Homework #1

(due: Sept. 21)

Show all the work/derivation with neat writing.

1. Consider the nonlinear system described by

$$\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + k_1\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} + k_2\sin\theta(t) + k_3u(t) = 0$$
$$y(t) = \theta(t)$$

   where $y(t)$ is output, $u(t)$ is input of the system, and k_1, k_2, k_3 are constants. Define the state of the system and write the nonlinear state equation of the following form.

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t))$$

2. Consider the n-th order linear differential equation

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + a_{n-2}y^{(n-2)}(t) + \cdots + a_1y^{(1)}(t) + a_0y(t) = bu(t)$$

   where $y(t)$ is output, $u(t)$ is input of the system, and $y^{(k)}(t)$ is defined as $y^{(k)}(t) = \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k}$. Define the state of the system and write the (linear) state equation for the following form.

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

3. Consider the state equation for a linear time-invariant system given by

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0$$
$$y(t) = Cx(t) + Du(t)$$

   Suppose that we apply output feedback defined by

$$\dot{z}(t) = A_zz(t) + B_zy(t),\ z(0) = z_0$$
$$u(t) = C_zz(t) + D_zy(t)$$

   to the system. Define the state and write the state equation for the resulting closed loop system.

4. Consider a vector space $\mathbb{V}$ defined by

$$\mathbb{V} = \left\{x \in \mathbb{R}^5 \mid x_1 = 3x_2 \text{ and } x_3 = 7x_4\right\}$$

   Find bases for $\mathbb{V}$.

5. Compute 1-norm, 2-norm, and ∞-norm of

   (a) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

6. Using the definition of the induced matrix norm, verify that

$$\|AB\|_2 \le \|A\|_2 \|B\|_2$$

for any $A, B \in \mathbb{R}^{n\times n}$.

7. Using the eigendecomposition, compute the matrix exponential e^M for the following matrices.

(a) $M = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & -2 & 3 \end{pmatrix}$

(b) $M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & -2 & 3 \end{pmatrix}$

8. Given a diagonal matrix Λ

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

verify that the matrix exponential is given by

$$e^{\Lambda} = \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix}.$$

9. Given a block upper triangular matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

where M_{11} and M_{22} are square matrices. Identify the <u>diagonal blocks</u> of e^M in terms of M_{11}, M_{12}, and M_{22} and show that e^M is block upper triangular.

10. For the state equation given by

$$\dot{x}(t) = Ax(t) + Bu(t),$$

verify that

$$x(t) = e^{At}x(0) + \int_{\tau=0}^{t} e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau$$

is a solution of the equation.

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Homework 1

1. Consider the non-linear system

$$\frac{d^2\theta(t)}{dt^2} + K_1\frac{d\theta(t)}{dt} + K_2 \sin\theta(t) + K_3 u(t) = 0$$

Define the state and write in the form

$$\dot{X}(t) = f(x(t), u(t)) \quad y(t) = g(x(t))$$

$$\ddot{\theta} = \frac{d^2\theta(t)}{dt^2}$$

$$\dot{\theta} = \frac{d\theta(t)}{dt}$$

$$\theta = \theta(t)$$

then:

$$\ddot{\theta} + K_1\dot{\theta} + K_2\sin\theta + K_3 u = 0$$

$$-\ddot{\theta} = K_1\dot{\theta} + K_2\sin\theta + K_3 u$$

Define the state system $X(t) = \begin{bmatrix}\theta(t)\\ \dot{\theta}(t)\end{bmatrix}$

then:

$$\dot{X}(t) = \begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\end{bmatrix} = f(x(t), u(t))$$

$$\rightarrow \dot{X} = \begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ -K_1\dot{\theta} - K_2\sin\theta - K_3 u\end{bmatrix} \quad y(t) = g(x(t)) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\theta(t)\\ \dot{\theta}(t)\end{bmatrix}$$

$$\rightarrow y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\theta(t)\\ \dot{\theta}(t)\end{bmatrix}$$

2) Consider the n^{th} order linear diff equation

$$y^{n}(t) + a_{n-1}y^{n-1}(t) + a_{n-2}y^{n-2}(t) + \dots + a_1 y^{1}(t) + a_0 y(t) = bu(t)$$

$y(t) \rightarrow$ output

$u(t) \rightarrow$ input

$$y^{K}(t) = \frac{d^{K}y(t)}{dt^{K}}$$

Define the state of the system and write the state equation $\rightarrow \dot{x}(t) = Ax(t)\, Bu(t)$, $y(t) = c\,x(t)$.

