1 The Real Number

1.1 Number Systems

Examples of number systems include:

- $\mathbb{N}$:= set of all **natural numbers** $= 1, 2, 3, \cdots$
- $\mathbb{Z}$:= set of all **integers** $= \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$
- $\mathbb{Q}$:= set of all **rational numbers** $= \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$
- $\mathbb{R}$:= set of all **real numbers**.

Theorem 1.1. $\sqrt{2}$ is an irrational number.

1.2 Natural Numbers

$\mathbb{N}$ has the *well-ordering property*:

Every non-empty subset S of $\mathbb{N}$ has a **minimum** element, i.e., there exists $m \in S$ such that $m \leq n$ for every $n \in S$.

Theorem 1.2 (Principle of Mathematical Induction)**.**
Let $S \subseteq \mathbb{N}$. If

(i) $1 \in S$, and

(ii) for every $k \in \mathbb{N}, k \in S \Rightarrow k + 1 \in S$,

then $S = \mathbb{N}$.

Remark: The Theorem can be rephrasesd as:
For each $n \in \mathbb{N}$, let $P(n)$ be a statement about n. Suppose that

(i) $P(1)$ is true, and

(ii) for every $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k + 1)$ is true.

Then $P(n)$ is true for every $n \in \mathbb{N}$.

1.3 The Algebraic Properties of $\mathbb{R}$

The binary operation **addition**(denoted by $+$) on the set $\mathbb{R}$ of real numbers satisfies the following properties:

(A1) **(Commutativity)** $a + b = b + a$ for all $a, b \in \mathbb{R}$.

(A2) **(Associativity)** $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{R}$

(A3) **(Existence of zero element)** There exists a zero element $0 \in \mathbb{R}$ such that $a + 0 = 0 + a = a$ for all $a \in \mathbb{R}$

(A4) **(Existence of inverse)** For each $a \in \mathbb{R}$, there exists an element $-a \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$.

Similarly, the binary operation **multiplication**(denoted by $\cdot$ or $\times$) on the set $\mathbb{R}$ satisfies the following properties:

(M1) **(Commutativity)** $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{R}$.

(M2) **(Associativity)** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathbb{R}$

(M3) **(Existence of unit element)** There exists a unit element $1 \in \mathbb{R}$ such that $1 \neq 0$ and $a \cdot 1 = 1 \cdot a = a$ for all $a \in \mathbb{R}$

(M4) **(Existence of inverse)** For each $a \neq 0$ in $\mathbb{R}$, there exists an element $\frac{1}{a} \in \mathbb{R}$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.

In addition, the two binary operations satisfy the following property:

(D) **(Distributivity of multiplication over addition)** $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{R}$.

Because of properties (A1)—(A4),(M1)—(M4) and (D), we say that $(\mathbb{R}, +, \cdot)$ forms a field.

Theorem 1.3 (Some properties of $\mathbb{R}$)**.**
Let $a, b, c \in \mathbb{R}$

(i) (*Uniqueness of additive inverse*) If $a + b = 0$, then $b = -a$.

(ii) (*Uniqueness of multiplicative inverse*) If $a \cdot b = 1$ and $b \neq 0$, then $b = \frac{1}{a}$.

(iii) If $a + b = b$, then $a = 0$.

(iv) If $b \neq 0$ and $a \cdot b = b$, then $a = 1$.

(v) $a \cdot 0 = 0$ for all $a \in \mathbb{R}$.

(vi) If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

(vii) (*Cancellative property*) If $a \neq 0$ and $a \cdot b = a \cdot c$, then $b = c$.

Theorem 1.4 (Order Properties of $\mathbb{R}$)**.**
There is a binary relation $>$ on $\mathbb{R}$ which has the following properties:
Let $a, b, c, d \in \mathbb{R}$.

(O1) If $a > b$, then $a + c > b + c$.

(O2) If $a > 0$ and $b > 0$, then $a \cdot b > 0$.

(O3) (**Trichotomy Property**) If $a, b \in \mathbb{R}$, then exactly one of the following holds:

$$a > b, \qquad a = b, \qquad b > a$$

(O4) (**Transitive Property**) If $a > b$ and $b > c$, then $a > c$.

The other oder relations $<, \geq, \leq$ can be defined in terms of $>$.

Because of the order relation satisfies (O1)—(O4), we say that $(\mathbb{R}, +, \cdot, >)$ forms an **ordered field**.

Theorem 1.5 (Some more properties of $\mathbb{R}$)**.**
Let $a, b, c \in \mathbb{R}$. Then the following statements hold:

(i) $a > b \Leftrightarrow a - b > 0$.
In particular, $c < 0 \Leftrightarrow -c > 0$

(ii) Exactly one of the following holds:

$$a > 0, \qquad a = 0, \qquad a < 0$$

(iii) If $a > b$ and $c > 0$, then $c \cdot a > c \cdot b$.
If $a > b$ and $c < 0$, then $c \cdot a < c \cdot b$.

(iv) If $a \geq b$ and $b \geq a$, then $a = b$.

Theorem 1.6.

(i) If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$

(ii) $1 > 0$

(iii) If $n \in \mathbb{N}$, then $n > 0$

(iv) If $a > 0$, then $\frac{1}{a} > 0$

Theorem 1.7. If $a \in \mathbb{R}$ is such that $0 \leq a < \epsilon$ for every positive number ϵ, then $a = 0$.

1.4 Intervals

An interval is a subset I of $\mathbb{R}$ with the following property:

$$\boxed{\begin{array}{c}\text{If } x, y \in I \text{ and } x < y, \text{ then} \\ x < t < y \Rightarrow t \in I\end{array}}$$

1.5 Solving Inequalities

THe following two rules are often useful in solving inequalities:

Rule 1 If $a \cdot b > 0$, then either

(i) $a > 0$ and $b > 0$, or
(ii) $a < 0$ and $b < 0$

Rule 2 If $a \cdot b < 0$, then either

(i) $a > 0$ and $b < 0$, or
(ii) $a < 0$ and $b > 0$

Theorem 1.8 (Bernoulli's inequality)**.** If $x \geq -1$, then

$$(1+x)^n \geq 1 + nx \ \forall n \in \mathbb{N}$$

Definition 1.1. Let $n \geq 2$ and let $a_1, a_2, \ldots, a_n$ be positive numbers.

