Trojan Asteroids: a computational investigation into Lagrangian Orbits

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1. Abstract

The aim of this project was to design and test a computer simulation of asteroids placed in Lagrangian orbits in order to examine the stability of the orbits and the effect of altering the mass of the system. The simulation agreed with the theoretical prediction that points L4 and L5 are stable, while L1, L2 and L3 are not. It was found that point L4 remained stable upon changing the mass of one of the bodies, until the ratio of two masses exceeded 0.038. This agreed with Gascheau's Value (μ_G=0.0385208)[1].

2. Introduction

Lagrangian points are named after French-Italian mathematician Joseph-Louis Lagrange. These are five special points in space where the gravitational force of two large bodies equals the centripetal force required for a smaller body to move in regular fashion with them[2]. This mathematical problem, known as the 'General Three-Body Problem', was partially solved by Lagrange in his paper '*Essai sur le Problème des Trois Corps'*. Lagrange found an equilateral triangle solution, to add to Euler's earlier elliptical solutions, where three bodies could orbit each other but stay in the same position relative to one another. This is an extremely important phenomenon for space exploration; a satellite or telescope placed at one of these points can observe the other two bodies from a fixed position at all times[3].

Figure 1 shows the relative positions of the five points for two masses m_1 and m_2. L4 and L5 form equilateral triangles with the two masses, whereas L1, L2 and L3 lie on a straight line running through the centre of both m_1 and m_2. The mathematics used by Lagrange shows that two of the points, L4 and L5, are stable. Because of their stability, dust and asteroids tend to accumulate in these regions. These asteroids are called Trojans after the asteroids Agamemnon, Achilles and Hector that are between Jupiter and the Sun[4].

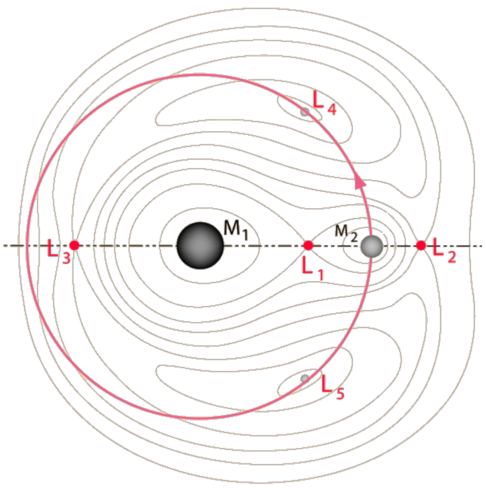


Figure 1: Position of the 5 Lagrangian Points relative to masses m_1 and m_2. Brown lines represent gravitational potential.[5]

3. Theory

The 'General Three-Body Problem' is to determine the motion of three masses m_1, m_2 and m_3 which attract each other according to Newton's Law of Gravitation. It was most convenient for physicists to work in the centre-of-mass system, with $\mathbf{x_i}$ denoting the position of mass m_i. The Newtonian equations of motion in this system are of the form;

$$\ddot{\boldsymbol{x}}_i = -Gm_j \frac{\boldsymbol{x_i} - \boldsymbol{x_j}}{\left|\boldsymbol{x_i} - \boldsymbol{x_j}\right|^3} - Gm_k \frac{\boldsymbol{x_i} - \boldsymbol{x_k}}{\left|\boldsymbol{x_i} - \boldsymbol{x_k}\right|^3} \tag{1}$$

where i, j and k stand for masses 1,2 and 3. These three second-order vector differential equations are equivalent to 18 first order scalar differential equations. In 1973 Broucke and Lass realised that the above equations could be written more symmetrically using relative position vectors $\mathbf{s_i}$ [6]:

$$\boldsymbol{s_i} = \boldsymbol{x_j} - \boldsymbol{x_k} \tag{2}$$

Using *Figure 2*, it was found that the sum of these vectors is equal to zero. *Equation (1)* can therefore be simplified to give;

$$\ddot{\boldsymbol{s}}_i = -GM\frac{\boldsymbol{s_i}}{{s_i}^3} + m_i\boldsymbol{G}, \qquad \boldsymbol{G} = \sum_{i=1}^{i=3} \frac{\boldsymbol{s_i}}{s_{\boldsymbol{i}}^{\boldsymbol{3}}} \tag{3}$$

