Dependable Systems and Networks

HW #3

Due Nov. 9, 2020

Each question is 10 points.

1. The lifetime (measured in years) of a processor is exponentially distributed, with a mean lifetime of 2 years. You are told that a processor failed sometime in the interval [4, 8] years. Given this information, what is the conditional probability that it failed before it was 5 years old?

Solution:

2.1. We are told that the system failed between 4 and 8 years. We need to find the probability that it failed before 5 years. That we need the conditional probability

P(T<5 | 4<T<8) → failed between 4 and 5 years

$$\frac{\Pr ob\{[T<5]\cap[4\le T\le 8]\}}{\Pr ob\{4\le T\le 8\}}=\frac{\Pr ob[4\le T<5]}{\Pr ob\{4\le T\le 8\}}=\frac{F(5)-F(4)}{F(8)-F(4)}$$

$$=\frac{(1-e^{-05*5)})-(1-e^{-0.5*4)})}{(1-e^{-05*8)})-(1-e^{-0.5*4)})}=0.455$$

2. A component with time to failure $T$ has constant failure rate $\lambda=2.5*10^{-5}$/hour

a. Determine the probability that the component survives a period of 2 months without failure (Assume each month has 30 days)

b. Find the MTTF of the component

c. Find the probability that the component survives its MTTF

a. $t = 2$ months $= 2\times 30\times 24$ hours $= 1440$ hours

$R(t) = e^{-\lambda t} = e^{-2.5\times10^{-5}\times 1440} = e^{-3.6\times10^{-2}} = 0.9646$

b. Constant failure rate means T is exponentially distributed.

So, $MTTF = \int_0^\infty R(t)dt = \int_0^\infty e^{-\lambda t}dt = \frac{1}{\lambda} = \frac{1}{2.5\times10^{-5}}$ hours

$\therefore MTTF = 4\times10^4$ hours

c. $R(t=MTTF) = e^{-\lambda t} = e^{-2.5\times10^{-5}\times4\times10^4} = e^{-1} \approx 0.3679$

3. The lifetime of a processor (measured in years) follows the Weibull distribution, with parameters $\lambda = 0.5$ and $\beta = 0.6$.
   (a) What is the probability that it will fail in its first year of operation?
   (b) Suppose it is still functional after t = 6 years of operation. What is the conditional

probability that it will fail in the next year?

(c) Repeat parts (a) and (b) for β = 2.

(d) Repeat parts (a) and (b) for β = 1.

Solution:

Second problem is straightforward --> use Weibull distribution

(a) 0.3934

(b) 0.1323

(c) 0.9984

(d) 0.3934

4. A component may fail due to two different causes, A and B. It has been shown that the time to failure TA caused by A is exponentially distributed with density function $f_A(t) = \lambda_A e^{-\lambda_A t}$ for t >= 0, while the time to failure TB caused by B is exponentially distributed with density function

$$f_B(t) = \lambda_B e^{-\lambda_B t}$$ for t >= 0.

a. Describe the rationale behind using $f(t) = pf_A(t) + (1-p)f_B(t)$ as the probability density function for the time to failure

b. Explain the meaning of $p$ in this model

a. Since there are only 2 different causes that can lead to the failure of the component, according to "Total Probability Theorem", we have:

$$f(t) = f(t \mid \text{cause} = A) \cdot P\{\text{cause} = A\} + f(t \mid \text{cause} = B) \cdot P\{\text{cause} = B\}$$

$$= f_A(t) \cdot P\{\text{cause} = A\} + f_B(t) \cdot P\{\text{cause} = B\}$$

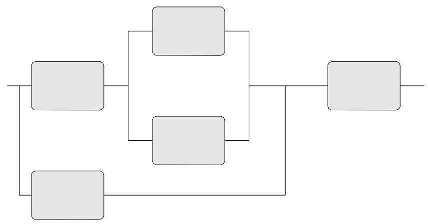
and $P\{\text{cause} = A\} + P\{\text{cause} = B\} = 1$.

If we let $P\{\text{cause} = A\} = p$, then

$$f(t) = p \cdot f_A(t) + (1-p) \cdot f_B(t)$$

b. $p$ denotes the probability that the failure of the component is caused by A.