- The *arithmetic mean* of $a_1, a_2, \ldots, a_n$ is defined as $A = \frac{a_1+a_2+\ldots+a_n}{n}$.
- The *geometric mean* of $a_1, a_2, \ldots, a_n$ is defined as $G = (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$
- The *harmonic mean* of $a_1, a_2, \ldots, a_n$ is defined as $H = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$

Theorem 1.9 (AM-GM-HM inequality)**.**
Let A, G, H be the arithmetic mean, the geometric mean and the harmonic mean of the positive numbers $a_1, a_2, \ldots, a_n$ respectively. Then

$$H \leq G \leq A$$

Moreover, the equalities hold if and only if $a_1 = a_2 = \cdots = a_n$.

1.6 Absolute Value

Definition 1.2. Let $a \in \mathbb{R}$. The **absolute value** of a is defined by

$$|a| = \begin{cases} x & \text{if } a > 0 \\ -x & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$$

Theorem 1.10 (Properties of absolute value)**.**
For all $a, b, c \in \mathbb{R}$, the following statements hold:

(i) $|a| \geq 0$, $a \leq |a|$ and $-a \leq |a|$

(ii) $|a| = 0 \Leftrightarrow a = 0$

(iii) $|-a| = |a|$

(iv) $|ab| = |a| \cdot |b|$

(v) $|a|^2 = a^2$

(vi) If $c \geq 0$, then $|a| \leq c \Leftrightarrow -c \leq a \leq c$

(vii) $-|a| \leq a \leq |a|$

Theorem 1.11 (Triangle inequality)**.**
For all $a, b \in \mathbb{R}$, we have

$$|a+b| \leq |a| + |b|$$

Corollary 1.1.
For all $a, b \in \mathbb{R}$, we have

(i) $\big||a| - |b|\big| \leq |a-b|$

(ii) $|a| - |b| \leq |a| + |b|$

Corollary 1.2. For $a_1, a_2, \ldots, a_n \in \mathbb{R}$, we have

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

2 The Completeness of Real Numbers

2.1 Boundedness

Definition 2.1. A non-empty set of real numbers $S \subset \mathbb{R}$ is said to be **bounded above** if there exists some $M \in \mathbb{R}$ such that

$$x \leq M \ \forall x \in S$$

Such an M is called a **upper bound** of S.

Definition 2.2. A non-empty set of real numbers $S \subset \mathbb{R}$ is said to be **bounded below** if there exists some $m \in \mathbb{R}$ such that

$$m \leq x \ \forall x \in S$$

Such an m is called a **lower bound** of S.

Definition 2.3. A non-empty set of real numbers S is said to be **bounded** if it is both bounded above and bounded below.

2.2 Maximum and Minimum of a set of real numbers

Definition 2.4 (Maximum)**.**
For a non-empty set $S \subset \mathbb{R}$, one defines the **maximum** of S to be the (necessarily unique) number M such that

max-i $M \in S$, and

max-ii $M \geq x$ for all $x \in S$

Definition 2.5 (Minimum)**.**
Similarly, the **minimum** of S to be the (necessarily unique) number m such that

min-i $M \in S$, and

min-ii $m \geq x$ for all $x \in S$

Remark:

1. If a set $S \subset \mathbb{R}$ has a maximum, then S is bounded above, since by (max-ii), $\max S$ is an upper bound of S. Similarly if a set $S \subset \mathbb{R}$ has a minimum, then S is bounded below.
2. If a set S has a maximum, then $\max S$ is unique.

2.3 Infimum and Supremum

Definition 2.6. Let E be a non-empty set of real numbers. A real number $M \in \mathbb{R}$ is called the least upper bound or **supremum** of E if

(S-i) M is an upper bound of E, i.e., $x \leq M \ \forall x \in E$, and

(S-ii) if M' is an upper bound of E, then $M' \geq M$.

Lemma 2.1. Let E be a non-empty set of real numbers. Then $M = \sup E$ if and only if M satisfies (S-i) and

(S-ii' for every $\epsilon > 0$, there exists $x_\epsilon \in E$ such that $x_\epsilon > M - \epsilon$

Remark:

(i) $\sup E$ is unique whenever it exists.

(ii) The main difference between $\sup E$ and $\max E$ is that $\sup E$ may not be an element of E, whereas $\max E$ must be an element of E, if it does exist.

(iii) If E has a maximum, then $\sup E = \max E$.

Definition 2.7. Let E be a non-empty set of real numbers. A real number $m \in \mathbb{R}$ is called the greatest lower bound or **infimum** of the set E if

(I-i) m is a lower bound of E, i.e., $m \leq x \ \forall x \in E$, and

(I-ii) if m' is a lower bound of E, then $m' \leq m$.

Lemma 2.2. Let E be a non-empty set of real numbers. Then $m = \inf E$ if and only if m satisfies (I-i) and

(I-ii') for every $\epsilon > 0$, there exists $x_\epsilon \in E$ such that $x_\epsilon < m + \epsilon$.

Remark:

(i) $\inf E$ is unique whenever it exists.

(ii) The main difference between $\inf E$ and $\min E$ is that $\inf E$ may not be an element of E, whereas $\min E$ must be an element of E, if it does exist.

(iii) If E has a minimum, then $\inf E = \min E$.

2.4 Completeness Property of $\mathbb{R}$

Theorem 2.1. Every non-empty subset of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.

2.4.1 Infimum property of $\mathbb{R}$

Theorem 2.2. Every non-empty subset of $\mathbb{R}$ which is bounded below has an infimum in $\mathbb{R}$.

2.4.2 Some consequences of the completeness of $\mathbb{R}$

Theorem 2.3 (Archimedian property of $\mathbb{R}$)**.**
For any $x \in \mathbb{R}$, there exists $n_x \in \mathbb{N}$ such that $x < n_x$.

Remark: The Archimedean property of $\mathbb{R}$ is equivalent to the statement that $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Corollary 2.1.

1. Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then $\inf A = 0$.
2. In particular, if $\epsilon > 0$, then there exists $n_\epsilon \in \mathbb{N}$ such that $0 < \frac{1}{n_\epsilon} < \epsilon$

Theorem 2.4. Let $c > 0$ and $k \in \mathbb{N}$. Then there exists a unique positive real number a with $a^k = c$.

