

Linear Algebra

[KOMS119602] - 2022/2023

2.1 - Algebra of Matrices

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Motivating example (1)

	Mon	Tue	Wed	Thu	Fri
John	30	10	20	9	14
Amy	10	9	7	19	25
Bob	20	7	0	10	20

A matrix of messages

Motivating example (2)

	Jan	Feb	Mar	Apr	May
Rent	1000	1000	1050	1050	1050
Grocery	300	250	350	310	305
Car	400	450	350	300	320

A matrix of expenses

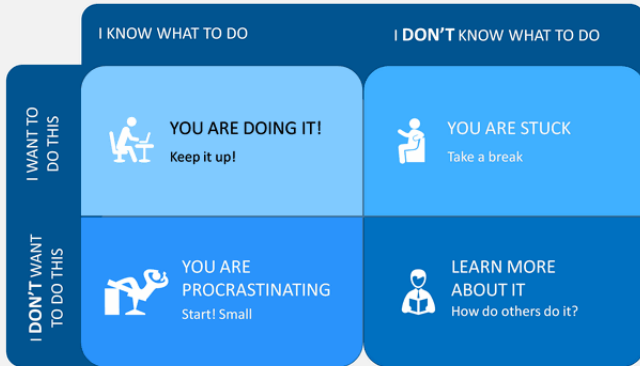
Motivating example (3)

	Boston	New York	London
Boston	0	187	3269
New York	187	0	3459
London	3269	3459	0

Motivating example (4)

MOTIVATION MATRIX

Enter your sub headline here



Then...what can you say about matrix?

Matrix is



Learning objectives

After this lecture, you should be able to:

1. Define and write the components of a matrix (row, column, diagonal, and entry) correctly.
2. Perform the operations between matrices, such as: scalar multiplication, matrix addition, matrix multiplication, transpose, powering of matrix, and polynomial of matrix.
3. Apply the properties of matrix operations to solve a problem.
4. Explain the concept and properties of square matrix.
5. Apply the concept of block matrices to solve matrix operation.

Part 1: Matrices and their operations

Formal definition of matrices

A **matrix** A over a field K (or simply a **matrix** A , when K is implicit), is a rectangular array of scalars:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The **rows** of matrix A are the m horizontal lists:

$$(a_{11}, a_{12}, \dots, a_{1n}), (a_{21}, a_{22}, \dots, a_{2n}), \dots, (a_{m1}, a_{m2}, \dots, a_{mn})$$

The **columns** of matrix A are the n vertical lists:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{bmatrix}, \dots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix}$$

Note: So, a matrix is composed by a set of *vectors*.

Formal definition of matrices

The element a_{ij} of matrix A (on row i , column j) is called *ij-entry* or *ij-element*.

We write the matrix as: $A = [a_{ij}]$.

A is a matrix of size $m \times n$.

- if $m = 1$ (only one row), then it is called *row matrix* or *row vector*;
- if $n = 1$ (only one column), then it is called *column matrix* or *column vector*.

A is called *zero matrix* if all entries of the matrix are zero.

Example

- Row matrix: $[1 \ 2 \ 3]$
- Column matrix: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
- Zero matrix: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- A 3×2 matrix: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Matrix operations

We are going to discuss:

1. Scalar multiplication
2. Matrix addition
3. Matrix multiplication
4. Transpose matrix
5. Power of matrix
6. Polynomial of matrix

1. Scalar multiplication

The **product** of matrix $A = [a_{ij}]$ with a scalar $k \in \mathbb{R}$ is defined as:

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

Moreover, $-A = (-1)A$.

2. Matrix addition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same size $m \times n$.
The **sum** of A and B is defined as:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

Moreover, $A - B = A + (-B)$.

Properties of matrices under addition and scalar multiplication

Theorem

Let A , B , and C be matrices with the same size, and $k, k' \in \mathbb{R}$. Then:

- $(A + B) + C = A + (B + C)$ (associativity)
- $A + B = B + A$ (commutativity)
- $A + 0 = A$ (0 is the identity elt over addition)
- $A + (-A) = 0$ (invers matrix over addition)
- $k(A + B) = kA + kB$ (distributivity)
- $(k + k')A = kA + k'A$ (distributivity w.r.t. scalar)
- $(kk')A = k(k'A)$ (associativity w.r.t. scalar)
- $1 \cdot A = A$ (1 is the identity elt over scalar multiplication)

Note: Hence, the sum $A_1 + A_2 + \cdots + A_n$ can be done in any order, and does not require any parenthesis.

Example

Given the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 5 & 5 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 2 \\ -1 & 0 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 8 & 7 \end{bmatrix}$$

Simplify the following matrix expression.

- $A + B$
- $B - C$
- $-3A + 2B$
- $5A + 2B - 3C$
- $3(A - C) + B$
- $A - A$

3. Matrix multiplication

Special case: the product of a row matrix and a column matrix having the same number of elements.

Let $A = [a_i]$ be a row matrix and $B = [b_i]$ be a column matrix. Then the product AB is defined as:

$$AB = [a_1, a_2, \dots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n = \sum_{i=1}^n a_i b_i$$

Note: the product of A and B is a scalar.