5. Write the expression for the reliability $R_{system}(t)$ of the series/parallel system shown in the figure below, assuming that each of the five modules has a reliability of R(t).



A 5-module series-parallel system.

Solution:

Third problem involves converting the system into series-parallel combinations

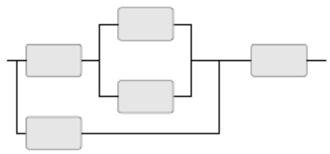


Figure 2.2: A 5-module series-parallel system.

Let us consider the parallel set in the top middle (I will label the units as C and D)
$R_{CD} = 1 - (1-R_C)(1-R_D)$

Now let us add the unit in front (say A)
$R_{ACD} = R_A * R_{CD} = R_A * [1 - (1-R_C)(1-R_D)]$

Now we have two parallel paths (with the bottom left, say B)
$R_{BACD} = 1- [(1-R_B)*\{R_A*(1 - (1-R_C)(1-R_D))\}]$

Finally we add the unit on the right (say E
$R_{ABCDE} = R_{BACD} * R_E$

$$R_{system} = R^5(t) - 3R^4(t) + 2R^3(t) + R^2(t)$$

6. Show that the MTTF of a parallel system of N modules, each of which suffers permanent failures at a rate $\lambda$, is $MTTF_P = \Sigma_{1 \le k \le N} (1/k\lambda)$

**Solution:**

Let the state of the system be the number of modules that are still functional and let $T_k$ be the time spent in state k. Then, $E[T_k] = \frac{1}{k\lambda}$. Since MTTF $= \sum_{k=1}^{N} E[T_k]$, the result follows immediately.

7. Consider a system consisting of 2 subsystems in series. For improved reliability, you can build subsystem **i** as a parallel system with $\mathbf{k_i}$ units, for **i** = 1, 2. Suppose permanent failures occur at a constant rate $\lambda$ per unit.

   (a) Derive an expression for the reliability of this system.

   (b) Obtain an expression for the MTTF of this system with $k_1 = 2$ and $k_2 = 3$.

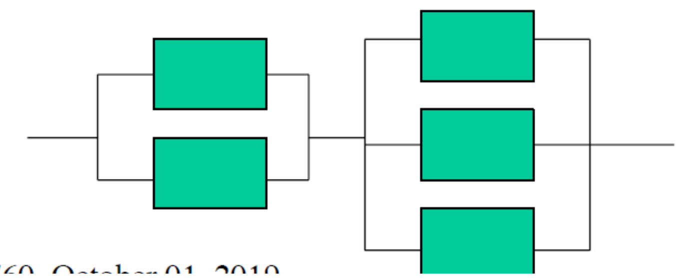
**Solution:**

(a) The reliability of a parallel system with k units is $R_p^{(k)}(t) = 1 - (1 - e^{-\lambda t})^k$. Hence, the reliability of the series system is given by

$$R_{series} = R_p^{(k_1)}(t) R_p^{(k_2)}(t)$$

(b)

$$\text{MTTF} = \int_{t=0}^{\infty} R_{series}(t)dt = \int_{t=0}^{\infty} \left(6e^{-2\lambda t} - 9e^{-3\lambda t} + 5e^{-4\lambda t} - e^{-5\lambda t}\right) dt = \frac{21}{20\lambda}$$



8. The system shown in the figure below consists of a TMR core with a single spare a which can serve as a spare only for module 1. Assume that modules 1 and a are active. When either of the two modules 1 or a fails, the failure is detected by the perfect comparator C, and the single operational module is used to provide an input to the voter.

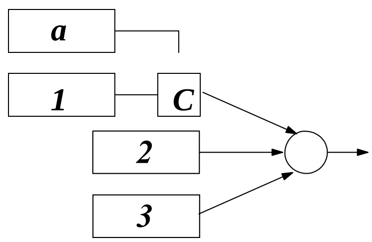
(a) Assuming that the voter is perfect as well, which one of the following expressions for the system reliability is correct (where each module has a reliability R and the modules are independent).

