FIN-405
PS3

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March 2021

1 Efficient Portfolios

1)
We wish to maximize the utility function $U(\omega)$ with respect to ω.

$$U(\omega) = E[R_p] - \frac{\gamma}{2}V[R_p] \tag{1}$$
$$= R_f + \omega^\top(\mu - R_f\mathbf{1}) - \frac{\gamma}{2}\omega^\top\Omega\omega, \tag{2}$$

where $\Omega \in \mathbb{R}^{N\times N}$ is the true covariance matrix of returns $R_i,\ i = 1,\dots,N$. Differentiating (2) with respect to ω and cancelling yields

$$\mu - R_f\mathbf{1} = \gamma\Omega\omega \tag{3}$$

whose l.h.s and r.h.s i-th component is

$$\mu_i - R_f = \gamma\sum_{j=1}^{N}\Omega_{i,j}\omega_j \tag{4}$$

which boils down to

$$\mu_i - R_f = \gamma Cov[R_i, R_p]. \tag{5}$$

2)
By setting $\gamma = \frac{\mu_p - R_f}{\sigma_p^2}$, we have

$$\mu_i - R_f = \beta_{i,p}(\mu_p - R_f) \tag{6}$$

since $\beta_{i,p} = Cov[R_i, R_p]/\sigma_p^2$. This holds because for an increase in the expected excess return of the portfolio, risk aversion has to increase to keep the utility level constant, and inversely for an increase in the variance of the portfolio return.

3)
From (6) we know that

$$\mu_i - R_f = \beta_{i,p}(\mu_p - R_f). \tag{7}$$

If we model $R_{i,t} - R_f$ as a simple linear function of $X_{i,t} = R_{p,t} - R_f$ for $i = 1, \dots, N$, $t = 1, \dots, T$ plus a zero mean Gaussian error term $\epsilon_{i,t} \perp\!\!\!\perp X_{i,t}$ as

$$R_{i,t} - R_f = \beta_{i,p}(R_{p,t} - R_f) + \epsilon_{i,t} \tag{8}$$

we get

$$R_{i,t} = R_f + \beta_{i,p}(R_{p,t} - R_f) + \epsilon_{i,t}. \tag{9}$$

By keeping the deterministic R_f in the l.h.s in (8), we know that the ordinary least squares estimate of $\beta_{i,p}$ is $Cov[R_i, R_p]/\sigma_p^2$. Then, by taking expectations we get exactly (6).

4)
We know that the optimal portfolio weights are

$$\omega_0 = \frac{1}{\gamma}\Omega(\mu - R_f\mathbf{1}) \tag{10}$$

and that the Sharpe ratio of the portfolio is defined as

$$SR_p = \frac{\mu_p - R_f}{\sigma_p} \tag{11}$$

where

$$\mu_p = R_f + \omega^\top(\mu - R_f\mathbf{1}), \tag{12}$$

$$\sigma_p = (\omega^\top\Omega^{-1}\omega)^{1/2}. \tag{13}$$

By evaluating both (12) and (13) at ω_0 and substituting both expressions into (11) we get

$$\begin{aligned}
SR_p &= \frac{\left[\frac{1}{\gamma}\Omega^{-1}(\mu - R_f\mathbf{1})\right]^\top(\mu - R_f\mathbf{1})}{\left[(\frac{1}{\gamma}\Omega^{-1}(\mu - R_f\mathbf{1}))^\top\Omega(\frac{1}{\gamma}\Omega^{-1}(\mu - R_f\mathbf{1}))\right]^{1/2}} \\
&= \frac{\frac{1}{\gamma}(\mu - R_f\mathbf{1})^\top\Omega^{-1}(\mu - R_f\mathbf{1})}{\frac{1}{\gamma}\left[(\mu - R_f\mathbf{1})^\top\Omega^{-1}(\mu - R_f\mathbf{1})\right]^{1/2}} \\
&= \left[(\mu - R_f\mathbf{1})^\top\Omega^{-1}(\mu - R_f\mathbf{1})\right]^{1/2},
\end{aligned}$$

which is a constant that does not depend on how much is invested in the optimal portfolio and the risk free asset.