Assignment 7

1. **APT (50 points)**

   Consider the following model for the stock return vector $R = [R_1 \dots R_N]$

   $$R = \alpha + BF + \epsilon$$

   where B is an N, K matrix of factor exposures and F is a K-dimensional vector of random factor realization with $E[F] = \mu_F$ and $V[F] = \Omega_F$ and ϵ is an N-dimensional vector of idiosyncratic risks with zero mean and $V[\epsilon] = \Omega_\epsilon$ a diagonal matrix. Suppose there is a risk-free asset with return R_0.

   (a) Assume $\Omega_\epsilon = 0$. Show that absence of arbitrage implies that there exists some λ such that $\alpha = R_0 + B(\lambda - \mu_F)$.

   (b) If $\Omega_\epsilon \neq 0$ then the APT implies the same restriction on α as in the previous question, should hold (a) for all stocks if $n \to \infty$, (b) for all stocks for any finite n, (c) for most stocks if $n \to \infty$, (d) for most stocks for any finite n. Which is right? Give a short explanation.

   (c) For the rest assume the APT restriction on α holds for all stocks. Write the equation for the vector or excess returns $R^e = R - R_0\mathbf{1}$. Define the variance of returns $V[R] = \Sigma = B\Omega_F B^\top + \Omega_\epsilon$.

   (d) Suppose the factors F_k are not traded asset returns. Define the 'factor-mimicking portfolio' weight (N,1) vector w_k to be the portfolio with excess returns $\hat{F}_k = w_k^\top R^e$, chosen to minimize the total risk of the portfolio subject to unit exposure to the k^{th} factor. Specifically the k^{th}-factor mimicking portfolio solves:

   $$\min_{w_k} \frac{1}{2} w_k^\top \Sigma w_k \;\; s.t. \;\; w_k^\top B = e_k^\top$$

   where e_k is a vector of zeros with a 1 in the k^{th} line. Solve for the w_k. Let's define the vector of expected excess returns and the covariance matrix of the factor mimicking portfolios by $\hat{\mu}_F = E[\hat{F}]$ and $\hat{\Omega}_F = V[\hat{F}]$.

   (e) Show that $\hat{\mu}_F = \lambda$ and
   show that $\hat{\Omega}_F = (B^\top \Sigma^{-1} B)^{-1} = \Omega_F + (B^\top \Omega_\epsilon^{-1} B)^{-1}$.

HINT: You may simply take the second equality in the second equation as given. However, if you wish to prove it use the "Woodbury matrix equality" which states that $(A + UCV)^{-1} = A^{-1} - A^{-1}U[C^{-1} + VA^{-1}U]^{-1}VA^{-1}$, for suitably sized matrices A, U, C, V.

(f) Prove that if the APT holds then, the mean-variance optimal portfolio w_{mve} with return $R_p = R_0 + w_{mve}^\top R^e$, which solves

$$\max_w E[R_P] - \frac{\gamma}{2} V[R_p]$$

can be rewritten as a linear combination of the K factor-mimicking portfolios. Specifically if we define $W = [w_1, w_2, \ldots, w_K]$ the (N, K) matrix of the K factor-mimicking portfolio weight vectors, then prove that

$$w_{mve} = W\eta$$

where η is K-dimensional vector of loadings on the various factor-mimicking portfolios.

(g) Show that η solves

$$\max_\eta \eta^\top \hat{\mu}_F - \frac{\gamma}{2} \eta^\top \hat{\Omega}_F \eta$$

and conclude that if the APT holds with a number of (possibly non-traded) factors $K < N$ then any mean-variance efficient portfolio consists of K-funds represented by the K-factor mimicking portfolios (there is K-fund separation).

(h) Give sufficient conditions for a mean-variance investor to choose not to invest in a factor-mimicking portfolio. Is it sufficient that $E[\hat{F}_k] = \lambda_k = 0$? Why?

2. **Beta and expected returns (50 points).** In this exercise we will test the CAPM using portfolio sorted based on beta.

Next week, we will continue with the same setup and test momentum and size-sorted portfolios. This two-part exercise will highlight (i) the difference between equal-weighting and value weighting returns, and (ii) the importance of avoiding forward-looking data in testing strategy performances.

(a) Download monthly stock returns from CRSP for all common stocks (share codes (shcd) 10 and 11) traded on NYSE (exchange codes (excd) 1 and 2) from 1980 to

December 31,2019. Also download a risk-free rate and the value-weighted CRSP market return. For the risk-free rate, use the same data that you also used in Problem Set 3. Use `select date, vwretd from crsp.msi` to obtain data on the CRSP value-weighted index return. In order to get the exchange codes you can for instance access CRSP's stock event file. You can do this by using a query of the following form:

```
select a.permno, a.date, b.shrcd, b.exchcd, ...  (etc.)
from crsp.msf as a left join crsp.msenames as b
on a.permno=b.permno and b.namedt<=a.date ...  (etc.)
```

Delete data of all stocks for which you have less than 480 observations on returns. This should leave you with 275 stocks that have been traded every single month from the beginning of 1980 to the end of 2019.

(b) Using the full sample, estimate the market beta for each stock. One way of doing this is to use `df.groupby()` to calculate the relevant moments for each stock,[1] merge these data with the original dataset (left join) and then create a new column in the original dataset that contains the market betas (you are however free to choose any procedure that works). For each month, sort stocks by beta into 10 decile portfolios. For each portfolio compute the equal-weighted average average return and compute the beta of the portfolio excess returns with respect to the market excess return for the full sample. Plot the 10 average portfolio returns for the full sample versus the portfolios' betas. If you fit a line through these points, how does the slope of that line compare to the average market excess return for the sample? Are these findings consistent with the CAPM?

(c) Notice that the previous results are forward looking in the sense that the strategy could not have been implemented in real time, since we used the full-sample to estimate the betas. Instead, we would like to have a test that does not suffer from look ahead bias. To that end, compute market betas using the period from 1980 to December 31, 1999. Then, starting in 2000, form 10 portfolios as in point b), but using the betas based on the period from 1980 to 1999. Compute the average returns (for the sample starting in 2000) and betas (for both samples: the first one, starting in 1980 and the second one, starting in 2000) for those portfolios.

Plot the average return to these 'out-of-sample-beta' decile portfolios in the second

[1]With the .goupby() method you can perform operations on subsets of your data separately, e.g. calculate covariances for each PERMNO.

sample period versus their average beta in the first sample period. Also plot the portfolios' betas in the second sample period against the portfolios' betas in the first sample period. How stable are the betas across periods? Are the out-of-sample findings consistent with the CAPM?

(d) In their Betting-against-Beta paper Frazzini and Pedersen argue that one can make abnormal profits by going long low-beta stocks and shorting high beta stocks, with sufficient leverage in the low-beta portfolio so as to have zero market exposure. To investigate their findings compute the mean, standard deviation, and sharpe ratio in the second period (post 2000) of a strategy that goes long $\frac{1}{\beta_L}$ of the lowest decile beta portfolio financed at the risk-free rate, and goes short $\frac{1}{\beta_H}$ of the highest decile beta portfolio financed at the risk-free rate, where β_L and β_H are the betas of the respective portfolios from the first period (before 2000). Are your findings consistent with the existence of a betting against beta factor? What might explain your findings?