Theorem 2.5 (The Density Theorem)**.**
For any two **real** number $x, y \in \mathbb{R}$ satisfying $x < y$, there exists a **rational** number $r \in \mathbb{Q}$ such that

$$x < r < y$$

Corollary 2.2. If $a, b \in \mathbb{R}$ is such that $a < b$, then there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$.

Corollary 2.3. If an interval $I \subseteq \mathbb{R}$ has at least two elements, then I contains infinitely many rational numbers and infinitely many irrational numbers.

Definition 2.8. A subset D of $\mathbb{R}$ is said to be **dense** in $\mathbb{R}$ if for any $a, b \in \mathbb{R}$ with $a < b$, $D \cap (a, b) \neq \varnothing$

Remark: Both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

2.5 More on Intervals

Theorem 2.6. Every interval belongs to one of the following 9 types:

$$\begin{aligned}
(a, b) &= \{x \in \mathbb{R} : a < x < b\} \\
[a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\} \\
[a, b) &= \{x \in \mathbb{R} : a \leq x < b\} \\
(a, b] &= \{x \in \mathbb{R} : a < x \leq b\} \\
[a, \infty) &= \{x \in \mathbb{R} : x \geq a\} \\
(a, \infty) &= \{x \in \mathbb{R} : x > a\} \\
(-\infty, b] &= \{x \in \mathbb{R} : x \leq b\} \\
(-\infty, b) &= \{x \in \mathbb{R} : x < b\} \\
(-\infty, \infty) &= \mathbb{R}
\end{aligned}$$

Definition 2.9. A sequence of intervals $I_n, n \in \mathbb{N}$, is said to be **nested** if $I_n \supseteq I_{n+1}$ for each $n \in \mathbb{N}$.

$$I_i \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$$

Theorem 2.7 (Nested Interval Property)**.**
If $I_n = [a_n, b_n], n \in \mathbb{N}$, is a nested sequence of non-empty closed bounded intervals, then there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$,i.e. $\cap_{n=1}^{\infty} I_n \neq \varnothing$.

Theorem 2.8. If $I_n = [a_n, b_n], n \in \mathbb{N}$, is a nested sequence of non-empty closed bounded intervals, and the length $b_n - a_n$ of I_n satisfy

$$\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$$

then $\cap_{n=1}^{\infty} I_n$ has exactly one element.

Theorem 2.9. Let $I = [0, 1]$. There does not ecists a bijection $f : \mathbb{N} \to I$ from $\mathbb{N}$ onto I.

3 Sequences

Definition 3.1.
A **sequence** in $\mathbb{R}$ is a real-valued function X with demain $\mathbb{N}$, that is

$$X : \mathbb{N} \to \mathbb{R}$$

The numbers $X(n)$ for $n = 1, 2, 3, \ldots$ are called the terms of the sequence X by (x_n).

Definition 3.2 (Limit of sequence)**.**
We say that x is the **limit** of (x_n) if for every $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon \forall n \geq K$$

Definition 3.3 (Convergence and Divergence)**.**
If x is the limit of (x_n), then we aso say that (x_n) **converges** to x, and we write

$$\lim_{n\to\infty} x_n = x$$

We say that a sequence (x_n) **converges**, if (x_n) converges to a **finite** limit $x \in \mathbb{R}$.
We say that a sequence (x_n) **diverges** if (x_n) does not converge.

Theorem 3.1. If (x_n) converges, then it has exactly one limit.

3.1 Limit Theorems

Definition 3.4. A sequence (x_n) is said to be **bounded** if there exists $M > 0$ such that

$$|x_n| \leq M \quad \forall n \in \mathbb{N}$$

Theorem 3.2. Every convergent sequence is bounded.

Theorem 3.3.
If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then

1. $\lim_{n\to\infty} (x_n + y_n) = x + y$
2. $\lim_{n\to\infty} x_n y_n = xy$
3. $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$, provided $y_n \neq 0\ \forall n \in \mathbb{N}$, and $y \neq 0$.

Corollary 3.1. If (x_n) converges and $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k$$

Theorem 3.4 (Squeeze Theorem)**.**
If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$. Then,

$$Limn \to \infty y_n = a$$

Remark: the conditon "$x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$" can be replaced by weaker condition $x_n \leq y_n \leq z_n$ for all natural numbers $n \geq K_0$.

Theorem 3.5. If $|x_n| \to 0$, then $x_n \to 0$.

Theorem 3.6. If $0 < b < 1$, then $\lim_{n\to\infty} b^n = 0$.

Theorem 3.7. If $c > 0$, then $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$

Theorem 3.8.
1. If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} |x_n| = |x|$.
2. If all $x_n \geq 0$ and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

Theorem 3.9.

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Theorem 3.10.
1. If $x_n \geq 0$ for all $n \in \mathbb{N}$, and (x_n) converges, then $\lim_{n\to\infty} x_n \geq 0$.
2. If (x_n) and (y_n) are convergent and $x_n \geq y_n$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n$$

3. If $a, b \in \mathbb{R}$, and $a \leq x_n \leq b$ for all n and (x_n) is convergent, then

$$a \leq \lim_{n\to\infty} x_n \leq b$$

3.2 Monotone Sequences

Definition 3.5. We say that the sequence (x_n) is

- **increasing** if
$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$$
- **decreasing** if
$$x_1 \geq g_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$$
- **monotone** if it is either increasing and decreasing

Theorem 3.11 (Monotone Convergence Theorem).
If (x_n) is monotone and bounded, then it is convergent. In this case,

$$\lim_{n\to\infty} x_n = \begin{cases} \sup\{x_n \mid n \in \mathbb{N}\} \text{ if } x_n \text{ is increasing} \\ \inf\{x_n \mid n \in \mathbb{N}\} \text{ if } x_n \text{ is decreasing} \end{cases}$$

Corollary 3.2. *1. If (x_n) is increasing and bounded above, then it converges and*

$$\lim_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$$

2. If (x_n) is decreasing and bounded below, then it converges and

$$\lim_{n\to\infty} x_n = \inf\{x_n \mid n \in \mathbb{N}\}$$

3.3 Subsequences and the Bolzano-Weierstrass Theorem

Definition 3.6 (Subsequence).
Let (x_n) be a sequence and let
$$n_1 \leq n_2 \leq \cdots \leq n_k \leq n_{k+1} \leq \cdots$$
be an increasing sequence of natural numbers. The sequence

$$(x_{n_k}) = (x_{n_1}, x_{n_2}, \cdots, x_{n_k}, \cdots)$$

is called a subsequence of (x_n).

