Homework #3

(due: Oct. 23)

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Show all the work/derivation with neat writing.

1. Check whether the following linear systems are controllable and/or stabilizable.

(a) $\dot{x}(t) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -2 \end{pmatrix} u(t)$

(b) $\dot{x}(t) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} u(t)$

(c) $\dot{x}(t) = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} u(t)$

2. Consider a linear system given by

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0$$

(a) Suppose that the system is controllable. Find a control input $u(t),\ 0 \leq t \leq T_\mathrm{f}$ such that $x(T_\mathrm{f}) = x_\mathrm{f}$ for any $x_\mathrm{f} \in \mathbb{R}^n$, where $T_\mathrm{f} > 0$ is a fixed time instant.

(b) Now suppose that the system is *NOT* controllable and $x_0 = 0$. We want to identify the set of points in the state space $\mathbb{R}^n$ to which we can drive the system's state from the origin. This is called the **reachable set** of the system. Assume that a point $x_\mathrm{f} \in \mathbb{R}^n$ satisfies

$$x_\mathrm{f} \in \mathrm{range}\left(\int_0^{T_\mathrm{f}} e^{A\tau} B B^T e^{A^T \tau}\, \mathrm{d}\tau \right) \tag{4}$$

find a control input $u(t),\ 0 \leq t \leq T_\mathrm{f}$ such that $x(T_\mathrm{f}) = x_\mathrm{f}$. *Hint:* Using a change of variables, first show that

$$\int_0^{T_\mathrm{f}} e^{A(T_\mathrm{f}-\tau)} Bu(\tau)\, \mathrm{d}\tau = \int_0^{T_\mathrm{f}} e^{A\tau'} Bu(T_\mathrm{f} - \tau')\, \mathrm{d}\tau'$$

Note: The condition in (4) is equivalent to

$$x_\mathrm{f} \in \mathrm{range}\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$$

3. In the class, we learned that if the pair (A, B) is controllable, then the system is stabilizable, i.e., one can design the state feedback $u(t) = Kx(t)$ such that all the eigenvalues of $A + BK$ have negative real parts.

(a) Check the controllability of the following linear systems and see if you can design stabilizing state feedback $u(t) = Kx(t)$ with $K = (k_1, k_2)$. (*Hint:* Find the state equation of resulting closed loop systems and see if you can select K that assigns eigenvalues with all negative real parts).

$$\dot{x}(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u(t)$$

and

$$\dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u(t)$$

(b) Now consider a linear system with order n: $\dot{x}(t) = Ax(t) + Bu(t)$ with $x(t) \in \mathbb{R}^n$. Suppose all the eigenvalues of A have positive real parts. Show that if one can choose K such that the eigenvalues of $A + BK$ have negative real parts, then (A, B) is controllable.

4. In Problem 5 of HW#2, we learned the discretization of continuous-time linear systems. In this problem, we are going to study how the controllability of a continuous-time linear system relates to that of its discretization. Given a linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5}$$

recall that if you apply a control signal defined as

$$u(t) = u_k, \; kT \leq t < (k+1)T$$

then we can derive a discrete-time linear system given by

$$x[k+1] = A_d x[k] + B_d u[k] \tag{6}$$

with $A_d = e^{AT}$ and $B_d = \int_0^T e^{A(T-\tau)} B \, \mathrm{d}\tau$. The controllability of the discrete-time linear system can be tested using the PBH test: the discrete-time linear system is controllable if

$$\operatorname{rank} \begin{pmatrix} A_d - \lambda_d I & B_d \end{pmatrix} = n$$

for every eigenvalue λ_d of A_d.

(a) Discuss if λ, v is an eigenvalue and eigenvector of A then $e^{\lambda T}, v$ is an eigenvalue and eigenvector of A_d.

(b) Validate that if (6) is controllable then (5) must be controllable. (*Note:* The converse is not true in general. See Problem 5 below.)

5. Consider a linear system given by

$$\begin{aligned} \dot{x}_1(t) &= -2\pi x_2(t), \; x_1(0) = 1, x_2(0) = 0 \\ \dot{x}_2(t) &= 2\pi x_1(t) + u(t) \end{aligned}$$

(a) Is the system controllable?

(b) Define the system matrix A and compute its matrix exponential e^{At}.

(c) Suppose digital control given by

$$u(t) = u_k, \; k \leq t < k+1$$

is applied to the system, where k is a non-negative integer and u_k is a real number. Using your solution from Problem 5 in HW#2, find a state equation for a resulting discrete-time linear system.

