Homework #4

(due: Nov. 16)

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<u>*Show all the work/derivation with neat writing.*</u>

1. Consider a (single-input) linear time-invariant system given by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with $A = \begin{pmatrix} -3 & -4 \\ 2 & 3 \end{pmatrix}$.

(a) Check controllability/stabilizability for the following two cases:

- Case 1: $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- Case 2: $B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(b) For each of the cases in (a), can you design state feedback $u(t) = Kx(t)$ such that the closed loop has eigenvalues at $-1, -5$? If the answer is yes, compute such K.

2. Check whether the following linear systems are observable and/or detectable.

(a)

$$\dot{x}(t) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} x(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} x(t)$$

(b)

$$\dot{x}(t) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} x(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix} x(t)$$

(c)

$$\dot{x}(t) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & -1 & -1 \end{pmatrix} x(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix} x(t)$$

3. Consider a ball with unit mass on a plane. Suppose the motion of the ball is governed by

$$\ddot{p}_x(t) = -k_1(p_x(t) - p_y(t)) - k_2\dot{p}_x(t) + u_x(t)$$
$$\ddot{p}_y(t) = -k_1(p_y(t) - p_x(t)) - k_2\dot{p}_y(t) + u_y(t)$$

where $(p_x(t), p_y(t))$ is the position of the ball and $(u_x(t), u_y(t))$ is the force applied to the ball.

   (a) Let the output be $y(t) = (p_x(t), p_y(t))$ and $k_1 = k_2 = 0$. Write the state equation and design Luenberger observer to estimate the system's state. How about if $y(t) = p_x(t)$? Can you design the observer?

   (b) Repeat (a) with $k_1 = k_2 = 1$.

4. Consider the state equation for a linear time-variant (LTI) system with noise, given by

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t),\ x(0) = 0$$
$$y(t) = Cx(t) + Du(t) + v(t)$$

where $w(t), v(t)$ are noise in the system.

   (a) Find the transfer function from $(u(t), w(t), v(t))$ to $y(t)$. In other words, find $H_{YU}(s), H_{YW}(s), H_{YV}(s)$ in the following equation using A, B, C, D:

$$Y(s) = H_{YU}(s)U(s) + H_{YW}(s)W(s) + H_{YV}(s)V(s)$$

   where $Y(s), U(s), W(s), V(s)$ are the Laplace transforms of $y(t), u(t), w(t), v(t)$, respectively.

   (b) Consider output feedback defined by

$$\dot{z}(t) = A_z z(t) + B_z y(t),\ z(0) = 0$$
$$u(t) = C_z z(t) + D_z y(t)$$

   Compute the transfer function of the output feedback $U(s) = H_{UY}(s)Y(s)$ in terms of A_z, B_z, C_z, D_z.

   (c) Now suppose that the output feedback is applied to the linear system. Can you express the transfer function of the close loop system in the following form?

$$Y(s) = \bar{H}_{YW}(s)W(s) + \bar{H}_{YV}(s)V(s)$$

   Specify $\bar{H}_{YW}(s), \bar{H}_{YV}(s)$ in terms of $H_{YU}(s), H_{YW}(s), H_{YV}(s), H_{UY}(s)$.

5. Consider the controller form of a single-input single-output linear system.

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{pmatrix} x(t)$$

   (a) Write down the transfer function $H(s) = \frac{N(s)}{D(s)}$, where $N(s)$ is the numerator and $D(s)$ is the denominator.

   (b) Suppose $N(s)$ and $D(s)$ are coprime, i.e., don't have a common factor other than 1, in which case the controller form is a minimal realization. Validate that $D(s) = \det(sI - A)$. Discuss whether poles of a transfer function and eigenvalues of its associated realization are identical if the realization is minimal.

Howework 4

David Felipe Alvear Goyes.

