FIN-405 — PS6

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1 APT

a) Assume that there is only 1 stock with positive alpha in B. Since F, $\epsilon_{B,j}$ are independently and identically distributed with zero mean and unit variance, we can calculate the followings:

$$\text{Systematic risk} = \beta_B^2 Var[F] = 0.5^2 \cdot 1 = 0.25$$

$$\text{Idiosyncratic risk} = \sigma_B^2 Var[\epsilon_{B,1}] = 0.3^2 \cdot 1 = 0.09$$

$$\text{Total risk} = 0.25 + 0.09 = 0.34$$

Sharpe Ratio $= \frac{E[R_{B,1} - R_0]}{\sigma_{B,1}} = \frac{\alpha_B + \beta_B(E[F] + \mu_F) + \sigma_B E[\epsilon_{B,1}]}{\sqrt{\text{Total risk}}} = \frac{0.03 + 0.5 \cdot 0.08}{\sqrt{0.34}} = 0.12$

b) Invest in all the zero-alpha stocks, since the factor exposure is 1, we have:

$$\sum_{i=1}^{N_A} w_i \beta_{A,i} = 1$$

Then with the given value, we have:

$$\sum_{i=1}^{N_A} w_i = 2$$

Expand the return on the factor bet portfolio:

$$R_f = \sum_{i=1}^{N_A} w_i R_{A,i} = \sum_{i=1}^{N_A} w_i (R_0 + \alpha_A + \beta_A(F + \mu F) + \sigma_A \epsilon_{A,i})$$

$$= 0.08 + \sum_{i=1}^{N_A} w_i \beta_A (F + \mu F) + \sum_{i=1}^{N_A} w_i \sigma_A \epsilon_{A,i}$$

The idiosyncratic risk is shown as follows:

$$\text{Idiosyncratic risk} = Var[\sum_{i=1}^{N_A} w_i \sigma_A \epsilon_{A,i}] = 0.09 \sum_{i=1}^{N_A} w_i^2 Var[\epsilon_{A,i}] = 0.09 \sum_{i=1}^{N_A} w_i^2$$

To minimize the Idiosyncratic risk with the above restriction, the Lagrange equation is set to be:

$$\mathcal{L} = 0.09 \sum_{i=1}^{N_A} w_i^2 + \lambda(\sum_{i=1}^{N_A} w_i - 2)$$

The first order conditions give $0.18 \sum_{i=1}^{N_A} w_i + \lambda N_A = 0$, and $0.18 w_i + \lambda = 0$. Plug in the value that we derived before, and therefore, the factor bet portfolios w satisfies $w_i = \frac{2}{N_A}$

c) To construct the zero cost portfolio, we have the positive alpha asset $R_{B,1}$, the factor bet portfolio R_F, and the risk free asset R_0. If you invest \$ 1 in asset $R_{B,1}$, you need to short \$ 0.5 in R_F, and thus to make it zero cost, need to borrow \$ (1-0.5) in the risk free asset. Then the portfolio is:

$$R_{zcp} = R_{B,1} - 0.5 R_F - (1 - 0.5) R_0$$

$$= R_0 + \alpha_B + \beta(F + \mu F) + \sigma_B \epsilon_{B,j} - \frac{0.5 \cdot 2}{N_A} \sum_{i=1}^{N_A} (R_0 + \alpha_A + \beta(F + \mu F) + \sigma_A \epsilon_{A,i}) - 0.5 R_0$$

$$= \alpha_B - 0.5 R_0 + + \sigma_B \epsilon_{B,j} - \frac{1}{N_A} \sum_{i=1}^{N_A} \sigma_A \epsilon_{A,i}$$

- The expected return $E[R_{zcp}] = \alpha_B - 0.5 R_0 = 0.03 - 0.5 \cdot 0.04 = 0.01$
- The systematic risk is zero.
- Thus, the total risk is the idiosyncratic risk.

$$risk = Var[R_{zcp}] = \sigma_B^2 Var[\epsilon_{B,j}] + \sigma_a^2 \frac{Var[\epsilon_{A,i}]}{N_A} = 0.09 + \frac{0.09}{N_A}$$

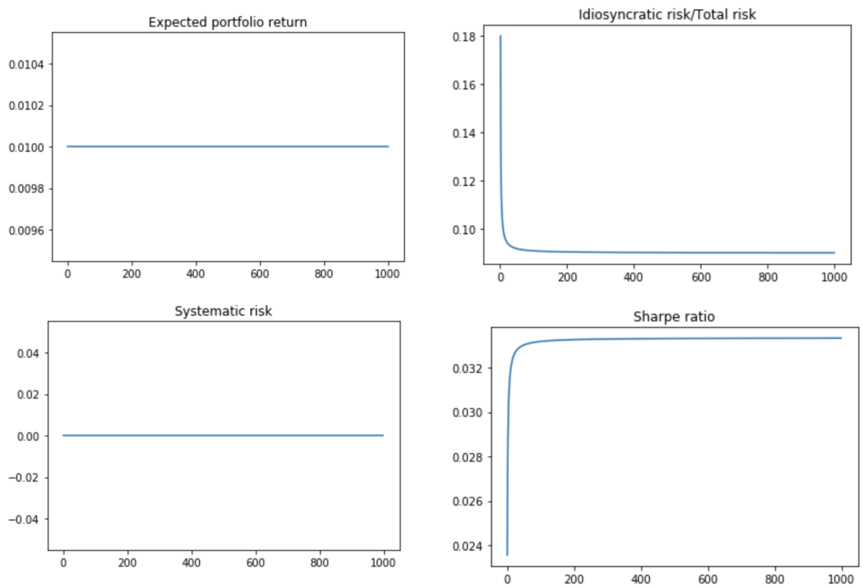

Figure 1: Figures for 1 positive α asset zero cost portfolio

- Sharpe ratio $= \frac{E[R_{zcp}]}{\sqrt{risk}} = \frac{0.01}{\sqrt{0.09+0.09/N_A}}$