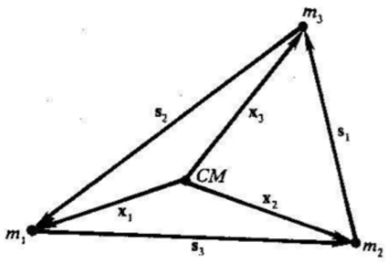


Figure 2: Diagrammatic representation of 'General Three-Body' problem.[7]

Lagrange found a solution when $\mathbf{G}=\mathbf{0}$ and the equations for $\mathbf{s_i}$ decouple. The three equations have the form of a two-body problem with elliptical solutions[7]. For $\mathbf{G}=\mathbf{0}$, $s_1=s_2=s_2$: the particles must sit at the vertices of an equilateral triangle at all times. Each particle orbits in an ellipse of the same eccentricity but orientated at different angles. Their motion is periodic, with the same period for all three particles. These triangular vertices are those shown in *Figure 1* labelled L4 and L5. The L1, L2 and L3 solutions were found by Euler but are generally referred to as 'Lagrangian points'[7].

The spatial coordinates of these points were calculated to be as follows by Lagrange and Euler. The values below are in a rotating coordinate system where the m_1 is at the origin and m_2 at a fixed distance 'R' along the 'x-axis'[6]:

$$L1\text{:}\left(R\left(1-\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right),0\right) \tag{4}$$

$$L2\text{:}\left(R\left(1+\left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right),0\right) \tag{5}$$

$$L3\text{:}\left(-R\left(1+\frac{5}{12}\alpha\right),0\right) \tag{6}$$

$$L4\text{:}\left(\frac{R}{2}\left(\frac{m_1-m_2}{m_1+m_2}\right),\frac{\sqrt{3}}{2}R\right) \tag{7}$$

$$L5\text{:}\left(\frac{R}{2}\left(\frac{m_1-m_2}{m_1+m_2}\right),-\frac{\sqrt{3}}{2}R\right) \tag{8}$$

$$\alpha \equiv \frac{m_2}{m_1+m_2} \tag{9}$$

Generally, Lagrange and Euler found that L4 and L5 are stable points but L1, L2 and L3 are not. The linear stability of the L4 and L5 points was first studied by Gascheau, who showed that the motion of three non-zero masses in a rotating equilateral configuration becomes linearly unstable if [1]:

$$(m_1 + m_2 + m_3)^2 < 27(m_1 m_2 + m_1 m_3 + m_2 m_3) \tag{10}$$

If the mass of the third object is assumed to be negligible, (i.e. m_3=0), the above reduces to:

$$\frac{m_2}{(m_1 + m_2)} > \mu_G = 0.0385 \dots \tag{11}$$

where μ_G later became known as 'Gascheau's Value'. The masses of both the Sun-Earth and Earth-Moon systems satisfy this condition. Throughout this investigation the mass of m_3 was assumed negligible: the mass of the asteroids in our simulation were assumed to be significantly less than that of the two planets.

4. Method & Code

To investigate the workings of the 'General Three-Body problem' we chose to simulate the Sun-Jupiter system, where m_1=m_{sun} and m_2=$m_{jupiter}$. The simulation below was performed in a 'Solar System Frame', where the Sun is fixed at the origin at all times. As explained above, the centre-of-mass is the chosen reference frame when working with many-body systems by theorists. However due to the fact that the mass of the Sun is significantly larger than both m_2 and m_3 in this set-up, the centre-of-mass of the system is effectively at the centre of the sun: the origin. Therefore, for simplicity, the simulation was performed in the 'Solar System Frame of Reference'. This choice also meant the simulation no longer required inertial force terms to account for the fact that a rotating frame is accelerating - the Solar System frame is an inertial frame of reference.

