Linear Algebra
[KOMS120301] - 2023/2024

3.1 - Linear System of Equations

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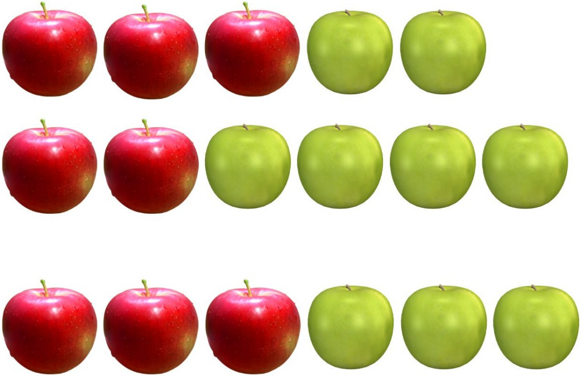
Motivating example



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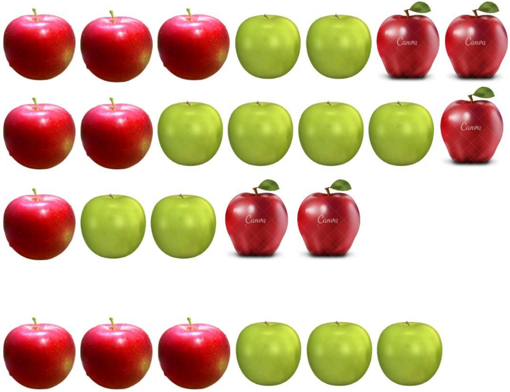
Motivating example



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Part 1: System of linear equations

(*We sometime call it "linear system"*)

Learning objectives

After this lecture, you should be able to:

1. analyze the components of a system of linear equations;
2. verify whether a given set is a solution of a linear system;
3. identify a homogeneous and non-homogeneous linear system;
4. formulate the coefficient matrix and augmented matrix of a given linear system;
5. showing that elementary row system gives an equivalent linear system;
6. analyze the geometric interpretation of a linear system with 1, 2, or 3 variables;
7. apply the elimination and substitution algorithms to solve a linear system;
8. explain the concept of linear system written in triangular matrix or in echelon form.

Terminology and notation (1)

Given unknowns variables $x_1, x_2, \ldots, x_n$, a linear equations on the variables is defined as:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$ (this can be replaced by another *field*).

A solution of equation (1) is a list of values for the unknowns or a vector u in $\mathbb{R}^n$.

$$x_1 = r_1,\ x_2 = r_2,\ \ldots,\ x_n = r_n \quad \text{or} \quad u = (r_1, r_2, \ldots, r_n)$$

This means that:

$$a_1r_1 + a_2r_2 + \cdots + a_nr_n = b \quad \text{is true}$$

In this case, we say that u satisfies equation (1).

Terminology and notation (2)

In equation (1):

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

We say that:

- the equation is written in the standard form
- the constant a_k is the coefficient of x_k
- b is the constant term of the equation

Note: If n is small, we use different letters to denote the variables, instead of using indexing.

Example: how many solutions are there?

Given an equation:

$$2x + 3y - z = 4$$

Can you find a solution for the equation?

How many solutions that you can find?

System of linear equations

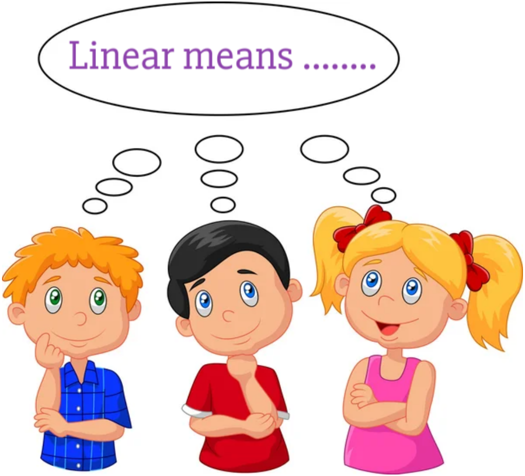
A system of linear equations is a list of linear equations: $L_1, L_2, \ldots, L_m$ with the same variables $x_1, x_2, \ldots, x_n$.

$$
\begin{align}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \tag{1}\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \tag{2}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \tag{3}\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \tag{4}
\end{align}
$$

where a_{ij} and b_i are constants.

