series coefficient is also real & even.

Case (ii)

$x(t)$ is real

$x(t) \longrightarrow X_n$ $x^*(t) \longrightarrow X^*_n$

$x(t) = x^*(t)$, $X_n = X^*_{-n}$

$x(t)$ is odd.

$x(t) = -x(-t)$

$X_n = -X_{-n}$

$$\Rightarrow \boxed{\begin{aligned} X_{-n} &= -X_n = X^*_n \\ -X_{-n} &= X_n = -X^*_n \end{aligned}}$$

is satisfied only when X_n is imaginary

$\therefore$ $X_n = -X_{-n} = X^*_n$

$\Rightarrow X_{-n} = -X_n = X^*_n$

$\cdot$ if $x(t)$ is real & odd, then its Fourier series coefficients are purely imaginary & odd

Case (III) $x(t)$ is real

$x(t) = x^*(t)$ $\qquad x(t) \to X_n$

$\qquad\qquad\qquad x^*(t) \to X^*_{-n}$

$\therefore X_n = X^*_{-n}$

$\Rightarrow \boxed{X_{-n} = X^*_n}$

even part of signal $x(t)$ is $x_e(t)$,

$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

Lets consider $x_e(t) \to X_{en}$

$\therefore X_{en} = \frac{1}{2}(X_n + X_{-n})$ using linearity property

$[\because x(t) \to X_n$ & $x(-t) \to X_{-n}]$

But $X_{-n} = X^*_n$ for $x(t)$ is real

$X_{en} = \frac{1}{2}(X_n + X^*_n)$

$= \frac{1}{2} \times 2 \, Re[X_n]$.

Odd part of signal $x(t)$ is $x_o(t)$.

$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

$\therefore X_{en} = \frac{1}{2}[X_n - X_{-n}]$

$= \frac{1}{2}[X_n - X^*_n]$

$= \frac{1}{2} \times 2 \times j \, Im[X_n]$

$= j \, Im[X_n]$

$X_{en} = j \, Im[X_n]$.

Q. A continuous-time periodic signal $x(t)$ is real valued & has a fundamental period $T = 8$. The non zero fourier series coefficients for $x(t)$ are $X_1 = 2$ & $X_3 = 4j$

Express $x(t) = \sum_{n=0}^{\infty} A_n \cos(\omega_0 t + \phi_n)$

Soln. $\Rightarrow$ $X_1 = X_{-1}^* = 2$ $\quad X_3 = X_{-3}^* = 4j$

$\therefore$ $\boxed{x(t) \text{ is real} \rightarrow X_{-n} = X_n^*}$

$X_{-3}^* = 4j$

$X_{-3} = -4j$ & $X_3 = +4j$.

$X_{-1} = (2)^* = 2$

$X_1 = 2$.

~~Trigonom~~ Exponential fourier series Expansion

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{8} = \pi/4$$

$$= \cdots X_{-3} e^{-j3\pi/4 t} + X_{-1} e^{-j\pi/4 t} + X_1 e^{j\pi/4 t} + X_3 e^{j3\pi/4 t} + \cdots$$

[with the given coefficients]

$$= -j4 e^{-j3\pi/4 t} + 2e^{-j\pi/4 t} + 2e^{j\pi/4 t} + 4j e^{j3\pi/4 t}$$

$$= j4\left(e^{j3\pi/4 t} - e^{-j3\pi/4 t}\right) + 2\left(e^{j\pi/4 t} + e^{-j\pi/4 t}\right)$$

$$= -8\frac{\left(e^{j3\pi/4 t} - e^{-j\frac{3\pi}{4} t}\right)}{2j} + 4\frac{\left(e^{j\pi/4 t} + e^{-j\pi/4 t}\right)}{2}$$

$\boxed{\phi_n = \frac{\pi}{6} n}$

$$= 4\cos(\pi/4\, t) - 8\sin(3\pi/4\, t)$$

$x(t) = 4\cos(\pi/4\, t) - 8\cos(3\pi/4\, t + \pi/2)$ which is of the form $x(t) = \sum_{n=0}^{\infty} A_n \cos(\omega_0 t + \phi_n) = \sum_{n=0}^{\infty} A_n \cos(\omega_0 n t + \phi_n)$

Fourier Transform

Let $x(t)$ is a continuous signal, does not necessarily to be periodic in nature.

Then fourier transform of $x(t)$ is given by the eqn.

$$F[x(t)] = X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

The given signal $x(t)$ is expressed in time domain, t representing the time index, whereas $X(j\Omega)$ a complex function represents the signal $x(t)$ in its frequency domain.

$\Omega \rightarrow$ represents the angular frequency in radians/sec

$\Omega = 2\pi f$, $f \rightarrow$ represents the cyclic frequency in Hertz i,e cycles per second.

(ii) So the ~~fourier~~ Fourier transform could also be expressed in terms of cyclic frequency f, as

$$X(jf) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt .$$

Condition for existence of Fourier Transform.

The Fourier transform of $x(t)$ exists if ~~the~~ it satisfies the following Dirichlet condition.

1) The signal $x(t)$ must be absolutely integrable

i,e $\int_{-\infty}^{\infty} x(t)\, dt < \infty$.

i,e while integrating the signal w.r.t time, it must give a finite value.

2) The signal $x(t)$ should have finite number of maxima & minima within any finite interval.

iii) The signal $x(t)$ can have a finite number of discontinuities within any interval.

Inverse Fourier transform

Inverse Fourier transform of $X(j\Omega)$ is defined as

$$x(t) = F^{-1}[X(j\Omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

So Inverse brings back the frequency domain back to the time domain.

$\therefore$ $x(t)$ is a signal in time domain

& $X(j\Omega)$ is also a signal representing $x(t)$ in frequency domain.

The Fourier transform pair is expressed as

$$x(t) \underset{F^{-1}}{\overset{F}{\longleftrightarrow}} X(j\Omega)$$

Frequency spectrum using Fourier transform.

$X(j\Omega)$ is a complex function, so it can be expressed as $X(j\Omega) = X_r(j\Omega) + j X_i(j\Omega)$

$X_r(j\Omega) \rightarrow$ real part $\quad X_i(j\Omega) =$ imaginary part of $X(j\Omega)$

Then magnitude spectrum could be given by

$$|X(j\Omega)| = \cancel{\sqrt{X_r^2(j\Omega) + j X_i^2(j\Omega)}} = \sqrt{X_r^2(j\Omega) + X_i^2(j\Omega)}$$

$$= \sqrt{X(j\Omega)\, X^*(j\Omega)}$$

$X^*(j\Omega) =$ complex conjugate of $X(j\Omega)$

& phase spectrum by $\angle X(j\Omega) = \tan^{-1}\left[\frac{X_i(j\Omega)}{X_r(j\Omega)}\right]$

Fourier transform of some important signals.

1) Impulse signal.

$$x(t) = \delta(t) = \infty \; ; \; t = 0 \quad \& \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$
$$= 0 \; ; \; t \neq 0$$

It is the signal that has infinite value at $t = 0$ & 0 when $t \neq 0$, but gives a unit area when $t \to 0$

So using the eqn of Fourier transform

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$
$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\Omega t}\, dt$$

$$\boxed{\begin{array}{l} \text{For } t \neq 0, \; \delta(t) = 0 \\ \text{but when } t = 0, \; \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \end{array}}$$

So using ~~these~~ these conditions

$$X(j\Omega) = \int_{-\infty \atop t \to 0}^{\infty} \delta(t)\, e^{-j\Omega t}\, dt$$
$$= \int_{-\infty \atop t \to 0}^{\infty} \delta(t)\, e^{0}\, dt = 1. \qquad \boxed{|X(j\Omega)| = 1}$$

Plot of impulse signal $\delta(t)$ & its magnitude spectrum $|X(j\Omega)|$ is given below:

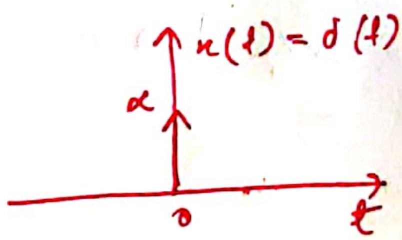


Fig: Impulse signal

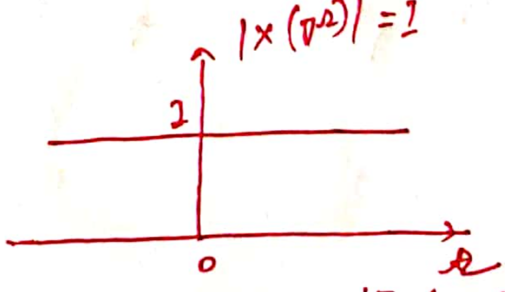


Fig: Magnitude spectrum of impulse signal b)

Single Sided Exponential Signal

$x(t) = Ae^{-at}$; for $t \geq 0$ or $Ae^{-at}u(t)$

$$F[x(t)] = F[Ae^{-at}] = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{\infty} Ae^{-at} e^{-j\Omega t}\, u(t)\, dt$$

$$= A\int_{0}^{\infty} e^{-(a+j\Omega)t}\, dt$$

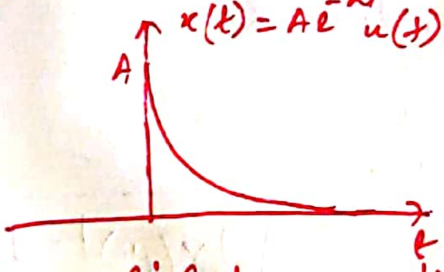
$$= A\left[\frac{e^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_0^{\infty}$$

$$= A\left[0 + \frac{1}{(a+j\Omega)}\right] = \frac{A}{a+j\Omega}$$

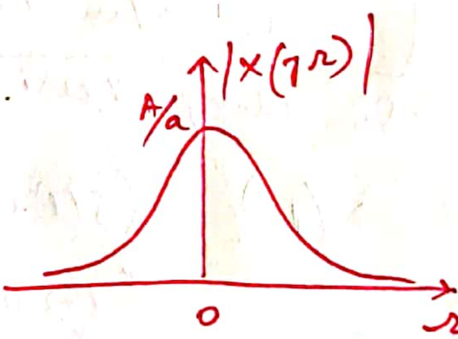
$$\boxed{\begin{array}{l} A\int_0^{\infty} e^{-at}\, dt = A\left[\frac{e^{-at}}{-a}\right]_0^{\infty} \\ = A\left[0+\frac{1}{a}\right] = \frac{A}{a} \\ \therefore x(t) \text{ is integrable} \end{array}}$$

$$X(j\Omega) = \frac{A}{a+j\Omega} \qquad |X(j\Omega)| = \frac{A}{\sqrt{a^2+\Omega^2}}$$

Plots:



Single sided exponential signal.



Magnitude Spectrum

$\Omega = 0$

$$|X(j\Omega)| = \frac{A}{\sqrt{a^2}} = \frac{A}{a}$$

when $\Omega \to \infty$

$|X(j\Omega)| \to 0$

Double sided exponential signal

$x(t) = Ae^{-a|t|}$; for all t

$= Ae^{-at}u(t) + Ae^{at}u(-t)$

or

$x(t) = Ae^{-at}$; $t \geq 0$ (0 to ∞)

$= Ae^{at}$; $t \leq 0$ ($-\infty$ to 0)

~~For + t &~~

For +ve t & -ve t

$x(t) = Ae^{-at}$

So for $t \geq 0$

$x(t) = Ae^{-at}$

& for $t \leq 0$

$x(t) = Ae^{at}$

$$F[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{0} A e^{at} e^{-j\Omega t} dt + \int_{0}^{\infty} A e^{-at} e^{-j\Omega t} dt$$

$$= A\int_{-\infty}^{0} e^{(a-j\Omega)t} dt + A\int_{0}^{\infty} e^{-(a+j\Omega)t} dt$$

$$= A\left[\frac{e^{(a-j\Omega)t}}{(a-j\Omega)}\right]_{-\infty}^{0} + A\left[\frac{e^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_{0}^{\infty}$$

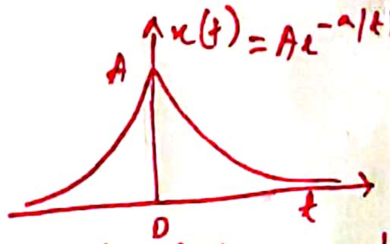
$$= A\left[\frac{1}{a-j\Omega} - 0\right] + A\left[0 + \frac{1}{a+j\Omega}\right]$$

$$= \frac{A}{a-j\Omega} + \frac{A}{a+j\Omega} = \frac{A[a+j\Omega + a-j\Omega]}{a^2+\Omega^2} = \frac{2aA}{a^2+\Omega^2}$$

$$X(j\Omega) = \frac{2aA}{a^2+\Omega^2}$$

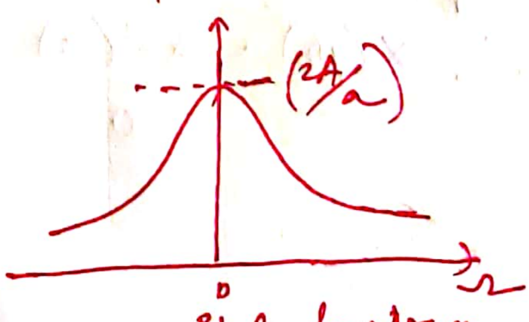
$$|X(j\Omega)| = \frac{2aA}{a^2+\Omega^2}$$

Plots



Double sided exponential

$|X(j\Omega)| = \frac{2aA}{a^2+\Omega^2}$



Magnitude Spectrum

Fourier transform of a constant.

let $x(t) = A$

checking the signal is integrable or not.

