Linear Algebra

[KOMS119602] - 2022/2023

12.2 - Linear Transformation

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Learning objectives

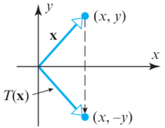
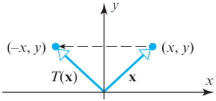
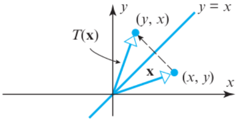
Basic Matrix Transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$

(page 259 of Elementary LA Applications book)

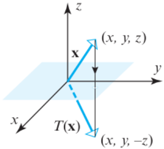
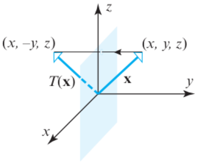
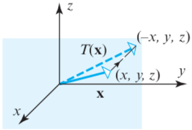
1. Reflection

Reflection operators on $\mathbb{R}^2$

Reflection operators are operators on $\mathbb{R}^2$ (or $\mathbb{R}^3$) that maps each point into its symmetric image about a fixed line or a fixed plane that contains the origin.

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$ $T(\mathbf{e}_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

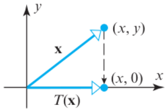
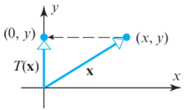
Reflection operators on $\mathbb{R}^3$

Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Reflection about the xy-plane $T(x, y, z) = (x, y, -z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x, y, z) = (x, -y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, -1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x, y, z) = (-x, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (-1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

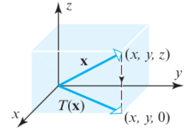
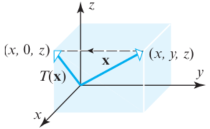
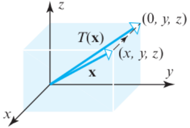
2. Projection

Projection operators on $\mathbb{R}^2$

Projection operators or orthogonal projection operators are matrix operators on $\mathbb{R}^2$ (or $\mathbb{R}^3$) that map each point into its orthogonal projection onto a fixed line or plane through the origin.

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Projection operators on $\mathbb{R}^3$

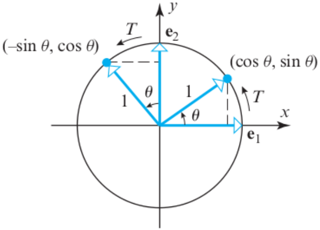
Operator	**Illustration**	**Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$**	**Standard Matrix**
Orthogonal projection onto the xy-plane $T(x, y, z) = (x, y, 0)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection onto the xz-plane $T(x, y, z) = (x, 0, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection onto the yz-plane $T(x, y, z) = (0, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Rotation

Rotation operators for $\mathbb{R}^2$

Rotation operators are matrix operators on $\mathbb{R}^2$ or $\mathbb{R}^3$ that move points along arcs of circles centered at the origin.

How to find the standard matrix for the rotation operator $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that moves points counterclockwise about the origin through a positive angle θ ?



$$T(\mathbf{e}_1) = T(1,0) = (\cos\theta, \sin\theta) \quad \text{and} \quad T(\mathbf{e}_2) = T(0,1) = (-\sin\theta, \cos\theta)$$

The standard transformation matrix for T is:

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Review on "angle"

Conversion from ^{o} to rad

Rotation operators for $\mathbb{R}^2$ (*cont.*)

The matrix:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

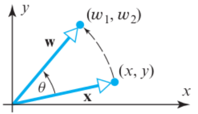
is called the rotation matrix for $\mathbb{R}^2$.

Let $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and $\mathbf{w} = (w_1, w_2)$ be its image under the rotation. Then:

$$\mathbf{w} = R_\theta \mathbf{x}$$

with:

$$\begin{aligned} w_1 &= x\cos\theta - y\sin\theta \\ w_2 &= x\sin\theta + y\cos\theta \end{aligned}$$

Operator	Illustration	Rotation Equations	Standard Matrix
Counterclockwise rotation about the origin through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Example: a rotation operator

Find the image of $\mathbf{x} = (1, 1)$ under a rotation of $\pi/6$ rad $(= 30^o)$ about the origin.