Evaluate in a determined situation $n=3$.

$$y^3 + a_2 y^2 + a_1 y^1 + a_0 y = bu(t)$$

$$y^3 = -a_2 y^2 - a_1 y^1 - a_0 y + bu(t)$$

For $n=3$ case the state is $X = [y \;\; y^1 \;\; y^2]^T$

The state equation in $n=3$ is:

$$\dot{X} = \begin{bmatrix}y^1\\ y^2\\ y^3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ -a_0 & -a_1 & -a_2\end{bmatrix}\begin{bmatrix}y\\ y^1\\ y^2\end{bmatrix} + \begin{bmatrix}0\\ 0\\ b\end{bmatrix}u(t) \longrightarrow y(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}y\\ y^1\\ y^2\end{bmatrix}$$

Now Generalizing to the n^{th} case.

State $X(t) = [y, y^1, y^2, \cdots y^{n-1}]^T$

$$y^n = -[a_{n-1} y^{n-1} + \cdots + a_1 y^1 + a_0 y] + bu(t)$$

write in the form $\dot{X} = AX(t) + BU(t)$

$$\dot{X}(t) = \begin{bmatrix} y^1 \\ y^2 \\ y^3 \\ \vdots \\ y^{n-1} \\ y^n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & \cdots & & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & & -a_{n-1} \end{bmatrix} \begin{bmatrix} y \\ y^1 \\ y^2 \\ \vdots \\ y^{n-2} \\ y^{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u(t).$$

and finally get $y(t) = CX(t)$

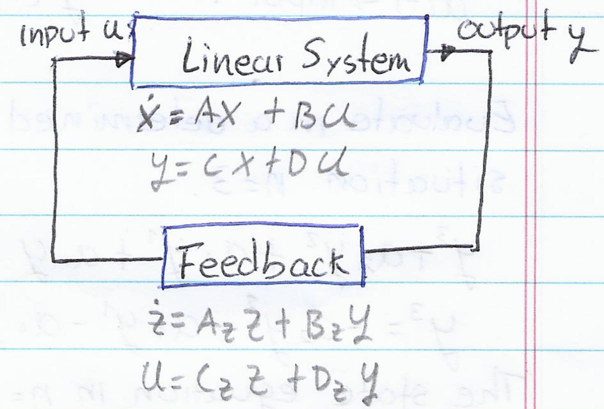
$$y(t) = [1\,0\,0 \cdots 0] \begin{bmatrix} y \\ y^1 \\ \vdots \\ y^{n-1} \end{bmatrix}$$

3. Consider the state equation for a linear-time invariant. System.

$$\dot{X}(t) = A\cancel{X}(t) + B(u(t))$$
$$\dot{X}(t) = AX(t) + BU(t) \qquad X(0) = X_0$$
$$y(t) = CX(t) + DU(t)$$

Suppose we apply output feedback defined by:

$$\dot{z} = A_z z(t) + B_z y(t) \qquad z(0) = z_0$$
$$u(t) = C_z z(t) + D_z y(t).$$



Define the state and write the state equation for the loop system.

Define the state for the closed loop system:

$$S(t) = \begin{bmatrix} x \\ z \end{bmatrix} \quad \dot{S} = \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix}$$

now include the linear system terms in the feedback.

$$\dot{z}(t) = A_z z(t) + B_z [Cx(t) + Du(t)]$$

$$\dot{z}(t) = A_z z(t) + B_z Cx(t) + B_z Du(t) \quad (1)$$

now connect the input to the linear system.

