Linear Algebra

[KOMS120301] - 2023/2024

15.1 - Diagonalization

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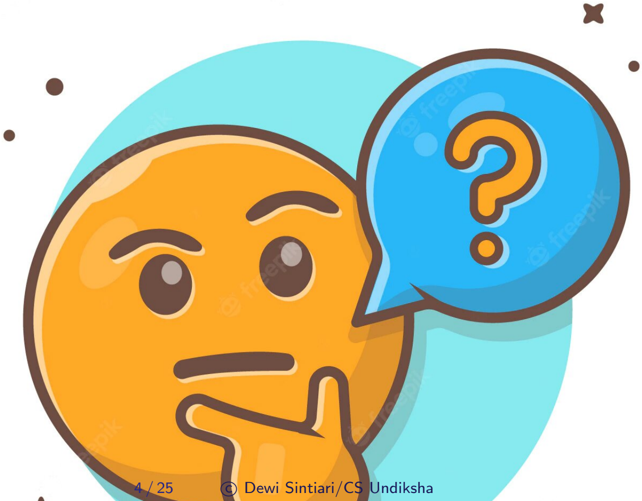
Learning objectives

After this lecture, you should be able to:

1. explain the concept of diagonalization on square matrix, and why diagonalization is useful in Linear Algebra;
2. analyze the characteristic of matrix that is diagonalizable;
3. perform diagonalization on square matrix (if possible).

Part 1: Diagonalization

Can you recall the definition of **diagonal matrix**?



Definition of diagonalization

Matrix diagonalization is the process of taking a square matrix and converting it into a diagonal matrix that shares the same fundamental properties of the underlying matrix.

Definition

Let A and P be an $n \times n$ matrix, such that P is invertible. Diagonalization of A is a process of transforming:

$$A \to P^{-1}AP$$

A square matrix A is said to be diagonalizable if there exists an invertible matrix P s.t. $P^{-1}AP$ is a diagonal matrix. In this case, the matrix P is said to diagonalize A.

Motivation of the usefulness of diagonalization (1)

Why do we need such a basis?

$\rightarrow$ Roughly speaking, if we have the diagonal form, **many properties** can be studied more easily.

We will see later what properties of a matrix that are preserved by diagonalization.

Definition

A similarity invariant is any property that is preserved by a similarity transformation.

Motivation of the usefulness of diagonalization (2)

Example (Determinant is a similarity invariant)

Matrix A and $P^{-1}AP$ satisfy:

$$\det(A) = \det(P^{-1}AP)$$

Proof:

$$\begin{aligned}\det(P^{-1}AP) &= \det(P^{-1})\det(A)\det(P) \\ &= \frac{1}{\det(P)}\det(A)\det(P) \\ &= \det(A)\end{aligned}$$

Can you propose another property that is a similarity invariant?

Try to check the following properties:

- size of matrix
- invers
- rank
- nullity
- trace
- characteristic polynomial
- eigenvalues

Similarity invariant

Table 1. Similarity invariant

Fig/similarity.png

Motivating question

How to find a basis for $\mathbb{R}^n$ consisting of eigenvectors of a matrix A of size $n \times n$?

Similar matrices

Let A and B be square matrices. Then we say that A similar to B if there is an invertible matrix P s.t. $B = P^{-1}AP$.

Lemma

If A similar to B, then B is similar to A.

Proof:

Since $B = P^{-1}AP$, then $PBP^{-1} = A$.

Define $Q = P^{-1}$. Then Q is a diagonal matrix, and:

$$Q^{-1}BQ = PBP^{-1} = A$$

Determining if a matrix is diagonalizable & finding a matrix P that performs the diagonalization

Theorem (1)

If A is an $n \times n$ matrix, the following statements are equivalent.

1. *A is diagonalizable.*
2. *A has n linearly independent eigenvectors.*

Theorem (2)

1. *If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a matrix A, and if $v_1, v_2, \ldots, v_k$ are corresponding eigenvectors, then $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set.*
2. *An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.*

What do Theorems 1 & 2 say about matrices that are diagonalizable, and the matrix that performs diagonalization?

- Theorem 1 $\rightarrow$ need to find n linearly independent eigenvectors to diagonalize a matrix A.
- Theorem 2 $\rightarrow$ such vectors might be the egienvectors of A (if there are n different eigenvectors).

