FIN-405 — PS5

Jon Kqiku and Mingchi Hou

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1 Mean-variance portfolio choice and leverage constraints

1) see *PS5-Code.ipynb*

2) see *PS5-Code.ipynb*

3)
Since every optimal portfolio is a linear combination of the minimum variance portfolio and the slope portfolio, and since every portfolio on the efficient frontier is also such a linear combination, we know that both the tangency portfolio and the zero beta portfolio are a linear combination of the minimum variance and the slope. Therefore, the optimal portfolio is also a linear combination of the tangency portfolio and the zero beta portfolio.

The investors optimality problem is

$$\begin{aligned}\max_x \mu_p - \frac{a}{2}\sigma_p^2 &= \max_x R_0 + x_t(R_t - R_0) + x_z(R_z - R_0) - \frac{a}{2}x^\top \Sigma x \\ &= \max_x R_0 + x^\top R_e - \frac{a}{2}x^\top \Sigma x\end{aligned}$$

such that

$$x^\top \mathbf{1} \leq m$$

where $x = (x_t, x_z)^\top$ are the weights invested in the tangency and zero beta portfolios, $R_e = (R_t - R - 0, R_z - R_0)^\top$ and $m = 1.3$.

The first order conditions are given by

$$\frac{\partial L}{\partial x} = R_e - a\Sigma x - \lambda \mathbf{1} = 0 \tag{1}$$

and

$$\frac{\partial L}{\partial \lambda} = m - x^\top \mathbf{1} = 0. \tag{2}$$

Solving (1) for x and plugging it into (2) yields

$$x^* = \frac{1}{a}\Sigma^{-1}(R_e - \lambda \mathbf{1} m) \tag{3}$$

which gives the optimal portfolio as a function of the risk aversion coefficient a, the Lagrange multiplier λ and the budget constraint m.

The resulting portfolio in terms of weights of the underlying securities is then given by

$$(w_0^*, w_1^*, w_2^*, w_3^*) =$$
$$\begin{pmatrix} x_t^* & x_z^* \end{pmatrix} \begin{pmatrix} 0 & w_{t1} & w_{t2} & w_{t3} \\ 0 & w_{z1} & w_{z2} & w_{z3} \end{pmatrix} + (1 - x^{*\top}) \begin{pmatrix} w_0 & 0 & 0 & 0 \end{pmatrix}$$

where w_{ti} and w_{zi} are the weights invested in risky asset i in the tangency and zero beta portfolios respectively.

2 Betting against Beta and information ratio

1)
Let w_0, w_L, w_H be the weights of the risk-free asset, the low beta and high beta portfolio respectively. If we fix $w_H = -1$, we can achieve a zero beta portfolio by appropriately choosing w_L as follows

$$\begin{aligned} w_L \beta_L + w_H \beta_H &= 0 \\ w_L = \beta_H / \beta_L &\cong 3.24. \end{aligned}$$

However, this portfolio is not zero cost since $w_L + w_H \neq 0$. We know that the risk-free asset has a market beta $\beta_0 = 0$, so that combining this portfolio with the risk-free asset still yields a zero beta portfolio. Thus, we choose w_0 such that

$$w_0 = -(w_L + w_H) = -2.24.$$

We finally have a zero cost and zero beta portfolio with weights $w_z = (w_0, w_L, w_H)$ $= (-2.24, 3.24, -1)$. Since $\beta_0 = 0$, $\alpha_0 = R_0 = 0.5\%$. We can now compute the alpha of the portfolio as

$$\alpha_z = w_z^\top \alpha = -2.24 \cdot 0.5\% + 3.24 \cdot 2.36\% - 1 \cdot (-4.36\%) = 10.89\%.$$

Since the assets in the zero beta portfolio are uncorrelated, the standard deviation of the zero beta portfolio is given by

$$\begin{aligned} \sigma_z &= \sqrt{w_L^2 \cdot \sigma_L^2 + w_H^2 \cdot \sigma_H^2} \\ &= \sqrt{3.24^2 \cdot 0.12^2 + (-1)^2 \cdot 0.12^2} \\ &= \sqrt{0.1656} \\ &= 40.69\%. \end{aligned}$$

