04 - Recursive Algorithm

[KOMS119602] & [KOMS120403]

Design and Analysis of Algorithm (2021/2022)

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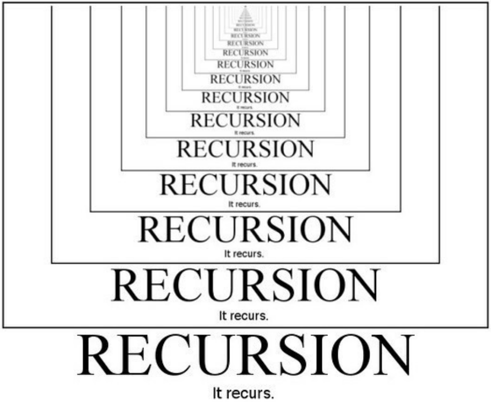
Table of contents



What is recursion or recursive algorithm?

1. The principal of recursive algorithm

A recursive algorithm is an algorithm which calls itself with "smaller (or simpler)" input values, and which obtains the result for the current input by applying simple operations to the returned value for the smaller (or simpler) input.

Characteristics of recursive algorithm:

1. It calls itself recursively
2. It has a base case
3. It must change it state and move towards the base case

A base case is the condition that allows the algorithm to stop recursing: a base case is typically a problem that is small enough to solve directly.

A change of state means that some data that the algorithm is using is modified. Usually the data that represents our problem gets smaller in some way.

Recursion versus Iteration

Iteration: A function repeats a defined process until a condition fails. This is usually done through a loop, such as a for or while loop with a counter and comparative statement making up the condition that will fail. An infinite loop for iteration occurs when the condition never fails.

Recursion: Instead of executing a specific process within the function, the function calls itself repeatedly until a certain condition is met (this condition being the base case). The base case is explicitly stated to return a specific value when a certain condition is met. An infinite recursive loop occurs when the function does not reduce its input in a way that will converge on the base case.

Simple examples of recursive algorithms

2.1 - Computing factorial (1)

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

The formula can be expressed recursively:

$$n! = \begin{cases} n \times (n-1)!, & \text{if } n > 1 \\ 1, & n = 1 \end{cases}$$

2.1 - Computing factorial (2)

Algorithm 1 Factorial of a number

```
procedure FACTORIAL(n)
    if n = 1 then
        return 1
    else
        temp = FACTORIAL(n − 1)
        return n ∗ temp
    end if
end procedure
```

- What is the base case?

2.1 - Computing factorial (2)

Algorithm 2 Factorial of a number

```
procedure FACTORIAL(n)
    if n = 1 then
        return 1
    else
        temp = FACTORIAL(n − 1)
        return n ∗ temp
    end if
end procedure
```

- What is the base case? $n = 1$
- What is the change of states?

2.1 - Computing factorial (2)

Algorithm 3 Factorial of a number

```
procedure FACTORIAL(n)
    if n = 1 then
        return 1
    else
        temp = FACTORIAL(n − 1)
        return n ∗ temp
    end if
end procedure
```

- What is the base case? $n = 1$
- What is the change of states? n decreases
- What is the complexity?

2.1 - Computing factorial (2)

Algorithm 4 Factorial of a number

```
procedure FACTORIAL(n)
    if n = 1 then
        return 1
    else
        temp = FACTORIAL(n − 1)
        return n ∗ temp
    end if
end procedure
```

- What is the base case? $n = 1$
- What is the change of states? n decreases
- What is the complexity? $\mathcal{O}(n)$

2.1 - Computing factorial (3)

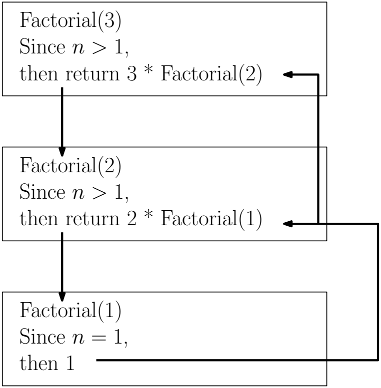


Figure: Illustration of recursive algorithm FACTORIAL with $n = 3$

2.1 - Computing factorial (4): proving correctness by induction

- Induction base: from line 1, we see that the function works correctly for $n = 1$.
- Hypothesis: suppose the function works correctly when it is called with $n = m$, for some $m \geq 1$.
- Induction step: We prove that it also works when it is called with $n = m + 1$. By the hypothesis, we know the recursive call works correctly for $n = m$ and computes $m!$. Subsequently, it is multiplied by $n = m + 1$, thus computes $(m + 1)!$. And this is the value correctly returned by the program.