Solution:

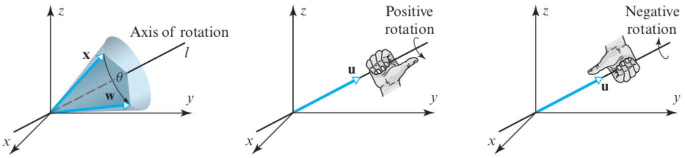
We know that: $\sin(\pi/6) = \frac{1}{2}$ and $\cos(\pi/6) = \frac{\sqrt{3}}{2}$.

By the previous formula:

$$R_{\pi/6}\mathbf{x} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix} \approx \begin{bmatrix} 0.37 \\ 1.37 \end{bmatrix}$$

Rotations in $\mathbb{R}^3$

Rotations in $\mathbb{R}^3$ is commonly described as axis of rotation and a unit vector **u** along that line.

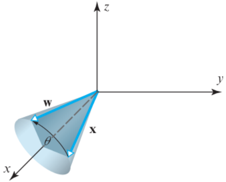
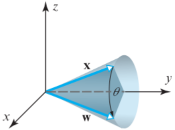
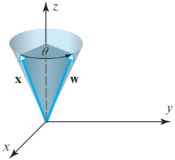


(a) Angle of rotation (b) Right-hand rule

Right-hand rule is used to establish a sign for the angle for rotation.

- If the axes are the axis x, y, or z, then take the unit vectors **i**, **j**, and **k** respectively.
- An angle of rotation will be *positive if it is counterclockwise* looking toward the origin along the positive coordinate axis and will be *negative if it is clockwise*.

Rotations in $\mathbb{R}^3$

Operator	**Illustration**	**Rotation Equations**	**Standard Matrix**
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. Dilation and contraction

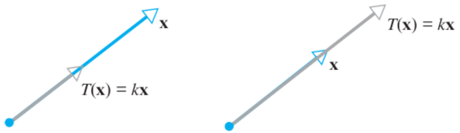
Dilation & contraction

Let $k \in \mathbb{R}, k \geq 0$. The operator:

$$T(\mathbf{x}) = k\mathbf{x}$$

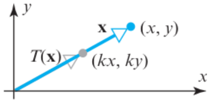
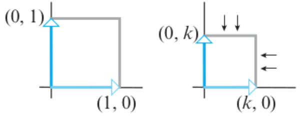
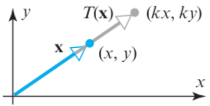
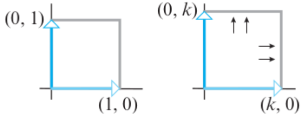
on $\mathbb{R}^2$ or $\mathbb{R}^3$ defines the increment or decrement of the length of vector $\mathbf{x}$ by a factor of k.

- If $k > 1$, it is called a dilation with factor k;
- If $0 \leq k \leq 1$, it is called a contraction with factor k.

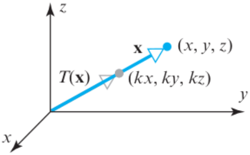
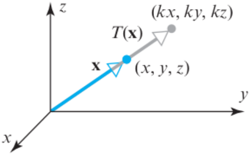


(a) $0 \leq k < 1$ (b) $k > 1$

Dilation & contraction on $\mathbb{R}^2$

Operator	**Illustration** $T(x, y) = (kx, ky)$	**Effect on the Unit Square**	**Standard Matrix**
Contraction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k in R^2 $(k > 1)$			

Dilation & contraction on $\mathbb{R}^3$

Operator	Illustration $T(x, y, z) = (kx, ky, kz)$	Standard Matrix
Contraction with factor k in R^3 $(0 \leq k < 1)$		$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k in R^3 $(k > 1)$		

5. Expansion and compression

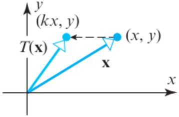
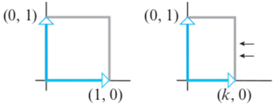
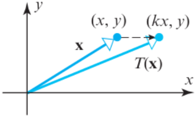
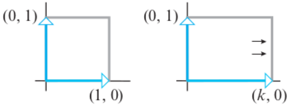
Expansion and compression

In a dilation or contraction of $\mathbb{R}^2$ or $\mathbb{R}^3$, **all coordinates** are multiplied by a non-negative factor k.

