Linear Algebra

Connections to Linear Equations

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Vector Independence

$\{\vec{v}_1, \dots, \vec{v}_N\}$ Linearly Independent if and only if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_N\vec{v}_N = \vec{0}$$

c_i variables

v_{ij} coeffs

only has ONE distinct solution $c_1 = c_2 = \dots = 0$

$$\vec{v}_j = \begin{pmatrix} v_{1j} \\ v_{2j} \\ \vdots \\ v_{Mj} \end{pmatrix}$$

$$\boxed{\begin{aligned} c_1 v_{11} + c_2 v_{12} + \dots + c_N v_{1N} &= 0 \\ c_1 v_{21} + c_2 v_{22} + \dots + c_N v_{2N} &= 0 \\ &\vdots \\ c_1 v_{M1} + c_2 v_{M2} + \dots + c_N v_{MN} &= 0 \end{aligned}}$$

Vector Independence

Are these vectors linearly independent?

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \quad x_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$+ (-1) x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$+ x_2 + \quad = 0$$

$x_2 = 0$

$\downarrow$

$x_3 = 0$

$\downarrow$

$x_1 = 0$

3-D span

Vector Spans

The three vectors form a basis for 3D-vector space ($\mathbb{R}^3$)

Is $\vec{w}$ in the span of $\{\vec{v}_1, \vec{v}_2\}$?

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 = \vec{w}$$

$\vec{w}$ is in Span $\{\vec{v}_1, \vec{v}_2\}$ iff this system has a solution.

$$\underline{x_1} \cdot v_{11} + \underline{x_2} v_{12} = w_1$$

$$\underline{x_2} \cdot v_{21} + \underline{x_2} v_{22} = w_2$$

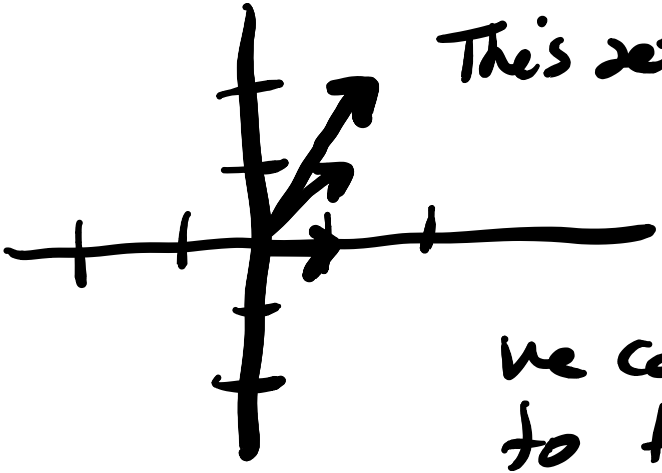
Systems of Linear Equations

$$\begin{array}{ccc} 1 & -2 & 1 \\ x_1 + & x_2 + & x_3 = 0 \\ 2x_1 + & x_2 & = 0 \end{array}$$

$\xrightarrow{\text{translate to a vector}}$ $x_1\begin{pmatrix}1\\2\end{pmatrix} + x_2\begin{pmatrix}1\\1\end{pmatrix} + x_3\begin{pmatrix}1\\0\end{pmatrix} = \vec{0}$

$x_2 = -2x_1$

$x_3 = x_1$



This set is linearly **DEpendent**.

This is true iff we can find a non-zero solution to the equations above!!!!

Matrix Equation

$$x_1 \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} + x_2 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} + x_3 \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \vec{0}$$

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ 2x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$