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- Theorem 1 $\rightarrow$ need to find n linearly independent eigenvectors to diagonalize a matrix A.
- Theorem 2 $\rightarrow$ such vectors might be the egienvectors of A (if there are n different eigenvectors).

$\Rightarrow$ An $(n \times n)$ matrix A is **diagonalizable** if A has n different eigenvalues.

$\Rightarrow$ Now, how to diagonalize A?

An algorithm to diagonalize a matrix

A Procedure for Diagonalizing an $n \times n$ Matrix

Step 1. Determine first whether the matrix is actually diagonalizable by searching for n linearly independent eigenvectors. One way to do this is to find a basis for each eigenspace and count the total number of vectors obtained. If there is a total of n vectors, then the matrix is diagonalizable, and if the total is less than n, then it is not.

Step 2. If you ascertained that the matrix is diagonalizable, then form the matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$ whose column vectors are the n basis vectors you obtained in Step 1.

Step 3. $P^{-1}AP$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ that correspond to the successive columns of P.

Example 1: Finding matrix P that diagonalizes matrix A

We want to find a matrix P that diagonalizes matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution:

1. Since A is of size 3×3, first check whether A has 3 different eigenvalues.
2. If yes, find the bases $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for the eigenspace of A.
3. Create matrix $P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3]$.
4. Check that $P^{-1}AP = D$ where D is a diagonal matrix with diagonal entries are eigenvalues of A.

Example 1 (*cont.*)

1. You should obtain the following characteristic equation of A:

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

2. Find the bases of the eigenspace:

$$\lambda = 2 \rightarrow \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_2 \rightarrow \mathbf{p}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

3. The matrix that diagonalizes A is

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

4. We verify that:

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

Example 2: A matrix that is not diagonalizable

Show that the matrix: $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalizable.

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Show that the matrix: $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalizable.

Solution:

The characteristic polynomial of A is:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2$$

The distinct eigenvalues are: $\lambda_1 = 1$ and $\lambda_2 = 2$.

We will find the bases for the eigenspace of A.

Example 2 (*cont.*)

For $\lambda = 1$

Solve:

$$(\lambda I - A)\mathbf{x} = \mathbf{0} \Leftrightarrow \begin{bmatrix} 1-1 & 0 & 0 \\ -1 & 1-2 & 0 \\ 3 & -5 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 3 & -5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can derive linear systems:

$$\begin{cases} -x_1 - x_2 = 0 \\ 3x_1 - 5x_2 - x_3 = 0 \end{cases}$$

This gives: $x_1 = t,\ x_2 = -t,\ x_3 = 8t$, or base: $\begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$.

Example 2 (*cont.*)

For $\lambda = 2$

Solve:

$$(\lambda I - A)\mathbf{x} = \mathbf{0} \iff \begin{bmatrix} 2-1 & 0 & 0 \\ -1 & 2-2 & 0 \\ 3 & -5 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & -5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can derive linear systems:

$$\begin{cases} x_1 = 0 \\ -x_1 = 0 \\ 3x_1 - 5x_2 = 0 \end{cases}$$

This gives: $x_1 = 0,\ x_2 = 0,\ x_3 = t$ with $t \in \mathbb{R} \setminus \{0\}$, or base: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Example 2 (*cont.*)

Hence, the bases of eigenspace of matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since the size of matrix A is 3×3, and there are only two basis vectors, then A is not diagonalizable.

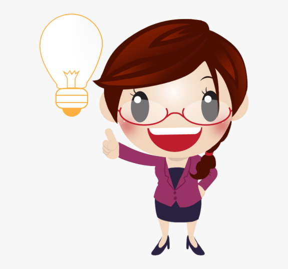
Exercises

Are the following matrices diagonalizable?

1. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

2. The triangular matrix: $B = \begin{bmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

So...what can you conclude of eigenvectors and eigenvalues?



Eigenvectors represent...

Eigenvalues represent...

Part 2: Applications of eigenvector

Applications of eigenvector

- https://www.quora.com/Why-are-eigenvectors-and-eigenvalues-important
- https://vitalflux.com/why-when-use-eigenvalue-eigenvector/