Linear Algebra

Introduction to Calculus with Matrices

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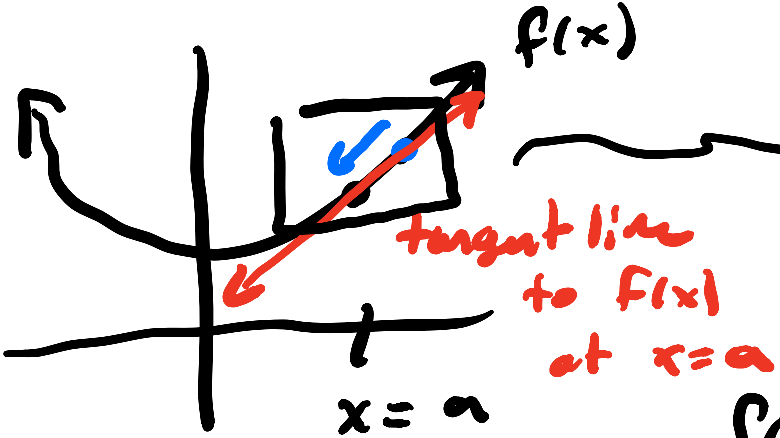
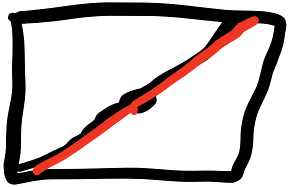
The Derivative at a point $x = a$.

$f(x) = y$

$$f'(a) = \lim_{x \to a} \boxed{\frac{f(x) - f(a)}{x - a}} \qquad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



local approx of f(x).



(around $x = a$)

$$f(x) \approx f'(x)(x-a) + f(a)$$

The Derivative

- polynomial $x^n = f(x)$

$f'(x) = n x^{n-1}$ $\boxed{\text{\underline{Deriv.} of a \underline{polynomial}}}$

- $f(x) = g(x) + h(x)$ $f'(x) = g'(x) + h'(x)$

- $f(x) = g(x)h(x)$ $f'(x) = g'(x)h(x) + g(x)h'(x)$

<u>Chain Rule</u>

- $f(x) = g(h(x))$ $f'(x) = g'(h(x))h'(x)$

The Derivative

A critical point of $f(x)$ is any point $x=a$ such that $f'(a)=0$.

~ Minimums/Maximums occur at crit. pt.s

Ex] $f(x) = x^2 + 1$ find crit. pt.

$$f'(x) = 2x$$

$$2x = 0 \longrightarrow \boxed{x=0}$$

The Partial Derivative

$$f(x_1, x_2, x_3, \ldots, x_N)$$

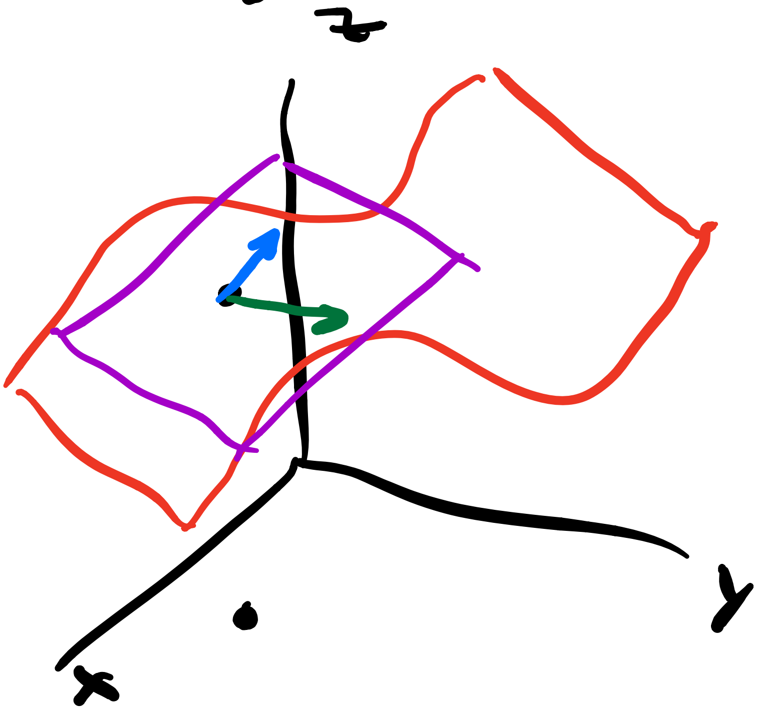
The partial derivative of f with respect to x_i is the derivative of f w.r.t. x_i holding all other variables constant.

<u>Ex)</u> $f(x, y) = 2xy + x^2$

$$\frac{\partial f}{\partial x} = 2y + 2x \qquad \frac{\partial f}{\partial y} = 2x$$

The Partial Derivative

$$z = f(x, y)$$



$$\frac{\partial f}{\partial x}(a)\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\frac{\partial f}{\partial y}(a)\begin{pmatrix}0\\1\\0\end{pmatrix}$$

The Jacobian

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

$$\boxed{f(\vec{x}) \approx J_f(\vec{p})(\vec{x}-\vec{p}) + f(\vec{p})}$$

$$f(f_1(x_1,\dots,x_m), f_2(x_1,\dots,x_m), \dots, f_n(x_1,\dots,x_m))$$

The <u>Jacobian</u> of f is given by

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & & & \\ \vdots & & & \\ \partial f_n/\partial x_1 & \cdots & \cdots & \partial f_n/\partial x_m \end{bmatrix} = J_f = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$$

Example

$$f: \mathbb{R}^2 \to \mathbb{R}^2 \quad (\underline{x^2y+2x},\ \underline{y^3+2xy})$$

$$J_f = \begin{bmatrix} 2xy+2 & x^2 \\ 2y & 3y^2+2x \end{bmatrix}$$

$J_f(0,0)$

$$J_f(1,2) = \begin{bmatrix} 2(1)(2)+2 & 1^2 \\ 2(2) & 3(2)^2+2(1) \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 14 \end{bmatrix}$$

Jacobian Properties

→ A point is a critical point if the rank of the Jacobian is <u>not maximal</u>.

→ For a square Jacobian matrix, full rank ⇒ invertibility of f. (locally)