Transformations

CS 432 Interactive Computer Graphics
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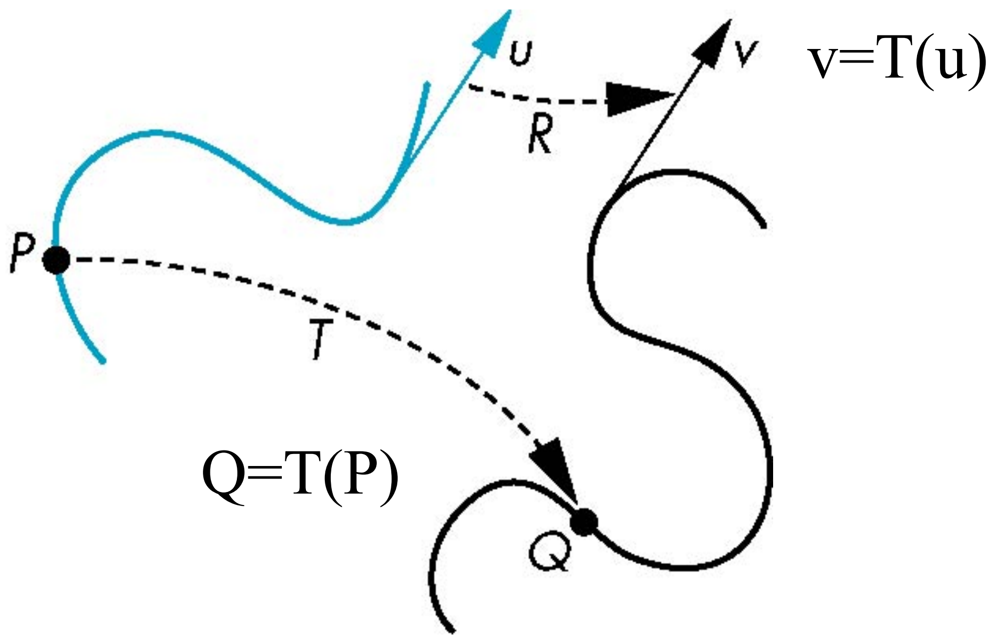
Objectives

- Introduce standard transformations
  - Rotation
  - Translation
  - Scaling
  - Shear
- Derive homogeneous coordinate transformation matrices
- Learn to build arbitrary transformation matrices from simple transformations

General Transformations

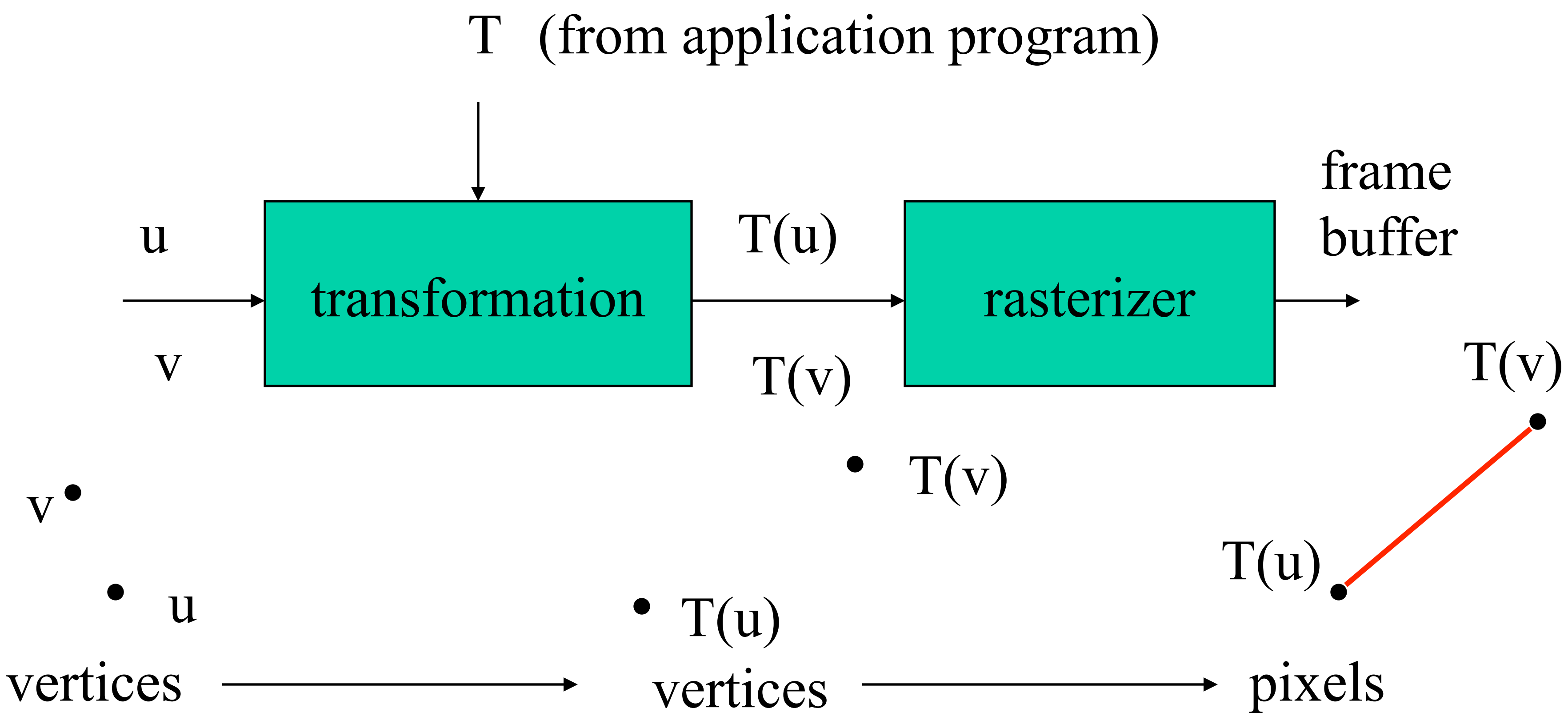
A transformation maps points to other points and/or vectors to other vectors