2.2 - Finding Maximum Element of an Array (1)

To compute the max of n elements for $n > 1$ recursively:

- Compute the max of $n - 1$ elements
- Compare with the last element to find the entire max

To compute the max of n elements for $n > 1$ recursively:

- Compute the max of $n - 1$ elements
- Compare with the last element to find the entire max

Algorithm 6 Finding maximum of an array

```
procedure MAX(A[0..n − 1], int n)
    if n = 1 then return A[0]
    else
        T = MAX(A, n − 1)
        if T < A[n − 1] then
            return A[n − 1]
        else
            return T
        end if
    end if
end procedure
```

2.2 - Finding Maximum Element of an Array (2)

- Complexity?
- Correctness?

2.3 - Computing sum of elements in an array (1)

Problem: Given an array of n elements $A[0..n-1]$. We want to compute the sum: $S = \sum_{i=0}^{n-1} A[i]$

Algorithm 7 Sum of an array

```
procedure SUM(A[0..n], int n)
    if n = 1 then return A[0]
    else
        S = SUM(A, n − 1)
        S = S + A[n − 1]
        if T < A[n − 1] then
            return S
        end if
    end if
end procedure
```

2.3 - Computing sum of elements in an array (2)

- Complexity?
- Correctness?

2.6. Recursive MAX, 2nd approach (1)

Problem: Given an array of n elements $A[0..n-1]$, we aim to find an element of maximum value of the array.

Approach: Divide the array into two halves sub-array. Find the max of each sub-array. Then compare the two max values.

Algorithm 8 Finding max of an array

procedure $\textsc{FindMax}(A[0..n-1], \text{int } n)$
    if $n = 1$ **then** **return** $A[S]$
    end if
    $T_1 = \textsc{FindMax}(A\left[0..\lfloor \frac{n}{2} \rfloor\right], \lfloor \frac{n}{2} \rfloor)$
    $T_2 = \textsc{FindMax}(A\left[(\lfloor \frac{n}{2} \rfloor + 1)..(n-1)\right], n - \lfloor \frac{n}{2} \rfloor)$
    if $T_1 \geq T_2$ **then return** T_1
    end if
    return T_2
end procedure

Remark. $\lfloor x \rfloor$ means the largest integer that is $\leq x$; example: $\lfloor 3.5 \rfloor = 3$

2.6. Recursive Max, 2nd approach (2)

Complexity analysis: Special case when $n = 2^k$

Let $f(n)$: the number of key-comparisons to find the max of an n-array, with $n = 2^k$ for some $k \in \mathbb{Z}^+$. Hence:

$$f(n) = \begin{cases} 0, & n = 1 \\ 2f(n/2) + 1, & n \geq 2 \end{cases}$$

By repeated substitution:

$$\begin{aligned} f(n) &= 1 + 2f(n/2) \\ &= 1 + 2[1 + 2f(n/4)] = 1 + 2 + 2f(n/4) \\ &= 1 + 2 + 4 + 8f(n/4) \\ &\vdots \\ &= 1 + 2 + 4 + \cdots + 2^{k-1} + 2^k f(n/2^k) \\ &= 1 + 2 + 4 + \cdots + 2^{k-1} \\ &= 2^k - 1/(2-1) = 2^k - 1 \\ &= n - 1 \end{aligned}$$

2.6. Recursive MAX, 2nd approach (3)

$f(n)$: the number of key-comparisons to find the max of an n-array, with $n = 2^k$ for some $k \in \mathbb{Z}^+$.

Complexity analysis: For general n

$$f(n) = \begin{cases} 0, & n = 1 \\ f(\lfloor \frac{n}{2} \rfloor) + f(n - \lfloor \frac{n}{2} \rfloor) + 1, & n \geq 2 \end{cases}$$

Prove that:

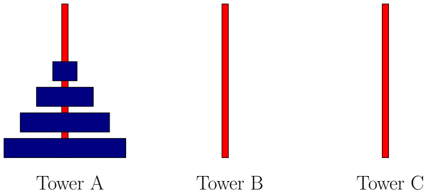
By induction, we obtain $f(n) = n - 1$. How?

