Homework #2

(due: Oct. 5)

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<u>*Show all the work/derivation with neat writing.*</u>

1. Consider a linear equation:

$$y = Mx$$

where $M \in \mathbb{R}^{m \times n}$. Show that the following two statements are true.

(a) If M has more columns than rows ($n > m$) and $\text{rank}(M) = m$, then the linear equation always has a solution. (*Hint:* Use the definition of vector space and basis.)

(b) If M has less columns than rows ($n < m$) and $\text{rank}(M) = n$, then the linear equation has a unique solution (if the solution exists). (*Hint:* Use the definition of linearly independent vectors.)

2. We learned that for a matrix $M \in \mathbb{R}^{m \times n}$, a linear equation $y = Mx$ has a solution x if $y \in \text{range}(M)$. Now suppose $y \notin \text{range}(M)$ and we want to compute the element $x \in \mathbb{R}^n$ that minimizes the difference between y and Mx (in the sense of 2-norm). In other words, find x that minimizes

$$\|y - Mx\|_2 . \tag{1}$$

Show that

(a) y can be expressed as $y = v + w$ where $v \in \text{range}(M)$ and $w \in \ker(M^T)$, and

(b) x satisfying $v = Mx$ minimizes (1).

3. Let

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Evaluate $e^{A_1 t}, e^{A_2 t}, e^{(A_1 + A_2)t}$. Does $e^{A_1 t} e^{A_2 t} = e^{A_2 t} e^{A_1 t} = e^{(A_1 + A_2)t}$ hold?
(Can you use: $\cos t = \frac{e^{jt} + e^{-jt}}{2}$ and $\sin t = \frac{e^{jt} - e^{-jt}}{j2}$)

For Questions 4-6, consider the state equation of a linear time-invariant (LTI) system given by

$$\dot{x}(t) = Ax(t) + Bu(t), \ x(0) = x_0 \tag{2a}$$
$$y(t) = Cx(t) \tag{2b}$$

4. Let $P \in \mathbb{R}^{n \times n}$ be an invertible matrix and let us define a *similarity transformation* $z(t) = P^{-1}x(t)$.

(a) Write the state equation with $z(t)$ as its state:

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}u(t)$$
$$y(t) = \bar{C}z(t)$$

Represent the output $y(t)$ of the new state equation in terms of the initial condition $z(0)$ and control input $u(t)$. Verify that the output of the new state equation is same as that of the original state equation.

(b) Discuss the exponential stability of the new state equation in terms of the exponential stability of (2).

5. In this question, we will learn discretization of continuous-time LTI systems. Suppose we apply a control signal defined as

$$u(t) = u_k, \ kT \leq t < (k+1)T$$

to the LTI system, where k is a nonnegative integer and T is a positive number.

   (a) Evaluate the solution $x(t)$ of the state equation at $t = kT$.

   (b) Define $x[k] = x(kT)$, can you express the state equation for $x[k]$ in the following form?

$$x[k+1] = A_d x[k] + B_d u[k] \tag{3a}$$
$$y[k] = C_d x[k] \tag{3b}$$

   Can you relate A_d, B_d, C_d to A, B, C?

   (c) Suppose (2) is exponentially stable. Show that with $u[k] = 0, \ k = 0, 1, \cdots,$

$$\|x[k]\|_2 \leq \alpha_d \beta_d^k \|x[0]\|_2$$

   where $\alpha_d > 0$ and $1 > \beta_d > 0$.

6. Now consider that $u(t) = 0$ and A is given by

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Assess the stability of the system using the Lyapunov stability theorem: Find a symmetric and positive definite matrix P satisfying

$$A^T P + PA < 0$$

7. Consider a LTI system given by

$$\dot{x}(t) = FAx(t)$$

where F is a $n \times n$ symmetric and positive-definite matrix. If A is such that $A + A^T$ is negative definite, show that the system is exponentially stable.

8. Consider the state equation of a nonlinear system given by

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} -x_1(t) + u_1(t) \\ -x_2(t) + u_2(t) \\ x_2(t)u_1(t) - x_1 u_2(t) \end{pmatrix}$$
$$y(t) = \sin x_1(t) + 2 \sin x_2(t) + 3 \sin x_3(t)$$

   (a) Compute the linearization around $x_1 = x_2 = x_3 = 0$ and $u_1 = u_2 = 0$.

   (b) Compute the linearization around $x_1 = x_2 = x_3 = 1$ and $u_1 = u_2 = 1$.

9. Consider the state equation of a N-th order nonlinear system given by

$$\dot{\tilde{\theta}}_i(t) = \frac{K}{N} \sum_{j=1}^{N} \sin\left(\tilde{\theta}_j(t) - \tilde{\theta}_i(t)\right) - \frac{K}{N} \sum_{j=1}^{N} \sin\left(\tilde{\theta}_j(t)\right), \ i = 1, \cdots, N$$

Discuss the stability of the equilibrium $\tilde{\theta}_1 = \cdots = \tilde{\theta}_N = 0$, i.e., does $\tilde{\theta}_i(t) \to 0$ as $t \to \infty$ for every $i = 1, \cdots, N$?

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Homework 2.

1. Consider a linear equation
$$y = MX \qquad M \in \mathbb{R}^{m\times n}$$
Show that the following statements are true.

a) If M has more columns than rows $(n > m)$ and $\operatorname{rank}(M) = m$, then the linear equation always has a solution.

$MX = Y \qquad M \in \mathbb{R}^{m\times n}$ $\qquad n \to$ Columns
$\qquad m \to$ Rows

we have that $Y \in \mathbb{R}^m \to Y$ belongs to the vector space $\mathbb{V} = \mathbb{R}^m$.