Theorem 3.12. If (x_n) converges to x, then any subsequence (x_{n_k}) also converges to x.

Corollary 3.3. If (x_n) has a subsequence which is divergent, then (x_n) diverges.

Corollary 3.4. If (x_n) has two convergent subsequences whose limits are not equal, then (x_n) diverges.

Theorem 3.13 (Monotone Subsequence Theorem).
Every sequence has a monotone subsequence.

Theorem 3.14 (Bolzano-Weierstrass Theorem).
Every bounded sequence has a convergent subsequence.

3.4 Limit superior and limit inferior

Definition 3.7. Let (x_n) be a sequence of real numbers. A real number x is called a **subsequential limit** of (x_n) if (x_n) has a subsequence (x_{n_k}) which converges to x, that is,
$$x_{n_k} \to x$$

Definition 3.8. Let $S(x_n)$ denote the set of all subsequential limits of (x_n).

Definition 3.9. 1. We define the **limit superior** of (x_n) to be

$$\limsup x_n := \sup S(x_n)$$

2. We define the **limit inferior** of (x_n) to be
$$\liminf x_n := \inf S(x_n)$$

Theorem 3.15. Let (x_n) be a bounded sequence and let $M = \limsup x_n$.

1. For each $\epsilon > 0$, there are at most finitely many n's such that $x_n \geq M + \epsilon$.
   Equivalently, for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$x_n < M + \epsilon \quad \forall n \geq K$$

2. For each $\epsilon > 0$, there are infinitely many n's such that $x_n > M - \epsilon$.

The converse is also true.

Theorem 3.16. Let (x_n) be a bounded sequence and let $m = \liminf x_n$.

1. For each $\epsilon > 0$, there are at most finitely many n's such that $x_n \leq m - \epsilon$.
   Equivalently, for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that
$$x_n > m - \epsilon \quad \forall n \geq K$$
2. For each $\epsilon > 0$, there are infinitely many n's such that $x_n < m + \epsilon$.

The converse is also true.

Theorem 3.17 (Equivalent definition of lim sup)**.**
If (x_n) is a bounded sequence of real numbers, then the following statements for a real number x^* are equivalent.

1. $x^* = \limsup(x_n)$. The limit superior of (x_n) is the infimum of the set V of $v \in \mathbb{R}$ such that $v < x_n$ for at most a finite number of $n \in \mathbb{N}$.
2. If $\epsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ such that $x^* + \epsilon < x_n$, but an infinite number of $n \in \mathbb{N}$ such that $x^* - \epsilon < x_n$.
3. If $u_m = \sup\{x_n \mid n \geq m\}$, then $x^* = \inf\{u_m \mid m \in \mathbb{N}\} = \lim(u_m)$.
4. If S is the set of subsequential limits of (x_n), then $x^* = \sup S$.

Theorem 3.18. Let (x_n) be a bounded sequence. Then (x_n) converges if and only if
$$\limsup x_n = \liminf x_n$$
Theorem 3.19. Let (x_n) and (y_n) be bounded sequences such that $x_n \leq y_n$ for every $n \in \mathbb{N}$. Then,
$$\limsup x_n \leq \limsup y_n$$
and
$$\liminf x_n \leq \liminf y_n$$

3.5 The Cauchy Criterion

Definition 3.10. A sequence (x_n) is called a **Cauchy sequennce** if for every $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ such that
$$|x_n - x_m| \leq \epsilon \quad \forall n, m \geq K$$
Theorem 3.20. Every convergent sequence is Cauchy.

Theorem 3.21. Every Cauchy sequence is bounded.

Theorem 3.22 (Cauchy Criterion)**.**
Every Cauchy sequence is convergent.

Definition 3.11. A sequence (x_n) is said to be **contractive** if there exists C with $0 < C < 1$ such that
$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n| \quad \forall n \in \mathbb{N}$$
Theorem 3.23. Every contractive sequence is Cauchy.

3.6 Properly Divergent Sequences

Definition 3.12. We say that a sequence (x_n) **tends to** ∞ if for every $M > 0$, there exists $K = K(M) \in \mathbb{N}$ such that
$$x_n > M \quad \forall n \geq K$$
In this case,we write
$$\lim_{n\to\infty} x_n = \infty$$
Remark: If a sequence (x_n) is increasing and unbounded, then $x_n \to \infty$.

Definition 3.13. We say that a sequence (x_n) **tends to** $-\infty$ if for every $M < 0$, there exists $K = K(M) \in \mathbb{N}$ such that
$$x_n < M \quad \forall n \geq K$$
In this case,we write
$$\lim_{n\to\infty} x_n = -\infty$$
Definition 3.14. We call a sequence (x_n) **properly divergent** if either $x_n \to \infty$ or $x_n \to -\infty$.

Definition 3.15 ($o(x_n)$)**.**
Let (x_n) and (y_n) be two sequences of positive numbers such that $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n = \infty$. We write,
$$x_n = o(y_n) \text{ and also } x_n \ll y_n \text{ if } \lim_{n\to\infty} \frac{x_n}{y_n} = 0$$
Remark: $n^k \ll a^n \ll n!$

4 Infinite Series

Definition 4.1. Given a series $\sum_{k=1}^{\infty} a_k$, its nth **partial sum** s_n is given by

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

The sequence (s_n) is called **the sequence of partial sums** of the series $\sum_{k=1}^{\infty} a_k$.

Definition 4.2. Consider the sequence of partial sums (s_n) of the series $\sum_{k=1}^{\infty} a_k$. If the sequence (s_n) converges to a number $S \in \mathbb{R}$, we say that the series $\sum_{k=1}^{\infty} a_k$ **converges** to S and write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = S$$

In this case, S is called the **sum** of the seriess $\sum_{k=1}^{\infty} a_k$.
If (s_n) diverges, then we say $\sum_{k=1}^{\infty} a_k$ **diverges**.

