Sky tesselation

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I. ICOSAHEDRON

The vertices of an icosahedron centered at the origin and with circumradius 1 are:

$$V_1 = (0, 0, 1) \tag{1a}$$
$$V_2 = (0, 0, -1) \tag{1b}$$
$$V_3 = \left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right) \tag{1c}$$
$$V_4 = \left(-\frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right) \tag{1d}$$
$$V_5 = \left(-\frac{5+\sqrt{5}}{10}, \sqrt{\frac{5-\sqrt{5}}{10}}, \frac{1}{\sqrt{5}}\right) \tag{1e}$$
$$V_6 = \left(-\frac{5+\sqrt{5}}{10}, -\sqrt{\frac{5-\sqrt{5}}{10}}, \frac{1}{\sqrt{5}}\right) \tag{1f}$$
$$V_7 = \left(\frac{5+\sqrt{5}}{10}, \sqrt{\frac{5-\sqrt{5}}{10}}, -\frac{1}{\sqrt{5}}\right) \tag{1g}$$
$$V_8 = \left(\frac{5+\sqrt{5}}{10}, -\sqrt{\frac{5-\sqrt{5}}{10}}, -\frac{1}{\sqrt{5}}\right) \tag{1h}$$
$$V_9 = \left(\frac{5-\sqrt{5}}{10}, \sqrt{\frac{5+\sqrt{5}}{10}}, \frac{1}{\sqrt{5}}\right) \tag{1i}$$
$$V_{10} = \left(\frac{5-\sqrt{5}}{10}, -\sqrt{\frac{5+\sqrt{5}}{10}}, \frac{1}{\sqrt{5}}\right) \tag{1j}$$
$$V_{11} = \left(-\frac{5-\sqrt{5}}{10}, \sqrt{\frac{5+\sqrt{5}}{10}}, -\frac{1}{\sqrt{5}}\right) \tag{1k}$$
$$V_{12} = \left(-\frac{5-\sqrt{5}}{10}, -\sqrt{\frac{5+\sqrt{5}}{10}}, -\frac{1}{\sqrt{5}}\right) \tag{1l}$$

II. TASKS

1. If we have a unit sphere centered on the origin O and 2 points $P_{1,2} = (x_{1,2}, y_{1,2}, z_{1,2})$ on it, find the Cartesian equation of the great circle which passes through both of these points.

2. We have a unit sphere centered on the origin O and 2 points $P_{1,2} = (x_{1,2}, y_{1,2}, z_{1,2})$ on it. Take a point M which lies on the line P_1P_2 and is inside the sphere, such that
$$\frac{P_1M}{MP_2} = \frac{a}{b}, a, b \in \mathbb{R}. \tag{2}$$
Find the coordinates of the point M in terms of the coordinates of P_1 and P_2, a, and b.
If the ray $O\vec{M}$ intersects the sphere at point Q, find the coordinates of Q.

3. Take an equilateral triangle ABC, divide it in smaller equilateral triangles by dividing its sides into $n = 2, 3, 4, ...$ smaller segments.

4. Take our icosahedron from before. For $n = 2$, on each face, construct n^2 smaller equilateral triangles. Find the projections of the vertices triangles onto the unit sphere. Record the vertices and faces of the new polyhedron in a file. Repeat for all $n <= 50$.

5. For each of the grids in the previous task, compute also the coordinates of the centroids of each triangle face, along with their projections on the sphere. Store these in the same files.

6. Find the (approximate) area of a face of the new polyhedron as a function of n.

7. Find the (approximate) area of a spherical triangle formed by the projection of a face of the polyhedron onto the unit sphere. For what $n = \tilde{n}$ is this area less than 1 $\deg^2$?

8. Find the area factor as a function of n:
$$F_A(n) = \frac{A_n}{4\pi}$$

9. Find the volume factor as a function of n:
$$F_V(n) = \frac{V_n}{4\pi/3}$$

10. Implement in a computer program the conversion from Cartesian coordinates (x, y, z) to spherical polar coordinates (r, θ, φ).

11. Implement in a computer program the conversion from spherical polar coordinates (θ, φ) to Mollweide coordinates (X, Y) (here we take $r = 1$ for the unit sphere).

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