Homework #3

(due: March 29)

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(P1) Recall the state equation (kinematic model) of the racecar.

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = \frac{v}{L}\tan u$$

Rewrite the state equation with $(x, y, \cos\theta, \sin\theta)$ as a new state, and using the provided Pybullet exercise, implement the MPC. Submit your answer for the state equation derivation and completed Pybullet exercise.

For the MPC implementation, you will first need to write a nonlinear optimization formulation (as we learned in the class). Then use the following to complete the exercise sections in `race_mpc_se2.py`.

$$\texttt{cost} \leftarrow f(\mathrm{x}) = \sum_{\tau=t}^{\tau=t+T_{\text{MPC}}} \left(w_x(x(\tau)-x_{\text{ref}}(\tau))^2 + w_y(y(\tau)-y_{\text{ref}}(\tau))^2 + w_{c\theta}(c\theta(\tau)-c\theta_{\text{ref}}(\tau))^2 + w_{s\theta}(s\theta(\tau)-s\theta_{\text{ref}}(\tau))^2 + w_v v^2(\tau) + w_u u^2(\tau)\right)$$

Solution:

MPC Design as Nonlinear Optimization

1. Time discretization, evaluate the state and input at $\tau = t, t+h \cdots t+T_{MPC}$

we have that the dynamics of the system are:

$$\left.\begin{aligned}\dot{x} &= v\cos\theta\\ \dot{y} &= v\sin\theta\\ \dot{\theta} &= \frac{v}{L}\tan u\end{aligned}\right\} \text{ define } \quad \frac{d}{dt}\cos\theta = -\dot{\theta}\sin\theta \qquad \frac{d}{dt}\sin\theta = \dot{\theta}\cos\theta \quad \rightarrow \text{where } \dot{\theta} = \frac{v}{L}\tan u$$

→ new formulation with new state $(x, y, \cos\theta, \sin\theta)$

$$C_\theta = \cos\theta \qquad S_\theta = \sin\theta \quad \rightarrow \quad \boxed{\begin{aligned}\dot{x} &= vC_\theta\\ \dot{y} &= vS_\theta\\ \dot{C}_\theta &= -wS_\theta\\ \dot{S}_\theta &= wC_\theta\end{aligned}} \qquad \boxed{\dot{\theta} = w = \frac{v}{L}\tan u}$$

MPC: Find $v(\tau), u(\tau)$, $\tau = t, t+h, \cdots, t+T_{MPC}$

Minimize $$f(\mathrm{x}) = \sum_{\tau=t}^{\tau=t+T_{\mathrm{MPC}}} \left(w_x(x(\tau) - x_{\mathrm{ref}}(\tau))^2 + w_y(y(\tau) - y_{\mathrm{ref}}(\tau))^2 + w_{c\theta}(c\theta(\tau) - c\theta_{\mathrm{ref}}(\tau))^2 \right.$$
$$\left. + w_{s\theta}(s\theta(\tau) - s\theta_{\mathrm{ref}}(\tau))^2 + w_v v^2(\tau) + w_u u^2(\tau) \right)$$

Subject to:
$$\dot{x} = v c_\theta$$
$$\dot{y} = v s_\theta$$
$$\dot{c}_\theta = -\omega s_\theta$$
$$\dot{s}_\theta = \omega c_\theta, \quad \omega = \frac{v}{L}\tan u, \quad \tau = t, t+h, \cdots t+T_{MPC} - h$$

2. Discretize the state equation: For $\tau = t, t+h \,..\, t+T_{MPC} - h$

Approximate $\rightarrow$ $\dot{x} = \dfrac{x(t+h) - x(t)}{h}$

$$\boxed{\begin{aligned} \dot{x} &= v c_\theta \\ \dot{y} &= v s_\theta \\ \dot{c}_\theta &= -\omega s_\theta \\ \dot{s}_\theta &= \omega c_\theta \end{aligned}} \longrightarrow \boxed{\begin{aligned} x(\tau+h) &= x(\tau) + h v(\tau) c_\theta(\tau) \\ y(\tau+h) &= y(\tau) + h v(\tau) s_\theta(\tau) \\ c_\theta(\tau+h) &= c_\theta(\tau) - h\omega s_\theta(\tau) \\ s_\theta(\tau+h) &= s_\theta(\tau) + h\omega c_\theta(\tau) \end{aligned}} \quad \omega = \frac{v(\tau)}{L}\tan u(\tau)$$