$\int_{-\infty}^{\infty} x(t)\,dt = \int_{-\infty}^{\infty} A\,dt = \infty$ [for any value of t, $x(t) = A$ so adding A for infinite times will give infinity].

So signal $x(t) = A$ can be expressed as a double sided exponential $e^{-a|t|}$, when $a \to 0$

$$\therefore\ x(t) = Ae^{-at} \quad t \geq 0$$
$$= Ae^{at} \quad t \leq 0 \quad \Big\} \text{ when } a \to 0$$

$$x(t) = A \lim_{a \to 0} e^{-a|t|}$$

Taking Fourier transform on $x(t)$

$$F[x(t)] = F\left[\lim_{a \to 0} e^{-a|t|}\right]$$
$$= \lim_{a \to 0} F\left[e^{-a|t|}\right]$$

$$X(j\Omega) = \lim_{a \to 0} \frac{2aA}{a^2+\Omega^2} \quad \left[F\left[e^{-a|t|}\right] = \frac{2aA}{a^2+\Omega^2}\right]$$

The above eqn is 0 for all values of Ω except at $\Omega = 0$

At $\Omega = 0$, the above eqn represents an impulse of magnitude "k"

$$\therefore\ X(j\Omega) = k\delta(\Omega) \quad ; \Omega = 0$$
$$= 0 \quad ; \Omega \neq 0$$

The magnitude "k" can be evaluated as

$$k = \int_{-\infty}^{\infty} \frac{2aA}{a^2+\omega^2}\,d\omega = 2aA\int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2}\,d\omega$$

$$= 2aA\left[\frac{1}{a}\tan^{-1}\left(\frac{\omega}{a}\right)\right]_{-\infty}^{\infty} \qquad \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= 2aA\left[\frac{1}{a}\tan^{-1}(+\infty) - \frac{1}{a}\tan^{-1}(-\infty)\right]$$

$$= 2aA\left[\frac{1}{a}\frac{\pi}{2} - \frac{1}{a}\left(-\frac{\pi}{2}\right)\right] = 2aA\left(\frac{\pi}{a}\right) = 2\pi A$$

$$\boxed{\therefore F[a] = 2\pi A\,\delta(\omega)}$$

The magnitude "k" can be evaluated as

$$k = \int_{-\infty}^{\infty} \frac{2aA}{a^2+\Omega^2} d\Omega = 2aA \int_{-\infty}^{\infty} \frac{1}{a^2+\Omega^2} d\Omega$$

$$= 2aA \left[\frac{1}{a} \tan^{-1}\left(\frac{\Omega}{a}\right) \right]_{-\infty}^{\infty} \quad \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \right]$$

$$= 2aA \left[\frac{1}{a}\tan^{-1}(+\infty) - \frac{1}{a}\tan^{-1}(-\infty) \right]$$

$$= 2aA \left[\frac{1}{a}\frac{\pi}{2} - \frac{1}{a}\left(-\frac{\pi}{2}\right) \right] = 2aA\left(\frac{\pi}{a}\right) = 2\pi A$$

$$\boxed{\therefore F[A] = 2\pi A \delta(\Omega)}$$

Fourier Transform of Signum Function

Signum function is defined as

$$x(t) = sgn(t) = 1 \; ; \; t > 0$$
$$= -1 \; ; \; t < 0$$

It can be expressed as a sum of two one sided exponential signal and taking limit $a \to 0$, as shown below.

$$x(t) = sgn(t) = \underset{a \to 0}{Lt} \left[e^{-at} u(t) - e^{at} u(-t) \right] = \underset{a \to 0}{Lt} e^{-a|t|}$$

By the definition of Fourier Transform

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{\infty} \underset{a \to 0}{Lt} \left[e^{-at} u(t) - e^{at} u(-t) \right] e^{-j\Omega t} dt$$

$$= \underset{a \to 0}{Lt} \left[\int_{-\infty}^{\infty} e^{-at} u(t) - e^{at} u(-t) \right] e^{-j\Omega t} dt$$

$$= \underset{a \to 0}{Lt} \left[\int_{0}^{\infty} e^{-at} e^{-j\Omega t} dt - \int_{-\infty}^{0} e^{at} e^{-j\Omega t} dt \right]$$

$$= \lim_{a \to 0} \left[\int_0^{\infty} e^{-(a+j\Omega)t}\, dt + \int_{-\infty}^{0} e^{(a-j\Omega)t}\, dt \right]$$

$$= \lim_{a \to 0} \left[\left[\frac{e^{-(a+j\Omega)t}}{-(a+j\Omega)} \right]_0^{\infty} + \left[\frac{e^{(a-j\Omega)t}}{(a-j\Omega)} \right]_{-\infty}^{0} \right]$$

$$= \lim_{a \to 0} \left[0 + \frac{1}{a+j\Omega} + \frac{(-)1}{a-j\Omega} - 0 \right]$$

$$= \lim_{a \to 0} \left[\frac{1}{a+j\Omega} - \frac{1}{a-j\Omega} \right]$$

$$= \frac{1}{j\Omega} + \frac{1}{j\Omega} = \frac{2}{j\Omega}$$

$$\therefore\ F[\mathrm{sgn}(t)] = \frac{2}{j\Omega}\ //.$$

$$X(j\Omega) = \frac{2}{j\Omega}, \quad |X(j\Omega)| = \frac{2}{\Omega}$$

Plot:

$x(t) = +1\ ;\ t > 0$
$\quad\ \ = -1\ ;\ t < 0$

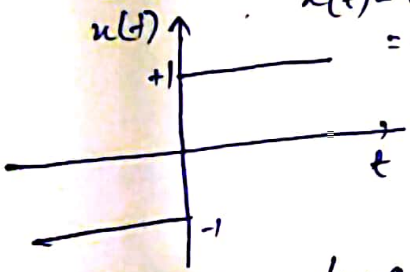


Fig: Signum function

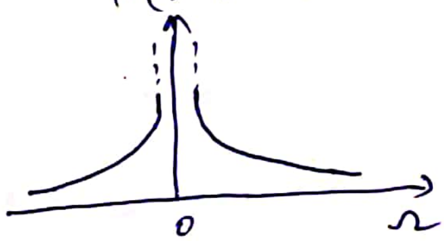


Fig: Magnitude Spectrum of Signum function.

Properties of Fourier Transform.

1) Linearity

Let $F[x_1(t)] = X_1(j\Omega)$ & $F[x_2(t)] = X_2(j\Omega)$

Then $F[a_1 x_1(t) + a_2 x_2(t)] = a_1 F[x_1(t)] + a_2 F[x_2(t)]$
$\qquad = a_1 X_1(j\Omega) + a_2 X_2(j\Omega)$

2) Time Shifting

If $F[x(t)] = X(j\Omega)$, then $F[x(t-t_0)] = e^{-j\Omega t_0} X(j\Omega)$

Proof:

We know that $F[x(t)] = X(j\Omega) = \int_{-\alpha}^{\alpha} x(t) e^{-j\Omega t} dt$

$\therefore \; F[x(t-t_0)] = \int_{-\alpha}^{\alpha} x(t-t_0) e^{-j\Omega t} dt$

Let $t - t_0 = \tau$ $\qquad t = \alpha \quad t = -\alpha$

$\qquad\qquad\qquad\qquad \tau = \alpha \quad \tau = -\alpha$.

$\frac{dt}{d\tau} - 0 = 1$

$dt = d\tau$

$F[x(t-t_0)] = \int_{-\alpha}^{\alpha} x(\tau) e^{-j\Omega(t_0+\tau)} d\tau$

~~Frequency~~ shifting in time domain

$= \int_{-\alpha}^{\alpha} x(\tau) e^{-j\Omega\tau} e^{-\Omega t_0 j} d\tau$

$= e^{-j\Omega t_0} \int_{-\alpha}^{\alpha} x(\tau) e^{-j\Omega\tau} d\tau$ [$\because$ τ is the variable not t_0]

$= e^{-j\Omega t_0} X(j\Omega)$

3. Time scaling

If $F[x(t)] = X(j\Omega)$

then $F[x(at)] = \frac{1}{|a|} X\left(\frac{j\Omega}{a}\right)$

4) Time Reversal

If $F[x(t)] = X(j\Omega)$

$F[x(-t)] = X(-j\Omega)$

5) Conjugation. $\qquad x^*(t) =$ Complex conjugate of $x(t)$

If $F[x(t)] = X(j\Omega)$

$F[x^*(t)] = X^*(-j\Omega)$

6 Frequency Shifting.

If $F[x(t)] = X(j\Omega)$

$$F[e^{j\Omega_0 t} x(t)] = X(j(\Omega - \Omega_0))$$

$\hookrightarrow$ Shifting in frequency domain

7) Time differentiation

If $F[x(t)] = X(j\Omega)$

Then $F\left[\frac{d\,x(t)}{dt}\right] = j\Omega X(j\Omega)$

8) Time integration

If $F[x(t)] = X(j\Omega)$ & $X(0) = 0$ Then

$$F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\Omega} X(j\Omega)$$

& if $X(0) \neq 0$

$$F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\Omega} X(j\Omega) + \pi X(0)\,\delta(\Omega)$$

9) Frequency differentiation.

if $F[x(t)] = X(j\Omega)$, Then

$$F[t\,x(t)] = j\frac{d\,X(j\Omega)}{d\Omega}.$$

10) Convolution Theorem

if $F[x_1(t)] = X_1(j\Omega)$ & $F[x_2(t)] = X_2(j\Omega)$

Then $F[x_1(t) * x_2(t)] = X_1(j\Omega)\,X_2(j\Omega)$

linear convolution $\rightarrow$ multiplication in frequency domain.

$$\boxed{x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau}$$

11) Frequency convolution

Let $F[x_1(t)] = X_1(j\Omega)$ & $F[x_2(t)] = X_2(j\Omega)$

then $F[x_1(t)\, x_2(t)] = X_1(j\Omega) * X_2(j\Omega)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\lambda)\, X_2(j(\Omega-\lambda))\, d\lambda$$

multiplication in time domain

convolution in frequency domain

12) Parseval's relation

if $F[x(t)] = X(j\Omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega$$

$|X(j\Omega)|^2$ represents the distribution of energy as function of angular frequency Ω, and so it is called energy density spectrum or energy spectral density of the signal $x(t)$.

& $\int_{-\infty}^{\infty} |x(t)|^2 dt$ = Energy of a continuous signal $x(t)$ in Joules.

13). Area under a time domain signal.

Area under a signal $x(t) = \int_{-\infty}^{\infty} x(t)\, dt$

if $x(t) \rightleftharpoons X(j\Omega)$ are Fourier transform pair

Then, $\int_{-\infty}^{\infty} x(t)\, dt = X(0)$

where $X(0) = \lim_{j\Omega \to 0} X(j\Omega)$

14.