1. Consider a (single-input) linear time-invariant system given by:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$A = \begin{pmatrix} -3 & -4 \\ 2 & 3 \end{pmatrix}$$

a) Check controllability/stabilizability for the following two cases:

Case 1: $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ To check the controllability of the pair (A, B) we use the controllability rank test. $\text{rank}(B \ AB) = n$

$$AB = \begin{vmatrix} -3 & -4 \\ 2 & 3 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} -3 \\ 2 \end{vmatrix} \rightarrow \text{rank} \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 2.$$

For this case the system is controllable and stabilizable

Case 2: $B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $AB = \begin{vmatrix} -3 & -4 \\ 2 & 3 \end{vmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-4 \\ -2+3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\text{rank} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = 1 \longrightarrow$ This system with $B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is not controllable.

$\rightarrow$ now we can check if the system is stabilizable.

Compute eigenvalues λ.

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -4 \\ 2 & 3-\lambda \end{vmatrix} = -(3+\lambda)(3-\lambda) + 8 = 0$$

$$= -(9-\lambda^2) + 8 = 0$$

$$\lambda^2 - 1 = 0 \rightarrow (\lambda-1)(\lambda+1) = 0$$

we can check if the system is stabilizable with the PBH for λ with non-negative Real part.

$\rightarrow \lambda = 1 \rightarrow \text{rank}(A - \lambda I \;\; B) = \begin{pmatrix} -3-1 & -4 & -1 \\ 2 & 3-1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & -4 & -1 \\ 2 & 2 & 1 \end{pmatrix} = 2$

Given that the PBH resulted in the order of the System, then the system is stabilizable.

b) For each of the cases in (a), can you design state feedback $u(t) = Kx(t)$ such that the closed loop has eigenvalues at $-1, -5$? compute such K.

Case a: we can design the linear state feedback $u(t) = Kx(t)$ given that the system is controllable.

$\rightarrow$ now compute K such that eigenvalues of $(A+BK)$ are $\lambda = -1$, $\lambda = -5$.

$$BK = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (K_1 \;\; K_2) = \begin{vmatrix} K_1 & K_2 \\ 0 & 0 \end{vmatrix} \qquad A + BK = \begin{vmatrix} -3+K_1 & -4+K_2 \\ 2 & 3 \end{vmatrix}$$

$$\det((A+BK) - \lambda I) = \begin{vmatrix} -3+K_1-\lambda & -4+K_2 \\ 2 & 3-\lambda \end{vmatrix} = (-3+K_1-\lambda)(3-\lambda) - 2(K_2 - 4) = 0$$

$\rightarrow -9 + \cancel{3\lambda} + 3K_1 - K_1\lambda \cancel{-3\lambda} + \lambda^2 - 2K_2 + 8 = 0$

$\lambda^2 - K_1\lambda + 3K_1 - 2K_2 - 1 = 0$

$\rightarrow$ we need $(\lambda+1)(\lambda+5) = 0 \rightarrow \lambda^2 + 5\lambda + \lambda + 5 = 0$

$\lambda = -1 \quad \lambda = -5$ $\qquad \boxed{\lambda^2 + 6\lambda + 5 = 0}$

then $\rightarrow -K_1 = 6 \rightarrow K_1 = -6$

$3K_1 - 2K_2 - 1 = 5 \rightarrow -18 - 2K_2 - 1 = 5$

$-18 - 1 - 5 = 2K_2 \rightarrow K_2 = -12$

$\rightarrow$ then $K = (-6, -12)$

$\lambda^2 + 6\lambda + 3(-6) - 2(-12) - 1 = 0$

$\lambda^2 + 6\lambda + 5 = 0 \rightarrow (\lambda+5)(\lambda+1) = 0$

$\rightarrow$ eigvals $\lambda = -5$, $\lambda = -1$

Case 2 → Given that the second case is not controllable but it's stabilizable we can find such K that the system is stabilizable with $u = Kx(t)$.