(d) Discuss whether one can compute $u_0, u_1, \cdots, u_{N-1}$ such that $x_1(N) = x_2(N) = 0$ for some positive integer N.

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Homework 3

1. Check whether the following linear systems are controllable and/or stabilizable.

a) $$\dot{x}(t) = \begin{vmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{vmatrix} x(t) + \begin{vmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -2 \end{vmatrix} u(t).$$

Check corollary 6.1 p 151, Linear System theory and design.

First → check if the system is controllable using the controllability rank test: $\text{rank}(B\ AB\ A^2B) = n$ → dimension of A.

$$AB = \begin{vmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{vmatrix}\begin{vmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 0 \\ 0 & -6 \\ 0 & 0 \end{vmatrix} \qquad A^2 = A.A = \begin{vmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{vmatrix}\begin{vmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

then $$A^2B = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}\begin{vmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{vmatrix}$$

$$\text{rank}\begin{vmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \end{vmatrix} = 4$$ → Given that the rank test results equal 4 that is the order of the matrix A. then the system is controllable

For a closed loop system → A+BK can have any eigenvalues.

b) $$\dot{x}(t) = \begin{vmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{vmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} u(t).$$

Checking the controllability of the system → use rank test to check $\text{rank}(B\ AB\ A^2B\ A^3B) = n$ → order of the system

$\text{rank}(B \; AB \; A^2B \; A^3B) = ?$

From previous exercise we find that $A^2 = \begin{vmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{vmatrix}$

Given that A is the same.

then $\rightarrow A^3 = (A^2)A = \begin{vmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{vmatrix}$

Then:

$AB = \begin{vmatrix} 0&2&0&0 \\ 0&0&0&0 \\ 0&0&0&3 \\ 0&0&0&0 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 0 \\ -2 \end{vmatrix} = \begin{vmatrix} 2 \\ 0 \\ -6 \\ 0 \end{vmatrix}$ and Given $A^3 = A^2 =$ zero 4x4 matrix $\rightarrow A^2B = A^3B = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$

$\text{rank}\begin{vmatrix} 0&2&0&0 \\ 1&0&0&0 \\ 0&-6&0&0 \\ -2&0&0&0 \end{vmatrix} = 2 < 4 \rightarrow$ This system is not controllable.

$\rightarrow$ now, we can check if the system is stabilizable

Can we find for $(A+BK)$ eigenvalues with negative real parts?

Using PBH $\rightarrow$ The system is stabilizable if and only if

$$\text{rank}(A - \lambda I \;\; B) = n.$$

Calculate eigvalues of A:

$\det(A - \lambda I) = \begin{vmatrix} -\lambda&2&0&0 \\ 0&-\lambda&0&0 \\ 0&0&-\lambda&3 \\ 0&0&0&-\lambda \end{vmatrix} = -\lambda(-\lambda(\lambda^2)) - 2(0) = 0$

$\lambda^4 = 0 \rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.$

then $\rightarrow \text{rank}(A - \lambda I \;\; B) = \text{rank}\begin{vmatrix} 0&2&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&3&0 \\ 0&0&0&0&-2 \end{vmatrix} = 3 < 4$

Given that the PBH resulted in $3 < 4$, then the system is not controllable, neither stabilizable.

c.

$$\dot{x}(t) = \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} u(t)$$

First → check if the system is controllable using the controllability rank test. rank$(B \; AB \; A^2B \; A^3B)$

$$A^2 = \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad B = \begin{vmatrix} 0 \\ 1 \\ 0 \\ -2 \end{vmatrix}$$

$$A^3 = \begin{vmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

→ now

$$AB = \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} 0 \\ 1 \\ 0 \\ -2 \end{vmatrix} = \begin{vmatrix} 2 \\ 0 \\ -6 \\ -2 \end{vmatrix} \qquad A^2B = \begin{vmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} 2 \\ 0 \\ -6 \\ -2 \end{vmatrix} = \begin{vmatrix} -2 \\ 0 \\ 0 \\ -2 \end{vmatrix}$$

$$A^3B = \begin{vmatrix} -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} 0 \\ 1 \\ 0 \\ -2 \end{vmatrix} = \begin{vmatrix} 2 \\ 0 \\ -6 \\ -2 \end{vmatrix}$$

Then: the controllability matrix C →

$$\text{rank}\begin{vmatrix} 0 & 2 & -2 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & -6 \\ -2 & -2 & -2 & -2 \end{vmatrix} = 3 < 4$$ → The system is not controllable.

now → to check if the system is stabilizable, we will use the PBH rank test. The system is stabilizable if and only if rank$(A-\lambda I \;\; B) = n$, $\forall \lambda$ of matrix A.