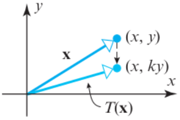
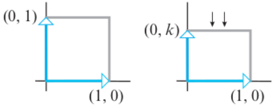
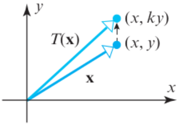
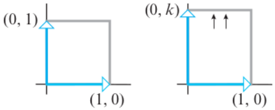
Now what if **only one coordinate** is multiplied by k ?

- If $k > 1$, it is called the expansion with factor k in the direction of a coordinate axis (x, y, or z);
- If $0 \leq k \leq 1$, it is called compression

Expansion and compression in $\mathbb{R}^2$ (in x-direction)

<table>
<tr><th>Operator</th><th>Illustration
$T(x, y) = (kx, y)$</th><th>Effect on the
Unit Square</th><th>Standard
Matrix</th></tr>
<tr><td>Compression in the
x-direction
with factor k in R^2
$(0 \leq k < 1)$</td><td>
</td><td>
</td><td rowspan="2">$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$</td></tr>
<tr><td>Expansion in the
x-direction
with factor k in R^2
$(k > 1)$</td><td>
</td><td>
</td></tr>
</table>

Expansion and compression in $\mathbb{R}^2$ (in y-direction)

Operator	**Illustration** $T(x, y) = (x, ky)$	**Effect on the Unit Square**	**Standard Matrix**
Compression in the y-direction with factor k in R^2 $(0 \leq k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion in the y-direction with factor k in R^2 $(k > 1)$			

6. Shear

Shear

A matrix operator of the form:

$$T(x, y) = (x + ky, y)$$

translates a point (x, y) in the xy-plane parallel to the x-axis by an amount ky that is proportional to the y-coordinate of the point.

This is called shear in the x-direction by a factor k.

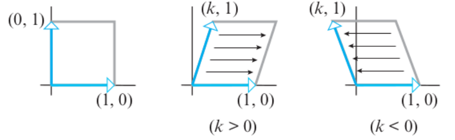
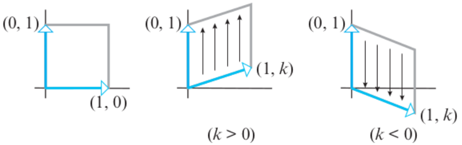
Similarly, a matrix operator:

$$T(x, y) = (x, y + kx)$$

is called shear in the y-direction by a factor k.

When $k > 0$, then the shear is in the positive direction. When $k < 0$, it is in the negative direction.

Shear

<table>
<tr><th>Operator</th><th>Effect on the Unit Square</th><th>Standard Matrix</th></tr>
<tr><td>Shear in the x-direction by a factor k in R^2
$T(x, y) = (x + ky, y)$</td><td>
</td><td>$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$</td></tr>
<tr><td>Shear in the y-direction by a factor k in R^2
$T(x, y) = (x, y + kx)$</td><td>
</td><td>$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$</td></tr>
</table>

Example

Describe the matrix operator whose standard matrix is as follows:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

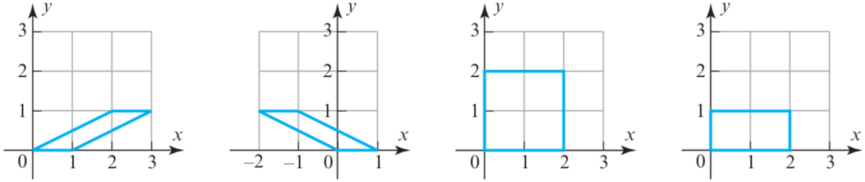
Solution:

From the tables on the previous slides, we can see that:

- A_1 corresponds to a shear in the x-direction by a factor 2;
- A_2 corresponds to a shear in the x-direction by a factor -2;
- A_3 corresponds to a dilation with factor 2;
- A_4 corresponds to an expansion in the x-direction with factor 2.

