Homework #5

(due: Dec. 4)

Show all the work/derivation with neat writing.

1. We learned that if a linear time-invariant system is controllable, then we can design state feedback for which the resulting closed loop can have its eigenvalues at any locations in the complex plane. We validate this using a single-input system.

   Consider a controllable single-input linear system given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{7}$$

   where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$. Let

$$\mathcal{C}^{-1} = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix} \qquad \text{and} \qquad P = \begin{pmatrix} M_n \\ M_n A \\ \vdots \\ M_n A^{n-1} \end{pmatrix}$$

   where $\mathcal{C} = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$ is the controllability matrix. Suppose P is invertible. (Note: M_n in P is same as the last row of $\mathcal{C}^{-1}$)

   (a) Validate that with the similarity transformation $z(t) = Px(t)$, we can derive

$$\dot{z}(t) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_{n-1} \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u(t) \tag{8}$$

   (*Hint:* Use $\mathcal{C}^{-1}\mathcal{C} = I_n$ and $A^n = \sum_{k=0}^{n-1} \alpha_k A^k$ by Cayley-Hamilton theorem.)

   (b) Can you find state feedback $u(t) = Kz(t)$ for (8) such that the eigenvalues of the resulting closed loop can be assigned to any points in the complex plane? (*Hint:* Use answers for Problem 5 in HW 4.)

   (c) Apply state feedback $u(t) = KPx(t)$ to (7). Using your answer in (b), validate that the eigenvalues of the resulting closed loop can be assigned to any points in the complex plane.

2. Consider a single-input single-output linear time-invariant system given by

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) - 4x_2(t) + 2u(t) \\ \dot{x}_2(t) &= x_1(t) + 3x_2(t) + u(t) \\ y(t) &= x_1(t) \end{aligned}$$

   (a) Design state feedback $u(t) = K \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ to stabilize the system. Place the eigenvalues of the closed loop at $(-1, -3)$.

(b) Design Luenberger observer to estimate the state of the system. Place the eigenvalues of the state equation for the estimation error at $(-1, -3)$.

(c) Discuss how you can use the answers from (a) and (b) to design output feedback that stabilizes the system..

3. Consider the linear system given by

$$
\begin{aligned}
\dot{x}_1(t) &= -x_2(t) \\
\dot{x}_2(t) &= x_1(t) + u(t) \\
y(t) &= x_1(t)
\end{aligned}
$$

with the initial condition $x_1(0) = 1$ and $x_2(0) = 0$.

(a) Validate that with $u(t) = 0$, the system exhibits the rotational motion with velocity 1 (rad/s) and radius 1.

(b) Using the Luenberger observer and state feedback, can you design output feedback that allows the system's state to track the rotational motion trajectory with velocity 1 (rad/s) and radius 2: $\bar{x}_1(t) = 2\cos t, \bar{x}_2(t) = 2\sin t$?

4. Consider the (nonlinear) state equation for a simple racing car (moving at constant velocity 1 m/s) given by

$$
\begin{aligned}
\dot{p}_x(t) &= \cos\theta(t) \\
\dot{p}_y(t) &= \sin\theta(t) \\
\dot{\theta}(t) &= \tan u(t)
\end{aligned}
$$

We aim to design state feedback control that allows the car to follow a racing track depicted below.

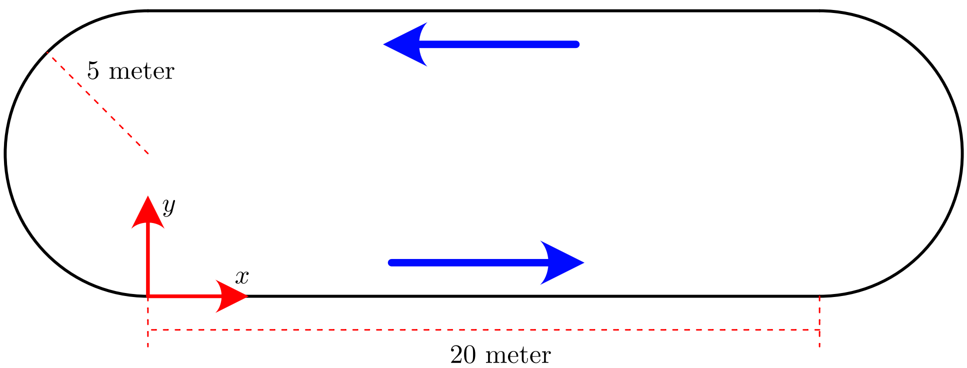
Let

$$
u(t) = \mu_{\text{Linear}}(p_y(t), \theta(t)) = K_1 \begin{pmatrix} p_y(t) \\ \theta(t) \end{pmatrix}
$$

be a controller that allows the car to track a straight line originating from the origin along the x-axis, and

$$
\begin{aligned}
u(t) &= \mu_{\text{Circular}}(p_x(t), p_y(t), \theta(t), p_{c,x}, p_{c,y}, r_0) \\
&= \arctan\left(\frac{1}{r_0}\right) + K_2 \begin{pmatrix} \sqrt{(p_x(t) - p_{c,x})^2 + (p_y(t) - p_{c,y})^2} - r_0 \\ \theta(t) - \left(\frac{\pi}{2} + \arctan\left(\frac{p_y(t) - p_{c,y}}{p_x(t) - p_{c,x}}\right)\right) \end{pmatrix}
\end{aligned}
$$

be a controller that allows the car to track a circular track centered at $(p_{c,x}, p_{c,y})$ with radius r_0. Use both of the controllers μ_{Linear} and μ_{Circular} to design state feedback control to maneuver the car along the below racing track. (No need to compute K_1, K_2.)



