Derivatives - FIN404
Project 1: Reverse Convertible Notes

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1 Documentation

Reverse Convertible Notes (RCN) are a short-term structured investment. A structured product is an easy way to access to derivatives. It is a package investment that typically includes assets linked to interest plus one or more derivatives (e.g. one zero-coupon bond and one call option on an underlying asset). Reverse Convertible Notes are therefore linked to an underlying, that could be one or several stocks. It has a predetermined maturity, that could be from few months to a couple of years (usually it is around 1 year or less since it is a short-term investment). Thus, RCN are closely related to traditional models of option pricing. It is usually issued by major financial institutions such as banks, and the company whose underlying is a stock is not involved at all.

RCN have a fixed coupon rate. Coupons are delivered to the investor generally on a quarterly, biannual or annual basis. Since it has a fixed rate, its value is not linked to the underlying performances. We the maturity is reached, there are two possibilities for a classic RCN:

- The underlying closing price at maturity is above its strike value. In this case, the investor got his initial investment back. Thus, he received his investment plus one (or several) coupons payment. The only thing that matters here is if the underlying final value is above the strike value. We do not care if its value gets under its strike value at some point between the investment and the maturity.
- The underlying closing price at maturity is under its strike value. In this case, the investor will still receive one (or several) coupon payments. However, he will not get his investment back. He will receive physical delivery of the underlying shares. Then, he will become a shareholder of the underlying shares company. The number of shares that the investor receives is predetermined. It is equal to the original investment amount divided by the initial price of the underlying asset. Then, it will naturally be less than the initial investment if the price of the underlying has fallen.

However, thanks to the coupon received, the return on the investment is always higher than the return on the underlying.

Barrier Reverse Convertible Notes (BRCN) work the same as RCN, except that they have a fixed barrier or a knock-in level. This barrier is a threshold (e.g. 70% of the initial value of the underlying) and the payment at maturity will depend on whether the price of the underlying has crossed it during the holding period. For BRCN, we differentiate three cases at maturity:

- The underlying closing price at maturity is above its strike value. Regardless of whether the stock price gets under the knock-in level during the holding period, the investor gets his investment back plus the coupon payment.
- The underlying closing price at maturity is between its strike value and the barrier, and the underlying never closed under the knock-in level during the holding period. In this case, the investor received the same amount as in the first case, i.e. his initial investment plus the coupon payment.
- The underlying closing price at maturity is under its strike value, and it has close under the knock-in level at some point during the holding period. This case includes the case where the underlying closing price at maturity is under the threshold. Here, the investor receives the physical delivery of the underlying shares as for a classic RCN, plus the coupon payment.

Reverse Convertible Notes and Barrier Reverse Convertible Notes each have an (auto)callable version. These products have auto-callable dates at which they can be auto-called or not. On each of these dates, there is a new coupon payment that could be issued. Usually, the closer the auto-callable date is to maturity, the higher the coupon payment amount is. For example, if we have a maturity of 3 years, we could have a 10% coupon at the date of 1 year, a 20% coupon at 2 years, and a 30% at the maturity date. The auto-callable dates are usually on a month, quarter, biannual, or annual basis. At an auto-callable date, two situations can occur:

- The underlying closing price is above the predefined strike corresponding to this callable date. Then, we say that the auto-callable (Barrier) Reverse Convertible Note is called. This means that the investor received his initial investment plus the corresponding coupon payment. Then, the product is over, nothing else will happen.
- The underlying closing price is not above the predefined corresponding strike and the maturity is not reached yet. Then, the product is not called and we wait until the next callable date.

When we reached maturity, the investor gets its money back (or not) such as in a classic (respectively a Barrier) Reverse Convertible Note, i.e. by checking if the underlying closing price is above the initial value (respectively the knock-in level for the BRCN). Finally, the difference between an RCN (or a BRCN) and an Auto-callable Reverse Convertible Note, ARCN (or an Auto-callable Barrier Reverse Convertible Note, i.e. an ABRCN) is that the product can be auto-called, and then the investor gets paid, before its maturity date.

Products such as RCN, BRCN, ARCN, or ABRCN are very interesting for sideways-moving or moderately increasing underlying assets. For small movements of the underlying asset, the performance of these products will be higher than the performance of the underlying (even in the case of a loss). They combine aspects of equities and bonds, have high interest and short maturity. Another good point of such products is that they have fixed coupons, that do not depend on the underlying performances. An investment in these products is also wise in the case of falling volatility of the underlying. Indeed, the coupon is a reflection of the underlying stock's volatility, the higher the volatility, the higher the coupon. In addition to that, the more time passes, the more there is a positive effect on the valuation of the product since we are closer to maturity.

