PS 11

Jon Kqiku, Mingchi Hou

May 2021

1 Bayesian updating

a)

Let $x_0 = \mu = \mu_0 + \nu_0 \sim \mathcal{N}(\mu_0, v_0^2)$ and $x_1 = \mu + \nu_1 = \mu_0 + \nu_0 + \nu_1 \sim \mathcal{N}(\mu_0, v_0^2 + v_1^2)$, assuming $\nu_0 \perp\!\!\!\perp \nu_1$. Then, $m = E\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \mu_0 \\ \mu_0 \end{pmatrix}$ and

$$\begin{aligned} Cov(x_0, x_1) &= Cov(\mu, \mu + \nu_1) \\ &= v_0^2 + Cov(\mu, \nu_1) = v_0^2. \end{aligned}$$

We now have

$$\Omega = Cov\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} v_0^2 & v_0^2 \\ v_0^2 & v_0^2 + v_1^2 \end{pmatrix}$$

so that $\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \sim \mathcal{N}(m, \Omega)$.
The Gaussian projection theorem states that

$$x_0 | x_1 = \mu_1 \sim \mathcal{N}(\hat{\mu}, \hat{v}^2)$$

with

$$\begin{aligned} \hat{\mu} &= \mu_0 + \frac{v_0^2}{v_0^2 + v_1^2}(\mu_1 - \mu_)) = \mu_0 + \beta(\mu_1 - \mu_)) \\ \hat{v}^2 &= v0^2 - \frac{v_0^4}{v_0^2 + v_1^2} = v_0^2 - \left(\frac{v_0^2}{v_0^2 + v_1^2}\right)^2 (v_0^2 + v_1^2) \\ &= v_0^2 - \beta(v_0^2 + v_1^2), \end{aligned}$$

where $\beta = \frac{v_0^2}{v_0^2 + v_1^2}$.

b)

$$
\begin{aligned}
\hat{\mu} &= \mu_0 + \frac{v_0^2}{v_0^2 + v_1^2}(\mu_1 - \mu_)\\
&= \frac{\mu_0 v_0^2 + \mu_0 v_1^2 + \mu_1 v_0^2 - \mu_0 v_0^2}{v_0^2 + v_1^2}\\
&= \frac{\mu_0 v_1^2 + \mu_1 v_0^2}{v_0^2 + v_1^2}\\
&= \frac{\frac{\mu_0 v_1^2 + \mu_1 v_0^2}{v_0^2 + v_1^2}}{\frac{v_0^2 + v_1^2}{v_0^2 v_1^2}}\\
&= \frac{\frac{1}{v_0^2}\mu_0 + \frac{1}{v_1^2}\mu_1}{\frac{1}{v_0^2} + \frac{1}{v_1^2}}.
\end{aligned}
$$

$$
\begin{aligned}
\hat{v}^2 &= v_0^2 - \frac{v_0^4}{v_0^2 + v_1^2}\\
&= \frac{v_0^4 + v_0^2 v_1^2 - v_0^4}{v_0^2 + v_1^2}\\
&= \frac{\frac{v_0^2 v_1^2}{v_0^2 v_1^2}}{\frac{v_0^2 + v_1^2}{v_0^2 v_1^2}}\\
&= \frac{1}{v_0^2 + v_1^2}.
\end{aligned}
$$

The interpretation of the posterior mean $\hat{\mu}$ is that it is a normalized weighted average of the prior mean μ_0 and an additional signal μ_1. The weights are such that more weight is put on the prior/signal that has smaller variance, i.e. lower uncertainty.

c)

Assume that the formula holds for $n - 1 \in \mathbf{N}$, i.e.

$$
E(x_0 | x_1 = \mu_1, \ldots, x_{n-1} = \mu_{n-1}) = \frac{\sum_{i=0}^{n-1} \frac{1}{v_i^2}\mu_i}{\sum_{i=0}^{n-1} \frac{1}{v_i^2}} =: \tilde{\mu}
$$

$$
Var(x_0 | x_1 = \mu_1, \ldots, x_{n-1} = \mu_{n-1}) = \frac{1}{\sum_{i=0}^{n-1} \frac{1}{v_i^2}} =: \tilde{\sigma}^2
$$

