Linear Algebra

Matrix Rank and Linear Maps

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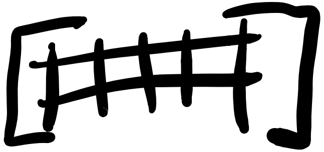
Column Rank



$$\boxed{\begin{array}{l}\operatorname{rank}(A) \le \min(M,N) \\ \quad A_{M\times N}\end{array}}$$

$\operatorname{colsp}(A) = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

<u>Definition</u>) The <u>column</u> (row) rank of A is dimension of the column (row) space

<u>Thm</u>) row rank = column rank

<u>Definition</u>) The <u>rank</u> is the dim of row/col sp.

Rank and RREF

- For a matrix in RREF, the rank of A is the # of pivots.

rank = 2

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Square Matrix</u>

$\underset{N \times N}{A} \xrightarrow{\text{RREF}} I_N \rightarrow \text{Rank} = N$ "full Rank"

$A \rightarrow \text{RREF} \; \cancel{I_N} \rightarrow \text{Rank} < N$

- A sq. matrix is invertible iff it has full Rank.

Matrix as a Linear Map

$\underset{M\times N}{A}\vec{x} = \vec{y}$ ← M-dim

$\vec{x}$: N-dim

span colums of A

$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_N\vec{v}_N$

column Space = Image of this Liner map

Rank(A) = Dimension of this Image

Nullspace

Definition The *Nullspace* of A is the set of vectors $\text{Nul}(A) = \{\vec{x} \text{ such that } A\vec{x} = \vec{0}\}$

The *Nullity* of A is the $\dim(\text{Nul}(A))$.

Rank - Nullity Thm

$A_{m \times n}$ $\qquad \text{Rank}(A) + \text{Nullity}(A) = n$

Square matrices: full rank $\Rightarrow$ Nullity $= 0$

Example

Find the Nullity of A.

(use RREF & Rank-Nullity)

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}_{2\times 2}$$

$$\downarrow R_2 \to -\tfrac{1}{2}R_1 + R_2$$

$$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix} \xrightarrow{\tfrac{1}{2}R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

RREF

- 1 pivot
- $\text{Rank}(A) = 1$

$1 + \text{Nullity} = 2$

$$\boxed{\text{Nullity} = 1}$$