- The system of linear equations is written in standard form
- The system is called an $m \times n$ system
- a_{ij} is the coefficient of variable x_j in the equation L_i
- the number b_i is the constant of the equation L_i

What does the word **“linear”** mean???



Solution of "system of linear equations"

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

A solution of the system is a list of values for the unknowns or a vector u in $\mathbb{R}^n$.

Example: verifying solution of a linear system

Given the following system of linear equations:

$$\begin{cases} x_1 + x_2 + 4x_3 + 3x_4 = 5 \\ 2x_1 + 3x_2 + x_3 - 2x_4 = 1 \\ x_1 + 2x_2 - 5x_3 + 4x_4 = 3 \end{cases}$$

- What is the value of m and n in the system?
- Determine whether the following are solutions of the system!
  1. $u = (-8, 6, 1, 1)$
  2. $v = (-10, 5, 1, 2)$

Part 2: Types of system of linear equations

Augmented and coefficient matrices of a system

The system of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

can be written in matrix form:

$$
\begin{bmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n \\ \cdots & & & \\ a_{m1}x_1 & a_{m2}x_2 & \cdots & a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}
$$

Augmented and coefficient matrices of a system

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

- the left matrix is called the coefficient matrix of the system;
- the right matrix is called the augmented matrix of the system.

Furthermore, the vector

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

is called the constant vector (or constant matrix) of the system.

Example: augmented matrix and coefficient matrix

Given the following system of equations:

$$\begin{cases} x_1 + x_2 + 4x_3 + 3x_4 = 5 \\ 2x_1 + 3x_2 + x_3 - 2x_4 = 1 \\ x_1 + 2x_2 - 5x_3 + 4x_4 = 3 \end{cases}$$

The coefficient matrix and the augmented matrix are as follows:

$$\begin{bmatrix} 1 & 1 & 4 & 3 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & -5 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 4 & 3 & 5 \\ 2 & 3 & 1 & -2 & 1 \\ 1 & 2 & -5 & 4 & 3 \end{bmatrix}$$

Homogeneous & non-homogeneous linear system

For the given system:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

It is called homogeneous if $b_i = 0, \forall i$. Otherwise, it is called non-homogeneous.

Every homogeneous linear system always has a solution. Can you guess what it is?

Degenerate and non-degenerate linear equations

A linear equation is degenerate if all coefficients are zero

$$0x_1 + 0x_2 + \cdots + 0x_n = b$$

Can you guess, **what is the condition s.t. the linear equation has a solution?**

Degenerate and non-degenerate linear equations

A linear equation is degenerate if all coefficients are zero

$$0x_1 + 0x_2 + \cdots + 0x_n = b$$

Can you guess, **what is the condition s.t. the linear equation has a solution?**

- If $b \neq 0$, then the equation has no solution.
- If $b = 0$, then every vector $u = (r_1, r_2, \ldots, r_n)$ in $\mathbb{R}^n$ is a solution.

Degenerate linear equations

Theorem

Let $\mathcal{L}$ be a system of linear equations that contains a degenerate equation L, with constant b.

1. *If $b \neq 0$, then the system $\mathcal{L}$ has no solution.*
2. *If $b = 0$, then L may be deleted from $\mathcal{L}$ without changing the solution set of $\mathcal{L}$.*

Leading unknown in a nondegenerate linear equation

Given a **non-degenerate** linear equation L.

- What can you say about *the coefficients of* L*?*

Leading unknown in a nondegenerate linear equation

Given a **non-degenerate** linear equation L.