Affine Transformations

- Line preserving
- Characteristic of many physically important transformations
  - Rigid body transformations: rotation, translation
  - Scaling, shear
- Importance in graphics is that we need only transform endpoints of line segments and let implementation draw line segment between the transformed endpoints

Pipeline Implementation



Notation

We will be working with both coordinate-free representations of transformations and representations within a particular frame

P,Q, R: points in an affine space

u, v, w: vectors in an affine space

α, β, γ: scalars

p, **q**, **r**: representations of points

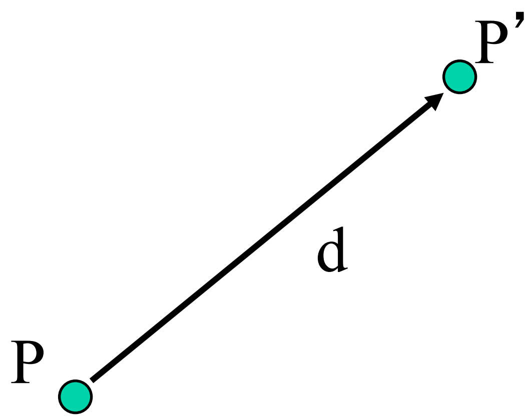
-array of 4 scalars in homogeneous coordinates

u, **v**, **w**: representations of points

-array of 4 scalars in homogeneous coordinates

Translation

- Move (translate, displace) a point to a new location

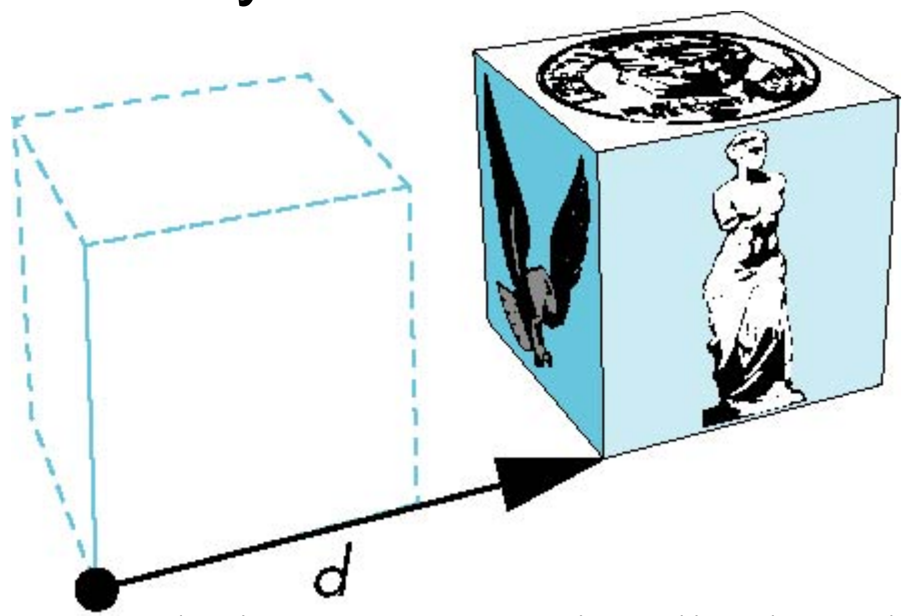


- Displacement determined by a vector d
  - Three degrees of freedom
  - P' = P+d

How many ways?

Although we can move a point to a new location in infinite ways, when we move many points there is usually only one way

object



translation: every point displaced by same vector

Translation Using Representations

Using the homogeneous coordinate representation in some frame

$\mathbf{p} = [\,x\ y\ z\ 1]^T$

$\mathbf{p}' = [x'\ \ y'\ \ z'\ \ 1]^T$

$\mathbf{d} = [dx\ dy\ dz\ 0]^T$

Hence $\mathbf{p}' = \mathbf{p} + \mathbf{d}$ or

$x' = x + d_x$

$y' = y + d_y$

$z' = z + d_z$

note that this expression is in four dimensions and expresses point = vector + point

Translation Matrix

We can also express translation using a
4 x 4 matrix $\mathbf{T}$ in homogeneous coordinates
$\mathbf{p}' = \mathbf{Tp}$ where

$$\mathbf{T} = \mathbf{T}(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This form is better for implementation because all affine transformations can be expressed this way and multiple transformations can be concatenated together

