Linear Algebra

[KOMS119602] - 2022/2023

7.2 - Relation between Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

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Learning objectives

After this lecture, you should be able to:

1. explain dot product between two vectors;
2. explain computing norm of a vector;
3. explain computing distance, angles, and projection of two vectors
4. explain cross product of vectors.

Part 1: **Inner Product & Norm**

Dot (inner) product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$:

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad \mathbf{v} = (v_1, v_2, \ldots, v_n)$$

The dot product or inner product or scalar product of $\mathbf{u}$ and $\mathbf{v}$ is defined by:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Algebraically, the dot product is the sum of the products of the corresponding entries of the two sequences of numbers.

Can we interpret dot product of two vectors geometrically?

Example

1. Let $\mathbf{u} = (1, -2, 3)$, $\mathbf{v} = (4, 5, -1)$, find $\mathbf{u} \cdot \mathbf{v}$.

$$\mathbf{u} \cdot \mathbf{v} = 1(4) + (-2)(5) + (3)(-1) = 4 - 10 - 3 = -9$$

2. Suppose $\mathbf{u} = (1, 2, 3, 4)$ and $\mathbf{v} = (6, k, -8, 2)$. Find k such that $\mathbf{u} \cdot \mathbf{v} = 0$.

$$\mathbf{u} \cdot \mathbf{v} = 1(6) + 2(k) + 3(-8) + 4(2) = -10 + 2k$$

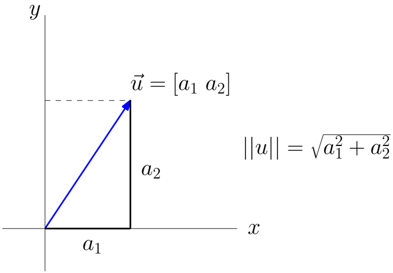
If $\mathbf{u} \cdot \mathbf{v} = 0$ then $-10 + 2k = 0$, meaning that $k = 5$.

Norm (length) of a vector

Norm (length) of a vector $\mathbf{u}$ in $\mathbb{R}^n$ is defined by:

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Illustration in 2D:



A vector $\mathbf{u}$ is a unit vector if $\|u\| = 1$.

Example

1. Let $\mathbf{u} = (1, -2, -4, 5, 3)$. Find $\|\mathbf{u}\|$.

   $$\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = 1^2 + (-2)^2 + (-4)^2 + 5^2 + 3^2 = 1 + 4 + 16 + 25 + 9 = 55$$

   Hence, $\|\mathbf{u}\| = \sqrt{55}$.

2. Given vectors $\mathbf{v} = (1, -3, 4, 2)$ and $w = (\frac{1}{2}, -\frac{1}{6}, \frac{5}{6}, \frac{1}{6})$. Determine which one of the two vectors is a unit vector?

   $$\|\mathbf{v}\| = \sqrt{1 + 9 + 16 + 4} = \sqrt{30} \quad \text{and} \quad \|w\| = \sqrt{\frac{9}{36} + \frac{1}{36} + \frac{25}{36} + \frac{1}{36}} = 1$$

   Hence, $\mathbf{w}$ is a unit vector, and $\mathbf{v}$ is not a unit vector.

Standard unit vector

The standard unit vector in $\mathbb{R}^n$ is composed of n vectors:

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$$

dimana:

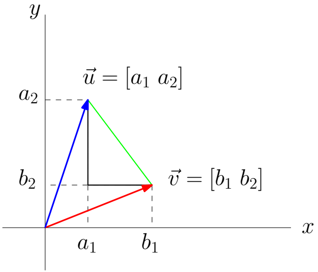
$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ \mathbf{e}_n = (0, 0, \ldots, 0, 1)$$

Part 2: **Distance, Angle, Projections**

Distance

The distance between vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is defined by:

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

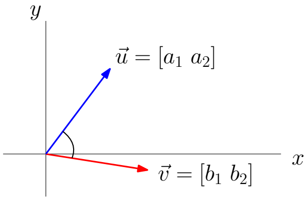


$$||u - v|| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

Angle between two vectors

The angle θ between vectors $u, \mathbf{v} \neq 0$ in $\mathbb{R}^n$ is defined by:

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$



Is this well defined? Remember that the value of cos range from -1 to 1. So the following should hold:

$$-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \leq 1$$

Exercise: prove the last inequality!

