

# Linear Algebra

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## 7.2 - Relation between Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

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# Learning objectives

After this lecture, you should be able to:

1. explain dot product between two vectors;
2. explain computing norm of a vector;
3. explain computing distance, angles, and projection of two vectors
4. explain cross product of vectors.

# Part 1: Inner Product & Norm

# Dot (inner) product

Let  $\mathbf{u}$  and  $\mathbf{v}$  be vectors in  $\mathbb{R}^n$ :

$$\mathbf{u} = (u_1, u_2, \dots, u_n) \quad \text{and} \quad \mathbf{v} = (v_1, v_2, \dots, v_n)$$

The **dot product** or **inner product** or **scalar product** of  $\mathbf{u}$  and  $\mathbf{v}$  is defined by:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Algebraically, the dot product is the sum of the products of the corresponding entries of the two sequences of numbers.

*Can we interpret dot product of two vectors geometrically?*

## Example

1. Let  $\mathbf{u} = (1, -2, 3)$ ,  $\mathbf{v} = (4, 5, -1)$ , find  $\mathbf{u} \cdot \mathbf{v}$ .

$$\mathbf{u} \cdot \mathbf{v} = 1(4) + (-2)(5) + (3)(-1) = 4 - 10 - 3 = -9$$

2. Suppose  $\mathbf{u} = (1, 2, 3, 4)$  and  $\mathbf{v} = (6, k, -8, 2)$ . Find  $k$  such that  $\mathbf{u} \cdot \mathbf{v} = 0$ .

$$\mathbf{u} \cdot \mathbf{v} = 1(6) + 2(k) + 3(-8) + 4(2) = -10 + 2k$$

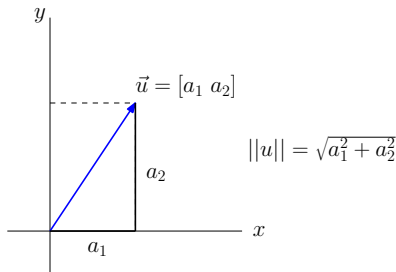
If  $\mathbf{u} \cdot \mathbf{v} = 0$  then  $-10 + 2k = 0$ , meaning that  $k = 5$ .

# Norm (length) of a vector

Norm (length) of a vector  $\mathbf{u}$  in  $\mathbb{R}^n$  is defined by:

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Illustration in 2D:



A vector  $\mathbf{u}$  is a **unit vector** if  $\|\mathbf{u}\| = 1$ .

## Example

1. Let  $\mathbf{u} = (1, -2, -4, 5, 3)$ . Find  $\|\mathbf{u}\|$ .

$$\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = 1^2 + (-2)^2 + (-4)^2 + 5^2 + 3^2 = 1 + 4 + 16 + 25 + 9 = 55$$

$$\text{Hence, } \|\mathbf{u}\| = \sqrt{55}.$$

2. Given vectors  $\mathbf{v} = (1, -3, 4, 2)$  and  $\mathbf{w} = (\frac{1}{2}, -\frac{1}{6}, \frac{5}{6}, \frac{1}{6})$ .  
Determine which one of the two vectors is a unit vector?

$$\|\mathbf{v}\| = \sqrt{1 + 9 + 16 + 4} = \sqrt{30} \quad \text{and} \quad \|\mathbf{w}\| = \sqrt{\frac{9}{36} + \frac{1}{36} + \frac{25}{36} + \frac{1}{36}} = 1$$

Hence,  $\mathbf{w}$  is a unit vector, and  $\mathbf{v}$  is not a unit vector.

# Standard unit vector

The **standard unit vector** in  $\mathbb{R}^n$  is composed of  $n$  vectors:

$$\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$$

dimana:

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, 0, \dots, 0, 1)$$

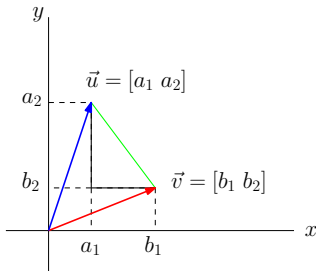


# Part 2: **Distance, Angle, Projections**

# Distance

The **distance** between vectors  $\mathbf{u} = (u_1, u_2, \dots, u_n)$  and  $\mathbf{v} = (v_1, v_2, \dots, v_n)$  in  $\mathbb{R}^n$  is defined by:

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2}$$

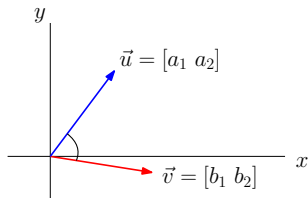


$$\|u - v\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

## Angle between two vectors

The angle  $\theta$  between vectors  $\mathbf{u}, \mathbf{v} \neq 0$  in  $\mathbb{R}^n$  is defined by:

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$



Is this well defined? Remember that the value of  $\cos$  range from  $-1$  to  $1$ . So the following should hold:

$$-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \leq 1$$

**Exercise:** prove the last inequality!

# Cauchy-Schwarz inequality

## Solution of the exercise:

If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in  $\mathbb{R}^n$ , then  $-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \leq 1$ .