Definition 4.3 (Geometric Series)**.**
Let $a \neq 0$ and r be fixed real numbers. Consider the geometric series

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + \cdots$$

Here r is called the **common ratio** of the series.
In summary, the **geometric series**

$$\sum_{k=1}^{\infty} ar^{k-1} = \begin{cases} \frac{a}{1-r}, & \text{if } |r| < 1 \\ \textbf{diverges} & \text{if } |r| \geq 1 \end{cases}$$

Theorem 4.1 (Linear Combination of convergent sequences is convergent)**.**

1. If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then the series $\sum_{n=1}^{\infty}(a_n+b_n)$ is also convergent and $\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
2. If the series $\sum_{n=1}^{\infty} a_n$ is convergent and $c \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} ca_n$ is also convergent and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.

Theorem 4.2. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Theorem 4.3 (The nth term divergence test)**.**

1. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.
2. If $\lim_{n\to\infty} a_n = 0$, then NO conclusion for $\sum_{n=1}^{\infty} a_n$ can be drawn.

Theorem 4.4 (Cauchy Criterion for Series)**.**
The series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ such that

$$|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon \quad \text{for all } m > n \geq K$$

4.1 Series with nonnegative terms

Definition 4.4. A series $\sum_{k=1}^{\infty} a_k$ is called an **eventually nonnegative series** if each term $a_k \geq 0$. Here $a_k \geq 0$ eventually means there exists $K \in \mathbb{N}$ such that $a_k \geq 0$ for all $k \geq K$.

Theorem 4.5. Let $\sum_{n=1}^{\infty} a_n$ be an (eventually) non-negative series. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequences (s_n) of partial sums if bounded.

Theorem 4.6. If $p > 1$, then p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Theorem 4.7 (Comparison Test)**.**
Consider 2 nonnegative series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$. Suppose that there exists $K \in \mathbb{N}$ such that

$$0 \leq a_k \leq b_k \quad \text{for all } k \geq K$$

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Theorem 4.8. If $p \leq 1$, then the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Theorem 4.9 (Limit Comparison Test).
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two eventually positive series, and suppose that the limit

$$\rho = \lim_{n\to\infty} \frac{a_n}{b_n}$$

exists.

1. If $\rho > 0$, then either the two series both converge or both diverge.
2. If $\rho = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 4.10 (Ratio Test).
Let $\sum_{n=1}^{\infty} a_n$ be eventually positive series and suppose that the limit

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists.

1. If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. Ig $\rho > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. No conclusion if $\rho = 1$.

Theorem 4.11 (Root Test).
Let $\sum_{n=1}^{\infty} a_n$ be an eventually non-negative series, and suppose $(a_n^{\frac{1}{n}})$ is a bounded sequence. Let

$$\rho = \limsup a_n^{\frac{1}{n}}$$

1. If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. Ig $\rho > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. No conclusion if $\rho = 1$.

Theorem 4.12 (Simplified Root Test).
Let $\sum_{n=1}^{\infty} a_n$ be an eventually non-negative series. Suppose

$$\rho = \lim_{n\to\infty} a_n^{\frac{1}{n}}$$

1. If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\rho > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. No conclusion if $\rho = 1$.

4.2 Alternating series

Definition 4.5. An **alternating series** is a series of the form

$$\sum_{k=1}^{\infty} (-1)^k a_k = a_1 - a_2 + a_3 - a_4 + \cdots, \quad \text{or}$$
$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = -a_1 + a_2 - a_3 + a_4 - \cdots$$

with each $a_k \geq 0$.

Theorem 4.13 (Alternating series Test).
Let $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ or $\sum_{n=1}^{\infty} (-1)^n a_n$ be an alternating series. Suppose that

1. $a_n \geq 0$ for all n,
2. (a_n) is decreasing, and
3. $\lim_{n\to infty} a_n = 0$.

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ or $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent.

4.3 Series with both positive and negative terms

Definition 4.6.

1. We say that the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.
2. We say that the series $\sum_{n=1}^{\infty} a_n$ **converges conditionally** if
   (a) $\sum_{n=1}^{\infty} a_n$ converges, and
   (b) $\sum_{n=1}^{\infty} |a_n|$ diverges.

Theorem 4.14. If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

Theorem 4.15. Every series is either absolutely convergent, conditionally convergent or divergent.

4.4 Grouping of series

Theorem 4.16. If the series $\sum_{n=1}^{\infty} a_n$ converges, then any series obtained by grouping the terms of $\sum_{n=1}^{\infty} a_n$ is also convergent and has the same sum as $\sum_{a=1}^{\infty} a_n$.

4.5 Rearrangement of series

Definition 4.7. A series $\sum_{n=1}^{\infty} b_n$ is a **rearrangement** of the series $\sum_{n=1}^{\infty} a_n$ if there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{f(n)}$ for all $n \in \mathbb{N}$.

Theorem 4.17. If the series $\sum_{n=1}^{\infty} a_k$ converges absolutely, then any rearrangement $\sum_{n=1}^{\infty} b_k$ of $\sum_{n=1}^{\infty} a_k$ also converges and has the same sum as $\sum_{n=1}^{\infty} a_n$.

5 Limits of Functions

5.1 Definition of Limits

Definition 5.1. Let $\varnothing \neq A \subset \mathbb{R}$. A number $c \in \mathbb{R}$ is said to be a **cluster point** of A if for every $\delta > 0$, the open interval $(c-\delta, c+\delta)$ contains a point of $A \setminus \{c\}$.

Theorem 5.1. A real number c is a cluster point of $\varnothing \neq A \subset \mathbb{R}$ if and only if there exists a sequence (a_n) in $A \setminus \{c\}$ converging to c.

Definition 5.2 (ϵ–δ definition of limit)**.**
Let $f : A \to \mathbb{R}$ be a function, where $\varnothing \neq A \subset \mathbb{R}$, and let c be a cluster point of A. We say that a real number L is the **limit** of f at $x = c$ and write

$$\lim_{x \to c} f(x) = L$$

if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{for all } x \in A \text{ satisfying } 0 < |x - c| < \delta$$

In this case, we also say that f **converges** to L at $x = a$.