Area under a frequency domain signal

A if $x(t) \rightleftharpoons X(j\Omega)$ Fourier transform pair

$$\text{Area under } X(j\Omega) = \int_{-\infty}^{\infty} X(j\Omega)\, d\Omega$$

then $\int_{-\infty}^{\infty} X(j\Omega)\, d\Omega = 2\pi\, x(0)$

$$x(0) = \lim_{t \to 0} x(t).$$

Problems.

$x(t) = A e^{j\Omega_0 t}$

Soln $F[x(t)] = F[A e^{j\Omega_0 t}]$

$= F[A]_{\Omega = \Omega - \Omega_0}$ [applying frequency shifting, $F[e^{j\Omega_0 t} x(t)] = X(j(\Omega - \Omega_0))$]

$F[A] = 2\pi\delta(\Omega) A$

$\therefore F[x(t)] = A 2\pi \delta(\Omega)\Big|_{\Omega = \Omega - \Omega_0}$

$= 2\pi A \delta(\Omega - \Omega_0)$

Similarly $F[e^{-j\Omega_0 t}] = 2\pi A \delta(\Omega + \Omega_0)$

$\boxed{\text{at } \Omega \neq \Omega_0 \quad |X(j\Omega)| = 0}$

Plot.

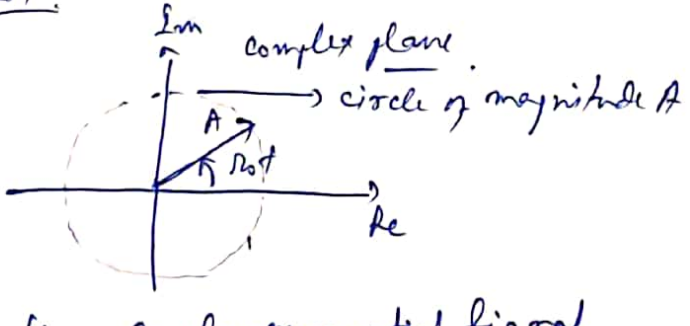


fig. complex exponential signal

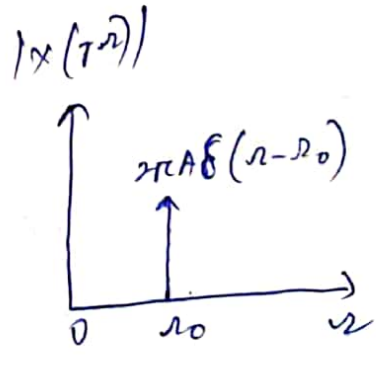


fig Magnitude spectrum of $A e^{+j\Omega_0 t} = |2\pi A \delta(\Omega - \Omega_0)|$

at $\Omega = \Omega_0$, $2\pi A \delta(0) = \underline{2\pi A \times 1}$

Fourier Transform of unit step signal.

We have seen that to perform Fourier transform of signum function (sgn(t)), we have expressed it in the form of exponential signal $e^{-a|t|}$ (with $a \to 0$), which is an integrable signal, to satisfy the required criteria for considering Fourier transform of a signal.

Similarly unit step signal which is not integrable, could be expressed in terms of sgn(t) as

$$u(t) = 1 \; ; \; t \geq 0 \qquad \text{[By definition of unit step signal]}$$
$$= 0 \; ; \; t < 0$$

Sgn(t) ~~can be~~ can be expressed in term of unit step signal

$$sgn(t) = 2u(t) - 1$$
$$\Rightarrow u(t) = \frac{1}{2}[1 + sgn(t)]$$

Now taking Fourier transform of the above eqn

$$F[u(t)] = F\left[\frac{1}{2}[1 + sgn(t)]\right]$$
$$= F\left[\frac{1}{2}\right] + \frac{1}{2}F[sgn(t)]$$
$$= 2\pi \times \frac{1}{2}\delta(\Omega) + \frac{1}{2}\left[\frac{2}{j\Omega}\right] \qquad \left[\because F[A] = 2\pi A\delta(\Omega) \;\&\; F[sgn(t)] = \frac{2}{j\Omega}\right]$$
$$= \pi\delta(\Omega) + \frac{1}{j\Omega}$$

$$\therefore F[u(t)] = \pi\delta(\Omega) + \frac{1}{j\Omega}$$

Plot: $u(t) = 1$ for $t \geq 0$; $= 0$ for $t < 0$

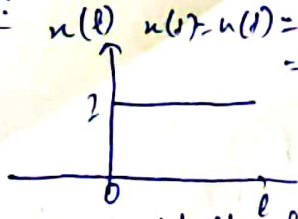


Fig. unit step signal

$|X(j\Omega)| = \pi\delta(\Omega) + \frac{1}{\Omega}$

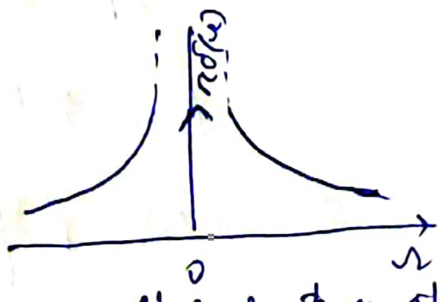


Fig. Magnitude spectrum of unit step signal

Inverse Fourier transform using partial fraction expansion technique.

Q.7 $X(j\Omega) = \dfrac{3(j\Omega)+14}{(j\Omega)^2+7(j\Omega)+12}$

Get signal $x(t)$ w.r.t the given function.

Soln
$$X(j\Omega) = \frac{3(j\Omega)+14}{(j\Omega)^2+7(j\Omega)+12}$$
$$= \frac{3(j\Omega)+14}{(j\Omega+3)(j\Omega+4)}$$

By partial fraction expansion technique.

$$X(j\Omega) = \frac{K_1}{(j\Omega+3)} + \frac{K_2}{(j\Omega+4)}$$

$$K_1 = \frac{3(j\Omega)+14}{(j\Omega+3)(j\Omega+4)} \times (j\Omega+3)\Bigg|_{j\Omega=-3}$$
$$= \frac{3(-3)+14}{-3+4} = 5$$

$$K_2 = \frac{(3(j\Omega)+14)\times(j\Omega+4)}{(j\Omega+3)(j\Omega+4)}\Bigg|_{j\Omega=-4}$$
$$= \frac{3(-4)+14}{-4+3} = -2$$

$$\therefore X(j\Omega) = \frac{5}{j\Omega+3} - \frac{2}{j\Omega+4}$$

We know that $F[e^{-at}u(t)] = \dfrac{1}{j\Omega+a}$

Then $x(t) = F^{-1}[X(j\Omega)] = F^{-1}\left[\dfrac{5}{j\Omega+3}\right] + F^{-1}\left[\dfrac{-2}{j\Omega+4}\right]$

$$= 5e^{-3t}u(t) - 2e^{-4t}u(t)$$

Q. Find the Inverse Fourier transform of

$$X(j\Omega) = \frac{j\Omega + 7}{(j\Omega + 3)^2}$$

Soln.

$$X(j\Omega) = \frac{j\Omega + 7}{(j\Omega + 3)^2}$$

By partial fraction technique

$$X(j\Omega) = \frac{j\Omega+7}{(j\Omega+3)^2} = \frac{K_1}{(j\Omega+3)^2} + \frac{K_2}{(j\Omega+3)}$$

$$K_1 = \frac{j\Omega+7}{(j\Omega+3)^2} \times (j\Omega+3)^2 \Big|_{j\Omega=-3} = -3+7 = 4$$

$$K_2 = \frac{d}{d(j\Omega)}\left[\frac{j\Omega+7}{(j\Omega+3)^2} \times (j\Omega+3)^2\right]\Big|_{j\Omega=-3} = \frac{d\,[j\Omega+7]}{d(j\Omega)}\Big|_{j\Omega=-3} = 1$$

$$\therefore X(j\Omega) = \frac{4}{(j\Omega+3)^2} + \frac{1}{(j\Omega+3)}$$

We know that $F[e^{-at}u(t)] = \dfrac{1}{j\Omega + a}$

& $F[t\,e^{-at}u(t)] = \dfrac{1}{(j\Omega+a)^2}$

[using Frequency differentiation property: $F[t\,x(t)] = j\dfrac{d\,X(j\Omega)}{d\Omega}$]

Thus $x(t) = F^{-1}\left[\dfrac{4}{(j\Omega+3)^2}\right] + F^{-1}\left[\dfrac{1}{j\Omega+3}\right]$

$$= 4t\,e^{-3t}u(t) + e^{-3t}u(t)$$

Q. Determine the convolution of $x_1(t) = e^{-2t}u(t)$ & $x_2(t) = e^{-6t}u(t)$, using Fourier transform.

Soln

By convolution property of Fourier transform

$$F[x_1(t) * x_2(t)] = X_1(j\Omega)\, X_2(j\Omega)$$

$$\Rightarrow x_1(t) * x_2(t) = F^{-1}[X_1(j\Omega)\, X_2(j\Omega)]$$

$$X_1(j\Omega) = F[e^{-2t}u(t)] = \frac{1}{j\Omega+2}$$

$$X_2(j\Omega) = F[e^{-6t}u(t)] = \frac{1}{j\Omega+6}$$

$$X(j\Omega) = X_1(j\Omega)\, X_2(j\Omega) = \frac{1}{(j\Omega+2)} \times \frac{1}{(j\Omega+6)}$$

By partial fraction technique

$$X(j\Omega) = \frac{1}{(j\Omega+2)} \frac{1}{(j\Omega+6)} = \frac{k_1}{(j\Omega+2)} + \frac{k_2}{(j\Omega+6)}$$

$$k_1 = \frac{1}{(j\Omega+2)(j\Omega+6)} \times (j\Omega+2)\Big|_{j\Omega=-2} = \frac{1}{-2+6} = \frac{1}{4}$$

$$k_2 = \frac{1}{(j\Omega+2)(j\Omega+6)} \times (j\Omega+6)\Big|_{j\Omega=-6} = \frac{1}{-6+2} = \frac{-1}{4}$$

Taking inverse Fourier transform of

$$X(j\Omega) = \frac{\frac{1}{4}}{(j\Omega+2)} + \frac{-\frac{1}{4}}{(j\Omega+6)}$$

$$x(t) = \frac{1}{4}\left(e^{-2t} - e^{-6t}\right) u(t)$$

Q. Determine the Fourier transform of the periodic impulse function show in fig.

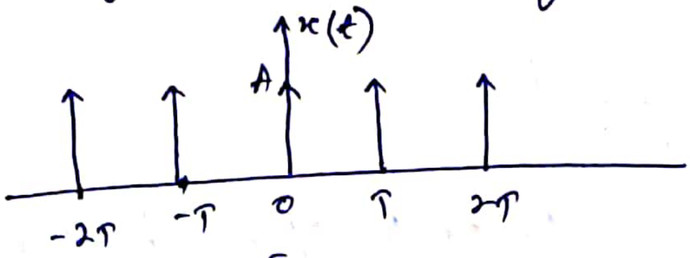


Fig.

Soln:

The given signal is periodic impulse function, which is periodic in nature with a period T.

So lets try to express the given signal in its exponential components using exponential Fourier series expansion as

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\Omega_0 t}$$, where Ω_0 is the fundamental angular frequency, $\Omega_0 = 2\pi/T$.

