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# Stability of Nonlinear Systems

Lyapunov 1857-1918

**Def** ① Consider a nonlinear system in $\mathbb{R}^n$

$$\dot{x}(t) = f(x(t))$$

vector field

$f(0) = 0 \rightarrow$ origin is equilibrium point

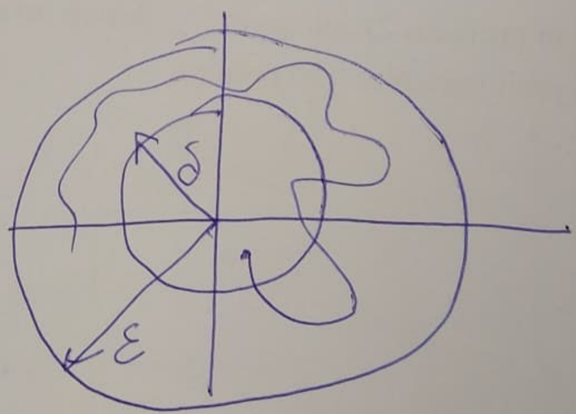
Null solution $\rightarrow x(t) = 0$

**Def** ②

$x(t) = 0$ is stable iff $\forall\, \varepsilon > 0$ given can find $\delta(\varepsilon) > 0$ such that

$\|x(t_0)\| < \delta$ implies $\|x(t)\| < \varepsilon$ $\forall\, t > t_0$

If not $\rightarrow$ unstable




**Def**: $x(t) = 0$ is asymptotically stable

iff 1. Stable as in def ②

2. There exist $\delta(\varepsilon) > 0$ such that

$\|x(t_0)\| < \delta$ implies $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$

# Lyapunov first method

Let $A = (a_{ij})$ where $a_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x=0}$ Linearization around equilibrium $x(t) = 0$

i.e. ~~the~~ linearized system

~~The~~ $\dot{x} = Ax$

nonlinear-system

1. is stable when A is Hurwitz (has all eigenvalues in open left half plane)
2. is unstable when there is at least one eigenvalue of A in open right half plane
3. Stability cannot be determined when there is at least one eigenvalue on $j\omega$ axis

weakness of 1st method

## Lyapunov second method

Given a symmetric matrix P

a (quadratic form) scalar function

$$V(x) = x^T P x = \sum_{i,j=1}^{n} p_{ij} x_i x_j$$

is called <u>positive definite</u> iff $V(x) > 0 \ \forall x \neq 0$

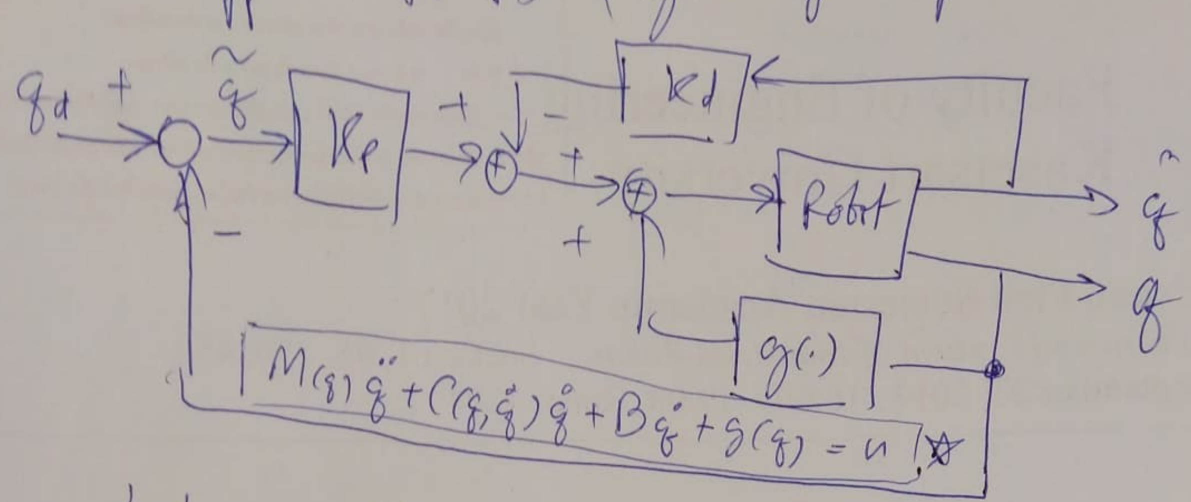
Positive definite $V(x)$ $\iff$ positive definite $P$ (all eigenvalues are positive)

Lyapunov 2nd method

1. Find $V(x) > 0$ Lyapunov function candidate
2. ~~Show~~ Show that $\dot{V} = \frac{\partial V}{\partial x} f(x) \leq 0$ stable

$\dot{V} < 0$ asymptotically stable

Applied to PD + gravity compensation




Let $q = \begin{bmatrix} \tilde q \\ \dot q \end{bmatrix}$ where $\tilde q = q_d - q$

Choose Lyapunov function

$$V(\dot q, \tilde q) = \frac{1}{2}\dot q^T \overset{PD}{M(q)} \dot q + \frac{1}{2}\tilde q^T \overset{PD}{K_p} \tilde q > 0 \qquad (1)$$

Note that $\dot{\tilde q} = -\dot q$

From (1)

$$\dot V = \frac{1}{2}\ddot q^T M(q)\dot q + \frac{1}{2}\dot q^T M(q)\ddot q + \frac{1}{2}\dot q^T \dot M(q)\dot q + \frac{1}{2}\overbrace{\left(\dot{\tilde q}^T K_p \tilde q + \tilde q^T K_p \dot{\tilde q}\right)}^{= -2\dot q^T K_p \tilde q}$$

$$= \dot q^T M(q)\ddot q + \frac{1}{2}\dot q^T \dot M(q)\dot q - \dot q^T K_p \tilde q \qquad (2)$$

From ☆

$$M(q)\ddot q = u - C(q,\dot q)\dot q - B\dot q - g(q)$$

Subs to (2)

$$\dot V = \dot q^T\left(u - C(q,\dot q)\dot q - B\dot q - g(q)\right) + \frac{1}{2}\dot q^T \dot M(q)\dot q - \dot q^T K_p \tilde q$$

$$\dot{V} = \dot{q}^T u - \dot{q}^T C(q,\dot{q})\dot{q} - \dot{q}^T B \dot{q} - \dot{q}^T g(q) + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} - \dot{q}^T K_p \tilde{q}$$

$$= \cancelto{0}{\frac{1}{2}\dot{q}^T \big(\underbrace{\dot{M}(q) - 2C}_{\text{Skew symmetric}}\big)\dot{q}} - \dot{q}^T B \dot{q} + \dot{q}^T\big(u - g(q) - K_p \tilde{q}\big)$$

Choose $u = g(q) + K_p \tilde{q} \rightarrow \dot{V} \leq 0$

② If choose $u = g(q) + K_p \tilde{q} - K_d \tilde{q}$

$$\dot{V} = -\dot{q}^T (B + K_d)\dot{q} \leq 0$$

Note $\dot{V} < 0$ if ~~the~~ choose $K_d$ PD