FIN405-PS10

Jon Kqiku, Mingchi Hou

May 2021

1 Mean-variance investing with fixed, linear-proportional, and quadratic transaction costs

a)

The first order condition is given by

$$\mu - \gamma\sigma^2 x_1 - sgn(\Delta x)b_1 - \lambda(\Delta x) = 0 \tag{1}$$

which yields the following optimal

$$x_1^* = \begin{cases} \dfrac{\mu - b1 + \lambda x_0}{\gamma\sigma^2 + \lambda} & \text{if } sgn(\Delta x) > 0 \text{ and } \dfrac{\mu - b1 + \lambda x_0}{\gamma\sigma^2 + \lambda}\mu - b_0 > x_0\mu - \dfrac{\gamma}{2}x_0\sigma^2 \quad (2) \\ \dfrac{\mu + b1 + \lambda x_0}{\gamma\sigma^2 + \lambda} & \text{if } sgn(\Delta x) < 0 \text{ and } \dfrac{\mu + b1 + \lambda x_0}{\gamma\sigma^2 + \lambda}\mu - b_0 > x_0\mu - \dfrac{\gamma}{2}x_0\sigma^2 \quad (3) \\ x_0 & \text{else} \quad (4) \end{cases}$$

The no-trade region is thus characterized by

$$\left(\left[\frac{\mu - b1 + \lambda x_0}{\gamma\sigma^2 + \lambda}, +\infty\right) \bigcup R_0^C\right) \bigcap \left(\left(-\infty, \frac{\mu + b1 + \lambda x_0}{\gamma\sigma^2 + \lambda}\right] \bigcup S_0^C\right), \tag{5}$$

where R_0 and S_0 correspond to the regions of x_0 characterized by the second inequalities in (2) and (3) respectively.
We can write

$$\begin{aligned} x_1 &= \frac{\mu - sgn(\Delta x)b1 + \lambda x_0}{\gamma\sigma^2 + \lambda} \\ &= \frac{\mu - b_1}{\gamma\sigma^2 + \lambda} + \frac{\lambda}{\gamma\sigma^2 + \lambda}x_0 \\ &= \tau aim + (1 - \tau)x_0 \end{aligned}$$

so that

$$\tau = \frac{\gamma\sigma^2}{\gamma\sigma^2 + \lambda} \tag{6}$$

$$aim = \frac{\mu - b_1}{\gamma\sigma^2}. \tag{7}$$

b)

If $b_0 = 0$, the no-trade region becomes

$$\left[\frac{\mu - b1 + \lambda x_0}{\gamma\sigma^2 + \lambda}, \frac{\mu - b1 + \lambda x_0}{\gamma\sigma^2 + \lambda}\right] \tag{8}$$

and there is no other changes in x_1^*.
If $b_1 = 0$, the optimal becomes

$$x_1^* = \begin{cases} \frac{\mu+\lambda x_0}{\gamma\sigma^2+\lambda} & \text{if } \frac{\mu+\lambda x_0}{\gamma\sigma^2+\lambda} - b_0 > x_0\mu - \frac{\gamma}{2}x_0^2\sigma^2 \\ x_0 & \text{else} \end{cases}$$

If $b_0 = b_1 = 0$ then

$$x_1^* = (1-\tau)x_0 + \tau\frac{\mu}{\gamma\sigma^2} \tag{9}$$

with $\tau = \frac{1}{1+\frac{\lambda}{\gamma}}$.
If $\lambda = 0$, the optimal becomes

$$x_1^* = \begin{cases} \frac{\mu-b_1}{\gamma\sigma^2} & \text{if } sgn(\Delta x) > 0 \text{ and } \frac{\mu-b1}{\gamma\sigma^2+\lambda}\mu - b_0 > x_0\mu - \frac{\gamma}{2}x_0\sigma^2 \\ \frac{\mu+b_1}{\gamma\sigma^2} & \text{if } sgn(\Delta x) < 0 \text{ and } \frac{\mu+b1}{\gamma\sigma^2+\lambda}\mu - b_0 > x_0\mu - \frac{\gamma}{2}x_0\sigma^2 \\ x_0 & \text{else} \end{cases}$$