From the investor's point of view, a BRCN is adding another security compared to an RCN, since there is this barrier. It allows the underlying to get a little under its initial value. Moreover, the (auto)callable version is interesting for people wishing to increase short-term returns, since it can be called before maturity.

However, there are always some risks for the investors. First of all, the losses could be high if the underlying undergoes a big drop. Secondly, the earning potential is limited compared to the one of the underlying. The maximum an investor can receive is his initial payment and the coupons. If the underlying undergoes a sharp rise, the investor would have missed a real opportunity by not investing directly in the underlying. In addition to that, these products are, such as most financial products, hard to predict because of the high number of factors involved in pricing: performance of the underlying, volatility, residual maturity, interest rates, dividend... Finally, there is also a default risk. This risk is that the company issuing the product is not able to pay the coupon and the final payoff.

From the bank issuing the RCN point of view, a BRCN is a security that they would give to a client, which then would be more likely to invest in the product. However, they allow the investor to receive back his investment even if the underlying closing price is under the strike price. Usually, a bank would lower down the coupon rate or increase the price of the initial investment for a BRCN compared to a RCN since it offers more security to the investor. A callable RCN would be even more interesting for an issuer. In the case where the underlying undergoes a sharp rise, the RCN would be called, and then the bank would not have to pay coupons at every time step. In the case of the underlying would not reach the strike price it would have the same effect for the issuer than for a noncallable RCN. However, if the underlying undergoes a sharp rise and then a big drop under the strike price, the investor would have called his product just after the rise, and would then have received his investment back, on the contrary of if he had invested in a noncallable RCN.

A Contingent Convertible, also called CoCo, is a security that is similar to a traditional Convertible Bond, however, the difference is that it will converts into stocks at a higher strike price, called the upside contingency. For the investor, it will guarantee him steady payments now, with a chance of cashing return in the future. Yet, there is high uncertainty and the investor may keep his CoCo for years, that will never convert into stocks. The difference with an RCN is that the CoCo will converts into stocks for a higher strike than the classic strike of an RCN (or the barrier for a BRCN), but it may never convert. Actually, there is no maturity in a CoCo. Finally, if the CoCo is not converted, this is a quite wise investment for the issuing bank.

2 Preliminary results

Q1

Let the terminal payoff of the RCN (excluding the last coupon payment) be

$$H(T) = 1 - \left(\alpha - \frac{I_T}{I_0}\right)^+. \tag{1}$$

We directly see that

$$H(I_T) = \begin{cases} 1 & \text{if } \; I_T \geq \alpha I_0 \\ 1 - \alpha + \frac{1}{I_0} I_T & \text{otherwise.} \end{cases} \tag{2}$$

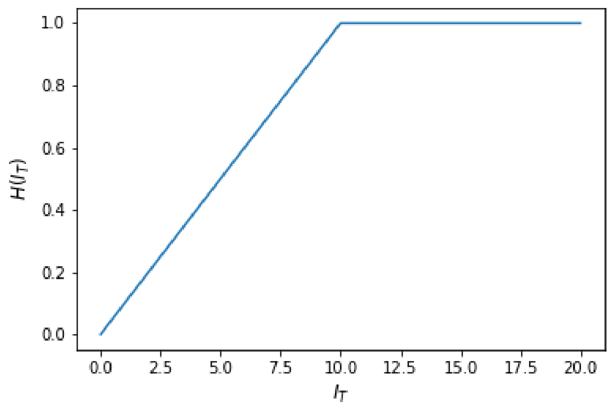

Figure 1: RCN terminal payoff with $\alpha = 1$ and $I_0 = 10$

As shown on figure 1, the terminal payoff of the simple RCN resembles that of a short vanilla put option. To replicate this payoff, we first consider $H(I_T) - 1$ which we use to represent the payoff of a short put excluding its price. Indeed, consider a European put with strike $K = \alpha I_0$. By shorting $\frac{1}{I_0}$ of that put, the terminal payoff is

$$\begin{aligned} -\frac{1}{I_0}(K - I_T)^+ &= \begin{cases} 0 & \text{if } I_T \geq K \\ -\frac{1}{I_0}\left(\alpha I_0 - I_T\right) & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & \text{if } I_T \geq K \\ -\left(\alpha - \frac{I_T}{I_0}\right) & \text{otherwise} \end{cases} \end{aligned} \tag{3}$$

which is exactly equal to $H(I_T) - 1$. Now, to replicate $H(I_T)$, we only need to receive 1 at the terminal date $t = T$, which can be achieved by being long one zero-coupon bond with face value 1. At the same time, the intermediary cash flows are also replicable by long positions in T different zero-coupon bonds with maturities ranging from 1 to T. Each of the latter bonds must be purchased in quantities of Δc for face values of 1 since they represent the coupon payments the investor will receive at times $0 < t \leq T$. Below is a table of the trades and their respective cash flows.

