MPCMath User Manual

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Contents

Chapter 1

Introduction

MPCMath is a mathematical library for real time implementation of process identification, model predictive control and other optimization tasks

Introduction to *MPCMath*

MPCMath is a library of C#/.NET routines designed for real time implementation of Process Identification packages, Model Predictive Control and Optimizations tasks. The foundation of *MPCMath* is a comprehensive implementation of Matrix and Vector calculus.

Development and test of mathematical algorithms are often based on interactive environments as Matlab or Python providing a high level mathematical language and efficient debugging facilities with high productivity. These packages are not suited for on-line applications running continuously controlling chemical plants or embedded systems where the controllers are implemented on small processors. During the recent years Microsoft C# language combined with the very comprehensive .NET libraries, has been a very popular platform for development of client and server solutions. *MPCMath* provides a set off tools enabling the programmers to implement process control task, MPC controllers and optimization tasks on the .NET platform providing intelligent objects and functionality enabling high productivity and quality solutions.

Many of *MPCMath* 's functions are based on Intel Math Kernel Library Intel a highly optimized implementation of the public domain computing packages BLAS (Basic Linear Algebra Subroutines) and LAPCK (Linear Algebra Package). This gives *MPCMath* a boost in performance resulting more than an order of magnitude increased computational speed.

MPCMath are aimed at organizations and companies developing equipment for process control, having programmers mastering C# and object oriented programming. *MPCMath* is callable from any .NET language as C# , Visual-Basic.NET and F#.

This chapter gives a walk through how to program tasks related to MPC control in *MPCMath* , starting with an introduction to the basic Vector and Matrix objects.

1.1 Vectors

Vector functionality is implemented in the Vector class. A simple example:

```
console.WriteLine("\MPCMath Introduction");
console.WriteLine();

// Vector definition
Vector x = new Vector(5, Math.PI);
Vector y = Vector.Random(5, -100.0, 1000.0);
Vector z;
z = x + y;
Vector.Show("x", x);
Vector.Show("y", y);
Vector.Show("z", z);
```

The console.Writeline writes a text string on the *MPCMath* console. The next statements defines a vector x with 5 elements all having the value pi = 3.141, a vector y with 5 elements having random values between -100.0 and 100.0. The fourth statement adds vector x and y storing the result in Vector z. The Vector.Show statements displays the three vectors in a *MPCMath* console:

```
x Vector
    3,1416    3,1416    3,1416    3,1416    3,1416
y Vector
  732,8023   51,3077  691,7185  895,7004  672,5596
z Vector
  735,9439   54,4493  694,8601  898,8420  675,7012
```

Vector functionality includes Add , Subtract, Multiply, Clone, Negate, Equal, Normalize, OutherProduct, Sort, Sum and other functions. Vectors support the operators +, - ,* as shown in the example above. Many functions are also implemented as methods

```
z.Add(x);
```

where the vector z is overwritten with z + x

1.2 Matrices

Matrices are implemented using the Matrix class:

```
// Matrix definition
Matrix A = new Matrix(MatrixForm.General, 3, 5);
Matrix B = Matrix.Random(3, 5, -100.0, 100.0);
Matrix C;

A[2,3] = 25.0;
C = A + B;
Matrix.Show("A", A);
Matrix.Show("B", B);
Matrix.Show("C", C);
```

These statements defines matrix A with 3 rows and 5 columns. Matrix B is defined as a 3 rows 5 columns matrix with random elements in the range -100.0 to 100.0. Element [2,3] of A is set to 25.0, and matrix C is the sum of matrix A and B. The Matrix.Show calls displays the matrices:

```
A Matrix.Form General[3,5]
        0         0         0         0         0
        0         0         0         0         0
        0         0         0   25,0000         0

B Matrix.Form General[3,5]
  -87,4640  -74,9052   50,3092  -14,7262  -73,6282
  -82,9797   27,8304    5,4292   -4,7578   61,5226
    7,0211   19,3501   98,1663   27,1058   58,8416

C Matrix.Form General[3,5]
  -87,4640  -74,9052   50,3092  -14,7262  -73,6282
  -82,9797   27,8304    5,4292   -4,7578   61,5226
    7,0211   19,3501   98,1663   52,1058   58,8416
```