- What can you say about *the coefficients of L?*

  <u>L has at least one *non-zero coefficient*</u>

Example

The following are non-degenerate linear equations.

$$0x_1 + 0x_2 + 5x_3 + 6x_4 + 0x_5 + 8x_6 = 7 \text{ and } 0x + 2y - 4z = 5$$

The zero coefficients are usually omitted.

$$5x_3 + 6x_4 + 8x_6 = 7 \text{ and } 2y - 4z = 5$$

Part 3: Elementary row operations

Linear combination

Given:

$$
\begin{align}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \tag{1}\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \tag{2}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \tag{3}\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \tag{4}
\end{align}
$$

Multiply the m equations by constants $c_1, c_2, \ldots, c_m$:

$$(c_1 a_{11} + \cdots + c_m a_{m1})x_1 + \cdots + (c_1 a_{1n} + \cdots + c_m a_{mn})x_n = c_1 b_1 + \cdots + c_m b_m$$

This is a linear combination of the equations in the system.

Example

Given a linear system:

$$\begin{cases} x_1 + \ \ x_2 + 4x_3 + 3x_4 = 5 \\ 2x_1 + 3x_2 + \ \ x_3 - 2x_4 = 1 \\ x_1 + 2x_2 - 5x_3 + 4x_4 = 3 \end{cases}$$

Then:

$$\begin{array}{rl} 3L1: & 3x_1 + 3x_2 + 12x_3 + 9x_4 \qquad = 15 \\ -2L_2: & -4x_1 - 6x_2 - 2x_3 + 4x_4 \qquad = -2 \\ 4L_1: & 4x_1 + 8x_2 - 20x_3 + 16x_4 \qquad = 12 \\ \hline \\ (\text{Sum})L: & 3x_1 + 5x_2 - 10x_3 + 29x_4 \qquad = 25 \end{array}$$

- L is a linear combination of L_1, L_2, and L_3
- Is $u = (-8, 6, 1, 1)$ a solution of the system?
- Is $u = (-8, 6, 1, 1)$ a solution of the linear combination?

What can you conclude?

Equivalent systems

Theorem

Given two systems of linear equations, say L_1 and L_2. They have the same solutions iff each equation in L_1 is a linear combination of the equations in L_2.

Definition

Two systems of linear equations are equivalent if they have the same solutions.

Elementary operations

Given a system of linear equations $L_1, L_2, \ldots, L_m$. The following operations are called elementary operations.

- **[E1]** Interchange two of the equations

$$\text{Interchange } L_i \text{ and } L_j \quad \text{or} \quad L_i \leftrightarrow L_j$$

- **[E2]** Replace an equation by a nonzero multiple of itself.

$$\text{Replace } L_i \text{ by } kL_i \quad \text{or} \quad kL_i \to L_i$$

- **[E3]** Replace an equation by the sum of a multiple of another equation and itself.

$$\text{Replace } L_j \text{ by } kL_i + L_j \quad \text{or} \quad kL_i + L_j \to L_j$$

Theorem

Given a system $\mathcal{L}$. Let $\mathcal{M}$ be the system obtained from $\mathcal{L}$ by a finite sequence of elementary operations.

Then $\mathcal{M}$ and $\mathcal{L}$ have the same solutions.

Note: Sometimes E_2 and E_3 can be applied in one step:

[E] Replace equation L_j by $kL_i + k'L_j$ (where $k, k' \neq 0$)

$$kL_i + k'L_j \to L_j$$

How to find a solution of a linear equations system?

- Use elementary operations to transform the given system into <u>an equivalent system whose solution can be easily obtained</u>

This is called **Gaussian Elimination** (will be discussed later).

Part 4: Small square systems of linear equations

Linear equation in **one variable**

Example

Solve the following linear system of one variable:

- $4x - 1 = x + 6$
- $2x - 5 - x = x + 3$
- $4 + x - 3 = 2x + 1 - x$

What can you conclude?

Linear equation in **one variable**

Example

Solve the following linear system of one variable:

- $4x - 1 = x + 6$
- $2x - 5 - x = x + 3$
- $4 + x - 3 = 2x + 1 - x$

What can you conclude?

Theorem

Given the system of unique linear equation $ax = b$.

