

Linear Algebra  
[KOMS119602] - 2022/2023

## 9(a) - Relation between Vectors in a Space

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# Learning objectives

After this lecture, you should be able to:

1. explain the concept of spanning set and linear combination of vectors;
2. explain the concept of basis and dimension of vector space;
3. find a basis and the dimension of a vector space.

# Subspace and Linear Combination

# Linear combination

Recall **linear combination of vectors** is defined as:

Let  $\mathbf{w} \in V$ . Then  $w$  is a linear combination of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$  if  $\mathbf{w}$  can be written as:

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n$$

where  $k_1, k_2, \dots, kn \in \mathbb{R}$ .

## Example

Let  $\mathbf{v}_1 = (3, 2, -1)$  and  $\mathbf{v}_2 = (2, -4, 3)$ . Then:

$$\mathbf{w} = 2\mathbf{v}_1 + 3\mathbf{v}_2 = 2(3, 2, -1) + 3(2, -4, 3) = (12, -8, 7)$$

is a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

## Defining linear combination of vectors

Given a vector  $(5, 9, 5)$ . How to represent the vector as a linear combination of vectors:

$$\mathbf{u} = (2, 1, 4), \mathbf{v} = (1, -1, 3), \text{ and } \mathbf{w} = (3, 2, 5)$$

**Solution:** Let  $k_1, k_2, k_3 \in \mathbb{R}$  be such that:

$$k_1 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + k_3 \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 5 \end{bmatrix}$$

This yields linear system:

$$\begin{cases} 2k_1 + k_2 + 3k_3 = 5 \\ k_1 - k_2 + 2k_3 = 9 \\ 4k_1 + 3k_2 + 5k_3 = 5 \end{cases}$$

By Gauss elimination, we obtain:

$$k_1 = 3, \quad k_2 = -4, \quad k_3 = 2$$

# Linear combination forms subspace

## Theorem

If  $S = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r\}$  is a set of vectors in a vector space  $V$ .  
Then:

1. The set  $W$  containing all linear combinations of vectors in  $S$  is a subspace of  $V$ .
2.  $W$  is the smallest subspace of  $V$  that contains vectors in  $S$ , i.e., all the other subspaces containing the vectors also contain  $W$ .

*Exercise: prove the correctness of the theorem.*

# Spanning Set

## Set of vectors forming subspace

- Let  $V$  be a vector space,  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$ .
- Let  $W$  be a subspace of  $V$  s.t.  $\forall \mathbf{w} \in W$ ,

$$\mathbf{w} = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_n \mathbf{v}_n$$

where  $k_1, k_2, \dots, k_n$ .

Hence,  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is said to **span**  $W$ .

$S$  is called **spanning set**, and is denoted as:

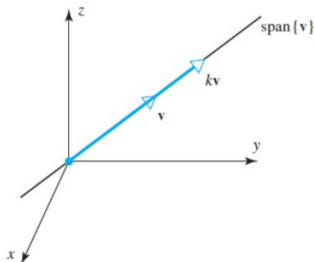
$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \text{ or } \text{span}(S)$$



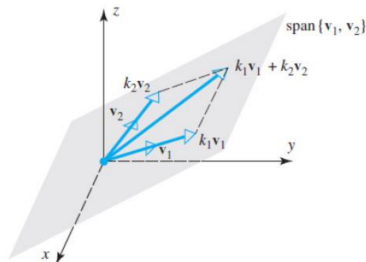
## Example: *space spanned by one of two vectors*

Let  $\mathbf{v}_1, \mathbf{v}_2$  are *noncollinear* vectors in  $\mathbb{R}^3$ , with their initial points at the origin, then:

- $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$  consisting all linear combinations  $k_1\mathbf{v}_1 + k_2\mathbf{v}_2$ , is the plane determined by vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .
- if  $\mathbf{v} \neq \mathbf{0}$  is a vector in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , then  $\text{span}\{\mathbf{v}\}$  consisting all scalar multiples  $k\mathbf{v}$ , is the line determined by  $\mathbf{v}$ .



(a)  $\text{Span}\{\mathbf{v}\}$  is the line through the origin determined by  $\mathbf{v}$ .