$$u = C_z z + D_z y$$

$$u = C_z z + D_z (Cx + Du) \qquad (2) \quad \boxed{u = (I - D_z D)^{-1} C_z z + (I - D_z D)^{-1} D_z D x}$$

$$u = C_z z + D_z Cx + D_z Du$$

$$u - D_z Du = C_z z + D_z Cx$$

$$(I - D_z D) u = C_z z + D_z Cx$$

introducing u in $\dot{x}(t)$ and $\dot{z}(t)$:

$$\dot{x}(t) = Ax(t) + B[(I - D_z D)^{-1} C_z z + (I - D_z D)^{-1} D_z Cx]$$

$$\rightarrow \dot{x}(t) = (A + B(I - D_z D)^{-1} D_z C) x(t) + B(I - D_z D)^{-1} C_z z(t)$$

$$\dot{z}(t) = A_z z + B_z Cx + B_z D(I - D_z D)^{-1} C_z z + B_z D(I - D_z D)^{-1} D_z Cx$$

$$\rightarrow \dot{z}(t) = (B_z C + B_z D(I - D_z D)^{-1} D_z C) x(t) + (A_z + B_z D(I - D_z D)^{-1} C_z) z(t)$$

integrating to obtain the state equation for the closed loop syst

$$\dot{S}(t) = \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + B(I - D_z D)^{-1} D_z C & B(I - D_z D)^{-1} C_z \\ B_z C + B_z D(I - D_z D)^{-1} D_z C & A_z + B_z D(I - D_z D)^{-1} C_z \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

4. Consider a vector space $\mathbb{V}$

$$\mathbb{V} = \{ x \in \mathbb{R}^5 \mid x_1 = 3x_2 \text{ and } x_3 = 7x_4 \}$$

Find bases

$$X_i = \begin{bmatrix} 3x_2 \\ x_2 \\ 7x_4 \\ x_4 \\ x_5 \end{bmatrix} \quad \bar{X}_1 = \begin{bmatrix} 0 \\ 0 \\ 7 \\ 1 \\ 5 \end{bmatrix} \quad \bar{X}_2 = \begin{bmatrix} 6 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \bar{X}_3 = \begin{bmatrix} 0 \\ 0 \\ 14 \\ 2 \\ 3 \end{bmatrix}$$

Since $x_1 = 3x_2$, $x_3 = 7x_4$, we have the freedom to to choice of x_2, x_4 and x_5, So we have three degrees of freedom this soggest a base to $\mathbb{V}$ of dimension 3.

5. Compute 1-norm, 2-norm, ∞-norm.

a)

$$A = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \qquad \|A\|_1 = \max_j \sum_{i=1}^{2} |a_{ij}| \rightarrow \text{Maximum absolute column Som norm}$$

$$\boxed{\|A\|_1 = 1}$$

Maximum absolute row sum norm

$$\|A\|_\infty = \max_i \sum_{j=1}^{2} |a_{ij}| \Rightarrow \boxed{\|A\|_\infty = 1}$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^TA)} \qquad \det(A^TA - \lambda I) = 0$$

$$A^TA = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \qquad \det\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} \Rightarrow -\lambda(1-\lambda) = 0 \longrightarrow \lambda_1 = 1, \ \lambda_0 = 0$$

$$\boxed{\|A\|_2 = \sqrt{1} = 1}$$

b)

$$A = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \qquad \boxed{\|A\|_1 = 4}, \boxed{\|A\|_\infty = 4}$$

$$\det(A) = 9 - 1 = 8$$

$$A^TA = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = \begin{vmatrix} 10 & 6 \\ 6 & 10 \end{vmatrix} = M$$

Find eigenvalues:

$$\det(M - \lambda I) = 0$$

$$\det\begin{vmatrix} 10-\lambda & 6 \\ 6 & 10-\lambda \end{vmatrix} = 0$$

$$(10-\lambda)(10-\lambda) - 36 = 0$$

$$100 - 20\lambda + \lambda^2 - 36 = 0$$

$$\lambda^2 - 20\lambda + 64 = 0$$

$$\lambda = \frac{20 \pm \sqrt{20^2 - 4(64)}}{2} \rightarrow \boxed{\lambda_1 = 16} \quad \lambda_2 = 4$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^TA)} = \sqrt{16} = 4$$

$$\boxed{\|A\|_2 = 4}$$

6. Using the definition of the induced matrix form
Verify $\|AB\|_2 \leq \|A\|_2 \|B\|_2$.

Given the matric $C \in \mathbb{R}^{m\times n}$ and the induced defined matrix norm is:

$$\|C\|_2 = \max \frac{\|Cx\|_2}{\|x\|_2} \rightarrow \boxed{\|Cx\|_2 \leq \|C\|_2 \|x\|_2}$$

then $\|AB\|_2 = \max \frac{\|(AB)x\|_2}{\|x\|_2} = \max\limits_{\|x\|=1} \|(AB)x\|_2$

$$\rightarrow \max_{\|x\|_2=1} \|(AB)x\|_2 = \max_{\|x\|_2=1} \|A(Bx)\|_2 \leq \max_{\|x\|_2=1} \|A\|_2 \|Bx\|_2 \leq \max_{\|x\|_2=1} \|A\|_2 \|B\|_2 \|x\|_2$$

$$\leq \max_{\|x\|_2=1} \|A\|_2 \|B\|_2 \|x\|_2 = \max_{x} \frac{\|A\|_2 \|B\|_2 \|x\|_2}{\|x\|_2} = \|A\|_2 \|B\|_2$$

Finally $\boxed{\|AB\|_2 \leq \|A\|_2 \|B\|_2}$

7. Using matrix eigendecomposition, compute matrix exponential e^M

b) $M = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix}$

1. Compute eigenvalues $\det(M - \lambda I) = 0$

$$\det \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & -2 & 3-\lambda \end{vmatrix} = (1-\lambda)\left[(2-\lambda)(3-\lambda)\right] - 2(0) + 0$$
$$= (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

the eigenvalues of M are: $\boxed{\lambda_1 = 1}$ $\boxed{\lambda_2 = 2}$ $\boxed{\lambda_3 = 3}$

2. Calculate the eigenvectors of M

$\boxed{\lambda = 1} \rightarrow (M - \lambda I)V = 0 \rightarrow \begin{vmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 2 \end{vmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$

$$\left\{\begin{aligned} 2V_2 &= 0 \\ V_2 &= 0 \\ V_1 - 2V_2 + 2V_3 &= 0 \end{aligned}\right\}$$

$\boxed{V_1 = -2V_3}$

$\boxed{V_2 = 0}$

For $\lambda_1 = 1$ $V_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 2 \rightarrow \begin{vmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{vmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$ $\quad -V_1 + 2V_2 = 0 \quad V_1 = 2V_2$

$V_1 - 2V_2 + V_3 = 0 \quad V_3 = 0$

for $\lambda = 2$ $V_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 3$ $\begin{vmatrix} -2 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & -2 & 0 \end{vmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$ $\quad -2V_1 + V_2 = 0$ $\quad V_2 = 0$

$V_2 = 0$ $\quad V_1 = 0$

$V_1 - 2V_2 = 0$

For $\lambda_3 = 3 \rightarrow V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

with the vectors $V_1, V_2\ V_3 \rightarrow V_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ $V_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

we could find the form $M = V \Lambda V^{-1}$

where

$$V = \begin{bmatrix} -2 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad V^{-1} = \begin{bmatrix} -2 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Then using the decomposition:

$$M = \begin{vmatrix} -2 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} \begin{vmatrix} -2 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}^{-1}$$

Then, the matrix exponential e^{Mt} is:

$$e^{Mt} = \begin{vmatrix} -2 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \begin{vmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{vmatrix} \begin{vmatrix} -2 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}^{-1}$$

$$= \begin{vmatrix} -2e^t & 2e^{2t} & 0 \\ 0 & e^{2t} & 0 \\ e^t & 0 & e^{3t} \end{vmatrix} \begin{vmatrix} -2 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}^{-1}$$

$$\det \begin{vmatrix} -2 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -2(1) - 2(0) + 0 = -2$$

$$= \begin{vmatrix} -2e^t & 2e^{2t} & 0 \\ 0 & e^{2t} & 0 \\ e^t & 0 & e^{3t} \end{vmatrix} \begin{vmatrix} -1/2 & 1 & 0 \\ 0 & 1 & 0 \\ 1/2 & -1 & 1 \end{vmatrix}$$

