02 - Computational Complexity Analysis

[KOMS119602] & [KOMS120403]

Design and Analysis of Algorithm (2021/2022)

Dewi Sintiari

Prodi S1 Ilmu Komputer
Universitas Pendidikan Ganesha

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Euclidean algorithm to compute gcd (1)

From last week...

Computing gcd:

- Input: two integers a and b
- Output: the greatest common divisor of m and n

Algorithm 1 Naive gcd algorithm of two integers

```
procedure GCD(a, b)
    r = 1
    x = min(a, b)
    for i = 1 to x do
        if a mod i == 0 and b mod i == 0 then r = i
        end if
    end for
end procedure
```

Complexity? homework!

Euclidean algorithm to compute gcd (2)

Example

Using the Euclidean algorithm, find the gcd of 210 and 45.

Solution:

$$
\begin{aligned}
210 &= 4 \cdot 45 + 30 \\
45 &= 1 \cdot 30 + 15 \\
30 &= 2 \cdot 15 + 0
\end{aligned}
$$

So $gcd(210, 45) = 15$

Euclidean algorithm to compute gcd (3)

Algorithm 2 Euclidean algorithm

procedure $\textsc{EuclidGcd}(a, b)$
  while $b \neq 0$ **do**
    $r = a \mod b$
    $a = b$
    $b = r$
  end while
  return a
end procedure

Why does it terminate?
Complexity? homework!

Computational complexity model (1)

Can you recall what is complexity of an algorithm, and why should we study it?

Computational complexity model (2)

A part of *algorithm analysis* is computing the *computational complexity* of an algorithm.

The computational complexity or simply complexity of an algorithm is the amount of resources (*time* and *memory*) required to run it.

- Time efficiency: how fast an algorithm is executed
- Space efficiency: how much memory needed to execute an algorithm

How do we compute the complexity of an algorithm?

Computational complexity model (3)

Example

Let a supercomputer executes an algorithm A, and a PC executes an algorithm B. Both computers have to sort an array of 1 million elements. The supercomputer can execute 100 million instructions in one second, while the PC is only able to execute 1 million instructions in one second.

Algorithm A needs $2n^2$ instructions to sort n elements, and algorithm B needs $50n \log n$ instructions. Compute the amount of time to sort 1 million elements in each computer!

Computational complexity model (4)

Solution:

- Supercomputer: $\frac{2 \cdot (10^6)^2 \text{ instructions}}{10^8 \text{ instructions / sec}} = 20000$ sec ≈ 5.56 hours
- PC: $\frac{50 \cdot 10^6 \log 10^6 \text{ instructions}}{10^6 \text{ instructions / sec}} \approx 1000 sec \approx 16.67$ minutes

Remark. So, the number of executions matters!

Computational complexity model (5)

What affects computational complexity?

Time (and space) complexity depends on lots of things like *hardware*, *OS*, *processors*, *programming language* and *compiler*, etc. But we don't consider any of these factors when analyzing the algorithm.

Remarks:

- Our focus on this subject will be on time complexity.
- We assume that our machine uses only one processor (i.e. *generic one-processor*).
- Time complexity is computed based on the number of operations/instructions
- The running time of an algorithm increases (or remains constant in case of constant running time) as the input size (n) increases.

Computational complexity model (6)

Algorithm 3 Average of an array of integers

```
1: procedure AVERAGE(A[1..n])
2:     sum ← 0
3:     for i = 1 to n do
4:         sum ← sum + A[i]
5:     end for
6:     avg ← sum/n
7: end procedure
```

The number of operations:

- Assignment: lines 2, 4, 6; with $1 + n + 1 = n + 2$ operations
- Summation: line 4, with n operations
- Division: line 6, with 1 operation

Complexity: $T(n) = (n + 2) + n = 2n + 2$ operations.

Computational complexity model (7)

Three measurements of resource usage:

- **Worst-case** ($T_{\max}(n)$): it measures the resources (e.g. running time, memory) that an algorithm requires in the worst case i.e. most difficult case, given an input of arbitrary size n (usually denoted in asymptotic notation).
- **Best-case** ($T_{\min}(n)$): describe an algorithm's behavior under optimal conditions.
- **Average-case** ($T_{\text{avg}}(n)$): the amount of computational time used by the algorithm, averaged over all possible inputs.