3. Define the optimization Variable

$$\mathrm{x} = (x(t), y(t), c_\theta(t), s_\theta(t), v(t), u(t), x(t+h), y(t+h), c_\theta(t+h), s_\theta(t+h), v(t+h), u(t+h),$$
$$\ldots, x(t+T_{MPC}), y(t+T_{MPC}), c_\theta(t+T_{MPC}), s_\theta(t+T_{MPC}), v(t+T_{MPC}), u(t+T_{MPC}))$$

then, we have:

MPC: Find $v(\tau), u(\tau)$, $\tau = t, t+h, \cdots, t+T_{MPC}$

Minimize $$f(\mathrm{x}) = \sum_{\tau=t}^{\tau=t+T_{\mathrm{MPC}}} \left(w_x(x(\tau) - x_{\mathrm{ref}}(\tau))^2 + w_y(y(\tau) - y_{\mathrm{ref}}(\tau))^2 + w_{c\theta}(c\theta(\tau) - c\theta_{\mathrm{ref}}(\tau))^2 \right.$$
$$\left. + w_{s\theta}(s\theta(\tau) - s\theta_{\mathrm{ref}}(\tau))^2 + w_v v^2(\tau) + w_u u^2(\tau) \right)$$

Subject to:
$$x(\tau+h) = x(\tau) + h v(\tau) c_\theta(\tau)$$
$$y(\tau+h) = y(\tau) + h v(\tau) s_\theta(\tau) \qquad \omega = \frac{v(\tau)}{L}\tan u(\tau)$$
$$c_\theta(\tau+h) = c_\theta(\tau) - h\omega s_\theta(\tau)$$
$$s_\theta(\tau+h) = s_\theta(\tau) + h\omega c_\theta(\tau) \qquad \tau = t, t+h, \cdots t+T_{MPC} - h$$

MPC formulation as non linear optimization.

MPC design as nonlinear optimization: Find x minimizing

$$\text{Minimize } f(\mathrm{x}) = \sum_{\tau=t}^{\tau=t+T_{\mathrm{MPC}}} \left(w_x(x(\tau)-x_{\mathrm{ref}}(\tau))^2 + w_y(y(\tau)-y_{\mathrm{ref}}(\tau))^2 + w_{c\theta}(c\theta(\tau)-c\theta_{\mathrm{ref}}(\tau))^2 \right.$$
$$\left. + w_{s\theta}(s\theta(\tau)-s\theta_{\mathrm{ref}}(\tau))^2 + w_v v^2(\tau) + w_u u^2(\tau)\right)$$

Subject to:

$$x(\tau+h) - x(\tau) - h v(\tau) C_\theta(\tau) \le 0 \rightarrow g_x^t(x) \le 0$$
$$-x(\tau+h) + x(\tau) + h v(\tau) C_\theta(\tau) \le 0 \rightarrow -g_x^t(x) \le 0$$
$$y(\tau+h) - y(\tau) - h v(\tau) S_\theta(\tau) \le 0 \rightarrow g_y^t(x) \le 0$$
$$-y(\tau+h) + y(\tau) + h v(\tau) S_\theta(\tau) \le 0 \rightarrow -g_y^t(x) \le 0$$
$$C_\theta(\tau+h) - C_\theta(\tau) + h\omega(\tau) S_\theta(\tau) \le 0 \rightarrow g_{c_\theta}^t(x) \le 0$$
$$-C_\theta(\tau+h) + C_\theta(\tau) - h\omega(\tau) S_\theta(\tau) \le 0 \rightarrow -g_{c_\theta}^t(x) \le 0$$
$$S_\theta(\tau+h) - S_\theta(\tau) - h\omega(\tau) C_\theta(\tau) \le 0 \rightarrow g_{s_\theta}^t(x) \le 0$$
$$-S_\theta(\tau+h) + S_\theta(\tau) + h\omega(\tau) C_\theta(\tau) \le 0 \rightarrow -g_{s_\theta}^t(x) \le 0$$