Call $Y = \ x_0 | x_1 = \mu_1, \ldots, x_{n-1} = \mu_{n-1} \ \sim N(\tilde{\mu}, \tilde{\sigma}^2)$. We know that $Cov(Y, x_n) =$

$Cov(x_0, x_n) = v_0^2$. Thus, $\begin{pmatrix} Y \\ x_n \end{pmatrix} \sim N(m, \Omega)$ with $m = (\tilde{\mu}, \mu_0)^\top$ and

$$\Omega = \begin{pmatrix} \tilde{\sigma}^2 & v_0^2 \\ v_0^2 & v_0^2 + v_n^2 \end{pmatrix}.$$

Applying the projection theorem gives us that

$$Y|x_n = \mu_n \;=\; x_0|x_1 = \mu_1, \ldots, x_n = \mu_n \sim N(\hat{\mu}, \hat{v}^2)$$

with

$$\begin{aligned}
\hat{\mu} &= \tilde{\mu} + \frac{v-0^2}{v_0^2 + v_n^2}(\mu_n - \mu_0) \\
&= \frac{\sum_{i=0}^{n-1} \frac{1}{v_i^2}\mu_i}{\sum_{i=0}^{n-1} \frac{1}{v_i^2}} + \frac{v-0^2}{v_0^2 + v_n^2}(\mu_n - \mu_0) \\
&= \frac{\sum_{0 \le i_1 \le \ldots \le i_n \le n-1} v_{i_1}^2 \cdot \ldots \cdot v_{i_{n-1}}^2 \mu_{i_n}}{\sum_{0 \le i_1 \le \ldots \le i_n \le n-1} v_{i_1}^2 \cdot \ldots \cdot v_{i_{n-1}}^2} + \frac{v_0^2}{v_0^2 + v_n^2}(\mu_n - \mu_0) \\
&= \frac{\sum_{0 \le j_1 \le \ldots \le j_n \le n} v_{j_1}^2 \cdot \ldots \cdot v_{j_n}^2 \mu_{j_{n+1}}}{\sum_{0 \le j_1 \le \ldots \le j_n \le n} v_{j_1}^2 \cdot \ldots \cdot v_{j_n}^2} \\
&= \frac{\sum_{i=0}^{n} \frac{1}{v_i^2}\mu_i}{\sum_{i=0}^{n} \frac{1}{v_i^2}}
\end{aligned}$$

where the 3rd equality results from multiplying and dividing by $\prod_{i=0}^{n-1} v_i^2$ and the last one from multiplying and dividing by $(\prod_{i=0}^{n} v_i)^{-1}$, and with

$$\begin{aligned}
\hat{v}^2 &= \tilde{\sigma}^2 - \frac{v_0^4}{v_0^2 + v_1^2} \\
&= \frac{1}{\sum_{i=0}^{n-1} \frac{1}{v_i^2}} - \frac{v_0^4}{v_0^2 + v_n^2} \\
&= \frac{v_0^2 \cdot v_1^2 \cdot \ldots \cdot v_{n-1}^2}{\sum_{0 \le i_1 \le \ldots \le i_n \le n-1} v_{i_1}^2 \cdot \ldots \cdot v_{i_{n-1}}^2} - \frac{v_0^4}{v_0^2 + v_n^2} \\
&= \frac{v_0^2 \cdot v_1^2 \cdot \ldots \cdot v_n^2}{\sum_{0 \le j_1 \le \ldots \le j_n \le n} v_{j_1}^2 \cdot \ldots \cdot v_{j_n}^2} \\
&= \frac{1}{\sum_{i=0}^{n} \frac{1}{v_i^2}},
\end{aligned}$$

where the 3rd equality results from multiplying and dividing by $\prod_{i=0}^{n-1} v_i^2$ and the last one from multiplying and dividing by $(\prod_{i=0}^{n} v_i)^{-1}$.
If $\lim_{n\to\infty} \sum_{i=0}^{n} v_i^2 = \infty$, then $\lim_{n\to\infty} \hat{v}^2 = 0$ which effectively means that the uncertainty on the posterior distribution goes to zero as we add information (signals) with non-vanishing uncertainty.

