(Theta, triangle)-free and (even hole, K4)-free graphs. Part 2: Bounds on treewidth

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Abstract

A *theta* is a graph made of three internally vertex-disjoint chordless paths $P_1 = a \dots b$, $P_2 = a \dots b$, $P_3 = a \dots b$ of length at least 2 and such that no edges exist between the paths except the three edges incident to a and the three edges incident to b. A *pyramid* is a graph made of three chordless paths $P_1 = a \dots b_1$, $P_2 = a \dots b_2$, $P_3 = a \dots b_3$ of length at least 1, two of which have length at least 2, vertex-disjoint except at a, and such that $b_1b_2b_3$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident to a. An *even hole* is a chordless cycle of even length. For three non-negative integers $i \leq j \leq k$, let $S_{i,j,k}$ be the tree with a vertex v, from which start three paths with i, j, and k edges respectively. We denote by K_t the complete graph on t vertices.

We prove that for all non-negative integers i, j, k, the class of graphs that contain no theta, no K_3, and no $S_{i,j,k}$ as induced subgraphs have bounded treewidth. We prove that for all non-negative integers i, j, k, t, the class of graphs that contain no even hole, no pyramid, no K_t, and no $S_{i,j,k}$ as induced subgraphs have bounded treewidth. To bound the treewidth, we prove that every graph of large treewidth must contain a large clique or a minimal separator of large cardinality.

1 Introduction

In this article, all graphs are finite, simple, and undirected. A graph H is an *induced subgraph* of a graph G if some graph isomorphic to H can be obtained

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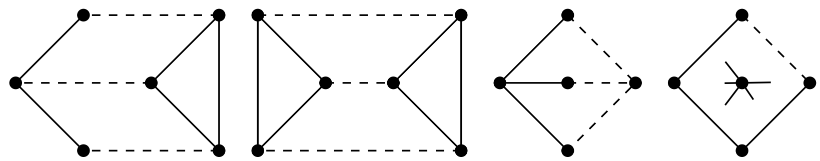

Figure 1: Pyramid, prism, theta, and wheel (dashed lines represent paths)

from G by deleting vertices. A graph G *contains* H if H is an induced subgraph of G. A graph is H*-free* if it does not contain H. For a family of graphs $\mathcal{H}$, G is $\mathcal{H}$*-free* if for every $H \in \mathcal{H}$, G is H-free.

A *hole* in a graph is a chordless cycle of length at least 4. It is *odd* or *even* according to its length (that is its number of edges). We denote by K_t the complete graph on t vertices.

A *theta* is a graph made of three internally vertex-disjoint chordless paths $P_1 = a \dots b$, $P_2 = a \dots b$, $P_3 = a \dots b$ of length at least 2 and such that no edges exist between the paths except the three edges incident to a and the three edges incident to b (see Fig. 1). Observe that a theta contains an even hole, because at least two paths in the theta have lengths of same parity and therefore induce an even hole.

We are interested in understanding the structure of even-hole-free graphs and theta-free graphs. Our motivation for this is explained in the first paper of this series [24], where we give a construction that we call layered wheel, showing that the cliquewith, the rankwidth, and the treewidth of (theta, triangle)-free and (even hole, K_4)-free graphs are unbounded. We also indicate questions, suggested by this construction, about the induced subgraphs contained in graphs with large treewidth.

In this second and last part, we prove that when excluding more induced subgraphs, there is an upper bound on the treewidth. Our results imply that the maximum independent set problem can be solved in polynomial time for some classes of graphs that are possibly of interest because they are related to several well known open questions in the field.

Results

We denote by P_k the path on k vertices. For three non-negative integers $i \leq j \leq k$, let $S_{i,j,k}$ be the tree with a vertex v, from which start three paths with i, j, and k edges respectively. Note that $S_{0,0,k}$ is a path of length k (so, is equivalent to P_{k+1}) and that $S_{0,i,j} = S_{0,0,i+j}$. The *claw* is the graph $S_{1,1,1}$. Note that $\{S_{i,j,k}; 1 \leq i \leq j \leq k\}$ is the set of all the subdivided claws and $\{S_{i,j,k}; 0 \leq i \leq j \leq k\}$ is the set of all subdivided claws and paths.

A *pyramid* is a graph made of three chordless paths $P_1 = a \dots b_1$, $P_2 = a \dots b_2$, $P_3 = a \dots b_3$ of length at least 1, two of which have length at least 2, vertex-disjoint except at a, and such that $b_1b_2b_3$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident to a (see Fig. 1).

We do not not recall here the definition of treewidth and cliquewidth. They are parameters that measure how complex a graph is. See [19, 17] for surveys about them.

Our main result states that for every fixed non-negative integers i, j, k, t, the following graph classes have bounded treewidth:

- (theta, triangle, $S_{i,j,k}$)-free graphs;
- (even hole, pyramid, K_t, $S_{i,j,k}$)-free graphs.

The exact bounds and the proofs are given in Section 4 (Theorems 4.6 and 4.7). In fact, the class on which we actually work is larger. It is a common generalization $\mathcal{C}$ of the graphs that we have to handle in the proofs for the two bounds above. Also, we do not exclude $S_{i,j,k}$, but some graphs that contain it, the so-called l-span-wheels for sufficiently large l. We postpone the definitions of $\mathcal{C}$ and of span wheels to Section 4.

To bound the treewidth, we prove that every graph of large treewidth must contain a large clique or a minimal separator of large cardinality. Let us define them.

For two vertices $s, t \in V(G)$, a set $X \subseteq V(G)$ is an *st-separator* if $s, t \notin X$ and s and t lie in different connected components of $G \setminus X$. An st-separator X is a *minimal st-separator* if it is an inclusion-wise minimal st-separator. A set $X \subseteq V(G)$ is a *separator* if there exist $s, t \in V(G)$ such that X is an st-separator in G. A set $X \subseteq V(G)$ is a *minimal separator* if there exist $s, t \in V(G)$ such that X is a minimal st-separator in G.

Our graphs have no large cliques by definition, and by studying their structure, we prove that they cannot contain large minimal separators, implying that their treewidth is bounded.

Note that from the celebrated grid-minor theorem, it is easy to see that every graph of large treewidth contains a subgraph with a large minimal separator (a column in the middle of the grid contains such a separator). But since we are interested in the induced subgraph containment relation, we cannot delete edges and we have to rely on our reinforcement.

