Assignment 8

1. **Price-dividend ratio, return predictability and the Campbell-Shiller decomposition (50 points)**

   Consider a stock (or portfolio) with price P_t which pays dividend D_t. Suppose that the dividend process is as follows:

   $$D_{t+1} = D_t e^{\bar{g} - \frac{1}{2}\sigma_g^2 + g_{t+1}}$$

   where the stochastic component of dividend growth follows an autoregressive process:

   $$g_{t+1} = \rho_g g_t + \sigma_g \epsilon_g(t+1)$$

   where $\epsilon_g(t)$ are i.i.d. standard normal random variables (with zero mean and unit variance). The gross return on the stock is defined by:

   $$R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t}$$

   We define the expected return to be

   $$E_t[R_{t+1}] = e^{\bar{k} + k_t}$$

   and the price dividend ratio and its logarithm to be:

   $$PD_t = \frac{P_t}{D_t} \quad \text{and} \quad pd_t = \log PD_t$$

   This exercise will help you establish relations between the variability of the price-dividend ratio and the predictability of dividend growth (g_t) and of expected returns (k_t).

   (a) Assume first that expected returns are constant, that is $k_t = 0 \ \forall t$ and that dividend growth is unpredictable, in that $\rho_g = 0$, which implies that $E_t[g_{t+1}] = 0$. Show that under these assumptions the price-dividend ratio is constant. Derive an explicit solution for PD. Is this model consistent with the fact that empirically, price volatility is typically 5 to 10 times larger than dividend volatility?

Hint: Two approaches may be used. First, you may use the 'brute force approach' of computing the price as the expectation of the sum of discounted dividends $P_t = \lim_{T\to\infty}\{E_t[\sum_{n=1}^{T} e^{-\bar{k}n} D_{t+n}] + E_t[e^{-\bar{k}T} P_{t+T}]\}$ *and assuming (and then verifying with your solution) that the no-bubble condition* $\lim_{T\to\infty} E_t[e^{-\bar{k}T} P_{t+T}] = 0$ *holds you can obtain an explicit solution for the price.*

Alternatively, you can use use the fact that one can rewrite $R_{t+1} = \frac{D_{t+1}}{D_t}\frac{(1+PD_{t+1})}{PD_t}$. *Then compute the expectation of the RHS using the moment generating function of a standard normal distribution* $E_t[e^{\lambda\epsilon_g(t+1)}] = e^{\frac{\lambda^2}{2}}$.

Both approaches should of course give the same answer. You may also note during your derivations that $\bar{k} > \bar{g}$ *is required for the price to be finite.*

(b) Now suppose that $\rho_g \neq 0$ and that expected returns are mean-reverting so that:

$$k_{t+1} = \rho_k k_t + \sigma_k \epsilon_k(t+1)$$

where $\epsilon_k(t)$ are i.i.d. standard normal random variables (with zero mean and unit variance), which we assume independent of $\epsilon_g(t)$ for simplicity. We want to find the equilibrium price dividend ratio function $PD_t = PD(g_t, k_t)$ which solves the fixed point:

$$E_t[R_{t+1}] = E_t[\frac{D_{t+1}}{D_t}\frac{(1+PD_{t+1})}{PD_t}]$$

Equivalently, we seek a function for the log of price dividend $pd_t = pd(g_t, k_t)$ that satisfies:

$$e^{\bar{k}+k_t} = E_t[e^{\bar{g}-\frac{1}{2}\sigma_g^2+g_{t+1}+\log(1+e^{pd_{t+1}})-pd_t}] \quad (\star)$$

To derive an approximate solution to for the log-price dividend ratio, we will use the Campbell-Shiller approximation, which consists in linearizing the non-linear term in the RHS exponential using the Taylor approximation of the function: $\log(1+e^x) \approx \log(1+e^{\bar{x}}) + \frac{e^{\bar{x}}}{1+e^{\bar{x}}}(x-\bar{x})$.