David Felipe Alvear Goyes

Homework 5

1. Consider a controllable single-input linear system

$$\dot{x}(t) = Ax(t) + Bu(t). \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}$$

Let:

$$C^{-1} = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_n \end{pmatrix} \qquad P = \begin{pmatrix} M_n \\ M_n A \\ \vdots \\ M_n A^{n-1} \end{pmatrix}$$

where $C = (B \ AB \cdots A^{n-1}B)$ is the controllability matrix

Suppose P is invertible.

a) Validate that with the similarity transformation $z(t) = Px(t)$ we can derive

$$\dot{z}(t) = \begin{vmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_{n-1} \end{vmatrix} z(t) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u(t)$$

$\rightarrow$ we have state eq $\rightarrow$ $\dot{x}(t) = Ax(t) + Bu(t)$

and $z(t) = Px(t)$, $\rightarrow$ $x(t) = P^{-1}z(t)$

then: $P^{-1}\dot{z}(t) = AP^{-1}z(t) + Bu(t)$

$$\boxed{\dot{z}(t) = PAP^{-1}z(t) + PBu(t)}$$

Demonstrate that $PAP^{-1} \rightarrow \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \end{vmatrix}$

$PA = \begin{vmatrix} M_n A \\ M_n A^2 \\ \vdots \\ M_n A^n \end{vmatrix} \longrightarrow$ Using Cayley-hamilton theorem

$$A^n = \sum_{k=0}^{n-1} \alpha_k A^k \rightarrow M_n A^n = \sum_{k=0}^{n-1} M_n A^{n-1}$$

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where $\rightarrow$ $PA = \begin{vmatrix} M_n A \\ M_n A^2 \\ \vdots \\ M_n A^n \end{vmatrix} = \underbrace{\begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & & 1 \\ \alpha_0 & \alpha_1 & \cdots & & \alpha_{n-1} \end{vmatrix}}_{M} \underbrace{\begin{vmatrix} M_n \\ M_n A \\ M_n A^2 \\ \vdots \\ M_n A^{n-1} \end{vmatrix}}_{P}$

We can define $PA = MP$

then $\rightarrow$ $PAP^{-1} = (MP)P^{-1} = MPP^{-1} = M$

$\rightarrow$ Now for PB $\rightarrow$ $PB = \begin{vmatrix} M_n B \\ M_n AB \\ \vdots \\ M_n A^{n-1} B \end{vmatrix}$ define $V = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$ $V \in \mathbb{R}^n$.

and $C = (B \ AB \ \cdots \ A^{n-1}B)$

We can define PB as:

$$PB = \begin{vmatrix} M_n B \\ M_n AB \\ \vdots \\ M_n A^{n-1} B \end{vmatrix} = \underbrace{(C^{-1} \cdot V)}\, C = (C^{-1} C) V = V$$

Recall $C^{-1} = (M_1 \ M_2 \cdots M_n)$

$C^{-1} \cdot V = M_n$

Therefore:

$\dot{z}(t) = PAP^{-1} z(t) + PBu(t)$

$\dot{z}(t) = MPP^{-1} z(t) + (C^{-1}C) V u(t)$ $\longrightarrow$ $M = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \end{vmatrix}$ $V = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

$\boxed{\dot{z}(t) = M z(t) + V u(t)}$

b) Can you find state feedback $u(t) = Kz(t)$ such that the eigenvalues of the resulting closed loop can be assigned to any points in the complex plane?

$\dot{z}(t) = M z(t) + V u(t)$ $\rightarrow$ $u(t) = Kz(t)$

$\dot{z}(t) = M z(t) + V K z(t) = (M + VK) z(t)$

$\dot{z}(t) = (M + VK) z(t)$

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For the System $\rightarrow$ $\dot{z}(t) = (M + VK) z(t)$

$$\dot{z}(t) = \left(\begin{vmatrix} 0 & 1 & 0 & 0 & & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & & \alpha_{n-1} \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{vmatrix} (K_1 \cdots K_n) \right) z(t).$$

$$\dot{z}(t) = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \alpha_0 + K_n & \cdots & & & \alpha_n + K_n \end{vmatrix} z(t)$$

We have:

For a single-input, single output linear System the transfer function is $\rightarrow$ $H(s) = \dfrac{N(s)}{D(s)} = C(sI - A)^{-1} B$.

$$\rightarrow H(s) = \frac{1}{\det(sI - A)} C(\operatorname{Adj}(sI - A)) B$$

we can find a state feedback $u(t) = Kz(t)$ if the System has a transfer function $H(s)$ where $N(s)$ and $D(s)$ are coprime, it means that the controller form is a minimal Realization which implies that the state equation is controllable and observable $\rightarrow$ Resulting in that we can select K that the closed loop system have negative eigenvalues.

c) Apply state feedback $u(t) = KPx(t)$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\rightarrow \dot{x}(t) = Ax(t) + BKPx(t).$$

$$\dot{x}(t) = (A + BKP)x(t). \quad (1)$$

$\rightarrow$ we have $x(t) = P^{-1} z(t)$ (2)

(1) $\wedge$ (2) $\therefore$ $P^{-1}\dot{z}(t) = (A + BKP)P^{-1} z(t)$

$$\dot{z}(t) = P(A + BKP)P^{-1} z(t)$$

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$\rightarrow \quad \dot{z}(t) = \dot{P}(A + BKP)\dot{P}^{-1} z(t)$

$\dot{z}(t) = (PAP^{-1} + PBK)\, z(t)$

$\rightarrow$ We have demostrated that $PAP^{-1} = M = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \end{vmatrix}$

and that $PB = V = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Then $\rightarrow \quad \dot{z}(t) = (PAP^{-1} + PBK)\, z(t) = (M + VK)\, z(t)$

$$\dot{z}(t) = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ \alpha_0 + K_n & \cdots & & & \alpha_{n-1} + K_n \end{vmatrix} z(t)$$

$\rightarrow$ Applying the state feedback control $u(t) = KPx(t)$ we can have the same Result that in b $\rightarrow$ this implies that the $u(t)$ applied can permit the selection of K to have determined eigenvalues for the closed loop System and due to the reasons explained in literal b.

