

Linear Algebra

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12.1 - Change of Basis

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Coordinates of general vector space

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Definition

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V , and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$$

Then the scalars c_1, c_2, \dots, c_n are called **coordinates vector of \mathbf{v} relative to the basis S** .

The vector $\{c_1, c_2, \dots, c_n\}$ in \mathbb{R}^n is called the **coordinates vector of \mathbf{v} relative to the basis S** , and is denoted by

$$(\mathbf{v})_S = (c_1, c_2, \dots, c_n)$$

Remark.

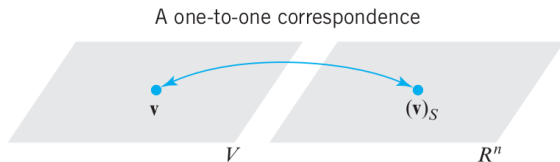
A basis S of a vector space V is a **set**. This means that the order in which those vectors in S are listed does not generally matter.

To deal with this, we define **ordered basis**, which is the basis in which the listing order of the basis vectors remains fixed.

Coordinates of general vector space

\mathbf{v}_S is a vector in \mathbb{R}^n .

Once an ordered basis S is given for a vector space V , the “Uniqueness Theorem” establishes a **one-to-one correspondence** between vectors in V and vectors in \mathbb{R}^n .



Example 1: coordinates relative to the standard basis for \mathbb{R}^n

For the vector space $V = \mathbb{R}^n$ and S is the standard basis, the coordinate vector $(\mathbf{v})_S$ and the vector \mathbf{v} are the same;

$$\mathbf{v} = (\mathbf{v})_S$$

Example

For $V = \mathbb{R}^3$, $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

The representation of vector $\mathbf{v} = (a, b, c)$ in the standard basis is:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

The coordinate vector relative to the basis S is $(\mathbf{v})_S = (a, b, c)$ (same as \mathbf{v}).

Example 2: coordinate vectors relative to standard bases

Find the coordinate vector for the **polynomial**:

$$\mathbf{p}(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

relative to the standard basis for the vector space P_n .

Solution:

The standard basis for P_n is: $= \{1, x, x^2, \dots, x^n\}$.

So, the coordinate vector for \mathbf{p} relative to S is:

$$(\mathbf{p})_S = (c_0, c_1, c_2, \dots, c_n)$$

Example 3: coordinate vectors relative to standard bases

Find the coordinate vector of:

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for M_{22} .

Solution:

The standard basis vectors for M_{22} is:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Hence,

$$B = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So, the coordinate vector of B relative to S is:

$$(B)_S = (a, b, c, d)$$

Exercise 1

Show that the following set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a basis of \mathbb{R}^3 .

$$\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 9, 0), \mathbf{v}_3 = (3, 3, 4)$$

Find the coordinate vector of $\mathbf{v} = (5, 1 - 9)$ relative to the basis S .

Exercise 1

Show that the following set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a basis of \mathbb{R}^3 .

$$\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 9, 0), \mathbf{v}_3 = (3, 3, 4)$$

Find the coordinate vector of $\mathbf{v} = (5, 1 - 9)$ relative to the basis S . **Solution:**

Question 1 (*skipped*)

Question 2:

We have to find the values c_1, c_2, c_3 s.t.:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

or, in this case:

$$(5, 1 - 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$$

from which we can extract the linear equations system:

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 5 \\ 2c_1 + 9c_2 + 3c_3 = -1 \\ c_1 + 4c_3 = 9 \end{cases}$$

Solving the system, we obtain (verify it!):

$$c_1 = 1, c_2 = -1, c_3 = 2$$

This means that: $(\mathbf{v})_S = (1, -1, 2)$.

Exercise 2

Find the vector \mathbf{v} in \mathbb{R}^3 whose coordinate vector relative to $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 9, 0), \mathbf{v}_3 = (3, 3, 4)$$

is $(\mathbf{v})_S = (-1, 3, 2)$.

Solution:

Let: $(c_1, c_2, c_3) = (-1, 3, 2)$. Hence,

$$\begin{aligned}\mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= (-1)(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4) \\ &= (11, 31, 7)\end{aligned}$$

So, the vector \mathbf{v} for which $(\mathbf{v})_S = (-1, 3, 2)$ is $(11, 31, 7)$.

Change of basis

Why change of basis needed?

- A basis that is suitable for one problem may not be suitable for another;
- ?
- ?

