

ÉCOLE NORMALE SUPÉRIEURE DE LYON
Laboratoire de l'Informatique du Parallélisme (LIP) - UMR 5668

THÈSE DE DOCTORAT DE L'UNIVERSITÉ DE LYON

Spécialité de doctorat: Théorie des Graphes

Discipline: Informatique

Width Parameters on Even-Hole-Free Graphs
Paramètres de largeur des graphes sans trous pairs

Auteure:

Ni Luh Dewi SINTIARI

Directeur de Thèse:

Dr. Nicolas TROTIGNON

*Une thèse soumise en vue d'obtenir le grade de
Docteur de l'Université de Lyon,
délivré par l'École Normale Supérieure de Lyon
au titre de*

École Doctorale en Informatique et Mathématiques de Lyon (ED 512)

Mars 2021

Acknowledgements

With my deep gratitude and appreciation to my thesis advisor, Nicolas Trotignon, for accepting me as his student, and for his continued support and encouragement throughout this research project. I am immensely grateful for our scientific discussions, for our friendly chats in the corridor, for his personal support in my academic business, for his assistance during my stay in Lyon, and for many other things which can not be listed here. Nicolas has inspired me in many aspects of my academic life. He is a great researcher, an excellent advisor, a person with lots of stories and jokes who can bring a comfortable atmosphere to those around him. I aspire to be as lively and energetic as he is.

My sincere thanks to the members of my thesis advisory and exam committee: Kathie Cameron, Professor Edy Tri Baskoro, Frédéric Havet, and Aurélie Lagoutte, for their precious time to offer me with helpful feedback on how to improve my work. I would also like to acknowledge all my collaborators for sharing their knowledge and expertise, for their help, and for completing our collaborative projects: Pierre Aboulker, Isolde Adler, Eun Jung Kim, Marcin Pilipczuk, Myriam Preismann, Stéphan Thomassé, and Kristina Vušković. Special thanks to Rémi Watrigant, Édouard Bonnet, and Nicolas Bousquet for their valuable guidance throughout my studies. To Jake Horsfield and Cléophee Robin, I can't thank you enough for the stimulating discussions and memorable moments that we shared together.

My gratitude also goes to ENS de Lyon for funding this three-year PhD program, through Contrats doctoraux spécifiques pour normaliens. Thanks to the Laboratoire de l'informatique du parallélisme (LIP) for accepting me as part of them: to MALIP, MILIP, and everyone I could not mention one by one, I very much appreciate your assistance and kind reception. To everyone in MC2, thank you for welcoming me warmly into the team. I am missing so much our cheerful chats over lunches in the CROUS cafeteria. I would also like to thank the members of the Département d'Informatique, particularly to Damien Stehlé and Daniel Hirschkoff who thoughtfully guided me throughout my studies, to all the lecturers who have given me the opportunity to become a teaching assistant and learn from them, and to all my teaching colleagues for their excellent cooperation. Last but not least I would like to mention Christophe Crespelle, thanks to whom I began my journey at ENS de Lyon in the first place.

My PhD time would not be so enjoyable without the support of my friends and found family. To PPI Lyon, the doctorants Indonésiens, and all my friends who cannot be listed here, thanks for sharing the beautiful moments, for the stories, and for cheering me up. Thanks to ibu-ibu Indonesia, who have been our mothers here in Lyon, I will definitely miss your delicious cooking. To all my lab-mates, thanks so much for the happy distractions to rest my mind outside of my research. To Nicolas' family, many thanks for your warm welcome, for the lunch and dinner invitations, and for the unforgettable moments we passed together during Christmas break in Usson-en-Forez.

Let me also take this moment to thank my beloved parents and sisters, for their unconditional love and unending support. To my dear Hendra, thanks for your sympathetic ears, for always being there, in my joy and in my sorrow. To my relatives, thank you for always wishing the best for me.

Finally, my deep gratitude to everyone who has put their hands to take care of me and brought me to this step. There are no words that can express how grateful I am to meet all of you. I will really miss the time we have spent together, and I hope there will be another moment to make new memories in the future.

Contents

Acknowledgements	iii
Résumé	1
Summary	5
1 Introduction	9
1.1 What is Graph Theory	9
1.1.1 A history of Graph Theory	10
1.1.2 Graph Theory is everywhere	12
1.2 Literature review	14
1.2.1 Hereditary classes of graphs	15
1.2.2 Dealing with large graphs: graph decomposition	16
1.2.3 Parameters for graph complexity	18
1.2.4 Why even-hole-free graphs?	22
From perfect graphs to even-hole-free graphs	23
1.3 Terminology	27
1.4 Main contributions of the thesis	29
Even-hole-free graphs with no large cliques	29
Forbidding more structures and its impacts on the tree-width	31
Publications	32
2 A survey on even-hole-free graphs	33
2.1 Decomposition of even-hole-free graphs	33
2.2 The use of cutsets for algorithms	37
2.2.1 Recognition algorithm	37
2.2.2 The good and the bad cutsets in the decomposition of even-hole-free graphs	39
2.2.3 Even-hole-free graphs with no star cutset	41
2.3 Widths of several subclasses of even-hole-free graphs	44
2.3.1 Planar case	44
2.3.2 Triangle-free case	45
2.3.3 Cap-free case	48
2.3.4 Diamond-free case	50
2.3.5 Pan-free case	51
2.3.6 Rings	55
3 Layered wheels	57
3.1 Summary of the main results	59
3.2 Construction and tree-width	60
3.3 Lower bound on rank-width	78
3.4 Upper bound	83
3.5 Discussion and open problems	86

4	A bound on the tree-width	87
4.1	Known results and summary of the main results	88
4.2	Tree-width and minimal separators	89
4.3	Nested 2-wheels	92
4.4	Bounding the tree-width	95
4.5	Discussion and open problems	101
5	Even-hole-free graphs of bounded degree	103
5.1	Subcubic case	103
5.2	(Even hole, pyramid)-free graphs of maximum degree 4	109
5.3	Discussion	119
6	Conclusion and open problems	123
6.1	What can be observed from layered wheels?	124
6.2	The grid-minor-like theorem	125
	Bibliography	129

Résumé

Un graphe est une structure mathématique (V, E) , où V est un ensemble fini (non vide) d'éléments appelés *sommets* (ou *nœuds*), et E est un ensemble fini d'éléments appelés *arêtes*, dont chacun a deux sommets associés. Les graphes sont utilisés pour modéliser toutes sortes d'objets interconnectés, comme les réseaux. Chaque objet du réseau est représenté par un sommet dans le graphe et les arêtes représentent la relation par paire entre ces objets. Il existe de nombreux problèmes pratiques qui peuvent être modélisés par des graphes, et les graphes ont été appliqués dans de nombreux domaines, tels que l'informatique, les réseaux informatiques, les sciences sociales, la physique et la chimie, ou même la linguistique.

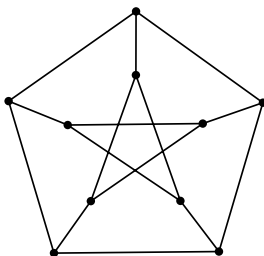


FIGURE 1: Un exemple de graphe

Au cours des dernières années, l'étude des graphes s'est considérablement développée. De nombreux sujets ont été explorés par de nombreux chercheurs du monde entier. L'un d'eux est la "théorie structurelle des graphes". Ce domaine de recherche établit des résultats qui décrivent finement les adjacences dans certaines classes de graphes. L'objectif principal est la conception d'algorithmes efficaces, ainsi que d'autres applications. Dans cette thèse, nous étudions un sujet particulier en théorie structurelle des graphes, qui est appelée "classes héréditaires de graphes". La principale préoccupation dans ce domaine est d'étudier comment l'exclusion de certaines configurations affecte la structure globale des graphes et quels types de structure permettent des algorithmes efficaces pour les classes de graphes. De nombreuses classes de graphes héréditaires sont étudiées, et dans cette thèse, nous étudions spécifiquement la classe des "graphes sans trous pairs". Nous allons maintenant la définir formellement.

Graphes sans trous pairs

Un graphe H est appelé un *sous-graphe induit* d'un graphe G si H peut être obtenu à partir de G en *supprimant* des sommets (la suppression d'un sommet v signifie que nous supprimons v et toutes les arêtes incidentes à v). Un graphe G est appelé *sans* H s'il ne contient pas H comme sous-graphe induit. Un *trou* dans un graphe G est un sous-graphe composé d'un nombre $n \geq 4$ de sommets v_1, v_2, \dots, v_n , tel que $v_i v_{i+1} \in E(G)$ pour $i \in \{1, 2, \dots, n-1\}$ et $v_1 v_n \in E(G)$, et il n'y a pas d'autres arêtes dans le graphe entre ces sommets. Un trou est *pair* ou *impair* selon la parité

de n . Par conséquent, les graphes sans trou pair sont simplement les graphes qui ne contiennent pas de trou pair comme sous-graphe induit.

L'étude des graphes sans trous pairs est initialement motivée par l'étude de la classe des *graphes de Berge*, dans une tentative de prouver la conjecture forte des graphes parfaits de Claude Berge en 1961. Il s'avère que la technique qui a été développée pour décomposer des graphes sans trous pairs a effectivement été appliquée sur les graphes de Berge, ce qui a ensuite conduit au théorème fort des graphes parfaits de Maria Chudnovsky, Neil Robertson, Paul Seymour et Robin Thomas, prouvé en 2002. Nous remarquons qu'il existe une sorte de relation de dichotomie entre la classe des graphes de Berge (ou de manière équivalente, appelés *graphes parfaits*¹) et la classe des graphes sans trous pairs. La classe des graphes de Berge ne contient pas de trous impairs et pas d'*anti-trou* impairs (un anti-trou est le complément d'un trou, et le complément d'un graphe est obtenu à partir du graphe original en remplaçant les arêtes par des non-arêtes et des non-arêtes par des arêtes). En plus d'exclure les trous pairs, la classe des graphes sans trous pairs exclut implicitement tous les anti-trous de longueur au moins 6, car chacun d'eux contient toujours un trou de longueur 4. Cependant, alors que les problèmes d'optimisation tels que la *coloration optimale*, la *clique maximale*, l'*ensemble indépendant maximal* et la *couverture de clique* sont résolubles en temps polynomial pour les graphes de Berge, ce n'est pas le cas pour les graphes sans trous pairs. Ces problèmes (sauf la clique maximale) sont encore ouverts de nos jours dans la classe des graphes sans trous pairs, et la résolution de cette question est devenue l'objectif principal dans ce domaine de recherche.

Paramètres de largeur

Les notions de "largeur de graphe", telles que *tree-width* (ou *largeur d'arbre*, désigné par tw), *rank-width* (ou *largeur de rang*, désigné par rw), *path-width* (ou *largeur de chemin*, désigné par pw), *clique-width* (ou *largeur de clique*, désigné par cw), et quelques autres largeurs ont reçu grande attention ces dernières années. Ces notions sont des paramètres mesurant la simplicité/complexité de la structure d'un graphe. Ils sont vraiment importants dans l'étude de la structure des graphes et ils ont de nombreuses applications algorithmiques. Le *tree-width*, par exemple, est un paramètre mesurant à quel point un graphe est proche d'être un *arbre* (un arbre est un graphe connexe sans cycle), et avoir une petite largeur d'arbre signifie que le graphe est proche de être un arbre. Comme nous le savons, presque tous les problèmes d'optimisation de graphes sont résolubles en temps polynomial pour les arbres, et le théorème de Courcelle (par Bruno Courcelle, 1990) indique que de nombreux problèmes d'optimisation de graphes (y compris les quatre problèmes que nous mentionnons dans la Section précédente) peuvent être décidés en linéaire temps sur des graphes de *tree-width* bornée. Il est donc intéressant d'étudier la *tree-width* des graphes lorsque l'on essaie de développer un algorithme pour les problèmes d'optimisation des graphes. Dans cette thèse, notre objectif est d'analyser le *tree-width*. Cependant, les paramètres de largeur mentionnés ci-dessus dans la liste ci-dessus sont liés les uns aux autres. Pour chaque graphe G , les éléments suivants sont valables [CR05; OS06]:

- $rw(G) \leq cw(G) \leq 2^{rw(G)+1}$;

¹Un graphe G est appelé *parfait* si pour chaque sous-graphe induit H de G , le nombre chromatique de H est égal à la taille du plus grand sous-graphe complet de H . Le théorème fort des graphes parfaits affirme qu'un graphe est parfait si et seulement si il s'agit de Berge.

- $\text{cw}(G) \leq 3 \cdot 2^{\text{tw}(G)} - 1$;
- $\text{tw}(G) \leq \text{pw}(G)$.

En termes de paramètres de largeur, nous remarquons que les graphes sans trous pairs en général ont une *tree-width non bornée*, car les graphes complets (c'est-à-dire les graphes dont tous les sommets deux à deux sont adjacents) sont sans trous pairs, et ils ont une *tree-width* arbitrairement grande. Par conséquent, le théorème de Courcelle n'est pas applicable dans ce cas. La question est maintenant de savoir ce qui permet structurellement des graphes sans trous pairs ayant une *tree-width* bornée (ou petite). Cameron et al. [Cam+18] prouvent qu'en excluant les triangles (c'est-à-dire graphes complets sur 3 sommets), les graphes sans trous pairs ont une *tree-width* au plus 5. La preuve est basée sur les résultats structurels complets obtenus par Conforti et al. [Con+00] ce qui montre que la structure des graphes sans trous pairs ni le triangle est "simple" dans un certain sens, ce qui conduit à une largeur d'arbre bornée. Il est alors naturel de se demander si cela est vrai en général, c'est-à-dire si l'exclusion de graphes complets sur n sommets donne une *tree-width* bornée. Cette question est formellement proposée par Cameron et al. [CCH18].

Le but de cette thèse est d'explorer certains paramètres de largeur sur plusieurs sous-classes de graphes sans trous pairs. Nous commençons par étudier la question posée par Cameron et al. mentionnée ci-dessus. Plus de détails sur le contenu de cette thèse, y compris les résultats que nous obtenons au cours de notre étude sont décrits ci-dessous.

Les grandes lignes de la thèse

- Dans le **Chapitre 1**, nous donnons une introduction générale aux problèmes que nous étudions dans cette thèse. Dans la Section 1.1 nous donnons une introduction générale sur ce qu'est un graphe et de ce qu'est la théorie des graphes. Dans la Section 1.2, nous donnons une revue de la littérature sur le domaine de la théorie structurelle des graphes. Dans la Section 1.3, nous introduisons la terminologie de base, et enfin nous décrivons nos contributions dans la Section 1.4.
- Dans le **Chapitre 2**, nous fournissons un aperçu de certains résultats antérieurs liés aux graphes sans trous pairs. Dans la Section 2.1, nous expliquons quelques théorèmes de décomposition connus des graphes sans trous pairs. Dans la Section 2.2, nous expliquons l'algorithme de reconnaissance pour les graphes sans trous pairs, et nous examinons les ensembles de coupes qui sont utilisés dans la décomposition des graphes sans trous pairs. Dans la Section 2.3, nous passons en revue quelques résultats sur les paramètres de largeur de plusieurs sous-classes de graphes sans trous pairs.
- Dans le **Chapitre 3**, nous présentons une construction de quelques familles de graphes sans trous pairs qui ont une *tree-width* arbitrairement grande. En particulier, nous prouvons que les graphes sans trou pair ni K_4 ont une *tree-width* non-bornée. Nous établissons une construction d'une famille de graphes que nous nommons "roues étagées" dont la *tree-width* croît avec le nombre d'étages. Nos résultats sont fortement basés sur l'étude d'une autre classe de graphes, à savoir la classe des graphes sans θ ni triangle, qui est fortement liée à la classe des graphes sans trou pair ni K_4 . La classe des

graphes sans θ est une superclasse de graphes sans trous pairs, donc la classe des graphes sans θ ni triangle intersecte la classe des graphes sans trou pair ni K_4 . Nous prouvons que cette classe a une tree-width illimitée. En effet, notre construction de roues étagées pour les graphes sans trou pair ni K_4 est inspirée par la construction pour cette classe. La Section 3.1 présente un résumé de ce chapitre. Les principaux résultats de ce chapitre sont traités dans la Section 3.2. Pour les deux classes, nous donnons un résultat plus fort, en montrant que la largeur de rang des deux classes est également non-bornée, ce qui est expliqué dans la Section 3.3. De plus, nous donnons une borne supérieure sur la tree-width des roues étagées dans la Section 3.4.

- Dans le **Chapitre 4**, nous expliquons comment majorer la tree-width de certaines sous-classes de graphes sans trous pairs, ainsi que des sous-classes de graphes sans θ ni triangle. Dans la Section 4.1, nous mentionnons quelques résultats connus sur les largeurs d'arbres de certaines classes de graphes qui sont liées à notre étude. Nous prouvons alors qu'en excluant certaines structures (à savoir une subdivision d'une griffe), nous pouvons limiter la tree-width des graphes sans trous pairs, ainsi que des graphes sans θ ni triangle. Pour la preuve, nous établissons une nouvelle méthode pour majorer la tree-width sur des classes de graphes avec petit nombre de clique et petit nombre de séparation. C'est le cœur de la Section 4.2. Certaines propriétés des classes étudiées qui conduisent à la majoration de la tree-width sont traitées dans la Section 4.3. Enfin, dans la Section 4.4, nous donnons la preuve de la borne supérieure sur la tree-width.
- Dans le **Chapitre 5**, nous discutons de la tree-width des graphes sans trous pairs avec un degré maximum borné, en particulier pour un degré maximum 3. Nous fournissons un théorème de structure complet pour les graphes de cette classe, ce qui conduit à la majoration de la tree-width. Ces résultats sont traités dans la Section 5.1. Nous présentons également le théorème de structure des graphes sans trous pairs de degré maximum 4, pour le cas sans pyramide, qui est donné dans la Section 5.2.
- Dans le **Chapitre 6**, nous donnons une conclusion et mentionnons quelques problèmes ouverts.

Summary

A graph is a mathematical data structure that is defined as a pair of sets (V, E) , where V is a finite (non-empty) set of elements called *vertices* (or *nodes*), and E is a finite set of elements called *edges*, each of which has two associated vertices. Graphs are used to model all sorts of interconnected things, such as networks. Every object in the network is represented by a vertex in the graph, and the edges represent a pairwise relation between those objects. There are many practical problems that can be modeled using graphs, and graphs have been applied in many areas of real-world systems, such as Computer Science, Computer Networks, Social Science, Physics and Chemistry, even Linguistics.

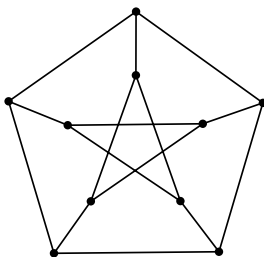


FIGURE 2: An example of graph

Since its introduction in the 1800s, the study of graphs has grown considerably. Many topics have been explored by many researchers all over the world. Among those topics, one that attracts many researchers is “Structural Graph Theory”. This area of research deals with establishing results that describe various properties of graphs. The aim is mostly to utilize them in the design of efficient algorithms, as well as in other applications. In this thesis, we investigate a particular subject in structural graph theory, which is called “hereditary classes of graphs”. The main concern in this field is to study how excluding certain configurations affects the overall structure of the graphs and what types of structure allows efficient algorithms for graph classes. There are many hereditary graph classes that are studied, and in this thesis, we specifically study the class of “even-hole-free graphs”. We will now formally define it.

Even-hole-free graphs

A graph H is called an *induced subgraph* of some graph G if H can be obtained from G by *deleting* vertices (deleting a vertex v means that we delete v and all edges that are adjacent to v). A graph G is called H -free if it does not contain H as an induced subgraph. A *hole* is a graph that is made of a number $n \geq 4$ of vertices v_1, v_2, \dots, v_n , such that $v_i v_{i+1} \in E(G)$ for $i \in \{1, 2, \dots, n-1\}$ and $v_1 v_n \in E(G)$, and no other edge between those vertices is in the graph. A hole is *even* or *odd* depending on the parity of n . Hence, even-hole-free graphs are simply the graphs that do not contain an even hole as an induced subgraph.

The study of even-hole-free graphs was initially motivated by the study of the so-called class of *Berge graphs*, in an attempt to prove the Claude Berge’s famous Strong Perfect Graph Conjecture. It turns out that the technique that was developed to decompose even-hole-free graphs was successfully applied on Berge graphs, which then led to the Strong Perfect Graph Theorem by Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas, proved in 2002. We remark that there is a sort of dichotomy relation between the class of Berge graphs (or equivalently, known as *perfect graphs*²) and the class of even-hole-free graphs. The class of Berge graphs does not contain odd holes or odd *antiholes* (an antihole is the complement of a hole, and complement of a graph is obtained from the original graph by replacing edges with non-edges, and replacing non-edges with edges). Besides excluding even holes, the class of even-hole-free graphs implicitly excludes all antiholes of length at least 6, because every such antihole always contains a hole of length 4. However, while optimization problems such as *optimal coloring*, *maximum clique*, *maximum independent set*, and *clique cover* are solvable in polynomial time on perfect graphs, it is not the case for even-hole-free graphs. These problems (except maximum clique) are still open nowadays in the class of even-hole-free graphs, and solving these problems has become the main objective in this area of research.

Width parameters

The notions of “graph widths”, such as *tree-width* (tw), *rank-width* (rw), *path-width* (pw), *clique-width* (cw), and some other widths have received high attention in the recent years. These notions are parameters measuring how simple/complex the structure of a graph is. They are really important in the study of graph structure and they have many algorithmic applications. Tree-width, for instance, is a parameter measuring how close is a graph from being a *tree* (a tree is a connected graph that does not contain any cycle), and having a small tree-width means that the graph is close to being a tree. As we know, many graph optimization problems are solvable in polynomial time for trees, and Courcelle’s theorem (by Bruno Courcelle, 1990) states that many graph optimization problems (including the four problems that we mention in the previous section) can be decided in linear time on graphs of bounded tree-width. It is therefore intriguing to study the tree-width of graphs when trying to develop an algorithm for graph optimization problems. In this thesis, our focus is to analyze tree-width. However, the width parameters mentioned above are related to each other. For every graph G , the followings hold [CR05; OS06]:

- $\text{rw}(G) \leq \text{cw}(G) \leq 2^{\text{rw}(G)+1}$;
- $\text{cw}(G) \leq 3 \cdot 2^{\text{tw}(G)} - 1$;
- $\text{tw}(G) \leq \text{pw}(G)$.

In terms of width parameters, we remark that even-hole-free graphs in general have *unbounded* tree-width, because complete graphs (that are graphs whose set of vertices are pairwise adjacent) are even-hole-free, and they have arbitrarily large tree-width. Hence, Courcelle’s theorem is not applicable in this case. The question now is what structurally allows even-hole-free graphs having bounded (or small)

²A graph G is called *perfect* if for every induced subgraph H of G , the chromatic number of H equals the size of the largest complete subgraph of H . The Strong Perfect Graph Theorem asserts that a graph is perfect if and only if it is Berge.

tree-width. Cameron et al. [Cam+18] proves that when excluding triangles (i.e. complete graphs on 3 vertices), even-hole-free graphs have tree-width at most 5. The proof is based on the full structural results obtained by Conforti et al. [Con+00], which shows that the structure of (even hole, triangle)-free graphs is “nice” in some sense, that leads to the boundedness on the tree-width. It is then natural to ask whether this holds in general, i.e. whether excluding complete graphs on n vertices yields bounded tree-width. This question is formally proposed by Cameron et al. [CCH18].

The goal of this thesis is to explore some width parameters on several subclasses of even-hole-free graphs. We begin by studying the question asked by Cameron et al. that is mentioned above. More details about the content of this thesis, including the results that we obtain during our study are described below. We postpone the definitions used throughout this outline into the next chapters.

Outline of the thesis

- In **Chapter 1**, we give a general introduction to the problems that we study in this thesis. In Section 1.1, we provide a general introduction to what graphs are and what Graph Theory is. In Section 1.2, we give a literature review on the area of structural graph theory. In Section 1.3, we introduce some basic terminology, and finally we outline our contributions in Section 1.4.
- In **Chapter 2**, we provide a survey of some prior results related to even-hole-free graphs. In Section 2.1, we explain some known decomposition theorems of even-hole-free graphs. In Section 2.2, we explain a recognition algorithm for even-hole-free graphs, and we examine the cutsets that are used in the decomposition of even-hole-free graphs. In Section 2.3, we survey some results on the width parameters of several subclasses of even-hole-free graphs.
- In **Chapter 3**, we present a construction of some families of even-hole-free graphs that have arbitrarily large tree-width. In particular, we prove that (even hole, K_4)-free graphs are of unbounded tree-width. We establish a construction of a family of graphs that we name “layered wheels”, which provides graphs of increasing arbitrarily high tree-width. Our results are heavily based on the study of another class of graphs, namely the class of (theta, triangle)-free graphs, which is highly related to the class of (even hole, K_4)-free graphs. The class of theta-free graphs is a superclass of even-hole-free graphs, so the class of (theta, triangle)-free graphs intersects the class of (even hole, K_4)-free graphs. We prove that this class is of unbounded tree-width. Indeed, our construction of layered wheels for (even hole, K_4)-free graphs is inspired by the construction of layered wheels in the class of (theta, triangle)-free graphs. Section 3.1 provides a summary of this chapter. The main results of this chapter are covered in Section 3.2. For the two classes, we give a stronger result, by showing that the rank-width of both classes are also unbounded, which is explained in Section 3.3. We moreover give an upper-bound on the tree-width of layered wheels in Section 3.4.
- In **Chapter 4**, we explain how to bound the tree-width of some subclasses of even-hole-free graphs, as well as subclasses of (theta, triangle)-free graphs. In Section 4.1, we mention some known results on the tree-widths of some classes of graphs that are related to our study. We then prove that by excluding some

structures (namely a subdivision of claw graph), we can bound the tree-width of even-hole-free graphs (parameterized by the clique number of the graph), as well as (theta, triangle)-free graphs. For the proof, we establish a new method to bound the tree-width on classes of graphs with small clique number and small separation number. This is the core of Section 4.2. Some properties of the classes being studied which lead to the boundedness of the tree-width are covered in Section 4.3. Finally, in Section 4.4, we give the proof of the upper-bound on the tree-width.

- In **Chapter 5**, we discuss the tree-width of even-hole-free graphs with bounded maximum degree, in particular for maximum degree 3. We provide a full structure theorem for graphs in this class, which leads to the boundedness of the tree-width. These results are covered in Section 5.1. We also present the structure theorem of even-hole-free graphs with maximum degree 4 for the pyramid-free case, which also leads to the boundedness of the tree-width. This is explained in Section 5.2.
- In **Chapter 6**, we give a conclusion and mention some open problems which are related to our discussion in this thesis.

Chapter 1

Introduction

1.1 What is Graph Theory

Explaining the idea of Graph Theory with just a few sentences could be a somewhat challenging task. In particular, when our interlocutors do not work in a field related to Mathematics or Computer Science. I still sometimes struggle to find a simple yet elegant answer whenever somebody posed this question:

What is your research domain?

Despite its name, the word “graph” that we use here is unrelated to the pictures of equations drawn in high school algebra courses (which is one of the most common usages of the word), and does not refer to any figures that we often find from statistics (like pie charts) as shown below.

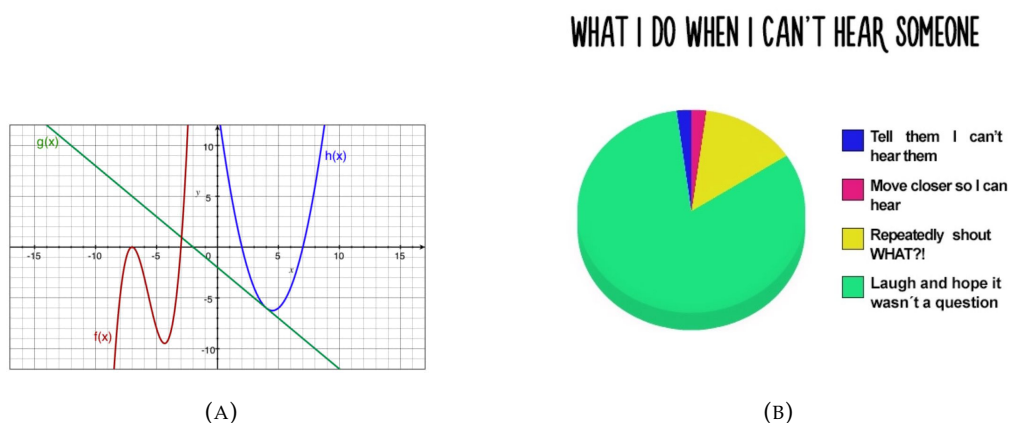


FIGURE 1.1: Graph of a function (A) and graph of a statistic (B) that we do not refer to (pictures are taken from google)

The notion of graphs that we are going to talk about in this thesis is an object as shown in Figure 1.2. To have some intuition, let us think of a problem. To organize our thinking and guide us to a solution, we often draw a *connect-the-dots* picture. We symbolize the objects related to the problem by “dots”, and we connect them with “lines” to represent the relationship between the dots. Such connect-the-dot structures are called *graphs* (we will soon see a formal definition of graphs).

Graphs might seem to be a very abstract and theoretical structure at first and might not seem like anything valuable that we could apply to the “real world” — which could be a reason why it evokes skepticism. We use the word “skepticism” because one might be curious, why would anybody spend years studying such a

thing? What is Graph Theory, and why does it matter? I believe that Graph Theory is one of the most beautiful of all human inventions. Many graph theorists would agree if I say that the mathematical beauty of graphs gives us pleasure once we start diving into it. Its abstractness, purity, simplicity, and depth are just beautiful. Nevertheless, graphs are everywhere and are indeed really useful. Let me try to convince the readers about some important aspects of Graph Theory by discussing it from scratch. We will later discuss more deeply some specific topics that we worked on during my doctoral program. A good start, perhaps, is to answer the fundamental question, that is taken as the title of this section: **What is Graph Theory?**

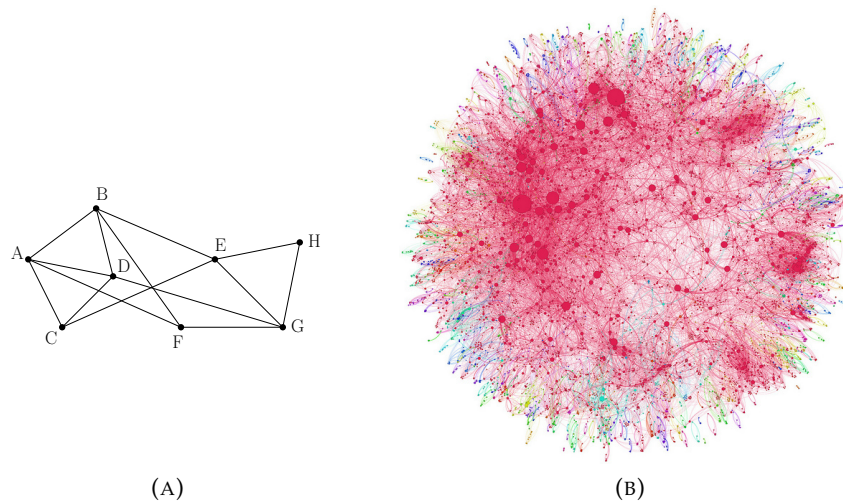


FIGURE 1.2: (A) Representation of a problem with a graph, which can be huge depending on the problem (B) Co-authorship network map of physicians publishing on hepatitis C by Andy Lamb ¹

In mathematics, Graph Theory refers to the study of graphs. In this context, graphs are mathematical structures that are widely used to model pairwise relations between objects. As has been written in the previous paragraph, a graph is made up of dots that are called *vertices* (or sometimes called *nodes*) which are connected by lines (not necessarily straight) that are called *edges* (or *links*).

1.1.1 A history of Graph Theory

The history of graphs takes us back in time to the 18th century, when Swiss mathematician Leonhard Euler was trying to solve a problem known as “The Seven Bridges of Königsberg”, which was a notable problem in Mathematics. The town of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, which flowed through the town, creating two islands. Geographically, the layout of the town is composed of four parts of the land, which are connected by a total of seven bridges as shown in Figure 1.3 (A). The inhabitants of the town were intrigued by the following question: *is it possible to take a walk through the town by visiting each area of the town and crossing each bridge only once?* In this context, reaching an island or mainland bank other than via one of the bridges, or accessing any bridge without crossing to its other end, are not allowed. The walk itself does not have to start and end at the same spot ².

¹Taken from: <https://www.flickr.com/photos/speedoflife/8273922515>

²However, another version of the problem states that the trip must end in the same place it began.

In 1735, Leonhard Euler proved that the problem has no solution. He recognized that the relevant constraints were the four parts of the land and the seven bridges. Euler represented the object as a structure that we now acknowledge as a “modern graph” (cf. Figure 1.3 (B)). Euler eventually extrapolated a general rule: to be able to walk in without repeating an edge (which is later known as an Eulerian walk), a graph can have none or two vertices of an odd degree³; and in this case, the Königsberg bridge’s graph representation does not have such property. This problem leads to the concept of “Eulerian Graph”. Since then, the field of Graph Theory has undergone remarkable growth over the past centuries. Nowadays, many researchers from all over the world explore various topics of study in Graph Theory, and its application is finally exploding.

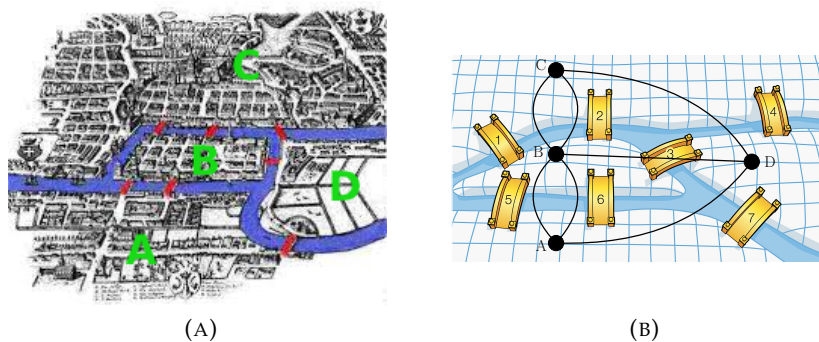


FIGURE 1.3: Königsberg bridge illustration⁴

Another insightful example. In our previous example, Euler drew a graph where the vertices represent different bodies of the town of Königsberg, and each edge linking two vertices represents whether there exists or not a bridge connecting the corresponding two cities. This shows how graphs can be used to model a problem. Now let us look at a different scenario.

During my stay in Lyon city, I lived in the 9th district, where the closest subway station to my flat is the station “Gare de Vaise”. Every day I had to commute by subway to LIP, the laboratory where I was working. The closest subway station to LIP is the station “Debourg”. Hence for efficiency, I had to find the shortest way to reach Debourg station from Gare de Vaise station. In the following figure, you can see a subway map containing all necessary information about the metro lines in Lyon (see Figure 1.4).

As you can observe from the figure, the subway map is represented as a graph where every vertex represents a subway stop, and edges represent connections between stops. On my first day in Lyon, I did not have any information about the time needed to go from one station to another one, so I just decided to take the route that passes through the minimum number of stops and connections. Later as I got used to the city, I knew exactly the amount of time that it took between two stops, and the amount of time needed to change a subway line (so in this case, to optimize the total time to travel, I took into account the time between stops).

In Graph Theory, the problem we are dealing with is related to what is known as *the shortest path problem* — which in the latter case, the edges are given a certain

³For the version where the walk starts and ends in the same place, all vertices must of even degrees.

⁴Taken from: https://www.maa.org/sites/default/files/images/cms_upload/Konigsberg_colour37936.jpg

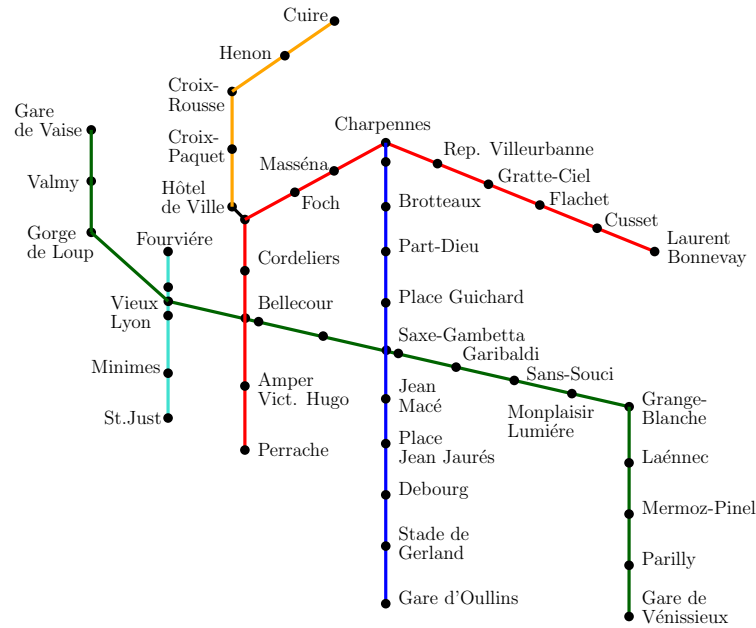


FIGURE 1.4: The graph representing Lyon subway network

weight. And hurrah! This problem is solvable in polynomial time, and several algorithms to solve it are known (even though in our case, we do not need to care too much because the graph we are dealing with is of “small” size).

1.1.2 Graph Theory is everywhere

Graphs are helpful because they can be used to model many different situations. In the examples that we have seen so far, the “underlying graphs” of the problems we are investigating are pretty simple (it contains only a few vertices and edges). However, we may go further. Instead of dealing with the Lyon subway network, we may deal with the European railway network (consisting of all cities in Europe that have a railway station), which yields a bigger underlying graph. Even further, in some situations, our graph can be tremendously huge. The internet, for example, is a vast graph, where every vertex is an individual webpage and edges telling us which websites are linked to which others. It results in a gigantic graph (there are not just tens or hundreds of websites out there, there are billions of them). Another example is social media, such as Facebook or Instagram; Graph Theory is also behind them. Graphs can easily model the friendship relation on Facebook: each person is a vertex, and an edge connects two people if they are friends (and there are over 1.6 billion users of Facebook, so the graph will be huge). Much research has been conducted related to those social media, and many of them lie on Graph Theory.

A roadmap, the internet, and social media are examples of networks. A *network* is a system of connected objects, and we have seen that it can be nicely visualized using a graph: *the objects in your network can be represented as vertices and the connections represented by edges*. This way, graphs can model all sorts of interconnected things. Graph Theory is everywhere because networks are used everywhere. Graphs enable us to model numerous problems that occur in everyday life. It is widely used in many areas: from links between web pages to friendships in social networks, even to connections between neurons in our brains. Graph Theory is used to model and study all kinds of things that affect our daily lives. Wouldn’t the reader agree that graphs have tremendous impacts on our world?

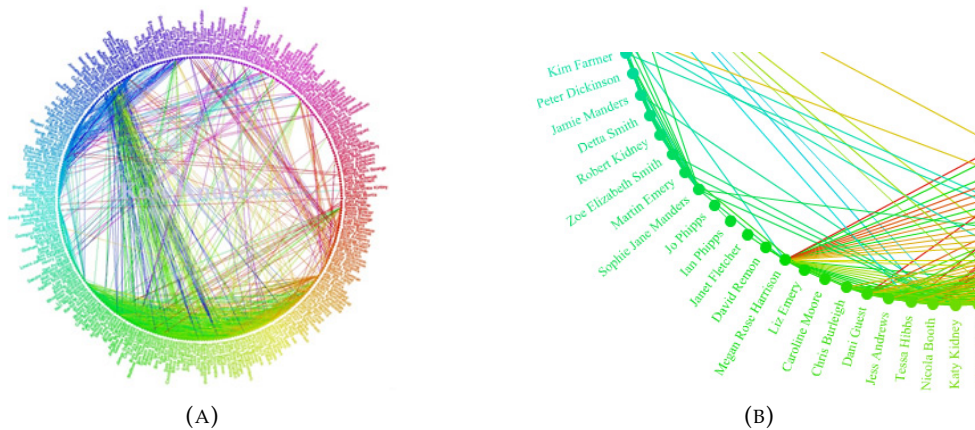


FIGURE 1.5: A friend wheel on Facebook, an example of graphs representing social network ⁵

A link to algorithm. Within this deluge of interconnected data, graphs often span billions of nodes and interactions between them. Hence, we may end up with all kinds of drawings of graphs, even huge, messy ones. Graph Theory involves the study concerning some specific graphs to understand their structures, including how we can find the most important structures and summarize them. Despite its wide applications in many real-world problems, as a mathematical field, Graph Theory leads us from the concrete to the abstract scenario. The motivation behind the study of some of the graph properties goes beyond its relevance to some real-world applications. Their natural relation to other mathematical preexisting concepts, or their mathematical beauty, is sometimes more fascinating to explore. Hence, our study is not limited to types of graphs actually arising in the real-world situation, but we could go far beyond that.

Furthermore, as an integral part of Computer Science topics, Graph Theory also concerns with algorithms. In this context, the area of *Structural Graph Theory* deals with establishing results that characterize various properties of graphs and utilizing them in the design of efficient algorithms and other applications. Graph optimization, for instance, deals with the problem of *maximizing* or *minimizing* some function relative to some set (such as the set of vertices, or edges, or a combination of the two), which allows comparison of the different choices for determining which could be the “best” solution. There are much interesting decision and optimization problems that people encounter while exploring Graph Theory. Among all topics studied in this area, some of them can be solved efficiently, i.e. in polynomial time (for instance, the shortest path problem that we discussed in the beginning). Nevertheless, many challenging problems are not polynomial-time solvable for the general graphs. Some known and well-studied problems of that kind are *coloring*, *maximum independent set*, *maximum clique*, and *minimum clique cover* problems (we will explain them hereafter). These problems play an important role in Graph Theory. However, they are NP-hard in general [Kar72], which means that it is unlikely that efficient/polynomial-time algorithms exist for these problems. Even worse, they are not approximable within $O(n^{1-\epsilon})$ for any $\epsilon > 0$ unless $P = NP$ [Zuc06].

⁵Taken from: <http://www.visualcomplexity.com/vc/project.cfm?id=501>

1.2 Literature review

We first of all remark that we postpone the definition of some basic terminology and notations that are used in this section to Section 1.3.

In some classes where the graphs are “well-structured”, the optimization problems mentioned above (namely coloring, maximum independent set, maximum clique, and minimum clique cover) become polynomial-time solvable. Some examples of the classes for which those problems are polynomial time solvable are the class of *forests* (i.e. graphs with no cycle), the class of *chordal graphs* (i.e. graphs that do not contain any hole), and the famous class of *perfect graphs* (that forbids odd holes and odd antiholes; we will discuss it a bit later). On the other hand, the coloring problem remains “difficult” when the graphs do not contain the triangle (i.e. clique on three vertices, cf. Figure 1.11), even though excluding the triangle seems to impose a lot of structure on the input graph. For example, determining whether a graph is 3-colorable remains NP-complete for triangle-free graphs with a maximum degree of 4 [MP96]. Therefore, one main concern in the area of Structural Graph Theory typically addresses the following question: *what structurally yields efficiency optimization algorithms for graph classes?* In these circumstances, we examine how forbidding certain substructures as an induced subgraph affects the overall structure of graphs in the class.

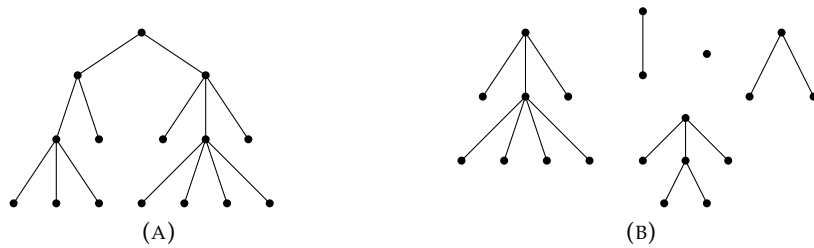


FIGURE 1.6: A tree and a forest, graph classes in which many optimization problems are efficiently solvable

Let us now explain in more detail the four optimization problems that we mentioned earlier. Along with it, we introduce three graph parameters that are related to those optimization problems. Computing any of them for general graphs is well-known to be NP-hard [Kar72].

- Recall that a clique in graph G is a set of pairwise adjacent vertices. The *clique number* of G , denoted by $\omega(G)$, is the size of the largest (in terms of cardinality) clique in G . The maximum clique problem asks for a maximum clique of the given input graph.
- An *independent set* or *stable set* of a graph G is a subset of vertices of G that are pairwise non-adjacent. The *independence number* (or *stability number*) of G , denoted by $\alpha(G)$, is the size of the largest independent set in G . In the maximum independent set problem, the input is a graph, and we want to find an independent set of the maximum cardinality in the graph.
- *Coloring* (our concern here is *vertex coloring*) is an assignment of labels called “colors” to the vertices of a graph so that no two adjacent vertices receive the same color (such a coloring is often called *proper*). The objective is to minimize the number of colors in a coloring of a graph. The *chromatic number* of G , denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices

of G properly. Equivalently, one can define the chromatic number of G that is the smallest k such that $V(G)$ can be partitioned into k independent sets. A coloring of G with $\chi(G)$ colors is called an *optimal coloring* of G .

- A *clique cover* is a partition of the vertices of a graph into cliques. A minimum clique cover is a clique cover that uses as few cliques as possible. The minimum size of the clique cover of graph G is denoted by $\theta(G)$. A clique cover of a graph G may be seen as a coloring of the complement graph of G . Note that colorings are partitions of the set of vertices but into independent sets (instead of cliques).

Another interesting yet very important combinatorial problem is *graph recognition*. We aim to find out if there exists an efficient algorithm to recognize certain graph classes (for example, whether the input graph belongs to some class, e.g. bipartite, perfect, etc.). For that purpose, we often need to detect the existence of some specific structures of the input graph, so it is crucial to understand what structure exists or not in the graphs of the class being studied.

1.2.1 Hereditary classes of graphs

In graph theory, a *graph property* or *graph invariant* is a property of graphs that depends only on the abstract structure, not on the graph representations such as particular labelings or drawings of the graph. A graph property P is *hereditary* if it is “inherited” by its *induced* subgraphs, that is, if every induced subgraph of a graph with property P also has property P . Hereditary properties provide a general perspective to study many graph properties, which can be a tool to understand what structural properties enable efficient recognition and optimization algorithms. In this section, we discuss a particular hereditary property of graphs.

Let H be a graph. A graph G is *H -free* if it *does not contain* H (as an induced subgraph). For a family of graphs \mathcal{H} , G is *\mathcal{H} -free* if for every $H \in \mathcal{H}$, G is H -free. A class of graphs that is \mathcal{H} -free for some \mathcal{H} is hereditary, or equivalently, the class is closed under taking induced subgraphs. For instance, a *class of graphs that do not contain any even hole*, or even-hole-free graphs (which is in the title of this thesis) is hereditary, because if G does not contain any even hole, then neither does any induced subgraph of G . The converse of the statement also holds: every hereditary class \mathcal{I} is defined by a set of (minimal) *forbidden structures*, i.e. there is a set of graphs \mathcal{H} such that \mathcal{I} is H -free for every $H \in \mathcal{H}$.

In recent years, many classes of graphs defined by excluding a family of induced subgraphs have been studied, perhaps initially motivated by the study of *perfect graphs* (will be explained in more detail in Section 1.2.4). Typical questions in this field were whether excluding induced subgraphs affects the global structure of the particular class in a way that allows us to bound some parameters such as χ and ω , or to construct combinatorial algorithms for problems such as maximum clique, maximum independent set, coloring, minimum clique cover, or the graph recognition problem.

A motivating example. An example of classical hereditary graph classes that have interesting properties is the class of *chordal graphs*, namely the class of graphs that do not contain *holes* (i.e. chordless cycles of length at least 4). It turns out that excluding holes yields a graph class possessing some nice structure, that are useful to solve algorithmic problems such as coloring, maximum independent set, maximum clique,

and minimum clique cover in polynomial time. If a connected graph is chordal, then either the graph is complete or it contains a complete subgraph whose removal disconnects the graph into two smaller pieces. Furthermore, chordal graphs can be characterized by the so-called *perfect elimination ordering*, namely an ordering of the vertices of the graph such that, for each vertex v , v and the neighbors of v that occur after v in the order form a clique. This characterization is helpful, for instance, to solve the maximum clique problem in linear time. Indeed, a chordal graph can have only linearly many maximal cliques (while non-chordal graphs may have exponentially many). We can list all maximal cliques of a chordal graph by finding a perfect elimination ordering of the graph, form a clique for each vertex v together with the neighbors of v that appear later than v in the perfect elimination ordering, and test whether each of the resulting cliques is maximal.

Are all hereditary families friendly with algorithmic purposes? Naturally, one might think that excluding “small” graphs would impose some structure on the class of graphs from where they being excluded. This often works, for instance, in the class of chordal graphs that we have just discussed: excluding all holes implies the graphs to have a simple structure that is suitable for algorithmic purposes. However, this does not always hold in general. As we have discussed earlier, excluding the triangle (i.e. K_3 or C_3) produces graphs that do not contain a large clique, which might be an indication that the graphs have a constant bound on the number of colors needed to (vertex) color the graphs. However, it is not the case because, in 1955, Jan Mycielski developed graphs that preserve the property of being triangle-free and increments the chromatic numbers. These are known as *Mycielski graphs*, they are triangle-free but may have an arbitrarily large chromatic number, i.e. $\chi(G) \leq f(\omega(G))$ does not hold for this family of graphs. Another example is the class of graphs that do not contain an independent set of size three. Graphs in this class are highly connected, but it turns out that computing $\omega(G)$ for some graph G cannot be done in polynomial time, unless $P = NP$ [Pol74]. We are interested in the hereditary families of graphs with properties that are easy to handle *structurally* and *algorithmically*.

1.2.2 Dealing with large graphs: graph decomposition

When working with “small” graphs, we can easily understand the structure of the graphs. Hence solving computational problems is straightforward. One then would ask: *what if we are dealing with massive and complex graphs?* An approach that has been proved to be robust for this purpose, especially on hereditary classes of graphs, is by applying the *graph decomposition* techniques. Roughly speaking, decomposing a graph means “cutting” the graphs along the so-called *cutsets* until they cannot be decomposed anymore.

Definition 1.2.1 (Separator or cutset). *Given a connected graph G , a *separator* or a *cutset* of G is a set of vertices $S \subseteq V(G)$ such that removing S from G disconnects G ; that is, the graph induced by $V(G) \setminus S$ contains at least two connected components.*

By decomposing a graph, we obtain a set of simpler graphs called *basic graphs* and a list \mathcal{L} of *graph compositions*. In this context, a decomposition theorem tells us that every graph in some class \mathcal{C} of graphs can be “broken down” in a *tree-like fashion*: internal nodes correspond to decompositions in \mathcal{L} and leaves correspond to the basic graphs. A decomposition theorem can be formulated as follows.

Theorem 1.2.2 (Structure theorem)

If G belongs to class \mathcal{C} , then either G is basic or G has a “special” cutset, meaning that it can be built from smaller graphs G' and G'' that are also in \mathcal{C} using a prescribed composition operation in \mathcal{L} .

Decomposition is a general concept that plays a vital role for theoretical purposes and algorithms for many classes of graphs. As pointed out by Vušković ([Vuš13]), decomposition allows us to understand complex structures of the graphs by breaking them down into simpler pieces of graphs that are easier to study. Once these more superficial structures are understood, this knowledge is propagated back to the original structure by understanding how their composition behaves. The proof of decomposition theorem usually consists of a sequence of structures present in the graph to be decomposed in a specific order. Once a structure is decomposed, one may assume that the graph does not contain the structure for the rest of the proof. The key of every decomposition theorem is to find such sequence/order.

Decomposition techniques have been widely used in the study of graph structure, for instance in the study of the celebrated perfect graphs that we discussed earlier. It is also used in the design of algorithms (such as answering decision problems or designing recognition algorithms) in a *divide-and-conquer* approach or *dynamic programming*. With this approach, the graph is recursively decomposed into a hierarchy of components, so that we obtain a set of simple enough graphs for which the problem we want to solve is “easy” to handle. The solutions of each of the components is then gradually pieced into larger components in a recursive way to give a solution to the problem on the original graph.

The key attribute that a problem must have in order this approach to be applicable is an optimal substructure, that is, when decomposing the graph, the result of the decomposition should be “easy” to handle. It is then essential to derive the best decomposition theorem (if any) that supports this goal. In this case, the choice of separators used to decompose the graph is crucial. Depending on class \mathcal{C} and the aim of the study, the basic graphs and separators must have some properties that fit our goal when using the decomposition theorem. In many cases, such as for *detection* algorithm, we sometimes require that the decomposition theorem is *class-preserving*, meaning that the graphs obtained from the decomposition are still in the class. Moreover, to support the divide-and-conquer approach, it is also necessary that the separators used to decompose the graph are of small size. Hence, there are two main concerns on choosing a separator: the *structure* and the *size* of the separator.

A classical decomposition. Let us now describe an example of a classical decomposition that has been widely studied and is often chosen as the first step to decompose the graph being studied. This is called *clique cutset decomposition*. Recall that a clique is a set of pairwise adjacent vertices in a graph, so a *clique separator* or a *clique cutset* is simply a separator that induces a clique. The clique cutset decomposition was first introduced by Tarjan in [Tar85].

Definition 1.2.3 (Clique cutset decomposition, [Tar85]). *Given a connected graph G that has a clique separator C . Decomposing G using C partition $V(G)$ into three vertex sets A, B, C such that no edges from A to B present in the graph. The graphs $G_A = G[A \cup C]$ and $G_B = G[B \cup C]$ are called *blocks* of the clique decomposition.*

Note that removing a clique from a graph may give several connected components (at least two of them) — so there are possibly several options of the blocks of

decomposition when we decompose a graph using some particular clique separator. We can repeat the decomposition process for every block of the decomposition as long as a clique separator exists, until no further clique decomposition is possible. We obtain a collection of subgraphs of G , each of which does not contain a clique separator — these are called *atoms*. Those atoms are joined together in a hierarchy that forms the entire graph G , and such a hierarchy can be represented using a binary tree that is called *binary decomposition tree* (cf. Figure 1.7). The leaves of the tree form a set of atoms, and the internal nodes form a set of separators used to decompose the graph. The binary decomposition tree that corresponds to the given graph is not unique (see the example in Figure 1.7).