Cauchy-Schwarz inequality

Solution of the exercise:

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then $-1 \leq \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1$.

Theorem (Schwarz inequality)

For any vectors $\mathbf{u}, \mathbf{v}$ *in* $\mathbb{R}^n$, $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

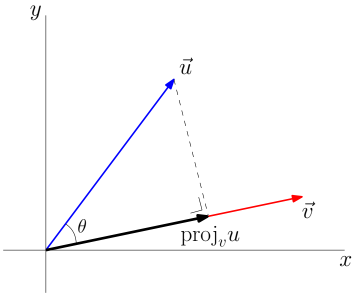
Proof.

See this paper `https://www.uni-miskolc.hu/~matsefi/Octogon/volumes/volume1/article1_19.pdf` for different proof alternatives. □

Projection

The projection of a vector **u** onto a **nonzero** vector **v** is defined by:

$$\text{proj}_v \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}$$



The length of vector $\text{proj}_v u$ is $\|\mathbf{u}\| \cos(\theta)$. So,

$$\begin{aligned}
\text{proj}_v \mathbf{u} &= \|\mathbf{u}\| \cos(\theta)\ \mathbf{v} \\
&= \|\mathbf{u}\|\ \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}\ \mathbf{v} \\
&= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|}\ \mathbf{v} \\
&= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\ \mathbf{v}
\end{aligned}$$

What is vector projection used for?

Orthogonality

Part 2: **Cross Product**

Cross product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$:

$$\mathbf{u} = (u_1, u_2, u_3) \quad \text{and} \quad \mathbf{v} = (v_1, v_2, v_3)$$

The cross product of $\mathbf{u}$ and $\mathbf{v}$ is defined by:

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)$$

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

This can be easily seen using the following method:

$$\begin{bmatrix} u_1 & \boxed{u_2 & u_3} \\ v_1 & v_2 & v_3 \end{bmatrix} \quad \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \quad \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Example

Given vectors:

$$\mathbf{u} = (0, 1, 7) \quad \text{and} \quad \mathbf{v} = (1, 4, 5)$$

The vectors can be represented as matrix: $\begin{bmatrix} 0 & 1 & 7 \\ 1 & 4 & 5 \end{bmatrix}$

Hence,

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \left(\begin{vmatrix} 1 & 7 \\ 4 & 5 \end{vmatrix}, -\begin{vmatrix} 0 & 7 \\ 1 & 5 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} \right) \\ &= (5 - 28,\ -(0 - 7),\ 0 - 1) \\ &= (-23, 7, -1) \end{aligned}$$

How does $\mathbf{u} \times \mathbf{v} = \mathbf{w}$ mean?

Given: $\mathbf{u} \times \mathbf{v} = \mathbf{w}$. This means that:

$$\mathbf{w} \perp \mathbf{u} \quad \text{and} \quad \mathbf{w} \perp \mathbf{v}$$

Example

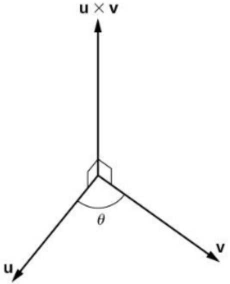
Given $\mathbf{u} = (0, 1, 7)$ and $\mathbf{v} = (1, 4, 5)$, and:

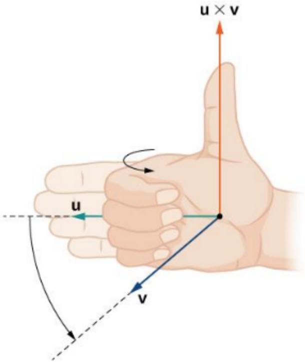
$$\mathbf{u} \times \mathbf{v} = \mathbf{w} = (-23, 7, -1)$$

Note that:

- $\mathbf{w} \cdot \mathbf{u} = (-23, 7, -1) \cdot (0, 1, 7) = 0 + 7 - 7 = 0$
- $\mathbf{w} \cdot \mathbf{v} = (-23, 7, -1) \cdot (1, 4, 5) = -23 + 28 - 5 = 0$

