L2. Basic Concepts and Properties of Probability

- Random Experiments and Outcome space
- Events and Probability
- Conditional Probability

I. Random Experiments and Outcome Space.

Probability Theory considers experiments / events for which outcomes are not certain, but stochastic.

$\boxed{\text{Ex 1}}$ Roll a die once

The outcome is not a certain number, but a collection of possible values $\{1, 2, 3, 4, 5, 6\}$ each with a chance $1/6$.

$\boxed{\text{Definations}}$

Random Experiment : experiments for which the outcomes are uncertain before the experiment is performed, and the collection of every possible outcome can be described.

Outcome Space : the collection of all the possible outcomes. Denote by S.

$\boxed{\text{Ex 1 (revisit)}}$:

The random experiment is to roll a die once

The outcome space is $S = \{1, 2, 3, 4, 5, 6\}$

$\boxed{\text{Ex 2}}$:

A. Flip a coin three times, what is the S ?

B. Flip three coins once, what is the S ?

II. Events and Properties of Probability

The def. and properties of probability are carried out by the $\boxed{\text{chance of the occurrance of an event}}$ when a random experiement is performed.

Definations

Event: An Event A is a subset of the outcome space S, i.e. $\boxed{A \subseteq S}$. It's a collection of some outcomes of a random experiment.

Review at home: Algebra of sets.

Probability: Probability is a function:

$$S \xrightarrow{\mathbb{P}} \mathbb{R}$$

For each event A:

$$A \xrightarrow{\mathbb{P}} \underbrace{\mathbb{P}(A)}$$

A real value that describes the chance of event A has occured.

The function $\mathbb{P}$ satisfies:

(1) $\mathbb{P}(A) \geq 0$ for any $A \subseteq S$

(2) $\mathbb{P}(S) = 1$

(3) If $A_1, A_2, A_3, \cdots$ are a sequence of $\boxed{\text{mutually exclusive}}$

events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Theorems

Thm 1: For any event A,

$$\mathbb{P}(A) = 1 - \mathbb{P}(A')$$

proof: $S = A \cup A'$, $A \cap A' = \phi$

so $\mathbb{P}(S) = \mathbb{P}(A \cup A') = \mathbb{P}(A) + \mathbb{P}(A')$

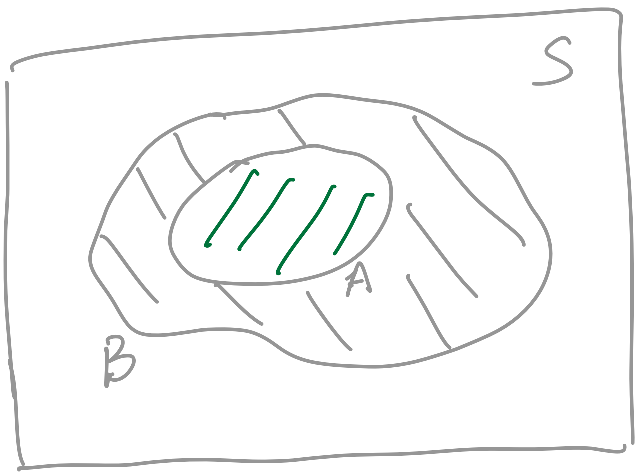
$= 1$

then $\mathbb{P}(A) = 1 - \mathbb{P}(A')$

Thm 2: $\mathbb{P}(\phi) = 0$

Thm 3: $\forall A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

Proof: use Venn Diagram



$$\mathbb{P}(B) = \mathbb{P}(A \cup (B \cap A'))$$

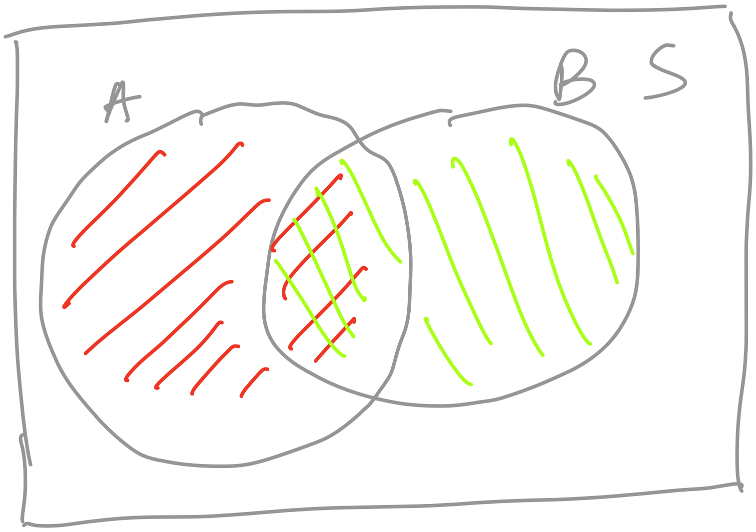
$$= \mathbb{P}(A) + \mathbb{P}(B \cap A')$$

$$\geq \mathbb{P}(A)$$

Thm 4: $\forall A$, $\mathbb{P}(A) \leq 1$

Thm 5: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

proof:



$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \cap A')$

$\mathbb{P}(B \cap A') = \mathbb{P}(B) - \mathbb{P}(A \cap B)$ why?

so $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Thm5: $\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)$

$- \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B)$

$+ \mathbb{P}(A \cap B \cap C)$

Practice at Home: Can you prove this?