(1) $R_{system} = R^4 + 4R^3(1-R) + 3R^2(1-R)^2$

(2) $R_{system} = R^4 + 4R^3(1-R) + 4R^2(1-R)^2$

(3) $R_{system} = R^4 + 4R^3(1-R) + 5R^2(1-R)^2$

(4) $R_{system} = R^4 + 4R^3(1-R) + 6R^2(1-R)^2$




A TMR with a spare.

(a) $2R^4(t) - 6R^3(t) + 5R^2(t) = (3)$

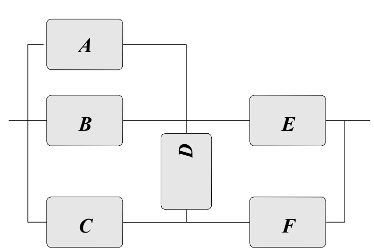
(b) Write an expression for the reliability of the system if instead of a perfect comparator for modules 1 and a there is a coverage factor c (c is the probability that a failure in one module is detected, the faulty module is correctly identified and the operational module is successfully connected to the voter which is still perfect).

(b) $R^4(t)(4c-2) + R^3(t)(2-8c) + R^2(t)(4c+1)$

9. Consider a triplex that produces a one-bit output. Failures that cause the output of a processor to be permanently stuck at 0 or stuck at 1 occur at constant rates $\lambda 0$ and $\lambda 1$, respectively. The voter never fails. At time t, you carry out a calculation the correct output of which should be 0. What is the probability that the triplex will produce an incorrect result? (Assume that stuck-at faults are the only ones that a processor can suffer from, and that these are permanent faults; once a processor has its output stuck at some logic value, it remains stuck at that value forever).

$f^3(t) + 3f^2(t)(1 - f(t))$

10. Write expressions for the upper and lower bounds and the exact reliability of the following non series/parallel system shown in Figure (denote by $R_i(t)$ the reliability of module $i$). Assume that $D$ is a bidirectional unit.



A 6-module non series/parallel system.

Solution

Upper bound:

$$1-[1-R_A(t)R_E(t)][1-R_A(t)R_D(t)R_F(t)[1-R_B(t)R_E(t)]$$
$$\cdot[1-R_B(t)R_D(t)R_F(t)][1-R_C(t)R_F(t)][1-R_C(t)R_D(t)R_E(t)]$$

Lower bound:

$$[1-[1-R_E(t)][1-R_F(t)][1-[1-R_A(t)][1-R_B(t)][1-R_C(t)]]$$
$$\cdot[1-[1-R_C(t)][1-R_D(t)][1-R_E(t)]][1-[1-R_A(t)][1-R_B(t)][1-R_D(t)][1-R_F(t)]]$$

Exact reliability: $R_D(t)[1-(1-R_A(t))(1-R_B(t))(1-R_C(t))][R_E(t)+R_F(t)-R_E(t)R_F(t)]$
$+(1-R_D(t))[1-(R_A(t)+R_B(t)-R_A(t)R_B(t))R_E(t)](1-R_C(t)R_F(t))$

11. (a) Your manager in the Reliability and Quality Department asked you to verify her calculation of the reliability of a certain system. The equation that she derived is

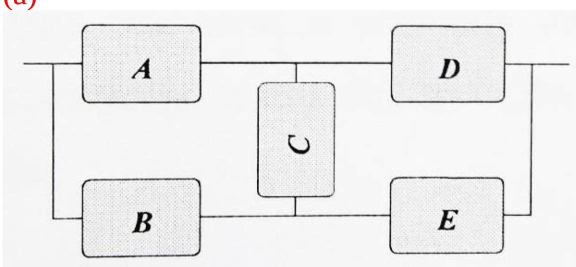
$$R_{system} = R_C\,[1-(1-R_A)(1-R_B)]\,[1-(1-R_D)(1-R_E)] + \quad (1)- R_C)\,[1-(1-R_A R_D)(1-R_B R_E)]$$

However, she lost the system diagram. Can you draw the diagram based on the expression above?

(b) Write expressions for the upper and lower bounds on the reliability of the system and calculate these values and the exact reliability for the case $R_A = R_B = R_C = R_D = R_E = R = 0.9$.