(c) Guess that $pd_t = pd(g_t, k_t) = A + Bg_t + Ck_t$ (and thus that $pd_{t+1} = A + Bg_{t+1} + Ck_{t+1}$), use the Campbell-Shiller Taylor approximation by linarizing around the long-run mean of the log-price dividend ratio $\bar{pd} = A$ (note that the unconditional means are $E[g_t] = E[k_t] = 0$) to find three non-linear equations for the coefficients A, B, C so that equation $(\star)$ is satisfied.

Hint: note that after using the Taylor approximation in the RHS side of $(\star)$*, the term in the exponential becomes linear in* k_{t+1}, g_{t+1} *which is thus a normally* *distributed random variable. The RHS expectation can thus be computed using the moment generating function of a normally distributed random variable. Then taking logs on both sides and matching the loadings on g_t, k_t and constants identify three equations that pin down the unknown coefficients.*

Alternatively, you could use the 'brute force' approach as suggested in the first question's hint. The brute force approach will give the exact solution in all cases. You can verify that the Campbell-Shiller approximation is in fact exact only if $\rho_g = \rho_k = 0$. When $\rho_g \neq \rho_k \neq= 0$, then the brute-force approach also becomes more difficult to implement. Therefore I don't recommend that you use it for this case.

(d) Use your solution to prove that the price-dividend ratio (and its log) is constant if $\rho_g = \rho_k = \sigma_k = 0$ (where you should recover the exact solution derived in the first question). Conclude that for prices to be more volatile than dividends (as we observe empirically) it is necessary that either dividends are predictable or expected returns are stochastic (or both).

Further, show that for the price-dividend ratio to be stochastic and mean-reverting (e.g., to be stationary) it is necessary that either expected returns are mean-reverting ($\rho_k \in (0, 1)$) or dividend growth is mean-reverting ($\rho_g \in (0, 1)$) or both. Since, empirically dividend growth seems to be close to i.i.d. ($\rho_g \approx 0$) and price-dividend ratio are clearly stochastic and appear to be mean-reverting, conclude then that stock expected returns should be predictable (i.e., mean-reverting).

2. **Market capitalization and expected returns (50 points).** In this exercise we will test the CAPM using portfolio sorted based on beta.

   (a) Download data on the same stock you used in Problem Set 7, i.e. the 275 stocks that have been traded each day between 1980 and December 31, 2019. Also download the share price (prc) as well as the number of shares outstanding (shrout). Download the same market return and risk-free rate that you also used in the last assignment.

   (b) Calculate the market capitalization of each stock in each month (price times number of shares outstanding). Make sure to always use the absolute value of the share price (CRSP sometimes reports negative values). Form 10 groups of stocks based on the market capitalization on December 31, 2019, with first decile being in group 1 and so on... Then, compute:

- the equally weighted decile portfolio returns for each month, the sample average excess return of each portfolio as well as the corresponding (equally weighted) alphas and betas, i.e. the regression coefficients of the portfolios' excess returns on the market risk premium.
- value-weighted decile portfolio returns for each month, the sample average excess return of each portfolio as well as the corresponding (value weighted) alphas and betas, i.e. the regression coefficients of the portfolios' excess returns on the market risk premium. **Note:** Value-weighting means computing a weighted average of the returns in each month, where the weights are proportional to the **lagged** market capitalizations.

(c) Based on the results in part b), how is market capitalization related to average excess returns? Can the capm explain the behavior of the stocks? Consider equally weighted and value-weighted returns separately.

(d) Instead of sorting stock based on market capitalization at the end of the sample, which implies a "look-ahead bias" in the portfolio formation, sort stocks in each month into 10 deciles based on their lagged market capitalization. As in part b), compute value-weighted and equally weighted returns for those 10 groups. Also compute the average excess return of each portfolio as well as the corresponding (value weighted) alphas and betas. How is market capitalization related to average excess returns? Can the capm explain the behavior of the stocks? Consider equally weighted and value-weighted returns separately. How do the results compare to the previous question results with "look-ahead bias"?