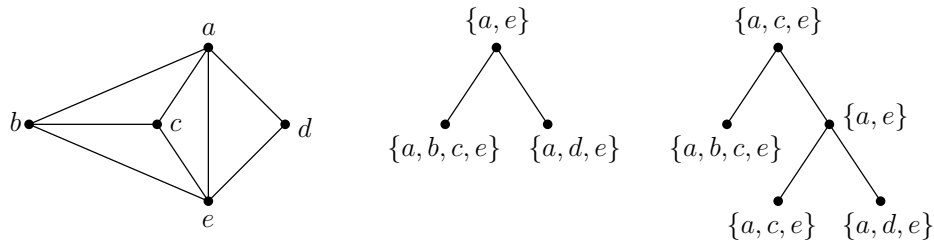


FIGURE 1.7: An example of graph and its corresponding binary decomposition trees (from the paper of Tarjan [Tar85])

In the paper of Tarjan [Tar85], an algorithm to find a decomposition by clique separators in time $\mathcal{O}(nm)$ is presented, where n and m respectively denote the size of the vertex-set and the size of the edge-set of the input graph. Note that graph combinatorial problems such as coloring, maximum independent set, and maximum clique can be solved efficiently using a divide-and-conquer approach by combining the solutions on the atoms to obtain a solution for the entire input graph.

We note that many other types of decompositions exist; each of them has its advantages and drawbacks that may not be suitable for our purpose. We have seen that clique cutset decomposition can be applied for solving graph problems in a divide-and-conquer fashion. However, we should note that having no clique separator does not mean that the graph has a “nice” structure that is easy to deal with. Moreover, clique separators (as well as other type of separators) are better when it has a small size. Indeed, if the overlap is large, then the divide-and-conquer approach is not so promising, because the graphs obtained by decomposing the original graph might be not significantly simpler than the original graph, i.e. they might be as complex as the original graph. Therefore, when decomposing a graph, we often need to use several types of separators and to be able to combine them systemically — this has also been one of the main concerns when one applies the graph decomposition techniques.

1.2.3 Parameters for graph complexity

Many combinatorial problems (such as coloring, maximum independent set, maximum clique, or minimum clique cover) which are NP-hard in general become polynomial-time solvable when the graph classes are pretty restricted (in the sense that they forbid many configurations as induced subgraphs). One question that might be of interest is the following: *is there a characterization of graph classes for which those aforementioned algorithmic problems are polynomial-time solvable?* Furthermore, it is known that many NP-hard problems can be solved efficiently on *trees*. Therefore,

it seems natural to ask the following question: *could it be true that if our graph resembles a tree, then some (if not all) graph problems are efficiently solvable for the graph?*. Throughout this section, we will investigate these questions.

The above-aforementioned questions are both answered affirmatively. One can define a parameter that measures how close a graph from being a tree (in some sense that we will explain later); and when our graph is “close” to being a tree, then many graph problems are polynomial-time solvable. Such a parameter is known as *tree-width*. Intuitively, a graph with low tree-width is “simple” and admits a tree-like structure (which we usually hope for). Robertson and Seymour popularized the notion of tree-width through their analysis of Graph Minor Theory⁶. Tree-width is an important graph parameter that is applicable for solving many algorithmic problems. Many results show that NP-hard problems can be solved in polynomial time on classes of graphs with bounded tree-width (see a survey given by Bodlaender [Bod93b]). For instance, Courcelle et al. [Cou90] described a unified approach to the efficient solution of many combinatorial problems on graph classes of bounded tree-width via the expressibility of the problems in terms of specific logical expression that is called *monadic second-order logic*.

Theorem 1.2.4 (Courcelle et al. [Cou90])

Every graph property definable in the monadic second-order logic of graphs can be decided in linear time on graphs of bounded tree-width.

However, it is NP-complete to determine whether a given graph G has tree-width at most a given variable k [ACP87]. Nevertheless, when k is any fixed constant, graphs with tree-width at most k can be recognized, and a width k tree decomposition can be constructed in linear time [Bod93a]. It is therefore of interest to bound the tree-width of certain classes of graphs. There are several other parameters to measure graph complexity that are closely related to tree-width, such as *path-width*, *rank-width*, and *clique-width* (there are many other parameters that are not in the scope of our discussion, see [HW17] for a list of them), and they are all bounded one by another.

Similar to tree-width, *clique-width* of a graph G is a parameter that describes the structural complexity of the graph. Courcelle, Engelfriet, and Rozenberg [CER93] formulated the concept in 1990. It turns out to be fruitful because it allows many hard problems to become tractable on graph classes of bounded clique-width [CMR00] (including coloring and maximum independent set). While bounded tree-width implies bounded clique-width, the converse is not true in general. Clique-width is closely related to tree-width, but unlike tree-width (for which boundedness requires graph classes to be sparse), clique-width can be bounded even for dense graphs (for instance, n -vertex complete graphs have clique-width 2 but tree-width $n - 1$). Hence, clique-width is particularly interesting in the study of algorithmic properties of hereditary graph classes.

A graph parameter that is equivalent to clique-width (in the sense that one is bounded if and only if the other is bounded) is called *rank-width*, where the following bound holds: $\text{rw}(G) \leq \text{cw}(G) \leq 2^{\text{rw}(G)+1}$. Oum and Seymour first introduced this notion in [OS06], where they use it to obtain an approximation algorithm for clique-width. They also show that rank-width and clique-width are equivalent, in the sense that a graph class has bounded rank-width if and only if it has bounded clique-width

⁶It is believed that the concept had been previously used by Halin in 1976, under a different name.

(see [HW17; DJP19] for surveys about them). Let us now define the three notions mentioned above formally.

Tree-width. We begin by defining the so-called *tree decomposition* of a graph G , that is, a pair $(T, \{X_t\}_{t \in V(T)})$, where T is a tree where every node t is assigned a vertex subset $X_t \subseteq V(G)$, called a *bag*, such that the following three conditions hold:

- (i) $\bigcup_{t \in V(T)} X_t = V(G)$, i.e., every vertex of G is in at least one bag.
- (ii) For every $uv \in E(G)$, there exists a node t of T such that bag X_t contains both u and v .
- (iii) For every $u \in V(G)$, the set $T_u = \{t \in V(T) : u \in X_t\}$, i.e., the set of nodes whose corresponding bags contain u , induces a connected subtree of T .

Example of graphs and their tree decompositions are given in Figure 1.8. The *width* of tree decomposition $(T, \{X_t\}_{t \in V(T)})$ equals $\max_{t \in V(T)} |X_t| - 1$, that is, one less from the maximum size of its bags. The *tree-width* of a graph G , denoted by $\text{tw}(G)$, is the minimum possible width of a tree decomposition of G . Note that a tree itself is a tree decomposition that has tree-width 1 (this is also the reason for the existence of minus 1 in the definition of tree-width; we want that the tree-width of trees to be 1).

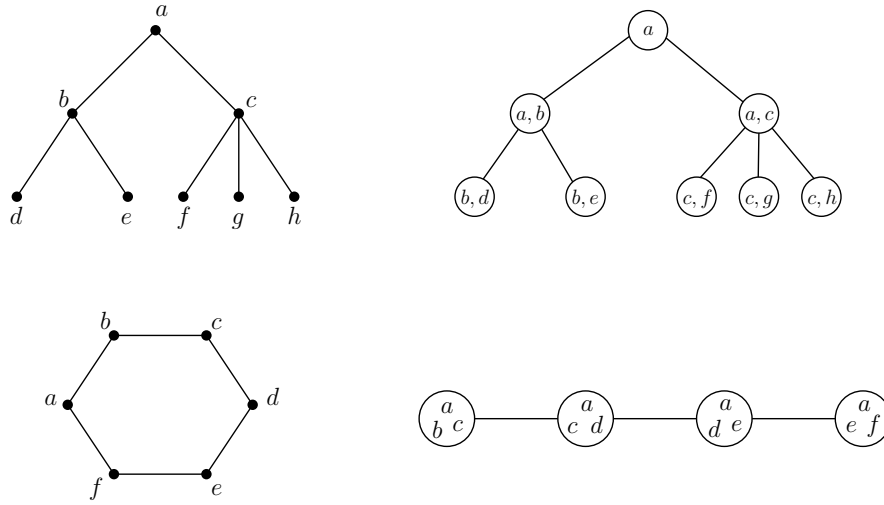


FIGURE 1.8: A tree graph (top), a cycle (bottom), and their corresponding optimal tree decompositions

To compute the tree-width of a graph, one can compute the *chordalization* of the graph. We say that a graph H is a chordalization of graph G , if H contains G as a subgraph, and H is chordal. The tree-width of G is defined as one less from the minimum over the clique number of the chordal graphs that contain G , i.e. the following holds:

$$\text{tw}(G) = \min_{H \text{ is a chordalization of } G} \omega(H) - 1$$

Despite its algorithmic implications, the motivation behind tree-width was not initially related to algorithms. The notion was invented when Robertson and Seymour were trying to solve Wagner's conjecture, which says that, in an infinite set of graphs, one of them is a minor of another. Tree-width has been effective for

profound hypothetical investigation of graph minor structure and algorithmic applications. Specifically, numerous NP-hard issues can be tackled productively if the input graph belongs to a class of graphs of bounded tree-width. However, graphs of bounded tree-width are usually limited to “sparse” graph classes. For dense graphs, a parameter similar to tree-width does exist, as we explain below.

Clique-width. The *clique-width* of a graph G (denoted by $\text{cw}(G)$) is defined as the minimum number of labels required to construct G by means of the following four operations: (i) creation of a new vertex v with label i ; (ii) joining by an edge every vertex labeled i to every vertex labeled j , where $i \neq j$; (iii) renaming label i to label j ; and (iv) taking disjoint union of two labeled graphs G and H . A sequence of these operations that constructs the graph using at most k labels is called a *k-expression*. In this thesis, however, we will never compute the clique-width of the graph classes being discussed (we define it to make sure that we are in total agreement). Having bounded clique-width is a weaker property than having bounded tree-width (recall the following inequality: $\text{cw}(G) \leq 3 \cdot 2^{\text{tw}(G)-1}$), but it still has nice algorithmic applications.

Rank-width. Generally speaking, the rank-width of a graph is the minimum integer k such that the graph can be decomposed into a tree-like structure with leaves correspond to the vertices of the original graph, by recursively splitting its vertex set so that each cut induces a matrix of rank at most k . Let us present some useful notion and definition about rank-width.

For a set X , let 2^X denote the set of all subsets of X . For sets R and C , an (R, C) -matrix is a matrix where the rows are indexed by elements in R and columns are indexed by elements in C . For an (R, C) -matrix M , if $X \subsetneq R$ and $Y \subsetneq C$, we let $M[X, Y]$ be the submatrix of M where the rows and the columns are indexed by X and Y respectively. For a graph $G = (V, E)$, let A_G denote the adjacency matrix of G over the binary field (i.e., A_G is the (V, V) -matrix, where an entry is 1 if the column-vertex is adjacent to the row-vertex, and 0 otherwise). The *cutrank function* of G is the function $\text{cutrk}_G : 2^V \rightarrow \mathbb{N}$, given by

$$\text{cutrk}_G(X) = \text{rank}(A_G[X, V \setminus X]),$$

where the rank is taken over the binary field.

A tree is a connected, acyclic graph. A *leaf* of a tree is a vertex incident to exactly one edge. For a tree T , we let $L(T)$ denote the set of all leaves of T . A tree vertex that is not a leaf is called *internal*. A tree is *cubic*, if it has at least two vertices and every internal vertex has degree 3.

A *rank decomposition* of a graph G is a pair (T, λ) , where T is a cubic tree and $\lambda : V(G) \rightarrow L(T)$ is a bijection. If $|V(G)| \leq 1$, then G has no rank decomposition. For every edge $e \in E(T)$, the connected components of $T \setminus e$ induce a partition (A_e, B_e) of $L(T)$. The *width of an edge* e is defined as $\text{cutrk}_G(\lambda^{-1}(A_e))$. The *width of (T, λ)* , denoted by $\text{width}(T, \lambda)$, is the maximum width over all edges of T . The *rank-width of G* , denoted by $\text{rw}(G)$, is the minimum integer k , such that there is a rank decomposition of G of width k . (If $|V(G)| \leq 1$, we let $\text{rw}(G) = 0$.)

The following remark will be used several times.

Remark 1. When computing the tree-width of a graph G , we may always assume that G has no clique cutset. Indeed, if G contains a clique cutset S , which partition the

graph induced by $V(G) \setminus S$ into a set of connected components G_1, G_2, \dots, G_k , then

$$tw(G) = \max_{1 \leq i \leq k} tw(G[G_i \cup S])$$

Let T_i be an optimal tree decomposition of the graph $G_i \cup S$, and let B_i be a bag of T_i which contains the vertices of S (note that such a bag exists because S is a clique). Then it is possible to obtain an optimal tree decomposition of G by “gluing” each T_i along B_i . For instance, such a gluing can be done by adding a bag B containing S , and adding an edge from the node B to the node B_i for every $1 \leq i \leq k$. Note that identifying two trees at a vertex yields a tree.

1.2.4 Why even-hole-free graphs?

Recall that a graph is called *even-hole-free*⁷ if it does not contain even hole (as an induced subgraph). At this point, one might wonder, *among many hereditary graph classes, why do we choose to study the class of even-hole-free graphs?* The reason why “even-hole-free graphs” is written on the title of this thesis is not just an arbitrary choice because nobody has examined it yet. This last section of this chapter is devoted to answering this principal question.

So, why is the class of even-hole-free graphs intriguing? To begin with, let us bring our attention back to the class of perfect graphs. The class of *perfect graphs* seems to be the hereditary class of graphs which drawn the most attention and has been widely studied over the past years. Recall the four graph parameters (χ , α , ω , and θ) that we mention at the beginning of this chapter. Note that the relation: $\chi(G) \geq \omega(G)$ holds for any graph G , since a clique of size k needs at least k different colors in any proper coloring of G (since the vertices of the clique have to be colored differently). Hence, the clique number gives a natural lower bound for the chromatic number of a graph. However, for which graphs does the equality hold? When the chromatic number of every induced subgraph equals the order of the largest clique of that subgraph, i.e. $\chi(H) = \omega(H)$ for every induced subgraph H of G , we say that the graph G is *perfect*. Obviously, not all graphs are perfect, because every odd hole C_{2k+1} satisfies $\omega(C_{2k+1}) = 2$, but $\chi(C_{2k+1}) = 3$, and by similar reasoning one can show that every odd antihole is also not perfect. Hence, odd holes and odd antiholes are *obstructions*⁸ to a graph being perfect. Are there any other obstructions?

A long-standing conjecture by Claude Berge in 1961 asserted that a graph is perfect *whenever the graph does not contain odd holes and odd antiholes*. This conjecture was known as the *Strong Perfect Graph Conjecture*, and any graph that forbids such configurations is called *Berge graph*. A long-term study of this class finally proved the conjecture, yielding an essential theorem in Structural Graph Theory, namely the celebrated *Strong Perfect Graph Theorem* that was proved by Chudnovsky, Robertson, Seymour, and Thomas in 2002 (the result was published in 2006, see [Chu+06]). The proof of the Strong Perfect Graph Theorem is long and technical, and is based on a deep structural decomposition of Berge graphs. They show that every Berge graph with the chromatic number k contains a clique on k vertices, which yields the following.

Theorem 1.2.5 (Strong Perfect Graph Theorem [Chu+06])

A graph is perfect if and only if it is a Berge graph.

⁷We will sometimes abbreviate it as *ehf*.

⁸An obstruction is a structure that is forbidden from belonging to a given graph family.

Recognizing perfect graphs can be done in polynomial-time [CLV03]. Also, in all perfect graphs, the optimal coloring, maximum clique, maximum independent set, and minimum clique cover problems are all known to be polynomial-time solvable [GLS88]. However, the proof relies on the so-called *ellipsoid method* from linear programming (which is impractical), and in some sense, uses less combinatorial structure of the class. Surprisingly, the structural description of perfect graphs does not seem to help much to obtain a purely combinatorial polynomial-time algorithm to solve any of the optimization problems. This is one of the fundamental questions in this area that remains open. Nevertheless, the *decomposition* technique used to prove the theorem has been successfully applied in other graph classes, and in particular for *claw-free graphs* (see Figure 1.11 for claw). There is a nice survey on the class of perfect graphs written by Trotignon (see [Tro13]). In Table 1.1, we give a summary of the complexity results of the five problems we mentioned above.

However for the class of odd-hole-free graphs (which is a superclass of the class of Berge graphs), most of those problems are NP-complete. The NP-completeness of computing ω follows from the result of Poljak [Pol74]. In the paper, Poljak proves that for a given graph G , and a graph G' obtained from G by subdividing twice every edge of G , we have the following relation: $\alpha(G') = \alpha(G) + |E(G)|$. Since computing α for any graph is NP-complete, computing α is also NP-complete for (C_4, C_5) -free graphs, because given any graph, one can obtain a (C_4, C_5) -free graph by subdividing every edge in the graph twice. A class of $(\overline{C_4}, \overline{C_5})$ -free graphs forms a subclass of odd-hole-free graphs. Computing χ is also NP-complete for odd-hole-free graphs. This is because computing θ is NP-complete for the class of planar $(K_4, \text{diamond}, C_4, C_5)$ -free graphs [Kr+01] (note that the complement of this class is a subclass of odd-hole-free graphs). Finally, computing α is also NP-complete for this class of graphs, which follows from the fact that coloring (C_3, C_4, C_5) -free graphs is NP-complete [LM17] (and again, the complement of this class forms a subclass of odd-hole-free graphs). Finally, recognition problem can be done in polynomial-time [Chu+20].

From perfect graphs to even-hole-free graphs

Knowing perfect graphs, out of curiosity one might think of the following dichotomy: if excluding *odd holes* and *odd antiholes* enforces some structure, what can we say about the class excluding *even holes* or *even antiholes*?

The study of even-hole-free graphs was initially motivated by perfect graphs when researchers were trying to develop a technique to “decompose” Berge graphs (that we presented in the previous subsection). The decomposition technique developed during the study of even-hole-free graphs led to proving the Strong Perfect Graph Theorem. The study establishes that these two classes have a similar decomposition (we postpone further discussion about the decomposition of the two classes into Chapter 2). Observe that by excluding a hole of length 4 (i.e square, cf. Figure 1.11), we implicitly exclude all antiholes of length at least 6 (because such an antihole always contains a square). Hence, compared to odd-hole-free graphs (which is a superclass of perfect graphs), there is a “closer similarity” between even-hole-free graphs and perfect graphs.

Despite its similarity with the class of perfect graphs, the class of even-hole-free graphs is also interesting on its own. The class has received much attention for the past years; Vušković [Vuš10] wrote a survey on problems on this class. The study of this topic is also motivated by their connection to *β -perfect graphs* introduced by Markossian et al. [MGR96]. For a graph G , define $\beta(G) = \max\{\delta_H + 1\}$ where H

is an induced subgraph of G , and δ_G is the minimum degree of a vertex in G . Consider a total ordering of vertices of G by repeatedly removing a vertex of minimum degree in the subgraph of vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. Coloring greedily on this order shows that $\chi(G) \leq \beta(G)$, and we say G is β -perfect if $\chi(H) = \beta(H)$ for every induced subgraph H of G . It is easy to see that β -perfect graphs are even-hole-free (note that $\beta(C_{2k}) = 3$ and $\chi(C_{2k}) = 2$ for $k \geq 2$). Indeed, Markossian et al. [MGR96] show that G (and \overline{G}) is β -perfect if and only if it does not contain an even hole or an even antihole, which is an interesting analog of the Strong Perfect Graph Theorem. This idea motivates us to ask what types of structural tools might give new insights for even-hole-free graphs.

A decomposition theorem and a recognition algorithm are known for this class (see Chapter 2 for further discussion). However, while the coloring, the maximum independent set, and the clique cover problems are polynomial-time solvable for perfect graphs, those problems are still open for even-hole-free graphs. On the other hand, finding a maximum clique of an even-hole-free graph can be done in polynomial time, since square-free graph has polynomial number of maximal cliques [Far89] and one can list them all in polynomial time. However, on the positive side, it is known that even-hole-free graphs are χ -bounded (i.e. there exists a function f s.t. for every induced subgraph H of an even-hole-free graph G , we have the relation $\chi(H) \leq f(\omega(H))$), in particular, $\chi(H) \leq 2\omega(H) - 1$ for every H [Add+08; Add+20].

	Perfect	OHF	EHF
Recognition	P [Chu+05]	P [Chu+20]	P [Con+02b]
ω	P [GLS88]	NPC [Pol74]	P [Far89]
α	P [GLS88]	?	?
χ	P [GLS88]	NPC [Kr+01]	?
θ	P [GLS88]	NPC [LM17]	?

TABLE 1.1: Complexity of algorithmic problems on perfect graphs, odd-hole-free graphs, and even-hole-free graphs

Truemper configurations, a link between perfect graphs and even-hole-free graphs

How are the class of perfect graphs and even-hole-free graphs related one to the other? In this part, we introduce some configurations that play an essential role in the class structure of even-hole-free graphs.

A *pyramid* is a graph made of three chordless paths $P_1 = x \dots a$, $P_2 = x \dots b$, $P_3 = x \dots c$, each of length at least 1, two of which have length at least 2, internally vertex-disjoint, and such that abc is a triangle and no edges exist between the paths except those of the triangle and the three edges incident to x . The vertex x is called the *apex* of the pyramid. Such a pyramid is also referred to as a $3PC(abc, x)$ or a $3PC(\Delta, \cdot)$ (3PC stands for *3-path-configuration*).

A *prism* is a graph made of three vertex-disjoint chordless paths $P_1 = a \dots a'$, $P_2 = b \dots b'$, $P_3 = c \dots c'$ of length at least 1, such that abc and $a'b'c'$ are triangles and no edges exist between the paths except those of the two triangles. Such a prism is also referred to as a $3PC(abc, a'b'c')$ or a $3PC(\Delta, \Delta)$.

A *theta* is a graph made of three internally vertex-disjoint chordless paths $P_1 = a \dots b$, $P_2 = a \dots b$, $P_3 = a \dots b$ of length at least 2 and such that no edges exist between the paths except the three edges incident to a and the three edges incident to b . Such a theta is also referred to as a $3PC(a, b)$ or a $3PC(\cdot, \cdot)$.

Observe that the lengths of the paths in the three definitions above are designed so that the union of any two of the paths induces a hole. A *wheel* is a graph formed from a hole H together with a vertex x that has at least three neighbors in the hole. Such a hole H is called the *rim*, and such a vertex x is called the *center* of the wheel. We denote by (H, x) , the wheel with rim H and centered at x .

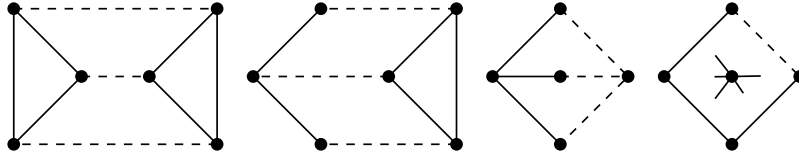


FIGURE 1.9: Prism, pyramid, theta, and wheel (solid lines represent edges and dashed lines represent paths of length at least one)

Some authors refer to K_4 (the complete graph on four vertices) as a wheel, but in this thesis we do not. A *3-path-configuration* is a graph isomorphic to a prism, a pyramid, or a theta. A *Truemper configuration* is a graph isomorphic to one of the 3-path-configurations or a wheel. They appear in a theorem of Truemper [Tru82] that characterizes graphs whose edges can be labeled so that all chordless cycles have prescribed parities (see [Tru82] for more details). The 3-path-configurations seem to have first appeared in a paper of Watkins and Mesner [WM67]).

The configurations that Truemper identified in the theorem play an essential role in understanding the structure of several objects, such as perfect graphs and even-hole-free graphs. A study of classes of graphs forbidding Truemper configurations has been done over these past years, a survey on research in this area is available, written by Vušković [Vuš13]. As explained in the survey, Truemper configurations play an essential role in analyzing several important hereditary graph classes. Note that in each of the 3-path configurations, at least two paths must have the same parity. Hence, pyramids always contain an odd hole, and thetas and prisms always contain an even hole. Moreover, every wheel that has an even number of spokes always contains an even hole. So, *perfect graphs are pyramid-free* while *even-hole-free graphs are (theta, prism, even wheel)-free*. Many decomposition theorems for graph classes are proved by studying how some Truemper configurations are contained in the graph attached to the rest of the graph. In particular, attachments to a Truemper configuration are often used to provide a contradiction when one is working with Berge graphs or even-hole-free graphs. A famous example is in the study of the class of perfect graphs, which is a class that excludes pyramids but may contain prisms. By studying how graphs are structured around prisms, in the sense that how the rest of the graph is attached to the prisms contained in the graph, one obtains the decomposition theorem for perfect graphs, which leads to the celebrated *Strong Perfect Graph Theorem* [Chu+06]. We will see that this is also the case when discussing even-hole-free graphs in Chapter 3.

Despite its relation to perfect graphs and even-hole-free graphs, classes of graphs that forbid some of the Truemper configurations are also intriguing. When a graph does not contain any of the Truemper configurations, it turns out that the graph is

structured very nicely — we call such a graph *universally signable*. This characterization of universally signable graphs is then used to obtain the following decomposition theorem (cf. Theorem 1.2.6), from which one can derive efficient algorithms for finding the size of the largest clique, or the largest independent set, or the smallest clique cover, or coloring the class. Hence, along with the goal mentioned in the previous paragraph, we would also like to understand better the classes defined by excluding Truemper configurations.

Theorem 1.2.6 (Universally signable graphs [Con+97])

A connected $(3PC(\cdot, \cdot), 3PC(\Delta, \cdot), 3PC(\Delta, \Delta), \text{wheel})$ -free graph is either a clique or a hole, or it has a clique cutset.

Among many classes that forbid Truemper configurations, the most exciting thing for us is the class of theta-free graphs. Indeed, theta-free graphs generalize claw-free graphs (since a theta contains claws each centered at the end of the three paths). Hence it is natural to ask whether it shares the essential features of claw-free graphs: a structural description (see [CS08]), a polynomial-time algorithm for the maximum independent set (see [FOS11]), an approximation algorithm for the chromatic number (see [Kin09]), and a polynomial χ -bounding function (see [Hal87]). Furthermore, note that this is a superclass of even-hole-free graphs. Chapter 3 discusses a specific subclass of theta-free graphs, that is, when the triangle are excluded.

Detecting Truemper configurations. Detecting the presence of some of the Truemper configurations in some given graph is of interest. Even though pyramids, prisms, and thetas seem to have similar features, testing for their presence in some graphs has different complexity. The table below gives a resume of the complexity for detecting some of the Truemper configurations.

Structure	Complexity	Reference
Theta	$\mathcal{O}(n^{11})$	[CS10]
Pyramid	$\mathcal{O}(n^9)$	[Chu+05]
Prism	NPC	[MT13]
Wheel	NPC	[DTT13]
Theta or pyramid	$\mathcal{O}(n^7)$	[MTV08]
Theta or prism	$\mathcal{O}(n^{35})$	[CK08]
Theta or wheel	$\mathcal{O}(n^4m)$	[RTV20]
Pyramid or prism	$\mathcal{O}(n^5)$	[MT13]
Pyramid or wheel	NPC	[DTT13]
Prism or wheel	NPC	[DTT13]
Theta, pyramid, or prism	$\mathcal{O}(n^7)$	[MT13; MTV08]
Theta, pyramid, or wheel	$\mathcal{O}(n^3m)$	[Dio+20]
Theta, prism, or wheel	$\mathcal{O}(n^4m)$	[Dio+20]
Pyramid, prism, or wheel	NPC	[DTT13]
Theta, pyramid, prism, or wheel	$\mathcal{O}(nm)$	[Con+97; Tar85]

TABLE 1.2: Complexity of detecting Truemper configurations (n and m respectively denote the number of vertices and edges in the input graph)

1.3 Terminology

This section introduces some formal mathematical notions about graphs that will be used throughout this thesis. Other advanced notions and terminology will be defined later in the relevant sections/subsections.

Graphs. As we have seen in the previous paragraphs, a graph is a mathematical structure consisting of *vertices* (singular: *vertex*) or *nodes*⁹ that are connected by *edges*. More formally, a graph G is a pair of sets (V, E) , where V is a finite non-empty set of vertices, and E is a finite set of edges, each of which has two associated vertices, i.e. every edge e is a set containing two vertices $\{u, v\}$. The sets V and E are the *vertex-set* and the *edge-set* of G , and are denoted by $V(G)$ and $E(G)$.

Sometimes to represent our data, we need to associate a direction with the edges to indicate a one-way relationship. For this purpose, we assign a direction to the edges of the graphs, which naturally defines *directed* graph. A graph that is not directed is called *undirected*. Sometimes we allow that more than one edge present between two vertices (this is called *multiple edges*), or allow the existence of an edge that ends in a single vertex (this is called *loop*). Graphs that do not have loops or multiple edges are called *simple* graphs. In this thesis, unless stated, we always refer to *undirected simple* graphs, and we will often denote a graph with the letter G . The *complement* of a graph G is the graph \bar{G} with vertex set $V(\bar{G}) = V(G)$ and edge set $E(\bar{G}) = [V(G)]^2 \setminus E(G)$, where $[V(G)]$ denotes the set of all possible edges of G .

Connectivity. Two vertices u and v of a graph that are connected by an edge are called *adjacent*. Vertex u is called a *neighbor* of v or is *adjacent* to v . In a graph G , the *neighborhood* of a vertex v is the set of vertices in G adjacent to v , and we denote it by $N_G(v)$. We also denote $N_G[v] = N_G(v) \cup \{v\}$, which is called the *closed neighborhood* of v . For a subset of vertices $A \subseteq V(G)$, the set of vertices in $V(G) \setminus A$ which consists of vertices with at least one neighbor in A is denoted with $N_G(A)$ (and we denote $N_G[A] = N_G(A) \cup A$). When the context is clear, we write $N(v)$, $N[v]$, $N(A)$, and $N[A]$ instead of $N_G(v)$, $N_G[v]$, $N_G(A)$, and $N_G[A]$ respectively. The *degree of a vertex* v in G is defined as $\deg_G(v) := |\{u \in V(G) \mid \{u, v\} \in E(G)\}|$. The *maximum degree of G* , $\Delta(G)$, is the *maximum degree* over all vertices of G .

For disjoint sets $A, B \subseteq V(G)$, we say that A is *complete* (resp. *anticomplete*) to B if all edges (resp. no edges) are present between A and B in G . If A consists of a single vertex a , then we say that a is complete (resp. anticomplete) to B .

Graph operations and graph drawings. Two graphs G and G' are *isomorphic* if there is a bijection f between the vertex sets of G and G' such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in G' . Simply put, two graphs being isomorphic means they are essentially the same but potentially not represented by the same drawing. A graph is *planar* if it can be embedded in the plane, i.e., it can be drawn on the plane so that its edges intersect only at its endpoints. Such a drawing is called a *planar embedding* of the graph.

The *edge subdivision* operation for an edge $uv \in E(G)$ is the deletion of uv from G and the addition of two edges uw and wv adjacent to the new vertex w . This operation generates a new graph $G' = (V \cup \{w\}, (E \setminus \{uv\}) \cup \{uw, wv\})$. A *subdivision*

⁹We will use the terms “vertex” and “node” interchangeably in this thesis.

of G is a graph that can be derived from G by a sequence of edge subdivision operations. An *edge contraction* is an operation which removes an edge from a graph while simultaneously merging the two vertices that it previously joined, that is, given an edge $uv \in E(G)$, replace vertices u and v with a single new vertex x adjacent to all vertices initially adjacent to u or v .

Substructures. A graph H is a *subgraph* of a graph G , if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ (i.e., H is obtained from G by a sequence of vertex deletions and edge deletions). For a set $X \subseteq V(G)$, the subgraph *induced* by X in G is the subgraph $G[X]$ of G with vertex set X , such that $e \in E(G[X])$ iff $e \in E(G)$ and e has both ends in X . A graph H is an *induced subgraph* of G , if $H = G[X]$ for some $X \subseteq V(G)$ (i.e., H is obtained from G by a sequence of vertex deletions). If G contains a graph isomorphic to some graph H as an induced subgraph, we say that *G contains H* . In this thesis, when we say that *G contains H* without specifying any particular relation, we mean that H is an induced subgraph of G . For a set $S \subseteq V(G)$ we let $G \setminus S := G[V(G) \setminus S]$ and if $S = \{v\}$ is a singleton set, then we write $G \setminus v$ instead of $G \setminus \{v\}$. A graph H is a *minor* of a graph G if a graph isomorphic to H can be constructed from G by repeated applications of vertex deletion, edge deletion, and edge contraction.

A *path* in G is a sequence P of distinct vertices $p_1 \dots p_n$, where $p_i p_{i+1} \in E(G)$ for $1 \leq i < n$. A path is called *chordless* (or an *induced*) path, if for $i, j \in \{1, \dots, n\}$, $p_i p_j \in E(G)$ if and only if $|i - j| = 1$. For two vertices $p_i, p_j \in V(P)$ with $j > i$, the path $p_i p_{i+1} \dots p_j$ is a subpath of P that is denoted by $p_i P p_j$. The subpath $p_2 \dots p_{n-1}$ is called the *interior* of P . The vertices p_1, p_n are the *ends* of the path, and the vertices in the interior of P are called the *internal* vertices of P . In this thesis, unless stated, by path we always mean a *chordless* path. Sometimes we still write *chordless path* instead of just *path* to emphasize.

A *cycle* is defined similarly, with the additional properties that $n \geq 4$ and $p_1 = p_n$. The *length* of a path P is the number of edges of P . The length of a cycle is defined similarly. As in path, we have the notion of the chordless or induced cycle. An edge $e \in E(G)$ is a *chord* of cycle C , if the endpoints of e are vertices of C that are not adjacent on C . We often denote a chordless cycle on n vertices with C_n . A *hole* is a chordless cycle of length at least 4. It is *odd* or *even* according to its length (that is its number of edges). An *antihole* is an induced subgraph H of G , such that \overline{H} is a hole of \overline{G} . Note that C_3 is a chordless cycle but it is not a hole. The *girth* of a graph is the length of the shortest cycle in the graph. See Figure 1.10 for an example of paths and holes.

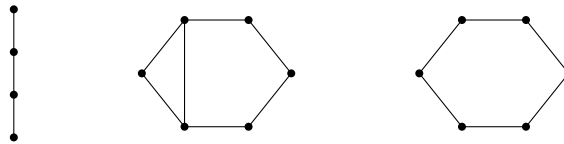


FIGURE 1.10: A path, a cycle with chord, and an (even) hole

Some particular graphs. Let us now present some particular graphs that will be often mentioned throughout this thesis.

Recall that a *tree* is a connected graph that does not contain any cycle (also called *acyclic*); trees can be considered as the simplest graphs. A disjoint union of trees is called *forest* (cf. Figure 1.6).

A *clique* in G is a set $X \subseteq V(G)$ of vertices such that $\{v, w\} \in E(G)$ for every pair $v, w \in X$ with $v \neq w$. A graph K is *complete*, if $V(K)$ is a clique in K . We denote complete graph on n vertices by K_n . The *triangle* is a graph isomorphic to K_3 (or C_3). The *square* is a graph isomorphic to C_4 (in this thesis we will use the terms square and 4-hole interchangeably). A *claw* is a graph with vertex set $\{v, x, y, z\}$ and edge set $\{vx, vy, vz\}$; the vertex v is called the *center* of the claw. An *independent set* in G is a set of pairwise non-adjacent vertices.

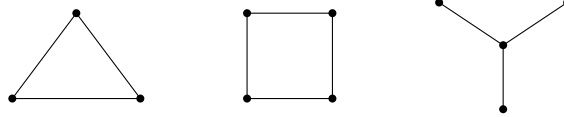


FIGURE 1.11: Triangle, square, and claw

A graph is *bipartite* if its vertex set can be partitioned into two independent sets. It is called *complete bipartite* if every pair of vertices of the different partite sets are adjacent. The complete bipartite graph whose bipartite components are of size p and q respectively is denoted by $K_{p,q}$.

1.4 Main contributions of the thesis

In this thesis, we study the structure of even-hole-free graphs. In particular, we study some width parameters (such as tree-width and rank-width of the graphs in the class). We now give a summary of our contribution that will be discussed further in Chapter 3, Chapter 4, and Chapter 5.

Even-hole-free graphs with no large cliques

In general, even-hole-free graphs have unbounded tree-width. This is because complete graphs are even-hole-free and complete graphs on n vertices have tree-width $n - 1$ for any integer n . However, when some configurations are excluded from the class, the tree-width of the graphs is bounded. In the next chapter, we give a survey about some known results on the width of some subclasses of even-hole-free graphs.

This work was initially motivated by a study that was conducted on the class of (even hole, triangle)-free graphs (recall that the triangle is clique on 3 vertices). Cameron, da Silva, Huang, and Vušković [Cam+18] prove that the class of even-hole-free graphs that do not contain the triangle has tree-width bounded by some constant (see Theorem 2.3.7). This is due to the structure theorem which exists for graphs in this class (see Subsection 2.3.2 for more details). Moreover, when the clique number of an even-hole-free graph G equals t , then excluding a configuration called *caps* and *pans* (a cap is made of a hole plus one additional vertex adjacent to two adjacent vertices in the hole, and a pan is made of a hole plus one pendant edge incident to a vertex of the hole) yields a bound on the tree-width in terms of t (see Section 2.3 of Chapter 2 for more detail). A natural question to ask, is that whether the following generalization of these results holds: *is the tree-width of an even-hole-free graph bounded by some function of its clique number, i.e. whether $tw(G) \leq f(\omega(G))$ is true for any graph G that is even-hole-free?* Another way to see this question is that whether bounding the size of the maximum clique yields an upper bound on the tree-width of the graph. This question was first asked by Cameron, Chaplick, and Hoàng in their paper [CCH18].

We remark that in the triangle-free case, wheels have an important role to the existence of the structure theorem. Indeed, (even hole, triangle)-free graphs are decomposed using the so-called star cutset, which arises from a wheel (see Subsection 2.1 for the definition). One important property of wheels in this class is that, when applying decomposition, the choice of wheel used in the decomposition does not matter, i.e. we can use any wheel to decompose the graph, in any order. This method is possible because in (even hole, triangle)-free graphs, the graph is nicely structured around every wheel in the graph. For every wheel $W = (H, x)$, the neighborhood of each connected component of $G \setminus W$ is contained in a unique sector of W . In particular, for any two distinct neighbors x_i and x_j of x on H , $\{x, x_i, x_j\}$ is a cutset (see Theorem 2.3.5 of Section 2.3). When the triangle is allowed, this is no more true because wheels in the graph interact in more complex ways. Two wheels of different centers might share the same rim H (which is not the case in the triangle-free case), and the neighbors of those two centers might cross in H (see an example in Figure 1.12). Further discussion about wheels in (even hole, triangle)-free graphs will be postponed to Chapter 4 when we discuss the class of (theta, triangle)-free graphs, a superclass of (even hole, triangle)-free graphs.

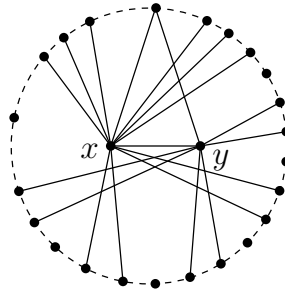


FIGURE 1.12: Two wheels in an even-hole-free graph with crossing spokes along the rim (dashed edges represent paths of odd length)

In Chapter 3, we construct a family of even-hole-free graphs with clique number 3 (i.e. the graphs are K_4 -free) for which the tree-width is unbounded. We call this family of graphs *layered wheels*. These constructions show that an even-hole-free graph with no clique of size 4 may have arbitrarily large tree-width. Indeed, for any integer $l \geq 1$, we can construct a family of (even hole, K_4 , pyramid)-free graphs with tree-width at least k . We furthermore explore another class of graphs for our study, namely the class of (triangle, theta)-free graphs. Recall that theta-free graphs form a superclass of even-hole-free graphs. Since we forbid the triangle in one class but allow its presence in the other class, these two classes intersect, and the class of (even hole, triangle)-free graphs lies in the intersection of the two (cf. Figure 1.13). The class of (theta, triangle)-free graphs is a “tool” that we use to study the class of (even hole, K_4)-free graphs. It turns out that when trying to prove some properties for the class of (even hole, K_4)-free graphs, it is “easier” to examine the property first in the class of (theta, triangle)-free graphs, then adapt the technique to be applied in the class of (even hole, K_4)-free graphs. The structure of wheels in the two classes of graphs have a similarity, as stated in Lemma 4.3.1 and Lemma 4.3.3 of Chapter 4. These inspire the construction of layered wheels in both classes of graphs which have similar properties. Thus, we have a similar result for the class of (theta, triangle)-free graphs.

The following theorem covers the results that we will explain further in Chapter 3.

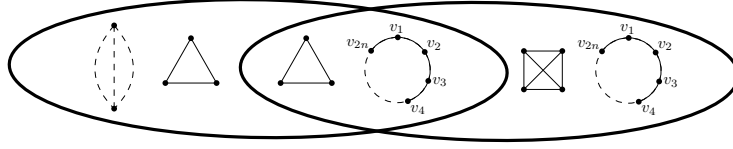


FIGURE 1.13: An illustration of the forbidden structures in (theta, triangle)-free graphs (left), (even hole, K_4)-free graphs (right) and (even hole, triangle)-free graphs in the intersection

Theorem 1.4.1

For every integers $l \geq 1$ and $k \geq 4$, there exists a graph $G_{l,k}$ that is theta-free, all cycles contained in the graph are of length at least k , and the tree-width of the graph is at least l .

Theorem 1.4.2

For every integers $l \geq 1$ and $k \geq 4$, there exists a graph $G_{l,k}$ that is (even hole, K_4 , pyramid)-free, all holes contained in the graph are of length at least k , and the tree-width of the graph is at least l .

Forbidding more structures and its impacts on the tree-width

We note that the family of layered wheels which provides a lower bound on the tree-width of (even hole, K_4)-free graphs needs a “huge” number of vertices to increase such a lower bound. More specifically, to get a lower bound l on the tree-width, a layered wheel must contain $\Omega(k^l)$ vertices for some $k \geq 3$. This suggests a conjecture that the tree-width of an (even hole, K_4)-free graph is of a logarithmic function of the size of the input graph (see Conjecture 6.1.2 of Chapter 6). This conjecture holds for the family of layered wheels (see Corollary 3.4.5 of Chapter 3).

In an attempt to answer this conjecture, we studied some subclasses of (even hole, K_4)-free graphs when some other configurations are excluded, and where the tree-width is bounded. Let $S_{i,j,k}$ be the tree with a vertex v , from which start three paths with i , j , and k edges respectively. We show that (even hole, pyramid, $S_{i,j,k}$)-free graphs have bounded tree-width, as shown in the following theorems. We study this property for both classes mentioned earlier, namely the class of (theta, triangle)-free graphs and the class of (even hole, K_4 , pyramid)-free graphs. These results will be discussed more clearly in Chapter 4. In the following, $R(\cdot, \cdot)$ denotes the Ramsey number.

Theorem 1.4.3

For $k \geq 1$, every (theta, triangle, $S_{k,k,k}$)-free graph G has tree-width at most $2(R(3, 4k-1))^3 - 1$.

Theorem 1.4.4

For $k \geq 1$, every (even hole, pyramid, K_t , $S_{k,k,k}$)-free graph G has tree-width at most $(t-1)(R(t, 4k-1))^3 - 1$.

We furthermore note that, in layered wheels, the presence of a large clique minor and high maximum degree are necessary to increase the bound on the tree-width. This fact motivates our study on the class of even-hole-free graphs with no large

clique minor and the class of even-hole-free graphs with bounded maximum degree. Studies show that both classes have bounded tree-width [ACV20; Abo+cs]. In Chapter 5, we present the tree-width of even-hole-free graphs of bounded degree. In particular, we study the case when the maximum degree of the graphs is 3 (i.e. when the graphs are subcubic). We also study the case when the maximum degree is 4 and the graph is pyramid-free. For the proof, we establish a full structure theorem of even-hole-free subcubic graphs, saying that every even-hole-free subcubic graph either is a basic graph (which in some sense has a simple structure) or has a “good” separator (cf. Theorem 5.1.2), which yields a constant bound on the tree-width. The case for the maximum degree 4 is proved similarly. The following theorems cover our results given in Chapter 5. The theorems are, in fact, proved for the class of (theta, prism)-free graphs that is a superclass of even-hole-free graphs.

Theorem 1.4.5

Every (theta, prism)-free subcubic graph (and therefore every even-hole-free subcubic graph) has tree-width at most 3.

Theorem 1.4.6

Every (even hole, pyramid)-free graph of maximum degree 4 has tree-width at most 4.

Publications

The main results in this thesis are covered in the following publications:

1. N.L.D. Sintiari and N. Trotignon. (Theta, triangle)-free and (even hole, K_4)-free graphs. Part 1 : Layered wheels. *ArXiv preprint*, <https://arxiv.org/abs/1906.10998> (appear in *Journal of Graph Theory*, early view).
2. M. Pilipczuk, N.L.D. Sintiari, S. Thomassé, and N. Trotignon. (Theta, triangle)-free and (even hole, K_4)-free graphs. Part 2 : bounds on treewidth. *ArXiv preprint*, <https://arxiv.org/abs/2001.01607> (appear in *Journal of Graph Theory*, early view).
3. P. Aboulker, I. Adler, E.J. Kim, N.L.D. Sintiari, and N. Trotignon. On the tree-width of even-hole-free graphs. *ArXiv preprint*, <https://arxiv.org/abs/2008.05504> (accepted, to appear in *European Journal of Combinatorics*).

We are now preparing a paper about *the structure of graphs when all holes in the graphs are of the same fixed length $k \geq 7$* . The goal of our study is to have a better view on the structure of the class, in the sense of a full structure theorem of the class (if any). Note that when k is odd, this class of graphs is a subclass of even-hole-free graphs, hence having knowledge about this class might give an insight about even-hole-free graphs in general. This is a joint work with Jake Horsfield, Myriam Preissmann, Cléopée Robin, Nicolas Trotignon, and Kristina Vučković. However, in this thesis, we do not discuss about this class of graphs.

Chapter 2

A survey on even-hole-free graphs

Recall the four combinatorial problems that we mention in Chapter 1: *optimal coloring*, *maximum clique*, *maximum independent set*, and *minimum clique cover* problems. For even-hole-free graphs, it is known that the maximum clique problem is polynomial-time solvable because a graph without a hole on four vertices has a polynomial number of maximal cliques [Far89], and one can list them all in polynomial time. The three other problems, however, are still open. We are particularly interested in understanding whether the decomposition theorem can be applied in the design of polynomial-time algorithms for all these combinatorial problems. Let us begin this chapter by presenting decomposition theorems of even-hole-free graphs.

2.1 Decomposition of even-hole-free graphs

The decomposition technique for even-hole-free graphs was first developed in the process of studying perfect graphs. It turned out that the tools that were used to decompose even-hole-free graphs could also be implemented for perfect graphs. In this section, we give a summary of the decomposition theorem of even-hole-free graphs.

Conforti, Cornuéjols, Kapoor, and Vusković in [Con+02a] first studied the structure of even-hole-free graphs, where they presented a decomposition theorem for this class using the so-called *2-joins* and *k-star cutsets* (we explain the notions below), that was used later to construct a polynomial-time recognition algorithm [Con+02b]. Conforti et al. [CCV04] proved the Strong Perfect Graph Conjecture for 4-hole-free graphs, by decomposing Berge graphs using star cutsets and 2-joins into bipartite graphs and *line graphs*¹ of bipartite graphs. Later, Chudsnovky et al. used a similar approach to prove the general case, i.e. when 4-holes are allowed, resulting in the Strong Perfect Graph Theorem. Let us now present the two main ingredients of the decomposition of even-hole-free graphs, namely the *cutsets* and the *basic graphs*.

Star cutset and 2-join

The decomposition theorem that we describe in the following uses two types of cutsets. A vertex cutset $S \subseteq V(G)$ is a *k-star cutset* of G if S is comprised of a clique C of size k and vertices with at least one neighbor in C , i.e. $C \subseteq S \subseteq N[C]$. We refer to C as the *center* of S . A 1-star is also referred to as a *star*, a 2-star as a *double star*, and 3-star as a *triple star*.

A *2-join*² In a graph G is a partition of the vertex set of G into (V_1, V_2) , with *special sets* (A_1, A_2, B_1, B_2) , such that the following holds:

¹The line graph of G is another graph $L(G)$ that represents the adjacencies between edges of G ; a formal definition is given in the next pages, when defining basic graphs.

²In particular, a 2-join is an *edge cutset*.

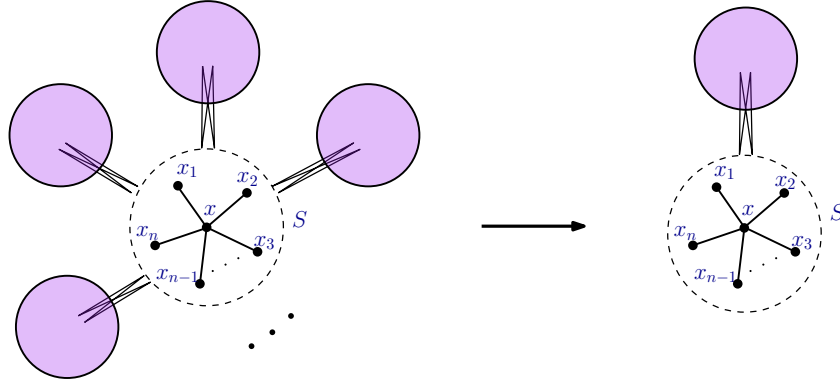


FIGURE 2.1: A star cutset and a block of decomposition

- (i) For $i = 1, 2$, $A_i \cup B_i \subseteq V_i$, and A_i and B_i are nonempty and disjoint.
- (ii) A_1 is complete to A_2 , B_1 is complete to B_2 , and these are the only adjacencies between V_1 and V_2 .
- (iii) For $i = 1, 2$, the graph induced by V_i , $G[V_i]$, contains a path with one end in A_i and the other in B_i . Furthermore, V_i does not induce a chordless path with one end in A_i , one end in B_i , and no internal vertex in $A_i \cup B_i$.

We note that slightly different definitions of 2-joins are used in different papers.

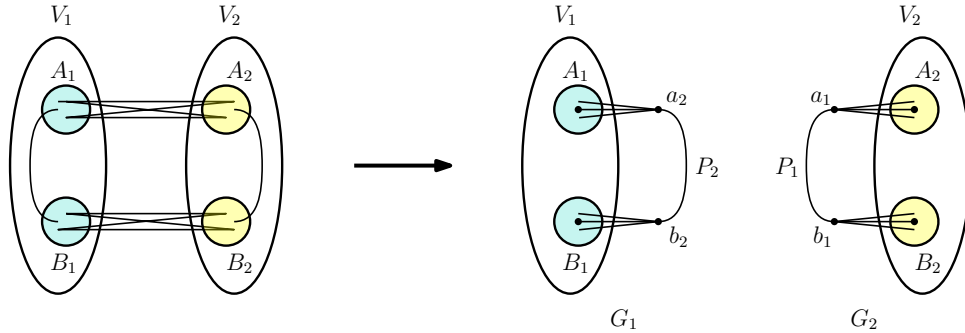


FIGURE 2.2: 2-join decomposition into two blocks of decomposition

The notion of star cutsets was introduced by Chvátal [Chv85] and 2-join was first introduced by Cornuéjols and Cunningham in [CC85]. The intuition behind the use of star cutsets and 2-joins in the decomposition of even-hole-free graphs is the necessity of “breaking” a hole of the graph (see Figure 2.3). In particular, for even-hole-free graphs, star cutsets are used to break wheels in the graph being decomposed.

When decomposing an even-hole-free graph G using a star cutset or a 2-join, we obtain a set of simpler graphs which is called the *blocks of decomposition*, which can be defined as follows:

- Given a graph G and a star cutset $S = \{x, x_1, x_2, \dots, x_n\}$ such that $G \setminus S$ contains connected components C_1, C_2, \dots, C_k . A block of the star cutset decomposition is the graph induced by $S \cup C_j$ for $j \in [1, n]$ (cf. Figure 2.1).
- Suppose that (V_1, V_2) is a 2-join of a graph G with special sets (A_1, A_2, B_1, B_2) . The blocks of decomposition w.r.t. the 2-join are the graphs G_1 and G_2 constructed in the following way: G_1 is the subgraph of G induced by the vertex

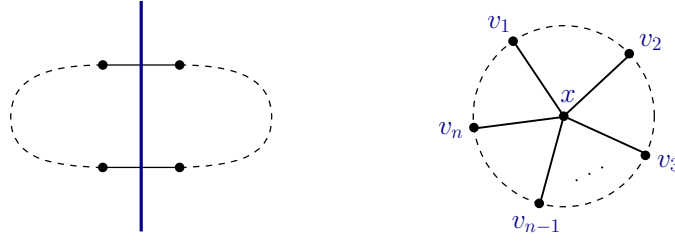


FIGURE 2.3: Key cutsets in the decomposition which are used to break a hole in the graph being decomposed

set V_1 plus a *marker path* $P_2 = a_2 \dots b_2$ that is a chordless path from a vertex a_2 complete to A_1 to a vertex b_2 complete to B_1 , and whose interior vertices are all of degree two in G_1 . Block G_2 is obtained in a similar way, by replacing V_1 with a marker path P_1 (cf. Figure 2.2). We note that for $i \in \{1, 2\}$, P_i is used to “encode” the chordless paths from A_{3-i} to B_{3-i} in V_{3-i} . Vertices a_{3-i} and b_{3-i} respectively represent vertices in A_{3-i} and B_{3-i} . The length of P_i depends on the class we are working with. In the case of even-hole-free graphs, note that all chordless paths in V_i from A_i to B_i with interior in $V_i \setminus (A_i \cup B_i)$ are of the same parity (for otherwise, one of the two paths in V_i together with a path in V_{3-i} from A_{3-i} to B_{3-i} would form an even hole). We can then assign an odd length $k_i \geq 3$ (resp. an even length $k_i \geq 4$) to P_i if the paths in V_i are of odd (resp. even) lengths.