Definition 5.3. If $h > 0$, then the h-**neighbourhood** of the point a is the set

$$V_h(a) = \{x : |x - a| < h\} = (a - h, a + h)$$

Define

$$V_h^*(a) = V_h(a) \setminus \{a\} = \{x : 0 < |x - a| < h\}$$

$V_h^*(a)$ is called a **deleted neighbourhood** of a.

Remark: Let $f : A \to \mathbb{R}$ be a function, and let c be a cluster point of A as before. Then the definition of limit can be restated as follows: $\lim_{x \to c} f(x) = L$ if and only if for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$f(A \cap V_\delta^*(c)) \subset V_\epsilon(L)$$

Theorem 5.2 (Sequential Criterion for Limits)**.**
Let $\varnothing \subsetneq A \subset \mathbb{R}$, and let c be a cluster point of A. Suppose that $f : A \to \mathbb{R}$ is a function and $L \in \mathbb{R}$. Then the following statements are equivalent.

1. $\lim_{x \to c} f(x) = L$
2. For every seqeuence (x_n) in $A \setminus \{c\}$ satisfying $\lim_{n \to \infty} x_n = c$, one has $\lim_{n \to \infty} f(x_n) = L$.

Theorem 5.3 (Uniqueness of Limits).
If $f : A \to \mathbb{R}$ is a function and c is a cluster point of A, then the limit of f at $x = c$ is unique, if it does exist. In other words, if $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$ also, then $L_1 = L_2$.

Theorem 5.4. Let $f : A \to \mathbb{R}$ be a function, and let c be a cluster point of A. Then $\lim_{x \to c} f(x) \neq L \Leftrightarrow$ there is a sequence (x_n) in $A \setminus \{c\}$ such that $x_n \to c$, but $f(x_n)$ does not converge to L.

Theorem 5.5 (Divergence Criteria).
Let $f : A \to \mathbb{R}$ be a function, and let c be a cluster point of A. To prove that $\lim_{x \to c} f(x)$ does not exist:

thod 1 Find a sequence (x_n) in $A \setminus \{c\}$ such that $x_n \to c$, but the sequence $(f(x_n))$ diverges.

thod 2 Find two sequences (x_n) and (y_n) in $A \setminus \{c\}$ such that $x_n \to c$, $y_n \to c$, but $\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n)$.

Theorem 5.6. Let $c \in \mathbb{R}$.

1. There exists a sequence (x_n) such that x_n is rational for all n, $x_n \neq c$ for each n and $x_n \to c$.
2. There exists a sequence (y_n) such that y_n is irrational for all n, $y_n \neq c$ for each n and $y_n \to c$.

5.2 Limit Theorems

Theorem 5.7. Let $f : A \to \mathbb{R}$ be a function, and let c be a cluster point of A. If $\lim_{x \to c} f(x)$ exists, then there exist constants $M, \delta > 0$ such that

$$|f(x)| \leq M \quad \forall x \in A \text{ satisfying } 0 < |x - c| < \delta$$

Definition 5.4. Consider two functionss, $f, g : A \to \mathbb{R}$, and let $k \in \mathbb{R}$. One can define associated functions $f + g$, $f - g$, kf and $f \cdot g$ given by

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(kf)(x) = kf(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$

In general, one can also define the function $\frac{f}{g}$ on $A \setminus \{x \in \mathbb{R} : g(x) = 0\}$ given by

$$\frac{f}{g}(x) = fracf(x)g(x)$$

Theorem 5.8. Let $f, g : A \to \mathbb{R}$ be two functions, and let c be a cluster point of A. Suppose that

$$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c} g(x) = M, \qquad \text{where } L, M \in \mathbb{R}$$

Also, let $k \in \mathbb{R}$ be a fixed constant. Then the following statements hold:

1. $\lim_{x \to c} (f + g)(x) = L + M$
2. $\lim_{x \to c} (f - g)(x) = L - M$
3. $\lim_{x \to c} (kf)(x) = kL$
4. $\lim_{x \to c} (f \cdot g)(x) = LM$
5. If, in addition, $g(x) \neq 0$ for all $x \in A$, and $M \neq 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$$

Remark: For any $k \in \mathbb{N}$,

$$\lim_{x \to c} [f(x)]^k = [\lim_{x \to c} f(x)]^k$$

Theorem 5.9 (Basic Principle). Let $f, g : A \to \mathbb{R}$ be two functions, and let c be a cluster point of A. Suppose that there exists a deleted neighbourhood $V_h^*(c)$ (with $h > 0$) such that $f(x) = g(x)$ for all $x \in A \cap V_h^*(c)$, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

Theorem 5.10. Let $f, g : A \to \mathbb{R}$ be two functions, and let c be a cluster point of A. Suppose that there exists a deleted neighbourhood $V_h^*(c)$ (with $h > 0$) such that $f(x) \leq g(x)$ for all $x \in A \cap V_h^*(c)$, and both $\lim_{x \to c} f(x) and \lim_{x \to c} g(x)$ exist, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$$

Theorem 5.11 (Squeeze Theorem).
Let $f, g, h : A \to \mathbb{R}$ be three functions and let c be a cluster point of A. Suppose that there exists a deleted neighbourhood $V_h^*(c)$ (with $h > 0$) such that $f(x) \leq g(x) \leq h(x)$ for all $x \in A \cap V_h^*(c)$, and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then

$$\lim_{x \to c} g(x) = L$$

Theorem 5.12. Let $f : A \to \mathbb{R}$ be a function and let c be a cluster point of A. If $\lim_{x \to c} f(x) = L$ exists and $L > 0$, then there exists $\delta > 0$ such that

$$f(x) > 0 \quad \text{for all } x \in A \cap V_\delta^*(c)$$

5.3 One-sided Limits

Definition 5.5. Let $\varnothing \subsetneq A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$.

1. Let c be a cluster point of $A \cap (c, \infty)$. We say that L is the **right-hand** limit of f at c if for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that
$$x \in A \text{ and } c < x < c + \delta \Rightarrow |f(x) - L| < \epsilon$$
In this case, we write
$$\lim_{x \to c^+} f(x) = L$$
2. Let c be a cluster point of $A \cap (-\infty, c)$. We say that L is the **left-hand** limit of f at c if for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that
$$x \in A \text{ and } c - \delta < x < c \Rightarrow |f(x) - L| < \epsilon$$
In this case, we write
$$\lim_{x \to c^-} f(x) = L$$
3. The limits $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ are called **one-sided** limits at the point $x = c$.