C_n is the exponential Fourier series coefficients given by

$$C_n = \frac{1}{T}\int_0^T x(t) e^{-jn\Omega_0 t} \quad \text{or} \quad C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\Omega_0 t}$$

Using the convenient one,

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jn\Omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} A\,(\delta(t)) e^{-jn\Omega_0 t}\, dt \quad \left[\because x(t) = A\delta(t) \text{ for } t = -T/2 \text{ to } T/2\right]$$

$$= \frac{A}{T} e^{-jn\Omega_0 t}\int_{-T/2}^{T/2} \delta(t)\, dt \Big|_{t\to 0}$$

$$= \frac{A}{T} \times 1 \times 1 = A/T$$

$\therefore$ The exponential Fourier expansion of signal $x(t)$ is given as

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\Omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} A/_T \, e^{jn\Omega_0 t}$$

Now as we have seen Fourier transform of complex exponential, so we can easily perform Fourier transform of the above signal as

$$F[x(t)] = F\left[\sum_{n=-\infty}^{\infty} A/_T \, e^{jn\Omega_0 t}\right]$$

$$= \frac{A}{T} F\left[\sum_{n=-\infty}^{\infty} e^{jn\Omega_0 t}\right] \quad [\because A/_T \text{ is constant}]$$

$$= \frac{A}{T} \sum_{n=-\infty}^{\infty} F[e^{jn\Omega_0 t}]$$

$$= \frac{A}{T} \sum_{n=-\infty}^{\infty} 2\pi \times 1 \, \delta(\Omega) \Big|_{\Omega = \Omega - n\Omega_0} \quad \left[\begin{array}{l} \because F[1] = 2\pi\,\delta(\Omega) \\ F[A] = 2\pi A\,\delta(\Omega) \end{array}\right]$$

$$= \frac{A}{T} \sum_{n=-\infty}^{\infty} 2\pi\,\delta(\Omega - n\Omega_0) \quad \left[F[x(t)e^{j\Omega_0 t}] = X(j(\Omega - \Omega_0))\right]$$

$$= A \sum_{n=-\infty}^{\infty} \frac{2\pi}{T}\,\delta(\Omega - n\Omega_0)$$

$$= \sum_{n=-\infty}^{\infty} A\Omega_0\,\delta(\Omega - n\Omega_0) \quad [\Omega_0 = 2\pi/_T]$$

$$\therefore X(j\Omega) = \sum_{n=-\infty}^{\infty} A\Omega_0\,\delta(\Omega - n\Omega_0)$$

$$|X(j\Omega)| = \sum_{n=-\infty}^{\infty} A\Omega_0\,\delta(\Omega - n\Omega_0)$$

Plot

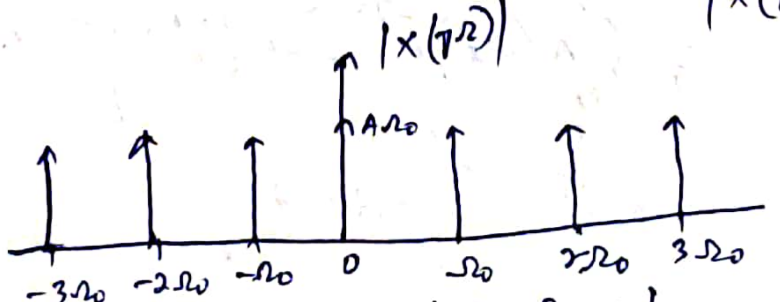


Fig magnitude spectrum

Q.

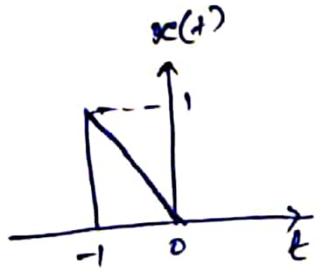


Fig. 1

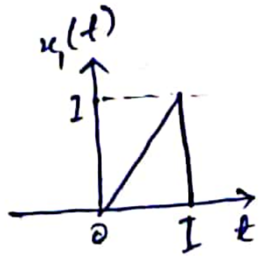


Fig 2.

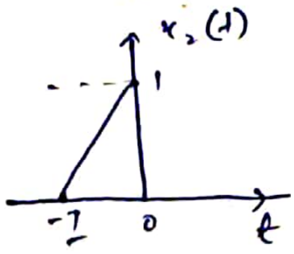


Fig 3

The Fourier transform of the signal shown in Fig 1 is $X(j\Omega) = \frac{1}{\Omega^2}\left(e^{j\Omega} - j\Omega e^{j\Omega} - 1\right)$. Using properties of Fourier transform find the Fourier transform of the signals shown in Fig 2 & Fig 3.

Soln. As we can see signal $x_1(t)$ is the folded version (or time reversal) of signal $x(t)$.

i.e $x_1(t) = x(-t)$.

Using time reversal property of Fourier transform.

i.e $F[x(t)] = X(j\Omega)$

$F[x(-t)] = X(-j\Omega)$

Given $F[x(t)] = X(j\Omega) = \frac{1}{\Omega^2}\left(e^{j\Omega} - j\Omega e^{j\Omega} - 1\right)$

$F[x_1(t)] = F[x(-t)] = X(-j\Omega) = \frac{1}{\Omega^2}\left(e^{-j\Omega} + j\Omega e^{-j\Omega} - 1\right)$

Then signal $x_2(t)$ is the shifted version of $x_1(t)$

i.e $x_2(t) = x_1(t+1)$

using shifting property of Fourier transform

$F[x(t \mp t_0)] = e^{\mp j\Omega t_0} X(j\Omega)$

$F[x_2(t)] = F[x_1(t+1)] = e^{j\Omega \times 1} \cdot \frac{1}{\Omega^2}\left(e^{-j\Omega} + j\Omega e^{-j\Omega} - 1\right)$

Property : **Duality** of Fourier transform.

If $F[x_1(t)] = X_1(j\Omega)$
$F[x_2(t)] = X_2(j\Omega)$

If $x_2(t) \equiv X_1(j\Omega)$ [i,e $x_2(t)$ and $X_1(j\Omega)$ are similar functions]

then $X_2(j\Omega) \equiv 2\pi\, x_1(t)\big|_{t=-j\Omega} = 2\pi\, x_1(-j\Omega)$

[i,e $X_2(j\Omega)$ and ~~$2\pi X_1($~~ $2\pi x_1(-j\Omega)$ are similar functions]

Q. If Fourier transform of $e^{-t}u(t)$ is $\frac{1}{1+j\Omega}$, then find the Fourier transform of $\frac{1}{1+t}$ using duality property.

Soln

Let $x_1(t) = e^{-t}u(t)$

then $F[x_1(t)] = X_1(j\Omega) = \frac{1}{1+j\Omega}$

& $x_2(t) = \frac{1}{1+t}$.

$x_2(t)$ & $X_1(j\Omega)$ are similar functions

i,e $x_2(t) \equiv X_1(j\Omega)$

Then $X_2(j\Omega) \equiv 2\pi\, x_1(t)\big|_{t=-j\Omega}$

$= 2\pi\left(e^{-t}u(t)\big|_{t=-j\Omega}\right)$

$= 2\pi\, e^{+j\Omega}u(-j\Omega)$

$\therefore X_2(j\Omega) = 2\pi\, e^{j\Omega}u(-j\Omega)$.

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Laplace transform

Complex frequency plane or s-plane

Complex frequency is defined/represented as

$$s = \sigma + j\Omega$$

where σ = Real part of s

Ω = Imaginary part of s

Ω is the angular frequency in radians/sec.

σ & Ω can take values from $-\infty$ to ∞.

A two dimensional complex plane with values of σ on horizontal axis and Ω on vertical axis as shown in figure 1. below is called as complex frequency plane or s-plane.

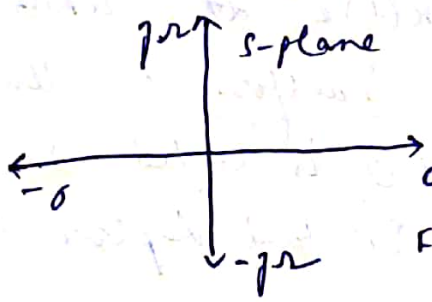


Fig 1: Complex frequency plane or s-plane.

The s-plane is used to represent various critical frequencies (represented by poles and zeros) of signals which are actually function of $s (= \sigma + j\Omega)$. It also represents the path taken by these critical frequencies when some parameters of the signals are varied. Such representation/study is useful to design/analyse systems (in continuous domain) for a ~~desired~~ desired response of a system.

Definition of Laplace transform.

Laplace transform of a continuous time signal $x(t)$ is expressed as

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \quad \text{— (1)}$$

where s represents the complex frequency as defined in the previous page, with $s = \sigma + j\Omega$.

Also represented as

$$L[x(t)] = X(s) \quad \text{Symbolically.}$$

If the signal $x(t)$ is a causal signal, i.e $x(t)$ is defined for $t \geq 0$, then

$$X(s) = \int_{0}^{+\infty} x(t)e^{-st}\,dt$$ which is also called as One sided Laplace transform or Unilateral Laplace transform, where eqn (1) is called as Two sided Laplace transform or Bilateral Laplace transform.

- Laplace transform actually converts a time domain signal $x(t)$ to a complex frequency domain. S-plane represents the frequency contents in the form of poles and zeros (values of s) of the signal $x(t)$.

Definition of Inverse Laplace transform

The s-domain signal (complex frequency signal) $X(s)$ can be transformed back to time domain signal $x(t)$ by using inverse Laplace transform.

Inverse Laplace transform of $X(s)$ is defined as

$$L^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\Omega}^{s=\sigma+j\Omega} X(s)e^{st}\,ds.$$

(*) we have ~~discus~~ discussed that condition for a signal $x(t)$ to be expanded by Fourier Series expansion, is that the signal should be periodic in nature i,e

$$x(t) = x(t+T) \quad . \, T \text{ is the time period.}$$

whereas a signal $x(t)$ must be integrable or ~~conver~~ the signal must converge i,e

$$\int_{-\alpha}^{\alpha} x(t)\, dt < \alpha$$

for the ~~signal~~ Fourier transform of the signal to exists. So to perform Fourier transform, ~~if~~ the signal is not necessarily to be periodic but requires to be integrable.

Fourier Series & Fourier transform enables a time domain signal $x(t)$ to be expressed in its frequency contents. So a signal $x(t)$ or a system represented by $h(t)$ could be analysed in a different approach (or dimension) by converting into its frequency contents with the help of Fourier Series & Fourier transform.

(*) But how to analyze a signal $(x(t))$ or a system $(h(t))$ which is not integrable. Laplace transform comes to this point where a ~~signal~~ function ($x(t)$ or $h(t)$) ~~is made~~ not integrable is still converted to complex frequency plane / s-plane, such that the signal or the system can still be analysed.

For instance consider an LTI system with impulse response

$h(t) = e^{at} u(t)$.

We can easily see that $h(t)$ is aperiodic in nature and also not integrable i.e

$$\int_{-\infty}^{\infty} h(t)\,dt = \int_{-\infty}^{\infty} e^{at} u(t)\,dt$$

$$= \int_{0}^{\infty} e^{at}\,dt = \left[\frac{e^{at}}{a}\right]_0^{\infty} = \infty.$$

Lets perform the operation of Laplace transform on the impulse response $h(t)$ to convert it in s-plane.

$$L[h(t)] = H(s) = \int_{-\infty}^{\infty} h(t)\,e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} e^{at} u(t) e^{-st}\,dt = \int_{0}^{\infty} e^{at} e^{-st}\,dt$$

$$= \int_{0}^{\infty} e^{(a-s)t}\,dt = \int_{0}^{\infty} e^{at} e^{-(\sigma+j\Omega)t}\,dt \quad [s=\sigma+j\Omega]$$

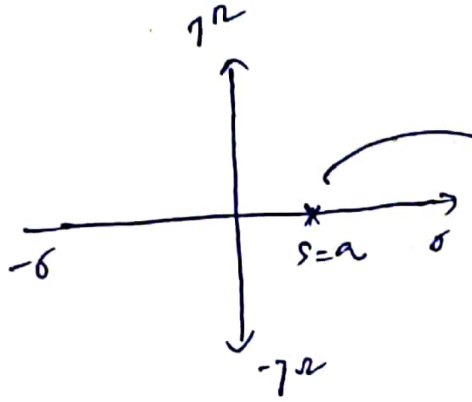
$$= \int_{0}^{\infty} e^{(a-\sigma)t} e^{-j\Omega t}\,dt$$

$$= \left[\frac{e^{(a-\sigma)t} e^{-j\Omega t}}{(a-\sigma)-j\Omega}\right]_0^{\infty} \longrightarrow (2)$$

(*) When $t=\infty$, $e^{(a-\sigma)t} = \infty$ if $a-\sigma > 0$
whereas when $t=\infty$, $e^{(a-\sigma)t} < \infty$ if $a-\sigma < 0$ i.e $\sigma > a$
where σ is the real part of $s=\sigma+j\Omega$.

$$= 0 - \frac{1}{a-\sigma-j\Omega} = -\frac{1}{a-(\sigma+j\Omega)} = \frac{-1}{a-s} = \frac{1}{s-a}.$$

$$\therefore H(s) = \frac{1}{s-a}$$



pole on right hand side
(+ve pole)

$H(s) = \frac{1}{s+a}$ is also called the transfer function of the system whose impulse response is given by $h(t)$.

