$x \in \textsc{Var}, \quad n \in \mathbb{Z}, \quad b \in \{true, false\}, \quad s \in \textsc{Share}, \quad p \in \textsc{Pred}, \quad bt \in \textsc{BaseType}, \quad t \in \textsc{Type}, \quad e \in \textsc{Term}$

$$s ::= n \mid b \mid x$$

$$p ::= (+\ p\ p) \mid (-\ p\ p) \mid (\leq\ \ p\ p) \mid (=\ \ p\ p) \mid (if\ p\ p\ p) \mid s$$

$$bt ::= int \mid bool$$

$$t ::= (:\ \ t\ p) \mid (\rightarrow x\ t\ t) \mid (\forall\ x\ t) \mid x \mid bt$$

$$e^* ::= s \mid (\lambda\ (x)\ e)$$

$$e ::= (let\ ((x\ e))\ e) \mid (letrec\ ((x\ e\ t))\ e) \mid (e^*\ e^*) \mid (inst\ e^*\ t) \mid (if\ e^*\ e\ e) \mid (as\ e\ t) \mid e^*$$

When looking up a type from the type environment we can strip off all it's refinements since they are already known from the context.

$\boxed{\Gamma \vdash s : t}$ **Shared Typing**

$$\frac{\Gamma \vdash x : (:\ \ t\ p)}{\Gamma \vdash x : t}\ (lookup\ refined) \qquad \frac{}{\Gamma; x : t \vdash x : t}\ (lookup) \qquad \frac{}{\Gamma \vdash n : int}\ (n) \qquad \frac{}{\Gamma \vdash b : bool}\ (b)$$

$\boxed{\Gamma \vdash p : bt}$ **Pred Typing**

$$\frac{\Gamma \vdash p_1 : int \quad \Gamma \vdash p_2 : int}{\Gamma \vdash (+\ p_1\ p_2) : int}\ (+) \qquad \frac{\Gamma \vdash p_1 : int \quad \Gamma \vdash p_2 : int}{\Gamma \vdash (-\ p_1\ p_2) : int}\ (-) \qquad \frac{\Gamma \vdash p_1 : int \quad \Gamma \vdash p_2 : int}{\Gamma \vdash (\leq\ \ p_1\ p_2) : bool}\ (\leq)$$

$$\frac{\Gamma \vdash p_1 : bt \quad \Gamma \vdash p_2 : bt}{\Gamma \vdash (=\ \ p_1\ p_2) : bool}\ (=) \qquad \frac{\Gamma \vdash p_1 : bool \quad \Gamma \vdash p_2 : bt \quad \Gamma \vdash p_2 : bt}{\Gamma \vdash (if\ p_1\ p_2\ p_3) : bt}\ (if)$$

$$\frac{\Gamma \vdash s : t \quad t = bt}{\Gamma \vdash s : bt}\ (pred\ shared)$$

$\boxed{\Gamma \vdash t : *}$ **Well Formedness**

$$\frac{}{\Gamma \vdash bt : *}\ (base) \qquad \frac{\Gamma;\ res : bt \vdash p : bool}{\Gamma \vdash (:\ \ bt\ p) : *}\ (refined\ base) \qquad \frac{\Gamma \vdash t : * \quad \Gamma \vdash p : bool}{\Gamma \vdash (:\ \ t\ p) : *}\ (refined)$$

$$\frac{\Gamma \vdash t_1 : * \quad \Gamma;\ x : t_1 \vdash t_2 : *}{\Gamma \vdash (\rightarrow\ \ x\ t_1\ t_2) : *}\ (\rightarrow) \qquad \frac{x \notin \Gamma \quad \Gamma; x : * \vdash t : *}{\Gamma \vdash (\forall\ x\ t) : *}\ (\forall)$$

$$c \in \textsc{Constraint} \quad c := p \mid (\Rightarrow\ \ p\ c) \mid (\forall\ x\ bt\ c)$$

$\boxed{\Gamma \vdash c}$ **Context to Constraints**

$$\frac{\Gamma \vdash (\forall\ x\ bt\ (\Rightarrow\ \ p\ c))}{\Gamma; x : (:\ \ bt\ p) \vdash c}\ (refined\ base) \qquad \frac{\Gamma; x : t \vdash (\Rightarrow\ \ p\ c)}{\Gamma; x : (:\ \ t\ p) \vdash c}\ (refined) \qquad \frac{\Gamma \vdash (\forall\ x\ bt\ c)}{\Gamma; x : bt \vdash c}\ (base)$$

$$\frac{\Gamma \vdash c}{\Gamma; x : t \vdash c}\ (other) \qquad \frac{\textbf{SMTValid}(c)}{\emptyset \vdash c}\ (emp)$$

When checking if a refined type is a sub-type of another type we can add it to the type context so that any further SMT checks can assume it's refinements hold, we then re-extract it from the context to strip it's refinements.

