

# Linear Algebra

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## 3.2 - Algorithm for Linear System

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# Learning objectives

After this lecture, you should be able to:

1. apply elimination algorithm and substitution algorithm to solve linear system in two variables;
2. understand the characteristics of linear system which is in triangular form, row echelon form, or reduced row echelon form.
3. detect if a linear system has a unique solution, has no solution, or has an infinitely many solution.

# Part 1: Algorithms to solve a system of linear system in **two variables**

# 1. Elimination algorithm (1)

Given:

$$\begin{cases} L_1 : x - y = -4 \\ L_2 : 3x + 2y = 12 \end{cases}$$

Solve the system!

- Multiply the first equation by 2.

$$\begin{cases} 2L_1 : 2x - 2y = -8 \\ L_2 : 3x + 2y = 12 \end{cases}$$

- Eliminate the variable  $y$ , by adding the two equations.

$$2L_1 + L_2 : 5x = 4 \Leftrightarrow x = \frac{4}{5}$$

- Substitute  $x = \frac{4}{5}$  back to  $L_1$  or  $L_2$  to find  $y$ .

$$x - y = -4 \Leftrightarrow y = x + 4 = \frac{4}{5} + 4 = \frac{24}{5}$$

# 1. Elimination algorithm (2)

Assume that the given system has a unique solution.

**Input:** Non-degenerate linear equations  $L_1$  and  $L_2$  in two variables.

## **Step 1:** *Forward elimination*

- Multiply each equation by a constant s.t. the resulting coefficients of one variable are equal (or negative of the other).
- Subtract (or add) the two equations to eliminate one of the variables.

## **Step 2:** *Back substitution*

- Substitute the value of the variable to an equation of linear system, to obtain the value of the other variable.

## 2. Substitution algorithm

Given:

$$\begin{cases} L_1 : x - y = -4 \\ L_2 : 3x + 2y = 12 \end{cases}$$

Solve the system!

- Represent  $x$  in  $y$ , in the equation  $L_1$ .

$$x = y - 4 \tag{1}$$

- Substitute the value of  $x$  in equation (1) to  $L_2$

$$3(y - 4) + 2y = 12 \Leftrightarrow 5y = 24 \Leftrightarrow y = \frac{24}{5}$$

- Substitute  $y = \frac{24}{5}$  to equation (1)

$$x = \frac{24}{5} - 4 = \frac{4}{5}$$

## 2. Substitution algorithm

**Input:** Non-degenerate linear equations  $L_1$  and  $L_2$

For simplification, suppose that the variables are  $x$  and  $y$ .

### Step 1:

- Represent one variable, say  $x$ , as an equation in  $y$  in equation  $L_1$ . Then substitute the value of  $x$  in  $L_1$  to  $L_2$ , to obtain the value of  $y$ .

### Step 2:

- Substitute the value of the variable  $y$  to equation  $L_1$  or  $L_2$ , to obtain the value of variable  $x$ .

## Exercise

Solve the following linear systems using the elimination and substitution algorithms.

1. Solve:

$$\begin{cases} L_1 : x - 3y = 4 \\ L_2 : -2x + 6y = 5 \end{cases}$$

2. Solve:

$$\begin{cases} L_1 : x - 3y = 4 \\ L_2 : -2x + 6y = -8 \end{cases}$$



## Exercise solution (1)

In Exercise 1, we can simplify the second equation, and obtain:

$$\begin{cases} L_1 : x - 3y = 4 \\ L_2 : x - 3y = 5 \end{cases}$$

This yields  $4 = 5$ , which is wrong. So, there is no values of  $x$  and  $y$  that satisfy the system.

## Exercise solution (2)

In Exercise 2, we can simplify the second equation, and obtain:

$$\begin{cases} L_1 : x - 3y = 4 \\ L_2 : x - 3y = 4 \end{cases}$$

The two linear equations are equivalent, which means that the lines that represent them **coincide** (intersect at all points on the coordinate system), and all points on the line satisfy both equations.

**How to represent the set of solutions?**

$$x - 3y = 4$$

Let  $y = t$  for  $t \in \mathbb{R}$ . Then  $x = 3y + 4 = 3t + 4$ .