Basic graphs

Recall that a graph is *basic* if it does not contain any cutset that is used in the decomposition.

Nontrivial basic graph. The following definition of the first basic graphs in the decomposition theorem of even-hole-free graphs was introduced by Conforti, Cornuéjols, Kapoor, and Vušković [Con+02a].

Given a graph G , the *line graph* of G is a graph $L(G)$ that represents the adjacencies between edges of G , i.e. there is a one-to-one correspondence between the edge set of G and the vertex set of $L(G)$, where two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G . Let L be the line graph of a tree (cf. Figure 2.4 for an example). Note that every edge of L belongs to exactly one maximal clique, and every node of L belongs to at most two maximal cliques. The nodes of L that belong to exactly one maximal clique are called *leaf nodes*. A clique of L is *big* if it is of size at least 3. In the graph obtained from L by removing all edges in big cliques, the connected components are chordless paths (possibly of length 0). Such a path P is an *internal segment* if it has its endnodes in distinct big cliques (when P is of length 0, it is called an internal segment when the node of P belongs to two big cliques). The other paths are called *leaf segments*. Note that one of the endnodes of a leaf segment is a leaf node.

A *nontrivial basic graph* R is defined as follows: R contains two adjacent nodes x and y , that are called the *special nodes*. The graph L induced by $R \setminus \{x, y\}$ is the line graph of a tree and contains at least two big cliques. In R , each leaf node of L is adjacent to exactly one of the two special nodes, and no other node of L is adjacent to any of the special nodes. The last condition for R is that no two leaf segments of L with leaf nodes adjacent to the same special node have their other endnode in the same big clique. The internal segments of R are the internal segments of L , and

cutsets and 2-joins into bipartite graphs and line graphs of bipartite graphs (Conforti, Cornuéjols, and Vušković [CCV04]). The general case was finally proved by Chudnovsky, Robertson, Seymour, and Thomas in [Chu+06], where they decompose Berge graphs using *skew cutsets*⁴, 2-joins, and their complements. In 4-hole-free graphs, a skew cutset reduces to a star cutset⁵ and a 2-join in the complement implies a star cutset. It is therefore natural to believe that a similar decomposition would work for even-hole-free graphs.

In [SV13], da Silva and Vušković implemented this idea to obtain a strengthening of the decomposition theorem (cf. Theorem 2.1.2). The basic graphs for this new decomposition are cliques, holes, long pyramids, and extended nontrivial basic graphs, where an *extended nontrivial basic graph* of G is a graph R' consisting of a nontrivial basic graph R and all nodes $v \in V(G) \setminus V(R)$ such that for some big clique K of R and for some $z \in \{x, y\}$, $N(v) \cap V(R) = V(K) \cup \{z\}$. In Figure 2.6, we give an example of an extended nontrivial basic graph.

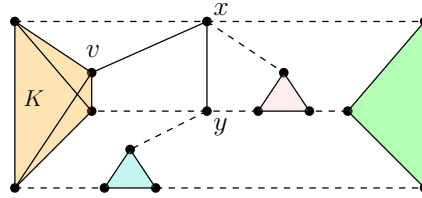


FIGURE 2.6: An extended nontrivial basic graph with special nodes $\{x, y\}$, $v \in V(G) \setminus V(R)$ with $N(v) \cap V(R) = V(K) \cup \{x\}$; dashed lines represent paths of length at least 1 which are the segments of R

Theorem 2.1.2 (Decomposition of EHF graphs strengthening [SV13])

A connected 4-hole-free odd-signable graph is either basic or it has a 2-join or a star cutset.

Theorem 2.1.2 was proved for a superclass of even-hole-free graphs. A simplified decomposition is obtained when the class is restricted to even-hole-free graphs. A graph is a *clique tree* if each of its maximal 2-connected components is a clique. A graph is an *extended clique tree* if it can be obtained from a clique tree by adding at most two vertices.

Corollary 2.1.3 ([SV13])

A connected even-hole-free graph is either an extended clique tree, or it has a 2-join or a star cutset.

2.2 The use of cutsets for algorithms

2.2.1 Recognition algorithm

Conforti, Cornuéjols, Kapoor, and Vušković [Con+02b] presented the first polynomial-time recognition algorithm for the class of even-hole-free graphs. This

⁴A skew partition (first defined by Chvátal [Chv85]) of a graph G is a partition (A, B) of $V(G)$ such that $G[A]$ is not connected, and $G[B]$ is not anticonnected. In this case, we say that B is a skew cutset.

⁵From the definition, it is clear that S is a skew cutset if S can be partitioned into (S_1, S_2) such that S_1 is complete to S_2 , and in the case of even-hole-free graphs, this yields that S_1 or S_2 must be a clique (for otherwise, there would exist a square).

algorithm was built based on the decomposition algorithm that is presented in [Con+02a] (see again Section 2.1), and it runs in time $\mathcal{O}(n^{40})$ where n is the size of the input graph. This recognition algorithm was later improved by da Silva and Vušković [SV13] into $\mathcal{O}(n^{19})$. Chang and Lu [CL15] improved this to $\mathcal{O}(n^{11})$ time. The best currently known algorithm is given by Lai, Lu, and Thorup [LLT20] which runs in $\mathcal{O}(n^9)$ time.

Let us now explain the ideas behind the decomposition-based recognition algorithm. As explained in [Con+02b], standard conditions which have to be satisfied in order to create a polynomial-time algorithm based on a decomposition theorem are the following:

- (i) checking whether a decomposition of G exists can be done in polynomial time;
- (ii) G is in the class if and only if all the decomposition blocks are (i.e. the decomposition is *class-preserving*);
- (iii) when the decomposition is applied recursively to the decomposition blocks, the total number of blocks created is polynomial.

To recognize a class of graphs \mathcal{C} based on a decomposition theorem, basic graphs need to be recognizable easily (in our case, in polynomial time). For even-hole-free graphs, this works for all basic graphs of the decomposition theorem explained in Section 2.1. A recognition algorithm takes a graph G as input and decomposes it using \mathcal{C} -preserving decomposition into a polynomial number of basic “blocks”, which are then checked, in polynomial time, whether they belong to \mathcal{C} . The property (ii) ensures that the original input graph is also in \mathcal{C} if all the blocks are in \mathcal{C} , and the polynomial-time complexity result then follows from the property (iii).

The recognition algorithm in [Con+02b] is based on the decomposition theorem explained in [Con+02a] (also explained in Section 2.1), where graphs in the class are decomposed using 2-join, and star, double-star, and triple-star cutsets. Whenever a 2-join or a k -star cutset is present in a graph G , the graph is decomposed into two or more simpler graphs, that are the *blocks of the decomposition*. For even-hole-free graphs, the decomposition using star cutsets and 2-joins which we describe in Section 2.1 satisfies (i). However, neither (ii) nor (iii) holds, so this scenario does not work for obtaining a recognition algorithm for even-hole-free graphs. The problem is that our current definition of blocks of star cutsets is not class-preserving. The 2-joins, on the other hand, are still class-preserving. We will explain more about the fact that star cutsets are not always class-preserving in the next section.

To handle the problem in item (ii), Conforti, Cornuéjols, Kapoor, and Vušković [Con+02b] implemented the so-called *cleaning* procedure. This technique was developed by Conforti and Rao [CR93], which was also the key to obtain a polynomial-time recognition algorithm for Berge graphs [Chu+05]. Given an input graph G , the cleaning procedure produces, in polynomial time, a clean graph G' , such that G is even-hole-free if and only if G' is even-hole-free, and if G contains an even hole then G' contains a (so-called) *clean even hole* (namely an even hole for which there are no vertices outside the hole that have problematic neighbors on the hole, which can be used as clique centers of star cutsets to break the hole). A clean graph G can be decomposed recursively into a family of blocks that have no k -star cutsets and satisfy the following property: (i) either G is identified as containing an even hole during the decomposition process or (ii) when the decomposition process is completed, all blocks in the family are even-hole-free graphs if and only if G is

even-hole-free. Hence, once the graph is clean, decomposition can be applied safely since the graph is now class-preserving.

To resolve the problem in item (iii), they take care of the order of cutsets used to decompose the graph. As was first observed by Chvátal [Chv85], a graph has a star cutset if and only if it has a dominated node or a full star cutset (a vertex u is *dominated* by v if u is adjacent to v and $N(u) \subseteq N[v]$, and a star cutset S is *full* if S is comprised of a clique and all vertices with at least one neighbor in the clique). Indeed, when u is dominated by v , the set $\{v\} \cup N(u)$ forms a star cutset (it separates u from the rest of the graph). For the complexity to be polynomial, Conforti et al. [Con+02b] noted that when applying decomposition using k -star cutset, it is important to handle dominated vertices properly. In this case, dominated vertices are removed before applying the decomposition. They proved that the total number of blocks generated by the recursive decomposition with k -star cutsets is polynomial if one first remove dominated vertices and use *full* star cutsets. For the recognition algorithm to work in polynomial, this problem is handled by first decomposing the graph with star cutsets, then decomposing it with 2-joins, and later dealing with the dominated vertices in a particular way. Indeed, when the graph is *clean*, removing dominated vertices preserves the graph being even-hole-free. Each of the details in this paragraph can be found in [Con+02b].

Finally as pointed out in [Con+02b] their recognition algorithm can be used to find an even hole in graph G , if one exists, in the following way. Let v_1, \dots, v_n denote the nodes of G and let $H = G$. In iteration i , test whether $H \setminus v_i$ contains an even hole. If the answer is yes, set $H = H \setminus v_i$ and otherwise keep H unchanged. By performing n iterations, when the algorithm terminates, the graph H is the desired even hole.

2.2.2 The good and the bad cutsets in the decomposition of even-hole-free graphs

Decomposition theorems are often used for proving theorems. For instance, the proof of “The Strong Perfect Graph Conjecture (SPGC)” is based on the decomposition of Berge graphs. This method is done by ensuring that being “perfect” can be proved easily for basic graphs. The cutsets used in the decomposition have properties that cannot occur in a minimum counter-example⁶ to the SPGC. Decomposition theorems can also be used for algorithms, such as for the recognition algorithms discussed in the previous section. In some cases, the graph decomposition approach can be a helpful tool for solving optimization problems. Nevertheless, does it always work?

This question cannot be answered surely, since there are some kinds of decomposition that do not seem to be friendly with optimization problems such as optimal coloring, maximum clique, maximum independent set, and minimum clique cover problems. For instance, in perfect graphs, this method has not worked so far. The proof that the aforementioned optimization problems are polynomial-time solvable for perfect graphs actually uses the ellipsoid method and is far from using the decomposition of perfect graphs. Similarly, the proof for the maximum clique problem on even-hole-free graphs is not based on the decomposition theorem of even-hole-free graphs. The proof follows from the fact that excluding the 4-hole causes the graphs to have a polynomial number of maximal cliques and that one can list them in polynomial time. Let us now explain why cutsets are crucial.

⁶A minimum counter-example to SPGC is a Berge graph, non perfect, and of minimum size.

Let S be a node cutset of a graph G , and let C_1, \dots, C_k be the connected components of $G \setminus S$. A standard way to construct blocks of decomposition w.r.t. a node cutset would be to define blocks to be graphs G_1, \dots, G_k , where $G_i = G[C_i \cup S]$ for $i = 1, \dots, k$ (see again Section 2.1 for the definition of blocks of the star-cutset decomposition and the 2-join decomposition). We have seen the clique cutset decomposition in Chapter 1 Section 1.2.2. In the context of item (ii) of the three main requirements for using a decomposition theorem to construct a polynomial-time algorithm (explained in Subsection 2.2.1), clique cutset is an example of a cutset that works well when it comes to algorithms. Note that when S is a clique cutset, G contains an even (or odd) hole if and only if there exists an i such that G_i contains an even (or odd) hole, i.e. G is class-preserving. However, such a definition of blocks is not always class-preserving for the class of odd-hole-free graphs or even-hole-free graphs.

We note that the decomposition theorem of perfect graphs uses skew cutsets, and the decomposition of even-hole-free graphs uses star cutsets. We have seen that the problem with star cutsets is that we do not know how to construct the blocks of decomposition that are class-preserving while guaranteeing the polynomial complexity of the decomposition tree (see Figure 2.7). Therefore, skew cutsets and star cutsets fail, for instance, when one tries to implement them for recognition algorithm (because we want it to be class-preserving). A graph that is not even-hole-free might yield blocks of decomposition that are even-hole-free, so this is not applicable when one wants to implement the decomposition algorithm for constructing a recognition algorithm. Furthermore, a star cutset can be very big (as big as all of the vertex set except for two vertices). Hence, on even-hole-free graphs, we might end up with an exponential number of blocks even when decomposing only with star cutsets (which is bad since we hope to build a polynomial-time algorithm). An example is when in the recursive process, the star cutset used to decompose is always of size $n - 2$, where n is the size of the current block of decomposition⁷.

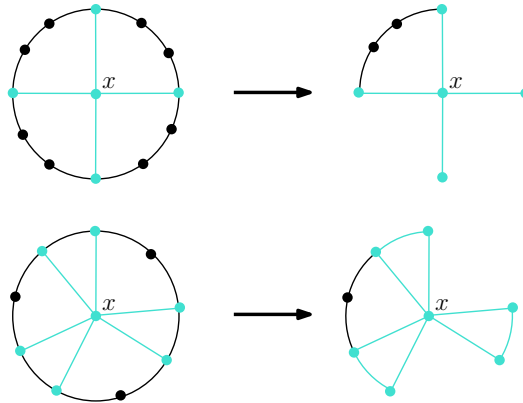


FIGURE 2.7: Star cutsets are not class-preserving; G contains an even/odd hole (left) but its block of decomposition does not (right)

On the contrary, 2-joins seem to be more applicable. Trotignon and Vušković [TV12] constructed combinatorial polynomial-time algorithms for solving the *weighted* version⁸ of the maximum clique, maximum independent set, and optimal coloring problems for a class of perfect graphs decomposable by 2-joins. They

⁷Nevertheless, this example is artificial, we have not yet succeeded in creating an example of a graph with this property.

⁸In the weighted version, the vertices of the input graph is assigned a certain weight (non-negative real or integer numbers).

also show that the techniques can also be applied to finding a maximum weighted independent set for another class of graphs known to be decomposable by 2-joins, namely the class of even-hole-free graphs that do not have a star cutset. However, as shown in [TV12], there is a class of graphs that is fully decomposable by 2-joins into bipartite graphs and line graphs, and for which finding a maximum independent set is NP-hard (note that the complexity of computing maximum independent set in the basic graphs is polynomial). This fact could indicate that having holes all of the same parity might give essential properties for the use of 2-joins in computing maximum independent sets.

For algorithmic purposes, we furthermore require that the set of cutsets used to decompose the graphs to be *non-crossing*. Two cutsets S_1 and S_2 that partition the graph into (A_1, B_1) and (A_2, B_2) with $A_1 \cap B_1 = S_1$ and $A_2 \cap B_2 = S_2$ are non-crossing if one of the following holds: $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$, or $A_1 \subseteq B_2$ and $A_2 \subseteq B_1$, or $B_1 \subseteq A_2$ and $B_2 \subseteq A_1$, or $B_1 \subseteq B_2$ and $A_2 \subseteq A_1$. However, star cutsets are very far from being non-crossing. On the other hand, a non-crossing 2-join decomposition exists when no star cutset is present in the graph [TV12]. In the case of 2-joins, the marker path used in the process of the decomposition always lies entirely in one side of every subsequent 2-join, that is, when G is decomposed into 2-join (V_1, V_2) , the edges between V_1 and V_2 do not belong to any marker path.

Recall that the decomposition of a graph is done recursively. When we finish decomposing the graph using clique cutsets, then for the rest of the proof we assume that the graph does not contain a clique cutset. Similarly, after finishing the decomposition using star cutsets, we will never see a star cutset again in the next step. So, at some point, we will end up with a graph having no star cutset, and it turns out that when this happens, the structure of the graph is less complex. Even-hole-free graphs satisfy this property when they do not contain star cutsets. In [TV12], Trotignon and Vušković constructed polynomial algorithms to find a maximum clique and maximum independent set in the subclasses of even-hole-free graphs and Berge graphs, which are fully decomposable by only 2-joins.

In [Chu+15], Chudnovsky et al. gave a generalization of such results, namely a polynomial algorithm to compute a maximum independent set on Berge graphs with no *balanced skew-partitions*⁹, and a polynomial algorithm to color them. In [Le18], it is proved that the structure of even-hole-free graphs with no star cutsets is simple in the sense that they have small rank-width, which yields a bound on the chromatic number in terms of clique number and the existence of a polynomial-time algorithm to color any graph in this class. We now explain more about this result of Le.

2.2.3 Even-hole-free graphs with no star cutset

The critical property of even-hole-free graphs with no star cutset is that they are fully decomposable using only 2-joins as shown by Trotignon and Vušković [TV12]. Even further, they admit 2-joins with the following property: one of its blocks of decomposition is a basic graph called an *extreme 2-join* in [TV12]. Moreover, the 2-join decomposition is class-preserving, meaning that the blocks of decomposition w.r.t. the 2-join are even-hole-free graphs with no star cutset.

⁹For a graph G , a partition (A, B) of $V(G)$ is balanced if every path of length at least 3, with ends in B and interior in A , and every antipath of length at least 3, with ends in A and interior in B has even length.

Lemma 2.2.1 ([TV12])

Let G be a connected even-hole-free graph with no star cutset and (V_1, V_2) a 2-join of G . Let G_1 and G_2 be blocks of decomposition w.r.t. this 2-join. Then G_1 and G_2 are connected even-hole-free graphs with no star cutset.

Lemma 2.2.2 ([TV12])

A connected even-hole-free graph with no star cutset is either basic or it has an extreme 2-join.

Furthermore, Le [Le18] showed that in the decomposition of even-hole-free graphs with no star cutset, it is possible to use non-crossing 2-joins. The sequence of the 2-join decomposition satisfying those properties can be represented using a *2-join decomposition tree*. In the following definition, we call a path P in G *flat* if all the internal vertices of P have degree 2 in G . We use the following notation for the 2-join decomposition, keeping the notation used by Le in his paper: when G is decomposed into 2-join (X_1, X_2) , the corresponding blocks of decomposition are denoted by G_1 and G_2 respectively, and the corresponding marker paths are denoted by P_2 and P_1 respectively (so, $G_1 = G[X_1 \cup V(P_2)]$ and $G_2 = G[X_2 \cup V(P_1)]$).

Definition 2.2.3. Let \mathcal{C} be a class of graphs and $\mathcal{B} \subseteq \mathcal{C}$ be the set of basic graphs in \mathcal{C} . For a given graph $G \in \mathcal{C}$, a tree \mathcal{T}_G is a 2-join decomposition tree for G if the following properties hold.

- (i) Every node of \mathcal{T}_G is a pair (H, S) , where H is a graph in \mathcal{C} , and S is a set of disjoint flat paths of H .
- (ii) The root of \mathcal{T}_G is (G, \emptyset) .
- (iii) Every non-leaf node of \mathcal{T}_G is a pair (G', S') , where G' has a 2-join (X_1, X_2) such that the edges between X_1 and X_2 do not belong to any flat path in S' . Let $S_1, S_2 \subseteq S'$ be the set of the flat paths of S' in $G'[X_1]$, $G'[X_2]$ respectively. The node (G', S') has two children, namely $(G_1, S_1 \cup \{P_2\})$ and $(G_2, S_2 \cup \{P_1\})$.
- (iv) Every leaf node of \mathcal{T}_G is (G', S') where $G' \in \mathcal{B}$.

Even-hole-free graphs with no star cutset have a simple structure in the following sense:

Theorem 2.2.4 ([Le18])

Every even-hole-free graph G with no star cutset has rank-width at most 3.

For a given graph G and a rank-decomposition (T, L) which corresponds to G , a subset $X \subseteq V(G)$ is said to be *separated* in (T, L) if there exists an edge e_X of T which corresponds to the partition $(X, V(G) \setminus X)$ of $V(G)$. Let d be an integer, we say that graph G has *property $\mathbb{P}(d)$* if for every set S of disjoint flat paths of length at least 3 in G , there is a rank-decomposition (T, L) of G such that the width of (T, L) is at most d and every flat path $P \in S$ is separated in (T, L) .

Lemma 2.2.5 ([Le18])

Let \mathcal{C} be a class of graphs and $\mathcal{B} \subseteq \mathcal{C}$ be the set of its basic graphs such that every graph $G \in \mathcal{C}$ has a 2-join decomposition tree. Furthermore, there exists an integer $d \geq 2$ such that every basic graph in \mathcal{B} has a property $\mathbb{P}(d)$. Then for

| every graph $G \in \mathcal{C}$, we have $\text{rw}(G) \leq d$.

The lemma is proved by showing that in a 2 join decomposition tree \mathcal{T}_G of G , every node (G', S') of \mathcal{T}_G admits a rank-decomposition (T, L) such that $\text{width}(T, L) \leq d$ and every flat path $P \in S'$ is separated in (T, L) . The lemma then follows because having the property for the root of \mathcal{T}_G implies $\text{rw}(G) \leq d$. Such a property is satisfied for every leaf of the tree decomposition which corresponds to the basic graphs. It then can be propagated along the tree in a bottom-up fashion, up to the root of the tree. In each step, a non-leaf node (G', S') having child nodes (G_1, S_1) and (G_2, S_2) (which actually correspond to the two blocks of decomposition of (G', S')) is considered. From the rank-decomposition of (G_1, S_1) and (G_2, S_2) (call them (T_1, L_1) and (T_2, L_2) respectively), it is possible to obtain a tree decomposition (T, L) of (G', S') by “merging” the two trees while maintaining the rank being at most d , and the flat paths are separated.

To do so, let P_2 and P_1 be the corresponding marker paths of the decomposition corresponding to the blocks G_1 and G_2 respectively. So, $P_2 \in S_1$ and $P_1 \in S_2$ because they are flat paths. By assumption, the path P_2 is separated in (T_1, L_1) by some edge $e_1 = u_1v_1$ of T_1 . So, $T_1 \setminus e_1$ is composed by two subtrees U_1 and V_1 rooted at u_1 and v_1 respectively, and without loss of generality the leaves of V_1 corresponds to the vertices of P_2 . The same holds for the tree decomposition (T_2, L_2) of G_2 (see Figure 2.8). The tree T is built from the two subtrees $T_1[V(U_1) \cup \{v_1\}]$ and $T_2[V(U_2) \cup \{v_2\}]$ at vertices by identifying the vertices u_1 with v_2 and u_2 with v_1 , and taking the mapping L as the union of two mappings restricted to the partition of the 2-join decomposition. Clearly T is a subcubic tree and the leaves of T correspond to $V(G')$. Every path of S' is separated in (T, L) because a flat path of G' is a flat path of G_1 or G_2 . Moreover, $\text{width}((T, L)) \leq d$ because the width of the identified edge of T is 2 (as it corresponds to the partition of the 2-join decomposition), and for the other edges, the rank is maintained, because it corresponds to a cut of G' separating a subset Z of X_i from $V(G') \setminus Z$, and we have $\text{cutrk}_{G'}(Z) = \text{cutrk}_{G_i}(Z)$.

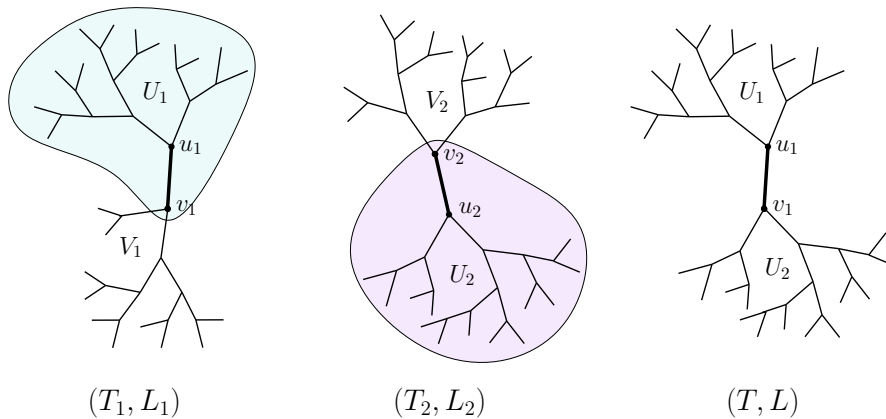


FIGURE 2.8: Rank-decomposition of the two blocks G_1 and G_2 and a rank-decomposition of G' obtained by identifying u_1v_1 and v_2u_2

Le pointed out that the Definition 2.2.3 and Lemma 2.2.5 are not restricted only to even-hole-free graphs with no star cutset but also applicable to a more general class of graphs. So, this technique might apply to some other classes of graphs with similar properties.

Because of Lemma 2.2.5, to prove Theorem 2.2.4, it is enough to prove that every basic graph of the class of even-hole-free graphs with no star cutset admits property $\mathbb{P}(d)$ for some integer d , which is indeed the case as shown by the following lemma.

Lemma 2.2.6 ([Le18])

Every basic even-hole-free graph with no star cutset satisfies the followings:

- (i) *For every set S of disjoint flat paths of length at least 3 in G , there is a rank-decomposition (T, L) of G such that the width of (T, L) is at most d ; and*
- (ii) *Every flat path $P \in S$ is separated in (T, L) .*

Note that a basic even-hole-free graph with no star cutset is either a clique, a hole, a long pyramid, or an extended nontrivial basic graph (see Figure 2.6 again). The lemma holds for cliques and holes because the rank-width of every clique is at most 1, and the rank-width of every hole is at most 2. The case for a long pyramid follows easily from the case where it is an extended nontrivial basic graph. However, the proof for an extended nontrivial basic graph is more involved. More details about computing the rank-width of this class of graphs can be found in [Le18]. Figure 2.9 shows an example of the tree decomposition of an extended nontrivial basic graph.

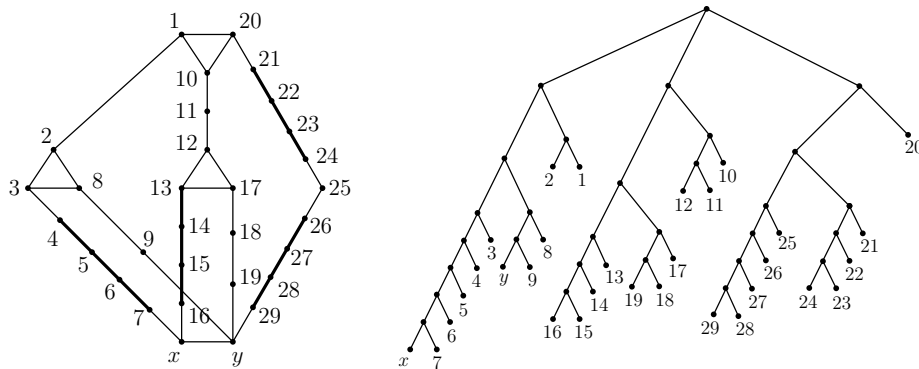


FIGURE 2.9: An extended nontrivial basic graph and its rank-decomposition of width 3 as explained in [Le18]

2.3 Widths of several subclasses of even-hole-free graphs

In general, even-hole-free graphs have unbounded tree-width (as well as clique-width or rank-width) because chordal graphs are even-hole-free, and those widths are unbounded for chordal graphs. Nonetheless, the tree-width and the rank-width become bounded when we restrict to some subclasses of even-hole-free graphs. This subject has been widely studied, and in this section, we will review some known results. For each subclass, we explain the known results on some width parameters. We also give a sketch of the idea of how such a result is obtained.

2.3.1 Planar case

The first known result regarding tree-width on subclasses of even-hole-free graphs was proved for the planar case. Silva, da Silva, and Sales prove that even-hole-free

planar graphs have tree-width at most 49 by showing that every graph in the class does not contain the so-called 9×9 -grid minor [SSS10]. Let us give an overview of this result.

Theorem 2.3.1 ([SSS10])

Every planar even-hole-free graph has tree-width at most 49.

The proof of Theorem 2.3.1 follows from the fact that when a planar graph contains a large grid minor, then it must contain an even hole.

Model of minor. Let G be a graph and $H \subseteq G$ be a minimal induced subgraph of G containing a graph M as a minor. Since H is minimal, M can be obtained from H only by edge contractions or deletions (so, no vertex deletions). Every vertex v_i of M was derived from a subset of vertices of H , denoted by V_i , which induces a connected subgraph of H . The pair (H, \mathcal{V}) where \mathcal{V} is the partition of $V(H)$ formed by the sets V_i is called a *model* of M in G , and V_i is called a *node* of the model. Note that vertices v_i and v_j of M are adjacent if there exists at least one edge of H whose one end in V_i and the other in V_j . Note that, however, the presence of such an edge between two nodes in H does not mean that the corresponding vertices are adjacent in M .

In the case of even-hole-free graphs, we are particularly interested in the existence of a *grid-minor* in the graphs. The $k \times l$ grid is the graph $G_{k \times l} = (V, E)$ where $V = \{v_{i,j} : 1 \leq i \leq k, 1 \leq j \leq l, i, j \in \mathbb{N}\}$ and $E = \{(v_{i,j}, v_{i',j'}) : |i - i'| + |j - j'| = 1\}$. The following theorem guarantees that planar even-hole-free graphs do not contain a $G_{k \times l}$ minor model for large k, l . The non-existence of a grid minor of big size is enough to guarantee that a graph has small tree-width.

Theorem 2.3.2 ([RST94])

If G is planar and does not contain a $(k \times k)$ -grid as a minor, then $tw(G) \leq 6k - 5$.

It turns out that when a planar even-hole-free graph contains a grid minor of some specific size, then it must contain a forbidden structure. So, graphs in the class cannot contain a big grid-minor. Hence the result follows from Theorem 2.3.2 below.

Theorem 2.3.3 ([SSS10])

Let (H, \mathcal{V}) be a $G_{9 \times 9}$ model, then H contains a theta or a prism. In particular, if G is a planar even-hole-free graph, then G has no (9×9) -grid minor.

2.3.2 Triangle-free case

The study of the structure of triangle-free even-hole-free graphs was initiated by Conforti, Cornuéjols, Kapoor, and Vušković [Con+00]. In their paper, they gave a decomposition theorem for triangle-free even-hole-free graphs. They proved the theorem for a more general class, namely the class of (triangle, theta, even wheel)-free graphs¹⁰. This decomposition theorem was then applied to construct a polynomial-time algorithm to recognize whether a given input triangle-free graph is odd-signable, in particular, to detect whether a triangle-free graph contains an even

¹⁰In the paper, the proof is indeed for odd-signable graphs, because being triangle-free implies prism-free, see Theorem 2.3.5.

hole. Triangle-free odd-signable graphs are decomposed using *clique-cutset* (as defined in Chapter 1) and a typical *star cutset* that arises from a wheel in the graph (this is what they call as *wheel decomposition*). We now discuss the wheel decomposition in more detail.

Definition 2.3.4. Let G be a connected triangle-free graph that contains a wheel (H, v) and let v_1, \dots, v_n be the neighbors of v in H , appearing in this order when traversing H . Then G can be decomposed with wheel (H, v) if the following holds:

- $G \setminus \{v, v_1, \dots, v_n\}$ contains exactly n connected components Q_1, \dots, Q_n .
- The intermediate nodes of the sector with endnodes v_i and v_{i+1} belong to Q_i and no node of Q_i is adjacent to v_j , $j \neq i, i+1$.

Informally speaking, every connected component obtained from G by removing $\{v, v_1, \dots, v_n\}$ attaches to a sector. Moreover, such a component is the only connected component attaching to that sector, and every sector occupies a connected component. The blocks of such a wheel decomposition is a set of graphs that are induced by $V(Q_i) \cup \{x, x_i, x_{i+1}\}$ for every $1 \leq i \leq n$. Furthermore, note that it follows from the definition, that given a wheel (H, x) in a triangle-free graph G , one can check in polynomial time whether G can be decomposed with (H, x) . The key property of the wheel decomposition for graphs in this class is that, any wheel in the graph can be used to decompose the graph (which is not the case in general for the class of even-hole-free graphs).

Cube is the graph formed from a hole of length 6, say $h_1 h_2 \dots h_6 h_1$ together with a vertex u adjacent to h_1, h_3, h_5 and a vertex v non-adjacent to u and adjacent to h_2, h_4, h_6 . A (triangle, theta, even wheel)-free graph has a substantial property related to containment of the cube as shown in [Con+00]: if it contains the cube, then the entire graph itself is the cube. The following is a decomposition theorem of graphs in this class.

Theorem 2.3.5 (Decomposition of triangle-free ehf graphs [Con+00])

For any (triangle, theta, even wheel)-free graph G , one of the following holds.

- G is either a K_1 , K_2 , a hole, or the cube.
- G has a clique cutset.
- G contains a wheel, and it can be decomposed with any arbitrarily chosen wheel.

It turns out that under certain circumstances, one can construct (triangle, theta, even wheel)-free graphs by “gluing” smaller (triangle, theta, even wheel)-free graphs along some particular 3-vertex path (see Theorem 3.1 of [Con+00] for more details). Based on this fruitful composition theorem, Conforti et al. give a procedure to construct all (triangle, theta, even wheel)-free graphs. The structure theorem says that, every (triangle, theta, even wheel)-free graph that is not isomorphic to the cube and has no clique separators can be built starting from a hole by iteratively gluing a path to the currently constructed graph along some particular 3-vertex path of the graph (under some particular conditions). Let us define them formally.

A chordless (x, z) -path P is an *ear* of the hole H if the internal vertices of P belong to $V(G) \setminus V(H)$, vertices x, z have a common neighbor y in H , and $(H \setminus y) \cup P$ induces

a hole H' . A graph G is said to be obtained from a graph G' by an *ear addition* if the vertices of $G \setminus G'$ are the internal vertices of an ear of some hole H in G' . Let G be obtained from G' by adding an (x, z) -ear P with $x, z \in V(G')$. Such ear addition is *good* if y has an odd number of neighbors in P , and the following holds.

- (i) G' contains no wheel (H, v) s.t. $x, y, z \in V(H)$ and $vy \in E(G)$;
- (ii) G' contains no wheel (H, y) s.t. $x, z \in V(H)$.

Theorem 2.3.6 (Structure theorem of (triangle, theta, even wheel)-free graphs, [Con+00])

Let G be a connected triangle-free graph that contains at least three vertices, such that G is not the cube, and it has no K_1 or K_2 separator. Then, G is (theta, even wheel)-free if and only if G can be obtained, starting from a hole, by a sequence of good ear additions.

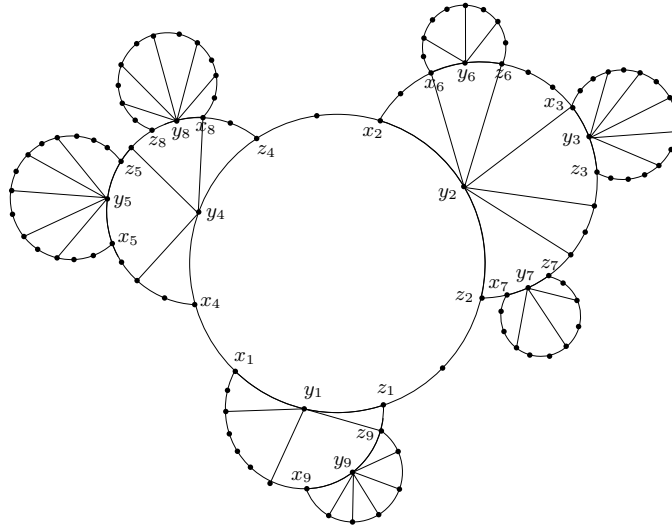


FIGURE 2.10: An illustration of ear addition

A bound on the tree-width

Applying the aforementioned structure theorem, Cameron et al. [Cam+18] proved that the tree-width of triangle-free even-hole-free graphs is bounded by some constant (it is in fact proved for a superclass, namely (triangle, theta, even wheel)-free graphs). The result of Corneil and Rotics [CR05] which says that the clique-width of a graph G is at most $3 \times 2^{\text{tw}(G)-1}$, gives an immediate implication of Theorem 2.3.7 to the clique-width of graphs in the class (cf. Corollary 2.3.8).

Theorem 2.3.7 (Bounded tree-width, [Cam+18])

Every (triangle, theta, even wheel)-free graph G satisfies $\text{tw}(G) \leq 5$.

Corollary 2.3.8 (Bounded clique-width, [Cam+18])

Every (triangle, theta, even wheel)-free graph G satisfies $\text{cw}(G) \leq 48$.

Proof. [Sketch of proof of Theorem 2.3.7]

We now sketch the proof of Theorem 2.3.7 based on the proof explained in [Cam+18]. Note that it is enough to bound the tree-width for graphs that contains no clique cutset because gluing along the clique cutsets that are used to decompose G preserves the tree-width (see Remark 1). One then can show that every (triangle, theta, even wheel)-free graph is contained in a chordal graph that has a clique number at most 6. This holds when the graph is the cube or it contains at most two vertices. So, we may assume that the graph is cube-free and contains at least three vertices. Hence, Theorem 2.3.6 holds, i.e. it admits the "ear-addition construction" as described in the previous paragraph. In particular, let P_1, \dots, P_q be the sequence of ears in the construction, with P_q being the last ear added. For each i , let H_i be the hole P_i is attached to, let x_i and z_i be the attachments of P_i in H_i , and let y_i be the common neighbor of x_i and z_i in H_i .

From G , it is possible to obtain a triangulation whose clique number is at most 6. Such a triangulated graph T of G can be constructed as follows. For each ear P_i , make x_i, y_i and z_i complete to $P_i \setminus \{x_i, z_i\}$, and add the edge $x_i z_i$. Let $H = v_1 v_2 \dots v_k v_1$, then choose any edge, say $v_1 v_2$ of H , and join $\{v_1, v_2\}$ to all vertices of $H \setminus \{v_1, v_2\}$. It can be shown that for $1 \leq i \leq q$, $S_i = \{x_i, y_i, z_i\}$ is a clique cutset in T that separates $H_i \setminus S_i$ from $P_i \setminus S_i$. From this observation, it suffices to show that $T[H]$, and for $1 \leq i \leq q$, $T[V(P_i) \cup \{y_i\}]$ are all chordal and have clique number at most 6.

To show that $T[H]$ is chordal, note that the edges that are present in H are either the edges of H itself or edges of type $x_i z_i$ (i.e. it is a short chord of H). If there were a hole C in $T[H]$, then because the two vertices $v_1, v_2 \in V(H)$ cannot be in any hole (they are indeed adjacent to all other vertices of H), the vertex v_i of C that are the closest to $\{v_1, v_2\}$ must be adjacent to v_{i+1} and v_{i+2} , hence creating a chord in the hole C , a contradiction. So, $T[H]$ is chordal, and clearly, $\omega(T[H]) \leq 5$ (since $\omega(T[H] \setminus \{v_1, v_2\}) = 3$ and $\{v_1, v_2\}$ is complete to $V(H) \setminus \{v_1, v_2\}$). We now consider an ear P_i . Define $G_i = G[V(H) \cup V(P_1) \cup \dots \cup V(P_i)]$, and $G_0 = G[H]$. If $T[V(P_i) \cup \{y_i\}]$ contains a hole C , then since $\{x_i, y_i, z_i\}$ forms a clique, $V(C) \subseteq V(P_i) \setminus \{x_i, z_i\}$. Since P_i is a chordless path, then by the rule of the triangulation, $T[P_i \setminus \{x_i, z_i\}]$ contains an edge $x_j z_j$. If $y_j \in P_i$, then $\{x_j, y_j, z_j\}$ forms a triangle in C , a contradiction. Moreover, by the ear-addition construction, $y_j \notin P_k$ for every $k > i$. So, y_j is a vertex of G_{i-1} , and it follows that $y_j = y_i$ (because y_i is the only vertex of G_{i-1} which can have a neighbor in the interior of P_i). Let H' be the hole obtained by augmenting H_i with P_i . Then the wheel (H', y_j) is contained in G_{j-1} and contradicts P_j being a good ear. \square

2.3.3 Cap-free case

A *cap* is a graph made of a hole together with a vertex, which is adjacent to exactly two adjacent vertices of the hole. (cf. Figure 2.11). In the same paper where a constant upper bound on tree-width of the triangle-free even-hole-free graphs is given, Cameron et al. [Cam+18] also come up with an upper bound on the clique-width and the tree-width for a more general subclass, where triangles are allowed, but caps are excluded. They give a structure theorem for the class of (even hole, cap)-free graphs, which yields that graphs in the class have clique-width bounded by a constant (and hence bounded rank-width), and tree-width bounded by a function of clique number of the graphs. The proof of the boundedness on the clique-width and tree-width relies heavily on the decomposition theorem of (cap, 4-hole)-free odd-signable graphs. Let us now explain this in more detail.

We say that the graph G' is obtained from a graph G by *blowing up* vertices of G into cliques if G' consists of the disjoint union of cliques K_u , for every $u \in V(G)$, and

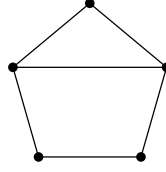


FIGURE 2.11: A cap

all edges between cliques K_u and K_v if and only if $uv \in E(G)$. This is also referred to as *substituting* clique K_u for vertex u (for all u). The graph G' is obtained from a graph G by adding a universal clique if G' consists of G together with a (possibly empty) clique K , and all edges between vertices of K and vertices of G . Note that both of these operations preserve being $(\text{cap}, 4\text{-hole})$ -free, i.e., G is $(\text{cap}, 4\text{-hole})$ -free if and only if G' is $(\text{cap}, 4\text{-hole})$ -free. The following is a decomposition theorem of $(\text{even hole}, \text{cap})$ -free graphs in [Cam+18].

Theorem 2.3.9 (A structure theorem, Theorem 3.6 of [Cam+18])

Let G be a $(\text{cap}, 4\text{-hole})$ -free graph that contains a hole and has no clique cutset. Let F be any maximal induced subgraph of G with at least 3 vertices that is triangle-free and has no clique cutset. Then G is obtained from F by first blowing up vertices of F into cliques, and then adding a universal clique. Furthermore, any graph obtained by this sequence of operations starting from a $(\text{triangle}, 4\text{-hole})$ -free graph with at least 3 vertices and no clique cutset is $(\text{cap}, 4\text{-hole})$ -free and has no clique cutset.

Theorem 2.3.9 tells us that every $(\text{cap}, 4\text{-hole})$ -free odd-signable graph with no clique cutset can be obtained starting from a $(\text{triangle}, 4\text{-hole})$ -free odd-signable graph F by first blowing up the vertices into cliques, then sequentially adding universal vertices. From Corollary 2.3.8, we know that the clique-width of F is at most 48. We remark that substituting a clique K for a vertex of a graph F gives a graph with clique-width at most the maximum of the clique-widths of K and F (we simply blow each pair of adjacent vertices into cliques, and make them complete). Since the clique-width of a clique on at least two vertices is 2, the blow-up of F has clique-width at most 48. Adding a universal vertex to a graph with at least one edge does not change the clique-width, so the theorem below follows.

Theorem 2.3.10 (Theorem 5.3 of [Cam+18])

If G is $(\text{cap}, 4\text{-hole})$ -free odd-signable graph with no clique cutset, then G has clique-width at most 48.

Applying the structure theorem and following a similar reasoning for the clique-width, we can prove that the tree-width of $(\text{cap}, 4\text{-hole})$ -free odd-signable graphs is also bounded by some function of the clique number of the graph. We consider the graph G constructed as stated in Theorem 2.3.9. We note that Theorem 2.3.7 tells us that the graph F of the structure theorem above has tree-width at most 5. In particular, there is a triangulation T of F with maximum clique size at most 6. We can obtain a triangulation T' of the blow-up F' of F by substituting the cliques K_v for the vertices v of T , which yields that the size of a largest clique in T' is at most $6 \max_v |K_v|$. Adding a set U of universal vertices, we obtain a triangulation T'' of G by adding to T' the clique induced by U and joining every vertex of U to every vertex of T' . Since the size of every clique K_v that was substituted cannot be larger

than $\omega(G) - |U|$, the largest clique in T'' has size at most $6(\omega(G) - |U|) + |U| = 6\omega(G) - 5|U| \leq 6\omega(G)$. Thus G has tree-width at most $6\omega(G) - 1$.

Theorem 2.3.11 (Theorem 5.4 of [Cam+18])

If G is a (cap, 4-hole)-free odd-signable graph with no clique cutset, then G has tree-width at most $6\omega(G) - 1$.

2.3.4 Diamond-free case

Diamond is the graph made of a clique on four vertices by removing one edge from the clique. In contrast to the other structures that we exclude above, excluding diamond does not have much effect on the structure of even-hole-free graphs, since [Adl+17], Adler et al. show that there exists a family of (even hole, diamond)-free graphs whose rank-width can be arbitrarily large. We remark that this class clearly has unbounded tree-width because complete graphs are (even hole, diamond)-free.

We now describe a construction of a family of (even hole, diamond)-free graphs $(G_d)_{d \geq 1}$ for which the rank-width is unbounded, as explained in the paper of Adler et al. [Adl+17]. For some integers $d \geq 1$ and $1 \leq k \leq d$, let $S^d = \bigcup_{k=1}^d S_k$, where:

$$S_k = \{(a_1, a_2, \dots, a_{k-1}, a_k) : a_1, a_2, \dots, a_{k-1} \in \{1, 3\}, a_k \in \{1, 2, 3, 4\}\}.$$

If $u \in S_k$, then we denote $l(u) = k$, and say that the *length* of u is k .

In S^d , let \preceq denote the lexicographical order defined as follows. For $a = (a_1, a_2, \dots, a_k) \in S^d$ and $b = (b_1, b_2, \dots, b_l) \in S^d$, $a \preceq b$ if and only if $k \leq l$ and $a_i = b_i$ for $1 \leq i \leq k$, or $t = \min\{i : a_i \neq b_i\}$ is well-defined and $a_t < b_t$.

Let P'_d denote the path on vertex set S^d connecting the vertices according to the lexicographic order, and let P_d be the path obtained from P'_d by subdividing every edge $uv \in E(P'_d)$ twice if $l(u) = l(v)$ and once, otherwise. Finally, let $W_d = \{v_1, v_2, \dots, v_d\}$ be a set of (new) vertices, such that v_k , for $1 \leq k \leq d$, is adjacent to all vertices of S_k and all other vertices of W_d . Then, G_d is the graph induced by the set $W_d \cup V(P_d)$. Figure 2.12 shows G_4 .

As proved in the paper [Adl+17], the graph G_d is (even hole, diamond)-free for all $d \geq 1$ and G_d has no clique cutset for all $d \geq 2$. Moreover, G_d contains a d -vertex clique, that is formed by the set W_d . It is also important to remark that the set S^d contains at least 2^{d+1} vertices, in particular, the size of the graph is exponential in d . The rank-width of G_d is bounded below by $\frac{d}{3}$ for every $d \geq 22$. Hence, the rank-width of G_d grows at least logarithmically with $|V(G_d)|$, since if $d \geq 22$ then $rw(G_d) > \frac{d}{3} \geq (\log 2 |V(G_d)| - 4)/3$.

Theorem 2.3.12 ([Adl+17])

The family of (even hole, diamond)-free graphs G_d , $d \geq 2$, without clique cutsets has unbounded rank-width.

Theorem 2.3.12 shows that excluding clique cutsets does not guarantee boundedness of the rank-width. Note that clique cutset is a particular case of star cutset, and this result is in contrast to the case when star cutsets are excluded because even-hole-free graphs with no star cutset have bounded rank-width [Le18] (see Subsection 2.2.2). However, $rw(G_d) \leq d + 1$ for all $d \in \mathbb{N}$, because there exists a rank-decomposition of G_d with width at most $d + 1$ (see Figure 2 of [Adl+17] for further explanation). We furthermore remark that the construction can be easily extended to (even hole, pyramid, diamond)-free graphs as explained in [Chu+19].

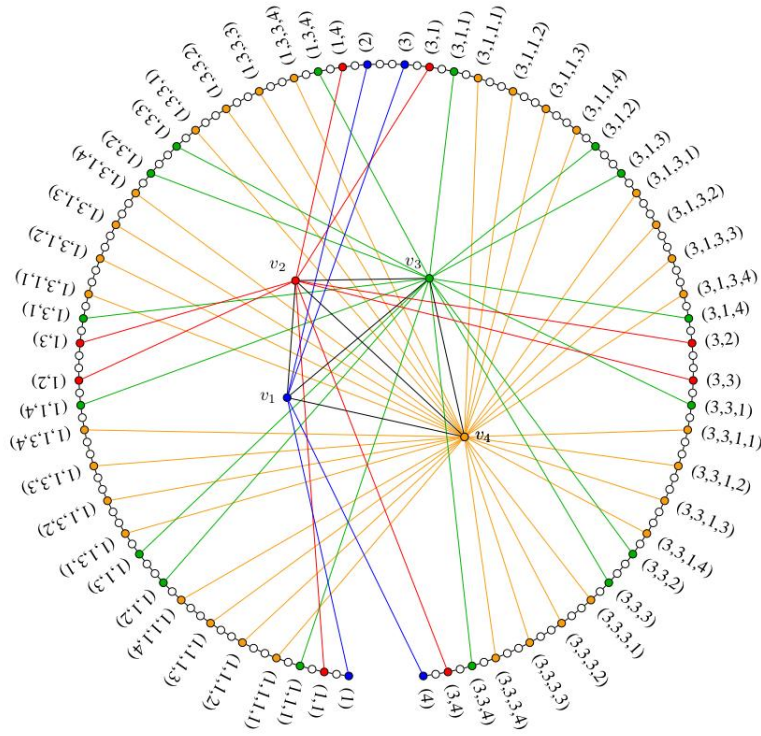


FIGURE 2.12: A diamond-free ehf graph that has arbitrarily large rank-width [Adl+17]

2.3.5 Pan-free case

A *pan* is a graph that consists of a hole and a single vertex with precisely one neighbor on the hole (cf. Figure 2.13). Cameron, Chaplick, and Hoáng [CCH18] showed that excluding pans yields boundedness on the tree-width of even-hole-free graphs. They proved that an (even hole, pan)-free graph can be decomposed by clique cutsets into essentially *unit circular-arc graphs*. Using this structure theorem, they showed that the tree-width of an (even hole, pan)-free graph is bounded by a linear function of its clique number. Let us now explain the structure theorem in more detail.

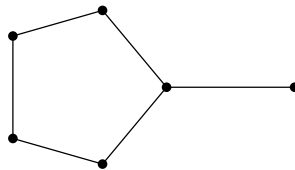


FIGURE 2.13: A pan

In order to analyze the width of graphs in this class, let us first present some useful notions. A graph G is a *circular-arc* graph if there exists a bijection between its vertices and a set A of *arcs* on a circle such that two vertices of G are adjacent if and only if the two corresponding arcs of A intersect (see Figure 2.14 for an example). A circular-arc graph is *proper* if no arc contains another. Additionally, G is a *unit* circular-arc graph if every arc of A has the same length. It is easy to see that unit circular-arc graphs are proper and that proper circular-arc graphs are claw-free and hence pan-free. Let G_1 and G_2 be two vertex-disjoint graphs. The *join* of G_1 and G_2

is the graph G obtained from G_1 and G_2 by adding every edge between the vertices of G_1 and those of G_2 .

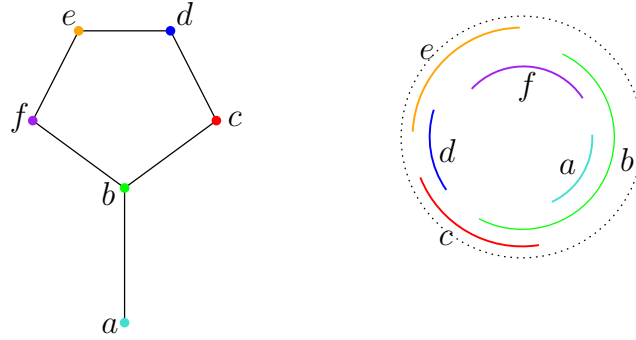


FIGURE 2.14: A (non-proper) circular-arc graph and a corresponding arc model

Definition 2.3.13. For $l \geq 5$, an l -*buoy* B is a collection of sets B_0, B_1, \dots, B_{l-1} of vertices of G such that each B_i induces a clique, each vertex in B_i has a neighbor in B_{i+1} and one in B_{i-1} , and there are no edges between B_i and $B \setminus (B_{i-1} \cup B_i \cup B_{i+1})$, with subscripts taken modulo l (cf. Figure 2.15 for an example); the sets B_i are called the *bags* of the buoy; a buoy is odd or even depending on whether the number of bags (l) is odd or even. We also refer to $G[B]$ as a *buoy*. A buoy B in a graph G is said to be *full* when it includes every vertex of G . Due to the cyclic structure of l -buoys, bag B_i of an l -buoy is referred to as the bag $B_{i \pmod l}$.

As explained in the paper, in an (even hole, pan)-free graph, the structure of l -buoys are quite restricted. For every bag B_i of a buoy B , the vertices of B_i admit domination ordering both on B_{i-1} and B_{i+1} , i.e. for every vertex $u, v \in B_i$, one of the following holds.

- $N_{B_{i-1}}(u) \subseteq N_{B_{i-1}}(v)$ and $N_{B_{i+1}}(u) \subseteq N_{B_{i+1}}(v)$; or
- $N_{B_{i-1}}(v) \subseteq N_{B_{i-1}}(u)$ and $N_{B_{i+1}}(v) \subseteq N_{B_{i+1}}(u)$.

On B_{i-1} (and similarly on B_{i+1}), if neither u dominates v nor v dominates u hold, then there would exist two vertices $u', v' \in B_{i-1}$ such that $uu', vv' \in E(B)$ and $uv', u'v \notin E(B)$, creating a C_4 because $uv, u'v' \in E(B)$. Hence, the first item listed above holds, and in particular, B is a so-called ring (see Subsection 2.3.6 for the definition). Moreover, if for some vertices $u, v \in B_i$, u dominates v in B_{i-1} but not in B_{i+1} , then there would exist vertices $u' \in B_{i-1}$, and $v'' \in B_{i+1}$ such that $uu', vv'' \in E(B)$ and $vu', uv'' \notin E(B)$. This yields a chordless path $u'uv''$, and together with a path from u' to v'' in $B \setminus B_i$, it creates an even hole.