Given $M = \begin{vmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & & & \\ M_{m1} & M_{m2} & & M_{mn} \end{vmatrix}$ Since, $n > m$, the matrix M have more columns than rows, and having $\operatorname{rank}(M) = m$, then M cannot have a pivot in each Column (it has at most one pivot per Row), then there are m columns Linear independent that could form a basis for $\mathbb{V} \in \mathbb{R}^m$.

Then we can partionate M by its columns
$M = [C_1\ C_2 \cdots C_n]$ where C_i form a vector space of $\mathbb{R}^m$ due to its linear independance. $C_i \in \mathbb{R}^m$

$$MX = x_1\begin{bmatrix} M_{11} \\ \vdots \\ M_{m1}\end{bmatrix} + x_2\begin{bmatrix} M_{12} \\ \vdots \\ M_{m2}\end{bmatrix} + \cdots + x_n\begin{bmatrix} M_{1n} \\ \\ M_{mn}\end{bmatrix} = Y \qquad \underset{\hookrightarrow \text{coefficients}}{X_i \in \mathbb{R}}$$

$$MX = x_1 C_1 + x_2 C_2 + \cdots + x_n C_n = Y \longrightarrow Y \in \mathbb{R}^m.$$

Since there exist a solution for $Y = MX$, then $Y \in \operatorname{Range}(M)$ and given the columns are a basis of $\mathbb{R}^m$, Y can be Represented as a linear Conbination of the basis of the vector space $V \in \mathbb{R}^m$

$y = x_1 C_1 + x_2 C_2 + \cdots + x_n C_n$, with coefficients x_i, $C_i \in \mathbb{R}^m$

then we always will have a solution of $y = MX$.

b. If M has less columns than rows $(n<m)$ and $\text{rank}(M) = n$, then the linear equation has unique solution (if the solution exists)

Given $M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & & & \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix}$ we could Represent the Matrix with the columns C_i

$C_1 \quad C_2 \quad \cdots \quad C_n$

$$M = [C_1, C_2, \cdots C_n] \quad ①$$

we define the Nullity space of M as:

$$\boxed{N(M) = \{X \in \mathbb{R}^n \mid MX = 0\}} \quad ②$$

Then we could find that

$MX = 0$

$$[C_1 C_2 \cdots C_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$$

then $\boxed{x_1 C_1 + x_2 C_2 + \cdots + x_n C_n = 0}$ ③

Given that $\text{Rank}(M) = n$ implies that the columns C_i $i \in \{1, 2, \cdots n\}$ are linear independent, therefore, the solution to ③:

$MX = 0$ is that $x_1, x_2, \cdots x_n = 0 \rightarrow X = 0$.

Then $\rightarrow$ $\text{Ker}(M) = \{0\}$

now $\rightarrow$ Suppose $y = MX_1$ and $y = MX_2$.

If X_1, X_2 Satisfy $MX_i = y$, then $(X_1 - X_2) \in \text{Range}(M)$

and $y - y = MX_1 - MX_2$

$\boxed{0 = M(X_1 - X_2)}$ $\rightarrow$ we have demostrated that given the $\text{rank}(M) = n$, the columns C_i of M are linear independent and the only solution to $MX = 0 \rightarrow X = 0$, then $(X_1 - X_2) = 0 \rightarrow \boxed{X_1 = X_2}$

Since $X_1 = X_2$, we could have an unique solution to $y = MX$

2. For a matrix M, $y = Mx$ has a solution x if $y \in Range(M)$. If $y \notin Range(M)$ find $x \in \mathbb{R}^n$ that minimize:

$$\| y - Mx \|_2 .$$

Show that:

a) y can be expressed $y = v + w$, where $v \in Range(M)$ $w \in Ker(M^T)$.

b) and $v = Mx$ minimize $\|y - Mx\|_2$.

$M \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$

a) $\rightarrow$ Given that $y \notin Range(M)$ and we have $k = rank(M)$. It means that we have a basis $\{b_1\ b_2 \cdots b_k\}$ of independent columns of M, $b_i \in \mathbb{R}^m$.

$rank(M) = k < m$

then:

we have k linear independant vectors $\{b_1\ b_2 \cdots b_k\}$ from M columns and a vector v can be Represented with the linear combination of $\{b_1, b_2, \cdots b_k\}$ then $v \in Range(M)$.

to Represent $y = v + w$, we defined $v \in Range(M)$ if $k = rank(M)$

now $w \in Ker(M^T) \rightarrow$ we have $m - rank(M) = m - k$ orthogonal linear independent vectors $\{b_{k+1}\ b_{k+2} \cdots b_m\}$ $b_i \in \mathbb{R}^m$

$v = c_1 b_1 + c_2 b_2 + \cdots + c_k b_k$

$u = a_1 b_{k+1} + a_2 b_{k+2} + \cdots + a_{m-k+1} b_m$

to minimize $\|y - Mx\|$, $u \rightarrow 0$ and $w = [a_1\ a_2 \cdots a_{m-k+1}]^T$ that $M^T w = 0$.

b) Suppose $y = v + w$, where $v \in \text{Range}(M)$, $w \in \text{Ker}(M^T)$ minimizing $\|y - Mx\|_2$ is the same that minimizing $\|y - Mx\|_2^2$:

$$\|y-Mx\|_2^2 = (y-Mx)^T(y-Mx)$$
$$= y^Ty - x^TM^Ty - y^TMx + x^TM^TMx$$
$$= y^Ty - x^TM^T(v+w) - (v+w)^TMx + x^TM^TMx$$
$$= y^Ty - x^TM^Tv - v^TMx + x^TM^TMx$$
$$= y^Ty - v^Tv + \|v - Mx\|^2$$

y^Ty and v^Tv are constants in the expression, then to minimize $\|y-Mx\|_2^2$ the term $\|v-Mx\|_2^2$ goes to zero if $v \in \text{Range}(M)$ and it means $v = Mx$.