$\rightarrow (A+BK) \rightarrow BK = \begin{pmatrix} -1 \\ 1 \end{pmatrix} (K_1 \;\; K_2) = \begin{pmatrix} -K_1 & -K_2 \\ K_1 & K_2 \end{pmatrix}$

$$A + BK = \begin{pmatrix} -3-K_1 & -4-K_2 \\ 2+K_1 & 3+K_2 \end{pmatrix}$$

→ Now compute K such that eigenvalues of $(A+BK)$ are $\lambda = -1, \lambda = -5$.

$$\det((A+BK) - \lambda I) = \begin{vmatrix} -3-K_1-\lambda & -4-K_2 \\ 2+K_1 & 3+K_2-\lambda \end{vmatrix} = (-3-K_1-\lambda)(3+K_2-\lambda) - (2+K_1)(-4-K_2) = 0$$

$$\rightarrow \overset{-1}{\cancel{-9}} - \underline{3K_2} + \cancel{3\lambda} - \underline{3K_1} - \cancel{K_1K_2} + \underline{K_1\lambda} \cancel{-3\lambda} - \underline{K_2\lambda} + \underline{\lambda^2}$$
$$+ \cancel{8} + \underline{2K_2} + \underline{4K_1} + \cancel{K_1K_2} = 0$$

$$\rightarrow \lambda^2 + \lambda(K_1 - K_2) - 1 - K_2 + K_1 = 0$$

→ Now we need the equation like $\lambda^2 + 6\lambda + 5 = 0$

$(\lambda+1)(\lambda+5) = 0$

$\lambda = -1 \quad \lambda = -5.$

① $(K_1 - K_2) = 6 \rightarrow K_1 = 6 + K_2.$

② $-1 - K_2 + K_1 = 5.$

① ∧ ② $\rightarrow K_1 = 6 \quad K_2 = 0$

$$\rightarrow \lambda^2 + \lambda(6-0) - 1 - (0) + 6 = 0$$
$$\lambda^2 + 6\lambda + 5 = 0 \rightarrow \lambda = -1 \quad \lambda = -5$$

Then $\boxed{K = (6 \;\; 0)}$

2. Check whether the following linear systems are observable and/or detectable.

a)

$$\dot{x}(t) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} x(t) \qquad y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} x(t).$$

We can use the PBH test for observability

$$\text{rank} \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = n$$

→ First calculate eigenvalues of A.

$$\det(A - \lambda I) = \begin{pmatrix} -\lambda & 2 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} \qquad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$$

now →

$$\text{rank} \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2 \neq 4$$

→ The system is not observable and not detectable because the PBH test for observability and detectability are the same given $\lambda_i = 0$, all eigenvalues zero.

b)

$$\dot{x}(t) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} x(t) \qquad y(t) = (1 \ 0 \ 2 \ 0)\, x(t)$$

Given that the matrix A is the same as (a) we have eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ then PBH test is the same for observability and Detectability.

$$\text{rank} \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} = 2 \neq 4$$

→ The system is not observable nor detectable.

c) $\dot{x}(t) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & -1 & -1 \end{pmatrix} x(t)$ $\quad y(t) = (1\ 0\ 2\ 0)\, x(t)$

we can check if the system is observable using the observability rank condition.

$$\text{rank}\begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & -1 & -1 \end{pmatrix} \quad A^2 = \begin{pmatrix} -2 & -2 & 4 & 0 \\ 1 & -1 & -2 & 6 \\ 3 & 0 & -3 & 3 \\ -1 & 2 & 1 & -2 \end{pmatrix}$$

$\rightarrow CA = (1\ -1\ 0\ -2)$

$CA^2 = (-3\ 3\ 0\ 2)$

$CA^3 = (-1\ -5\ 4\ -2)$

$$A^3 = \begin{pmatrix} 2 & -2 & -4 & 12 \\ 7 & 3 & -8 & -12 \\ -3 & 6 & 3 & -6 \\ -4 & -4 & 6 & 5 \end{pmatrix}$$

Finally $\text{rank}\begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & -1 & 0 & -2 \\ -3 & 3 & 0 & 2 \\ -1 & -5 & 4 & -2 \end{pmatrix} = 4 = n$ $\rightarrow$ Given that the system passed the observability rank test, the system is observable and detectable.