Example

$$[7, -4, 5] \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = 7(3) + (-4)(2) + 5(-1) = 21 - 8 - 5 = 8$$

Matrix multiplication

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of size $m \times p$ and $p \times n$ respectively. Then the product of A and B is a matrix AB of size $m \times n$ defined by:

$$\begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \cdots & \vdots \\ a_{i1} & \cdots & a_{ip} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mp} \end{bmatrix} \times \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \cdots & \vdots \\ \vdots & c_{ij} & \vdots \\ \vdots & \cdots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^p a_{ik}b_{kj}$

Example

Find AB where $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$.

Example

Find AB where $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$.

Multiply each row of A with each column of B .

Since A is of size 2×2 and B is of size 2×3 , then AB is of size 2×3 .

$$AB = \begin{bmatrix} 2 + 15 & 0 - 6 & -4 + 18 \\ 4 - 5 & 0 + 2 & -8 - 6 \end{bmatrix} = \begin{bmatrix} 17 & -6 & 14 \\ -1 & 2 & -14 \end{bmatrix}$$

Relation between matrix addition and matrix multiplication

Theorem

Let A , B , and C be matrices. Then whenever the products and sums are defined,

- $(AB)C = A(BC)$ (associative)
- $A(B + C) = AB + AC$ (left distributive)
- $(B + C)A = BA + CA$ (right distributive)
- $k(AB) = (kA)B = A(kB)$ where $k \in \mathbb{R}$
- $0A = 0$ and $A0 = 0$, where 0 is the zero matrix

Transpose matrix

The **transpose** of a matrix A , denoted by A^T , is the the matrix obtained by writing the columns of A , in order, as rows.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Note: If A has size $m \times n$, then A^T has size $n \times m$.

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -3 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$$

Powers of Matrices, Polynomials in Matrices

Let A be an n -square matrix over \mathbb{R} (or over other fields). Powers of A are defined as:

$$A^2 = AA, \quad A^3 = A^2A, \quad \dots, \quad A^{n+1} = A^nA, \quad \dots, \quad \text{and} \quad A^0 = I$$

We can also define **polynomials in the matrix A** . For any polynomial:

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n, \quad \text{where } a_i \in \mathbb{R},$$

Polynomial $f(A)$ is defined as:

$$f(A) = a_0I + a_1A + a_2A^2 + \dots + a_nA^n$$

Note: If $f(A) = 0$ (the zero matrix), then A is called a *zero* or *root* of $f(x)$.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$. Then:

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}, \text{ and}$$

$$A^3 = A^2 A = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -11 & 38 \\ 57 & -106 \end{bmatrix}$$

Suppose $f(x) = 2x^2 - 3x + 5$, then:

$$f(A) = 2 \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} + 3 \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & -18 \\ -27 & 61 \end{bmatrix}$$

Exercise

1. Form a group of 3 students;
2. Do the following exercises (Howard Anton's book):
 - Number 1 & 2 (2 questions @)
 - Number 3-6 (3 questions @)
 - Number 9-10 (choose 1 or 2 columns)

Part 2: Square matrices

Square matrices

A **square** matrix is a matrix with the same number of rows and columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{nn} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Diagonal and Trace

Let $A = [a_{ij}]$ be an n -square matrix. The **diagonal** or **main diagonal** of A consists of the elements with the same subscripts, that is:

$$a_{11}, a_{22}, \dots, a_{nn}$$

The **trace** of A , denoted by $\text{tr}(A)$ is the sum of the diagonal elements of A .

$$\text{tr}(A) = a_{11} + a_{22} + \dots + a_{nn} = \sum_{i=1}^n a_{ii}$$

Theorem (Properties of trace)

- $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
- $\text{tr}(kA) = k\text{tr}(A)$
- $\text{tr}(A^T) = \text{tr}(A)$
- $\text{tr}(AB) = \text{tr}(BA)$ (recall that $AB \neq BA$ is not always correct)

Identity matrix, scalar matrices

The **identity** or **unit** matrix, denoted by I_n (or simply I) is the square matrix $n \times n$, with 1's on the diagonal, and 0's elsewhere.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

I has a similar role as the scalar 1 for \mathbb{R} .

Important property: When it is well-defined,

$$IA = A$$

For some scalar $k \in \mathbb{R}$, the matrix kI is called **scalar matrix** corresponding to scalar k .

Special types of square matrices

A matrix $D = [d_{ij}]$ is a **diagonal matrix** if its nondiagonal entries are all zero.

$$D = \text{diag}(d_{11}, d_{22}, \dots, d_{nn})$$

where some or all the d_{ii} may be zero.

Example

$$\begin{bmatrix} 3 & 0 & \cdots & 0 \\ 0 & -5 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 9 \end{bmatrix}$$

Hence, identity matrices and scalar matrices are also diagonal matrices.

Upper and lower triangular matrices

A square matrix $A = [a_{ij}]$ is **upper triangular**, if all entries below the (main) diagonal are equal to 0.