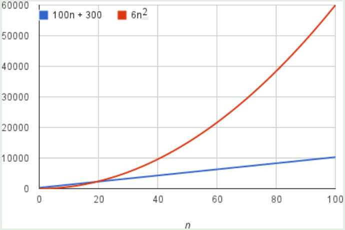
Asymptotic notation and order of magnitude (1)

- The running time of an algorithm is measured as a *function of the size of its input*.
- Rate of growth of the running time measures how fast a function grows with the input size. Asymptotically means the function matters *only for large values of n*.
- The order of magnitude function describes the part of the function that increases the fastest as the value of n increases.

Asymptotic notation and order of magnitude (2)

Example

Suppose that an algorithm, running on an input of size n, takes $6n^2 + 100n + 300$.



We only keep the most significant term. We say that the function $6n^2$ has a higher order of magnitude than $100n + 300$.

Big-$\mathcal{O}$ notation: asymptotic upper-bound (1)

Worst-case complexity measures the resources an algorithm needs in the *worst-case*. It gives an upper bound on the resources required by the algorithm.

Why learn worst-case complexity?

- provides information about the maximum resource requirements
- naturally, it often happens in a system

Big-$\mathcal{O}$ notation: asymptotic upper-bound (2)

Big-O ($\mathcal{O}(\cdot)$) notation: a mathematical notation that describes the limiting behavior of a function when the argument tends towards a particular value or infinity.

Definition

$g(n) \in \mathcal{O}(f(n))$ if $\exists\ k > 0$ and n_0 s.t. $g(n) \leq k \cdot f(n), \ \ \forall n \geq n_0$.

Big-$\mathcal{O}$ notation: asymptotic upper-bound (3)

Definition

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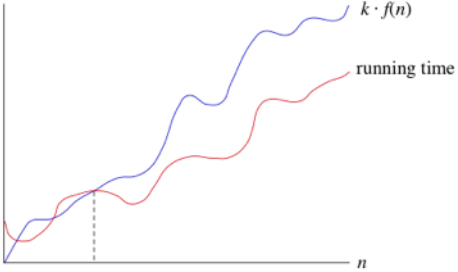


Big-$\mathcal{O}$ notation (linear and polynomial functions)

Example

Show that $g(n) = 5n + 3$ is in $\mathcal{O}(n)$.

Solution:

Note that $5n + 3 \leq 5n + 3n = 8n$ for all $n \geq 1$. In this case, $k = 8$ and $n_0 = 1$. So, $g(n) \in \mathcal{O}(n)$.

Big-$\mathcal{O}$ notation: asymptotic upper-bound (4)

Example

Show that $g(n) = 3n^2 - 5n + 6$ is in $\mathcal{O}(n^2)$.

Big-$\mathcal{O}$ notation: asymptotic upper-bound (5)

Solution:

Note that $3n^2 - 5n + 6 \leq 3n^2 + 0 + 6n^2 = 9n^2$ for all $n \geq 1$. In this case, $k = 9$ and $n_0 = 1$. So, $g(n) \in \mathcal{O}(n^2)$.

Big-$\mathcal{O}$ notation: asymptotic upper-bound (6)

We denote by $T(n)$ a function of time complexity.

Theorem (Big-O of a polynomial complexity)

If $T(n) = a_m n^m + a_{m-1} n^{m-1} + \cdots + a_1 n + a_0$ is a polynomial of order m, then $T(n) \in \mathcal{O}(n^m)$.

Theorem (Arithmetic operations on Big-O)

Let $T_1(n) \in \mathcal{O}(f(n))$ and $T_2(n) \in \mathcal{O}(g(n))$, then:

1. $T_1(n) + T_2(n) \in \mathcal{O}(f(n)) + \mathcal{O}(g(n)) \in \mathcal{O}(\max(f(n), g(n)))$
2. $T_1(n) T_2(n) \in \mathcal{O}(f(n))\mathcal{O}(g(n)) \in \mathcal{O}(f(n)g(n))$
3. $\mathcal{O}(cf(n)) \in \mathcal{O}(f(n))$, *where c is a constant*
4. $f(n) \in \mathcal{O}(f(n))$

Proof: homework!