3) Let $A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Evaluate $e^{A_1 t}$, $e^{A_2 t}$, $e^{(A_1+A_2)t}$

Does $e^{A_1 t} e^{A_2 t} = e^{A_2 t} e^{A_1 t} = e^{(A_1+A_2)t}$ hold?

Evaluate the exponential matrix $e^{A_1 t}$, $e^{A_2 t}$, $e^{(A_1+A_2)t}$.

$e^{A_2 t} \to$ Calculate eigenvalues of A_2

$$\det \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2) = 0 \quad \lambda = 0 \to \text{eigen value}$$

Using the characteristic polynomial of A_2, we could evaluate the exponential matrix using its characteristic Polynomial.

define: $f(\lambda) = e^{\lambda t} = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2$.

A_2 has eigen value $\lambda = 0$ then find $\beta_0, \beta_1, \beta_2$:

$e^{0t} = \beta_0 \to \beta_0 = 1$ ①

to calculate β_1, β_2, differentiate $f(\lambda)$

$f'(\lambda) = t e^{\lambda t} = \beta_1 + 2\beta_2 \lambda$ ②

② $\to$ $t e^{0} = \beta_1 + 2\beta_2(0)$

$\boxed{\beta_1 = t}$

now to calculate $e^{A_2 t}$ use the function $f(\lambda) \to \lambda = A_2$.

$f(A_2) = \beta_0 I + \beta_1 A_2 + \beta_2 A_2^2$

$f(A_2) = e^{A_2 t} = \beta_0 I + \beta_1 A_2$.

$$A_2 . A_2 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$e^{A_2 t} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$e^{A_2 t} = \begin{vmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Now to calculate $e^{A_1 t}$ use Matrix diagonalization:

$A_1 = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$ $\rightarrow$ we could substract the matrix $M = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$

Using eigen decomposition $M = V \Lambda V^{-1}$ $V = (v_{\lambda_1}, v_{\lambda_2})$

$\det \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \rightarrow \lambda^2 = -1 \quad \lambda = \pm i \quad i = \sqrt{-1}$

eigenvalues of $M \Rightarrow \lambda = \pm i$

Find $V \rightarrow$ $\lambda_1 = i$ $\begin{vmatrix} -i & -1 \\ 1 & -i \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = 0$ $\quad -v_1 i - v_2 = 0$
$v_1 - v_2 i = 0 \rightarrow \boxed{v_1 = v_2 i}$

$\lambda_2 = -i$ $\begin{vmatrix} i & -1 \\ 1 & i \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = 0$ $\quad V = \begin{vmatrix} i & 1 \\ 1 & i \end{vmatrix}$ $\quad V^{-1} = \begin{vmatrix} -i/2 & 1/2 \\ \frac{1}{2} & -i/2 \end{vmatrix}$

$v_1 i - v_2 = 0 \rightarrow v_2 = v_1 i$

$M = V \Lambda V^{-1} = \begin{vmatrix} i & 1 \\ 1 & i \end{vmatrix} \begin{vmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{vmatrix} \begin{vmatrix} -i/2 & 1 \\ 1 & -i/2 \end{vmatrix} = \begin{vmatrix} ie^{it} & e^{-it} \\ e^{it} & ie^{-it} \end{vmatrix} \begin{vmatrix} -i/2 & 1/2 \\ 1/2 & -i/2 \end{vmatrix}$

$$\boxed{\begin{array}{l} \frac{ie^{it} - ie^{-it}}{2} = \frac{e^{-it} - e^{it}}{2i} = -\sin t \\ \frac{e^{it} + e^{-it}}{2} = \cos t \end{array}}$$

$= \begin{vmatrix} \frac{e^{it}}{2} + \frac{e^{-it}}{2} & \frac{ie^{t} - ie^{it}}{2} \\ \frac{-ie^{it} + ie^{-it}}{2} & \frac{e^{it} + e^{-it}}{2} \end{vmatrix}$

$M = \begin{vmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{vmatrix}$

then, ensambling M in $A_1 \rightarrow$

$e^{A_1 t} = \begin{vmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Now $\rightarrow$ $(A_1 + A_2) = A = \begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

Calculate eigen values: $\det \begin{vmatrix} -\lambda & -1 & 1 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2) + 1(-\lambda) + 1(0)$

$= -\lambda(\lambda^2) - \lambda = 0$

$\lambda(\lambda^2 + 1) = 0$

eigenvalues of $A \Rightarrow \lambda = 0 \quad \lambda = i \quad \lambda = -i$

using Cayley-Hamilton theorem:

write the Function $F(\lambda) = e^{\lambda t}$

where: $F(\lambda) = e^{\lambda t} = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2$

To find $\beta_0\ \beta_1, \beta_2$ evaluate the eigen values

① $F(0) = \beta_0 = e^0 = 1$

② $F(i) = 1 + \beta_1 i - \beta_2 = e^{it}$

③ $F(-i) = 1 - \beta_1 i - \beta_2 = e^{-it}$

$B_2 = 1 + \beta_1 i - e^{it}$

$\cancel{1} - \beta_1 i - \cancel{1} - \beta_1 i + e^{it} = e^{-it}$

$-2\beta_1 i + e^{it} = e^{-it}$

$\boxed{B_1 = \dfrac{e^{it} - e^{-it}}{2i} = \sin t}$

$B_2 = 1 + \dfrac{e^{it} - e^{-it}}{2} - e^{it}$

$B_2 = \dfrac{2}{2} - \dfrac{(e^{it} + e^{-it})}{2} \rightarrow \boxed{B_2 = 1 - \cos t}$

Now $\rightarrow$ $F(A) = e^{At} = I + \sin t\, A + (1 - \cos t) A^2$

$e^{At} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & -\sin t & \sin t \\ \sin t & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \cos t - 1 & 0 & 0 \\ 0 & \cos t - 1 & 1 - \cos t \\ 0 & 0 & 0 \end{vmatrix}$

Calculate A^2.