Treewidth and cliquewidth of some classes of graphs

We now survey results about the treewidth in classes of graphs related to the present work. Complete graphs provide trivial examples of even-hole-free graphs of arbitrarily large treewidth. In [11], it is proved that (even hole, triangle)-free graphs have bounded treewidth (this is based on a structural description from [13]). In [10], it is proved that for every positive integer t, (even hole, pan, K_t)-free graphs have bounded treewidth (where a *pan* is any graph that consists of a hole and a single vertex with precisely one neighbor on the hole). It is proved in [24] that the treewidth of (theta, triangle)-free graphs and (even hole, pyramid, K_4)-free graphs are unbounded. Growing the treewidth in [24]

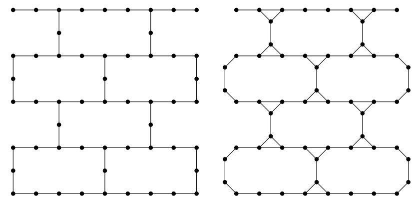

Figure 2: A subdivision of a wall and its line graph

requires introducing in the graph a large clique minor and vertices of large degree. It is therefore natural to ask whether these two conditions are really needed, and the answer is yes for both of them, because in [2] it is proved that even-hole-free graphs with no K_t-minor have bounded treewidth, and in [1] it is proved that even-hole-free graphs with maximum degree t have bounded treewidth.

Since having bounded cliquewidth is a weaker property than having bounded treewidth but still has nice algorithmic applications, we survey some results about the cliquewidth in classes related to the present work.

It is proved in [11] that (even hole, cap)-free graphs with no clique separator have bounded cliquewidth (where a *cap* is any graph that consists of a hole and a single vertex with precisely two adjacent neighbors on the hole, and a *clique separator* is a separator that is a clique). It is proved in [24], that (triangle, theta)-free and (even hole, pyramid, K_4)-free graphs have unbounded cliquewidth. It is proved in [3], that (even hole, diamond)-free graphs with no clique separator have unbounded cliquewidth (the *diamond* is the graph obtained from K_4 by deleting an edge). The construction can be easily extended to (even hole, pyramid, diamond)-free graphs as explained in [12]. It is easy to provide (theta, K_4, $S_{1,1,1}$)-free graphs (or equivalently (claw, K_4)-free graphs) of unbounded cliquewidth. To do so, consider a *wall* W, subdivide all edges to obtain W', and take the line graph $L(W')$ (see [24] for a definition and Fig. 2).

The results mentioned in this paragraph are extracted from [15] (but some of them were first proved in other works). Let $\mathcal{H}_U = \{P_7, S_{1,1,4}, S_{2,2,2}\}$ and $\mathcal{H}_B = \{P_6, S_{1,1,3}\}$. If H contains a graph from $\mathcal{H}_U$ as an induced subgraph, then the class of (triangle, H)-free graph has unbounded cliquewidth (see Theorem 7.ii.6 in [15]). If H is contained in a graph from $\mathcal{H}_B$, then the class of (triangle, H)-free graphs has bounded cliquewidth (see Theorem 7.i.3 in [15]).

The cliquewidth of (triangle, $S_{1,2,2}$)-free graphs is bounded, see [7] or [16].

Algorithmic consequences

It is proved in [14] that in every class of graphs of bounded treewidth, many problems can be solved in polynomial time. Our result has therefore applications to several problems, but we here focus on one because the induced subgraphs that are excluded in the most classical results and open questions about it seem to be related to our classes.

An *independent set* in a graph is a set of pairwise non-adjacent vertices. Our results imply that computing an independent set of maximum cardinality can be performed in polynomial time for (theta, triangle, $S_{i,j,k}$)-free graphs and (even hole, pyramid, K_t, $S_{i,j,k}$)-free graphs.

Finding an independent set of maximum cardinality is polynomial time solvable for (even hole, triangle)-free graphs [11] and (even hole, pyramid)-free graphs [12]. Its complexity is not known for (even hole, K_4)-free graphs and for (theta, triangle)-free graphs. Determining its complexity is also a well known question for $S_{i,j,k}$-free graphs. It is NP-hard for the class of H-free graphs whenever H is not an induced subgraph of some $S_{i,j,k}$ [4]. It is solvable in polynomial time for H-free graphs whenever H is contained in P_k for $k = 6$ (see [20] for $H = P_5$ and [18] for $H = P_6$) or contained in $S_{i,j,k}$ with $(i, j, k) \leq (1, 1, 2)$ (see [5] and [22] for the weighted version). It is solvable in polynomial time for (P_7, triangle)-free graphs [8] and for ($S_{1,2,4}$, triangle)-free graphs [9]. The complexity is not known for H-free graphs whenever H is some $S_{i,j,k}$ that contains either P_7, $S_{1,1,3}$, or $S_{1,2,2}$.

Bounding the number of minimal separators

One possible method to find maximum weight independent sets for a class of graphs is by proving that every graph in the class has polynomially many minimal separators (where the polynomial is in the number of vertices of the graph). This was for instance successfully applied to (even hole, pyramid)-free graphs in [12]. Therefore, our result on (even hole, pyramid, K_t, $S_{i,j,k}$)-free graphs does not settle a new complexity result for the Maximum Independent Set problem (but it still can be applied to other problems).

Note that bounding the number of minimal separators cannot be applied to (even hole, K_4)-free graphs and to (theta, triangle)-free graphs since there exist graphs in both classes that contain exponentially many minimal separators. These graphs are called *k-turtle* and *k-ladder*, see Fig 3. It is straightforward to check that they have exponentially many minimal separators (the idea is that a separator can be built by making a choice in each horizontal edge, and there are k of them). Moreover, k-turtles are (theta, triangle)-free (provided that the outer cycle is sufficiently subdivided) and k-ladders are (even hole, K_4)-free.

Open questions

It is not known whether (even hole, K_4, diamond)-free graphs have bounded treewidth (or cliquewidth). Also, for every fixed integer $t \geq 4$, it is not known whether (theta, triangle)-free graphs of maximum degree t have bounded treewidth (for $t = 1, 2$, the treewidth is trivially bounded and for $t = 3$ it follows from Corollary 4.3 in [2]). It is not known whether (triangle, $S_{1,2,3}$)-free graphs have bounded cliquewidth, see [16] for other open problems of the same flavor.

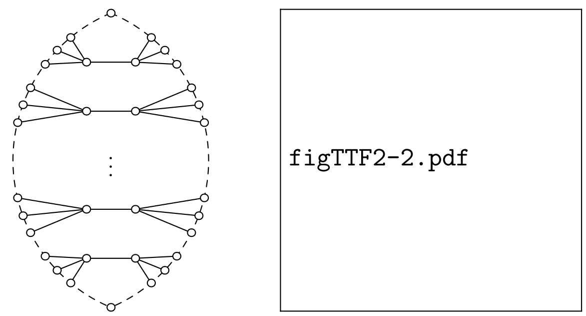

Figure 3: k-turtle and k-ladder (dashed lines represent paths)

Outline of the paper

In Section 2, we explain our method to bound the treewidth. In Section 3, we give two technical lemmas that highlight structural similarities between (theta, triangle)-free and (even hole, pyramid)-free graphs. These will be used in Section 4 where we prove that graphs in our classes do not contain minimal separators of large cardinality, implying that their treewidth is bounded.

