

Linear Algebra

[KOMS119602] - 2022/2023

5.1 - Determinants of Matrices

Dewi Sintiar

Computer Science Study Program
Universitas Pendidikan Ganesha

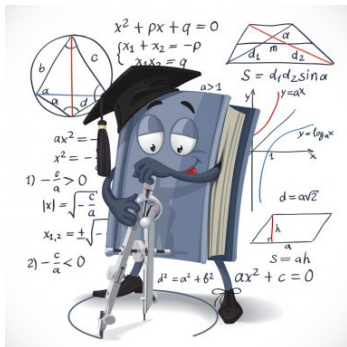
Week 7-11 February 2022

Learning objectives

After this lecture, you should be able to:

1. explain the concept of determinant of a matrix;
2. compute the determinant of (2×2) matrices;
3. compute the determinant of (3×3) matrices;
4. explain the geometric interpretation of determinant of (2×2) matrices;
5. explain the geometric interpretation of determinant of (3×3) matrices;
6. explain the use of determinant in the system of linear equations;
7. use permutation to compute determinants;

Good math skills are developed by doing lots of problems.



Part 1: Formal definition of determinant

Formal definition of determinant matrix

Given a square matrix $A = [a_{ij}]$ of size $n \times n$.

We can assign a *scalar* to matrix A , as a function of the entries of the square matrix. This is called the **determinant** of matrix A .

The determinant of matrix A is denoted by $|A|$, and often written as:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Determinants of orders 1 and 2

For $n = 1, 2$, the determinants are defined as:

$$|a_{11} = a_{11}| \quad \text{and} \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Example

Find the determinant of the following matrices:

$$\begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & -5 \\ -6 & 3 \end{bmatrix}$$

Part 2: Determinants of 2×2 matrices

Determinants of 2×2 matrices

Given a matrix:

$$A = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}$$

In high school, you might have learned that the **determinant** of the matrix (size 2×2) is defined as

$$A_1 B_2 - A_2 B_1$$

and is denoted by:

$$|A| = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$$

Motivating example: *an important application of determinant*

Recall that, given a system of linear equations in two variables:

$$A_1x + B_1y = C_1$$

$$A_2x + B_2y = C_2$$

- The system has exactly one solution when $A_1B_2 - A_2B_1 \neq 0$
- The system has no solution or infinitely many solutions when $A_1B_2 - A_2B_1 = 0$

Motivating example: *an important application of determinant*

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- The system has exactly one solution when $A_1B_2 - A_2B_1 \neq 0$
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The coefficient matrix $\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}$ has determinant $= A_1B_2 - A_2B_1$.

Remark. Determinant of the coefficient matrix determines the number of solutions of the given system. The system has a unique solution iff $D \neq 0$.

Application to linear equations

Solving the system by variable elimination:

$$A_1 B_2 x + B_1 B_2 y = B_2 C_1$$

$$A_2 B_1 x + B_1 B_2 y = B_1 C_2$$

$$(A_1 B_2 - A_2 B_1)x = B_2 C_1 - B_1 C_2$$

$$x = \frac{B_2 C_1 - B_1 C_2}{A_1 B_2 - A_2 B_1}$$

We have:

$$B_2 C_1 - B_1 C_2 = \begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix} = N_x \text{ and } A_1 B_2 - A_2 B_1 = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = D$$

Hence, $x = \frac{N_x}{D}$

Application to linear equations

Similarly, we can find the value of y :

$$A_1 A_2 x + A_2 B_1 y = A_2 C_1$$

$$A_1 A_2 x + A_1 B_2 y = A_1 C_2$$

$$(A_2 B_1 - A_1 B_2)y = A_2 C_1 - A_1 C_2$$

$$y = \frac{A_2 C_1 - A_1 C_2}{A_2 B_1 - A_1 B_2} = \frac{A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1}$$

We have:

$$A_1 C_2 - A_2 C_1 = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = N_y \text{ and } A_1 B_2 - A_2 B_1 = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = D$$

$$\text{Hence, } y = \frac{N_y}{D}$$

Example

Solve the following system using determinants:

$$\begin{cases} 3x - 4y = -10 \\ -x + 2y = 2 \end{cases}$$

Solution:

$$N_x = \begin{vmatrix} -10 & -4 \\ 2 & 2 \end{vmatrix} = -20 - (-8) = -12$$

$$N_y = \begin{vmatrix} 3 & -10 \\ -1 & 2 \end{vmatrix} = 6 - 10 = -4$$

$$D = \begin{vmatrix} 3 & -4 \\ -1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Hence, $x = \frac{-12}{2} = -6$ and $y = \frac{-4}{2} = -2$.