$A^2 = \begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

$A^2 = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{vmatrix}$

$e^{At} = \begin{pmatrix} \cos t & -\sin t & \sin t \\ \sin t & \cos t & 1 - \cos t \\ 0 & 0 & 1 \end{pmatrix}$

now check: $e^{A_1t}e^{A_2t}$ and $e^{A_2t}e^{A_1t}$

$$\begin{vmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos t & -\sin t & t\cos t \\ \sin t & \cos t & t\sin t \\ 0 & 0 & 1 \end{vmatrix} = e^{A_1t}e^{A_2t}$$

$$\begin{vmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos t & -\sin t & t \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = e^{A_2t}e^{A_1t}.$$

Given that $e^{A_1t}e^{A_2t} \neq e^{A_2t}e^{A_1t} \neq e^{(A_1+A_2)t}$

Then, the condition not hold, additionally, if we check that A_1, A_2 conmute, we find that $A_1A_2 \neq A_2A_1$

therefore: $e^{(A_1+A_2)t} \neq e^{A_1t}e^{A_2t}$

4. State equation of a Linear time-invariant (LTI) system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

$$y(t) = Cx(t).$$

$P \in \mathbb{R}^{n\times n}$ $\rightarrow$ define similarity transformation

$$z(t) = P^{-1}x(t)$$

a) write the state equation with $z(t)$ as its state:

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}u(t)$$

$$y(t) = \bar{C}z(t)$$

Represent $y(t)$ in terms of $z(0)$ and $u(t)$, Verify that thew output is same as the original equation.

Express the state equation in terms of the new state $z(t)$:

$$\dot{z}(t) = P^{-1}\dot{x}(t) \; ; \; z(t) = P^{-1}x(t) \rightarrow x(t) = Pz(t)$$

$$\dot{z}(t) = P^{-1}\dot{x}(t) \rightarrow \dot{x}(t) = P\dot{z}(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t),$$

$$P\dot{z}(t) = A(Pz(t)) + Bu(t)$$

$$\dot{z}(t) = P^{-1}APz(t) + P^{-1}Bu(t)$$

$$\boxed{\dot{z}(t) = \tilde{A}z(t) + \tilde{B}u(t)} \rightarrow \text{where } \boxed{\begin{array}{l}\tilde{A} = P^{-1}AP \\ \tilde{B} = P^{-1}B\end{array}}$$

and: $y(t) = Cx(t)$

$$y(t) = C(Pz(t)) \rightarrow y(t) = CPz(t)$$

$$\boxed{y(t) = \tilde{C}z(t)} \quad \tilde{C} = CP$$

Given that $\rightarrow$ $x(t) = e^{At}x(0) + \int_{\tau=0}^{t} e^{A(t-\tau)} Bu(\tau)\,d\tau$

and:

$z(t) = P^{-1}x(t) \rightarrow$ at $x(0) = x_0$ $z(0) = P^{-1}x_0$

The solution to the new state equation with $z(t) = P^{-1}x(t)$ is:

$$Pz(t) = e^{At}Pz(0) + \int_{\tau=0}^{t} e^{A(t-\tau)} Bu(\tau)\,d\tau$$

$$z(t) = P^{-1}e^{At}Pz(0) + \int_{\tau=0}^{t} P^{-1}e^{A(t-\tau)} Bu(t)$$

$$\boxed{z(t) = P^{-1}e^{At}Pz(0) + \int_{\tau=0}^{t} P^{-1}e^{A(t-\tau)} Bu(\tau)\,d\tau}$$

Verify that the output of the new state equation $y(t)$ is equal to the original state equation.

Given that $y(t) = \tilde{C}z(t) \rightarrow$ where $\tilde{C} = CP$

now verify if the solution $z(t)$ gives the same output $y(t)$

$$y(t) = CP[P^{-1}e^{At}Pz(0) + \int_{\tau=0}^{t} P^{-1}e^{A(t-\tau)} Bu(\tau)\,d\tau$$

$$y(t) = C[\underbrace{\cancel{PP^{-1}}}_{I} e^{At}Pz(0) + P\int_{\tau=0}^{t} P^{-1}e^{A(t-\tau)} Bu(\tau)\,d\tau] = C[e^{At}Pz(0) + \int_{\tau=0}^{t} \underbrace{\cancel{PP^{-1}}}_{I} e^{A(t-\tau)} Bu(\tau)\,d\tau$$

Given that $x(0) = Pz(0)$ $\therefore$

$$y(t) = C[e^{At}x(0) + \int_{\tau=0}^{t} e^{A(t-\tau)} Bu(\tau)\,d\tau]$$

then $y(t) = Cx(t)$ $\rightarrow$ the output of the new state equation is equal to the output to the original state equation.

b) Discuss the exponential stability of the new state equations in terms of the exponential stability of 2

we have that a system is exponentially stable if there are $\alpha > 0$, $\beta > 0$ such that:

$$\|x(t)\| \leq \alpha e^{-\beta t} \|x_0\| \quad \forall t \geq 0.$$

For the new system we have: $z(t) = P^{-1} x(t)$

$\rightarrow x(t) = P z(t)$.