Notation

By *path* we mean chordless (or induced) path. When a and b are vertices of a path P, we denote by aPb the subpath of P with ends a and b.

When $A, B \subseteq V(G)$, we denote by $N_B(A)$ the set of vertices from $B \setminus A$ that have at least one neighbor in A and $N(A)$ means $N_{V(G)}(A)$. Note that $N_B(A)$ is disjoint from A. We write $N(a)$ instead of $N(\{a\})$ and $N[a]$ for $\{a\} \cup N(a)$. We denote by $G[A]$ the subgraph of G induced by A. To avoid too heavy notation, since there is no risk of confusion, when H is an induced subgraph of G, we write N_H instead of $N_{V(H)}$.

A vertex x is *complete* (resp. *anticomplete*) to A if $x \notin A$ and x is adjacent to all vertices of A (resp. to no vertex of A). We say that A is *complete* (resp. *anticomplete*) to B if every vertex of A is complete (resp. anticomplete) to B (note that this means in particular that A and B are disjoint).

2 Treewidth and minimal separators

If a graph has large treewidth, then it contains some sub-structure that is highly connected in some sense (grid minor, bramble, tangle, see [19]). Theorem 2.1 seems to be a new statement of that kind. It says that graphs of large treewidth must contain either a large clique or a minimal separator of large size. However, its converse is false, as shown by $K_{2,t}$ that has treewidth 2 (it is a series-parallel graph) and contains a minimal separator of size t.

A variant of the following theorem can be obtained from the celebrated excluded grid theorem of Robertson and Seymour. The idea is to use a large grid to obtain a large minimal separator. But there are technicalities because we are not allowed to delete edges, so the grid might contain many crossing edges. To find two vertices that cannot be separated by a small separator, one needs to clean the grid. We do not include the details since the following provides a better bound.

Theorem 2.1. *Let G be a graph and let $k \geq 2$ and $s \geq 1$ be positive integers. If G does not contain a clique on k vertices nor a minimal separator of size larger than s, then the treewidth of G is at most $(k-1)s^3 - 1$.*

Before proving Theorem 2.1, let us introduce some terminology and state results due to Bouchitté and Todinca [6]. For a graph G we denote by $\textsc{cc}(G)$ the set of all connected components of G (viewed as subsets of $V(G)$). A set $F \subseteq \binom{V(G)}{2} \setminus E(G)$ is a *fill-in* or *chordal completion* if $G+F = (V(G), E(G) \cup F)$ is a chordal graph. A fill-in F is *minimal* if it is inclusion-wise minimal. If $X \subseteq V(G)$, then every connected component $D \in \textsc{cc}(G \setminus X)$ with $N(D) = X$ is called a component *full* to X. Observe that a set $X \subseteq V(G)$ is a minimal separator if and only if there exist at least two connected components of $G \setminus X$ that are full to X. An important property of minimal separators is that no new minimal separator appears when applying a minimal fill-in.

Lemma 2.2 (see [6])**.** *For every graph G, minimal fill-in F, and minimal separator X in $G+F$, X is a minimal separator in G as well. Furthermore, the families of components $\textsc{cc}((G+F) \setminus X)$ and $\textsc{cc}(G \setminus X)$ are equal (as families of subsets of $V(G)$).*

A set $\Omega \subseteq V(G)$ is a *potential maximal clique (PMC)* if there exists a minimal fill-in F such that Ω is a maximal clique of $G+F$. A PMC is surrounded by minimal separators.

Lemma 2.3 (see [6])**.** *For every PMC Ω in G and every component $D \in \textsc{cc}(G \setminus \Omega)$, the set $N(D)$ is a minimal separator in G with D being a full component.*

The following characterizes PMCs.

Theorem 2.4 (see [6])**.** *A set $\Omega \subseteq V(G)$ is a PMC in G if and only if the following two conditions hold:*

(i) for every $D \in \textsc{cc}(G \setminus \Omega)$ we have $N(D) \subsetneq \Omega$;

(ii) for every $x, y \in \Omega$ either $x = y$, $xy \in E(G)$, or there exists $D \in \textsc{cc}(G \setminus \Omega)$ with $x, y \in N(D)$.

In the second condition of Theorem 2.4, we say that a component D *covers* the nonedge xy.

Lemma 2.5. *Let G be a graph, $k \geq 2$ and $s \geq 1$ be integers, and let Ω be a PMC in G with $|\Omega| > (k-1)s^3$. Then there exists in G either a clique of size k or a minimal separator of size larger than s.*

Proof. By Lemma 2.3, we may assume that for every $D \in \textsc{cc}(G \setminus \Omega)$ we have $|N(D)| \leq s$.

Assume first that for every $x \in \Omega$ the set of non-neighbors of x in Ω (i.e., $\Omega \setminus N[x]$) is of size less than s^3. Let $A_0 = \Omega$ and consider the following iterative process. Given A_i for $i \geq 0$, pick $x_i \in A_i$, and set $A_{i+1} = A_i \cap N(x_i)$. The process terminates when A_i becomes empty. Clearly, the vertices $x_0, x_1, \ldots$ induce a clique. Furthermore, by our assumption, $|A_i \setminus A_{i+1}| \leq s^3$. Therefore this process continues for at least k steps, giving a clique of size k in G.

Thus we are left with the case when there exists $x \in \Omega$ with the set $\Omega \setminus N[x]$ of size at least s^3. Let $Y = \{x\} \cup (\Omega \setminus N[x])$; we have $|Y| > s^3$, $Y \subseteq \Omega$, and $G[Y]$ is disconnected.

Consider the following iterative process. At step i, we will maintain a partition $\mathcal{A}_i$ of Y into at least two parts and for every $A \in \mathcal{A}_i$ a set $\mathcal{D}_i(A) \subseteq \textsc{cc}(G \setminus \Omega)$ with the following property: the sets $\{A \cup \bigcup_{D \in \mathcal{D}_i(A)} D \mid A \in \mathcal{A}_i\}$ is the partition of $G[Y \cup \bigcup_{A \in \mathcal{A}_i} \bigcup_{D \in \mathcal{D}_i(A)} D]$ into vertex sets of connected components. In particular, for every $A \in \mathcal{A}_i$ and $D \in \mathcal{D}_i(A)$ we have $N(D) \cap Y \subseteq A$. We start with $\mathcal{A}_0 = \textsc{cc}(G[Y])$ and $\mathcal{D}_0(A) = \emptyset$ for every $A \in \mathcal{A}_0$.

The process terminates when there exists $A \in \mathcal{A}_i$ of size larger than s^2. Otherwise, we perform a step as follows. Pick two distinct $A, B \in \mathcal{A}_i$ and vertices $a \in A$, $b \in B$. By the properties of $\mathcal{A}_i$, $ab \notin E(G)$. By Theorem 2.4, there exists $D \in \textsc{cc}(G \setminus \Omega)$ with $a, b \in N(D)$. Let $\mathcal{A} = \{C \in \mathcal{A}_i \mid N(D) \cap C \neq \emptyset\}$. Note that $A, B \in \mathcal{A}$. Furthermore, since $|N(D)| \leq s$, we have $2 \leq |\mathcal{A}| \leq s$.