Furthermore, for every buoy B with bags B_0, B_1, \dots, B_{l-1} , either $B_{i-1} \cup B_i$ or $B_i \cup B_{i+1}$ is a clique. For otherwise, if both $B_{i-1} \cup B_i$ and $B_i \cup B_{i+1}$ are not cliques, then there would exist a vertex $v \in B_i$ such that v has non-neighbors v' in B_{i-1} and v'' in B_{i+1} . Such a vertex v exists by the domination ordering of B_i in $B_{i-1} \cup B_{i+1}$. Now consider a hole in B that goes through v', v'' , and some vertex $u \in B_i$ (such a hole exists by the definition of buoy). Note that the hole uses exactly one vertex of each bag of the buoy, and vertex v has exactly one neighbor in the hole (namely vertex u), hence creating a pan.

Theorem 2.3.14 ([CCH18])

An l -buoy in an (even hole, pan)-free graph is a unit circular-arc graph.

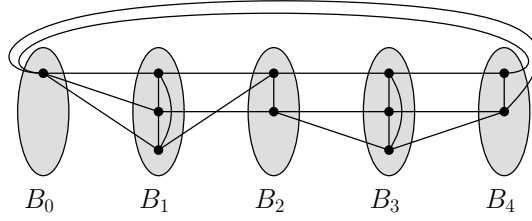


FIGURE 2.15: An example of 5-buoy

We give a sketch of the unit circular-arc representation of (even hole, pan)-free graphs as explained in the proof of Theorem 2.3.14 in [CCH18] (see Theorem 5 and Theorem 6 of [CCH18] for more details).

To represent an l -buoy B as a circular arc graph, a circle can be partitioned into l arcs where each of them corresponds to a bag of B . The vertices of every buoy B_i admit a domination ordering, so we can write $B_i = \bigcup_{j=1}^{t_i} B_{i,j}$, that is B_i can be partitioned into t_i subsets $B_{i,1}, \dots, B_{i,t_i}$, such that for every $v, v' \in B_{i,j}$, $N_{B_{i+1}}(v) = N_{B_{i+1}}(v')$, and for $u \in B_{i,j}$ and $v \in B_{i,j+1}$, $N_{B_{i+1}}(u) \subsetneq N_{B_{i+1}}(v)$. For every $i \in \{0, 1, \dots, l-1\}$, the arc from i to $i+1$ is equally partitioned into t_i arcs by putting t_i equally spaced points $\{(i, 1), \dots, (i, t_i)\}$, where:

- for each vertex $b_j \in B_{i,j}$, we use the arc from (i) to (i, j) ; and
- for each vertex $b'_j \in B_{i+1,j}$, we use the arc from (i, j) to $(i+1)$

In order to obtain a *unit* circular-arc representation, the following additional condition is needed:

- for each bag B_i , an arc A_i of length $\epsilon > 0$ is assigned;
- for each $i \in \{0, 1, \dots, l-1\}$, an arc A_{i+} is assigned in the following way: when $B_i \cup B_{i+1}$ is a clique, the length of A_{i+} is two units, and otherwise, it has length one unit;
- when the arc between (i) and $(i+1)$ has length 2, only a half of the arc is equally partitioned into t_i partitions;
- these arcs are arranged as $A_0, A_0+, A_1, A_1+, \dots, A_{l-1}, A_{l-1}+$ around the circle so that the circle is covered and consecutive arcs intersect in precisely one point.

An illustration corresponds to this setting is depicted in Figure 2.16. The top figure illustrates the case when $B_i \cup B_{i+1}$ does not induce a clique, which means that each of $B_{i-1} \cup B_i$ and $B_{i+1} \cup B_{i+2}$ induces a clique. The bottom figure shows the case when $B_i \cup B_{i+1}$ induces a clique. Every arc in the circular-arc graph has length $2 + \epsilon$.

Theorem 2.3.15 ([CCH18])

If G is a connected (even hole, pan)-free graph with. Then either:

- (i) G is a clique;
- (ii) G contains a clique cutset; or
- (iii) for every maximal buoy B of G , either B is a full buoy of G , or G is the join of B and a clique.

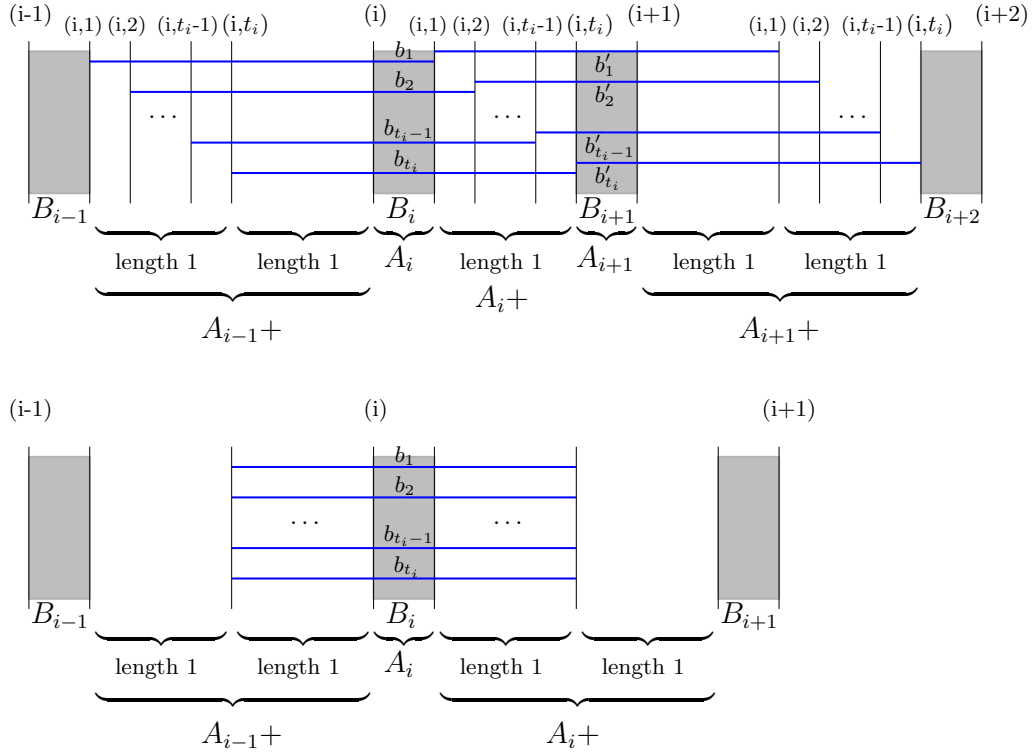


FIGURE 2.16: Unit circular-arc representation of buoy

From the structure theorem, the following theorem of the tree-width of (even hole, pan)-free graphs is derived.

Theorem 2.3.16 ([CCH18])

Every (even hole, pan)-free graph G has $tw(G) \leq 1.5\omega(G) - 1$.

Proof. First of all, note that for any graph G that is the join of a graph B and a clique K , $tw(G) = tw(B) + |V(K)|$. Moreover, recall that the tree-width of G is the maximum of the tree-width of every block of the clique decomposition of G . Now consider an (even hole, pan)-free graph G . By our previous observation, we may assume that G has no clique cutset. By Theorem 2.3.15, it is then enough to show that $tw(B) \leq 1.5\omega(B)$ for any buoy in G .

Recall that for every bag B_i of a buoy B , either $B_i \cup B_{i+1}$ is a clique or $B_{i-1} \cup B_i$ is a clique. Consider the unit circular arc representation U of B (as shown in Figure 2.16). Construct a path representation P of B based on its unit circular-arc representation U as follows: each node $v \in \bigcup_{i \in [0, l-1]} \{(i, 1), \dots, (i, t_i), (i)\}$ of P is a bag containing vertices which intersect at point v in U ; two nodes of P are adjacent if the points corresponding to those nodes are consecutive in U ; the extremities of T both correspond to point (i) such that B_i is the bag of the buoy B which is of the smallest size (so the two extremities of P are bags containing the same set of vertices). Now by adding all vertices of B_i to the internal nodes of P , we obtain a tree decomposition of B . Note that every bag node of P contains at most $\omega(B) \leq \omega(G)$ vertices. Moreover $|B_i| \leq \frac{1}{2}\omega(G)$ because B_i is the smallest bag in B . Hence the largest bag in this tree decomposition has size $\omega(G) + |B_i| \leq 1.5\omega(G)$. \square

2.3.6 Rings

For an integer $n \geq 3$, a *ring on n sets* is a graph G whose vertex set can be partitioned into n cliques X_1, \dots, X_n , with the following additional properties:

- for all $i \in \{1, \dots, n\}$ and all $x, x' \in X_i$, either $N(x) \subseteq N(x')$ or $N(x') \subseteq N(x)$;
- for all $i \in \{1, \dots, n\}$ and all $x \in X_i$, $N(x) \subseteq X_{i-1} \cup X_i \cup X_{i+1}$ (where the addition of subscripts is modulo n); and
- for all $i \in \{1, \dots, n\}$, there exists a vertex $x \in X_i$ that is adjacent to all vertices of $X_{i-1} \cup X_{i+1}$.

We remark that in a ring, every hole is of same length. In particular, for n odd, rings are even-hole-free.

It is shown by Hoàng and Trotignon in [HT20] that for some fixed integer $n \geq 3$, there exist rings on n sets with arbitrarily large rank-width (or equivalently, clique-width). In fact, their result is not restricted only to rings. For the proof, they construct the so-called *carousels*, which are more general than rings, but in some specific cases are rings. We do not include the details of the construction here, and interested readers may refer to Section 4 of [HT20] for further explanation. Furthermore, as noted in [HT20], when $n \geq 5$ is an odd integer, rings provides a new construction of even-hole-free graphs with arbitrarily large rank-width. The following theorem is a more restricted formulation of Theorem 4.1 of [HT20].

Theorem 2.3.17 ([HT20])

For all integers $n \geq 3$ and $r \geq 1$, there exists rings on n sets with rank-width at least r .

We furthermore remark that rings on n sets can be colored in polynomial time [MPV20].

Chapter 3

Layered wheels

The main motivation of this chapter is a better understanding of even-hole-free graphs. We note that there are some classical perfect graphs which have arbitrarily large tree-width (or even rank-width), such as bipartite graphs, or their line graphs. On the other hand, for even-hole-free graphs, apart from complete graphs, it is not trivial to find classical graphs of large tree-width (or other widths). We have seen in Chapter 2, Section 2.3, that there exists a family of (even hole, diamond)-free graphs with unbounded rank-width [Adl+17]. However, every existing construction of even-hole-free graphs of arbitrarily large tree-width (or rank-width) contains large cliques. As explained in Chapter 2, excluding the triangle (i.e. the clique on 3 vertices, which means that we exclude all the large cliques) yields bounded tree-width on even-hole-free graphs [Cam+18]. Recall also that when pans and the clique on t vertices are excluded, even-hole-free graphs have bounded tree-width [CCH18].

As mentioned in the end of Chapter 2, the work presented in this chapter was originally motivated by a question asked by Cameron, Chaplick, and Hoàng in [CCH18]: *is the tree-width (or clique-width) of an even-hole-free graph bounded by a function of its clique number?*. In this chapter, we give a negative answer to the aforementioned question. In particular, we describe a construction called *layered wheel* which is a family of even-hole-free graphs with no clique of size four which may have arbitrarily large tree-width. Later, we prove a stronger result that under certain conditions, one might obtain layered wheels with arbitrarily large rank-width (or equivalently, clique-width).

We note that (even hole, pyramid)-free graphs received some attention lately (see [Chu+19]). It is therefore worth noting that even-hole-free layered wheels are pyramid-free (see Theorems 3.2.10 and 3.2.11). We note that it is also possible to obtain a variant of even-hole-free layered wheel that does contain pyramids. We omit giving the details of this construction, it is of interest because it might give some ideas of how an even-hole-free graph can be decomposed (or not) around a pyramid.

We note that for the classes where we prove unbounded tree-width, the clique-width (and therefore the rank-width) is also large (see Theorems 3.2.15 and 3.3.16).

We postpone the formal definition of layered wheels to Section 3.2 although we use the term several times before then. Other sources of motivation for this chapter are the following:

- When considering the induced subgraph relation (instead of the minor relation), is there a theorem similar to the celebrated grid-minor theorem of Robertson and Seymour?
- A better understanding of the classes defined by excluding Truemper configurations, particularly the class of theta-free graphs.

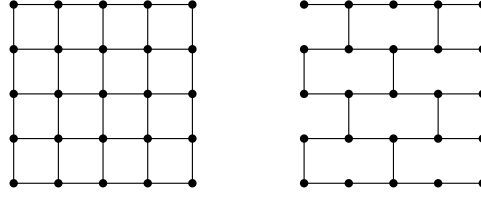


FIGURE 3.1: A grid and a wall

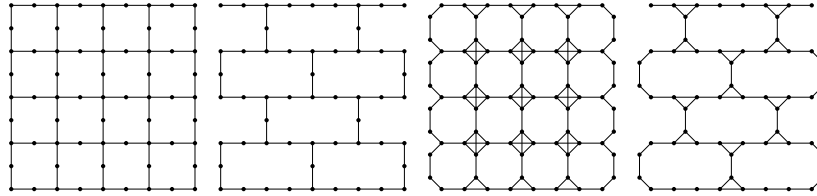


FIGURE 3.2: A subdivision of a grid, of a wall, and the line graphs of the former

In the following, we give more details on each of the two items listed above.

The grid-minor theorem

The $(k \times k)$ -grid is the graph on the set of vertices $\{(i, j) : 1 \leq i, j \leq k\}$ where two distinct ordered pairs (i, j) and (i', j') are adjacent whenever exactly one of the following holds: $|i - i'| = 1$ and $j = j'$, or $i = i'$ and $|j - j'| = 1$ (see Figure 3.1). Robertson and Seymour [RS86] proved that there exists a function f such that every graph with tree-width at least $f(k)$ contains a $(k \times k)$ -grid as a minor (see Theorem 4.2.2 given by Chuzhoy [Chu16] for the best function known so far). This is called the *grid-minor theorem*. The $(k \times k)$ -wall is the graph obtained from the $(k \times k)$ -grid by deleting all edges with form $(2i + 1, 2j) - (2i + 1, 2j + 1)$ and $(2i, 2j + 1) - (2i, 2j + 2)$.

Let G be a graph. *Subdividing an edge $e = uv$ of G k times*, where $k \geq 1$, means deleting e from G and adding a path $uw_1 \dots w_kv$. The *k -subdivision* of a graph G is the graph obtained from G by subdividing all its edges k -times (simultaneously). Note that replacing “grid” by a more specific graph in the grid-minor theorem, such as k -subdivision of a $(k \times k)$ -grid, $(k \times k)$ -wall, or k -subdivision of a $(k \times k)$ -wall provides statements that are formally weaker (at the expense of a larger function), because a large grid contains a large subdivision of a grid, a large wall, and a large subdivision of a wall. However, these trivial corollaries are in some sense stronger, because walls, subdivisions of walls, and subdivision of grids are graphs of large tree-width that are more sparse than grids. So they somehow certify a large tree-width with less information. Since one can always subdivide more, there is no “ultimate” theorem in this direction.

One big open question is whether a theorem similar to the celebrated grid-minor theorem exists in terms of “induced subgraph” instead of “minor”. Simply replacing “minor” with “induced subgraph” in the statement is trivially false, and here is a list of known counter-examples: the complete graph K_k , the complete bipartite graph $K_{k,k}$, subdivisions of walls, line graphs of subdivisions of walls (see Figure 3.2). Each of those graphs has a large tree-width, yet does not contain a grid of big size. One of our results implicitly show that the previously-given list is not complete. Layered wheels have large tree-width and do not contain long holes, which implies that they

contain no K_4 , no $K_{2,2}$, and no line graphs of subdivisions of walls. Moreover, layered wheels contain no subdivisions of $(3,5)$ -grids (this is explained after Lemma 3.2.3). In this direction, our objective is to find a complete list to be in the “grid-minor-like theorem”.

Theta-free graphs

We have pointed out that among the classes forbidding Truemper configuration, the one which has drawn our attention the most is the class of theta-free graphs, because it generalizes the class of even-hole-free graphs. In the attempt of finding a structural description of theta-free graphs, a seemingly easy case is when triangles are also excluded, because then, every vertex of degree at least 3 is the center of a claw (therefore a possible start for a theta), so excluding thetas and triangles should force some structure. Supporting this idea, Radovanović and Vušković [RV13] proved that every (theta, triangle)-free is 3-colorable. They gave a structural characterization of this class of graphs, from which a coloring algorithm with time complexity $\mathcal{O}(nm)$ is derived (where n denotes the number of vertices and m the number of edges of the input graph). Hence, we believed when starting this work that (theta, triangle)-free graphs have bounded tree-width. But this turned out to be false: layered wheels are (theta, triangle)-free graphs of arbitrarily large tree-width.

Outline of the chapter

In Section 3.1, we give a summary of the main results in this chapter. In Section 3.2, we describe the construction of layered wheels for two classes of graphs: (theta, triangle)-free graphs and (even hole, K_4)-free graphs (in fact, we prove it for a more restricted class namely (even hole, K_4 , pyramid)-free graphs). We prove that the constructions actually yield graphs in the corresponding classes (this is non-trivial, see Theorems 3.2.5, 3.2.10, and 3.2.11). We then prove that layered wheels have unbounded tree-width (see Theorem 3.2.12) and unbounded clique-width (see Theorem 3.2.15).

In Section 3.3, we recall the definition of rank-width. We exhibit (theta, triangle)-free graphs and (even hole, K_4)-free graphs with large rank-width. Having unbounded rank-width trivially follows from Theorem 3.2.15. However, in this section, we give a more rigorous computation of the lower bound of the rank-width (see Theorem 3.3.16).

In Section 3.4, we give an upper bound on the tree-width of layered wheels. We prove a stronger result: the so-called *path-width* of layered wheels is bounded by some linear function of the number of its layers (see Theorem 3.4.4).

3.1 Summary of the main results

The tree-width, clique-width, rank-width, and path-width of a graph G are denoted by $\text{tw}(G)$, $\text{cw}(G)$, $\text{rw}(G)$, and $\text{pw}(G)$ respectively. The following lemma is well-known.

Lemma 3.1.1 (See [CR05] and [OS06])

For every graph G , the followings hold:

- $\text{rw}(G) \leq \text{cw}(G) \leq 2^{\text{rw}(G)+1}$;
- $\text{cw}(G) \leq 3 \cdot 2^{\text{tw}(G)-1}$;

- $\text{tw}(G) \leq \text{pw}(G)$.

The first item of the lemma is proved in [CR05], and the second item is proved in [OS06]. The third item follows because path-width is a special case of tree-width (see Section 3.4). All results presented in this chapter can be summarized in the next two theorems.

Theorem 3.1.2

For all integers $l \geq 1$ and $k \geq 4$, there exists a graph $G_{l,k}$ such that the followings hold:

- $G_{l,k}$ is theta-free and has girth at least k (in particular, $G_{l,k}$ is triangle-free);
- $l \leq \text{tw}(G_{l,k}) \leq \text{pw}(G_{l,k}) \leq 2l$;
- $l \leq \text{rw}(G_{l,k}) \leq \text{cw}(G_{l,k}) \leq 3 \cdot 2^{\text{tw}(G)-1} \leq 3 \cdot 2^{2l-1} \leq |V(G_{l,k})|$.

Theorem 3.1.3

For all integers $l \geq 1$ and $k \geq 4$, there exists a graph $G_{l,k}$ such that the followings hold:

- $G_{l,k}$ is (even hole, K_4 , pyramid)-free and every hole in $G_{l,k}$ has length at least k ;
- $l \leq \text{tw}(G_{l,k}) \leq \text{pw}(G_{l,k}) \leq 2l$;
- $l \leq \text{rw}(G_{l,k}) \leq \text{cw}(G_{l,k}) \leq 3 \cdot 2^{\text{tw}(G)-1} \leq 3 \cdot 2^{2l-1} \leq |V(G_{l,k})|$.

3.2 Construction and tree-width

In this section, we describe the construction of layered wheels for two classes of graphs, namely the class of (theta, triangle)-free graphs and the class of (even hole, K_4)-free graphs. We also give a lower bound on their tree-width.

(Theta, triangle)-free layered wheels

We now present *ttf-layered-wheels* which are theta-free graphs of girth at least k , containing K_{l+1} as a minor, for all integers $l \geq 1, k \geq 4$ (see Figure 3.3).

Construction 3.2.1. Let $l \geq 0$ and $k \geq 4$ be integers. An (l, k) -*ttf-layered-wheel*, denoted by $G_{l,k}$, is a graph consisting of $l + 1$ layers, which are paths P_0, P_1, \dots, P_l . The graph is constructed as follows.

- (A1) $V(G_{l,k})$ is partitioned into $l + 1$ vertex-disjoint paths P_0, P_1, \dots, P_l . So, $V(G_{l,k}) = V(P_0) \cup \dots \cup V(P_l)$. The paths are constructed in an inductive way.
- (A2) The path P_0 consists of a single vertex.
- (A3) For every $0 \leq i \leq l$ and every vertex u in P_i , we call *ancestor of u* any neighbor of u in $V(P_0) \cup \dots \cup V(P_{i-1})$. The *type of u* is the number of its ancestors (as we will see, the

construction implies that every vertex has type 0 or 1). Observe that the unique vertex of P_0 has type 0. We will see that the construction implies that for every $1 \leq i \leq l$, the ends of P_i are vertices of type 1.

- (A4) Suppose inductively that $l \geq 1$ and layers P_0, P_1, \dots, P_{l-1} are constructed. The l^{th} -layer P_l is built as follows.

For any $u \in P_{l-1}$ we define a path BOX_u (which will be a subpath of P_l), in the following way (here, for some indexes i and j , $u_i \dots u_j$ denotes a path with ends u_i and u_j):

- if u is of type 0, BOX_u contains three neighbors of u , namely u_1, u_2, u_3 , in such way that $\text{BOX}_u = u_1 \dots u_2 \dots u_3$.
- if u is of type 1, let v be its unique ancestor. BOX_u contains six neighbors of u , namely u_1, \dots, u_6 , and three neighbors of v , namely v_1, v_2, v_3 , in such a way that

$$\text{BOX}_u = u_1 \dots u_2 \dots u_3 \dots v_1 \dots v_2 \dots v_3 \dots u_4 \dots u_5 \dots u_6.$$

The neighbors of u and the neighbors of v in BOX_u are of type 1, the other vertices of BOX_u are of type 0. We now specify the lengths of the boxes and how they are connected to form P_l .

- (A5) The path P_l goes through the boxes of P_l in the same order as vertices in P_{l-1} . For instance, if uvw is a subpath of P_{l-1} , then P_l goes through BOX_u , BOX_v , and BOX_w , in this order along P_l . Note that the vertices of P_l that are in none of the boxes are of type 0. Note that for $u \neq v$, we have $\text{BOX}_u \cap \text{BOX}_v = \emptyset$.
- (A6) Let w, w' be vertices of type 1 in P_l (so vertices from the boxes), and consecutive in the sense that the interior of wP_lw' contains no vertex of type 1. Then wP_lw' is a path of length at least $k - 2$.
- (A7) Observe that every vertex in P_l has type 0 or 1.
- (A8) There are no other vertices or edges apart from the ones specified above.

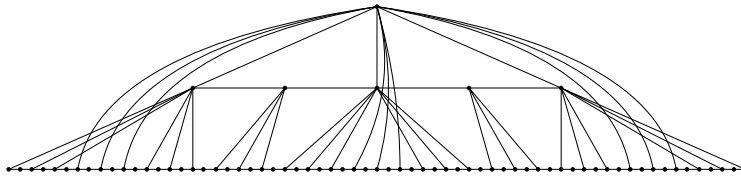


FIGURE 3.3: A ttf-layered-wheel $G_{2,4}$

Observe that the construction is not fully deterministic because in (A6), we just indicate a lower bound on the length of wP_lw' , so there may exist different ttf-layered-wheels $G_{l,k}$ for different values of l and k . This flexibility will be convenient below to exhibit ttf-layered-wheels of arbitrarily large rank-width.

Lemma 3.2.2

For $0 \leq i \leq l - 1$ and $i + 1 \leq j \leq l$, every vertex $u \in V(P_i)$ has at least 3^{j-i} neighbors in P_j .

Proof. We prove the lemma by induction on j . If $j = i + 1$, then (A4) implies that for every $0 \leq i \leq l - 1$ and every vertex u in P_i , u has three or six neighbors in P_{i+1} . If $j > i + 1$, then by the induction hypothesis, every vertex $u \in V(P_i)$ has at least 3^{j-1-i} neighbors in P_{j-1} . Hence by (A4), it has at least $3 \cdot 3^{j-1-i} = 3^{j-i}$ neighbors in P_j . \square

Lemma 3.2.2 implies in particular that every vertex of layer i has neighbors in all layers $i + 1, \dots, l$. Construction 3.2.1 is in fact the description of an inductive algorithm that constructs $G_{l,k}$. So, the next lemma is clear.

Lemma 3.2.3

For all integers $l \geq 0$ and $k \geq 4$, there exists an (l, k) -tff-layered-wheel.

We now prove that Construction 3.2.1 produces a theta-free graph with arbitrarily large girth and tree-width. Observe that any subdivision of the (3,5)-grid contains a theta. Thus, Theorem 3.2.5 implies that a tff-layered-wheel does not contain any subdivision of (3,5)-grid as mentioned in the introduction.

The next lemma is useful to prove Theorem 3.2.5. For a theta consisting of three paths P_1, P_2, P_3 , the common ends of those paths are called the *apexes* of the theta. Let G be graph containing a path P . The path P is *special* if

- there exists a vertex $v \in V(G \setminus P)$ such that $|N_P(v)| \geq 3$; and
- in $G \setminus v$, every vertex of P has degree at most 2.

Note that in the next lemma, we make no assumption on G , which in particular may contain triangles.

Lemma 3.2.4

Let G be a graph containing a special path P . For any theta that is contained in G (if any), every apex of the theta is not in P .

Proof. Let v be a vertex satisfying the properties as in the definition of special path. For a contradiction, suppose that P contains some vertex u which is an apex of some theta Θ in G . Note that u must have degree 3, and is therefore a neighbor of v . Consider two subpaths of P , u_1Pu_2 and u_2Pu_3 such that $u \in \{u_1, u_2, u_3\} \subseteq N(v)$ and both u_1Pu_2 , u_2Pu_3 have no neighbors of v in their interior. This exists since $|N_P(v)| \geq 3$. Since u is an apex, either $H_1 = vu_1Pu_2v$ or $H_2 = vu_2Pu_3v$ is a hole of Θ . Without loss of generality suppose that $V(H_1) \subseteq V(\Theta)$. Hence the other apex of Θ must be also contained in H_1 . Since $u_1v, u_2v \in E(G)$ and all vertices of $H_1 \setminus \{u_1, v, u_2\}$ have degree 2, u_1, u_2 must be the two apexes of Θ . Since $d(u_2) = 3$, $V(u_2Pu_3) \subseteq \Theta$. But then v has degree 3 in Θ while not being an apex, a contradiction. This completes the proof. \square

Theorem 3.2.5

For all integers $l \geq 0$ and $k \geq 4$, every (l, k) -tff-layered-wheel $G_{l,k}$ is theta-free graph with girth at least k .

Proof. We first show by induction on l that $G_{l,k}$ has girth at least k . This is clear for $l \leq 1$, so suppose that $l \geq 2$ and let H be a cycle in $G_{l,k}$ whose length is less than k . We may assume that layer P_l contains some vertex of H , for otherwise H is a cycle in $G_{l-1,k}$, so it has length at least k by the induction hypothesis. Let $P = u \dots v$ be a path such that $V(P) \subseteq V(H) \cap V(P_l)$ and with the maximum length among such possible paths. Note that P contains at least two vertices. Indeed, if P contains a

single vertex, then such a vertex must have at least two ancestors, since it has degree 2 in H , which is impossible by the construction of $G_{l,k}$. So $u \neq v$. Moreover, note that as P is contained in a cycle, both u and v must have an ancestor. Let u' and v' be the ancestor of u and v respectively. By (A6) of Construction 3.2.1 P has length at least $k - 2$. Hence $u'uvv'$ has length at least k , so H has length at least k . This completes the proof.

Now we show that $G_{l,k}$ is theta-free. For a contradiction, suppose that it contains a theta. Let Θ be a theta with minimum number of vertices, and having u and v as apexes. As above, without loss of generality, we may assume that P_l contains some vertex of Θ . Note that every vertex of P_l is contained in a special path of $G_{l,k}$. Hence, by Lemma 3.2.4, $u, v \notin V(P_l)$. In particular, every vertex of $V(P_l) \cap V(\Theta)$ has degree 2 in Θ .

Let $P = x \dots y$ for some $x, y \in P_l$, be a path such that $V(P) \subseteq V(\Theta) \cap V(P_l)$ and it is inclusion-wise maximal w.r.t. this property. Since every vertex of P_l has at most one ancestor, $x \neq y$. Moreover, both x and y must have an ancestor, because every vertex of Θ has degree 2 or 3 in Θ . Let x' and y' be the ancestor of x and y respectively. By the maximality of P , both x' and y' are also in Θ . Note that no vertex in the interior of P is adjacent to x' or y' , since otherwise such a vertex would have degree 3 in Θ , meaning that it is an apex, a contradiction.

Claim 1. We have $x' \neq y'$, $x'y' \notin E(G_{l,k})$, and some internal vertex of P is of type 1.

Proof of Claim 1. Otherwise, $x' = y'$ or $x'y' \in E(G_{l,k})$, or every internal vertex of P is of type 0. In the last case, we also have $x' = y' \in V(P_{l-1})$ or $x'y' \in E(G_{l,k})$ by the construction of $G_{l,k}$. Hence, in all cases, $V(P) \cup \{x', y'\}$ induces a hole in Θ , that must contain both u and v . Since $u, v \notin V(P_l)$, we have $u, v \in \{x', y'\}$. But this is not possible as $x' = y'$ or $x'y' \in E(G_{l,k})$. This proves Claim 1.

We now set $P' = x'xP_l y'y'$ (which is a path by Claim 1).

Claim 2. There exists no vertex of type 0 in P_{l-1} that has a neighbor in the interior of P .

Proof of Claim 2. For a contradiction, let $t \in V(P_{l-1})$ be of type 0 that has neighbors in the interior of P . Note that $t \notin V(\Theta)$ because internal vertices of P have degree 2 in Θ . Let Q be the shortest path from x' to y' in $G_{l,k}[V(P') \cup \{t\}]$. Note that Q is shorter than P' , because it does not go through one vertex of $N_P(t)$. So, P' can be substituted for Q in Θ , which provides a theta from u to v with fewer vertices, a contradiction to the minimality of Θ . This proves Claim 2.

Claim 3. We may assume that:

- $x' \in V(P_{l-1})$ and x' has type 0.
- $y' \notin V(P_{l-1})$.
- y' has a neighbor w in P_{l-1} and $x'w \in E(G_{l,k})$.
- Every vertex in P has type 0, except x, y , and three neighbors of w . Observe that w has type 1 and has three more neighbors in P_l that are not in P .

Proof of Claim 3. Suppose first that x', y' are both in P_{l-1} . Then by Claim 1, the path $x'P_{l-1}y'$ has length at least two. Moreover, by Claim 2, all its internal vertices are of type 1, because they all have neighbors in the interior of P . It follows that $x'P_{l-1}y'$ has length exactly two. We denote by z its unique internal vertex. Substituting $x'zy'$ for

P' , we obtain a theta that contradicts the minimality of Θ . Observe that the ancestor of z is not in $V(\Theta)$, because it has three neighbors in P . This proves that x', y' are not both in P_{l-1} .

So up to symmetry, we may assume that $y' \notin V(P_{l-1})$. Since y' has neighbor in P_l , it must be that y' has a neighbor $w \in V(P_{l-1})$, and that along P_l , one visits in order three neighbors of w , then y and two other neighbors of y' , and then three other neighbors of w .

Let w' be a neighbor of w in P_{l-1} , chosen so that w' has neighbors in P . Since w' has type 0, by Claim 2, we have $w' = x'$. Hence, as claimed, $x' \in V(P_{l-1})$ and $x'w \in E(G)$. This proves Claim 3.

Let a, b, c, a', b', c' be the six neighbors of w in P_l appearing in this order along P_l , in such a way that $a, b, c \in V(P)$ and $a', b', c' \notin V(P)$. We have $\{a', b', c'\} \cap V(\Theta) \neq \emptyset$, since otherwise we obtain a shorter theta from u to v by replacing P' with $x'wy'$, a contradiction to the minimality of Θ . Let y'' be the neighbor of y' in yP_la' closest to a' along yP_la' . Since $w \notin V(\Theta)$, $V(y'y''P_lc') \subseteq V(\Theta)$.

If $y' \notin \{u, v\}$, then by replacing $x'P'y'y''P_lc'$ with $x'wc'$, we obtain a theta, a contradiction to the minimality of Θ . So, $y' \in \{u, v\}$. Without loss of generality, we may assume that $y' = v$.

If $u \neq x'$, then by replacing $V(x'P'y'y''P_lc')$ with $\{x', w, y', c'\}$ in Θ , we obtain a theta from w to u which contains fewer vertices than Θ , a contradiction to the minimality of Θ . So, $u = x'$.

Recall that x' has type 0. Let $z \neq w$ be the neighbor of x' in P_{l-1} . Moreover, let z' and z'' be the neighbor of z and x' in P_l respectively, such that all vertices in the interior of $z'P_lz''$ have degree 2. Since Θ goes through P , $w \notin V(\Theta)$. Therefore $z, z', z'' \in V(\Theta)$. This implies the hole $zx'z''P_lz'z$ is a hole of Θ , a contradiction because the other apex $v = y'$ is not in the hole. This completes the proof that $G_{l,k}$ is theta-free. \square

Even-hole-free layered wheels

Recall that (even hole, triangle)-free graphs have tree-width at most 5 (see [Cam+18]), and as we will see, ttf-layered-wheels of arbitrarily large tree-width exist. Hence, some ttf-layered-wheels contain even holes (in fact, it can be checked that they contain even wheels). We now provide a construction of layered wheels that are (even hole, K_4)-free, but contain triangles (see Figure 3.5). Its structure is similar to ttf-layered-wheels, but slightly more complicated.

The construction of ehf-layered-wheels that we are going to discuss emerges from the structure of wheels that may exist in a graph of the studied class (namely, even-hole-free graphs with no K_4). In the class of even-hole-free graphs, wheels with the same rim may have different centers. Those centers may be adjacent or not. In Figure 3.4, we give examples of wheels that may exist in an even-hole-free graph. Formally, we do not need to prove that these wheels are even-hole-free, and therefore we omit the (straightforward) proof.

Now we are ready to describe the construction of ehf-layered-wheel.

Construction 3.2.6. Let $l \geq 1$ and $k \geq 4$ be integers. An (l, k) -ehf-layered-wheel, denoted by $G_{l,k}$, consists of $l + 1$ layers, which are paths P_0, P_1, \dots, P_l . We view these paths as oriented from left to right. The graph is constructed as follows.

(B1) $V(G_{l,k})$ is partitioned into $l + 1$ vertex-disjoint paths P_0, P_1, \dots, P_l . So, $V(G_{l,k}) = V(P_0) \cup \dots \cup V(P_l)$. The paths are constructed in an inductive way.

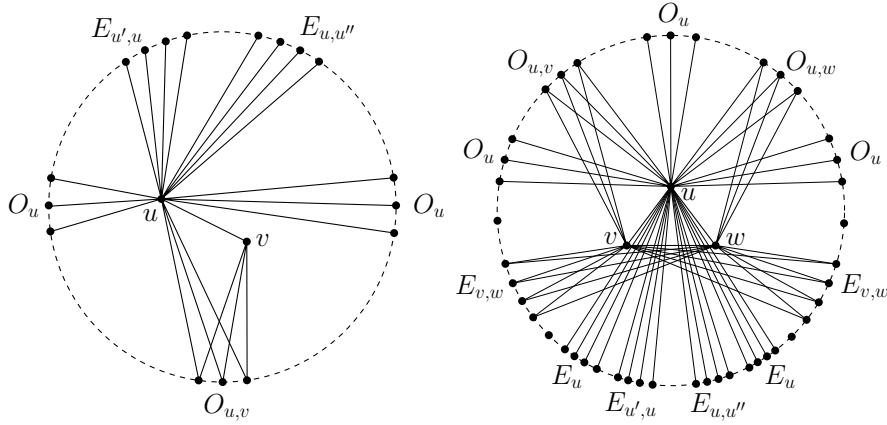


FIGURE 3.4: Wheels in an even-hole-free graph whose centers induce an edge or a triangle, with the corresponding zones as described in Construction 3.2.6 (dashed lines between two vertices represent paths of odd length)

(B2) The first layer P_0 consists of a single vertex r . The second layer P_1 is a path such that $P_1 = r_1 P_1 r_2 P_1 r_3$, where $\{r_1, r_2, r_3\} = N_{P_1}(r)$ and for $j = 1, 2$, $r_j P_1 r_{j+1}$ is of odd length at least $k - 2$.

(B3) For every $0 \leq i \leq l$ and every vertex u in P_i , we call **ancestor of u** any neighbor of u in $G_{l,k}[P_0 \cup \dots \cup P_{i-1}]$. The **type of u** is the number of its ancestors (as we will see, the construction implies that every vertex has type 0, 1, or 2). Observe that the unique vertex of P_0 has type 0, and P_1 consists only of vertices of type 0 or type 1. Moreover, we will see that if u is of type 2, then its ancestors are adjacent. Also, the construction implies that for every $1 \leq i \leq l$, the ends of P_i are vertices of type 1.

(B4) Suppose inductively that $l \geq 2$ and P_0, P_1, \dots, P_{l-1} are constructed. The l^{th} -layer P_l is built as follows.

For all $0 \leq i \leq l - 1$, any vertex $u \in V(P_i)$ has an odd number of neighbors in P_l , that are into subpaths of P_l that we call **zones**. These zones are labeled by E_u or O_u according to their parity: a zone labeled E_u contains four neighbors of u , and a zone labeled O_u contains three neighbors of u . All these four or three neighbors are of type 1, and all the other vertices of the zone are of type 0.

There are also zones that contain common neighbors of two vertices u, v . We label them $E_{u,v}$ (or $O_{u,v}$). A zone $E_{u,v}$ (resp. $O_{u,v}$) contains four (resp. three) common neighbors of u and v . All these four or three neighbors are of type 2, and all the other vertices of the zone are of type 0.

The ends of a zone E_u (resp. O_u) are neighbors of u . The ends of a zone $E_{u,v}$ (resp. $O_{u,v}$) are common neighbors of u and v . Distinct zones are disjoint.

(B5) For any $u \in P_{l-1}$, we define the box BOX_u , which is a subpath of P_l , as follows:

- If u is of type 0 (so it is an internal vertex of P_{l-1}), then let u' and u'' be the neighbors of u in P_{l-1} , so that $u'uu''$ is a subpath of P_{l-1} . In this case, BOX_u goes through three zones $E_{u',u}$, O_u , $E_{u,u''}$ that appear in this order along P_l (see Figure 3.5).
- If u is of type 1, then let $v \in P_i$, $i < l - 1$ be its ancestor.

If u is an internal vertex of P_{l-1} , then let u' and u'' be the neighbors of u in P_{l-1} , so that $u'uu''$ is a subpath of P_{l-1} . In this case, BOX_u is made of five zones $E_{u',u}$, O_u , $O_{u,v}$, O_u , $E_{u,u''}$ (see Figure 3.5).

If u is the left end of P_{l-1} , then let u'' be the neighbor of u in P_{l-1} . In this case, BOX_u is made of four zones O_u , $O_{u,v}$, O_u , $E_{u,u''}$.

If u is the right end of P_{l-1} , then let u' be the neighbor of u in P_{l-1} . In this case, BOX_u is made of four zones $E_{u',u}$, O_u , $O_{u,v}$, O_u .

- If u is of type 2 (so it is an internal vertex of P_{l-1}), then let $v \in P_i$ and $w \in P_j$, $j \leq i$ be its ancestors. If $i = j$, we suppose that v and w appear in this order along P_i (viewed from left to right). It turns out that either w is an ancestor of v , or v, w are consecutive along some path P_i (because as one can check, all vertices of type 2 that we create satisfy this statement). In this case, BOX_u is made of 11 zones, namely $E_{u',u}$, E_u , $E_{v,w}$, O_u , $O_{u,v}$, O_u , $O_{u,w}$, O_u , $E_{v,w}$, E_u , and $E_{u,u''}$ (see Figure 3.5).

Note that for any two adjacent vertices $u, v \in P_{l-1}$, BOX_u and BOX_v are not disjoint.

- (B6) The path P_l visits all the boxes BOX_- of P_l in the same order as vertices in P_{l-1} . For instance, if uvw is a subpath of P_{l-1} , then BOX_u , BOX_v , and BOX_w appear in this order along P_l .
- (B7) Let u and v be two vertices of P_l , both of type 1 or 2, and consecutive in the sense that every vertex in the interior of uP_lv is of type 0. If u and v have a common ancestor, then uP_lv has odd length, at least $k - 2$. If u and v have no common ancestor, then uP_lv has even length, at least $k - 2$.
- (B8) Observe that every vertex in P_l has type 0, 1, or 2. Moreover, as stated, every vertex of type 2 has two adjacent ancestors.
- (B9) There are no other vertices or edges apart from the ones specified above.

For the same reason as for ttf-layered-wheels, we allow flexibility in Construction 3.2.6, by just giving lower bounds for the lengths of paths described in (B7). So there may exist different ehf-layered-wheels $G_{l,k}$ for the same value of l and k .

Lemma 3.2.7

For $0 \leq i \leq l - 1$ and $i + 1 \leq j \leq l$, every vertex $u \in V(P_i)$ has at least 3^{j-i} neighbors in P_j .

Proof. We omit the proof since it is similar to the proof of Lemma 3.2.2. \square

Lemma 3.2.7 implies that every vertex of layer i has neighbors in all layers $i + 1, \dots, l$. The next lemma is clear.

Lemma 3.2.8

For all integers $l \geq 1$ and $k \geq 4$, there exists an (l, k) -ehf-layered-wheel.

We need some properties of lengths of some paths in ehf-layered-wheel. It is convenient to name specific subpaths of boxes first (see Figure 3.5).

- Suppose that u is a vertex in P_{l-1} (of any type).
If u is not an end of P_{l-1} , then a subpath of BOX_u is a *shared part of BOX_u* if it is either the zone $E_{u',u}$ or the zone $E_{u,u''}$. The *private part of BOX_u* is the path from the rightmost vertex of $E_{u',u}$ to the leftmost vertex of $E_{u,u''}$.

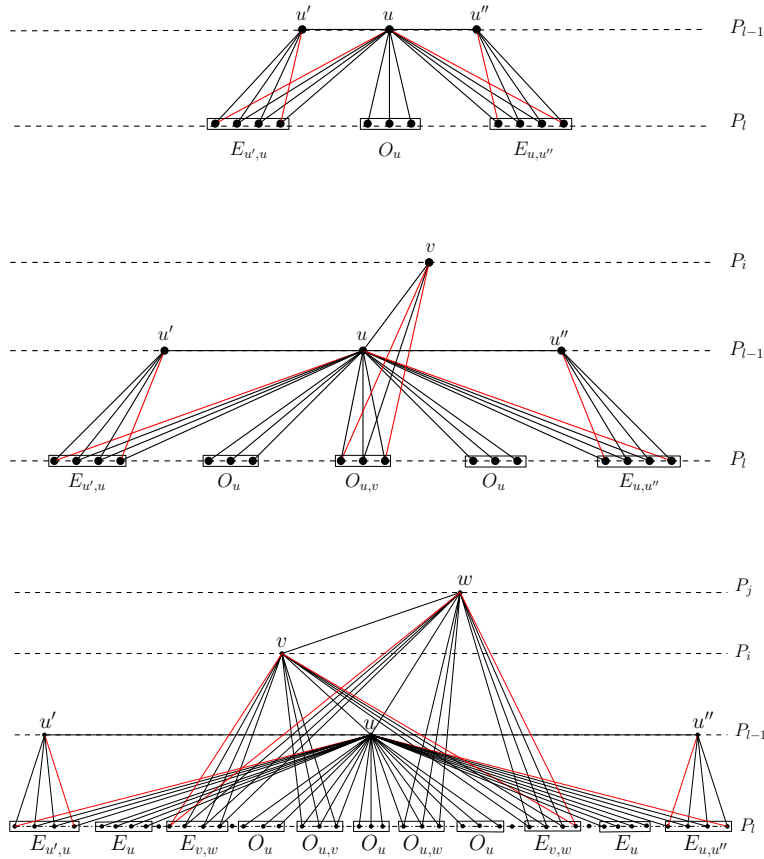


FIGURE 3.5: The neighborhood of a type 0, type 1, or type 2 vertex $u \in V(P_{l-1})$ in P_l (dashed lines between two vertices in P_l represent paths of odd length, and red edges represent non-internal edges as in the proof of Theorem 3.2.10)

Otherwise, if u is the left end of P_{l-1} (and therefore of type 1), then u has only one shared part, that is the zone $E_{u,u''}$, where $u'' \in N_{P_{l-1}}(u)$. The private part of u is the path from the leftmost vertex of the leftmost zone O_u to the leftmost vertex of $E_{u,u''}$.

Similarly, if u is the right end of P_{l-1} , then u has only one shared part, that is the zone $E_{u',u}$, where $u' \in N_{P_{l-1}}(u)$. The private part of u is the path from the rightmost vertex of $E_{u',u}$ to the rightmost vertex of the rightmost zone O_u .

Observe that BOX_u is edgewise partitioned into a private part and some shared parts (namely zero if $l = 1$ and u is the unique vertex of layer P_0 , one if $l > 1$ and u is an end of P_{l-1} , two otherwise).

- Suppose that u is of type 1 and v is its ancestor.

If u is not the left end of P_{l-1} , then the *left escape of v in BOX_u* is the subpath of BOX_u from the rightmost vertex of $E_{u',u}$ to the leftmost vertex of $O_{u,v}$.

If u is not the right end of P_{l-1} , then the *right escape of v in BOX_u* is the subpath of BOX_u from the rightmost vertex of $O_{u,v}$ to the leftmost vertex of $E_{u,u''}$.

- Suppose that u is of type 2 and v, w are its ancestors as in Construction 3.2.6. Note that u is not an end of P_{l-1} .

The *left escape of v (resp. of w) in BOX_u* is the subpath of BOX_u from the rightmost vertex of $E_{u',u}$ to the leftmost vertex of the zone $E_{v,w}$ that is the closest to $E_{u',u}$.

The *right escape of v (resp. of w) in BOX_u* is the subpath of BOX_u from the rightmost vertex of the zone $E_{v,w}$ that is the closest to $E_{u,u''}$, to the leftmost vertex of $E_{u,u''}$.

Lemma 3.2.9

Let $G_{l,k}$ be an ehf-layered-wheel with $l \geq 1$ and u be a vertex in the layer P_{l-1} . Then the following hold:

- Shared parts of BOX_u are paths of odd length.
- The private part of BOX_u is a path of even length if u is not an end of P_{l-1} ; and it is of odd length otherwise.
- If u has type 1 or 2, then all the left and right escapes of its ancestors in BOX_u are paths of even length.

Proof. To check the lemma, it is convenient to follow the path BOX_u in Figure 3.5 from left to right. In this proof, we refer to Construction 3.2.6, and we follow the notation given in Figure 3.5.

By (B7), shared parts of BOX_u have obviously odd length.

If u has type 0, then along the private part of BOX_u , one meets 1 common neighbor of u and u' , then 3 private neighbors of u , and then 1 common neighbor of u and u'' . In total, from the leftmost neighbor of u to its rightmost neighbor, one goes through 4 subpaths of BOX_u , each of odd length by (B7) (2 of the paths are in zones, while 2 of them are between zones). The private part of BOX_u has therefore even length.

If u has type 1, then the proof is similar. If it is not an end of P_{l-1} , then along the private part of BOX_u , one visits 10 subpaths (6 in zones, 4 between zones), each of odd length by (B7). Otherwise, one visits 9 subpaths (6 in zones, 3 between zones), each of odd length by (B7).

If u has type 2 then u is not an end of P_{l-1} . Now there are more details to check. Along the private part of BOX_u , one visits 32 subpaths. Among them, 22 are in zones and have odd length by (B7), and 10 are between zones. But 4 of the subpaths between zones have even length by (B7), namely, the paths linking E_u to $E_{v,w}$ (because $\{u\} \cap \{v,w\} = \emptyset$), $E_{v,w}$ to O_u, O_u to $E_{v,w}$, and $E_{v,w}$ to E_u . The 6 remaining subpaths between zones have odd length by (B7). In total, the private part of BOX_u has even length as claimed.

For the left and right escapes, the proof is similar. If u is of type 1, then the escape is made of 4 paths each of odd length. If u is of type 2, then the escape is made of the path between zones $E_{v,w}$ and E_u that is of even length, three paths in zone E_u each of an odd length, and the path between zone E_u and $E_{u',u}$ or $E_{u,u''}$ that is of odd length. So, every left and every right escape is of even length. \square

Theorem 3.2.10

For all integers $l \geq 1$ and $k \geq 4$, every (l,k) -ehf-layered-wheel $G_{l,k}$ is (even hole, K_4)-free and every hole in $G_{l,k}$ has length at least k .

Proof. It is clear from the construction that $G_{l,k}$ does not contain K_4 . Moreover, it follows from (B7) that apart from triangles, any chordless cycle in $G_{l,k}$ is of length at

least k (we omit the formal proof that is similar to the proof that ttf-layered-wheels have girth at least k).

For a contradiction, consider an ehf-layered-wheel $G_{l,k}$ that contains an even hole H . Suppose that l is minimal, and under this assumption that H has minimum length. Hence, layer P_l contains some vertex of H , for otherwise $G_{l,k}[P_0 \cup \dots \cup P_{l-1}]$ would be a counterexample. Let us start by the following claim.

Let x be a vertex in P_i where $0 \leq i < l$, and y be a neighbor of x in P_l . We say that xy is an *internal edge* (see Figure 3.5) if one of the following holds:

- $i = l - 1$ and y is an internal vertex of BOX_x .
- $i < l - 1$, x is an ancestor of $x' \in V(P_{l-1})$, x' has type 1 or 2, y is in $\text{BOX}_{x'}$ and y is neither the leftmost neighbor of x in $\text{BOX}_{x'}$ nor the rightmost neighbor of x in $\text{BOX}_{x'}$.

Claim 1. H contains no internal edge.

Proof of Claim 1. Let xy be an internal edge as in the definition and suppose for a contradiction that xy is an edge of H . Let $Q = y \dots z$ be the path of H that is included in P_l and that is maximal w.r.t. this property. Let z' be the ancestor of z that is in H (it exists by the maximality of Q).

Suppose first that x is in P_{l-1} . We then set $x = u$ and observe that u has type 0, 1 or 2 (see Figure 3.5). If u has type 0, then since uy is internal edge, y is either in the zone O_u , or is among the three rightmost vertices of zone $E_{u',u}$, or is among the three leftmost vertices of zone $E_{u,u''}$ (where u' and u'' are the left and the right neighbors of u respectively in P_{l-1} as shown in Figure 3.5). Since no internal vertex of Q is adjacent to u because H is a hole, we have $zu \in E(G)$ and $z' = u$. So, $H = uyQzu$ and H has odd length by the axiom (B7), a contradiction. If u has type 1 (and ancestor v as represented in Figure 3.5), the proof is similar (note in this case that $z' \neq v$ for otherwise the triangle uvy would be in H , a contradiction).

If u has type 2 (and ancestors v, w as represented in Figure 3.5) the proof is similar with some other possibilities. For instance, it can be that y is the rightmost vertex of the leftmost zone $Z = E_u$. In this case, z can be either the leftmost vertex of the zone $E_{v,w}$ that is next to Z , or the leftmost vertex of the zone O_u that is closest to Z . In the first case, $z' = v$ or $z' = w$ (say $z' = v$ up to symmetry), so $H = uyQzvu$ and H has odd length by (B7); in the second case, $H = uyQzu$ and H has again odd length by (B7), a contradiction. Similar situations are when y is the leftmost vertex of the leftmost zone O_u , when y is the rightmost vertex of the rightmost zone O_u and when y is the leftmost vertex of the rightmost zone E_u . We omit the details of each situation.

Suppose now that x is not in P_{l-1} . Since x has neighbor in P_l , x is the ancestor of some vertex u from P_{l-1} . If u is of type 1 with ancestor v , then $x = v$. We observe that y must be the middle vertex of the zone $O_{u,v}$. Hence, $H = vyQzv$ and H has odd length by (B7), a contradiction.

So, u has type 2 and ancestors v, w . Up to symmetry, we may assume that $x = v$. As in the previous cases, regardless of the position of y in BOX_u , we must have either $H = vyQzv$, or $H = vyQzuv$, or $H = vyQzvw$ (when y is the rightmost vertex of $O_{u,v}$ and z is the leftmost vertex of $O_{u,w}$). In all cases, H has odd length, a contradiction. This proves Claim 1.

Now let $P = s \dots t$ be a subpath of H in P_l such that P is inclusion-wise maximal. So both s and t have an ancestor that is in H . If P contains a single vertex (i.e., $s = t$), then s must have two ancestors, say, s_1 and s_2 , which are adjacent by (B3)

of Construction 3.2.6. Thus $\{s, s_1, s_2\}$ forms a triangle in H , which is not possible. So P contains at least two vertices and $s \neq t$. Let u and v be ancestors of s and t respectively, such that $u, v \in V(H)$ (possibly $u = v$, or $uv \in E(G)$).

Recall that all layers are viewed as oriented from left to right. We suppose that s and t appear in this order, from left to right, along P_l .

Claim 2. For every vertex $p \in V(P_{l-1})$, $N(p) \cap V(P_l) \not\subseteq V(P)$.

Proof of Claim 2. Suppose that $p \in V(P_{l-1})$ and $N(p) \cap V(P_l) \subseteq V(P)$. So, $p \notin V(H)$. Note that p is an internal vertex of P_{l-1} , for otherwise, s or t is an end of P_l and has degree 2, while having two neighbors in $V(H) \cap V(G_{l,k} \setminus p)$, a contradiction.