So a system with impulse response $h(t) = e^{at}u(t)$ cannot be converted into its frequency domain with the help of fourier series expansion or fourier transform an thus cannot be analysed.

But Laplace transform transform can still convert $h(t) = e^{at}u(t)$ into its complex frequency domain ie s-plane represented by $H(s) = \frac{1}{s-a}$, & still can be ~~analysed~~ analysed,

it have a +ve pole of $s = a$.

* In other words we ~~has~~ can say that Laplace transform generalizes fourier transform.

$$F[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt$$

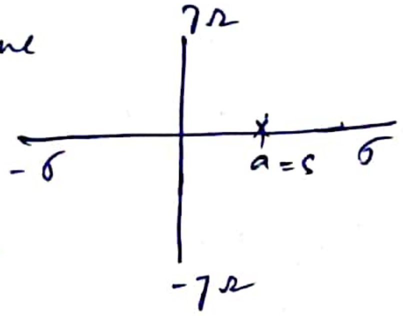
$$L[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\Omega)t}\,dt$$

Laplace transform has $e^{-\sigma t}$ component that is multiplied with $x(t)$ to give $x(t)e^{-\sigma t}$, to give $\lim_{t\to\infty} e^{-\sigma t}x(t) = 0$ as in equation (2) where $x(t) = e^{at}$

$h(t) = e^{at} u(t)$ is an unstable system since

$$\int_{-\alpha}^{\alpha} |h(t)|\, dt = \alpha \int_{0}^{\alpha} e^{at}\, dt = \alpha$$

In S-plane



pole $s=a$ lies on right hand side represents an unstable system.

∴ In S-plane a system $h(t) = e^{at} u(t)$, can be be expressed as $H(s) = \frac{1}{s+a}$, that can be analysed as an ~~us~~ unstable system, But the ~~sample~~ same system can not be analysed by Fourier series since it is not periodic. And also not Fourier transform since $h(t) = e^{at} u(t)$ is not integrable.

Continuation of Laplace transform

Region of Convergence:

The Laplace transform of a continuous time signal $x(t)$ is given by $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$. The values of s for which the integral $\int_{-\infty}^{\infty} x(t) e^{-st} dt$ converges/exists is called Region of Convergence (ROC). The ROC for the following three types of signals are discussed here.

case i : Right sided (causal) signal
case ii : Left sided (anticausal) signal
case iii : Two sided signal.

case i: Right sided (causal) signal

eg. $x(t) = e^{-at} u(t)$, where $a > 0$

$= e^{-at}$ for $t \geq 0$

Laplace transform of $x(t)$ is given by

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt$$

$$= \int_0^{\infty} e^{-at} \cdot e^{-st} dt = \cancel{\int_0^{\infty} \frac{e^{-(s+a)t}}{-(s+a)}}$$

$$= \int_0^{\infty} e^{-(s+a)t} dt = \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty}$$

$$= \left[\frac{e^{-(\sigma+a)t} e^{-j\Omega t}}{-(s+a)} \right]_0^{\infty} \quad [\because s = \sigma + j\Omega]$$

at $t = \infty$

(i) $e^{-(\sigma+a)t} e^{-j\Omega t} = \infty$ if $\sigma + a > 0$ i,e $\sigma > -a$

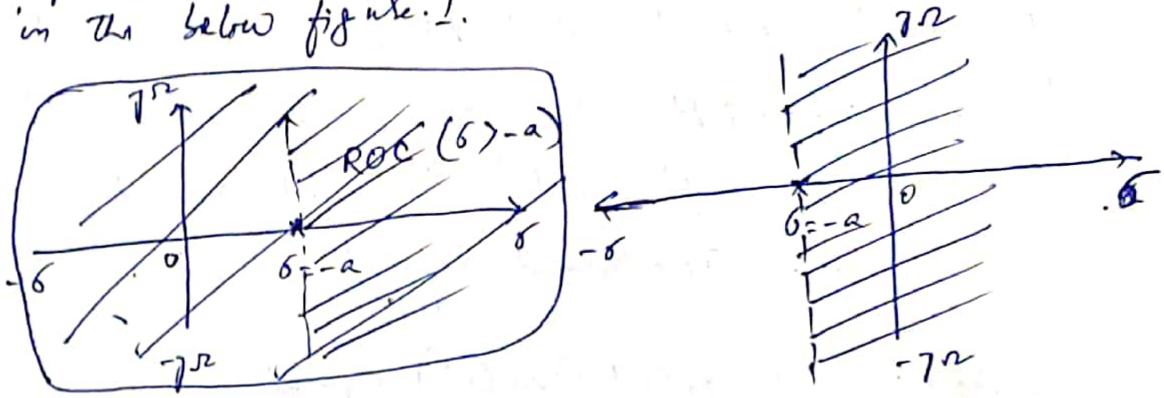
(ii) $e^{-(\sigma+a)t} e^{-j\Omega t} = 0$ if $\sigma + a < 0$ i,e $\sigma < -a$

So if $\sigma > -a$,

the Laplace transform continuous as converges and

$$= -0 - \frac{-1}{(s+a)} = \frac{1}{s+a}$$

$\therefore X(s) = \frac{1}{s+a}$ & the ROC for a causal signal includes all points on the s-plane to the right of pole $s = -a$ as shown in the below figure. I.



$-a$ is also called the abscissa of convergence.

Point to be noted, for $\sigma < -a$, $X(s)$ does not converge.

Case ii: Left sided (anticausal) signal.

eg. $x(t) = e^{-bt}u(-t) = e^{-bt}$ for $t \leq 0$ where $b > 0$

Laplace transform of $x(t)$ is given by,

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} e^{-bt}u(-t)e^{-st}dt$$

$$= \int_{-\infty}^{0} e^{-bt}e^{-st}dt = \left[\frac{e^{-(s+b)t}}{-(s+b)}\right]_{-\infty}^{0}$$

$$= \left[\frac{e^{-(\sigma+b)t}e^{-j\Omega t}}{-(s+b)}\right]_{-\infty}^{0}$$

at $t = -\infty$,

* $e^{-(\sigma+b)t}e^{-j\Omega t} = 0$, if $\sigma + b < 0$, $\sigma < -b$
* $e^{-(\sigma+b)t}e^{-j\Omega t} = \infty$, if $\sigma + b > 0$, $\sigma > -b$.

Hence $X(s)$ converges when $\sigma < -b$ & does not converge when $\sigma > -b$

$$= -\frac{1}{s+b} - (-0) = -\frac{1}{s+b}$$

$$\therefore X(s) = -\frac{1}{s+b}$$

And for an anticausal signal the ROC includes all points on the s-plane to the left of abscissa of convergence $(-b)$ as shown in fig. 2.

* $x(t) = e^{-at}u(t)$

  $X(s) = \frac{1}{(s+a)}$, ROC $\sigma > -a$

* $x(t) = e^{-at}u(-t)$

  $X(s) = -\frac{1}{(s+a)}$, ROC $\sigma < -a$

Same signal e^{-at} has same $X(s)$ with difference in -ve sign & ROC $> -a$ & ROC $< -a$

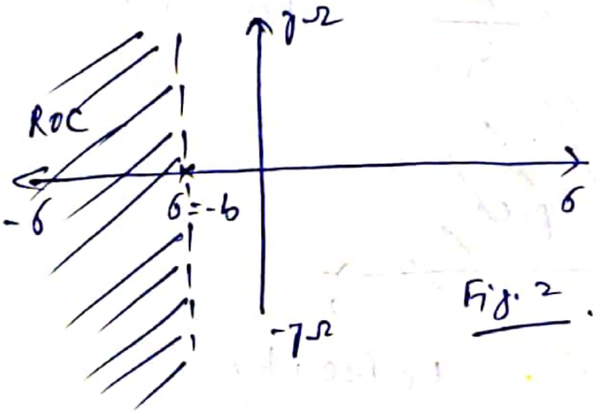


Fig. 2

iii) Two sided signal (Non causal signal).

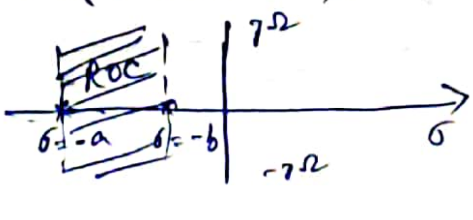
Let $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$

Using the conclusions in case i & case ii

$$X(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

ROC: $\sigma > -a$; ROC: $\sigma < -b$.

$\therefore X(s)$ will converge only when ROC is in the range of $-a < \sigma < -b$ i.e $a > b$



Problems

1) $x(t) = A\,u(t)$

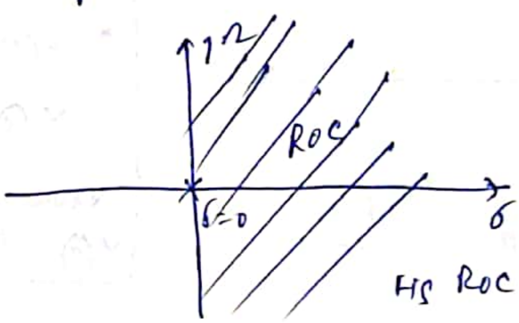
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} A\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} A\, e^{-st}\, dt = A \int_{0}^{\infty} e^{-st}\, dt = A \int_{0}^{\infty} e^{-\sigma t}\, e^{-j\Omega t}\, dt$$

$$= A \left[\frac{e^{-\sigma t}\, e^{-j\Omega t}}{-(\sigma + j\Omega)} \right]_0^{\infty} = A \left[\frac{e^{-\infty}\, e^{-\infty}}{-(\sigma + j\Omega)} - (-)\frac{1 \times e^{0}}{(\sigma + j\Omega)} \right]$$

$$= A \frac{1}{\sigma + j\Omega} = \frac{A \times 1}{s} = \frac{A}{s}$$

ROC: $\sigma > 0$



Hs ROC of $A\,u(t)$

ROC: all points in s-plane to the right of line passing through $\sigma = 0$. (or ROC is right half of s-plane).

2) $x(t) = t\,u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} t\, u(t)\, e^{-st}\, dt = \int_{0}^{\infty} t\, e^{-st}\, dt$$

$$= \left[t\, \frac{e^{-st}}{-s} \right]_0^{\infty} - \int 1 \times \frac{e^{-st}}{-s}\, dt = \left[t\, \frac{e^{-st}}{-s} \right]_0^{\infty} - \left[\frac{e^{-st}}{s^2} \right]_0^{\infty}$$

$$= \left[\frac{t\, e^{-(\sigma + j\Omega)t}}{-s} \right]_0^{\infty} - \left[\frac{e^{-(\sigma + j\Omega)t}}{s^2} \right]_0^{\infty}$$

$$= \left[\infty\, \frac{e^{-(\sigma + j\Omega)\infty}}{-s} - 0 \times \frac{e^{0}}{-s} - \frac{e^{-(\sigma + j\Omega)\infty}}{s^2} + \frac{e^{0}}{s^2} \right]$$

$$=$$

When $\sigma > 0$ (i,e when σ is +ve) , $e^{-\alpha \times \sigma} = e^{-\alpha} = 0$

When $\sigma < 0$ (i,e when σ is -ve) , $e^{-\sigma \times \alpha} = e^{\alpha} = \alpha$

So $X(s)$ converges when $\sigma > 0$

When $\sigma > 0$

$$X(s) = \left[\alpha \times \frac{e^{-\sigma \times \alpha} e^{-j\Omega \times \alpha}}{-s} - \frac{e^{-\sigma \times \alpha} e^{-j\Omega \times \alpha}}{s^2} + \frac{1}{s^2} \right]$$

$$= \left[\alpha \times \frac{0 \times e^{-j\Omega \times \alpha}}{-s} - \frac{0 \times e^{-j\Omega \times \alpha}}{s^2} + \frac{1}{s^2} \right]$$

$$= \frac{1}{s^2}$$

$\therefore L[t\,u(t)] = \frac{1}{s^2}$, with ROC is right half of s-plane

Porperties of Laplace transform

1) Amplitude Scaling.

$$L[x(t)] = X(s)$$

Then $L[A\,x(t)] = A\,X(s)$

eg. $x(t) = e^{-at}u(t)$

$$L[x(t)] = X(s) = \frac{1}{s+a}$$

then $L[4e^{-at}u(t)] = 4L[e^{-at}u(t)] = 4\frac{1}{s+a}$

2) Linearity.

$$L[x_1(t)] = X_1(s) \qquad L[x_2(t)] = X_2(s)$$

$$L[a_1x_1(t) \pm a_2x_2(t)] = a_1L[x_1(t)] \pm a_2L[x_2(t)]$$

$$x(t) = 2e^{-2t}u(t) + 3e^{-3t}u(-t)$$

$$L[x(t)] = 2L[e^{-2t}u(t)] + 3L[e^{-3t}u(-t)]$$

$$= 2\frac{1}{s+2} + 3\,\frac{-1}{s+3}$$

$$= \frac{2}{s+2} - \frac{3}{s+3}.$$

3) Time differentiation.

$$L\left[\frac{d\,x(t)}{dt}\right] = sX(s) - x(0)$$

Eg. $\frac{dx(t)}{dt} + x(t) = 4$ [A differential eqn].

