Graphs with all holes the same length

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Abstract

A graph is ℓ-*holed* if all its induced cycles of length at least four have length exactly ℓ. We give a complete description of the ℓ-holed graphs for each $\ell \geq 7$.

1 Introduction

A *hole* in a graph is an induced cycle of length at least four (graphs in this paper are finite and have no loops or parallel edges). Berge graphs and even-hole-free graphs have decomposition theorems that are deep and useful (see [3, 5]), but not fully satisfactory, and we do not know explicit constructions that will generate all the graphs of either type. What if we restrict the lengths of holes much more severely; can we then give explicit constructions? In this direction, the simplest class is the class where holes of all lengths are excluded. This is the class of chordal graphs, and it has a structural description that is fully understood [6]. The next simplest is to exclude holes of all lengths except one, and this is what we study here. As we will see, the description is complex, but it fully describes the structure of all the graphs in the class.

If $\ell \geq 4$ is an integer, we say a graph is ℓ-*holed* if all its holes have length exactly ℓ. How can we make the most general ℓ-holed graph? There are a few cases that come to mind immediately: chordal graphs; cycles; pyramids in which every path from apex to base has the same length; thetas in which every path between the two ends of the theta has the same length; and prisms in which every path between the two triangles of the prism has the same length. (We will define all these terms later.) And we can enlarge these in trivial ways, for instance by overlapping them on clique cutsets, but finding further examples is not so easy. It turns out that the most general example has a family resemblance to these easy ones.

Before we give the construction, let us digress a little: how can we test if a graph is ℓ-holed in polynomial time? This is easy for each given ℓ, but obtaining an algorithm with running time a polynomial in $|G|$ and independent of ℓ is not so clear. But here is one simple way. There is an algorithm due to Berger, Seymour and Spirkl [1] that, given two vertices s, t of a graph G, tests whether there is an induced path between s, t with length more than the distance between s, t. (Its running time is $\mathcal{O}(|G|^{18})$.) For each three-vertex induced path a-b-c of G, delete b and all its neighbours except a, c from G, forming G' say, and first test whether there is an $a - c$ path in G' (if not, move on to the next three-vertex path); check that the distance in G' between a, c is $\ell - 2$ (if not, G is not ℓ-holed and we stop); and use the algorithm of [1] to check that there is no induced path in G' between s, t with length more than $\ell - 2$ (if there is such a path, the graph is not ℓ-holed and we stop). If after processing all three-vertex paths, we still have not determined that G is not ℓ-holed, then it is ℓ-holed and we stop. This has running time $\mathcal{O}(|G|^{21})$. It can be done faster with more complication, making use of the structure theorem proved in this paper, and such an algorithm will appear in Jake Horsfield's thesis.

Let us return to constructing ℓ-holed graphs. One way to make a larger ℓ-holed graph from two smaller ones is via clique cutsets. We say that $X \subseteq V(G)$ is a *clique cutset* of G if $G[X]$ is a clique and $G \setminus X$ is disconnected. ($G[X]$ denotes the subgraph of G induced on X, and $G \setminus X$ is the subgraph of G obtained by deleting X.) If G_1, G_2 are two ℓ-holed graphs, and X_i is a clique of G_i for $i = 1, 2$, both of the same cardinality and with $|X_i| < |G_i|$, and we identify each vertex of X_1 with a vertex of X_2 bijectively, the graph G we produce is ℓ-holed, and admits a clique cutset; and every graph with a clique cutset can be built by this operation. Consequently, to understand ℓ-holed graphs in general, it suffices to understand those with no clique cutset.

There is another "trivial" way to make larger ℓ-holed graphs from smaller ones; add a new vertex adjacent to all old vertices. We call a vertex of G adjacent to all the other vertices of G a *universal vertex*. Thus, we would like to describe all ℓ-holed graph with no clique cutset and no universal vertex. We were not able to do this for $\ell = 4, 5, 6$, but we have a complete description for all $\ell \geq 7$.

There is a third way to enlarge ℓ-holed graphs to larger ones; choose a vertex and replace it by a set of pairwise adjacent vertices, each with the same neighbours as the original vertex (and one another). If two adjacent vertices have exactly the same neighbours, we call them *adjacent twins*. We could assume there are no adjacent twins, without loss of generality, but it makes little difference, so usually we will not do so.

Some terminology: two disjoint subsets X, Y of a graph are *complete* if every vertex of X is adjacent to every vertex in Y, and *anticomplete* if there are no edges between X, Y. Incidentally, we will sometimes use expressions such as "G-adjacent" and "G-neighbour" when we want to make clear which graph we are using, as we will often have different graphs with the same vertex set.

Let X, Y be disjoint subsets of $V(G)$; we denote by $G[X, Y]$ the bipartite subgraph of G with vertex set $X \cup Y$ and edge set the set of edges of G between X, Y. A *half-graph* is a bipartite graph with no induced two-edge matching. Thus, a graph with bipartition (X, Y) is a half-graph if and only if we can order $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ such that for all i, i', j, j' with $1 \le i \le i' \le m$ and $1 \le j \le j' \le n$, if $x_{i'} y_{j'}$ is an edge then $x_i y_j$ is an edge. Such a half-graph together with the orderings of X, Y is called an *ordered half-graph*.

Half-graphs will be of great importance in this paper. For instance, let G be an ℓ-holed graph where $\ell \ge 5$, and let X, Y be disjoint cliques of G. Since there is no 4-hole with two vertices in X and two in Y (a *k-hole* means a hole of length k), it follows that $G[X, Y]$ is a half-graph. If X, Y, Z are disjoint cliques of G, we say that $G[X, Y], G[X, Z]$ are *compatible* if $G[X, Y \cup Z]$ is a half-graph. If Y, Z are anticomplete, then $G[X, Y], G[X, Z]$ are compatible if and only if there is no induced four-vertex path in $G[X \cup Y \cup Z]$ with first vertex in Y, second and third in X, and fourth in Z, and this latter property is important for keeping all holes the same length. This is also equivalent to asking that there are orderings of X, Y and Z such that $G[X, Y], G[X, Z]$ are both ordered half-graphs.

Let G be a graph with vertex set partitioned into sets $W_1, \ldots, W_\ell$, with the following properties:

- $W_1, \ldots, W_\ell$ are non-null cliques;

- for $1 \le i \le \ell$, $G[W_{i-1}, W_i]$ is a half-graph (reading subscripts modulo ℓ);

- for all distinct $i, j \in \{1, \ldots, \ell\}$, if there is an edge between W_i, W_j then $j = i \pm 1$ (modulo ℓ); and

- for $1 \le i \le \ell$, the graphs $G[W_i, W_{i+1}], G[W_i, W_{i-1}]$ are compatible.

(See figure 1.) We call such a graph a *blow-up of an ℓ-cycle*.
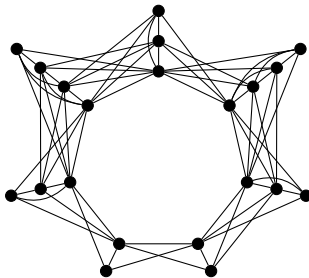

Figure 1: A blow-up of a 7-cycle.

To describe our main result we will need arborescences. An *arborescence* is a tree with its edges directed in such a way that no two edges have a common head; or equivalently, such that for some vertex $r(T)$ (called the *apex*), every edge is directed away from $r(T)$. A *leaf* is a vertex different from the apex, with outdegree zero, and $L(T)$ denotes the set of leaves.

Our theorem says that if G is ℓ-holed, and $\ell \geq 7$, and G has no clique cutset or universal vertex, then either G is a blow-up of an ℓ-cycle, or G is an instance of a construction we will describe. The construction is rather complicated, however, and we will give the description in stages. The cases of ℓ odd and ℓ even are different, and the case when ℓ is odd is simpler, so let us begin with that. The underlying structure is what we call an *ℓ-framework*, and is best described with a figure.

Let us describe the important features of figure 2. There are 19 vertices $a_0, \ldots, a_{18}$ and 18 vertices $b_1, \ldots, b_{18}$ (these could be any two numbers $k+1$ and k). For $1 \leq i \leq k$ there is a vertical path P_i of length $(\ell - 3)/2$ between a_i, b_i. (In the case of the figure, $\ell = 9$). The numbers $0, \ldots, k$ break into two intervals $\{0, \ldots, m\}$ and $\{m+1, \ldots, k\}$ (in the figure $m = 10$).

Let us call the grey shaded areas "tents". The tents are disjoint subsets of the plane, and each of the (four, in the figure) upper tents contains one vertex in $\{a_0, \ldots, a_m\}$ called its "apex", and contains a nonempty interval of $\{a_{m+1}, \ldots, a_k\}$ called its "base". Each of $a_{m+1}, \ldots, a_k$ belongs to the base of an upper tent. The lower tents do the same with left and right switched. There can be any positive number of tents, but there must be a tent with apex a_0. (There is an odd number of tents in the figure, but there could be an even number.) Possibly $m = 0$, and if so there are no lower tents. The way the upper and lower tents interleave is important; for each upper tent (except the innermost when there is an odd number of tents), the leftmost vertex of its base is some a_i, and b_i is the apex of one of the lower tents; and for each lower tent (except the innermost when there is an even number of tents), the rightmost vertex of its base corresponds to the apex for one of the upper tents. (This gives a sort of spiral running through all the apexes, in the figure with vertices

$$a_0\text{-}a_{17}\text{-}b_{17}\text{-}b_3\text{-}a_3\text{-}a_{16}\text{-}b_{16}\text{-}b_6\text{-}a_6\text{-}a_{14}\text{-}b_{14}\text{-}b_{10}\text{-}a_{10},$$

which might be helpful for visualization.)

Each tent is meant to be an arborescence with the given apex and with set of leaves the base of the tent, and with its other vertices not drawn. (We call such an arborescence a *tent-arborescence*.) For each $i \in \{1, \ldots, m\}$, if a_{i-1} is the apex of an upper tent-arborescence T_{i-1} say, there is a directed edge from some nonleaf vertex of T_{i-1} (possibly from a_{i-1}) to a_i; and if a_{i-1} is not the apex of a tent, there is a directed edge from a_{i-1} to a_i. So all these upper tent-arborescences and all the vertices $a_0, \ldots, a_m$, are connected up in a sequence to form one big arborescence T with apex a_0, and with set of leaves either $\{a_{m+1}, \ldots, a_k\}$ or $\{a_m, \ldots, a_k\}$. There is a directed path of T that contains $a_0, a_1, \ldots, a_m$ in order, possibly containing other vertices of T between them. Similarly for each $i \in \{m+1, \ldots, k-1\}$, if b_{i+1} is an apex of a lower tent-arborescence S_{i+1}, there is a directed edge from some nonleaf vertex of S_{i+1} to b_i, and otherwise there is a directed edge $b_{i+1}b_i$. So similarly the lower tent-arborescences, and the vertices $b_{m+1}, \ldots, b_k$, are joined up to make one arborescence S with apex b_k and with set of leaves either $\{b_1, \ldots, b_m\}$ or $\{b_1, \ldots, b_{m+1}\}$

Thus the figure describes a graph in which some of the edges are directed: each directed edge belongs to one of two arborescences T, S and each undirected edge belongs to one of the paths P_i. We call such a graph an *ℓ-framework*. (We will explain later how ℓ-frameworks describe the structure of ℓ-holed graphs.) We observe:
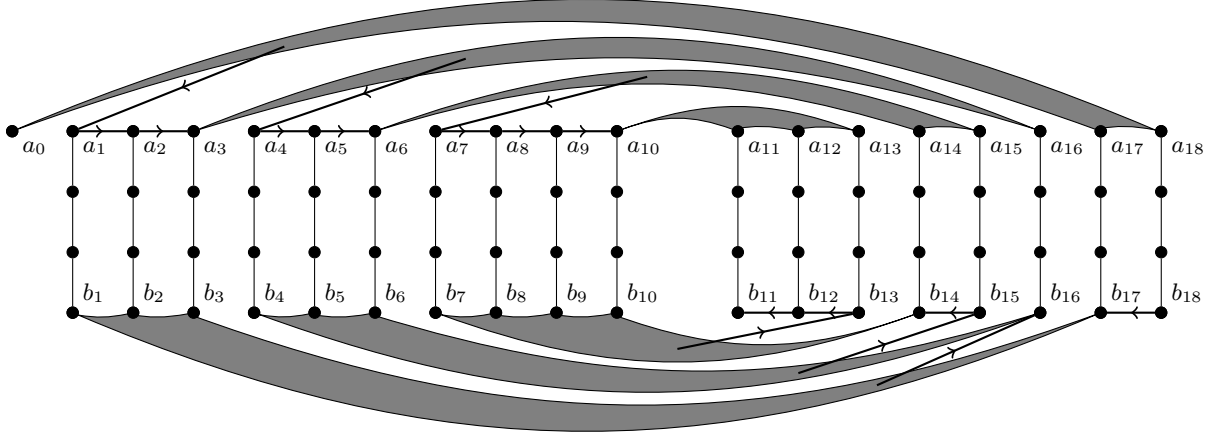
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Figure 2: An 18-bar 9-framework.

1.1 *Let $\ell \geq 5$ be odd and let F be an ℓ-framework, with notation as above. For $1 \leq i < j \leq k$, either there is a directed path of T between a_i, a_j, or there is a directed path of S between b_i, b_j and not both.*

Next we will describe a similar object for when ℓ is even, but we need another concept. Let T be an arborescence. For $v \in V(T)$, let D_v be the set of all vertices $w \in L(T)$ for which there is a directed path of T from v to w. Let S be a tree with $V(S) = L(T)$. We say that T *lives in S* if for each $v \in V(T)$, the set D_v is the vertex set of a subtree of S. Let T, T' be arborescences with $L(T) = L(T')$. We say they are *coarboreal* if there is a tree S with $V(S) = L(T) = L(T')$ such that T, T' both live in S. For instance, the first pair of arborescences in figure 3 (with leaf set the four black vertices) are coarboreal, but the second pair are not. Finally, let T, T' be arborescences and let ϕ be a bijection from $L(T)$ onto $L(T')$. We say that T, T' are *coarboreal under ϕ* if identifying each vertex of $L(T)$ with its image under ϕ gives a coarboreal pair.
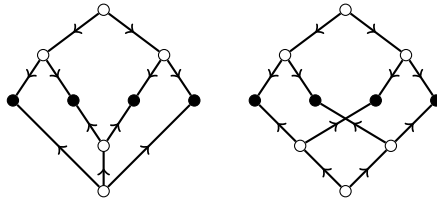

Figure 3: The first pair are coarboreal, the second pair are not.

The structure we need when ℓ is even is shown in figure 4. We have vertices $a_0, \ldots, a_k$ ($k = 18$ in the figure) and $b_1, \ldots, b_k$, but now there is an extra vertex b_0. There are paths P_i between a_i, b_i of length $\ell/2 - 1$ for $1 \leq i \leq m$, and length $\ell/2 - 2$ for $m + 1 \leq i \leq k$. ($\ell = 8$ and $m = 8$ in the figure.) There are upper and lower tents as before, but now all the tents have apex on the left. There must be an upper tent with apex a_0, and one with apex a_m, although $m = 0$ is permitted. The upper

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tents are paired with the lower tents; for each upper tent with base $\{a_i, \ldots, a_j\}$ there is also a lower tent with base $\{b_i, \ldots, b_j\}$, and vice versa. But the apexes shift by one; if an upper tent has apex a_i, the paired lower tent has apex b_{i+1} (or b_0 when $i = m$). An important condition, not shown in the figure, is:

- for each upper tent-arborescence T_i say, with apex a_i, the paired lower tent-arborescence S_{i+1} with apex b_{i+1} (or b_0, if $i = m$) must be coarboreal with T_i under the bijection that maps a_j to b_j for each leaf a_j of T_i.
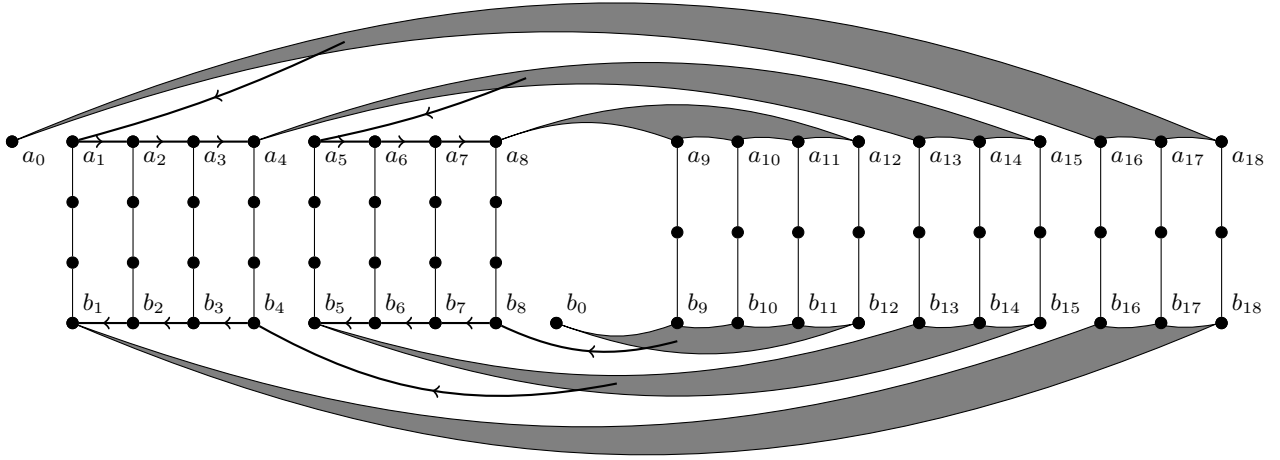

Figure 4: An 18-bar 8-framework.

As before, for each $i \in \{1, \ldots, m\}$, if a_{i-1} is the apex of an upper tent-arborescence T_{i-1} say, there is a directed edge from some nonleaf vertex of T_{i-1} (possibly from a_{i-1}) to a_i; and if a_{i-1} is not the apex of a tent, there is a directed edge from a_{i-1} to a_i. So the upper tent-arborescences are connected up to form an arborescence T with apex a_0, and with set of leaves $\{a_{m+1}, \ldots, a_k\}$. Also, for each $i \in \{1, \ldots, m-1\}$, if b_{i+1} is the apex of a lower tent-arborescence S_{i+1} say, there is a directed edge from some nonleaf vertex of S_{i+1} (possibly from b_{i+1}) to b_i; and if b_{i+1} is not the apex of a tent, there is a directed edge from b_{i+1} to b_i. Finally, there is a directed edge from some nonleaf vertex of the tent-arborescence S_0 with apex b_0 (possibly from b_0 itself) to b_m. So the lower tent-arborescences are connected up to form an arborescence S with apex b_0, and with set of leaves $\{b_{m+1}, \ldots, b_k\}$. We call this graph an ℓ-framework. We observe:

1.2 *Let $\ell \geq 5$ be even, and let F be an ℓ-framework, with notation as above. Let $1 \leq i < j \leq k$.*

- *If $i, j \leq m$, then there is a directed path of T between a_i, a_j, and there is a directed path of S between b_i, b_j;*

- *If $i \leq m < j$, either there is a directed path of T between a_i, a_j, or there is a directed path of S between b_i, b_j, and not both;*

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- *If $m < i < j$, there is no directed path of T between a_i, a_j, and there is no directed path of i S between b_i, b_j.*

The *transitive closure* $\overrightarrow{T}$ of an arborescence T is the undirected graph with vertex set $V(T)$ in which vertices u, v are adjacent if and only if some directed path of T contains both of u, v. Let F be an ℓ-framework (here, ℓ may be odd or even). Let $P_1, \ldots, P_k, T, S$ and so on be as in the definition of an ℓ-framework. Let $D = \overrightarrow{T} \cup \overrightarrow{S} \cup P_1 \cup \cdots \cup P_k$. Thus $V(D) = V(F)$, and distinct $u, v \in V(D)$ are D-adjacent if either they are adjacent in some P_i, or there is a directed path of one of S, T between u, v. We say a graph G is a *blow-up* of F if

- D is an induced subgraph of G, and for each $t \in V(D)$ there is a clique W_t of G, all pairwise disjoint and with union $V(G)$; $W_t \cap V(D) = \{t\}$ for each $t \in V(D)$, and $W_t = \{t\}$ for each $t \in V(D) \setminus V(P_1 \cup \cdots \cup P_k)$.

- For each $t \in V(D)$, there is a linear ordering of W_t with first term t, say $(x_1, \ldots, x_n)$ where $x_1 = t$. It has the property that for every $v \in V(G) \setminus W_t$, and every $j \in \{1, \ldots, n\}$, if v is G-adjacent to x_j then v is G-adjacent to $x_1, \ldots, x_j$. (It follows that for all distinct $t, t' \in V(D)$, if t, t' are not D-adjacent then $W_t, W_{t'}$ are anticomplete, and if t, t' are D-adjacent then $G[W_t, W_{t'}]$ is an ordered half-graph.)

- If $t, t' \in \{a_1, \ldots, a_k\}$ or $t, t' \in \{b_1, \ldots, b_k\}$, and t, t' are D-adjacent, then W_t is complete to $W_{t'}$.

- For each $t \in V(T)$, if $0 \le i \le m$ and a_i, t are D-adjacent, then W_t is complete to W_{a_i}. For each $t \in V(S)$, if either ℓ is odd and $i \in \{m+1, \ldots, k\}$, or ℓ is even and $i \in \{0, \ldots, m\}$, and b_i, t are D-adjacent, then W_t is complete to W_{b_i}.

- For each upper tent-arborescence T_j with apex a_j say, let $t \in L(T_j)$ and let the path Q of T from a_0 to t have vertices

$$a_0 = y_1\text{-}\cdots\text{-}y_p\text{-}a_j\text{-}z_1\text{-}\cdots\text{-}z_q = t$$

in order. Then W_t is complete to $\{y_1, \ldots, y_p, a_j\}$; W_t is anticomplete to $W(T \setminus V(Q))$; and $G[W_t, \{z_1, \ldots, z_{q-1}\}]$, with the given order of W_t and the order $z_1, \ldots, z_{q-1}$ of $\{z_1, \ldots, z_{q-1}\}$, is an ordered half-graph. The same holds for lower tent-arborescences with T, a_0 replaced by S, b_0.

Our main theorem states:

1.3 *Let G be a graph with no clique cutset and no universal vertex, and let $\ell \ge 7$. Then G is ℓ-holed if and only if either G is a blow-up of a cycle of length ℓ, or G is a blow-up of an ℓ-framework.*

We will often have two graphs F, G, and a clique W_t of G for each $t \in V(F)$, pairwise vertex-disjoint. For $C \subseteq F$, we denote $\bigcup_{t \in V(C)} W_t$ by $W(C)$.

The paper is organized as follows. In the next section we show the easier "if" half of 1.3, that graphs with the structure specified in the theorem are ℓ-holed. The remainder of the paper concerns the "only if" half. In section 3 we show that every ℓ-holed graph that has no clique cutset or universal vertex, and that contains no theta, pyramid or prism (defined later), is a blow-up of a cycle. Then we turn to ℓ-holed graphs with no clique cutset or universal vertex that do contain a theta, pyramid or prism; and they will turn out to be blow-ups of ℓ-frameworks.

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An ℓ-framework consists of the paths $P_1, \ldots, P_k$, and the transitive closure of two arborescences, related somehow according to whether ℓ is odd or even. If G is a blow-up of an ℓ-framework F, then with notation as before, G can be thought of as consisting of three parts, induced respectively on $W(T)$, $W(P_1 \cup \cdots \cup P_k)$, and $W(S)$. The first and third of these are disjoint and anticomplete, but the first and second overlap, as do the second and third. In sections 4 to 8, we show that if a graph G is ℓ-holed, and has no clique cutset or universal vertex, and contains a theta, pyramid or prism, then G decomposes into three parts in this way, where the second is as it should be, and the intersection of first and second, and between second and third, is as it should be, but we know little about the interiors of the first and third. Then in section 9 we show that the first and third parts are both blow-ups of transitive closures of arborescences. Finally in sections 10–12 we ask how these two arborescences must be related, and this depends on whether ℓ is odd or even.

The work reported in this paper is the product of two groups of researchers, working independently. Much of it forms part of the PhD thesis [4] of Linda Cook; and also much of it will appear in the PhD theses of Cléophée Robin [7] and of Jake Horsfield.