3. Consider a ball with unit mass on a plane.
Suppose the motion of the ball is governed by.

$$\ddot{P}_x = -K_1(P_x(t) - P_y(t)) - K_2 \dot{P}_x(t) + U_x(t)$$

$$\ddot{P}_y = -K_1(P_y(t) - P_x(t)) - K_2 \dot{P}_y(t) + U_y(t)$$

where $(P_x(t), P_x(t))$ is the position of the ball and $(U_x(t), U_y(t))$ is the force applied to the ball.

a) Let the output by $y(t) = (P_x(t), P_y(t))$ and $K_1 = K_2 = 0$.
write the state equation and design Luenberg observer to estimate the system's state.
How about if $y(t) = P_x(t)$? Can you design the observer?

Solution:

we can define the state as $X(t) = \begin{pmatrix} P_x(t) \\ \dot{P}_x(t) \\ P_y(t) \\ \dot{P}_y(t) \end{pmatrix}$

$$\ddot{P}_x = -K_1 P_x + K_1 P_y - K_2 \dot{P}_x + U_x$$

$$\ddot{P}_y = -K_1 P_y + K_1 P_x - K_2 \dot{P}_y + U_y$$

and the state equation:

$$\dot{x}(t) = \begin{pmatrix} \dot{P}_x(t) \\ \ddot{P}_x(t) \\ \dot{P}_y(t) \\ \ddot{P}_y(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -K_1 & -K_2 & K_1 & 0 \\ 0 & 0 & 0 & 1 \\ K_1 & 0 & -K_1 & -K_2 \end{pmatrix} \begin{pmatrix} P_x(t) \\ \dot{P}_x(t) \\ P_y(t) \\ \dot{P}_y(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U_x(t) \\ U_y(t) \end{pmatrix}$$

$$Y(t) = (P_x, P_y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} X(t)$$

With $K_1 = K_2 = 0 \rightarrow$

$$\dot{X}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} X(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} U(t)$$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} X(t)$$

To design a Luenberger observer we need to check if this system is observable, and Detectable.

Using the observability rank condition: $\text{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^3 \end{pmatrix} = ?$

$CA = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $CA^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $CA^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

then

$\text{rank}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 4$ - It's observable Given that the order of the system is $n=4$

If the system is observable then we can design a Luenberger observer.

We define $\tilde{x} = x(t) - \hat{x}(t)$

where $\dot{\hat{x}}(t) = A\hat{x}(t) + L(y(t) - C\hat{x}(t))$, $\hat{x}(0) = \hat{x}_0$

then: $\dot{\tilde{x}} = Ax(t) - A\hat{x}(t) - L(y(t) - (\hat{x}(t))$

$\dot{\tilde{x}} = A\underbrace{(x(t) - \hat{x}(t))}_{\tilde{x}(t)} - L(Cx(t) - C\hat{x}(t))$

$\rightarrow \dot{\tilde{x}} = A\tilde{x}(t) - LC\tilde{x}(t)$ $\qquad \tilde{x}_0(0) = x_0 - \hat{x}_0$

$$\boxed{\dot{\tilde{x}} = (A - LC)\tilde{x}(t)}$$

Define $L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \\ L_5 & L_6 \\ L_7 & L_8 \end{bmatrix}$ $\rightarrow LC = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \\ L_5 & L_6 \\ L_7 & L_8 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} L_1 & 0 & L_2 & 0 \\ L_3 & 0 & L_4 & 0 \\ L_5 & 0 & L_6 & 0 \\ L_7 & 0 & L_8 & 0 \end{bmatrix}$

$\rightarrow (A - LC) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} - LC = \begin{vmatrix} -L_1 & 1 & -L_2 & 0 \\ -L_3 & 0 & -L_4 & 0 \\ -L_5 & 0 & -L_6 & 1 \\ -L_7 & 0 & -L_8 & 0 \end{vmatrix}$