2 → Consider a single input linear time-invariant system given by:

$$\dot{x}_1(t) = -x_1(t) - 4x_2(t) + 2u(t)$$
$$\dot{x}_2(t) = x_1(t) + 3x_2(t) + u(t)$$
$$y(t) = x_1(t)$$

a. Design state feedback $u(t) = K\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ to stabilize the system. place the eigvalues of the closed loop at $(-1, -3)$

1 → Define the state $X(t) = (x_1(t), x_2(t))^T$

2 → Write down the state equation.

$$\dot{X}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = (1 \ \ 0)\, X(t).$$

3 → First we want to check if the system is stabilizable.

we want to check if the system is controllable.

using controllability rank condition

$\text{rank}(B \ AB) \Rightarrow AB = \begin{vmatrix} -1 & -4 \\ 1 & 3 \end{vmatrix}\begin{vmatrix} 2 \\ 1 \end{vmatrix} = \begin{vmatrix} -6 \\ 5 \end{vmatrix}$

$\text{rank}\begin{vmatrix} 2 & -6 \\ 1 & 5 \end{vmatrix} = 2$ → Given that the system is controllable then the system is stabilizable

4 → Design state feedback $u(t) = KX(t)$ that stabilize the system

$$\dot{X}(t) = AX(t) + Bu(t)$$
$$\dot{X}(t) = AX(t) + BKX(t)$$
$$\dot{X}(t) = (A + BK)X(t)$$

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Compute $BK \rightarrow BK = \begin{pmatrix} 2 \\ 1 \end{pmatrix} (K_1 \;\; K_2) = \begin{pmatrix} 2K_1 & 2K_2 \\ 1K_1 & 1K_2 \end{pmatrix}$

Then the Result of state equation + Feedback design.

$$\dot{X}(t) = (A + BK) X(t) = \begin{vmatrix} -1+2K_1 & -4+2K_2 \\ 1+K_1 & 3+K_2 \end{vmatrix} X(t).$$

5 → Compute the K-Valuest to have negative eigenvalues → $\lambda = -1,\ \lambda = -3$

define → $M = (A + BK)$

then → $\det(M - \lambda I) = \begin{vmatrix} -1+2K_1-\lambda & -4+2K_2 \\ 1+K_1 & 3+K_2-\lambda \end{vmatrix}$

$$= (-1+2K_1-\lambda)(3+K_2-\lambda) - (-4+2K_2)(1+K_1)$$

$$= -3 - K_2 + \lambda + 6K_1 + \cancel{2K_1K_2} - 2K_1\lambda - 3\lambda - K_2\lambda + \lambda^2 - (-4 - 4K_1 + 2K_2 + \cancel{2K_1K_2})$$

$$= \lambda^2 + (-2 - 2K_1 - K_2)\lambda + (1 + 10K_1 - 3K_2)$$

we want $\lambda = -1,\ \lambda = -3 \rightarrow (\lambda+1)(\lambda+3) = \underline{\lambda^2 + 4\lambda + 3 = 0}$

$-3\lambda - 1\lambda + 3$

we want to have this equation

$\rightarrow -2 - K_2 - 2K_1 = 4$

① $K_2 + 2K_1 = -6 \rightarrow K_2 = -6 - 2K_1$

$\rightarrow 1 + 10K_1 - 3K_2 = 3$

② $1 + 10K_1 + 18 + 6K_1 = 3 \rightarrow 16K_1 = -16$

$K_1 = -1 \rightarrow \boxed{K_1 = -1} \;\; \boxed{K_2 = -4}$

With $K = (-1, -4)$ → System is stable.

b) Design Luenberger observer to estimate the state of the system. Place the eigenvalues of the state equation for the estimation error at (-1, -3)

Given the system

$$\dot{x}(t) = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

1 → We want to check if the system is observable to finally design the Luenberger observer. Using the observability rank condition.

$$\text{rank} \begin{pmatrix} C \\ CA \end{pmatrix} \rightarrow CA = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -4 \end{pmatrix}$$

$$\text{rank} \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix} = 2 \rightarrow$$ Then the system is observable we can create a luenberger observer

2 → Design Luenberger observer

estimate

We define $\tilde{x}(t) = x(t) - \hat{x}(t)$

$$\dot{\hat{x}} = A\hat{x}(t) + L(y(t) - C\hat{x}(t))$$

$$\dot{\tilde{x}} = \underbrace{A\,x(t) - A\hat{x}(t)}_{\tilde{x}(t)} - L(y(t) - C\hat{x}(t))$$

$$\tilde{x}(t)$$

$$\dot{\tilde{x}} = A\,\tilde{x}(t) - L(Cx(t) - C\hat{x}(t))$$

$$\dot{\tilde{x}} = A\,\tilde{x}(t) - LC\tilde{x}(t)$$

$$\boxed{\dot{\tilde{x}}(t) = (A - LC)\,\tilde{x}(t)}$$

Define:

$$L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \rightarrow LC = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} L_1 & 0 \\ L_2 & 0 \end{pmatrix}$$