## Theorem (Schwarz inequality)

For any vectors  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{R}^n$ ,  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ .

## Proof.

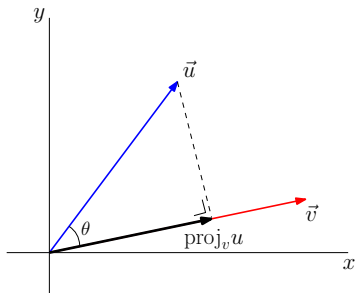
See this paper [https://www.uni-miskolc.hu/~matsefi/Octagon/volumes/volume1/article1\\_19.pdf](https://www.uni-miskolc.hu/~matsefi/Octagon/volumes/volume1/article1_19.pdf) for different proof alternatives. □

# Projection

The **projection** of a vector  $\mathbf{u}$  onto a **nonzero** vector  $\mathbf{v}$  is defined by:

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}$$

The length of vector  $\text{proj}_{\mathbf{v}} \mathbf{u}$  is  $\|\mathbf{u}\| \cos(\theta)$ .  
So,



$$\begin{aligned} \text{proj}_{\mathbf{v}} \mathbf{u} &= \|\mathbf{u}\| \cos(\theta) \mathbf{v} \\ &= \|\mathbf{u}\| \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \mathbf{v} \\ &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} \mathbf{v} \\ &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} \end{aligned}$$

# What is vector projection used for?

- Browse on the internet about “the reasons why vector projection operations are needed/used”.
- Present the results of your group discussion to other colleagues.

# Orthogonality

In the previous section, we discussed that the angle formed by the two vectors  $\mathbf{u}$  and  $\mathbf{v}$  can be calculated by:

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

Note that:

$$\theta = \frac{\pi}{2} \text{ jika dan hanya jika } \mathbf{u} \cdot \mathbf{v} = 0$$

## Definition (Vektor-vektor yang ortogonal)

The two vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$  are said to be **orthogonal** (or **perpendicular**, or **perpendicular**) if  $\mathbf{u} \cdot \mathbf{v} = 0$ .

**Note:** in this case, the vector **zero** is always orthogonal to every vector in  $\mathbb{R}^n$ .

## Example

1. Show that the vectors:  $\mathbf{u} = (-2, 3, 1, 4)$  and  $\mathbf{v} = (1, 2, 0, -1)$  are orthogonal in  $\mathbb{R}^4$ .
2. Let  $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  be the standard unit vector in  $\mathbb{R}^3$ . Show that the three vectors are orthogonal to each other.



# Part 2: **Cross Product**

## Cross product

Let  $\mathbf{u}$  and  $\mathbf{v}$  be vectors in  $\mathbb{R}^3$ :

$$\mathbf{u} = (u_1, u_2, u_3) \quad \text{and} \quad \mathbf{v} = (v_1, v_2, v_3)$$

The **cross product** of  $\mathbf{u}$  and  $\mathbf{v}$  is defined by:

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$$

$$\mathbf{u} \times \mathbf{v} = \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

This can be easily seen using the following method:

$$\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \quad \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \quad \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

## Example

Given vectors:

$$\mathbf{u} = (0, 1, 7) \quad \text{and} \quad \mathbf{v} = (1, 4, 5)$$

The vectors can be represented as matrix:  $\begin{bmatrix} 0 & 1 & 7 \\ 1 & 4 & 5 \end{bmatrix}$

Hence,

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \left( \begin{vmatrix} 1 & 7 \\ 4 & 5 \end{vmatrix}, -\begin{vmatrix} 0 & 7 \\ 1 & 5 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} \right) \\ &= (5 - 28, -(0 - 7), 0 - 1) \\ &= (-23, 7, -1) \end{aligned}$$