2 Problem 2

2.1 1 risky asset and risk free asset

When $X_1 > X_0$, i.e. it is favored to buy more risky assets. From the first order criterion of the given optimization problem, we have

$$0 = \mu - R_f - \gamma\Sigma X_1 - b$$

Then the optimal solution for X_1 is

$$X_1^* = (\gamma\Sigma)^{-1}(\mu - R_f - b) \tag{10}$$

As long as $X_0 < X_1^*$, it is optimal to increase to X_1^*. In other words, $X_L = X_1^* = (\gamma\Sigma)^{-1}(\mu - R_f - b)$.
When $X_1 < X_0$, i.e. it is favored to sell more risky assets. From the first order criterion of the given optimization problem, we have

$$0 = \mu - R_f - \gamma\Sigma X_1 + b$$

Then the optimal solution for X_1 is

$$X_1^{**} = (\gamma\Sigma)^{-1}(\mu - R_f + b) \tag{11}$$

As long as $X_0 > X_1^{**}$, it is optimal to decrease to X_1^{**}. In other words, $X_H = X_1^{**} = (\gamma\Sigma)^{-1}(\mu - R_f + b)$.
Therefore, the No trade region is $[(\gamma\Sigma)^{-1}(\mu - R_f - b)\ (\gamma\Sigma)^{-1}(\mu - R_f + b)]$
Taking the given values into consideration, we generate the following plot (For the details of the plots, *see PS10-Code.ipynb*):

Figure 1: Trading region for 1 risky and 1 risk free assets

In the figure, the No trade area is between the two dotted lines. The dot indicates $X_1^*, and X_1**$ respectively.

2.2

With Buy(B), Sell(S), No trade(N) three options, there will be 9 regions for 2 risky assets and 1 risk free asset. Observing $X_1^*, and X_1**$, they differ only by the factor in front of b.

- Buy Asset 1 and Buy Asset 2
  From the FOC in the previous task, we can derive the optimal solution for both assets separately:

$$X_{BB1} = (\gamma\Sigma)^{-1}(\mu - R_f - b)$$

$$X_{BB2} = (\gamma\Sigma)^{-1}(\mu - R_f - b)$$

As long as $X_{10} < X_{BB1}$, $X_{20} < X_{BB2}$, it is optimal to increase the position of both assets. Then the region covered by BB is denoted as

$$BB = \{X_0 \in R^2 | X_{10} \leq X_{BB1} \& X_{20} \leq X_{BB2}\} \tag{12}$$

- Buy Asset 1 and Sell Asset 2
  From the FOC in the previous task, we can derive the optimal solution for both assets separately:

$$X_{BS1} = (\gamma\Sigma)^{-1}(\mu - R_f - b)$$
$$X_{BS2} = (\gamma\Sigma)^{-1}(\mu - R_f + b)$$

  As long as $X_{10} < X_{BS1}$, $X_{20} > X_{BS2}$, it is optimal to increase the position of Asset 1, and decrease the position of Asset 2. Then the region covered by BS is denoted as

$$BS = \{X_0 \in R^2 | X_{10} \leq X_{BS1} \& X_{20} \geq X_{BS2}\} \tag{13}$$

- Sell Asset 1 and Buy Asset 2
  Same derivation as above, and we get

$$X_{SB1} = (\gamma\Sigma)^{-1}(\mu - R_f + b)$$
$$X_{SB2} = (\gamma\Sigma)^{-1}(\mu - R_f - b)$$
$$SB = \{X_0 \in R^2 | X_{10} \geq X_{SB1} \& X_{20} \leq X_{SB2}\} \tag{14}$$