$\| P z(t)\| \leq \alpha e^{-\beta t} \| x_0 \|$ but $z(0) = P^{-1} x(0) = P^{-1} x_0$

$\| P z(t)\| \leq \alpha e^{-\beta t} \| P z_0 \|$ $\quad z(0) = z_0 = P^{-1} x_0$

Given that $\|AB\| \leq \|A\| \|B\|$

$\| P z(t) \| \leq \|P\| \| z(t) \| \leq \alpha e^{-\beta t} \| P z_0 \|$

and: $\|P\| \| z(t) \| \leq \alpha e^{-\beta t} \| P z_0 \| \leq \alpha e^{-\beta t} \|P\| \| z_0 \|$

then $\cancel{\|P\|} \| z(t) \| \leq \alpha e^{-\beta t} \cancel{\|P\|} \| z_0 \|$

$$\boxed{\| z(t) \| \leq \alpha e^{-\beta t} \| z_0 \|}$$

The new state equation after defining and applying the similarity transformation, preserve the exponential stability for the new state $z(t)$.

5. Discretization of continuous-time LTI systems
Suppose we apply a control signal defined as:

$$u(t) = u_K \qquad KT \leq t < (K+1)T \qquad K > 0$$
$$T > 0$$

a) Evaluate the solution $x(t)$ of the state equation at $t = KT$

The solution is defined as:

$$x(t) = e^{At}x(0) + \int_{\tau=0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau.$$

Then: $$x(KT) = e^{AKT} x(0) + \int_{\tau=0}^{KT} e^{A(KT-\tau)} B u(\tau)\, d\tau \quad (1)$$

b) Define $x[K] = x(KT)$. Can you express the state equation for $x[K]$ in the following form?

$$x[K+1] = A_d\, x[K] + B_d\, u[K]$$
$$y[K] = C_d\, x[K]$$

we have $$x[(K+1)T] = e^{A(K+1)T} x(0) + \int_{\tau=0}^{(K+1)T} e^{A((K+1)T-\tau)} B u(\tau)\, d\tau.$$

$$x[(K+1)T] = e^{AKT} e^{AT} x(0) + \int_{\tau=0}^{KT} e^{A(KT-\tau+T)} B u(\tau) d\tau + \int_{KT}^{(K+1)T} e^{A((K+1)T-\tau)} B u(\tau) d\tau$$

$$x[(K+1)T] = e^{AKT} e^{AT} x(0) + \int_{\tau=0}^{KT} e^{A(KT-\tau)} e^{AT} B u(\tau)\, d\tau + \int_{KT}^{(K+1)T} e^{A(KT-\tau+T)} B u(\tau)\, d\tau.$$

$$x[(K+1)T] = e^{AT}\left[e^{AKT} x(0) + \int_{\tau=0}^{KT} e^{A(KT-\tau)} B u(\tau)\, d\tau \right] + \int_{KT}^{(K+1)T} e^{A(KT+T-\tau)} B u(\tau)\, d\tau.$$

$$x[(K+1)T] = e^{AT} x[KT] + \int_{0}^{T} e^{A\alpha} d\alpha \; B u[KT] \qquad \alpha = KT + T - \tau$$

$$x[K+1] = A_d\, x[K] + B_d\, u[K] \quad \text{where } A_d = e^{AT} \quad B_d = \int_0^T e^{A\alpha} d\alpha\, B$$

now evaluate $y[K] = C_d X[K]$

$$y[K] = y(KT) = C_d X(KT)$$

we have $X[K] = e^{AKT} X(0) + \int_{\tau=0}^{KT} e^{A(KT-\tau)} B u(\tau) d\tau$

$$y(KT) = C X(KT)$$
$$y(KT) = C e^{AKT} X(0) + C \int_{\tau=0}^{KT} e^{A(KT-\tau)} B u(\tau) d\tau$$

$$y(KT) = C_d \left[e^{AKT} X(0) + \int_{\tau=0}^{KT} e^{A(KT-\tau)} B u(\tau) d\tau \right] \qquad C_d = C$$

$$y[K] = C_d X[K]$$

c) Suppose the state equation is exponentially stable. Show $u[K] = 0$, $K = 0, 1 \cdots$ then:

$$\|X[K]\|_2 \leq \alpha_d \beta_d^K \|X[0]\|_2 \qquad \alpha_d > 0, \quad 1 > \beta_d > 0$$

System exponentially stable if $\|X(t)\| \leq \alpha e^{-\beta t} \|X_0\| \quad \forall t > 0$.

then $\to$ $u[K] = 0$: $X[K] = e^{AKT} X(0) + \cancel{\int_{\tau=0}^{KT} e^{A(KT-\tau)} B \overset{0}{u}(\tau) d\tau}$

$$\boxed{e^{AKT} X_0 = X[0]}$$

then:

$$\|X[KT]\| \leq \alpha e^{-K\beta T} \|X[0]\|_2$$
$$\|X[KT]\|_2 \leq \alpha (e^{-\beta T})^K \|X[0]\|_2$$
$$\|X[KT]\|_2 \leq \alpha_d \beta_d^K \|X[0]\|_2 \quad \to \quad \alpha_d = \alpha$$
$$\beta_d = e^{-\beta T}$$

$\alpha_d > 0$, $\beta T > 0 \to -\beta T < 0 \Rightarrow e^{-\beta T} < 1 \to \beta_d < 1$

6) Now consider $u(t) = 0$
and A:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Assess the stability of the system using Lyapunov stability theorem: Find a symmetric and positive definite matrix P satisfying:

$$A^T P + PA < 0$$

If $u(t) = 0 \rightarrow \dot{x}(t) = Ax(t) \quad x(0) = x_0$

If the system is exponentially stable, we can always find a function $V(x) = x^T P x \rightarrow$ where P is a symetric and positive definite matrix, given that $V(x) \geq 0$, and $\frac{d}{dt} V(x) < 0$

we have:

$A^T P + PA < 0$, equivalently, if exist a solution P for a Q matrix (p.sd)

$$A^T P + PA = -Q$$

then:

$$A^T P + PA = -I$$

$$\begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{vmatrix} + \begin{vmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{vmatrix} \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = -I$$

$$\begin{pmatrix} -P_{11} & -P_{12} \\ P_{11} - P_{12} & P_{12} - P_{22} \end{pmatrix} + \begin{pmatrix} -P_{11} & P_{11} - P_{12} \\ -P_{12} & P_{12} - P_{22} \end{pmatrix} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

1 $-P_{12} + P_{11} - P_{12} = 0 \rightarrow P_{11} = 2P_{12}$.