By (B5), ancestors of p (if any) and the neighbors of p in P_{l-1} must also have neighbors in P . Thus, all of such vertices do not belong to H because P is a subpath of H . By Lemma 3.2.9, the path $\text{BOX}_p = p' \dots p''$ has an even length. Indeed BOX_p consists of two shared parts (each of odd length) and one private part (of even length). It follows that BOX_p and $p'pp''$ have the same parity, and hence replacing BOX_p in H with $p'pp''$ yields an even hole with length strictly less than the length of H , a contradiction to the minimality of H . This proves Claim 2.

Claim 3. Exactly one of u and v is in P_{l-1} .

Proof of Claim 3. Suppose that both u and v are not in P_{l-1} . Since u and v have neighbors in P , each of them has a neighbor in P_{l-1} (where such neighbors also have some neighbor in P). Let u' and v' be the respective neighbors of u and v in P_{l-1} .

If $u' = v'$, then u' is a type 2 vertex in P_{l-1} . So, H is a hole of form $usPtvu$, and it has odd length by construction. So $u' \neq v'$, and by construction, the interior of $u'P_{l-1}v'$ must contain a vertex w of type 0. It follows that $N_{P_l}(w)$ is all contained in P , a contradiction to Claim 2.

Suppose now that both u and v are in P_{l-1} . By Claim 2, no vertex of P_{l-1} has all its neighbors in P . So the interior of $uP_{l-1}v$ contains at most two vertices.

If $u = v$, then by (B7) P is of odd length, and since $V(H) = \{u\} \cup V(P)$, H is also of odd length, a contradiction. Similarly if $uv \in E(G)$, then by (B7), P is of even length, $V(H) = \{u, v\} \cup V(P)$, and H has odd length, again a contradiction.

If the interior of $uP_{l-1}v$ contains a single vertex, then let w be this vertex. Let w_1 (resp. w_2) be the neighbor of w in P that is closest to s (resp. t). Note that by (B5), $s = w_1, t = w_2$ because both u and v are adjacent to w in P_{l-1} . So, sPt is the private part of BOX_w , and by Lemma 3.2.9, it has even length (the parity of the length is the same as uwv , a subpath of P_{l-1}). Moreover, by Claim 1, $\{s\} = V(E_{u,w}) \cap V(H)$ and $\{t\} = V(E_{w,v}) \cap V(H)$. Also, if w has an ancestor, then such an ancestor must have neighbors in P , and hence it does not belong to H . Altogether, we see that $N_H(w) \subseteq V(usPt v)$. So, replacing $usPt v$ in H with uwv gives an even hole with length strictly less than the length of H , a contradiction to the minimality of H .

So the interior of $uP_{l-1}v$ contains two vertices. We let $uP_{l-1}v = uww'v$, and w_1 (resp. w'_2) be the neighbor of w (resp. w') in P that is closest to s (resp. t). By (B5), $s = w_1, t = w'_2$. So, sPt is edgewise partitioned into the private part of w , the part shared between w and w' , and the private part of w' . By Lemma 3.2.9, sPt has therefore odd length. In particular, the length of $usPt v$ has the same parity as the length of $uww'v$. Moreover, by Claim 1, $\{s\} = V(E_{u,w}) \cap V(H)$ and $\{t\} = V(E_{w',v}) \cap V(H)$. Also, if w or w' has an ancestor, then such an ancestor must have neighbors in P , and hence it does not belong to H . Altogether, we see that $N_H(\{w, w'\}) \subseteq V(usPt v)$. So, replacing $usPt v$ in H with $uww'v$ gives an even hole that is shorter than H , again a contradiction to the minimality of H . This proves Claim 3.

By Claim 3 and up to symmetry, we may assume that $u \in V(P_{l-1})$ and $v \notin V(P_{l-1})$. So, v has a neighbor v' in P_{l-1} such that $t \in \text{BOX}_{v'}$. Note that $v' \notin H$, because by construction v' has some neighbor in P . Hence, $v' \neq u$ (because $u \in H$). If the path $uP_{l-1}v'$ has length at least three, then some vertex in the interior of $uP_{l-1}v'$ contradicts Claim 2.

If $uP_{l-1}v'$ has length two, so $uP_{l-1}v' = uww'$ for some vertex $w \in V(P_{l-1})$, then w is of type 0 because v' is not of type 0. Hence, P is edgewise partitioned into the private part of w , the part of $\text{BOX}_{v'}$ shared between w and v' and the left escape of v in $\text{BOX}_{v'}$. Let w' be the rightmost vertex of the shared zone $E_{w,v'}$. By Lemma 3.2.9, $usPw'$ has even length, as uww' . Moreover, by Claim 1, $\{s\} = V(E_{u,w}) \cap V(H)$, and since w has type 0, we see that $N_H(w) \subseteq V(usPw')$. So, replacing $usPw'$ in H with uww' gives an even hole with length strictly less than the length of H , a contradiction to the minimality of H .

Hence, $uP_{l-1}v'$ has length one: $uP_{l-1}v' = uv'$. So, P is the left escape of v in $\text{BOX}_{v'}$. By Lemma 3.2.9, P has even length. By Claim 1, $\{s\} = V(E_{u,v'}) \cap V(H)$. Recall that $v' \notin H$. If $N_H(v') \subseteq V(usPtv)$, then replacing $usPtv$ in H with $uv'v$ gives an even hole with length strictly less than the length of H , a contradiction to the minimality of H .

So, v' has neighbors in H that are not in $usPtv$. Note that if v' is of type 2, the ancestor of v' that is different from v is not in H (because it is adjacent to t and to v). Also, by Claim 1, the neighbors of v' in $E_{u,v'} \setminus s$ are not in H .

We denote by v'' the right neighbor of v' in P_{l-1} . Note that v'' has type 0, since v' has type 1 or 2. Let s' and t' be vertices such that $t'P_1s'$ is the right escape of v in $\text{BOX}_{v'}$, t' is adjacent to v , and s' is adjacent to v'' . Note that s' is the leftmost vertex of $E_{v',v''}$ and t' is the rightmost vertex of the zone $O_{v',v}$ (when v' is of type 1) or of the rightmost zone $E_{v,w}$ (when v' is of type 2, and w is the other ancestor of v').

Let us see which vertex can be a neighbor of v' in $H \setminus usPtv$. We already know it cannot be an ancestor of v' or be in $E_{u,v'} \setminus s$. Suppose it is v'' . Then, H must contain two edges incident to v'' , and none of them can be an internal edge by Claim 1. Note that $s'v''$ must be an edge of H , for otherwise, the two only available edges are $v''v'''$ and $v''s''$ (where v''' is the right neighbor of v'' in P_{l-1} and s'' is the rightmost neighbor of v'' in P_l), and this yields a contradiction because $v'''s'' \in E(G)$. Since $s'v'' \in E(H)$, H goes through the path $R = usPtv't'P_1s'v''$. This path has even length, and contains all vertices of $N_H(v')$. So, we may replace R by $uv'v''$ in H , to obtain an even hole that contradicts the minimality of H . Now we know that $v'' \notin V(H)$.

Since v' has a neighbor in $H \setminus usPtv$, and since this neighbor is not an ancestor of v' , is not v'' , and is not in $E_{u,v'}$, it must be in $\text{BOX}_{v'} \setminus (V(P) \cup E_{u,v'})$. By Claim 1, the only way that H can contain some vertex of $\text{BOX}_{v'} \setminus (V(P) \cup E_{u,v'})$ is if H goes through the edge vt' , in particular through the right escape of v in $\text{BOX}_{v'}$. Let t'' be the rightmost vertex of $E_{v',v''}$. Hence, H must go through the path $S = usPtv't'P_1t''$ (see Figure 3.6). This path has even length, and contains all vertices of $N_H(v')$. So, we may replace S by $uv't''$ in H , to obtain an even hole that contradicts the minimality of H . \square

Let us now prove that every ehf-layered-wheel is pyramid-free.

Theorem 3.2.11

For all integers $l \geq 1, k \geq 4$, every (l, k) -ehf-layered-wheel $G_{l,k}$ is pyramid-free.

Proof. Recall that all layers are viewed as oriented from left to right. For a contradiction, suppose that an ehf-layered-wheel $G_{l,k}$ contains a pyramid $\Pi = 3PC(\Delta, x)$. (Here we denote by Δ the triangle of Π , and call the apex the only vertex of degree 3

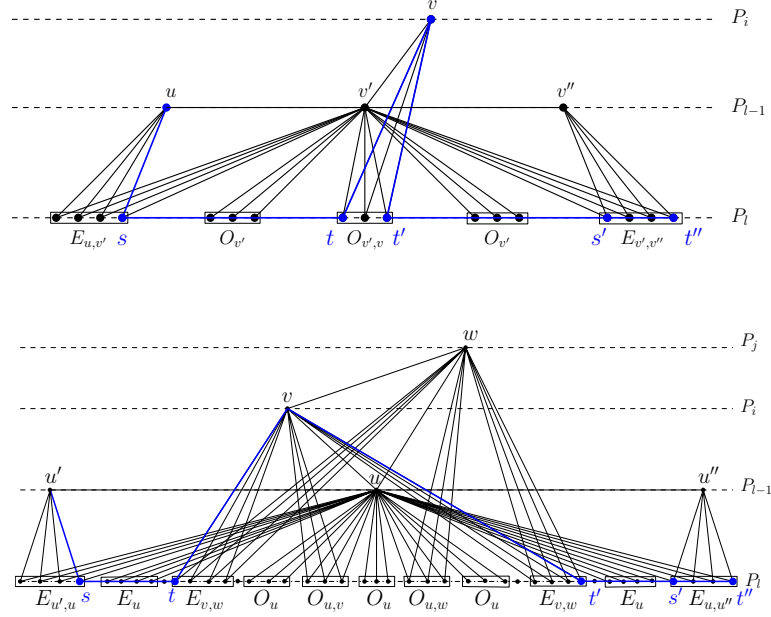


FIGURE 3.6: The proof of Theorem 3.2.10: in blue is the path $S = usPtvt'Pt''$, when v' is of type 1 (top) and when v' is of type 2 (bottom)

in $\Pi \setminus \Delta$ which in this case is the vertex x .) Suppose that l is minimal, and under this assumption that Π contains the minimum number of vertices among all pyramids in $G_{l,k}$. Clearly $l \geq 3$, and layer P_l contains some vertex of Π , for otherwise $G_{l,k}[P_0 \cup \dots \cup P_{l-1}]$ would be a counterexample.

The next claim is trivially correct, so we omit the proof.

Claim 1. Any hole in Π contains the apex and two vertices of Δ .

Claim 2. If a vertex of Δ is in P_l , then it is not in the interior of some zone.

Proof of Claim 2. Suppose that some vertex a of Δ is in P_l and is in the interior of some zone Z . Then a is of type 2. If $Z = E_{u',u}$ for some $u', u \in P_{l-1}$, then $\Delta = auu'$, and we see that the left or the right neighbor of a in P_l is in Π . Let $Q = a \dots b$ be the subpath of P_l that contains a , that is included in Π , and that is maximal w.r.t. these properties. We see that b is adjacent to u and u' , so that Π contains a diamond, a contradiction. The proof is the same for all other kinds of zones (namely $E_{u,u''}$, $E_{v,w}$, $O_{u,v}$, or $O_{u,w}$). This proves Claim 2.

Claim 3. The apex x is not in P_l .

Proof of Claim 3. Let us see that $x \in P_l$ yields a contradiction. Since x has degree 3 in Π , it is a vertex of type 1 or 2, so it belongs to some zone.

Suppose first that x is in the interior of some zone Z . If $Z = E_{u,u'}$ for some $u, u' \in P_{l-1}$, then since x has degree 3 in Π and is not in a triangle of Π , we see that the two neighbors of x in P_l are in Π . Also, exactly one ancestor y of x must be in Π . Let Q be the subpath of P_l that contains x , that is included in Π , and that is maximal w.r.t. these properties. We see that the ends of Q are adjacent to y , so that Q and y form a cycle with a unique chord in a pyramid, while not containing a triangle, a contradiction. When Z is another zone, say $E_{u,u''}$, O_u , $O_{u,v}$, etc, the proof is exactly the same.

Suppose now that x is an end of some zone Z . Again, the two neighbors of x in P_l and an ancestor u of x are in Π . So, Π contains the path Q from x to the vertex y with ancestor u that is next to x along Z . Note that y is in the interior of Z . So, Q and

u form a hole of Π . Apart from x , y , and u , every vertex of H has degree 2, so uy is an edge of Δ , a contradiction to Claim 2. This proves Claim 3.

Claim 4. If $u \in P_{l-1}$ has type 0 or 1 and is in Π , then no internal vertex of BOX_u is in Π .

Proof of Claim 4. Suppose $a \in \Pi$ is an internal vertex of BOX_u . Let Q be the subpath of P_l that contains a , is included in Π , and maximal w.r.t. this property. Since a is an internal vertex of BOX_u and u has type 0 or 1, Q and u form a hole H , that must contain the apex. Since by Claim 3, the apex is not in P_l , it must be u , and since every internal vertex of Q has degree 2, the two neighbors of u in H are in Δ , a contradiction since they are non-adjacent. This proves Claim 4.

Claim 5. No vertex of Δ is in P_l .

Proof of Claim 5. Suppose for a contradiction that a is a vertex of Δ in P_l . So a has type 2, and in particular, it is not an end of P_l . As every internal vertex of P_l , a is in the interior of some box BOX_u . If u is of type 0 or 1, it must be part of Δ , so a contradicts Claim 4. Hence, u is of type 2.

We denote by $P = a \dots p$ a subpath of Π included in BOX_u and maximal with this property. We will now analyze every possible zone where a belongs to, and we will see that each of the cases yields a contradiction.

Suppose first that a is in a shared zone Z . If $Z = E_{u',u}$, and therefore, $\Delta = au'u$, then by Claim 2, a is the rightmost vertex of $E_{u',u}$ (since it is in the interior of BOX_u). Since u' is of type 0 (because u is of type 2), Claim 4 applied to $\text{BOX}_{u'}$ implies that a is the only vertex of Π in $E_{u',u}$, so p must be the leftmost vertex of the zone E_u that is next to $E_{u',u}$. So P and u form a hole H of Π , and since u is in Δ , the apex x must be in P_l , a contradiction to Claim 3. The proof is similar when $Z = E_{u,u''}$.

If $Z = O_{u,v}$, then $\Delta = auv$, and by Claim 2, a is either the leftmost or the rightmost vertex of $O_{u,v}$. If a is the leftmost vertex of $O_{u,v}$, then p is either the rightmost vertex of the zone O_u (that is on the left of $O_{u,v}$) or p is the vertex of type 2 next to a along $O_{u,v}$. In either case, P and u form a hole H of Π , and since u is in the triangle of Π , the apex x must be in P_l , a contradiction to Claim 3. If a is the rightmost vertex of $O_{u,v}$, the proof is similar. By symmetry, the case when $Z = O_{u,w}$ yields a similar contradiction.

When a is the rightmost vertex of the leftmost zone $E_{v,w}$ (that is between E_u and O_u when oriented from left), we have $\Delta = avw$ and so $u \notin \Pi$. The proof is again the same, with a hole H that goes through v . The case when a is the leftmost vertex of the rightmost zone $E_{v,w}$ (that is between O_u and E_u when oriented from left) can be done in the similar way.

We are left with the case when a is the leftmost vertex of the leftmost zone $E_{v,w}$, or the rightmost vertex of the rightmost zone $E_{v,w}$. These two cases are symmetric, so we may assume that a is the leftmost vertex of the leftmost zone $E_{v,w}$.

It then follows that $\Delta = avw$. Note that $u \notin \Pi$ because a pyramid has only one triangle. If P goes in the interior of the zone $E_{v,w}$, then Π contains a diamond, a contradiction. So, P goes through the zone E_u that is left to $E_{v,w}$ and contains the rightmost vertex of $E_{u',u}$. There are two cases: P contains the zone $E_{u',u}$ (so p is the leftmost vertex of $E_{u',u}$ and $u' \notin \Pi$), or P contains only the rightmost vertex of $E_{u',u}$ (so p is the rightmost vertex of $E_{u',u}$ and $u' \in \Pi$). In the first case, we remove $P \setminus p$ from Π and put instead the edge up ; in the second case, we remove P from Π completely and put the edge uu' . We obtain a pyramid (with triangle uvw) that is of smaller size than Π — a contradiction, unless u has some neighbor in $\Pi \setminus (P \cup \{u', v, w\})$. Hence, we now suppose such a neighbor z exists.

Let q be the leftmost vertex of the leftmost zone O_u (that is first met when traversing the layer from left to right), and r be the rightmost vertex of $E_{u,u''}$. Consider the path $Q = qP_l r$. Observe that z is in $Q \cup \{u''\}$ because $Q \cup \{u''\}$ contains all possible neighbors of u in $\Pi \setminus (P \cup \{u', v, w\})$.

Suppose that some vertex of Q is in Π . Let z' be the vertex of Π in Q that is the closest to q along Q . Note that by Claim 3, z' has degree 2 in Π . Since z' is the closest vertex to q , it has a neighbor in $\Pi \setminus Q$. In particular, z' is a type 1 or type 2 vertex, and exactly one of its ancestor is in Π . Since $u \notin \Pi$, such an ancestor is v or w , or possibly u'' if $u'' \in \Pi$ (and only one of them). If $z' \in O_{u,v}$, or $z' \in O_{u,w}$, or $z' \in E_{v,w}$, then there exists a vertex $z'' \in Q$ such that $vz'P_l z''v$, or $wz'P_l z''w$, or $vz'P_l z''wv$ is a hole of Π , which in each case contradicts Claim 3. So $z' \in E_{u,u''}$ and the ancestor of z' in Π must be u'' (in particular $u'' \in \Pi$). But then, the right neighbor of z' in P_l is an internal vertex of $\text{BOX}_{u''}$ that belongs to Π , a contradiction to Claim 4. Therefore, $\Pi \cap Q = \emptyset$.

This means that $z = u''$. Note that the neighbors of u'' in Π cannot contain u (because $u \notin \Pi$), cannot be in $E_{u,u''}$ (because $E_{u,u''}$ is subpath of Q), cannot be in the interior of $\text{BOX}_{u''}$ (because u'' has type 0 and by Claim 4), so they are precisely the right neighbor u''' of u'' in P_{l-1} and the rightmost vertex b of $E_{u'',u'''}$. But then, $u''u'''b$ is a triangle in Π , a contradiction. This proves Claim 5.

The rest of the proof is quite similar to the proof of Theorem 3.2.10, that ehf-layered-wheel contains no even hole.

Let $P = s \dots t$ be a subpath of Π in P_l such that P is inclusion-wise maximal (and s, t appear in this order from left to right). By Claims 3 and 5, every vertex of P has degree 2 in Π . Moreover by the maximality of P , each of s and t has an ancestor which is also in Π . Note that $s \neq t$, for otherwise s would be of type 2, and together with its ancestors, it forms a triangle, which contradicts Claim 5. Let u and v be the ancestors of s and t respectively, such that $u, v \in V(\Pi)$. By Claims 1, 3, and 5, $u \neq v$ and $uv \notin E(G)$.

Claim 6. For every vertex $p \in V(P_{l-1})$, $N(p) \cap V(P_l) \not\subseteq V(P)$.

Proof of Claim 6. Suppose that $p \in V(P_{l-1})$ and $N(p) \cap V(P_l) \subseteq V(P)$. So, $p \notin V(\Pi)$. Note that p is an internal vertex of P_{l-1} , for otherwise, s or t is an end of P_l and has degree 2, while having two neighbors in $V(\Pi) \cap V(G_{l,k} \setminus p)$, a contradiction.

By (B5), the ancestors of p (if any) and the neighbors of p in P_{l-1} must also have neighbors in P . Thus, no such vertices belong to Π because P is a subpath of Π . Hence, replacing $\text{BOX}_p = p' \dots p''$ in Π with $p'pp''$ yields a pyramid with strictly fewer vertices than Π , a contradiction to the minimality of Π . This proves Claim 6.

Let a be a vertex in P_i for some $0 \leq i < l$, and p be a neighbor of a in P_l . We say that ap is an *internal edge* if one of the following holds:

- $i = l - 1$ and p is an internal vertex of BOX_a .
- $i < l - 1$, a is an ancestor of some $a' \in V(P_{l-1})$, a' has type 1 or 2, p is in $\text{BOX}_{a'}$ and p is neither the leftmost neighbor of a in $\text{BOX}_{a'}$ nor the rightmost neighbor of a in $\text{BOX}_{a'}$.

Claim 7. No internal edge is an edge of Π .

Proof of Claim 7. Suppose that $p \in P_l$ is the end of an internal edge e that is also an edge of Π . If the other end of e is in P_{l-1} , we set $e = pu$ and observe that p is in the interior of BOX_u . Otherwise, the other end of e is in P_i , with $i < l - 1$, we set $e = px$

and observe that x has a neighbor u in P_{l-1} . Again, p is an internal vertex of BOX_u . Observe that x is either v or w as represented in Figure 3.5.

By Claims 3 and 5, p has degree 2 in Π , so p has a unique neighbor in $\Pi \cap P_l$. Let $P = p \dots p'$ be the subpath of P_l included in Π , containing p , and maximal with respect to this property.

It can be checked in Figure 3.5 that P together with u, v, w, uv, uw , or vw forms a hole, that contains the apex and two vertices of Δ (by Claim 1), a contradiction to Claims 3 and 5. This proves Claim 7.

Claim 8. Exactly one of u and v is in P_{l-1} .

Proof of Claim 8. Suppose that both u and v are not in P_{l-1} . Since u and v have neighbors in P , each of them has a neighbor u' and v' respectively in P_{l-1} , such that $s \in \text{BOX}_{u'}$ and $t \in \text{BOX}_{v'}$.

If $u' = v'$, then u' is a type 2 vertex in P_{l-1} . By construction, $usPtvu$ is then a hole of Σ , so it must contain the apex and two vertices of Δ , contradicting Claim 3 or Claim 5, since u and v are the only vertices of the hole that are not in P_l .

Since u' and v' are vertices with ancestors, by construction, the interior of $u'P_{l-1}v'$ contains a vertex w of type 0. It yields that $N_{P_l}(w)$ is all contained in P , a contradiction to Claim 6.

Suppose now that both u and v are in P_{l-1} . By Claim 6, no vertex of P_{l-1} has all its neighbors in P . So the interior of $uP_{l-1}v$ contains at most two vertices.

If $u = v$, then $usPtu$ is a hole of Π . Since u is the only vertex in the hole that is not in P_l , by Claim 1, P contains the apex or a vertex of Δ , a contradiction to Claims 3 or 5. Similarly if $uv \in E(G)$, then $usPtvu$ is a hole of Π , this again yields a contradiction.

If the interior of $uP_{l-1}v$ contains a single vertex, then let w be this vertex. Let w_1 (resp. w_2) be the neighbor of w in P that is closest to s (resp. t). It follows by construction, that $s = w_1, t = w_2$ (because both u and v are adjacent to w in P_{l-1}). By Claim 7, $\{s\} = V(E_{u,w}) \cap V(\Pi)$ and $\{t\} = V(E_{w,v}) \cap V(\Pi)$. Also, if w has an ancestor, then such an ancestor must have neighbors in P , and hence it does not belong to Π . Altogether, we see that $N_{\Pi}(w) \subseteq V(usPtv)$. So, replacing $usPtv$ in Π with uwv gives a pyramid with fewer vertices than Π , a contradiction to the minimality of Π .

So the interior of $uP_{l-1}v$ contains two vertices. We let $uP_{l-1}v = uw'v'$, and w_1 (resp. w'_2) be the neighbor of w (resp. w') in P that is closest to s (resp. t). Similar to in the previous case, we know that $s = w_1, t = w'_2$; and by Claim 7, $\{s\} = V(E_{u,w}) \cap V(\Pi)$ and $\{t\} = V(E_{w',v'}) \cap V(\Pi)$. Also, if w or w' has an ancestor, then such an ancestor must have neighbors in P , and hence it does not belong to Π . Altogether, we see that $N_{\Pi}(\{w, w'\}) \subseteq V(usPtv)$. So, replacing $usPtv$ in Π with $uw'w'v$ gives a pyramid with fewer vertices than Π , again a contradiction to the minimality of Π . This proves Claim 8.

By Claim 8 and up to symmetry, we may assume that $u \in V(P_{l-1})$ and $v \notin V(P_{l-1})$. So, v has a neighbor v' in P_{l-1} such that $t \in \text{BOX}_{v'}$. Note that $v' \neq u$, for otherwise $usPtvu$ is a hole of Σ , so it contains the apex and two vertices of Δ , a contradiction to Claim 3 or Claim 5. Furthermore, note that the path $uP_{l-1}v'$ has length at most two, for otherwise some vertex in the interior of $uP_{l-1}v'$ contradicts Claim 6.

Suppose that $uP_{l-1}v'$ has length two, so $uP_{l-1}v' = uw'v'$ for some vertex $w \in V(P_{l-1})$. Then w is of type 0 because v' is not of type 0. Let w' be the rightmost vertex of the shared zone $E_{w,v'}$. By Claim 7, $\{s\} = V(E_{u,w}) \cap V(\Pi)$, and since w has type 0, we see that $N_{\Pi}(w) \subseteq V(usPw')$. So, replacing $usPw'$ in Π with $uw'w'$ gives a pyramid with fewer vertices than Π , a contradiction to the minimality of Π .

Hence, $uP_{l-1}v'$ has length one, i.e. $uP_{l-1}v' = uv'$. By Claim 7, $\{s\} = V(E_{u,v'}) \cap V(\Pi)$. Observe that P is the left escape of v in $\text{BOX}_{v'}$. So, P goes through the zone $O_{v'}$ (when v' has type 1) or through the zone $E_{v'}$ (when v' has type 2). In particular $v' \notin \Pi$.

If $N_{\Pi}(v') \subseteq V(usPtv)$, then replacing $usPtv$ in Π with $uv'v$ gives a pyramid with fewer vertices than Π , a contradiction to the minimality of Π . So, v' has neighbors in Π that are not in $usPtv$. Note that if v' is of type 2, the ancestor of v' that is different from v is not in Π (because it is adjacent to t and to v , but $t \notin \Delta$ by Claim 5).

We denote by v'' the right neighbor of v' in P_{l-1} . Note that v'' has type 0, since v' has type 1 or 2. Let s' and t' be vertices such that $t'P_1s'$ is the right escape of v in $\text{BOX}_{v'}$, t' is adjacent to v and s' is adjacent to v'' . Note that s' is the leftmost vertex of $E_{v',v''}$ and t' is the rightmost vertex of the zone $O_{v',v}$ (when v' is of type 1) or of the rightmost zone $E_{v,w}$ (when v' is of type 2, and w is the other ancestor of v').

Let us see which vertex can be a neighbor of v' in $\Pi \setminus usPtv$. We already know it cannot be an ancestor of v' or be a vertex of $E_{u,v'} \setminus s$. Suppose it is v'' . Then, Π must contain two edges incident to v'' , and none of them can be an internal edge by Claim 7. Note that $s'v''$ must be an edge of Π , for otherwise, the two only available edges are $v''v'''$ and $v''s''$ (where v''' is the right neighbor of v'' in P_{l-1} and s'' is the rightmost neighbor of v'' in P_l), and this is a contradiction because $v'''s'' \in E(G)$. Since $s'v'' \in E(\Pi)$, Π goes through the path $R = usPtv't'P_1s'v''$. This path contains all vertices of $N_{\Pi}(v')$. Note that $v \notin \Delta$, because if so, one of t or t' should be in Δ , a contradiction to Claim 5. But v can be the apex. If v is not the apex, we may replace R by $uv'v''$ in Π , to obtain a pyramid that contradicts the minimality of Π . If v is the apex, then we may replace $R \setminus v$ by $uv'v''$ in Π , to obtain a pyramid with apex v' that contradicts the minimality of Π . Now we know that $v'' \notin V(\Pi)$.

Since v' has a neighbor in $\Pi \setminus usPtv$, and since this neighbor is not an ancestor of v' , is not v'' , and is not in $E_{u,v'}$, it must be in $\text{BOX}_{v'} \setminus (V(P) \cup E_{u,v'})$. By Claim 7, the only way that Π can contain some vertex of $\text{BOX}_{v'} \setminus (V(P) \cup E_{u,v'})$ is that Π goes through the edge vt' , in particular through the right escape of v in $\text{BOX}_{v'}$ and through the zone $E_{v',v''}$. Let t'' be the rightmost vertex of $E_{v',v''}$. Hence, Π must go through the path $S = usPtv't'P_1t''$. This path contains all vertices of $N_{\Pi}(v')$. Note that $v \notin \Delta$, because if so, one of t or t' should be in Δ , a contradiction to Claim 5. But v can be the apex. If v is not the apex, we may replace S by $uv't''$ in Π , to obtain a pyramid that contradicts the minimality of Π . If v is the apex, then we may replace $S \setminus v$ by $uv't''$ in Π , to obtain a pyramid with apex v' that contradicts the minimality of Π . \square

Tree-width and clique-width

For any $l \geq 0$, ttf-layered-wheels and ehf-layered-wheels on $l + 1$ layers contain K_{l+1} as a minor. To see this, note that each vertex in layer P_i , $i < l$, has neighbors in all layers $i + 1, \dots, l$ (see Lemma 3.2.2 and Lemma 3.2.7). Hence, by contracting each layer into a single vertex, a complete graph on $l + 1$ vertices is obtained. Since when H is a minor of G we have $\text{tw}(H) \leq \text{tw}(G)$, and since for $l \geq 1$, a complete graph on l vertices has tree-width $l - 1$, we obtain the following.

Theorem 3.2.12

For any $l \geq 0$, ttf-layered-wheels and ehf-layered-wheels on $l + 1$ layers have tree-width at least l .

Gurski and Wanke [GW00] proved that the tree-width is in some sense equivalent to the clique-width when some complete bipartite graph is excluded as a subgraph. Let us state and apply this formally (thanks to Sang-il Oum for pointing this out to us).

Theorem 3.2.13 (Gurski and Wanke [GW00])

If a graph G contains no $K_{3,3}$ as a subgraph, then $\text{tw}(G) \leq 6 \text{cw}(G) - 1$.

Lemma 3.2.14

A layered wheel (ttf or ehf) contains no $K_{3,3}$ as a subgraph.

Proof. Suppose that a ttf-layered-wheel G contains $K_{3,3}$ as a subgraph. Then, either it contains a theta (if $K_{3,3}$ is an induced subgraph of G) or it contains a triangle (if $K_{3,3}$ is not an induced subgraph of G). In each case, there is a contradiction.

Suppose that an ehf-layered-wheel G contains $K_{3,3}$ as a subgraph. If one side of the $K_{3,3}$ is a clique, then G contains a K_4 . Otherwise, each side of $K_{3,3}$ contains a non-edge, so G contains $K_{2,2}$, that is isomorphic to a C_4 . In both cases, there is contradiction. \square

Theorem 3.2.15

For any integers $l \geq 2, k \geq 4$, the clique-width of a layered wheel $G_{l,k}$ is at least $\frac{l+1}{6}$.

Proof. Follows from Lemma 3.2.14 and Theorems 3.2.13 and 3.2.12. \square

Observations and open questions

It should be pointed out that by carefully subdividing, one may obtain bipartite ttf-layered-wheels on any number l of layers. This is easy to prove by induction on l . We just sketch the main step of the proof: when building the last layer, assuming that the previous layers induce a bipartite graph, only the vertices with ancestors are assigned to one side of the bipartition (and only to one side, since a vertex has at most one ancestor in a ttf-layered-wheel). The parity of the paths linking vertices with ancestors can be adjusted to produce a bipartite graph.

It is easy to see that every prism, every theta, and every even wheel contains an even hole. Therefore, by Theorem 3.2.10 and Theorem 3.2.11, ehf-layered-wheels are (prism, pyramid, theta, even wheel)-free, which is not obvious from their definitions. Note that ehf-layered-wheels contain diamonds (see Conjecture 6.1.4).

However, we note that it is possible to modify Construction 3.2.6 in such a way that we obtain a layered wheel that is even-hole-free but contains a pyramid. Such a construction might be of interest to see what structure one can get in a even-hole-free graphs by studying how the graph attaches to a pyramid. The construction is done by modifying axiom (B5) where the two zones E_u 's are deleted. More specifically, if u is of type 2 (so it is an internal vertex of P_{l-1}), then let $v \in P_i$ and $w \in P_j$, $j \leq i$ be its ancestors. In this case, BOX_u is made of only 9 zones, namely $E_{u',u}, E_{v,w}, O_u, O_{u,v}, O_u, O_{u,w}, O_u, E_{v,w}$, and $E_{u,u''}$ (see Figure 3.7). The fact that this modified construction keeps the property of the layered wheel being even-hole-free can be proved similarly to Theorem 3.2.10. Notice that Lemma 3.2.9 also remains true for this modified construction. We remark that a corresponding wheel that is even-hole-free (similar to the one in Figure 3.4) exists considering this modified pattern of zones.

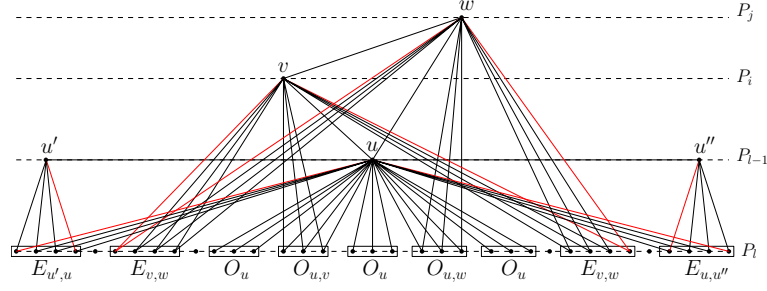


FIGURE 3.7: A modified construction of ehf-layered-wheel $G_{l,k}$ which contain pyramids (dashed lines between two vertices in P_l represent paths of odd length)

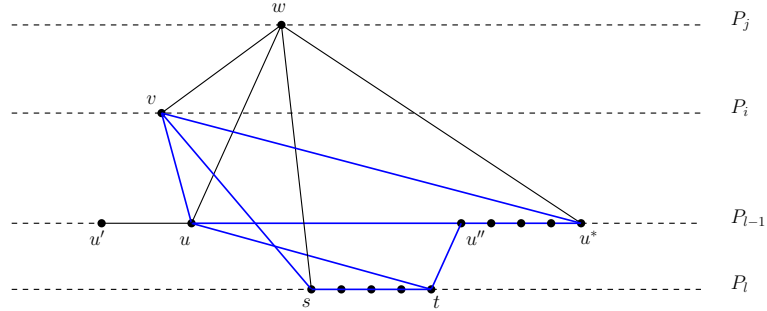


FIGURE 3.8: A pyramid (in blue) that is contained in a modified ehf-layered-wheel $G_{l,k}$, for some integers l, k

An example of a pyramid that may be found in such a modified ehf-layered-wheel is given in Figure 3.8. In the figure, $u \in P_{l-1}$ is a type 2 vertex with ancestors $v \in P_i$ and $w \in P_j$, $j < i$, u^* is a common neighbor of v and w in P_{l-1} such that u and u^* are consecutive common neighbors of v and w in some vw -zone in P_{l-1} , s is the rightmost vertex of a zone labeled $E_{v,w} \subseteq \text{BOX}_u$ in P_l ; and t is the leftmost vertex of the zone labeled $E_{u,u''} \subseteq \text{BOX}_u$ in P_l where $u'' \neq u'$ is adjacent to u in P_{l-1} . The pyramid has triangle utu'' and apex v .

3.3 Lower bound on rank-width

In this section, we prove that there exist ttf-layered-wheels and ehf-layered-wheels with arbitrarily large rank-width. This follows directly from Theorem 3.2.15 and Lemma 3.1.1, but by a direct computation, we provide a better bound. The next lemma follows directly from the definition of the rank-width defined in Chapter 1 Section 1.2.3.

Lemma 3.3.1

Let G be a graph and H be an induced subgraph of G . Then $\text{rw}(H) \leq \text{rw}(G)$.

A class \mathcal{C} of graphs has *bounded rank-width* if there exists a constant $k \in \mathbb{N}$, such that every $G \in \mathcal{C}$ satisfies $\text{rw}(G) \leq k$. If such a constant does not exist, then \mathcal{C} has *unbounded rank-width*. In the following lemmas, we present some basic properties related to rank-width. Let T be a tree, we call an edge $e \in E(T)$ *balanced*, if the

partition (A_e, B_e) of $L(T)$ satisfies $\frac{1}{3}|L(T)| \leq |A_e|$ and $\frac{1}{3}|L(T)| \leq |B_e|$. The following is well-known (we include a proof for the sake of completeness).

Lemma 3.3.2

Every cubic tree has a balanced edge.

Proof. Let T be a cubic tree with n leaves. We may assume that $n \geq 3$, for otherwise, T is a path of length 1, and the only edge of T is balanced.

Let $e = ab$ be an edge of T such that the set of leaves A_e of the connected component of $T \setminus e$ that contains a , satisfies $|A_e| \geq |L(T)|/3$. Suppose that a and b are chosen subject to the minimality of $|A_e|$. If $|A_e| \leq 2|L(T)|/3$, then e is balanced. Otherwise, $|A_e| > 2|L(T)|/3 \geq 2$ so a has two neighbors a', a'' different from b . Let A' (resp. A'') be the set of leaves of the connected component of $T \setminus aa'$ (resp. $T \setminus aa''$) that contains a' (resp. a''). Since $|A_e| > 2|L(T)|/3$ and $A_e = A' \cup A''$, either $|A'| > |L(T)|/3$ or $|A''| > |L(T)|/3$. Hence, one of A' or A'' contradicts the minimality of $|A_e|$. \square

Let us now introduce a notion that is useful to describe how we can represent the structure of layered wheels into a matrix. An $n \times n$ matrix M is *fuzzy triangular* if $m_{1,1} = 1$ and for every $i \in \{2, \dots, n\}$, $m_{i,i} = 1$ and either $m_{1,i} = m_{2,i} = \dots = m_{i-1,i} = 0$ or $m_{i,1} = m_{i,2} = \dots = m_{i,i-1} = 0$.

Lemma 3.3.3

Every $n \times n$ fuzzy triangular matrix has rank n .

Proof. Let M be an $n \times n$ fuzzy triangular matrix. We prove by induction on n , that $\text{rank}(M) = n$. For $n = 1$, this trivially holds. Suppose that $n \geq 2$. If $m_{1,n} = m_{2,n} = \dots = m_{n-1,n} = 0$, we show that rows r_1, \dots, r_n of M are linearly independent. Let $\lambda_1, \dots, \lambda_n \in [0, 1]$ be such that $\sum_{i=1}^n \lambda_i r_i = \mathbf{0}$ (where $\mathbf{0}$ is the zero vector of length n). Since $m_{n,n} = 1$, we have $\lambda_n = 0$. This implies that $\sum_{i=1}^{n-1} \lambda_i r'_i = \mathbf{0}$, where r'_i is the row obtained from r_i by deleting its last entry. Since r'_1, \dots, r'_{n-1} are the rows of an $(n-1) \times (n-1)$ fuzzy triangular matrix, they are linearly independent by the induction hypothesis, so $\lambda_1 = \dots = \lambda_{n-1} = 0$.

We can prove in the same way that, if $m_{n,1} = m_{n,2} = \dots = m_{n,n-1} = 0$, then the set of n columns of M is linearly independent. This shows that $\text{rank}(M) = n$. \square

Let G be a graph and (X, Y) be a partition of $V(G)$. A path P in G is *separated by (X, Y)* if $V(P) \cap X$ and $V(P) \cap Y$ are both non-empty. Note that when P is separated by (X, Y) , there exists a *separating edge* xy of P whose end-vertices are $x \in X$ and $y \in Y$.

Lemma 3.3.4

Let (T, λ) be a rank decomposition of width at most r of a layered wheel with layers P_0, P_1, \dots, P_l . Let e be an edge of T , and (X, Y) be the partition of $V(G)$ induced by $T \setminus e$. Then there are at most r paths among $\{P_0, P_1, \dots, P_l\}$ that are separated by (X, Y) .

Proof. Suppose for a contradiction that $P_{i_1}, \dots, P_{i_{r+1}}$ are layers that are all separated by (X, Y) , where $1 \leq i_1 < \dots < i_{r+1} \leq l$. For each integer i_j , consider a separating edge $x_{i_j}y_{i_j}$ of P_{i_j} such that $x_{i_j} \in X$ and $y_{i_j} \in Y$. Set $S_X = \{x_{i_1}, \dots, x_{i_{r+1}}\}$ and $S_Y = \{y_{i_1}, \dots, y_{i_{r+1}}\}$.

Consider $M[S_X, S_Y]$, the adjacency matrix whose rows are indexed by S_X and columns are indexed by S_Y . The definition of layered wheels (see (A6) and (B7)) says

that when two vertices in a layer are adjacent, at most one of them has ancestors. It follows that $M[S_X, S_Y]$ is fuzzy triangular. By Lemma 3.3.3, $M[S_X, S_Y]$ has rank $r + 1$, a contradiction, because

$$\text{width}(T, \lambda) \geq \text{cutrk}_G(X) = \text{rank}(M[X, Y]) \geq \text{rank}(M[S_X, S_Y]) = r + 1.$$

□

We need the following lemma in our proof.

Lemma 3.3.5 (See [Adl+17])

Let G be a graph and (T, λ) be a rank decomposition of G whose width is at most r . Let P be an induced path of G and (X, Y) be the partition of $V(G)$ induced by $T \setminus e$ where $e \in E(T)$. Then each of $P[X]$ and $P[Y]$ contains at most $r + 1$ connected components.

Now we are ready to describe layered wheels for which we can prove that the rank-width is unbounded. Let us first define some terminology that is used in the proof. Recall Construction 3.2.1 of ttf-layered-wheels. Let u and v be two vertices that are adjacent in a layer P_i for some $i \in \{1, \dots, l - 1\}$, and they appear in this order (from left to right) along P_i . Let a be the rightmost vertex of BOX_u and b be the leftmost vertex of BOX_v in P_{i+1} . Let a' (resp. b') be the neighbor of a (resp. b) in $P_{i+1} \setminus \text{BOX}_u$ (resp. $P_{i+1} \setminus \text{BOX}_v$). The path $a'P_{i+1}b'$ is called the *uv-bridge*. An edge pq in $a'P_{i+1}b'$ is called *the middle edge of the bridge* if the length of the paths $a'P_{i+1}p$ and $qP_{i+1}b'$ are equal.

We have a similar definition for ehf-layered-wheel. For adjacent vertices u and v in P_{i+1} , the *uv-bridge* in P_{i+1} is the zone labelled $E_{u,v} \subseteq \text{BOX}_u \cap \text{BOX}_v$ (that we called in the previous section a shared part). Observe that in both layered wheels, every internal vertex of some layer yields two bridges, and each end of a layer yields one bridge. We say that a layered wheel is *special* if every bridge in all layers has odd length (and therefore admits a middle edge). The following lemmas are a direct consequence of Construction 3.2.1 and Construction 3.2.6.

Lemma 3.3.6

For all integers $l \geq 1$ and $k \geq 4$, there exists a special (l, k) -ttf-layered-wheel.

Proof. The result follows because by (A6) of Construction 3.2.1, the path between BOX_u and BOX_v is of length at least $k - 2$. So for any two adjacent vertices in a layer, the *uv-bridge* can have any odd length, at least $k - 4$. □

Lemma 3.3.7

For all integers $l \geq 1$ and $k \geq 4$, any (l, k) -ehf-layered-wheel is special.

Proof. The result follows from the fact that the shared parts have odd length (see Lemma 3.2.9). □

Let $G_{l,k}$ be a layered wheel that is special. Let uv be an edge of some layer P_i , where $1 \leq i < l$, such that u and v appear in this order (from left to right) along P_i . Then we denote by $r_u l_v$ the middle edge of the *uv-bridge* (again, r_u and l_v appear in this order from left to right).

For any vertex $v \in P_i$, $1 \leq i < l$, the *domain of v* (or the *v-domain*), denoted by $\text{DOM}(v)$ is defined as follows:

- if $v \in V(P_0)$, then $\text{DOM}(v) = V(P_1)$;
- if v is an internal vertex of P_i , then $\text{DOM}(v) = V(l_v P_{i+1} r_v)$;
- if v is the left end of P_i , then $\text{DOM}(v) = V(p P_{i+1} r_v)$, where p is the leftmost vertex of BOX_v ; and
- if v is the right end of P_i , then $\text{DOM}(v) = V(l_v P_{i+1} q)$, where q is the rightmost vertex of BOX_v .

Note that for ttf-layered-wheels, BOX_v is completely contained in the v -domain, which is not the case for ehf-layered-wheels. We are now ready to describe the layered wheels that we need.

Definition 3.3.8. For some integer m , a special layered wheel $G_{l,k}$ is *m-uniform*, if for every vertex $v \in V(P_i)$, $0 \leq i \leq l-1$, $\text{DOM}(v)$ contains exactly m vertices.

Observe that by definition, any m -uniform layered wheel is special.

Lemma 3.3.9

For all integers $l \geq 1$ and $k \geq 4$ and M , there exists an integer $m \geq M$ and an (l, k) -ttf-layered-wheel that is m -uniform.

Proof. We construct an m -uniform ttf-layered-wheel $G_{l,k}$ by adjusting the length obtained in step (A6) of Construction 3.2.1. \square

Lemma 3.3.10

For all integers $l \geq 1$ and $k \geq 4$ and M , there exists an integer $m \geq M$ and an (l, k) -ehf-layered-wheel that is m -uniform.

Proof. We construct an m -uniform ehf-layered-wheel $G_{l,k}$ by adjusting the length obtained in step (B7) of Construction 3.2.6. \square

For a vertex $v \in P_i$, $0 \leq i \leq l$ and an integer $0 \leq d \leq l-i$, the *v-domain of depth d* , denoted by $\text{DOM}^d(v)$ is defined as follows.

- $\text{DOM}^0(v) = \{v\}$ and $\text{DOM}^1(v) = \text{DOM}(v)$;
- $\text{DOM}^d(v) = \bigcup_{x \in \text{DOM}(v)} \text{DOM}^{d-1}(x)$ for $d \geq 1$.

Observation 3.3.11. For every $v \in P_i$ with $0 \leq i \leq l$, and for any $0 \leq d \leq l-i$, we have $\text{DOM}^d(v) \subseteq V(P_{i+d})$, where the equality holds when $i = 0$.

Lemma 3.3.12

For every $0 \leq i \leq l$ and $0 \leq d \leq i$, $V(P_i) = \bigcup_{v \in P_{i-d}} \text{DOM}^d(v)$. Moreover, for any distinct $u, v \in V(P_{i-d})$, $\text{DOM}^d(u) \cap \text{DOM}^d(v) = \emptyset$.

Proof. The statement simply follows by induction on d . \square

Lemma 3.3.13

For some integers l, k, m , let $G_{l,k}$ be an m -uniform layered wheel. For every $0 \leq i \leq l-1$, $v \in P_i$, and $1 \leq d \leq l-i$, we have $|\text{DOM}^d(v)| = m^d$.

Proof. The statement simply follows from Lemma 3.3.12 and the m -uniformity: for any vertex v , $|\text{DOM}^1(v)| = m$ and $|\text{DOM}^d(v)| = m \cdot |\text{DOM}^{d-1}(v)|$. \square

Lemma 3.3.14

For some integers l, k, m , let $G_{l,k}$ be an m -uniform layered wheel. Denote by $G_{i,k}$, the subgraph induced by the first $i + 1$ layers P_0, P_1, \dots, P_i of $G_{l,k}$. Then $|V(G_{i,k})| < \frac{1}{m-1}|V(P_{i+1})|$ for $0 \leq i \leq l - 1$.

Proof. Recall that $V(P_i) = \text{DOM}^i(r)$ for every $1 \leq i \leq l$, with $r \in V(P_0)$. So by Lemma 3.3.13, $|V(P_i)| = m^i$. Moreover, $|V(G_{i,k})| = \sum_{d=0}^i |\text{DOM}^d(r)| = \frac{m^{i+1}-1}{m-1}$. Hence, the result follows directly. \square

Lemma 3.3.15

Let $l \geq 2, k \geq 4$, and $m \geq 15$ be integers, and (T, λ) be a rank decomposition of an m -uniform layered wheel $G_{l,k}$ of width at most r . Let e be a balanced edge in T , and (X, Y) be the partition of $V(G_{l,k})$ induced by e . Then P_l is separated by (X, Y) , and each of X and Y contains an induced subpath of P_l , namely P_X and P_Y where:

$$|V(P_X)|, |V(P_Y)| \geq \left\lfloor \frac{|V(P_l)|}{3.5(r+1)} \right\rfloor.$$

Proof. Let first prove that P_l is separated by (X, Y) . By Lemma 3.3.14, we know that $|V(P_l)| > (m-1)|V(G_{l-1,k})|$ where $G_{l-1,k} = G_{l,k} \setminus P_l$. Since $m-1 \geq 14$, we have $|V(P_l)| > \frac{14}{15}|V(G_{l,k})|$. Hence, P_l cannot be fully contained in X , for otherwise $|Y| < \frac{1}{15}|V(G_{l,k})|$ that would contradict the fact that (X, Y) is a balanced decomposition. By the same reason, P_l is not fully contained in Y . This proves the first statement.

For the second statement, we will only prove the existence of P_X (for P_Y , the proof is similar). Since e is a balanced edge of T , we have $|X| \geq \frac{1}{3}|V(G_{l,k})|$. Clearly,

$$|V(P_l) \cap X| \geq \frac{1}{3}|V(G_{l,k})| - |V(G_{l-1,k})| = \frac{1}{3}(|V(P_l)| - 2|V(G_{l-1,k})|).$$

By Lemma 3.3.5, X contains at most $r + 1$ connected components of P_l . Hence:

$$\begin{aligned} |V(P_X)| &\geq \frac{|V(P_l) \cap X|}{r+1} \\ &> \frac{|V(P_l)| - \frac{2}{m-1}|V(P_l)|}{3(r+1)} \end{aligned} \quad (1)$$

$$\begin{aligned} &= \frac{m-3}{3(m-1)(r+1)} |V(P_l)| \\ &\geq \frac{2}{7(r+1)} |V(P_l)| \end{aligned} \quad (2)$$

Inequality (1) is obtained from Lemma 3.3.14, and (2) follows because $m \geq 15$. \square

The following theorem is the main result of this section.

Theorem 3.3.16

For $l \geq 2, k \geq 4$, there exists an integer m such that the rank-width of an m -uniform layered wheel $G_{l,k}$ is at least l .

Proof. Set $M = 15$ and consider an integer m as in Lemma 3.3.9 (or Lemma 3.3.10), and let $G_{l,k}$ be m -uniform.

Suppose for a contradiction that $\text{rw}(G_{l,k}) = r$ for some integer $r \leq l - 1$. Let (T, λ) be a rank decomposition of $G_{l,k}$ of width r , and e be a balanced edge of T that partitions $V(G_{l,k})$ into (X, Y) . Let $\mathcal{P} = \{P_0, P_1, \dots, P_l\}$ be the set of layers in the layered wheel, and \mathcal{S} be the set of paths in \mathcal{P} that are separated by (X, Y) . By Lemma 3.3.4, $|\mathcal{S}| \leq r$.

Note that $P_0 \notin \mathcal{S}$ because it contains a single vertex. So, $\mathcal{P} \setminus \mathcal{S} \neq \emptyset$. Let $P_j \in \mathcal{P} \setminus \mathcal{S}$, i.e., the vertices of P_j are completely contained either in X or Y . Without loss of generality, we may assume that $V(P_j) \subseteq X$.

Claim 1. There exists some j such that $1 \leq j < l$.

Proof of Claim 1. Note that $l - r \geq 1$, because $r \leq l - 1$. So it is enough to prove that such a $j \geq l - r$ exists. We know that $|\mathcal{S}| \leq r \leq l - 1$. If every path $P_j \in \mathcal{P} \setminus \mathcal{S}$ has index $j < l - r$, then $|\mathcal{P} \setminus \mathcal{S}| \leq l - r$. This implies $|\mathcal{S}| \geq (l + 1) - (l - r) = r + 1$, a contradiction, so the left inequality of the statement holds (the bound is tight when $\mathcal{S} = \bigcup_{l-r+1 \leq i \leq l} \{P_i\}$). Furthermore, by Lemma 3.3.15, $P_l \in \mathcal{S}$, so for every P_j that satisfies the left inequality, we know that $j < l$. This proves Claim 1.

Now by Lemma 3.3.15, there exists a subpath P_Y of P_l , such that $V(P_Y) \subseteq Y$ and $|V(P_Y)| \geq \left\lfloor \frac{|V(P_l)|}{3.5(r+1)} \right\rfloor$, with $|V(P_l)| = m^l$ (because $|V(P_l)| = \text{DOM}^l(r)$ where $r \in P_0$).

Let P' be the set of vertices in P_j such that $N(v) \cap V(P_Y) \neq \emptyset$ for every $v \in P'$. Note that the order (left to right) of the domains of $V(P_j)$ in layer P_l appear in the same order as the order of $V(P_j)$ in P_j , and by Lemma 3.3.12, for every $v \neq v' \in P_j$, we have $\text{DOM}^{l-j}(v) \cap \text{DOM}^{l-j}(v') = \emptyset$. So P' induces a path. Moreover, for each vertex $v \in P'$, we can fix a vertex $y_v \in V(P_Y) \cap \text{DOM}^{l-j}(v)$, such that $vy_v \in E(G)$. Thus for any $v \neq v' \in P'$, we have $y_v \neq y_{v'}$, and in particular, $vy_v, v'y_{v'} \in E(G)$ and $v'y_v, vy_{v'} \notin E(G)$. Let us denote $S_X = V(P')$ and $S_Y = \{y_v \mid v \in S_X\}$. Observe that there is a bijection between S_X and S_Y , so $M[S_X, S_Y]$ is the identity matrix of size $|S_X|$.

