Linear Algebra

[KOMS120301] - 2023/2024

13.2 - Jenis Transformasi Linier

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Tujuan pembelajaran

Setelah perkuliahan ini, Anda diharapkan mampu:

1. menjelaskan konsep berbagai jenis transformasi linier antar vektor dalam ruang vektor;
2. melakukan transformasi linier (refleksi, proyeksi, rotasi, dilatasi, ekspansi, geser) pada suatu vektor dalam ruang vektor.

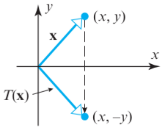
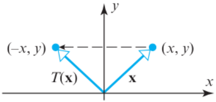
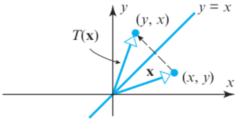
Transformasi Matriks Dasar di $\mathbb{R}^2$ dan $\mathbb{R}^3$

(page 259 of Elementary LA Applications book)

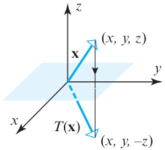
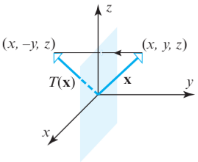
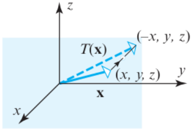
1. Refleksi

Operator refleksi pada $\mathbb{R}^2$

Operator refleksi adalah operator pada $\mathbb{R}^2$ (atau $\mathbb{R}^3$) yang memetakan setiap titik ke dalam gambar simetrisnya terhadap garis atau bidang yang memuat titik asal.

Operator	**Illustration**	**Images of $\mathbf{e}_1$ and $\mathbf{e}_2$**	**Standard Matrix**
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$ $T(\mathbf{e}_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

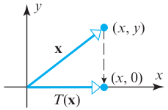
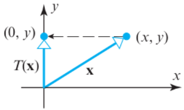
Operator refleksi pada $\mathbb{R}^3$

Operator	**Illustration**	**Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$**	**Standard Matrix**
Reflection about the xy-plane $T(x, y, z) = (x, y, -z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x, y, z) = (x, -y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, -1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x, y, z) = (-x, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (-1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

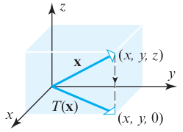
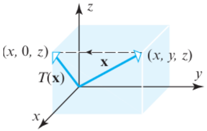
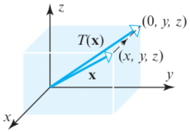
2. Proyeksi

Operator proyeksi pada $\mathbb{R}^2$

Operator proyeksi atau operator proyeksi ortogonal adalah operator matriks pada $\mathbb{R}^2$ (atau $\mathbb{R}^3$) yang memetakan setiap titik ke dalam proyeksi ortogonalnya ke garis tetap atau bidang yang melalui titik asal.

Operator	**Illustration**	**Images of $\mathbf{e}_1$ and $\mathbf{e}_2$**	**Standard Matrix**
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

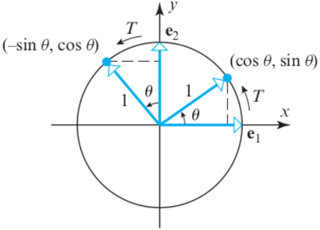
Operator proyeksi pada $\mathbb{R}^3$

Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Orthogonal projection onto the xy-plane $T(x, y, z) = (x, y, 0)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection onto the xz-plane $T(x, y, z) = (x, 0, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection onto the yz-plane $T(x, y, z) = (0, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Rotasi

Operator rotasi adalah operator matriks pada $\mathbb{R}^2$ atau $\mathbb{R}^3$ yang memindahkan titik sepanjang busur lingkaran yang berpusat di asal.

Cara mencari matriks standar untuk operator rotasi $T : \mathbb{R}^2 \to \mathbb{R}^2$ yang memindahkan titik berlawanan arah jarum jam di sekitar titik asal melalui sudut positif θ ?



$$T(\mathbf{e}_1) = T(1,0) = (\cos\theta, \sin\theta) \quad \text{and} \quad T(\mathbf{e}_2) = T(0,1) = (-\sin\theta, \cos\theta)$$

Matriks transformasi standar untuk T adalah:

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Konversi dari ^{o} ke rad

- $180^o = 1\pi$ rad
- $1^o = \frac{\pi}{180}$ rad

Operator rotasi pada $\mathbb{R}^2$ (*cont.*)

Matriks:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

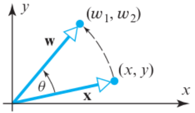
disebut matriks rotasi untuk $\mathbb{R}^2$.