Right-hand system





Properties of cross product

Theorem

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *be vectors in* $\mathbb{R}^3$, *and* $k \in \mathbb{R}$. *Then:*

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
4. $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
5. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times u = \mathbf{0}$
6. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Properties of dot product and cross product

Theorem

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *be vectors in* $\mathbb{R}^3$*. Then:*

1. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$ *($\mathbf{u} \times \mathbf{v}$ is orthogonal to u)*
2. $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$ *($\mathbf{u} \times \mathbf{v}$ is orthogonal to v)*
3. $||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2$ *(Lagrange's identity)*
4. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$
5. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$

Exercise

Prove the following identity:

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \; ||\mathbf{v}|| \sin\theta$$

where θ is the angle between **u** and **v**.

Answer:

$$\begin{aligned}
||\mathbf{u} \times \mathbf{v}||^2 &= ||\mathbf{u}||^2 \; ||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\
&= ||\mathbf{u}||^2 \; ||\mathbf{v}||^2 - (||\mathbf{u}|| \; ||\mathbf{v}|| \cos\theta)^2 \\
&= ||\mathbf{u}||^2 \; ||\mathbf{v}||^2 - (||\mathbf{u}||^2 \; ||\mathbf{v}||^2 \cos^2\theta) \\
&= ||\mathbf{u}||^2 \; ||\mathbf{v}||^2 (1 - \cos^2\theta) \\
&= ||\mathbf{u}||^2 \; ||\mathbf{v}||^2 \sin^2\theta
\end{aligned}$$

Dengan demikian, $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \; ||\mathbf{v}|| \sin\theta$

Cross product of standard unit vectors

The standard unit vectors in $\mathbb{R}^3$:

$$\mathbf{i} = (1,0,0) \quad \mathbf{j} = (0,1,0) \quad \mathbf{k} = (0,0,1)$$

The cross product between **i** and **j** is given by:

$$\mathbf{i} \times \mathbf{j} = \left(\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \; -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \; \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (0,0,1) = \mathbf{k} \right)$$

The cross product between **i**, **j**, and **k**:

- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
- $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$
- $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
- $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
- $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
- $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Cross product of two vectors

Given:

- $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
- $\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

Using the cofactor expansion:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

Example of *cofactor expansion for cross product*

From the previous example:

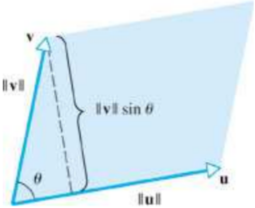
- $\mathbf{u} = (0,1,7) = \mathbf{j} + 7\mathbf{k}$
- $\mathbf{v} = (1,4,5) = \mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

Then:

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 7 \\ 1 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 7 \\ 4 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 7 \\ 1 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} \mathbf{k} \\
&= (5-28)\mathbf{i} \; - \; (0-7)\mathbf{j} \; + \; (0-1)\mathbf{k} \\
&= -23\mathbf{i} + 7\mathbf{j} - \mathbf{k}
\end{aligned}$$

Geometric interpretation of cross product (in $\mathbb{R}^2$)

The cross product of two vectors **u** and **v** in $\mathbb{R}^2$ is equal to the area of the parallelogram determined by the two vectors.

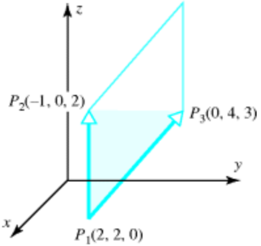


$$\begin{aligned}\text{Area} &= \text{base} \times \text{height} \\ &= ||\mathbf{u}||\ ||\mathbf{v}||\sin\theta \\ &= ||\mathbf{u}\times\mathbf{v}||\end{aligned}$$

Example

Determine the area of the triangle determined by the points:

$$P_1 = (2, 2, 0), \quad P_2 = (-1, 0, 2), \quad \text{and} \quad P_3 = (0, 4, 3)$$