2 $P_{12} - P_{22} + P_{12} - P_{22} = -1$

$2P_{12} - 2P_{22} = -1 \longrightarrow \frac{1}{2} - 2P_{22} = -1 \rightarrow 1 + \frac{1}{2} = 2P_{22}$.

3 $-P_{11} - P_{11} = -1$

$\boxed{P_{11} = \frac{1}{2}}$ $\boxed{P_{12} = \frac{1}{4}}$ $\boxed{P_{22} = \frac{3}{4}}$

Concluding:

$$P = \begin{vmatrix} 1/2 & 1/4 \\ 1/4 & 3/4 \end{vmatrix} > 0$$

P is a symmetric and positive definite matrix, and P satisfy $A^T P + PA < 0$, we can conclude by the Lyapunov stability theorem that the system is stable.

7. Consider a LTI system given that

$$\dot{X}(t) = FAX(t)$$

where F is a $n \times n$ symmetric and positive-definite matrix. If A is such that $A + A^T$ is negative definite, show that the system is exponentially stable.

Given that: $\dot{X}(t) = FAX(t)$

where F is a positive and definite matrix, (p.s.d)

Evaluate if the system is exponentially stable with the Lyapunov stability theorem:

Define $\tilde{A} = FA$

→ Apply the condition that exist P (p.s.d) that satisfy

$$A^T P + PA < 0.$$

In our case → $\tilde{A}P + P\tilde{A} < 0$

$$= (FA)^T P + P(FA) < 0$$

$$A^T F^T P + PFA < 0.$$

Given that F is symmetric and positive definite matrix → $F^T = F$.

$$A^T FP + PFA < 0.$$

Due to F is invertible, we can select $P = F^{-1}$. F^{-1} is also a positive definite matrix, then:

$$A^T FP + PFA = A^T \underbrace{\cancel{FF^{-1}}}_{I} + \underbrace{\cancel{F^{-1}F}}_{I} A < 0$$

Concluding, the system is exponetially stable if $(A^T + A) < 0$.

8. Consider the state equation of a nonlinear system given by:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} -x_1(t) + u_1(t) \\ -x_2(t) + u_2(t) \\ x_2(t)\,u_1(t) - x_1 u_2(t) \end{pmatrix}, \quad y(t) = \sin x_1(t) + 2\sin x_2(t) + 3\sin x_3(t).$$

a) Compute the linearization around $x_1 = x_2 = x_3 = 0$ and $u_1 = u_2 = 0$.

b) Compute the linearization around $x_1 = x_2 = x_3 = 1$ and $u_1 = u_2 = 1$.

a) $\rightarrow$ Defining the state vector as $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$

To compute the linearization, where $x_1(t) = x_2(t) = x_3(t) = 0$ and $u_1(t) = u_2(t) = 0$; we define:

$$\tilde{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \tilde{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now; the linearization of the non-linear system in a small neighbor of $(\tilde{x}, \tilde{u})$ can be computed as:

$$f(x(t), u(t)) = f(\tilde{x}, \tilde{u}) + \nabla_x f(\tilde{x}, \tilde{u})(x(t) - \tilde{x}) + \nabla_u f(\tilde{x}, u)(u(t) - \tilde{u})$$

$$\nabla_x f(x,u) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -u_2(t) & u_1(t) & 0 \end{bmatrix} \quad \nabla_u f(x,u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ x_2(t) & -x_1(t) \end{bmatrix}$$

The linear approximation around $\tilde{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\tilde{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \nabla_x f(\tilde{x}, \tilde{u}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \nabla_u f(\tilde{x}, \tilde{u}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Then → The linearization of $f(x,u)$ around $(\tilde{x},\tilde{u})$

Results:

$$f(x,u) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

Given the output $y(t) = \sin x_1(t) + 2\sin x_2(t) + 3\sin x_3(t)$.

The linearization of $y(t)$ can be expressed as:

$g(x(t)) = g(\tilde{x}) + \nabla_x g(\tilde{x})(x(t) - \tilde{x})$ around $\tilde{x}$.

$$\nabla_x g(x) = \left[\cos(x_1(t)),\ 2\cos(x_2(t)),\ 3\cos(x_3(t))\right]$$

For the case $\tilde{x}$, we have → $g(\tilde{x}) = y(\tilde{x}(t)) = 0$ $\quad \tilde{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

and

$$g(x(t)) = \left[\cos(0), 2\cos(0), 3\cos(0)\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$$g(x(t)) = [1, 2, 3] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

a)

Then, The linearization of the nonlinear system around

$\tilde{x} = [0,0,0]^T$ $\quad u = [0,0,0]^T$ is:

$$f(x,u) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$y(x(t)) = [1, 2, 3] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

b) For literal b, we have to compute the linearization around: $\tilde{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\tilde{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

we have: $f(x,u) = f(\tilde{x},\tilde{u}) + \nabla_x f(\tilde{x},\tilde{u})(x(t)-\tilde{x}) + \nabla_u f(\tilde{x},\tilde{u})(u(t)-\tilde{u})$

$$f(\tilde{x},\tilde{u}) = \begin{pmatrix} -1+1 \\ -1+1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \nabla_x f(\tilde{x},\tilde{u}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \quad \nabla_u f(\tilde{x},\tilde{u}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Now, the linear approximation of f around $\tilde{x}, \tilde{u}$, Result in:

$$f(x,u) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t)-1 \\ x_2(t)-1 \\ x_3(t)-1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1(t)-1 \\ u_2(t)-1 \end{bmatrix}$$

For the linearization of $y(t) = \sin x_1(t) + 2\sin x_2(t) + 3\sin x_3(t)$.