In order to simulate the motion of the asteroids, it was first necessary to set up the Sun-Jupiter system. As mentioned above, the mass of the asteroids was considered negligible and thus was ignored when examining the gravitational forces acting on Jupiter. The force between the Sun and Jupiter is given by Newton's Law of Universal Gravitation[8];

$$\boldsymbol{F} = -\frac{G M_{sun} M_{jupiter}}{r^2}\hat{\boldsymbol{r}} \tag{12}$$

Using Newton's First Law, this can be simplified into a second order differential equation. To solve this equation, Python's built-in scipy.integrate.odeint library was used. A function was set up using this integrator, such that when Jupiter was given initial spatial coordinates and velocity, its elliptical orbit around the Sun was found and plotted. For the purposes of the simulation, Jupiter's perihelion distance was used as its initial position (741 million km)[9].

To simulate the asteroids' motion, it was necessary to include force terms from both the Sun and Jupiter. The below equation is expressed in the Solar System coordinate system, with the Sun at the origin, see *Figure 3*.

$$\boldsymbol{a} = -\frac{G M_{sun}}{r^3}\boldsymbol{r} - \frac{G M_{jupiter}}{{r_a}^3}\boldsymbol{r_a} \quad , \boldsymbol{r_a} = \boldsymbol{r} - \boldsymbol{r_{jupiter}} \tag{13}$$

The above expression represents a simplified version of *Equation 3,* where m_1 is fixed at the origin and m_3 has negligible mass. Because the position of Jupiter, and therefore $\mathbf{r_a}$, are time-dependent it was necessary to use the interpolation feature of the numpy library to allow the integrator to estimate the value of $\mathbf{r_a}$ for each time of the integral. Upon integration, the simulator gave the spatial coordinates of the Sun, Jupiter and asteroid at all times. It was therefore possible to plot the orbits of the asteroids and examine their stability. The code allows the user to manually choose which Lagrangian orbit to simulate by having a library of the coordinates of each point stored, see *Equations 4-9*.

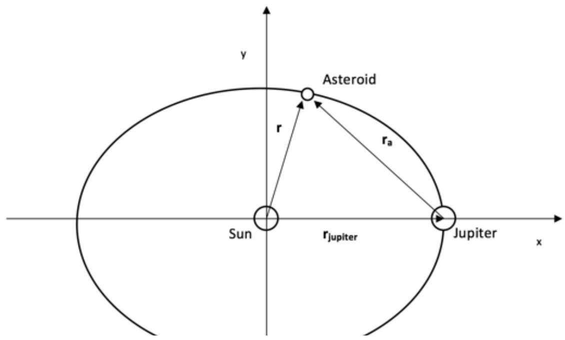


Figure 3: Diagrammatic explanation of coordinate system and vector notation for Equation 13.

The stability of the orbits was quantitatively determined using the 'wandering' distance of the asteroids from their equilibrium position.
Throughout this investigation the simulated asteroids were released at rest in the rotating frame. In the Solar System frame, they were therefore moving with the same angular velocity and period as Jupiter around the Sun. This angular velocity 'ω' was determined using the period of Jupiter's orbit (11.86 earth years)[9] to be:

$$\omega = 1.68 \text{ x } 10^{-8}\ rad.s^{-1} \quad (14)$$

The asteroids were released at rest in the rotating frame, at varying initial positions around the equilibrium position. Their velocity in the Solar System Frame was determined simply using:

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} \quad (15)$$

where **r** is the position vector of the asteroids. Examining how the asteroids 'wandered' depending on their initial displacement from equilibrium enabled a conclusion to be made about the stability of the Lagrangian points. If a point was stable, it would be expected that the asteroid would oscillate about the equilibrium position and eventually return to it. The lengths of the asteroids' position vectors were plotted against time in order to examine whether the asteroids were moving closer to equilibrium separation (stable orbit), or further away (unstable), over time.

The second aim of this investigation was to explore the effect of changing the mass of Jupiter on the stability of the orbits. Given that L4 is generally a stable position, this was the point that was used for this simulation. The simulation was carried out by looping through several different masses for the Planet, and for each one, plotting the positions of the asteroids and their maximum displacements. Although the mass of the planet was changed, its displacement was kept constant at Jupiter's average orbital radius (741 million km)[9]. In this simulation the asteroids were all released from the exact location of the L4 point, with zero velocity in the rotating frame. Orbital velocity in the Solar System reference frame, calculated above, is independent of the mass of the Planet and thus did not need to be modified for this part of the investigation.