Taking Laplace transform on the given eqn.

$sX(s) - x(0) + X(s) = \frac{4}{s}$ // [$x(0)$ is the value of $x(t)$ at $t=0$].

Continuation.

$$L\left[\frac{d^n x(t)}{dt^n}\right] = s^n X(s) - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1} x(t)}{dt^{k-1}}\Bigg|_{t=0}$$

eg.

$$L\left[\frac{d^2 x(t)}{dt^2}\right] = s^2 X(s) - \sum_{k=1}^{2} s^{n-k} \frac{d^{k-1} x(t)}{dt^{k-1}}\Bigg|_{t=0}$$

$$= s^2 X(s) - \left[s^{2-1} \frac{d^{1-1}}{dt^{1-1}} x(t) + s^{2-2} \frac{d^{2-1}}{dt^{2-1}} x(t)\right]_{t=0}$$

$$= s^2 X(s) - \left[s\, x(t) + \frac{d x(t)}{dt}\right]_{t=0}$$

$$= s^2 X(s) - s x(0) + \frac{d x(t)}{dt}\Bigg|_{t=0}$$

4) Time Integration

$$L\left[\int x(t)\,dt\right] = \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]\Big|_{t=0}}{s}$$

5) Frequency Shifting

$$L[x(t)] = X(s)$$

$$L[e^{\pm at} x(t)] = X(s \mp a)$$

eg. $x(t) = \cos \Omega_0 t$

$$L[x(t)] = L[\cos \Omega_0 t] = \frac{s}{s^2 + \Omega_0^2}$$

Then $$L[e^{-2t}\cos \Omega_0 t] = \frac{s}{s^2 + \Omega_0^2}\Bigg|_{s = s+2}$$

$$= \frac{s+2}{(s+2)^2 + \Omega_0^2}$$

6) Time shifting

$L[x(t)] = X(s)$

$L[x(t \pm \alpha)] = e^{\pm \alpha s} X(s)$

eg. $x(t) = e^{-2t} u(t)$

$X(s) = \frac{1}{s+2}$ [ROC: $s > -2$ / $\sigma > -2$]

$L[x(t-2)] = L[e^{-2(t-2)} u(t-2)]$

$= e^{-2s} \times \frac{1}{s+2} = \frac{e^{-2s}}{s+2}$ //

7) Frequency differentiation

$L[x(t)] = X(s)$

$L[t\,x(t)] = -\frac{d X(s)}{ds}$

eg. $x(t) = e^{-2t} u(t)$

$L[x(t)] = X(s) = \frac{1}{s+a}$

$L[t\,x(t)] = L[t \cdot e^{-2t} u(t)] = -\frac{d X(s)}{ds} = -\frac{d \; 1/s+a}{ds}$

$= -\frac{(0-1)}{(s+a)^2} = \frac{1}{(s+a)^2}$

$\therefore L[t e^{-2t} u(t)] = \frac{1}{(s+a)^2}$

8) Time scaling

$L[x(t)] = X(s)$

$L[x(at)] = \frac{1}{|a|} X(s/a)$.

eg. $x(t) = e^{-t} u(t)$, then $L[x(2t)] = L[e^{-2 \times t} u(t)]$

$L[x(t)] = X(s) = \frac{1}{s+1}$,

$= \frac{1}{|2|} X(\frac{s}{2}) = \frac{1}{2} \frac{1}{\frac{s}{2}+1} = \frac{1}{s+2}$

9 **Periodicity**

$L[x_1(t)] = X_1(s)$

If $x(t)$ is a periodic signal with time period T

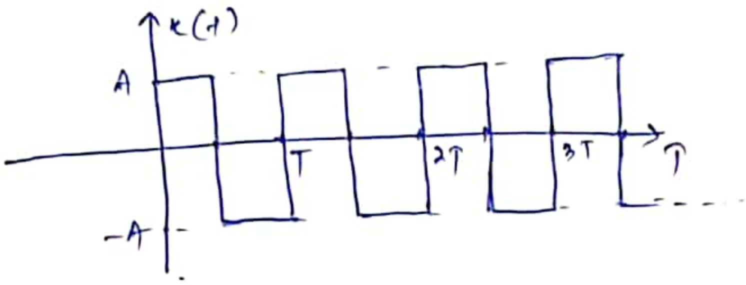
i.e. $x(t) = x(t + nT)$ $n \to 0, 1, 2 ----$

Then $L[x(t)] = \frac{1}{1-e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$

Where $x_1(t)$ is one period of signal $x(t)$.

eg

Q. Determine the Laplace transform of periodic square wave shown in fig I.



Soln The given waveform is periodic with time period T.

Let $x_1(t)$ be one period of $x(t)$. The equation of one period ($x_1(t)$) of the given periodic waveform can be expressed as

$x_1(t) = A$, for $t = 0$ to $T/2$

$= -A$, for $t = T/2$ to T

Then using the periodic property.

$$L[x(t)] = \frac{1}{1-e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$$

$$= \frac{1}{1-e^{-sT}}\left[\int_0^{T/2} A e^{-st}\,dt + \int_{T/2}^{T} (-A) e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-sT}}\left\{A\left[\frac{e^{-st}}{-s}\right]_0^{T/2} + (-A)\left[\frac{e^{-st}}{-s}\right]_{T/2}^{T}\right\}$$

$$= \frac{1}{1-e^{-sT}}\left\{A\left(\frac{e^{-sT/2}}{-s} - \frac{1}{-s}\right) - A\left(\frac{e^{-sT}}{-s} - \frac{e^{-sT/2}}{-s}\right)\right\}$$

$$= \frac{1}{1-e^{-sT}}\left\{\frac{A}{s}\left(1 + e^{-sT} - 2e^{-sT/2}\right)\right\}$$

$$= \frac{1}{1-e^{-sT}}\left\{\frac{A}{s}\left(1 - e^{-sT/2}\right)^2\right\}$$

$$= \frac{1}{(1-e^{-sT/2})(1+e^{-sT/2})}\left[\frac{A}{s}(1-e^{-sT/2})^2\right] \quad [a^2-b^2=(a+b)(a-b)]$$

$$= \frac{A}{s}\left[\frac{1-e^{-sT/2}}{1+e^{-sT/2}}\right]$$

10) Initial value theorem

$$L[x(t)] = X(s)$$

$$x(0) = \lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s)$$

11) Final value Theorem

$$L[x(t)] = X(s)$$

$$x(\infty) = \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Q: $X(s) = \frac{1}{s(s-1)}$ $X(s) = L[x(t)]$. Determine the initial value, $x(0)$ & the final value, $x(\infty)$, for the ~~following~~ given signal using initial & final value theorem.

Soln Given $X(s) = \frac{1}{s(s-1)}$

Initial value, $x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\times\frac{1}{s(s-1)} = \lim_{s\to\infty}\frac{1}{(s-1)} = 0$

Final value, $x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s\times\frac{1}{s(s-1)} = -1$

11) Convolution theorem.

$$L[x_1(t) * x_2(t)] = X_1(s)\, X_2(s)$$

where $x_1(t) * x_2(t) = \int_{-\alpha}^{\alpha} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$ is the linear convolution of $x_1(t)$ & $x_2(t)$

& $L[x_1(t)] = X_1(s)$ & $L[x_2(t)] = X_2(s)$.

Eg. Perform convolution of $x_1(t)$ & $x_2(t)$ using convolution theorem of Laplace Theorem.

$x_1(t) = u(t+5)$ & $x_2(t) = \delta(t-7)$

Soln.

$$L[x_1(t)] = L[u(t+5)]$$

$$= e^{5s} L[u(t)] \quad [L[x(t \pm a)] = e^{\pm as} X(s)]$$

$$X_1(s) = e^{5s} \times \frac{1}{s}. \quad [\because L[u(t)] = \frac{1}{s}]$$

$$L[x_2(t)] = L[\delta(t-7)]$$

$$= e^{-7s} L[\delta(t)]$$

$$X_2(s) = e^{-7s} \times 1 = e^{-7s} \quad [\because L(\delta(t)) = 1]$$

using convolution theorem of Laplace transform

$$L[x_1(t) * x_2(t)] = X_1(s) \times X_2(s)$$

$$\Rightarrow x_1(t) * x_2(t) = L^{-1}[X_1(s)\, X_2(s)]$$

$$\therefore x_1(t) * x_2(t) = L^{-1}[X_1(s) * X_2(s)]$$

$$= L^{-1}\left[e^{5s} \frac{1}{s} \times e^{-7s}\right]$$

$$= L^{-1}\left[e^{-2s} \times \frac{1}{s}\right]$$

$$= L^{-1}\left[e^{-2s} \times \frac{1}{s} \right]$$

$$= u(t-2)$$ [By applying time shifting property]

$$L[u(t)] = \frac{1}{s}$$

$$L[u(t-2)] = e^{-2s}\frac{1}{s}$$

$\therefore \quad x_1(t) * x_2(t) = u(t-2)$ //

Representation of poles & zeroes in s-plane

We know that complex frequency, $s = \sigma + j\Omega$

where σ is the real part of s denoted on the Horizontal axis of the s-plane, Ω is the imaginary part of s, denoted usually in the vertical axis of the s-plane. Hence the s-plane is a complex plane as show in Fig. I, where the zeros are marked by small circle "o" and the ~~plots~~ poles are marked by letter "x".

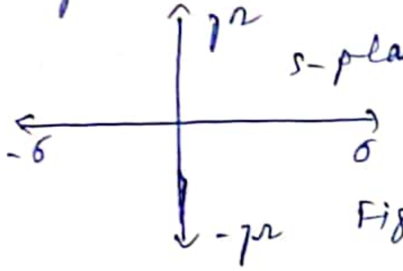


Fig I: s-plane.

For example consider the rational function of s shown below.

$$X(s) = \frac{(s+2)(s+5)}{s(s^2+6s+13)} \quad \text{———} \quad (a)$$

The roots of quadratic eqn $s^2+6s+13=0$ are

$$s = \frac{-6 \pm \sqrt{36 - 4\times 13}}{2\times 1} = \frac{-6 \pm j4}{2} = -3+j2, -3-j2$$

$$\therefore s^2+6s+13 = (s+3-j2)(s+3+j2)$$

$$\& \quad X(s) = \frac{(s+2)(s+5)}{s(s+3-j2)(s+3+j2)}$$

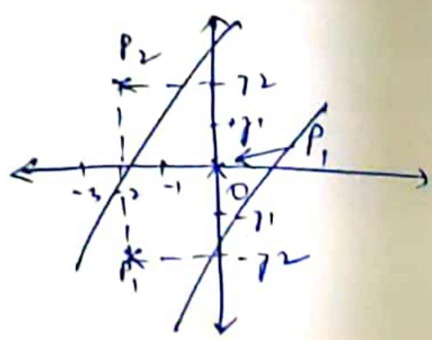
The zeros of the above function are,

$z_1 = -2$ & $z_2 = -5$

& The poles of the above function are,

$p_1 = 0$ $\quad p_3 = -3-j2$

$p_2 = -3+j2$



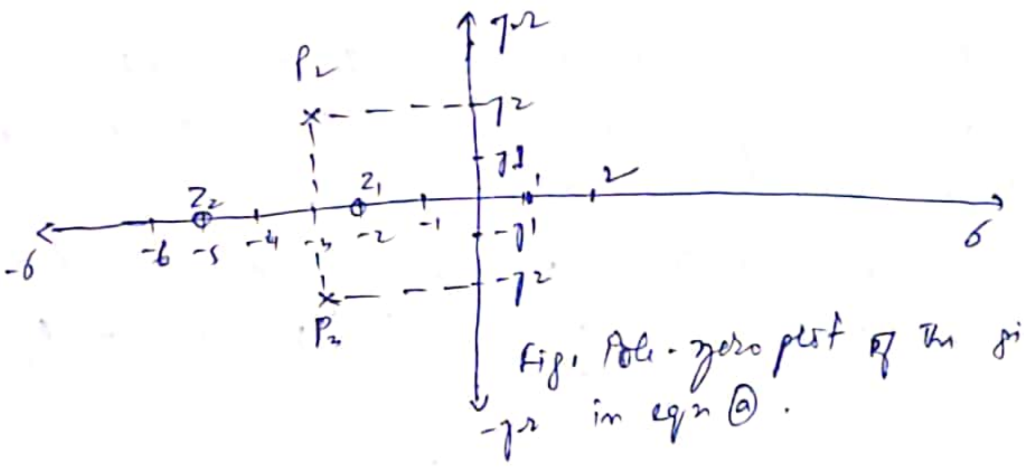


Fig. Pole-zero plot of the given function in eqn (a).