2 The "if" half

In this section we prove the "if" half of 1.3. We will need the following lemma:

2.1 *Let R be a tree, and let $X \subseteq V(R)$, with $|X| = n$ where n is even. Let us say a partition $\{X_1, \ldots, X_{n/2}\}$ of X into sets of size two is* feasible *(in R) if there are $n/2$ vertex-disjoint paths $P_1, \ldots, P_{n/2}$ of R, such that, for $1 \le i \le n/2$, the ends of P_i are the two members of X_i. There is at most one feasible partition of X.*

Proof. We proceed by induction on $|V(R)|$. If $|V(R)| \le 2$ the result is clear, so we assume that $|V(R)| \ge 3$. If some leaf t of R does not belong to X, we may delete t without affecting which partitions are feasible, and the result follows from the inductive hypothesis; so we may assume that every leaf belongs to X. Let t be an end of the longest path of R, and let s be its neighbour in R. Thus every neighbour of s is a leaf except possible one. If $s \notin X$, let $X' = (X \setminus \{t\}) \cup \{s\}$; then for every feasible partition $\{X_1, \ldots, X_{n/2}\}$ of X, with $t \in X_1$ say, the partition $\{(X_1 \setminus \{t\}) \cup \{s\}, X_2, \ldots, X_{n/2}\}$ is a feasible partition of X', and the claim follows from the inductive hypothesis applied to $R \setminus t$ and X'. Thus we may assume that $s \in X$. If some neighbour of s different from t is a leaf, and hence belongs to X, there are no feasible partitions; so we may assume that s has degree exactly two (since $|V(R)| \ge 3$). Consequently $R' = R \setminus \{s, t\}$ is a tree. If $\{X_1, \ldots, X_{n/2}\}$ is a feasible partition of X in R, then $\{s, t\}$ is one of its sets, say $X_1 = \{s, t\}$, and $\{X_2, \ldots, X_{n/2}\}$ is a feasible partition of $X' = X \setminus \{s, t\}$ in R'; and hence it is unique from the inductive hypothesis. This proves 2.1. ∎

In this section we prove:

2.2 *For $\ell \ge 5$, if G is a blow-up of a cycle of length ℓ, or G is a blow-up of an ℓ-framework, then G is ℓ-holed.*

Proof. The first statement is proved in 3.1, but since the proof is very short we give a proof anyway. We assume first that G is a blow-up of an ℓ-cycle; let $W_1, \ldots, W_\ell$ be as in the definition of a blow-up of an ℓ-cycle, and let C be a hole of G. Suppose that $|V(C) \cap W_i| \ge 2$ for some i, say $i = 2$. Since W_2 is a clique, there is a four-vertex path c_1-c_2-c_3-c_4 of C such that $c_2, c_3 \in W_2$ and $c_1, c_4 \notin W_2$.

Consequently $c_1, c_4 \in W_1 \cup W_3$. Since $G[W_1, W_2]$ is a half-graph, not both $c_1, c_4 \in W_1$, and similarly they are not both in W_3; so we may assume that $c_1 \in W_1$ and $c_4 \in W_3$. But this contradicts that $G[W_2, W_3], G[W_2, W_1]$ are compatible. Thus $|V(C) \cap W_i| \le 1$ for $1 \le i \le \ell$. Since C is a hole, it follows that $|V(C) \cap W_i| = 1$ for $1 \le i \le \ell$, and hence C has length ℓ, as required.

Now we assume that G is a blow-up of an ℓ-framework. Let F be an ℓ-framework, and let $P_1, \ldots, P_k, T, S$ and so on be as in the definition of an ℓ-framework. Thus $F = T \cup S \cup P_1 \cup \cdots \cup P_k$. Let $D = \overrightarrow{T} \cup \overrightarrow{S} \cup P_1 \cup \cdots \cup P_k$, and for each $t \in V(F)$ let W_t be as in the definition of a blow-up of an ℓ-framework. Let $A = G[\{a_1, \ldots, a_k\}]$ and $B = G[\{b_1, \ldots, b_k\}]$. From the definition, we have

(i) D is an induced subgraph of G, and for each $t \in V(D)$ there is a clique W_t of G, all pairwise disjoint and with union $V(G)$; $W_t \cap V(D) = \{t\}$ for each $t \in V(D)$, and $W_t = \{t\}$ for each $t \in V(D) \setminus V(P_1 \cup \cdots \cup P_k)$.

(ii) For each $t \in V(D)$, there is a linear ordering of W_t with first term t, say $(x_1, \ldots, x_n)$ where $x_1 = t$. It has the property that for every $v \in V(G) \setminus W_t$, and every $j \in \{1, \ldots, n\}$, if v is G-adjacent to x_j then v is G-adjacent to $x_1, \ldots, x_j$.

(iii) If $t, t' \in \{a_1, \ldots, a_k\}$ or $t, t' \in \{b_1, \ldots, b_k\}$, and t, t' are D-adjacent, then W_t is complete to $W_{t'}$.

(iv) For each $t \in V(T)$, if $0 \le i \le m$ and a_i, t are D-adjacent, then W_t is complete to W_{a_i}. For each $t \in V(S)$, if either ℓ is odd and $i \in \{m + 1, \ldots, k\}$, or ℓ is even and $i \in \{0, \ldots, m\}$, and b_i, t are D-adjacent, then W_t is complete to W_{b_i}.

(v) For each upper tent-arborescence T_j with apex a_j say, let $t \in L(T_j)$ and let the path Q of T from a_0 to t have vertices
$$a_0 = y_1\text{-}\cdots\text{-}y_p\text{-}a_j\text{-}z_1\text{-}\cdots\text{-}z_q = t$$
in order. Then W_t is complete to $\{y_1, \ldots, y_p, a_j\}$; W_t is anticomplete to $W(T \setminus V(Q))$; and $G[W_t, \{z_1, \ldots, z_{q-1}\}]$, with the given order of W_t and the order $z_1, \ldots, z_{q-1}$ of $\{z_1, \ldots, z_{q-1}\}$, is an ordered half-graph. The same holds for lower tent-arborescences with T, a_0 replaced by S, b_0.

Now suppose that C is a hole of G of length different from ℓ, and choose C with $V(C) \setminus V(D)$ minimal.

(1) $|V(C) \cap W_t| \le 1$ for each $t \in V(D)$.

Suppose that $|V(C) \cap W_t| \ge 2$ for some $t \in V(D)$. As before, there is a four-vertex path $c_1\text{-}c_2\text{-}c_3\text{-}c_4$ of C such that $c_2, c_3 \in W_t$ and $c_1, c_4 \notin W_t$. We may assume that c_2 is earlier than c_3 in the ordering of W_t; but c_4 is adjacent to c_3 and not to c_2, contrary to (ii) above. This proves (1).

(2) C is a hole of D.

Suppose not; then there exists $t \in V(D)$ such that $t \notin V(C)$ and $V(C) \cap W_t \ne \emptyset$. Choose $t \notin V(A \cup B)$ if possible. Let $v \in V(C) \cap W_t$, and let u, w be the neighbours of v in C. Since $|W_t| \ge 2$ it follows that $t \in V(P_i)$ for some $i \in \{1, \ldots, k\}$. From (ii) above, u, w are adjacent to t. From the minimality of $V(C) \setminus V(D)$, we cannot replace v by t to obtain a hole, and so t has a neighbour $x \in V(C)$, where

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$x \neq u, v, w$. Let $u \in W_r$, and $w \in W_s$, and $x \in W_q$. By (1), q, r, s, t are all different. Since u, w, x all have neighbours in W_t it follows that t is D-adjacent to each of q, r, s, and so has degree at least three in D; and hence $t \in \{a_i, b_i\} \in V(A \cup B)$. From the choice of t it follows that every vertex of C not in $V(D)$ belongs to $W(A \cup B)$.

We assume that $t = a_i$ (the argument when $t = b_i$ is similar and we omit it). From the symmetry between r, s, we may assume that $s \notin V(P_i)$. Thus $s \in V(T)$. Suppose first that $q \notin V(T)$. Thus q is the neighbour of $t = a_i$ in P_i. Since $x \in V(C) \setminus W(A \cup B)$, it follows that $x \in V(D)$ and so $x = q$; and so x is complete to W_t, a contradiction since x, v are not adjacent. Thus $q \in V(T)$. Since x is mixed on W_t, it follows from **(iv)** above that $i > m$, and so $t = a_i$ is a leaf of T.

Let Q be the path of T from a_0 to t. Since $q, s \in V(T)$, and each is $\overrightarrow{T}$-adjacent to t, it follows that $q, s \in V(Q)$. By **(v)** above, since w is adjacent to v and x is not adjacent to v, it follows that q lies in the interior of the path of Q between s, t, and so w G-dominates x (from **(iv)** above), which is impossible since $w, x \in V(C)$. This proves (2).

Since $\overrightarrow{T}, \overrightarrow{S}$ do not contain a 4-vertex induced path or a 4-hole, it follows that C is not a subgraph of either of them. For $1 \leq i \leq k$, each internal vertex of P_i has degree two in D, and so if some such vertex belongs to C then so does the whole of P_i. Hence C is the concatenation of paths of G, in alternation equal to one of $P_1, \ldots, P_k$ or to a subpath of one of $\overrightarrow{T}, \overrightarrow{S}$ with length one or two (because $\overrightarrow{T}, \overrightarrow{S}$ contain no induced path of length three). Let us write

$$C = P_{d_1} \cup Q_1 \cup P_{d_2} \cup Q_2 \cup P_{d_3} \cup Q_3 \cup \cdots \cup P_{d_n} \cup Q_n$$

where $d_1, d_2, \ldots, d_n \in \{1, \ldots, k\}$ are distinct, and for i odd Q_i is a path of $\overrightarrow{T}$ with ends $a_{d_i}, a_{d_{i+1}}$, and for i even Q_i is a path of $\overrightarrow{S}$ with ends $b_{d_i}, b_{d_{i+1}}$, where $d_{n+1} = d_1$. Thus $n \geq 2$ and n is even.

Suppose first that ℓ is odd. If $n \geq 4$, then there are no edges of G between P_{d_1}, P_{d_3} by 1.1, contradicting that F is an ℓ-framework. So $n = 2$; and exactly one of Q_1, Q_2 has length one, from the definition of an ℓ-framework, and the other has length two; and since P_{d_1}, P_{d_2} both have length $(\ell - 3)/2$ from the definition of an ℓ-framework, it follows that C has length ℓ, a contradiction.

Thus ℓ is even. Note that in this case there is a symmetry in the definition of an ℓ-framework that exchanges S and T (although the bars have to be renumbered). This will be helpful to reduce the number of cases we need to examine.

Suppose that $n = 2$. If P_{d_1}, P_{d_2} both have length $\ell/2 - 1$, then Q_1, Q_2 both have length one (from 1.2 and the definition of an ℓ-framework) and so C has length ℓ; if exactly one of P_{d_1}, P_{d_2} has length $\ell/2 - 1$, then exactly one of Q_1, Q_2 has length one (again from 1.2 and the definition) and so C has length ℓ; and if neither of P_{d_1}, P_{d_2} has length $\ell/2 - 1$, then neither of Q_1, Q_2 has length one (again from 1.2 and the definition) and so C has length ℓ, in all cases a contradiction. So $n \geq 4$.

Suppose that P_{d_1} has length $\ell/2 - 1$. There are no edges between $V(P_{d_1})$ and $V(P_{d_3})$, and in particular a_{d_1}, a_{d_3} are nonadjacent and b_{d_1}, b_{d_3} are nonadjacent, contrary to 1.2. Thus all of $P_{d_1}, \ldots, P_{d_n}$ have length $\ell/2 - 2$, and therefore all of $Q_1, \ldots, Q_n$ have length two. Let q_i be the middle vertex of Q_i for $1 \leq i \leq n$.

Each of $a_{d_1}, \ldots, a_{d_n}$ belongs to the base of one of the tent-arborescences $T_0, \ldots, T_m$, and each of $b_{d_1}, \ldots, b_{d_m}$ belongs to the base of one of $S_0, \ldots, S_m$.

(3) *There exists $i \in \{0, \ldots, m\}$ such that $a_{d_1}, \ldots, a_{d_n}$ all belong to the base of T_i and $b_{d_1}, \ldots, b_{d_n}$ all belong to the base of S_{i+1}.*

Suppose not; then from the symmetry between S, T, we may assume that $a_{d_1} \in V(T_g)$ and $a_{d_2} \in V(T_h)$ where $g < h$. Hence $b_{d_1} \in S_{g+1}$ and $b_{d_2} \in S_{h+1}$. Since q_1 is G-adjacent (and hence $\overrightarrow{T}$-adjacent) to both a_{d_1}, a_{d_2}, it follows that q_1 belongs to the path of T between a_0, a_{g+1}. Consequently q_1 is G-adjacent to every vertex in $V(T_{g+1}) \cup \cdots \cup V(T_m)$; and so none of $a_{d_3}, \ldots, a_{d_n}, q_3, q_5, \ldots, q_{n-1}$ belong to $V(T_{g+1}) \cup \cdots \cup V(T_m)$. Let $a_{d_3} \in T_f$; it follows that $f \leq g$. Moreover, $b_{d_3} \in S_{f+1}$, and since q_2 is adjacent to both b_{d_2}, b_{d_3}, it follows that q_2 belongs to the path of S between b_0 and b_h. But then q_2 is adjacent to b_{d_1}, a contradiction. This proves (3).

Choose i as in (3). Since q_1 has a neighbour and a non-neighbour in the base of T_i, it follows that $q_1 \in V(T_i)$, and similarly $q_1, q_3, q_5, \ldots, q_{n-1} \in V(T_i)$ and $q_2, q_4, q_6, \ldots, q_n \in V(S_{i+1})$. From the definition of an ℓ-framework, the arborescences T_i, S_{i+1} are coarboreal under the bijection that maps a_j to b_j for each leaf a_j of T_i. Let S' be the arborescence obtained from S_{i+1} by replacing b_j by a_j for each leaf a_j of T_i. Thus T_i, S' are coarboreal; let R be a tree with vertex set $L(T_i)$ in which they both live. For $j \in \{1, \ldots, n\}$ with j odd, since q_j is adjacent to $a_{d_j}, a_{d_{j+1}}$ and to no other vertices in $\{a_{d_1}, \ldots, a_{d_n}\}$, and the set of G-neighbours of q_j in $V(R)$ is the vertex set of a subtree of R, there is a path Q_j of R between $a_{d_j}, a_{d_{j+1}}$, such that all its vertices are $\overrightarrow{T_i}$-adjacent to q_j, and

$$V(Q_j) \cap \{a_{d_1}, \ldots, a_{d_n}\} = \{a_{d_j}, a_{d_{j+1}}\}.$$

Similarly, for $j \in \{1, \ldots, n\}$ with j even, there is a path Q_j of R between $a_{d_j}, a_{d_{j+1}}$, such that all its vertices are $\overrightarrow{S'}$-adjacent to q_j, and

$$V(Q_j) \cap \{a_{d_1}, \ldots, a_{d_n}\} = \{a_{d_j}, a_{d_{j+1}}\},$$

where $d_{n+1} = d_1$. The paths $Q_1, Q_3, \ldots, Q_{n-1}$ are pairwise vertex-disjoint, since $q_1, q_3, q_5, \ldots, q_{n-1}$ are pairwise nonadjacent and therefore no two of them have a common neighbour in $L(T)$; and similarly $Q_2, Q_4, \ldots, Q_n$ are pairwise vertex-disjoint. It follows that

$$\{\{a_{d_1}, a_{d_2}\}, \{a_{d_3}, a_{d_4}\}, \ldots, \{a_{d_{n-1}}, a_{d_n}\}\}$$

is a feasible partition of $\{a_{d_1}, \ldots, a_{d_n}\}$ in R, and so is

$$\{\{a_{d_2}, a_{d_3}\}, \{a_{d_4}, a_{d_5}\}, \ldots, \{a_{d_n}, a_{d_1}\}\};$$

and they are different since $n \geq 4$, contrary to 2.1. This proves 2.2. ∎

3 Blow-ups of cycles

Our strategy to prove 1.3 is to choose a maximal "apexed ℓ-frame", and a maximal blow-up of it contained in our graph, and to analyze how the remainder of the graph can attach to it. But this only works in graphs that contain apexed ℓ-frames, and here we handle the graphs that do not.

Here are some types of graph that we will need:

- A *theta* is a graph that is the union of three paths R_1, R_2, R_3, each with the same pair of ends, each of length more than one, and pairwise vertex-disjoint except for their ends.

- A *pyramid* is a graph that is the union of three paths R_1, R_2, R_3 and three additional edges b_1b_2, b_2b_3, b_3b_1, where R_i has ends $a \neq b_i$ for $i = 1, 2, 3$, and R_1, R_2, R_3 are pairwise vertex-disjoint except for their common end a, and at least two of R_1, R_2, R_3 have length at least two.

- A *prism* is a graph that is the union of three paths R_1, R_2, R_3 and six additional edges $a_1a_2, a_2a_3, a_3a_1, b_1b_2, b_2b_3, b_3b_1$, where R_i has ends $a_i \neq b_i$ for $i = 1, 2, 3$, and R_1, R_2, R_3 are pairwise vertex-disjoint.

- A *proper wheel* is a graph consisting of a cycle C of length at least four, and one additional vertex v, where v has at least three neighbours and at least one non-neighbour in $V(C)$, and if v has exactly three neighbours in $V(C)$ then they do not induce a path.
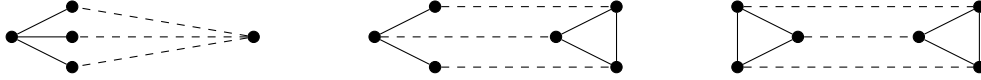

Figure 5: A theta, a pyramid and a prism (dashed lines mean paths of arbitrary positive length)

In this paper, we say G *contains* H to mean G has an induced subgraph isomorphic to H. The following is a corollary of theorem 1.6 of a paper [2] by V. Boncompagni, I. Penev, and K. Vušković:

3.1 *If a graph G has no clique cutset or universal vertex, and G contains no theta, pyramid, prism, proper wheel, or 4-hole, then G is a blow-up of a cycle.*

Since every proper wheel has two holes of different lengths, 3.1 has the following immediate consequence:

3.2 *Let $\ell \geq 5$, and let G be a non-null ℓ-holed graph, not containing a theta, pyramid or prism, and with no clique cutset or universal vertex. Then G is a blow-up of a cycle of length ℓ.*

4   Frames

This section is to define an "ℓ-frame". A *threshold graph* is a graph that does not contain P_4, C_4 or $\overline{C_4}$. (The notation $\overline{G}$ means the complement graph of G.) Let $\ell \geq 5$ be odd. Let $k \geq 3$ be an integer, and take distinct vertices $a_1, \ldots, a_k, b_1, \ldots, b_k$. For $1 \leq i \leq k$ let P_i be a path of length $(\ell - 3)/2$ with ends a_i, b_i, pairwise vertex-disjoint. Let the subgraphs A and B induced on $\{a_1, \ldots, a_k\}$ and on $\{b_1, \ldots, b_k\}$ respectively be threshold graphs, and for $1 \leq i < j \leq k$, let b_i, b_j be adjacent if and only if a_i, a_j are nonadjacent. Moreover, let A either be disconnected or two-connected, and the same for B. (See figure 6.) For ℓ odd, a graph F constructible in this way is called an ℓ-*frame*; all its holes have length ℓ. We call $P_1, \ldots, P_k$ the *bars* of the frame, and A, B are its *sides*.
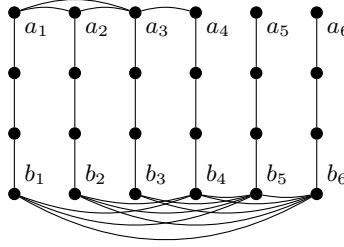
Figure 6: A 9-frame.

We remark that, since the subgraphs induced on $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are complementary threshold graphs, one of them has a vertex of degree 0 (such as a_6 in the figure). Thus all ℓ-frames when ℓ is odd have one-vertex clique cutsets.

Now the case when ℓ is even. Let $m, n \geq 0$ be integers with $n \geq 2$ and $m + n \geq 3$; and let

$$a_1, \ldots, a_n, c_1, \ldots, c_m$$

$$b_1, \ldots, b_n, d_1, \ldots, d_m$$

all be distinct. For $1 \leq i \leq n$ let P_i be a path with ends a_i, b_i of length $\ell/2 - 2$, and for $1 \leq i \leq m$ let Q_i be a path between c_i, d_i of length $\ell/2 - 1$, all pairwise vertex-disjoint. Let $\{c_1, \ldots, c_m\}$ and $\{d_1, \ldots, d_m\}$ be cliques; and let the bipartite subgraph with bipartition $(\{a_1, \ldots, a_n\}, \{c_1, \ldots, c_m\})$ be a half-graph. For $1 \leq i \leq n$ and $1 \leq j \leq m$, let b_i, d_j be adjacent if and only if a_i, c_j are nonadjacent. Let one of $a_1, \ldots, a_n$ and one of $b_1, \ldots, b_n$ have degree one in F. There are no other edges (thus $\{a_1, \ldots, a_n\}$ is a stable set, and so is $\{b_1, \ldots, b_n\}$). Let us call such a graph F an ℓ-frame. (See figure 7.) Every hole in an ℓ-frame has length ℓ. Let A, B be the subgraphs induced on $\{a_1, \ldots, a_n, c_1, \ldots, c_m\}$ and on $\{b_1, \ldots, b_n, d_1, \ldots, d_m\}$ respectively. (It follows that A, B are both disconnected threshold graphs.) We call $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ the *bars* of the frame, and A, B its *sides*.
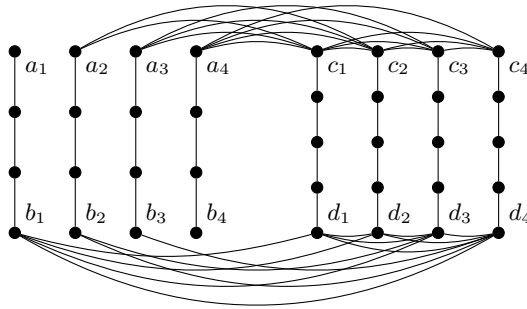

Figure 7: A 10-frame.

5 Gates

Let $k \geq 3$ be an integer. A *k-bar semigate* in a graph G is a $(k + 2)$-tuple $(A, B, P_1, \ldots, P_k)$ of induced subgraphs of G, with the following properties:

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- A, B are vertex-disjoint connected subgraphs of G;

- for $1 \leq i \leq k$, P_i is a path with ends a_i, b_i, and $V(P_i \cap A) = \{a_i\}$ and $V(P_i \cap B) = \{b_i\}$;

- $V(P_i \cap P_j) = \{a_i, b_i\} \cap \{a_j, b_j\}$ for $1 \leq i < j \leq k$;

- each vertex of A has at most one neighbour in $V(B)$, and vice versa;

- $A \cup B \cup P_1 \cup \cdots \cup P_k$ is an induced subgraph of G.

We call A, B the *sides* of the semigate, and $P_1, \ldots, P_k$ its *bars*. (Sadly for the metaphor, we will draw semigates with the bars vertical and the sides horizontal, because they fit better on the page that way.) Note that a theta makes a 3-bar semigate with $|A| = |B| = 1$; and similarly a pyramid makes one with $|A| = 1$ and $|B| = 3$, and a prism makes one with $|A| = |B| = 3$.

Let $(A, B, P_1, \ldots, P_k)$ and $(A', B', P_1', \ldots, P_k')$ be k-bar semigates in G. We say the second is *better* than the first if $A' \subseteq A$ and $B' \subseteq B$ and $P_i \subseteq P_i'$ for $1 \leq i \leq k$, and *strictly better* if either $A' \neq A$ or $B' \neq B$. (Note that the sides are getting smaller and the bars are getting bigger.) A k-bar semigate is a *k-bar gate* if there is no strictly better k-bar semigate. For every k-bar semigate, there is a k-bar gate which is better than it. Given a gate in this notation, we want to see what we can say about A and the vertices $a_1, \ldots, a_k \in V(A)$, and the same in B, when G is ℓ-holed.

In general, if u, v are vertices of some graph H, $d_H(u, v)$ denotes the distance in H between u, v. We begin with the trivial but very useful:

5.1 Let $(A, B, P_1, \ldots, P_k)$ be a k-bar gate in G. Then for $1 \leq i \leq k$, either a_i has at least two neighbours in A, or $a_i = a_j$ for some $j \in \{1, \ldots, k\} \setminus \{i\}$.

Proof. Suppose that $a_1 \neq a_2, \ldots, a_k$ say, and a_1 has degree at most one in A. Since A is connected and $a_2 \in V(A)$, it follows that a_1 has a neighbour $u \in V(A)$. But then $(A \setminus \{a_1\}, B, P_1', P_2, \ldots, P_k)$ is a strictly better k-bar semigate, where P_1' is the path obtained by adding u and the edge $a_1 u$ to P_1, a contradiction. This proves 5.1. ∎

A *cut-vertex* in a graph A is a vertex $v \in V(A)$ such that $A \setminus \{v\}$ is disconnected.

5.2 Let $(A, B, P_1, \ldots, P_k)$ be a k-bar gate in G, and suppose that A has a cut-vertex. There exist distinct $g, h, i, j \in \{1, \ldots, k\}$ such that

$$d_A(a_g, a_h) + d_A(a_i, a_j) + 2 \leq d_A(a_g, a_i) + d_A(a_h, a_j) = d_A(a_g, a_j) + d_A(a_h, a_i).$$

Proof. Let v be a cut-vertex of A, and let C, D be distinct components of $A \setminus \{v\}$.