Tower of Hanoi

Problem: there are three towers A, B, and C. Initially, there are n disks of varying sizes stacked on tower A, ordered by their size, with the largest disk on the bottom and the smallest one on the top. The object is to move all disc to the 2nd tower by keeping their order.

- Only one disk may be moved at a time in a restricted manner, from the top of one tower to the top of another tower.
- A larger disk must never be placed on top of a smaller disk.

Check `https://www.mathsisfun.com/games/towerofhanoi.html` for an illustration of the problem



3. Tower of Hanoi problem (2)

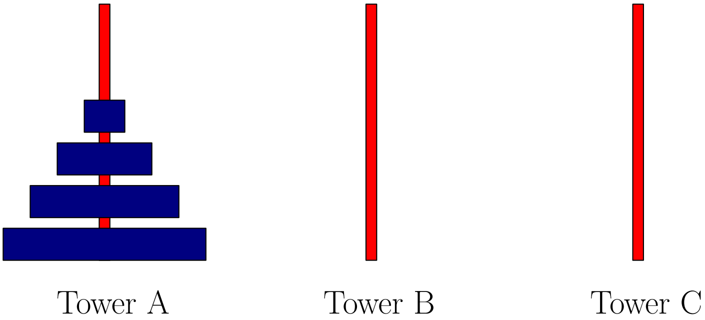


Figure: Initial configuration with 4 disks on Tower 0

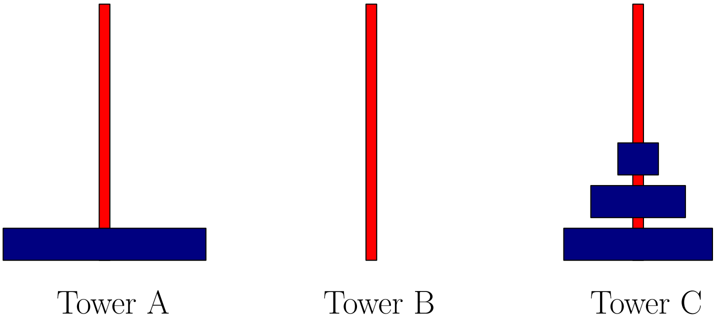


Figure: After recursively moving the top 3 disks from Tower 0 to Tower 2

3. Tower of Hanoi problem (2)

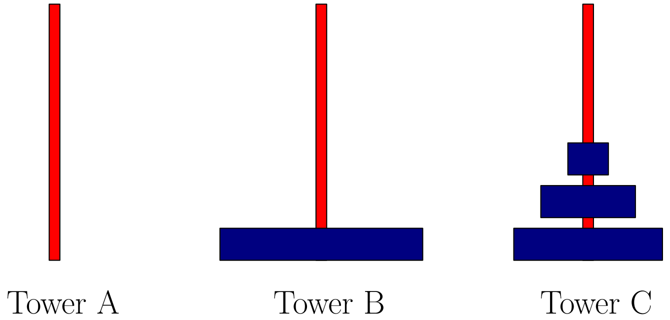


Figure: After moving the bottom disk from Tower 0 to Tower 1

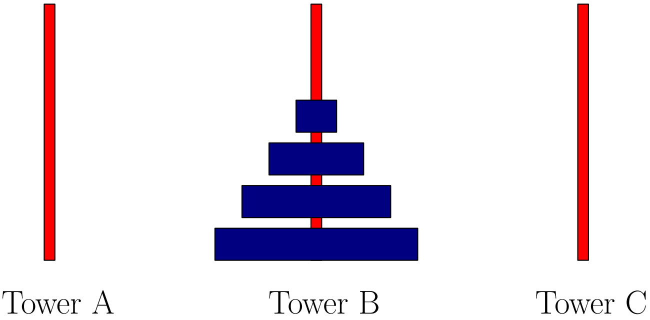


Figure: After recursively moving back 3 disks from Tower 2 to Tower 1.

3. Tower of Hanoi problem (3)

Task: Write the pseudocode of the Tower of Hanoi problem!

Task: Write the pseudocode of the Tower of Hanoi problem!