ROC of Rational function of s

case (1) Right sided (causal) signal.

eg. $x(t) = e^{-a_1 t}u(t) + e^{-a_2 t}u(t) + e^{-a_3 t}u(t)$; where $-a_1 < -a_2 < -a_3$

Now Laplace transform of $x(t)$ is,

$$X(s) = \frac{1}{s+a_1} + \frac{1}{s+a_2} + \frac{1}{s+a_3} = \frac{N(s)}{(s+a_1)(s+a_2)(s+a_3)}$$

where $N(s) = (s+a_2)(s+a_3) + (s+a_1)(s+a_3) + (s+a_1)(s+a_2)$

The poles of $X(s)$ are,

$p_1 = -a_1$; $p_2 = -a_2$; $p_3 = -a_3$

we know that σ = Real part of s.

so that the Region of Convergence of $X(s)$ may be described as

$\sigma > -a_1$; $\sigma > -a_2$; $\sigma > -a_3$

Since $-a_1 < -a_2 < -a_3$, the $X(s)$ converges when $\sigma > -a_3$

& does not converge for $\sigma < -a_3$

$\therefore$ Abscissa of ~~const~~ convergence, $\sigma_c = -a_3$

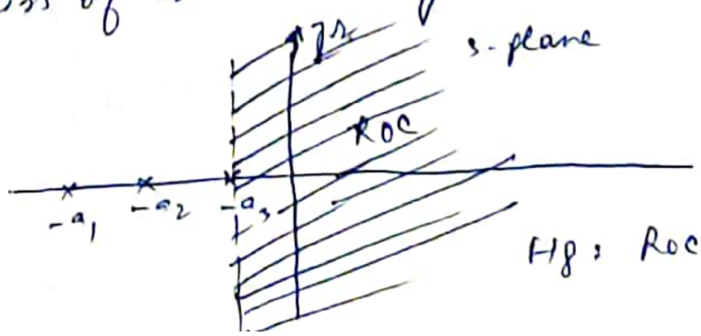


Fig: Roc of a causal signal

Thus the ROC of $X(s)$ is all points to the right of abscissa of convergence $(-a_3)$ or right of the line passing through $-a_3$ as shown in the above fig.

In terms of poles of $X(s)$ we can say that the ROC is right of right most pole of $X(s)$, (i.e right of the pole with largest real part).

Case ii) Left sided (anticausal signal)

eg. $x(t) = e^{-a_1 t} u(-t) + e^{-a_2 t} u(-t) + e^{-a_3} u(-t)$; where $-a_1 < -a_2 < -a_3$

Taking Laplace transform of the signal $x(t)$

$$X(s) = \frac{1}{s+a_1} + \frac{1}{s+a_2} + \frac{1}{s+a_3}$$

$$= \frac{(s+a_2)(s+a_3) + (s+a_1)(s+a_3) + (s+a_1)(s+a_2)}{(s+a_1)(s+a_2)(s+a_3)}$$

Poles of $X(s)$ are : $P_1 = -a_1$, $P_2 = -a_2$, $P_3 = -a_3$

Now the Region of convergence criteria can be described as ($\sigma \to$ Real part of s) $\sigma < -a_1$, $\sigma < -a_2$, $\sigma < -a_3$ [for anticausal signals]

Since it is given that $-a_1 < -a_2 < -a_3$, the $X(s)$ converges when $\sigma < -a_1$ and does not converge for $\sigma > -a_1$ as a whole.

$\therefore$ Abscissa of convergence , $\sigma_c = -a_1$

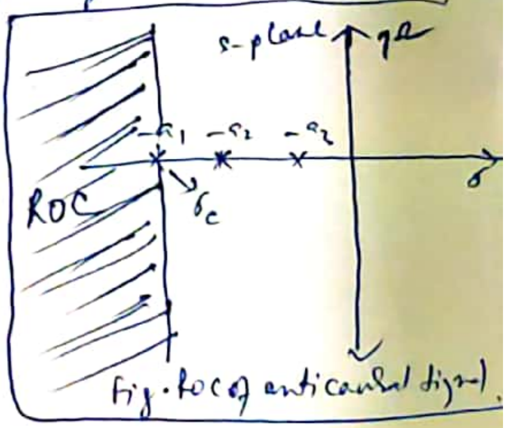


Fig. ROC of anticausal signal.

Therefore ROC of $X(s)$ is all points to the left of abscissa of convergence (or left of the line passing through $-a_1$) in s plane as shown in fig.

In terms of poles of $X(s)$, we can say that the ROC is left of left most pole of $X(s)$ (i.e left of pole with smallest real part).

Case iii Two sided signal.

eg. $x(t) = e^{-a_1 t}u(t) + e^{-a_2 t}u(t) + e^{-a_3 t}u(-t) + e^{-a_4 t}u(-t)$;

where $-a_1 < -a_2 < -a_3 < -a_4$

Now. the Laplace transform of $x(t)$ is

$$X(s) = \frac{1}{s+a_1} + \frac{1}{s+a_2} + \frac{1}{s+a_3} + \frac{1}{s+a_4}$$

$$= \frac{(s+a_2)(s+a_3)(s+a_4) + (s+a_1)(s+a_3)(s+a_4) + (s+a_1)(s+a_2)(s+a_4) + (s+a_1)(s+a_2)(s+a_3)}{(s+a_1)(s+a_2)(s+a_3)(s+a_4)}$$

The poles of $X(s)$ are

$p_1 = -a_1$, $p_2 = -a_2$, $p_3 = -a_3$, $p_4 = -a_4$

Let σ = Real part of s.

Now, the Region of Convergence criteria for $X(s)$ can be described as.

$\sigma > -a_1$, $\sigma > -a_2$, $\sigma < -a_3$, $\sigma < -a_4$ [where σ is real part of s]

& given that $-a_1 < -a_2 < -a_3 < -a_4$, the function $X(s)$ converges, when σ lies between $-a_2$ and $-a_3$ (i.e. $-a_2 < \sigma < -a_3$) and does not converge for $\sigma < -a_2$ and $\sigma > -a_3$

$\therefore$ Abscissa of convergence, $\sigma_{c_1} = -a_2$ & $\sigma_{c_2} = -a_3$.

Hence ROC of $X(s)$ is all points in the region in between two abscissa of convergence (or region in between the two lines passing through $-a_2$ & $-a_3$) in s-plane as shown in fig.

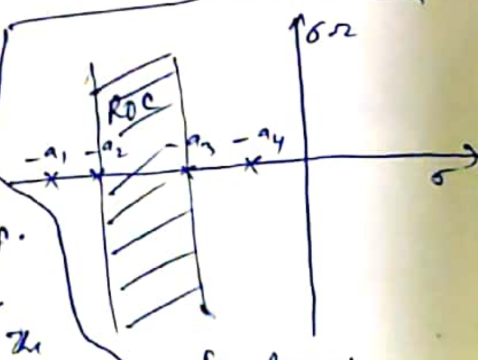


Fig. ROC of a non causal signal

Now for a non causal signal which contains causal part & non-causal part. Let a_x be the magnitude of the largest pole of causal part of the signal (a_1 in this example) and a_y be the magnitude of smallest pole of anticausal part

of the signal (a_3 in the example) and let $a_x < a_y$ (i.e $a_2 < a_3$).
So in term of poles of $X(s)$. we can say that the ROC is the region of convergence (ROC) is the region between two lines passing through a_x (a_2) and a_y (a_3) where $a_x < a_y$ ($a_2 < a_3$).

Inverse Laplace transform.

The Inverse Laplace transform of $X(s)$ is defined as,

$$L^{-1}[X(s)] = x(t) = \frac{1}{2\pi j} \int_{s=\sigma-j\Omega}^{s=\sigma+j\Omega} X(s)\, e^{st}\, ds$$

Performing ~~so~~. Inverse Laplace transform using the above definition is tedious and complex. But taking Inverse using partial fraction expansion method will be much easier.

Inverse Laplace transform by Partial fraction Method.

Let $L[x(t)] = X(s)$. The s-domain signal $X(s)$ will be a ratio of two polynomials in s (i.e rational function of s)
The roots of the denominator polynomial are called poles.
The roots of numerator polynomials are called zeros.
In Signals & Systems, three different types of s-domain signals are encountered. They are

case i: Signals/Systems with separate poles.
Case ii: " with multiple poles.
Case iii: " " complex conjugate poles.

Case-i: When s-Domain Signal X(s) has Distinct poles.

Let, $X(s) = \frac{k}{s(s+P_1)(s+P_2)}$

By partial fraction expantion technique, it can be expanded as

$$X(s) = \frac{k}{s(s+P_1)(s+P_2)} = \frac{k_1}{s} + \frac{k_2}{(s+P_1)} + \frac{k_3}{(s+P_2)}$$

$$k_1 = X(s) \times s \Big|_{s=0} \qquad k_2 = X(s) \times (s+P_1) \Big|_{s=-P_1}$$

$$\cancel{k_3 = X(s) \times s} \qquad k_3 = X(s) \times (s+P_2) \Big|_{s=-P_2}$$

Using $L[u(t)] = \frac{1}{s}$, $L[e^{-at}u(t)] = \frac{1}{s+a}$

$$L^{-1}[X(s)] = L^{-1}\left[\frac{k_1}{s} + \frac{k_2}{s+P_1} + \frac{k_3}{s+P_2}\right]$$

$$\therefore x(t) = k_1 L^{-1}\left[\frac{1}{s}\right] + k_2 L^{-1}\left[\frac{1}{s+P_1}\right] + k_3 L^{-1}\left[\frac{1}{s+P_2}\right]$$

$$= k_1 u(t) + k_2 e^{-P_1 t} u(t) + k_3 e^{-P_2 t} u(t)$$

Q. Determine the inverse of Laplace transform of

$$X(s) = \frac{2}{s(s+1)(s+2)}$$

Soln. $X(s) = \frac{2}{s(s+1)(s+2)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$

$$k_1 = X(s) \times s \Big|_{s=0} = \frac{2}{s(s+1)(s+2)} \times s \Big|_{s=0} = \frac{2}{1 \times 2} = 1$$

$$k_2 = X(s) \times (s+1) \Big|_{s=-1} = \frac{2}{s(s+2)} \Big|_{s=-1} = \frac{2}{-1(-1+2)} = -2.$$

$$k_3 = X(s) \times (s+2) \Big|_{s=-2} = \frac{2}{s(s+1)} \Big|_{s=-2} = \frac{2}{-2(-2+1)} =$$

$$X(s) = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

Now $x(t) = L^{-1}[X(s)] = L^{-1}\left[\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}\right]$

$= u(t) - 2e^{-t}u(t) + e^{-2t}u(t)$

$= (1 - 2e^{-t} + e^{-2t})u(t) = (1 - e^{-t})^2 u(t)$.

Case ii) When s-domain signal $X(s)$ has Multiple poles.