Misalkan $\mathbf{x} = (x, y) \in \mathbb{R}^2$ dan $\mathbf{w} = (w_1, w_2)$ menjadi bayangannya saat diputar. Kemudian:

$$\mathbf{w} = R_\theta \mathbf{x}$$

with:

$$\begin{aligned} w_1 &= x\cos\theta - y\sin\theta \\ w_2 &= x\sin\theta + y\cos\theta \end{aligned}$$

Operator	**Illustration**	**Rotation Equations**	**Standard Matrix**
Counterclockwise rotation about the origin through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Contoh: operator rotasi

Carilah bayangan $\mathbf{x} = (1, 1)$ dengan rotasi $\pi/6$ rad $(= 30^o)$ terhadap titik asal.

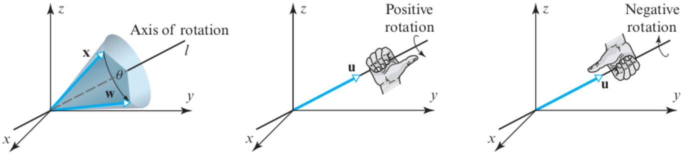
Solusi:

Kita tahu bahwa $\sin(\pi/6) = \frac{1}{2}$ dan $\cos(\pi/6) = \frac{\sqrt{3}}{2}$.

Dari formula sebelumnya:

$$R_{\pi/6}\mathbf{x} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix} \approx \begin{bmatrix} 0.37 \\ 1.37 \end{bmatrix}$$

Rotasi dalam $\mathbb{R}^3$ umumnya digambarkan sebagai sumbu rotasi dan vektor satuan **u** di sepanjang garis tersebut.

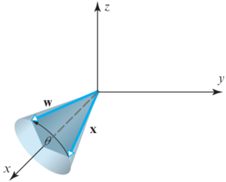
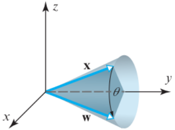
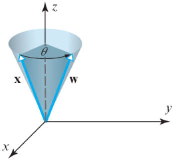


(*a*) Angle of rotation (*b*) Right-hand rule

Aturan tangan kanan digunakan untuk menentukan tanda sudut rotasi.

- Jika sumbunya adalah sumbu x, y, atau z, maka ambillah vektor satuan **i**, **j**, dan **k**.
- Sudut rotasi akan menjadi *positif jika berlawanan arah jarum jam* menghadap ke titik asal sepanjang sumbu koordinat positif dan akan menjadi *negatif jika searah jarum jam*.

Rotasi di $\mathbb{R}^3$

Operator	Illustration	Rotation Equations	Standard Matrix
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

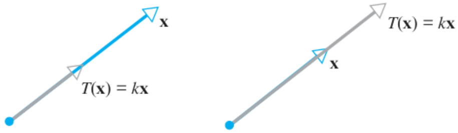
4. Dilatasi dan kontraksi

Misalkan $k \in \mathbb{R}, k \geq 0$. Operatornya:

$$T(\mathbf{x}) = k\mathbf{x}$$

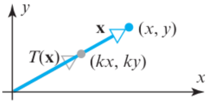
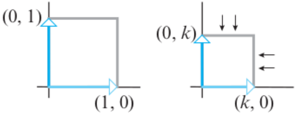
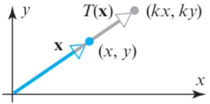
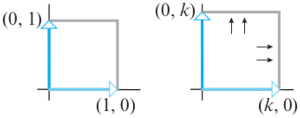
pada $\mathbb{R}^2$ atau $\mathbb{R}^3$ mendefinisikan penambahan atau pengurangan panjang vektor $\mathbf{x}$ dengan faktor k.

- Jika $k > 1$, disebut dilatasi dengan faktor k;
- Jika $0 \leq k \leq 1$, maka disebut kontraksi dengan faktor k.

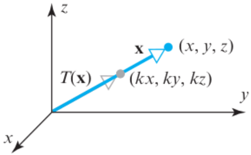
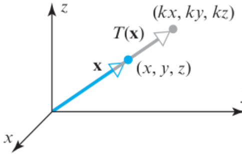


(a) $0 \leq k < 1$ (b) $k > 1$

Dilatasi & kontraksi di $\mathbb{R}^2$

Operator	Illustration $T(x, y) = (kx, ky)$	Effect on the Unit Square	Standard Matrix
Contraction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k in R^2 $(k > 1)$			