Area of $\triangle = 1/2$ Area of *parallelogram*

Two vectors that determine the parallelogram:

$$\begin{aligned}
\mathbf{u} &= \vec{P_1P_2} = \vec{OP_2} - \vec{OP_1} \\
&= (-1, 0, 2) - (2, 2, 0) = (-3, -2, 2) \\
\mathbf{v} &= \vec{P_1P_3} = \vec{OP_3} - \vec{OP_1} \\
&= (0, 4, 3) - (2, 2, 0) = (-2, 2, 3)
\end{aligned}$$

Hence: $\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} -2 & 2 \\ 2 & 3 \end{vmatrix}, - \begin{vmatrix} -3 & 2 \\ -2 & 3 \end{vmatrix}, \begin{vmatrix} -3 & -2 \\ -2 & 2 \end{vmatrix} \right) = (-10, 5, -10)$

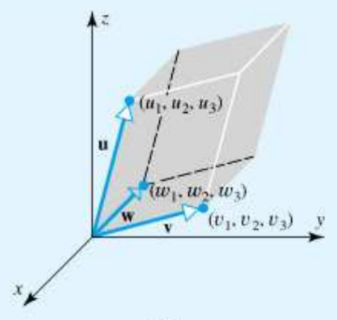
So, the area of the parallelogram is:

$$||\mathbf{u} \times \mathbf{v}|| = \sqrt{(-10)^2 + (5)^2 + (-10)^2} = \sqrt{225} = 15$$

and the area of the triangle is $15/2 = 7.5$.

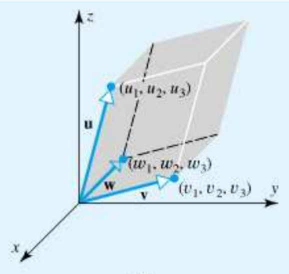
Geometric interpretation of cross product (in $\mathbb{R}^3$)

The cross product of three vectors **u**, **v**, and **w** in $\mathbb{R}^3$ is equal to the volume of the parallelepide determined by the three vectors.



$$
\begin{aligned}
\text{Volume} &= \text{area of base} \times \text{height} \\
&= ||\mathbf{v} \times \mathbf{w}|| \cdot (||\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}||) \\
&= ||\mathbf{v} \times \mathbf{v}|| \cdot \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{||\mathbf{v} \times \mathbf{w}||} \\
&= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|
\end{aligned}
$$

Geometric interpretation of cross product (in $\mathbb{R}^3$)



$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{u}\cdot\left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\mathbf{k}\right)$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} u_1 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} u_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} u_3$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

which is the determinant of matrix whose first row is composed of elements of **u** and the 2nd and 3rd rows are composed with the elements of **v**

The volume of the parallelepide is equal to $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$

Example

Find the volume of the *parallelepide* formed by three vectors:

$$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k},\ \ \mathbf{v} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k},\ \ \mathbf{w} = 3\mathbf{j} + 2\mathbf{k}$$

Solution:

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} \\
&= 3 \begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} \\
&= 60 + 4 - 15 \\
&= 49
\end{aligned}$$

Exercise 1

Find the area of parallelogram that is formed by two vectors:

$$\mathbf{u} = 4\mathbf{i} + 3\mathbf{j} \quad \text{and} \quad \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$$

Solution:

$$\det\left(\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}\right) = \begin{vmatrix} 4 & 3 \\ 3 & -4 \end{vmatrix} = -16 - 9 = -25$$

Hence, the area of the parallelogram is $|-25| = 25$.

Exercise 2

Given three vectors:

$$\mathbf{u} = (1, 1, 2),\ \mathbf{v} = (1, 1, 5),\ \mathbf{v} = (3, 3, 1)$$

Find the volume of the parallelepide formed by the three vectors!

Solution:

$$\begin{aligned}
\begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 5 \\ 3 & 3 & 1 \end{vmatrix} &= (1)\begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix} - (1)\begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} \\
&= (1)(-14) - (-1)(-14) + (2)(0) \\
&= -14 + 14 + 0 \\
&= 0
\end{aligned}$$

A recap

We have learned:

- the definition of vectors in Linear Algebra;
- some operations on vectors:
  - vector addition and scalar multiplication;
  - linear combination;
  - dot product between two vectors;
  - computing norm of a vector;
  - computing distance, angles, and projection of two vectors

Task: write a summary about our discussion, and do the exercises!

to be continued...