# Linear Algebra

## Sets of Vectors

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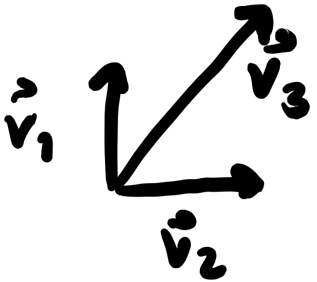
# Overview

- Vector Spans
- Linear In/dependence
- Orthonormal Basis

# Vector Span

$$\left\{ \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix} \right\} \qquad \left\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \right\}$$




We say that a $\vec{w}$ is a linear combination of these three vectors if there exist scalar values $c_1, c_2, c_3$ such that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{w}$$

$$1 \begin{pmatrix}0\\1\end{pmatrix} + (-7)\begin{pmatrix}1\\0\end{pmatrix} + 2\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix}$$

# Vector Span

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N\}$

Span( ) = The collection of all linear combinations of these vectors.

$$= \left\{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_N\vec{v}_N \;\middle|\; c_1, c_2, \ldots, c_N \in \mathbb{R} \right\}$$

$$= \left\{ \sum_{i=1}^{N} c_i \vec{v}_i \;\middle|\; c_i \in \mathbb{R} \right\}$$

# Vector Span

$\underbrace{\begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix}}, \begin{pmatrix}2\\2\end{pmatrix}$

All 2D vectors

$\underline{\text{Span}}\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}$

$\vec{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix} = v_1\begin{pmatrix}1\\0\end{pmatrix} + v_2\begin{pmatrix}0\\1\end{pmatrix}$

$2\begin{pmatrix}0\\1\end{pmatrix} + 2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$

redundant

# Vector Span

# Vector Independence

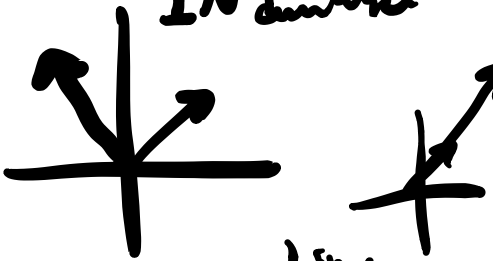
A set of vectors is <u>linearly independent</u> if no vector in the set can be written as a linear combination of the others.

- If this is NOT true then the set is linearly dependent.

# Vector Independence

When are **two** vectors linearly dependent? $\{\vec{v}_1, \vec{v}_2\}$

linear INdependence



Linear DEpendence

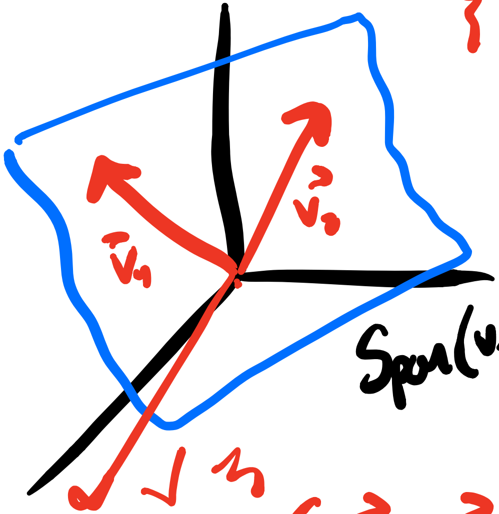
$$\vec{v}_1 = c\,\vec{v}_2$$

So if this is true, then $\{v_1, v_2\}$ is linearly dependent

# Vector Independence

linearly independent

$\{\vec{v}_1, \vec{v}_2\}$




The span of $v_1$ + $v_2$

$$\text{Span}(v_1, v_2) = \{c_1 v_1 + c_2 v_2 \mid c_1, c_2 \in \mathbb{R}\}$$

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{All 3D vectors}$$

Linearly independent

# Vector Independence $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N\}$

A set of vectors is linearly independent if the following equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_N \vec{v}_N = \vec{0}$$

Only has ONE solution $c_1 = c_2 = \cdots = c_N = 0$

$$c_1 \vec{v}_1 + \cdots + c_{N-1} \vec{v}_{N-1} = -c_N \vec{v}_N$$

# Basis

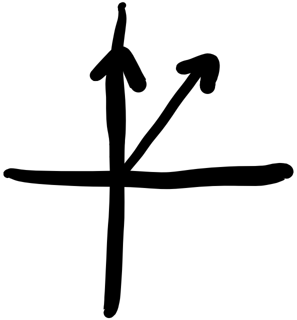
A set of vectors is a <u>basis</u> for a vector space if they linearly independent and their span is equal to the space.

Basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Basis for 2D vector space



$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (v_2 - v_1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} v_1 \\ v_1 + v_2 - v_1 \end{pmatrix} = \vec{v}$$

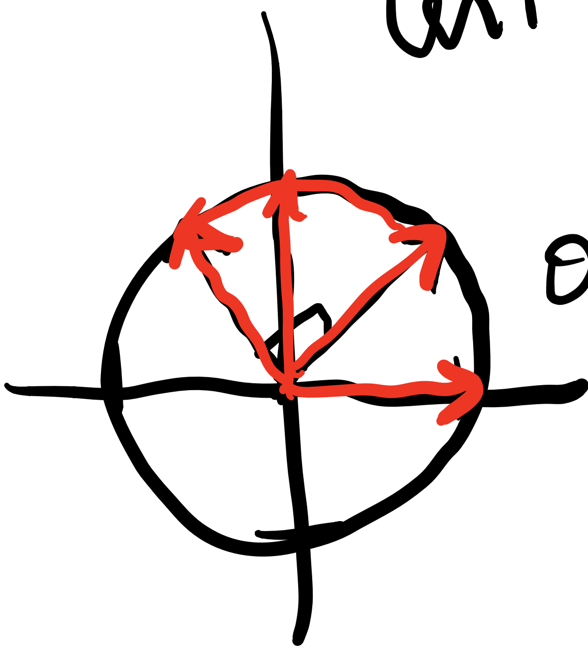
Basis

A set of vectors forms an <u>orthonormal basis</u> for a vector space if they are a basis and the vectors are all mutually orthogonal and have norm $= 1$.

$N$-dimensional vector space $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$

$$\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

unit circle



orthonormal basis

Different o. bases as
Different orientations.