# BioE 135/235 Homework 5

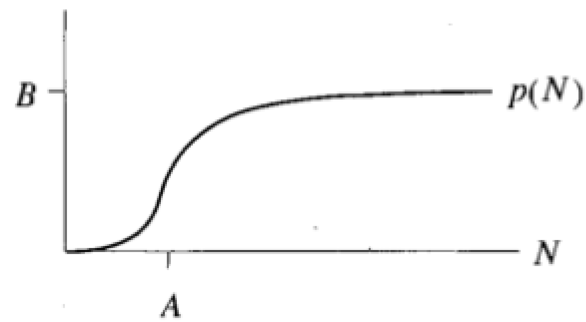
Due Date: Friday, 4/30/2021, 5:00pm PST to Gradescope
Late work will be deducted 20% for each day it is late.

### 1. Population Dynamics

Ludwig et. al. (1978) proposed a model for the interaction between spruce budworms and the balsam fir tree in Eastern Canadian forests. The proposed model for budworm population dynamics is:

$$\frac{d}{dt}N = RN\left(1 - \frac{N}{K}\right) - p(N)$$

In the absence of predators, the budworm population $N(t)$ is assumed to grow logistically with growth rate $R$ and carrying capacity $K$. The term $p(N)$ represents the death rate due to predation, mainly due to birds, and is assumed to have the sigmoidal shape shown below:




When the budworm population is low, there is almost no predation. Once the population exceeds the threshold level at $N = A$, the predation turns on sharply and then saturates. The specific form of the predation term in the model is:

$$p(N) = B\frac{N^2}{A^2 + N^2}$$

where $A > 0, B > 0$. The full model, then, is:

$$\frac{d}{dt}N = RN\left(1 - \frac{N}{K}\right) - B\frac{N^2}{A^2 + N^2}$$

(a) The model contains four parameters: $R, K, A,$ and $B$. Following the same procedure as in Homework 3, nondimensionalize the system so that it arrives in the following form:

$$\frac{d}{d\tau}x = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2}$$

What do the dimensionless quantities $r$ and $k$ represent?

(b) By inspection, the system has a steady state at $x = 0$. What is the stability of this point?

(c) The remaining steady states are given by solutions to the following equation:

$$r\left(1 - \frac{x}{k}\right) = \frac{x}{1 + x^2}$$

It turns out that, depending on the values of $r$ and $k$, the above equations can have one, two, or three intersections. Graphically depict each of these three cases.

(d) In addition to the equation above, what other equation must be satisfied for the system to have two steady states (not counting the steady state at $x = 0$)? From these two equations, derive a relationship between $k$ and $x$ for the system in this situation. Given that $k > 0$, how can we restrict $x$?

## 2. Lotka-Volterra Dynamics

In this question, we will extend some of the ideas about trapping regions that we introduced in the previous homework, to give a better sense about what the existence of a trapping region does—and does *not*—imply.

To begin, consider the following system:

$$\frac{d}{dt}x = x - \frac{1}{10}xy$$
$$\frac{d}{dt}y = \frac{1}{20}xy - \frac{1}{2}y$$

(a) Plot out the nullclines and vector field for the system. Use a window of $x \in [0, 60]$ and $y \in [0, 40]$.

(b) Draw an appropriate trapping region for this system.

(c) Determine the fixed point(s) of the system and their stability. Does this appear to come in conflict with the trapping region you just drew? If not, why not? If so, why?

(d) Now, numerically simulate the system with each of the following initial conditions: $(x_0, y_0) = (30, 30); (10, 15); (20, 5)$. What do you notice about these systems? Does this make sense, given your answer to the previous part?