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Then the luenberger observer can be expressed:

$$\dot{\tilde{x}}(t) = (A - LC)\tilde{x} = \left(\begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} L_1 & 0 \\ L_2 & 0 \end{pmatrix} \right) = \begin{pmatrix} -1-L_1 & -4 \\ 1-L_2 & 3 \end{pmatrix}$$

3 → Compute the L_i Values to have negative eigenvalues

$\lambda_1 = -1 \quad \lambda_2 = -3$

→ we want the characteristic equation of $(A-LC)$ to be: $(\lambda+1)(\lambda+3) = 0$

$$\lambda^2 + 3\lambda + \lambda + 3 = 0 \rightarrow \boxed{\lambda^2 + 4\lambda + 3 = 0}$$

Then:

define → $M = A - LC = \begin{pmatrix} -1-L_1 & -4 \\ 1-L_2 & 3 \end{pmatrix}$

Then →

$$\det(M - \lambda I) = \begin{vmatrix} -1-L_1-\lambda & -4 \\ 1-L_2 & 3-\lambda \end{vmatrix}$$

$$= (-1-L_1-\lambda)(3-\lambda) - (-4)(1-L_2)$$

$$= \underline{\lambda} - \underline{3} - 3L_1 + L_1\lambda - 3\lambda + \lambda^2 - (-4 + 4L_2)$$

$$= \lambda^2 + (L_1 - 2)\lambda + (1 - 3L_1 - 4L_2)$$

→ $L_1 - 2 = 4 \rightarrow \boxed{L_1 = 6}$ → $1 - 3L_1 - 4L_2 = 3$

$$1 - 18 - 4L_2 = 3 \rightarrow \boxed{L_2 = -5}$$

With $L = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$ we have a Luenberger observer for the estimation error.

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c) Discuss how you can use the answers from (a) and (b) to design output feedback that stabilizes the system.

we just proved that the system is controllable and obsevable, after that we have designed the state feedback Giving negative eigenvalues to $(A+BK)$ to make the system stable. Similarly, we designed a Luenberger observer to define an estimation error, and makeng the systen $(A-LC)$ estable Giving negative eigenvalues

i) from state feedback design $\rightarrow$ $u(t) = K\hat{x}(t)$

$\uparrow$ estimation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\dot{x}(t) = Ax(t) + BK\hat{x}(t)$$

ii) From observer design $\tilde{x}(t) = x(t) - \hat{x}(t)$

$\tilde{x}$: estimation Error; $\hat{x}$: Estimation

$$\dot{\tilde{x}}(t) = (A-LC)\tilde{x}(t)$$

To create an output feedback we can design a custom state to combine $x(t)$ and $\tilde{x}(t)$, and create a new state equation:

$\rightarrow$ state: $x'(t) = (x(t), \tilde{x}(t))'$

$\hat{x}(t) = x(t) - \tilde{x}(t)$

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{pmatrix} = \begin{pmatrix} Ax(t) + BK\hat{x}(t) \\ (A-LC)\tilde{x}(t) \end{pmatrix} = \begin{pmatrix} (A+BK)x(t) - BK\tilde{x}(t) \\ (A-LC)\tilde{x}(t) \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{pmatrix} = \begin{pmatrix} (A+BK) & -BK \\ 0 & A-LC \end{pmatrix} \begin{pmatrix} x(t) \\ \tilde{x}(t) \end{pmatrix}$$ output feedback design

The observer and state feedback estabilize the system $\rightarrow$ $(A+BK)$ with negative eigenvalues

$\rightarrow$ $(A-LC)$ with negative eigenvalues.

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3 → Consider the linear system given by

$$\dot{x}_1(t) = -x_2(t)$$
$$\dot{x}_2(t) = x_1(t) + u(t)$$
$$y(t) = x_1(t)$$

with initial condition $x_1(0) = 1$ and $x_2(0) = 0$

a) Validate that with $u(t) = 0$, the system exhibits the rotational motion with velocity 1 (rad/s) and radius 1.

1 → First we define the state. $X(t) = (x_1(t), x_2(t))'$

2 → State equation

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \quad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$y(t) = (1 \quad 0) X(t)$$

→ With $u(t) = 0$:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}^{A} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

→ Find the solution to this equation.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \quad \rightarrow \text{eigvalues } \lambda = i \; \lambda = -i$$

Eigenvectors →

$\lambda = i \rightarrow (A - \lambda I)V = 0 \quad \begin{vmatrix} -i & -1 \\ 1 & -i \end{vmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow -iV_1 - V_2 = 0$

$V_2 = -iV_1$

$\lambda = -i \rightarrow (A - \lambda I)V = 0 \quad \begin{vmatrix} i & -1 \\ 1 & i \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad V_{\lambda_1} = (1, -i)$

$V_{\lambda_2} = (1, i)$

$iV_1 - V_2 = 0$

then → Matrix exponential

$$e^{At} = \begin{vmatrix} 1 & 1 \\ -i & i \end{vmatrix} \begin{vmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{vmatrix} \begin{vmatrix} 1 & 1 \\ -i & i \end{vmatrix}^{-1} = \begin{vmatrix} e^{it} & e^{-it} \\ -ie^{it} & ie^{-it} \end{vmatrix} \begin{vmatrix} 1/2 & i/2 \\ 1/2 & -i/2 \end{vmatrix}$$

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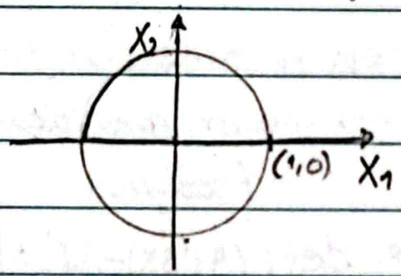
$$\to e^{At} = \begin{vmatrix} \frac{e^{it}+e^{-it}}{2} & \frac{-e^{it}+e^{-it}}{2i} \\ \frac{e^{it}-e^{-it}}{2} & \frac{e^{it}+e^{-it}}{2} \end{vmatrix} = \begin{vmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{vmatrix}$$

$\to$ If $u(t)=0 \to$ the solution to the linear system is

$$X(t) = e^{At}X(0)$$

$$X(t) = \begin{vmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$ $\to$ This is a rotational motion.



b) Using the Luenberger observer and state feedback, can you design output feedback that allows the system's state to track the rotational motion trajectory with velocity 1 (rad/s) and radius 2
$\bar{X}_1(t) = 2\cos t \quad \bar{X}_2(t) = 2\sin t$.