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

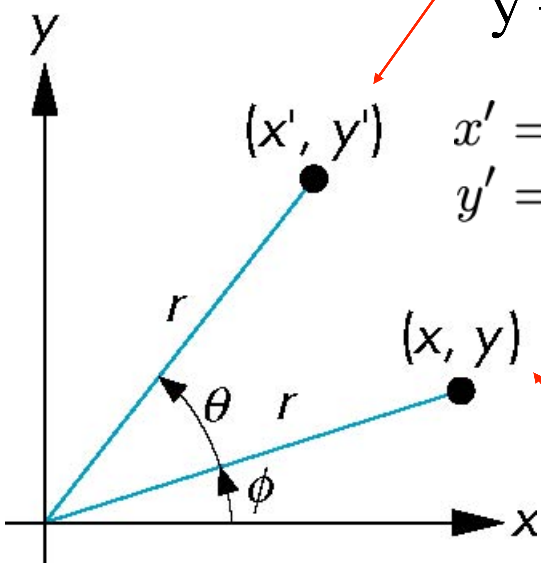
$$x' = x + d_x$$
$$y' = y + d_y$$
$$z' = z + d_z$$

Rotation (2D)

Consider rotation about the origin by θ degrees

- radius stays the same, angle increases by θ

$$x = r \cos(\phi + \theta)$$
$$y = r \sin(\phi + \theta)$$



$$x' = r \cdot cos\ (\theta + \phi) = r \cdot cos\ \phi \cdot cos\ \theta - r \cdot sin\ \phi \cdot sin\ \theta$$
$$y' = r \cdot sin\ (\theta + \phi) = r \cdot cos\ \phi \cdot sin\ \theta + r \cdot sin\ \phi \cdot cos\ \theta$$

$$x = r \cos \phi$$
$$y = r \sin \phi$$

$$\boxed{\begin{aligned} x' &= x \cos\theta - y \sin\theta \\ y' &= x \sin\theta + y \cos\theta \end{aligned}}$$

Rotation about the z axis

- Rotation about z axis in three dimensions leaves all points with the same z
  - Equivalent to rotation in two dimensions in planes of constant z

$$x' = x \cos\theta - y \sin\theta$$
$$y' = x \sin\theta + y \cos\theta$$
$$z' = z$$

  - or in homogeneous coordinates

$$\mathbf{p}' = \mathbf{R}_{\mathbf{z}}(\theta)\mathbf{p}$$

Rotation Matrix

$$\mathbf{R} = \mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation Matrix

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$x' = x \cos\theta - y \sin\theta$$
$$y' = x \sin\theta + y \cos\theta$$
$$z' = z$$

Rotation about x and y axes

- Same argument as for rotation about *z* axis
  - For rotation about x axis, x is unchanged
  - For rotation about y axis, y is unchanged

$$\mathbf{R} = \mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

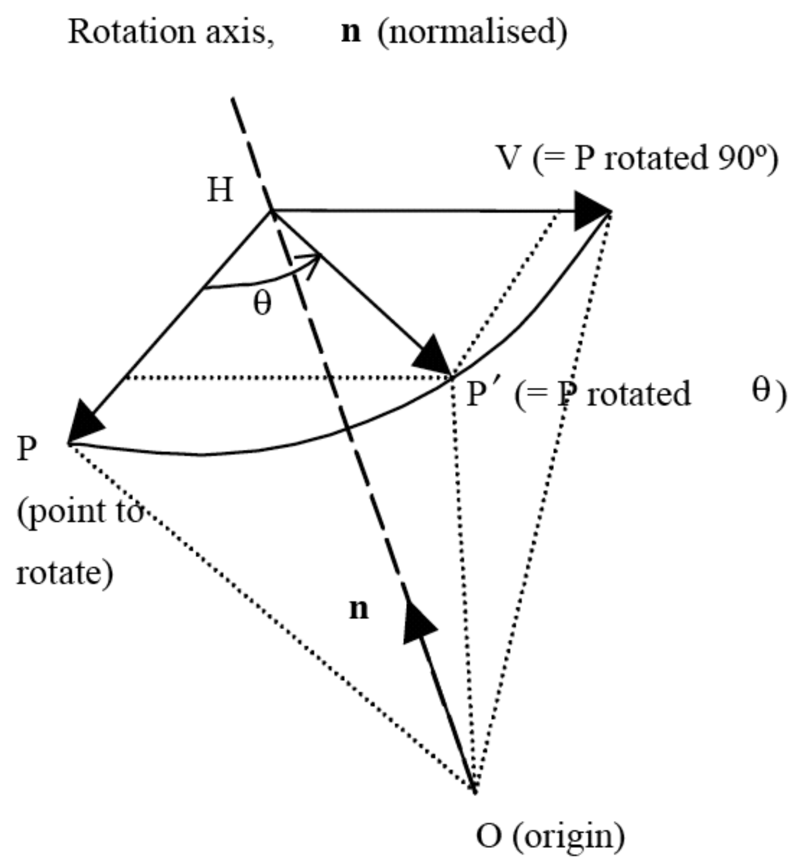
$$\mathbf{R} = \mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation Around an Arbitrary Axis

- Rotate a point P around axis ***n*** (x,y,z) by angle θ

$$R = \begin{bmatrix} tx^2 + c & txy + sz & txz - sy & 0 \\ txy - sz & ty^2 + c & tyz + sx & 0 \\ txz + sy & tyz - sx & tz^2 + c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- c = cos(θ)
- s = sin(θ)
- t = (1 - c)