We define $\mathcal{A}_{i+1} = (\mathcal{A}_i \setminus \mathcal{A}) \cup \{\bigcup_{C \in \mathcal{A}} C\}$. For every $C \in \mathcal{A}_{i+1} \cap \mathcal{A}_i$ we keep $\mathcal{D}_{i+1}(C) = \mathcal{D}_i(C)$. Furthermore, we set $\mathcal{D}_{i+1}(\bigcup_{C \in \mathcal{A}} C) = \{D\} \cup \bigcup_{C \in \mathcal{A}} \mathcal{D}_i(C)$. It is straightforward to verify the invariant for $\mathcal{A}_{i+1}$ and $\mathcal{D}_{i+1}$.

Furthermore, since every set $C \in \mathcal{A}_i$ is of size at most s^2 while $|Y| > s^3$ we have that $|\mathcal{A}_i| > s$. Since $2 \leq |\mathcal{A}| \leq s$, we have $2 \leq |\mathcal{A}_{i+1}| < |\mathcal{A}_i|$. Consequently, the process terminates after a finite number of steps with $\mathcal{A}_i$ of size at least 2, $\mathcal{D}_i$, and some $A \in \mathcal{A}_i$ of size greater than s^2.

Let $X = A \cup \bigcup_{D \in \mathcal{D}_i(A)} D$ and let $y \in Y \setminus A$. Note that $G[X]$ is connected by the invariant on $\mathcal{A}_i$ and $\mathcal{D}_i$, y exists as $|\mathcal{A}_i| \geq 2$, and y is anticomplete to X. We use Theorem 2.4: for every $a \in A$ fix a component $D_a \in \textsc{cc}(G \setminus \Omega)$ covering the nonedge ya. Since $|N(D_a)| \leq s$ while $|A| > s^2$, the set $\mathcal{D} = \{D_a \mid a \in A\}$ is of size greater than s. Since $G[X]$ is connected and y is anticomplete to X, there exists a minimal separator S with y in one full side and X in the other full side. However, then $S \cap D \neq \emptyset$ for every $D \in \mathcal{D}$. Hence, $|S| \geq |\mathcal{D}| > s$. This finishes the proof of the lemma. $\square$

Proof of Theorem 2.1.

Let G be a graph such that it does not contain a clique on k vertices and a minimal separator of size larger than s. Let F be a minimal chordal completion of G. By Lemma 2.5, every maximal clique of $G + F$ is of size at most $(k-1)s^3$. Therefore a clique tree of $G + F$ is a tree decomposition of G of width at most $(k-1)s^3 - 1$, as desired. $\square$

3 Nested 2-wheels

Let $k \geq 0$ be an integer. A *k-wheel* is a graph formed by a hole H called the *rim* together with a set C of k vertices that are not in $V(H)$ called the *centers*, such that each center has at least three neighbors in the rim. We denote such a k-wheel by (H, C). Observe that a 0-wheel is a hole. A 1-wheel is called a *wheel* (see Fig. 1). We often write (H, u) instead of $(H, \{u\})$.

A 2-wheel $(H, \{u, v\})$ is *nested* if H contains two vertices a and b such that all neighbors of u in H are in one path of H from a to b, while all the neighbors of v are in the other path of H from a to b. Observe that a and b may be adjacent to both u and v. As we will see in this section, the properties of 2-wheels highlight structural similarities between (theta, triangle)-free graphs and (even hole, pyramid)-free graphs, in the sense that in both classes, apart from few exceptions, every 2-wheel with non-adjacent centers is nested.

For a center u of a k-wheel (H, C), a *u-sector* of H is a subpath of H of length at least 1 whose ends are adjacent to u and whose internal vertices are not. However, a u-sector may contain internal vertices that are adjacent to v for some center $v \neq u$. Observe that for every center u, the rim of a wheel is edgewise partitioned into its u-sectors.

In (theta, triangle)-free graphs

The *cube* is the graph formed from a hole of length 6, say $h_1h_2 \cdots h_6h_1$ together with a vertex u adjacent to h_1, h_3, h_5 and a vertex v non-adjacent to u and adjacent to h_2, h_4, h_6. Note that the cube is a non-nested 2-wheel with non-adjacent centers.

Lemma 3.1. *Let G be a (theta, triangle)-free graph. If $W = (H, \{u, v\})$ is a 2-wheel in G such that $uv \notin E(G)$, then W is either a nested wheel or the cube.*

Proof. Suppose that W is not a nested wheel. We will prove that W is the cube.

Claim 1. *Every u-sector of H contains at most one neighbor of v and every v-sector of H contains at most one neighbor of u.*

Proof of Claim 1. For otherwise, without loss of generality, some u-sector $P = x \dots y$ of H contains at least two neighbors of v. Let x', y' be neighbors of v closest to x, y respectively along P. Note that $x'y' \notin E(G)$ because G is triangle-free. Since W is not nested, $H \setminus P$ contains some neighbors of v. Note also that $H \setminus P$ contains some neighbors of u.

So, let $Q = z \dots z'$ be the path of $H \setminus P$ that is minimal length and such that $uz \in E(G)$ and $vz' \in E(G)$. Note that z' is adjacent to either x or y, for otherwise $uzQz'v$, $uxPx'v$, and $uyPy'v$ form a theta from u to v. So suppose up to symmetry that z' is adjacent to y. So, v is not adjacent to y since G is triangle-free. It then follows that the three paths $vz'y$, $vy'Py$, and $vx'Pxuy$ form a theta, a contradiction. This proves Claim 1.

Claim 2. *u and v have no common neighbors in H.*

Proof of Claim 2. Otherwise, let x be such a common neighbor. Consider a subpath $x \dots y$ of H of maximum length with the property of being a u-sector or a v-sector, and suppose up to symmetry that it is a u-sector. By its maximality, it contains a neighbor of v different from x. So in total it contains at least two neighbors of v, a contradiction to Claim 1. This proves Claim 2.

Claim 1 and 2 prove that $|N_H(u)| = |N_H(v)|$ and the neighbors of u and v alternate along H. So, let $x, y, z \in N_H(u)$ and $x', y', z' \in N_H(v)$ be distinct vertices in H with x, x', y, y', z, z' appearing in this order along H. If $V(H) = \{x, y, z, x', y', z'\}$, then $V(H) \cup \{u, v\}$ induces the cube, so suppose $\{x, y, z, x', y', z'\} \subsetneq V(H)$. Hence, up to symmetry, we may assume that x, x', y, y', z and z' are chosen such that: $xz' \notin E(G)$. But then the three paths $vz'(H \setminus x)z$, $vy'(H \setminus y)z$, and $vx'(H \setminus y)xuz$ form a theta, a contradiction. □

The following lemma of Radovanović and Vušković shows that the presence of the cube in a (theta, triangle)-free graph entails some structure.