1 $\to$ Define trajectory Reference $\bar{x}(0) = (2,0)^T$

$$\bar{X} = \begin{pmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{pmatrix} = \begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix} \qquad \dot{\bar{X}} = \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix}$$

2 $\to$ State equation for trajectory reference

$$\dot{\bar{X}}(t) = \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{u} \quad \to \bar{u} = 0$$

3 $\to$ Define $\tilde{x}(t) = X(t) - X_{ref}(t) \qquad \tilde{u}(t) = u(t) - \bar{u} \to$ But $\bar{u} = 0$.

$$\begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) - \bar{x}_1(t) \\ x_2(t) - \bar{x}_2(t) \end{pmatrix}$$

4→ Design a Luenberger observer

Define $x'(t) = \tilde{x}(t) - \hat{x}(t)$

where $\hat{x}(t) \rightarrow$ is the estimation of the state and

$x'(t) \rightarrow$ is the error in the estimation.

→ To design a Luenberger observer we must check if the system is observable.

$$\text{rank}\begin{pmatrix} C \\ AC \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{= n}{\underset{\downarrow}{2}} \qquad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

↳ The system is observable, then we can compute a Luenberger observer.

we define → $L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$

and $\dot{x}' = (A - LC)\, x'(t)$. $\qquad LC = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{vmatrix} l_1 & 0 \\ l_2 & 0 \end{vmatrix}$

→ $\dot{x}'(t) = \begin{vmatrix} -l_1 & -1 \\ 1-l_2 & 0 \end{vmatrix} x'(t)$ $\qquad M = (A - LC) = \begin{vmatrix} -l_1 & -1 \\ 1-l_2 & 0 \end{vmatrix}$

→ Compute $\det(M - \lambda I)$ → to set l_1, l_2 that $(A-LC)$ has eigenvalues with negative real parts

$$\det(M - \lambda I) = \begin{vmatrix} -l_1-\lambda & -1 \\ 1-l_2 & -\lambda \end{vmatrix} = \lambda(l_1 + \lambda) + (1 - l_2) = 0$$

$$= \lambda^2 + l_1\lambda + (1 - l_2) = 0$$

→ we would like to have $\lambda = -1 \rightarrow (\lambda+1)^2 = \lambda^2 + 2\lambda + 1 = 0$

$\therefore\ l_1 = 2 \quad (1 - l_2) = 1 \rightarrow l_2 = 0$

Finally $L = (2, 0)$

5→ Design state feedback that $\tilde{x}(t) \rightarrow 0$ as $t \rightarrow \infty$

$$\dot{\tilde{x}}(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\tilde{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\tilde{u}(t) \qquad \tilde{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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→ we would like to design a state feedback

$\boxed{u(t) = \bar{u}(t) + K\hat{x}(t)}$ → where $\hat{x}(t)$ is the estimation of the state.

$\bar{u} = 0 \rightarrow u(t) = K\hat{x}(t)$

→ then: $\dot{\tilde{x}}(t) = A\tilde{x} + Bu(t)$

$\dot{\tilde{x}}(t) = A\tilde{x} + BK\hat{x}(t)$ → we have defined

→ $\dot{\tilde{x}}(t) = A\tilde{x} + BK\tilde{x} - BKx'(t)$ $\qquad x'(t) = \tilde{x}(t) - \hat{x}(t)$

$\boxed{\dot{\tilde{x}}(t) = (A+BK)\tilde{x} - BKx'(t)}$ $\qquad \hat{x}(t) = \tilde{x}(t) - x'(t)$

where $x'(t)$ → Error in estimation

First, we have to check if the system is controllable to be able to implement $u(t) = K\hat{x}(t)$

→ $\operatorname{rank}(B \ AB) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2 = n$ → order of the system, then we can design the state feedback.

Define: $\qquad BK = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(K_1 \ K_2) = \begin{pmatrix} 0 & 0 \\ K_1 & K_2 \end{pmatrix}$

$K = (K_1, K_2)$

→ $(A+BK) = \begin{pmatrix} 0 & -1 \\ 1+K_1 & K_2 \end{pmatrix}$ → $\det((A+BK) - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1+K_1 & K_2-\lambda \end{vmatrix}$

$= \lambda^2 - K_2\lambda + (1+K_1)$

∴ we want negative eigenvalues $\lambda = -1, \lambda = -1$

$(\lambda+1)^2 = 0 \rightarrow \lambda^2 + 2\lambda + 1 = 0.$

→ $-K_2 = 2 \quad K_2 = -2$

$K_1 + 1 = 1 \quad K_1 = 0 \rightarrow K = (0, -2).$

Finally → $\dot{\tilde{x}}(t) = (A+BK)\tilde{x} - BKx'(t) = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}\tilde{x}(t) - \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}x'(t)$

For state feedback design

→ $\dot{x}'(t) = (A - LC)\dot{x}'(t) = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}\dot{x}'(t)$