Now we can find L_i such that $(A-LC)$ has eigen-
-values with negative real parts.

$$\det(A-LC-\lambda I)= \begin{vmatrix} -L_1-\lambda & 1 & -L_2 & 0 \\ -L_3 & -\lambda & -L_4 & 0 \\ -L_5 & 0 & -L_6-\lambda & 1 \\ -L_7 & 0 & -L_8 & -\lambda \end{vmatrix}$$

If $L_5 = L_7 = 0$, and $L_3 = L_8 = 1$ → we have 4 variables to set-up.
triangular matrix.

$$\det(A-LC-\lambda I)= \begin{vmatrix} -L_1-\lambda & 1 & -L_2 & 0 \\ -1 & -\lambda & -L_4 & 0 \\ 0 & 0 & -L_6-\lambda & 1 \\ 0 & 0 & -1 & -\lambda \end{vmatrix}$$

→ Polynomial $\lambda^4 + L_6\lambda^3 + L_1\lambda^3 + L_1L_6\lambda^2 + 2\lambda^2 + L_6\lambda + L_1\lambda + 1 = 0$

→ Simplyfing: $(1+\lambda(L_1+\lambda))(1+\lambda(L_6+\lambda)) = 0$

$(1+L_1\lambda+\lambda^2)(1+L_6\lambda+\lambda^2) = 0$

→ we can set $L_1 = L_6 = 2$ to have eigenvals $= -1$
with the form $\lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)^2 = 0$

$\boxed{L_2 = L_4 = 1}$

L_2, L_4 can have any value.

Then:

$L = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$ → Luenberger observer if $y(t) = P_x(t) \rightarrow$

the system: $\dot{x}(t) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} x(t) + \begin{vmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{vmatrix} u(t)$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x(t)$$

To design a Luenberger observer we need to check if the system is observable and detectable using observability rank condition $\text{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^3 \end{pmatrix}$

$CA = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$ $CA^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $CA^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\text{rank}\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 2 < 4 \rightarrow$ This system is not observable.

we can check that the matrix A has eigenvalues equal to zero, and it implies that the PBH test for observability is the same as PBH test for Detectability, then if the system is not observable then it is not detectable, which means that we can not compute L matrix Luenberger observer.

PBH test

$\text{rank}\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$

b) Repeat a with $K_1 = K_2 = 1$

the new state equation is:

$\dot{x}(t) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{vmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} u(t)$ $x(t) = \begin{vmatrix} P_x(t) \\ \dot{P}_x(t) \\ P_y(t) \\ \dot{P}_y(t) \end{vmatrix}$ $u(t) = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t)$$

First we can check if the system is observable using observability rank condition:

$$CA=\begin{pmatrix}0&1&0&0\\0&0&0&1\end{pmatrix} \quad CA^2=\begin{pmatrix}-1&-1&1&0\\1&0&-1&-1\end{pmatrix} \quad CA^3=\begin{pmatrix}1&0&-1&1\\-1&1&1&0\end{pmatrix}$$

then: $$\text{rank}\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\\-1&-1&1&0\\1&0&-1&-1\\1&0&-1&1\\-1&1&1&0\end{vmatrix}=4$$ $\rightarrow$ The system is observable then we can design a Luenberger observer.

$\rightarrow \dot{\hat{x}}=(A-LC)\hat{x}(t) \qquad \tilde{x}_0(0)=x_0-\hat{x}_0$ (we had previously)

Define $L=\begin{bmatrix}l_1&l_2\\l_3&l_4\\l_5&l_6\\l_7&l_8\end{bmatrix} \qquad LC=\begin{bmatrix}l_1&0&l_2&0\\l_3&0&l_4&0\\l_5&0&l_6&0\\l_7&0&l_8&0\end{bmatrix}$

$$A-LC=\begin{vmatrix}-l_1-\lambda&1&-l_2&0\\-l_3-1&-\lambda-1&1-l_4&0\\-l_5&0&-l_6-\lambda&1\\1-l_7&0&-1-l_8&-\lambda-1\end{vmatrix}$$