(b)  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$  is the plane through the origin determined by  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

## Exercise 1

*The following standard unit vectors span  $\mathbb{R}^3$ .*

$$\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$$

## Exercise 1

*The following standard unit vectors span  $\mathbb{R}^3$ .*

$$\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$$

This is because, every vector  $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$  can be represented as linear combination:

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

In this case,  $\mathbb{R}^3 = \text{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ .

## Exercise 2

*Polynomials  $1, x, x^2, \dots, x^n$  span the vector space  $P_n$*

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This is because, every polynomial  $\mathbf{p} \in P_n$  can be written as:

$$\mathbf{p} = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

which is a linear combination of  $1, x, x^2, \dots, x^n$ .

In this case,  $P_n = \text{span}\{1, x, x^2, \dots, x^n\}$ .

## Exercise 3

*Determine whether following vectors span  $\mathbb{R}^3$  !*

$$\mathbf{v}_1 = (2, -1, 3), \mathbf{v}_2 = (4, 1, 2), \mathbf{v}_3 = (8, -1, 8)$$

## Exercise 3

*Determine whether following vectors span  $\mathbb{R}^3$  !*

$$\mathbf{v}_1 = (2, -1, 3), \mathbf{v}_2 = (4, 1, 2), \mathbf{v}_3 = (8, -1, 8)$$

Let  $\mathbf{u} = (u_1, u_2, u_3)$  be a vector in  $\mathbb{R}^3$ , and  $k_1, k_2, k_3 \in \mathbb{R}$ .

If the set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  span  $\mathbb{R}^3$ , then it should be:

$$(u_1, u_2, u_3) = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

We will check if the following linear system has a solution.

$$2k_1 + 4k_2 + 8k_3 = u_1, -k_1 + k_2 - k_3 = u_2, 3k_1 + 2k_2 + 8k_3 = u_3$$

## Exercise 4 (*cont.*)

The linear system has coefficient matrix:

$$A = \begin{bmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{bmatrix}$$

Note that:

$$\det(A) = 2 \begin{vmatrix} 1 & -1 \\ 2 & 8 \end{vmatrix} - 4 \begin{vmatrix} -1 & -1 \\ 3 & 8 \end{vmatrix} + 8 \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = 20 + 20 - 40 = 0$$

Hence, there is no solution for the linear system, meaning that  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  does not span  $\mathbb{R}^3$ .



# Linear Independence

## Linear independence in $\mathbb{R}^2$ and $\mathbb{R}^3$

Let  $V$  be a vector space. The set  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$  is said **linearly independent** iff the linear equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n = 0 \quad (1)$$

has **exactly one solution**, which is the **trivial solution**:

$$k_1 = 0, \quad k_2 = 0, \quad \dots, \quad k_n = 0$$

Conversely, the set  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$  is said **not linearly independent** or **linearly dependent**, iff the linear combination (1) has a **non-trivial solution** (i.e., a solution other than  $k_1 = 0, \quad k_2 = 0, \quad \dots, \quad k_n = 0$ ).

## Example of linearly independent set

*The vectors  $\mathbf{i} = (1, 0, 0)$ ,  $\mathbf{j} = (0, 1, 0)$ , and  $\mathbf{k} = (0, 0, 1)$  are linearly independent vectors in  $\mathbb{R}^3$ .*

### Why?

Note that for scalars  $k_1, k_2, k_3 \in \mathbb{R}$ , we have:  $k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = \mathbf{0}$ ,  
that is equivalent to

$$k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (0, 0, 0) \Leftrightarrow (k_1, k_2, k_3) = (0, 0, 0)$$

Clearly, there is no solution other than  $k_1 = 0$ ,  $k_2 = 0$ , and  $k_3 = 0$ .

This means that  $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  is linearly independent.

Similarly, we can show that:

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \text{ and } \mathbf{e}_n = (0, 0, 0, \dots, 1)$$

are linearly independent vectors.

## Example of linearly dependent sets (1)

Determine whether the vectors:

$$\mathbf{v}_1 = (2, -1, 0, 3), \mathbf{v}_2 = (1, 2, 5, -1), \text{ and } \mathbf{v}_3 = (7, -1, 5, 8)$$

are linearly independent or not!

