

Linear Algebra

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12.2 - Fundamental spaces: row, column, and null spaces

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Row vectors and column vectors

Given an $m \times n$ matrix A :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- **Row vector**: vector formed from a row of A
- **Column vector**: vector formed from a column of A

Row vectors and column vectors

The row vectors of A are:

$$\mathbf{r}_1 = [a_{11} \ a_{12} \ \cdots \ a_{1n}]$$

$$\mathbf{r}_2 = [a_{21} \ a_{22} \ \cdots \ a_{2n}]$$

$$\vdots = \quad \quad \quad \vdots$$

$$\mathbf{r}_m = [a_{m1} \ a_{m2} \ \cdots \ a_{mn}]$$

The column vectors of A are:

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Let A be an $(m \times n)$ matrix.

- The subspace of \mathbb{R}^n formed by row vectors of A is called **row space** of matrix A .
- Subspace of \mathbb{R}^m formed by column vectors of A is called **column space** of matrix A .
- The solution space of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ (which is a subspace of \mathbb{R}^n) is called **null space** of matrix A .

Relationship

Question 1. What relationships exist among the solutions of a linear system $A\mathbf{x} = \mathbf{b}$ and the row space, column space, and null space of the coefficient matrix A ?

Question 2. What relationships exist among the row space, column space, and null space of a matrix?

Column space

Consider the system $A\mathbf{x} = \mathbf{b}$ where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ be the column vectors of A . The system can be written as:

$$A\mathbf{x} = \mathbf{b}$$

$$\Leftrightarrow x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}$$

Hence, the system has a solution if and only if \mathbf{b} can be expressed as a linear combination of the column vectors of A .

Theorem

A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if \mathbf{b} is in the column space of A .

Example of column space

Given a linear system $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -9 & -3 \end{bmatrix}$$

Show that \mathbf{b} is in the column space of A by expressing it as a linear combination of the column vectors of A .

Solution:

Steps:

- Solve the system by Gaussian elimination:

$$x_1 = 2, x_2 = -1, x_3 = 3$$

- This yields:

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

i.e.,

$$x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + x_3 \mathbf{c}_3 = \mathbf{b}$$

Null space

Given matrix:

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & 1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix}$$

To determine the null space of A , solve the homogeneous linear system $A\mathbf{x} = \mathbf{0}$:

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & 1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the system by Gauss elimination, we obtain:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s - t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The solution of the system can be written in matrix equation:

$$\mathbf{x} = s\mathbf{v}_1 + t\mathbf{v}_2$$

where $s, t \in \mathbb{R}$, $\mathbf{v}_1 = (-1, 1, 0, 0, 0)$ and $\mathbf{v}_2 = (-1, 0, -1, 0, 1)$.

Determine the basis of null
space

Properties of row/column space and null space

Theorem

*Elementary row operations do not change the **row space** of a matrix.*

Theorem

*Elementary row operations do not change the **null space** of a matrix.*

How to determine the basis of row space, column space, and null space?

Let A be an $(m \times n)$ matrix. How to determine the basis of row space, column space, and null space of matrix A ?

1. Perform elementary row operations to obtain the reduced-row echelon form matrix R ;
2. The basis of the row space of A is all row vectors that contain leading 1 * of matrix R ;
3. The basis of column space of A is all column vectors of matrix A that correspond with the column vector of matrix R that contains leading 1.

*Leading 1 is the leading entry in each nonzero row is 1

Intuition behind the algorithm

Example 1: determining the basis for row space and column space

Determine the basis of row space, column space, and null space of matrix:

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim ERO \sim \begin{bmatrix} \color{red}{1} & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & \color{red}{1} & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & \color{red}{1} & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

The basis of the row space is:

$$\mathbf{r}_1 = [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4]$$

$$\mathbf{r}_2 = [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6]$$

$$\mathbf{r}_3 = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5]$$

Example 1 (*cont.*)

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

So, the basis of the column space is:

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \quad \mathbf{c}_2 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} \quad \mathbf{c}_3 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

Example 2: determining the basis of null space

To determine the basis of null space, solve the equation $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 0 \\ 2 & -6 & 9 & -1 & 8 & 2 & 0 \\ 2 & -6 & 9 & -1 & 9 & 7 & 0 \\ -1 & 3 & -4 & 2 & -5 & -4 & 0 \end{bmatrix} \sim ERO \sim \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 0 \\ 0 & 0 & 1 & 3 & -2 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The linear system correspond to the last augmented matrix is:

$$\begin{cases} x_1 - 3x_2 + 4x_3 - 2x_4 + 5x_5 + 4x_6 = 0 \\ x_3 + 3x_4 - 2x_5 - 6x_6 = 0 \\ x_5 + 5x_6 = 0 \end{cases}$$

from which we can extract the following:

$$x_5 = -5x_6$$

$$x_3 = -3x_4 + 2x_5 + 6x_6 = -3x_4 + 2(-5x_6) + 6x_6 = -3x_4 - 4x_6$$

$$x_1 = -3x_2 - 4x_3 + 2x_4 - 5x_5 - 4x_6$$

$$= -3x_2 - 4(-3x_4 - 4x_6) + 2x_4 - 5(-5x_6) - 4x_6$$

$$= -3x_2 + 14x_4 + 22x_6$$

Example 2 (cont.)