Example (Arithmetic operations on Big-O)

1. Let $T_1(n) \in \mathcal{O}(n)$ and $T_2(n) \in \mathcal{O}(n^2)$, then:

$$T_1(n) + T_2(n) \in \mathcal{O}(\max(n, n^2)) \in \mathcal{O}(n^2)$$

2. Let $T_1(n) \in \mathcal{O}(n)$ and $T_2(n) \in \mathcal{O}(n^2)$, then:

$$T_1(n) T_2(n) \in \mathcal{O}(n \cdot n^2) = \mathcal{O}(n^3)$$

3. $\mathcal{O}(5n^2) \in \mathcal{O}(n^2)$
4. $n^2 \in \mathcal{O}(n^2)$

Big-$\mathcal{O}$ notation: logarithmic function (1)

Review logarithms and exponents

$$\log_b a = c \Leftrightarrow b^c = a$$

- $a > 0$ is the power
- $b > 0$ is the base
- c is the exponent

Remark. If the base $b = 2$, then it is called binary logarithm. The base is often omitted.

Big-$\mathcal{O}$ notation: logarithmic function (2)

In Computer Science, we usually use base-two logarithm complexity by default. Why?

- It is common to work with binary numbers or divide input data in half
- In Big-O notation (upper bound growth), all logarithms are *asymptotically equivalent* (the only difference is there multiplicative constant factor)
- So, we do not specify the base, and only write it as $\mathcal{O}(\log n)$

Some properties of logarithmic function

- $\log_b 1 = 0$ for any $b \geq 0$
- **Change of bases:** $\log_b a = \frac{\log_p a}{\log_p b}$
- **Addition:** $\log_p m + \log_p n = \log_p mn$
- **Subtraction:** $\log_p m - \log_p n = \log_p \frac{m}{n}$
- **Power:** $\log_p a^x = x \cdot \log_p a$
- **Inverse:** $\log_p \frac{1}{a} = -log_p a$
- Many others...

Big-$\mathcal{O}$ notation: logarithmic function (4)

Example

Show that $g(n) = (n+3)\log(n^2+1) + 2n^2$ is in $\mathcal{O}(n^2)$

Solution:

Note that:
$\log(n^2+1) \leq \log(2n^2) = \log 2 + \log n^2 \leq 2\log n^2 = 4\log n$.
So, $\log(n^2+1) \in \mathcal{O}(\log n)$.

Since $n+3 \in \mathcal{O}(n)$, then
$(n+3)\log(n^2+1) \in \mathcal{O}(n) \cdot \mathcal{O}(\log n) \in \mathcal{O}(n \log n)$.

Since $2n^2 \in \mathcal{O}(n^2)$, and $\max(n\log n, n^2) = n^2$, then $g(n) \in \mathcal{O}(n^2)$.

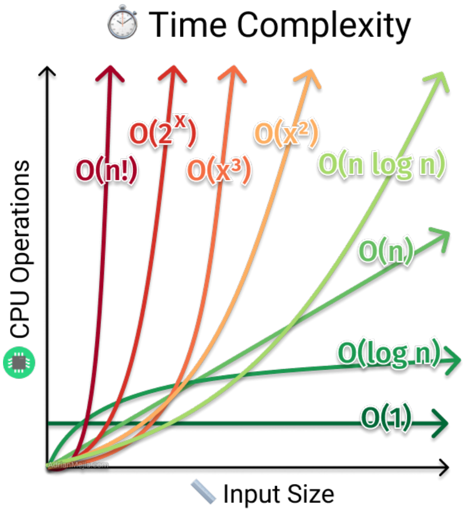
Big-$\mathcal{O}$ notation: classification of algorithms (1)

The classification of algorithms based on the worst-time complexity

Complexity	**Class**
$\mathcal{O}(1)$	constant
$\mathcal{O}(\log n)$	logarithmic
$\mathcal{O}(n)$	linear
$\mathcal{O}(n \log n)$	quasi-logarithmic
$\mathcal{O}(n^2)$	square
$\mathcal{O}(n^3)$	cubic
$\mathcal{O}(n^k),\ k \geq 2$	polynomial
$\mathcal{O}(2^n)$	exponential
$\mathcal{O}(n!)$	factorial

$$\underbrace{\mathcal{O}(1) < \mathcal{O}(\log n) < \mathcal{O}(n) < \mathcal{O}(n \log n) < \mathcal{O}(n^2) < \mathcal{O}(n^3) < \cdots <}_{\text{polynomial algorithms}} \underbrace{\mathcal{O}(2^n) < \mathcal{O}(n!)}_{\text{exponential algorithms}}$$

Big-$\mathcal{O}$ notation: classification of algorithms (2)