Lemma 3.2 (see [23])**.** *Let G be a (theta, triangle)-free graph. If G contains the cube, then either it is the cube, or it has a clique separator of size at most 2.*

In even-hole-free graphs

Let us consider a classical generalization of even-hole-free graphs.

A *prism* is a graph made of three vertex-disjoint chordless paths $P_1 = a_1 \dots b_1$, $P_2 = a_2 \dots b_2$, $P_3 = a_3 \dots b_3$ of length at least 1, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles and no edges exist between the paths except those of the two triangles (see Fig. 1). An *even wheel* is a wheel (H, u) such that u has an even number of neighbors in H. A *square* is a hole of length 4.

It is easy to see that all thetas, prisms, even wheels, and squares contain even holes. The class of (theta, prism, even wheel, square)-free graphs is therefore a generalization of even-hole-free graphs that capture the structural properties that we need here.

A proof of the following lemma can be found in [12] (where it relies on many lemmas). We include here our self-contained proof for the sake of completeness. Call a wheel *proper* if it is not pyramid. A *cousin wheel* is a 2-wheel made of a hole $H = h_1h_2 \dots h_nh_1$ and two non-adjacent centers u and v, such that $N_H(u) = \{h_1, h_2, h_3\}$ and $N_H(v) = \{h_2, h_3, h_4\}$.

Lemma 3.3. *Let G be a (theta, prism, pyramid, even wheel, square)-free graph. If $W = (H, \{u, v\})$ is a 2-wheel in G such that $uv \notin E(G)$, then W is either a nested or a cousin wheel. Moreover, if W is nested then $|N_H(u) \cap N_H(v)| \leq 1$.*

Proof. In the case where $W = (H, \{u, v\})$ is nested, it must be that $|N_H(u) \cap N_H(v)| \leq 1$, for otherwise G would contain a square. Since G contains no even wheel, it is sufficient to consider the following cases.

Case 1: $N_H(u) = 3$ or $N_H(v) = 3$.

Assume that W is not a nested wheel. We will prove that W is a cousin wheel. Without loss of generality, we may assume that $|N_H(u)| = 3$, and let

$N_H(u) = \{x, y, z\}$. We denote by $P_x = y \dots z$, $P_y = x \dots z$ and $P_z = x \dots y$ the three u-sectors of H.

Suppose xyz is a path of H. Then v must be adjacent to y, for otherwise W is nested, a contradiction. Since $V(H) \cup \{u\}$ and $V(H \setminus y) \cup \{u, v\}$ do not induce an even wheel, v has exactly two neighbors in P_y. Moreover, the two neighbors of v in P_y are adjacent, for otherwise $H \setminus y$, u, and v form a theta. Since (H, v) is not a pyramid, this means that one of x or z is a neighbor of v. Therefore, W is a cousin wheel.

Now suppose that $\{x, y, z\}$ does not induce a path. So xy, yz, and zx are non-edges. Note that v is adjacent to at most one of x, y, or z, because G contains no square. Up to symmetry, assume that $vx \notin E(G)$. Let R be the v-sector of H which contains x (in its interior). Since $(H, \{u, v\})$ is not a nested wheel, the ends of R are not both in P_x, or both in P_y, or both in P_z. So assume that $R = y' \dots z'$ with z' is in the interior of P_z and y' is not in P_z. If y' is in P_x, then R, u, and v form a theta from x to z, a contradiction. Hence, y' is not in P_x, so y' is in the interior of P_y.

Call x' the neighbor of v in H different from y' and z'. If x' is not in the interior of P_x, then P_x is contained in the v-sector $x'Hz'$. Thus, there exists a v-sector S which contains P_x. In particular, the hole made of S and v contains two non adjacent neighbors of u, namely y and z. Hence, S, u, and v form a theta from y to z. So, x' is in the interior of P_x.

This means x, y', z, x', y, z' appear in this order along H. If $x'z \notin E(G)$, then the paths $x'(H \setminus y)z$, $x'(H \setminus z)yuz$, and $x'vy'(H \setminus x)z$ form a theta from x' to z, a contradiction. So, $x'z \in E(G)$. By symmetry, $x'y \in E(G)$. But then, $\{u, y, x'z\}$ induces a square, a contradiction.

Case 2: $N_H(u) \geq 5$ and $N_H(v) \geq 5$

For a contradiction, suppose that $(H, \{u, v\})$ is not a nested wheel. First of all, we have $N_H(u) \neq N_H(v)$, for otherwise u, v, and two non-adjacent vertices of $N_H(u)$ would form a square. So in H, there exists a neighbor of v that is not adjacent to u. It is therefore well defined to consider the u-sector $P = x \dots y$ of H whose interior contains $k \geq 1$ neighbors of v, and to choose such a sector with k minimum. We denote by x' the neighbor of x in $H \setminus P$, by y' the neighbor of y in $H \setminus P$ and by $Q = x' \dots y'$ the path $H \setminus P$.

Note that u has some neighbor in the interior of Q, because u has at least 5 neighbors in H. We now show that v also has some neighbor in the interior of Q. Suppose that it is not the case. Then, the neighborhood of v in H is completely contained in $V(P) \cup \{x', y'\}$. Since $(H, \{u, v\})$ is not a nested wheel, v is adjacent to x' or y' — and in fact to both of them, for otherwise the hole $uxPyu$ would contain an even number (at least 4) of neighbors of v, thus inducing an even wheel, a contradiction. Now since $\{u, v, x, y\}$ does not induce a square, up to symmetry we may assume that $vx \notin E(G)$. Since $|N_H(v)| \geq 5$, v has at least 2 neighbors in the interior of P, and so $k \geq 2$. Note that u is adjacent to x', for otherwise, x' would be the unique neighbor of v in the interior of a u-sector, contradicting the minimality of k. Since $\{u, v, x', y'\}$ does not induce a square, we know that u is not adjacent to y'. But then, y' is the unique neighbor of v

in the interior of some u sector, a contradiction to the minimality of k. This proves that v has some neighbor in the interior of Q.

By the fact that each of u and v has some neighbor in the interior of Q, a path S from u to v whose interior is in the interior of Q exists. Let x'' (resp. y'') be the neighbor of v in P closest to x (resp. y) along P. If $x'' = y''$, then x'' is an internal vertex of P, and so S and P form a theta from u to x''. If $x''y'' \in E(G)$, then S and P form a pyramid. If $x'' \neq y''$ and $x''y'' \notin E(G)$, then S, $uxPx''v$, and $uyPy''v$ form a theta from u to v. Each of the cases yields a contradiction; this completes the proof. $\square$

4 Bounding the treewidth

In this section, we prove that the treewidth is bounded in (theta, triangle, $S_{i,j,k}$)-free graphs and in (even hole, pyramid, K_t, $S_{i,j,k}$)-free graphs.