- Sell Asset 1 and Sell Asset 2
  Same derivation as above, and we get

$$X_{SS1} = (\gamma\Sigma)^{-1}(\mu - R_f + b)$$
$$X_{SS2} = (\gamma\Sigma)^{-1}(\mu - R_f + b)$$
$$SS = \{X_0 \in R^2 | X_{10} \geq X_{SS1} \& X_{20} \geq X_{SS2}\} \tag{15}$$

- Buy Asset 1 and not trade on Asset 2
  Since there is no trade on Asset 2, the terminal position is the same as the initial position. i.e. $X_{21} = X_{20}$. Then the optimal trade can be derived from solving the optimization

$$max_{X_{11}} R_f + X_{11}(\mu_1 - R_f - b) + X_{20}(\mu_1 - R_f) - \frac{\gamma}{2}(\sigma_1^2 X_{11}^2 + \sigma_2^2 X_{20}^2 + 2\rho\sigma_1\sigma_2 X_{11} X_{20}) \tag{16}$$

  FOC gives $0 = \mu_1 - R_f - b - \gamma\sigma_1^2 X_{11} - \rho\sigma_1\sigma_2 X_{20}$

$$X_{11BN} = \frac{\mu_1 - R_f - b}{\gamma\sigma_1^2} - \frac{\rho\sigma_2}{\gamma\sigma_1} X_{20}$$

if $\rho \neq 0$, the optimal trade has a linear relationship with the position of the position of Asset 2. Combining the constraints of buy A1 and no trade in A2, we have

$$BN = \{X_0 \in R^2 | X_{10} \leq X_{11BN} \& X_{BB2} \leq X_{20} \leq X_{BS2}\} \tag{17}$$

- Sell Asset 1 and no trade in Asset 2
  Similarly, there should be just a sign change in front of b. So, we get

$$X_{11SN} = \frac{\mu_1 - R_f + b}{\gamma \sigma_1^2} - \frac{\rho \sigma_2}{\gamma \sigma_1} X_{20}$$

$$SN = \{X_0 \in R^2 | X_{10} \geq X_{11SN} \& X_{SB2} \leq X_{20} \leq X_{SS2}\} \tag{18}$$

- No trade in Asset 1 and Buy Asset 2
  Since there is no trade on Asset 1, the terminal position is the same as the initial position. i.e. $X_{11} = X_{10}$. Then the optimal trade can be derived from solving the optimization

$$max_{X_{21}} R_f + X_{21}(\mu_2 - R_f - b) + X_{10}(\mu_2 - R_f) - \frac{\gamma}{2}(\sigma_1^2 X_{10}^2 + \sigma_2^2 X_{21}^2 + 2\rho\sigma_1\sigma_2 X_{10} X_{21}) \tag{19}$$

FOC gives $0 = \mu_2 - R_f - b - \gamma \sigma_2^2 X_{21} - \rho \sigma_1 \sigma_2 X_{10}$

$$X_{21NB} = \frac{\mu_2 - R_f - b}{\gamma \sigma_2^2} - \frac{\rho \sigma_1}{\gamma \sigma_2} X_{10}$$

if $\rho \neq 0$, the optimal trade has a linear relationship with the position of the position of Asset 1. Combining the constraints of buy A2 and no trade in A1, we have

$$NB = \{X_0 \in R^2 | X_{BB1} \leq X_{10} \leq X_{SB1} \& X_{20} \leq X_{21NB}\} \tag{20}$$

- No trade in Asset 1 and Sell Asset 2
  Similarly, there should be just a sign change in front of b. So, we get

$$X_{21NS} = \frac{\mu_2 - R_f + b}{\gamma \sigma_2^2} - \frac{\rho \sigma_1}{\gamma \sigma_2} X_{10}$$

$$NS = \{X_0 \in R^2 | X_{BS1} \leq X_{10} \leq X_{SS1} \& X_{20} \geq X_{21NS}\} \tag{21}$$