Let $X(s) = \frac{K}{s(s+P_1)(s+P_2)^2}$

By partial fraction expansion technique,

$$X(s) = \frac{K_1}{s} + \frac{K_2}{s+P_1} + \frac{K_3}{(s+P_2)^2} + \frac{K_4}{(s+P_2)}$$

$K_1 = X(s) \times s\big|_{s=0}$; $K_2 = X(s) \times (s+P_1)\big|_{s=-P_1}$

$K_3 = X(s) \times (s+P_2)^2\big|_{s=-P_2}$; $K_4 = \frac{d}{ds}[X(s) \times (s+P_2)^2]\Big|_{s=-P_2}$

We know that $L[u(t)] = \frac{1}{s}$, $L[e^{-at}u(t)] = \frac{1}{s+a}$, $L[te^{-at}u(t)] = \frac{1}{(s+a)^2}$

Then by using the above standard Laplace transform pairs, the inverse Laplace transform of the given function $X(s)$ can be obtained as

$$L^{-1}[X(s)] = L^{-1}\left[\frac{K_1}{s} + \frac{K_2}{s+P_1} + \frac{K_3}{(s+P_2)^2} + \frac{K_4}{(s+P_2)}\right]$$

$\Rightarrow x(t) = K_1 L^{-1}\left[\frac{1}{s}\right] + K_2 L^{-1}\left[\frac{1}{s+P_1}\right] + K_3 L^{-1}\left[\frac{1}{(s+P_2)^2}\right] + K_4\left[\frac{1}{s+P_2}\right]$

$= K_1 u(t) + K_2 e^{-P_1 t}u(t) + K_3 te^{-P_2 t}u(t) + K_4 e^{-P_2 t}u(t)$

In general if the pole P_2 in the given function $X(s)$ has multiplicity of q, as shown below,

$$X(s) = \frac{k}{s(s+P_1)(s+P_2)^q}$$

$$= \frac{k_1}{s} + \frac{k_2}{s+P_1} + \frac{K_{0P_2}}{(s+P_2)^q} + \frac{K_{1P_2}}{(s+P_2)^{q-1}} + \cdots \frac{K_{rP_2}}{(s+P_2)^{q-r}} + \cdots \frac{K_{(q-1)P_2}}{(s+P_2)}$$

$$K_{rP_2} = \frac{1}{r!}\frac{d^r[X(s)\times(s+P_2)^q]}{ds^r}\Bigg|_{s=-P_2} \quad ; r = 1, 2, 3 \dots q-1$$

If $q = 3$

$r = 1, 2.$

eg. Determine the inverse Laplace transform of

$$X(s) = \frac{2}{s(s+1)(s+2)^2}$$

soln

$$X(s) = \frac{2}{s(s+1)(s+2)^2} = \frac{k_1}{s} + \frac{k_2}{(s+1)} + \frac{K_{02}}{(s+2)^{2-0}} + \frac{K_{12}}{(s+2)^{2-1}}$$

$$= \frac{k_1}{s} + \frac{k_2}{(s+1)} + \frac{K_{02}}{(s+2)^2} + \frac{K_{12}}{(s+2)}$$

$$k_1 = X(s)\times s\Big|_{s=0}$$

$$= \frac{2}{\cancel{s}(s+1)(s+2)^2}\times\cancel{s}\Bigg|_{s=0} = \frac{2}{1\times 2^2} = 0.5$$

$$k_2 = X(s)\times(s+1)\Big|_{s=-1} = \frac{2}{s(s+2)^2}\Bigg|_{s=-1} = \frac{2}{-1(-1+2)^2} = -2$$

$$K_{02} = \frac{d^0[X(s)\times(s+2)^2]}{ds^0}\Bigg|_{s=-2} = \frac{2}{s(s+1)}\Bigg|_{s=-2} = \frac{2}{-2(-2+1)} = 1$$

$$K_{12} = \frac{d^1[X(s)\times(s+2)^2]}{ds^1}\Bigg|_{s=-2} = \frac{d}{ds}\left(\frac{2}{s(s+1)}\right)\Bigg|_{s=-2} = \frac{-2(2s+1)}{s^2(s+1)^2}\Bigg|_{s=-2}$$

$$= 1.5$$

$\therefore \; X(s) = \dfrac{2}{s(s+1)(s+2)^2} = \dfrac{0.5}{s} - \dfrac{2}{s+1} + \dfrac{1}{(s+2)^2} + \dfrac{1.5}{(s+2)}$

$x(t) = L^{-1}\left[\dfrac{0.5}{s} - \dfrac{2}{s+1} + \dfrac{1}{(s+2)^2} + \dfrac{1.5}{(s+2)}\right]$

$x(t) = 0.5\,u(t) - 2e^{-t}u(t) + te^{-2t}u(t) + 1.5e^{-2t}u(t)$.

$= \left(0.5 - 2e^{-t} + te^{-t} + 1.5e^{-2t}\right)u(t)$.

Case-iii. When s-domain signal $X(s)$ has complex conjugate poles.

Let $X(s) = \dfrac{k}{(s+p_1)(s^2+bs+c)}$

By partial fraction expansion technique,

$X(s) = \dfrac{k}{(s+p_1)(s^2+bs+c)} = \dfrac{k_1}{(s+p_1)} + \dfrac{k_2 s + k_3}{s^2+bs+c}$

$= \dfrac{k_1}{(s+p_1)} + \dfrac{k_2 s + k_3}{s^2 + 2\times\frac{b}{2}\times s + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2}$ [Arranging s^2+bs, in the form of $(x+y)^2$ where $(x+y)^2 = x^2+2xy+y^2$]

$= \dfrac{k_1}{(s+p_1)} + \dfrac{k_2 s + k_3}{\left(s+\frac{b}{2}\right)^2 + \left[c - \left(\frac{b}{2}\right)^2\right]}$

$= \dfrac{k_1}{(s+p_1)} + \dfrac{k_2 s + k_3}{(s+a)^2 + \Omega_0^2}$ (Put) $\left[a = \frac{b}{2} \;\&\; \Omega_0 = c - \left(\frac{b}{2}\right)^2\right]$

$= \dfrac{k_1}{(s+p_1)} + \dfrac{k_2\left[s + \frac{k_3}{k_2}\right]}{(s+a)^2 + \Omega_0^2} = \dfrac{k_1}{(s+p_1)} + \dfrac{k_2\left[s + a + \frac{k_3}{k_2} - a\right]}{(s+a)^2 + \Omega_0^2}$

$= \dfrac{k_1}{(s+p_1)} + \dfrac{k_2(s+a)}{(s+a)^2+\Omega_0^2} + \dfrac{k_2 k_4}{\Omega_0} \times \dfrac{\Omega_0}{(s+a)^2+\Omega_0^2}$ (Put) $\left[\frac{k_3}{k_2} - a = k_4\right]$

$$= \frac{k_1}{(s+P_1)} + \frac{k_2(s+a)}{(s+a)^2+\Omega_0^2} + k_5\frac{\Omega_0}{(s+a)^2+\Omega_0^2} \quad \left[\text{Put } \frac{k_2 k_4}{\Omega_0} = k_5\right]$$

we know that.

$$L[e^{-at}u(t)] = \frac{1}{s+a} \quad ; \quad L[e^{-at}\cos\Omega_0 t\, u(t)] = \frac{s+a}{(s+a)^2+\Omega_0^2}$$

$$L[e^{-at}\sin\Omega_0 t\, u(t)] = \frac{\Omega_0}{(s+a)^2+\Omega_0^2} \quad ; \quad L[\cos\Omega_0 t] = \frac{s}{s^2+\Omega_0^2} \quad ; \quad L[\sin\Omega_0 t] = \frac{\Omega_0}{s^2+\Omega_0^2}$$

using property $L[e^{-at}x(t)] = X(s+a)$.

$$\therefore x(t) = L^{-1}[X(s)] = L^{-1}\left[\frac{k_1}{s+P_1}\right] + L^{-1}\left[\frac{k_2(s+a)}{(s+a)^2+\Omega_0^2}\right] + L^{-1}\left[\frac{k_5\Omega_0}{(s+a)^2+\Omega_0^2}\right]$$

$$= k_1 e^{-P_1 t}u(t) + k_2 e^{-at}\cos\Omega_0 t\, u(t) + k_5 e^{-at}\sin\Omega_0 t\, u(t)$$

eg. Determine the inverse Laplace transform of $X(s) = \frac{1}{(s+2)(s^2+s+1)}$

Soln Given

$$X(s) = \frac{1}{(s+2)(s^2+s+1)} = \frac{k_1}{(s+2)} + \frac{k_2 s + k_3}{s^2+s+1}$$

$$k_1 = X(s)\times(s+2)\Big|_{s=-2} = \frac{1}{(s+2)(s^2+s+1)}\times(s+2)\Big|_{s=-2} = \frac{1}{(-2)^2-2+1} = \frac{1}{3}$$

Now to solve k_2 & k_3

$$\frac{1}{(s+2)(s^2+s+1)} = \frac{k_1}{(s+2)} + \frac{k_2 s + k_3}{s^2+s+1}$$

$$\Rightarrow 1 = k_1(s^2+s+1) + (k_2 s + k_3)(s+2) \quad [k_1 = 1/3]$$

$$\Rightarrow 1 = \frac{1}{3}(s^2+s+1) + k_2 s^2 + 2k_2 s + k_3 s + 2k_3$$

$$\Rightarrow 1 = \frac{s^2}{3} + \frac{s}{3} + \frac{1}{3} + k_2 s^2 + 2k_2 s + k_3 s + 2k_3$$

$$\Rightarrow 1 = \left(\frac{1}{3}+k_2\right)s^2 + \left(\frac{1}{3}+2k_2+k_3\right)s + \frac{1}{3} + 2k_3$$

Equating coefficients of s^2 terms

$$0 = \frac{1}{3} + k_2 \Rightarrow k_2 = -1/3$$

Equating coefficients of s terms

$$0 = \frac{1}{3} + 2k_2 + k_3 \Rightarrow K_3 = -\frac{1}{3} - 2k_2 = -\frac{1}{3} - 2\times\left(-1/3\right) = 1/3$$

$$\therefore X(s) = \frac{1}{(s+2)(s^2+s+1)} = \frac{1/3}{s+2} + \frac{-1/3 s + 1/3}{s^2+s+1} = \frac{\frac{1}{3}}{s+2} + \frac{-1/3(s-1)}{s^2+s+1}$$

$$= 1/3 \times \frac{1}{s+2} - \frac{1}{3}\frac{s-1}{\left[s^2 + 2\times\frac{1}{2}s + \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{2}\right)^2\right]}$$ [Arranging s^2 + s in the form of $\left(s+\frac{1}{2}\right)^2$]

$$= \frac{1}{3}\times\frac{1}{(s+2)} - \frac{1}{3}\frac{(s-1)}{\left(s+\frac{1}{2}\right)^2 + 0.75} = \frac{1}{3}\times\frac{1}{s+2} - \frac{1}{3}\frac{(s-1)}{(s+0.5)^2 + 0.75}$$

$$= \frac{1}{3}\times\frac{1}{s+2} - \frac{1}{3}\frac{s+0.5-1-0.5}{(s+0.5)^2 + (\sqrt{0.75})^2} = \frac{1}{3}\times\frac{1}{(s+2)} - \frac{1}{3}\frac{(s+0.5)}{(s+0.5)^2+(\sqrt{0.75})^2} + \frac{1}{3}\times\frac{1.5}{\sqrt{0.75}}\times\frac{\sqrt{0.75}}{(s+0.5)^2+(\sqrt{0.75})^2}$$

$$= \frac{1}{3}\times\frac{1}{(s+2)} - \frac{1}{3}\frac{(s+0.5)}{(s+0.5)^2+(0.866)^2} + 0.577\times\frac{0.866}{(s+0.5)^2+0.866^2}$$ $\left[\sqrt{0.75} = 0.866\right]$

$$x(t) = L^{-1}\left[\frac{1}{3}\times\frac{1}{s+2}\right] - L^{-1}\left[\frac{s+0.5}{(s+0.5)^2+0.866^2}\right] + L^{-1}\left[0.577\times\frac{0.866}{(s+0.5)^2+0.866^2}\right]$$

$$= \frac{1}{3}e^{-2t}u(t) - e^{-0.5t}\cos 0.866t\, u(t) + 0.577e^{-0.5t}\sin 0.866t\, u(t)$$