(1) *There exist distinct $i, j \in \{1, \ldots, k\}$ such that $a_i, a_j \in V(C)$ and $d_A(a_i, a_j) < d_A(a_i, v) + d_A(a_j, v)$.*

Let I be the set of $i \in \{1, \ldots, k\}$ with $i \in V(C)$, and for each $i \in I$, let Q_i be an induced path of $A[V(C) \cup \{v\}]$ between a_i, v of length $d_A(a_i, v)$, and let X be the union of the vertex sets of the paths Q_i ($i \in I$). Since

$$((A \setminus V(C)) \cup G[X], B, P_1, \ldots, P_k)$$

is not a strictly better k-bar semigate, it follows that $X = V(C)$. In particular, since $V(C) \neq \emptyset$, it follows that $I \neq \emptyset$. Let $i \in I$ say. If $a_i = a_j$ for some $j \in \{1, \ldots, k\} \setminus \{i\}$, then (1) holds; so we may

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assume there is no such j. By 5.1, a_i has degree at least two in A. Not both these neighbours lie in $V(Q_i)$ since Q_i is induced; let $u \in V(A) \setminus V(Q_i)$ be A-adjacent to a_i. Thus $u \in X$, and so $u \in V(Q_j)$ for some $j \in I \setminus \{i\}$. But then $Q_i \cup Q_j$ is not an induced path, and so $d_A(a_i, a_j) < d_A(a_i, v) + d_A(a_j, v)$. This proves (1).

From (1) we may assume that $a_g, a_h \in V(C)$ and $d_A(a_g, a_h) < d_A(a_g, v) + d_A(a_h, v)$. Similarly we may assume that $a_i, a_j \in V(D)$ and $d_A(a_i, a_j) < d_A(a_i, v) + d_A(a_j, v)$. Summing, we deduce that

$$d_A(a_g, a_h) + d_A(a_i, a_j) \le d_A(a_g, v) + d_A(a_h, v) + d_A(a_i, v) + d_A(a_j, v) - 2.$$

But $d_A(a_g, a_i) = d_A(a_g, v) + d_A(a_i, v)$ since every path of A between a_g, a_i contains v, and there are three similar equations; and we deduce that the theorem holds. This proves 5.2. ∎

5.3 *Let $\ell \ge 5$, let G be ℓ-holed, and let $(A, B, P_1, \ldots, P_4)$ be a 4-bar gate in G. Define $d(1, 2, 3, 4) = d_A(a_1, a_2) + d_A(a_3, a_4)$, and define $d(1, 3, 2, 4)$ and $d(1, 4, 2, 3)$ similarly. Then two of*

$$d(1, 2, 3, 4), d(1, 3, 2, 4), d(1, 4, 2, 3)$$

are equal and the third is at most one more.

Proof. The graph A is connected. If it has a cut-vertex then the claim follows from 5.2, so we may assume that it is 2-connected or has at most two vertices. In particular, $V(A) = \{a_1, a_2, a_3, a_4\}$ since any other vertex in A could be deleted to obtain a strictly better 4-bar semigate. For each $i \in \{1, \ldots, 4\}$ either $a_i = a_j$ for some $j \in \{1, \ldots, 4\} \setminus \{i\}$, or a_i is adjacent to at least two other members of $\{a_1, \ldots, a_4\}$, by 5.1; so one of the following holds:

- $|V(A)| = 1$; then $d(1, 2, 3, 4), d(1, 3, 2, 4), d(1, 4, 2, 3)$ are all equal to zero.

- $|V(A)| = 2$; then we may assume that $a_1 = a_2 \ne a_3 = a_4$, and so $d(1, 2, 3, 4) = 0$ and $d(1, 3, 2, 4) = d(1, 4, 2, 3) = 2$.

- $|V(A)| = 3$; then we may assume that $a_1 = a_2$, and a_1, a_3, a_4 are distinct, and all three pairs of these vertices are adjacent since A is 2-connected; but then $d(1, 2, 3, 4) = 1$ and $d(1, 3, 2, 4) = d(1, 4, 2, 3) = 2$.

- $|V(A)| = 4$; then A has a cycle of length four, since it is 2-connected, and since G is ℓ-holed, this cycle is not induced. Consequently A has at least five edges, and we may assume that every two of a_1, a_2, a_3, a_4 are adjacent except possibly a_1, a_4. Thus $d(1, 2, 3, 4) = d(1, 3, 2, 4) = 2$ and $2 \le d(1, 4, 2, 3) \le 3$.

This proves 5.3. ∎

5.4 *Let $\ell \ge 5$, let G be ℓ-holed, and let $(A, B, P_1, \ldots, P_4)$ be a 4-bar gate in G. Define $d(1, 2, 3, 4) = d_A(a_1, a_2) + d_A(a_3, a_4)$, and define $d(1, 3, 2, 4)$ and $d(1, 4, 2, 3)$ similarly. Then every two of*

$$d(1, 2, 3, 4), d(1, 3, 2, 4), d(1, 4, 2, 3)$$

differ by at most one.

Proof. Suppose that $d(1,2,3,4) \leq d(1,3,2,4) - 2$ say. By 5.3, $d(1,3,2,4) = d(1,4,2,3)$. Let $d'(1,2,3,4) = d_B(b_1, b_2) + d_B(b_3, b_4)$, and define $d'(1,3,2,4)$ and $d'(1,4,2,3)$ similarly. For $1 \leq i \leq 4$, let P_i have length n_i. Let $1 \leq i < j \leq 4$. Since there is a hole formed by the union of P_i, P_j with a shortest path of A between a_i, a_j, and a shortest path in B between b_i, b_j, it follows that

$$d_A(a_i, a_j) + d_B(b_i, b_j) + n_i + n_j = \ell.$$

In particular,

$$d(1,2,3,4) + d'(1,2,3,4) = d_A(a_1, a_2) + d_A(a_3, a_4) + d_B(b_1, b_2) + d_B(b_3, b_4) = 2\ell - n_1 - n_2 - n_3 - n_4,$$

and similarly

$$d(1,3,2,4) + d'(1,3,2,4) = d(1,4,2,3) + d'(1,4,2,3) = 2\ell - n_1 - n_2 - n_3 - n_4.$$

Since $d(1,3,2,4) = d(1,4,2,3)$, it follows that $d'(1,3,2,4) = d'(1,4,2,3)$, and since $d(1,2,3,4) \leq d(1,3,2,4) - 2$ it follows that $d'(1,2,3,4) \geq d'(1,3,2,4) + 2$, contrary to 5.3 applied in B. This proves that every two of

$$d(1,2,3,4), d(1,3,2,4), d(1,4,2,3)$$

differ by at most one. This proves 5.4. ∎

Next we extend this to semigates:

5.5 *Let $\ell \geq 5$, let G be ℓ-holed, and let $(A, B, P_1, \ldots, P_4)$ be a 4-bar semigate in G. Define $d(1,2,3,4) = d_A(a_1, a_2) + d_A(a_3, a_4)$, and define $d(1,3,2,4)$ and $d(1,4,2,3)$ similarly. Then every two of*

$$d(1,2,3,4), d(1,3,2,4), d(1,4,2,3)$$

differ by at most one.

Proof. Choose a 4-bar gate $(A', B', P_1', P_2', P_3', P_4')$ that is better than $(A, B, P_1, \ldots, P_4)$, and let P_i' have ends $a_i' \in V(A')$ and $b_i' \in V(B')$ for $1 \leq i \leq 4$. Define $d'(1,2,3,4) = d_{A'}(a_1', a_2') + d_{A'}(a_3', a_4')$, and define $d'(1,3,2,4)$ and $d'(1,4,2,3)$ similarly. For $1 \leq i \leq 4$, P_i is a subpath of P_i'; let the subpath of P_i' between a_i, a_i' have length n_i. Let $1 \leq i < j \leq 4$. Since every induced path of A between a_i, a_j has the same length (because they can both be extended to a hole via P_i, P_j and the same path of B between b_i, b_j), it follows that

$$d_A(a_i, a_j) = n_i + n_j + d_{A'}(a_i', a_j').$$

Consequently

$$d(1,2,3,4) = n_1 + n_2 + n_3 + n_4 + d'(1,2,3,4),$$

and similar equations hold for $d(1,3,2,4)$ and $d(1,4,2,3)$. Since every two of

$$d'(1,2,3,4), d'(1,3,2,4), d'(1,4,2,3)$$

differ by at most one by 5.4, it follows that every two of

$$d(1,2,3,4), d(1,3,2,4), d(1,4,2,3)$$

differ by at most one. This proves 5.5. ∎

5.6 *Let $\ell \geq 5$, let G be ℓ-holed, and let $(A, B, P_1, \ldots, P_k)$ be a k-bar gate in G. Then:*

- *either $|V(A)| = 1$, or $|V(A)| \geq 3$ and A is 2-connected;*

- $V(A) = \{a_1, \ldots, a_k\}$;

- *A is a threshold graph;*

- *at most one vertex of A appears in the list $a_1, \ldots, a_k$ more than once; that is, there do not exist distinct $g, h, i, j \in \{1, \ldots, k\}$ with $a_g = a_h \neq a_i = a_j$.*

These statements also hold with A replaced by B.

Proof. Suppose that $|V(A)| = 2$, and $V(A) = \{u, v\}$ say. By 5.1 we may assume that $a_1 = a_2 = u$ and $a_3 = a_4 = v$; and $(A, B, P_1, P_2, P_3, P_4)$ is a 4-bar semigate, contrary to 5.5. This proves that $|V(A)| \neq 2$. If $|V(A)| = 1$ then all four bullets of the theorem hold; so we may assume that $|V(A)| \geq 3$. Suppose that there exist distinct $g, h, i, j \in \{1, \ldots, k\}$ such that

$$d_A(a_g, a_h) + d_A(a_i, a_j) + 2 \leq d_A(a_g, a_i) + d_A(a_h, a_j) = d_A(a_g, a_j) + d_A(a_h, a_i).$$

Then $(A, B, P_g, P_h, P_i, P_j)$ is a 4-bar semigate and 5.5 is violated. So there are no such g, h, i, j; and so by 5.2, this shows that A has no cut-vertex, and hence it is 2-connected. This proves the first bullet.

If there exists $v \in V(A) \setminus \{a_1, \ldots, a_k\}$, then since $A \setminus \{v\}$ is connected, it follows that $(A \setminus \{v\}, B, P_1, \ldots, P_k)$ is a strictly better k-bar semigate, a contradiction. This proves the second bullet.

Suppose that A contains P_4; say with vertices a_1, a_2, a_3, a_4 in order. Thus $d_A(a_1, a_2) + d_A(a_3, a_4) = 2$, and $d_A(a_1, a_3) + d_A(a_2, a_4) = 4$, contrary to 5.5 applied to the 4-bar semigate $(A, B, P_1, P_2, P_3, P_4)$. Suppose that A contains $\overline{C_4}$; then we may assume that $a_1 a_2$ and $a_3 a_4$ are edges, and $\{a_1, a_2\}$ and $\{a_3, a_4\}$ are anticomplete. Then $d_A(a_1, a_2) + d_A(a_3, a_4) = 2$ and $d_A(a_1, a_3) + d_A(a_2, a_4) \geq 4$, again contrary to 5.5. Certainly A does not contain C_4 since G is ℓ-holed. Thus A is a threshold graph. This proves the third bullet.

Finally, suppose that $a_1 = a_2 \neq a_3 = a_4$ say. Then $d_A(a_1, a_2) + d_A(a_3, a_4) = 0$ and $d_A(a_1, a_3) + d_A(a_2, a_4) \geq 2$, again contrary to 5.5. This proves the fourth bullet, and so proves 5.6. ∎

We can say more about k-bar gates, but we need to treat the ℓ even and ℓ odd cases separately. First:

5.7 *Let $\ell \geq 5$ be odd, let G be ℓ-holed, and let $(A, B, P_1, \ldots, P_k)$ be a k-bar gate in G. Then, possibly after exchanging A, B, there exists $J \subseteq \{1, \ldots, k\}$ with $|J| \geq 2$ such that all the vertices $(a_i : i \in J)$ are equal, equal to a_0 say, and the following hold:*

- *the vertices a_0 and a_i $(i \in \{1, \ldots, k\} \setminus J)$ are all distinct;*

- *$b_1, \ldots, b_k$ are all distinct;*

- *the vertices b_i $(i \in J)$ are pairwise adjacent, and P_i has length $(\ell - 1)/2$ for each $i \in J$;*

- *for each $i \in \{1, \ldots, k\} \setminus J$, a_0 is adjacent to a_i, and b_j is adjacent to b_i for all $j \in J$, and P_i has length $(\ell - 3)/2$;*

- *for all distinct $i, j \in \{1, \ldots, k\} \setminus J$, a_i, a_j are adjacent if and only if b_i, b_j are nonadjacent.*
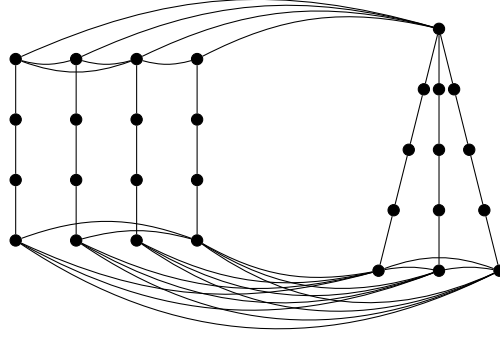
(See figure 8.)

Figure 8: A 9-holed 7-bar gate.

Proof. Let P_i have length n_i for each i. Since A is a threshold graph (by 5.6), and so does not contain P_4, and A is connected, it follows that every two vertices in A have distance at most two in A; and the same in B. For all $i, j \in \{1, \ldots, k\}$, let us define $d(i, j) = d_A(a_i, a_j) + d_B(b_i, b_j)$. We observe first:

(1) $n_i + n_j + d(i, j) = \ell$ for $1 \le i < j \le k$.

Take a path of A between a_i, a_j of length $d_A(a_i, a_j)$, and a similar path in B; then their union with P_i and P_j is a hole of length ℓ. This proves (1).

(2) *There do not exist distinct $i, j \in \{1, \ldots, k\}$ with $a_i = a_j$ and $b_i = b_j$.*

Suppose that $a_1 = a_2$ and $b_1 = b_2$ say. Thus $n_1 + n_2 = \ell$. From (1), and since $k \ge 3$, it follows that $n_i + n_3 + d(i, 3) = \ell$ for $i = 1, 2$, and since $d(1, 3) = d(2, 3)$, it follows that $n_1 = n_2$, and so $\ell = n_1 + n_2$ is even, a contradiction. This proves (2).

(3) $n_i \le (\ell + 1)/2$ for $1 \le i \le k$.

Suppose that $n_1 > (\ell + 1)/2$ say, and so $n_1 \ge (\ell + 3)/2$. From (1) and (2), $n_1 + n_i + 1 \le \ell$ for $i = 2, 3$; and hence $n_2 + n_3 \le \ell - 5$. But since $d(2, 3) \le 4$, (1) implies that $n_2 + n_3 \ge \ell - 4$, a contradiction. This proves (3).

(4) $n_i \ge (\ell - 5)/2$ for $1 \le i \le k$.

Suppose that $n_1 \le (\ell - 7)/2$ say. For $2 \le i \le k$, since $n_i \le (\ell + 1)/2$ by (3), (1) implies that $d(1, i) \ge 3$. In particular, $a_i \ne a_1$. By 5.1, a_1 has at least two neighbours in A, say a_2, a_3; and since $d(1, i) \le 3$ for $i = 2, 3$, it follows from (1) that $n_i \ge (\ell + 1)/2$ for $i = 2, 3$. But this is impossible since $n_i + n_j \le \ell$ by (1). This proves (4).

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(5) $n_i \leq (\ell - 1)/2$ *for* $1 \leq i \leq k$.

Suppose that $n_1 = (\ell + 1)/2$ say. Suppose first that some vertex in A has distance two from a_1; say a_2. Since $n_2 \geq (\ell - 5)/2$ by (3), (1) implies that $d(1, 2) \leq 2$, and since $d_A(a_1, a_2) = 2$ we deduce that $b_1 = b_2$, and $n_2 = (\ell - 5)/2$. For $3 \leq i \leq k$, $n_2 + n_i \geq \ell - 4$ by (1), and so $n_i \geq (\ell - 3)/2$, and therefore a_i is equal or adjacent to a_1. Since $d_A(a_1, a_2) = 2$, it follows that a_2 has a neighbour $a_3 \in V(A)$ adjacent to a_1. Choose an induced path between a_3, b_1 consisting of P_3 and a path of B. Then this path, together with P_1, P_2 and the edges $a_1 a_3, a_2 a_3$, forms a theta with ends a_3, b_1, and so G has an even hole, a contradiction.

This proves that a_1 is adjacent to every other vertex in A, and similarly b_1 is adjacent to every other vertex in B. Suppose that $n_2 = (\ell - 5)/2$ say. From (1) it follows that $a_1 \neq a_2$ and $b_1 \neq b_2$. By 5.1 there exists $i \in \{3, \ldots, k\}$ such that a_i is equal or adjacent to a_2, and so $d(2, i) \leq 3$. But (1) implies that $n_2 + n_i = \ell - d(2, i)$; and from (1) again, $n_1 + n_i = \ell - d(1, i)$. Consequently $3 = n_1 - n_2 = d(2, i) - d(1, i)$. But this is impossible since $d(2, i) \leq 3$ and $d(1, i) \geq 1$.

Thus $n_i \geq (\ell - 3)/2$ for $2 \leq i \leq k$. Since $d(1, i) \geq 1$ by (2), (1) implies that $n_1 + n_2 \leq \ell - 1$, and so $n_i = (\ell - 3)/2$ for $2 \leq i \leq k$. From (1), $d(1, i) = 1$, and so either $a_1 = a_i$ or $b_1 = b_i$, for each $i \in \{2, \ldots, k\}$. Consequently $d(2, 3) \leq 2$, and so $n_2 + n_3 \geq \ell - 2$ by (1), a contradiction. This proves (5).

(6) $n_i \geq (\ell - 3)/2$ *for* $1 \leq i \leq k$.

Suppose that $n_1 = (\ell - 5)/2$ say. If $a_i = a_1$ for some $i \in \{2, \ldots, k\}$, then $d(1, i) \leq 2$, and so (1) implies that $n_i \geq \ell - n_1 - 2 \geq (\ell + 1)/2$ contrary to (5). Choose a neighbour a_2 say of a_1. Since $d(1, i) \leq 3$ and $n_2 \leq (\ell - 1)/2$, (1) implies that equality holds in both; that is, b_1, b_2 are nonadjacent and $n_2 = (\ell - 1)/2$. Choose a neighbour b_3 of b_1 say. Then similarly a_1, a_3 are nonadjacent and $n_3 = (\ell - 1)/2$. So $a_2 \neq a_3$ and $b_2 \neq b_3$, and so $d(2, 3) \geq 2$, contrary to (1). This proves (6).

In summary, we have now shown that $n_i \in \{(\ell - 3)/2, (\ell - 1)/2\}$ for each $i \in \{1, \ldots, k\}$. Let I be the set of all $i \in I$ with $n_i = (\ell - 3)/2$, and $J = \{1, \ldots, k\} \setminus I$.

(7) *There do not exist* $i \in I$ *and* $j \in J$ *with* $a_i = a_j$.

Suppose that $1 \in I$ and $2 \in J$ and $a_1 = a_2$ say. By (1), $d_B(b_1, b_2) = 2$, so we may assume that b_3 is adjacent to both b_1, b_2. Let Q be an induced path consisting of P_3 and a path of A from a_3 to a_1; then the union of Q, P_1, P_2 and the edges $b_1 b_2, b_2 b_3$ is a theta, a contradiction. This proves (7).

(8) *If* $i, j \in I$ *are distinct then* P_i, P_j *are vertex-disjoint, and* a_i, a_j *are adjacent if and only if* b_i, b_j *are nonadjacent.*

By (1), $d(i, j) = 3$, and both the statements follow. This proves (8).

(9) *If* $i \in I$ *and* $j \in J$ *then* $a_i a_j$ *and* $b_i b_j$ *are edges.*

From (7), $a_i \neq a_j$ and $b_i \neq b_j$; but by (1), $d(i,j) = 2$. This proves (9).

(10) $|J| \geq 2$.

Suppose that J is empty. Since A, B are threshold graphs and hence cographs, and one is isomorphic to the complement of the other by (8), it follows that one of A, B is disconnected, a contradiction. Thus $J \neq \emptyset$. Now suppose that $J = \{1\}$ say. Again, the graphs $A \setminus \{a_1\}$, $B \setminus \{b_1\}$ are complementary threshold graphs, with at least two vertices since $k \geq 3$, and so one of them is disconnected; and so one of A, B has a cut-vertex, contrary to 5.6. This proves (10).

(11) *Either all the vertices a_j $(j \in J)$ are equal or all the vertices b_j $(j \in J)$ are equal.*

Suppose not; then by Kőnig's theorem, there exist distinct $i, j \in J$ with $a_i \neq a_j$ and $b_i \neq b_j$, and so $d(i,j) \geq 2$, contrary to (1). This proves (11).

From (1), (2), (7), (10) and (11), this proves 5.7. ∎

These graphs look like they are getting complicated, but there is a better way to think of them. Remove the vertex a_0 from A, and add to A all the neighbours of a_0 instead, forming A'. All the $A' - B$ paths have the same length and are pairwise vertex-disjoint. With a_0 deleted, this has become what we called an ℓ-frame in the previous section; the graph A' is not connected, but the extra vertex a_0 is adjacent to every vertex in A', and B is two-connected by the first bullet of 5.6.

We can say this more precisely as follows. Let F be an ℓ-frame, with ℓ odd, and with sides A, B and bars $P_1, \ldots, P_t$. The graphs A, B are complementary threshold graphs, and so exactly one of them is disconnected. If A is disconnected, add a new vertex a_0 adjacent to every vertex of A, and with no other neighbours. Let us call this new vertex the *apex*. Let A^+ be the subgraph induced on $V(A) \cup \{a_0\}$, and let $B^+ = B$. If B is disconnected, add a new vertex b_0 adjacent to every vertex of B, and with no other neighbours; and we call this new vertex the *apex*. Let B^+ be the subgraph induced on $V(B) \cup \{b_0\}$, and let $A^+ = A$.

The graph obtained from F by adding the apex is called an *apexed ℓ-frame*, we call $P_1, \ldots, P_t$ its *bars*, and A^+, B^+ its *sides*. From 5.6 and 5.7, it follows that:

5.8 *Let $\ell \geq 5$ be odd, let G be ℓ-holed, and let $(A', B, P'_1, \ldots, P'_k)$ be a k-bar gate in G. Let P'_i have ends a'_i, b_i for $1 \leq i \leq k$, where $a'_i \in V(A')$. Then, possibly after exchanging A', B, there exists $a_0 \in V(A')$ with the following property. For $1 \leq j \leq k$, if $a'_j \neq a_0$ let $a_j = a'_j$ and $P_j = P'_j$, and if $a'_j = a_0$ let a_j be the neighbour of a'_j in P_j, and let $P_j = P'_j \setminus \{a'_j\}$. Let A^+ be the subgraph of G induced on $\{a_0, a_1, \ldots, a_k\}$, and let $B^+ = B$. Then*

$$A' \cup B \cup P'_1 \cup \cdots \cup P'_k = A^+ \cup B^+ \cup P_1 \cup \cdots \cup P_k$$

is an apexed ℓ-frame, with sides A^+, B^+, bars $P_1, \ldots, P_k$ and apex a_0.

For the ℓ even case, we have:

5.9 *Let $\ell \geq 5$ be even, let G be ℓ-holed, and let $(A, B, P_1, \ldots, P_k)$ be a k-bar gate in G. Then either $|A| = |B| = 1$ and $P_1, \ldots, P_k$ all have length $\ell/2$, or there is a partition (I, J, K, L) of $\{1, \ldots, k\}$ into four sets with the following properties:*

- $J, K, L \neq \emptyset$ *(possibly $I = \emptyset$); the vertices a_i ($i \in K$) are all equal, with common value a_0 say; the vertices a_i ($i \in I \cup J \cup L \cup \{0\}$) are all distinct; the vertices b_i ($i \in L$) are all equal, with common value b_0 say; and the vertices b_i ($i \in I \cup J \cup K \cup \{0\}$) are all distinct;*

- *$\{a_i : i \in I \cup L\}$ and $\{b_i : i \in I \cup K\}$ are stable sets;*

- *$\{a_i : i \in J \cup K\}$ and $\{b_i : i \in J \cup L\}$ are cliques;*

- *a_0 is adjacent to a_i for all $i \in I \cup J \cup L$, and b_0 is adjacent to b_i for all $i \in I \cup J \cup K$;*

- *the graph $G[\{a_i : i \in I \cup L\}, \{a_j : j \in J \cup K\}]$ is a half-graph; and for each $i \in I \cup L$ and each $j \in J \cup K$, a_i, a_j are adjacent if and only if b_i, b_j are nonadjacent; and*

- *P_i has length $\ell/2 - 2$ for each $i \in I$, and P_i has length $\ell/2 - 1$ for each $i \in J \cup K \cup L$.*

(See figure 9.)
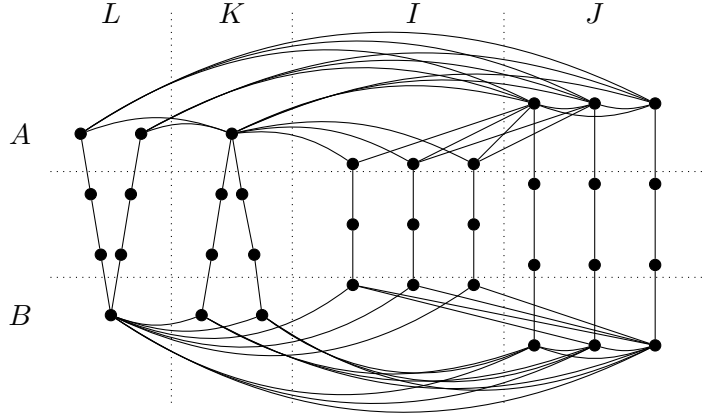
Figure 9: An 8-holed 10-bar gate.