$$\det\begin{vmatrix} -1-\lambda & 2 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -1-\lambda & 3 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = -1-\lambda\left(-\lambda\left(-(\lambda+1)(\lambda-1)\right)\right) = -(\lambda+1)\left(\lambda(\lambda+1)(\lambda-1)\right) = 0$$

eigenvalues of A → $\lambda_1 = 0$ $\lambda_2 = 1$ $\lambda_3 = -1$

Check PBH test for non-negative eigenvalues:

$\lambda = 0$

$$\text{rank} \begin{vmatrix} -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{vmatrix} = 4$$

$$\lambda = 1 \quad \text{rank} \begin{vmatrix} -2 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix} = 4$$

The PBH test claims that the system is stabilizable if and only if $\text{rank}(A - \lambda I \;\; B) = n$, for every eigenvalue λ of A that are non-negative.

Since the matrix A has eigenvalues with non-negative real parts $\lambda = 0$ and $\lambda = 1$, and the PBH test result in the order of the system $(n = 4)$. The described system is not controllable but it's stabilizable.

2. Consider a linear system given by:

$$\dot{X}(t) = AX(t) + Bu(t), \quad X(0) = X_0$$

a) Suppose that the system is controllable. Find $u(t)$ $0 \le t \le T_f$, such that $X(T_f) = X_f$ for any $X_f \in \mathbb{R}^n$.

$T_f > 0$ is a fixed time instant.

$$\dot{X} = AX + Bu$$
$$y = CX$$

Find $u(t)$ that $X(T_f) = X_f$

$$X(t) = e^{At} X_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau \rightarrow$$

System is controllable if given T_f for any X_0 exists $u(t)$ that: $t \in [0, T_f]$

$$X(T_f) = e^{AT_f}x_0 + \int_0^{T_f} e^{A(T_f-\tau)}Bu(\tau) = x_f$$

$$\int_0^{T_f} e^{A(T_f-\tau)} Bu(\tau)d\tau = x_f - e^{AT_f}x_0$$

$$\int_0^{T_f} e^{AT_f}e^{-A\tau} Bu(\tau)d\tau = x_f - e^{AT_f}x_0$$

$$e^{AT_f}\int_0^{T_f} e^{-A\tau}Bu(\tau)d\tau = x_f - e^{AT_f}x_0 \rightarrow \boxed{\int_0^{T_f} e^{-A\tau}Bu(\tau)d\tau = e^{-AT_f}x_f - x_0}$$

From lectures $\rightarrow$ $\displaystyle\int_0^{T_f} e^{-AT_f} Bu(\tau)d\tau = \sum_{K=1}^{N}\frac{T_f}{N}e^{-AT_f}Bu(t_K)$ $\quad t_K = \frac{T_f}{N}K$

$K \in \{1, 2 .. N\}$

$$= \underbrace{\left(\frac{T_f}{N}e^{-At_1}B \quad \frac{T_f}{N}e^{-At_2}B \cdots \frac{T_f}{N}e^{-At_N}B\right)}_{M} \underbrace{\begin{pmatrix} u(t_1) \\ u(t_2) \\ \vdots \\ u(t_N) \end{pmatrix}}_{u}$$

linear equation where $u(t)$ is solution.

If $x_f e^{-AT_f} - x_0 \in \text{Range}(M)$.

then:

$$e^{-AT_f}x_f - x_0 = \underbrace{\left(\frac{T_f}{N}e^{-At_1}B \cdots\cdots \frac{T_f}{N}e^{-At_N}B\right)}_{= M}\begin{pmatrix} u(t_1) \\ u(t_2) \\ \vdots \\ u(t_N) \end{pmatrix}$$

$$\begin{pmatrix} u(t_1) \\ u(t_2) \\ \vdots \\ u(t_N) \end{pmatrix} = M^T(MM^T)^{-1}(e^{-AT_f}x_f - x_0)$$

for $t \in \{t_1 \cdots t_N\}$ $u(t) = B^T e^{-A^T t}\left(\frac{T_f}{N}\sum_{K=1}^{N} e^{-At_K}BB^T e^{-A^T t_K}\right)^{-1}(e^{-AT_f}x_f - x_0)$

when $N \to \infty$: $u(t) = B^T e^{-A^T t}\left(\int_0^{T_f} e^{-A\tau}BB^T e^{-A^T\tau}d\tau\right)^{-1}(e^{-AT_f}x_f - x_0)$

Defining $\rightarrow$ $\tilde{x}_0 = e^{-AT_f}x_f - x_0$

Then $\rightarrow$ $\boxed{u(t) = B^T e^{-A^T t}\left(\int_0^{T_f} e^{-A\tau}BB^T e^{-A^T\tau}d\tau\right)\tilde{x}_0}$

2 b) System is not controllable., $X_0 = 0$

Find Reachable set of the system.