Therefore, the information ratio of the portfolio is

$$IR_z = \frac{\alpha_z}{\sigma_z} = \frac{0.1089}{0.4069} = 0.2676.$$

2)
Let X_M, X_{BAB}, and X_0 be the weights of the Market portfolio, BAB portfolio, and risk free portfolio respectively. Then we have $X_0 = 1 - X_{BAB} - X_M$. In order to find the optimal, we need to maximize the Sharpe Ratio. With the equation given in the lecture, for risk aversion level a, we have the weights as shown below:

$$X_{BAB} = \frac{\alpha_z}{a\sigma_z^2}$$

$$X_M = \frac{\mu_M - R_0}{a\sigma_M^2}$$

Then we want to find out the ratio of BAB portfolio with respect to all risky portfolio.

$$w_{BAB} = \frac{X_{BAB}}{X_{BAB} + X_M} = \frac{\frac{\alpha_z}{\sigma_z^2}}{\frac{\alpha_z}{\sigma_z^2} + \frac{\mu_M - R_0}{\sigma_M^2}} = \frac{\alpha_z \sigma_M^2}{\alpha_z \sigma_M^2 + (\mu_M - R_0)\sigma_z^2}$$

$$= \frac{0.1089 \cdot 0.14^2}{0.1089 \cdot 0.14^2 + 0.075 \cdot 0.1656} = 0.1467$$

Therefore, if we want to invest \$1 in risky asset, we should invest \$0.15 in BAB, and \$0.85 in the Market portfolio. More specifically, we should go long $\frac{0.15}{\beta_L} =$ \$0.0825 in low beta firm, and go short $\frac{0.15}{\beta_H} =$ \$0.267 in high beta firm.

3)
From what we found above, we have

$$R_p - R_0 = X_M(R_M - R_0) + X_{BAB}R_{BAB} = \frac{1}{a}(SR_M^2 + IR_{BAB}^2)$$

$$\sigma_p = \sqrt{X_M^2\sigma_M^2 + X_{BAB}^2\sigma_{BAB}^2} = \frac{1}{a}\sqrt{SR_M^2 + IR_z^2}$$

With the given information, we can calculate the Sharpe ratio of the market, which is $\frac{0.075}{0.14} = 0.536$. Then we can calculate the Sharpe ratio of the portfolio:

$$Sh_p = \frac{\mu_p - R_0}{\sigma_p} = \sqrt{SR_M^2 + IR_{BAB}^2} = \sqrt{0.536^2 + 0.296^2} = 0.599$$

The Sharpe Ratio of the portfolio is larger than that of the Market portfolio, which shows that it is more favorable for investors, i.e. with the same level of risk, there is higher return.

3 APT

1)
According to APT the expected return of stock i should be

$$E(R_i) = \sum_{k=1}^{K} B_{ik} E(F_k)$$

which implies the restriction that $\alpha_i = 0$.

2)
If APT holds we have

$$E(R_i) = \sum_{k=1}^{K} B_{ik} E(F_k)$$

and if CAPM holds we have

$$E(R_i) = \beta_i E(R_m).$$

But if APT spans the market portfolio it implies that there exits $w \in \mathbb{R}^K$ such that

$$E(R_m) = \sum_{k=1}^{K} w_k E(F_k).$$

Then CAPM can be rewritten as

$$E(R_i) = \beta_i \sum_{k=1}^{K} w_k E(F_k)$$

and for CAPM to hold, APT has to hold with

$$B_{ik} = \beta_i w_k \quad \forall\, i, k.$$

3)
This new formulation under constant risk factors returns is consistent with APT if and only if the all error terms have zero mean, i.e. $E(e_i) = 0 \,\forall\, i$. In that case,

$$E(R_i - R_0) = \sum_{k=1}^{K} B_{ik} \lambda_k.$$

Otherwise, we could interpret $E(e_i) = \alpha_i \neq 0$ as an alpha over the returns explained by the APT risk factors which would come from the asset's idiosyncratic risk. And as the theory states, idiosyncratic risk should not be compensated as it can be diversified away.