$g(x(t)) = g(\tilde{x}) + \nabla_x g(\tilde{x})(x(t)-\tilde{x})$ $\quad \nabla_x g(x(t)) = [\cos x_1, 2\cos x_2, 3\cos x_3]$

$g(\tilde{x}) = \sin(1) + 2\sin(1) + 3\sin(1)$

$g(\tilde{x}) = 6\sin(1)$ $\quad \nabla_x g(\tilde{x}) = [\cos 1, 2\cos 1, 3\cos 1]$

$\nabla_x g(\tilde{x}) = \cos(1)\,[1, 2, 3]$

then: $g(x(t)) = 6\sin(1) + \cos(1)\,[1,2,3] \begin{bmatrix} x_1(t)-1 \\ x_2(t)-1 \\ x_3(t)-1 \end{bmatrix}$

For small angles we can approximate. $\sin(\theta) = \theta$ and $\cos(\theta) = 1$

The linear approximation of the non-linear System around $(\tilde{x}, \tilde{u})$ is:

$$f(x(t),u(t)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t)-1 \\ x_2(t)-1 \\ x_3(t)-1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1(t)-1 \\ u_2(t)-1 \end{bmatrix}$$

$$g(x(t)) = 6 + \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1(t)-1 \\ x_2(t)-1 \\ x_3(t)-1 \end{bmatrix}$$

9. Consider the state equation of a N^{th} order non linear system given by:

$$\dot{\tilde{\theta}}_i(t) = \frac{K}{N}\sum_{J=1}^{N} \sin(\tilde{\theta}_j(t) - \tilde{\theta}_i(t)) - \frac{K}{N}\sum_{J=1}^{N} \sin(\hat{\theta}_j(t)) \quad i = 1, 2 \cdots N$$

Discuss the stability of the equilibrium $\tilde{\theta}_1 = \cdots = \tilde{\theta}_N = 0$ does $\tilde{\theta}_i(t) \to 0$ as $t \to \infty$ for every $i = 1, \cdots, N$?

To discuss if $\tilde{\theta}_i(t) \to 0$ as $t \to \infty$ for $i = 1 \cdots N$, we have to linearize the state equation of the non-linear system, and check if $\tilde{\theta}_i(t) \to 0$ near to equilibrium point $\tilde{\theta}_i = 0$

$$X(t) = \begin{pmatrix} \tilde{\theta}_1(t) \\ \tilde{\theta}_2(t) \\ \vdots \\ \tilde{\theta}_N(t) \end{pmatrix} \qquad \dot{X}(t) = f(X(t)) \qquad \hat{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Using the linearization definition: $f(x) = f(\hat{x}) + \nabla_\theta f(\hat{x})(x - \hat{x})$ we can approximate the non-linear System.

Evaluating the particular case:

$$\dot{\tilde{\theta}}_1 = \frac{K}{N}\left(\sin(\cancel{\tilde{\theta}_1 - \tilde{\theta}_1}) + \sin(\tilde{\theta}_2 - \tilde{\theta}_1) + \cdots + \sin(\tilde{\theta}_N - \tilde{\theta}_1)\right) - \frac{K}{N}(\sin\theta_1 + \sin\theta_2 + \cdots + \sin\theta_N)$$

$$\dot{\tilde{\theta}}_3 = \frac{K}{N}\left(\sin(\tilde{\theta}_1 - \tilde{\theta}_3) + \sin(\tilde{\theta}_2 - \tilde{\theta}_3) + \sin(\cancel{\tilde{\theta}_3 - \tilde{\theta}_3}) \cdots\right) - \frac{K}{N}(\sin\theta_1 + \sin\theta_2 + \cdots + \sin\theta_N)$$

To calculate $\nabla_\theta f(\hat{x}) \to$

$$\begin{bmatrix} \frac{\partial f_1}{\partial\theta_1} & \frac{\partial f_1}{\partial\theta_2} & \cdots & \frac{\partial f_1}{\partial\theta_N} \\ \frac{\partial f_2}{\partial\theta_1} & \frac{\partial f_2}{\partial\theta_2} & & \frac{\partial f_2}{\partial\theta_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_N}{\partial\theta_1} & \frac{\partial f_N}{\partial\theta_2} & \cdots & \frac{\partial f_N}{\partial\theta_N} \end{bmatrix}$$

to find $\nabla_\Theta f(\tilde{x})$, evaluate particular cases:

$$\frac{\partial f_1}{\partial \Theta_1} = \frac{K}{N}\left(-\cos(\tilde{\Theta}_2 - \tilde{\Theta}_1) - \cos(\hat{\Theta}_3 - \hat{\Theta}_1) - \cdots - \cos(\tilde{\Theta}_N - \tilde{\Theta}_1)\right) - \frac{K}{N}\cos(\tilde{\Theta}_1)$$

$$\frac{\partial f_1}{\partial \Theta_2} = \frac{K}{N}\left(\cos(\hat{\Theta}_2 - \hat{\Theta}_1)\right) + \frac{K}{N}(\cos\hat{\Theta}_2) = \frac{K}{N}\left(\cos(\tilde{\Theta}_2 - \hat{\Theta}_1) + \cos(\Theta_2)\right)$$

$$\frac{\partial f_3}{\partial \Theta_1} = \frac{K}{N}\left(\cos(\hat{\Theta}_1 - \tilde{\Theta}_3)\right) - \frac{K}{N}\left(\cos(\tilde{\Theta}_1)\right) = \frac{K}{N}\left(\cos(\tilde{\Theta}_1 - \hat{\Theta}_3) - \cos\hat{\Theta}_1\right)$$

$$\frac{\partial f_3}{\partial \Theta_3} = \frac{K}{N}\left(-\cos(\tilde{\Theta}_1 - \hat{\Theta}_3) - \cos(\tilde{\Theta}_2 - \hat{\Theta}_3) - \cos(\tilde{\Theta}_4 - \tilde{\Theta}_3) - \cdots - \cos(\tilde{\Theta}_N - \hat{\Theta}_3)\right) - \frac{K}{N}(\cos(\Theta_3)).$$