5. Results

Stable Orbits (L4 and L5)

It was apparent from the simulation that the motion of asteroids placed at L4 and L5 was highly symmetrical: they followed identical orbits spatially separated on either side of Jupiter. Therefore, the following analysis which was generated by examining the asteroids at L4 is also representative of the motion of asteroids at L5.

As explained previously, to examine the stability of the Lagrangian point, asteroids were released with zero velocity in the rotating frame at different positions around the point. For this simulation, asteroids were released from different positions along the straight line connecting the Sun and L4; from points between 1.01R and 1.05R, where R is the distance between the Sun and the Lagrangian point, see *Figure 5*. Distances in the figures below are in units of Jupiter aphelion distance, taken to be 817 million km[9].

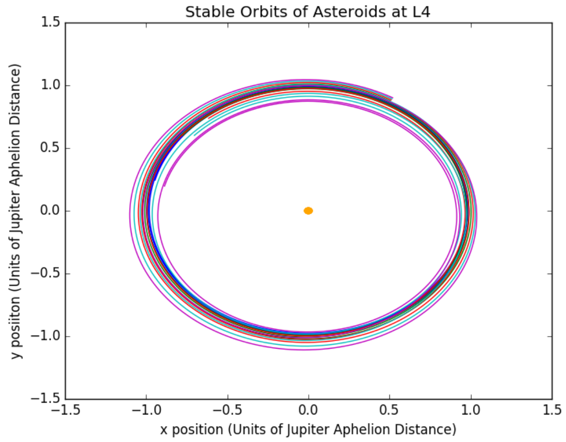


Figure 4: Trajectories of five asteroids displaced slightly from L4 over a period of 3 Jupiter years. For visual purposes the Sun has been enlarged by a factor of 25.

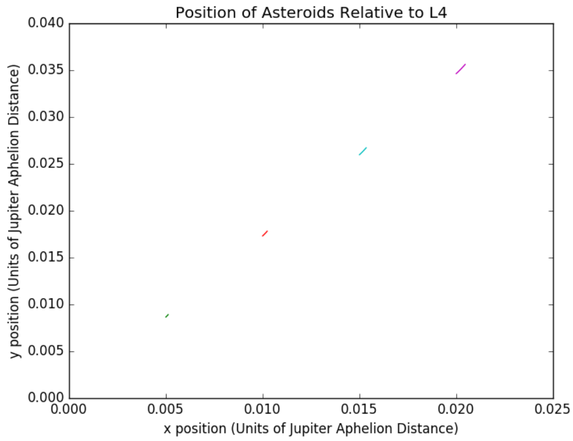


Figure 5: Initial displacement of asteroids in reference frame of an observer at L4.

Figure 4 shows the asteroids following regular and periodic orbits. It can be observed that the orbits are of similar eccentricity and length as that of Jupiter: the elliptical orbits all share a semi-major axis of roughly 1 in Jupiter aphelion distance units. When the simulation was run for tens of years, the asteroids eventually all followed identical orbits. *Figure 6* shows a form of oscillatory motion; when displaced from equilibrium the asteroids oscillated around the Lagrangian point. By combining these two results it was concluded that L4, and therefore L5, are stable Lagrangian points: when the asteroids were displaced from L4 they oscillated around it before eventually returning to it.

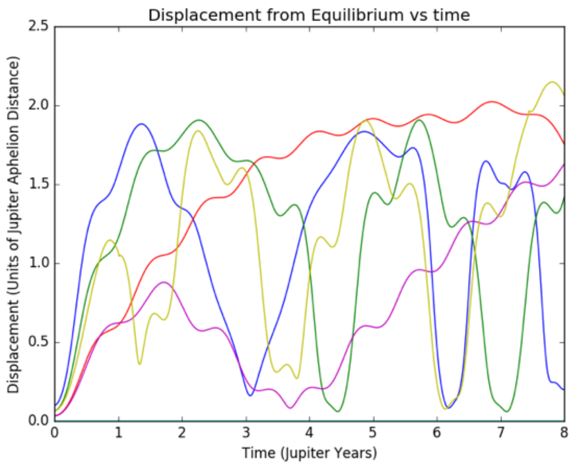


Figure 6: Displacement of asteroids from L4 point over 8 Jupiter years.