According to different N_A, the plot is shown as above:

d) Given that in these $N_A + 1$ assets, it holds that $\alpha_i = 0$, we can construct the maximum sharpe ratio portfolio as $R_{max} = x_0 R_0 + x_F R_F^e$. Then we want to maximize

$$\max E[R_{max}] - \frac{a}{2} Var[R_{max}]$$

$$= \max_{x_F} R_0 + x_F(\mu_f - R_0) - \frac{a}{2}(x_F^2 \sigma_F^2)$$

The first order conditions give $x_F = \frac{\mu_F - R_0}{a\sigma_F^2} = \frac{0.04}{a}$, $x_0 = 1 - x_F$

The the Sharpe ratio is calculated as follows:

$$\text{Sharpe ratio} = \frac{E[R_{max}]}{\sqrt{Var[R_{max}]}} = \frac{R0 + x_F(\mu_F - R_0)}{\sqrt{x_F^2 \sigma_F^2}} = \frac{0.04 + 0.04^2/a}{0.04/a} = a{+}0.04$$

With a given risk aversion ratio, these $N_A + 1$ assets will have a fixed Sharpe ratio for the maximum sharpe ratio portfolio. The expected return and the sharpe ratio of this maximum sharpe ratio portfolio will be larger than the zero cost portfolio.

Furthermore, in order to construct the maximum sharpe ratio portfolio, we should add up all the positive alpha assets, so that the returns will be really high compared to the risk. Therefore, I do not think that you can construct a strategy investing in these $N_A + 1$ assets that has zero cost and infinite sharpe ratio in the limit as $N \longrightarrow \infty$.

e)For these 2N assets, it all satisfies the model:

$$R_i^e = \alpha_i + \beta_i R_F^e + \sigma_i \epsilon_i \forall i = 1, ...2N$$

Where $R_i^e = R_i - R_0$, and $R_F^e = F + \mu_F$

The zero investment portfolio has return:

$$r_i = R_i^e - \beta_i R_F^e = \alpha_i + \sigma_i \epsilon_i$$

Therefore, the expected return $E[r_i] = \alpha_i$, the variance $Var[r_i] = \sigma_i^2 Var[\epsilon_i]$

Then we construct the zero cost portfolio to be

$$R_p = \frac{1}{2N} \sum_{i=1}^{2N} sign[\alpha_i] r_i$$

Here, as we have $N_A = N_B = N$, we have plus signs in front of the B assets, and minus signs in front of the A assets.

- The expected return is:

$$E[R_p] = \frac{1}{2N} \sum_{i=1}^{2N} |\alpha_i| = \frac{1}{2N} N \cdot 0.03 = 0.015$$

- Since the systematic risk is zero, The total risk is the same as the idiosyncratic risk:

$$Var[R_p] = \frac{1}{4N^2} \sum_{i=1}^{2N} \sigma_i^2 Var[\epsilon_i] = \frac{1}{4N^2} 2N \cdot 0.09 = \frac{0.45}{N}$$

- The sharpe ratio is Sharpe ratio $= \frac{E[R_p]}{\sqrt{Var[R_p]}} = \frac{0.015}{\sqrt{0.45/N}}$

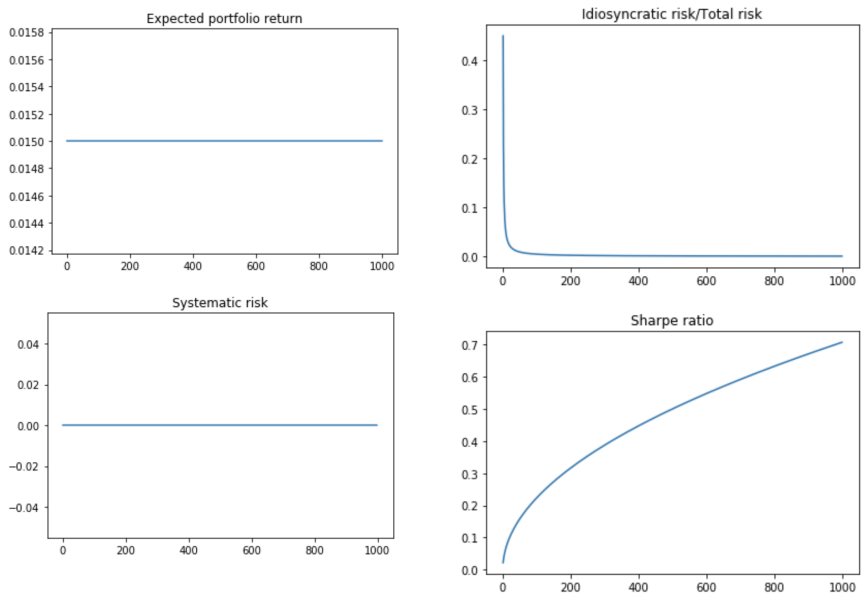

Figure 2: Figures with $N_A = N_B$

The plots are shown above:

N_B cannot be very large, because when it is infinitely large, the variance of the combination of these positive alpha assets becomes so large so that it is not neglectable anymore. Then since total risk is decreasing faster than the increase of the sharpe ratio, then even with infinitely large N, the sharpe ratio cannot go to infinity.

2 Understanding Warren Buffett's performance

see *PS6-Code.ipynb*