Algorithm 10 Tower of Hanoi

```
procedure TOWERS(A, B, C, n)
    if n = 1 then
        MOVEONE(A, B)
        return
    end if
    TOWERS(A, C, B, n − 1)
    MOVEONE(A, B)
    TOWERS(C, B, A, n − 1)
end procedure
```

- TOWERS(A, B, C, n): move n disks from A to B, using A, B, C
- MOVEONE(A, B): move one disk from A to B

3. Tower of Hanoi problem (4)

Correctness: by induction

- Base case: For $n = 1$, a single move is made from A to B. So the algorithm works correctly for $n = 1$.
- For any $n \geq 2$, suppose the algorithm works correctly for $n - 1$.
- Then, by the hypothesis, the recursive call of line 6 works correctly and moves the top $n - 1$ disks to C, leaving the bottom disk on tower A.
- The next step, line 7, moves the bottom disk to B.
- Finally, the recursive call of line 8 works correctly by the hypothesis and moves back $n - 1$ disks from C to B.
- Thus, the entire algorithm works correctly for n.

Recurrence equation to analyze time complexity

Let $f(n)$: the number of single moves to solve the problem for n disks

Hence we have the following relation:

$$f(n) = \begin{cases} 1, & \text{if } n = 1 \\ 2f(n-1)+1, & n \geq 2 \end{cases}$$

To obtain the explicit formula of $f(n)$, we have to solve the recurrence equation for $f(n)$.

Method 1: Repeated substitution

$$
\begin{aligned}
f(n) &= 1 + 2 \cdot f(n-1) \\
&= 1 + 2 + 4 \cdot f(n-2) \\
&= 1 + 2 + 4 + 8 \cdot f(n-3) \\
&= \cdots \\
&= 1 + 2 + 2^2 + \cdots + 2^{n-1} \cdot f(1)
\end{aligned}
$$

Substituting the base case $f(1) = 1$ and by the geometric sum formula, we obtain:

$$f(n) = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

Method 2: Guess the solution and prove by induction

Suppose our guess is "$f(n)$ is exponential"

Guess: $f(n) = a \cdot 2^n + b$

Inductive proof:

- Induction base: $n = 1$
  - $f(1) = 1$ (from the reccurence)
  - $f(1) = 2a + b$ (from the solution form)

  So we have $2a + b = 1$
- Induction: Suppose that the solution is correct for some $n \geq 1$.

$$f(n) = a \cdot 2^n + b$$

  Then we have to prove the correctness for $n + 1$

$$f(n+1) = a \cdot 2^{n+1} + b$$

3. Tower of Hanoi problem (8)

- From the recurrence relation, we obtain:

$$\begin{aligned} f(n+1) &= 2f(n)+1 \\ &= 2(a\cdot 2^{+}b)+1 \\ &= a\cdot 2^{n+1}+(2b+1) \end{aligned}$$

- From the two equations, we obtain:

$$a\cdot 2^{n+1}+b = a\cdot 2^{n+1}+(b+1) \Leftrightarrow 2b+1=b \Leftrightarrow b=-1$$

  Hence, $2a+b=1 \Leftrightarrow a=1$.

- So, $b=-1$ and $a=1$ and $f(n)=a\cdot 2^n+b=2^n-1$.

Binary Search

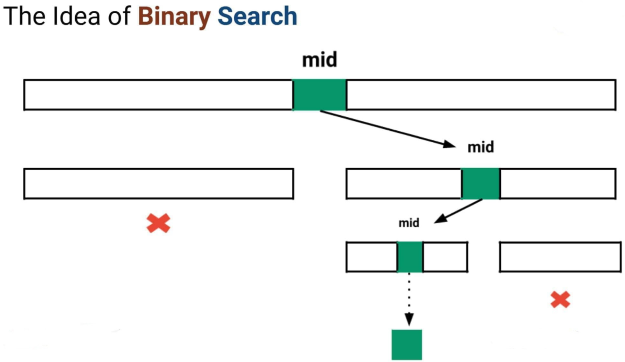
4. Binary search algorithm (1)

Problem: Given a *sorted* array $A[0..n-1]$ and a search key KEY. The algorithm does the following:

- If $KEY = A[m]$, then return m
- If $KEY < A[m]$, then recursively search the left half of the array
- If $KEY > A[m]$, then recursively search the right half of the array

In each step, the size of the search is reduced by half.

