

Linear Algebra

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10.2 - Basis and Dimension

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Basis of vector space

Intuitive example

In $\mathbb{R}^3 \rightarrow$ Let $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$

Every vector $\mathbf{v} = (a, b, c) \in \mathbb{R}^3$ can be expressed as a linear combination of the vector basis, namely:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

In $\mathbb{R}^n \rightarrow$ This can be generalized for the Euclidean vector space \mathbb{R}^n

Let: $\mathbf{e}_1 = (1, 0, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)$, $\mathbf{e}_3 = (0, 0, 0, \dots, 1)$

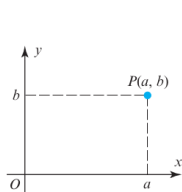
Every vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ can be expressed as:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \dots + v_n\mathbf{e}_n$$

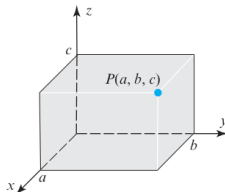
Can a vector space have more than one basis?

What about the basis of general vector space V ?

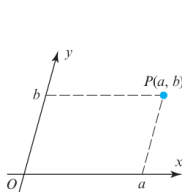
Rectangular and non-rectangular linear system



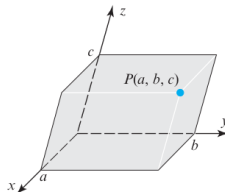
Coordinates of P in a rectangular coordinate system in 2-space.



Coordinates of P in a rectangular coordinate system in 3-space.



Coordinates of P in a nonrectangular coordinate system in 2-space.



Coordinates of P in a nonrectangular coordinate system in 3-space.

*In linear algebra, coordinate systems are commonly specified using **vectors** rather than coordinate axes.*

Formal definition of basis

If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a set of vectors in V , then S is called a **basis** for V if the following two conditions hold:

1. S is linearly independent;
2. S spans V .

Example 1: standard basis for \mathbb{R}^n

The standard basis for \mathbb{R}^n is the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, where:

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, 0, 0, \dots, 1)$$

This means that: $\forall \mathbf{v} \in V$, then $\exists k_1, k_2, \dots, k_n \in \mathbb{R}$, s.t.:

$$\mathbf{v} = k_1 \mathbf{e}_1 + k_2 \mathbf{e}_2 + \dots + k_n \mathbf{e}_n$$

Example (specific case, in \mathbb{R}^3)

In \mathbb{R}^3 , we have the standard basis:

$$\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$$

Example 2: standard basis for P_n

Show that the set $S = \{1, x, x^2, \dots, x^n\}$ is a standard basis for vector space P_n of polynomials.

Solution:

By the theorem, it should be showed that the polynomials in S are linearly independent, and span P_n .

Denote the polynomials by vectors:

$$\mathbf{p}_0 = 1, \mathbf{p}_1 = x, \mathbf{p}_2 = x^2, \dots, \mathbf{p}_n = x^n$$

We showed (in the previous discussion) that the vectors span P_n , and they are linearly independent.

Example 2: another basis for \mathbb{R}^3

Show that the vectors:

$$\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 9, 0), \text{ and } \mathbf{v}_3 = (3, 3, 4)$$

form a basis for \mathbb{R}^3 .

Solution:

It must be showed that the vectors are **linearly independent** and **span** \mathbb{R}^3 .

- **Linear independence**: the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_3\mathbf{v}_3 = \mathbf{0}$$

has only the trivial solution.

- **Span the vector space \mathbb{R}^3** : every vector $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ can be expressed as:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_3\mathbf{v}_3 = \mathbf{b}$$

Example 2 (*cont.*)

The vector equations can be expressed as linear systems:

$$\left\{ \begin{array}{l} c_1 + 2c_2 + 3c_3 = 0 \\ 2c_1 + 9c_2 + 3c_3 = 0 \\ c_1 \quad \quad + 4c_3 = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} c_1 + 2c_2 + 3c_3 = b_1 \\ 2c_1 + 9c_2 + 3c_3 = b_2 \\ c_1 \quad \quad + 4c_3 = b_3 \end{array} \right.$$

To show that the homogeneous linear system (*left*) has only trivial solution and the system (*right*) has a unique solution, is equivalent to showing that the coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

has nonzero determinant.

Task: Prove that $\det(A) \neq 0$.

Uniqueness of basis representation

Theorem (Uniqueness)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis of a vector space V , then every vector \mathbf{v} in V can be expressed in the following form, in exactly one way.

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

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$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$$

Proof.

Suppose that \mathbf{v} can be expressed in another linear combination, say:

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n$$

Subtracting two equations gives:

$$\mathbf{v} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \dots + (c_n - k_n)\mathbf{v}_n$$

Since vectors in S are linearly independent, then:

$$c_1 - k_1 = 0, \quad c_2 - k_2 = 0, \quad \dots, \quad c_n - k_n = 0$$

meaning that: $c_1 = k_1, \quad c_2 = k_2, \quad \dots, \quad c_n = k_n$

Dimension

The number of vectors in a basis

A vector space may have more than one basis which are of the same size.

Theorem (Size of Basis)

All bases for a finite-dimensional vector space have the same number of vectors.

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Theorem

Let V be an n -dimensional vector space, and let $S = \{v_1, v_2, \dots, v_n\}$ be any basis.

- 1. If a set in V has more than n vectors, then it is linearly dependent.*
- 2. If a set in V has fewer than n vectors, then it does not span V .*

Proof.

The statements follow because the vectors in S are linearly independent.



Dimension

The **dimension** of a finite-dimensional vector space V is defined to be the number of vectors in a basis for V .

The zero vector space is defined to have dimension zero.

Example (Dimensions of some familiar vector spaces)

$$\dim(\mathbb{R}^n) = n \quad [\text{the standard basis has } n \text{ vectors}]$$

$$\dim(P_n) = n + 1 \quad [\text{the standard basis has } n + 1 \text{ vectors}]$$

$$\dim(M_{mn}) = mn \quad [\text{the standard basis has } mn \text{ vectors}]$$

Task: *What is the standard basis for each of the vector space?*

Example 1: dimension of $\text{span}(S)$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be the set of linearly independent vectors.

Prove that $\dim(\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}) = n$.

Solution:

Every vector in $\text{span}(S)$ can be expressed as a linear combination of the vectors in S .

Hence, S is the basis of $\text{span}(S)$.

By the “Size of Basis” theorem,

$$\dim(\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}) = n$$

Example 2: dimension of a solution space

Given the following linear system:

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0 \\ 5x_3 + 10x_4 + 15x_5 = 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0 \end{cases}$$

Find the dimension of the solution space of the linear system.

Solution:

- Find the solution of the system:

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

- In vector form:

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5, x_6) &= (-3r - 4s - 2t, r, -2s, s, t, 0) \\ &= r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0) \end{aligned}$$

Example 2 (*cont.*)

- So the following vectors span the vector space:

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

- Check that the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. It should be showed that the vector equation:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

has only trivial solution, i.e. $c_1 = 0, c_2 = 0, c_3 = 0$.

Verify it!

- If it is, then S is a basis of the solution space, and $\dim(S) = 3$.

Dimension of subspace

Theorem

If W is a subspace of a finite-dimensional vector space V , then:

1. *W is finite-dimensional;*
2. $\dim(W) \leq \dim(V)$;
3. *$W = V$ if and only if $\dim(W) = \dim(V)$.*

Proof.

See page 225 of “Elementary Linear Algebra Applications Version (Howard Anton, Chris Rorres - Edisi 1 - 2013)”. □

to be continued...