Theorem 5.13. Let c be a cluster point of both $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then $\lim_{x \to c} f(x) = L$ exists if and only if both $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ exist and
$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

Theorem 5.14 (Basic Principle)**.** Let $f, g : A \to \mathbb{R}$ be two functions, and let c be a cluster point of $A \cap (c, c + h)$ for some $h > 0$. If $f(x) = g(x)$ for all $x \in A \cap (c, c + h)$, then
$$\lim_{x \to c^+} f(x) = \lim_{x \to c^+} g(x)$$
provided one of these limits exists.

Definition 5.6 (The greatest integer function)**.**
For $x \in \mathbb{R}$, we let
$$[x] = \text{greatest integer less than or equal to } x$$
So for each $n \in \mathbb{Z}$,
$$[x] = n \qquad \text{if } x \in [n, n + 1)$$

Theorem 5.15 (Sequential Criterion for right-hand limits)**.**
Let $f : A \to \mathbb{R}$ be a function, and let c be a cluster point of $A \cap (c, \infty)$. Then the following statements are equivalent:

$Limx \to c^+ f(x) = L$

For every sequence (x_n) in $A \cap (c, \infty)$ satisfying $\lim_{n \to \infty} x_n = c$, one has $\lim_{x \to \infty} f(x_n) = L$.

Remark: Limit theorems and squeeze theorem for single sided limit is similar and thus not introduced here.

5.4 Infinite Limit

Definition 5.7. Let $f : A \to \mathbb{R}$ be a function, and let c be a cluster point of A.

1. We say that $f(x)$ tends to ∞ as $x \to c$ if for every $M > 0$, there exists $\delta = \delta(M) > 0$ such that
$$x \in A \text{ and } 0 < |x - c| < \delta \Rightarrow f(x) > M$$
In this case, $\lim_{x \to c} f(x) = \infty$
2. We say that $f(x)$ tends to $-\infty$ as $x \to c$ if for every $M < 0$, there exists $\delta = \delta(M) > 0$ such that
$$x \in A \text{ and } 0 < |x - c| < \delta \Rightarrow f(x) < M$$
In this case, $\lim_{x \to c} f(x) = -\infty$

Theorem 5.16 (Sequential criterion for infinite limits)**.**
Let $f : A \to \mathbb{R}$ be a function and let c be a cluster point of A. Then the following are equivalent:

1. $\lim_{x \to c} f(x) = \infty$
2. For every sequence $\{x_n\}$ in $A \setminus \{c\}$ satisfying $\lim_{x \to \infty} x_n = c$, one has
$$\lim_{x \to \infty} f(x_n) = \infty$$

5.5 Limits and Infinity

Definition 5.8. Let $\varnothing \subsetneq A \subset \mathbb{R}$, and let $f : A \to \mathbb{R}$ be a function.

1. Suppose that A is **not** bounded above.
   We say that L is the **limit** of f as $x \to \infty$ if for any given $\epsilon > 0$, there exists $M = M(\epsilon) > 0$ such that
   $$x \in A \textbf{ and } x > M \Rightarrow |f(x) - L| < \epsilon$$

2. Suppose that A is **not** bounded below.
   We say that L is the **limit** of f as $x \to \infty$ if for any given $\epsilon > 0$, there exists $M = M(\epsilon) < 0$ such that
   $$x \in A \textbf{ and } x < M \Rightarrow |f(x) - L| < \epsilon$$

In this case, we write $\lim_{x\to\infty} f(x) = L$

Theorem 5.17 (Sequential Criterion for limits at Infinity).
Let $f : A \to \mathbb{R}$ be a function,a nd suppose A is not bounded above. Then the following statements are equivalent:

1. $\lim_{x\to\infty} f(x) = L$.
2. For any sequence (x_n) in A such that $x_n \to \infty$, one has $f(x_n) \to L$.

Remark: The limit theorems and squeeze theorem are similar and not stated here.

5.6 Infinite Limits at Infinity

Definition 5.9. Let $f : A \to \mathbb{R}$ be a function, and suppose that A is **not** bounded above.
We say that $f(x)$ tends to ∞ as $x \to \infty$, if for any $M > 0$, there exists $K = K(M) > 0$ such that
$$x \in A \text{ and } x > K \Rightarrow f(x) > M$$
In this case, we write
$$\lim_{x\to\infty} f(x) = \infty$$

6 Continuous Functions

Definition 6.1 ($\epsilon - \delta$ definition of continuity).
Let $\varnothing \subsetneq A \subset \mathbb{R}$ and $a \in A$. A function $f : A \to \mathbb{R}$ is said to be **continuous at** $x = a$ if for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon, a) > 0$ such that
$$|f(x) - f(a)| < \epsilon \quad \text{for all } x \in A \text{ satisfying } |x - a| < \delta$$
(or equivalently, $x \in A$ and $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$)
(or equivalently, $f(A \cap V_\delta(a)) \subset V_\epsilon(f(a))$)
If f is **not** continuous at a, we say that f is **discontinuous** at $x = a$.
If f is continuous at **every** point in A, we say that f is **continuous** on A.

Remark: If $a \in A$ is **not** a cluster point of A, then f is *always* continuous at a, since there exists $\delta > 0$ such that
$$A \cap (a - \delta, a + \delta) = \{a\}$$

Theorem 6.1 (Continuity in terms of Limits).
If $a \in A$ is a cluster point of A, then
$$f \text{ is continuous at } x = a \Leftrightarrow \lim_{x\to a} f(x) = f(a)$$

Theorem 6.2 (Sequential Criterion for Continuity).
Let $\varnothing \subsetneq A \subset \mathbb{R}$, $a \in A$ and $f : A \to \mathbb{R}$. Then the following conditions are equivalent:

1. f is continuous at a.
2. For every sequence (x_n) in A satisfying $\lim_{n\to\infty} x_n = a$, one has $\lim_{n\to\infty} f(x_n) = f(a)$.

Remark: By this theorem, if we can find a sequence (x_n) of numbers in A such that
$$x_n \to a, \text{ but } f(x_n) \not\to f(a)$$
then f is not continuous at $x = a$.