Furthermore, by Lemmas 3.3.12 and 3.3.13, we have $|S_X| \geq \left\lfloor \frac{|V(P_Y)|}{|\text{DOM}^{l-j}(v)|} \right\rfloor = \left\lfloor \frac{|V(P_Y)|}{m^{l-j}} \right\rfloor$. By Claim 1, Lemma 3.3.15, and taking $m \geq 4l^2$, the following holds.

$$|S_X| \geq \left\lfloor \frac{m^l}{3.5(r+1)m^{l-j}} \right\rfloor \geq \left\lfloor \frac{m^j}{3.5(r+1)} \right\rfloor \geq \left\lfloor \frac{m}{3.5l} \right\rfloor \geq \left\lfloor \frac{3.5l^2}{3.5l} \right\rfloor \geq l$$

which yields a contradiction, because

$$r \geq \text{width}(T, \lambda) \geq \text{cutrk}_{G_{l,k}}(X) = \text{rank}(M[X, Y]) \geq \text{rank}(M[S_X, S_Y]) = |S_X| \geq l.$$

□

3.4 Upper bound

Layered wheels have an exponential number of vertices in terms of the number of layers l . In Section 3.2, we have seen that the tree-width of layered wheels is lower-bounded by l . In this section, we give an upper bound of the tree-width of layered wheels. As mentioned in the introduction, we indeed prove a stronger result: the so-called *path-width* of layered wheels is upper-bounded by some linear function of l . Since layered wheels $G_{l,k}$ contain an exponential number of vertices in terms of the number of layers, this implies that $\text{tw}(G_{l,k}) = \mathcal{O}(\log |V(G_{l,k})|)$. Beforehand, let us state some useful notions.

Path-width

A *path decomposition* of a graph G is defined similarly to a tree decomposition except that the underlying tree is required to be a path. Similarly, the width of the path decomposition is the size of a largest bag minus one, and the path-width is the minimum width of a path decomposition of G . The *path-width* of a graph G is denoted by $\text{pw}(G)$. As outlined in the introduction, path decomposition is a special case of tree decomposition. We restate the following lemma that was already mentioned in Lemma 3.1.1.

Lemma 3.4.1

For any graph G , $\text{tw}(G) \leq \text{pw}(G)$.

Let P be a path, and P_1, \dots, P_k be subpaths of P . The *interval graph* associated to P_1, \dots, P_k is the graph whose vertex set is $\{P_1, \dots, P_k\}$ with an edge between any pair of paths sharing at least one vertex. So, interval graphs are intersection graphs of a set of subpaths of a path.

Lemma 3.4.2 (See Theorem 7.14 of [Cyg+15])

Let G be a graph, and I be an interval graph that contains G as a subgraph (possibly not induced). Then $\text{pw}(G) \leq \omega(I) - 1$, where $\omega(I)$ is the size of the maximum clique of I .

Now, for every layered wheel $G_{l,k}$, we describe an interval graph $I(G_{l,k})$ such that $G_{l,k}$ is a subgraph of $I(G_{l,k})$. We define the *scope* of a vertex. This is similar to its domain, but slightly different (the main difference is that scopes may overlap while domains do not). For $v \in V(P_i)$, where $0 \leq i \leq l-1$, the scope of v , denoted by $\text{SCP}(v)$, is defined as follows.

For a ttf-layered-wheel:

- if $v \in P_0$, $\text{SCP}(v) = V(P_1)$;
- if v is in the interior of P_i , then $\text{SCP}(v) = V(L) \cup \text{BOX}_v \cup V(R)$, where u and w are the left and the right neighbors of v in P_i respectively, and L is the uv -bridge and R is the vw -bridge;
- if v is the left end of P_i , then $\text{SCP}(v) = \text{BOX}_v \cup V(R)$ where w is the right neighbor of v in P_i and R is the vw -bridge;
- if v is the right end of P_i , then $\text{SCP}(v) = V(L) \cup \text{BOX}_v$, where u is the left neighbor of v in P_i and L is the uv -bridge.

For an ehf-layered-wheel:

- $\text{SCP}(v) = \text{BOX}_v$ for every $v \in P_i$, $0 \leq i \leq l-1$.

For $d \geq 0$, we also define the *depth- d scope* of each vertex in the layered wheel, which will be denoted by $\text{SCP}^d(v)$. We define $\text{SCP}^0(v) = \{v\}$, and

$$\text{SCP}^d(v) = \bigcup_{x \in \text{SCP}(v)} \text{SCP}^{d-1}(x) \text{ for } 1 \leq d \leq l-i.$$

For a layered wheel $G_{l,k}$, we define the interval graph $I(G_{l,k})$. For every vertex $v \in G_{l,k}$, define path $P(v)$ associated to v as follows:

- if $v \in P_l$ is not the right end of P_l , then $P(v) = vw$ where w is the right neighbor of v ;
- if v is the right end of P_l , then $P(v) = \{v\}$;
- if $v \in P_i$ with $i < l$, then $P(v) = P_l \left[\text{SCP}^{l-i}(v) \right]$.

Note that $P(v)$ is a subpath of P_l . The graph $I(G_{l,k})$ is the interval graph associated to $\{P(v) \mid v \in V(G_{l,k})\}$.

Lemma 3.4.3

For any layered wheel $G_{l,k}$ and the corresponding interval graph $I(G_{l,k})$, $G_{l,k}$ is a subgraph (possibly not induced) of $I(G_{l,k})$.

Proof. It is clear by definition that there is a bijection between $V(I(G_{l,k}))$ and $V(G_{l,k})$. We show that $E(G_{l,k}) \subseteq E(I(G_{l,k}))$: for any two vertices $u, v \in G_{l,k}$, if $uv \in E(G_{l,k})$ then the corresponding paths $P(u)$ and $P(v)$ share at least one vertex (i.e. $V(P(u)) \cap V(P(v)) \neq \emptyset$).

For $u, v \in P_l$ where u is on the left of v , this property trivially holds, because by definition, $P(u)$ and $P(v)$ both contain v .

If $u \in P_i$ for some $i < l$ and $v \in P_l$, then $V(P(v)) \subseteq V(P(u)) = \text{SCP}^{l-i}(u)$. The argument is similar when $v \in P_i$ for some $i < l$ and $u \in P_l$.

If $u, v \in P_i$ for some $i < l$, then by definition, $\text{SCP}(u) \cap \text{SCP}(v) \neq \emptyset$ (they both contain the uv -bridge). Let $x \in \text{SCP}(u) \cap \text{SCP}(v)$. Note that for $1 \leq d \leq l-i$, $\text{SCP}^d(u)$ and $\text{SCP}^d(v)$ both contain $\text{SCP}^{d-1}(x)$. If $u \in P_i$ and $v \in P_j$ where $1 \leq i < j < l$, then $\text{SCP}(v) \subseteq \text{SCP}^{j-i+1}(u)$. So $\text{SCP}^d(v) \subseteq \text{SCP}^{d+j-i}(u)$ for every $1 \leq d \leq l-j$. The argument is similar when $u \in P_j$ and $v \in P_i$ where $1 \leq i < j < l$. Hence, $V(P(u)) \cap V(P(v)) \neq \emptyset$. \square

Theorem 3.4.4

For all integers $l \geq 2$ and $k \geq 4$, we have $\text{tw}(G_{l,k}) \leq \text{pw}(G_{l,k}) \leq 2l$.

Proof. By Lemmas 3.4.1 (third item), 3.4.2 and 3.4.3, it is enough to show that $\omega(I(G_{l,k})) \leq 2l + 1$.

Claim 1. Let u and v be non-adjacent vertices in P_i for some $1 \leq i \leq l-1$. Then for any $1 \leq d \leq l-i$, we have $\text{SCP}^d(u) \cap \text{SCP}^d(v) = \emptyset$.

Proof of Claim 1. Let u and v be non-adjacent vertices in P_i , where $1 \leq i \leq l-1$ and without loss of generality, they appear in this order (from left to right) along P_i . We prove the statement by induction on d .

For $d = 1$, it follows from the definition that $\text{SCP}^1(u) \cap \text{SCP}^1(v) = \emptyset$ for every possible i . Suppose for induction that $\text{SCP}^d(u) \cap \text{SCP}^d(v) = \emptyset$ for some $1 \leq d \leq l-i-1$. Note that $\text{SCP}^d(u)$ and $\text{SCP}^d(v)$ appear in this order along P_{i+d} . Moreover, the right end of $\text{SCP}(u)$ and the left end of $\text{SCP}(v)$ are also non-adjacent (because they both are vertices with an ancestor). So for any $x \in \text{SCP}(u)$ and $y \in \text{SCP}(v)$, we have $xy \notin E(G_{l,k})$. It then follows by construction, that for every $d \geq 2$, for any $x \in \text{SCP}^d(u)$ and $y \in \text{SCP}^d(v)$, we have $xy \notin E(G_{l,k})$, so the induction hypothesis holds for the pair x and y . We need to show that $\text{SCP}^{d+1}(u) \cap \text{SCP}^{d+1}(v) = \emptyset$. Indeed:

$$\text{SCP}^{d+1}(u) \cap \text{SCP}^{d+1}(v) = \bigcup_{x \in \text{SCP}(u)} \text{SCP}^d(x) \cap \bigcup_{y \in \text{SCP}(v)} \text{SCP}^d(y) = \emptyset,$$

which completes our induction. This proves Claim 1.

Let K be a maximum clique in $I(G_{l,k})$. By definition, for every $u, v \in P_l$ that are non-adjacent, we have $V(P(u)) \cap V(P(v)) = \emptyset$. So no edge exists between P_u and P_v in $I(G_{l,k})$. Similarly for non-adjacent vertices $u, v \in P_i$ where $1 \leq i \leq l-1$, it follows from Claim 1, that $V(P(u)) \cap V(P(v)) = \emptyset$. Therefore, K contains at most two vertices of every layer P_i , with $1 \leq i \leq l$. Since K may also contain the unique vertex in P_0 , then $\omega(I(G_{l,k})) \leq 2l + 1$ as desired. \square

The following is immediate.

Corollary 3.4.5

For any integers $l \geq 2$ and $k \geq 4$, we have $\text{tw}(G_{l,k}) = \mathcal{O}(\log |V(G_{l,k})|)$.

Proof. By Lemma 3.2.2 and Lemma 3.2.7, we know that $G_{l,k}$ contains at least $c \cdot 3^l$ vertices for some integer $c \geq 3$. Hence by Theorem 3.4.4, we have $\text{tw}(G_{l,k}) \leq 2l \leq c' \cdot \log |V(G_{l,k})|$ for some constant $c' > 0$. \square

3.5 Discussion and open problems

High lower bound on the tree-width of layered wheels requires the presence of large clique minor. Moreover, the existence of a large clique minor in a layered wheel forces the graph to contain a vertex of high degree. These facts arise the following two principal questions:

- Do bounds on the tree-width of even-hole-free graphs exist when the graphs have no large clique minor?
- Do bounds on the tree-width of even-hole-free graphs exist when the maximum degree is bounded?

Those two questions are both answered positively [Abo+cs; ACV20]. We give an overview of the theorem answering the first question.

Excluding big clique minor. Recall that even-hole-free graphs with no clique of size 4 may have arbitrarily large tree-width, and the construction of layered wheels that we presented in Chapter 3 must contain a clique minor of big size in order to increase the bound on the tree-width. It is known that planar even-hole-free graphs have bounded tree-width [SSS10], and planar graphs do not contain K_ℓ as a minor for $\ell \geq 5$. It is then natural to ask if this condition is necessary, i.e. *does excluding large clique minor imply bounded tree-width on even-hole-free graphs?* A positive answer to the aforementioned question is given in a paper I wrote jointly with Aboulker, Adler, Kim, and Trotignon [Abo+cs], where the following theorem is proved. In this thesis, we do not explain the details of this result. Interested readers should refer to [Abo+cs] for further explanation.

Theorem 3.5.1 ([Abo+cs])

There is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that every even-hole-free graph not containing K_ℓ as a minor has tree-width at most $f(\ell)$.

In [ACV20] it is proved that even-hole-free graphs with maximum degree t have bounded tree-width. A discussion related to the class of even-hole-free graphs with bounded maximum degree will be given in Chapter 5.

Chapter 4

A bound on the tree-width

In the previous chapter, we have seen a construction that we call *layered wheel*, showing that the rank-width and the tree-width of (theta, triangle)-free graphs and (even hole, K_4)-free graphs are unbounded. We have also seen that the family of layered wheels for both classes share a similar characteristic in some sense. Our main concerns in this chapter are the following:

- What are the structures that enforce the unboundedness of the tree-width (as well as the rank-width) of graphs in the two classes, in the sense that excluding them bounds the tree-width?
- Understanding the structure of even-hole-free graphs and theta-free graphs.

In this chapter, we prove that when excluding more induced subgraphs, there is an upper bound on the tree-width. Our results imply that the maximum independent set problem can be solved in polynomial time for some classes of graphs that are possibly of interest because they are related to several well known open questions in the field.

Notation

Let us give some definitions that will be used throughout the chapter.

For two vertices $s, t \in V(G)$, a set $X \subseteq V(G)$ is an *st-separator* if $s, t \notin X$, and s and t lie in different connected components of $G \setminus X$. An *st-separator* X is a *minimal st-separator* if it is an inclusion-wise minimal *st-separator*. A set $X \subseteq V(G)$ is a *separator* if there exist $s, t \in V(G)$ such that X is an *st-separator* in G . A set $X \subseteq V(G)$ is a *minimal separator* if there exist $s, t \in V(G)$ such that X is a minimal *st-separator* in G .

When $A, B \subseteq V(G)$, we denote by $N_B(A)$ the set of vertices of $B \setminus A$ that have at least one neighbor in A . Note that $N_B(A)$ is disjoint from A . We write $N(a)$ instead of $N(\{a\})$ and $N[a]$ for $\{a\} \cup N(a)$. To avoid too heavy notation, since there is no risk of confusion, when H is an induced subgraph of G , we write N_H instead of $N_{V(H)}$.

A vertex x is *complete* (resp. *anticomplete*) to A if $x \notin A$ and x is adjacent to all vertices of A (resp. to no vertex of A). We say that A is *complete* (resp. *anticomplete*) to B if every vertex of A is complete (resp. anticomplete) to B (note that this means in particular that A and B are disjoint).

Outline of the chapter

In Section 4.1, we address some known results on the clique-widths of some hereditary graph classes related to the classes that we study, and we state the main results of our work. In Section 4.2, we explain our method to bound the tree-width. In

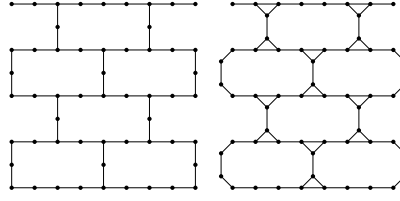


FIGURE 4.1: A subdivision of a wall and its line graph

Section 4.3, we give two technical lemmas that highlight structural similarities between (theta, triangle)-free graphs and (even hole, pyramid)-free graphs. These will be used in Section 4.4 where we prove that graphs in our classes do not contain minimal separators of large cardinality, implying that their tree-width is bounded.

4.1 Known results and summary of the main results

We denote by P_k the path on k vertices. For three non-negative integers $i \leq j \leq k$, let $S_{i,j,k}$ be the tree with a vertex v , from which start three paths with i , j , and k edges respectively. Note that $S_{0,0,k}$ is a path of length k (so, is equivalent to P_{k+1}) and that $S_{0,i,j} = S_{0,0,i+j}$, and the claw is the graph $S_{1,1,1}$. Note that $\{S_{i,j,k}; 1 \leq i \leq j \leq k\}$ is the set of all the subdivided claws and $\{S_{i,j,k}; 0 \leq i \leq j \leq k\}$ is the set of all subdivided claws and paths.

The following results are extracted from [DP16] (but some of them were first proved in other works). Let $\mathcal{H}_U = \{P_7, S_{1,1,4}, S_{2,2,2}\}$ and $\mathcal{H}_B = \{P_6, S_{1,1,3}\}$. If H contains a graph from \mathcal{H}_U as an induced subgraph, then the class of (triangle, H)-free graph has unbounded clique-width (see Theorem 7.ii.6 in [DP16]). If H is contained in a graph from \mathcal{H}_B , then the class of (triangle, H)-free graphs has bounded clique-width (see Theorem 7.i.3 in [DP16]). Moreover, the clique-width of (triangle, $S_{1,2,2}$)-free graphs is bounded, see [BMM16] or [DDP17].

It is easy to provide (theta, K_4 , $S_{1,1,1}$)-free graphs (or equivalently (claw, K_4)-free graphs) of unbounded clique-width. To do so, consider a wall W , subdivide all edges to obtain W' , and take the line graph $L(W')$ (see Figure 4.1).

Our main result states that for all fixed non-negative integers i, j, k, t , the following graph classes have bounded tree-width:

- (theta, triangle, $S_{i,j,k}$)-free graphs;
- (even hole, pyramid, K_t , $S_{i,j,k}$)-free graphs.

The exact bounds and the proofs are given in Section 4.4 (Theorems 4.4.6 and 4.4.7). In fact, the class on which we actually work is larger. It is a common generalization \mathcal{C} of the graphs that we have to handle in the proofs for the two bounds above. Also, we do not exclude $S_{i,j,k}$, but some graphs that contain it, namely the so-called l -span-wheels for sufficiently large l . We postpone the definitions of \mathcal{C} and of span wheels to Section 4.4. To bound the tree-width, we prove that every graph of large tree-width must contain a large clique or a minimal separator of large cardinality, which were defined above.

Our graphs have no large cliques by definition, and by studying their structure, we prove that they cannot contain large minimal separators, implying that their tree-width is bounded. Note that from the celebrated grid-minor theorem, it is easy to see that every graph of large tree-width contains a subgraph with a large minimal

separator (a column in the middle of the grid contains such a separator). But since we are interested in the induced subgraph containment relation, we cannot delete edges, so the aforementioned separator does not always work. In this case, we have to rely on our reinforcement to obtain a large minimal separator (see Theorems 4.2.3 and 4.2.4).

Algorithmic consequences

It is proved in [Cou90] that in every class of graphs with bounded tree-width, many algorithmic problems (such as the optimal coloring or the maximum independent set problems) can be solved in polynomial time. Therefore, our result has applications to several problems, but we here focus on one because the induced subgraphs that are excluded in the most classical results and open questions about it seem to be related to our classes. Our results imply that computing an independent set of maximum cardinality can be performed in polynomial time for $(\theta, \text{triangle}, S_{i,j,k})$ -free graphs and (even hole, pyramid, K_t , $S_{i,j,k}$)-free graphs.

Finding an independent set of maximum cardinality is polynomial-time solvable for (even hole, triangle)-free graphs [Cam+18] and (even hole, pyramid)-free graphs [Chu+19]. The complexity of this problem is not known for (even hole, K_4)-free graphs and for $(\theta, \text{triangle})$ -free graphs. Determining its complexity is also a well known question for $S_{i,j,k}$ -free graphs. It is NP-hard for the class of H -free graphs whenever H is not an induced subgraph of some $S_{i,j,k}$ [Ale83]. It is solvable in polynomial time for H -free graphs whenever H is contained in P_k for $k = 6$ (for $H = P_5$ see [LVV14] and for $H = P_6$ see [Grz+19]) or contained in $S_{i,j,k}$ with $(i, j, k) \leq (1, 1, 2)$ (see [Ale04] and [LM08] for the weighted version). It is solvable in polynomial time for $(P_7, \text{triangle})$ -free graphs [BM18a] and for $(S_{1,2,4}, \text{triangle})$ -free graphs [BM18b]. The complexity is not known for H -free graphs whenever H is some $S_{i,j,k}$ that contains either P_7 , $S_{1,1,3}$, or $S_{1,2,2}$.

4.2 Tree-width and minimal separators

If a graph has large tree-width, then it contains some sub-structure that is highly connected in some sense (grid minor, bramble, tangle, see [HW17]). Theorem 4.2.4 seems to be a new statement of that kind. It says that graphs of large tree-width must contain either a large clique or a minimal separator of large size. However, its converse is false, as shown by $K_{2,t}$ that has tree-width 2 (it is a series-parallel graph) and contains a minimal separator of size t .

The main theorem in this section was suggested by Stéphan Thomassé and improved by Marcin Pilipczuk. We will give the proof of both variants. The first variant of the theorem is obtained from the celebrated excluded grid minor theorem of Robertson and Seymour. The idea is to use a large grid to obtain a large minimal separator. However, there are technicalities because we are not allowed to delete edges, so the grid might contain many crossing edges. To find two vertices that cannot be separated by a small separator, one needs to clean the grid. In the following, we give a proof of such a theorem (cf. Theorem 4.2.3). To support the proof, we first need to mention the following variant of Menger's theorem.

Lemma 4.2.1

If $A, B \subseteq V(G)$ are two disjoint sets which are anticomplete to each other, $G[A]$, $G[B]$ are connected, and there exist k internally vertex disjoint $A - B$ paths, then

there exists an induced subgraph H of G admitting a minimal separator of size at least k .

Proof. Let C be a cutset in G such that one connected component A' of $G \setminus C$ contains A , and another one B' contains B , and suppose C is minimal w.r.t. this property. Note that C exists since $V(G) \setminus (A \cup B)$ is such a cutset. Every vertex in C has a neighbor in both A' , B' , for otherwise, a vertex from C with no neighbor in A' or B' could be removed from C , contradicting its minimality.

We now set $H = G[A' \cup C \cup B']$, and we observe that C is a minimal separator that separates A' and B' in H . Since C must contain at least one vertex in each of the internally vertex disjoint paths from A to B , we have $|C| \geq k$. \square

As stated above, we furthermore rely on the grid-minor theorem. The best function for the grid-minor theorem known so far is the following, given by Chuzhoy.

Theorem 4.2.2 ([Chu16])

If G has tree-width at least $f(k)$ with $f(k) = \Omega((k \text{ poly log } k)^{19})$, then G contains a $(k \times k)$ -grid as a minor.

Theorem 4.2.3

Let G be a graph and k be a positive integer. If G does not contain a clique on $2k$ vertices or an induced subgraph that admits a minimal separator of size k , then the tree-width of G is $\mathcal{O}((2k \text{ poly log } 2k)^{19})$.

Proof. Suppose that $\text{tw}(G) = \Omega((2k \text{ poly log } 2k)^{19})$. By Theorem 4.2.2, G contains a $k \times 2k$ grid minor. So, G contains model for it, namely $k \times 2k$ induced subgraphs $H_{i,j}$, for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, 2k\}$, and there exists an edge between $H_{i,j}$ and $H_{i',j'}$ when (i, j) and (i', j') are adjacent points in the integer grid.

Consider two distinct columns A and B of the grid. So, $A = \bigcup_{j=1}^k V(H_{a,j})$ and $B = \bigcup_{j=1}^k V(H_{b,j})$. If there are no edges between A and B , then they form disjoint connected induced subgraphs of G , and obviously, there exist k internally disjoint paths linking them (each of these paths go across the k rows of the grid). So, we are done by Lemma 4.2.1. Therefore, we may assume there exists an edge between every pair of columns of the grid. Hence, G contains K_{2k} as a minor.

Let $H_i = \bigcup_{j=1}^k V(H_{i,j})$ for $i = 1, \dots, 2k$. Note that, H_1, \dots, H_{2k} is a model of K_{2k} in G . Suppose that $H_1 \cup \dots \cup H_{2k}$ is minimal w.r.t. this property. If for all $i = 1, \dots, 2k$ we have $|V(H_i)| = 1$, then G contains a clique of size $2k$ as a subgraph, so suppose up to symmetry that $|V(H_{2k})| \geq 2$. Let $u, v \in V(H_{2k})$ be two leaves of some spanning tree of H_{2k} . It follows that $H_{2k} \setminus u$ and $H_{2k} \setminus v$ are both connected. By the minimality of $H_1 \cup \dots \cup H_{2k}$, $H_{2k} \setminus v$ is anticomplete to H_i for some $i \in \{1, \dots, 2k-1\}$. Also, $H_{2k} \setminus u$ is anticomplete to $H_{i'}$ for some $i' \in \{1, \dots, 2k\}$. Since $N_G(H_{2k}) \cap V(H_j) \neq \emptyset$ for all $j \in \{1, \dots, 2k-1\}$, we know that one of $H_{2k} \setminus u$ or $H_{2k} \setminus v$ contains vertices with neighbors in at least k sets among $V(H_1), \dots, V(H_{2k-1})$. Up to a relabelling, we may assume that $N(H_{2k} \setminus v)$ has a non-empty intersection with each of H_1, \dots, H_k . Moreover, we suppose $i = k+1$, that is, $N(H_{2k} \setminus v)$ has an empty intersection with H_{k+1} .

Now, $A = V(H_{2k} \setminus v)$ and $B = V(H_{k+1})$ are two connected subsets, and they can be linked by k internally vertex-disjoint paths (one through each H_j , $j = 1, \dots, k$). By Lemma 4.2.1, G contains an induced subgraph with a minimal separator of size at least k . This completes the proof. \square

Thanks to Marcin Pilipczuk, we have a better bound than the one given in Theorem 4.2.3. The proof lies heavily on the so-called “potential maximum clique”. Let us define it and explain the proof in more detail.

Theorem 4.2.4

Let G be a graph and let $k \geq 2$ and $s \geq 1$ be positive integers. If G does not contain a clique on k vertices or a minimal separator of size larger than s , then the tree-width of G is at most $(k - 1)s^3 - 1$.

Before proving Theorem 4.2.4, let us introduce some terminology and state results due to Bouchitté and Todinca [BT01]. For a graph G we denote by $\text{CC}(G)$ the set of all connected components of G (viewed as subsets of $V(G)$). A set $F \subseteq [V(G)]^2 \setminus E(G)$ is a *fill-in* or *chordal completion* if $G + F = (V(G), E(G) \cup F)$ is a chordal graph. A fill-in F is *minimal* if it is inclusion-wise minimal. If $X \subseteq V(G)$, then every connected component $D \in \text{CC}(G \setminus X)$ with $N(D) = X$ is called a component *full* to X . Observe that a set $X \subseteq V(G)$ is a minimal separator if and only if there exist at least two connected components of $G \setminus X$ that are full to X . An important property of minimal separators is that no new minimal separator appears when applying a minimal fill-in.

Lemma 4.2.5 (see [BT01])

For every graph G , minimal fill-in F , and minimal separator X in $G + F$, X is a minimal separator in G as well. Furthermore, the families of components $\text{CC}((G + F) \setminus X)$ and $\text{CC}(G \setminus X)$ are equal (as families of subsets of $V(G)$).

A set $\Omega \subseteq V(G)$ is a *potential maximal clique (PMC)* if there exists a minimal fill-in F such that Ω is a maximal clique of $G + F$. A PMC is surrounded by minimal separators.

Lemma 4.2.6 (see [BT01])

For every PMC Ω in G and every component $D \in \text{CC}(G \setminus \Omega)$, the set $N(D)$ is a minimal separator in G with D being a full component.

The following characterizes PMCs.

Theorem 4.2.7 (see [BT01])

A set $\Omega \subseteq V(G)$ is a PMC in G if and only if the following two conditions hold:

- (i) for every $D \in \text{CC}(G \setminus \Omega)$ we have $N(D) \subsetneq \Omega$;
- (ii) for every $x, y \in \Omega$ either $x = y$, $xy \in E(G)$, or there exists $D \in \text{CC}(G \setminus \Omega)$ with $x, y \in N(D)$.

In the second condition of Theorem 4.2.7, we say that a component D *covers* the nonedge xy .

Lemma 4.2.8

Let G be a graph, $k \geq 2$ and $s \geq 1$ be integers, and let Ω be a PMC in G with $|\Omega| > (k - 1)s^3$. Then there exists in G either a clique of size k or a minimal separator of size larger than s .

Proof. By Lemma 4.2.6, we may assume that for every $D \in \text{CC}(G \setminus \Omega)$ we have $|N(D)| \leq s$.

Assume first that for every $x \in \Omega$ the set of non-neighbors of x in Ω (i.e., $\Omega \setminus N[x]$) is of size less than s^3 . Let $A_0 = \Omega$ and consider the following iterative process. Given A_i for $i \geq 0$, pick $x_i \in A_i$, and set $A_{i+1} = A_i \cap N(x_i)$. The process terminates when A_i becomes empty. Clearly, the vertices x_0, x_1, \dots induce a clique. Furthermore, by our assumption, $|A_i \setminus A_{i+1}| \leq s^3$. Therefore this process continues for at least k steps, giving a clique of size k in G .

Thus we are left with the case when there exists $x \in \Omega$ with the set $\Omega \setminus N[x]$ of size at least s^3 . Let $Y = \{x\} \cup (\Omega \setminus N[x])$; we have $|Y| > s^3$, $Y \subseteq \Omega$, and $G[Y]$ is disconnected.

Consider the following iterative process. At step i , we will maintain a partition \mathcal{A}_i of Y into at least two parts and for every $A \in \mathcal{A}_i$ a set $\mathcal{D}_i(A) \subseteq \text{CC}(G \setminus \Omega)$ with the following property: the sets $\{A \cup \bigcup_{D \in \mathcal{D}_i(A)} D \mid A \in \mathcal{A}_i\}$ is the partition of $G[Y \cup \bigcup_{A \in \mathcal{A}_i} \bigcup_{D \in \mathcal{D}_i(A)} D]$ into vertex sets of connected components. In particular, for every $A \in \mathcal{A}_i$ and $D \in \mathcal{D}_i(A)$ we have $N(D) \cap Y \subseteq A$. We start with $\mathcal{A}_0 = \text{CC}(G[Y])$ and $\mathcal{D}_0(A) = \emptyset$ for every $A \in \mathcal{A}_0$.

The process terminates when there exists $A \in \mathcal{A}_i$ of size larger than s^2 . Otherwise, we perform a step as follows. Pick two distinct $A, B \in \mathcal{A}_i$ and vertices $a \in A, b \in B$. By the properties of \mathcal{A}_i , $ab \notin E(G)$. By Theorem 4.2.7, there exists $D \in \text{CC}(G \setminus \Omega)$ with $a, b \in N(D)$. Let $\mathcal{A} = \{C \in \mathcal{A}_i \mid N(D) \cap C \neq \emptyset\}$. Note that $A, B \in \mathcal{A}$. Furthermore, since $|N(D)| \leq s$, we have $2 \leq |\mathcal{A}| \leq s$.

We define $\mathcal{A}_{i+1} = (\mathcal{A}_i \setminus \mathcal{A}) \cup \{\bigcup_{C \in \mathcal{A}} C\}$. For every $C \in \mathcal{A}_{i+1} \cap \mathcal{A}_i$ we keep $\mathcal{D}_{i+1}(C) = \mathcal{D}_i(C)$. Furthermore, we set $\mathcal{D}_{i+1}(\bigcup_{C \in \mathcal{A}} C) = \{D\} \cup \bigcup_{C \in \mathcal{A}} \mathcal{D}_i(C)$. It is straightforward to verify the invariant for \mathcal{A}_{i+1} and \mathcal{D}_{i+1} .

Furthermore, since every set $C \in \mathcal{A}_i$ is of size at most s^2 while $|Y| > s^3$ we have that $|\mathcal{A}_i| > s$. Since $2 \leq |\mathcal{A}| \leq s$, we have $2 \leq |\mathcal{A}_{i+1}| < |\mathcal{A}_i|$. Consequently, the process terminates after a finite number of steps with \mathcal{A}_i of size at least 2, \mathcal{D}_i , and some $A \in \mathcal{A}_i$ of size greater than s^2 .

Let $X = A \cup \bigcup_{D \in \mathcal{D}_i(A)} D$ and let $y \in Y \setminus A$. Note that $G[X]$ is connected by the invariant on \mathcal{A}_i and \mathcal{D}_i , y exists as $|\mathcal{A}_i| \geq 2$, and y is anticomplete to X . We use Theorem 4.2.7: for every $a \in A$ fix a component $D_a \in \text{CC}(G \setminus \Omega)$ covering the nonedge ya . Since $|N(D_a)| \leq s$ while $|A| > s^2$, the set $\mathcal{D} = \{D_a \mid a \in A\}$ is of size greater than s . Since $G[X]$ is connected and y is anticomplete to X , there exists a minimal separator S with y in one full side and X in the other full side. However, then $S \cap D \neq \emptyset$ for every $D \in \mathcal{D}$. Hence, $|S| \geq |\mathcal{D}| > s$. This finishes the proof of the lemma. \square

Proof. [Proof of Theorem 4.2.4]

Let G be a graph such that it does not contain a clique on k vertices and a minimal separator of size larger than s . Let F be a minimal chordal completion of G . By Lemma 4.2.8, every maximal clique of $G + F$ is of size at most $(k-1)s^3$. Therefore a clique tree of $G + F$ is a tree decomposition of G of width at most $(k-1)s^3 - 1$, as desired. \square

4.3 Nested 2-wheels

Let $k \geq 0$ be an integer. A k -wheel is a graph formed by a hole H called the *rim* together with a set C of k vertices that are not in $V(H)$ called the *centers*, such that each center has at least three neighbors in the rim. We denote such a k -wheel by (H, C) .

Observe that a 0-wheel is a hole. A 1-wheel is precisely a *wheel* (see Figure 1.9). We often write (H, u) instead of $(H, \{u\})$.

A 2-wheel $(H, \{u, v\})$ is *nested* if H contains two vertices a and b such that all neighbors of u in H are in one path of H from a to b , while all the neighbors of v are in the other path of H from a to b . Observe that a and b may be adjacent to both u and v . As we will see in this section, the properties of 2-wheels highlight structural similarities between (theta, triangle)-free graphs and (even hole, pyramid)-free graphs, in the sense that in both classes, apart from few exceptions, every 2-wheel with non-adjacent centers is nested.

For a center u of a k -wheel (H, C) , a u -sector of H is a subpath of H of length at least 1 whose ends are adjacent to u and whose internal vertices are not. However, a u -sector may contain internal vertices that are adjacent to v for some center $v \neq u$. Observe that for every center u , the rim of a wheel is edgewise partitioned into its u -sectors.

In (theta, triangle)-free graphs

Recall that the *cube* is the graph formed from a hole of length 6, say $h_1 h_2 \cdots h_6 h_1$ together with a vertex u adjacent to h_1, h_3, h_5 , and a vertex v non-adjacent to u and adjacent to h_2, h_4, h_6 . Note that the cube is a non-nested 2-wheel with non-adjacent centers.

Lemma 4.3.1

Let G be a (theta, triangle)-free graph. If $W = (H, \{u, v\})$ is a 2-wheel in G such that $uv \notin E(G)$, then W is either a nested wheel or the cube.

Proof. Suppose that W is not a nested wheel. We will prove that W is the cube.

Claim 1. Every u -sector of H contains at most one neighbor of v and every v -sector of H contains at most one neighbor of u .

Proof of Claim 1. For otherwise, without loss of generality, some u -sector $P = x \dots y$ of H contains at least two neighbors of v . Let x', y' be neighbors of v closest to x, y respectively along P . Note that $x'y' \notin E(G)$ because G is triangle-free. Since W is not nested, $H \setminus P$ contains some neighbors of v . Note also that $H \setminus P$ contains some neighbors of u .

So, let $Q = z \dots z'$ be the path of $H \setminus P$ that is minimal length and such that $uz \in E(G)$ and $vz' \in E(G)$. Note that z' is adjacent to either x or y , for otherwise $uzQz'v$, $uxPx'v$, and $uyPy'v$ form a theta from u to v . So suppose up to symmetry that z' is adjacent to y . So, v is not adjacent to y since G is triangle-free. It then follows that the three paths $vz'y$, $vy'Py$, and $vx'Pxuy$ form a theta, a contradiction. This proves Claim 1.

Claim 2. u and v have no common neighbors in H .

Proof of Claim 2. Otherwise, let x be such a common neighbor. Consider a subpath $x \dots y$ of H of maximum length with the property of being a u -sector or a v -sector, and suppose up to symmetry that it is a u -sector. By its maximality, it contains a neighbor of v different from x . So in total it contains at least two neighbors of v , a contradiction to Claim 1. This proves Claim 2.

Claims 1 and 2 prove that $|N_H(u)| = |N_H(v)|$ and the neighbors of u and v alternate along H . So, let $x, y, z \in N_H(u)$ and $x', y', z' \in N_H(v)$ be distinct vertices in H with x, x', y, y', z, z' appearing in this order along H . If $V(H) = \{x, y, z, x', y', z'\}$,

then $V(H) \cup \{u, v\}$ induces the cube, so suppose $\{x, y, z, x', y', z'\} \subsetneq V(H)$. Hence, up to symmetry, we may assume that x, x', y, y', z and z' are chosen such that: $xz' \notin E(G)$. Note that by the argument mentioned in the beginning of this paragraph, u and v have no other neighbor in the interior of $x'(H \setminus y)x$, $y'(H \setminus y)z$, and $z(H \setminus x)z'$. So the three paths $vz'(H \setminus x)z$, $vy'(H \setminus y)z$, and $vx'(H \setminus y)xuz$ form a theta, a contradiction. \square

The following lemma of Radovanović and Vušković shows that the presence of the cube in a (theta, triangle)-free graph entails some structure.

Lemma 4.3.2 (see [RV13])

Let G be a (theta, triangle)-free graph. If G contains the cube, then either it is the cube, or it has a clique separator of size at most 2.

In even-hole-free graphs

Let us consider a classical generalization of even-hole-free graphs. Recall that all thetas, prisms, even wheels, and square contain even holes. The class of (theta, prism, even wheel, square)-free graphs is therefore a generalization of even-hole-free graphs that captures the structural properties that we need here.

A proof of the following lemma can be found in [Chu+19] (where it relies on many lemmas). We include here our self-contained proof for the sake of completeness. Call a wheel *proper* if it is not a pyramid. A *cousin wheel* is a 2-wheel made of a hole $H = h_1h_2 \dots h_nh_1$ and two non-adjacent centers u and v , such that $N_H(u) = \{h_1, h_2, h_3\}$ and $N_H(v) = \{h_2, h_3, h_4\}$.

Lemma 4.3.3

Let G be a (theta, prism, pyramid, even wheel, square)-free graph. If $W = (H, \{u, v\})$ is a 2-wheel in G such that $uv \notin E(G)$, then W is either a nested or a cousin wheel. Moreover, if W is nested then $|N_H(u) \cap N_H(v)| \leq 1$.

Proof. In the case where $W = (H, \{u, v\})$ is nested, it must be that $|N_H(u) \cap N_H(v)| \leq 1$, for otherwise G would contain a square. Since G contains no even wheel, it is sufficient to consider the following cases.

Case 1: $N_H(u) = 3$ or $N_H(v) = 3$.

Assume that W is not a nested wheel. We will prove that W is a cousin wheel. Without loss of generality, we may assume that $|N_H(u)| = 3$, and let $N_H(u) = \{x, y, z\}$. We denote by $P_x = y \dots z$, $P_y = x \dots z$ and $P_z = x \dots y$ the three u -sectors of H .

Suppose xyz is a path of H . Then v must be adjacent to y , for otherwise W is nested, a contradiction. Since $V(H) \cup \{u\}$ and $V(H \setminus y) \cup \{u, v\}$ do not induce an even wheel, v has exactly two neighbors in P_y . Moreover, the two neighbors of v in P_y are adjacent, for otherwise $H \setminus y, u$, and v form a theta. Since (H, v) is not a pyramid, this means that one of x or z is a neighbor of v . Therefore, W is a cousin wheel.

Now suppose that $\{x, y, z\}$ does not induce a path. So xy, yz , and zx are non-edges. Note that v is adjacent to at most one of x, y , or z , because G contains no square. Up to symmetry, assume that $vx \notin E(G)$. Let R be the v -sector of H which contains x (in its interior). Since $(H, \{u, v\})$ is not a nested wheel, the ends of R are not both in P_x , or both in P_y , or both in P_z . So assume that $R = y' \dots z'$ with z' is in

the interior of P_z and y' is not in P_z . If y' is in P_x , then R, u , and v form a theta from x to z , a contradiction. Hence, y' is not in P_x , so y' is in the interior of P_y .

Call x' the neighbor of v in H different from y' and z' . If x' is not in the interior of P_x , then P_x is contained in the v -sector $x'Hx'$. Thus, there exists a v -sector S which contains P_x . In particular, the hole made of S and v contains two non adjacent neighbors of u , namely y and z . Hence, S, u , and v form a theta from y to z . So, x' is in the interior of P_x .

This means x, y', z, x', y, z' appear in this order along H . If $x'z \notin E(G)$, then the paths $x'(H \setminus y)z$, $x'(H \setminus z)yuz$, and $x'vy'(H \setminus x)z$ form a theta from x' to z , a contradiction. So, $x'z \in E(G)$. By symmetry, $x'y \in E(G)$. But then, $\{u, y, x'z\}$ induces a square, a contradiction.

Case 2: $N_H(u) \geq 5$ and $N_H(v) \geq 5$

For a contradiction, suppose that $(H, \{u, v\})$ is not a nested wheel. First of all, we have $N_H(u) \neq N_H(v)$, for otherwise u, v , and two non-adjacent vertices of $N_H(u)$ would form a square. So in H , there exists a neighbor of v that is not adjacent to u . It is therefore well defined to consider the u -sector $P = x \dots y$ of H whose interior contains $k \geq 1$ neighbors of v , and to choose such a sector with k minimum. We denote by x' the neighbor of x in $H \setminus P$, by y' the neighbor of y in $H \setminus P$ and by $Q = x' \dots y'$ the path $H \setminus P$.

Note that u has some neighbor in the interior of Q , because u has at least 5 neighbors in H . We now show that v also has some neighbor in the interior of Q . Suppose that this is not the case. Then, the neighborhood of v in H is completely contained in $V(P) \cup \{x', y'\}$. Since $(H, \{u, v\})$ is not a nested wheel, v is adjacent to x' or y' — and in fact to both of them, for otherwise the hole $uxPyu$ would contain an even number (at least 4) of neighbors of v , thus inducing an even wheel, a contradiction. Now since $\{u, v, x, y\}$ does not induce a square, up to symmetry we may assume that $vx \notin E(G)$. Since $|N_H(v)| \geq 5$, v has at least 2 neighbors in the interior of P , and so $k \geq 2$. Note that u is adjacent to x' , for otherwise, x' would be the unique neighbor of v in the interior of a u -sector, contradicting the minimality of k . Since $\{u, v, x', y'\}$ does not induce a square, we know that u is not adjacent to y' . But then, y' is the unique neighbor of v in the interior of some u sector, a contradiction to the minimality of k . This proves that v has some neighbor in the interior of Q .

By the fact that each of u and v has some neighbor in the interior of Q , a path S from u to v whose interior is in the interior of Q exists. Let x'' (resp. y'') be the neighbor of v in P closest to x (resp. y) along P . If $x'' = y''$, then x'' is an internal vertex of P , and so S and P form a theta from u to x'' . If $x''y'' \in E(G)$, then S and P form a pyramid. If $x'' \neq y''$ and $x''y'' \notin E(G)$, then $S, uxPx''v$, and $uyPy''v$ form a theta from u to v . Each of the cases yields a contradiction; this completes the proof. \square

4.4 Bounding the tree-width

In this section, we prove that the tree-width is bounded in (theta, triangle, $S_{i,j,k}$)-free graphs and in (even hole, pyramid, K_t , $S_{i,j,k}$)-free graphs.

For (theta, triangle)-free graphs, by Lemma 4.3.2, we may assume that the graphs we work on are cube-free since the cube itself has small tree-width, and clique separators of size at most 2 in some sense preserve the tree-width (this will be formalized in the proofs). For (even hole, pyramid)-free graphs, recall that we first work in a superclass, namely (theta, prism, pyramid, even wheel, square)-free graphs.

Since our proof is the same for $(\theta, \text{triangle}, S_{i,j,k})$ -free graphs and (even hole, pyramid, K_t , $S_{i,j,k}$)-free graphs, to avoid duplicating it, we introduce a class \mathcal{C} that contains all the graphs that we need to consider while entailing the structural properties that we need.

Call *butterfly* a wheel (H, v) such that $N_H(v) = \{a, b, c, d\}$ with $ab \in E(G)$, $bc \notin E(G)$, $cd \in E(G)$ and $da \notin E(G)$. Let \mathcal{C} be the class of all $(\theta, \text{prism}, \text{pyramid}, \text{butterfly})$ -free graphs such that every 2-wheel with non-adjacent centers is either a nested or a cousin wheel.

Lemma 4.4.1

If G is a $(\theta, \text{triangle}, \text{cube})$ -free graph or a $(\theta, \text{prism}, \text{pyramid}, \text{even wheel}, \text{square})$ -free graph, then $G \in \mathcal{C}$.

Proof. If G is a $(\theta, \text{triangle}, \text{cube})$ -free graph, then G is θ -free and $(\text{prism}, \text{pyramid}, \text{butterfly})$ -free (because prisms, pyramids, and butterflies contain triangles). Furthermore, every 2-wheel with non-adjacent centers is a nested wheel by Lemma 4.3.1.

If G is a $(\theta, \text{prism}, \text{pyramid}, \text{even wheel}, \text{square})$ -free graph, then G is $(\theta, \text{prism}, \text{pyramid})$ -free and *butterfly*-free (because a butterfly is an even wheel). Furthermore, every 2-wheel with non-adjacent centers is either a nested or a cousin wheel by Lemma 4.3.3.

Hence $G \in \mathcal{C}$ as claimed. \square

6 proof, we need a special type of k -wheel. A *k -span-wheel* is a k -wheel (H, C) that satisfies the following properties.

- There exist two non-adjacent vertices x, y in H and we denote by $P_A = a_1 \dots a_\alpha$ and $P_B = b_1 \dots b_\beta$ the two paths of H from x to y , with $x = a_1 = b_1$ and $y = a_\alpha = b_\beta$.
- $C \cup \{x, y\}$ is an independent set.
- There exists an ordering of vertices in C , namely v_1, v_2, \dots, v_k .
- Every vertex of C has neighbors in the interiors of both P_A and P_B (and at least 3 neighbors in H since (H, C) is a k -wheel).
- For every $1 \leq i < j \leq k$ and $1 \leq i', j' \leq \alpha$, if $v_i a_{i'} \in E(G)$ and $v_j a_{j'} \in E(G)$ then $i' \leq j'$.
- For every $1 \leq i < j \leq k$ and $1 \leq i', j' \leq \beta$, if $v_i b_{i'} \in E(G)$ and $v_j b_{j'} \in E(G)$ then $i' \leq j'$.

Informally, a k -span-wheel is such that, walking from x to y along both P_A and P_B , one first meets all the neighbors of v_1 , then all neighbors of v_2 , and so on until v_k . Observe that a 1-span-wheel is a wheel, 2-span-wheel is a nested 2-wheel. Note that distinct v_i and v_j may share common neighbors on H (it is even possible that $N_{P_A}(v_1) = \dots = N_{P_A}(v_k) = \{a_i\}$).

Observe that in the following theorem, θ s, pyramids, prisms, and butterflies have to be excluded, since they do not satisfy the conclusion.

Lemma 4.4.2

Let G be a connected graph in \mathcal{C} . Let C be a minimal separator in G of size at least 2 that is furthermore an independent set, and A and B be connected

components of $G \setminus C$ that are full to C . Then:

1. There exist two vertices x and y in C , a path P_A from x to y with interior in A , and a path P_B from x to y with interior in B such that all vertices in $C \setminus \{x, y\}$ have neighbors in the interior of both P_A and P_B . Note that $V(P_A) \cup V(P_B)$ induces a hole that we denote by H .
2. $(H, C \setminus \{x, y\})$ is a $(|C| - 2)$ -span-wheel.

Proof. We first prove 1, by induction on $k = |C|$. If $k = 2$, then x, y, P_A , and P_B exist from the connectivity of A and B , and the conditions on $C \setminus \{x, y\}$ vacuously hold. So suppose the result holds for some $k \geq 2$, and let us prove it for $k + 1$. Let z be any vertex from C , and apply the induction hypothesis to $C \setminus z$ in $G \setminus z$. This provides two vertices x, y in $C \setminus z$ and two paths P_A and P_B . We denote by H the hole formed by P_A and P_B .

Claim 1. Every vertex in $C \setminus \{x, y, z\}$ has neighbors in the interior of both P_A and P_B .

Proof of Claim 1. Follows directly from the induction hypothesis. This proves Claim 1.

Since z has a neighbor in A and A is connected, there exists a path $Q_A = z \dots z_A$ in $A \cup \{z\}$, such that z_A has a neighbor in the interior of P_A . A similar path Q_B exists. We set $Q = z_A Q_A z Q_B z_B$. We suppose that x, y, P_A, P_B, Q_A , and Q_B are chosen subject to the minimality of Q .

Observe that Q is a chordless path by its minimality and the fact that A and B are anticomplete. The minimality of Q implies that the interior of Q is anticomplete to the interior of P_A and to the interior of P_B .

Claim 2. We may assume that Q has length at least 1.

Proof of Claim 2. Otherwise, $z = z_A = z_B$, so z has neighbors in the interior of both P_A and P_B . Hence, by Claim 1, x, y, P_A , and P_B satisfy 1. This proves Claim 2.

Let a (resp. a') be the neighbor of z_A in P_A closest to x (resp. to y) along P_A . Let b (resp. b') be the neighbor of z_B in P_B closest to x (resp. to y) along P_B .

Claim 3. If $a \neq a'$ and $aa' \notin E(G)$, then $z = z_A$. If $b \neq b'$ and $bb' \notin E(G)$, then $z = z_B$.

Proof of Claim 3. We give a proof only for the statement of a , since the proof for b is similar.

For suppose $a \neq a'$, $aa' \notin E(G)$, and $z \neq z_A$, let z' be the neighbor of z_A in Q . Set $P'_A = x P_A a z_a a' P_A y$ and $Q' = z' Q z_B$. Let us prove that x, y, P'_A, P_B , and Q' contradict the minimality of Q . Obviously, Q' is shorter than Q , so we only have to prove that every vertex in $C \setminus \{z\}$ has neighbors in the interior of both P'_A and P_B . For P_B , it follows from Claim 1. So suppose for a contradiction that a vertex $c \in C \setminus \{z\}$ has no neighbor in the interior of P'_A . Since by Claim 1 c has a neighbor c' in the interior of P_A , c' is an internal vertex of $a P_A a'$. Since G is theta-free, (H, z_A) is a wheel. Note that $(H, \{c, z_A\})$ is not nested because of c' and some neighbor of c in the interior of P_B (i.e. the neighborhood of c in H is not contained in a unique z_A -sector). Since $G \in \mathcal{C}$, by Lemma 4.3.3, $(H, \{c, z_A\})$ is a cousin wheel. Since c has neighbors in the interiors of both P_A and P_B , this means that x or y is a common neighbor of c and z_A , a contradiction to C being an independent set. The proof for the latter statement (with b) is similar. This proves Claim 3.

Claim 4. We may assume that x has neighbors in the interior of Q and y has no neighbor in the interior of Q .

Proof of Claim 4. We show that if it is not the case, then there is a contradiction. For suppose both x and y have a neighbor in the interior of Q , then a path of minimal length from x to y with interior in the interior of Q forms a theta together with P_A and P_B , a contradiction.

Now suppose that neither x nor y has a neighbor in the interior of Q . Recall that Claim 2 tells us that $z_A \neq z_B$. So either $z \neq z_A$ or $z \neq z_B$. Up to symmetry, we may assume that $z \neq z_A$. Hence by Claim 3, either $a = a'$ or $aa' \in E(G)$.

Suppose $a = a'$. This implies that a is in the interior of P_A . If $b = b'$, then b is in the interior of P_B , so H and Q form a theta from a to b ; if $bb' \in E(G)$, then H and Q form a pyramid; and if $b \neq b'$ and $bb' \notin E(G)$, then $aP_AxP_Bbz_B$, $aP_AyP_Bb'z_B$, az_AQz_B form a theta from a to z_B (note that az_AQz_B has length at least 2 because $z_A \neq z_B$), a contradiction. So, $aa' \in E(G)$.

Suppose that $bb' \in E(G)$. Note that $|\{a, a'\} \cap \{b, b'\} \cap \{x, y\}| \neq 2$, because x and y are not adjacent. Moreover, $|\{a, a'\} \cap \{b, b'\} \cap \{x, y\}| \neq \emptyset$, for otherwise H and Q form a prism. So, $|\{a, a'\} \cap \{b, b'\} \cap \{x, y\}| = 1$. In this last case, we suppose up to symmetry that $x = a = b$. So, z is in the interior of Q since it is non-adjacent to x — in particular Q has length at least 2. Hence, H and Q form a butterfly (with $x = a = b$ being the center), a contradiction.

So, $bb' \notin E(G)$. If $b = b'$, then b is in the interior of P_B ; thus P_A , P_B , and Q form a pyramid (i.e. $3PC(az_Aa', b)$), a contradiction. So, $b \neq b'$, and hence by Claim 3, $z_B = z$. This means that $b \neq x$ and $b' \neq y$ (because C is an independent set). Therefore, aP_AxP_Bbz , $a'P_AyP_Bb'z$, and z_AQz form a pyramid (i.e. $3PC(aa'z_A, z)$), a contradiction.

So, each case leads to a contradiction. Hence, exactly one of x or y has neighbors in the interior of Q , and up to symmetry we may assume it is x . This proves Claim 4.

Claim 5. $a'x \in E(G)$ and $b'x \in E(G)$.

Proof of Claim 5. First, suppose z_A is adjacent to x , i.e. $a = x$. Then, $z_A \neq z$ since C is an independent set. Note that $a = a'$ is impossible since z_A has neighbors in the interior of P_A . So, by Claim 3, $a'x \in E(G)$.

Now suppose z_A is not adjacent to x . By Claim 4, x has a neighbor in the interior of Q , so we choose such a neighbor x' closest to z_A along Q . Note that by the minimality of Q , no vertex in the interior of Q has neighbor in the interior of P_A and in the interior of P_B . Since y is not adjacent to x' (by Claim 4), x' has no neighbors in $(P_A \cup P_B) \setminus \{x\}$. We set $R = xx'Qz_A$ and observe that R has length at least 2. If $a \neq a'$ and $aa' \notin E(G)$, then xP_Aaz_A , $xP_ByP_Aa'z_A$, and R form a theta from x to z_A . If $aa' \in E(G)$, then P_A , P_B , and R form a pyramid. Therefore $a = a'$. Note that $xa \in E(G)$, for otherwise, P_A , P_B , and R form a theta. Hence, $a'x \in E(G)$.

The proof for $b'x \in E(G)$ is similar. This proves Claim 5.

To conclude the proof of 1, set $P'_A = zQz_Aa'P_Ay$ and $P'_B = zQz_Bb'P_By$. By Claim 5, x has neighbors in the interior of both P'_A and P'_B (these neighbors are a' and b'). Note that since $a'x, b'x \in E(G)$, the interiors of P_A and P_B are included in the interiors of P'_A and P'_B respectively. Hence, by Claim 1, every vertex of $C \setminus z$ has neighbors in the interior of both P'_A and P'_B .