Big-$\mathcal{O}$ notation: determining asymptotic complexity (1)

1. **Assignment of values** (*comparison, arithmetic operations, read, write*) needs $\mathcal{O}(1)$
2. **Accessing** an element of an array, or selecting a field from a record needs $\mathcal{O}(1)$

Example

- read(x) $\rightarrow \mathcal{O}(1)$
- $x : x + a[k] \rightarrow \mathcal{O}(1)$
- print(x) $\rightarrow \mathcal{O}(1)$

Big-$\mathcal{O}$ notation: determining asymptotic complexity (2)

3. **If-Else condition:** IF C THEN A1 ELSE A2 needs time: $T_C + \max(T_{O1}, T_{O2})$

Example

```
read(x)
if x mod 2 = 0 then
    x := x + 1
    print("Even")
else
    print("Odd")
end if
```

Asymptotic TC: $\mathcal{O}(1) + \mathcal{O}(1) + \max(\mathcal{O}(1) + \mathcal{O}(1), \mathcal{O}(1)) \in \mathcal{O}(1)$

Big-$\mathcal{O}$ notation: determining asymptotic complexity (3)

4. **For loop:** the time complexity is the number of iterations multiplied with the time complexity of the *body loop* (i.e. *loop statements*)

Example (Single for loop)

---

for $i = 1$ to n **do**
    sum:= sum + a[1]
end for

Asymptotic TC: $n \cdot \mathcal{O}(1) = \mathcal{O}(n)$

Example (Two nested for loops with one instruction)

```
for i = 1 to n do
    for j = 1 to n do
        a[i, j] := i + j
    end for
end for
```

Asymptotic TC: $n \cdot \mathcal{O}(n) = \mathcal{O}(n^2)$

Big-$\mathcal{O}$ notation: determining asymptotic complexity (5)

Example (Two nested for loops with two instructions)

```
for i = 1 to n do
    for j = 1 to i do
        a := a + 1
        b := b − 1
    end for
end for
```

The outer loop is executed n times, and the inner loop is executed i times for each j. The number of iterations: $1 + 2 + \cdots + n = \frac{n(n+1)}{2} \in \mathcal{O}(n^2)$.

The body loop needs $\mathcal{O}(1)$-time.

Asymptotic time complexity: $\mathcal{O}(n^2)$

5. **While loop:** WHILE C DO A; and REPEAT A UNTIL C.
   Time complexity $=$ # iterations $\times$ T_{body}

Example (Single loop with $n-1$ iterations)

```
i := 2
while i ≤ n do
    sum:= sum + a[i]
    i := i + 1
end while
```

Asymptotic TC:
$\mathcal{O}(1) + (n-1)(\mathcal{O}(1) + \mathcal{O}(1) + \mathcal{O}(1)) = \mathcal{O}(1) + \mathcal{O}(n-1) \in \mathcal{O}(n)$

Example (Infinite loop)

```
x := 0
while x < 5 do
    x := 1
    x := x + 1
end while
```

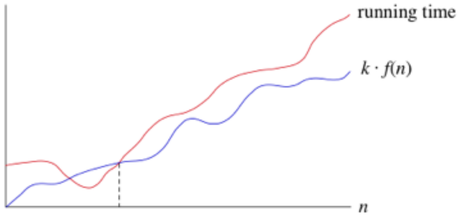
In this situation, x will never be greater than 5, since at the start of the while loop, x is given the value of 1, thus, the loop will always end in 2 and the loop will never break.

We can also say that an algorithm takes *at least a certain amount of time*, without providing an upper bound.

Big-Omega ($\Omega(\cdot)$) notation

Definition

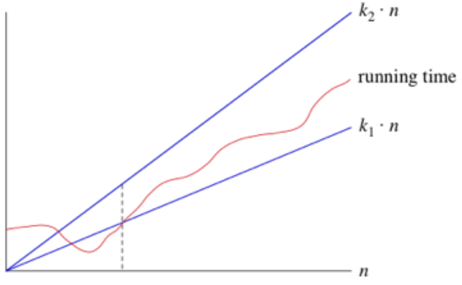
$g(n) \in \Omega(f(n))$ if $\exists\ k > 0$ and n_0 s.t. $g(n) \geq k \cdot f(n),\ \ \forall n \geq n_0$.