## How does $\mathbf{u} \times \mathbf{v} = \mathbf{w}$ mean?

Given:  $\mathbf{u} \times \mathbf{v} = \mathbf{w}$ . This means that:

$$\mathbf{w} \perp \mathbf{u} \text{ and } \mathbf{w} \perp \mathbf{v}$$

### Example

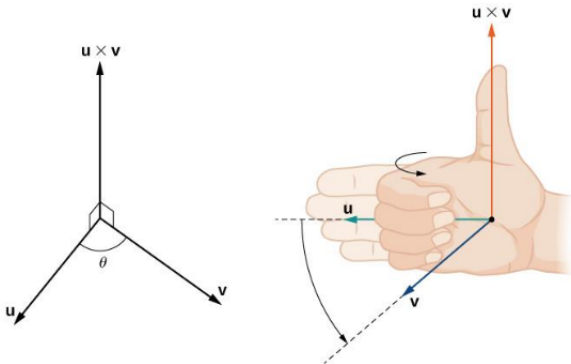
Given  $\mathbf{u} = (0, 1, 7)$  and  $\mathbf{v} = (1, 4, 5)$ , and:

$$\mathbf{u} \times \mathbf{v} = \mathbf{w} = (-23, 7, -1)$$

Note that:

- $\mathbf{w} \cdot \mathbf{u} = (-23, 7, -1) \cdot (0, 1, 7) = 0 + 7 - 7 = 0$
- $\mathbf{w} \cdot \mathbf{v} = (-23, 7, -1) \cdot (1, 4, 5) = -23 + 28 - 5 = 0$

# Right-hand system



# Properties of cross product

## Theorem

Let  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  be vectors in  $\mathbb{R}^3$ , and  $k \in \mathbb{R}$ . Then:

1.  $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2.  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3.  $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
4.  $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
5.  $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
6.  $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

# Properties of dot product and cross product

## Theorem

Let  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  be vectors in  $\mathbb{R}^3$ . Then:

1.  $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$   $(\mathbf{u} \times \mathbf{v}$  is orthogonal to  $u$ )
2.  $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$   $(\mathbf{u} \times \mathbf{v}$  is orthogonal to  $v$ )
3.  $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$   $(\text{Lagrange's identity})$
4.  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$
5.  $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$

## Exercise

Prove the following identity:

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \ ||\mathbf{v}|| \sin \theta$$

where  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .

**Answer:**

$$\begin{aligned} ||\mathbf{u} \times \mathbf{v}||^2 &= ||\mathbf{u}||^2 \ ||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\ &= ||\mathbf{u}||^2 \ ||\mathbf{v}||^2 - (||\mathbf{u}|| \ ||\mathbf{v}|| \cos \theta)^2 \\ &= ||\mathbf{u}||^2 \ ||\mathbf{v}||^2 - (||\mathbf{u}||^2 \ ||\mathbf{v}||^2 \cos^2 \theta) \\ &= ||\mathbf{u}||^2 \ ||\mathbf{v}||^2 (1 - \cos^2 \theta) \\ &= ||\mathbf{u}||^2 \ ||\mathbf{v}||^2 \sin^2 \theta \end{aligned}$$

Dengan demikian,  $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \ ||\mathbf{v}|| \sin \theta$



## Cross product of standard unit vectors

The standard unit vectors in  $\mathbb{R}^3$ :

$$\mathbf{i} = (1, 0, 0) \quad \mathbf{j} = (0, 1, 0) \quad \mathbf{k} = (0, 0, 1)$$

The cross product between  $\mathbf{i}$  and  $\mathbf{j}$  is given by:

$$\mathbf{i} \times \mathbf{j} = \left( \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right) = (0, 0, 1) = \mathbf{k}$$

The cross product between  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ :

- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
- $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
- $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
- $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$
- $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
- $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

# Cross product of two vectors

Given:

- $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
- $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

Using the **cofactor expansion**:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

## Example of *cofactor expansion* for cross product

From the previous example:

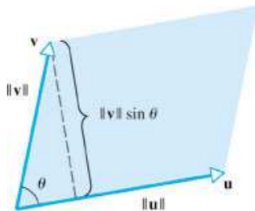
- $\mathbf{u} = (0, 1, 7) = \mathbf{j} + 7\mathbf{k}$
- $\mathbf{v} = (1, 4, 5) = \mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

Then:

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 7 \\ 1 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 7 \\ 4 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 7 \\ 1 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} \mathbf{k} \\ &= (5 - 28)\mathbf{i} - (0 - 7)\mathbf{j} + (0 - 1)\mathbf{k} \\ &= -23\mathbf{i} + 7\mathbf{j} - \mathbf{k}\end{aligned}$$

# Geometric interpretation of cross product (in $\mathbb{R}^2$ )

The cross product of two vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^2$  is equal to the area of the parallelogram determined by the two vectors.