Unstable Orbits (L1, L2 and L3)

The symmetry argument used above to compare the motion around L4 and L5 cannot be used for L1, L2 and L3. These three points are all at different distances from the Sun and Jupiter and therefore will experience different gravitational forces at any one time. The stability of the three points was explored using a similar method to the one explained above for stable orbits. The asteroids were released with zero velocity in the rotating frame from points around each of the Lagrangian points and their resulting motion was analysed. They were released from various points along the line connecting the Sun and L1, see *Figure 8*.

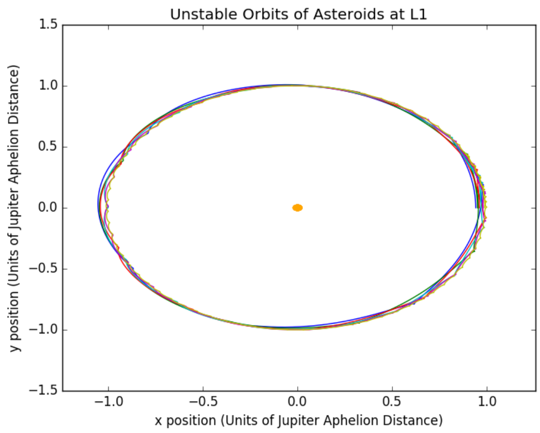


Figure 7: Trajectories of asteroids displaced slightly from L1 over a period of 1 Jupiter year.

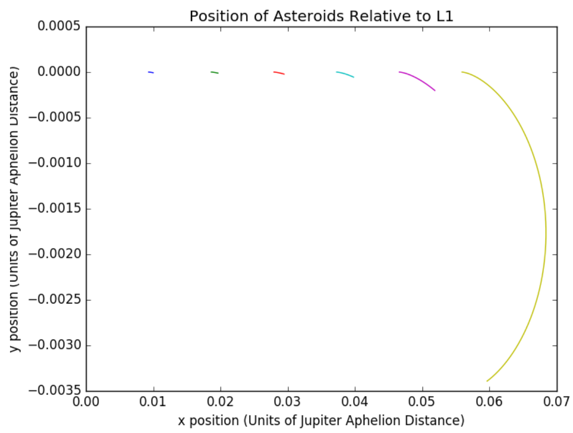


Figure 8: Initial displacements of asteroids in the reference frame of an observer at L1.

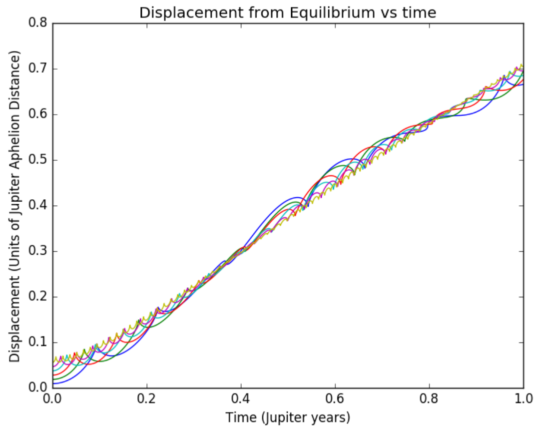


Figure 9: Displacement of asteroids from L1 over a period of 1 Jupiter year.

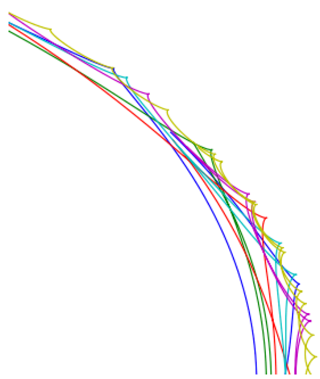


Figure 10: Close up of Figure 8, chaotic and disordered motion of asteroids.

The orbits of these asteroids are disordered and chaotic. *Figure 10* shows a close-up portion of the orbits: it is clear that they are not following simple elliptical paths; their motion is unpredictable and irregular. *Figure 9* shows that over time, the asteroids are moving further and further away from equilibrium. This is in direct contrast with *Figure 6* where the asteroids released from near L4 oscillate around the equilibrium position. It was therefore concluded that L1 is an unstable Lagrangian point. *Figures 11 and 12* show the path of the asteroids in the reference frame of an observer at L4 and L1 respectively. The stability of the two points can be determined by direct comparison. The asteroids in *Figure 11* move around the equilibrium point at the origin in a regular oscillatory fashion: they are in a stable arrangement. The asteroids in *Figure 12* spiral away from the equilibrium point at the origin in a chaotic fashion: they are in an unstable arrangement.