Thm 6: Let S be a discrete and finite sample space, i.e. $S = \bigcup_{i=1}^{n} \{s_i\}$. If the members of S are $\boxed{\text{equally likely}}$, then

$$\mathbb{P}(s_i) = \frac{1}{n}.$$

$\boxed{\text{Ex 3}}$ Roll a die twice.

A. What is the sample space S?

B. What is the prob. of the event: sum of the outcomes is 7?

C. If the sum is 7, what is the prob.

of getting an outcome $\{(2,5)\}$?

III. Conditional Probability.

$\boxed{\text{Continued Ex 3 (c)}}$

Event A: getting outcome (2, 5)

Event B: sum is 7.

$$P(A \mid B) = \frac{1}{6} = \frac{\#\text{ in }\{A \cap B\}}{\#\text{ in }\{B\}}$$

$\boxed{\text{Ex 4}}$: Voting

candidates

Voters		A	B
	F	10	20
	M	15	10

For a randomly selected ticket,

we know it's voted for candidate A,

what is the prob. that it's a female

voter ?

$$P(F|A) = \frac{10}{25} = \frac{P(A \cap F)}{P(A)}$$

$\boxed{\text{Definations}}$

The Conditional Probability of an event A given

that event B has occured, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Practice at home:

If we denote $\mathbb{P}_B(A) = \mathbb{P}(A|B)$, is $\mathbb{P}_B$ a probability measure function itself?

Hint: Check the defination of probability function for $\mathbb{P}_B$:

(1) $\mathbb{P}_B(A) \geq 0$ for $\forall A \subseteq S$?

(2) $\mathbb{P}_B(S) = 1$ for S?

(3) $\mathbb{P}_B(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} \mathbb{P}_B(A_i)$ for mutually exclusive sets $A_1, \cdots$?

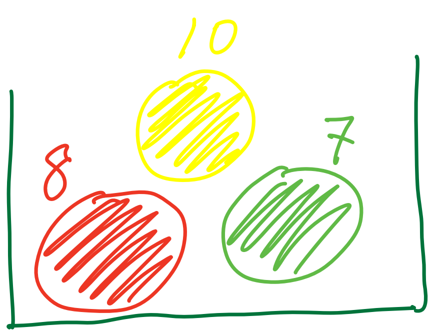
If so, all the properties described in Thm 1-6 apply:

For ex. $\mathbb{P}_B(A \cup C) = \underline{\mathbb{P}(A \cup C \mid B)}$

$= \mathbb{P}_B(A) + \mathbb{P}_B(C) - \mathbb{P}_B(A \cap C)$

$= \underline{\mathbb{P}(A \mid B) + \mathbb{P}(C \mid B) - \mathbb{P}(A \cap C \mid B)}$

$\boxed{\text{Ex 5}}$:



Draw twice without replacement.

A. If we know the first ball is yellow, what is the prob. of the second is yellow?

$$\mathbb{P}(\text{2nd is yellow} \mid \text{1st is yellow}) = \frac{\mathbb{P}(A \cap B) \ (?)}{\mathbb{P}(B) \ (\checkmark)}$$

At the 2nd draw $= \dfrac{\#\text{ of Yellow}}{\#\text{ of total}} = \dfrac{9}{24}$

consider the nature of the experiment

B. What is the prob. that both draws are yellow?

$\mathbb{P}(\text{1st is Yellow} \cap \text{2nd is Yellow})$

$= \mathbb{P}(A \cap B) = \underbrace{\dfrac{10}{25} \cdot \dfrac{9}{24}}_{}= \mathbb{P}(A) \cdot \mathbb{P}(A|B)$

Review: Methods of Enumeration

Thm: $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(A|B)$ **Correction: =P(A)*P(B|A)**

$= \mathbb{P}(B) \cdot \mathbb{P}(B|A)$ **Correction: =P(B)*P(A|B)**

Ex. Drawing cards!!

Draw cards one by one without replacement..

What is the prob. of the 3^{rd} spade appears on the 6^{th} draw?

A : Two spades in the first five cards

B : A spade on the sixth card

$$P(A) = \frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}}$$

$$P(B|A) = \frac{11}{47}$$

$$P(A \cap B) = P(B|A) \cdot P(A)$$

$\boxed{\text{Take Aways}}$

- Review the basics of random experiment and probability function → Foundation for random variable and distributions

- Conditional probability $\longrightarrow$ Independence

Bayes's thm