

# Linear Algebra

[KOMS119602] - 2022/2023

## 6.1 - Inverses of matrices

Dewi Sintiar

Computer Science Study Program  
Universitas Pendidikan Ganesha

Week 6 (October 2022)

# Learning objectives

After this lecture, you should be able to:

1. investigate if a matrix inverse exists;
2. compute the inverse of a *small size* matrix (if exists);
3. compute the inverse of an  $n \times n$  matrix (if exists);
4. explain the concepts of *minor*, *cofactor*, *adjoint*;
5. explain the properties of matrix inverse;
6. analyze if a matrix is orthogonal;
7. analyze if a set of vectors is orthonormal.

# Part 1: Inverse of matrices

# Inverse

A square matrix  $A$  is said to be **invertible** or **nonsingular** if  $\exists B$  s.t.:

$$AB = BA = I \quad \text{where } I \text{ is the identity matrix}$$

**Note:** Such a matrix  $B$  is **unique**, and it is called the **inverse** of  $A$ , and is denoted by  $A^{-1}$ . The relation of  $A$  and  $B$  is symmetric:

*If  $B$  is the inverse of  $A$ , then  $A$  is the inverse of  $B$ , i.e.*

$$(A^{-1})^{-1} = A$$

---

## Example

Let  $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$  Then

$$AB = \begin{bmatrix} 6 - 5 & -10 + 10 \\ 3 - 3 & -5 + 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Why do we need to find **inverse** of a matrix?

1. 'Primarily, "division" does not exist for matrices, instead, we do "inverse".

*Given a matrix  $A$  and  $B$  such that  $B = AX$ .*

*How do we find  $X$ ?  $\Rightarrow X = BA^{-1}$*

2. Applications:

- solving a system of linear equations;
- used to encrypt/decrypt message codes;
- etc.

## How to compute the inverse of $2 \times 2$ matrices?

*Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , what is  $A^{-1}$ ?*

Let  $A^{-1} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ . We have:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**We solve the linear system:**

$$\begin{cases} ax_1 + by_1 = 1 \\ cx_1 + dy_1 = 0 \end{cases} \quad \text{and} \quad \begin{cases} ax_2 + by_2 = 0 \\ cx_2 + dy_2 = 1 \end{cases}$$

## Inverse of $2 \times 2$ matrices

It gives:

$$x_1 = \frac{d}{ad - bc}, \quad y_1 = \frac{-c}{ad - bc}, \quad x_2 = \frac{-b}{ad - bc}, \quad y_2 = \frac{a}{ad - bc}$$

Note that  $ad - bc = |A|$  (the *determinant* of  $A$ ).

When  $|A| \neq 0$ , the values  $x_1$ ,  $y_1$ ,  $x_2$ , and  $y_2$  exist.

Hence,

$$A^{-1} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} d/|A| & -b/|A| \\ -c/|A| & a/|A| \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

# Inverse of $2 \times 2$ matrices

## Conclusion:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

When  $|A| \neq 0$ , the inverse of a  $2 \times 2$  matrix  $A$  may be obtained from  $A$  as follows:

1. Interchange the two elements on the diagonal ( $a$  and  $d$ );
2. Take the negatives of the other two elements ( $b$  and  $c$ );
3. Multiply the resulting matrix by  $\frac{1}{|A|}$  or, equivalently, divide each element by  $|A|$ .

**Note:** If  $|A| = 0$ , then  $A$  is not invertible.



## Example

Find the inverse of  $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$$|A| = 2(5) - 3(4) = 10 - 12 = -2$$

Since  $|A| \neq 0$ , then  $A$  is invertible.

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

Furthermore,  $|B| = 1(6) - 3(2) = 0$ , so  $B$  is not invertible.

## Part 2: Computing inverse from adjoint

## Inverse of $n \times n$ matrices

### Note:

If  $A$  is an  $n \times n$  matrices,  $A^{-1}$  can be obtained as above, by finding the solution of the  $n \times n$  linear system equations.