Hence, the vertices z, y and the paths P'_A and P'_B show that 1 is satisfied.

Let us now prove 2. Note that $(H, C \setminus \{x, y\})$ is $(|C| - 2)$ -wheel (this follows because G is theta-free, every vertex in $C \setminus \{x, y\}$ has at least three neighbors in H). It remains to prove that it is a $(|C| - 2)$ -span-wheel. Note that it is clearly true if $|C| \leq 3$. We set $P_A = a_1 \dots a_\alpha$ and $P_B = b_1 \dots b_\beta$ with $x = a_1 = b_1$ and $y = a_\alpha = b_\beta$,

as in the definition of a k -span-wheel. We just have to exhibit an ordering of the vertices of $C \setminus \{x, y\}$ that satisfies the rest of the definition.

We first define v_1 , the smallest vertex in the order we aim to construct. Note that no vertex $v \in C \setminus \{x, y\}$ is adjacent to x or y , because C is an independent set. We let v_1 be a vertex of C that is adjacent to a_i with i minimum. Let j be the smallest integer such that v_1 is adjacent to b_j . We suppose that v_1 is chosen subject to the minimality of j . Let i', j' be the greatest integers such that v_1 is adjacent to $a_{i'}$ and $b_{j'}$. Note that $1 < i \leq i' < \alpha$ and $1 < j \leq j' < \beta$.

Claim 6. For every $w \in C \setminus \{x, y, v_1\}$, we have $N_H(w) \subseteq V(a_{i'}P_AyP_Bb_{j'})$.

Proof of Claim 6. We first note that the 2-wheel $(H, \{v_1, w\})$ is not a cousin wheel, because this may happen only when $x \in N(v_1)$ or $y \in N(v_1)$ (recall that if it was a cousin wheel, $N_H(v_1)$ would induce a 3-vertex path in H).

Hence, $(H, \{v_1, w\})$ is a nested wheel. Suppose that $N_H(w) \not\subseteq V(a_{i'}P_Aa_\alpha) \cup V(b_{j'}P_Bb_\beta)$. This means that w has a neighbor z in $a_{i'-1}P_AxP_Bb_{j'-1}$. Since $(H, \{v_1, w\})$ is a nested wheel, $N_H(w)$ is contained in a v_1 -sector Q of (H, v_1) . Moreover, since w has a neighbor in the interior of both P_A and P_B , we have $Q = a_iP_AxP_Bb_j$. Since H and w form a wheel, w has neighbor in the interior of Q . This contradicts the minimality of i or j . This proves Claim 6.

The order of $C \setminus \{x, y\}$ is now constructed as follows: we remove v_1 from C , define v_2 as we defined v_1 (minimizing i , and then minimizing j), then remove v_2 , define v_3 , and so on. This iteratively constructs an ordering of $C \setminus \{x, y\}$ showing that $(H, C \setminus \{x, y\})$ is a $(|C| - 2)$ -span-wheel. \square

For integers $t, k \geq 1$, the Ramsey number $R(t, k)$ is the smallest integer n such that any graph on n vertices contains either a clique of size t , or an independent set of size k .

Theorem 4.4.3

An $(l$ -span-wheel, K_t)-free graph $G \in \mathcal{C}$ has tree-width at most $(t - 1)(R(t, l + 2) - 1)^3 - 1$.

Proof. Suppose for a contradiction that the tree-width of G is at least $(t - 1)(R(t, l + 2) - 1)^3$. Since G is K_t -free, by Theorem 4.2.4, G admits a minimal separator D of size at least $R(t, l + 2)$. Let A and B be two connected components of $G \setminus D$ that are full to D . By the definition of Ramsey number, $G[D]$ contains an independent set C of size $l + 2$. We define $G' = G[A \cup C \cup B]$, and observe that C is a minimal separator of G' . Hence by Lemma 4.4.2 applied to G' , the graph contains an l -span-wheel, a contradiction. \square

The following shows that in \mathcal{C} , an l -span-wheel with large l contains $S_{i,j,k}$ with large i, j, k .

Lemma 4.4.4

If a butterfly-free graph G contains a $(4k + 1)$ -span-wheel with $k \geq 0$, then it contains $S_{k+1, k+1, k+1}$.

Proof. Consider a $(4k + 1)$ -span-wheel in G , with x, y, P_A , and P_B be as in the definition of span-wheel given in the beginning of the current section. Let v_1, \dots, v_{4k+1} be the centers of the span wheel. For each $i = 1, \dots, 4k + 1$, let a_i (resp. a'_i) be the neighbor of v_i in P_A closest to x (resp. to y) along P_A . Let b_i (resp. b'_i) be the neighbor of v_i in P_B closest to x (resp. to y) along P_B . We set $P_i = a_iP_AxP_Bb_i$ and $Q_i = a'_iP_AyP_Bb'_i$.

Claim 1. P_i has length at least $i + 1$ and Q_i has length at least $4k + 3 - i$.

Proof of Claim 1. We prove this by induction on i for P_i . It is clear that P_1 has length at least 2 since x is not adjacent to v_1 . Suppose the claim holds for some fixed $i \geq 1$, and let us prove it for $i + 1$. From the induction hypothesis, P_i has length at least $i + 1$, and since v_i has a neighbor in the interior of P_{i+1} (because it has at least three neighbors in H), the length of P_{i+1} is greater than the length of P_i , so P_{i+1} has length at least $i + 2$.

The proof for Q_i is similar, except we start by proving that Q_{4k+1} has length at least 2, and that the induction goes backward down to Q_1 . This proves Claim 1.

We set $l = 2k + 1$. So, by Claim 1, P_l and Q_l both have length at least $2k + 2$. We set $v = v_l$, $P = P_l$, $Q = Q_l$, $a = a_l$, $a' = a'_l$, $b = b_l$ and $b' = b'_l$. Since G is butterfly-free, we do not have $aa' \in E(G)$ and $bb' \in E(G)$ simultaneously. So, up to symmetry we may assume that either $a = a'$; or $a \neq a'$ and $aa' \notin E(G)$.

If $a = a'$, let u, u' , and u'' be three distinct vertices in P such that a, u, u' , and u'' appear in this order along P , aPu has length $k + 1$ and bPu'' has length $k - 1$ (which is possible because P has length at least $2k + 1$). Let w be in Q and such that aQw has length $k + 1$ (which is possible because Q has length at least $2k + 1$). The three paths aPu , $avbPu''$, and aQw form an $S_{k+1,k+1,k+1}$.

If $a \neq a'$ and $aa' \notin E(G)$, then let u, u' , and u'' be three distinct vertices in P such that a, u, u' , and u'' appear in this order along P , aPu has length k and bPu'' has length k . Let w be in Q and such that $a'Qw$ has length k . The three paths $vaPu$, $vbPu''$ and $va'Qw$ form an $S_{k+1,k+1,k+1}$. \square

Recall the following classical result on tree-width that is given in Chapter 1 (cf. Remark 1).

Lemma 4.4.5

The tree-width of a graph G is the maximum tree-width of an induced subgraph of G that has no clique separator.

Theorem 4.4.6

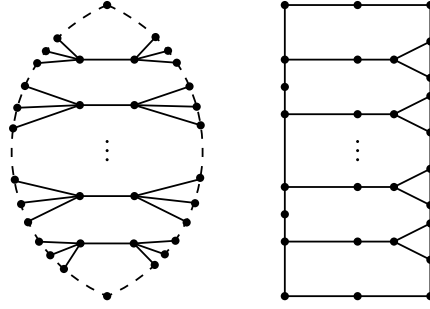
For $k \geq 1$, every (theta, triangle, $S_{k,k,k}$)-free graph G has tree-width at most $2(R(3, 4k - 1))^3 - 1$.

Proof. By Lemma 4.4.5, it is enough to consider a graph G that does not have a clique separator. If G contains the cube, then Lemma 4.3.2 tells us that G itself is the cube. By classical results on tree-width, the tree-width of the cube is 3 (but the trivial bound 8 would be enough for our purpose), which in particular achieves the given bound. We may therefore assume that G is cube-free. Moreover, by Lemma 4.4.1, G is in \mathcal{C} . Since G is $S_{k,k,k}$ -free, by Lemma 4.4.4, G contains no $(4k - 3)$ -span-wheel. Moreover, G contains no K_3 by assumption. Hence, by Theorem 4.4.3, G has tree-width at most $2(R(3, 4k - 1))^3 - 1$. \square

Theorem 4.4.7

For $k \geq 1$, every (even hole, pyramid, K_t , $S_{k,k,k}$)-free graph G has tree-width at most $(t - 1)(R(t, 4k - 1))^3 - 1$.

Proof. Since all thetas, prisms, even wheels, and square contain even holes, G is (theta, prism, pyramid, even wheel, square)-free. So, by Lemma 4.4.1, G is in \mathcal{C} . Since G is $S_{k,k,k}$ -free, by Lemma 4.4.4, G contains no $(4k - 3)$ -span-wheel. Moreover,

FIGURE 4.2: k -turtle and k -ladder (dashed lines represent paths)

G contains no K_t by assumption. Hence, by Theorem 4.4.3, G has tree-width at most $(t-1)(R(t, 4k-1))^3 - 1$. \square

4.5 Discussion and open problems

One possible method to find a maximum weight independent set for a class of graphs is by proving that every graph in the class has polynomially many minimal separators (where the polynomial is in the number of vertices of the graph). This was for instance successfully applied to (even hole, pyramid)-free graphs in [Chu+19]. Therefore, our result on (even hole, pyramid, K_t , $S_{i,j,k}$)-free graphs does not settle a new complexity result for the maximum independent set problem (but it might still be applicable to other problems).

Note that bounding the number of minimal separators cannot be applied to (even hole, K_4)-free graphs and to (theta, triangle)-free graphs since there exist graphs in both classes that contain exponentially many minimal separators. These graphs are called *k -turtle* and *k -ladder*, see Fig 4.2. It is straightforward to check that they have exponentially many minimal separators (the idea is that a separator can be built by making a choice in each edge between two centers of wheel (drawn as “horizontal” edges in Figure 4.2), and there are k of them). Moreover, k -turtles are (theta, triangle)-free (provided that the outer cycle is sufficiently subdivided) and k -ladders are (even hole, K_4)-free.

Chapter 5

Even-hole-free graphs of bounded degree

From the construction of layered wheels explained in Chapter 3, we observe that to obtain a large bound on the tree-width, a layered wheel must contain a large clique minor. Moreover, the existence of a large clique minor forces the layered wheel to contain vertices of high degree — this can be observed for instance from the unique vertex on the layer 0 (i.e. the topmost layer, see again Figure 3.5). These results suggest the following conjecture.

Conjecture 5.0.1. *There is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that every even-hole-free graph of degree at most d has tree-width at most $f(d)$.*

Outline of the chapter

In Section 5.1, we discuss a special case of Conjecture 5.0.1, that is when $d = 3$, which is covered in [Abo+cs]. We present subcubic (even hole, K_4)-free graphs, where we give a structure theorem of the class and prove a constant upper bound on the tree-width of graphs in the class. In Section 5.2, we present (even hole, K_4)-free graphs with maximum degree 4 and where pyramids are excluded. As for the subcubic case, we also give a structure theorem of the graphs in the class and prove that they have bounded tree-width. In Section 5.3, we give conclusion and discuss some open problems.

5.1 Subcubic case

In this section, we prove that even-hole-free subcubic graphs can be described by a structure theorem, that implies tree-width at most 3. In fact, our result is for a more general class: (theta, prism)-free subcubic graphs. Let us now present the basic graphs and the separators for the decomposition of graphs in the class.

A wheel that is not a pyramid is a *proper wheel*. A *sector* of a wheel (H, x) is a subpath of H whose endnodes are adjacent to x , and whose internal vertices are not.

An *extended prism* is a graph made of five vertex-disjoint chordless paths of length at least one, namely $A = a \dots x$, $A' = x \dots a'$, $B = b \dots y$, $B' = y \dots b'$, $C = c \dots c'$ such that abc is a triangle, $a'b'c'$ is a triangle, xy is an edge and no edges exist between the paths except xy and those of the two triangles (see Figure 5.1).

Recall that a subset (possibly empty) of vertices $S \subseteq V(G)$ is a *separator* of G if $G \setminus S$ contains at least two connected components. A *clique separator* is a separator S that is a clique.

A *proper separation* in a graph G is a triple $(\{a, b\}, X, Y)$ satisfying the following.

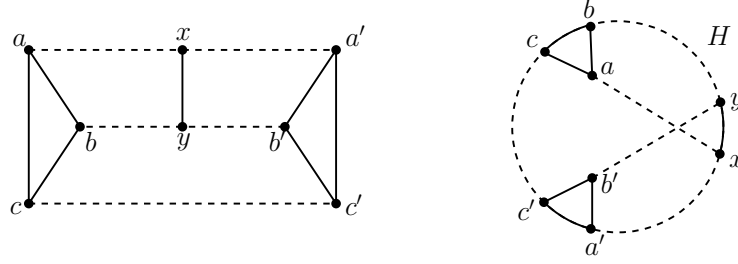
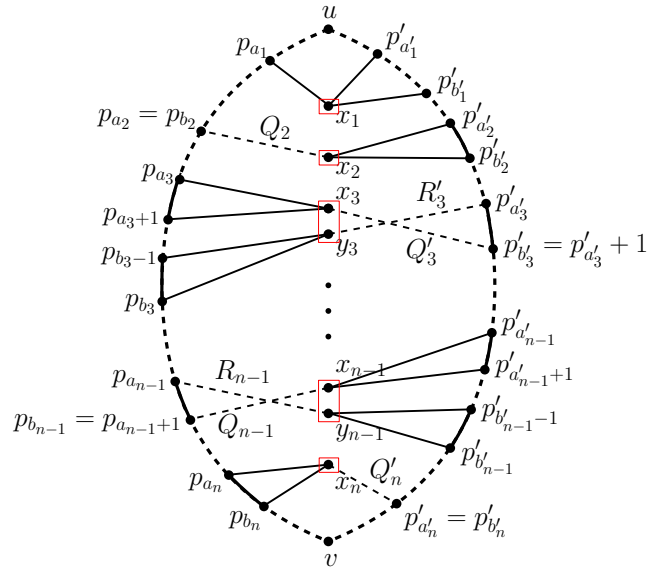


FIGURE 5.1: Two different drawings of an extended prism

- (i) $\{a, b\}$, X , Y are disjoint, non-empty and $V(G) = \{a, b\} \cup X \cup Y$.
- (ii) There are no edges from X to Y .
- (iii) a and b are non-adjacent.
- (iv) a and b have exactly two neighbors in X .
- (v) a and b have exactly one neighbor in Y .
- (vi) There exists a path from a to b with interior in X , and there exists a path from a to b with interior in Y .
- (vii) $G[Y \cup \{a, b\}]$ is not a chordless path from a to b .

A *proper separator* of G is a pair $\{a, b\} \subseteq V(G)$ such that there exists a proper separation $(\{a, b\}, X, Y)$.

Let \mathcal{C} be the class of (theta, prism)-free subcubic graphs. The *cube* is the graph made of a hole $v_1v_2 \dots v_6v_1$ and two non-adjacent vertices x and y such that $N_H(x) = \{v_1, v_3, v_5\}$ and $N_H(y) = \{v_2, v_4, v_6\}$. Call a graph in \mathcal{C} *basic* if it is isomorphic to a chordless cycle, a clique of size at most 4, the cube, a proper wheel, a pyramid, or an extended prism. An example of graph in \mathcal{C} that is not basic is given in Figure 5.2.

FIGURE 5.2: An example of non-basic graph in \mathcal{C}

We need the following lemma.

Lemma 5.1.1

Let G be a theta-free subcubic graph, let H be a hole in G , and $v \in G \setminus H$. Then v has at most three neighbors in H , and if v has exactly two neighbors in H , then they are adjacent.

Proof. Let $v \in G \setminus H$. Since G is subcubic, $d_H(v) \leq 3$. If v has exactly two neighbors in H , but they are non-adjacent then $G[H \cup \{v\}]$ would induce a theta, a contradiction. \square

Recall that a *clique separator* in a graph is a separator of the graph that induces a clique. The main theorem of this section is the following.

Theorem 5.1.2

Let G be a (theta, prism)-free subcubic graph. Then one of the following holds:

- G is a basic graph;
- G has a clique separator of size at most 2;
- G has a proper separator.

Proof. Let G be a (theta, prism)-free subcubic graph. We may assume that G has no clique separator (and is in particular connected for otherwise the empty set is a clique separator).

Claim 1. We may assume that G is (K_4, cube) -free.

Proof of Claim 1. If G contains K_4 , then since G is a subcubic connected graph, $G = K_4$, so G is basic. The proof is similar when G contains the cube. This proves Claim 1.

Claim 2. We may assume that G does not contain a proper wheel.

Proof of Claim 2. Let $W = (H, x)$ be a proper wheel in G . Let a, b, c , be the three neighbors of x . We call A (B, C , resp.) the path of H from b to c (from a to c , from a to b , resp.) that does not contain a (b, c , resp.). Observe that, since G is subcubic, no vertex of $G \setminus W$ has a neighbor in $\{x, a, b, c\}$.

Suppose that some vertex y of $G \setminus W$ has neighbors in the three sectors of W , say a' in A , b' in B , and c' in C . Hence, a, c', b, a', c , and b' appear in this order along H . If $ac' \notin E(G)$, then $xaBb'$, $xcBb'$, and $xbCc'yb'$ induce a theta, so $ac' \in E(G)$. Symmetrically, $c'b, ba', a'c, b'c$, and $b'a$ are all in $E(G)$, so H, x , and y induce the cube, a contradiction to (1). It follows that every vertex has neighbors in at most two sectors of W .

If $G = W$, then G is basic, so suppose that $G \neq W$. If every component of $G \setminus W$ attaches to a unique sector or a clique, then G contains a proper separator, that is the ends of some sector. So, we may assume that $G \setminus W$ contains a connected component L whose neighbors in W intersects at least two sectors of W .

Since L is connected, it contains a path $P = u \dots v$ such that u has neighbors in a sector of W (say C up to symmetry), and v has neighbors in another sector of W (say A up to symmetry). Suppose that P is minimal w.r.t. this property. Then either $u = v$ and by the second paragraph of this proof, u has no neighbor in B ; or $u \neq v$ and, by minimality of P , u has neighbors only in C , v has neighbors only in A , and the interior of P is anticomplete to W . In each case, we let u' be the neighbor of u in C closest to a along C and we let v' be the neighbor of v in A closest to c along A . Note that $u' \neq a, b$ and $v' \neq b, c$ because a, b , and c have degree 3 in W . Moreover, because u' and v' exist, $ab \notin E(G)$ and $bc \notin E(G)$. This implies, $ac \notin E(G)$ for

otherwise, (W, x) would form a pyramid and be a non-proper wheel. Now, the three paths axc , B , and $aCu'uPvv'Ac$ form a theta, a contradiction. This proves Claim 2.

Claim 3. We may assume that G does not contain an extended prism.

Proof of Claim 3. Let W be an extended prism in G , with notation as in the definition. Suppose that some vertex z of $G \setminus W$ has neighbors in three distinct paths among A, A', B, B' , and C , and call Q, R, S these three paths (so $\{Q, R, S\} \subseteq \{A, A', B, B', C\}$). Since G is subcubic, it follows that z has exactly one neighbor in each of Q, R, S , and these neighbors are in interiors of these paths. It is easy to check that some hole H of W contains Q and R . By Lemma 5.1.1, z must have three neighbors in H , so H and z form a proper wheel, a contradiction to (2).

If $G = W$, then G is basic, so suppose that $G \neq W$. If every component of $G \setminus W$ is contained in only $V(A), V(A'), V(B), V(B')$, or $V(C)$, then G contains a proper separator, that is the ends the path. So, we may assume that $G \setminus W$ contains a connected component L whose neighbors in W intersects at least two paths of $\{A, A', B, B', C\}$.

Since L is connected, it contains a path $P = u \dots v$ such that u has neighbors in a path $Q \in \{A, A', B, B', C\}$ and v has neighbors in another path $R \in \{A, A', B, B', C\}$. Suppose that P is minimal w.r.t. this property. So by the minimality of P , either $u = v$ and by the second paragraph of this proof, $u = v$ has no neighbor in $\{A, A', B, B', C\} \setminus \{Q, R\}$; or $u \neq v$ and u has neighbors only in Q , v has neighbor only in R and the interior of P is anticomplete to W .

Note that each of $N_Q(u)$ and $N_R(v)$ is a vertex or an edge, because u and v have maximum degree 3 in G . For otherwise, suppose that u has two non-adjacent neighbors in Q (resp. in R). Since G is subcubic and Q (resp. R) can be completed to a hole J of W , by Lemma 5.1.1, u has three pairwise non-adjacent neighbors in J , so G contains a proper wheel, a contradiction to (2). We may now break into four cases.

Case 1: $\{Q, R\} = \{A, A'\}$ or $\{Q, R\} = \{B, B'\}$. Up to symmetry, we suppose $Q = A$ and $R = A'$. Then, P can be used to find a path from a to a' that does not contain x , and that together with B, B' and C form a prism, a contradiction.

Case 2: $\{Q, R\} = \{A, B\}$ or $\{Q, R\} = \{A', B'\}$. Up to symmetry, we suppose $Q = A$ and $R = B$. If u has two adjacent neighbors in A , then A, A', C , a subpath of B , and P form a prism. So, u has exactly one neighbor in A , and symmetrically, v has exactly one neighbor in B . So, A, B , and P form a theta.

Case 3: $\{Q, R\} = \{A, B'\}$ or $\{Q, R\} = \{B, A'\}$. Up to symmetry, we suppose $Q = A$ and $R = B'$. If u has two adjacent neighbors in A , then A, A', C , a subpath of B' , and P form a prism. So, u has exactly one neighbor in A , and symmetrically, v has exactly one neighbor in B' . So, A, B', C , and P form a theta.

Case 4: $\{Q, R\}$ is one of $\{A, C\}, \{A', C\}, \{B, C\}$ or $\{B', C\}$. Up to symmetry, we suppose $Q = A$ and $R = C$. If v has two adjacent neighbors in C , then C, B, B' , a subpath of A and P form a prism. So, v has exactly one neighbor in C . So, C, B, A' , a subpath of A , and P form a theta. This proves Claim 3.

Claim 4. We may assume that G does not contain a pyramid.

Proof of Claim 4. Let W be a pyramid with notation as in the definition (so, abc is the triangle, and x is the apex). First note that a vertex $v \in V(G \setminus W)$ cannot have neighbors in the three paths P_1, P_2 , and P_3 , for otherwise there exists a theta from v to x .

If $G = W$, then G is basic, so suppose that $G \neq W$. If every component of $G \setminus W$ attaches to a unique sector, then G contains a proper separator, that is the ends of

some sector. We may therefore assume that $G \setminus W$ contains a connected component L whose neighbors in W intersects at least two paths among P_1, P_2 , and P_3 .

Since L is connected, it contains a path $P = u \dots v$ such that u has neighbors in a path P_i (say P_1 up to symmetry), and v has neighbors in another path P_j (say P_2 up to symmetry). Suppose that P is minimal w.r.t. this property. So by minimality, either $u = v$ and by the first paragraph of this proof, $u = v$ has no neighbor in P_3 ; or $u \neq v$ and u has neighbors only in P_1 , v has neighbor only in P_2 , and the interior of P is anticomplete to W .

Note that each of $N_{P_1}(u)$ and $N_{P_2}(v)$ is a vertex or an edge. If $u = v$, this is because G contains no proper wheel by (2). If $u \neq v$, this is because u and v have degree at most 3 and we apply Lemma 5.1.1.

If $N_{P_1}(u)$ and $N_{P_2}(v)$ are both edges, then $u \neq v$ (because G is subcubic), so P_1, P_2 , and P form a prism. If each of $N_{P_1}(u)$ and $N_{P_2}(v)$ is a vertex, then P_1, P_2 , and P form a theta. So, up to symmetry, $N_{P_1}(u)$ is a vertex u' , $N_{P_2}(v)$ is an edge yz (where x, y, z, b appear in this order along P_2). If $u'x$ is not an edge, then $V(P) \cup V(W) \setminus V(zP_2b)$ induces a theta from u' to x , so $u'x$ is an edge. Hence, W and P form an extended prism, a contradiction to (3). This proves Claim 4.

Claim 5. We may assume that G does not contain a hole.

Proof of Claim 5. Let W be a hole in G . First note that a vertex $v \in V(G \setminus W)$ cannot have three neighbors in W , for otherwise v and W would form a proper wheel or a pyramid, contradicting 2 or 4. So, by Lemma 5.1.1, every vertex of $G \setminus W$ has at most one neighbor in W , or exactly two neighbors in W that are adjacent.

If $G = W$, then G is basic, so suppose that $G \neq W$. If for every component of $G \setminus W$, its neighborhood is included in some edge of W , then G has a clique separator, so suppose that for some connected component L of $G \setminus W$, there exist $a, b \in V(W)$ that are non-adjacent and that both have neighbors in L . Since L is connected, there exists a path $P = u \dots v$, such that u is adjacent to a and v is adjacent to b . We suppose that a, b, u, v and P are chosen subject to the minimality of P . Note that $u \neq v$ since a vertex in $G \setminus W$ cannot have two non-adjacent neighbors in W .

Suppose that some internal vertex of P has a neighbor x in W . So x must be adjacent to a , for otherwise a subpath of P from u to a neighbor of x in P contradicts the minimality of P . Similarly, x is adjacent to b . If a and b have two common neighbors in W , say x and y (so $W = axbya$), and x and y both have neighbors in the interior of P , then the vertices x and y together with a subpath of P contradict the minimality of P . Hence, x is the unique vertex of W with neighbors in the interior of P . If u and v each has exactly two adjacent neighbors in W , then W and P form an extended prism, a contradiction to (3). If exactly one of u or v has exactly two neighbors in W , then W and a subpath of P form a pyramid, a contradiction to (4). So, u and v both have a unique neighbor in W . Now, P and W form a proper wheel, a contradiction to (2).

So, the interior of P is anticomplete to W . Hence, P and W form a theta, a prism or a pyramid, in every case a contradiction to $G \in \mathcal{C}$, or to (4). This proves Claim 5.

Claim 6. We may assume that G does not contain a triangle.

Proof of Claim 6. Let $W = abc$ be a triangle in G . If $G = W$, then G is basic, so suppose that L is a connected component of $G \setminus W$. If $|N(L)| \leq 2$, then G has a clique separator of size at most 2, so suppose that $N(L) = \{a, b, c\}$.

Let $P = u \dots v$ be a path in L such that u is adjacent to a , v is adjacent to b , and suppose P is minimal. If $u \neq v$, then P, a , and b form a hole, a contradiction to (5), so $u = v$. By (1), u is non-adjacent to c . Hence, a path in L from u to a neighbor of c , together with a , would form a hole, a contradiction to (5). This proves Claim 6.

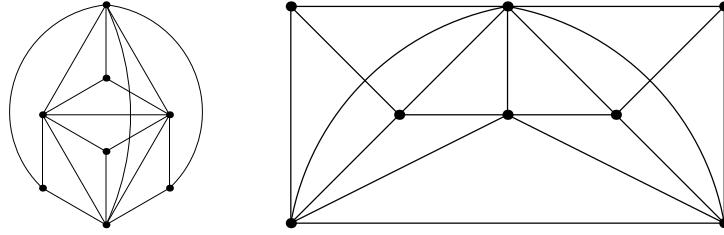


FIGURE 5.3: Two chordal graphs with clique number 4

Now, by (5) and (6), G has no cycle. So, G is a tree. It is therefore a complete graph on at most two vertices (that is basic) or it has a clique separator of size 1. \square

Let us point out that Theorem 5.1.2 is a full structural description of the class of subcubic (theta, prism)-free graphs, in the sense that every graph in the class can be obtained from basic graphs by repeatedly applying some operations: gluing along a (possibly empty) clique, and an operation called proper gluing that we describe now.

Consider two graphs G_1 and G_2 . Suppose that G_1 contains two non-adjacent vertices a_1 and b_1 of degree 3, and such that a path P_1 from a_1 to b_1 with internal vertices all of degree 2 exists in G_1 . Suppose that G_2 contains two non-adjacent vertices a_2 and b_2 of degree 2, and such that a path P_2 from a_2 to b_2 with internal vertices all of degree 2 exists in G_2 . Let G be the graph obtained from the disjoint union of G_1 and G_2 by removing the internal vertices of P_1 and P_2 , by identifying a_1 and a_2 , and by identifying b_1 and b_2 . We say that G is obtained from G_1 and G_2 by a *proper gluing*.

We omit the details of the proof and just sketch it. We apply Theorem 5.1.2. If G is basic, there is nothing to prove. If G has a clique separator, it is obtained by two smaller graphs by gluing along a clique. If G has a proper separation, then it is obtained from smaller graphs by a proper gluing. The example of non-basic graph given in Figure 5.2 for instance, can be obtained by properly gluing pyramids, proper wheels, and extended prisms.

Corollary 5.1.3

Every subcubic (theta, prism)-free graph (and therefore every even-hole-free subcubic graph) has tree-width at most 3.

Proof. The proof is by induction. Let us first prove that all basic graphs have tree-width at most 3. First observe that contracting an edge with one vertex of degree 2 preserves the tree-width. It follows that all basic graphs, except the cube and the extended prisms, have tree-width at most the tree-width of K_4 , that is 3. In Figure 5.3, we show a chordal graph J with $\omega(J) = 4$ that contains the cube or the smallest extended prism as a subgraph, showing that here again the tree-width is at most 3.

Now we explain that the two operations namely gluing along a clique and proper gluing do not increase the tree-width. The fact that gluing along a clique preserves the tree-width directly follows from Remark 1. The explanation for proper gluing is similar. If G is a graph obtained by proper gluing two graphs G_1 and G_2 along some paths P_1 of G_1 and P_2 of G_2 , then we can obtain an optimal tree decomposition of G by combining an optimal tree decomposition of G_1 and G_2 . Indeed, it is straightforward to see that an optimal tree decomposition T_1 containing a path decomposition of P_1 (formed by bags of edges of P_1) exists for G_1 . Similarly, such a tree

decomposition T_2 exists for G_2 . It is also straightforward to see that we can make the path decomposition of P_1 in T_1 and the path decomposition of P_2 in T_2 to be of the same length (simply by copying one bag in the shorter path several times until the lengths are equal). Now, gluing those optimal tree decompositions yields a tree decomposition T of G , and the width of T is the maximum of the widths of T_1 and T_2 (we may even delete the bags in the interior of the path and preserves the width). This shows that proper gluing does not increase the tree-width.

Since all basic graphs have tree-width at most 3, the result trivially follows. \square

Note also that all graphs in \mathcal{C} can be proved to be planar easily by induction. All basic graphs are planar, and gluing along clique separator of size at most 2 or along proper separator preserves the planarity of the graph. As example, we present in Figure 5.4 a planar representation of the graph shown in Figure 5.2.

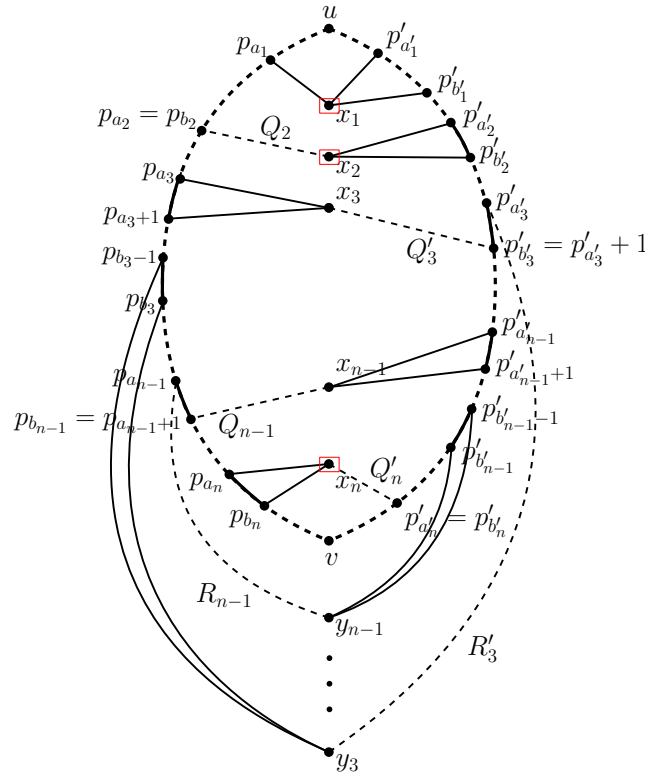


FIGURE 5.4: A planar representation of the graph shown in Figure 5.2

5.2 (Even hole, pyramid)-free graphs of maximum degree 4

In this section, we investigate a possible structure theorem describing even-hole-free graphs with maximum degree at most 4. We call *patterns*, the graphs that are represented on Figure 5.5 and Figure 5.8. Say that a graph is *basic* if it is a complete graph or a chordless cycle, or it can be obtained from one of the patterns, by replacing dashed lines with paths of length at least two. We believe that an even-hole-free graph with maximum degree 4 must be either basic or decomposable with a clique separator or a 2-join that we define below. In the end of this chapter we propose a conjecture about the decomposition of graphs in this class.

Recall that a 2-join in a graph G is a partition of $V(G)$ into two sets V_1, V_2 each of size at least 3, such that for $i = 1, 2$, V_i contains two non-empty disjoint sets A_i, B_i ,

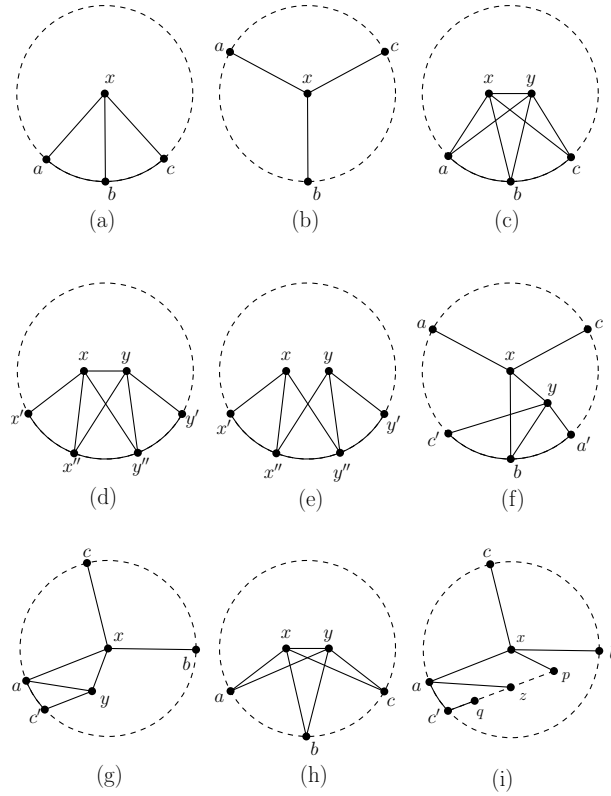


FIGURE 5.5: The wheel family: patterns in the pyramid-free case (solid lines represent edges and dashed edges represent paths of length at least two)

A_1 is complete to A_2 , B_1 is complete to B_2 , and there are no other edges between V_1 and V_2 . Moreover, for $i = 1, 2$, V_i does not consist of a path with one end in A_i , one end in B_i and no internal vertex in $A_i \cup B_i$.

We now study the structure of even-hole-free graphs of maximum degree 4 when pyramids are excluded. Let us denote by \mathcal{C} , the class of (even hole, pyramid)-free graphs with $\Delta(G) \leq 4$. A basic graph in \mathcal{C} is one of the following: a chordless cycle, a clique of size at most 5, or one of the graphs shown in Figure 5.5 that we refer to as *wheel family*. In this section, we will prove that every graph in the class \mathcal{C} either is basic, or has a clique separator, or admits the so-called *proper separator for \mathcal{C}* that we now define.

For a graph G in the class \mathcal{C} , a *proper separation for \mathcal{C}* in G is defined as in the subcubic case explained in Section 5.1, except that the items (iv) and (v) in the definition are replaced by the following condition:

- a and b have at least one neighbor in X , and in Y .

If $(\{a, b\}, X, Y)$ is a proper separation for \mathcal{C} of G , then $\{a, b\}$ is called a *proper separator for \mathcal{C}* . Note that proper separation for \mathcal{C} is in particular, a 2-join.

The *wheel family* is a set of wheels as shown in Figure 5.5. Each graph in the figure is made of a hole H and a vertex x that has three neighbors in H (so that, $V(H) \cup \{x\}$ forms a wheel), together with a component disjoint from the wheel that contains a single vertex y (for figure (a) to (h)) or a chordless path P that has neighbors in the wheel (H, x) (for figure (i)). They are called wheel of *type a to type-i* which correspond to the names represented on the figures. The hole H is called the *rim* of the wheel, and for wheels of type-a to type-h, vertices x and y are called the *centers*.

Basic lemmas

Let us describe some lemmas that will be used in proving the structure theorem for the class of (even hole, pyramid)-free graphs of maximum degree 4. First of all, we recall the following lemma that appears in [Chu+19] that will be used in our proof. Let H be a hole in a graph and let u be a vertex not in H . We say that x is major w.r.t. H if $N_H(x)$ is not included in a 3-vertex path of H . The following theorem by Chudnovsky et al. will be used several times.

Lemma 5.2.1 (see [Chu+19])

Let G be a (square, prism, pyramid, theta, even wheel)-free graph, H a hole in G , and x a major vertex w.r.t. H . If Q is a connected component of $G \setminus N[x]$, then there exists an x -sector $S = x' \dots x''$ of H such that $N(Q) \subseteq \{x', x''\} \cup (N(x) \setminus V(H))$.

Lemma 5.2.2

Let $G \in \mathcal{C}$, H be a hole in G , and v be a vertex in $G \setminus H$. Then v has at most three neighbors in H . Moreover, the following holds:

- (i) If v has two neighbors, then its neighbors induce an edge.
- (ii) If v has three neighbors, then $V(H) \cup \{v\}$ induces a wheel of type-a or type-b.

Proof. Since G contains no even wheel, v has at most three neighbors in H . If v has exactly two neighbors in H but they are non-adjacent, then $V(H) \cup \{v\}$ induces a theta, a contradiction. If v has three neighbors, then $V(H) \cup \{v\}$ induces a proper wheel (i.e. type-a or type-b), because G contains no pyramid. \square

Lemma 5.2.3

Let $G \in \mathcal{C}$, H be a hole in G , and $P = u \dots v$ be a path of length at least one in $G \setminus H$, such that both u and v have neighbors in H , and $N_H(u) \cap N_H(v) = \emptyset$. If P^* is anticomplete to H , then $N_H(P)$ induces a clique.

Proof. Suppose that $P = u \dots v$ is a path in $G \setminus H$ that satisfies the premise of the lemma. Suppose that $N_H(P)$ does not induce a clique.

Suppose that u has one neighbor in H , say u' . If v has one (resp. two neighbors), then $V(H) \cup \{u, v\}$ induces a theta (resp. a pyramid). If v has three neighbors, then $G[H \cup P]$ contains a theta from u' to v .

Now suppose that u has two neighbors in H , say u' and u'' . Note that $u'u'' \in E(G)$, because G is theta-free. By the symmetric case of the previous paragraph, v has two or three neighbors in H . If v has two neighbors, then $V(H) \cup V(P)$ induces a prism (when $N_H(u) \cap N_H(v) = \emptyset$) or an even wheel (when $N_H(u) \cap N_H(v) \neq \emptyset$). If v has three neighbors, then $V(H) \cup \{v\}$ induces a wheel, and $N_H(u)$ is contained in a long sector of the wheel (H, v) , so $G[H \cup P]$ contains a $3PC(uu'u'', v)$.

So by symmetry, u and v have three neighbors in H . We may assume that all neighbors of v are in a unique u -sector of H . Indeed, if u is not major, this is obvious, and if u is major, this follows from Lemma 5.2.1 (applied to the component of $G \setminus N[v]$ that contains v). Hence, either $G[W \cup P]$ contains a theta from u to v (when $uv \notin E(G)$) or an even wheel (when $uv \in E(G)$). \square

Lemma 5.2.4

Let $G \in \mathcal{C}$, $W = (H, x)$ be a wheel of type-b in G , and v be a vertex in $G \setminus W$ be such that $N_W(v) \neq \emptyset$. Then one of the following holds:

- (i) $N_W(v)$ induces a clique.
- (ii) $vx \notin E(G)$ and $N_W(v) \subseteq S$ for some sector S of W .
- (iii) $vx \in E(G)$ and $W \cup \{v\}$ induces either a wheel of type-f, type-g, or type-h.

Proof. Let $W = (H, x)$ be a wheel of type-b in G . Let a, b, c be the three neighbors of x , appearing in this order along H . We call A (resp. B, C) the path of H from b to c (resp. from a to c , from a to b) that does not contain a (resp. b, c). Let v be a vertex in $G \setminus W$. Suppose that $N_W(v)$ does not induce a clique.

If v has neighbors in the interior of three sectors of W , say a' in A , b' in B , and c' in C , then a, c', b, a', c , and b' appear in this order along H . If $ac' \notin E(G)$, then $xaBb'$, $xcBb'$, and $xbCc'vb'$ induce a theta (when $vx \notin E(G)$), or $xaBb'$, $xcBb'$, and xvb' induce a theta (when $vx \in E(G)$). So $ac' \in E(G)$. Symmetrically, $c'b \in E(G)$, so $\{x, a, c', b\}$ induces a square, a contradiction. So up to symmetry, we may assume that v has no neighbor in the interior of A .

Suppose that v has no neighbor in the interior of B and in the interior of C . If v is adjacent to a, b , and c , then $V(H) \cup \{x, v\}$ induces wheel of type-h. If v is adjacent to only two vertices among $\{a, b, c\}$, then $V(H) \cup \{v\}$ induces a theta, a contradiction. If v is adjacent to only a, b , or c , then $N_W(v)$ induces a clique. So, we may assume that v has neighbors in the interior of C .

If v is not adjacent to x , then by Lemma 5.2.1, the neighborhood of v in H is contained in two vertices among $\{a, b, c\}$, and since $V(H) \cup \{v\}$ cannot induce a theta, v is adjacent only to a, b , or c . Hence, $N_W(v)$ induces a clique.

So we may assume that v is adjacent to x . Recall that v has neighbor in the interior of C . Note that v cannot have three neighbors in C , because G contains no even wheel, and it also cannot have only one neighbor, because G contains no theta. So v has two neighbors in C , and in particular, $V(C) \cup \{x, v\}$ induces a wheel. Suppose that it induces a wheel of type-a. If v has no neighbor in the interior of B , then $V(H) \cup \{x, v\}$ induces a wheel of type-g. Otherwise, v has one neighbor in the interior of B , and consequently, $N_W(v) = \{x, a, b', c'\}$, where b' and c' are respectively the neighbor of a in B and C , hence $V(H) \cup \{x, v\}$ induces a wheel of type-f. \square

The proof of the structure theorem

We now ready to prove our main theorem in this section. Let us give an overview of the proof. We consider a graph G that is in the class \mathcal{C} . Recall that the basic graphs are the followings: chordless cycles, cliques of size at most 5, or one of the graphs in the wheel family (see Figure 5.5). At each step of the proof, we repeatedly pick a basic graph H , and we prove that if G contains H , then either G is equal to H itself, or G has a clique separator or a proper separator for \mathcal{C} . Afterwards, we may assume that G does not contain H . We finish the proof when all basic graphs are considered.

Theorem 5.2.5

Let G be an (even hole, pyramid)-free graph with $\Delta(G) \leq 4$. Then one of the following holds:

- G is a basic graph;

- G has a clique separator of size at most 3;
- G has a proper separator for \mathcal{C} .

Proof. Let G be an (even hole, pyramid)-free graph with $\Delta(G) \leq 4$. We may assume that G is connected, for otherwise the empty set is a clique separator. Moreover, we may assume that G is K_5 -free (because in K_5 , all vertices are of degree 4, and so if G contains K_5 , G must be K_5 and thus is basic).

Claim 1. We may assume that G does not contain a wheel of type-c and a wheel of type-d.

Proof of Claim 1. Let $W = (H, \{x, y\})$ be a wheel of type-c or a wheel of type-d in G , and S be the long sector of W . If $G = W$, then G is a basic graph. So, suppose that $G \setminus W \neq \emptyset$. Since all vertices that are not in S are of degree 4, a component of $G \setminus W$ can only attach to S . So, the ends of S form a proper separator for \mathcal{C} . This proves Claim 1.

Claim 2. We may assume that G does not contain a wheel of type-e.

Proof of Claim 2. Let $W = (H, \{x, y\})$ be a wheel of type-e in G , with notation as in Figure 5.5. We denote by $S = x'H'y'$, the long sector of W . We first study how a vertex in $G \setminus W$ attaches to W . Let v be a vertex of $G \setminus W$ that has neighbors in W . Note that v can only have neighbors in $S \cup \{x, y\}$, because the other vertices of W have degree 4. We show that v attaches to either a clique of size at most 2 of W or only to the sector S .

Suppose that $N_W(v) \not\subseteq V(S)$. So v is adjacent to x or y . Note that v cannot be adjacent to both x and y , because G does not contain a square. Hence, without loss of generality, we may assume that v is adjacent to x and not adjacent to y . Set $H' = xx'Sy'y''x$. If v has exactly one neighbor in S , then by Lemma 5.2.2, the neighbor of x in S is x' , so $N_W(x)$ induces a clique. So we may assume that $V(H') \cup \{v\}$ induces a wheel. If (H', v) is a wheel of type-a, then $N_W(v) = \{x, x', x'''\}$, where x''' is the neighbor of x' in S . Note that $xvx'''Sy'y''x$ is a hole, and x' has four neighbors in the hole (namely, x, v, x''' , and x''), inducing an even wheel, a contradiction. Hence, (H', v) is a wheel of type-b. Let v' and v'' be the neighbor of v in S , note that $v'v'' \notin E(G)$. Hence, v has two neighbors in H that are non-adjacent, contradicting Lemma 5.2.2.

Now, if $G = W$, then G is basic, so suppose that $G \neq W$. We may furthermore assume that for every connected component Q of $G \setminus W$, $N_W(Q)$ does not induce a clique, for otherwise G has a clique separator. If every component of $G \setminus W$ attaches to a unique sector, then G contains a proper separator for \mathcal{C} , that is the ends of some sector. We may therefore assume that $G \setminus W$ contains a connected component Q whose neighborhood in W contains x or y . Since Q is connected, it contains a path $P = u \dots v$ such that both u and v have neighbors in W , where the union of their neighbors is not a clique, and u is adjacent to x or y . Suppose that P is minimal w.r.t. this property. By the second paragraph of this proof, $u \neq v$.

So, without loss of generality, we may assume that $N_W(u) \subseteq \{x, x'\}$, and v has some neighbor in $(S \setminus x') \cup y$. Suppose that P^* is anticomplete to W . If v has some neighbor in $S \setminus x'$, then the attachment of P in the hole $xx'Sy''x$ contradicts Lemma 5.2.3. So, $N_W(v) = \{y\}$. In this case, if u is adjacent to x' , then the attachment of P in the hole $yy'Sx'x''y$ contradicts Lemma 5.2.3. So, $N_W(u) = \{x\}$. Consequently, the hole $x'x''yy'Sx'$, P and x form a 3PC($xx'x''$, y), a contradiction.

Therefore, some internal vertex of P has neighbors in W . By the minimality of P , $N_W(P^*) = \{x'\}$. Hence, there exists only one vertex in P^* that has neighbor in W ,

because $d_W(x') = 3$. Let us call w , such an internal vertex. By Lemma 5.2.3, the neighborhood of the paths uPw on H , wPv on H , and wPv on the hole $yx''x'Sy'y$ must induce a clique. Hence, we know that $N_W(u) = \{x\}$, $N_W(w) = \{x'\}$, and $N_W(v) = \{x'''\}$, where x''' is the neighbor of x' in S . Note that $xuPvx'''Sy'yx''x$ is a hole, and x' has four neighbors in the hole (namely, x , w , x''' , and x''), forming an even wheel. This proves Claim 2.

Claim 3. We may assume that G does not contain a wheel of type-f and type-h.

Proof of Claim 3. Let $W = (H, \{x, y\})$ be a wheel of type-f in G (we keep the notation as in Figure 5.5). Note that x is a major vertex w.r.t. H , and $N(x) \setminus V(H) = \{y\}$. Now, if $G = W$, then G is basic, so let Q be a component of $G \setminus W$. By Lemma 5.2.1, there exists an x -sector $S = x' \dots x''$ such that $N_W(Q) \subseteq \{x', x''\} \cup \{y\}$. Since y has degree 4 in W , then $N_W(Q) \subseteq \{x', x''\}$, so $\{x', x''\}$ is a proper separator for \mathcal{C} .

The proof for type-h is similar. This proves Claim 3.

Claim 4. We may assume that G does not contain a wheel of type-g.

Proof of Claim 4. Let $W = (H, \{x, y\})$ be a wheel of type-g in G , with notation as in Figure 5.5. We denote by $A = bHc$, $B = aHc$, and $C = aHb$, the three sectors of (H, x) . If $G = W$, then G is basic, so let Q be a connected component of $G \setminus W$. If $N_W(Q)$ does not contain y , then Lemma 5.2.1 implies $N_W(Q) \subseteq A$, $N_W(Q) \subseteq B$, or $N_W(Q) \subseteq C$, implying that G has a proper separator for \mathcal{C} . So suppose that N_Q contains y .

If N_Q contains some vertex of C , then again Lemma 5.2.1 implies $N_W(Q) \subseteq C$. Otherwise, since G is connected, there exists a path $P = u \dots v$ in Q such that u is adjacent to y and v has neighbors in $A \cup B$. Let u and v be chosen subject to the minimality of P . So, the interior of P is anticomplete to W , and possibly $u = v$. If v has one neighbor (say v'), then H , P , and y form a $3PC(ac'y, v')$; if v has two neighbors (say v' and v''), then $v'v'' \in E(G)$, thus H , P , and y form a $3PC(ac'y, vv'v'')$; and if v has three neighbors, then the graph induced by H , P , and y contains a $3PC(ac'y, v)$, a contradiction. This proves Claim 4.

Claim 5. We may assume that G does not contain a wheel of type-a.

Proof of Claim 5. Let $W = (H, x)$ be a wheel of type-a in G , and $N_H(x) = \{a, b, c\}$ be such that $ab, bc \in E(G)$. Denote by $S = aHc$, the long sector of W , and set $H' = xaScx$. Note that W can be seen as a wheel of form (H', b) , which is a wheel of type-a. We study how a vertex in $G \setminus W$ attaches to W . Let v be a vertex in $G \setminus W$ that has neighbor in W . We show that either v attaches to a clique of W , or the neighbors of v in W are contained in S . We suppose that the latter does not hold, i.e. $N_W(v) \not\subseteq V(S)$. Hence, v is adjacent to x or b .

Suppose first that v is adjacent to both x and b . If v has at most one neighbor in S , then it follows from Lemma 5.2.2, that $N_W(v) \subseteq \{x, b, a\}$ or $N_W(v) \subseteq \{x, b, c\}$, which yields that $N_W(v)$ induces a clique. So, we may assume that v has two neighbors in S , hence it has three neighbors in H ; so $V(H) \cup \{v\}$ induces a wheel. If (H, v) is a wheel of type-a, then $V(W) \cup \{v\}$ induces either a wheel of type-c or type-d, a contradiction to Claim 1. Otherwise, (H, v) is a wheel of type-b, so $W \cup \{v\}$ induces wheel of type-f, a contradiction to Claim 3.

So, v is adjacent only to x or only to b , and by the symmetry of (H, x) and (H', b) , we may assume that v is adjacent to x , and not adjacent to b . Note that v has at most two neighbors in S , for otherwise (H', v) is an even wheel. If v has at most one neighbor in S , then by Lemma 5.2.2, $N_W(v) \subseteq \{x, a\}$ or $N_W(v) \subseteq \{x, c\}$, which yields that $N_W(v)$ induces a clique. So, v has two neighbors in S , yielding $V(H') \cup \{v\}$ induces a wheel. In this case, by Lemma 5.2.2 applied to the hole H , the two

neighbors of v in S are adjacent. So, (H, v) is a wheel of type-a; and in particular, $N_W(v) = \{x, a, a'\}$, where a' is the neighbor of a in S . Hence, $W \cup \{v\}$ induces a wheel of type-e (with rim H' and centers b and v). Hence, our claim that a vertex of $G \setminus W$ always attaches to a clique of W or is included in the long sector of W holds.

Now, if $G = W$, then G is basic, so suppose that $G \neq W$. We may assume that for every connected component Q of $G \setminus W$, $N_W(Q)$ does not induce a clique, for otherwise G has a clique separator. If for every connected component of $G \setminus W$ is included in S , then $\{a, c\}$ is a proper separator for \mathcal{C} of G . We may therefore assume that $G \setminus W$ contains a connected component Q whose neighborhood in W contains x or b . Since Q is connected, it contains a path $P = u \dots v$ such that both u and v have neighbors in W , where the union of their neighbors is not a clique, and u is adjacent to x or b . Suppose that P is minimal w.r.t. this property. By what we proved in the previous paragraph, $u \neq v$.

Suppose that u is adjacent to both x and b . So, $N_W(u) \subseteq \{x, a, b\}$ or $N_W(u) \subseteq \{x, b, c\}$, and $N_W(P \setminus u) \subseteq S$. In particular, $N_W(u) \cap N_W(v) = \emptyset$ (because b and c are of degree 3 in W). If P^* is anticomplete to W , then by Lemma 5.2.3, $N_H(P)$ induces a clique, which implies that $N_W(u) = \{x, b\}$, and $N_W(v) = \{a\}$ or $N_W(v) = \{c\}$. This yields a contradiction to the assumption that $N_W(P)$ does not induce a clique.