Proof. The proof is very much like that of 5.7, and we use the same notation without redefining it. In particular, any two vertices in A have distance at most two in A, and the same in B. We still have (with the same proof):

(1) $n_i + n_j + d(i, j) = \ell$ for $1 \leq i < j \leq k$.

(2) $n_i \leq \ell/2$ for $1 \leq i \leq k$.

Suppose that $n_1 \geq \ell/2 + 1$ say. For $2 \leq i \leq k$, if $a_i = a_1$ and $b_i = b_1$ then $n_i = \ell/2 - 1$, and so there is at most one such value of i. Since $k \geq 3$ it follows that one of $|A|, |B| > 1$, and we assume that $|A| > 1$ without loss of generality. Since A is therefore 2-connected by 5.6, a_1 has at least two neighbours in A, and so we may assume that a_2, a_3 are both adjacent to a_1. By (1), $n_1 + n_i + d(1, i) = \ell$, and so $n_i \leq \ell/2 - 1 - d(1, i)$ for $i = 2, 3$. But $n_2 + n_3 = \ell - d(2, 3)$ by (1), so $d(1, 2) + d(1, 3) \leq d(2, 3) - 2$, which is impossible by the triangle inequality. This proves (2).

(3) $n_i \geq \ell/2 - 2$ for $1 \leq i \leq k$.

Suppose that $n_1 \leq \ell/2 - 3$ say. For $2 \leq i \leq k$, (1) implies that $n_i \geq \ell/2 + 3 - d(1, i)$, and so by (2), $d(1, i) \geq 3$ for $2 \leq i \leq k$. In particular, $a_i \neq a_1$, and so by 5.6 we may assume that a_2, a_3 are distinct neighbours of a_1. Since $d(1, i) \leq 3$ for $i = 2, 3$ (because a_1, a_i are adjacent and any two vertices in B have distance at most two in B), (1) implies that $n_i = \ell/2$ for $i = 2, 3$; but $n_2 + n_3 < \ell$ by (1) since $a_2 \neq a_3$, a contradiction. This proves (3).

(4) We may assume that $n_i \leq \ell/2 - 1$ for $1 \leq i \leq k$.

Suppose that $n_1 \geq \ell/2$ say, and so $n_1 = \ell/2$ by (2). Suppose first that some vertex $a_2 \in V(A)$ is different from and nonadjacent to a_1. Thus $d(1, 2) \geq 2$; but $n_2 \geq \ell/2 - 2$ by (3), and so equality holds in both by (1). In particular $b_2 = b_1$ and $n_2 = \ell/2 - 2$. Choose a_3 adjacent to both a_1, a_2, and let Q be an induced path from a_3 to b_1 consisting of P_3 and a path of B between b_3, b_1. Then the union of P_1, P_2, Q and the edges $a_1 a_2, a_2 a_3$ is a theta with ends a_3, b_1, and not all its paths have the same length, a contradiction. Thus a_1 is adjacent to every other vertex in A, and the same for b_1 in B.

Suppose that $n_2 = \ell/2 - 2$ say. By (1), $d(1, 2) = 2$, so $a_2 \neq a_1$ and $b_2 \neq b_1$. Since A is 2-connected by 5.6, there exists $a_3 \neq a_1, a_2 \in V(A)$ adjacent to a_2; and also adjacent to a_1 since a_1 is adjacent to all other vertices in A. Thus $n_3 = \ell/2 - d(1, 3)$ by (1), and $n_3 = \ell/2 + 2 - d(2, 3)$ by (1); and hence $d(2, 3) = d(1, 3) + 2$. But $d(2, 3) \leq 3$ and $d(1, 3) \geq 1$, so equality holds in both. Since $d(1, 3) = 1$, it follows that $b_3 = b_1$, and so b_3, b_2 are adjacent, and so $d(2, 3) = 2$, a contradiction. This proves that $n_i \geq \ell/2 - 1$, and hence $d(1, i) \leq 1$, for $2 \leq i \leq k$.

Suppose that $|A| > 1$; then we may choose adjacent a_2, a_3 both different from a_1, by 5.6; and both are adjacent to a_1 since a_1 is adjacent to all the other vertices of A. Since $d(1, i) \leq 1$ for $i = 2, 3$, it follows that $b_2 = b_3 = b_1$; but then the union of P_1, P_2, P_3 and the three edges $a_1 a_2, a_2 a_3, a_3 a_1$ is a pyramid and so G has an odd hole, a contradiction.

This proves that $|A| = 1$ and similarly $|B| = 1$, and so $n_i = \ell/2$ for all $i \in \{1, \ldots, k\}$ and the theorem holds. This proves (4).

So $n_i \in \{\ell/2 - 1, \ell/2 - 2\}$ for $1 \leq i \leq k$. Let I be the set of $i \in \{1, \ldots, k\}$ with $n_i = \ell/2 - 2$. From (4), there do not exist distinct $i, j \in \{1, \ldots, k\}$ with $d(i, j) \leq 1$; and in particular, either $a_i \neq a_j$ or $b_i \neq b_j$ for all distinct i, j.

(5) If $i \in I$ and $j \in \{1, \ldots, k\} \setminus I$ then $a_i \neq a_j$ and $b_i \neq b_j$; and a_i, a_j are adjacent if and only if b_i, b_j are nonadjacent.

By (1), $d(i, j) = 3$; and both claims follow. This proves (5).

(6) If $i, j \in I$ are distinct, then a_i, a_j are distinct and nonadjacent, and b_i, b_j are distinct and nonadjacent. If $i, j \in \{1, \ldots, k\} \setminus I$ are distinct, then either $a_i a_j, b_i b_j$ are both edges, or $a_i = a_j$ and b_i, b_j are distinct and nonadjacent, or $b_i = b_j$ and a_i, a_j are distinct and nonadjacent.

If $i, j \in I$ are distinct, then $n_i + n_j = \ell - 4$, and so (1) implies that $d(i, j) = 4$. If $i, j \in \{1, \ldots, k\} \setminus I$

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are distinct, then $n_i + n_j = \ell - 2$, and so (1) implies that $d(i,j) = 2$. This proves (6).

(7) $|A|, |B| > 1$.

Suppose that $|B| = 1$. Since $n_i \le \ell/2 - 1$ for each i, (1) implies that $d(i,j) \ge 2$ for all distinct $i, j \in \{1, \ldots, k\}$; and since $b_1 = \cdots = b_k$, it follows that $a_1, \ldots, a_k$ are all distinct and nonadjacent, contradicting that A is connected. This proves (7).

If there exist distinct $i, j \in \{1, \ldots, k\}$ with $a_i = a_j$, let $a_0 = a_i$, and let K be the set of all $h \in \{1, \ldots, k\}$ with $a_h = a_0$. If there are no such i, j, since A is a connected threshold graph, it has a vertex (indeed, at least two such vertices, since it is 2-connected) adjacent to all other vertices in A. Choose some such vertex a_i, and let $K = \{i\}$ and define $a_0 = a_i$. Define b_0, L similarly. In the case when no two of the paths $P_1, \ldots, P_k$ share an end, it follows that $|K| = |L| = 1$, and in this case there are two choices for b_0, since at least two vertices in B are adjacent to all the others; so in this case we can additionally choose b_0 such that $K \ne L$.

(8) $n_i = \ell/2 - 1$ for each $i \in K$; a_0 is adjacent to all other vertices in A; and the set $\{b_i : i \in I \cup K\}$ is stable. Similarly b_0 is adjacent to all other vertices in B, and $\{a_i : i \in I \cup L\}$ is stable.

Suppose first that $|K| \ge 2$; then $n_i = \ell/2 - 1$ for each $i \in K$, by (5) and (6). Suppose that a_0 is nonadjacent to some other vertex in A; a_0 is not adjacent to a_1 say, where $a_1 \ne a_0$. We may assume that a_2 is adjacent to both a_0, a_1. Since $|K| \ge 2$, we may assume that $3, 4 \in K$. Since $b_3 \ne b_4$, we may assume that $b_3 \ne b_1$; then by (6), it follows that $1 \in I$. By (5), $b_1 \ne b_4$, and b_1 is adjacent to b_3, b_4. By (6), $2 \notin I$, and b_2 is different from b_3, b_4; by (6) again, b_2 is adjacent to b_3, b_4; and by (5), b_1, b_2 are nonadjacent. Also b_3, b_4 are distinct and nonadjacent by (6), and $\{b_1, b_2, b_3, b_4\}$ induces a 4-hole, a contradiction. So in this case, a_0 is adjacent to all other vertices in A; and $\{b_i : i \in K\}$ is stable by (6), and $\{b_i : i \in I\}$ is stable by (6), and $\{b_i : i \in K\}$ and $\{b_i : i \in I\}$ are anticomplete by (5). So in this case (8) holds.

Now we assume that $|K| = 1$, $K = \{1\}$ say, and so $a_1, \ldots, a_k$ are all distinct. It follows that a_1 is adjacent to all other vertices in A from the definition of a_0. Suppose that $1 \in I$. Then $I = \{1\}$ by (6). Choose b_2 say in B, adjacent to b_1 (this is possible by (7) and since B is connected). Then $a_1 a_2$ and $b_1 b_2$ are both edges, contrary to (5). Thus $1 \notin I$, and so again (8) holds. This proves (8).

(9) $K \cap L = \emptyset$.

Suppose that $1 \in K \cap L$. If $|K| \ge 2$ and $2 \in K$ say, then b_1, b_2 are distinct and nonadjacent by (6), and yet $b_1 = b_0$ is adjacent to all other vertices in B by (8), a contradiction. So $|K| = |L| = 1$; but in this case we were careful to choose b_0, L such that $K \ne L$, a contradiction. This proves (9).

Let $J = \{1, \ldots, k\} \setminus (I \cup K \cup L)$.

(10) $J, K, L \ne \emptyset$.

From their definitions, $K, L \ne \emptyset$. Let $1 \in L$ say. By (8), a_1 has no neighbour in $\{a_i : i \in I \cup L\}$, and has only one in $\{a_i : i \in K\}$ since all these vertices are equal; and since a_1 has degree at least two in

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A (from (7) and 5.6), it has a neighbour in $\{a_i : i \in J\}$. Thus $J \neq \emptyset$. This proves (10).

Since A is a threshold graph by 5.6, it follows that $G[\{a_1 : i \in I \cup L\}, \{a_j : j \in J \cup K\}]$ is a half-graph. From 5.6, (5), (6), (8), (9) and (10), this proves 5.9. ∎

Again, there is a better way to think of these graphs. Let A' be obtained from A by deleting a_0 and adding the neighbours of a_0 in each path P_i $(i \in K)$, and define B' similarly. Then all the $A' - B'$ paths are pairwise vertex-disjoint, and with a_0, b_0 deleted, it becomes an ℓ-frame; and a_0 is adjacent to every vertex in A', and b_0 to every vertex in B'.

More precisely, let $\ell \geq 6$ be even, let F be an ℓ-frame, with sides A, B and bars $P_1, \ldots, P_k$. Add two new vertices a_0, b_0, where a_0 is adjacent to each vertex of A, and b_0 adjacent to each vertex of B. Let A^+ be the subgraph induced on $V(A) \cup \{a_0\}$, and define B^+ similarly. The enlarged graph we produce is called an *apexed ℓ-frame*, the two new vertices are called its *apexes*, $P_1, \ldots, P_k$ are *bars*, and A^+, B^+ are its *sides*. From 5.6 and 5.9 we have:

5.10 *Let $\ell \geq 6$ be even, let G be ℓ-holed, and let $(A', B', P'_1, \ldots, P'_k)$ be a k-bar gate in G. Let P'_i have ends a'_i, b'_i for $1 \leq i \leq k$, where $a'_i \in V(A')$. Then there exist $a_0 \in V(A')$ and $b_0 \in V(B')$ with the following properties. For $1 \leq j \leq k$, if $a'_j \neq a_0$ let $a_j = a'_j$ and otherwise let a_j be the neighbour of a'_j in P_j; and if $b'_j \neq b_0$ let $b_j = b'_j$ and otherwise let b_j be the neighbour of b'_j in P_j; let P_j be the subpath of P'_j between a_j, b_j. Let A^+ be the subgraph induced on a_0 and all its neighbours, and define B^+ similarly. Then*

$$A' \cup B' \cup P'_1 \cup \cdots \cup P'_k = A^+ \cup B^+ \cup P_1, \ldots, P_k$$

is an apexed ℓ-frame, with sides A^+, B^+, bars $P_1, \ldots, P_k$ and apexes a_0, b_0.

We remark that, whether ℓ is odd or even, if F is an apexed ℓ-frame, then in the usual notation, if a_0 exists then A is disconnected, and if a_0 does not exist then A is two-connected; and the same for B, b_0.

6   Attachments to a maximal frame

In this section we prove the following:

6.1 *Let $\ell \geq 7$, and let G be an ℓ-holed graph, containing an apexed ℓ-frame. Choose an apexed ℓ-frame F in G, with as many bars as possible, with sides A^+, B^+ and bars $P_1, \ldots, P_k$. Let a_0 be the vertex of A^+ not in $P_1 \cup \cdots \cup P_k$, if there is one, and define b_0 similarly. Then either*

- *there exist $x \in V(G) \setminus V(F)$ and $u, v \in V(F)$, such that u, v are nonadjacent, both are adjacent to x, and not both are in $V(A^+)$, and not both are in $V(B^+)$; or*

- *there exists $x \in V(G) \setminus V(F)$ such that either a_0 exists and the set of neighbours of x in $V(F)$ is $V(A^+)$, or b_0 exists and the set of neighbours of x in $V(F)$ is $V(B^+)$; or*

- *for every component C of $G \setminus V(F)$, either $N(C) \subseteq V(A^+)$, or $N(C) \subseteq V(B^+)$, or $N(C)$ is a clique, where $N(C)$ denotes the set of vertices in $V(F)$ with a neighbour in $V(C)$.*

Proof. Let $A = A^+ \setminus \{a_0\}$ if a_0 exists, and let $A^+ = A$ otherwise; and define B similarly. We observe first that:

(1) *If $X \subseteq V(F)$, such that X is not a clique and $X \not\subseteq V(A^+)$ and $X \not\subseteq V(B^+)$, then there exist nonadjacent $u, v \in X$, not both in $V(A^+)$ and not both in $V(B^+)$.*

Let $u, v \in X$ be nonadjacent. If u or v belongs to $V(F) \setminus (V(A^+) \cup V(B^+))$, then u, v satisfy (1); so we assume that $u, v \in V(A^+) \cup V(B^+)$. If $u \in V(A^+)$ and $v \in V(B^+)$, again (1) holds; so we may assume that $u, v \in V(A^+)$. Choose $x \in X \setminus V(A^+)$. It follows that x is nonadjacent to one of u, v, say v; but then x, v satisfy (1). This proves (1).

We may assume that the first bullet is false, and so, from (1):

(2) *For each $x \in V(G) \setminus V(F)$, the set of vertices of F adjacent to x is either a clique, or a subset of $V(A^+)$, or a subset of $V(B^+)$.*

We may also assume that the third bullet is false, and so by (1), there is a component C of $G \setminus V(F)$ and nonadjacent $u, v \in N(C)$, such that $\{u, v\} \not\subseteq V(A^+)$ and $\{u, v\} \not\subseteq V(B^+)$. Consequently there is a minimal connected induced subgraph S of $G \setminus V(F)$, such that there are two nonadjacent vertices in $V(F)$ both with neighbours in $V(S)$, and not both in $V(A^+)$ and not both in $V(B^+)$. From the minimality of S, it follows that S is an induced path $s_1\text{-}\cdots\text{-}s_n$ say. Let $N(S)$ denote the set of vertices of F with a neighbour in $V(S)$. Thus $N(S)$ is not a clique, and not a subset of $V(A^+)$, and not a subset of $V(B^+)$. By (2), $n \geq 2$.

(3) *One of s_1, s_n has a neighbour in $V(F) \setminus (V(A^+) \cup V(B^+))$.*

Suppose that all neighbours of s_1 in $V(F)$ and all neighbours of s_n in $V(F)$ belong to $V(A^+) \cup V(B^+)$. Since neither of s_1, s_n has a neighbour in $V(A^+)$ and one in $V(B^+)$, we may assume that all neighbours of s_1 in $V(F)$ belong to $V(A^+)$, and all neighbours of s_n in $V(F)$ belong to $V(B^+)$.

Suppose first that some vertex $w \in V(F)$ is adjacent to an internal vertex of S. From the minimality of S, some P_i, say P_1, has length two, and w is its middle vertex, and a_1 is the only neighbour of s_1 in $V(F)$, and b_1 is the only neighbour of s_n in $V(F)$. There is an induced path R between a_1, b_1 consisting of a one- or two-edge path of A^+, the path P_2, and a one- or two-edge path of B^+; and the union of R and the path $a_1\text{-}s_1\text{-}s_2\text{-}\cdots\text{-}s_n\text{-}b_1$ is a hole. Also the union of R, P_1 is a hole; so the paths $a_1\text{-}s_1\text{-}s_2\text{-}\cdots\text{-}s_n\text{-}b_1$ and P_1 have the same length, and hence $n = 1$, a contradiction.

Thus no internal vertex of S has a neighbour in $V(F)$. Let A' be the subgraph induced on $V(A^+) \cup \{s_1\}$, and define B' similarly; then

$$(A', B', S, P_1, \ldots, P_k)$$

is a $(k+1)$-bar semigate, and so G contains a $(k+1)$-bar gate, and hence contains an apexed ℓ-frame with $k+1$ bars by 5.8 and 5.10, contrary to the maximality of k. This proves (3).

From (3) we may assume that some neighbour of s_1 is an internal vertex of P_1 say. By (2), all neighbours of s_1 in $V(F)$ belong to $V(P_1)$, and either there is just one such neighbour, or there are two and they are adjacent. From the minimality of S, every vertex in $V(F)$ with a neighbour in

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$V(S) \setminus \{s_n\}$ belongs to $V(P_1)$. Let P_1 have vertices $q_1\text{-}\cdots\text{-}q_m$ in order, where $q_1 = a_1$ and $q_m = b_1$, and $\ell/2 - 1 \leq m \leq \ell/2$ (because P_1 has length between $\ell/2 - 2$ and $\ell/2 - 1$).

(4) *Some neighbour of s_n in $V(F)$ does not belong to $V(P_1)$.*

Suppose that all neighbours of s_n in $V(F)$ belong to $V(P_1)$. Thus $N(S) \subseteq V(P_1)$. By (2), s_n has at most two neighbours in $V(F)$ and they form a clique. Choose $g, h \in \{1, \ldots, m\}$ minimum and maximum respectively such that $q_g, q_h \in N(S)$. Thus $h \geq g + 2$, since $N(S)$ is not a clique. From the minimality of S, no internal vertex of S is adjacent to q_g or to q_h, and so we may assume that s_1 is adjacent to q_g and s_n to q_h. There is a hole C containing $P_1 \cup P_2$; and by replacing the path $q_g\text{-}\cdots\text{-}q_h$ of this hole by the path $q_g\text{-}s_1\text{-}\cdots\text{-}s_n\text{-}q_h$, we obtain another hole, which therefore has the same length. Thus $h - g = n + 1$. Since $n \geq 2$, it follows that $h - g \geq 3$. From the minimality of S, and because $h - g \geq 3$, it follows that none of $s_2, \ldots, s_{n-1}$ have a neighbour in $V(F)$. If s_1 is adjacent to q_{g+1} let $g' = g + 1$, and otherwise let $g' = g$; and if s_n is adjacent to q_{h-1} let $h' = h - 1$, and otherwise let $h' = h$. Thus $h' > g'$. There is a hole

$$s_1\text{-}\cdots\text{-}s_n\text{-}q_{h'}\text{-}q_{h'-1}\text{-}\cdots\text{-}q_{g'}\text{-}s_1,$$

and its length is $n + 1 + h' - g'$; and so $n + 1 + h' - g' = \ell$. But $h - g = n + 1$, and so $h - g + h' - g' = \ell$, and therefore $h - g \geq \ell/2$. But $h - g$ is at most the length of P_1, which is at most $\ell/2 - 1$, a contradiction. This proves (4).

(5) *No neighbour of s_n is an internal vertex of any of $P_1, \ldots, P_k$.*

By (4) and (2), no neighbour of s_n is an internal vertex of P_1. Suppose that s_n is adjacent to some internal vertex of P_2, say. By (2), all neighbours of s_2 in $V(F)$ belong to $V(P_2)$ and form a clique; and from the minimality of S, none of $s_2, \ldots, s_{n-1}$ has any neighbour in $V(F)$. Let P_2 have vertices $r_1\text{-}\cdots\text{-}r_s$ in order, where $r_1 = a_2$, and $\ell/2 - 1 \leq s \leq \ell/2$. Choose $g \in \{1, \ldots, m\}$ minimum such that s_1, q_g are adjacent, and if s_1 is adjacent to q_{g+1} let $g' = g + 1$, and otherwise let $g' = g$. Similarly choose $h \in \{1, \ldots, s\}$ minimum such that s_n, r_h are adjacent, and if s_n is adjacent to r_{h+1} let $h' = h + 1$, and otherwise let $h' = h$. Since A^+ is a connected threshold graph, there is an induced path T of A^+ of length at most two between a_1, a_2. Since $n \geq 2$, the cycle

$$q_1\text{-}\cdots\text{-}q_g\text{-}s_1\text{-}\cdots\text{-}s_n\text{-}r_h\text{-}r_{h-1}\text{-}r_1\text{-}T\text{-}q_1$$

is a hole, so the path

$$q_1\text{-}\cdots\text{-}q_g\text{-}s_1\text{-}\cdots\text{-}s_n\text{-}r_h\text{-}r_{h-1}\text{-}\cdots\text{-}r_1$$

has length at least $\ell - 2$. Similarly the path

$$q_m\text{-}\cdots\text{-}q_{g'}\text{-}s_1\text{-}\cdots\text{-}s_n\text{-}r_{h'}\text{-}\cdots\text{-}r_s$$

has length at least $\ell - 2$. Consequently one of the paths

$$q_1\text{-}\cdots\text{-}q_g\text{-}s_1\text{-}\cdots\text{-}s_n\text{-}r_{h'}\text{-}\cdots\text{-}r_s$$

$$q_m\text{-}\cdots\text{-}q_{g'}\text{-}s_1\text{-}\cdots\text{-}s_n\text{-}r_h\text{-}\cdots\text{-}r_1$$

has length at least $\ell - 2$. But each of these can be completed to a hole by adding P_3 and at least two further edges (note that, since either A is 2-connected or a_0 exists, there is a path of A^+ between a_1, a_3 that does not pass through a_2, and similarly for the other three paths we need), and this hole has length at least $(\ell - 2) + 2 + |E(P_3)| > \ell$, a contradiction. This proves (5).

From (5), and the symmetry between A, B, we may assume that every neighbour of s_n in $V(F)$ belongs to $V(A^+)$. Let I be the set of $i \in \{0, \ldots, k\}$ such that s_n, a_i are adjacent (thus $0 \in I$ only if a_0 exists). By (4), $I \nsubseteq \{1\}$. Let J be the set of $i \in \{2, \ldots, k\}$ such that a_1, a_i are nonadjacent. Let $g', g \in \{1, \ldots, m\}$ be minimum and maximum respectively such that s_1 is adjacent to $q_{g'}, q_g$. Since s_1 has a neighbour in the interior of P_1 it follows that $g \geq 2$.

(6) *No vertex of $V(F)$ has a neighbour in the interior of S except possibly a_1, and a_1 is adjacent to at least one of $s_1, \ldots, s_{n-1}$, and $g = 2$.*

Suppose that some internal vertex of S has a neighbour $v \in V(F)$. From the minimality of S, v is adjacent or equal to every neighbour of s_1 in $V(F)$, and so $v \in V(P_1)$; and v is adjacent or equal to every neighbour of s_n in $V(F)$, and so $v = a_1$ since $I \nsubseteq \{1\}$. This proves the first statement.