$$\vdots \qquad\qquad \vdots$$

$$x(\tau+T_{MPC}) - x(\tau+T_{MPC}-h) - h v(\tau+T_{MPC}-h) C_\theta(\tau+T_{MPC}-h) \le 0 \rightarrow g_x^{t+T_{MPC}-h}(x) \le 0$$
$$-x(\tau+T_{MPC}) + x(\tau+T_{MPC}-h) + h v(\tau+T_{MPC}-h) C_\theta(\tau+T_{MPC}-h) \le 0 \rightarrow -g_x^{t+T_{MPC}-h}(x) \le 0$$
$$y(\tau+T_{MPC}) - y(\tau+T_{MPC}-h) - h v(\tau+T_{MPC}-h) S_\theta(\tau+T_{MPC}-h) \le 0 \rightarrow g_y^{t+T_{MPC}-h}(x) \le 0$$
$$-y(\tau+T_{MPC}) + y(\tau+T_{MPC}-h) + h v(\tau+T_{MPC}-h) S_\theta(\tau+T_{MPC}-h) \le 0 \rightarrow -g_y^{t+T_{MPC}-h}(x) \le 0$$
$$C_\theta(\tau+T_{MPC}) - C_\theta(\tau+T_{MPC}-h) + h\omega(\tau+T_{MPC}-h) S_\theta(\tau+T_{MPC}-h) \le 0 \rightarrow g_{C_\theta}^{t+T_{MPC}-h}(x) \le 0$$
$$-C_\theta(\tau+T_{MPC}) + C_\theta(\tau+T_{MPC}-h) - h\omega(\tau+T_{MPC}-h) S_\theta(\tau+T_{MPC}-h) \le 0 \rightarrow -g_{C_\theta}^{t+T_{MPC}-h}(x) \le 0$$
$$S_\theta(\tau+T_{MPC}) - S_\theta(\tau+T_{MPC}-h) - h\omega(\tau+T_{MPC}-h) C_\theta(\tau+T_{MPC}-h) \le 0 \rightarrow g_{S_\theta}^{t+T_{MPC}-h}(x) \le 0$$
$$-S_\theta(\tau+T_{MPC}) + S_\theta(\tau+T_{MPC}-h) + h\omega(\tau+T_{MPC}-h) C_\theta(\tau+T_{MPC}-h) \le 0 \rightarrow -g_{S_\theta}^{t+T_{MPC}-h}(x) \le 0$$

Where $\rightarrow$ $\omega(\tau) = \dfrac{V(\tau)}{L} \tan u(\tau)$, $\quad \omega(\tau+T_{MPC}-h) = \dfrac{V(\tau+T_{MPC}-h)}{L} \tan u(\tau+T_{MPC}-h)$

and $\rightarrow$ $x = \big(x(t), y(t), C_\theta(t), S_\theta(t), V(t), u(t), x(t+h), y(t+h), C_\theta(t+h), S_\theta(t+h), V(t+h), u(t+h,$
$\quad \ldots, x(t+T_{MPC}), y(t+T_{MPC}), C_\theta(t+T_{MPC}), S_\theta(t+T_{MPC}), V(t+T_{MPC}), u(t+T_{MPC})\big)$

Compute gradient, hessian, and Jacobian.