Solution

(a)




(c)Rsys=0.97848

12. Write the expression for the reliability of a 5MR system and calculate its MTTF. Assume that failures occur as a Poisson process with rate $\lambda$ per node, that failures are independent and permanent, and that the voter is failure-free.

**Solution:**

Denote by $r(t) = e^{-\lambda t}$ the reliability of an individual node. The reliability of the 5MR system is

$$\begin{aligned} R_{5MR}(t) &= \sum_{i=3}^{5} \binom{5}{i} r^i(t)(1-r(t))^{5-i} \\ &= 10r^3(t) - 15r^4(t) + 6r^5(t) \\ &= 10e^{-3\lambda t} - 15e^{-4\lambda t} + 6e^{-5\lambda t} \end{aligned}$$

and the MTTF is

$$\int_{t=0}^{\infty} R_{5MR}(t)dt = \frac{1}{\lambda}\left(\frac{10}{3} - \frac{15}{4} + \frac{6}{5}\right) = \frac{47}{60\lambda}$$

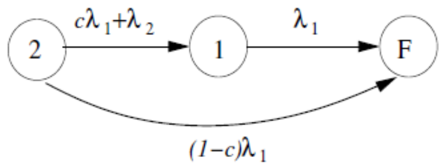
13. A duplex system consists of a switching circuit and two computing units: an active unit with a failure rate of $\lambda_1$ and a standby idle unit which has a lower failure rate $\lambda_2 < \lambda_1$ while idle. The switching circuit frequently tests the active unit and when a fault is detected, the faulty unit is switched out, the second unit is switched in and becomes fully operational with a failure rate $\lambda_1$. The probability that upon a failure, the fault is correctly detected and the fault-free idle unit resumes the computation successfully, is denoted by c (the coverage factor). Note that when a coverage failure occurs, the entire system fails.

(a) Show the Markov model for this duplex system (hint: three states are sufficient).

(b) Write the differential equations for the Markov model. Derive an expression for the reliability of the system.

**Solution:**

The states are $1, 2, F$: $1, 2$ refer to the number of functional processors; $F$ is the system-failure state. In state 2, both processors are up and the overall failure rate is $\lambda_1 + \lambda_2$. In state 1, only the active processor is up and the failure rate is $\lambda_1$. From state 2, we can go directly to state 0 if there is a coverage failure. Based on these observations, we can draw the Markov chain shown in Figure 2.10.




The differential equations can be written from inspection of the Markov chain.

$$\frac{dP_2(t)}{dt} = -(\lambda_1 + \lambda_2)P_2(t)$$
$$\frac{dP_1(t)}{dt} = (c\lambda_1 + \lambda_2)P_2(t) - \lambda_1 P_1(t)$$
$$\frac{dP_F(t)}{dt} = (1-c)\lambda_1 P_2(t) + \lambda_1 P_1(t)$$

We also have

$$P_F(t) = 1 - P_1(t) - P_2(t).$$

Starting with the initial condition $P_2(0) = 1$, we obtain $P_2(t) = e^{-(\lambda_1+\lambda_2)t}$. Plugging this expression into the differential equation for $P_1(t)$, we have

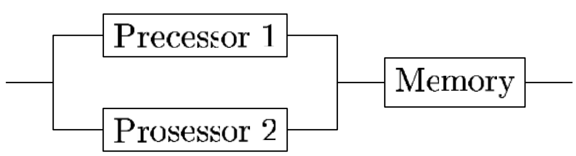
$$\frac{dP_1(t)}{dt} + \lambda_1 P_1(t) = (c\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)t}$$

Together with the initial condition $P_1(0) = 0$, we obtain

$$P_1(t) = \left(\frac{c\lambda_1 + \lambda_2}{\lambda_2}\right)\left(e^{-\lambda_1 t} - e^{-(\lambda_1+\lambda_2)t}\right)$$

The reliability is given by R(t) = P1(t) + P2(t).

14.