$\boxed{\Gamma \vdash t \prec: t}$ **Sub-typing**

$$\frac{\Gamma; x_{fresh} : t_1 \vdash p[res := x_{fresh}] \quad \Gamma \vdash t_1 \prec: t_2}{\Gamma \vdash t_1 \prec: (:\ t_2\ p)}\ (refined\ right)$$

$$\frac{\Gamma; x_{fresh} : t_1 \vdash x_{fresh} : t_1' \quad \Gamma; x_{fresh} : t_1 \vdash t_1' \prec: t_2}{\Gamma \vdash t_1 \prec: t_2}\ (refined\ left)$$

$$\frac{\Gamma \vdash t_{12} \prec: t_{11} \quad \Gamma; x_2 : t_{21} \vdash t_{12}[x_1 := x_2] \prec: t_{22}}{\Gamma \vdash (\rightarrow\ x_1\ t_{11}\ t_{12}) \prec: (\rightarrow\ x_2\ t_{21}\ t_{22})}\ (\rightarrow)$$

$$\frac{\Gamma; x_{fresh} : * \vdash t_1[x_1 := x_{fresh}] \prec: t_2[x_2 := x_{fresh}]}{\Gamma \vdash (\forall\ x_1\ t_1) \prec: (\forall\ x_2\ t_2)}\ (\forall) \qquad \frac{}{\Gamma \vdash bt \prec: bt}\ (bt) \qquad \frac{}{\Gamma \vdash x \prec: x}\ (x)$$

Note that $t[x := (\lambda\ (x_1)\ e)] ::= t$, since x has a function type t couldn't have referred to it anyways. $t[x := s]$ is defined to do regular substitution which is valid since s can appear inside of PRED.

$\boxed{\Gamma \vdash e \Rightarrow t}$ **Inference**

$$\frac{\Gamma \vdash s : bt}{\Gamma \vdash s \Rightarrow (:\ bt\ (=\ res\ s))}\ (shared\ base) \qquad \frac{\Gamma \vdash s : t}{\Gamma \vdash s \Rightarrow t}\ (shared) \qquad \frac{\Gamma \vdash t : * \quad \Gamma \vdash e \Leftarrow t}{\Gamma \vdash (as\ e\ t) \Rightarrow t}\ (as)$$

$$\frac{\Gamma \vdash e_1^* \Rightarrow (\rightarrow\ x\ t_1\ t_2) \quad \Gamma \vdash e_1^* \Leftarrow t_1}{\Gamma \vdash (e_1^*\ e_2^*) \Rightarrow t_2[x := e_1^*]}\ (app) \qquad \frac{\Gamma \vdash e_1^* \Rightarrow (\forall\ x\ t_2)}{\Gamma \vdash (inst\ e_1^*\ t_1) \Rightarrow t_2[x := t_1]}\ (inst)$$

$\boxed{\Gamma \vdash e \Leftarrow t}$ **Checking**

$$\frac{\Gamma \vdash e_1^* \Leftarrow bool \quad \Gamma; \_ : (:\ int\ e_1^*) \vdash e_2 \Leftarrow t \quad \Gamma; \_ : (:\ int\ (if\ e_1^*\ false\ true)) \vdash e_3 \Leftarrow t}{\Gamma \vdash (if\ e_1^*\ e_2\ e_3) \Leftarrow t}\ (if)$$

$$\frac{\Gamma; x : t_1 \vdash e \Leftarrow t_2}{\Gamma \vdash (\lambda\ (x)\ e) \Leftarrow (\rightarrow\ x\ t_1\ t_2)}\ (\lambda) \qquad \frac{x \notin \Gamma \quad \Gamma; x : * \vdash (\lambda\ (x_1)\ e) \Leftarrow t}{\Gamma \vdash (\lambda\ (x_1)\ e) \Leftarrow (\forall\ x\ t)}\ (\lambda\forall)$$

$$\frac{\Gamma \vdash e_1 \Rightarrow t_1 \quad \Gamma; x : t_1 \vdash e_2 \Leftarrow t_2}{\Gamma \vdash (let\ (x\ e_1)\ e_2) \Leftarrow t_2}\ (let)$$

$$\frac{t_1 = (\rightarrow\ x_1\ t_{11}\ t_{12}) \quad \Gamma; x : t_1 \vdash e_1 \Leftarrow t_1 \quad \Gamma; x : t_1 \vdash e_2 \Leftarrow t_2}{\Gamma \vdash (letrec\ (x\ e_1\ t_1)\ e_2) \Leftarrow t_2}\ (letrec)$$

$$\frac{\Gamma \vdash e \Rightarrow t_1 \quad \Gamma \vdash t_1 \prec: t_2}{\Gamma \vdash e \Leftarrow t_2}\ (check\ infer)$$