## Example of linearly dependent sets (1)

Determine whether the vectors:

$$\mathbf{v}_1 = (2, -1, 0, 3), \mathbf{v}_2 = (1, 2, 5, -1), \text{ and } \mathbf{v}_3 = (7, -1, 5, 8)$$

are linearly independent or not!

**Solution:**

Note that:  $3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$  (*show it!*).

This means that  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is **not** linearly independent.

## Example of linearly dependent sets (2)

Determine if the polynomials:

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \text{and} \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly independent or not!

## Example of linearly dependent sets (2)

Determine if the polynomials:

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \text{and} \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly independent or not!

**Solution:**

Note that  $3\mathbf{p}_1 - \mathbf{p}_2 + 2\mathbf{p}_3 = \mathbf{0}$  (*show it!*).

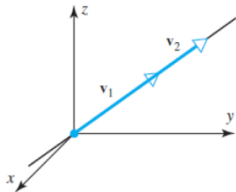
Hence, the vectors are linearly dependent.

# Exercises

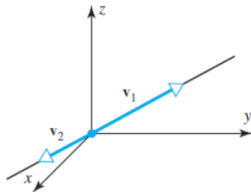
*Do the relevant exercises in the Howard Anton's nook.*



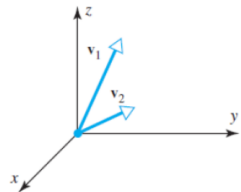
## Geometric interpretation of linear independence in $\mathbb{R}^2$ and $\mathbb{R}^3$



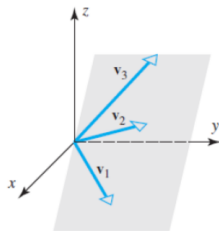
(a) Linearly dependent



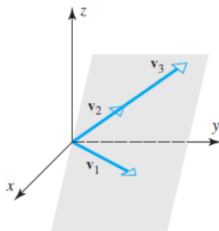
(b) Linearly dependent



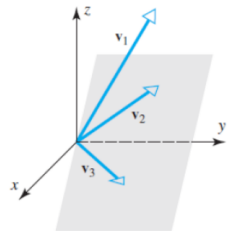
(c) Linearly independent



(a) Linearly dependent



(b) Linearly dependent



(c) Linearly independent

# Determining linear independence/dependence (1)

Determine the linear dependence of the vectors:

$$\mathbf{v}_1 = (1, -2, 3), \mathbf{v}_2 = (5, 6, -1), \text{ and } \mathbf{v}_3 = (3, 2, 1)$$

**Solution:**

We check if the vector equation  $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$  has a solution in  $\mathbb{R}$ .

The equation is equivalent to:

$$\begin{aligned} k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) &= (0, 0, 0) \\ (k_1 + 5k_2 + 3k_3, -2k_1 + 6k_2 + 2k_3, 3k_1 - k_2 + k_3) &= (0, 0, 0) \end{aligned}$$

Solve the system:

$$\begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ 2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases}$$

Solving the system using Gaussian elimination, we get:

$$k_1 = -\frac{1}{2}t, \quad k_2 = -\frac{1}{2}t, \quad k_3 = t, \quad t \in \mathbb{R}$$

Hence, the system has a non-trivial solution, so the set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent.

## Determining linear independence/dependence (2)

Show that the polynomials form a linearly independent set of vectors in  $P_n$ .

$$1, x, x^2, \dots, x^n$$

## Determining linear independence/dependence (2)

Show that the polynomials form a linearly independent set of vectors in  $P_n$ .

$$1, x, x^2, \dots, x^n$$

**Solution:**

Let  $a_0, a_1, \dots, a_n$  be such that:

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = \mathbf{0}$$

We must show that the only solution of the polynomial for  $x \in (-\infty, \infty)$  is:

$$a_0 = a_1 = a_2 = \dots = a_n = 0$$

From Algebra, we know that:

### Theorem

*Every nonzero polynomial of degree  $n$  has at most  $n$  roots.*

This implies that  $a_0 = a_1 = \dots = a_n$  (or, the polynomial is zero polynomial).

Otherwise, it is a nonzero polynomial, having infinite number of roots (that is,  $x \in (-\infty, \infty)$ ), contradicting the theorem.

# Exercises

*Do the relevant exercises in Howard Antons' book.*