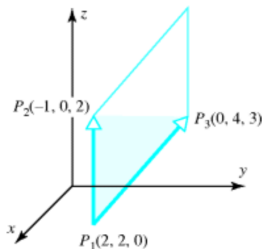


$$\begin{aligned}\text{Area} &= \text{base} \times \text{height} \\ &= \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta \\ &= \|\mathbf{u} \times \mathbf{v}\|\end{aligned}$$

## Example

Determine the area of the triangle determined by the points:

$$P_1 = (2, 2, 0), \quad P_2 = (-1, 0, 2), \quad \text{and} \quad P_3 = (0, 4, 3)$$



$$\text{Area of } \triangle = 1/2 \text{ Area of } \textit{parallelogram}$$

Two vectors that determine the parallelogram:

$$\begin{aligned} \mathbf{u} &= P_1 \vec{P}_2 = \vec{OP}_2 - \vec{OP}_1 \\ &= (-1, 0, 2) - (2, 2, 0) = (-3, -2, 2) \end{aligned}$$

$$\begin{aligned} \mathbf{v} &= P_1 \vec{P}_3 = \vec{OP}_3 - \vec{OP}_1 \\ &= (0, 4, 3) - (2, 2, 0) = (-2, 2, 3) \end{aligned}$$

$$\text{Hence: } \mathbf{u} \times \mathbf{v} = \left( \begin{vmatrix} -2 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} -3 & 2 \\ -2 & 3 \end{vmatrix}, \begin{vmatrix} -3 & -2 \\ -2 & 2 \end{vmatrix} \right) = (-10, 5, -10)$$

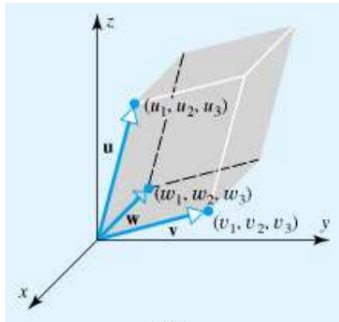
So, the area of the parallelogram is:

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-10)^2 + (5)^2 + (-10)^2} = \sqrt{225} = 15$$

and the area of the triangle is  $15/2 = 7.5$ .

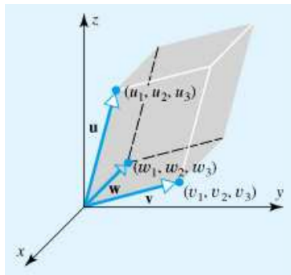
# Geometric interpretation of cross product (in $\mathbb{R}^3$ )

The cross product of three vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  in  $\mathbb{R}^3$  is equal to the volume of the parallelepiped determined by the three vectors.



$$\begin{aligned}\text{Volume} &= \text{area of base} \times \text{height} \\ &= \|\mathbf{v} \times \mathbf{w}\| \cdot (\|\text{proj}_{\mathbf{v} \times \mathbf{w}} \mathbf{u}\|) \\ &= \|\mathbf{v} \times \mathbf{w}\| \cdot \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{v} \times \mathbf{w}\|} \\ &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|\end{aligned}$$

# Geometric interpretation of cross product (in $\mathbb{R}^3$ )



$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{u} \cdot \left( \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \right) \\ &= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} u_1 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} u_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} u_3 \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\end{aligned}$$

which is the determinant of matrix whose first row is composed of elements of  $\mathbf{u}$  and the 2nd and 3rd rows are composed with the elements of  $\mathbf{v}$

The volume of the parallelepiped is equal to  $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$

## Example

Find the volume of the *parallelepiped* formed by three vectors:

$$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}, \mathbf{v} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}, \mathbf{w} = 3\mathbf{j} + 2\mathbf{k}$$

**Solution:**

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} \\ &= 3 \begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} \\ &= 60 + 4 - 15 \\ &= 49\end{aligned}$$



## Exercise 1

Find the area of parallelogram that is formed by two vectors:

$$\mathbf{u} = 4\mathbf{i} + 3\mathbf{j} \quad \text{and} \quad \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$$

**Solution:**

$$\det \left( \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix} \right) = \begin{vmatrix} 4 & 3 \\ 3 & -4 \end{vmatrix} = -16 - 9 = -25$$

Hence, the area of the parallelogram is  $|-25| = 25$ .

## Exercise 2

Given three vectors:

$$\mathbf{u} = (1, 1, 2), \mathbf{v} = (1, 1, 5), \mathbf{w} = (3, 3, 1)$$

Find the volume of the parallelepiped formed by the three vectors!

**Solution:**

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 5 \\ 3 & 3 & 1 \end{vmatrix} &= (1) \begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix} - (1) \begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} \\ &= (1)(-14) - (-1)(-14) + (2)(0) \\ &= -14 + 14 + 0 \\ &= 0 \end{aligned}$$

# A recap

*We have learned:*

- the definition of vectors in Linear Algebra;
- some operations on vectors:
  - vector addition and scalar multiplication;
  - linear combination;
  - dot product between two vectors;
  - computing norm of a vector;
  - computing distance, angles, and projection of two vectors

**Task:** write a summary about our discussion, and do the exercises!

*to be continued...*