Conclusion

Given:

$$A_1x + B_1y = C_1$$

$$A_2x + B_2y = C_2$$

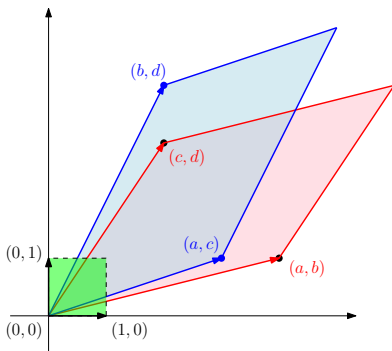
with the coefficient matrix $\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}$ having non-zero determinant (meaning that, the system has a unique solution).

The solution is given by:

$$x = \frac{N_x}{D} \quad \text{and} \quad y = \frac{N_y}{D}$$

$$\text{where } N_x = \begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix}, \quad N_y = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}, \quad \text{and } D = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}.$$

Geometric interpretation



Matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be viewed as an “arrangement” of:

- row vectors:
 $\begin{bmatrix} a & b \end{bmatrix}$ and $\begin{bmatrix} c & d \end{bmatrix}$
- or, column vectors:
 $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$

The matrix defines the so-called *linear transformation* of the unit square (in green) formed by the *basis vectors* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, with respect to:

- the **row vectors**, shown by the **red** parallelogram; or
- the **column vectors**, shown by the **blue** parallelogram

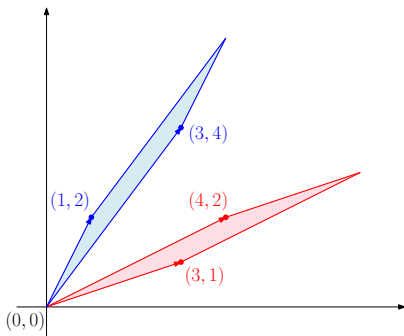
Both parallelograms have the same area.

Example

Given a matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

Draw two parallelograms that define the transformation of the unit square w.r.t. the row vectors and the column vectors, respectively.

Solution:



Part 3: Determinants of 3×3 matrices

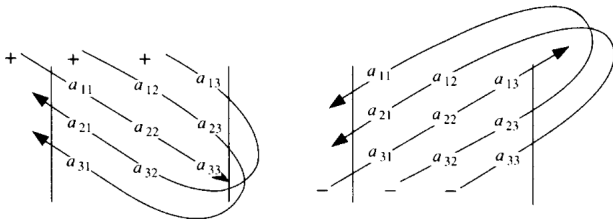
Determinants of matrices of order 3 (i.e., size 3×3)

Given a matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The **determinant** of the matrix above is defined as:

$$\begin{aligned} \det(A) = & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \end{aligned}$$



Alternative form for the determinant of an order-3 matrix

The **determinant** of the matrix above is defined as:

$$\begin{aligned}\det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\end{aligned}$$

This formula can be illustrated as follows:

$$a_{11} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example

Find the determinant of matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{bmatrix}$

Solution:

- *Using the diagram*

$$\begin{aligned}\det(A) &= 3(5)(4) + 2(-1)(2) + (1)(-4)(-3) - 1(5)(2) - 2(-4)4 - 3(-1)(-3) \\ &= 60 - 4 + 12 - 10 + 32 - 9 = 81\end{aligned}$$

- *Using the alternative form*

$$\begin{aligned}\begin{vmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{vmatrix} &= 1 \begin{vmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{vmatrix} \\ &= 1 \begin{vmatrix} 5 & -1 \\ -3 & 4 \end{vmatrix} - 2 \begin{vmatrix} -4 & -1 \\ 2 & 4 \end{vmatrix} + 3 \begin{vmatrix} -4 & 5 \\ 2 & -3 \end{vmatrix} \\ &= 1(20 - 3) - 2(-16 + 2) + 3(12 - 10) = 17 + 28 - 6 = 39\end{aligned}$$

Applications to linear equations system

Given the following linear system:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$$

We can perform similar computations as in the case (2×2) matrix, in order to find a solution of the system.