Dilatasi & kontraksi di $\mathbb{R}^3$

Operator	Illustration $T(x, y, z) = (kx, ky, kz)$	Standard Matrix
Contraction with factor k in R^3 $(0 \leq k < 1)$		$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k in R^3 $(k > 1)$		

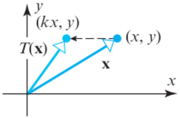
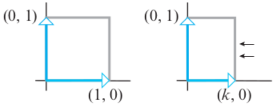
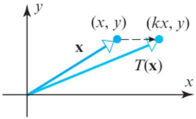
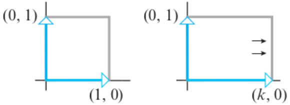
5. Ekspansi dan kompresi

Dalam dilatasi atau kontraksi $\mathbb{R}^2$ atau $\mathbb{R}^3$, **semua koordinat** dikalikan dengan faktor non-negatif k.

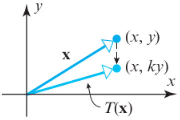
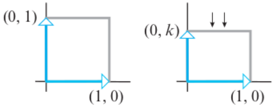
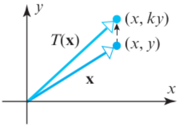
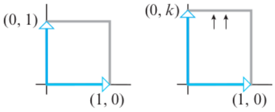
Sekarang bagaimana jika **hanya satu koordinat** dikalikan dengan k ?

- Jika $k > 1$, disebut ekspansi dengan faktor k searah sumbu koordinat (x, y, atau z);
- Jika $0 \leq k \leq 1$, disebut kompresi

Ekspansi dan kompresi in $\mathbb{R}^2$ (pada arah x)

Operator	Illustration $T(x, y) = (kx, y)$	Effect on the Unit Square	Standard Matrix
Compression in the x-direction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion in the x-direction with factor k in R^2 $(k > 1)$			

Ekspansi dan kompresi in $\mathbb{R}^2$ (pada arah y)

Operator	Illustration $T(x, y) = (x, ky)$	Effect on the Unit Square	Standard Matrix
Compression in the y-direction with factor k in R^2 $(0 \le k < 1)$			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion in the y-direction with factor k in R^2 $(k > 1)$			

6. Shear

Shear

Operator matriks berbentuk:

$$T(x, y) = (x + ky, y)$$

menerjemahkan titik (x, y) pada bidang xy yang sejajar dengan sumbu x dengan jumlah ky yang sebanding dengan koordinat y titik tersebut.

Ini disebut geser ke arah x dengan faktor k.

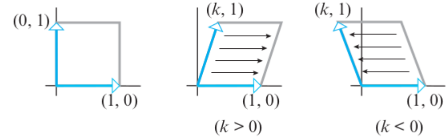
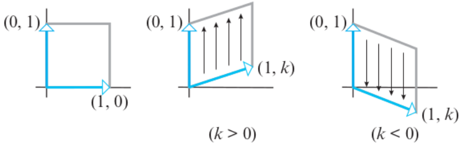
Demikian pula, operator matriks:

$$T(x, y) = (x, y + kx)$$

disebut geser ke arah y dengan faktor k.

Jika $k > 0$, maka gesernya ke arah positif. Ketika $k < 0$, arahnya negatif.

Shear

Operator	Effect on the Unit Square	Standard Matrix
Shear in the x-direction by a factor k in R^2 $T(x, y) = (x + ky, y)$		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear in the y-direction by a factor k in R^2 $T(x, y) = (x, y + kx)$		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Contoh

Jelaskan operator matriks yang matriks standarnya sebagai berikut:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

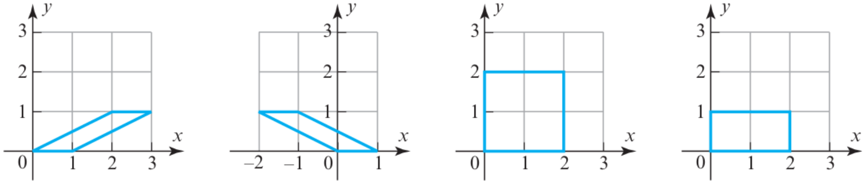
Solusi:

Dari tabel pada slide sebelumnya kita dapat melihat bahwa:

- A_1 berhubungan dengan pergeseran ke arah x dengan faktor 2;
- A_2 berhubungan dengan pergeseran ke arah x dengan faktor -2;
- A_3 berhubungan dengan pelebaran dengan faktor 2;
- A_4 berhubungan dengan perluasan ke arah x dengan faktor 2.

Contoh (*cont.*)

Jelaskan secara geometris hasil transformasinya:



A_1 A_2 A_3 A_4

Latihan

bersambung..