Find $u(t) \rightarrow 0 \le t \le T_f \quad X(T_f) = X_f$.

$$X(T_f) = e^{AT_f} \underset{\substack{\uparrow \\ X_0 = 0}}{x_0} + \int_0^{T_f} e^{A(T_f - \tau)} B u(\tau)\, d\tau$$

$$X(T_f) = \int_0^{T_f} e^{A(T_f-\tau)} B u(\tau)\, d\tau = X_f \quad \rightarrow \text{ if } \tau' = T_f - \tau \quad d\tau' = -d\tau$$

when $\tau = T_f \rightarrow \tau' = 0$

$\tau = 0 \rightarrow \tau' = T_f$.

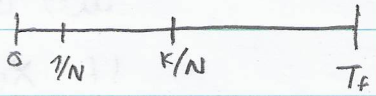
$$X(T_f) = -\int_{T_f}^{0} e^{A\tau'} B u(T_f - \tau')\, d\tau' \qquad -\int_b^a f(x)\,dx = \int_a^b f(x)\,dx \rightarrow \text{From calculus}$$

$$\boxed{X_f = \int_0^{T_f} e^{A\tau'} B u(T_f - \tau')\, d\tau'}$$

We need to find $u(t) \rightarrow 0 \le t \le T_f$

Recall that $t_k = \frac{T_f}{N} k \rightarrow k \in \{1, 2, \dots N\}$

$$\int_0^{T_f} e^{A\tau'} B u(T_f - \tau')\, d\tau' = \sum_{k=1}^{N} \frac{T_f}{N} e^{At_k'} B u(T_f - t_k')$$



$$X_f = \sum_{k=1}^{N} \frac{T_f}{N} e^{At_k'} B u(T_f - t_k') \qquad t_k' = T_f - t_k$$

$$X_f = \underbrace{\left(\frac{T_f}{N} e^{At_1'} B \quad \frac{T_f}{N} e^{At_2'} B \cdots \frac{T_f}{N} e^{At_N'} B \right)}_{M \text{ matrix}} \begin{pmatrix} u(T_f - t_1') \\ u(T_f - t_2') \\ \vdots \\ u(T_f - t_N') \end{pmatrix}$$

$X_f = M u(T_f - t')$

If $X_f \in \text{Range}(M)$ then exists input u that satisfy $X_f = Mu$, and the solution:

$$u = M^T (MM^T)^{-1} X_f$$

$$u(T_f - t') = B^T e^{A^T t'} \left(\int_0^{T_f} e^{A\tau'} B B^T e^{A^T \tau'} d\tau' \right)^{-1} X_f.$$

If $T' = T_f - T$

then $dT' = -dT$

$$\boxed{u(t) = B^T e^{A^T(T_f-t)}\left(\int_0^{T_f} e^{A(T_f-t)} B B^T e^{A^T(T_f-t)} dT\right)^{-1} X_f.}$$

3. If (A,B) pair is controllable, then the system is stabilizable. One can design $u = KX \rightarrow (A+BK)$ that has negative real parts.

a) Check controllability of the system

i) $$\dot{X}(t) = \underbrace{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}}_{A} X(t) + \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{B} u(t)$$

Using controllability rank test

$$AB = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} -1 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

$$\text{rank}(B \ \ AB) = \text{rank}\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = 2 = n$$

Given that the result of the test is 2 that is the order of the system $(n=2)$. This system is controllable.

By definition, if the system is controllable then it's stabilizable. We can design input feedback $u = KX$ that stabilize the system.

$$\dot{X}(t) = AX(t) + Bu(t) \xrightarrow{u=KX} \dot{X}(t) = AX(t) + BKX(t)$$

$$\boxed{\dot{X}(t) = (A+BK)X(t)}$$

then: $$BK = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{vmatrix} -K_1 & -K_2 \\ K_1 & K_2 \end{vmatrix}$$

$$(A+BK)X = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} X(t) + \begin{vmatrix} -K_1 & -K_2 \\ K_1 & 1+K_2 \end{vmatrix} X(t) = \begin{vmatrix} 1-K_1 & 1-K_2 \\ K_1 & 1+K_2 \end{vmatrix} X(t)$$

then:

$$\boxed{\dot{X}(t) = \overbrace{\begin{vmatrix} 1-k_1 & 1-k_2 \\ k_1 & 1+k_2 \end{vmatrix}}^{\tilde{A}} X(t)}$$

The new matrix $\tilde{A}$ can have any eigenvalues, setting k_1, k_2. Then, this system is stable if $\tilde{A}$ has eigenvalues with negative real parts.