A **lower triangular** matrix is a square matrix whose entries above the diagonal are all zero.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Upper and lower triangular matrices

Theorem

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times n$ triangular matrices. Then:

$$A + B, \quad kA, \quad AB$$

are triangular matrices w.r.t. diagonals:

$$(a_{11} + b_{11}, \dots, a_{nn} + b_{nn}), \quad (ka_{11}, \dots, ka_{nn}), \quad (a_{11}b_{11}, \dots, a_{nn}b_{nn})$$

Symmetric matrices

A matrix A is **symmetric** if $A^T = A$, i.e. $a_{ij} = a_{ji}$ for every $i, j \in \{1, 2, \dots, n\}$.

It is **skew-symmetric** if $A^T = -A$.

Example

$$A = \begin{bmatrix} 2 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & -8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$$

A is a symmetric matrix, and B is a skew-symmetric matrix.

Can you find other examples? Find an example of matrix that is neither symmetric nor skew-symmetric.

Normal matrices

A matrix A is **normal** if $AA^T = A^T A$.

Example

Let $A = \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix}$. Then:

$$AA^T = \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 45 & 0 \\ 0 & 45 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 45 & 0 \\ 0 & 45 \end{bmatrix}$$

Since $AA^T = A^T A$, the matrix A is normal.

Exercise of square matrices

- Create 3 groups;
- Each group discusses about the application of the following matrices in CS:
 - Upper (lower) triangular matrices;
 - Symmetric matrices;
 - Normal matrices.
- Write your discussion's result on a piece of paper (i.e., 1 page), and submit it.

Part 4: Block matrices

Block matrices

Using a system of horizontal and vertical (dashed) lines, a matrix A can be partitioned into submatrices called **blocks** (or **cells**) of A .

Example

$$\left(\begin{array}{cc|cc|c} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array} \right) \quad \left(\begin{array}{cc|cc|c} 1 & -2 & 0 & 1 & 3 \\ \hline 2 & 3 & 5 & 7 & -2 \\ 3 & 1 & 4 & 5 & 9 \\ \hline 4 & 6 & -3 & 1 & 8 \end{array} \right) \quad \left(\begin{array}{ccc|cc} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array} \right)$$

Operations on block matrices

Let $A = [A_{ij}]$ and $B = [B_{ij}]$ are block matrices with the same numbers of row and column blocks, and suppose that corresponding blocks have the same size.

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} & \cdots & A_{1n} + B_{1n} \\ A_{21} + B_{21} & A_{22} + B_{22} & \cdots & A_{2n} + B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} + B_{m1} & A_{m2} + B_{m2} & \cdots & A_{mn} + B_{mn} \end{bmatrix}$$

and

$$kA = \begin{bmatrix} kA_{11} & kA_{12} & \cdots & kA_{1n} \\ kA_{21} & kA_{22} & \cdots & kA_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ kA_{m1} & kA_{m2} & \cdots & kA_{mn} \end{bmatrix}$$

Square block matrices

A block matrix M is called a **square block matrix** if:

1. M is a square matrix.
2. The blocks form a square matrix.
3. The diagonal blocks are also square matrices.

Example

$$A = \left(\begin{array}{cc|cc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ \hline 9 & 8 & 7 & 6 & 5 \\ \hline 4 & 4 & 4 & 4 & 4 \\ 3 & 5 & 3 & 5 & 3 \end{array} \right) \quad B = \left(\begin{array}{cc|cc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ \hline 9 & 8 & 7 & 6 & 5 \\ 4 & 4 & 4 & 4 & 4 \\ \hline 3 & 5 & 3 & 5 & 3 \end{array} \right)$$

Which one of the matrices is a square block matrix?

Block diagonal matrices

A **block diagonal matrix** is a square block matrix $M = [A_{ij}]$ s.t. the non-diagonal blocks are zero matrices.

Example

$$\left(\begin{array}{cc|cc|c} 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 7 & 6 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 \end{array} \right)$$

A block diagonal matrix is often denoted as $M = \text{diag}(A_{11}, A_{22}, \dots, A_{rr})$

Exercise

(This will be discussed during the lecture)

1. Find an algorithm for matrix multiplication

Given two matrices:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -2 \\ 3 & 1 & 9 \\ 4 & 6 & 8 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 & 1 & 4 \\ -1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 4 \end{pmatrix}$$

- Compute $A \times B$.
- Describe the step-by-step procedure to compute $A \times B$ for any matrix $A_{m \times k}$ and $B_{k \times n}$.
- Write the procedure in algorithm (you may write it as a pseudocode).

2. How to solve matrix multiplication using block matrix?

Given two matrices:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -2 \\ 3 & 1 & 9 \\ 4 & 6 & 8 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 & 1 & 4 \\ -1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 4 \end{pmatrix}$$

Compute $A \times B$.

What if the two matrices are written in block matrices?

$$A = \left(\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & 3 & -2 \\ \hline 3 & 1 & 9 \\ 4 & 6 & 8 \end{array} \right) \quad B = \left(\begin{array}{cc|cc} 0 & 2 & 1 & 4 \\ -1 & 1 & 0 & 0 \\ \hline 2 & 3 & -1 & 4 \end{array} \right)$$

Can you derive the step-by-step of block matrix multiplication?