Exercise 1

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1 EECS 491: Probabilistic Graphical Models Assignment 4

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2 Exercise 1

In this notebook we will implement the EM algorithm to tune the means of a 2D Gaussian Mixture Model with fixed covariance matrices.

First let's load up our data and import the python packages we will be using:

```
In [2]: import csv, copy, gzip, pickle

        import numpy as np
        import matplotlib.pyplot as plt

        from scipy.stats import chi2

        %matplotlib inline
```

```
In [3]: with open('data/faithful.txt', 'rt') as csvfile:
            dataReader = csv.reader(csvfile, delimiter=' ')
            # initialize an empty array
            data = []
            for row in dataReader:
                data.append(np.array(row).astype(np.float))
            # convert data into a numpy array
            data = np.asarray(data)
```

Now let's initialize our Gaussian Mixture Model. First let's set how many Gaussian distributions we will have in our mixture:

```
In [4]: ngmm = 2 # quantity of Gaussian Mixture Model
```

Now let's initialize the mean and covariance for each Gaussian distribution. The covariance matrix must be positive-definite.

```
In [5]: mu = np.asarray([[4, 80],
                         [8, 80]]
                       ).astype('float')

        sigma = np.asarray([[[1, 0],
                             [0, 3]],
                            [[0.5, 0.2],
                             [0.2, 0.6]]]
                          ).astype('float')
```

The following function checks covariance matrices for validity if we decide to change up the covariance matrix.

```
In [6]: def covmatIsLegal(sigma):
            for covmat in sigma:
                if not(np.allclose(covmat, covmat.T)) or np.any(np.linalg.eigvals(covmat) <= 0)
                    return False
            return True

        print("Convariance Matrices are Legal? : %r" % covmatIsLegal(sigma))

Convariance Matrices are Legal? : True
```

With all of our Gaussian distributions set up, let's create a dictionary that will store all of our Gaussians.

```
In [9]: gmm = [{'mean': mu[m], 'covariance': sigma[m], 'prior': 1.0/ngmm} for m in range(ngmm)]
```

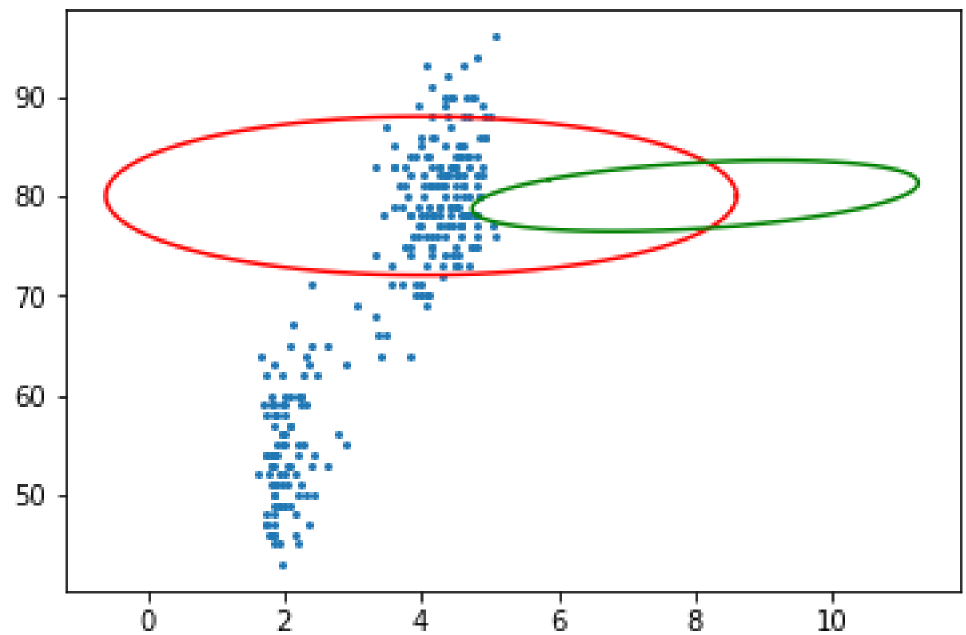
The following function will plot our Gaussian Mixture Model. It is taken from the demo.

```
In [10]: def plotGaussianModel2D(mu, sigma, pltopt='k'):
             if sigma.any():
                 # calculate ellipse constants
                 c = chi2.ppf(0.9, 2) # use confidence interval 0.9
                 # get eigen vector and eigen values
                 eigenValue, eigenVector = np.linalg.eig(sigma)
                 # calculate points on ellipse
                 t = np.linspace(0, 2*np.pi, 100) # draw 100 points
                 u = [np.cos(t), np.sin(t)]
                 w = c * eigenVector.dot(np.diag(np.sqrt(eigenValue)).dot(u))
                 z = w.T + mu
             else:
                 z = mu
             # plot ellipse by connecting sample points on curve
             plt.plot(z[:,0], z[:,1], pltopt)

         def colorPicker(index):
             colors = 'rgbcmyk'
```

```
        return colors[np.remainder(index, len(colors))]

    def gmmplot(data, gmm):
        # plot data points
        plt.scatter(data[:, 0], data[:, 1], s=4)
        # plot Gaussian model
        color = 'rgb'
        for index, model in enumerate(gmm):
            plotGaussianModel2D(model['mean'], model['covariance'], colorPicker(index))
```