Computing eigenvalues of $\tilde{A}$:

$$\det(\tilde{A}-\lambda I) = \det\begin{vmatrix} 1-k_1-\lambda & 1-k_2 \\ k_1 & 1+k_2-\lambda \end{vmatrix} = (1-k_1-\lambda)(1+k_2-\lambda)-(1-k_2)k_1 = 0$$

$$= (1-k_1)(1+k_2) - \lambda(1-k_1) - \lambda(1+k_2) + \lambda^2 - (1-k_2)k_1 = 0$$

$$= \lambda^2 - 2\lambda + \lambda k_1 - \lambda k_2 + 1 + k_2 - \cancel{k_1 k_2} - k_1 - k_1 + \cancel{k_1 k_2} = 0$$

$$= \lambda^2 + \lambda(k_1 - k_2 - 2) + 1 + k_2 - 2k_1 = 0$$

. For obtain $\lambda = -1$ $\rightarrow$ this equation should have the form

$$(\lambda+1)^2 = 0 \rightarrow \lambda^2 + 2\lambda + 1 = 0.$$

Then: $2 = k_1 - k_2 - 2 \rightarrow 4 + k_2 = k_1$

$1 = 1 + k_2 - 2k_1 \rightarrow 1 + k_2 - 8 - 2k_2 = 1$

$$\boxed{k_2 = -8} \quad \boxed{k_1 = -4}$$

The system is stabilizable if we select $k_1 = -4$ $k_2 = -8$.

ii) now for: $\dot{X}(t) = \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{B} u(t)$

Using controllability rank test $\rightarrow$ $\text{rank}(B\ AB) = \text{rank}\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} = 1 < 2$.

$$AB = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} -1 \\ 1 \end{vmatrix} = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

Given that the order of the system $n=2$, this system is not controllable. Controllability rank test failed.

we can check PBH rank test to determine stabilizability of the system.

eigenvalues of $A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \Rightarrow \lambda = 1$

Then PBH rank test $\begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1 < 2 \rightarrow$ The system is not stabilizable.

as a Result:

$$\dot{x}(t) = Ax(t) + Bu(t) \longrightarrow \text{Applying } u(t) = Kx(t)$$

$$\dot{x}(t) = (A + BK)\,x(t)$$

$$BK = \begin{bmatrix} -1 \\ 1 \end{bmatrix} [K_1 \; K_2] = \begin{vmatrix} -K_1 & -K_2 \\ K_1 & K_2 \end{vmatrix}$$

and. $\dot{x}(t) = \begin{vmatrix} 1-K_1 & -K_2 \\ K_1 & 1+K_2 \end{vmatrix} x(t)$.

Given that PBH for stabilizability failed, then it's not possible to find K_1, K_2 such that the system. stabilize in $t \rightarrow \infty$.

3 b. If (A,B) controllable then we can use $u(t) = Kx(t)$ that. $(A+BK) \rightarrow \lambda < 0$ (eigenvalues with negative Real Parts).

now → in general

$$x(t) = Ax(t) + Bu(t) \qquad x(t) \in \mathbb{R}^n$$

All eigenvalues of A, $\lambda > 0$. show that eigenvalues of $(A+BK)$, $\lambda < 0$, then (A,B) is controllable.

To check the controllability of the system, we can Perform the PBH test. Given that all eigenvalues of A are positive, the test is not only going to give a result if the system is controllable or not.

It will also give the result if the system is stabilizable by selecting a feedback control $u(t) = Kx(t)$ that applied to:

$$\dot{x}(t) = Ax(t) + Bu(t) \rightarrow \text{Results in}$$
$$\dot{x}(t) = (A+BK)x(t)$$

where the system is stabilizable if exists K such that $(A+BK)$ matrix has negative eigenvalues.

As a result, given that the PBH test for controllability and stabilizability is the same, when A has all eigenvalues with positive real parts. Then if exist K such that $(A+BK)$ has eigenvalues with negative real parts:

$$\text{rank}(A-\lambda I \;\; B) = n$$

for all λ of A, as a result A is stabilizable, and in the same path (A,B) controllable.