1. *If $a \neq 0$, then $x = \frac{b}{a}$ is a unique solution of the system.*
2. *If $a = 0$, but $b \neq 0$, then the system has no solution.*
3. *If $a = 0$ and $b = 0$, then every scalar k is a solution of $ax = b$.*

Example

Example

Solve the following linear system of one variable:

- $4x - 1 = x + 6$ (Theorem 7 (1))
  In standard form: $3x = 7$. Then $x = \frac{7}{3}$ is the unique solution.
- $2x - 5 - x = x + 3$ (Theorem 7 (2))
  In standard form: $0x = 8$. The equation has no solution.
- $4 + x - 3 = 2x + 1 - x$ (Theorem 7 (3))
  In standard form: $0x = 0$. Then every scalar k is a solution.

System of two linear equations in **two variables**

Given a system of <u>two non-degenerate linear equations</u> in two variables:

$$A_1x + B_1y = C_1$$
$$A_2x + B_2y = C_2$$

Example

Solve the following system of linear equations:

$$\begin{cases} L_1: \ x - y = -4 \\ L_2: \ 3x + 2y = 12 \end{cases} \quad \begin{cases} L_1: \ x + 3y = 3 \\ L_2: \ 2x + 6y = -8 \end{cases} \quad \begin{cases} L_1: \ x + 2y = 4 \\ L_2: \ 2x + 4y = 8 \end{cases}$$

What can you conclude?

The number of solutions of (2×2)-system

1. The system has exactly one solution.

$$
\begin{aligned}
L_1: &\quad x - y = -4 \\
L_2: &\quad 3x + 2y = 12
\end{aligned}
$$

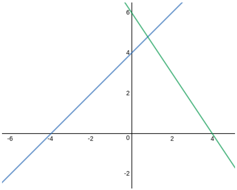
2. The system has no solution.

$$
\begin{aligned}
L_1: &\quad x + 3y = 3 \\
L_2: &\quad 2x + 6y = -8
\end{aligned}
$$

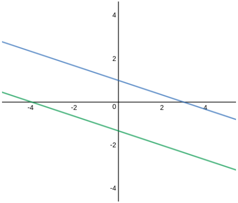
3. The system has an infinite number of solutions.

$$
\begin{aligned}
L_1: &\quad x + 2y = 4 \\
L_2: &\quad 2x + 4y = 8
\end{aligned}
$$

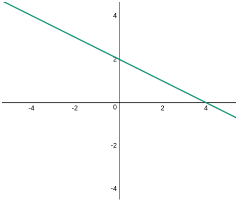
Geometric interpretation



(a) Exactly one solution



(b) No solution



(c) Infinitely many solution

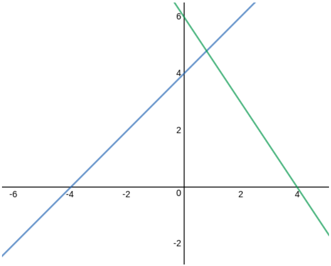
1. System with exactly one solution

- Given:

$$A_1x + B_1y = C_1$$
$$A_2x + B_2y = C_2$$

- Both lines have distinct slopes

$$\frac{A_1}{A_2} \neq \frac{B_1}{B_2} \quad \text{or} \quad A_1B_2 - A_2B_1 \neq 0$$



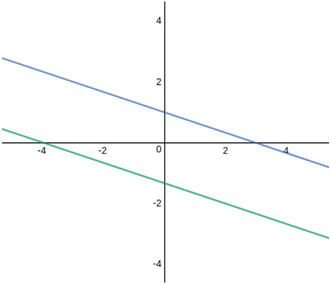
2. System with no solution

- Given:

$$A_1x + B_1y = C_1$$
$$A_2x + B_2y = C_2$$

- Both lines are parallel (have the same slope)

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2} \qquad \text{here} \quad A_1B_2 - A_2B_1 = 0$$



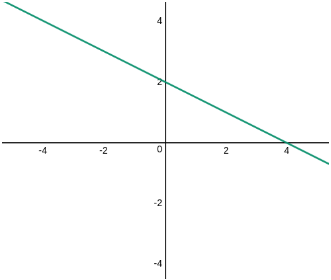
3. System with infinitely many solutions

- Given:

$$A_1x + B_1y = C_1$$
$$A_2x + B_2y = C_2$$

- Both lines have the same slopes and same y-intercepts

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \qquad \text{here} \quad A_1B_2 - A_2B_1 = 0$$



Recap

- The system has exactly one solution when $A_1B_2 - A_2B_1 \neq 0$
- The system has no solution of infinitely many solutions when $A_1B_2 - A_2B_1 = 0$

The value $A_1B_2 - A_2B_1$ is called determinant of order two

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$$

Q: Can you relate the solution of system of linear equations to determinant?