gradient_cost_x $\leftarrow \frac{\partial f}{\partial x} = 2w_x(x - x_{ref})$

gradient_cost_y $\leftarrow \frac{\partial f}{\partial y} = 2w_y(y - y_{ref})$

gradient_cost_ctheta $\leftarrow \frac{\partial f}{\partial c_\theta} = 2w_{c\theta}(c_\theta - c_{\theta ref})$

gradient_cost_stheta $\leftarrow \frac{\partial f}{\partial s_\theta} = 2w_{s\theta}(s_\theta - s_{\theta ref})$

gradient_cost_v $\leftarrow \frac{\partial f}{\partial v} = 2w_v v$

gradient_cost_u $\leftarrow \frac{\partial f}{\partial u} = 2w_u u$

hessian_cost_xx $\leftarrow \frac{\partial^2 f}{\partial x^2} = 2w_x$

hessian_cost_yy $\leftarrow \frac{\partial^2 f}{\partial y^2} = 2w_y$

hessian_cost_ctct $\leftarrow \frac{\partial^2 f}{\partial c_\theta^2} = 2w_{c\theta}$

hessian_cost_stst $\leftarrow \frac{\partial^2 f}{\partial s_\theta^2} = 2w_{s\theta}$

hessian_cost_vv $\leftarrow \frac{\partial^2 f}{\partial v^2} = 2w_v$

hessian_cost_uu $\leftarrow \frac{\partial^2 f}{\partial u^2} = 2w_u$

jacobian_g_x_t $\leftarrow (1, -1, -hv(\tau), -hc_\theta(\tau))$

jacobian_g_y_t $\leftarrow (1, -1, -hv(\tau), -hs_\theta(\tau))$

jacobian_g_ctheta_t $\leftarrow (1, -1, hw(\tau), \frac{h}{L}\tan u(t) s_\theta(t), \frac{h}{L} v(\tau)\sec^2 u(\tau) s_\theta(\tau))$

jacobian_g_stheta_t $\leftarrow (1, -1, -hw(\tau), -\frac{h}{L}\tan u(t) c_\theta(\tau), -\frac{h}{L} v(\tau)\sec^2 u(\tau) c_\theta(\tau))$

where $w(\tau) = \frac{v(\tau)}{L}\tan u(\tau)$

hessian_g_x_t_ctv $\leftarrow \frac{\partial^2 g_x^t(\mathrm{x})}{\partial c\theta(t)\partial v(t)} \to -h$

hessian_g_y_t_stv $\leftarrow \frac{\partial^2 g_y^t(\mathrm{x})}{\partial s\theta(t)\partial v(t)} \to -h$

hessian_g_ctheta_t_stv $\leftarrow \frac{\partial^2 g_{c\theta}^t(\mathrm{x})}{\partial s\theta(t)\partial v(t)} \to \frac{h}{L}\tan u(\tau)$

hessian_g_ctheta_t_stu $\leftarrow \frac{\partial^2 g_{c\theta}^t(\mathrm{x})}{\partial s\theta(t)\partial u(t)} \to \frac{h}{L} v(t)\sec^2 u(\tau)$

hessian_g_ctheta_t_vu $\leftarrow \frac{\partial^2 g_{c\theta}^t(\mathrm{x})}{\partial v(t)\partial u(t)} \to \frac{h}{L}\sec^2 u(\tau) s_\theta(\tau)$

hessian_g_ctheta_t_uu $\leftarrow \frac{\partial^2 g_{c\theta}^t(\mathrm{x})}{\partial u(t)\partial u(t)} \to \frac{2h}{L} v(\tau)\sec^2 u(\tau)\tan u(\tau) s_\theta(\tau)$

hessian_g_stheta_t_ctv $\leftarrow \frac{\partial^2 g_{s\theta}^t(\mathrm{x})}{\partial c\theta(t)\partial v(t)} \to -\frac{h}{L}\tan u(\tau)$

hessian_g_stheta_t_ctu $\leftarrow \frac{\partial^2 g_{s\theta}^t(\mathrm{x})}{\partial c\theta(t)\partial u(t)} \to -\frac{h}{L} v(t)\sec^2 u(\tau)$

hessian_g_stheta_t_vu $\leftarrow \frac{\partial^2 g_{s\theta}^t(\mathrm{x})}{\partial v(t)\partial u(t)} \to -\frac{h}{L}\sec^2 u(t) c_\theta(t)$

hessian_g_stheta_t_uu $\leftarrow \frac{\partial^2 g_{s\theta}^t(\mathrm{x})}{\partial u(t)\partial u(t)} \to -\frac{2h}{L} v(t)\sec^2 u(t)\tan u(\tau) c_\theta(\tau)$