4. We are going to study how the controllability of a continuous-time linear system relates to that of its discretization.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Applying control signal:

$$u(t) = u_K \qquad Kt \le t < (K+1)T$$

then we can derive a discrete-time linear system:

$$x[K+1] = A_d x[K] + B_d u[K]$$

with $A_d = e^{AT}$ and $B_d = \int_0^T e^{A(T-\tau)} B \, d\tau$.

The controllability of the discrete-time linear system can be tested using PBH test: It is controllable if

$$\text{rank}(A_d - \lambda_d I \quad B_d) = n$$

for every eigenvalue $\lambda_d \in$ eigenvals A_d

a) Discuss if λ, v is an eigenvalue and eigenvector of A then $e^{\lambda T}, v$ is an eigenvale and eigenvector of A_d.

We can define e^{At} with taylor expansion as follows:

$e^{At} = \sum_{K=0}^{\infty} \frac{(At)^K}{K!}$, If v is an eigenvector of A and A_d means that $\boxed{\begin{array}{l} Av = \lambda v \\ A_d v = e^{\lambda T} v \end{array}}$

then using taylor expantion:

$$e^{AT} \cdot v = \sum_{K=0}^{\infty} \frac{(AT)^K}{K!} \cdot v = \sum_{K=0}^{\infty} \frac{A^K T^K v}{K!} = \sum_{K=0}^{\infty} \frac{A^K \cdot v \cdot T^K}{K!}$$

If $A \cdot v = \lambda \cdot v \rightarrow A^2 \cdot v = A(A \cdot v) = A(\lambda \cdot v) = \lambda(A \cdot v) = \lambda(\lambda \cdot v) = \lambda^2 v$

then $A^2 \cdot v = \lambda^2 v$, and in general $\boxed{A^K \cdot v = \lambda^K v}$

From last term we have that:

if λ and v are eigenvalue and eigenvector of A,

then $A^K v = \lambda^K . v$.

now → returning to the definition of e^{AT} with taylor expantion:

$$e^{AT} = \sum_{K=0}^{\infty} \frac{(A^K T^K).v}{K!} = \sum_{K=0}^{\infty} \frac{(A^K.v)T^K}{K!} \underset{A^K.v = \lambda^K.v}{\overset{\uparrow}{=}} \sum_{K=0}^{\infty} \frac{(\lambda^K.v)T^K}{K!}$$

$$\rightarrow \sum_{K=0}^{\infty} \frac{(\lambda^K v)T^K}{K!} = \sum_{K=0}^{\infty} \frac{(\lambda^K T^K)}{K!}.v = \sum_{K=0}^{\infty} \frac{(\lambda T)^K}{K!}.v = e^{\lambda T}.v$$

we just proved that → $e^{AT}.v = e^{\lambda T}.v$, where it means that having λ, v eigenvalue and eigenvector of A, the eigenvalue and eigenvector of $A_d = e^{AT}$ is $e^{\lambda T}, v$ respectively.

b) Validate that if → $x[K+1] = A_d x[K] + B_d u[K]$ ① is controllable, then $\dot{x}(t) = Ax(t) + Bu(t)$ ② must be controllable

To prove that; if ① is controllable then ② must be controllable we have to prove that: (using PBH test)

if $\mathrm{rank}(A_d - \lambda_d I \quad B) = n$ then $\mathrm{rank}(A - \lambda I \quad B) = n$

→ To prove this we will use the contradiction that if. $\mathrm{rank}(A - \lambda I \quad B) < n$ then $\mathrm{rank}(A_d - \lambda_d I \quad B) < n$.

Given Matrix $M = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_n \end{pmatrix}$, if $\mathrm{rank}(M) < n$, it means that exist vector $w \in \mathrm{Ker}(M)$ that is different null vector (zero entry vector).

→ $w^T M = 0$

For the case of rank$(A-\lambda I \;\; B) < n$, then exists vector $w \in$ Kernel$(A-\lambda I \;\; B)$, $w \neq 0$.

then:

$$w^T \cdot (A-\lambda I \;\; B) = 0 \longrightarrow \boxed{w^T(A-\lambda I)=0,\; w^T B = 0} \quad (3)$$

now → For the case of rank$(A_d - \lambda_d I \;\; B) < n$, there exist a vector $w \in$ Kernel$(A-\lambda I \;\; B)$, $w \neq 0$.

then:

$$w^T \cdot (A_d - \lambda_d I \;\; B_d) = 0$$

$$\rightarrow w^T(A_d - \lambda_d I) = 0 \;,\; w^T B_d = 0$$

In (a) we proved that $\lambda_d = e^{\lambda T}$ for matrix $A_d = e^{AT}$.