Graphics Gems I, p. 466 & 498

Rotation Around an Arbitrary Axis

- Also can be expressed as the Rodrigues Formula

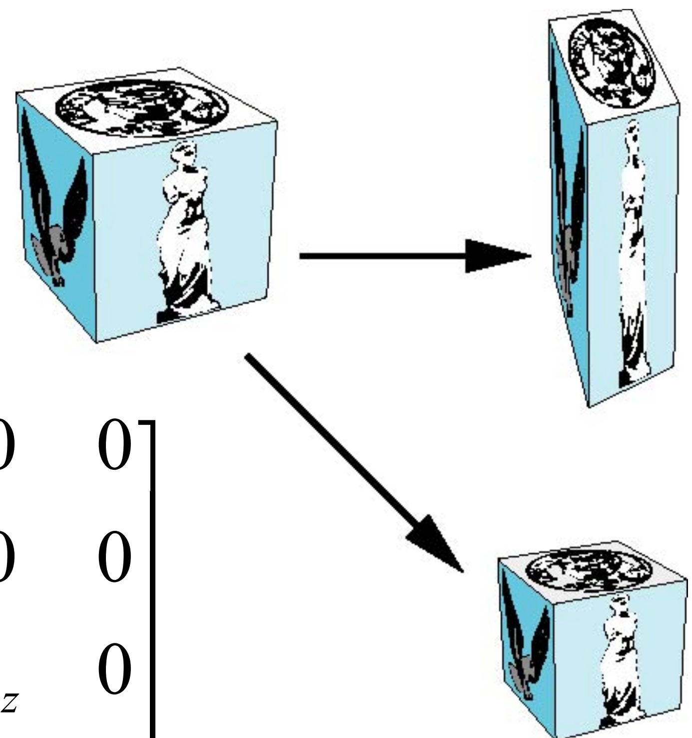
$$P_{rot} = P\cos(\vartheta) + (\mathbf{n} \times P)\sin(\vartheta) + \mathbf{n}(\mathbf{n} \cdot P)(1 - \cos(\vartheta))$$

Scaling

Expand or contract along each axis (fixed point of origin)

$$x' = s_x x$$
$$y' = s_y y$$
$$z' = s_z z$$

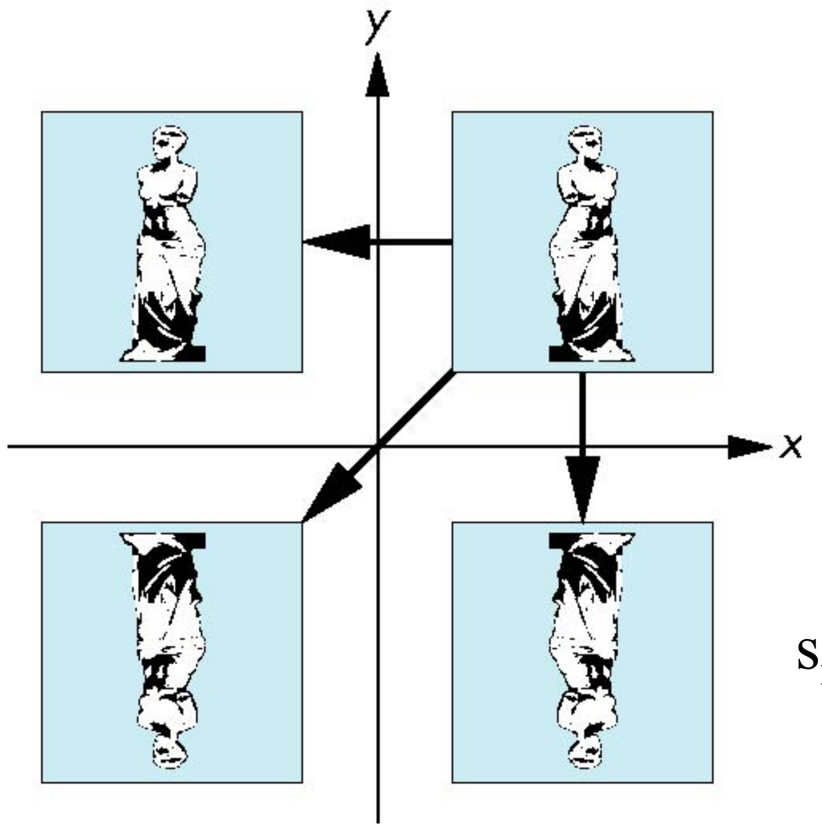
$$\mathbf{p}' = \mathbf{S}\mathbf{p}$$



$$\mathbf{S} = \mathbf{S}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflection

corresponds to negative scale factors



$s_x = -1\ s_y = 1$

original

$s_x = -1\ s_y = -1$

$s_x = 1\ s_y = -1$

Inverses

- Although we could compute inverse matrices by general formulas, we can use simple geometric observations
  - Translation: $\mathbf{T}^{-1}(d_x, d_y, d_z) = \mathbf{T}(-d_x, -d_y, -d_z)$
  - Rotation: $\mathbf{R}^{-1}(\theta) = \mathbf{R}(-\theta)$
    - Holds for any rotation matrix
    - Note that since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta)=-\sin(\theta)$
      $\mathbf{R}^{-1}(\theta) = \mathbf{R}^{T}(\theta)$
  - Scaling: $\mathbf{S}^{-1}(s_x, s_y, s_z) = \mathbf{S}(1/s_x, 1/s_y, 1/s_z)$