Suppose that a_1 has no neighbour in $V(S) \setminus \{s_n\}$, and in particular $g' \geq 2$. Let A' be the subgraph induced on $V(A^+) \cup \{s_n\}$, let B' be the subgraph induced on $V(B^+) \cup \{s_1, q_{g'}, q_g, \ldots, q_m\}$, and let P_1' be the path $q_1 \cdot \cdots \cdot q_{g'}$. Then $(A', B', S, P_1', P_2, \ldots, P_k)$ is a $(k+1)$-bar semigate, a contradiction. Hence a_1 has a neighbour in $V(S) \setminus \{s_n\}$; and by the minimality of S it follows that a_1 is adjacent to every neighbour of s_1 in $V(F)$, and consequently $g = 2$. This proves (6).

(7) $n = 2$, *and* $I \cap \{2, \ldots, k\} \subseteq J$. *If* $0 \in I$ *then* $I \cap \{2, \ldots, k\} = J$.

First we show that $I \cap \{2, \ldots, k\} \subseteq J$, and if $I \cap \{2, \ldots, k\} \neq \emptyset$ then $n = 2$. Both are true if $I \cap \{2, \ldots, k\} = \emptyset$, so we may assume that $i \in I \cap \{2, \ldots, k\}$. There is an induced path R between q_2 and a_i consisting of $q_2 \cdot \cdots \cdot q_m$, a one- or two-edge path of B^+, and the path P_i. This can be completed to a hole via $q_2 \cdot s_1 \cdot \cdots \cdot s_n \cdot a_i$, and also via a path of length at most three between q_2, a_i with interior consisting of a_1 and possibly one other vertex of A^+. Consequently both these completions have the same length, and so $n = 2$ and $i \in J$ as required.

It remains to show that if $I \cap \{2, \ldots, k\} = \emptyset$ then $n = 2$, and if $0 \in I$ then $I \cap \{2, \ldots, k\} = J$. But if $I \cap \{2, \ldots, k\} = \emptyset$ then a_0 exists and $0 \in I$, since $I \nsubseteq \{1\}$. Consequently, for both statements we may assume that a_0 exists and $0 \in I$, and under that assumption we must show that $I \cap \{2, \ldots, k\} = J$ and $n = 2$. Since a_0 exists, A is not connected and so there exists $i \in J$. Choose $i \in J \setminus I$ if possible. If $i \in I$, then $I \cap \{2, \ldots, k\} = J$ from the choice of i, and $n = 2$ from what we already proved, since $I \cap \{2, \ldots, k\} \neq \emptyset$. Thus we may assume that $i \notin I$. There is a path R between q_2, a_0 consisting of $q_2 \cdot \cdots \cdot q_m$, a one- or two-edge path of B^+, the path P_i, and the edge $a_i a_0$. Consequently the paths $q_2 \cdot s_1 \cdot \cdots \cdot s_n \cdot a_0$ and $q_2 \cdot q_1 \cdot a_0$ have the same length, a contradiction since $n \geq 2$. This proves (7).

Suppose that $0 \notin I$. Then $I \cap J \neq \emptyset$ by (7) since $I \nsubseteq \{1\}$; choose $j \in I \cap J$. Moreover, A^+ is a connected threshold graph, and so there exists $i \in \{0, \ldots, k\}$ such that a_i is adjacent to all other vertices of A^+. Since $J \neq \emptyset$, it follows that $i \neq 1$; and $i \notin J$ since a_1, a_i are adjacent. So $i \in \{0, 2, \ldots, k\} \setminus J$, and therefore $i \notin I$ by (7). By (6), a_1 is adjacent to one of s_1, s_2; so one of $a_1 \cdot a_i \cdot a_j \cdot s_2 \cdot a_1$, $a_1 \cdot a_i \cdot a_j \cdot s_2 \cdot s_1 \cdot a_1$ is a hole of length four or five, a contradiction.

This proves that a_0 exists and $0 \in I$. By (7), $I \setminus \{1\} \subseteq J \cup \{0\} \subseteq I$. Suppose that there exists $i \in \{2, \ldots, k\} \setminus J$. There is a path R between q_2, a_i consisting of $P_1 \setminus \{a_1\}$, P_i and a one- or two-edge path of B^+. Moreover, a_i is nonadjacent to s_1, s_2, and there is a induced path q_2-s_1-s_2-a_0-a_i of length four, and another q_2-a_1-a_i of length two, and for both these paths, their union with R is a hole, a contradiction. This proves that $J = \{2, \ldots, k\}$. If $1 \notin I$, there is a 4-hole or 5-hole with vertex set a subset of $\{a_1, q_2, s_1, s_2, a_0\}$, a contradiction; so $I = \{0, \ldots, k\}$, and the second bullet of the theorem holds. This proves 6.1. ∎

7   Blowing up a frame

To prove 1.3 we need to look at blow-ups of apexed ℓ-frames. We will handle the ℓ odd and ℓ even cases together, so the next result lists the properties of apexed ℓ-frames that we will use; they hold whether ℓ is odd or even.

7.1 *Let k, ℓ be integers with $k \geq 3$ and $\ell \geq 7$. Let F be an apexed ℓ-frame with sides A^+, B^+ and bars $P_1, \ldots, P_k$, and apexes a_0, b_0 (if they exist). For $1 \leq i \leq k$ let P_i have ends $a_i \in V(A^+)$ and $b_i \in V(B^+)$. Let $A = A^+ \setminus \{a_0\}$ if a_0 exists, and $A = A^+$ otherwise, and define B similarly. Then:*

- *$V(A^+), V(B^+)$ are anticomplete, and A^+, B^+ are connected;*

- *A, B are threshold graphs with $|A| = |B| = k$, such that either A is 2-connected and a_0 does not exist, or A is not connected and a_0 exists; and the same for B.*

- *$V(A) = \{a_1, \ldots, a_k\}$, and $V(B) = \{b_1, \ldots, b_k\}$.*

- *For $1 \leq i \leq k$, P_i has length at least $\ell/2 - 2$ and at most $\ell/2 - 1$, with no internal vertex in $V(A \cup B)$, and such that $P_1, \ldots, P_k$ are pairwise vertex-disjoint.*

- *F is ℓ-holed.*

Apexed ℓ-frames have the following convenient property:

7.2 *If F is an apexed ℓ-frame and $u, v \in V(F)$, some hole of F contains both u and v.*

Proof. We use the notation of 7.1. For $1 \leq i < j \leq k$, there is a hole of F containing $P_i \cup P_j$ (using a_0 if A is not connected, and b_0 if B is not connected); so we may assume that $u = a_0$ say. Hence A is not connected; let $I \subseteq \{1, \ldots, k\}$ such that $I \neq \emptyset, \{1, \ldots, k\}$, and there are no edges between $\{a_i : i \in I\}$ and $\{a_i : i \in \{1, \ldots, k\} \setminus I\}$. If $v \in V(P_i)$ for some $i \in \{1, \ldots, k\}$, choose $j \in \{1, \ldots, k\}$ with exactly one of i, j in I; then there is a hole containing P_i, P_j and a_0, as required. So we may assume that also $v = b_0$; and so B is not connected. Consequently the set $\{b_i : i \in I\}$ is not complete to $\{b_j : j \in \{1, \ldots, k\} \setminus I\}$; choose $i \in I$ and $j \in \{1, \ldots, k\} \setminus I$ with b_i, b_j nonadjacent. Since also a_i, a_j are nonadjacent, there is a hole contain a_0, b_0, P_i and P_j. This proves 7.2. ∎

Let us say two holes C, C' in an ℓ-holed graph G are *close* if $|V(C \cap C')| \geq 4$; and *equivalent* if there is a sequence of holes
$$C = C_1, \ldots, C_n = C'$$
such that C_i, C_{i+1} are close for $1 \leq i < n$. We need:

7.3 *Let $\ell \geq 7$, and let F be an apexed ℓ-frame. Then every two holes in F are equivalent.*

Proof. We use the notation of 7.1. Since A^+, B^+ are threshold graphs, every hole includes one (and hence two) of $P_1, \ldots, P_k$. Each of $P_1, \ldots, P_k$ has length at least $\ell/2 - 2 \geq 3/2$, and hence at least two.

(1) *Let C_1, C_2 be holes of F with some $P_i \subseteq C_1 \cap C_2$. Then C_1, C_2 are equivalent.*

We may assume that $i = 3$. Suppose that C_1, C_2 are not equivalent. In particular, they are not close, so P_3 has length two, and $C_1 \cap C_2 = P_3$. We may assume that $P_1 \subseteq C_1$ and $P_2 \subseteq C_2$. Let P_i have ends $a_i \in V(A)$ and $b_i \in V(B)$ for $i = 1, 2, 3$.

Suppose that a_1, a_3 are nonadjacent. If a_2 is adjacent to both a_1, a_3, then there is a hole close to C_1 containing P_1, P_3 and a_2, and this hole is also close to C_2, a contradiction. Thus a_2 is nonadjacent to at least one of a_1, a_3. Since A^+ is a connected threshold graph, it has a vertex a adjacent to all the others; and $a \neq a_1, a_2, a_3$. There is a hole close to C_1 containing P_1, P_3 and a, and so we may assume that $a \in V(C_1)$. If a_2, a_3 are nonadjacent, there is a hole containing P_2, P_3, a, and this hole is close to both C_1, C_2, a contradiction. Thus a_2, a_3 are adjacent, and so a_1, a_2 are nonadjacent (because a_2 is nonadjacent to at least one of a_1, a_3). There is a hole containing P_1, P_2 and a, and it is close to C_1, so it is not close to C_2, and therefore P_2 has length two; and so C_2 has length at most 7, and therefore $\ell = 7$. We have shown then that if a_1, a_3 are nonadjacent, then a_2, a_3 are adjacent, and a_1, a_2 are nonadjacent, and P_2 has length two, and $\ell = 7$.

Let us continue to assume that a_1, a_3 are nonadjacent. Since P_2, P_3 both have length two and a_2, a_3 are adjacent, and $\ell = 7$, it follows that b_2, b_3 are nonadjacent. By the argument of the paragraph above, with A, B exchanged and P_1, P_2 exchanged, it follows that b_1, b_2 are nonadjacent. But then a hole containing P_1, P_2 has length at least eight, a contradiction.

This proves that a_1, a_3 are adjacent, and similarly $a_2 a_3, b_1 b_3, b_2 b_3$ are edges. Consequently P_1, P_2 both have length at least three, since C_1, C_2 have length $\ell \geq 7$; but there is a hole containing P_1, P_2, and it is close to both C_1, C_2, since P_1, P_2 have length at least three, a contradiction. This proves (1).

For each pair of distinct $i, j \in \{1, \ldots, k\}$ let $C_{i,j}$ be a hole containing $P_i \cup P_j$. By repeated application of (1) it follows that all these holes are equivalent; but every hole contains some pair P_i, P_j and so is equivalent to $C_{i,j}$. This proves 7.3. ∎

Let F be an apexed ℓ-frame, and let A^+, B^+ and so on be as in 7.1. For each $t \in V(F)$, let $N_F(t)$ denote the set of vertices in F that are adjacent to t in F, and $N_F[t] = N_F(t) \cup \{t\}$. Let G be a graph, and for each $t \in V(F)$, let W_t be an ordered non-null clique of G. Suppose that for all distinct $s, t \in V(F)$:

- $W_s \cap W_t = \emptyset$;

- if s, t are not F-adjacent then W_s, W_t are anticomplete in G;

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- if s, t are F-adjacent then $G[W_s, W_t]$ is an ordered half-graph, and each of its vertices has positive degree; and

- if s, t both have degree at least three in F, and s, t are F-adjacent, then W_s is complete to W_t in G.

Thus, if s, t are F-adjacent, then W_s is complete to W_t unless either st is an edge of some P_i, or one of s, t equals a_0 and the other is a vertex of A with degree zero in A, or the same in B. We say that the subgraph of G induced on the union of the sets W_t ($t \in V(F)$) is a *blow-up* of F, with *bags* W_t ($t \in V(F)$). We will choose a maximal blow-up of F in G and analyze how the remainder of G is attached to this blow-up.

There is a convenient notational simplification, as follows. Given W_t ($t \in V(F)$) as above, for each $t \in V(F)$ let w_t be the first term of the ordering of W_t. Then the map that sends t to w_t for each $t \in V(F)$ is an isomorphism from F onto an induced subgraph F' of G; and it is convenient to index the bags not by vertices of F but by the corresponding vertex of F'. Thus, we say that the subgraph of G induced on the union of the sets W_t ($t \in V(F)$) is a *self-centred blow-up* of F, if it is a blow-up of F, and in addition:

- F is an induced subgraph of G; and

- $t \in W_t$, and t is the first term of the ordering of W_t, for each $t \in V(F)$.

Whenever we have a blow-up of some apexed ℓ-frame F, it is also a self-centred blow-up of some graph F' isomorphic to F, and working with a self-centred blow-up is often more convenient than working with a general one.

We begin with:

7.4 *Let $k \geq 3$ and $\ell \geq 7$ be integers. Let F be an apexed ℓ-frame, let G be an ℓ-holed graph, and let H be a blow-up of F that is contained in G. Then every two holes of H are equivalent, and every vertex of H is in a hole of H.*

Proof. We use the notation of 7.1. By replacing F by an isomorphic graph, we may assume that the blow-up is self-centred. Let W_t ($t \in V(F)$) be the bags of H.

(1) *If C is a hole of H then $|W_t \cap V(C)| \leq 1$ for each $t \in V(F)$.*

Suppose that $|W_t \cap V(C)| \geq 2$ for some $t \in V(F)$. Since W_t is a clique, there are exactly two vertices of C in W_t, say u, v; let u'-u-v-v' be a path of C. Let $u' \in W_s$ and $v' \in W_r$ say; then the edges uu', vv' are an induced two-edge matching in $G[W_t, W_r \cup W_s]$, contradicting that $G[W_t, W_r]$ and $G[W_t, W_s]$ are ordered half-graphs. This proves (1).

(2) *If C is a hole of H then $\{t : W_t \cap V(C) \neq \emptyset\}$ induces a hole of F.*

Let C have vertices v_1-$\cdots$-v_ℓ-v_1 in order, and let $v_i \in W_{t_i}$ for $1 \leq i \leq \ell$. By (1) $t_1, \ldots, t_\ell$ are all distinct, and t_1-t_2-$\cdots$-t_ℓ-t_1 are the vertices in order of a cycle D of F. Suppose that it is not induced, and t_1 is adjacent in F to t_i say, where $3 \leq i \leq \ell - 1$. Hence $G[W_{t_1}, W_{t_i}]$ is not complete

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bipartite, since v_1, v_i are nonadjacent in H; but t_1, t_i both have degree at least three in F (because they have degree two in D), a contradiction. This proves (2).

Let us call D as in (2) the *shadow* of C.

(3) *If C is a hole of H then it is equivalent to its shadow.*

Let D be the shadow of C. To show that C, D are equivalent we use induction on the number of vertices of C not in $V(D)$. Let v be such a vertex, with $v \in W_t$ say; then C is close to the hole C' obtained from C by replacing v by t, since $\ell \geq 5$; and C', D are equivalent from the inductive hypothesis. This proves (3).

From (3) and 7.3, we deduce that all holes in H are equivalent in H. This proves the first assertion of the theorem.

For the second, let $v \in V(H)$, with $v \in W_t$ say; we will show that v belongs to a hole of H. From 7.2, t belongs to some hole D of F. Let C' be obtained from C by replacing t by v; then C' is a hole of H containing v. This proves 7.4. ∎

7.5 *Let $k \geq 3$ and $\ell \geq 7$ be integers, and let F be an apexed ℓ-frame. Let G be an ℓ-holed graph, and let H be a maximal blow-up of F contained in G. Let the bags of H be W_t $(t \in V(F))$. Then for every vertex $x \in V(G) \setminus V(H)$, either:*

- *the set of neighbours of x in $V(H)$ is a clique; or*

- *x is adjacent to every vertex of H; or*

- *the set of neighbours of x in $V(H)$ is a subset of one of $W(A^+), W(B^+)$, in the usual notation.*

Proof. By replacing F by an isomorphic graph, we may assume that the blow-up is self-centred. We use the notation of 7.1. Let X be the set of neighbours of x in $V(H)$, and let M be the set of $t \in V(F)$ with $X \cap W_t \neq \emptyset$.

(1) *We may assume that for all $s, t \in M$, if s, t are nonadjacent then they have a common neighbour in M.*

Suppose that there exist $s, t \in M$ with distance at least three in F. By 7.2 there is a hole D of F containing s, t. Choose $x_s \in X \cap W_s$ and $x_t \in X \cap W_t$, and let C be the hole of G induced on $\{x_s, x_t\} \cup (V(D) \setminus \{s, t\})$. Thus C has length ℓ, and x_s, x_t have distance at least three in C. Since G is ℓ-holed it follows that $V(C) \subseteq X$. But if two holes of H are close and x is adjacent to all vertices of one, then it has at least four neighbours in the other and so is adjacent to all vertices of the other. By 7.4 it follows that x is adjacent to every vertex of H that is in a hole. But every vertex of H is in a hole of H, by 7.2, and so the second bullet of the theorem holds. This proves that every two vertices in M have distance at most two in F.

Suppose that $s, t \in M$ have distance two in F; we claim they have a common neighbour in M. Let $r \in V(F)$ be adjacent to them both. Choose $x_s \in X \cap W_s$ and $x_t \in X \cap W_t$. Since $\{x, x_s, x_t, r\}$

does not induce a 4-hole, it follows that $r \in X$, and so $r \in M$. This proves (1).

(2) *There exists $s \in V(F)$ such that $M \subseteq N_F[s]$.*

Choose $s \in M$ adjacent to as many members of M as possible. We may assume that there exists $s_1 \in M$ different from and nonadjacent to s. By (1) there exists $t_1 \in M$ adjacent to both s_1, s. Since t_1 is not a better choice than s, there exists $t_2 \in M$ adjacent to s and not to t_1. Since s-t_1-s_1-t_2 is not a 4-hole of F it follows that s_1, t_2 are nonadjacent. Hence s_1-t_1-s-t_2 is an induced path P. By (1), there exists $s_2 \in M$ adjacent to both s_1, t_2. Hence $s_2 \neq s, t_1$ since each of the latter is nonadjacent to one of s_1, s_2, and s_2 is adjacent to both s, t_1 since F has no hole of length four or five; so s_2 is adjacent to every vertex of the induced path s_1-t_1-s-t_2. But apexed ℓ-frames have no four-vertex induced path in the neighbourhood of a vertex, a contradiction. This proves (2).

(3) *We may assume that M is not a clique.*

Suppose that M is a clique; and we may assume that there exist nonadjacent $u, v \in X$, since otherwise the first bullet of the theorem holds. Let $u \in W_s$ and $v \in W_t$ say; thus $s, t \in M$, and so they are adjacent. Let C be a hole of F containing s, t (this exists by 7.2). None of its vertices different from s, t belong to M since M is a clique. There is an induced path of H between u, v, with internal vertices the vertices of $C \setminus \{s, t\}$; and none of its internal vertices are adjacent to x, since the corresponding vertices of C are not in M. Adding x to this path gives a hole of length $\ell + 1$, a contradiction. This proves (3).

(4) *If r-s-t is an induced path of F with $r, t \in M$, then $s \in M$ and $W_s \subseteq X$.*

Suppose that $W_s \nsubseteq X$, and choose $y \in W_s \setminus X$. Both x, y have a neighbour in the clique W_r, so there is an induced path between them with length two or three and with interior in W_r; and similarly there is such a path with interior in W_t. But the union of these paths makes a hole of length four, five or six, a contradiction. This proves (4).

Choose s as in (2). If s is an apex of F, then by (2), the third bullet of the theorem holds, so we may assume that $s \in V(P_1)$ say.

(5) $s \in \{a_1, b_1\}$.

Suppose that s is an internal vertex of P_1, and let r, t be its neighbours in P_1. Thus $X \subseteq W_r \cup W_s \cup W_t$. By (3), $X \cap W_r \neq \emptyset$, and $X \cap W_t \neq \emptyset$. By (4), $W_s \subseteq X$. Now $G[W_r, W_s \cup \{x\}]$ is a half-graph with positive minimum degree, since G has no 4-hole; and similarly so is $G[W_t, W_s \cup \{x\}]$. We claim that $G[W_s \cup \{x\}, W_r]$ and $G[W_s \cup \{x\}, W_t]$ are compatible; because if not there is a four-vertex path a-b-x-c where $a \in W_r, b \in W_s$ and $c \in W_t$. This is impossible because there is a hole D of F containing P_1 and hence containing r, s, t; and the subgraph induced on

$$\{a, b, c, x\} \cup (V(D) \setminus \{r, s, t\})$$

is a hole of G of length $\ell + 1$. So $G[W_s \cup \{x\}, W_r]$ and $G[W_s \cup \{x\}, W_t]$ are compatible.

Let $q \in V(F)$ be adjacent to r, with $q \neq s$; we claim that $G[W_r, W_s \cup \{x\}]$ and $G[W_r, W_q]$ are compatible. If not then there is an induced path x-a-b-c where $a, b \in W_r$ and $c \in W_q$; but then, let D be a hole of F containing q, r, s, and then the subgraph induced on

$$\{a, b, c, x\} \cup (V(D) \setminus \{q, r, s\})$$

is a hole of G of length $\ell + 1$. Similarly $G[W_t, W_s \cup \{x\}]$ and $G[W_t, W_q]$ are compatible for every $q \neq s$ adjacent to t in F. Consequently we can add x to W_s and obtain a blow-up of F larger than H, a contradiction. (Note that this blow-up might not be self-centred.) This proves (5).

From (5) and the symmetry, we may assume that $s = a_1$. Let p be the neighbour of a_1 in P_1. If $X \cap W_p = \emptyset$ then the third bullet of the theorem holds, so we assume that $X \cap W_p \neq \emptyset$. Let $u \in X \cap W_p$.

(6) $W_{a_1} \subseteq X$.

By (3), $X \not\subseteq W_s \cup W_p$, and so at least one vertex of $A^+ \setminus \{a_1\}$ belongs to M, say a_i. Consequently s, a_i are adjacent in F, since $M \subseteq N_F[s]$. But then $W_s \subseteq X$ by (4) since p-s-a_i is an induced path of F. This proves (6).

(7) a_0 *does not exist.*

Suppose that a_0 exists. Suppose first that a_1 is nonadjacent in F to all of $a_2, \ldots, a_k$. Thus $a_0 \in M$, since M is not a clique. We claim that adding x to W_{a_1} gives a larger blow-up of F, and to show this we must just check that

- $H[W_p, W_{a_1} \cup \{x\}]$ and $H[W_{a_1} \cup \{x\}, W_{a_0}]$ are half-graphs (this is true, because for each the two parts of its bipartition are cliques of G);

- $H[W_p, W_{a_1} \cup \{x\}]$ is compatible with $H[W_p, W_q]$, where q is the neighbour of p in P_1 different from a_1 (this is true since otherwise there is an induced path x-b-c-d with $b, c \in W_p$ and $d \in W_q$, which could be extended to a hole of length $\ell + 1$ of G); and

- $H[W_{a_1} \cup \{x\}, W_p]$ is compatible with $H[W_{a_1} \cup \{x\}, W_{a_0}]$ (this is true since otherwise there is an induced path a-b-c-d with $a \in W_p, b, c \in W_{a_1} \cup \{x\}$ and $d \in W_{a_0}$, which could be extended to a hole of length $\ell + 1$ in G).

This contradicts the maximality of H.

Consequently a_1 is adjacent to at least one of $a_2, \ldots, a_k$, and so has degree at least three in F; and so W_{a_1} is complete to W_{a_i} for each $i \in \{0, 2, 3, \ldots, k\}$ such that a_1, a_i are adjacent in F.

Suppose that $a_0 \notin M$; then we may assume that $a_2 \in M$, and so a_1, a_2 are adjacent. Choose $v_2 \in W_{a_2} \cap X$. Since a_0 exists, A is disconnected, so we may assume that a_3 is nonadjacent to a_1, a_2, and hence $a_3 \notin M$. Then there are induced paths u-a_1-a_0-a_3 and u-x-v_2-a_0-a_3, and they can both be extended to holes by taking their union with the same path, but their lengths are different, a contradiction.

So $a_0 \in M$. Choose $v_0 \in W_{a_0} \cap X$. We claim that $W_{a_i} \subseteq X$ for each $i \in \{2, \ldots, k\}$ such that a_1, a_i are adjacent in F. Suppose not; then we may assume that a_1, a_2 are adjacent, and $v_2 \in W_{a_2}$

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is not adjacent to x. There are paths u-a_1-v_2 and u-x-v_0-v_2, giving a contradiction as usual. Thus $W_{a_i} \subseteq X$ for each $i \in \{2, \ldots, k\}$ such that a_1, a_i are adjacent in F.