4. Binary search algorithm (2)



source: `https://www.enjoyalgorithms.com/blog/binary-search-algorithm`

Algorithm 11 Binary search algorithm

```
procedure BinSearch(A[ ], int Left, int Right, KEY)
    if Left > Right then
        return −1
    end if
    m = ⌊(Left+Right)/2⌋
    if KEY = A[m] then
        return m
    else
        if KEY < A[m] then
            return BinSearch(A, Left, m − 1, KEY)
        else
            return BinSearch(A, m + 1, Right, KEY)
        end if
    end if
end procedure
```

4. Binary search algorithm (4)

Complexity analysis: special case when $n = 2^k$

$$f(n) = \begin{cases} 1, & n = 1 \\ 1 + f(n/2), & n \geq 2 \end{cases}$$

By repeated substitution:

$$\begin{aligned} f(n) &= 1 + f(n/2) \\ &= 1 + 1 + f(n/4) \\ &= 1 + 1 + 1 + f(n/8) \\ &\vdots \\ &= k + f(n/2^k) \\ &= k + f(1) \\ &= k + 1 \\ &= \log n + 1 \end{aligned}$$

Complexity analysis: general case n

$$f(n) = \begin{cases} 1, & n = 1 \\ 1 + f(\lfloor \frac{n}{2} \rfloor), & n \geq 2 \end{cases}$$

By induction, we obtain $f(n) = \lfloor \log n \rfloor + 1$

How?

4. Binary search algorithm (6)

- Induction base: $n = 1$:
  From the recurrence, $f(1) = 1$, and the claimed solution $f(1) = \lfloor \log 1 \rfloor + 1 = 1$. Correct.
- Inductive proof: Suppose that the formula is correct for all smaller values.

  $$f(m) = \lfloor \log m \rfloor, \quad \forall m < n$$

  Every integer n can be expressed as (for some integer k):

  $$2^{k-1} \leq \lfloor n/2 \rfloor < 2^k$$

  So, $\lfloor \log \lfloor n/2 \rfloor \rfloor = k - 1$.

  By the recursive function:

  $$f(\lfloor n/2 \rfloor) = \lfloor \log \lfloor n/2 \rfloor \rfloor + 1 = (k-1) + 1 = k = \lfloor \log n \rfloor$$

  Then:

  $$f(n) = f(\lfloor n/2 \rfloor) + 1 = k + 1 = \lfloor \log n \rfloor + 1$$

A more advanced example: Recursive powering

5. Recursive powering (1)

Problem: Given X and an integer n. We want to compute X^n.

Algorithm 12 Recursive powering (brute force)

```
procedure POWER1(X, n)
    T = X
    for i = 2 to n do
        T = T * X
    end for
end procedure
```

Complexity: $\mathcal{O}(n)$. Why?

5. Recursive powering (2): Approach

Idea: $X^{16} = ((((X^2)^2)^2)^2$

Given $n = 2^k$, we can do repeated squaring.

Algorithm 13 Recursive powering

procedure $\textsc{Power2}(X, n = 2^k)$
    $T = X$
    for $i = 2$ to k **do**
        $T = T * T$
    end for
end procedure

Complexity: $\mathcal{O}(\log n)$. Why?

Generalize the case for n: Computing X^n for $n \in \mathbb{Z}^+$

- Compute $X^2 = X * X$
- Compute $X^3 = X^2 * X$
- Compute $X^6 = X^3 * X^3$
- Compute $X^{12} = X^6 * X^6$
- Compute $X^{13} = X^{12} * X$

Basic idea: Divide n by 2, $n = n/2 + n/2$. So

$$X^n = X^{(n/2+n/2)} = X^{n/2} \cdot X^{n/2}$$

- For $n = 0$, then $X^n = 1$
- For $n > 0$, then:
  - If n is even, then $X^n = X^{n/2} \cdot X^{n/2}$
  - If n is odd, then $X^n = X^{\lfloor n/2 \rfloor} \cdot X^{\lfloor n/2 \rfloor} \cdot X$

5. Recursive powering (5): Pseudocode

Algorithm 14 Recursive powering

procedure $\text{POWER3}(X, n)$
    if $n = 1$ **then**
        return X
    end if
    $T = \text{POWER}(X, \lfloor \frac{n}{2} \rfloor)$
    $T = T * T$
    if $n \mod 2 = 1$ **then**
        $T = T * X$
        return T
    end if
end procedure

Complexity: ?