Concatenation

- We can form arbitrary affine transformation matrices by multiplying together rotation, translation, and scaling matrices
- Because the same transformation is applied to many vertices, the cost of forming a matrix $\mathbf{M}=\mathbf{ABCD}$ is not significant compared to the cost of computing $\mathbf{Mp}$ for many vertices $\mathbf{p}$
- The difficult part is how to form a desired transformation from the specifications in the application

Order of Transformations

- Note that matrix on the right is the first applied
- Mathematically, the following are equivalent

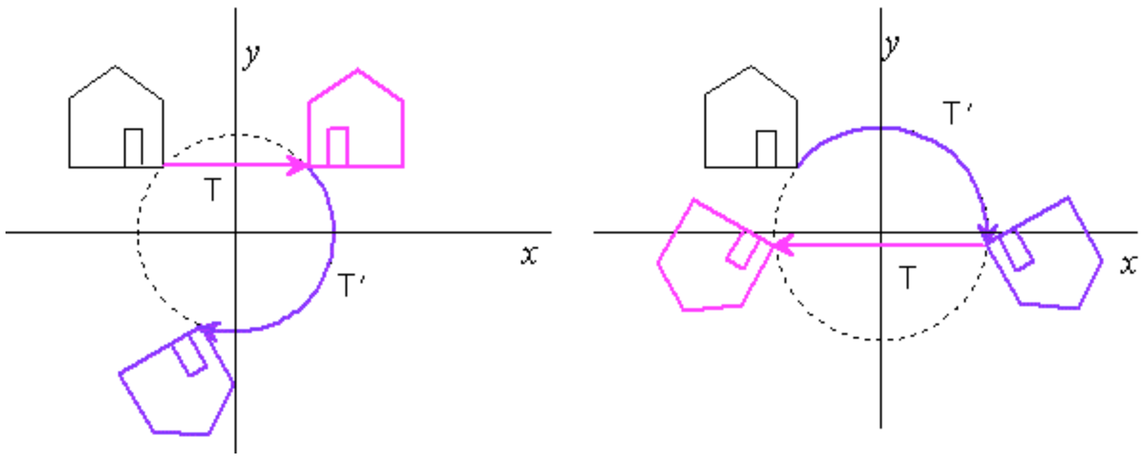
$$\mathbf{p}' = \mathbf{ABCp} = \mathbf{A}(\mathbf{B}(\mathbf{Cp})) \quad \text{// pre-multiply}$$

- Note many references use column matrices to represent points. In terms of column matrices

$$\mathbf{p}'^{T} = \mathbf{p}^{T}\mathbf{C}^{T}\mathbf{B}^{T}\mathbf{A}^{T} \quad \text{// post-multiply}$$

Properties of Transformation Matrices

- Note that matrix multiplication is not commutative
- i.e. in general $\mathbf{M_1 M_2} \neq \mathbf{M_2 M_1}$



- T – reflection around y axis
- T’ – rotation in the plane

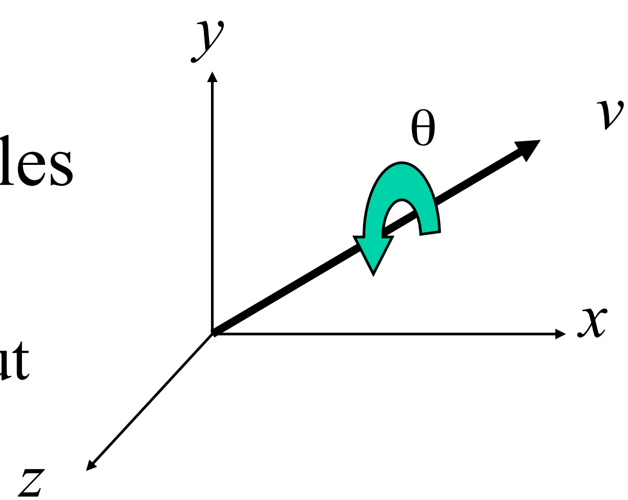
General Rotation About the Origin

A rotation by θ about an arbitrary axis can be decomposed into the concatenation of rotations about the *x*, *y*, and *z* axes

$$\mathbf{R}(\theta) = \mathbf{R}_z(\theta_z)\, \mathbf{R}_y(\theta_y)\, \mathbf{R}_x(\theta_x)$$

$\theta_x\ \theta_y\ \theta_z$ are called the Euler angles

Note that rotations do not commute.
We can use rotations in another order but with different angles.