Recap

- The system has exactly one solution when $A_1B_2 - A_2B_1 \neq 0$
- The system has no solution of infinitely many solutions when $A_1B_2 - A_2B_1 = 0$

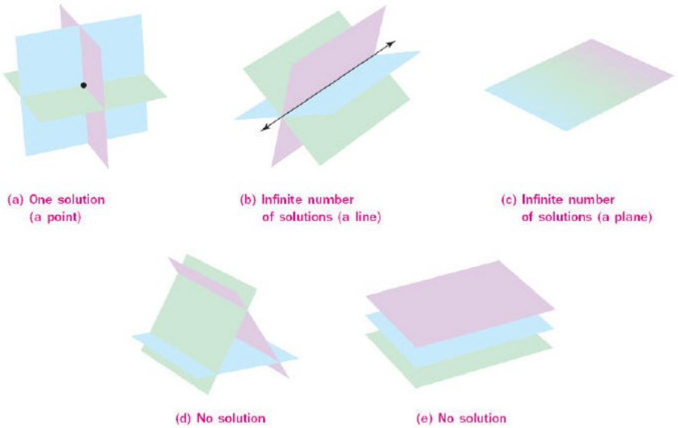
The value $A_1B_2 - A_2B_1$ is called determinant of order two

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$$

Q: Can you relate the solution of system of linear equations to determinant?

Remark: A system has a *unique solution iff the determinant of its coefficients is not zero*.

The number of solutions of (3×3)-system



Example 1: unique solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{array}\right] \xrightarrow{\text{Gaussian elimination}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

from which we can derive the set of solution:

$$x_1 = 1,\ x_2 = 0,\ x_3 = -1$$

Example 2: infinitely many solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & -1 & 1 & 2 \\ 1 & 2 & 3 & 6 \end{array}\right] \xrightarrow{\text{Gaussian elimination}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the last row, we can derive the equation:

$$0x_1 + 0x_2 + 0x_3 = 0$$

which can be satisfied by many value of x. The solution can be written in parametric form:

- Let $x_3 = k$, with $k \in \mathbb{R}$
- Then $x_2 = 2 - k$ and $x_1 = 4 - x_2 - 2x_3 = 4 - (2 - k) - 2k = 2 - k$

This means that there are an infinitely many solutions, because there are infinitely many possible values of k.

Example 3: no solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & -1 & 1 & 2 \\ 1 & 2 & 3 & 7 \end{array}\right] \xrightarrow{\text{Gaussian elimination}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

From the last row, we can derive the equation:

$$0x_1 + 0x_2 + 0x_3 = 1 \tag{1}$$

Clearly, no possible value of $x_1, x_2, x_3 \in \mathbb{R}$ that can satisfy equation (1).

What about a system with more than 3 variables?

Remark

- For a linear system with more than 3 variables, it's hard to interpret it geometrically.
- However we can check the possible number of solutions by looking at **the shape of the reduced echelon form**.

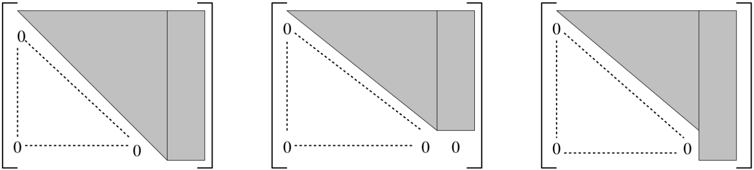


Figure: Left (unique solution), middle (many solutions), right (no solution) — *source: lecture notes of Rinaldi Munir, ITB* *to be continued...*