We claim that $W_{a_0} \subseteq X$. Suppose not, and let $w_0 \in W_{a_0} \setminus X$. Since a_1 is adjacent to at least one of $a_2, \ldots, a_k$, we may assume that a_1, a_2 are adjacent, and so $W_{a_2} \subseteq X$. As before, since a_0 exists, A is disconnected, so we may assume that a_3 is nonadjacent to a_1, a_2, and hence $a_3 \notin M$. There are induced paths u-a_1-w_0-a_3 and u-x-a_2-w_0-a_3, giving a contradiction as usual (note that w_0, a_3 are adjacent since the blowup H is self-centred).

This proves that $W_t \subseteq X$ for each $t \in N_F[a_1] \setminus \{p\}$. We claim that adding x to W_{a_1} gives a larger blow-up of F, and to show this we must just check that $G[W_p, W_{a_1} \cup \{x\}]$ is a half-graph (this is true, because the two parts of its bipartition are cliques of G); and that $G[W_p, W_{a_1} \cup \{x\}]$ is compatible with $G[W_p, W_q]$, where q is the neighbour of p in P_1 different from a_1. The latter holds since otherwise there is an induced path x-b-c-d with $b, c \in W_p$ and $d \in W_q$, which could be extended to a hole of length $\ell + 1$ in G, a contradiction. This proves (7).

Define $N = N_F[a_1] \setminus \{a_1, p\}$.

(8) *For all distinct adjacent $t, t' \in N$, either $W_t \cup W_{t'} \subseteq X$, or $(W_t \cup W_{t'}) \cap X = \emptyset$.*

Suppose not; then we may assume that $\{t, t'\} = \{a_2, a_3\}$, and $v_2 \in W_{a_2} \setminus X$ and $v_3 \in W_{a_3} \cap X$. But then there are two induced paths u-a_1-v_2 and u-x-v_3-v_2, a contradiction as before. This proves (8).

Since A is a 2-connected threshold graph (because a_0 does not exist by (7)), there are at least two vertices of A that are adjacent to all other vertices of A, and so one of them belongs to N. Consequently $F[N]$ is connected. Since $X \not\subseteq W_s \cup W_p$ by (3), it follows that $M \cap N \neq \emptyset$; and by (8), since $F[N]$ is connected, it follows that $W_t \subseteq X$ for all $t \in N$. But then we can add x to W_{a_1} as before, a contradiction to the maximality of H. This proves 7.5. ∎

8 Putting pieces together

Let J be a graph, and let A be a threshold graph contained in J with $|A| \geq 3$. For each $t \in V(A)$ let W_t be a non-null ordered clique of J, called a *bag*, all pairwise disjoint, such that if st is an edge of A then W_s is complete to W_t in J and otherwise they are anticomplete. Suppose that

- J is a connected chordal graph;

- each vertex in $V(J) \setminus W(A)$ has two nonadjacent neighbours in $V(A)$;

- for every induced path P of J with length at least three and with both ends in $W(A)$, some internal vertex of P belongs to the same bag as one of the ends of P;

- for each $t \in V(A)$ and each $v \in V(J) \setminus W_t$, let W_t be ordered as $\{x_1, \ldots, x_n\}$; then $t = x_1$, and for $1 \leq i < j \leq n$, if v, x_j are adjacent then v, x_i are adjacent (briefly, each vertex of $J \setminus W_t$ is adjacent to an *initial segment* of W_t).

In this case we call $(J, A, (W_t : t \in V(A)))$ a *border*. (Later, we will discuss how to construct borders.)

We were previously working with blow-ups of apexed ℓ-frames, but now we want blow-ups of ℓ-frames without the apexes. With ℓ odd or even, let F be an ℓ-frame, in the usual notation. For each $t \in V(F)$, let W_t be an ordered clique where t is the first term of the ordering of W_t, all pairwise disjoint, and we will define a graph H with vertex set the union of these cliques. For every edge uv of $A \cup B$ we make W_u complete to W_v in H. For every other edge uv of F let $H[W_u, W_v]$ be an ordered half-graph. We call such a graph H a *blow-up of an ℓ-frame*. (Note that here we are including the "self-centred" condition in the definition of a blow-up.)

Now take a graph H that is a blow-up of an ℓ-frame. Choose J, K such that $(J, A, (W_t : t \in V(A)))$ is a border, with $V(J) \cap V(H) = W(A)$, and $(K, B, (W_t : t \in V(B)))$ is a border, with $V(J) \cap V(H) = W(B)$, and $V(K) \cap V(J) = \emptyset$ and $V(J)$ is anticomplete to $V(K)$. We call the graph $H \cup J \cup K$ a *bordered blow-up of an ℓ-frame*, and that G is the *composition* of H, J, K.

We say a vertex v is *mixed* on a set C if $v \notin C$ and v has a neighbour and a non-neighbour in C. By combining several of the previous theorems, we obtain the following:

8.1 *Let $\ell \geq 7$, and let G be ℓ-holed, with no clique cutset or universal vertex. Then either G is a blow-up of an ℓ-cycle, or a bordered blow-up of an ℓ-frame.*

Proof. By 3.2, we may assume that G contains a theta, pyramid or prism, and so G contains a 3-bar gate. Choose k maximum such that G contains a k-bar gate. By 5.10 and 5.8, G contains an apexed ℓ-frame F^+ with sides A^+, B^+ and bars $P_1, \ldots, P_k$ say. Choose a maximal blow-up H of F^+ contained in G. By replacing F^+ by an isomorphic graph, we may assume the blow-up is self-centred. If some vertex of A^+ is not in $V(P_1 \cup \cdots \cup P_k)$, it is unique; call it a_0 (and otherwise a_0 is undefined). Define b_0 similarly. As usual, let $A = A^+ \setminus \{a_0\}$ if a_0 exists, and $A = A^+$ otherwise, and define B similarly. Let H have bags W_t ($t \in V(F^+)$). Let Z be the set of all vertices in $V(G) \setminus V(H)$ that are adjacent to every vertex of H. Since H has two nonadjacent vertices and G has no 4-hole, it follows that Z is a clique.

(1) *If $x \in V(G) \setminus (V(H) \cup Z)$, and $N_H(x)$ denotes the set of vertices in $V(H)$ adjacent to x, then either $N_H(x)$ is a clique, or $N_H(x)$ is a subset of one of $W(A^+), W(B^+)$.*

This is immediate from 7.5, since x is not adjacent to every vertex of H by definition of Z.

(2) *If $x \in V(G) \setminus (V(H) \cup Z)$, and $N_H(x)$ denotes the set of vertices in $V(H)$ adjacent to x, and a_0 exists, then $N_H(x)$ is disjoint from one of $W_{a_1}, \ldots, W_{a_k}$.*

Suppose not. By (1) we may assume that $N_H(x) \subseteq W(A^+)$; and as in step (4) of the proof of 7.5, it follows that $W_{a_0} \subseteq N_H(x)$. We claim that adding x to W_{a_0} gives a larger blow-up of F^+, a contradiction. To show this, we must check that:

- for $1 \leq i \leq k$, if x is not complete to W_{a_i} then a_i has no neighbour in $A^+ \setminus \{a_0\}$;

- for $1 \leq i \leq k$, the graph $G[W_{a_0} \cup \{x\}, W_{a_i}]$ is a half-graph;

- for $1 \leq i < j \leq k$, the graphs $G[W_{a_0} \cup \{x\}, W_{a_i}]$ and $G[W_{a_0} \cup \{x\}, W_{a_j}]$ are compatible.

For the first, suppose that x is not complete to W_{a_1} say, and $a_1 a_2$ is an edge of F^+. Since a_0 exists, A is disconnected; let (K, L) be a partition of $V(A)$ into two nonempty sets such that there are no edges of F^+ between K, L, where $a_1 \in K$. Consequently $a_2 \in K$; choose $a_3 \in L$, and for $i = 1, 2, 3$ let $v_i \in W_{a_i}$ be adjacent to x. Choose $u \in W_{a_1}$ nonadjacent to x. Let p be the neighbour of a_1 in P_1. Then there are induced paths p-v_1-x-v_3 and p-u-v_2-x-v_3 of different lengths between p, v_3, yielding a contradiction as usual. This proves the first bullet. The second holds since $W_{a_0} \cup \{x\}, W_{a_i}$ are cliques and G has no 4-hole.

For the third bullet, suppose it is false; then there is an induced path a-b-c-d where $b, c \in W_{a_0} \cup \{x\}$, and $a \in W_{a_i}$, and $d \in W_{a_j}$ (and so one of b, c equals x). Hence there are induced paths a-a_0-d and a-b-c-d between a, d of different lengths, yielding a contradiction as usual. This proves the third bullet, and so proves (2).

(3) *If C is a component of $G \setminus (V(H) \cup Z)$, and $N_H(C)$ denotes the set of vertices in $V(H)$ with a neighbour in $V(C)$, then $N_H(C)$ is a subset of one of $W(A^+), W(B^+)$.*

Suppose not. We claim that there exist $u, v \in N_H(C)$, nonadjacent, and not both in $W(A^+)$ and not both in $W(B^+)$. The proof is like that of step (1) of the proof of 6.1. Since G has no clique cutset, the set of vertices of $G \setminus V(C)$ with a neighbour in $V(C)$ is not a clique; let u, v be nonadjacent vertices in this set. Thus $u, v \in V(H) \cup Z$, and $u, v \notin Z$ since every vertex of Z is adjacent to all other vertices of $V(H) \cup Z$. Thus $u, v \in N_H(C)$. We claim that we can choose $u, v \in N_H(C)$, nonadjacent, and not both in $W(A^+)$ and not both in $W(B^+)$. We have already seen that we can choose them in $N_H(C)$. If u or v belongs to $V(H) \setminus (W(A^+) \cup W(B^+))$, or if $u \in W(A^+)$ and $v \in W(B^+)$, then u, v satisfy the requirement; so we assume that $u, v \in W(A^+)$. Choose $w \in N_H(C) \setminus W(A^+)$. It follows that w is nonadjacent to one of u, v, say v; but then w, v satisfy the requirement. This proves that we may choose $u, v \in N_H(C)$, nonadjacent, and not both in $W(A^+)$ and not both in $W(B^+)$. Let $u \in W_r$ and $v \in W_s$, where $r, s \in V(F^+)$. It follows that not both r, s are in $V(A^+)$ and not both are in $V(B^+)$.

We claim that we can choose u, v, r, s as just described, with r, s not F^+-adjacent. Because suppose that r, s are F^+-adjacent. Since not both r, s are in $V(A^+)$ and not both are in $V(B^+)$, it follows that rs is an edge of one of $P_1, \ldots, P_k$, say P_1. Since P_1 has length at least two, we may assume that s is an internal vertex of P_1. Let r-s-t be a subpath of P_1. If $N_H(C)$ is not a subset of $W_r \cup W_s \cup W_t$, then we may choose $q \in V(F^+)$, F^+-nonadjacent to s, such that $N_H(C) \cap W_q \neq \emptyset$ and the claim holds. Thus we assume that $N_H(C) \subseteq W_r \cup W_s \cup W_t$. If $N_H(C) \cap W_t \neq \emptyset$ then again the claim holds, with r, t, u and a vertex of $N_H(C) \cap W_t$; so we assume that $N_H(C) \subseteq W_r \cup W_s$. But there is a hole D of F^+ containing P_1; let q-r-s-t be a path of D. By deleting r, s from D and adding u, v and the edges uq, vt, we obtain an induced path between u, v of length $\ell - 1$; and the union of this path with an induced path between u, v with interior in C makes a hole of length more than ℓ, a contradiction.

This proves that we can choose u, v, r, s with r, s not F^+-adjacent. Because r, s are not F^+-adjacent, it follows that (in the usual notation), the subgraph induced on $(\{u, v\} \cup V(F^+)) \setminus \{r, s\}$ is an apexed ℓ-frame F' with sides A'^+, B'^+ and apexes a'_0, b'_0 say (if they exist). Thus a'_0 exists if and only if a_0 exists, and either $a'_0 = a_0$, or one of $r, s = a_0$, and the same for b_0; and either $A'^+ = A^+$, or one of $r, s \in A^+$ (say r) and $A'^+ = (A^+ \cup \{u\}) \setminus r$, and the same for B'^+. By 6.1, applied to the subgraph of G induced on $V(C) \cup V(F')$, either

- there exist $x \in V(C)$ and $u', v' \in V(F')$, such that u', v' are nonadjacent, both are adjacent to x, and not both are in $V(A'^+)$, and not both are in $V(B'^+)$; or

- there exists $x \in V(C)$ such that either a_0 exists and the set of neighbours of x in $V(F')$ is $V(A'^+)$, or b_0 exists and the set of neighbours of x in $V(F')$ is $V(B'^+)$; or

- either $N_{F'}(C) \subseteq V(A'^+)$, or $N_{F'}(C) \subseteq V(B'^+)$, or $N_{F'}(C)$ is a clique, where $N_{F'}(C)$ denotes the set of vertices in $V(F')$ with a neighbour in $V(C)$.

The first is false, because of (1). The third is false, because $u, v \in N_{F'}(C)$. Thus the second holds; and we may assume that a'_0 (and hence a_0) exists, and there exists $x \in V(C) \setminus V(F')$ such that the set of neighbours of x in $V(F')$ is $V(A'^+)$. But then x has a neighbour in each of $W_{a_1}, \ldots, W_{a_k}$, contrary to (2). This proves (3).

For every component C of $G \setminus (V(H) \cup Z)$, $N_H(C) \neq \emptyset$, since G has no clique cutset. Let X' be the union of the vertex sets of all components C of $G \setminus (V(H) \cup Z)$ such that $N_H(C) \subseteq W(A^+)$, and define Y' similarly for $W(B^+)$. Let $X = X' \cup W_{a_0}$ if a_0 exists, and $X = X'$ otherwise; and define Y similarly. By (3), X', Y', Z is a partition of $V(G) \setminus V(H)$; and X, Y are anticomplete, also by (3) and since $N_H(C) \neq \emptyset$ for each C.

We will show that $Z = \emptyset$, and every vertex in X has two nonadjacent neighbours in $V(A)$, and the same for Y. Let V be the set of all $x \in X$ that have two nonadjacent neighbours in $W(A)$, and $U = X \setminus V$. If a_0 exists then A is disconnected, and so every vertex in W_{a_0} has two nonadjacent neighbours in $W(A)$; and so $W_{a_0} \subseteq V$.

(4) *V is complete to Z.*

Let $x \in V$, and let $z \in Z$. Since $x \in V$, there exist nonadjacent $u, v \in W(A)$ both adjacent to x. But they are also both adjacent to z, and since G has no 4-hole, it follows that x, z are adjacent. This proves (4),

(5) *If Q is an induced path of G with both ends in $W(A)$ and with interior in X, then Q has length one or two.*

Let Q have ends u, v. We may assume that u, v are nonadjacent, since otherwise Q has length one. Let $u \in W_{a_1}$; then $v \notin W_{a_1}$ since W_{a_1} is a clique, so we may assume that $v \in W_{a_2}$. We claim that a_1, a_2 are nonadjacent in F^+. Suppose they are adjacent; then W_{a_1} is not complete to W_{a_2}, since u, v are not adjacent, and so one of a_1, a_2 has degree two in F^+, say a_1. Consequently A^+ is not 2-connected, and so a_0 exists, but then a_1 has at least three neighbours in F^+ (a_2, a_0, and the second vertex of P_1), a contradiction. This proves that a_1, a_2 are nonadjacent in F^+. Now there is a vertex in A^+ adjacent to all other vertices of A^+, say a_i; and $a_i \neq a_1, a_2$. There is a path R of F^+ between a_1, a_2 consisting of P_1, P_2 and a one- or two-edge path of B^+. Since the union of R with the path a_1-a_i-a_2 is a hole of F^+, it follows that R has length $\ell - 2$. There is a path of H between u, v with interior $V(R) \setminus \{a_1, a_2\}$; and it also has length $\ell - 2$. But its union with Q is a hole, so Q has length two. This proves (5).

If we can prove that $U = \emptyset$, then (4) implies that Z is complete to X, and similarly Z is complete

to Y; but then the vertices in Z are all universal vertices, and so $Z = \emptyset$. Thus, we next need to show that $U = \emptyset$.

Suppose that $U \neq \emptyset$, and let C be a component of $G[U]$. Since G has no clique cutset, there exist nonadjacent $u, v \in V(G) \setminus U$, both with neighbours in $V(C)$. Consequently $u, v \in W(A) \cup V \cup Z$; and by (4), $u, v \notin Z$. Thus $u, v \in W(A) \cup V$.

Either $u, v \in W(A)$, or one is in $W(A)$ and the other in V, or they are both in V. We handle these cases separately. Choose an induced path P with ends u, v and interior in $V(C)$.

(6) *Not both u, v belong to $W(A)$.*

This is immediate from (5), since if P has length two then its middle vertex is in V by definition of V, a contradiction. This proves (6).

(7) *Neither of u, v is in $W(A)$.*

Suppose that $u \in W(A)$ say; and so $v \in V$, and so v has nonadjacent neighbours $x, y \in W(A)$. Since u, v are nonadjacent and G has no 4-hole, it follows that u is nonadjacent to one of x, y, say x. The path obtained by adding x and the edge xv to P might not be induced, but its vertex set includes that of an induced path between u, x, which therefore has length two by (5); and consequently the neighbour of u in P is adjacent to x, a contradiction, since this neighbour is in U. This proves (7).

(8) *Not both u, v are in V.*

Suppose that $u, v \in V$, and so they both have two nonadjacent neighbours in $W(A)$; say x, y for u, and x', y' for v. Choose x, y, x', y' not all distinct if possible. Now v is not adjacent to both x, y, since G has no 4-hole; so we may assume that v is nonadjacent to x, and similarly we may assume that u is nonadjacent to x'. So either x, y, x', y' are different, or $y = y'$. Suppose first that $y = y'$. Since G has no 5-hole it follows that x, x' are nonadjacent; and so by (5) there is an induced path of length two between x, x' with middle vertex in $V(P)$, a contradiction. (This middle vertex cannot be u or v, since u is not adjacent to x' and v is not adjacent to x; and cannot belong to the interior of P since all those vertices are in U.)

This proves that x, y, x', y' are all different; and it is not possible to choose them not all different. Consequently, if $z \in \{x, y\}$ and $z' \in \{x', y'\}$ are nonadjacent, then neither of u, v is adjacent to both z, z'. If both of x, y are adjacent to both of x', y' then G has a 4-hole; so we may assume that some $z \in \{x, y\}$ is nonadjacent to some $z' \in \{x', y'\}$; and so neither of u, v is adjacent to both z, z'. But then by (5), there is an induced path between z, z' of length two with middle vertex in $V(P)$, and since this middle vertex is not in U, and not u or v, this is a contradiction. This proves (8).

From (6), (7), (8) it follows that $U = \emptyset$ and so $Z = \emptyset$. It remains to show that $(G[X \cup W(A)], A, (W_t : t \in V(A)))$ is a border (and the same for Y, B). First we show:

(9) *For every induced path P of $G[X \cup W(A^+)]$ of length at least three with both ends in $W(A)$, either the first two vertices or the last two vertices of P belong to the same bag W_t for some $t \in V(A)$.*

Let P have ends $u \in W_{a_1}$ and $v \in W_{a_2}$ say. Thus a_1, a_2 are not A-adjacent. Since $H \setminus W(A^+)$

is connected, and u, v both have neighbours in it, there is an induced path R between u, v with all its internal vertices in $V(H) \setminus W(A^+)$. There is a vertex a_j say of A^+ adjacent to all other vertices of A^+; and so $j \neq 1, 2$. Hence there is an induced path u-a_j-v. The union of this path and R is a hole, so R has length $\ell - 2$. It follows that $P \cup R$ is not a hole; so some internal vertex of P belongs to one of W_{a_1}, W_{a_2}. Since W_{a_1}, W_{a_2} are cliques and P is induced, this proves (9).

(10) *There is no induced path v_1-v_2-v_3-v_4 of G with $v_2, v_3 \in W_t$ for some $t \in V(A)$.*

Suppose that there is such a path. Since W_t is a clique, it follows that $v_1, v_4 \notin W_t$, and since they are both mixed on W_t, it follows that $v_1, v_4 \notin W(A)$. Suppose that $v_1 \notin X$; so we may assume that $t = a_1$ and $v_1 \in W_q$ where a_1-q-r is a subpath of P_1. Since v_1, v_4 are not adjacent, it follows that $v_4 \notin W_q$, and so $v_4 \in X$. Consequently v_4 has two nonadjacent neighbours $x, y \in W(A)$ (since $U = \emptyset$). Since $\{x, y, v_2, v_4\}$ does not induce a 4-hole, one of x, y is nonadjacent to v_2, say x; and hence x is also nonadjacent to v_3, since $v_2, v_3 \in W_{a_1}$ and $x \in W(A)$. Let $x \in W_{a_2}$ say. There is an induced path Q of F^+ between r, a_2 consisting of $P_1 \setminus \{a_1, q\}, P_2$ and an induced path of B^+ with length one or two; and so there is an induced path R of G between r, x with interior in $V(Q) \setminus \{a_2\}$. This can be completed to a hole via r-q-v_3-v_4-x or via r-v_1-v_2-v_3-v_4-x, a contradiction.

It follows that $v_1, v_4 \in X$. Consequently v_1 has two nonadjacent neighbours x_1, y_1 in $W(A)$ (since $U = \emptyset$), and similarly v_4 has two nonadjacent neighbours $x_4, y_4 \in W(A)$. Not both x_1, y_1 have a neighbour in $\{v_3, v_4\}$, since otherwise G contains a 4-hole or 5-hole; so we assume that x_1 is nonadjacent to both v_3, v_4, and in particular $x_1 \neq v_2$. Similarly we may assume that x_4 is nonadjacent to v_1, v_2. Thus $x_1, v_1, v_2, v_3, v_4, x_4$ are all distinct, and $v_1 v_2, v_2 v_3, v_3 v_4$ are edges, and all other pairs of these six vertices are nonadjacent except possible $x_1 v_2, x_4 v_3$ and $x_1 x_4$. If x_1, x_4 are adjacent then G contains a 4-,5- or 6-hole, a contradiction. Consequently there is an induced path with ends x_1, x_4 of length three, four or five, contradicting (9). This proves (10).

(11) *We may assume that for each $t \in V(A)$, every vertex in X is adjacent to an initial segment of W_t.*

From (10), there is an ordering of W_t (not necessarily with t as its first term) such that every vertex in $V(G) \setminus W_t$ is adjacent to an initial segment of the ordering. Thus, by replacing F^+ with an isomorphic graph, we may assume that the claim of (11) holds. This proves (11).

Let F be the ℓ-frame obtained from F^+ by deleting any apexes.

(12) *Every vertex in X has two nonadjacent neighbours in $V(A)$.*

Let $v \in X$; then v has two nonadjacent neighbours $x, y \in W(A)$. Let $x \in W_s$ and $y \in W_t$, where $s, t \in V(F)$. Since x, y are nonadjacent, it follows that $s \neq t$, and s, t are not adjacent; and by (11), v is adjacent to both s, t. This proves (12).

Let H' be obtained from H by deleting W_{a_0}, W_{b_0} (if they exist). Then H' is a blow-up of the ℓ-frame F. From (9), (11), (12), this proves 8.1. ∎

We remark that, when ℓ is odd, the structure of 8.1 is a sort of characterization of ℓ-holed graphs, because one can show that every bordered blow-up of an ℓ-frame is ℓ-holed. When ℓ is even this is not true.

9 Border construction

Now we turn to the second part of the proof of 1.3, showing how to make a border. Let G be a graph and let $u, v \in V(G)$ be distinct. We say that u G-*dominates* v if u, v are G-adjacent and every G-neighbour of v is equal or G-adjacent to u. If T is an arborescence, then for every vertex v of T different from the apex, there is a unique vertex u of T such that v is adjacent from u; we call u the T-*inneighbour* of v.

Let $(J, A, (W_t : t \in V(A)))$ be a border. We say an arborescence T is a *basis* for $(J, A, (W_t : t \in V(A)))$ if:

- $V(T) \subseteq V(J) \subseteq V(T) \cup W(A)$, and $V(T) \cap W(A) = V(A)$;

- $L(T) \subseteq V(A)$;

- $J[V(T)] = \overrightarrow{T}$;

- for each $t \in L(T)$, let S be the path of T from $r(T)$ to the T-inneighbour of t; then for each $v \in W_t$ there exists a subpath S_v of S with first vertex $r(T)$, containing all vertices of $V(S) \cap V(A)$, such that the set of J-neighbours of v in $V(J) \setminus W_t$ is $V(S_v) \cup W(S_v \cap A)$;

- for each $t \in V(A) \setminus L(T)$, all vertices in W_t have the same neighbours in $V(J) \setminus W_t$ as t;

- for each $t \in V(A) \setminus \{r(T)\}$, the T-inneighbour of t either belongs to $V(A)$ or has T-outdegree at least two;

- $r(T) \in V(A)$ if and only if A is connected.