2 Black Litterman

See *PS11-Code.ipynb* for calculations.

a)

We know that $R = \mu + \epsilon_R$ with $\epsilon_r \sim N(0, \Sigma)$ and $\mu = \mu_0 + \epsilon_0 = \mu_0$ such that $\mu \sim N(\mu_0, \Sigma)$. We want to find μ_0 such that $(\gamma\Sigma)^{-1}\mu_o = w_{eq}$. We thus find $\mu_0 = \gamma\Sigma w_{eq}$.

b)

We now have $R = \mu + \epsilon_R \sim N(\mu_0, \Sigma_0 = (1+\tau)\Sigma)$ such that the optimal portfolio given our prior is

$$w_0 = (\gamma(1+\tau)\Sigma)^{-1}\mu_0.$$

c)

We can express the views as

$$\begin{cases} 0.045 = \mu_4 - a\mu_3 - b\mu_6 + \epsilon_{Q_1} \\ 0.02 = \mu_2 - \mu_7 + \epsilon_{Q_2} \end{cases}$$

where a and b are the relative market weights of France and UK, and μ_i denotes the i-th component of μ, i.e. the prior mean of country i. We set the relative market weights as $a = w_{0,3}/(w_{0,3} + w_{0,6})$ and $b = w_{0,6}/(w_{0,3} + w_{0,6})$ with $w_{0,i}$ denoting the i-th component of w_0. We can write the views more compactly as

$$q = P\mu + \epsilon_Q$$

with

$$\begin{aligned} \epsilon_Q &\sim N\left(0, \begin{pmatrix} \Omega_{11} & 0 \\ 0 & \Omega_{22} \end{pmatrix}\right) \\ q &= \begin{pmatrix} 0.045 & 0.02 \end{pmatrix} \\ P &= \begin{pmatrix} 0 & 0 & -a & 1 & 0 & -b & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

d)

We know that

$$
\begin{aligned}
w^* &= (\gamma\hat{\Sigma})^{-1}\hat{\mu} \\
&= \frac{1}{\gamma}\left[\Sigma + \left(\frac{1}{\tau}\Sigma^{-1} + P^\top\Omega^{-1}P\right)^{-1}\right]^{-1} \cdot \\
&\quad \left(\frac{1}{\gamma}\Sigma^{-1} + P^\top\Omega^{-1}P\right)^{-1}\left(\frac{1}{\tau}\Sigma^{-1}\mu_0 + \frac{1}{\tau}P^\top\Sigma^{-1}q\right)
\end{aligned} \tag{1}
$$

and

$$
w_0 = \frac{1}{\gamma}\frac{1}{1+\tau}\Sigma^{-1}\mu_0.
$$

We want to find Λ such that

$$
\begin{aligned}
w^* &= w_0 + P^\top\Lambda \\
&= \frac{1}{\gamma}\frac{1}{1+\tau}\Sigma^{-1}\mu_0 + P^\top\Lambda
\end{aligned} \tag{2}
$$

Therefore, the Λ that satisfies (1) is given by

$$
\Lambda = \frac{1}{\gamma}\left[P\Sigma P^\top + \left(1 + \frac{1}{\tau}\right)\Omega\right]^{-1}\left(q - \frac{1}{1+\tau}P\mu_0\right)
$$

and thus the individual view scalar weights i are given by

$$
\lambda_i = \frac{1}{\gamma}\left[P_i\Sigma P_i^\top + \left(1 + \frac{1}{\tau}\right)\Omega\right]^{-1}\left(q_i - \frac{1}{1+\tau}P_i\mu_0\right).
$$

f)

See PS11-Code.ipynb