L3. Conditional Probability and its Applications

- Independent Events
- Bayes's Thm

1. Independent Events

Hint: Under the $\boxed{\text{same}}$ outcome space S of an experiment, the occurance of event A $(A \subseteq S)$ may not change the probability of the occurance of B $(B \subseteq S)$.

$\boxed{\text{Ex 1}}$ Flip a coin twice

$A = \{$ heads on the first flip $\}$

$B = \{$ tails on the second flip $\}$

Mathematically:

$\boxed{\text{Defination}}$

Independent Events: Events A and B are $\boxed{\text{independent}}$ if and only if

$$P(A \cap B) = P(A)P(B)$$

If independent, $P(A|B) = P(A)$, $P(B|A) = P(B)$ $\Rightarrow P(A\cap B) = P(A)\cdot P(B)$

$\boxed{\text{Revisit Ex 2}}$ check the math.

$S = \{HH, HT, TH, TT\}$

$A = \{HH, HT\}$

$B = \{HT, TT\}$

$A \cap B = \{HT\}$

$P(A\cap B) = \frac{1}{4} = P(A)\cdot P(B)$

Think: ① use the example: are "independent" and "mutualy exclusive" events the same? are they related in some way?

② (Later) How "independent events",

"independent random variables" and

"independent samples" related ?

Extend: multiple independence; A` and B`

II. Bayes's Theorem

$\boxed{\text{Ex 2}}$ Medical Test for patients.

A company created a test for a type of cancer.

From experiments / clinical trials, they have known

the $\boxed{\text{accuracy of the test}}$:

		Cancer (Known)	
		Yes	No
Test	Positive	✓ TP	✗ FP
	Negative	✗ FN	✓ TN

precision: For patients with cancer, what proportion (sensitivity) was identified by the test?

True Positive Rate

$$\mathbb{P}(\text{Positive} \mid \text{Cancer}) = 0.85$$

consequently, the $\boxed{\text{false negative}}$

$$\mathbb{P}(\text{Negative} \mid \text{Cancer}) = 0.15$$

specificity: $\mathbb{P}(\text{Negative} \mid \text{No Cancer}) = 0.9$

True Negative Rate

consequently, the $\boxed{\text{false positive}}$

$$\mathbb{P}(\text{Positive} \mid \text{No Cancer}) = 0.1$$

We have one more information, in US population

$$\mathbb{P}(\text{Cancer}) = 0.0001$$

Question: the company put the test on the market, a patient took it and tested positive. What is the prob. that the patient has cancer?

$\boxed{\text{Bayes's Thm}} \longrightarrow \boxed{\text{Bayesian Inference}}$

To use the prior information ($\mathbb{P}$(cancer))

and posterior information ($\mathbb{P}$(Test | cancer))

evidence

to predict new observation's outcome.

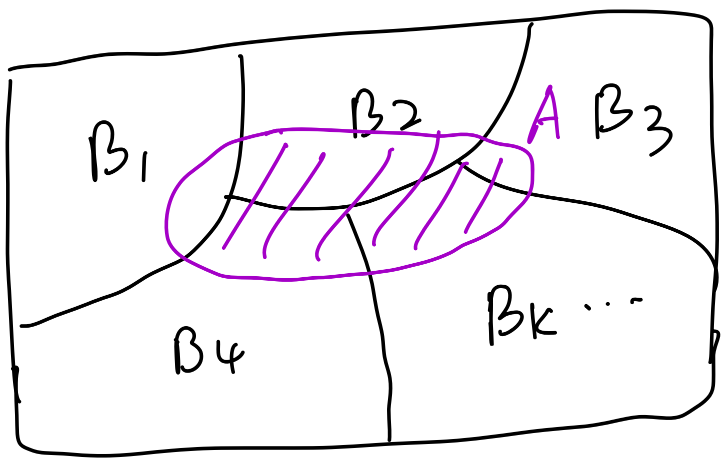
($\mathbb{P}$(cancer | Test) New patient)

$\boxed{\text{Bayes's Thm}}$

(a) $\mathbb{P}(B|A) = \dfrac{\mathbb{P}(B)\cdot\mathbb{P}(A|B)}{\mathbb{P}(A)}$

proof ?