Coordinate maps

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for a finite-dimensional vector space V . Let the coordinate vector of \mathbf{v} relative to S be:

$$(\mathbf{v})_S = (c_1, c_2, \dots, c_n)$$

The one-to-one correspondence (mapping) between vectors in V and vectors in the Euclidean vector space \mathbb{R}^n is defined as;

$$\mathbf{v} \rightarrow (\mathbf{v})_S$$

This is called the **coordinate map relative to S from V to \mathbb{R}^n** .

We will use column matrix to represent the coordinate vectors:

$$[\mathbf{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The Change-of-Basis Problem

Problem: If \mathbf{v} is a vector in a finite-dimensional vector space V , and we change the basis for V from a basis B to another basis B' , how are the coordinate vector $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ related?

- In the literature, B is usually called the **old basis** and B' is called the **new basis**.
- For convenience, I will use the terms **first basis** and **second basis**.

Solution of the Change-of-Basis problem (in 2-dimensional space)

Let

$$B = \{\mathbf{u}_1, \mathbf{u}_2\} \quad \text{and} \quad B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$$

and the coordinate vectors for the 2nd basis relative to the 1st basis is:

$$[\mathbf{u}'_1]_B = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad [\mathbf{u}'_2]_B = \begin{bmatrix} c \\ d \end{bmatrix}$$

i.e., the following relation holds:

$$\mathbf{u}'_1 = a\mathbf{u}_1 + b\mathbf{u}_2 \tag{1}$$

$$\mathbf{u}'_2 = c\mathbf{u}_1 + d\mathbf{u}_2 \tag{2}$$

Problem: Given a vector $\mathbf{v} \in V$, with

$$[\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

How to find the coordinate vector of \mathbf{v} relative to B ?

Solution (*cont.*)

Since the coordinate vector of \mathbf{v} relative to B' is

$$[\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

this means that:

$$\mathbf{v} = k_1 \mathbf{u}'_1 + k_2 \mathbf{u}'_2$$

By the relation (1) and (2) in the previous slide, we have:

$$\begin{aligned} \mathbf{v} &= k_1(a\mathbf{u}_1 + b\mathbf{u}_2) + k_2(c\mathbf{u}_1 + d\mathbf{u}_2) \\ &= (k_1a + k_2c)\mathbf{u}_1 + (k_1b + k_2d)\mathbf{u}_2 \end{aligned}$$

So, the coordinate vector of \mathbf{v} relative to B is:

$$[\mathbf{v}]_B = \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix}$$

Finding transition matrices

The vector $[\mathbf{v}]_B = \begin{bmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{bmatrix}$ can be written as:

$$[\mathbf{v}]_B = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} [\mathbf{v}]_{B'}$$

Let $P = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. This means that:

the coordinate vector $[\mathbf{v}]_B$ can be obtained by multiplying the coordinate vector $[\mathbf{v}]_{B'}$ on the left by matrix P .

Solution of the Change-of-Basis Problem

Theorem

Let V be an n -dimensional space. If we want to change the basis for V from basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ to another basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_n\}$.

Then for each vector $\mathbf{v} \in V$, we have the following relation between $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$, as follows:

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$$

where P is the matrix whose columns are the coordinate vectors of B' relative to B , i.e., the columns of P are:

$$[\mathbf{u}'_1]_B, [\mathbf{u}'_2]_B, \dots, [\mathbf{u}'_n]_B$$

P is called the **transition matrix from B' to B** , and is denoted by $P_{B' \rightarrow B}$.

$$P_{B' \rightarrow B} = [[\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \mid \dots \mid [\mathbf{u}'_n]_B] \quad (1)$$

$$P_{B \rightarrow B'} = [[\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid \dots \mid [\mathbf{u}_n]_{B'}] \quad (2)$$

Example 1: finding transition matrices

Given the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for \mathbb{R}^2 , where:

$$\mathbf{u}_1 = (1, 0), \mathbf{u}_2 = (0, 1), \mathbf{u}'_1 = (1, 1), \mathbf{u}'_2 = (2, 1)$$

1. Find the transition matrix $P_{B' \rightarrow B}$ from B' to B .
2. Find the transition matrix $P_{B \rightarrow B'}$ from B to B' .

