FIN405-PS4

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1 Risk-Decomposition

- Annualized idiosyncratic risk:
  $v_i^2 = Var(\epsilon_i) = (\sqrt{52}\ 0.03451)^2 = 0.0619$
  (Standard deviation of error on Bloomberg is calculated weekly and displayed in percentage.)
  $v_i = 24.88\%$
- Annualized systematic risk:
  $\beta_i^2 \sigma_m^2 = 1.577^2 \cdot 0.0223 = 0.055$
  $\sqrt{\beta_i^2 \sigma_m^2} = 23.45\%$
- Annualized total risk:
  $Var(R_i) = \beta_i^2 \sigma_m^2 + v_i^2 = 0.055 + 0.0619 = 0.1169$
  $\sigma_i = 34.19\%$
- Volatility of market portfolio:
  $\sigma_m^2 = \frac{R^2 v_i^2}{\beta_i^2 (1-R^2)} = \frac{0.472 \cdot 0.0619}{1.577^2 (1-0.472)} = 0.0223$
  $\sigma_m = 14.92\%$

2 Portfolio choice with liabilities

2.1

We want to compute the optimal portfolio weights w^* such that

$$w^* = \arg\max_w R_0 + w^{'}(\mu - R_0 \mathbf{1}) - \frac{a}{2} w^{'} \Sigma w \tag{1}$$

Let f be the function of w we want to maximize, then

$$\frac{df}{dw} = \mu - R_0 \mathbf{1} - a \Sigma w = 0 \tag{2}$$

yields

$$w^* = (a\Sigma)^{-1}(\mu - R_0 \mathbf{1}). \tag{3}$$

We know that the tangency portfolio weights are

$$w_{tan} = ((B - AR_0)\Sigma)^{-1}(\mu - R_0\mathbf{1}) \tag{4}$$

and we immediately see that w^* and w_{tan} only differ by a factor. If the risk aversion coefficient a equals $B - AR_0$, then the optimal portfolio is simply the tangency portfolio, and it is trivial to show that the weight invested in the risk free asset is 0. On the other hand, if a tends to infinity, w^* will tend to zero, which means that nothing will be invested in the risk tangency portfolio, and therefore everything will be invested in the risk free asset.

2.2

In presence of the liability, the expected return and variance are as follows:

$$E[R_p - L] = R_0 + w^{'}(\mu - R_0\mathbf{1}) - \mu_L \tag{5}$$

$$Var(R_p - L) = w^{'}\Sigma w + \sigma_L^2 - 2w^{'}\Sigma_L \tag{6}$$

Set up the Lagrangian for this investment problem

$$L = R_0 + w^{'}(\mu - R_0\mathbf{1}) - \mu_L - \frac{a}{2}(w^{'}\Sigma w + \sigma_L^2 - 2w^{'}\Sigma_L) \tag{7}$$

The first order condition is

$$\frac{\partial L}{\partial w} = \mu - R_0\mathbf{1} - a\Sigma w + a\Sigma_L = 0 \tag{8}$$

The investor's optimal portfolio is therefore

$$w^* = \frac{1}{a}\Sigma^{-1}(\mu - R_0\mathbf{1}) + \Sigma^{-1}\Sigma_L \tag{9}$$

2.3

From the above investor's optimal portfolio in presence of the liability, we can find that the weights of the risky portfolio is the combination of two parts, the first part is the tangency weight, as is shown in **2.1**. The second part is the liability weight, since this weight is dependent of the covariances between the return and liability. There is also the risk free asset, the weight of which is $(1 - w^{'}\mathbf{1})$.

When risk-aversion becomes infinitely large, a becomes infinitely large. That is to say, the investor would not invest in the tangency asset, but only in the liability asset and the risk free asset. In this way the risk is minimized. The optimal weight under this circumstance is

$$w^* = \Sigma^{-1}\Sigma_L \tag{10}$$

Due to the existence of the liability constraint, even for extremely risk-aversion investors, they have to comply with the constraint. In other words, risk-aversion investors cannot only invest in risk free assets, but have to invest in both liability asset and the risk free asset. In the extreme case, the investor will invest in liability asset with a weight of $\Sigma^{-1}\Sigma_L$, and invest in risk free asset with a weight of $1 - \Sigma^{-1}\Sigma_L$.