If so, it follows from the definition of a border that, with t, S, and S_v ($v \in W_t$) as in the fourth bullet above, if $\{x_1, \ldots, x_n\}$ is the ordering of the ordered clique W_t, then the length of S_{x_j} monotonely (non-strictly) decreases as j increases.

The first main result of this section is the following:

9.1 *Let* $(J, A, (W_t : t \in V(A)))$ *be a border. Then there is a basis* T *for* $(J, A, (W_t : t \in V(A)))$.

Thoughout the remainder of this section, we will assume that $(J, A, (W_t : t \in V(A)))$ is a border, and J' is the subgraph of J induced on $V(A) \cup (V(J) \setminus W(A))$.

9.2 *J' does not contain P_4.*

Proof. From the definition of a border, we have

(1) *No induced path of J' with length at least three has both ends in $V(A)$.*

We deduce

(2) *There is no four-vertex induced path v_1-v_2-v_3-v_4 of J' with $v_1 \in V(A)$.*

Suppose there is such a path. By (1), $v_4 \in V(J) \setminus V(A)$ and so has two nonadjacent neighbours $x, y \in V(A)$.

Thus $v_1 \neq x, y$, since v_1, v_4 are nonadjacent. If both x, y have a neighbour in $\{v_1, v_2\}$, then J' contains a 4-hole or 5-hole, a contradiction; so we may assume that neither of v_1, v_2 is adjacent to x, and so $x \neq v_3$. By (1), v_1-v_2-v_3-v_4-x is not an induced path, and so v_3 is adjacent to x. This contradicts (1) applied to v_1-v_2-v_3-x, and so proves (2).

Now suppose that v_1-v_2-v_3-v_4 is a copy of P_4 in J'. By (2), $v_1, v_4 \notin V(A)$. Hence v_4 has two nonadjacent neighbours x, y in $V(A)$. As before we may assume that v_1, v_2 are nonadjacent to x, and hence $x \neq v_3$. But then one of v_1-v_2-v_3-x, v_2-v_3-v_4-x violates (2). This proves 9.2. ∎

We use the following theorem of Wolk [8]:

9.3 *Let J' be a graph. Then J' is the transitive closure of some arborescence if and only if J' is non-null, connected, and does not contain P_4 or C_4 as an induced subgraph.*

The graph J' does not contain P_4 or C_4, by 9.2, so by 9.3, there is an arborescence T such that $J' = \overrightarrow{T}$. Thus, $V(T) = V(J')$, and for every two vertices in $V(T)$, they are J'-adjacent if and only if they are joined by a directed path of T.

Here is a useful observation (the proof is clear, since every vertex has indegree at most one):

9.4 *If P, Q are directed paths of T, and the last vertex of P belongs to $V(Q)$, then $P \cup Q$ is a directed path of T.*

9.5 $L(T) \subseteq V(A)$.

Proof. Suppose that $v \in L(T) \setminus V(A)$. From the definition of a border, there are $x, y \in V(A)$, not J'-adjacent to each other, and J'-adjacent to v. Consequently $x, y \in V(T)$, and each of them is joined to v by a directed path of T (since v has zero outdegree in T), say P and Q respectively. By 9.4, $P \cup Q$ is a directed path, and so x, y are adjacent in $\overrightarrow{T}$, a contradiction. This proves 9.5. ∎

Let us say the *big cost* of T is the sum, over all $t \in V(A)$, of the T-distance between $r(T)$ and t; and the *little cost* of T is the sum, over all $v \in W(A) \setminus V(A)$ and all $u \in V(T)$ J-adjacent to v, of the T-distance between $r(T), u$. There may be several choices of the arborescence T that have the same transitive closure $\overrightarrow{T}$. Let us choose T with apex in $V(A)$ if possible; subject to that, with minimum big cost; and subject to that, with minimum little cost. Such a choice of T is said to be *optimal*.

9.6 *If T is optimal then:*

- $r(T) \in V(A)$ *if and only if A is connected.*

- *If $r(T) \notin V(A)$ and $r(T)$ has a unique T-outneighbour, this outneighbour does not belong to $V(A)$.*

- *For each $t \in V(A) \setminus \{r(T)\}$, let v be the T-inneighbour of t; then either $v \in V(A)$ or the T-outdegree of v is at least two.*

- *For each $t \in L(T)$, let S be the path of T between $r(T)$ and the T-inneighbour of t; then for each $v \in W_t$, there exists a subpath S_v of S with first vertex $r(T)$, containing all vertices of $V(S) \cap V(A)$, such that $V(S_v)$ is the set of J-neighbours of v in $V(T) \setminus \{t\}$.*

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Proof. Since $r(T)$ is $\overrightarrow{T}$-adjacent to all other vertices of T, it follows that if $r(T) \in V(A)$ then A is connected. Now suppose that $r(T) \notin V(A)$, and A is connected. Consequently there exists $a \in V(A)$ A-adjacent to every other vertex of A, since A is a threshold graph. Since every vertex $z \in V(T) \setminus V(A)$ has two nonadjacent neighbours in $V(A)$, and J' contains no 4-hole, it follows that a is J'-complete to $V(T) \setminus \{a\}$; and so $r(T), a$ are adjacent twins of J'. There is an automorphism of J' exchanging $r(T), a$ and fixing all other vertices of J'. Let T' be the image of T under this automorphism. Then T' is an arborescence with $\overrightarrow{T'} = \overrightarrow{T}$, and with apex in $V(A)$, contrary to the optimality of T. This proves the first assertion of the theorem.

For the second, suppose that $r(T) \notin V(A)$ and that $r(T)$ has a unique T-outneighbour a_1 say, and $a_1 \in V(A)$. Thus a_1 is $\overrightarrow{T}$-adjacent to all other vertices in A, and so A is connected, contrary to the first assertion.

For the third assertion, let $t \in V(A) \setminus \{r(T)\}$, and let v be the T-inneighbour of a_i. We may assume that $v \notin V(A)$. Suppose that t is the unique T-outneighbour of v. Consequently t, v are adjacent J'-twins; and replacing T by the image of T under the automorphism of J' that exchanges v, t contradicts the optimality of T. This proves the third bullet.

Finally, let $t \in L(T)$ and $v \in W_t \setminus \{t\}$, and let S be the path of T between $r(T)$ and the T-inneighbour of t. Since t J-dominates v and all J-neighbours of t in $V(T)$ belong to $V(S)$, it follows that all J-neighbours of v in $V(T)$ belong to $V(S)$. We claim that this set of neighbours, excepting t, is the vertex set of a subpath of S starting at $r(T)$. Suppose not; then there exist $u, w \in V(S) \setminus \{t\}$, such that w is a T-outneighbour of u, and v is J-adjacent to w and not to u.

Suppose that u has T-outdegree more than one, and let $w' \in V(T)$ be a T-outneighbour of u different from w. Let $t' \in L(T)$ such that there is a directed path of T from w' to t'. It follows that w, t' and t, t' are not J-adjacent, since there is no directed path of T between them; and hence v, t' are not J-adjacent, since t' is not mixed on W_t. But $t' \in V(A)$ by 9.5; and so v-w-u-t' is an induced path of J, contradicting the definition of a border. Thus w is the unique T-outneighbour of u.

Now suppose that there exists $v' \in W(A) \setminus V(A)$ that is J-adjacent to u and not to w. If v, v' are J-adjacent, then $\{v, v', u, w\}$ induces a 4-hole, and if not then v-w-u-v' violates the definition of a border. Thus there is no such v'. Let T' be the image of T under the automorphism of J' that exchanges u, w. The big cost of T' equals that of T, but its little cost is strictly smaller, contrary to the optimality of T.

This proves that there is a subpath S_v of S with first vertex $r(T)$, such that $V(S_v)$ is the set of J-neighbours of v in $V(T) \setminus \{t\}$. Since each vertex in $V(S) \cap V(A)$ is adjacent to t and is not mixed on W_t, it follows that each such vertex belongs to S_v. This proves 9.6. ∎

9.7 *Let $t \in V(A)$. If there exists $v \in V(J) \setminus W_t$ mixed on W_t, then $t \in L(T)$.*

Proof. Suppose that $v \in V(J) \setminus W_t$ is mixed on W_t. Hence v is J-adjacent to t, and J-nonadjacent to some $u \in W_t$. It follows that $v \notin W(A)$, from the definition of a border. Hence there exist nonadjacent $x, y \in V(A)$ adjacent to v. Not both x, y are adjacent to u, since otherwise $\{x, y, u, v\}$ induces a 4-hole. We assume that x, u are nonadjacent. Consequently x, t are nonadjacent, since x is not mixed on W_t (because $x \in V(A)$).

Suppose that $t \notin L(T)$, and let $s \in L(T)$ such that there is a directed path of T from t to s. Consequently t J'-dominates s, and therefore u, s are adjacent since $u, s \in W(A)$. Since $x, v, t, s \in V(J')$, and J' contains no four-vertex induced path (by 9.2) and no 4-hole, it follows that v, s are

A-adjacent. But then x-v-s-u is an induced path that violates the definition of a border. This proves 9.7. ∎

Now we can prove 9.1, which we restate:

9.8 *Let $(J, A, (W_t : t \in V(A)))$ be a border. Then there is a basis T for $(J, A, (W_t : t \in V(A)))$.*

Proof. With J' as before, choose an optimal T. We must check that:

1. $V(T) \subseteq V(J) \subseteq V(T) \cup W(A)$, and $V(T) \cap W(A) = V(A)$;

2. $L(T) \subseteq V(A)$;

3. $J[V(T)] = \overrightarrow{T}$;

4. for each $t \in V(A) \setminus L(T)$, all vertices in W_t have the same neighbours in $V(J) \setminus W_t$ as t;

5. for each $t \in L(T)$, let S be the path of T from $r(T)$ to the T-inneighbour of t; then for each $v \in W_t$ there exists a subpath S_v of S with first vertex $r(T)$, containing all vertices of $V(S) \cap V(A)$, such that the set of J-neighbours of v in $V(J) \setminus W_t$ is $V(S_v) \cup W(S_v \cap A)$;

6. for each $t \in V(A) \setminus \{r(T)\}$, the T-inneighbour of t either belongs to $V(A)$ or has T-outdegree at least two; and

7. $r(T) \in V(A)$ if and only if A is connected.

The first holds since $V(T) = V(J')$, and the second by 9.5. The third holds since $J' = \overrightarrow{T}$. The fourth follows from 9.7. For the fifth, let $t \in L(T)$, let S be the path of T between $r(T)$ and the T-inneighbour of t, and let $v \in W_t$. By 9.6, there exists a subpath S_v of S with first vertex $r(T)$, containing all vertices of $V(S) \cap V(A)$, such that $V(S_v)$ is the set of J-neighbours of v in $V(T) \setminus \{t\}$. From the fourth bullet above, the set of J-neighbours of v in $V(J) \setminus W_t$ equals $V(S_v) \cup W(S_v \cap A)$. The sixth and seventh statements follows from 9.6. This proves 9.8. ∎

A *split* of a graph G is a partition (X, Y) of $V(G)$ into a clique X and a stable set Y (graphs that admits splits are called *split graphs*). (The split graphs are the graphs that do not contain C_5, C_4 or $\overline{C_4}$, so our graph A is a split graph.) If (X, Y) is a split of A, we say an (X, Y)-*spine* is a directed path R of T, with first vertex $r(T)$ and last vertex in $X \cup \{r(T)\}$, such that $X = V(A) \cap V(R)$ and $Y = L(T) \setminus V(R)$.

9.9 *Let T be a basis for a border $(J, A, (W_t : t \in V(A)))$. If (X, Y) is a split of A, then there is an (X, Y)-spine.*

Proof. Choose a directed path R of T with first vertex $r(T)$ and last vertex in $X \cup \{r(T)\}$, as long as possible. Let x be the last vertex of R. We claim that $X \subseteq V(R)$; for suppose that there exists $x' \in X \setminus V(R)$. Since x, x' are $\overrightarrow{T}$-adjacent, there is a directed path Q of T between x, x', and by 9.4 $Q \cup R$ is a directed path, contrary to the maximality of the length of R. This proves that $X \subseteq V(R)$.

We claim that $Y = L(T) \setminus V(R)$. Certainly if $t \in L(T) \setminus V(R)$, then $t \in V(A)$ by 9.5, and so $t \in Y$, since $t \notin V(R) \supseteq X$. Now suppose that there exists $t \in Y \setminus (L(T) \setminus V(R))$. Since $t \neq x$ (because $x \notin Y$) it follows that $t \notin L(T)$. Choose a leaf s of T such that there is a directed path of T from t to s; then $s \notin V(R)$ since $t \notin V(R)$, and so $s \in Y$. But $t \neq s$, and t, s are $\overrightarrow{T}$-adjacent, and Y is stable in $\overrightarrow{T}$, a contradiction. This proves 9.9. ∎

We would like it to be true that for each $t \in X$, no vertex of J is mixed on W_t. From the definition of a basis, this is true if $t \notin L(T)$, but we must be careful if the last vertex of R is a leaf of T. We will have two applications of 9.9, and in both we will have to check separately that if the last vertex of R is a leaf of T, then no vertex of J is mixed on W_t.

10   The case when ℓ is odd.

In this section we complete the proof of 1.3 when ℓ is odd. If t is a vertex of a graph A, we say that t is A-*non-terminal* (or just "non-terminal") if there exists $s \in V(A) \setminus \{t\}$ such that t A-dominates s, and A-*terminal* (or just "terminal") otherwise. We need:

10.1 *Let A be a threshold graph with $|A| > 1$. Then $t \in V(A)$ is A-terminal if and only if t is not $\overline{A}$-terminal. Consequently every two A-terminal vertices are nonadjacent, and every two A-non-terminal vertices in A are adjacent.*

Proof. Let N be the set of neighbours of t in A, and M its set of non-neighbours. Thus $\{t\}, N, M$ partition $V(A)$. Suppose that there exist $n_1, n_2 \in N$ and $m_1, m_2 \in M$ such that $m_1 n_1$ and $m_2 n_2$ are edges of A, and m_1 is not A-adjacent to n_2, and m_2 is not A-adjacent to n_1. Since m_1-n_1-t-n_2 is not an induced path it follows that n_1, n_2 are A-adjacent; but then $A[\{m_1, n_1, m_2, n_2\}]$ is a copy of P_4 or C_4, a contradiction. Thus there are no such m_1, n_1, m_2, n_2, and so the bipartite graph $A[N, M]$ is a half-graph.

Consequently either there exists $m \in M$ adjacent to every vertex in N, or there exists $n \in N$ with no neighbour in M. In the first case, t is A-terminal, since m is adjacent to all neighbours of t and not adjacent to t; and t is $\overline{A}$-non-terminal, since it $\overline{A}$-dominates m. In the second case, the same argument applied in $\overline{A}$ shows that t is $\overline{A}$-terminal and t is A-non-terminal. This proves the first claim.

Suppose that $t_1, t_2 \in V(A)$ are A-terminal and A-adjacent. Since neither A-dominates the other, there exists $s_1, s_2 \in V(A) \setminus \{t_1, t_2\}$, such that s_1 is adjacent to t_1 and not to t_2, and s_2 is adjacent to t_2 and not to t_1. In particular $s_1 \neq s_2$; and $\{s_1, s_2, t_1, t_2\}$ induces P_4 or C_4, a contradiction. This proves that A-terminal vertices are nonadjacent. The final statement follows by taking complements. This proves 10.1. ∎

By 2.2, to complete the proof of 1.3 when ℓ is odd, it suffices to prove the following:

10.2 *Let $\ell \geq 7$ be odd, and let G be an ℓ-holed graph with no clique cutset or universal vertex. Then G is either a blow-up of a cycle of length ℓ, or a blow-up of an ℓ-framework.*

Proof. By 8.1, we may assume that G is the bordered blow-up of an ℓ-frame, and G is the composition of H, J, K where:

- H is a blow-up of the ℓ-frame F, where F has sides A, B;

- $(J, A, (W_t : t \in V(A)))$ and $(K, B, (W_t : t \in V(B)))$ are borders; and

- $V(H \cap J) = W(A)$, and $V(H \cap K) = W(B)$, and $V(J), V(K)$ are disjoint and anticomplete.

Let F have k bars $P_1, \ldots, P_k$, where P_i has ends $a_i \in V(A)$ and $b_i \in V(B)$, and so $|A| = |B| = k \geq 3$. Since A, B are complementary threshold graphs, exactly one is disconnected, say A. By 9.1, there is a basis T for $(J, A, (W_t : t \in V(A)))$, and a basis S for $(K, B, (W_t : t \in V(B)))$; and $r(T) \notin V(A)$, since A is not connected, and $r(S) \in V(B)$. We will show that $S \cup T \cup P_1 \cup \cdots \cup P_k$ is an ℓ-framework, and G is a blow-up of this framework.

Let X be the set of all non-terminal vertices of A, and $Y = V(A) \setminus X$; thus (X, Y) is a split of A, by 10.1. By 9.9, there is an (X, Y)-spine R of T. Let $a_0 = r(T)$, and let the vertices of R in $V(A) \cup \{a_0\}$ be $\{a_0, a_1, \ldots, a_m\}$, numbered in order on R. Thus $X = \{a_1, \ldots, a_m\}$ since $a_0 \notin V(A)$. Let $X' = \{b_1, \ldots, b_m\}$, and $Y' = V(B) \setminus X'$. Since a_i, a_j are adjacent if and only if b_i, b_j are nonadjacent for $1 \leq i < j \leq k$, 10.1 implies that Y' is the set of non-terminal vertices of B. Let R' be a (Y', X')-spine. Hence $V(R') \cap V(B) = \{b_{m+1}, \ldots, b_k\}$, and so we may assume (by renumbering $b_{m+1}, \ldots, b_k$) that b_k is the apex of S and $b_k, b_{k-1}, \ldots, b_{m+1}$ are in order in R'.

For $0 \leq i \leq m$ let I_i be the set of vertices $v \in \{a_{m+1}, \ldots, a_k\}$ such that there is a directed path of T from a_i to v, not using a_{i+1} if $i < m$. For $m + 1 \leq j \leq k$, let L_j be the set of $v \in \{b_1, \ldots, b_m\}$ such that there is a directed path of S from b_j to v, not using b_{j-1} if $j > m + 1$. The sets $I_0, \ldots, I_m$ are pairwise disjoint and have union $\{a_{m+1}, \ldots, a_k\}$, the set of terminal vertices of A; and $L_{m+1}, \ldots, L_k$ are pairwise disjoint and have union $\{b_1, \ldots, b_m\}$, the set of terminal vertices of B.

(1) *For $1 \leq i \leq m$ and $m + 1 \leq p \leq k$, $a_p \in I_0 \cup I_1 \cup \cdots \cup I_{i-1}$ if and only if $b_i \in L_{m+1} \cup \cdots \cup L_p$.*

a_i and a_p are adjacent if and only if $a_p \in I_i \cup I_{i+1} \cup \cdots \cup I_m$; and b_p, b_i are adjacent if and only if $b_i \in L_{m+1} \cup L_{m+2} \cup \cdots \cup L_p$. Since a_i, a_p are adjacent if and only if b_i, b_p are nonadjacent, this proves (1).

(2) *For $0 \leq i < j \leq m$, if $a_p \in I_i$ and $a_q \in I_j$ then $p > q$, and consequently $I_0, I_1, \ldots, I_m$ are (possibly null) intervals of $\{a_{m+1}, \ldots, a_k\}$. Similarly, for $m + 1 \leq p < q \leq k$, if $b_i \in L_p$ and $b_j \in L_q$ then $i > j$, and so $L_{m+1}, \ldots, L_k$ are (possibly null) intervals of $\{b_1, \ldots, b_m\}$. (See figure 2.)*

Let $0 \leq i < j \leq m$, and $a_p \in I_i$ and $a_q \in I_j$. Since $a_q \notin I_0 \cup \cdots \cup I_{j-1}$, and $a_p \in I_0 \cup \cdots \cup I_{j-1}$, (1) implies that $b_j \notin L_{m+1} \cup \cdots \cup L_q$ and $b_j \in L_{m+1} \cup \cdots \cup L_p$, and so $p > q$. This proves the first statement.

Now let $m + 1 \leq p < q \leq k$ and $b_i \in L_p$ and $b_j \in L_q$. Thus $i, j \geq 1$. Since $b_j \notin L_{m+1} \cup \cdots \cup L_p$, and $b_i \in L_{m+1} \cup \cdots \cup L_p$, (1) implies that $a_p \notin I_0 \cup I_1 \cup \cdots \cup I_{j-1}$ and $a_p \in I_0 \cup I_1 \cup \cdots \cup I_{i-1}$, and so $i > j$. This proves the second statement, and so proves (2).

(3) *$I_0 \neq \emptyset$. Let $0 \leq i \leq m$ with $I_i \neq \emptyset$, and let I_i be the interval $\{a_p, \ldots, a_q\}$. Then $L_{p+1}, \ldots, L_q$ are all empty, and $L_p \neq \emptyset$ unless $p = m + 1$.*

Since $a_0 \notin V(A)$, the definition of a basis implies that the subpath of R from a_0 to a_1 has a vertex different from a_1 with T-outdegree at least two. Consequently $I_0 \neq \emptyset$. Now let $0 \leq i \leq m$ with $I_i \neq \emptyset$, let I_i be the interval $\{a_p, \ldots, a_q\}$, and suppose that $p < q' \leq q$ and $L_{q'} \neq \emptyset$. Let $b_j \in L_{q'}$. If $i < j$, then
$$a_p \in I_0 \cup \cdots \cup I_i \subseteq I_0 \cup \cdots \cup I_{j-1}$$
and $b_j \notin L_{m+1} \cup \cdots \cup L_p$, contrary to (1). If $i \geq j$, then $a_{q'} \notin I_0 \cup \cdots \cup I_{j-1}$ and $b_j \in L_{m+1} \cup \cdots \cup L_{q'}$,

contrary to (1). This proves that $L_{p+1}, \ldots, L_q$ are all empty.

Now suppose that $p > m + 1$. Since a_{m+1} belongs to $I_0 \cup \cdots \cup I_m$ and $a_{m+1} \notin I_i$, it follows from (2) that $i < m$. Since $a_{p-1} \notin I_0 \cup \cdots \cup I_i$, and $a_p \in I_0 \cup \cdots \cup I_i$, it follows from (1) that $b_{i-1} \notin L_{m+1} \cup \cdots \cup L_{p-1}$, and $b_{i-1} \in L_{m+1} \cup \cdots \cup L_p$. Consequently $L_p \neq \emptyset$. This proves (3).

(4) *Let* $m + 1 \leq p \leq k$ *with* $L_p \neq \emptyset$, *and let* L_p *be the interval* $\{b_i, \ldots, b_j\}$. *Then* $I_i, \ldots, I_{j-1}$ *are all empty, and* $I_j \neq \emptyset$ *unless* $j = m$.

Let $i \leq i' < j$, and suppose that $I_{i'} \neq \emptyset$. Let $a_q \in I_{i'}$. If $p \leq q$, then $a_q \notin I_0 \cup \cdots \cup I_{i'-1}$, and $b_{i'} \in L_{m+1} \cup \cdots \cup L_q$, contrary to (1). If $p > q$, then $a_q \in I_0 \cup \cdots \cup I_{j-1}$, and $b_j \notin L_{m+1} \cup \cdots \cup L_q$, contrary to (1). Thus $I_i, \ldots, I_{j-1}$ are all empty. Now suppose that $j \neq m$. Since $b_{j+1} \in L_{m+1} \cup \cdots \cup L_{p-1}$ and $b_j \notin L_{m+1} \cup \cdots \cup L_{p-1}$, (1) implies that $a_p \in I_0 \cup \cdots \cup I_j$, and $a_p \notin I_0 \cup \cdots \cup I_{j-1}$; and so $I_j \neq \emptyset$. This proves (4).

From (2), (3), (4) it follows that $S \cup T \cup P_1 \cup \cdots \cup P_k$ is an ℓ-framework.

(5) *For each* $t \in X$, *no vertex of* J *is mixed on* W_t; *and for each* $t \in Y'$, *no vertex of* K *is mixed on* W_t.

From the choice of R it follows that $t \in V(R)$; and the claim is true from the definition of a basis if $t \notin L(T)$; so we assume that $t \in L(T)$, and hence is the last vertex of R, and therefore $m > 0$ and $t = a_m$. Since $a_m \in X$, it follows that a_m is not terminal in A, and so a_m A-dominates some a_i where $0 \leq i < m$ and $a_i \in V(A)$ (since $L(T)$ is stable in A); and consequently $a_{m-1} \in V(A)$ and a_m A-dominates a_{m-1}. Let u be the T-inneighbour of a_m, and suppose that $u \neq a_{m-1}$. From the definition of a basis, u has T-outdegree at least two, and so there is a leaf $s \neq a_m$ of T such that there is a directed path of T from u to s. But then a_{m-1}, s are $\overrightarrow{T}$-adjacent and hence A-adjacent, and a_m, s are not A-adjacent since they both belong to $L(T)$. This contradicts that a_m A-dominates a_{m-1}.