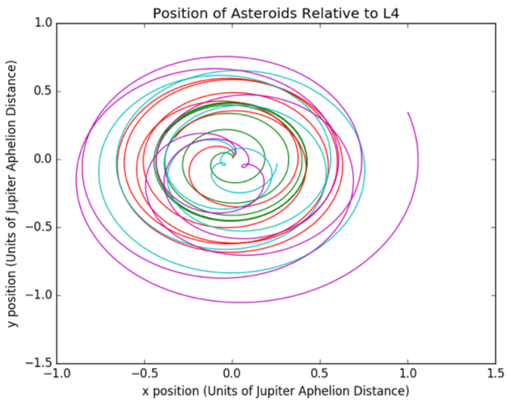


Figure 11: L4 asteroids in the reference frame of an observer at L4.

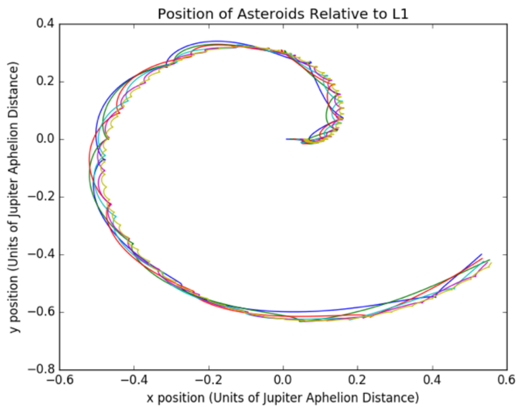


Figure 12: L1 asteroids in the reference frame of an observer at L1.

Similar analysis showed that both L2 and L3 are also unstable

Varying Planetary Mass

A secondary aim of this investigation was the explore the effect of changing the mass of the Planet. As explained in the theory section, the paths of the asteroids were plotted for a range of Planet masses. Through trial and error, the range used was shortened down to between 0.005 Solar masses and 0.06 Solar Masses. The paths of the asteroids are displayed in *Figure 13*.

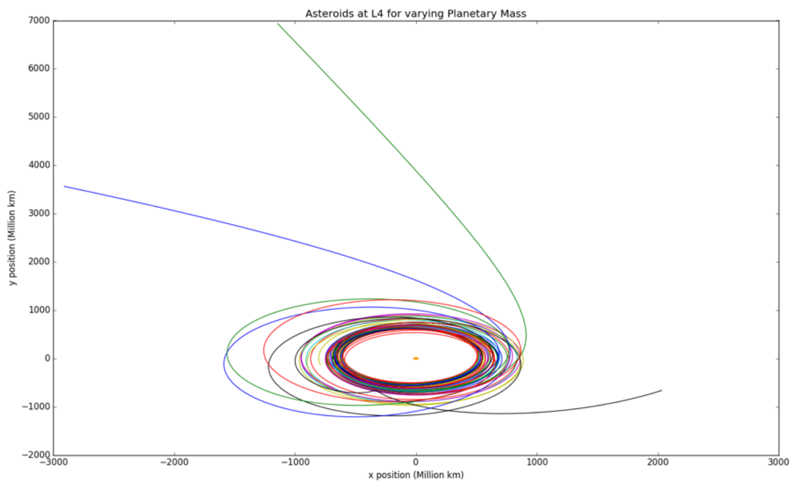
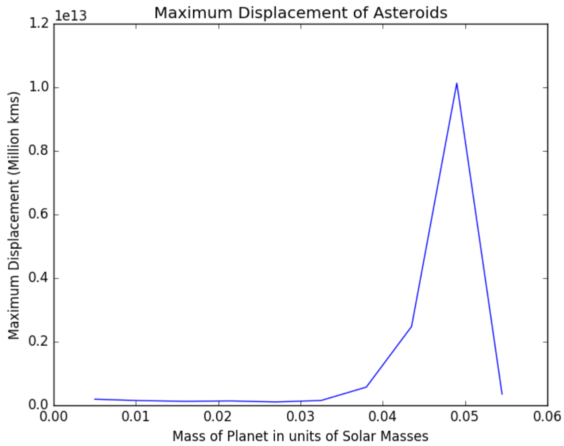


Figure 13: Asteroids released from L4 for different Planetary masses.