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6 → Create output feedback → combine the Luerenberger observer and the state feedback design.

i) State feedback design.

$$\dot{\tilde{x}}(t) = (A+BK)\tilde{x} - BKx'(t) \rightarrow x'(t): \text{ estimation error.}$$

ii) From observer design

$$\dot{x}'(t) = (A-LC)x'(t)$$

→ To combine we define: $X_\alpha(t) = (\tilde{x}(t), x'(t))'$

where $\tilde{x}(t) = x(t) - \bar{x}(t)$ and $x'(t) = \tilde{x}(t) - \hat{x}(t)$

$\bar{x}(t)$ → Xreference; $\hat{x}(t)$ → state estimation

Then: $$\begin{pmatrix} \dot{\tilde{x}}(t) \\ \dot{x}'(t) \end{pmatrix} = \begin{pmatrix} (A+BK)\tilde{x}(t) - BKx'(t) \\ (A-LC)x'(t) \end{pmatrix}$$

$$\begin{pmatrix} \dot{\tilde{x}}(t) \\ \dot{x}'(t) \end{pmatrix} = \begin{pmatrix} (A+BK) & -BK \\ 0 & (A-LC) \end{pmatrix} \begin{pmatrix} \tilde{x}(t) \\ x'(t) \end{pmatrix}$$

where $(A+BK) = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$ $BK = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$

$$A-LC = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$$

Given that $(A+BK)$ and $(A-LC)$ have negative eigenvalues then → the system is exponentially stable.

4 → Consider the (nonlinear) state equation for a simple racing car (moving at constant velocity 1 m/s) given by:

$$\dot{P}_x(t) = \cos\theta(t)$$
$$\dot{P}_y(t) = \sin\theta(t)$$
$$\dot{\theta}(t) = \tan u(t)$$

We aim to design state feedback control that allows the car to follow a racing track depicted below.

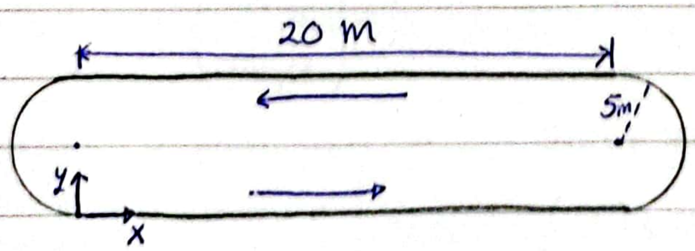
Let:

$$u(t) = \mu_{linear}\left(P_y(t), \theta(t)\right) = K_1 \begin{pmatrix} P_y(t) \\ \theta(t) \end{pmatrix}$$

be a controller that allows the car to follow a (racing track) straight line ~~orienta~~ originating from the origin along x-axis, and

and $$u(t) = \mu_{circular}\left(P_x(t), P_x(t), \theta(t), P_{c,x}, P_{c,y}, r_0\right)$$
$$= \arctan\left(\frac{1}{r_0}\right) + K_2 \begin{pmatrix} \sqrt{(P_x(t) - P_{c,x})^2 + (P_y(t) - P_{c,y})^2} - r_0 \\ \theta(t) - \left(\pi/2 + \arctan\left(\frac{P_y(t) - P_{c,y}}{P_x(t) - P_{c,x}}\right)\right) \end{pmatrix}$$

be a controller that allows the car to track a circular track centered at $(P_{c,x}, P_{c,y})$ with radius r_0. Use both controllers μ_{linear}, $\mu_{circular}$ to design state feedback control to maneuver the car along the below racing track → (No need to compute K_1, K_2).



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First, we will get concentrated on the straight line State feedback control design.

$$\dot{P}_x(t) = \cos\theta(t) \qquad P_x(0) = P_y(0) = 0 \quad \theta(0) = 0$$
$$\dot{P}_y(t) = \sin\theta(t)$$
$$\dot{\theta}(t) = \tan u(t)$$

1 → Design a Reference trajectory to follow the line:

$$\begin{cases} P_y^r(t) = 0 \\ \theta^r(t) = 0 \end{cases}$$

Define the state vector → $X(t) = \begin{pmatrix} P_y(t) \\ \theta(t) \end{pmatrix}$

2 → Create new state equation

$$\left.\begin{aligned} \tilde{P}_y(t) &= P_y(t) - P_y^r(t) \\ \tilde{\theta}(t) &= \theta(t) - \theta^r(t) \end{aligned}\right\} \rightarrow \tilde{X}(t) = X(t) - \underbrace{X_{ref}(t)}_{\text{Reference Trajectory}}$$

$$\dot{\tilde{P}}_y(t) = \sin\theta(t) - \dot{P}_y^r(t) = \sin\theta(t)$$
$$\dot{\tilde{\theta}}(t) = \tan u(t) - \dot{\theta}^r(t) = \tan u(t)$$

3 → Linearization around $\tilde{\theta} = 0, \tilde{P}_y(t) = 0 \rightarrow \tilde{X}(t) = 0$

$$\dot{\tilde{X}}(t) = \underbrace{\begin{pmatrix} \sin\theta(t) \\ \tan u(t) \end{pmatrix}}_{f(x,u)} \rightarrow \nabla_x f(x,u) = \begin{vmatrix} 0 & \cos\theta(t) \\ 0 & 0 \end{vmatrix}$$

$$\nabla_u f(x,u) = \begin{vmatrix} 0 \\ \frac{1}{1+u^2} \end{vmatrix}$$

$$f(\bar{x}, \bar{u}) = \begin{pmatrix} 0 \\ \tan\bar{u} \end{pmatrix} = 0 \rightarrow \text{Implies that } \bar{u} = 0$$

After Linearization: $\dot{\tilde{X}}(t) = \nabla_x f(\bar{x},\bar{u})(\tilde{X} - \cancel{\bar{x}}) + \nabla_u f(\bar{x},\bar{u})(u - \cancel{\bar{u}})$

$$\dot{\tilde{X}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{X}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

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4 → Check if the linearized system is controllable.