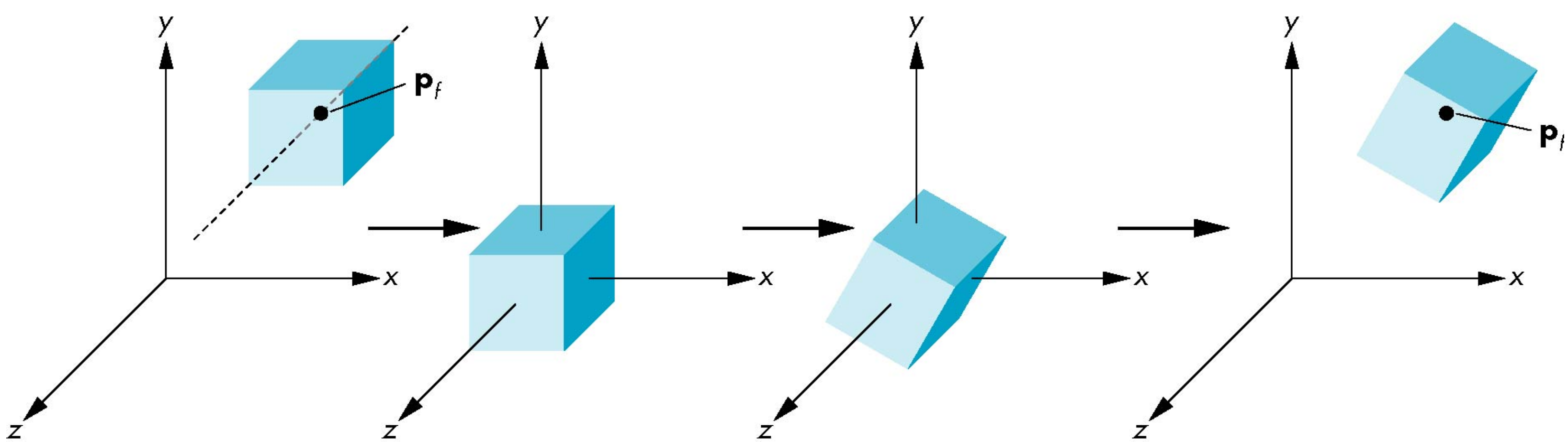
Rotation About a Fixed Point other than the Origin

Move fixed point to origin

Rotate

Move fixed point back

$\mathbf{M} = \mathbf{T}(p_f)\,\mathbf{R}(\theta)\,\mathbf{T}(-p_f)$



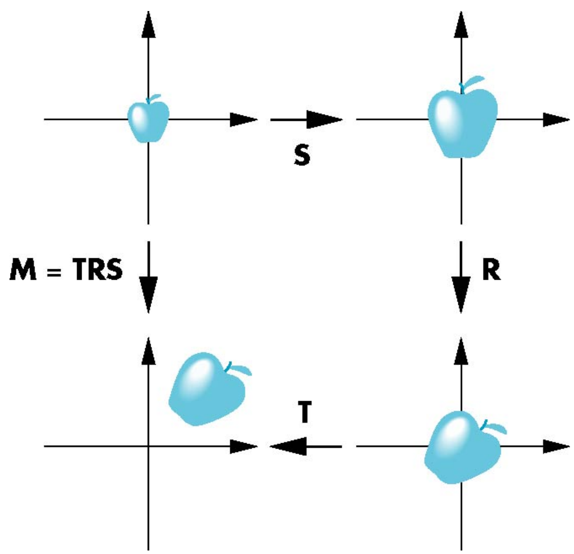
Instancing

- In modeling, we often start with a simple object centered at the origin, oriented with the axis, and at a standard size
- We apply an *instance transformation* to its vertices to

    Scale

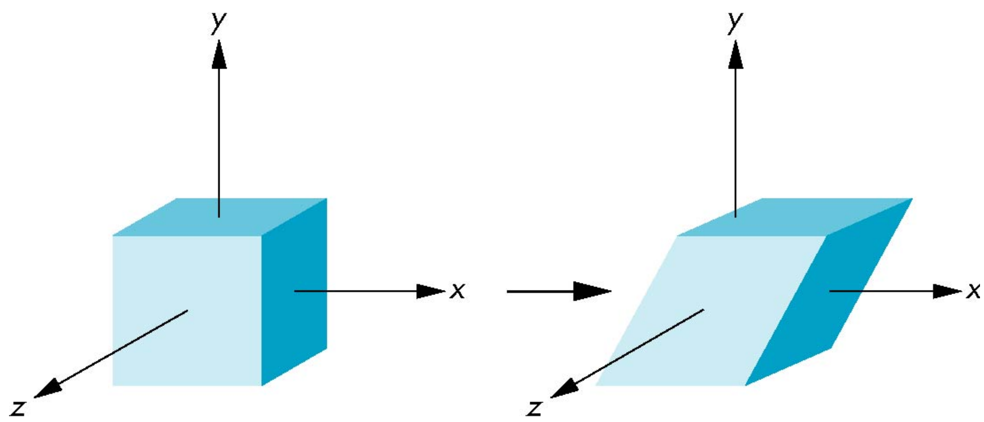
    Orient

    Locate



Shear

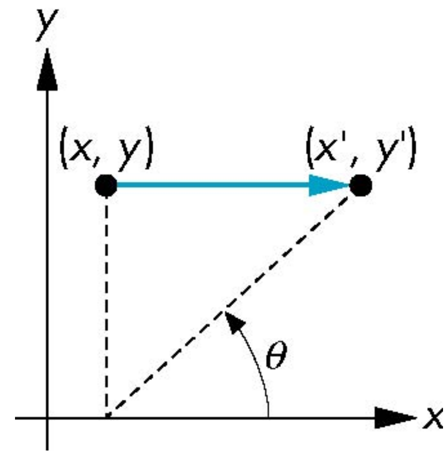
- Helpful to add one more basic transformation
- Equivalent to pulling faces in opposite directions