$\rightarrow l_3=-1 \quad l_5=0 \quad l_7=1 \quad l_8=-1$ setting this elements we can obtain a upper triangular matrix.

$$\rightarrow A-LC=\begin{vmatrix}-l_1-\lambda&1&-l_2&0\\0&-\lambda-1&1-l_4&0\\0&0&-l_6-\lambda&1\\0&0&0&-\lambda-1\end{vmatrix}$$

$\rightarrow$ the diagonal ~~of~~ of the matrix contains the eigen-values.

to have all the eigenvalues of $(A-LC)$ equal to -1 we assign $l_1=l_6=1$

l_2 and l_4 can take any value

$l_2=l_4=1$

then $\rightarrow L=\begin{vmatrix}1&1\\-1&1\\0&1\\1&-1\end{vmatrix}$

* For the case $y(t) = \mathbb{P}x(t)$
we can check that the system is observable.

$$A = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{vmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \operatorname{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^3 \end{pmatrix}$$

$$\operatorname{rank}\begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 4$$

→ Given that the observability test is passed, we can design a Luenberger observer.

$$L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \\ L_5 & L_6 \\ L_7 & L_8 \end{pmatrix} \rightarrow LC = \begin{vmatrix} L_1 & 0 & 0 & 0 \\ L_3 & 0 & 0 & 0 \\ L_5 & 0 & 0 & 0 \\ L_6 & 0 & 0 & 0 \\ L_7 & 0 & 0 & 0 \end{vmatrix}$$

$$\det(A - LC) = \begin{vmatrix} -L_1-\lambda & 1 & 0 & 0 \\ -1-L_3 & -1-\lambda & 1 & 0 \\ -L_5 & 0 & -\lambda & 1 \\ 1-L_7 & 0 & -1 & -1-\lambda \end{vmatrix}$$

→ we can set L_1, L_3, L_5, L_7 to obtain negative eigvalues in the matrix $(A-LC)$

4. Consider the state equation for LTI (Linear Time-Variant) System with noise:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t), \quad x(0) = 0$$
$$y(t) = Cx(t) + Du(t) + v(t)$$

$w(t), v(t) \rightarrow$ noise of the system.

a) Find the transfer function from $(u(t), w(t), v(t))$ to $y(t)$

Find $H_{yu}(s)$, $H_{yw}(s)$, $H_{yv}(s)$ using A, B, C, D.

$$Y(s) = H_{yu}(s)\,U(s) + H_{yw}(s)\,W(s) + H_{yv}(s)\,V(s)$$

where $Y(s), U(s), W(s), V(s)$ are the laplace transforms of $y(t), u(t), w(t), v(t)$.

Solution:

$\rightarrow$ $sX(s) = AX(s) + Bu(s) + w(s)$

$sX(s) - AX(s) = Bu(s) + w(s)$

$(sI - A)X(s) = Bu(s) + w(s)$

$X(s) = (sI - A)^{-1}B\,u(s) + (sI - A)^{-1}w(s)$

now $\rightarrow$ $y(s) = CX(s) + Du(s) + V(s)$

$y(s) = C((sI - A)^{-1}B\,u(s) + (sI - A)^{-1}w(s)) + D\,u(s) + V(s)$

$y(s) = C(sI - A)^{-1}B\,u(s) + C(sI - A)^{-1}w(s) + Du(s) + V(s)$

$y(s) = [C(sI - A)^{-1}B + D]\,u(s) + C(sI - A)^{-1}w(s) + V(s)$

Transfer function. $H_{yu}(s) = C(sI - A)^{-1}B + D$

$H_{yw}(s) = C(sI - A)^{-1}$

$H_{yv}(s) = I$

b) Consider output feedback defined by:

$$\dot{z}(t) = A_z z(t) + B_z y(t)$$
$$u(t) = C_z z(t) + D_z y(t).$$

Obtain laplace transform of $\dot{z}(t)$:

$\rightarrow$ $s z(t) = A_z z(s) + B_z Y(s)$

$(sI - A_z) z(s) = B_z Y(s)$

① $z(s) = (sI - A_z)^{-1} B_z Y(s)$

$\longrightarrow$ $U(s) = C_z (sI - A_z)^{-1} B_z Y(s) + D_z Y(s)$

$U(s) = [C_z (sI - A_z)^{-1} B_z + D_z] Y(s)$

Transfer function of the output feedback $H_{uy}(s) = [C_z (sI - A_z)^{-1} B_z + D_z]$

c) Now suppose that the output feedback is applied to the linear system, Can you express the transfer function of the close loop system in the following form?

$$Y(s) = \bar{H}_{yw} W(s) + H_{yv}(s) V(s)$$

specify $H_{yw}(s)$, $\bar{H}_{yv}(s)$ in terms of $H_{yu}(s)$, $H_{yw}(s)$ $H_{yv}(s)$, $H_{uy}(s)$

In a. we had $Y(s) = H_{yu}(s) U(s) + H_{yw}(s) W(s) + H_{yv}(s) V(s)$.

then:

$Y(s) = H_{yu}(s) (H_{uy}(s) Y(s)) + H_{yw}(s) W(s) + H_{yv} V(s)$

$Y(s) = H_{yu}(s) H_{uy}(s) Y(s) + H_{yw}(s) W(s) + H_{yv} V(s)$

$Y(s) - H_{yu}(s) H_{uy}(s) Y(s) = H_{yw}(s) W(s) + H_{yv}(s) V(s)$

$(I - H_{yu}(s) H_{uy}(s)) Y(s) = H_{yw}(s) W(s) + H_{yv}(s) V(s)$

$$\longrightarrow Y(s) = (I - H_{yu}(s) H_{uy}(s))^{-1} H_{yw}(s) W(s) + (I - H_{yu}(s) H_{uy}(s))^{-1} H_{yv}(s) V(s)$$

5. Consider the controller form of a single-input single-output linear system.

$$\dot{X}(t) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{pmatrix} X(t) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = (\beta_0 \ \beta_1 \ \cdots \ \beta_{n-1}) X(t)$$

a) write down the transfer function $H(s) = \frac{N(s)}{D(s)}$ where $N(s)$ is the numerator and $D(s)$ is the Denominator.

$$H(s) = \frac{\beta_{n-1} s^{n-1} + \beta_{n-2} s^{n-2} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$ Single input single output case.

b) Suppose $N(s)$ and $D(s)$ are coprime. don't have a common factor other than 1, In which case the controller form is a minimal realization. Validate the $D(s) = \det(sI - A)$. Discuss whether poles of a transfer function and eigenvalues of its associated realization are identical if the realization is minimal.

From the linear system with the form

$$\dot{X}(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t).$$

we have that the transfer function is defined as:

$$H(s) = \frac{N(s)}{D(s)} = C(sI - A)^{-1}B \rightarrow (sI - A)^{-1} = \frac{1}{\det(sI - A)} (Adj(sI - A))$$

then $$H(s) = \frac{1}{\det(sI - A)} C(Adj(sI - A))B = \frac{C(Adj(sI - A))B}{\det(sI - A)}$$

$$\longrightarrow D(s) = \det(sI - A)$$

→ we can express $N(s) = \bar{N}(s)\, R(s)$ and $D(s) = \bar{D}(s)\, R(s)$, where $R(s)$ is a polynomial called a greatest common divisor of $D(s)$ and $N(s)$.

Since $N(s)$ and $R(s)$ are coprime, means that $R(s)$ is a non-zero constant (equal to one in this case in base on the problem statement) with a polynomial of degree zero. This implies that given $D(s) = \det(sI - A)$, poles of the transfer function $H(s)$ are the solutions to $D(s) = 0$ and $D(s) = \det(sI - A) = 0$, where this yields to the eigenvalues of A. Concluding, if the controller form is a minimal Realization, the eigenvalues of A are the poles of the transfer function $H(s)$.