Generallizing to obtain $\nabla_\Theta f(x)$:

If $i \neq J$ (Entry not member of the diagonal):

$$\rightarrow \frac{\partial f_J}{\partial \Theta_i} = \frac{K}{N}\left(\cos(\tilde{\Theta}_i - \tilde{\Theta}_J) + \cos(\Theta_i)\right)$$

$J \rightarrow$ # Rows
$i \rightarrow$ # Columns

If $i = J$ (diagonal members of the matrix):

$$\frac{\partial f_i}{\partial \Theta_i} = -\frac{K}{N}\left[\sum_{J=1}^{N}\cos(\tilde{\Theta}_J - \hat{\Theta}_i)\right] + \underbrace{\frac{K}{N}\cos(\tilde{\Theta}_i - \hat{\Theta}_i)}_{} - \frac{K}{N}\cos\hat{\Theta}_i$$

adding this term because in the sum we will have $-\frac{K}{N}\cos(\tilde{\Theta}_i - \tilde{\Theta}_i)$, and it's necessary to Remove, because it's the Result of $\frac{\partial f_i}{\partial \Theta_i}$ where originally $\sin(\tilde{\Theta}_i - \tilde{\Theta}_i) \rightarrow$ which is zero.

$$\nabla_\Theta f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial \Theta_1} & \cdots \frac{\partial f_i}{\partial \Theta_1} & \cdots \frac{\partial f_N}{\partial \Theta_1} \\ \vdots & & \\ \frac{\partial f_1}{\partial \Theta_i} & \frac{\partial f_i}{\partial \Theta_i} & \vdots \\ \vdots & & \\ \frac{\partial f_1}{\partial \Theta_N} & \cdots & \frac{\partial f_N}{\partial \Theta_N} \end{bmatrix}$$

we have defined the linearization around equilibrium

$\tilde{\theta}_1 = \tilde{\theta}_2 = \cdots \tilde{\theta}_N = 0 \rightarrow \tilde{X} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\tilde{X} \in \mathbb{R}^N$.

then evaluating $\nabla_\theta f(\tilde{X})$:

→ If $i \neq J$:

$$\frac{\partial f_J}{\partial \theta_i} = \frac{K}{N}\left(\cos(\tilde{\theta}_J - \tilde{\theta}_i) - \cos\tilde{\theta}_i\right) = \frac{K}{N}\left(\cos(0) - \cos(0)\right) = 0$$

$$\boxed{\frac{\partial f_J}{\partial \theta_i} = 0}$$

$J \neq i$

→ If $i = J$:

$$\frac{\partial f_i}{\partial \theta_i} = -\frac{K}{N}\left[\sum_{J=1}^{N} \cos(\tilde{\theta}_J - \tilde{\theta}_i)\right] + \frac{K}{N}\cos(\tilde{\theta}_i - \tilde{\theta}_i) - \frac{K}{N}\cos(\tilde{\theta}_i)$$

$$\left.\frac{\partial f_i}{\partial \theta_i}\right|_{\theta=0} = -\frac{K}{N}\sum_{J=1}^{N}\cos(0) + \frac{K}{N}\cos(0) - \frac{K}{N}\cos(0)$$

$$\left.\frac{\partial f_i}{\partial \theta_i}\right|_{\theta=0} = -\frac{K}{N}\sum_{J=1}^{N} 1 + \cancel{\frac{K}{N}} - \cancel{\frac{K}{N}} = \frac{K}{\cancel{N}}[\cancel{N}] = K \rightarrow \boxed{\left.\frac{\partial f_i}{\partial \theta_i}\right|_{\theta=0} = -K}$$

then we have to evaluate $f(\tilde{X})$:

Given that $\dot{\tilde{\theta}}_i = \frac{K}{N}\sum_{J=1}^{N}\sin(\tilde{\theta}_J(t) - \tilde{\theta}_i(t)) - \frac{K}{N}\sum_{J=1}^{N}\sin(\tilde{\theta}_J)$

and $\tilde{\theta}_i = 0 \rightarrow \sin(0) = 0$, therefore $f(\tilde{X}) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Finally, ensambling the linearization: $f(x) = f(\tilde{x}) + \nabla_\theta f(\tilde{x})(x - \tilde{x})$

$f(x(t)) = \nabla_\theta f(\tilde{x})(X - \tilde{X})$ $\quad \nabla_\theta f(\tilde{X}) \in \mathbb{R}^{N \times N}$ $\quad f(\tilde{X}) = 0 \in \mathbb{R}^N$

$\tilde{X} = 0 \in \mathbb{R}^N$

$$f(x(t)) = \begin{bmatrix} -K & 0 & 0 & \cdots & 0 \\ 0 & -K & 0 & & \vdots \\ 0 & 0 & -K & & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & -K \end{bmatrix}\begin{bmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \vdots \\ \tilde{\theta}_i \\ \vdots \\ \tilde{\theta}_N \end{bmatrix}$$

$\rightarrow f(x(t)) = \dot{x}(t) = AX(t)$

By definition of the stability analysis, The Linearization of the system around $\tilde{X} = 0 \in \mathbb{R}^N$, Results in a linearized system $\dot{x} = AX(t)$, and given that A is a diagonal Matrix with eigenvalues with Negative Real parts, then the system is exponentially stable at $\tilde{\theta}_i \rightarrow 0$, $t \rightarrow \infty$.