This is not so practical to be solved using the substitution/elimination method. A method will be discussed later.

## Review on minors and cofactors

Let  $A = [a_{ij}]$  be an  $n$ -square matrix.

Define  $M_{ij}$  as the  $(n - 1)$ -square matrix obtained from  $A$  by deleting the  $i$ -th row and the  $j$ -th column of  $A$ .

The **minor of the element  $a_{ij}$  of  $A$**  is defined as:

$$\text{minor}(A) = \det(M_{ij})$$

The **cofactor of  $a_{ij}$**  is defined as the **signed minor** of  $a_{ij}$ , and denoted by:

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

# Adjoint

We can form a **matrix of cofactors**

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

where  $C_{ij}$  is the cofactor of  $a_{ij}$ .

The **adjoint of matrix  $A$**  is defined as:

$$\text{adj}(A) = C^T$$

## Example of adjoint

Given matrix:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

The cofactors of  $A$  are:

- $C_{11} = 12$
- $C_{12} = 6$
- $C_{13} = -16$
- $C_{21} = 4$
- $C_{22} = 2$
- $C_{23} = 16$
- $C_{31} = 12$
- $C_{32} = -10$
- $C_{33} = 16$

The matrix of cofactors and the adjoint of  $A$  are:

$$C = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

# matrix inverse from adjoint

## Theorem

Let  $A$  be an **invertible** matrix. Then:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

*Proof can be read on the Howard Anton book, page 134.*

## Example

From the *previous example*, we have:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix} \qquad \text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$\det(A) = 0 + 12 + 4 - (-12 - 36 + 0) = 16 - (-48) = 64$$

Hence,

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & \frac{-10}{64} \\ \frac{-16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$



# Part 3: Properties of matrix inverse

# Basic properties of matrix inverse

Let  $A$  be an **invertible** matrix. The followings hold.

1.  $(A^{-1})^{-1} = A$
2.  $(kA)^{-1} = k^{-1}A^{-1}$  for a scalar  $k \neq 0 \in \mathbb{R}$
3.  $(A^T)^{-1} = (A^{-1})^T$
4.  $\det(A^{-1}) = (\det(A))^{-1}$

## Exercises:

*Prove the properties of matrix inverse.*

*Give an example for each property to check the correctness of the theorem.*

# Basic properties of matrix inverse

## Theorem

*If  $A$  and  $B$  are invertible, then  $AB$  is invertible.*

## Proof.

Consider  $B^{-1}A^{-1}$ . Then:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Hence,  $AB$  is invertible, and  $(AB)^{-1} = B^{-1}A^{-1}$ . □

## Generalization:

If  $A_1, A_2, \dots, A_k$  are invertible matrices, then:

$$(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$$

# Exercise

*will be given during the lecture*

Numbers 4, 5, 6, page 76 Howard Anton Reference Book

# Part 4: Orthogonal matrices

# Orthogonal matrices

A matrix is called **orthogonal** if  $A^T = A^{-1}$ , i.e.,  $AA^T = A^T A = I$  (the identity matrix).

**Note:**  $A$  is orthogonal *only if*  $A$  is square and invertible matrix.

## Example

$$\text{Let } A = \begin{bmatrix} \frac{1}{9} & \frac{8}{9} & -\frac{4}{9} \\ \frac{4}{9} & -\frac{4}{9} & -\frac{7}{9} \\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9} \end{bmatrix}$$

Is  $A$  orthogonal? What is the result of  $AA^T$ ?

# Orthonormality

Vectors  $u_1, u_2, \dots, u_m$  in  $\mathbb{R}^n$  are said to form an **orthonormal** set of vectors if the vectors are unit vectors and are orthogonal to each other; i.e.,

$$u_i \cdot u_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

## Theorem

*Let  $A$  be a real matrix. Then the following are equivalent:*

- *$A$  is orthogonal.*
- *The rows of  $A$  form an orthonormal set.*
- *The columns of  $A$  form an orthonormal set.*

*to be continued...*