Example (*cont.*)

Describe geometrically the result of the transformation:



A_1 $\quad$ A_2 $\quad$ A_3 $\quad$ A_4

Exercise

Properties of Matrix Transformations

(page 270 of Elementary LA Applications book)

Compositions of matrix transformation

Let:

- T_A: a matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^k$
- T_B: a matrix transformation from $\mathbb{R}^k$ to $\mathbb{R}^m$

Let $\mathbf{x} \in \mathbb{R}^n$, and defines transformation:

$$\mathbf{x} \xrightarrow{T_A} T_A(\mathbf{x}) \xrightarrow{T_B} T_B(T_A(\mathbf{x}))$$

defines the transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

It is called the composition of T_B with T_A and is denoted by $T_B \circ T_A$. So:

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))$$

Compositions of matrix transformation

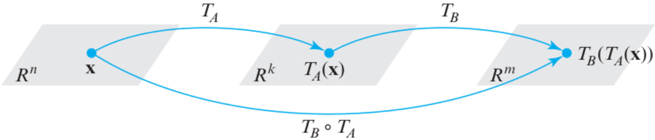
The composition is a matrix transformation, since:

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})) = B(T_A(\mathbf{x})) = B(A\mathbf{x}) = (BA)\mathbf{x}$$

meaning that the result of the composition to $\mathbf{x}$ is obtained by multiplying $\mathbf{x}$ with BA on the left.

This is denoted by:

$$T_B \circ T_A = T_{BA}$$



Composition of three transformations

Compositions can be defined for any finite succession of matrix transformations whose domains and ranges have the appropriate dimensions. For instance, given:

$$T_A : \mathbb{R}^n \to \mathbb{R}^k, \;\; T_B : \mathbb{R}^k \to \mathbb{R}^\ell, T_C : \mathbb{R}^\ell \to \mathbb{R}^m$$

we can define the composition:

$$(T_C \circ T_B \circ T_A) : \mathbb{R}^n \to \mathbb{R}^m$$

by:

$$(T_C \circ T_B \circ T_A)(\mathbf{x}) = T_C(T_B(T_A(\mathbf{x})))$$

It can be shown that this is a matrix transformation with standard matrix CBA, and:

$$T_C \circ T_B \circ T_A = T_{CBA}$$

Notation

We can write the standard matrix for transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ without specifying the name of the standard matrix.

It is often written as $[T]$.

For instance,

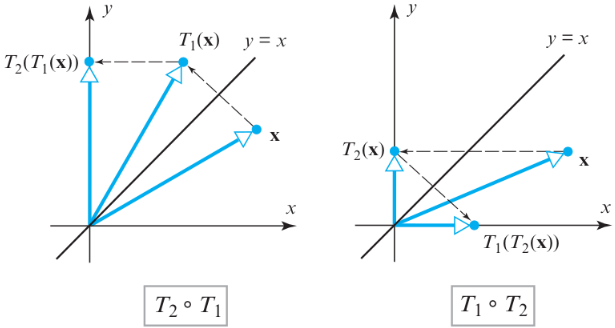
- $T(\mathbf{x}) = [T]\mathbf{x}$
- $[T_2 \circ T_1] = [T_2][T_1]$
- $[T_3 \circ T_2 \circ T_1] = [T_3][T_2][T_1]$

Composition is not commutative

Example

Let:

- $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line $y = x$;
- $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the y-axis.



Geometrically, both transformations have different effect on $\mathbf{x}$

Composition is not commutative (*cont.*)

Algebraically, we can compute:

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Clearly, $[T_1 \circ T_2] \neq [T_2 \circ T_1]$.

Composition of rotation is commutative

Example

Given :

$$T_1 : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{and} \quad T_2 : \mathbb{R}^2 \to \mathbb{R}^2$$

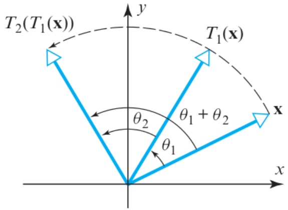
the matrix operators that rotate vectors about the origin through the angles θ_1 and θ_2 respectively.

So, the operation:

$$T_2 \circ T_1(\mathbf{x}) = T_2(T_1(\mathbf{x}))$$

first rotates x through the angle θ_1, then rotates $T_1(\mathbf{x})$ through the angle θ_2.

Hence, $(T_2 \circ T_1)(\mathbf{x})$ defines rotation of $\mathbf{x}$ through the angle $\theta_1 + \theta_2$.