Let $x_2 = r$, $x_4 = s$, and $x_6 = t$, then the solution of $A\mathbf{x} = \mathbf{0}$ is:

$$x_1 = -3x_2 + 14x_4 + 22x_6 = -3r + 14s + 22t$$

$$x_3 = -3x_4 - 4x_6 = -3s - 4t$$

$$x_5 = -5t$$

This can be written as vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r + 14s + 22t \\ r \\ -3s - 4t \\ s \\ -5t \\ t \end{bmatrix} = \begin{bmatrix} -3r \\ r \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 14s \\ 0 \\ -3s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 22t \\ 0 \\ -4t \\ 0 \\ -5t \\ t \end{bmatrix} = r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 14 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 22 \\ 0 \\ -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

The basis of the null space is:

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \mathbf{v}_2 = (14, 0, -3, 1, 0, 0), \mathbf{v}_3 = (22, 0, -4, 0, -5, 0)$$

Rank and Nullity

In Example 1, we found that the row space and column space of matrix:

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

both contain three vectors. Hence, they are both **three-dimensional spaces**.

Does this hold for other matrices?

Dimension of row space and column space

Theorem

The row space and the column space of a matrix A have the same dimension.

Proof.

- The elementary row operations do not change the dimension of the row space and column space of a matrix.
- Let R be any row echelon form of A , then:

$$\dim(\text{row space of } A) = \dim(\text{row space of } R)$$

$$\dim(\text{column space of } A) = \dim(\text{column space of } R)$$

- $\dim(\text{row space of } R) = \text{the number of nonzero rows in } R$; and
- $\dim(\text{column space of } R) = \text{the number of leading 1's in } R$.

Since in R , $\text{the number of nonzero rows} = \text{the number of leading 1's}$,
hence $\dim(\text{row space of } A) = \dim(\text{column space of } A)$. □

Rank and nullity

The dimension of the row space (and column space) of a matrix A is called the **rank of A** , and denoted by **$\text{rank}(A)$** .

The dimension of the *null space* of A is called the **nullity of A** , and denoted by **$\text{nullity}(A)$** .

Theorem (Dimension Theorem for Matrices)

If A is a matrix with n columns, then:

$$\text{rank}(A) + \text{nullity}(A) = n$$

Example

Find the rank and nullity of the matrix (size 4×6):

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Solution:

- **Rank**

The reduced row echelon form of A is (verify it!):

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are two rows with leading 1, then:

$$\dim(\text{row space of } A) = \dim(\text{column space of } A) = 2$$

Example (*cont.*)

- Nullity

To find the nullity, solve the linear system: $A\mathbf{x} = \mathbf{0}$.

From the reduced echelon form of A , we obtain the following linear system:

$$\begin{cases} x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0 \\ x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0 \end{cases}$$

Solving these equations for the *leading variables* yields:

$$x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6$$

$$x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6$$

So, the solution of the system is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example (*cont.*)

Hence, the vectors:

$$\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a **basis** for the solution space, then:

$$\text{nullity}(A) = 4$$

Remark. Observed that:

$$\begin{aligned} \text{rank}(A) + \text{nullity}(A) &= n \\ 2 + 4 &= 6 \end{aligned}$$

Conclusion

Theorem

If A is an $(m \times n)$ matrix, then:

1. $\text{rank}(A)$ = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$.
2. $\text{nullity}(A)$ = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

Exercise:

Find the rank and nullity of the matrix:

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Solution of exercise

The reduced echelon form of the matrix is the following:

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three nonzero rows in the matrix, so $\text{rank}(A) = 3$.

By the “Dimension Theorem”, $\text{nullity}(A) = n - \text{rank}(A) = 6 - 3 = 3$

Solution of exercise (*cont.*)

To prove that $\text{nullity}(A) = 5$, we solve the linear system: $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 0 \\ 2 & -6 & 9 & -1 & 8 & 2 & 0 \\ 2 & -6 & 9 & -1 & 9 & 7 & 0 \\ -1 & 3 & -4 & 2 & -5 & -4 & 0 \end{bmatrix} \sim ERO \sim \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 0 \\ 0 & 0 & 1 & 3 & -2 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the reduced augmented matrix, we get the linear system:

$$\begin{cases} x_1 - 3x_2 + 4x_3 - 2x_4 + 5x_5 + 4x_6 = 0 \\ x_3 + 3x_4 - 2x_5 - 2x_6 = 0 \\ x_5 + 5x_6 = 0 \end{cases}$$

Solving the system for the leading 1's yields:

$$x_5 = -5x_6$$

$$x_3 = -3x_4 - 8x_6$$

$$x_1 = 3x_2 + 14x_4 + 57x_6$$

Solution of exercise (*cont.*)

Hence, the solution of the system can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3r + 14s + 57t \\ s \\ -3s - 8t \\ s \\ -5t \\ t \end{bmatrix} = r \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 57 \\ 0 \\ -8 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

where $r, s, t \in \mathbb{R}$.

Hence, the basis of the null space of A is:

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 57 \\ 0 \\ -8 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$$

which means that $\text{nullity}(A) = 3$.

Equivalent statements

If A is an $(n \times n)$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $Ax = \mathbf{0}$ has only the trivial solution.
3. The reduced row echelon form of A is I_n .
4. A is expressible as a product of elementary matrices.
5. $Ax = \mathbf{0}$ is consistent for every $(n \times 1)$ matrix b .
6. $Ax = \mathbf{0}$ has exactly one solution for every $(n \times 1)$ matrix b .
7. $\det(A) \neq 0$.
8. The column vectors of A are linearly independent.
9. The row vectors of A are linearly independent.
10. The column vectors of A span \mathbb{R}^n .
11. The row vectors of A span \mathbb{R}^n .
12. The column vectors of A form a basis for \mathbb{R}^n .
13. The row vectors of A form a basis for \mathbb{R}^n .
14. A has rank n .
15. A has nullity 0.