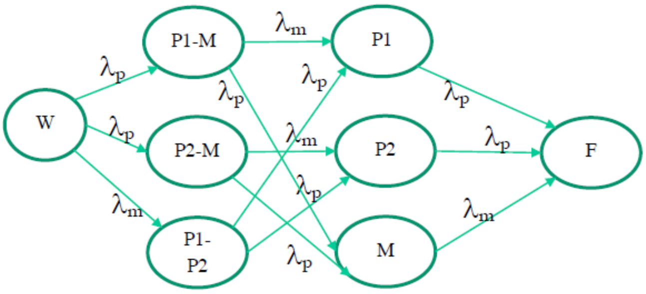



Construct the Markov model of the three-component system shown in Figure above. Assume that the components are independent and non-repairable. The failure rate of the processors 1 and 2 is $\lambda_p$. The failure rate of the memory is $\lambda_m$.

Derive and solve the system of state transition equations representing this system. Compute the reliability of the system.

**Solution**. Since we have 3 components, we can use the following states

Working (all components working)
P1-M: Processor 1 and Memory working (P2 failed)
P2-M: Processor 2 and Memory working (P1 failed)
P1-P2: Processors 1 and 2 working (Memory failed)
P1: P1 is working (but P2 and M failed)
P2: P2 is working (but P1 and M failed)
M: Memory is working (P1 and P2 failed)
Failed; All units failed.

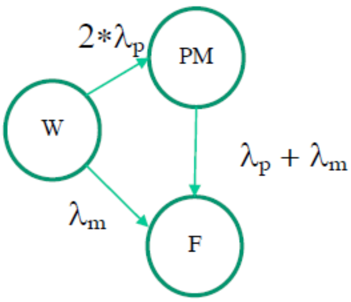



We can also simplify this as

W: all working
PM: One processor and Memory working
M: Only memory working
2P: both processors working
P: only one processor is working

Since a system cannot function without memory, we can further simplify this as

W: all working:
PM: one processor and memory working
Failed: Either memory failed or both processors failed




$$\frac{dW}{dt} = -(2\lambda_p + \lambda_m)$$

$$\frac{dPM}{dt} = -(\lambda_p + \lambda_m) * P_{PM} + 2\lambda_p P_W$$

$$\frac{dF}{dt} = \lambda_m * P_W + (\lambda_p + \lambda_m) P_W$$

Initial condition: $P_W(0) = 1$; $P_{PM}(0) = 0$

$$P_w = e^{-(2\lambda_p + \lambda_m)}$$

$$P_{PM} = 2e^{-\lambda_p} - 2e^{-(2\lambda_p + \lambda_m)}$$

$$P_F = 1 - P_W - P_{PM}$$

$$= 1 - e^{-(2\lambda_p + \lambda_m)} - 2e^{-\lambda_p}$$

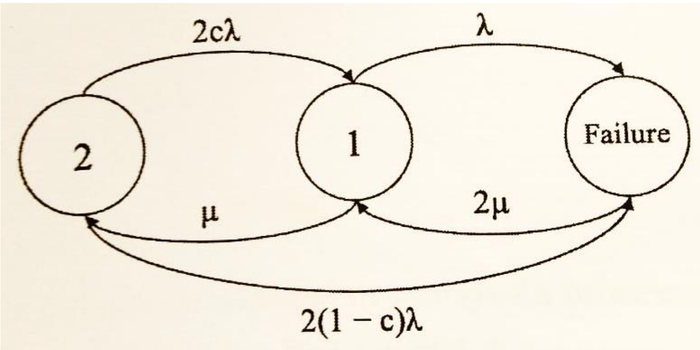
15. A duplex system consists of two active units and a comparator. Assume that each unit has a failure rate of $\lambda$ and a repair rate of $\mu$. The outputs of the two active units are compared, and when a mismatch is detected, a procedure to locate the faulty unit is performed. The probability that upon a failure, the faulty unit is correctly identified and the fault-free unit (and consequently, the system) continues to run properly is the coverage factor $c$. Note that when a coverage failure occurs, the entire system fails and both units have to be repaired (at a rate $\mu$ each). When the repair of one unit is complete, the system becomes operational and the repair of the second unit continues, allowing the system to return to its original state.

(a) Show the Markov model for this duplex system.

(b) Derive an expression for the long-term availability of the system assuming that $\mu = 2\lambda$.

Solution

(a)




(b)

$$R_{sys} = \frac{8}{11 - 2c}$$