For (theta, triangle)-free graphs, by Lemma 3.2, we may assume that the graphs we work on are cube-free since the cube itself has small treewidth, and clique separators of size at most 2 in some sense preserve the treewidth (this will be formalized in the proofs). For (even hole, pyramid)-free graphs, recall that we work from the start in a superclass, namely (theta, prism, pyramid, even wheel, square)-free graphs.

Since our proof is the same for (theta, triangle, $S_{i,j,k}$)-free graphs and (even hole, pyramid, K_t, $S_{i,j,k}$)-free graphs, to avoid duplicating it, we introduce a class $\mathcal{C}$ that contains all the graphs that we need to consider while entailing the structural properties that we need.

Call *butterfly* a wheel (H, v) such that $N_H(v) = \{a, b, c, d\}$ with $ab \in E(G)$, $bc \notin E(G)$, $cd \in E(G)$ and $da \notin E(G)$. Let $\mathcal{C}$ be the class of all (theta, prism, pyramid, butterfly)-free graphs such that every 2-wheel with non-adjacent centers is either a nested or a cousin wheel.

Lemma 4.1. *If G is a (theta, triangle, cube)-free graph or a (theta, prism, pyramid, even wheel, square)-free graph, then $G \in \mathcal{C}$.*

Proof. If G is a (theta, triangle, cube)-free graph, then G is theta-free and (prism, pyramid, butterfly)-free (because prisms, pyramids, and butterflies contain triangles). Furthermore, every 2-wheel with non-adjacent centers is a nested wheel by Lemma 3.1.

If G is a (theta, prism, pyramid, even wheel, square)-free graph, then G is (theta, prism, pyramid)-free and butterfly-free (because a butterfly is an even wheel). Furthermore, every 2-wheel with non-adjacent centers is either a nested or a cousin wheel by Lemma 3.3.

Hence $G \in \mathcal{C}$ as claimed. $\square$

For our proof, we need a special kind of k-wheel. A *k-span-wheel* is a k-wheel (H, C) that satisfies the following properties.

- There exist two non-adjacent vertices x, y in H and we denote by $P_A = a_1 \dots a_\alpha$ and $P_B = b_1 \dots b_\beta$ the two paths of H from x to y, with $x = a_1 = b_1$ and $y = a_\alpha = b_\beta$.
- $C \cup \{x, y\}$ is an independent set.
- There exists an ordering of vertices in C, namely $v_1, v_2, \cdots, v_k$.
- Every vertex of C has neighbors in the interiors of both P_A and P_B (and at least 3 neighbors in H since (H, C) is a k-wheel).
- For every $1 \leq i < j \leq k$ and $1 \leq i', j' \leq \alpha$, if $v_i a_{i'} \in E(G)$ and $v_j a_{j'} \in E(G)$ then $i' \leq j'$.
- For every $1 \leq i < j \leq k$ and $1 \leq i', j' \leq \beta$, if $v_i b_{i'} \in E(G)$ and $v_j b_{j'} \in E(G)$ then $i' \leq j'$.

Informally, a k-span-wheel is such that, walking from x to y along both P_A and P_B, one first meets all the neighbors of v_1, then all neighbors of v_2, and so on until v_k. Observe that a 1-span-wheel is a wheel, 2-span-wheel is a nested 2-wheel. Note that distinct v_i and v_j may share common neighbors on H (it is even possible that $N_{P_A}(v_1) = \cdots = N_{P_A}(v_k) = \{a_i\}$).

Observe that in the following theorem, thetas, pyramids, prisms, and butterflies have to be excluded, since they do not satisfy the conclusion.

Lemma 4.2. *Let G be a graph in $\mathcal{C}$. Let C be a minimal separator in G of size at least 2 that is furthermore an independent set, and A and B be components of $G \setminus C$ that are full to C. Then:*

1. *There exist two vertices x and y in C, a path P_A from x to y with interior in A, and a path P_B from x to y with interior in B such that all vertices in $C \setminus \{x, y\}$ have neighbors in the interior of both P_A and P_B. Note that $V(P_A) \cup V(P_B)$ induces a hole that we denote by H.*
2. *$(H, C \setminus \{x, y\})$ is a $(|C| - 2)$-span-wheel.*

Proof. We first prove 1, by induction on $k = |C|$.

If $k = 2$, then x, y, P_A, and P_B exist from the connectivity of A and B, and the conditions on $C \setminus \{x, y\}$ vacuously hold. So suppose the result holds for some $k \geq 2$, and let us prove it for $k + 1$. Let z be any vertex from C, and apply the induction hypothesis to $C \setminus z$ in $G \setminus z$. This provides two vertices x, y in $C \setminus z$ and two paths P_A and P_B. We denote by H the hole formed by P_A and P_B.

Claim 1. *Every vertex in $C \setminus \{x, y, z\}$ has neighbors in the interior of both P_A and P_B.*

Proof of Claim 1. Follows directly from the induction hypothesis. This proves Claim 1.

Since z has a neighbor in A and A is connected, there exists a path $Q_A = z \dots z_A$ in $A \cup \{z\}$, such that z_A has a neighbor in the interior of P_A. A similar path Q_B exists. We set $Q = z_A Q_A z Q_B z_B$. We suppose that x, y, P_A, P_B, Q_A, and Q_B are chosen subject to the minimality of Q.

Observe that Q is a chordless path by its minimality and the fact that A and B being anticomplete. The minimality of Q implies that the interior of Q is anticomplete to the interior of P_A and to the interior of P_B.

Claim 2. *We may assume that Q has length at least 1.*

Proof of Claim 2. Otherwise, $z = z_A = z_B$, so z has neighbors in the interior of both P_A and P_B. Hence, by Claim 1, x, y, P_A, and P_B satisfy 1. This proves Claim 2.

Let a (resp. a') be the neighbor of z_A in P_A closest to x (resp. to y) along P_A. Let b (resp. b') be the neighbor of z_B in P_B closest to x (resp. to y) along P_B.

Claim 3. *If $a \neq a'$ and $aa' \notin E(G)$, then $z = z_A$. If $b \neq b'$ and $bb' \notin E(G)$, then $z = z_B$.*

Proof of Claim 3. We give a proof only for the statement of a, since the proof for b is similar.