Trade	$t = 0$	$0 < t < T$	$t = T$
Short $\frac{1}{I_0}$ put with $K = \alpha I_0$	$\frac{p_0}{I_0}$	0	$-\frac{1}{I_0}(K - I_T)^+$
Long 1 unit ZCB maturity T + Δc unit ZCBs maturity $m = 1, \ldots, T$	-b_0	Δc	$1 + \Delta c$

Table 1: Replicating strategy with payoffs of simple RCN

The price at $t = 0$ of 1 put is denoted by p_0, and b_0 denotes the sum the prices at $t = 0$ of the $T + 1$ bonds purchased. It is easy to verify that these positions satisfy the same cumulative cash flow process as RCN's, i.e

$$\sum_{j=1}^{N} \mathbf{1}_{\{t \geq T_j\}} \Delta c + \mathbf{1}_{\{t \geq T\}} \left[1 - \left(\alpha - \frac{I_T}{I_0} \right)^+ \right]. \tag{RCN}$$

Q2

Consider going **short $\frac{1}{I_0}$ units of a down-and-in put with strike price αI_0 and barrier level βI_0**. The terminal payoff at terminal date T is

$$-\frac{1}{I_0} \mathbf{1}_{\{\tau_\beta \leq T\}} \left[1 - (\alpha I_T - I_0)^+ \right] = -\mathbf{1}_{\{\tau_\beta \leq T\}} \left(\alpha - \frac{I_T}{I_0} \right)^+$$

If we go **long 1 unit of a bond with face value \$1 and \$$\Delta c$ coupons paid at dates** $\{T_j\}_{j=1}^N$, the cumulative cash flows to the holder at date t are

$$\sum_{j=1}^{N} \mathbf{1}_{\{t \geq T_j\}} \Delta c + \mathbf{1}_{\{t \geq T\}}$$

The total cumulative cash flows of this portfolio to the holder at date t are

$$\sum_{j=1}^{N} \mathbf{1}_{\{t \geq T_j\}} \Delta c + \mathbf{1}_{\{t \geq T\}} \left[1 - \mathbf{1}_{\{\tau_\beta \leq T\}} \left(\alpha - \frac{I_T}{I_0} \right)^+ \right]$$

This portfolio replicates the Barrier-RCN.

Note that we use this way of replicating the the simple as well as the barrier RCN to price both of these types of notes. Indeed, given the law of one price, it must hold that the price of the RCN is the negative of the sum of cash flows of the replicating portfolio at the initial date.

Q3

Suppose $\beta \in [a, 1]$.

If $I_T \leq \alpha I_0$, then $I_T \leq \beta I_0$ i.e. $\tau_\beta \leq T$. In that case, the terminal payoff is exactly $1 - \alpha + \frac{I_T}{I_0}$ for the Barrier and the simple RCN since $\mathbf{1}_{\{\tau_\beta \leq T\}} = 1$.

If $I_T > \alpha I_0$, then $\left(\alpha - \frac{I_T}{I_0} \right)^+ = 0$. In that case, the terminal payoff is exactly 1 for the Barrier and the simple RCN.

Q4

In order to replicate the callable RCN, we consider the portfolio made up of the ZCB and the put option as in question 2. Additionally, we need to short the bermudean option on a non-callable barrier RCN, starting at the time t_0, when the bermudean is exercised.

More precisely, the underlying derivative of the bermudean exercised at time t_0 pays out $\text{RCN}_{t_0} - 1$, where RCN_{t_0} is the non-callable barrier RCN starting from time $t = t_0 > 0$ instead of time $t = 0$. By shorting such bermudean option, we assure the payout of a principle (by that -1), while countering the remaining part of the non-callable barrier RCN replicated with the bonds and the put as in question 2.

The non-callable RCN in the bermudian can thus also be replicated via the bonds and the down-and-in put, it is just neccessary to note, that this down-and-in put considers also if the barrier was hit before the exercise time of the bermudian, t_0.