The coefficient matrix of the system is given by: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The system has a unique solution only if $D = \det(A) \neq 0$.

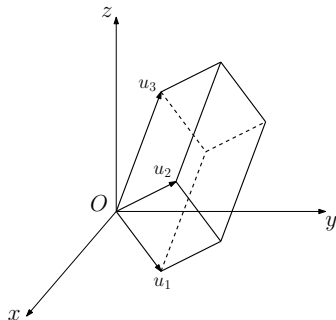
The solution is given by:

$$x = \frac{N_x}{D}, \quad y = \frac{N_y}{D}, \quad z = \frac{N_z}{D}$$

where N_x , N_y , and N_z is obtained by replacing the 1st, 2nd, and 3rd

column of A by the constant vector $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Geometric interpretation



In \mathbb{R}^3 , the vectors u_1 , u_2 , and u_3 determine the parallelepiped,

which is the result of transforming the unit cube using the vectors $\{u_1, u_2, u_3\}$.

Remark.

Let u_1, u_2, \dots, u_n be vectors in \mathbb{R}^n . Then:

$$S = \{a_1 u_1 + a_2 u_2 + \dots + a_n u_n : 0 \leq a_i \leq 1 \text{ for } i = 1, \dots, n\}$$

$$V(S) = \text{absolute value of } \det(A)$$

Part 4: Determinants of arbitrary order (*a combinatorial way*)

Pattern in the determinant formulas

Can you find a pattern of the following determinant formulas?

- For 2×2 matrix: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

- For 3×3 matrix: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\begin{aligned} \det(A) = & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \end{aligned}$$

We will study these patterns!

Sign (parity) of a permutation

Given a sequence of elements: $\sigma = j_1 j_2 \dots j_n$, a **permutation** of σ is defined as an arrangement of the objects in σ in a definite order.

The set of all permutations of n objects is denoted by S_n .

An **inversion** in σ is a pair of integers (i, k) , such that $i > k$ but i precedes k in σ .

σ is called:

- **even permutation**, if there are an even number of inversions in σ ;
- **odd permutation**, otherwise.

The **sign** or **parity** of the permutation σ is defined by:

$$\text{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Example: *sign of a permutation*

Given a permutation $\sigma = 35412$ in S_5 . What is the sign of σ ?

Solution:

- 3 numbers (3, 4 and 5) precede 1;
- 3 numbers (3, 4 and 5) precede 2;
- 1 number (5) precedes 4;
- no number that precedes 3 or 4

Since $3 + 3 + 1 = 7$ is odd, then σ is an odd permutation. Hence

$$\text{sgn}(\sigma) = -1$$

Exercises:

1. Find the sign of the permutation: $\epsilon = 123 \dots n$ in S_n .
2. Find the sign of each permutation in S_2 and S_3 .
3. Is it true that in S_n , half of the permutations are even, and half of them are odd?

Using permutation in computing determinants (1)

Given an $n \times n$ matrix $A = [a_{ij}]$ over a field K .

Consider a product of n elements of A (here, $j_1 j_2 \dots j_n$ is a permutation of $1 2 3 \dots n$):

$$a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

such that:

- *one and only one* element comes from each row of A ; and
- *one and only one* element comes from each column of A .

Q: How many different products of form $a_{1j_1} a_{2j_2} \dots a_{nj_n}$ are there?

A: There are $n!$ such products, because there are $n!$ permutations of $j_1 j_2 \dots j_n$.

Using permutation in computing determinants (2)

The determinant of the $n \times n$ matrix $A = [a_{ij}]$ is defined as:

the sum of all the $n!$ products $a_{1j_1} a_{2j_2} \dots a_{nj_n}$, where each product is multiplied by the sign of $\sigma = j_1 j_2 \dots j_n$.

$$|A| = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

or, this can be written as:

$$|A| = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

Using permutation in computing determinants (3)

1. Given $A = [a_{11}]$, then $\det(A) = a_{11}$.
2. Given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.
3. Given $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then:

$$\begin{aligned}\det(A) = & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}\end{aligned}$$

to be continued...