- Last but not least, trade neither assets
  So we can take the Union of all the sets above, and the find the complement. That will be the non trading region.

$$NN = \{X_0 \in R^2 | (BB \bigcup BS \bigcup SB \bigcup SS \bigcup BN \bigcup SN \bigcup NB \bigcup NS)^c\} \tag{22}$$

2.3

- Effect of ρ as shown in Figure 2
  When $\rho = 0$, $X_{11BN}, X_{11SN}, X_{21BN}, X21SN$ are constants. In other words the edges of the no-trade region are either horizontal or vertical line segments. Then the no trade region is a rectangle. In addition, according to the position, we can get directly the optimal strategy and for how much we should trade.

  When ρ increases, the no trade region becomes a parallelogram. As X_{20} increases, X_{11BN}, X_{11SN} decreases. As X_{10} increases, X_{21NB}, X_{21NS} decreases. With a larger ρ, the speed of decrease becomes larger. In other words, with larger ρ the height of the parallelogram decreases. And the function for edge of the region has steeper slope.

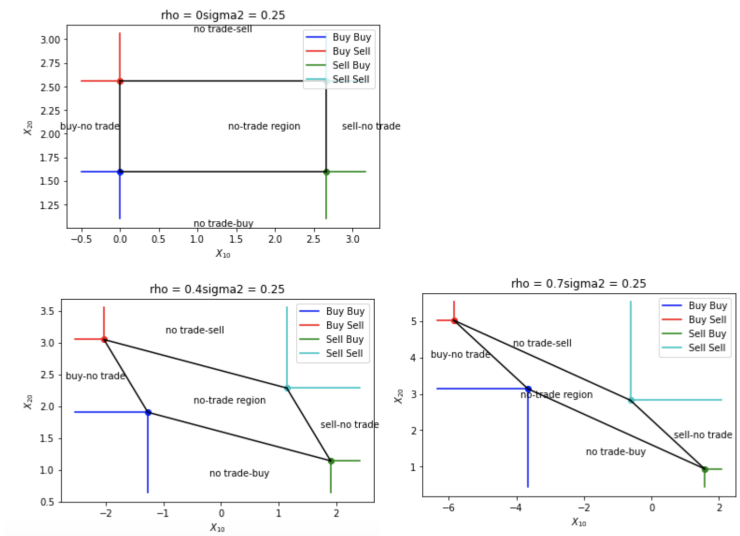

Figure 2: No trade region with different correlation

- Effect of σ_2 shown in Figure 3
  For σ_2, as X_{20} increases, X_{11BN}, X_{11SN} decrease. With larger σ_2, they decrease faster. As X_{10} increases, X_{21NB}, X_{21NS} decreases. Since $0 < \sigma_2^2 < \sigma_2 < 1$, the speed of the decrease is going down. In conclusion, with riskier asset 2, the parallelogram has steeper slopes on the edge and the overall surface of the No trade region becomes smaller.

For the details of the plots, *see PS10-Code.ipynb*

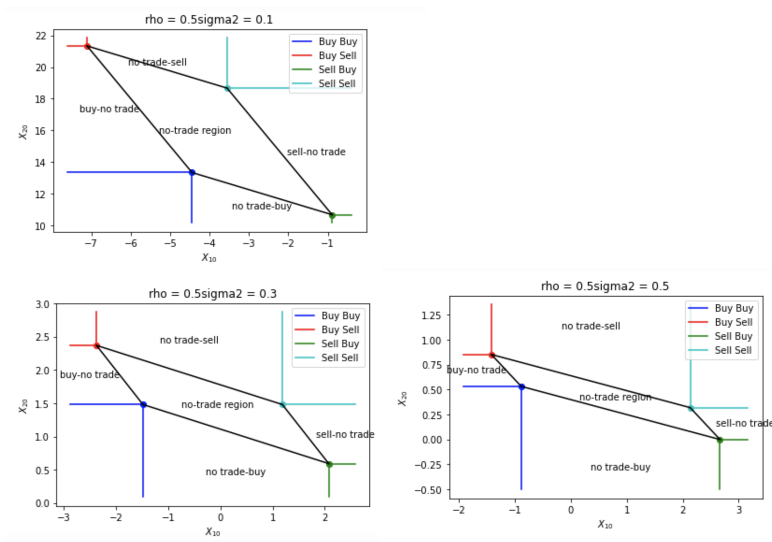

Figure 3: No trade region with different risk level of Asset 2

3 Optimal Dynamic trading of a single asset with linear-proportional price impact

a)