For suppose $a \neq a'$, $aa' \notin E(G)$, and $z \neq z_A$, let z' be the neighbor of z_A in Q. Set $P'_A = xP_A a z_a a' P_A y$ and $Q' = z' Q z_B$. Let us prove that x, y, P'_A, P_B, and Q' contradict the minimality of Q. Obviously, Q' is shorter than Q, so we only have to prove that every vertex in $C \setminus \{z\}$ has neighbors in the interior of both P'_A and P_B. For P_B, it follows from Claim 1. So suppose for a contradiction that a vertex $c \in C \setminus \{z\}$ has no neighbor in the interior of P'_A. Since by Claim 1 c has a neighbor c' in the interior of P_A, c' is an internal vertex of $aP_A a'$. Since G is theta-free, (H, z_A) is a wheel. Note that $(H, \{c, z_A\})$ is not nested because of c' and some neighbor of c in the interior of P_B (i.e. the neighborhood of c in H is not contained in a unique z_A-sector). Since $G \in \mathcal{C}$, by Lemma 3.3, $(H, \{c, z_A\})$ is a cousin wheel. Since c has neighbors in the interiors of both P_A and P_B, this means that x or y is a common neighbor of c and z_A, a contradiction to C being an independent set. The proof for the latter statement (with b) is similar. This proves Claim 3.

Claim 4. *We may assume that x has neighbors in the interior of Q and y has no neighbor in the interior of Q.*

Proof of Claim 4. We show that if it is not the case, then there is a contradiction. For suppose both x and y have a neighbor in the interior of Q, then a path of minimal length from x to y with interior in the interior of Q form a theta together with P_A and P_B, a contradiction.

Now suppose that none of x and y has a neighbor in the interior of Q. Recall that Claim 2 tells us that $z_A \neq z_B$. So either $z \neq z_A$ or $z \neq z_B$. Up to symmetry, we may assume that $z \neq z_A$. Hence by Claim 3, either $a = a$ or $aa' \in E(G)$.

Suppose $a = a'$. This implies that a is in the interior of P_A. If $b = b'$, then b is in the interior of P_B — so H and Q form a theta from a to b; if $bb' \in E(G)$, then H and Q form a pyramid; and if $b \neq b'$ and $bb' \notin E(G)$, then $aP_AxP_Bbz_B$, $aP_AyP_Bb'z_B$, az_AQz_B form a theta from a to z_B (note that az_AQz_B has length at least 2 because $z_A \neq z_B$), a contradiction. So, $aa' \in E(G)$.

Suppose that $bb' \in E(G)$. Note that $|\{a, a'\} \cap \{b, b'\} \cap \{x, y\}| \neq 2$, because x and y are not adjacent. Moreover, $|\{a, a'\} \cap \{b, b'\} \cap \{x, y\}| \neq \emptyset$, for otherwise H and Q form a prism. So, $|\{a, a'\} \cap \{b, b'\} \cap \{x, y\}| = 1$. In this last case, we suppose up to symmetry that $x = a = b$. So, z is in the interior of Q since it is non-adjacent to x — in particular Q has length at least 2. Hence, H and Q form a butterfly (with $x = a = b$ being the center), a contradiction.

So, $bb' \notin E(G)$. If $b = b'$, then b is in the interior of P_B; thus P_A, P_B, and Q form a pyramid (i.e. $3PC(az_Aa', b)$, a contradiction. So, $b \neq b'$, and hence by Claim 3, $z_B = z$. This means that $b \neq x$ and $b' \neq y$ (because C is an independent set). Therefore, aP_AxP_Bbz, $a'P_AyP_Bb'z$, and z_AQz form a pyramid (i.e. $3PC(aa'z_A, z)$), a contradiction.

So, each case leads to a contradiction. Hence, exactly one of x or y has neighbors in the interior of Q, and up to symmetry we may assume it is x. This proves Claim 4.

Claim 5. *$a'x \in E(G)$ and $b'x \in E(G)$.*

Proof of Claim 5. First, suppose z_A is adjacent to x, i.e. $a = x$. Then, $z_A \neq z$ since C is an independent set. Note that $a = a'$ is impossible since z_A has neighbors in the interior of P_A. So, by Claim 3, $a'x \in E(G)$.

Now suppose z_A is not adjacent to x. By Claim 4, x has a neighbor in the interior of Q, so we choose such a neighbor x' closest to z_A along Q. Note that by the minimality of Q, no vertex in the interior of Q has neighbor in the interior of P_A and in the interior of P_B. Since y is not adjacent to x' (by Claim 4), x' has no neighbors in $(P_A \cup P_B) \setminus \{x\}$. We set $R = xx'Qz_A$ and observe that R has length at least 2. If $a \neq a'$ and $aa' \notin E(G)$, then xP_Aaz_A, $xP_ByP_Aa'z_A$, and R form a theta from x to z_A. If $aa' \in E(G)$, then P_A, P_B, and R form a pyramid. Therefore $a = a'$. Note that $xa \in E(G)$, for otherwise, P_A, P_B, and R form a theta. Hence, $a'x \in E(G)$.

The proof for $b'x \in E(G)$ is similar. This proves Claim 5.

To conclude the proof of 1, set $P'_A = zQz_Aa'P_Ay$ and $P'_B = zQz_Bb'P_By$. By Claim 5, x has neighbors in the interior of both P'_A and P'_B (these neighbors are a' and b'). Note that since $a'x, b'x \in E(G)$, the interiors of P_A and P_B are included in the interiors of P'_A and P'_B respectively. Hence, by Claim 1, every vertex of $C \setminus z$ has neighbors in the interior of both P'_A and P'_B.

Hence, the vertices z, y and the paths P'_A and P'_B show that 1 is satisfied.

Let us now prove 2. Note that $(H, C \setminus \{x, y\})$ is $(|C| - 2)$-wheel (this follows because G is theta-free, every vertex in $C \setminus \{x, y\}$ has at least three neighbors in H). It remains to prove that it is a $(|C| - 2)$-span-wheel. Note that it is clearly true if $|C| \leq 3$. We set $P_A = a_1 \dots a_\alpha$ and $P_B = b_1 \dots b_\beta$ with $x = a_1 = b_1$ and

$y = a_\alpha = b_\beta$, as in the definition of a k-span-wheel. We just have to exhibit an ordering of the vertices of $C \setminus \{x, y\}$ that satisfies the rest of the definition.

We first define v_1, the smallest vertex in the order we aim to construct. Note that no vertex $v \in C \setminus \{x, y\}$ is adjacent to x or y, because C is an independent set. We let v_1 be a vertex of C that is adjacent to a_i with i minimum. Let j be the smallest integer such that v_1 is adjacent to b_j. We suppose that v_1 is chosen subject to the minimality of j. Let i', j' be the greatest integers such that v_1 is adjacent to $a_{i'}$ and $b_{j'}$. Note that $1 < i \leq i' < \alpha$ and $1 < j \leq j' < \beta$.

Claim 6. *For every $w \in C \setminus \{x, y, v_1\}$, we have $N_H(w) \subseteq V(a_{i'} P_A y P_B b_{j'})$.*

Proof of Claim 6. We first note that the 2-wheel $(H, \{v_1, w\})$ is not a cousin wheel, because this may happen only when $x \in N(v_1)$ or $y \in N(v_1)$ (recall that if it was a cousin wheel, $N_H(v_1)$ would induce a 3-vertex path in H).

