Linear Algebra

Reduced Row Echelon Form

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RREF : Gauss - Jordan Elimination

Defn) A matrix is in RREF provided:

1) The 1^{st} non-zero entry of each row is one. (pivot)
2) All values above/below a pivot are zero.
3) The rows are arranged by their pivots
i.e. The top row has the leftmost pivot, all rows of zeros are at the bottom.

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Every matrix has a <u>unique</u> RREF!!

RRREF

(iff ⟷ if and only if)

- a square matrix is invertible iff its RREF is the identity matrix
- The RREF of a matrix has the same row space. (column)
- The dimension of the row space of a matrix is the # of pivots in its RREF.

Elementary Row Operations: Manipulate the rows of A w/o changing the row space.

1) Swap any two rows

2) Multiply any row by a (non-zero) scalar (non-zero)

3) Add a multiple of one row to another.

All these operations preserve the row space.

Elementary Row Operations

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{r} 1 \quad 2 \quad 0 \\ +2(-1 \ -1 \ \ 2) \\ \hline \end{array}$$

1) Row Swap: $R_1 \leftrightarrow R_2 \rightarrow \begin{bmatrix} -1 & -1 & -2 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

2) Scalar Multiple: $R_1 \rightarrow 2R_1 \rightarrow \begin{bmatrix} 2 & 4 & 0 \\ -1 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

3) Add a row: $R_1 \rightarrow R_1 + 2R_2$

$$\rightarrow \begin{bmatrix} -1 & 0 & 4 \\ -1 & -1 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

RREF Idea: Systematically make each element of a column zero until we reach a pivot (or not) from the bottom-up.

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & -2 \\ 0 & 1 & 0 \end{bmatrix} R_2 \to R_2 + R_1 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{bmatrix} R_3 \to R_3 + \left(-\tfrac{1}{3}\right) R_2$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2/3 \end{bmatrix} R_2 \to \tfrac{1}{3} R_2 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 2/3 \end{bmatrix} R_1 \to R_1 + -2R_2$$

$$\begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 2/3 \end{bmatrix} R_3 \to \tfrac{3}{2} R_3 \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 + {-3/4}\, R_3 \\ R_2 \to R_2 + {3/2}\, R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

RREF

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix} R_3 \to R_3 + 2R_1 \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 0 & 7 & -2 \end{bmatrix} R_2 \to R_2 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 7 & -2 \end{bmatrix} R_3 \to R_3 + R_2 \begin{bmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & 0 \end{bmatrix} R_2 \to -\tfrac{1}{7} R_2$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2/7 \\ 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & 4/7 \\ 0 & 1 & -2/7 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{RREF}$$

Matrix Inverse

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Augmented Matrix

$$\underbrace{(B_3 B_2 B_1) A}_{I_3} \; \underbrace{(B_3 B_2 B_1) I_3}_{A^{-1}} \to B_3 B_2 B_1$$

$$A^{-1} A = I_3$$

Row Swap

$$\overset{B}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & -2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -2 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_1 \Leftrightarrow R_2$

Matrix Inverse

$$\overset{A}{\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

AA^{-1}

$A^{1}A$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -1 & -2 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right]$$

A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -1 & -2 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$