The possible exercise dates of this bermudian are all T_j, where $j = 1, \ldots, N$, with expiration date T_N.

Q5

Denote by P_t the value of the derivative at date $t \in \{T_j\}_{j=1}^N$. The seller can either exercise in which case he pays $1 + \Delta c$, or continue in which case his position is worth $-N_t - \Delta c$ (coupon is going to be paid no matter what), where

$$N_t = e^{-r\Delta t}\mathbb{E}_t^{\mathbb{Q}}\left[P_{t+1}\right]$$

is the continuation value of the derivative.

The seller will therefore exercise on the set $\{1 \leq N_t\}$ and otherwise take it to the next period.

The value of the derivative can be computed as

$$P_t = \mathbf{1}_{\{1\leq N_t\}}(1+\Delta c) + \mathbf{1}_{\{1>N_t\}}(N_t + \Delta c) = \min\left\{1, e^{-r\Delta t}\mathbb{E}_t^{\mathbb{Q}}\left[P_{t+1}\right]\right\} + \Delta c$$

3 Valuation code

See the provided python scripts; `RCN.py` *and* `RCN_callable.py`*.*

4 Model Calibration

See the python notebook `part5_analysis.ipynb` *for the actual calibration.*

To estimate r and δ, the put-call parity was used. More precisely, the following equation:

$$c_t - p_t = S_t - \mathcal{D}_t(T) - Ke^{-r(T-t)},$$

where c_t, p_t are the price of the call and put option at time t, respectively. We assume that the dividend is payed at $t = T$ such that its discounted value is $\mathcal{D}_t(T) = S_t\delta e^{-r(T-t)}$, where δ is the annualized dividend yield. The term S_t is the asset price at time t and K is the strike price. Our data is given as initial prices of options, so at time $t = 0$. Also, T is one year, so $T - t = 1$. Thus, the equation is as follows:

$$c_0 - p_0 = S_0 - S_0\delta e^{-r} - Ke^{-r}.$$

If we note $\alpha := S_0 - S_0\delta e^{-r}$ and $\beta := -e^{-r}$, we have that

$$c_0 - p_0 = \alpha + \beta K.$$

In a separate file we estimated α and β, from which we then inferred the estimates

$$\hat{r} = -0.00780,$$

$$\hat{\delta} = 0.0272.$$

5 Analysis

See the python notebook `part5_analysis.ipynb` *for the price estimation, the figures and the analysis.*

		c	0.05	0.07	0.09	0.11	0.13	0.15	0.17	0.19	0.21	0.23	0.25
α	$\frac{\beta}{\alpha}$	Callable											
0.2	0.2	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.4	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.6	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.8	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	1.0	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
0.4	0.2	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.4	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.6	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.8	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	1.0	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
0.6	0.2	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.4	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.6	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.8	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	1.0	NC	1.0579	1.0779	1.0980	1.1181	1.1382	1.1583	1.1784	1.1984	1.2185	1.2386	1.2587
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
0.8	0.2	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.4	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.6	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.8	NC	1.0558	1.0759	1.0960	1.1161	1.1361	1.1562	1.1763	1.1964	1.2165	1.2366	1.2567
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	1.0	NC	1.0443	1.0644	1.0844	1.1045	1.1246	1.1447	1.1648	1.1849	1.2050	1.2250	1.2451
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
1.0	0.2	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.4	NC	1.0580	1.0781	1.0982	1.1183	1.1384	1.1585	1.1786	1.1986	1.2187	1.2388	1.2589
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.6	NC	1.0553	1.0754	1.0955	1.1156	1.1357	1.1558	1.1758	1.1959	1.2160	1.2361	1.2562
		C	1.0048	1.0065	1.0082	1.0098	1.0115	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	0.8	NC	1.0043	1.0244	1.0444	1.0645	1.0846	1.1047	1.1248	1.1449	1.1649	1.1850	1.2051
		C	0.9772	0.9883	0.9959	1.0031	1.0085	1.0132	1.0148	1.0165	1.0182	1.0198	1.0215
	1.0	NC	0.9627	0.9828	1.0029	1.0230	1.0430	1.0631	1.0832	1.1033	1.1234	1.1435	1.1636
		C	0.9569	0.9731	0.9868	0.9968	1.0050	1.0115	1.0148	1.0165	1.0182	1.0198	1.0215

Table 2: Prices of the RCN for different sets of parameters