Composition of rotation is commutative (*cont.*)

In this case, we have:

$$[T_1] = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \quad \text{and} \quad [T_2] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

We show that: $[T_2 \circ T_1] = [T_1][T_1]$

$$[T_2 \circ T_1] = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

Furthermore:

$$\begin{aligned} [T_2][T_1] &= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -(\cos\theta_2\sin\theta_1 + \sin\theta_2\cos\theta_1) \\ \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1 & -\sin\theta_2 sin\theta_1 + \cos\theta_2\cos\theta_1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \\ &= [T_2 \circ T_1] \end{aligned}$$

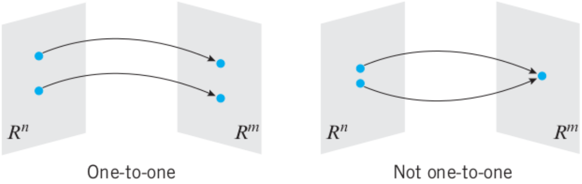
It can be easily seen that $[T_2 \circ T_1] = [T_1 \circ T_2]$ (hence, commutative).

Exercise

Read Example 3 and Example 4 (page 272-273)

One-to-one matrix transformation

A matrix transformation $T_A \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if T_A maps distinct vectors (points) in $\mathbb{R}6n$ into distinct vectors (points) in $\mathbb{R}^m$.



Equivalent statements:

- T_A is one-to-one if $\forall \mathbf{b}$ in the range of A, there is exactly one vector $\mathbf{x} \in \mathbb{R}^n$, s.t. $T_A\mathbf{x} = \mathbf{b}$.
- T_A is one-to-one if the equality $T_A(\mathbf{u}) = T_A(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$.

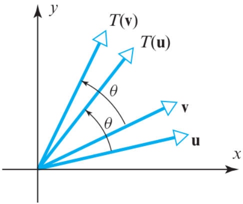
Examples: one-to-one and not one-to-one transformations

Rotation operators on $\mathbb{R}^2$ are one-to-one.

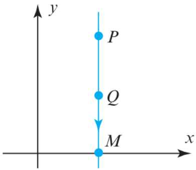
since distinct vectors that are rotated through the same angle have distinct images.

The orthogonal projection of $\mathbb{R}^2$ onto the x-axis is not one-to-one.

since it maps distinct points on the same vertical line into the same point.



▲ Figure 4.10.6 Distinct vectors $\mathbf{u}$ and $\mathbf{v}$ are rotated into distinct vectors $T(\mathbf{u})$ and $T(\mathbf{v})$.



▲ Figure 4.10.7 The distinct points P and Q are mapped into the same point M.

Kernel and range

If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation, then the set of all vectors in RR^n that T_A maps into 0 is called the kernel of T_A and is denoted by $\ker(T_A)$, i.e.:

$$\ker(T_A) = \{\mathbf{x} \in \mathbb{R}^n \text{ s.t. } A\mathbf{x} = \mathbf{0}\}$$

The set of all vectors in $\mathbb{R}^m$ that are images under this transformation of at least one vector in $\mathbb{R}^n$ is called the range of T_A and is denoted by $R(T_A)$, i.e.:

$$R(T_A) = \{\mathbf{b} \in \mathbb{R}^m \text{ s.t. } \exists \mathbf{x} \in \mathbb{R}^n, \text{ where } A\mathbf{x} = \mathbf{b}\}$$

In brief:

$$\ker(T_A) = \text{null space of } A$$
$$R(T_A) = \text{column space of } A$$

Matrix - linear system - transformation

Let A be an $(m \times n)$ matrix.

Three ways of viewing the same subspace of $\mathbb{R}^n$:

- **Matrix view:** the null space of A
- **System view:** the solution space of $A\mathbf{x} = 0$
- **Transformation view:** the kernel of T_A

Three ways of viewing the same subspace of $\mathbb{R}^m$:

- **Matrix view:** the column space of A
- **System view:** all $\mathbf{b} \in \mathbb{R}^m$ for which $A\mathbf{x} = \mathbf{b}$ is consistent
- **Transformation view:** the range of T_A

Exercise

Read Example 5 and Example 6 on page 275.

One-to-one matrix operator

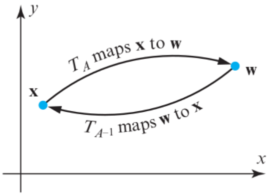
Let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be a one-to-one matrix operator. So, A is invertible.