Let's test it:

```
In [11]: gmmplot(data, gmm)
```



Now let's optimize our GMM. As stated above we will be using the Expectation-Maximization algorithm to tune our mean.

In our Expectation step we will compute:

$$p_{n,k} = p(c_k \mid x^{(n)}, \theta_{1:K}) = \frac{p(x^{(n)} \mid c_k, \theta_k)p(c_k)}{\sum_k p(x^{(n)} \mid c_k, \theta_k)p(c_k)}$$

where

$$p(x^{(n)} \mid c_k, \theta_k) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} exp\{-\frac{1}{2}(\mathbf{x} - ^-)^T\Sigma^{-1}(\mathbf{x} - ^-)\}$$

and the prior is given in the parameters.

In our Maximization step we will compute the mean for the next cycle using:

$$p_k = \frac{\sum_n p_{n,k}}{n}$$

$$\mu_k \leftarrow \frac{\sum_n p_{n,k} x^{(n)}}{n p_k}$$

In [12]:
```
def expectation(data, gmmcp):
    rows,dims = data.shape
    cols = len(gmmcp)
    posterior = np.zeros((rows,cols))
    for row in range(rows):
        for col in range(cols):
            gmm = gmmcp[col]
            mean = gmm['mean']
            covariance = gmm['covariance']
            prior = gmm['prior']
            num = multivariate_gaussian(dims, mean, covariance, data[row]) * prior
            denom = 0
            for j in range(cols):
                gmm = gmmcp[j]
                mean = gmm['mean']
                covariance = gmm['covariance']
                prior = gmm['prior']
                g = multivariate_gaussian(dims, mean, covariance, data[row])
                denom += g * prior
            posterior[row, col] = num/denom
    return posterior
```

In [16]:
```
def multivariate_gaussian(dims, mean, covariance, example):
    A = 1/((2*np.pi)**(dims/2)*np.linalg.det(covariance)**.5)
    B = np.exp(-.5*(example - mean).T.dot(np.linalg.inv(covariance).dot((example - mea
    return A * B
```

In [17]:
```
def maximization_mean(posterior, data, gmmcp):
    update_gmm = gmmcp
    pk = np.zeros(2)
    rows, cols = posterior.shape
    for col in range(cols):
        for row in range(rows):
            pk[col] += posterior[row, col]
    for col in range(cols):
        gmm = update_gmm[col]
        mean = 0
        for row in range(rows):
            mean += posterior[row,col]*(data[row])/pk[col]
        gmm['mean'] = mean
    return gmmcp
```

Now let's run our algorithm:

```
In [18]: # make a true copy of our model
         gmmcp = copy.deepcopy(gmm)

         # create figure
         plt.figure(figsize=(16, 8))
         # improve model with EM-Algorithm
         for i in range(5):
             # plot current status
             plt.subplot(231 + i)
             gmmplot(data, gmmcp)
             # excute EM-Algorithm
             for j in range(10):
                 posterior = expectation(data, gmmcp)
                 gmmcp = maximization_mean(posterior, data, gmmcp)
         # plot final status
         plt.subplot(236)
         gmmplot(data,gmmcp)
```