Shear Matrix

Consider simple shear along x axis

$$x' = x + y \cot \theta$$
$$y' = y$$
$$z' = z$$



$$\mathbf{H}(\theta) = \begin{bmatrix} 1 & \cot\theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

OpenGL Transformations

Objectives

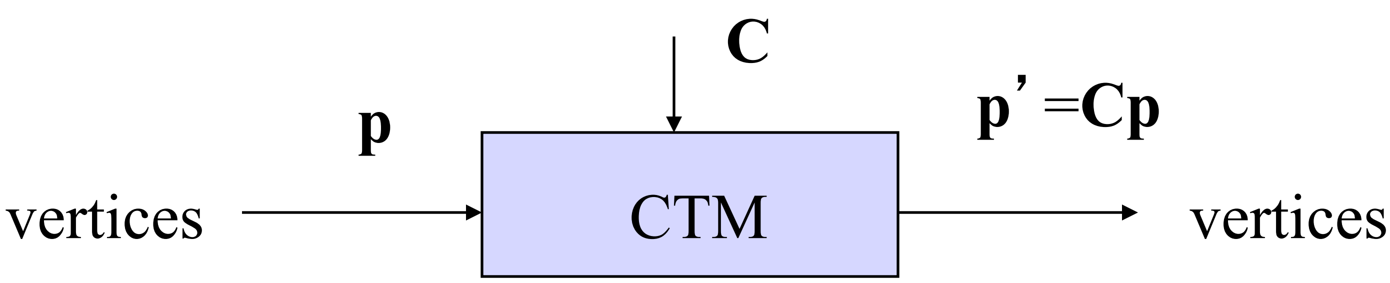
- Learn how to carry out transformations in OpenGL
  - Rotation
  - Translation
  - Scaling
- Introduce mat.h and vec.h transformations
  - Model-view
  - Projection

Pre 3.1 OpenGL Matrices

- In OpenGL matrices were part of the state
- Multiple types
  - Model-View (**`GL_MODELVIEW`**)
  - Projection (**`GL_PROJECTION`**)
  - Texture (**`GL_TEXTURE`**)
  - Color(**`GL_COLOR`**)
- Single set of functions for manipulation
- Select which to manipulated by
  - **`glMatrixMode(GL_MODELVIEW);`**
  - **`glMatrixMode(GL_PROJECTION);`**

Current Transformation Matrix (CTM)

- Conceptually there was a 4 x 4 homogeneous coordinate matrix, the *current transformation matrix* (CTM) that is part of the state and is applied to all vertices that pass down the pipeline
- The CTM was defined in the user program and loaded into a transformation unit



Rotation about a Fixed Point

Start with identity matrix: $\mathbf{C} \leftarrow \mathbf{I}$
Move fixed point to origin: $\mathbf{C} \leftarrow \mathbf{CT}$
Rotate: $\mathbf{C} \leftarrow \mathbf{CR}$
Move fixed point back: $\mathbf{C} \leftarrow \mathbf{CT}^{-1}$

Result: $\mathbf{C} = \mathbf{TR}\,\mathbf{T}^{-1}$ which is **backwards**.

This result is a consequence of doing postmultiplications.
Let's try again.

Reversing the Order

We want $\mathbf{C} = \mathbf{T}^{-1}\,\mathbf{R}\,\mathbf{T}$
so we must do the operations in the following order

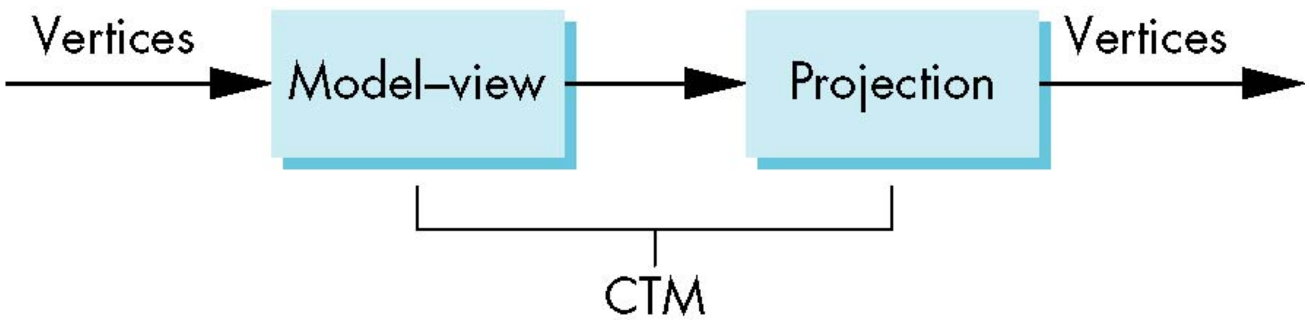
$\mathbf{C} \leftarrow \mathbf{I}$
$\mathbf{C} \leftarrow \mathbf{CT}^{-1}$
$\mathbf{C} \leftarrow \mathbf{CR}$
$\mathbf{C} \leftarrow \mathbf{CT}$

Note that the last operation specified is the first executed in the program

Recall $\mathbf{p}' = \mathbf{ABCp} = \mathbf{A}(\mathbf{B}(\mathbf{Cp}))$ // pre-multiply