$$A^T = \begin{vmatrix} -2 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad A^{-1} = \frac{1}{-2} \begin{vmatrix} 1 & -2 & 0 \\ 0 & -2 & 0 \\ -1 & 2 & -2 \end{vmatrix}$$

$$A^{-1} = \begin{vmatrix} -1/2 & 1 & 0 \\ 0 & 1 & 0 \\ 1/2 & -1 & 1 \end{vmatrix}$$

$$\boxed{e^{Mt} = \begin{vmatrix} e^t & -2e^t + 2e^{2t} & 0 \\ 0 & e^{2t} & 0 \\ \frac{e^{3t} - e^t}{2} & e^t - e^{3t} & e^{3t} \end{vmatrix}}$$

8) Given a diagonal matrix Λ

$$\Lambda = \begin{vmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & \lambda_n \end{vmatrix}$$

Verify that the matrix exponential is:

$$e^{\Lambda} = \begin{vmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & \cdots & e^{\lambda_n} \end{vmatrix}$$

Compute Matrix exponential.

by taylor expansion we have:

$$e^{a} = \sum_{K=0}^{\infty} \frac{(a)^K}{K!} \quad \text{and} \quad \underbrace{\Lambda.\Lambda.\cdots\Lambda}_{K\text{times}} = \Lambda^K = \begin{vmatrix} \lambda_1^K & \cdots & 0 \\ 0 & \lambda_2^K & \vdots \\ \vdots & & \ddots \\ 0 & \cdots & \lambda_n^K \end{vmatrix}$$

Since Λ is a diagonal matrix

$$e^{\Lambda} = \sum_{K=0}^{\infty} \frac{(\Lambda)^K}{K!} = \sum_{K=0}^{\infty} \frac{1}{K!} \Lambda^K$$

$$e^{\Lambda} = \sum_{K=0}^{\infty} \frac{1}{K!} \begin{bmatrix} \lambda_1^K & 0 & \cdots & 0 \\ 0 & \lambda_2^K & & \\ \vdots & & & \\ 0 & \cdots & \cdots & \lambda_n^K \end{bmatrix} = \begin{bmatrix} \sum_{K=0}^{\infty} \frac{\lambda_1^K}{K!} & 0 & \cdots & 0 \\ 0 & \sum_{K=0}^{\infty} \frac{\lambda_2^K}{K!} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \sum_{K=0}^{\infty} \frac{\lambda_n^K}{K!} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & 0 & \cdots 0 \\ 0 & 0 & e^{\lambda_3} & \\ \vdots & & & \ddots \\ 0 & \cdots & \cdots & e^{\lambda_n} \end{bmatrix}$$

then:

$$e^{\Lambda} = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & & \\ \vdots & & \ddots & \\ 0 & & & e^{\lambda_n} \end{bmatrix}$$

9. $M = \begin{vmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{vmatrix}$ M_{11}, M_{12}, M_{22} are square matrices.

Identify the diagonal blocks of e^M in terms of M_{11}, M_{12}, M_{22} and show that e^M is block upper right

By definition we have $e^A = \sum_{K=0}^{\infty} \frac{(A)^K}{K!}$

then:

$$\boxed{e^M = \sum_{K=0}^{\infty} \frac{M^K}{K!}} \qquad M.M = \begin{vmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{vmatrix} \begin{vmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{vmatrix} = \begin{vmatrix} M_{11}^2 & M_{11}M_{12} + M_{12}M_{22} \\ 0 & M_{22}^2 \end{vmatrix}$$

$$M^3 = (M.M).M = \begin{vmatrix} M_{11}^2 & M_{11}M_{12} + M_{12}M_{22} \\ 0 & M_{22}^2 \end{vmatrix} \begin{vmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{vmatrix} = \begin{vmatrix} M_{11}^3 & M_{11}^2 M_{12} + M_{11}M_{12}M_{22} + M_{12}M_{22}^2 \\ 0 & M_{22}^3 \end{vmatrix}$$

$$M^4 = \begin{vmatrix} M_{11}^4 & M_{11}^3 M_{12} + M_{11}^2 M_{12} M_{22} + M_{11} M_{12} M_{22}^2 + M_{12} M_{22}^3 \\ 0 & M_{22}^4 \end{vmatrix}$$