Big-Θ notation: asymptotically tight-bound

Definition

$g(n) \in \Theta(f(n))$ if $\exists\ k_1, k_2 > 0$ and n_0 s.t.
$k_1 \cdot f_n \leq g(n) \leq k_2 \cdot f(n), \ \ \forall n \geq n_0$.



QUIZ

Exc 1: Growth of function in Big-$\mathcal{O}$ (1)

Which kind of growth best characterizes each of these functions?

	Constant	Linear	Polynomial	Exponential
$(3/2)^n$				
1				
$(3/2)\,n$				
$2n^3$				
2^n				
$3n^2$				
1000				
$3n$				

Which kind of growth best characterizes each of these functions?

	Constant	**Linear**	**Polynomial**	**Exponential**
$(3/2)^n$				✓
1	✓			
$(3/2)\,n$		✓		
$2n^3$			✓	
2^n				✓
$3n^2$			✓	
1000	✓			
$3n$		✓		

Exc 2: Comparing function growth (1)

Match each function with an equivalent function that satisfies $g(n) = \Theta(f(n))$.

$g(n)$	$f(n)$
$n + 30$	$n^2 + 3n$
$n^2 + 2n - 10$	n^4
$n^3 * 3n$	$\log_2 2x$
$\log_2 x$	$3n - 1$

Exc 2: Comparing function growth (2)

Recall that $g(n) \in \Theta(f(n))$ if $\exists k_1, k_2 > 0$ s.t. for all sufficiently large n, we have

$$k_1 \cdot f(n) \leq g(n) \leq k_2 \cdot f(n)$$

We drop the constants and lower order terms (i.e. only keep the most significant term).

$g(n)$	**simplified**	$f(n)$	**simplified**
$n + 30$	n	$n^2 + 3n$	n^2
$n^2 + 2n - 10$	n^2	n^4	n^4
$n^3 * 3n$	n^4	$\log_2 2x$	$\log x$
$\log_2 x$	$\log x$	$3n - 1$	n

Two functions match if the corresponding simplified functions are equal.

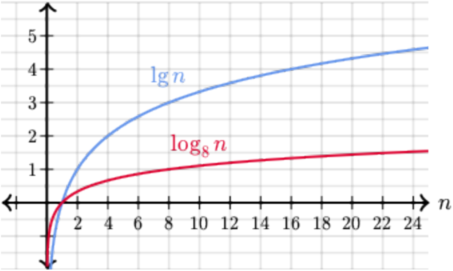
Exc 3: Asymptotic notation (1)

For the functions $\log_2 n$ and $\log_8 n$, what is the asymptotic relationship between these functions?

- $\log_2 n$ is in $\mathcal{O}(\log_8 n)$
- $\log_2 n$ is in $\Omega(\log_8 n)$
- $\log_2 n$ is in $\Theta(\log_8 n)$

Exc 3: Asymptotic notation (2)

Both $\log_2 n$ and $\log_8 n$ are functions with logarithmic growth, with their base as the only difference.



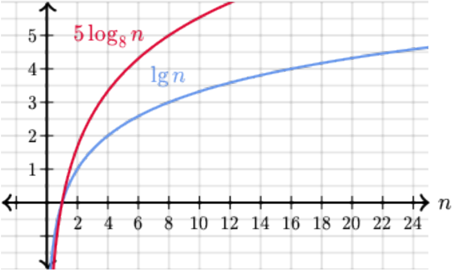
Exc 3: Asymptotic notation (3)

- Is $\log_2 n$ in $\mathcal{O}(\log_8 n)$?

Recall that $\log_a n = \frac{\log_b n}{\log_b a}$.

So, $\log_8 n = \frac{\log_2 n}{\log_2 8} = \frac{\log_2 n}{3} = \frac{1}{3} \cdot \log_2 n$.

We can take $k = 5$, so that: $\log_2 n \leq 5 \log_8 n$.



- Is $\log_2 n$ in $\Omega(\log_8 n)$?

Since $\log_8 n = \frac{1}{3} \cdot \log_2 n$, then $\log_2 n \geq \log_8 n$ for all $n \geq 1$.

So, $\log_2 n \in \Omega(\log_8 n)$

- Is $\log_2 n$ in $\Theta(\log_8 n)$?

Clearly, $\log_8 n \leq \log_2 n \leq 5 \cdot \log_8 n$ for all $n > 1$.

So, $\log_2 n \in \Theta(\log_8 n)$.