CTM in OpenGL

- OpenGL had a model-view and a projection matrix in the pipeline which were concatenated together to form the CTM
- We will emulate this process



vec.h and mat.h

Define basic types

- vec2, vec3, vec4
  - 2, 3 & 4 element arrays
- mat2, mat3, mat4
  - 2x2, 3x3 & 4x4 matrices
- Overloaded operators
  - +, -, *, /, +=, -=, *=, /=, []
- Generators
  - RotateX, RotateY, RotateZ, Translate, Scale
  - Ortho, Ortho2D, Frustum, Perspective, LookAt

Rotation, Translation, Scaling

Create an identity matrix:

```
mat4 m = identity();
```

Multiply on right by rotation matrix of **`theta`** in degrees where (**`vx, vy, vz`**) define axis of rotation

```
mat4 r = Rotate(theta, vx, vy, vz)
m = m*r;
```

Do same with translation and scaling:

```
mat4 s = Scale( sx, sy, sz)
mat4 t = Translate(dx, dy, dz);
m = m*s*t;
```

Example

- Rotation about z axis by 30 degrees with a fixed point of (1.0, 2.0, 3.0)

```
mat4 m = identity();
m = Translate(1.0, 2.0, 3.0)*
    Rotate(30.0, 0.0, 0.0, 1.0)*
    Translate(-1.0, -2.0, -3.0);
```

- Remember that last matrix specified in the program is the first applied

Arbitrary Matrices

- Can load and multiply by matrices defined in the application program
- Matrices are stored as one dimensional array of 16 elements which are the components of the desired 4 x 4 matrix stored by columns
- OpenGL functions that have matrices as parameters allow the application to send the matrix or its transpose

Matrix Stacks

- In many situations we want to save transformation matrices for use later
  - Traversing hierarchical data structures (Chapter 8)
  - Avoiding state changes when executing display lists
- Pre 3.1 OpenGL maintained stacks for each type of matrix
- Easy to create the same functionality with a simple stack class

Using Transformations

- Example: use idle function to rotate a cube and mouse function to change direction of rotation
- Start with a program that draws a cube in a standard way
  - Centered at origin
  - Sides aligned with axes
  - Will discuss modeling in next lecture

main.c

```
void main(int argc, char **argv)
{
    glutInit(&argc, argv);
    glutInitDisplayMode(GLUT_DOUBLE | GLUT_RGB |
       GLUT_DEPTH);
    glutInitWindowSize(500, 500);
    glutCreateWindow("colorcube");
    glutReshapeFunc(myReshape);
    glutDisplayFunc(display);
    glutIdleFunc(spinCube);
    glutMouseFunc(mouse);
    glEnable(GL_DEPTH_TEST);
    glutMainLoop();
}
```

Idle and Mouse callbacks

```
void spinCube()
{
 theta[axis] += 2.0;
 if( theta[axis] > 360.0 ) theta[axis] -= 360.0;
 glutPostRedisplay();
}

void mouse(int btn, int state, int x, int y)
{
    if(btn==GLUT_LEFT_BUTTON && state == GLUT_DOWN)
            axis = 0;
    if(btn==GLUT_MIDDLE_BUTTON && state == GLUT_DOWN)
            axis = 1;
    if(btn==GLUT_RIGHT_BUTTON && state == GLUT_DOWN)
            axis = 2;
}
```

Display callback

We can form matrix in application and send to shader and let shader do the rotation or we can send the angle and axis to the shader and let the shader form the transformation matrix and then do the rotation

More efficient than transforming data in application and resending the data

```
void display()
{
   glClear(GL_COLOR_BUFFER_BIT | GL_DEPTH_BUFFER_BIT);
   glUniform(…); //or glUniformMatrix
   glDrawArrays(…);
   glutSwapBuffers();
}
```

Using the Model-view Matrix

- In OpenGL the model-view matrix is used to
  - Position the camera
    - Can be done by rotations and translations but is often easier to use a LookAt function
  - Build models of objects
- The projection matrix is used to define the view volume and to select a camera lens
- Although these matrices are no longer part of the OpenGL state, it is usually a good strategy to create them in our own applications

Smooth Rotation

- From a practical standpoint, we often want to use transformations to move and reorient an object smoothly
  - Problem: find a sequence of model-view matrices $\mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_n$ so that when they are applied successively to one or more objects we see a smooth transition
- For orientating an object, we can use the fact that every rotation corresponds to part of a great circle on a sphere
  - Find the axis of rotation and angle
  - Virtual trackball (see text)

Incremental Rotation

- Consider the two approaches
  - For a sequence of rotation matrices $\mathbf{R}_0,\mathbf{R}_1, \ldots..,\mathbf{R}_n$, find the Euler angles for each and use $\mathbf{R}_i = \mathbf{R}_{iz}\,\mathbf{R}_{iy}\,\mathbf{R}_{ix}$
    - Not very efficient
  - Use the final positions to determine the axis and angle of rotation, then increment only the angle
- Quaternions can be more efficient than either

Quaternions

- Extension of imaginary numbers from two to three dimensions
- Requires one real and three imaginary components **i**, **j**, **k**

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

- Quaternions can express rotations on sphere smoothly and efficiently. Process:
  - Model-view matrix → quaternion
  - Carry out operations with quaternions
  - Quaternion → Model-view matrix