The inverse operator or the inverse of T_A is defined as:

$$T_{A^{-1}} : \mathbb{R}^n \to \mathbb{R}^n$$

In this case:

$$T_A(T_{A^{-1}}(\mathbf{x})) = AA^{-1}\mathbf{x} = I\mathbf{x} = \mathbf{x} \text{ or, equivalently } T_A \circ T_{A^{-1}} = T_{AA^{-1}} = T_I$$
$$T_{A^{-1}}(T_A(\mathbf{x})) = A^{-1}A\mathbf{x} = I\mathbf{x} = \mathbf{x} \text{ or, equivalently } T_{A^{-1}} \circ T_A = T_{A^{-1}A} = T_I$$



T_A maps $\mathbf{x}$ to $\mathbf{w}$ and $T_{A^{-1}}$ maps $\mathbf{w}$ back to $\mathbf{x}$, i.e., $T_{A^{-1}}(\mathbf{w}) = T_{A^{-1}}(T_A(\mathbf{x})) = \mathbf{x}$

Exercise

Read Example 7 and Example 8 on page 276.

Conclusion

THEOREM 4.10.2 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

(g) $\det(A) \neq 0$.

(h) The column vectors of A are linearly independent.

(i) The row vectors of A are linearly independent.

(j) The column vectors of A span R^n.

(k) The row vectors of A span R^n.

(l) The column vectors of A form a basis for R^n.

(m) The row vectors of A form a basis for R^n.

(n) A has rank n.

(o) A has nullity 0.

(p) The orthogonal complement of the null space of A is R^n.

(q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.

(r) The kernel of T_A is $\{\mathbf{0}\}$.

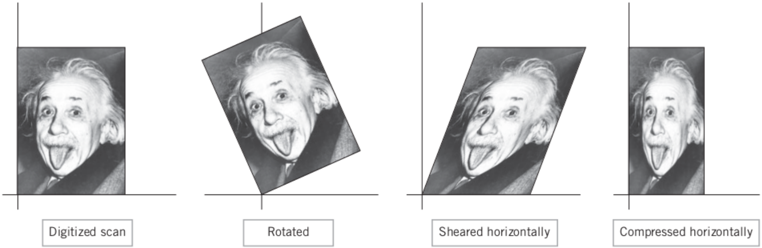
(s) The range of T_A is R^n.

(t) T_A is one-to-one.

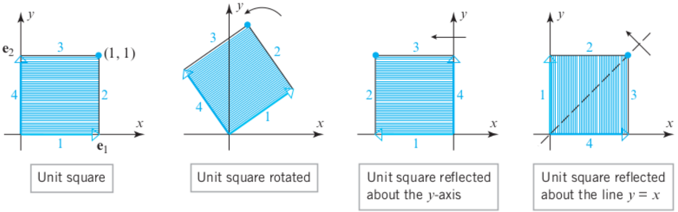
Geometry of Matrix Operators on $\mathbb{R}^2$

(page 280 of Elementary LA Applications book)

to be continued...



Digitized scan
Rotated
Sheared horizontally
Compressed horizontally



y
x
e2
e1
(1, 1)
1
2
3
4
Unit square
Unit square rotated
Unit square reflected about the y-axis
Unit square reflected about the line y = x

Exercise

Given a transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is multiplication by an invertible matrix. Determine the image of:

1. A straight line
2. A line through the origin
3. Parallel lines
4. The line segment joining points P and Q
5. Three points lie on a line

Task:

Divide yourselves into 5 groups, and examine each of the question!

Exercises

Question 1

Given a transformation matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

Find the image of line $y = 2x + 1$ under the transformation.

Question 2

Given a transformation matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

Find the image of the unit square on the *first quadrant* under the transformation.

Exercises

Determine the image of the unit square under the following transformation:

- Reflection about the y-axis
- Reflection about the x-axis
- Reflection about the line $y = x$
- Rotation about the origin through a positive angle θ
- Compression in the x-direction with factor k with $0 < k < 1$
- Compression in the y-direction with factor k with $0 < k < 1$
- Expansion in the x-direction with factor k with $k > 1$
- Expansion in the y-direction with factor k with $k > 1$
- Shear in the x-direction with factor k with $k > 0$
- Shear in the x-direction with factor k with $k < 0$
- Shear in the y-direction with factor k with $k > 0$
- Shear in the y-direction with factor k with $k < 0$

This is the end of slide...