Midterm Exam

Only one sheet of paper is authorized. All other documents are forbidden.
The exercises are not necessarily of increasing difficulty and -unless otherwise specified- questions can be treated even if the previous ones have not been treated.

Exercise 1 - False proof [1.5 pt]
Let (P) be a maximization LP and (D) its dual. We denote by z_p and z_d the optimal values of (P) and (D). (We assume that they exist) A student proposes the following proof of the Strong Duality theorem:
"*By the Weak Duality theorem we have $z_d \geq z_p$. And since the dual of the dual is the primal we have $z_p = (z_d)_d \geq z_d$ (by the Weak Duality theorem). So $z_p = z_d$.*"
Can you help me to find the mistake in his/her proof?

Exercise 2 - Simplex algorithm. [5 pts]
Consider the following LP:

$$\begin{aligned} & \max 2x_1 + 3x_2 \\ \text{subject to } & 4x_1 + 8x_2 \leq 12 \\ & 2x_1 + x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(a) Solve it using the Simplex algorithm.
(b) What is the dual (D) of this LP. Give an optimal solution of (D).
(c) Using complementary slackness, check the optimality of your solution of (P).

Exercise 3 - Interval matrices. [7 pts]
An *interval matrix* is a $\{0, 1\}$-matrix where, on each row, all the 1 coefficients are consecutive. In other words, if for all i and $j \leq k \leq \ell$ such that $a_{i,j} = a_{i,\ell} = 1$ implies $a_{i,k} = 1$. Let M be an interval matrix.

(a) Let B_t be the following $t \times t$ matrix:

$$B_t = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Let N be a squared submatrix of M of size $t \times t$. Prove that $B_t N^t$ is the incidence matrix of a directed graph.

(b) Deduce that any interval matrix is totally unimodular.

(c) Formulate the maximum independent set problem as an ILP.

(d) An *interval graph* is an intersection graph of intervals in the line. In other words, vertices correspond to intervals and there is an edge between two intervals if the corresponding intervals intersect.
Give another representation of the Maximum Independent Set problem as an ILP using the geometric representation.

(e) Prove that the constraint matrix of (d) is a TU matrix. Conclude.

Exercise 4 - Farkas' Lemma alternative. [2.5 pts]
First recall the Farkas' Lemma. Let $m, n \in \mathbb{N}$. Let A be a matrix of size $n \times m$ and b be a vector of $\mathbb{R}^n$. Exactly one of the following two holds:

(i) There exists a vector $x \geq 0$ in $\mathbb{R}^m$ such that $Ax = b$.

(ii) There exists a vector $y \in \mathbb{R}^n$ such that $y^t b < 0$ and $y^t A \geq 0$.

Using it prove that the following holds. Let A be a matrix of size $n \times m$ and b be a vector of $\mathbb{R}^n$. Exactly one of the following two holds:

(i) There exists a vector $x \geq 0$ in $\mathbb{R}^m$ such that $Ax \leq b$.

(ii) There exists a vector $y \in \mathbb{R}^n$ such that $y^t b < 0$ and $y^t A \geq 0$ and $y \geq 0$.

Hint: How do you transform inequalities into equalities in LPs?

Exercise 5 - Alternative proof of the Strong Duality Theorem. [5 pts]
Assume that the Weak Duality theorem holds. Let (P) be a LP of the form:

$$\begin{aligned} &\max c^t x \\ \text{subject to } &Ax \leq b \\ &x \geq 0 \end{aligned}$$

We assume that (P) is feasible with a bounded optimal value z^*, and that the dual (D) is feasible. Consider the following matrices, for some scalar $\gamma \geq 0$:

$$A_\gamma = \begin{pmatrix} A \\ -c^t \end{pmatrix} \qquad b_\gamma = \begin{pmatrix} b \\ -\gamma \end{pmatrix}$$

(a) Show that the system $A_\gamma x' \leq b_\gamma$ with $x' \geq 0$ has a solution if and only if (P) has a solution of value at least γ.

(b) Let $\epsilon > 0$ and $\gamma = z^* + \epsilon$. Using Exercise 4, show that there exists a vector $\begin{pmatrix} y \\ w \end{pmatrix}$ (where w is a scalar) such that:

$$y^t A \geq w \cdot c \quad \text{and} \quad b^t y < (z^* + \epsilon) w \quad \text{and} \quad y, w \geq 0.$$

(c) Let us prove that $w > 0$. Assume by contradiction that $w = 0$. Prove that (D) has a solution of value less than z^*. Conclude.

(d) Deduce that, for every $\epsilon > 0$, (D) admits a solution of value at most $z^* + \epsilon$. Conclude.