then:

$$w^T(e^{AT} - e^{\lambda T} I) = 0 \rightarrow w^T\left(\sum_{K=0}^{\infty} \frac{(AT)^K}{K!} - e^{\lambda T} I\right) = 0$$

In (a) we proved that $e^{AT} \cdot V = e^{\lambda T} \cdot V$ $\rightarrow w^T\left(\sum_{K=0}^{\infty} \frac{(AT)^K}{K!} - e^{\lambda T} I\right) = 0$

$$\rightarrow w^T\left(\sum_{K=0}^{\infty} \frac{(\lambda T)^K}{K!} I - e^{\lambda T} I\right) = 0$$

$$\text{then } \boxed{e^{\lambda T} = \sum_{K=0}^{\infty} \frac{(\lambda T)^K}{K!}}$$

now for → $w^T B_b = 0$

we have $B_b = \int_0^T e^{A(T-\tau)} B \, d\tau$

Then $w^T B_b = w^T \int_0^T e^{A(T-\tau)} B \, d\tau = \int_0^T w^T e^{A(T-\tau)} B \, d\tau$

$$w^T e^{A(T-\tau)} = \sum_{K=0}^{\infty} \frac{w^T A^K (T-\tau)^K}{K!} = \sum_{K=0}^{\infty} \frac{\lambda^K (T-\tau)^K w^T}{K!} = e^{\lambda(T-\tau)} w^T$$

then $w^T B_b = \int_0^T e^{\lambda(T-\tau)} \underbrace{w^T B}_{} \, d\tau = 0$

↳ from (3) $w^T B = 0$ if rank$(A-\lambda I \;\; B) < n$.

5. Consider a linear system given:

$\dot{x}_1 = -2\pi x_2(t)$ $\qquad x_1(0) = 1$

$\dot{x}_2 = 2\pi x_1(t) + u(t)$ $\qquad x_2(0) = 0$.

Rewritting $\rightarrow$ $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

then $\rightarrow$ $\dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \underbrace{\begin{vmatrix} 0 & -2\pi \\ 2\pi & 0 \end{vmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{vmatrix} 0 \\ 1 \end{vmatrix}}_{B} u(t)$

a) $\rightarrow$ is this system controllable?

Check if the system is controllable $\qquad BA = \begin{vmatrix} 0 & -2\pi \\ 2\pi & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} -2\pi \\ 0 \end{vmatrix}$

rank $(B\ AB)$ = rank $\begin{vmatrix} 0 & -2\pi \\ 1 & 0 \end{vmatrix} = 2$

So $\rightarrow$ The controllability rank test resulted equal to the order of the system $(n=2)$, then it's controllable.

b) $\rightarrow$ Define matrix A and compute matrix exponential.

$A = \begin{vmatrix} 0 & -2\pi \\ 2\pi & 0 \end{vmatrix}$ $\rightarrow$ Compute eigenvalues

$\det(A - \lambda I) = \det \begin{vmatrix} -\lambda & -2\pi \\ 2\pi & -\lambda \end{vmatrix} = \lambda^2 + 4\pi^2 = 0$

$\rightarrow \lambda^2 = -4\pi^2$

$\boxed{\lambda = \pm 2\pi i}$

$\rightarrow$ Compute eigenvectors

$\lambda = 2\pi i \rightarrow (A - \lambda I)V = 0 \qquad \rightarrow -2\pi i V_1 - 2\pi V_2 = 0$

$\begin{vmatrix} -2\pi i & -2\pi \\ 2\pi & -2\pi i \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \end{vmatrix} = 0 \qquad -iV_1 = V_2$

$\boxed{V_1 = \begin{vmatrix} 1 \\ -i \end{vmatrix} \text{ for } \lambda = 2\pi i}$

For $\lambda = -2\pi i$

$(A-\lambda I)V=0 \rightarrow \begin{vmatrix} 2\pi i & -2\pi \\ 2\pi & 2\pi i \end{vmatrix}\begin{vmatrix} V_1 \\ V_2 \end{vmatrix} = 0$