$\text{rank}(B \ AB) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$ → Then the system is controllable.

We can design a state feedback control that stabilizes the system → $u(t) = K_1 \tilde{x}(t)$

5 → State feedback design → Define $K_1 = (K_a, K_b)$

$$\dot{\tilde{x}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (K_a, K_b)\, \tilde{x}(t)$$

$$\dot{\tilde{x}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{x}(t) + \begin{pmatrix} 0 & 0 \\ K_a & K_b \end{pmatrix} \tilde{x}(t) = \begin{pmatrix} 0 & 1 \\ K_a & K_b \end{pmatrix} \tilde{x}(t)$$

This controller is used starting from the origen until reach the 20 m of the final stage of the straight line.

Second → we need to maneuver the car in a circular trajectory.

→ we have:

$\dot{P}_x(t) = \cos\theta(t)$

$\dot{P}_y(t) = \sin\theta(t)$

$\dot{\theta}(t) = \tan u(t)$

we will use polar coordinates to rewrite the state equation.

①

→ Define $\tilde{r}(t) = \sqrt{\bar{P}_x^2(t) + \bar{P}_y^2(t)} - r_0$ (Reference) where $\bar{P}_x = P_x(t) - P_{cx}$

$\tilde{\theta}(t) = \theta(t) - \left(\frac{\pi}{2} + \arctan\left(\frac{\bar{P}_y}{\bar{P}_x}\right)\right)$

$\bar{P}_y = P_y(t) - P_{cy}$

$\underbrace{(P_{cx}, P_{cy})}$, Coordinates of the Center of the circle.

2 → Define new state equation

$\dot{\tilde{r}}(t) = -\sin\tilde{\theta}(t)$

$\dot{\tilde{\theta}}(t) = \frac{-1}{\tilde{r}(t) + r_0} \cos\tilde{\theta}(t) + \tan u(t)$

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$\rightarrow X(t) = \begin{pmatrix} r(t) \\ \theta(t) \end{pmatrix}$

$\tilde{X}(t) = X(t) - X_{ref}(t) \rightarrow$ where $X_{ref}(t) = \begin{pmatrix} r_0 \\ \theta_r \end{pmatrix}$

$$\dot{\tilde{X}}(t) = \left.\begin{pmatrix} -\sin\tilde{\theta}(t) \\ \frac{-1}{\tilde{r}(t)+r_0}\cos\tilde{\theta}(t) + \tan u(t) \end{pmatrix}\right\} f(x,u)$$

3 $\rightarrow$ Linearize $\rightarrow$ we need that $\tilde{r}(t)=0$ and $\tilde{\theta}(t)=0$

$\bar{X}(t) = (0,0)$

$$\rightarrow \nabla_x f(\bar{x},\bar{u}) = \left.\begin{pmatrix} 0 & -\cos\tilde{\theta}(t) \\ (\tilde{r}+r_0)^{-2}\cos\tilde{\theta}(t) & (\tilde{r}+r_0)\sin\tilde{\theta}(t) \end{pmatrix}\right|_{\substack{\bar{x}=(0,0) \\ \bar{u}}}$$

$$\nabla_x f(\bar{x},\bar{u}) = \begin{pmatrix} 0 & -1 \\ 1/r_0 & 0 \end{pmatrix} \quad f(\bar{x},\bar{u}) = \begin{pmatrix} 0 \\ -\frac{1}{\tilde{r}+r_0} + \tan\bar{u} \end{pmatrix}$$

$$\nabla_u f(\bar{x},\bar{u}) = \left.\begin{pmatrix} 0 \\ \frac{1}{1+u^2} \end{pmatrix}\right|_{u=\bar{u}} \qquad \hookrightarrow \bar{u} = \arctan\left(\frac{1}{\tilde{r}+r_0}\right)$$

$$\boxed{\bar{u} = \arctan\left(\frac{1}{r_0}\right)}$$

$$\nabla_u f(\bar{x},\bar{u}) = \left.\begin{pmatrix} 0 \\ \frac{(\tilde{r}+r_0)^2}{1+(\tilde{r}+r_0)^2} \end{pmatrix}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} = \begin{pmatrix} 0 \\ \frac{r_0^2}{1+r_0^2} \end{pmatrix}$$

Linearization $\rightarrow$ $\dot{\tilde{x}}(t) = \nabla_x f(\bar{x},\bar{u})(x-\bar{x}) + \nabla_u f(\bar{x},\bar{u})(u-\bar{u})$

$$\dot{\tilde{x}}(t) = \begin{pmatrix} 0 & -1 \\ 1/r_0^2 & 0 \end{pmatrix}\tilde{x}(t) + \begin{pmatrix} 0 \\ \frac{r_0^2}{1+r_0^2} \end{pmatrix}\tilde{u}(t) \rightarrow \tilde{u} = u(t) - \bar{u}$$

4 $\rightarrow$ we can check that the system is controllable

$$\text{rank}(B\ AB) = \begin{pmatrix} 0 & -\frac{r_0^2}{1+r_0^2} \\ \frac{r_0^2}{1+r_0^2} & 0 \end{pmatrix} = 2 \rightarrow$$ System is controllable.