5. Recursive powering (6): Example of implementation

Example: Computing 3^{16}

$$
\begin{aligned}
3^{16} = 3^8 \cdot 3^8 &= (3^8)^2 \\
&= ((3^4)^2)^2 \\
&= (((3^2)^2)^2)^2 \\
&= ((((3^1)^2)^2)^2)^2 \\
&= ((((3^0) \cdot 3)^2)^2)^2)^2 \\
&= (((1 \cdot 3)^2)^2)^2)^2 \\
&= (((3)^2)^2)^2)^2 \\
&= (((9)^2)^2)^2 \\
&= ((81)^2)^2 \\
&= (6561)^2 \\
&= 43,046,721
\end{aligned}
$$

5. Recursive powering (7): Correctness (*informal proof*)

Algorithm 14 Power by multiplications

```
procedure POWER3(X, n)
    if n = 1 then
        return X
    end if
    T = POWER(X, ⌊n/2⌋)
    T = T * T
    if n mod 2 = 1 then
        T = T * X
        return T
    end if
end procedure
```

Let $n = 2m + r$, where $r \in \{0, 1\}$. The algorithm first makes a recursive call to compute $T = X^{2m}$. Then it squares T to get $T = X^{2m}$. If $r = 0$, this is returned. Otherwise, when $r = 1$, the algorithm multiplies T by X, to result in $T = X^{2m+1}$.

$f(n)$: the worst-case number of multiplication steps to compute X^n.

- The recursive call takes $f(\lfloor \frac{n}{2} \rfloor)$ multiplications.
- Then it is followed by one more multiplication. In the worst case, when n is odd, one additional multiplication is performed.

Hence,

$$f(n) = \begin{cases} 0, & \text{if } n = 1 \\ f(\lfloor \frac{n}{2} \rfloor) + 2, & \text{if } n \geq 2, n \text{ odd} \\ f(\lfloor \frac{n}{2} \rfloor) + 1, & \text{if } n \geq 2, n \text{ even} \end{cases}$$

Show that $f(n) = 2\lfloor \log n \rfloor$.

$$f(n) = \begin{cases} 0, & \text{if } n = 1 \\ f(\lfloor \frac{n}{2} \rfloor) + 2, & \text{if } n \geq 2, n \text{ odd} \\ f(\lfloor \frac{n}{2} \rfloor) + 1, & \text{if } n \geq 2, n \text{ even} \end{cases}$$

The last two cases have small difference. So we can approximate the function above with the following function to simplify the computation:

$$f(n) = \begin{cases} 0, & \text{if } n = 1 \\ f(\lfloor \frac{n}{2} \rfloor) + 2, & \text{if } n \geq 2 \end{cases}$$

5. Recursive powering (9): Explicit complexity function

Show that $f(n) = 2\lfloor \log n \rfloor$.

- Induction base ($n = 1$): From the recurrence, $f(1) = 0$, and from the formula, $f(1) = 2\lfloor \log 1 \rfloor = 0$. Correct.
- Inductive proof: Suppose that the formula is correct for all smaller values.

  $$f(m) = 2\lfloor \log m \rfloor, \quad \forall m < n$$

  Every integer n can be expressed as (for some integer k):

  $$2^k \leq n < 2^{k+1}$$

  So, $\lfloor \log n \rfloor = k$, and $\lfloor \frac{\log n}{2} \rfloor = k - 1$. By the recursive function:

  $$f(n) = f(\lfloor \frac{n}{2} \rfloor) + 2 = 2(k-1) + 2 = 2k = 2\lfloor \log n \rfloor$$

Remark. This gives a better complexity than the brute force approach ($\mathcal{O}(n)$).

Redundancy in recursive algorithm

Redundancy (1): Recursive powering

Algorithm 14 Power by multiplications

```
procedure POWER3(X, n)
    if n = 1 then
        return X
    end if
    T = POWER(X, ⌊n/2⌋)
    T = T * T
    if n mod 2 = 1 then
        T = T * X
        return T
    end if
end procedure
```

Should we store $\textsc{Power}(X, \lfloor \frac{n}{2} \rfloor)$ in T?

Redundancy (2): Recursive powering

Algorithm 15 Recursive powering

procedure $\text{POWER4}(X, n)$
    if $n = 1$ **then**
        return X
    end if
    return $\text{POWER}(X, \lfloor \frac{n}{2} \rfloor) * \text{POWER}(X, \lfloor \frac{n}{2} \rfloor)$
end procedure

- Is the algorithm correct? *What is the complexity?*

Redundancy (3): Recursive powering

The algorithm is correct.