We have that

$$V(T, n_{T-1}) = \max_{n_T} n_T\mu - \frac{\lambda}{2}(n_T - n_{T-1})^2 - \frac{\gamma}{2}n_T^2\sigma^2 \tag{23}$$

whose FOC is

$$\mu - \lambda(n_T - n_{T-1}) - \gamma\sigma^2 n_T = 0 \tag{24}$$

yielding the optimal

$$n_T^* = \frac{\mu + \lambda n_{T-1}}{\lambda + \gamma\sigma^2} \tag{25}$$

Inserting (25) in (23) yields

$$
\begin{aligned}
V(T,n) &= \frac{\mu+\lambda n}{\lambda+\gamma\sigma^2} - \frac{\lambda}{2}\left(\frac{\mu+\lambda n}{\lambda+\gamma\sigma^2} - n\right)^2 - \frac{\gamma}{2}\left(\frac{\mu+\lambda n}{\lambda+\gamma\sigma^2}\right)^2\sigma^2 \\
&\vdots \\
&= -\left[\frac{\lambda^2\left(\frac{\lambda}{2}+\frac{\gamma}{2}\right)}{(\lambda+\gamma\sigma^2)^2} + \left(\frac{\lambda^2}{\lambda+\gamma\sigma^2} + \frac{\lambda}{2}\right)\right] n^2 \\
&\quad + \left[\frac{2\mu\lambda\left(\frac{\lambda}{2}+\frac{\gamma}{2}\right)}{(\lambda+\gamma\sigma^2)^2} + \frac{\lambda(\mu+1)}{\lambda+\gamma\sigma^2}\right] n \\
&\quad + \frac{\mu^2}{\lambda+\gamma\sigma^2} - \frac{\mu^2\left(\frac{\lambda}{2}+\frac{\gamma}{2}\right)}{(\lambda+\gamma\sigma^2)^2}
\end{aligned}
$$

with

$$
Q_t = \frac{1}{2}\left[\frac{\lambda^2\left(\frac{\lambda}{2}+\frac{\gamma}{2}\right)}{(\lambda+\gamma\sigma^2)^2} + \left(\frac{\lambda^2}{\lambda+\gamma\sigma^2} + \frac{\lambda}{2}\right)\right] \tag{26}
$$

$$
q_T = \left[\frac{2\mu\lambda\left(\frac{\lambda}{2}+\frac{\gamma}{2}\right)}{(\lambda+\gamma\sigma^2)^2} + \frac{\lambda(\mu+1)}{\lambda+\gamma\sigma^2}\right] \tag{27}
$$

$$
c_T = \frac{\mu^2}{\lambda+\gamma\sigma^2} - \frac{\mu^2\left(\frac{\lambda}{2}+\frac{\gamma}{2}\right)}{(\lambda+\gamma\sigma^2)^2}. \tag{28}
$$

b)

Using the assumption that $V(t+1,n) = -\frac{1}{2}n^2Q_{t+1} + nq_{t+1} + c_{t+1}$, we have that

$$
V(t,n_{t-1}) = \max_{n_t}\left\{n_t\mu - \frac{\lambda}{2}(n_t-n_{t-1})^2 - \frac{\gamma}{2}n_t^2\sigma^2 + \rho\left(-\frac{1}{2}n_t^2Q_{t+1} + n_tq_{t+1} + c_{t+1}m\right)\right\} \tag{29}
$$