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5 → Then we can design a state feedback control

$$\tilde{u}(t) = K_2 (X(t) - X_{ref}) + \arctan\left(\frac{1}{r_0}\right)$$

$$\tilde{u}(t) = K_2 \tilde{X}(t) + \bar{u}$$

Define $K_2 = (K_c, K_d)$

$$\dot{\tilde{X}} = A\tilde{X}(t) + B\tilde{u}(t) = A\tilde{X}(t) + BK_2\tilde{X}(t) + B\bar{u}$$

$$\dot{\tilde{X}} = (A + BK_2)\tilde{X}(t) + B\bar{u}$$

$$\dot{\tilde{X}}(t) = \begin{pmatrix} 0 & -1 \\ \frac{1}{r_0^2} & 0 \end{pmatrix} \tilde{X}(t) + \begin{pmatrix} 0 \\ \frac{r_0^2}{1+r_0^2} \end{pmatrix} (K_c, K_d)\, \tilde{X}(t) + \begin{pmatrix} 0 \\ \frac{r_0^2}{1+r_0^2} \end{pmatrix} \bar{u}$$

$$\dot{\tilde{X}}(t) = \begin{pmatrix} 0 & -1 \\ \frac{K_c}{1+r_0^2} & \frac{r_0^2 K_d}{1+r_0^2} \end{pmatrix} \tilde{X}(t) + \begin{pmatrix} 0 \\ \frac{r_0^2}{1+r_0^2} \end{pmatrix} \bar{u}$$

where → $\bar{u} = \arctan\left(\frac{1}{r_0}\right)$

$$\tilde{X}(t) = X(t) - X_{ref}(t)$$

$$\tilde{X}(t) = \begin{pmatrix} r(t) - r_0 \\ \theta(t) - \left(\frac{\pi}{2} + \arctan\left(\frac{\bar{P}_x}{\bar{P}_y}\right)\right) \end{pmatrix}$$

$\bar{P}_x = P_x(t) - P_{cx}$

$\bar{P}_y = P_y(t) - P_{c,y}$.

→ For the circular trajectories The system developed Govern the motion. where $r_0 = 5$ m.

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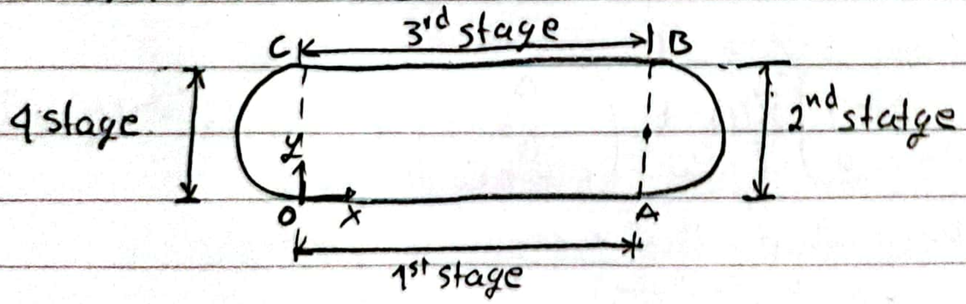
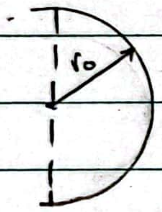
we have 2 controllers → one designed to follow a straight line, and the other for control a circular motion.



If the car starts in the origen of our coordinate system we devided the task in 4 stages, 2 should be for straighline control motion and the circular trajectories should be used the circular control design.

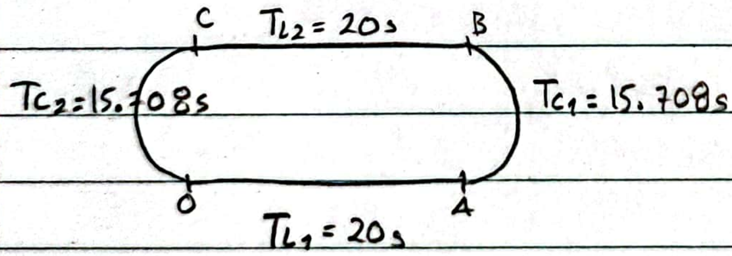
→ For stage 1 and 3 we have to make the race car to follow an straigth line, the distance is 20 m. The car has constant velocity of 1 m/s, then each stage takes 20 s to be completed → $T_L = 20s$

→ For the 2^{nd} and 4^{th} stage we have 2 half-circle trajectory $r_0 = 5m$, The distance in this curve is → $D = \frac{2\pi r_0}{2} = \pi r_0 = 5\pi m$. The car should travel distance D.



$D = \pi r_0$ → Circunference of the half-circle.

Given that the car is moving at constant velocity the car make the circular trajectory in the time $T_c = 15.708 s$



Total time

$T = T_{L1} + T_{c1} + T_{L2} + T_{c2}$

$\boxed{T = 71.41 s}$

To apply the controllers we can use the time:

$$u(t) = \begin{cases} K_1 \tilde{X}(t) = K_1 \begin{pmatrix} \tilde{P}_y(t) \\ \tilde{\theta}(t) \end{pmatrix} & \rightarrow \text{if } \; 0 \leq t \leq 20 \;(\text{stage } 1) \\ & \quad\;\; 35.70 \leq t \leq 55.70 \;(\text{stage } 3) \\ K_2 \tilde{X}(t) + \bar{u} = K_2 \begin{pmatrix} r(t) - r_0 \\ \theta(t) - \left(\frac{\pi}{2} + \tan^{-1}\left(\frac{-P_x}{P_y}\right)\right) \end{pmatrix} + \bar{u} & \text{if } \; 20 < t \leq 35.70 \;(\text{stage } 2) \\ & \quad\;\; 55.70 < t \leq 71.41 s \;(\text{stage } 4) \end{cases}$$

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