$$M^5 = \begin{vmatrix} M_{11}^5 & M_{11}^4 M_{12} + M_{11}^3 M_{12} M_{22} + M_{11}^2 M_{12} M_{22}^2 + M_{11} M_{12} M_{22}^3 + M_{12} M_{22}^4 \\ 0 & M_{22}^5 \end{vmatrix}$$

$$M5 = \begin{vmatrix} M_{11}^5 & M_{11}^4 M_{12} M_{22}^0 + M_{11}^3 M_{12} M_{22}^1 + M_{11}^2 M_{12} M_{22}^2 + M_{11} M_{12} M_{22}^3 + M_{11}^0 M_{12} M_{22}^4 \\ 0 & M_{22}^5 \end{vmatrix}$$

in general $K>0$

$$M^K = \begin{vmatrix} M_{11}^K & \sum_{i=1}^{K} M_{11}^{K-i} M_{12} M_{22}^{i-1} \\ 0 & M_{22}^K \end{vmatrix}$$

once we have defined M^K lets define e^M

$$e^M = \sum_{K=0}^{\infty} \frac{M^K}{K!} = \sum_{K=0}^{\infty} \begin{vmatrix} M_{11}^K/K! & \left[\sum_{i=1}^{K} M_{11}^{K-i} M_{12} M_{22}^{i-1}\right]/K! \\ 0 & M_{22}^K/K! \end{vmatrix}$$

$$e^M = \begin{vmatrix} \sum_{K}^{\infty} M_{11}^K/K! & \sum_{K=0}^{\infty} \sum_{i=1}^{K} \frac{M_{11}^{K-i} M_{12} M_{22}^{i-1}}{K!} \\ 0 & \sum_{K}^{\infty} \frac{M_{22}^K}{K!} \end{vmatrix} = \begin{vmatrix} e^{M_{11}} & \sum_{K=0}^{\infty} \sum_{i=1}^{K} \frac{\left[M_{11}^{K-i} M_{12} M_{22}^{i-1}\right]}{K!} \\ 0 & e^{M_{22}} \end{vmatrix}$$

As a result, the diagonal elements of e^M are:

$$e^{M_{11}} = \sum_{K=0}^{\infty} \frac{M_{11}^K}{K!} \quad \text{and} \quad e^{M_{22}} = \sum_{K=0}^{\infty} \frac{M_{22}^K}{K!}$$

and e^M is block upper triangular with the following Result.

$$e^M = \begin{vmatrix} e^{M_{11}} & \sum_{K=0}^{\infty} \sum_{i=1}^{K} M_{11}^{K-i} M_{12} M_{22}^{i-1} \\ 0 & e^{M_{22}} \end{vmatrix}$$

10. For the state equation given by:

$$\dot{X}(t) = AX(t) + BU(t)$$

Verify that:

$$X(t) = e^{At} X(0) + \int_{\tau=0}^{t} e^{A(t-\tau)} BU(\tau) d\tau$$

is a solution of the equation.

To verify that $X(t)$ is a solution of the state equation, we will have to differentiate $X(t)$ to satisfy the equation $\dot{X}(t)$

then:

$$\frac{dX(t)}{dt} = \frac{d}{dt}\left[e^{At} X(0) + \int_{\tau=0}^{t} e^{A(t-\tau)} BU(\tau) d\tau\right]$$

$$\dot{X} = Ae^{At} X(0) + \frac{d}{dt}\left[\int_{\tau=0}^{t} e^{A(t-\tau)} BU(\tau) d\tau\right]$$

$$\dot{X} = Ae^{At} X(0) + \int_{0}^{t} Ae^{A(t-\tau)} BU(\tau) d\tau + e^{A(t-t)} BU(t)$$

$$\dot{X} = Ae^{At} X(0) + A\int_{0}^{t} e^{A(t-\tau)} BU(\tau) d\tau + BU(t)$$

$$\dot{X} = A\left[e^{At} X(0) + \int_{0}^{t} e^{A(t-\tau)} BU(\tau) d\tau\right] + BU(t)$$

$\boxed{\dot{X}(t) = AX(t) + BU(t)}$ → The defined $X(t)$ is a solution to the State equation $\dot{X}(t)$