The number of recursive calls:

$$f(n) = \begin{cases} 0, & \text{if } n = 1 \\ f(\lfloor \frac{n}{2} \rfloor) + f(\lfloor \frac{n}{2} \rfloor) + 1, & \text{if } n \geq 2 \end{cases}$$

By induction, we can prove that $f(n) = n - 1$.

What can you conclude?

Redundancy (3): Recursive powering

The algorithm is correct.

The number of recursive calls:

$$f(n) = \begin{cases} 0, & \text{if } n = 1 \\ f(\lfloor \frac{n}{2} \rfloor) + f(\lfloor \frac{n}{2} \rfloor) + 1, & \text{if } n \geq 2 \end{cases}$$

By induction, we can prove that $f(n) = n - 1$.

What can you conclude?
POWER4 is not efficient, because we make two recursive calls for the same function $f(\lfloor \frac{n}{2} \rfloor)$

Redundancy (4): Fibonacci sequence

The Fibonacci sequence is defined as follows. Build an algorithm to compute the Fibonacci sequence!

$$f(n) = \begin{cases} 1, & n = 1 \\ 1, & n = 2 \\ F_{n-1} + F_{n-2}, & n \geq 3 \end{cases}$$

Redundancy (4): Fibonacci sequence

The Fibonacci sequence is defined as follows. Build an algorithm to compute the Fibonacci sequence!

$$f(n) = \begin{cases} 1, & n = 1 \\ 1, & n = 2 \\ F_{n-1} + F_{n-2}, & n \geq 3 \end{cases}$$

Complexity: $\mathcal{O}(n)$ with naive algorithm. How?

Redundancy (4): Fibonacci sequence

The Fibonacci sequence is defined as follows. Build an algorithm to compute the Fibonacci sequence!

$$f(n) = \begin{cases} 1, & n = 1 \\ 1, & n = 2 \\ F_{n-1} + F_{n-2}, & n \geq 3 \end{cases}$$

Complexity: $\mathcal{O}(n)$ with naive algorithm. How?
By looping

Redundancy (5): Fibonacci sequence

Algorithm 16 Fibonacci sequence

procedure $\textsc{Fib}(n)$
  if $n \le 2$ **then** **return** 1
  end if
  return $(\textsc{Fib}(n-1) + \textsc{Fib}(n-2))$
end procedure

Redundancy (5): Fibonacci sequence

Algorithm 17 Fibonacci sequence

```
procedure FIB(n)
    if n ≤ 2 then return 1
    end if
    return (FIB(n − 1) + FIB(n − 2))
end procedure
```

This program makes recursive calls with a great deal of overlapping computations, causing a huge inefficiency.

Algorithm 18 Fibonacci sequence

```
procedure FIB(n)
    if n ≤ 2 then return 1
    end if
    return (FIB(n − 1) + FIB(n − 2))
end procedure
```

This program makes recursive calls with a great deal of overlapping computations, causing a huge inefficiency.

Complexity:

$$T(n) = \begin{cases} 0, & n = 1 \\ 0, & n = 2 \\ T(n-1) + T(n-2) + 1, & n \geq 3 \end{cases}$$

Prove that: the explicit function: $T(n) \geq (1.618)^{n-2}$.

Advantages and drawbacks of recursive algorithm (1)

Advantages

- Recursion adds clarity and reduces the time needed to write and debug code (since it reduce the length of code).
- To solve such problems which are naturally recursive such as tower of Hanoi.
- Recursion can reduce time complexity (*sometimes counter-intuitive*).
- Reduce unnecessary calling of function.
- Extremely useful when applying the same solution.

Advantages and drawbacks of recursive algorithm (2)

Drawbacks

- Recursive functions are generally slower than non-recursive function.
- It may require a lot of memory space to hold intermediate results on the system stacks.
- Hard to analyze or understand the code.
- It is not more efficient in terms of space and time complexity (can be slow).
- The computer may run out of memory if the recursive calls are not properly checked.

Sum up...

What have we learned today?

1. Reviewing brute force approach
2. Understanding the principal of recursive approach
3. Some examples of recursive algorithms
4. Recurrence equation to analyze time complexity
5. Redundancy in recursion: be careful when writing the pseudocode
6. Binary search algorithm