Solution of Example 1

Solution 1: The transition matrix $P_{B' \rightarrow B}$ from B' to B .

$$\mathbf{u}'_1 = \mathbf{u}_1 + \mathbf{u}_2$$

$$\mathbf{u}'_2 = 2\mathbf{u}_1 + \mathbf{u}_2$$

Hence,

$$[\mathbf{u}'_1]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{u}'_2]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So,

$$P_{B' \rightarrow B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Solution of Example 1 (*cont.*)

Solution 2: The transition matrix $P_{B \rightarrow B'}$ from B to B' .

$$\mathbf{u}_1 = -\mathbf{u}'_1 + \mathbf{u}'_2$$

$$\mathbf{u}_2 = 2\mathbf{u}'_1 - \mathbf{u}'_2$$

Hence,

$$[\mathbf{u}_1]_{B'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{u}_2]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

So,

$$P_{B \rightarrow B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Example 2: computing coordinate vectors

Problem:

Given the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for \mathbb{R}^2 , where:

$$\mathbf{u}_1 = (1, 0), \mathbf{u}_2 = (0, 1), \mathbf{u}'_1 = (1, 1), \mathbf{u}'_2 = (2, 1)$$

Find the vector $[\mathbf{v}]_B$ given that $[\mathbf{v}]_{B'} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

Solution:

$$[\mathbf{v}]_B = P_{B' \rightarrow B} [\mathbf{v}]_{B'} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Invertibility of transition matrices

What happen if we multiply $P_{B' \rightarrow B}$ with $P_{B \rightarrow B'}$?

- We first map the B -coordinates of \mathbf{v} into its B' -coordinates;
- then map the B' -coordinates of \mathbf{v} into its B -coordinates;
- This yields that \mathbf{v} is back to its B -coordinates.

$$P_{B' \rightarrow B} P_{B \rightarrow B'} = P_{B \rightarrow B} = I$$

Example

Read again Example 1.

$$(P_{B' \rightarrow B})(P_{B \rightarrow B'}) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Theorem

$P_{B' \rightarrow B}$ is invertible, and its inverse is $P_{B \rightarrow B'}$.

A procedure for computing $P_{B \rightarrow B'}$

Procedure:

1. Form the matrix $[B' \mid B]$;
2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form;
3. The resulting matrix will be $[I \mid P_{B \rightarrow B'}]$; *Extract the matrix $P_{B \rightarrow B'}$ from the right side of the matrix in Step 3.*

Diagram:

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}]$$

(1)

Exercise

In Example 1, we are given the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for \mathbb{R}^2 , where:

$$\mathbf{u}_1 = (1, 0), \mathbf{u}_2 = (0, 1), \mathbf{u}'_1 = (1, 1), \mathbf{u}'_2 = (2, 1)$$

Use formula (1) of the previous slide to find:

1. The transition matrix from B' to B .
2. The transition matrix from B to B' .

Solution of exercise

Question 1. Old basis is B' and new basis is B . Then:

$$[B \mid B'] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array} \right]$$

Since the left side is already the identity matrix, no reduction is needed.
Hence,

$$P_{B' \rightarrow B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Question 2. Old basis is B and new basis is B' . Then:

$$[B' \mid B] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right]$$

By reducing the matrix, we obtain:

$$[I \mid \text{transition from } B \text{ to } B'] = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array} \right]$$

$$P_{B \rightarrow B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Exercise (at home)

Given a basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\}$ for \mathbb{R}^3 , where:

$$\mathbf{u}_1 = (2, 1, 1), \mathbf{u}_2 = (2, -1, 1), \mathbf{u}_3 = (1, 2, 1)$$

$$\mathbf{u}'_1 = (3, 1, -5), \mathbf{u}'_2 = (1, 1, -3), \mathbf{u}'_3 = (-1, 0, 2)$$

1. Find the transition matrix from B to B' .
2. Find the transition matrix from the standard basis of \mathbb{R}^3 to B .
3. Find the transition matrix from the standard basis of \mathbb{R}^3 to B' .
4. Find the coordinate vector \mathbf{w} relative to basis B , if the coordinate vector \mathbf{w} relative to the standard basis S is $[\mathbf{w}]_S = (-5, 8, -5)$.

Task

Create a computer program to transform a 2-dimensional vector from one basis to another basis.

Specification:

1. It takes input from a user: two basis B and B' , and a vector \mathbf{v} relative to the basis B .
2. It must output a new vector coordinate $'$ which is the vector coordinate of \mathbf{v} relative to the basis B'

to be continued...