$2\pi i V_1 - 2\pi V_2 = 0$

$2\pi V_1 + 2\pi i V_2 = 0$

$\rightarrow i V_1 = V_2.$

then $\rightarrow$ $\boxed{V_2 = \begin{vmatrix} 1 \\ i \end{vmatrix} \text{ for } \lambda = -2\pi i}$

now $\rightarrow$

$$e^{At} = \begin{vmatrix} 1 & 1 \\ -i & i \end{vmatrix}\begin{vmatrix} e^{2\pi i t} & 0 \\ 0 & e^{-2\pi i t} \end{vmatrix}\begin{vmatrix} 1 & 1 \\ -i & i \end{vmatrix}^{-1}$$

$$\begin{vmatrix} 1 & 1 \\ -i & i \end{vmatrix}^{-1} = \frac{1}{2i}\begin{vmatrix} i & -1 \\ i & 1 \end{vmatrix} = \begin{vmatrix} 1/2 & -1/2i \\ 1/2 & 1/2i \end{vmatrix}$$

so $\rightarrow$

$$e^{At} = \begin{vmatrix} e^{2\pi i t} & e^{-2\pi i t} \\ -ie^{2\pi i t} & ie^{-2\pi i t} \end{vmatrix}\begin{vmatrix} 1/2 & -1/2i \\ 1/2 & 1/2i \end{vmatrix} = \begin{vmatrix} \frac{e^{2\pi i t}+e^{-2\pi i t}}{2} & \frac{-e^{2\pi i t}+e^{-2\pi i t}}{2i} \\ \frac{-ie^{2\pi i t}+ie^{-2\pi i t}}{2} & \frac{e^{2\pi i t}+e^{-2\pi i t}}{2} \end{vmatrix}$$

$$\downarrow \quad \frac{e^{2\pi i t}-e^{-2\pi i t}}{2i}$$

finally:

$$\boxed{e^{At} = \begin{vmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{vmatrix}}$$

matrix exponential.

c) Suppose digital control $u(t) = u_k \quad k \leq t \leq k+1$

Find a state equation for a discrete-time linear System:

For a discrete time linear System:

$$X[k+1] = A_d X[k] + B_d u[k] \qquad A_d = e^{AT} \rightarrow T = 1$$

then:

$$A_d = \begin{vmatrix} \cos(2\pi) & -\sin(2\pi) \\ \sin(2\pi) & \cos(2\pi) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$B_d = \int_0^1 e^{At}\,dt\, B = \int_0^1 \begin{vmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{vmatrix}\begin{vmatrix} 0 \\ 1 \end{vmatrix} dt.$$

$$B_d = \int_0^1 \begin{bmatrix} -\sin(2\pi t) \\ \cos(2\pi t) \end{bmatrix} dt \Rightarrow B_d = \frac{1}{2\pi}\left[\begin{vmatrix} \cos(2\pi) \\ \sin(2\pi) \end{vmatrix} - \begin{vmatrix} \cos(0) \\ \sin(0) \end{vmatrix} \right]$$

Finally for a discrete-time linear system:

$$\boxed{X[K+1] = \begin{vmatrix} \cos(2\pi) & -\sin(2\pi) \\ \sin(2\pi) & \cos(2\pi) \end{vmatrix} X[K] + \int_0^1 \begin{vmatrix} -\sin(2\pi t) \\ \cos(2\pi t) \end{vmatrix} dt \; U[K]}$$

d) Can we find controls $U_0, U_1 \cdots U_{N-1}$ such that $X_1(N) = X_2(N) = 0$?

Find $U_0, U_1 \cdots U_{N-1}$

$$\rightarrow X[N] = \begin{vmatrix} \cos(2\pi) & -\sin(2\pi) \\ \sin(2\pi) & \cos(2\pi) \end{vmatrix} X[N-1] + \underbrace{\int_0^1 \begin{vmatrix} -\sin(2\pi t) \\ \cos(2\pi t) \end{vmatrix} dt \; U[N-1]}_{\text{control of the system.}}$$

We want to calculate:

$U[N-1]$ such that $X[N] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Let's check the term $\rightarrow$ Control of the system

$$\int_0^1 \begin{vmatrix} -\sin(2\pi t) \\ \cos(2\pi t) \end{vmatrix} dt \; U[N-1] = \left. \begin{vmatrix} \frac{\cos(2\pi t)}{2\pi} \\ \frac{\sin(2\pi t)}{2\pi} \end{vmatrix} \right|_0^1 U[N-1]$$

$$= \frac{1}{2\pi}\left[\begin{pmatrix} \cos(2\pi) \\ \sin(2\pi) \end{pmatrix} - \begin{pmatrix} \cos(0) \\ \sin(0) \end{pmatrix} \right] U[N-1]$$

$$= \frac{1}{2\pi}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] U[N-1] = 0$$

Given that the term of "the control of the system" goes to zero, it's not possible to find controls $u_0, u_1 \dots u_{N-1}$ that $X[N]=0$. The matrix A_d resulted is a clockwise rotation matrix by 2π; then $X[N]$ will never be zero.