Linear Algebra

Solving Linear Equations I

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Linear Systems

$$\begin{cases} 2x + 3y = 4 \\ y - z = 10 \\ x + z = 2 \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{x}$$

$$\begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = A$$

$$\begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix} = \vec{b}$$

$$\underset{3\times 3}{A}\,\underset{3\times 1}{\vec{x}} = \underset{3\times 1}{\vec{b}}$$

RREF

$$\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$2x + y = 1$$
$$3x - y = -2$$

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & -1 & -2 \end{array}\right)$$

$$R_2 \to -\tfrac{3}{2}R_1 + R_2$$

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & -5/2 & -7/2 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 7/5 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 2 & 0 & -2/5 \\ 0 & 1 & 7/5 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 1 & 0 & -1/5 \\ 0 & 1 & 7/5 \end{array}\right) \to \begin{array}{l} x = -1/5 \\ y = 7/5 \end{array}$$

RREF

$$A\vec{x} = \vec{b}$$

$\rightarrow$ Multiplying both sides by matrices

$$I_N \vec{x} = \vec{b}'$$

$\rightarrow$ In the prev. ex. A is invertible.

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$\rightarrow \boxed{\vec{x} = A^{-1}\vec{b}}$$

$$\boxed{\text{when } A_{N\times N} \text{ has full rank then } A\vec{x} = \vec{b} \text{ has a unique sol'n.}}$$

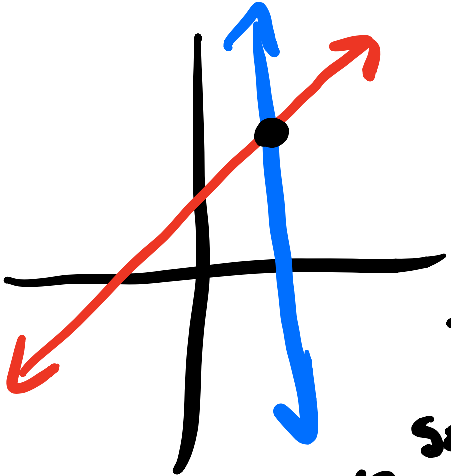
Geometric Intuition

$2x + y = 1$

$3x - y = -2$

Slope $\longleftrightarrow$ row vectors of A

Sol'n $\rightarrow$ any pt. (x, y) on both lines



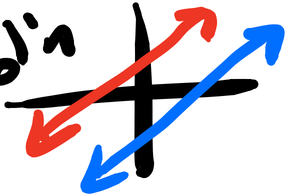
$\rightarrow$ full rank $\rightarrow$ lin. ind.

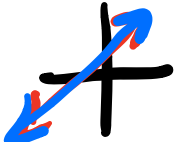
$\rightarrow$ lines intersect

$\rightarrow$ not full rank $\rightarrow$ lin. dep.

$\rightarrow$ lines parallel

No sol'n





infinite sol'n

Geometric Intuition

In general for any linear system $A\vec{x} = \vec{b}$ we either have

- 1 sol'n
- No sol'n
- inf. sol'n

$$3x + y + z = 0$$
$$y - z = 1$$
$$x + y \quad = 2$$