So the set of solutions is  $\{x = 3t + 4, y = t, \text{ where } t \in \mathbb{R}\}$

# Part 2: Linear systems in triangular and echelon forms

# Triangular form

The following system is said to be in **triangular form**.

$$\begin{cases} 2x_1 - 3x_2 + 5x_3 - 2x_4 = 9 \\ \quad 5x_2 - x_3 + 3x_4 = 1 \\ \qquad 7x_3 - x_4 = 3 \\ \qquad\qquad 2x_4 = 8 \end{cases} \quad (1)$$

Recall that a **triangular matrix** has one of the following shapes:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

What can you observe?

# Triangular form

A system of linear equations is in triangular form if [the corresponding coefficient matrix is an upper triangular matrix or a lower triangular matrix](#), i.e.:

1. The matrix is a square matrix;
2. The entries below the main diagonal (resp. above the main diagonal, for the upper triangular matrix) are 0;

## Remark:

- We do not care of the values in the main diagonals (they can be 0)

## Solving system in (*upper*) triangular form

$$\begin{cases} 2x_1 - 3x_2 + 5x_3 - 2x_4 = 9 \\ \quad 5x_2 - x_3 + 3x_4 = 1 \\ \quad \quad 7x_3 - x_4 = 3 \\ \quad \quad \quad 2x_4 = 8 \end{cases} \quad (2)$$

### Algorithm to solve the system:

1. Solve the last equation to get  $x_4$ ;
2. Substitute  $x_4$  to the third equation to obtain  $x_3$ ;
3. Substitute  $x_3$  and  $x_4$  to the second equation to obtain  $x_2$ ;
4. Substitute  $x_2$ ,  $x_3$ , and  $x_4$  to the first equation to obtain  $x_1$ .

**Exercise:** Find the solution of the system!

## Solution of the exercise

- From the last equation, we get:  $x_4 = 4$
- From the third equation:

$$x_3 = \frac{x_4 + 3}{7} = \frac{4 + 3}{7} = 1$$

- From the second equation:

$$x_2 = \frac{x_3 - 3x_4 + 1}{5} = \frac{1 - 3(4) + 1}{5} = \frac{-10}{5} = -2$$

- From the first equation:

$$\begin{aligned} x_1 &= \frac{3x_2 - 5x_3 + 2x_4 + 9}{2} = \frac{3(-2) - 5(1) + 2(4) + 9}{2} \\ &= \frac{-6 - 5 + 8 + 9}{2} = \frac{6}{2} = 3 \end{aligned}$$

So, the solution is:  $x_1 = 3, x_2 = -2, x_3 = 1, x_4 = 4$

## Echelon form

Now, what if the coefficient matrix is not a square matrix ???





## Echelon form

$$\begin{cases} 2x_1 - 4x_2 + 5x_3 - 2x_4 + x_5 = 9 \\ \quad 2x_2 - 2x_3 + 4x_4 - 2x_5 = 1 \\ \quad \quad 1x_3 - x_4 = 3 \end{cases}$$

The system is said to be in **echelon form**, that is:

1. All rows consisting of **only zeroes** are at the **bottom**.
2. The leading coefficient (also called the pivot) of a nonzero row is always **strictly to the right** of the leading coefficient of the row above it.

Characteristics:

- The leading variables ( $x_1, x_2, x_3$ ) in the system are called **pivot**;
- The other variables ( $x_4$  and  $x_5$ ) are **free** variables.

## Echelon form (general form)

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \cdots + a_{1n}x_n = b_1$$

$$a_{2j_2}x_{j_2} + a_{2j_2+1}x_{j_2+1} + \cdots + a_{2n}x_n = b_2$$

.....

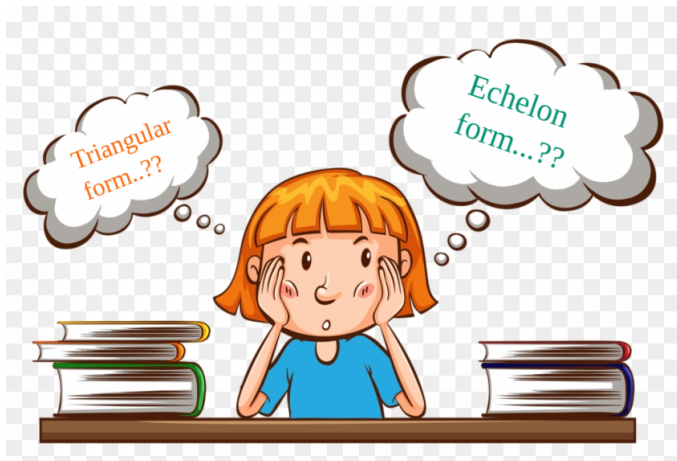
$$a_{rj_r}x_{j_r} + \cdots + a_{rn}x_n = b_r$$

where  $1 < j_2 < \cdots < j_r$  and  $a_{11}, a_{2j_2}, \dots, a_{rj_r} \neq 0$ .