(b)

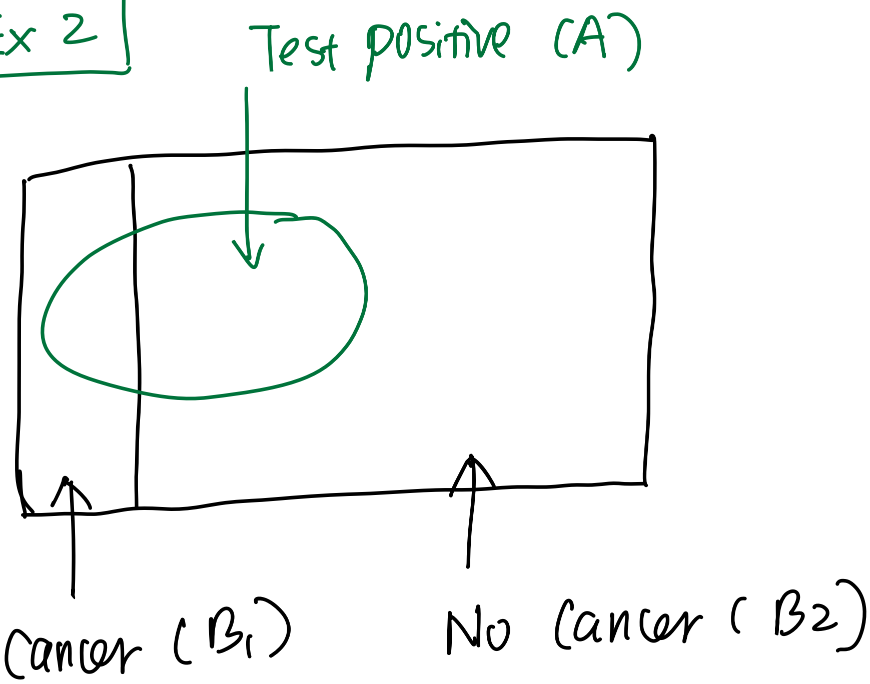


$$\mathbb{P}(B_K|A) = \frac{\mathbb{P}(B_K)\cdot\mathbb{P}(A|B_K)}{\mathbb{P}(A)}$$

$$= \frac{\mathbb{P}(B_K)\cdot\mathbb{P}(A|B_K)}{\sum_{i=1}^{m}\mathbb{P}(A\cap B_i)}$$

$$= \frac{P(B_k) \cdot P(A \mid B_k)}{\sum_{i=1}^{K} P(B_i) P(A \mid B_i)}$$

$\boxed{\text{Revisit Ex 2}}$



$\mathbb{P}(\text{Cancer} \mid \text{Test Positive})$

$= \mathbb{P}(B_1 \mid A)$ ← Posterior Prob.

$$= \frac{\overset{\text{Prior Prob.}}{\mathbb{P}(B_1)} \cdot \overset{\text{Test Accuracy}}{\mathbb{P}(A|B_1)}}{\mathbb{P}(B_1)\cdot \mathbb{P}(A|B_1) + \underset{\substack{\mathbb{P}(\text{No cancer}) \\ = 1 - \mathbb{P}(\text{cancer})}}{\mathbb{P}(B_2)} \cdot \underset{\text{Test Accuracy}}{\mathbb{P}(A|B_2)}}$$

$$= \frac{(0.0001)(0.85)}{(0.0001)(0.85) + (1-0.0001)(0.1)}$$

$$= 0.00085$$

Problem?

- False positive rate is too high
- prob. of disease is too small