Hence, $(H, \{v_1, w\})$ is a nested wheel. Suppose that $N_H(w) \not\subseteq V(a_{i'} P_A a_\alpha) \cup V(b_{j'} P_B b_\beta)$. This means that w has a neighbor z in $a_{i'-1} P_A x P_B b_{j'-1}$. Since $(H, \{v_1, w\})$ is a nested wheel, $N_H(w)$ is contained in a v_1-sector Q of (H, v_1). Moreover, since w has a neighbor in the interior of both P_A and P_B, we have $Q = a_i P_A x P_B b_j$. Since H and w form a wheel, w has neighbor in the interior of Q. This contradicts the minimality of i or j. This proves Claim 6.

The order of $C \setminus \{x, y\}$ is now constructed as follows: we remove v_1 from C, define v_2 as we defined v_1 (minimizing i, and then minimizing j), then remove v_2, define v_3, and so on. This iteratively constructs an ordering of $C \setminus \{x, y\}$ showing that $(H, C \setminus \{x, y\})$ is a $(|C| - 2)$-span-wheel. □

For integers $t, k \geq 1$, the Ramsey number $R(t, k)$ is the smallest integer n such that any graph on n vertices contains either a clique of size t, or an independent set of size k.

Theorem 4.3. *An (l-span-wheel, K_t)-free graph $G \in \mathcal{C}$ has treewidth at most $(t-1)(R(t, l+2) - 1)^3 - 1$.*

Proof. Suppose for a contradiction that the treewidth of G is at least $(t-1)(R(t, l+2) - 1)^3$. Since G is K_t-free, by Theorem 2.1 G admits a minimal separator D of size at least $R(t, l+2)$. Let A and B be two connected components of $G \setminus D$ that are full to D. By the definition of Ramsey number, $G[D]$ contains an independent set C of size $l+2$. We define $G' = G[A \cup C \cup B]$, and observe that C is a minimal separator of G'. Hence by Lemma 4.2 applied to G', the graph contains an l-span-wheel, a contradiction. □

The following shows that in $\mathcal{C}$, an l-span-wheel with large l contains $S_{i,j,k}$ with large i, j, k.

Lemma 4.4. *If a butterfly-free graph G contains a $(4k+1)$-span-wheel with $k \geq 0$, then it contains $S_{k+1,k+1,k+1}$.*

Proof. Consider a $(4k+1)$-span-wheel in G, with x, y, P_A, and P_B be as in the definition of span-wheel given in the beginning of the current section. Let

$v_1, \dots, v_{4k+1}$ be the centers of the span wheel. For each $i = 1, \dots, 4k+1$, let a_i (resp. a'_i) be the neighbor of v_i in P_A closest to x (resp. to y) along P_A. Let b_i (resp. b'_i) be the neighbor of v_i in P_B closest to x (resp. to y) along P_B. We set $P_i = a_i P_A x P_B b_i$ and $Q_i = a'_i P_A y P_B b'_i$.

Claim 1. *P_i has length at least $i+1$ and Q_i has length at least $4k+3-i$.*

Proof of Claim 1. We prove this by induction on i for P_i. It is clear that P_1 has length at least 2 since x is not adjacent to v_1. Suppose the claim holds for some fixed $i \geq 1$, and let us prove it for $i+1$. From the induction hypothesis, P_i has length at least $i+1$, and since v_i has a neighbor in the interior of P_{i+1} (because it has at least three neighbors in H), the length of P_{i+1} is greater than the length of P_i, so P_{i+1} has length at least $i+2$.

The proof for Q_i is similar, except we start by proving that Q_{4k+1} has length at least 2, and that the induction goes backward down to Q_1. This proves Claim 1.

We set $l = 2k+1$. So, by Claim 1, P_l and Q_l both have length at least $2k+2$. We set $v = v_l$, $P = P_l$, $Q = Q_l$, $a = a_l$, $a' = a'_l$, $b = b_l$ and $b' = b'_l$. Since G is butterfly-free, we do not have $aa' \in E(G)$ and $bb' \in E(G)$ simultaneously. So, up to symmetry we may assume that either $a = a'$; or $a \neq a'$ and $aa' \notin E(G)$.

If $a = a'$, let u, u', and u'' be three distinct vertices in P such that a, u, u', and u'' appear in this order along P, aPu has length $k+1$ and bPu'' has length $k-1$ (which is possible because P has length at least $2k+1$). Let w be in Q and such that aQw has length $k+1$ (which is possible because Q has length at least $2k+1$). The three paths aPu, $avbPu''$, and aQw form an $S_{k+1,k+1,k+1}$.

If $a \neq a'$ and $aa' \notin E(G)$, then let u, u', and u'' be three distinct vertices in P such that a, u, u', and u'' appear in this order along P, aPu has length k and bPu'' has length k. Let w be in Q and such that $a'Qw$ has length k. The three paths $vaPu$, $vbPu''$ and $va'Qw$ form an $S_{k+1,k+1,k+1}$. □

The following is a classical result on treewidth and we omit its proof.

Lemma 4.5 ([21])**.** *The treewidth of a graph G is the maximum treewidth of an induced subgraph of G that has no clique separator.*

Theorem 4.6. *For $k \geq 1$, a (theta, triangle, $S_{k,k,k}$)-free graph G has treewidth at most $2(R(3, 4k-1))^3 - 1$.*

Proof. By Lemma 4.5, it is enough to consider a graph G that does not have a clique separator. If G contains a cube, then Lemma 3.2 tells us that G itself is the cube. By classical results on treewidth, the treewidth of the cube is 3 (but the trivial bound 8 would be enough for our purpose), which in particular achieves the given bound. We may therefore assume that G is cube-free. Moreover, by Lemma 4.1, G is in $\mathcal{C}$. Since G is $S_{k,k,k}$-free, by Lemma 4.4, G contains no $(4k-3)$-span-wheel. Moreover, G contains no K_3 by assumption. Hence, by Theorem 4.3, G has treewidth at most $2(R(3, 4k-1))^3 - 1$. □

Theorem 4.7. *For $k \geq 1$, an (even hole, pyramid, K_t, $S_{k,k,k}$)-free graph G has treewidth at most* $(t-1)(R(t, 4k-1))^3 - 1$.

Proof. Since all thetas, prisms, even wheels, and squares contain even holes, G is (theta, prism, pyramid, even wheel, square)-free. So, by Lemma 4.1, G is in $\mathcal{C}$. Since G is $S_{k,k,k}$-free, by Lemma 4.4, G contains no $(4k-3)$-span-wheel. Moreover, G contains no K_t by assumption. Hence, by Theorem 4.3, G has treewidth at most $(t-1)(R(t, 4k-1))^3 - 1$. $\square$

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