Consequently a_{m-1} is the T-inneighbour of a_m. But then, from the definition of a basis, every vertex in W_t is J-adjacent to $V(S) \cup W(A \cap S)$, where S is the path of T from a_0 to a_{m-1}, and has no other J-neighbours in $V(J) \setminus W_t$. This proves the first assertion, and the second follows similarly. This proves (5).

Since G is the composition of H, J, K, and T is a basis for $(J, A, (W_t : t \in V(A)))$ (and the same for K), (5) implies that G is a blow-up of the ℓ-framework $S \cup T \cup P_1 \cup \cdots \cup P_k$. This proves 10.2. ∎

11 A structure theorem for laminar families

Now we turn to the case of 1.3 when ℓ is even. We will need a theorem about bipartite graphs that is proved in this section.

Let G be a bipartite graph, with bipartition (A, B). For $A' \subseteq A$, we say A' *is laminar in* G if for all distinct $a, a' \in A'$, either $N(a) \subseteq N(a')$, or $N(a') \subseteq N(a)$, or $N(a) \cap N(a') = \emptyset$, where $N(v)$ denotes the set of neighbours of a vertex $v \in A$.

The result we will use later is:

11.1 *Let G be a bipartite graph with bipartition (A, B), and let A be the union of disjoint sets A_1, A_2. Suppose that A_1, A_2 are both laminar in G. Then the following are equivalent:*

- *every hole of G has length four;*

- *there is a tree T with $V(T) = B$, such that for each $a \in A$, $N(a)$ is the vertex set of a subtree of T.*

That is a consequence of the following (proved later in this section):

11.2 *Let G be a bipartite graph with bipartition (A, B), where $B \neq \emptyset$ and every vertex in A has positive degree. Then the following are equivalent:*

- *for every hole C of G of length at least six, some vertex in B has at least three neighbours in $V(C)$;*

- *there is a tree T with $V(T) = B$, such that for each $a \in A$, $N(a)$ is the vertex set of a subtree of T.*

We say that G admits a *1-join* (V_1, V_2) if V_1, V_2 is a partition of $V(G)$ with $|V_1|, |V_2| \geq 2$, and there are subsets $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$, such that X_1 is complete to X_2, and there are no other edges between V_1, V_2. Let us define $N[a] = N(a) \cup \{a\}$, where $N(a)$ is the set of neighbours of a. To prove 11.2, we need the following (thanks to Maria Chudnovsky for this short proof; our original proof was much longer.)

11.3 *Let G be a connected bipartite graph, with bipartition (A, B), and with $|A|, |B| \geq 2$, that admits no 1-join. Then there exists $a \in A$ such that $G \setminus N[a]$ is connected.*

Proof. If G is complete bipartite, then it admits a 1-join (A, B), since $|A|, |B| \geq 2$. Consequently there exist $a \in A$ and $b \in B$ with distance at least three. Hence there exist $A_1 \subseteq A$ and $B_1 \subseteq B$ with $G[A_1 \cup B_1]$ connected and with A_1, B_1 nonnull, such that some vertex in $A \setminus A_1$ has no neighbour in B_1. Choose A_1, B_1 with this property, with $A_1 \cup B_1$ maximal.

Let A_2, B_2 be respectively the sets of vertices in $A \setminus A_1$ and in $B \setminus B_1$ that have a neighbour in $A_1 \cup B_1$. Let $A_3 = A \setminus (A_1 \cup A_2)$ and $B_3 = B \setminus (B_1 \cup B_2)$. By hypothesis, $A_3 \neq \emptyset$. From the maximality of $A_1 \cup B_1$ it follows that $A_2 = \emptyset$ (because otherwise we could add A_2 to A_1). If some vertex $b_2 \in B_2$ is nonadjacent to some vertex in A_3, then some vertex in $A \setminus A_1$ has no neighbour in $B_1 \cup \{b_2\}$; so we can add b_2 to B_1, contrary to the maximality of $A_1 \cup B_1$. So B_2 is complete to A_3. The only edges between $A_3 \cup B_3$ and $A_1 \cup A_2 \cup B_1 \cup B_2$ are those between A_3 and B_2. Since $|A_1 \cup B_1| \geq 2$ and G admits no 1-join, it follows that $|A_3 \cup B_3| \leq 1$. Since $A_3 \neq \emptyset$, it follows that $|A_3| = 1$, $A_3 = \{a_3\}$ say, and $B_3 = \emptyset$. But then $G \setminus N[a_3]$ is connected. This proves 11.3. $\blacksquare$

We deduce 11.2, which we restate:

11.4 *Let G be a bipartite graph with bipartition (A, B), where $B \neq \emptyset$ and every vertex in A has positive degree. Then the following are equivalent:*

- *for every hole C of G of length at least six, some vertex in B has at least three neighbours in $V(C)$;*

- *there is a tree T with $V(T) = B$, such that for each $a \in A$, $N(a)$ is the vertex set of a subtree of T.*

Proof. Suppose that the second bullet holds; we will prove the first. Let T be a tree with $V(T) = B$. Let C be a hole of G of length at least six, with vertices

$$b_1\text{-}a_1\text{-}b_2\text{-}a_2\text{-}\cdots\text{-}b_k\text{-}a_k\text{-}b_1$$

for some $k \geq 3$, where $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_k \in B$. For $1 \leq i \leq k$ let T_i be the subtree of T with vertex set $N(a_i)$. Thus for $1 \leq i \leq k$, b_i belongs to the trees T_i, T_{i-1} but to no other of $T_1, \ldots, T_k$. (We read the subscripts modulo k.) We need to show that some vertex of T belongs to three of the trees $T_1, \ldots, T_k$, to prove that the first bullet holds.

Let P be the path of T with ends b_1, b_2. Thus $P \subseteq T_1$. Let e be an edge of P. If e is not an edge of any of $T_2, \ldots, T_k$, then each of $T_2, \ldots, T_k$ is a subtree of one of the two components of $T \setminus e$, and so $\{T_2, \ldots, T_k\}$ may be partitioned into two nonempty subsets X, Y, with $T_k \in X$ (because it contains b_1) and $T_2 \in Y$ (because it contains b_2), such that each tree in the first subset is disjoint from each tree in the second, a contradiction. Thus each edge of P belongs to one of $T_2, \ldots, T_k$. They do not all belong to the same tree T_i where $2 \leq i \leq k$, since otherwise $b_1, b_2 \in V(T_i)$ contradicting that $k \geq 3$. So there are two consecutive edges of P that belong to different trees in the list $T_2, \ldots, T_k$. But then the common end of these two edges belongs to three of the trees $T_1, \ldots, T_k$, and so the first bullet holds.

We prove the converse implication by induction on $|V(G)|$. Let G be a bipartite graph with bipartition (A, B), where $B \neq \emptyset$ and every vertex in A has a neighbour in B, such that for every hole C of G of length at least six, some vertex in B has at least three neighbours in $V(C)$. We must show that the second bullet holds. From the inductive hypothesis we may assume that G is connected. The result is easy if $|A| \leq 1$ or $|B| = 1$, so we assume that $|A|, |B| \geq 2$.

(1) *We may assume that G does not admit a 1-join.*

Suppose that G admits a 1-join (V_1, V_2). Let $X_i \subseteq V_i$ for $i = 1, 2$, such that X_1 is complete to X_2 and there are no other edges between V_1, V_2. Thus $X_1, X_2 \neq \emptyset$ since G is connected; and so, since G is bipartite, X_1 is a subset of one of A, B, and X_2 of the other. We may assume that $X_1 \subseteq A$ and $X_2 \subseteq B$.

Take a new vertex x_2 and add it to the graph $G[V_1]$, making it adjacent to the vertices in X_1. Let this graph just made be G_1; it admits a bipartition $(V_1 \cap A, (V_1 \cap B) \cup \{x_2\})$, and it is connected since G is connected. Since G_1 is isomorphic to a subgraph of G, and $(V_1 \cap B) \cup \{x_2\} \neq \emptyset$ and every vertex in $V_1 \cap A$ has a neighbour in $(V_1 \cap B) \cup \{x_2\}$ (since G_1 is connected), the inductive hypothesis implies that there is a tree T_1 with vertex set $(V_1 \cap B) \cup \{x_2\}$, such that for each $a \in V_1 \cap A$, $N_{G_1}(a)$ is the vertex set of a subtree of T_1.

Take a new vertex x_1 and add it to the graph $G[V_2]$, making it adjacent to the vertices in X_2. Let this graph just made be G_2; it is connected, and admits a bipartition $((V_2 \cap A) \cup \{x_1\}, V_2 \cap B)$. Since it is isomorphic to a subgraph of G, and $V_2 \cap B \neq \emptyset$ (because it contains X_2) and every vertex in $(V_2 \cap A) \cup \{x_1\}$ has a neighbour in $V_2 \cap B$, the inductive hypothesis implies that there is a tree T_2 with vertex set $V_2 \cap B$, such that for each $a \in (V_2 \cap A) \cup \{x_1\}$, $N_{G_2}(a)$ is the vertex set of a subtree of T_2. Let T be the tree obtains from the disjoint union of T_1, T_2 by identifying x_2 ($\in V(T_1)$) and some vertex (x_2' say) of X_2 ($\subseteq V(T_2)$). Then for each $a \in A$, $N(a)$ is the vertex set of a subtree of

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T. To see this, it is clear if $a \in V_2 \cap A$, or $a \in (V_1 \cap A) \setminus X_1$. If $a \in X_1$, its neighbour set in T_1 forms a subtree of T_1 containing x_2, and its neighbour set X_2 in T_2 is the vertex set of a subtree of T_2; and the union of these (after identifying x_2, x_2') is a subtree of T. This proves (1).

From 11.3 and (1), there exists $b \in B$ such that $G \setminus N[b]$ is connected. We may assume that every vertex in A has degree at least two, because if one of them has degree one, the result follows easily by deleting it and applying the inductive hypothesis.

(2) *For all distinct $a, a' \in N(b)$, there is a vertex in $B \setminus \{b\}$ adjacent to them both.*

Suppose that $a, a' \in N(b)$ have no common neighbour in $B \setminus \{b\}$. But a, a' each have a neighbour in $B \setminus \{b\}$; and since the graph $G \setminus N[b]$ is connected, there is a path P with interior in $V(G) \setminus N[b]$ and with ends a, a'. Take the shortest such path; then it is induced, and has length at least four, since a, a' have no common neighbour in $B \setminus \{b\}$. Now adding b and the edges ba, ba' to P makes a hole in G of length at least six; and so from the hypothesis, some vertex $b' \in B$ has at least three neighbours in this hole, and hence in $V(P)$. But $b' \neq b$, since the interior of P is a subset of $V(G) \setminus N[b]$; and rerouting through b' the subpath of P between the first and last neighbours on P, gives a path shorter than P, a contradiction. This proves (2).

Since $B \setminus \{b\} \neq \emptyset$ and every vertex in A has a neighbour in $B \setminus \{b\}$, the inductive hypothesis implies that there is a tree S with vertex set $B \setminus \{b\}$, such that for each $a \in A$, its neighbour set in $B \setminus \{b\}$ is the vertex set of a subtree of S. Let this subtree be S_a.

Since the subtrees S_a ($a \in N(b)$) pairwise intersect by (2), the Helly property of subtrees of a tree implies that there exists $b' \in B \setminus \{b\}$ that belong to all of them. Let T be obtained from S by adding the new vertex b and an edge bb'; then T satisfies the second bullet. This proves 11.4. ∎

Now we deduce 11.1, which we restate:

11.5 *Let G be a bipartite graph with bipartition (A, B), and let A be the union of disjoint sets A_1, A_2. Suppose that A_1, A_2 are both laminar in B. Then the following are equivalent:*

- *every hole of G has length four;*

- *there is a tree T with $V(T) = B$, such that for each $a \in A$, $N(a)$ is the vertex set of a subtree of T.*

Proof. Let T be a tree with $V(T) = B$, such that for each $a \in A$, $N(a)$ is the vertex set of a subtree of T. We claim that the first bullet holds. Suppose that C is a hole in G of length at least six. By 11.2, some vertex $b \in B$ has at least three neighbours in $V(C)$; and so we may assume that at least two of them belong to A_1, say. Let $a, a' \in V(C) \cap A_1$ both be adjacent to b. Since A_1 is laminar, and a, a' have a common neighbour in B, we may assume that $N(a) \subseteq N(a')$. Consequently both neighbours of a in C are adjacent to a', a contradiction since C is a hole of length at least six.

For the converse, assume the first bullet of 11.5 holds. Then trivially the first bullet of 11.2 holds, so by 11.2 the second bullet of 11.5 holds. This proves 11.5. ∎

12 The case when ℓ is even.

In this section we complete the proof of 1.3 when ℓ is even. In view of 2.2 it suffices to prove the following:

12.1 *Let $\ell \geq 7$ be even, and let G be an ℓ-holed graph with no clique cutset or universal vertex. Then G is either a blow-up of a cycle of length ℓ, or a blow-up of an ℓ-framework.*

Proof. By 8.1, we may assume that G is a bordered blow-up of an ℓ-frame. Let $G = H \cup J \cup K$, where:

- H is a blow-up of the ℓ-frame F, where F has sides A, B;

- $(J, A, (W_t : t \in V(A)))$ and $(K, B, (W_t : t \in V(B)))$ are borders; and

- $V(H \cap J) = W(A)$, and $V(H \cap K) = W(B)$, and $V(J), V(K)$ are disjoint and anticomplete.

Let the bars of F be $P_1, \ldots, P_k$ where $P_1, \ldots, P_m$ have length $\ell/2 - 1$, and $P_{m+1}, \ldots, P_k$ have length $\ell/2 - 2$, and $m \leq k - 2$. Let each bar P_i have ends $a_i \in V(A)$ and $b_i \in V(B)$. For $1 \leq i \leq k$, we say that a_i is the *mate* of b_i and vice versa. Let $M = \{1, \ldots, m\}$ and $N = \{m+1, \ldots, k\}$, and A_M be the set $\{a_i : i \in M\}$, and define A_N, B_M, B_N similarly. Thus A_N, B_N are stable sets of F, A_M, B_M are cliques of F, and for $i \in N$ and $j \in M$, a_i, a_j are F-adjacent if and only if b_i, b_j are not F-adjacent. From the definition of an ℓ-frame, it follows that some vertex in A_N has no F-neighbour in $V(A)$.

From 9.1, there is a basis T for $(J, A, (W_t : t \in V(A)))$ and a basis S for $(K, B, (W_t : t \in V(B)))$. We will show that $S \cup T \cup P_1 \cup \cdots \cup P_k$ is an ℓ-framework, and G is a blow-up of it.

From 9.9, there is an (A_M, A_N)-spine R in T, and a (B_M, B_N)-spine R' in S. Since neither of A, B is connected, it follows that $r(T) \notin V(A)$, and $r(S) \notin V(B)$. Let $a_0, a_1, \ldots, a_m$ be the vertices of $V(A) \cup \{r(T)\}$ in order in R, where $a_0 = r(T)$. For $0 \leq i \leq m$ let T_i be the subarborescence of T induced on the set of vertices v of T such that there is a directed path of T from a_i to v, not using a_{i+1} if $i < m$. Thus the sets $V(A) \cap V(T_0), V(A) \cap V(T_1), \ldots, V(A) \cap V(T_m)$ partition $V(A)$. Let $V(A) \setminus V(R) = \{a_{m+1}, \ldots, a_k\}$, numbered such that for $0 \leq i < i' \leq m$, if $a_j \in V(T_i)$ and $a_{j'} \in V(T_{i'})$ then $j > j'$. Let $V(B) \cap V(R') = \{c_1, \ldots, c_m\}$ where $r(S) = c_0, c_1, \ldots, c_m$ are in order in R'. Since R' is a (B_M, B_N)-spine, it follows that $\{c_1, \ldots, c_m\} = \{b_1, \ldots, b_m\}$. For $0 \leq i \leq m$ let S_i be the subarborescence of S induced on the set of vertices v of S such that there is a directed path of S from c_i to v, not using c_{i+1} if $i < m$.

(1) *T_0, T_m both have at least one leaf, and so the last vertex of R is not a leaf of T. Similarly, S_0, S_m both have a leaf, and so the last vertex of R' is not a leaf of S.*

From the definition of an ℓ-frame, there exists $b_i \in B_N$ with no neighbours in B_M, and hence $a_i \in A_N$ is complete to A_M, and in particular is J-adjacent to a_m; and consequently $a_i \in T_m$. Thus T_m has a leaf. Since $a_0 \notin V(A)$, it follows from the definition of a basis that the T-inneighbour of a_1 has outdegree at least two, and so T_0 has a leaf. The second claim follows similarly. This proves (1).

Let $0 = i_0 < i_1 < \cdots < i_p = m$ be the values of $i \in \{0, \ldots, m\}$ with $L(T_i) \neq \emptyset$, and let $0 = i'_0 < i'_1 < \cdots < i'_{p'} = m$ be the values of $i \in \{0, \ldots, m\}$ with $L(S_i) \neq \emptyset$.

(2) $p = p'$; and for $0 \leq q \leq p$, the vertices in $L(T_{i_q}) \cap V(A)$ are the mates of the vertices in $L(S_{i'_{p-q}}) \cap V(B)$.

Let us say two vertices in $L(T) = A_N$ are *equivalent* if they have the same J-neighbours in $\{a_0, \ldots, a_m\}$. The equivalence classes are the nonempty sets of the form $L(T_i) \cap V(A)$ for some $i \in \{0, \ldots, m\}$, that is, the sets

$$L(T_{i_0}) \cap V(A), L(T_{i_1}) \cap V(A), \ldots, L(T_{i_p}) \cap V(A).$$

But two vertices $a_i, a_j \in A_N$ have the same neighbours in A_M if and only if b_i, b_j have the same neighbours in B_M; and consequently $p = p'$, and there is a bijection ϕ from $\{i_0, \ldots, i_p\}$ onto $\{i'_0, \ldots, i'_p\}$ such that for $0 \leq q \leq p$, $L(S_{\phi(i_q)}) \cap V(B)$ is the set of mates of the vertices in $L(T_{i_q}) \cap V(A)$.

Suppose that there exist q, q' with $0 \leq q < q' \leq p$ such that $\phi(i_q) < \phi(i_{q'})$. Let $a_j \in L(T_{i_q})$, and so $b_j \in L(S_{\phi(i_q)})$. Since $i_q < i_{q'}$, a_j is nonadjacent to $a_{i_{q'}}$. But since $\phi(i_q) < \phi(i_{q'})$, b_j is nonadjacent to $b_{\phi(i_{q'})}$, a contradiction. Thus there are no such q, q'; and so $\phi(i_q) = i'_{p-q}$ for $0 \leq q \leq p$. This proves (2).

(3) $i'_{p-q} = m - i_q$ for $0 \leq q \leq p$.

Let $0 \leq q < p$. The vertices in $\{a_1, \ldots, a_m\}$ that are complete to $L(T_{i_{q+1}}) \cap V(A)$ and anticomplete to $L(T_{i_q}) \cap V(A)$ are the vertices a_j for $j \in \{i_q + 1, \ldots, i_{q+1}\}$; and so by (2), their mates are the vertices in $\{c_1, \ldots, c_m\}$ that are anticomplete to $L(S_{i'_{p-q-1}}) \cap V(B)$ and complete to $L(S_{i'_{p-q}}) \cap V(A)$, that is, the vertices $c_{i'_{p-q-1}+1}, \ldots, c_{i'_{p-q}}$. Consequently $i_{q+1} - i_q = i'_{p-q} - i'_{p-q-1}$, that is, $i_{q+1} + i'_{p-q-1} = i_q + i'_{p-q}$, for $0 \leq q < p$. Since $i_0 = 0$ and $i'_p = m$, we deduce that $i_q + i'_{p-q} = m$ for $0 \leq q \leq p$. By induction on q it follows that $i'_{p-q} = m - i_q$ for $0 \leq q \leq p$. This proves (3).

Let $1 \leq q < p$. As in (3), the mates of the vertices $a_{i_q+1}, \ldots, a_{i_{q+1}}$ are the vertices

$$c_{i'_{p-q-1}+1}, \ldots, c_{i'_{p-q}}$$

that is, $c_{m-i_{q+1}+1}, \ldots, c_{m-i_q}$. But $c_{m-i_{q+1}+1}, \ldots, c_{m-i_q}$ are all adjacent K-twins; and so we may choose $S, c_0, \ldots, c_m$ such that $b_j = c_{m-j+1}$ for $i_q < j \leq i_{q+1}$. By repeating this for all values of q we may therefore assume that $b_j = c_{m-j+1}$ for $0 < j \leq m$.

In summary, there exist $0 = i_0 < i_1 < \cdots < i_p = m$ with the following properties:

- For $0 \leq i \leq m$ let T_i be the subarborescence of T induced on the set of vertices v of T such that there is a directed path of T from a_i to v, not using a_{i+1} if $i < m$. Then $|T_i| > 1$ if and only if $i \in \{i_0, i_1, \ldots, i_m\}$.

- Let S_0 be the subarborescence of S induced on the set of vertices v of S such that there is a directed path of S from b_0 to v, not using b_m if $m > 0$. For $1 \leq i \leq m$ let S_i be the subarborescence of S induced on the set of vertices v of S such that there is a directed path of S from $b_i = c_{m-i+1}$ to v, not using $b_{i-1} = c_{m-i}$ if $i > 0$. Then $|S_i| > 1$ if and only if $i \in \{0, i_0 + 1, i_1 + 1, \ldots, i_{m-1} + 1\}$.

- For $0 \leq q < p$, the leaves of T_{i_q} in $V(A)$ are the mates of the leaves of S_{i_q+1} in $V(B)$, and the leaves of T_{i_p} in $V(A)$ are the mates of the leaves of S_0 in $V(B)$.

To complete the proof of 12.1, it suffices to check the condition about being coarboreal. Let (x, y) be one of the pairs $(i_0, i_0 + 1), \ldots, (i_{p-1}, i_{p-1} + 1), (i_p, 0)$. Thus there are arborescences T_x, S_y defined, and the set of leaves of T_x in $V(A)$ are the mates of the leaves of S_y in $V(B)$. We must show that T_x, S_y are coarboreal under the bijection that sends each vertex in $L(T_x) \cap V(A)$ to its mate. Let the leaves of T_x be $\{a_r, \ldots, a_s\}$; so the leaves of S_y are $\{b_r, \ldots, b_s\}$.

Let $X = V(T_x) \setminus L(T_x)$ and $Y = V(T_y) \setminus L(T_y)$. Make a bipartite graph D with bipartition $(X \cup Y, \{r, \ldots, s\})$, in which each vertex $v \in X$ is adjacent to $i \in \{r, \ldots, s\}$ if and only if v is J-adjacent to a_i, and each vertex $v \in Y$ is adjacent to $i \in \{r, \ldots, s\}$ if and only if v is K-adjacent to b_i. Then X, Y are both laminar in D, so we can apply 11.1. If the second bullet of 11.1 holds, then T_x, S_y are coarboreal as required, so we assume that the first bullet of 11.1 holds, and so there is a hole C of D with length more than four.

If $u, v \in V(C) \cap (X \cup Y)$, with distance two in C, then each of them has a neighbour in C not adjacent to the other; and since they have a common neighbour in C, and X, Y are both laminar in D, it follows that one of u, v is in X and the other in Y. Hence the length of C is a multiple of four, and we may label its vertices in order as

$$v_1\text{-}j_1\text{-}v_2\text{-}j_2\text{-}\cdots\text{-}v_t\text{-}j_t\text{-}v_1$$

where $t \geq 4$ is even, and $v_1, v_3, v_5, \ldots \in X$, and $v_2, v_4, v_6, \ldots \in Y$, and $j_1, j_2, \ldots, j_t \in \{r, \ldots, s\}$. The set $\{v_1, v_3, v_5, \ldots\}$ is laminar in D, and so no two of its members have a common neighbour in $\{r, \ldots, s\}$; and consequently $v_1, v_3, v_5, \ldots$ are pairwise not J-adjacent, and similarly $v_2, v_4, v_6, \ldots$ are pairwise not K-adjacent. We recall that for $1 \leq i \leq k$, P_i is the bar of the ℓ-frame F with ends a_i, b_i. Then

$$v_1\text{-}a_{j_1}\text{-}P_{j_1}\text{-}b_{j_1}\text{-}v_2\text{-}b_{j_2}\text{-}P_{j_2}\text{-}a_{j_2}\text{-}v_3\text{-}\cdots\text{-}a_{j_t}\text{-}v_1$$

is a hole of G. Since each P_i has length $\ell/2 - 2$, this hole has length $(\ell/2)t$, a contradiction. This proves that T_x, S_y are coarboreal under the bijection that sends each vertex in $L(T_x) \cap V(A)$ to its mate. Consequently this proves 12.1, and so completes the proof of 1.3. ∎

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