6.1 Combinations of continuous functions

Theorem 6.3. Let $A \subseteq \mathbb{R}$ and let $f, g : A \to \mathbb{R}$ be functions that are continuous at $a \in A$. Let $k \in \mathbb{R}$.

1. Then the functions $f + g$, $f - g$, kf and $f \cdot g$ are all continuous at $x = a$, and
2. If $g(a) \neq 0$, then the function $\frac{f}{g}$ is also continuous at $x = a$.

Definition 6.2 (Composite functions)**.**
Suppose that $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subseteq B$.
We define the **composite function** $g \circ f : A \to \mathbb{R}$ by

$$(g \circ f)(x) = g(f(x)) \forall x \in A$$

Theorem 6.4.
Let A, B be subsets of $\mathbb{R}$ and let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that $f(A) \subset B$. If f is continuous at $a \in A$ and g is continuous at $f(a)$, then the composite function $g \circ f$ is continuous at a.

6.2 Continuous functions on Intervals

Definition 6.3. A function $f : A \to \mathbb{R}$ is said to be **bounded** on A if the image $f(A)$ is a bounded set, i.e., there exists $M > 0$ such that

$$|f(x)| < M \forall x \in A$$

Similarly, f is said to be **bounded** above(resp. below) on A if the image $f(A)$ is bounded above(resp. below).

Theorem 6.5.
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed bounded interval $[a, b]$. Then f is bounded on $[a, b]$.

Definition 6.4 (Absolute Maximum/Minimum)**.**
Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$.

1. We say that f has an **absolute maximum** on A if there exists $x^* \in A$ such that

   $$f(x^*) \geq f(x) \forall x \in A$$

   In this case, x^* is called an **absolute maximum point** for f on A, and we have

   $$f(x^*) = \max f(A)$$

2. We say that f has an **absolute minimum** on A if there exists $x_* \in A$ such that

   $$f(x^*) \leq f(x) \forall x \in A$$

   In this case, x_* is called an **absolute maximum point** for f on A, and we have

   $$f(x_*) = \max f(A)$$

Theorem 6.6 (Extreme Value Theorem)**.**
Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function on the closed bounded interval $[a, b]$. Then f has a absolute maximum and an absolute minimum on $[a, b]$, i.e., there exists $c_1, c_2 \in [a, b]$ such that

$$f(c_1) \leq f(x) \leq f(c_2) \forall x \in [a, b]$$

Theorem 6.7 (Intermediate Value Theorem)**.**
Suppose that a function $f : [a, b] \to \mathbb{R}$ is continuous on the closed bounded interval $[a, b]$.
Then for any number L between $f(a)$ and $f(b)$, there exists $c \in (a, b)$ such that $f(c) = L$.

Theorem 6.8. Let $a, b \in \mathbb{R}$ with $a < b$. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$. Then the image $f([a, b])$ is the closed bounded interval $[m, M]$, where

$$m = \min f([a, b]), \quad M = \max f([a, b])$$

i.e., we have $f([a, b]) = [m, M]$.

Theorem 6.9 (Preservation of Intervals)**.**
Let I be an interval in $\mathbb{R}$, and suppose a function $f : I \to \mathbb{R}$ is continuous on I. Then $f(I)$ is an interval.

Remark: THe image interval $f(I)$ may not have the same form as the domain interval I.

6.3 Monotone and Inverse functions on Intervals

Definition 6.5. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function.

1. f is said to be **increasing** on A if
$$x_1, x_2 \in A \text{ and } x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$
   f is said to be **strictly increasing** on A if
$$x_1, x_2 \in A \text{ and } x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

2. f is said to be **decreasing** on A if
$$x_1, x_2 \in A \text{ and } x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$
   f is said to be **strictly decreasing** on A if
$$x_1, x_2 \in A \text{ and } x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

A function is said to be **monotone** on A if it is either increasing or decreasing on A.
A function is said to be **strictly monotone** on A if it is either strictly increasing or strictly decreasing on A.

Theorem 6.10. Let $f : I \to \mathbb{R}$ be increasing on I. If $c \in I$ is not an endpoint of I, then

1. $\lim\limits_{x \to c^-} f(x) = \sup\{f(x) : x \in I, x < c\}$
2. $\lim\limits_{x \to c^+} f(x) = \int\{f(x) : x \in I, x > c\}$.
3. $\lim\limits_{x \to c^-} f(x) \leq f(c) \leq \lim\limits_{x \to c^+} f(x)$.

Theorem 6.11 (Continuous Inverse Theorem)**.**
Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a strictly monotone function. If f is continuous on I and $J = f(I)$, then its inverse function $f^{-1} : J \to \mathbb{R}$ is strictly monotone and continuous on J.

6.4 Uniform Continuity

Definition 6.6 (Uniform continuity)**.**
Let $\varnothing \subsetneq A \subseteq \mathbb{R}$. A function $f : A \to \mathbb{R}$ is said to be **uniformly continuous** on A if for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that
$$x, u \in A \text{ and } |x - u| < \delta \Rightarrow |f(x) - f(u)| < \epsilon$$
(or equivalently,)
$$|f(x) - f(u)| < \epsilon \quad \forall x, u \in A \text{ such as } |x - u| < \delta$$

Theorem 6.12 (Sequential Criterion for Uniform Continuity)**.**
The function $f : A \to \mathbb{R}$ is uniformly continuous on A if and only if for any two sequences (x_n) and (u_n) in A such that $x_n - u_n \to 0$. we have $f(x_n) - f(u_n) \to 0$.

Theorem 6.13. The function $f : A \to \mathbb{R}$ is not uniformly continuous on A if and only if there exist two sequences (x_n) and (u_n) in A such that $x_n - u_n \to 0$ but $f(x_n) - f(u_n) \not\to 0$.

Theorem 6.14.
Let $f : [a, b] \to \mathbb{R}$ be continuous on a closed bounded interval $[a, b]$. Then f is uniformly continuous on $[a, b]$.

Theorem 6.15.
Let $A \subseteq \mathbb{R}$, and let $f : A \to \mathbb{R}$ be uniformly continuous on A. If (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

Theorem 6.16 (Continuous Extension Theorem)**.**
A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on the open interval (a, b) if and only if f can be defined at the endpoints $x = a$ and $x = b$, so that the extended function is continuous on $[a, b]$.