The FOC yields the optimal

$$
n_t^* = \frac{\mu+\rho q_{t+1}}{\lambda+\rho Q_{t+1}+\gamma\sigma^2}. \tag{30}
$$

Inserting the optimal back into $V(t,n_{t-1})$ yields an equation of the same form as $V(t+1,n)$, namely

$$
V(t,n) = -\frac{1}{2}n^2Q_t + nq_t + c_t, \tag{31}
$$

where

$$
Q_t = \lambda, \tag{32}
$$

$$
q_t = \lambda\left(\frac{\mu+\rho q_{t+1}}{\lambda\rho Q_{t+1}+\gamma\sigma^2}\right) \tag{33}
$$

and

$$c_t = \mu \frac{\mu + \rho q_{t+1}}{\lambda \rho Q_{t+1} + \gamma \sigma^2} - \frac{\lambda}{2} \left(\frac{\mu + \rho q_{t+1}}{\lambda \rho Q_{t+1} + \gamma \sigma^2} \right)^2 - \frac{\gamma}{2} \sigma^2 \left(\frac{\mu + \rho q_{t+1}}{\lambda \rho Q_{t+1} + \gamma \sigma^2} \right)^2 + \rho \left[-\frac{1}{2} \left(\frac{\mu + \rho q_{t+1}}{\lambda \rho Q_{t+1} + \gamma \sigma^2} \right)^2 Q_{t+1} + \frac{\mu + \rho q_{t+1}}{\lambda \rho Q_{t+1} + \gamma \sigma^2} q_{t+1} + c_{t+1} \right]. \quad (34)$$

c)

We have

$$n_t = \frac{\mu + \rho q_{t+1}}{\lambda + \rho Q_{t+1} + \gamma \sigma^2}$$

and

$$\begin{aligned} n_{t+1} &= \frac{\mu + \rho q_{t+2}}{\lambda + \rho Q_{t+2} + \gamma \sigma^2} \\ &= \frac{\mu + \rho \lambda n_t}{\lambda + \rho \lambda + \gamma \sigma^2} \\ &= \frac{\mu}{\lambda + \rho \lambda + \gamma \sigma^2} + \frac{\rho \lambda}{\lambda + \rho \lambda + \gamma \sigma^2} n_t \\ &= \tau_t aim_t + (1 - \tau_t) n_t \end{aligned}$$

so that

$$\tau_t = \frac{\lambda + \gamma \sigma^2}{\lambda + \rho \lambda + \gamma \sigma^2} \quad (35)$$

$$aim_t = \frac{\mu}{\lambda + \gamma \sigma^2}. \quad (36)$$

d)
Since $\mu = 0$, the aim portfolio is also equal to zero for all t which implies in particular that the investor would want to hold zero shares of stock at T.
When $\mu = 0$, $n_{t+1} = (1 - \tau_t) n_t$. But since τ_t is also constant, we have that

$$n_t = (1 - \tau)^{t+1} n_{-1},$$

which shows that $\lim_{(1-\tau)\to 0} n_T = 0$ or equivalently $\lim_{\gamma \to \infty} n_T = 0$, which would lead us to think that the calibrated γ would be infinite. However, there is a trade-off between the the extent to which the constraint that n_T be close to zero (which is increasing in γ) is satisfied and the costs of liquidating the portfolio (which is also increasing in γ). *see PS10-Code.ipynb*
e)

It is the optimal thing to do since shocks are centered around zero, trading based on shocks would only lead to higher transaction costs since it would only lead to a repeated buy/sell pattern. For shocks to be considered in the trading decisions, ϵ_t would need to enter in the value function.
f)

We could add this variability of λ by modeling the slippage costs as

$$(2 - \mathbf{1}_{\{t<1200 \text{ or } t>1400\}})\lambda(n_{t+1} - n_t)^2$$

which effectively double between 12pm and 2pm.