# Linear Algebra
## [KOMS119602] - 2022/2023

# 6.1 - Inverses of matrices

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# Learning objectives

After this lecture, you should be able to:

1. investigate if a matrix inverse exists;
2. compute the inverse of a *small size* matrix (if exists);
3. compute the inverse of an $n \times n$ matrix (if exists);
4. explain the concepts of *minor*, *cofactor*, *adjoint*;
5. analyze if a matrix is orthogonal;
6. analyze if a set of vectors is orthonormal;
7. explain the properties of matrix inverse.

# **Part 1:** Inverse of matrices

# Inverse

A square matrix $A$ is said to be invertible or nonsingular if $\exists B$ s.t.:

$$AB = BA = I \qquad \text{where } I \text{ is the identity matrix}$$

**Note:** Such a matrix $B$ is unique, and it is called the inverse of $A$, and is denoted by $A^{-1}$. The relation of $A$ and $B$ is symmetric:

*If $B$ is the inverse of $A$, then $A$ is the inverse of $B$, i.e.*

$$(A^{-1})^{-1} = A$$

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## Example

Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ Then

$$AB = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Why do we need to find **inverse** of a matrix?

1. 'Primarily, "division" does not exist for matrices, instead, we do "inverse".

   *Given a matrix $A$ and $B$ such that $B = AX$.*
   *How do we find $X$?* $\Rightarrow X = BA^{-1}$

2. Applications:
   - solving a system of linear equations;
   - used to encrypt/decrypt message codes;
   - etc.

# How to compute the inverse of $2 \times 2$ matrices?

*Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *what is* $A^{-1}$*?*

Let $A^{-1} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$. We have:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**We solve the linear system:**

$$\begin{cases} ax_1 + by_1 & = 1 \\ cx_1 + dy_1 & = 0 \end{cases} \quad \text{and} \quad \begin{cases} ax_2 + by_2 & = 0 \\ cx_2 + dy_2 & = 1 \end{cases}$$

# Inverse of $2 \times 2$ matrices

It gives:

$$x_1 = \frac{d}{ad - bc}, \quad y_1 = \frac{-c}{ad - bc}, \quad x_2 = \frac{-b}{ad - bc}, \quad y_2 = \frac{a}{ad - bc}$$

Note that $ad - bc = |A|$ (the *determinant* of $A$).

When $|A| \neq 0$, the values $x_1$, $y_1$, $x_2$, and $y_2$ exist.

Hence,

$$A^{-1} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} d/|A| & -b/|A| \\ -c/|A| & a/|A| \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

# Inverse of $2 \times 2$ matrices

**Conclusion:**

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

When $|A| \neq 0$, the inverse of a $2 \times 2$ matrix $A$ may be obtained from A as follows:

1. Interchange the two elements on the diagonal ($a$ and $d$);
2. Take the negatives of the other two elements ($b$ and $c$);
3. Multiply the resulting matrix by $\frac{1}{|A|}$ or, equivalently, divide each element by $|A|$.

**Note:** If $|A| = 0$, then $A$ is not invertible.

# Example

Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$$|A| = 2(5) - 3(4) = 10 - 12 = -2$$

Since $|A| \neq 0$, then $A$ is invertible.

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

Furthermore, $|B| = 1(6) - 3(2) = 0$, so $B$ is not invertible.

# **Part 2:** Computing inverse from adjoint

# Inverse of $n \times n$ matrices

**Note:**

If $A$ is an $n \times n$ matrices, $A^{-1}$ can be obtained as above, by finding the solution of the $n \times n$ linear system equations.

This is not so practical to be solved using the substitution/elimination method. A method will be discussed later.

# Review on minors and cofactors

Let $A = [a_{ij}]$ be an $n$-square matrix.

Define $M_{ij}$ as the $(n-1)$-square matrix obtained from $A$ by deleting the $i$-th row and the $j$-th column of $A$.

The minor of the element $a_{ij}$ of $A$ is defined as:

$$\text{minor}(A) = \det(M_{ij})$$

The cofactor of $a_{ij}$ is defined as the signed minor of $a_{ij}$, and denoted by:

$$A_{ij} = (-1)^{i+j} \, |M_{ij}|$$

# Adjoint

We can form a matrix of cofactors

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

where $C_{ij}$ is the cofactor of $a_{ij}$.

The adjoint of matrix $A$ is defined as:

$$\operatorname{adj}(A) = C^T$$

# Example of adjoint

Given matrix:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

The cofactors of $A$ are:

- $C_{11} = 12$
- $C_{12} = 6$
- $C_{13} = -16$
- $C_{21} = 4$
- $C_{22} = 2$
- $C_{23} = 16$
- $C_{31} = 12$
- $C_{13} = -10$
- $C_{33} = 16$

The matrix of cofactors and the adjoint of $A$ are:

$$C = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix} \qquad \mathrm{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

# matrix inverse from adjoint

**Theorem**

*Let $A$ be an* **invertible** *matrix. Then:*

$$A^{-1} = \frac{1}{\det(A)}\ adj(A)$$

*Proof can be read on the Howard Anton book, page 134.*

# Example

From the *previous example*, we have:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix} \qquad \operatorname{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$\det(A) = 0 + 12 + 4 - (-12 - 36 + 0) = 16 - (-48) = 64$$

Hence,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & \frac{-10}{64} \\ \frac{-16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

# **Part 3:** Properties of matrix inverse

# Basic properties of matrix inverse

Let $A$ be an **invertible** matrix. The followings hold.

1. $(A^{-1})^{-1} = A$
2. $(kA)^{-1} = k^{-1}A^{-1}$ for a scalar $k \neq 0 \in \mathbb{R}$
3. $(A^T)^{-1} = (A^{-1})^T$
4. $\det(A^{-1}) = (\det(A))^{-1}$

**Exercises:**

*Prove the properties of matrix inverse.*

*Give an example for each property to check the correctness of the theorem.*

# Basic properties of matrix inverse

Theorem

*If $A$ and $B$ are invertible, then $AB$ is invertible.*

Proof.

Consider $B^{-1}A^{-1}$. Then:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Hence, $AB$ is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$. $\square$

**Generalization:**

If $A_1, A_2, \ldots, A_k$ are invertible matrices, then:

$$(A_1 A_2 \ldots A_k)^{-1} = A_k^{-1} \ldots A_2^{-1} A_1^{-1}$$

# Exercise

*will be given during the lecture*

Numbers 4, 5, 6, page 76 Howard Anton Reference Book

# **Part 4:** Orthogonal matrices

# Orthogonal matrices

A matrix is called orthogonal if $A^T = A^{-1}$, i.e., $AA^T = A^T A = I$ (the identity matrix).

**Note:** $A$ is orthogonal *only if* $A$ is square and invertible matrix.

### Example

Let $A = \begin{bmatrix} \frac{1}{9} & \frac{8}{9} & -\frac{4}{9} \\ \frac{4}{9} & -\frac{4}{9} & -\frac{7}{9} \\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9} \end{bmatrix}$

Is $A$ orthogonal? What is the result of $AA^T$?

# Orthonormality

Vectors $u_1, u_2, \ldots, u_m$ in $\mathbb{R}^n$ are said to form an orthonormal set of vectors if the vectors are unit vectors and are orthogonal to each other; i.e.,

$$u_i \cdot u_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

### Theorem

*Let $A$ be a real matrix. Then the following are equivalent:*

- *$A$ is orthogonal.*
- *The rows of $A$ form an orthonormal set.*
- *The columns of $A$ form an orthonormal set.*

*to be continued...*