So, some internal vertex of P has neighbors in W ; and by the minimality of P , $N_W(P^*) = \{a\}$ or $N_W(P^*) = \{c\}$. Hence, there exists only one internal vertex of P that has neighbor in W . Let w be such internal vertex, and up to symmetry assume that w is adjacent to c . By Lemma 5.2.3, the neighborhood of the paths uPw and wPv respectively on H also on H' induces a clique. Hence, $N_W(u) = \{x, b\}$ and $N_W(v) = \{c'\}$, where c' is the neighbor of c in $H \setminus b$. Consequently, $V(W) \cup V(P)$ induces a wheel of type-f (with rim $buPvc'Sab$ and centered at x and c), a contradiction to Claim 3. So u is adjacent only to x or only to b .

Up to symmetry, we may therefore assume that u is adjacent to x and not adjacent to b . Suppose that v is not adjacent to b . So, $N_W(u) \subseteq \{x, a\}$ or $N_W(u) \subseteq \{x, c\}$, and $N_W(v) \subseteq S$. In particular, $N_W(u) \cap N_W(v) = \emptyset$ (because a and c are of degree 3 in W). Suppose up to symmetry, that $N_W(u) \subseteq \{x, a\}$. Suppose that P^* is anticomplete to H . By Lemma 5.2.3, $N_H(P)$ induces a clique. Since $N_W(P)$ does not induce a clique, we know that $N_H(v) \not\subseteq \{a\}$, i.e. v has some neighbor in $S \setminus \{a\}$. But then the attachment of P on H' contradicts Lemma 5.2.3.

Hence, some internal vertex of P has a neighbor in W . In this case, by the minimality of P , $N_W(P^*) \subseteq \{a, b\}$ or $N_W(P^*) \subseteq \{b, c\}$. So there exists a unique vertex w in P that has a neighbor in W (because b has degree 3 in W). By the symmetry of a and c , we may assume that $N_W(P^*) \subseteq \{a, b\}$. Note that w cannot be adjacent to both a and b , for otherwise, $N_W(u) \cup N_W(v) = \{x\}$, which is not possible because x already has degree 3 in W . If $N_W(w) = \{b\}$, then $N_W(v) = \{c\}$ because $N_H(wPv)$ induces a clique by Lemma 5.2.3. Since $N_W(P)$ does not induce a clique, $N_W(u) = \{x, a\}$. But then, then attachment of the path P on H' contradicts Lemma 5.2.3. So, $N_W(w) = \{a\}$. By Lemma 5.2.3, the paths uPw and wPv respectively attach on a clique of the hole H and H' . Hence, $N_W(u) = \{x\}$ and $N_W(v) = a'$, where a' is the neighbor of a in $H \setminus a$. Consequently, $V(W) \cup V(P)$ induces wheel of type-g (with rim $xuPva'Scx$ and centers a and b), a contradiction to Claim 4.

So, v is adjacent to b . Hence, by what we proved in the beginning, $N_W(u)$ induces a clique and $N_W(v)$ induces a clique. Up to symmetry, we may assume that $\{x\} \subseteq N_W(u) \subseteq \{x, a\}$ and $\{b\} \subseteq N_W(v) \subseteq \{b, c\}$. Suppose that P^* is anticomplete to W . If $a \notin N_W(u)$ or $c \notin N_W(v)$, then $N_W(P)$ induces a clique, a contradiction to the assumption. Otherwise, both $a \in N_W(u)$ and $c \in N_W(v)$, and the attachment of P in H contradicts Lemma 5.2.3.

Hence, some internal vertex of P has a neighbor in W . Note that in this case, $N_W(P^*) = \{a\}$ or $N_W(P^*) = \{c\}$. So, there is only one internal vertex of P that has a neighbor in W . Let $w \in P^*$ be that vertex, and up to symmetry, assume that w is adjacent to a . By Lemma 5.2.3, the neighborhood of the path uPw on the hole H' and the neighborhood of the path wPv on H respectively induces a clique. Hence, $N_W(u) = \{x\}$, $N_W(w) = \{a\}$, and $N_W(v) = \{b\}$, so the definition of P is contradicted (i.e. $N_W(P)$ induces a clique). This proves Claim 5.

Claim 6. We may assume that G does not contain a wheel of type-i.

Proof of Claim 6. As shown in Figure 5.5, a wheel of type-i is made of a wheel of type-b, say (H, x) , with an additional path $p \dots q$ that attaches to the wheel, in the way as shown in the figure. In Figure 5.6, we present two isomorphic drawings of the wheel. We denote: $A = b \dots c$, $B = p' \dots c$, $C = c' \dots b$, $L_1 = p \dots z$, and $L_2 = c' \dots z$. Set $H' = xpL_1zL_2qc'Cb$.

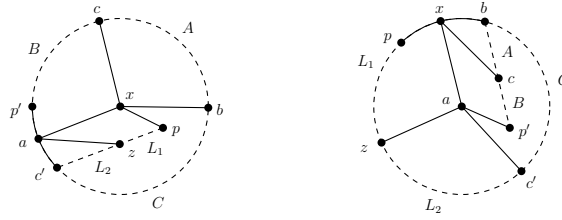


FIGURE 5.6: Two isomorphic drawings of wheel of type-i (with rim H and center at x ; and with rim H' and center at a)

Now if $G = W$, then G is basic, so let Q be a connected component of $G \setminus W$. Note that x is a major vertex w.r.t. H , and a is a major vertex w.r.t. H' . Suppose that $N_W(Q)$ does not induce a clique. If $N_W(Q)$ contains some vertex of A , then $Q' = Q \cup A$ is a component of $G \setminus N[x]$. Hence, Lemma 5.2.1 implies that $N(Q') \subseteq \{b, c, p\}$. Suppose that $N(Q')$ contains p , which means that $N_W(Q)$ contains p . Since Q is connected, there exists a path $P = u \dots v$ in Q such that u is adjacent to p and v has some neighbor v' in A . Hence, the graph containing Q , p and v' is a connected component of $G \setminus N[a]$ (recall that a is a major vertex w.r.t. H'), thus contradicting Lemma 5.2.1. Hence, $N_W(Q')$ does not contain p , so $N_W(Q) \subseteq A$.

Similarly, we can prove that $N_W(Q) \subseteq B$ when $N_W(Q)$ contains some vertex of B . Also, by the isomorphism of the two wheels presented in Figure 5.6, we have a similar consequence when $N_W(Q)$ contains some vertex of L_1 or L_2 . The remaining case is therefore when $N_W(Q)$ contains only vertices of C , which implies that $N_W(Q) \subseteq C$.

Hence, the neighborhood of Q in W is contained in either A , B , C , L_1 , or L_2 , which respectively yields that either $\{b, c\}$, $\{p', c\}$, This proves Claim 6.

Claim 7. We may assume that G does not contain a wheel of type-b.

Proof of Claim 7. Let $W = (H, x)$ be a wheel of type-b in G . Let a, b, c , be the three neighbors of x , appearing in this order along H (see Figure 5.5). We call A (resp. B , C) the path of H from b to c (resp. from a to c , from a to b) that does not contain a (resp. b , c). By Lemma 5.2.4, Claim 3, and Claim 4, for any vertex v in $G \setminus W$ that has neighbors in W , either $N_W(v)$ induces a clique or $N_W(v)$ are included in a unique sector of W (particularly when v is not adjacent to x).

Now, if $G = W$, then G is basic, so suppose that $G \neq W$. We may furthermore assume that for every connected component Q of $G \setminus W$, $N_W(Q)$ does not induce a clique, for otherwise G has a clique separator. If every component of $G \setminus W$ attaches to a unique sector of W , then G contains a proper separator for C , that is the ends of

some sector. We may therefore assume that $G \setminus W$ contains a connected component Q whose neighborhood in W contains x or $N_W(Q)$ intersects at least two sectors of W .

First suppose that $N_W(Q)$ contains x . Since Q is connected, there exists a path $P = u \dots v$ in Q such that u is adjacent to x , and v is adjacent to some vertex of H , and $N_W(P)$ does not induce a clique. By the first paragraph of this proof, we know that $N_W(u)$ induces a clique, and $N_W(v)$ is contained in a sector of W . This implies $N_W(u) \cap N_W(v) = \emptyset$, because the only vertex in W that can be a common neighbor of u and v is a , b , or c , but they all have degree 3 in W (so each of them can only have at most one more neighbor in $G \setminus W$). Let P be chosen such that its length is minimal. By Lemma 5.2.3, we know that some internal vertex of P has neighbors in H . Also by the minimality of P , we know that $N_W(P^*) = \{a\}$, or $N_W(P^*) = \{b\}$, or $N_W(P^*) = \{c\}$. Hence, there exists only one vertex in P^* that has a neighbor in H . Let w be the vertex, and up to symmetry, assume that $N_W(w) = \{a\}$. By Lemma 5.2.3, we know that $N_W(uPw)$ and $N_W(wPv)$ respectively induces a clique. So, $N_W(u) = \{x\}$ and $N_W(v) = \{c'\}$, where c' is a neighbor of a in H . This means that $V(W) \cup V(P)$ induces a wheel of type-i, a contradiction to Claim 6.

So $N_W(Q)$ does not contain x . It then follows from Lemma 5.2.1, that $N_W(Q)$ is contained in a sector of W . This proves Claim 7.

Note that now we may assume that G does not contain a theta, a prism, a pyramid, or a wheel, so G is the so-called universally signable graph that was discussed in Subsection 1.2.4 of Chapter 1. Hence, by using the characterization of universally signable graphs (cf. Theorem 1.2.6), we may assume that G is either a hole or a clique on 5 vertices (because otherwise, G has a clique separator and the theorem holds). In the following, we give more rigorous proof of this characterization of graphs.

Claim 8. We may assume that G does not contain a hole.

Proof of Claim 8. Let W be a hole in G . First note that a vertex $v \in V(G \setminus W)$ cannot have three neighbors in W , for otherwise v and W would form a wheel of type-b or a pyramid. So, by Lemma 5.2.2, every vertex of $G \setminus W$ has at most one neighbor in W , or exactly two neighbors in W that are adjacent.

If $G = W$, then G is basic, so suppose that Q is a component of $G \setminus W$. If $N_W(Q)$ is included in some edge of W , then G has a clique separator, so suppose that there exist $a, b \in V(W)$ that are non-adjacent and that both have neighbors in Q . Since Q is connected, there exists a path $P = u \dots v$, such that u is adjacent to a and v is adjacent to b . We suppose that a, b, u, v and P are chosen subject to the minimality of P . Note that $u \neq v$ since a vertex in $G \setminus W$ cannot have two non-adjacent neighbors in W .

If some internal vertex of P has a neighbor x in W , then x must be a common neighbor of a and b , for otherwise a subpath of P contradicts the minimality of P . If a and b have two common neighbors in H , say x and y (so $W = axbya$), and x and y both have neighbors in the interior of P , then the vertices x and y together with a subpath of P contradict the minimality of P . Hence, x is the unique vertex of W with neighbors in P^* . If u has exactly two adjacent neighbors in W , then W and a subpath of P form a pyramid. So, u has a unique neighbor in W , and symmetrically, so does v . Now, P and H form a wheel of type-b, a contradiction to (7).

So, P^* is anticomplete to W . Hence, P and W form a theta, a prism, or a pyramid, in every case a contradiction to $G \in \mathcal{C}$. This proves Claim 8.

Claim 9. We may assume that G does not contain a K_4 .

Proof of Claim 9. Let $W = abcd$ be a K_4 in G . If $G = W$, then G is basic, so suppose that Q is a component of $G \setminus W$. If $|N_W(Q)| \leq 3$, then G has a clique separator of size at most 3, so suppose that $N_W(Q) = \{a, b, c, d\}$.

Let $P = u \dots v$ be a path in Q such that u is adjacent to a , v is adjacent to b , and suppose P is minimal. If $u \neq v$, then P , a , and b form a hole, a contradiction to (8), so $u = v$. Since G is K_5 -free, u is non-adjacent either to c or d . Up to symmetry, we may assume that $uc \notin E(G)$. Hence, a path in C from u to a neighbor of c , together with a , would form a hole, a contradiction to (8). This proves Claim 9.

Claim 10. We may assume that G does not contain a triangle.

Proof of Claim 10. Let $W = abc$ be a triangle in G . If $G = W$, then G is basic, so suppose that Q is a component of $G \setminus W$. If $|N_W(Q)| \leq 2$, then G has a clique separator of size at most 2, so suppose that $N_W(Q) = \{a, b, c\}$.

Let $P = u \dots v$ be a path in Q such that u is adjacent to a , v is adjacent to b , and suppose P is minimal. If $u \neq v$, then P , a , and b form a hole, a contradiction to (8), so $u = v$. By Claim 9, u is non-adjacent to c . Hence, a path in Q from u to a neighbor of c , together with a and c , would form a hole, a contradiction to Claim 8. This proves Claim 10.

Now, by Claim 8 and Claim 10, G has no cycle. So, G is a tree. It is therefore a complete graph on at most two vertices (that is basic) or it has a clique separator of size 1. \square

Note that Theorem 5.2.5 implies that (even hole, pyramid)-free graphs of maximum degree 4 that contains no pyramid have tree-width bounded by some constant. In [Abo+cs], the following theorem is given.

Theorem 5.2.6 ([Abo+cs])

There exists some constant c , such that every (even hole, pyramid)-free graph of maximum degree 4 has tree-width less than c .

We remark that Theorem 5.2.6 is stated differently in the paper (see Theorem of [Abo+cs]). The proof relies on the fact that even-hole-free graphs with no clique minor of size n have tree-width bounded in terms of some function that depends on n , and moreover, it is proved in the paper that (even hole, pyramid)-free graphs with maximum degree 4 contain no K_6 -minor. The constant c here depends on the function used in the proof of the minor-freeness of subcubic even-hole-free graphs (Theorem 1.1 of [Abo+cs]), which is computable, and it relies heavily on the result obtained by Fomin, Golovach, and Thilikos (see Corollary 3.3 of [Abo+cs]).

Nevertheless, we can compute the exact tree-width of graphs in the class, in a similar way as for the subcubic case that we discuss in Section 5.1.

Corollary 5.2.7

Every (even hole, pyramid)-free graph of maximum degree 4 has tree-width at most 4.

Proof. Note that the two operations gluing along a clique and proper gluing do not increase the tree-width. It is then enough to show that every basic graph in the class has tree-width at most 4. This would complete the proof.

This property is trivially satisfied when the basic graph is either a chordless cycle or cliques of size at most 5. We now check that every graph in the wheel family shown in Figure 5.5 satisfies this property. Recall that contracting an edge with one vertex of degree 2 preserves the tree-width. So, we need to show that for every wheel W of a certain type in the wheel family (see Figure 5.5), and W' be the graph obtained by contracting all edges whose an end is of degree 2 in W , there exist a

chordal graph of clique number at most 5 that contains W' . In Figure 5.7, we provide a list of chordal graphs of clique number 5. The graph shown in Figure 5.7 is a chordal graph that contains W' as a subgraph (the naming of each figure corresponds to the type of the corresponding wheel from which W' is obtained). To check that the given graphs on Figure 5.7 are chordal, we make use of a characterization of chordal graphs, namely: every chordal graph admits a “perfect elimination ordering”. In the following, we give a perfect elimination ordering for every “contracted” wheel.

- Type a or type b: (a, b, c, x)
- Type c or type-h: (a, b, c, x, y)
- Type d or type e: (y'', x'', x', x, y, y')
- Type g or type-i: (c, c', a, y, a, b)
- Type f: (x, a', c, y, a, b, c')

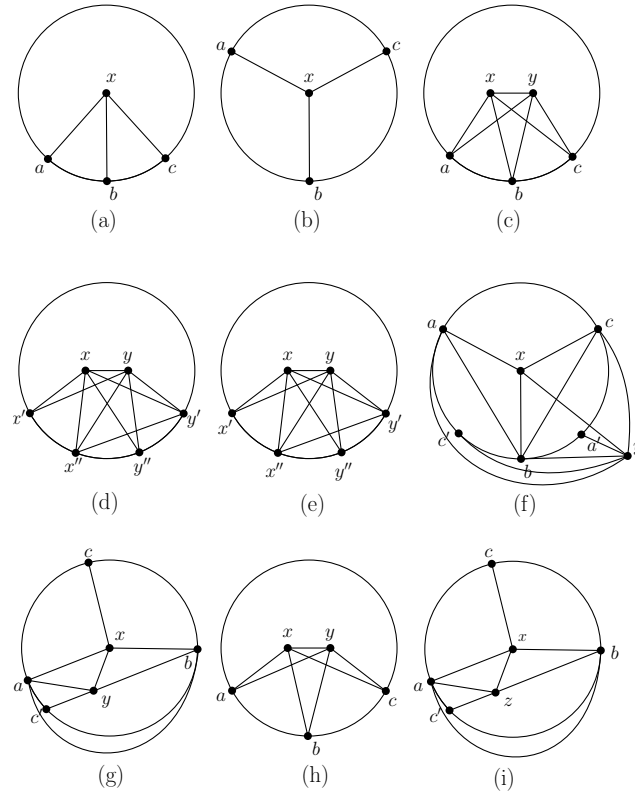


FIGURE 5.7: Chordal graphs containing the contraction of the wheel family

□

5.3 Discussion

The structure of even-hole-free graphs of maximum degree 4 when pyramids are allowed is more complex. We now present an approach to obtain a structure theorem for even-hole-free graphs of maximum degree 4 when pyramids are allowed. From

our initial study, we find that there are many possibilities of “basic graphs” when pyramids are allowed. In order to describe the structure in a more convenient way, we need to introduce some definition.

Strip system

A graph G is an (X, X', Y) -strip if:

- (i) $V(G) = Y \cup X \cup X'$.
- (ii) Y , X , and X' are disjoint.
- (iii) X and X' are non-empty (possibly Y is empty).
- (iv) For every vertex v of Y , v is in some chordless path with one end is in X , and the other end is in X' , and no interior vertex is in $X \cup X'$ (such a chordless path is called an XX' -rung).
- (v) Every vertex of X (resp. X') is contained in a chordless path of G , where one end is in X , the other is in X' , and no interior vertex is in $X \cup X'$ (possibly, this path is an edge).

An s -graph is a triple $S = (V, E, F)$ such that $(V, E \cup F)$ is a graph and $E \cap F = \emptyset$. We say that E is the set of *non-subdivisible edges* and F is the set of *subdivisible edges of S* . The graph $(V, E \cup F)$ is the *underlying graph of S* and we denote it by G_S . Any graph obtained from G_S by subdividing edges from F is a *realization of S* . A path in the realization that is obtained by subdividing a subdivisible edge of S is also called a *rung*. A *strip system obtained from S* is any graph Π with the following properties.

- For each vertex $v \in V(S)$, there is a non-empty set $X_v \subseteq V(\Pi)$ (we call it the *blob* of v).
- For each edge $e \in F(S)$, there is a (possibly empty) set $Y_e \subseteq V(\Pi)$.
- The sets X_v with $v \in V(S)$ and Y_e with $e \in E(S)$ are disjoint, and $V(\Pi) = \bigcup_{v \in V(S)} X_v \cup \bigcup_{e \in F(S)} Y_e$.
- For every non-subdivisible edge $uv \in E(S)$, X_u is complete to X_v .
- For every subdivisible edge $uv \in F(S)$, $\Pi[X_u \cup X_v \cup Y_{uv}]$ is an (X_u, X_v, Y_{uv}) -strip.
- For every $u \in V(S)$, and $v \in V(S)$ such that $uv \notin E(S) \cup F(S)$, X_u is anticomplete to X_v .
- For every $v \in V(S)$, and $e \in F(S)$ such that $v \notin e$, X_v is anticomplete to Y_e .
- For every distinct $e, f \in F(S)$, Y_e is anticomplete to Y_f .

We remark that any realization of an s -graph S can be viewed as a strip system (where every blob is of size 1). Let Π be a strip system that is obtained from S , and G be a graph that is obtained from Π by the following operations:

- pick one vertex v from every blob X of Π ;
- for every vertices u and v taken from some blobs X and X' , if X is complete to X' , then $uv \in E(G)$; otherwise if $X \cup X'$ is in some (X, X', Y) -strip, we pick an XX' -rung R_{uv} with end-vertices u and v from the strip and include it to G .

Then G is a subgraph of Π induced by the union of the sets $\{u : X_u \text{ is a blob of } \Pi\}$ and $\{R_{uv} : (X_u, X_v, Y_{uv}) \text{ is a strip of } \Pi\}$. Moreover, G is a realization of S . We say that G is *extracted* from S .

Note that for every strip system Π that is obtained from S , there exists a graph J that is extracted from Π and is a realization of S . In particular, for every rung R (resp. for every strong triangle T) of Π , there exists such a graph J such that $V(R) \subseteq V(J)$ (resp. $V(T) \subseteq V(J)$).

A *pattern* is a realization of one of the s-graphs. In Figure 5.5, we give a list of some possible patterns for even-hole-free graphs with maximum degree 4 when the graph contains a pyramid. Other patterns can be obtained from a pattern in Figure 5.8 by contracting dashed edges. Observe that for every pattern, every vertex in the pattern belongs to some pyramid that is contained in the pattern.

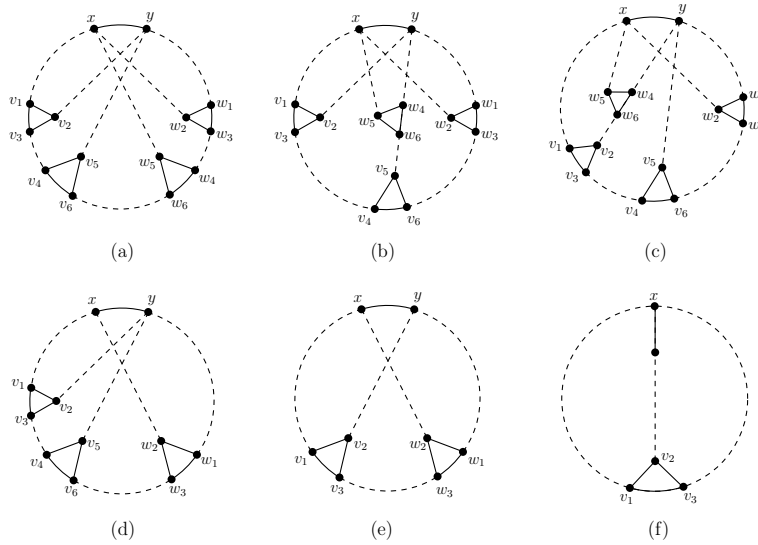


FIGURE 5.8: Some patterns that contain pyramid, dashed edges represent paths of length at least two

However, we are not sure that our list of patterns is complete for our class, but we believe that the real list is close to it and, above all, is finite. This should imply that the tree-width is bounded. We conjecture that every even-hole-free graph of maximum degree 4 is either a strip system obtained from a realization of one of the patterns listed on Figure 5.8, or it has either a clique separator or a “variant of” proper separator. We have made a trial to prove the conjecture, and we suspect that the length of the proof is about 20 pages long. The proof is done by the case by case analysis as what we did for the pyramid-free case. It is still in progress, and we do not include it in this thesis, .

Furthermore, we wonder whether a similar approach can be extended to even-hole-free graphs of maximum degree d for any fixed integer d . Observe that for $d = 3$, this is what we actually do in Theorem 5.1.2, since the list of basic graphs can be seen as obtained by a finite list of patterns and proper separator is a special case of 2-join. For $d \geq 5$, rings (already defined in Chapter 2) become a problem, but an extension of the notion of 2-join might lead to a true statement. We suspect a structure theorem of the following fashion might be true.

Conjecture 5.3.1. *Let G be a (theta, prism, even wheel, square)-free graph with maximum degree d , then one of the following holds.*

- (i) G is a clique on at most d vertices;

- (ii) G is a strip system obtained from some s -graph;
- (iii) G has a clique separator.

However, from our study of the maximum degree 3 and 4, we observe that the number of cases to be checked in the case analysis increase as the maximum degree d increases. Hence we suspect that in order to prove Conjecture 5.3.1 for any possible value of d might be difficult to be done by hand, and a computer program that can check every possible case (in a reasonable time) as what we did in the proofs of maximum degree 3 and 4, would be very useful. We need a computer program with the following specification: given any graph as an input, it can check whether the graph contains any of the forbidden configurations (i.e. theta, prism, even wheel, square). Starting from a hole, we check what kind of attachments (attaching a vertex or a path with internal vertices are all of degree 2) yield a strip system. If by attaching a vertex or a path, we obtain a graph that is in the class, but it has no clique separator and it does not belong to any of the existing strip systems, then we define a new strip system for such a graph. This way, we obtain a list of strip systems for which the conjecture above holds, or possibly, so many patterns are discovered that they suggest an infinite list of patterns, hence disproving the conjecture.

Chapter 6

Conclusion and open problems

In this last chapter, we review what we have discussed throughout the thesis, and we state some open questions.

In Chapter 3, we discuss how excluding big clique affects the general structure of even-hole-free graphs. Specifically, we prove that (even hole, K_4)-free graphs have unbounded tree-width. We actually prove it for a more restricted class, because layered wheels are pyramid-free. This work was initially motivated by the result of Cameron et al. [Cam+18] who proved that (even hole, triangle)-free graphs have tree-width bounded by 5. We also give an upper bound on the tree-width of the layered wheels, which is logarithmic in the size of the layered wheels. This motivates a conjecture whether the tree-width of (even hole, K_4)-free graphs are bounded by some logarithmic function in the size of the input graphs.

In addition to (even hole, K_4)-free graphs, in Chapter 3 we also study the class of (theta, triangle)-free graphs. Note that theta-free graphs form a superclass of even-hole-free graphs, so the class of (theta, triangle)-free graphs and the class of (even hole, K_4)-free graphs intersect. The intersection of these two classes forms the class of (even hole, triangle)-free graphs, and for this class, the tree-width is bounded. Our study indicates that the class of (theta, triangle)-free graphs is in some sense, similar to the class of (even hole, K_4)-free graphs. The two classes share similar properties. The results which we prove for (even hole, K_4)-free graphs are first proved for (theta, triangle)-free graphs, and by mimicking the technique used in the proof, we prove a similar result for (even hole, K_4)-free graphs. Finally, we remark that all results that are mentioned in the previous paragraph also hold for (theta, triangle)-free graphs. We also propose similar conjectures for (theta, triangle)-free graphs.

In Chapter 4, we still study the class of (even hole, K_4)-free graphs and the class of (theta, triangle)-free graphs. We show that when excluding more induced subgraphs, there is an upper bound on the tree-width. This study was begun when we were trying to answer the “logarithmic tree-width” conjecture mentioned above. We prove that excluding “subdivision of claw” (which we denote by $S_{i,j,k}$ in the corresponding chapter) yields graphs with tree-width bounded in terms of the size of the subdivided claw. For this, we derive a new method to bound the tree-width: we prove that every graph of large tree-width must contain a large clique or a minimal separator of large cardinality — which was applicable in the class we studied, and possibly for other classes of graphs.

Finally, in Chapter 5, we study the class of even-hole-free graphs when the maximum degree is bounded, in particular, for maximum degree equals 3. This study was motivated by an observation made in the construction of layered wheels. In layered wheels, we observe that in order to increase the lower bound on the tree-width, the constructed graph needs to contain a big clique minor. The existence of a large clique minor in a layered wheel forces the graph to contain a vertex of high degree. This raises two principal questions: the existence of bounds on the tree-width of

even-hole-free graphs, in the first case, when the graphs have no large clique minor; and in the second case, when the maximum degree is bounded. We have discussed the subcubic case and the case of maximum degree 4 that contains no pyramid. For each class, we give a full structure theorem that leads to proving that the tree-width of the class is bounded. In the end of Chapter 5, we propose a conjecture for the general case, namely the class of even-hole-free graphs of maximum degree 4.

In the following section, we mention more specifically some conjectures related to the tree-width of some hereditary graph classes. We note that some other open problems related to the three chapters we describe above are mentioned at the end of the corresponding chapters.

6.1 What can be observed from layered wheels?

Layered wheels provide a family of even-hole-free graphs with unbounded tree-width. However, on the positive side, we note that layered wheels need many vertices to increase the tree-width. More specifically, a layered wheel G is made of $l + 1$ layers, where l is an integer. Every layer is a path and $|V(G)| \geq 2^l$ (see Lemma 3.2.2), $l \leq \text{tw}(G) \leq 2l$ (see Theorems 3.2.12 and 3.4.4). So, the tree-width of a layered wheel is “small” in the sense that it is logarithmic in the size of its vertex set. We wonder whether such characteristic is general in the sense of the following conjecture.

Conjecture 6.1.1. *There exists a constant c such that for any $(\theta, \text{triangle})$ -free graph G , the tree-width of G is at most $c \log |V(G)|$.*

For this class of graphs, we also need a large number of vertices to grow the tree-width, so we propose the following conjecture.

Conjecture 6.1.2. *There exists a constant c such that for any $(\text{even hole}, K_4)$ -free graph G , the tree-width of G is at most $c \log |V(G)|$.*

If Conjecture 6.1.2 holds, then the maximum independent set problem is polynomial-time solvable for even-hole-free graphs with no K_4 , due to the following theorem. The same consequence holds for $(\theta, \text{triangle})$ -free graphs if Conjecture 6.1.1 is true.

Theorem 6.1.3 ([Bod88])

For any graph G , given a tree decomposition of width w , the Weighted Maximum Independent Set can be solved in time $\mathcal{O}(2^w \cdot n)$.

In an attempt to answer this conjecture, we suspect that the existence of a family of graphs \mathcal{F}_l with the following properties is sufficient:

- for every $H \in \mathcal{F}_l$, we have $|V(H)| \geq r^l$ for some $r > 1$; and
- every $(\text{even hole}, K_4, \mathcal{F}_l)$ -free graph has tree-width at most $t \cdot l$ for some $t > 0$.

We also observe that even-hole-free layered wheels contain diamonds. Recall that each of $(\text{even hole}, K_4)$ -free graphs and $(\text{even hole}, \text{diamond})$ -free graphs have unbounded rank-width. We therefore propose the following conjecture.

Conjecture 6.1.4. *Even-hole-free graphs with no K_4 and no diamonds have bounded tree-width.*

In Chapter 5, we have seen that even-hole-free graphs of maximum degree at most 3 have tree-width at most 3. Abrishami, Chudnovsky, and Vušković [ACV20] proved the following generalization of such result ¹.

Theorem 6.1.5 ([ACV20])

For every $d \geq 0$, there exists an integer k such that C_4 -free odd-signable graphs with maximum degree at most d have tree-width at most k .

Open questions. For every fixed integer $t \geq 4$, it is not known whether (theta, triangle)-free graphs of maximum degree t have bounded tree-width. For $t = 1, 2$, the tree-width is trivially bounded; and for $t = 3$, it follows from Corollary 4.3 in [Abo+cs], which says that every subcubic even-hole-free graph has tree-width at most 3. Furthermore, Conjecture 5.3.1 is still open. It is now less interesting to prove this conjecture, because even-hole-free graphs of bounded maximum degree are known to have bounded tree-width [ACV20]. However, it is still intriguing, since it would give another insight into how even-hole-free graphs are structured as the maximum degree increases.

6.2 The grid-minor-like theorem

One key result in the graph-minors seminal project of Robertson and Seymour is the celebrated “Grid-Minor Theorem”. The theorem states that for every grid H , every graph whose tree-width is large enough in terms of the size of $V(H)$, contains H as a minor. The following is a theorem similar to the Grid-Minor Theorem, in terms of subgraph containment.

Theorem 6.2.1 (Robertson and Seymour [RS86])

There is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that every graph of tree-width at least $f(k)$ contains a $(k \times k)$ -wall as a subgraph.

One big open question in the area of classes of graphs characterized by forbidden induced subgraphs is the relation between graphs having large tree-width with the existence of a list of forbidden configurations, i.e. whether a theorem similar to the grid-minor theorem exists in the context of induced subgraphs:

Does there exist a list \mathcal{H} of graphs such that for some function f , every graph with tree-width at least $f(k)$ contains a graph $H \in \mathcal{H}$ as an induced subgraph?

Theorem 6.2.1 cannot be strengthened to finding walls as *induced* subgraphs in general, because the complete graph K_n has tree-width $n - 1$ and only contains complete graphs as induced subgraphs.

A possible list of induced subgraphs that might be contained in a graph with large tree-width, as informally proposed by Zdeněk Dvořák is the following: K_k , $K_{k,k}$, a subdivision of the $(k \times k)$ -wall, or the line graph of some subdivision of the $(k \times k)$ -wall. Each of those graphs is not contained in any of the other and has tree-width which grows as a function of k . Note that the two first graphs imply the

¹Recall that odd-signable graphs are equivalent to (theta, prism, even wheel)-free graphs and it forms a superclass of even-hole-free graphs.

existence of a big clique minor and those four graphs all imply the existence of a big grid minor. Layered wheels are graphs for which tree-width grows logarithmically in terms of the size of the graphs. They moreover contain none of the graphs in \mathcal{H} as an induced subgraph: they contain no K_k for large k because of being K_4 -free, and no $K_{k,k}$, a subdivision of the $(k \times k)$ -wall, or its line graph because of being even-hole-free. This means that the aforementioned list \mathcal{H} given by Dvořák is not complete. A “new” graph that could be included in the list \mathcal{H} might be highly related to layered wheels. In the next paragraphs, we give some variants of Dvořák’s question.

Conjecture 6.1.1 and Conjecture 6.1.2 that are given in the end of Chapter 3 reflect our belief that constructions similar to layered wheel must have an exponential number of vertices (exponential in the tree-width). It suggests the following variant of Dvořák’s question:

Is it true that for some constant $c > 1$ and some function f , every graph with tree-width at least $f(k)$ contains either K_k , $K_{k,k}$, a subdivision of the $(k \times k)$ -wall, the line graph of some subdivision of the $(k \times k)$ -wall, or has at least $c^{f(k)}$ vertices?

In the paper of Aboulker et al. [Abo+cs], where the proof of Theorem 3.5.1 is given, the following stronger result which implies Theorem 3.5.1 is proved. Theorem 6.2.2 implies that *(theta, prism)-free* graphs (which is a superclass of even-hole-free graphs) excluding a fixed minor have bounded tree-width because a graph excluding a theta and a prism cannot contain a subdivision of a $(k \times k)$ -wall or the line graph of a chordless subdivision of a $(k \times k)$ -wall as an induced subgraph, as those two graphs respectively contain thetas and prisms. We remark that in the paper, an “ $(k \times k)$ -wall” is defined differently than the definition we use in this thesis. In the following theorem, a subdivision of a wall is *chordless* if no cycle with a chord is contained in the graph.

Theorem 6.2.2 (Induced-grid theorem for minor-free graphs [Abo+cs])

For every graph H , there is a function $f_H: \mathbb{N} \rightarrow \mathbb{N}$ such that every H -minor-free graph of tree-width at least $f_H(k)$ contains a subdivision of a $(k \times k)$ -wall or the line graph of a chordless subdivision of a $(k \times k)$ -wall as an induced subgraph.

This theorem can also be seen as an advancement towards the question about the “induced” version of the grid-minor theorem that we mention in the previous paragraph. Recall that the construction of layered wheels requires the presence of a high maximum degree to increase the lower bound on the tree-width and that even-hole-free graphs of bounded degree have bounded tree-width [ACV20]. This suggests the following conjecture:

Conjecture 6.2.3. *There is a function f such that if $\text{tw}(G) > f(k)$, either G contains a subdivision of a $(k \times k)$ -wall, the line graph of a subdivision of a $(k \times k)$ -wall, or a vertex of degree at least k .*

Kristina Vušković observed that $K_{k,k}$ is a (prism, pyramid, wheel)-free graph, or equivalently an *only-theta graph* (because thetas are the only Truemper configuration contained in $K_{k,k}$). Moreover, walls are only-theta graphs, line graphs of subdivisions of walls are only-prism graphs, and triangle-free layered wheels are only-wheel graphs. Observe that complete graphs contain no Truemper configuration, so they are simultaneously only-prism, only-wheel, and only-theta. One may

wonder whether a graph with large tree-width should contain an induced subgraph of large tree-width with a restricted list of induced subgraphs isomorphic to one of the Truemper configurations. Hence, we propose the following stronger conjecture.

Conjecture 6.2.4. *There is a function f such that if $\text{tw}(G) > f(k)$, either G contains a K_k , $K_{k,k}$, a subdivision of a $(k \times k)$ -wall, the line graph of a subdivision of a $(k \times k)$ -wall, or a wheel with at least k spokes.*

Bibliography

- [Abo+cs] P. Aboulker, I. Adler, E. J. Kim, N. L. D. Sintiari, and N. Trotignon. “On the tree-width of even-hole-free graphs”. In: *CoRR abs/2008.05504* (2020, to appear in *European Journal of Combinatorics*). arXiv: 2008 . 05504.
- [ACP87] S. Arnborg, D. G. Corneil, and A. Proskurowski. “Complexity of finding embeddings in a k -tree”. In: *SIAM Journal of Discrete Mathematics* 8.2 (1987), pp. 277–284.
- [ACV20] T. Abrishami, M. Chudnovsky, and K. Vušković. “Even-hole-free graphs with bounded degree have bounded treewidth”. In: *CoRR abs/2009.01297* (2020). arXiv: 2009.01297.
- [Add+08] L. Addario-Berry, M. Chudnovsky, F. Havet, B. A. Reed, and P. D. Seymour. “Bisimplicial vertices in even-hole-free graphs”. In: *J. Comb. Theory, Ser. B* 98.6 (2008), pp. 1119–1164.
- [Add+20] L. Addario-Berry, M. Chudnovsky, F. Havet, B. A. Reed, and P. D. Seymour. “Corrigendum to “Bisimplicial vertices in even-hole-free graphs””. In: *J. Comb. Theory, Ser. B* 142 (2020), pp. 374–375.
- [Adl+17] I. Adler, N.-K. Le, H. Müller, M. Radovanović, N. Trotignon, and K. Vušković. “On rank-width of even-hole-free graphs”. In: *Discret. Math. Theor. Comput. Sci.* 19.1 (2017).
- [Ale04] V. E. Alekseev. “Polynomial algorithm for finding the largest independent sets in graphs without forks”. In: *Discrete Applied Mathematics* (2004).
- [Ale83] V. E. Alekseev. “On the local restrictions effect on the complexity of finding the graph independence number”. In: *Combinatorial-algebraic methods in applied mathematics* (1983).
- [BM18a] A. Brandstädt and R. Mosca. “Maximum Weight Independent Sets for $(P_7, \text{triangle})$ -free graphs in polynomial time”. In: *Discret. Appl. Math.* 236 (2018), pp. 57–65.
- [BM18b] A. Brandstädt and R. Mosca. “Maximum Weight Independent Sets for $(S_{1,2,4}, \text{triangle})$ -free graphs in polynomial time”. In: *CoRR abs/1806.09472* (2018). arXiv: 1806.09472.
- [BMM16] A. Brandstädt, S. Mahfud, and R. Mosca. “Bounded clique-width of $(S_{1,2,2}, \text{triangle})$ -free graphs”. In: *CoRR abs/1608.01820* (2016). arXiv: 1608.01820.
- [Bod88] H. L. Bodlaender. “Dynamic programming on graphs with bounded treewidth”. In: *Automata, Languages and Programming, 15th International Colloquium, ICALP88, Tampere, Finland, July 11-15, 1988, Proceedings*. Ed. by T. Lepistö and A. Salomaa. Vol. 317. Lecture Notes in Computer Science. Springer, 1988, pp. 105–118.

- [Bod93a] H. L. Bodlaender. “A linear time algorithm for finding tree-decompositions of small treewidth”. In: *Proceedings of the Twenty-Fifth Annual ACM Symposium on Theory of Computing, May 16-18, 1993, San Diego, CA, USA*. Ed. by S. R. Kosaraju, D. S. Johnson, and A. Aggarwal. ACM, 1993, pp. 226–234.
- [Bod93b] H. L. Bodlaender. “A tourist guide through treewidth”. In: *Acta Cybern.* 11.1-2 (1993), pp. 1–21.
- [BT01] V. Bouchitté and I. Todinca. “Treewidth and minimum fill-in: grouping the minimal separators”. In: *SIAM J. Comput.* 31.1 (2001), pp. 212–232.
- [BW00] U. Brandes and D. Wagner, eds. *Graph-theoretic concepts in Computer Science, 26th International Workshop, WG 2000, Konstanz, Germany, June 15-17, 2000, Proceedings*. Vol. 1928. Lecture Notes in Computer Science. Springer, 2000.
- [Cam+18] K. Cameron, M. V. G. da Silva, S. Huang, and K. Vušković. “Structure and algorithms for (cap, even hole)-free graphs”. In: *Discret. Math.* 341.2 (2018), pp. 463–473.
- [CC85] G. Cornuéjols and W. H. Cunningham. “Compositions for perfect graphs”. In: *Discret. Math.* 55.3 (1985), pp. 245–254.
- [CCH18] K. Cameron, S. Chaplick, and C. T. Hoàng. “On the structure of (pan, even hole)-free graphs”. In: *Journal of Graph Theory* 87.1 (2018), pp. 108–129.
- [CCV04] M. Conforti, G. Cornuéjols, and K. Vušković. “Square-free perfect graphs”. In: *J. Comb. Theory, Ser. B* 90.2 (2004), pp. 257–307.
- [CER93] B. Courcelle, J. Engelfriet, and G. Rozenberg. “Handle-rewriting hyper-graph grammars”. In: *J. Comput. Syst. Sci.* 46.2 (1993), pp. 218–270.
- [Chu+05] M. Chudnovsky, G. Cornuéjols, X. Liu, P. D. Seymour, and K. Vušković. “Recognizing Berge graphs”. In: *Comb.* 25.2 (2005), pp. 143–186.
- [Chu+06] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. “The strong perfect graph theorem”. In: *Annals of Mathematics* 164.1 (2006), pp. 51–229.
- [Chu+15] M. Chudnovsky, N. Trotignon, T. Trunck, and K. Vušković. “Coloring perfect graphs with no balanced skew-partitions”. In: *J. Comb. Theory, Ser. B* 115 (2015), pp. 26–65.
- [Chu+19] M. Chudnovsky, S. Thomassé, N. Trotignon, and K. Vušković. “Maximum independent sets in (pyramid, even hole)-free graphs”. In: *CoRR abs/1912.11246* (2019). arXiv: 1912.11246.
- [Chu+20] M. Chudnovsky, A. Scott, P. D. Seymour, and S. Spirkl. “Detecting an odd hole”. In: *J. ACM* 67.1 (2020), 5:1–5:12.
- [Chu16] J. Chuzhoy. “Improved bounds for the excluded grid theorem”. In: *CoRR abs/1602.02629* (2016). arXiv: 1602.02629.
- [Chv85] V. Chvátal. “Star-cutsets and perfect graphs”. In: *J. Comb. Theory, Ser. B* 39.3 (1985), pp. 189–199.
- [CK08] M. Chudnovsky and R. Kapadia. “Detecting a theta or a prism”. In: *SIAM J. Discret. Math.* 22.3 (2008), pp. 1164–1186.
- [CL15] H.-C. Chang and H.-I. Lu. “A faster algorithm to recognize even-hole-free graphs”. In: *J. Comb. Theory, Ser. B* 113 (2015), pp. 141–161.

- [CLV03] G. Cornuéjols, X. Liu, and K. Vušković. “A polynomial algorithm for recognizing perfect graphs”. In: *44th Symposium on Foundations of Computer Science (FOCS 2003), 11-14 October 2003, Cambridge, MA, USA, Proceedings*. IEEE Computer Society, 2003, pp. 20–27.
- [CMR00] B. Courcelle, J. A. Makowsky, and U. Rotics. “Linear time solvable optimization problems on graphs of bounded clique-width”. In: *Theory Comput. Syst.* 33.2 (2000), pp. 125–150.
- [Con+00] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković. “Triangle-free graphs that are signable without even holes”. In: *Journal of Graph Theory* 34.3 (2000), pp. 204–220.
- [Con+02a] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković. “Even-hole-free graphs part I: Decomposition theorem”. In: *Journal of Graph Theory* 39.1 (2002), pp. 6–49.
- [Con+02b] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković. “Even-hole-free graphs part II: Recognition algorithm”. In: *Journal of Graph Theory* 40.4 (2002), pp. 238–266.
- [Con+97] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković. “Universally signable graphs”. In: *Combinatorica* 17.1 (1997), pp. 67–77.
- [Con+99] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković. “Even and odd holes in cap-free graphs”. In: *Journal of Graph Theory* 30.4 (1999), pp. 289–308.
- [Cou90] B. Courcelle. “The Monadic Second-Order Logic of graphs. I. Recognizable sets of finite graphs”. In: *Inf. Comput.* 85.1 (1990), pp. 12–75.
- [CR05] D. G. Corneil and U. Rotics. “On the relationship between clique-width and treewidth”. In: *SIAM J. Comput.* 34.4 (2005), pp. 825–847.
- [CR93] M. Conforti and M. R. Rao. “Testing balancedness and perfection of linear matrices”. In: *Math. Program.* 61 (1993), pp. 1–18.
- [CS08] M. Chudnovsky and P. D. Seymour. “Claw-free graphs. IV. Decomposition theorem”. In: *J. Comb. Theory, Ser. B* 98.5 (2008), pp. 839–938.
- [CS10] Maria Chudnovsky and Paul D. Seymour. “The three-in-a-tree problem”. In: *Comb.* 30.4 (2010), pp. 387–417.
- [Cyg+15] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized algorithms*. Springer, 2015.
- [DDP17] K. K. Dabrowski, F. Dross, and D. Paulusma. “Colouring diamond-free graphs”. In: *J. Comput. Syst. Sci.* 89 (2017), pp. 410–431.
- [Dio+20] Emilie Diot, Marko Radovanovic, Nicolas Trotignon, and Kristina Vuskovic. “The (theta, wheel)-free graphs Part I: Only-prism and only-pyramid graphs”. In: *J. Comb. Theory, Ser. B* 143 (2020), pp. 123–147.
- [DJP19] K. K. Dabrowski, M. Johnson, and D. Paulusma. “Clique-width for hereditary graph classes”. In: *CoRR abs/1901.00335* (2019). arXiv: 1901.00335.
- [DP16] K. K. Dabrowski and D. Paulusma. “Clique-width of graph classes defined by two forbidden induced subgraphs”. In: *Comput. J.* 59.5 (2016), pp. 650–666.

- [DTT13] E. Diot, S. Tavenas, and N. Trotignon. “Detecting wheels”. In: *CoRR* abs/1308.6433 (2013). arXiv: 1308.6433.
- [Far89] M. Farber. “On diameters and radii of bridged graphs”. In: *Discret. Math.* 73.3 (1989), pp. 249–260.
- [FOS11] Y. Faenza, G. Oriolo, and G. Stauffer. “An algorithmic decomposition of claw-free graphs leading to an $O(n^3)$ -algorithm for the weighted stable set problem”. In: *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2011, San Francisco, California, USA, January 23–25, 2011*. Ed. by Dana Randall. SIAM, 2011, pp. 630–646.
- [GLS88] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric algorithms and combinatorial optimization*. Vol. 2. Algorithms and Combinatorics. Springer, 1988.
- [Grz+19] A. Grzesik, T. Klimosová, M. Pilipczuk, and M. Pilipczuk. “Polynomial-time algorithm for Maximum Weight Independent Set on P_6 -free graphs”. In: *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6–9, 2019*. Ed. by Timothy M. Chan. SIAM, 2019, pp. 1257–1271.
- [GW00] F. Gurski and E. Wanke. “The tree-width of clique-width bounded graphs without $K_{n,n}$ ”. In: *Graph-Theoretic Concepts in Computer Science, 26th International Workshop, WG 2000, Konstanz, Germany, June 15–17, 2000, Proceedings*. Ed. by Ulrik Brandes and Dorothea Wagner. Vol. 1928. Lecture Notes in Computer Science. Springer, 2000, pp. 196–205.
- [Hal87] R. Halin. “Problems from the world surrounding perfect graphs”. In: *Zastowania Matematyki Applicationes Mathematicae vol. 19* (1987), pp. 413–441.
- [HT20] C. Hoàng and N. Trotignon. “A class of graphs of large rankwidth”. In: *CoRR* abs/2007.11513 (2020). arXiv: 2007.11513.
- [HW17] D. J. Harvey and D. R. Wood. “Parameters tied to treewidth”. In: *Journal of Graph Theory* 84.4 (2017), pp. 364–385.
- [Kar72] R. M. Karp. “Reducibility among combinatorial problems”. In: *Proceedings of a symposium on the Complexity of Computer Computations, held March 20–22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York, USA*. Ed. by Raymond E. Miller and James W. Thatcher. The IBM Research Symposia Series. Plenum Press, New York, 1972, pp. 85–103.
- [Kin09] A. King. “Claw-free graphs and two conjectures on ω , Δ , and χ ”. PhD thesis. McGill University, 2009.
- [Krá+01] D. Král, J. Kratochvíl, Z. Tuza, and G. J. Woeginger. “Complexity of coloring graphs without forbidden induced subgraphs”. In: *Graph-Theoretic Concepts in Computer Science, 27th International Workshop, WG 2001, Boltenhagen, Germany, June 14–16, 2001, Proceedings*. Ed. by Andreas Brandstädt and Van Bang Le. Vol. 2204. Lecture Notes in Computer Science. Springer, 2001, pp. 254–262.
- [Le18] N.-K. Le. “Coloring even-hole-free graphs with no star cutset”. In: *CoRR* abs/1805.01948 (2018). arXiv: 1805.01948.

- [LLT20] K.-Y. Lai, H.-I. Lu, and M. Thorup. “Three-in-a-tree in near linear time”. In: *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2020, Chicago, IL, USA, June 22-26, 2020*. Ed. by Konstantin Makarychev, Yury Makarychev, Madhur Tulsiani, Gautam Kamath, and Julia Chuzhoy. ACM, 2020, pp. 1279–1292.
- [LM08] V. V. Lozin and M. Milanič. “A polynomial algorithm to find an independent set of maximum weight in a fork-free graph”. In: *J. Discrete Algorithms* 6.4 (2008), pp. 595–604.
- [LM17] V. V. Lozin and D. S. Malyshev. “Vertex coloring of graphs with few obstructions”. In: *Discret. Appl. Math.* 216 (2017), pp. 273–280.
- [LVV14] D. Lokshantov, M. Vatschelle, and Y. Villanger. “Independent set in P_5 -free graphs in polynomial time”. In: *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014*. Ed. by Chandra Chekuri. SIAM, 2014, pp. 570–581.
- [MGR96] S. E. Markossian, G. S. Gasparian, and B. A. Reed. “ β -Perfect graphs”. In: *J. Comb. Theory, Ser. B* 67.1 (1996), pp. 1–11.
- [MP96] F. Maffray and M. Preissmann. “On the NP-completeness of the k -colorability problem for triangle-free graphs”. In: *Discrete Mathematics* (1996).
- [MPV20] F. Maffray, I. Penev, and K. Vušković. “Coloring rings”. In: *CoRR* abs/1907.11905 (2020). arXiv: 1907.11905.
- [MT13] F. Maffray and N. Trotignon. “Algorithms for perfectly contractile graphs”. In: *CoRR* abs/1309.0435 (2013). arXiv: 1309.0435.
- [MTV08] F. Maffray, N. Trotignon, and K. Vušković. “Algorithms for square-3PC(., .)-free berge graphs”. In: *SIAM J. Discret. Math.* 22.1 (2008), pp. 51–71.
- [OS06] S.-i. Oum and P. D. Seymour. “Approximating clique-width and branch-width”. In: *J. Comb. Theory, Ser. B* 96.4 (2006), pp. 514–528.
- [Pol74] S. Poljak. “A note on stable sets and colorings of graphs”. In: *Commentationes Mathematicae Universitatis Carolinae* 15.2 (1974), pp. 307–309.
- [RS86] N. Robertson and P. D. Seymour. “Graph minors. V. Excluding a planar graph”. In: *J. Comb. Theory, Ser. B* 41.1 (1986), pp. 92–114.
- [RST94] N. Robertson, P. D. Seymour, and R. Thomas. “Quickly excluding a planar graph”. In: *J. Comb. Theory, Ser. B* 62.2 (1994), pp. 323–348.
- [RTV20] Marko Radovanovic, Nicolas Trotignon, and Kristina Vuskovic. “The (theta, wheel)-free graphs Part II: Structure theorem”. In: *J. Comb. Theory, Ser. B* 143 (2020), pp. 148–184.
- [RV13] M. Radovanović and K. Vušković. “A class of three-colorable triangle-free graphs”. In: *Journal of Graph Theory* 72.4 (2013), pp. 430–439.
- [SSS10] A. Silva, A. A. da Silva, and C. L. Sales. “A bound on the treewidth of planar even-hole-free graphs”. In: *Discret. Appl. Math.* 158.12 (2010), pp. 1229–1239.
- [SV13] M. V. G. da Silva and K. Vušković. “Decomposition of even-hole-free graphs with star cutsets and 2-joins”. In: *J. Comb. Theory, Ser. B* 103.1 (2013), pp. 144–183.

- [Tar85] R. E. Tarjan. "Decomposition by clique separators". In: *Discret. Math.* 55.2 (1985), pp. 221–232.
- [Tro13] N. Trotignon. "Perfect graphs: a survey". In: *CoRR* abs/1301.5149 (2013). arXiv: 1301.5149.
- [Tru82] K. Truemper. "Alpha-balanced graphs and matrices and GF(3)-representability of matroids". In: *Journal of Combinatorial Theory, Series B* 32 (1982), pp. 112–139.
- [TV12] N. Trotignon and K. Vušković. "Combinatorial optimization with 2-joins". In: *J. Comb. Theory, Ser. B* 102.1 (2012), pp. 153–185.
- [Vuš10] K. Vušković. "Even-hole-free graphs: a survey". In: *Applicable Analysis and Discrete Mathematics* 10.2 (2010), pp. 219–240.
- [Vuš13] K. Vušković. "The world of hereditary graph classes viewed through Truemper configurations". In: *Surveys in Combinatorics 2013*. Vol. 409. London Mathematical Society Lecture Note Series. Cambridge University Press, 2013, pp. 265–326.
- [WM67] M. E. Watkins and D. M. Mesner. "Cycles and connectivity in graph". In: *Canadian Journal of Mathematics* 19 (1967), pp. 1319–1328.
- [Zuc06] D. Zuckerman. "Linear degree extractors and the inapproximability of max clique and chromatic number". In: *Proceedings of the 38th Annual ACM Symposium on Theory of Computing, Seattle, WA, USA, May 21-23, 2006*. Ed. by Jon M. Kleinberg. ACM, 2006, pp. 681–690.