Linear Algebra

Solving Linear Equations II

Michael Ruddy

Square Linear System

$$A_{N\times N}\vec{x} = \vec{b}$$

→ N equations and N variables

$$A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}$$

→ This system has a unique soln iff A has full rank.

Square Linear System

$$x - y = 1$$
$$-x + y = -1$$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ -1 & 1 & -1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

Not full Rank

$$\boxed{\begin{array}{c} x - y = 1 \\ 0 = 0 \end{array}} \rightarrow x - y = 1$$

infinitely Sol'n

$$x - y = 1$$
$$-x + y = 2$$

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 3 \end{array}\right)$$

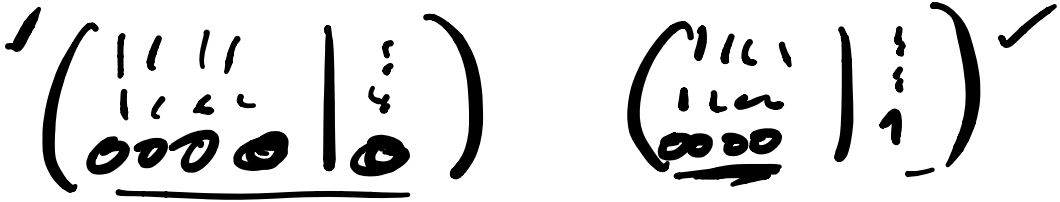
$$\boxed{\begin{array}{c} x - y = 1 \\ 0 = 3 \end{array}}$$

no solution

Consistency

Definition A linear system is **consistent** if it has a sol'n & inconsistent if it does not.

→ $\text{rank}(A) = \text{rank}(A|b)$ iff consistent



Overdetermined System

- There are more equations than variables

$$2x - y = 1$$
$$-x + y = 2$$
$$x - y = -2$$

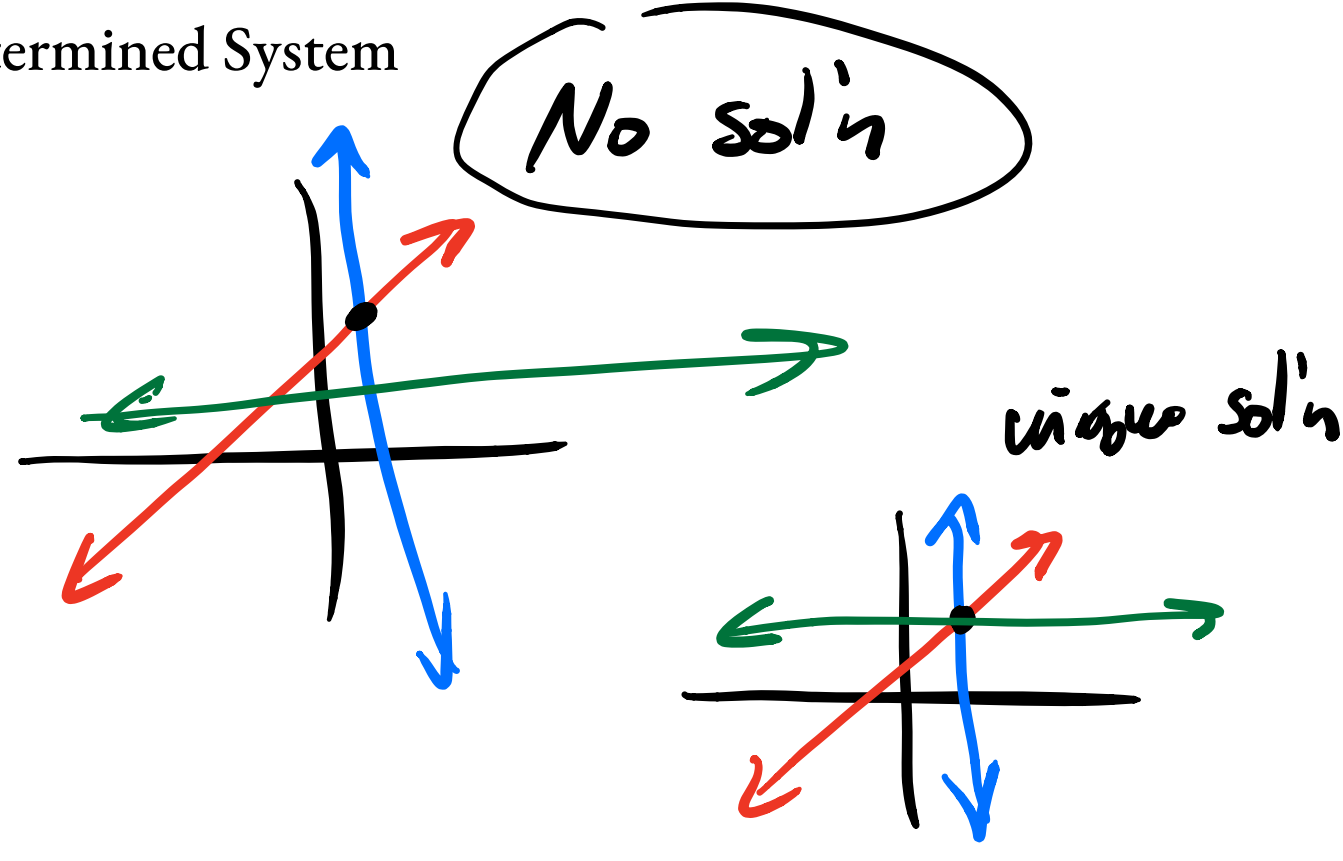
$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$\text{Rank}(A) \leq 2$

$$\left(\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -2 \end{array}\right) \leftarrow$$

$$0 \quad 0 \quad ?$$
$$0 \leftarrow \text{consistent}$$
$$? \leftarrow \text{inconsistent}$$
$$?$$

Overdetermined System

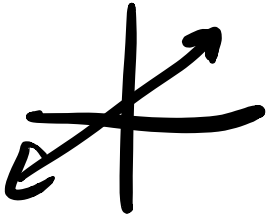


Underdetermined System

- More variables than equations
  - infintely many sol'n
  - No sol'n

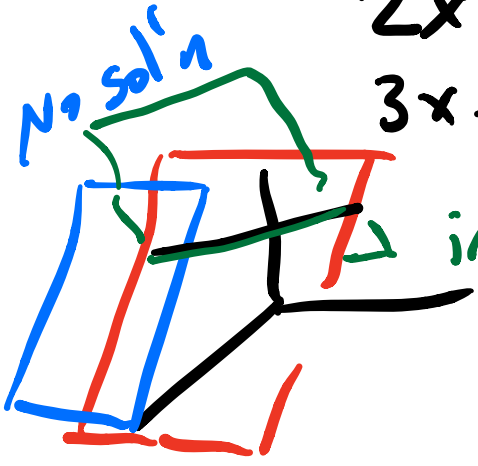
$2x+3y=1$

→ inf. sol'n



$2x+z=0$

$3x-y=2$



Underdetermined System