The **pivot** variables are:  $x_1, x_{j_2}, \dots, x_{j_r}$

**Remark:** in order for the system to have a solution, it must be  $r \leq n$ .

Then...is there any difference between  
triangular form and echelon form?



## Clarification

The following matrix **is echelon, but not triangular**

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

The following matrix **is triangular, but not echelon**

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The following matrix **is echelon and triangular** (LEFT), and **not echelon and not triangular** (RIGHT)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

**Remark.** For non-singular square matrices, “row echelon” and “upper triangular” are equivalent.

# Part 3: How to determine the number of solutions?

# How to know the number of solutions?

Given a system of linear equations with  $r$  equations in  $n$  variables.

Determine the conditions such that:

- the system has a **unique solution**?
- the system has **no solution**?
- the system has an **infinite number of solutions**?



# How to know the number of solutions?

Given a system of linear equations with  $r$  equations in  $n$  variables.

Then:

- the system has a **unique solution**
  - when  $r = n$  (where no equation is a linear combination of another)
- the system has **no solution**
  - when  $r > n$ , and no equation is a linear combination of another
- the system has an **infinite number of solutions**
  - when  $r < n$

## How to write the solutions when there are infinitely many? (case when $r < n$ )

Given:

$$\begin{cases} x_1 - 4x_2 + 5x_3 - 2x_4 + x_5 = 9 \\ x_2 - 2x_3 + 4x_4 - 2x_5 = 1 \\ x_3 - x_4 = 3 \end{cases}$$

- Pivot variables:  $x_1, x_2, x_3$
- Free variables:  $x_4, x_5$

### Algorithm to solve the system:

1. Assign *parameters* to the free variables;

$$x_4 = a \quad \text{and} \quad x_5 = b$$

2. Substitute the variables back to obtain the value of pivot variables.



# 1. Solution in parametric form

- From the third equation:

$$x_3 = x_4 + 3 = a + 3$$

- From the second equation:

$$\begin{aligned}x_2 &= 2x_3 - 3x_4 + 2x_5 + 1 \\&= 2(a + 3) - 4a + 2b + 1 = -2a + 2b + 7\end{aligned}$$

- From the first equation:

$$\begin{aligned}x_1 &= 3x_2 - 5x_3 + 2x_4 - x_5 + 9 \\&= 3(-2a + 2b + 7) - 5(a + 3) + 2a - b + 9 \\&= -9a + 5b + 15\end{aligned}$$

**Set of solutions:**

$$\{-9a + 5b + 15, -2a + 2b + 7, a + 3, a, b\}$$

## 2. Solution in free-variable form

Use back-substitution to solve the system, and obtain the pivot variables.

$$\begin{cases} x_1 &= 4x_2 - 5x_3 + 2x_4 - x_5 - 9 \\ x_2 &= 2x_3 - 4x_4 + 2x_5 + 1 \\ x_3 &= x_4 + 3 \\ x_4 &= \text{free variable} \\ x_5 &= \text{free variable} \end{cases}$$

**Set of solutions:**

$$\{(4x_2 - 5x_3 + 2x_4 - x_5 - 9), (2x_3 - 4x_4 + 2x_5 + 1), (x_4 + 3), x_4, x_5\}$$

# Part 4: The reduced row echelon form

# The reduced row echelon form

A matrix is in **reduced row echelon form** (also called **row canonical form**), if it satisfies the following conditions are satisfied:

1. It is in **row echelon form**.
2. The leading entry in each nonzero row is a 1 (called a leading 1).
3. Each **column** containing a leading 1 has zeros in **all** its other entries.

## Which matrix is in a reduced row echelon form?

- $A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

- $B = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

- $C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

# How to transform the coefficient matrix into a triangular or (reduced) row echelon form?

Apply the **elementary row operations**.

*In the next lecture, we will learn*

*how to solve a linear system by transforming the coefficient matrix to a reduced row echelon form.*

*to be continued...*