The lines in pink and red represent lower Planetary mass, these asteroids seem to follow roughly stable orbits as seen above in *Figure 4* for L4. However, the lines representing higher mass, blue and green in the above diagram, follow less regular orbits: diverging away from the Sun.

Figure 14 represents the maximum displacement of each asteroid from L1 over a period of 10 Jupiter years. The first region of the graph, a horizontal line marginally above 0, implies that these asteroids do not wander far from equilibrium. However, above a mass ratio of around 0.04 the asteroids rapidly move further and further away from equilibrium. This data agrees with the conclusion drawn from *Figure 13* which was that at higher mass the asteroids diverge away from the Sun. By further analysis the point at which this change from little to large maximum displacement was found to be 0.038 Solar Masses.



It was concluded that above this mass, the orbits of asteroids released from L4 become unstable and chaotic. Thus the mass ratio of 0.038 represents a 'limit' on Lagrangian's solutions; points L4 and L5 are only stable below this ratio.

Figure 14: Maximum displacement of asteroids against the simulated Planetary mass.

6. Discussion/ Evaluation

The central aim of this investigation was to design and test a computer simulation. By comparing the above results to accepted values and theoretical predictions it was possible to evaluate the accuracy of the simulation.

As explained in the theory section, Lagrange and Euler found that L1, L2 and L3 are unstable but L4 and L5 are stable. The computer simulation agreed with these findings. However, the simulation struggled to cope with some of the unstable orbits. When simulated for a period of time greater than around a year, the chaotic nature of the L1 asteroids' orbits caused the computer problems due to singularities and very large distances.

When the mass of the Planet was varied, the value determined for the mass ratio stability cut-off was 0.038. This is very close to the accepted value of 0.0385. However, there was some associated error with the determined value. Gascheau's value of 0.0385 comes from a theoretical derivation of the linear stability of the fourth Lagrangian point. However, the simulated value was found by looking at the maximum displacement of asteroids from the Sun, see *Figure 14.* The general shape of *Figure 14* is a very good indicator of how the stability of the point changes as Planetary mass increases. However, it is difficult to identify the exact point the orbits transition from stable to unstable using just this graph. As a result, the value found in this project comes with an associated uncertainty. Gascheau's work also involved making the distinction between linear instability and fully chaotic orbits. For mass ratios above Gascheau's value but below a value μ_∞=0.0463, he found that L4 and L5 are globally stable in the sense that asteroids released at those points remain within roughly the vicinity of the points over time. However, above μ_∞ he found that the asteroid orbits would eventually become completely chaotic[1]. Our simulation was limited in this regard; it was not able to distinguish between different types of instablility.

The simulation's performance over large times was also taken into account. For stable orbits, running the code over tens of years presented no problems. However, asteroids in unstable orbits would occasionally cause the simulator to crash when run for long times. It was hypothesised that this was due to the fact that as the asteroids moved further away from the simulated ‘Solar System’, the distances became too large and the forces too small for the integrator to handle: scipy.integrate.odeint uses a ‘brute-force’ method to carry out integrations.

It was therefore concluded that the simulator functions effectively and accurately when used to examine the general motion of asteroids at all five Lagrangian points. It was, however, limited at large times and did not provide a very precise method of determining Gascheau’s value.

7. Conclusion

The aim of this project was to design and test a computer simulation of asteroids placed in Lagrangian orbits in order to examine the stability of the orbits and the effect of altering the mass of the system. The results produced by the simulation agreed with theoretical and accepted findings that L4 and L5 are stable Lagrangian points, but L1, L2 and L3 are not. It was determined that asteroids released at L4 will remain in a stable orbit of the Sun until the mass ratio of the Planet and the Sun exceeds a value of 0.038. This agrees with the theoretical predictions made by Gascheau. However, several limiting features in the code led to the conclusion that the simulation did not provide a precise method of measuring Gascheau’s value.

8. Bibliography

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