The matrix standard functionality includes Add, Sub, Mul, Transpose, MulTransposed, Clone, Negate, Invert, Eigenvalues, SVD decompositions. As for Vectors the Matrix objects support +, -, * operators

```
int dim = 5;
x = Vector.Random(dim, -50.0, 50.0);
A = Matrix.Random(dim, dim, -50.0, 50.0);
B = Matrix.Random(dim, dim, -50.0, 50.0);

y = (A + B) * x;

Vector.Show("x", x);
Matrix.Show("A", A);
Matrix.Show("B", B);
Vector.Show("y", y);
```

with the following output to the *MPCMath* console

```
x Vector
   -3,3038  -47,6437   27,9289    6,2165   -8,0053
A Matrix.Form General[5,5]
   13,7145  -22,0332   24,2230   29,2905   43,0788
  -45,0651  -43,9516   46,3690    6,4084   -3,0715
  -33,1575    5,9085  -21,6215  -43,2811   22,5939
   44,6817   36,3091  -38,1826  -22,8073   11,0903
   37,0345   -5,7952   27,6559   32,9791    2,6289

B Matrix.Form General[5,5]
  -40,3434   23,3530  -48,6669    3,6267    1,2521
   43,3688  -14,1737   41,6677  -30,7539  -14,8256
   20,0622  -34,2421    8,8289   16,4055  -48,6315
   30,4909   -1,4764  -24,4871    4,8784  -46,8946
   28,3770  -43,4818  -14,0516   18,4139  -49,8697

y Vector
 -807,8519 5225,6046 1077,2666 -3483,0406 3209,2488
```

MPCMath implements the following more advanced functionalities:

- LU decomposition
- QR decomposition
- Eigenvalue decomposition giving eigenvalues and Eigenvectors
- Linear equation solvers based on LU decompositions and Cholesky decompositions.
- And other functions ...

MPCMath keeps track of the form of the Matrix object as

- Null,
- ConstantDiagonal,
- Diagonal,
- General

enabling *MPCMath* to exploit matrix forms and an automatic speeding up of calculations.

1.3 Block vectors and matrices

Block vectors and matrices stores vector and matrices in their cells, enabling a high level programming style. Block vectors and matrices are implemented using the BVector and BMatrix classes.

The block matrix below, were A and I are submatrices

$$\mathbf{AS} = \begin{pmatrix} A & I & 0 & 0 & 0 \\ 0 & A & I & 0 & 0 \\ 0 & 0 & A & I & 0 \\ 0 & 0 & 0 & A & I \\ 0 & 0 & 0 & 0 & A \end{pmatrix}$$

is implemented with the code

```
dim = 5;
int n = 3;
A = Matrix.Random(n, n, -100.0, 100.0);
Matrix I = Matrix.UnityMatrix(n);
BMatrix AS = new BMatrix(MatrixForm.General, dim);
for (int pos = 0; pos < dim; pos++)
{
    AS[pos, pos] = A;
    if (pos < dim - 1)
    {
        AS[pos, pos + 1] = I;
    }
}

Matrix.Show("A", A);
BMatrix.Show("AS", AS);
```

```
A Matrix.Form General[3,3]
    6,8836  -50,9906   -9,3580
    8,8937   -5,6039  -33,8479
  -29,8827   18,3255  -89,7406

AS Matrix.Form General[15,15]
    6,8836  -50,9906   -9,3580         1         0         0         0         0         0         0         0         0         0         0         0
    8,8937   -5,6039  -33,8479         0         1         0         0         0         0         0         0         0         0         0         0
  -29,8827   18,3255  -89,7406         0         0         1         0         0         0         0         0         0         0         0         0
         0         0         0    6,8836  -50,9906   -9,3580         1         0         0         0         0         0         0         0         0
         0         0         0    8,8937   -5,6039  -33,8479         0         1         0         0         0         0         0         0         0
         0         0         0  -29,8827   18,3255  -89,7406         0         0         1         0         0         0         0         0         0
         0         0         0         0         0         0    6,8836  -50,9906   -9,3580         1         0         0         0         0         0
         0         0         0         0         0         0    8,8937   -5,6039  -33,8479         0         1         0         0         0         0
         0         0         0         0         0         0  -29,8827   18,3255  -89,7406         0         0         1         0         0         0
         0         0         0         0         0         0         0         0         0    6,8836  -50,9906   -9,3580         1         0         0
         0         0         0         0         0         0         0         0         0    8,8937   -5,6039  -33,8479         0         1         0
         0         0         0         0         0         0         0         0         0  -29,8827   18,3255  -89,7406         0         0         1
         0         0         0         0         0         0         0         0         0         0         0         0    6,8836  -50,9906   -9,3580
         0         0         0         0         0         0         0         0         0         0         0         0    8,8937   -5,6039  -33,8479
         0         0         0         0         0         0         0         0         0         0         0         0  -29,8827   18,3255  -89,7406
```

The Matrix object keeps track of the form of its submatrices, as shown by the statement:

```
BMatrix.ShowStructure("AS Structure", AS);
```

with the following output:

```
AS Structure Matrix.Form General
General matrix[5][5]

General          ConstantDiagonal Null             Null             Null
Null             General          ConstantDiagonal Null             Null
Null             Null             General          ConstantDiagonal Null
Null             Null             Null             General          ConstantDiagonal
Null             Null             Null             Null             General
```

that AS is a structured 5x5 matrix where the diagonal A matrices are general matrices, the superdiagonal I matrices are diagonal with a constant value and all the other matrices are null matrices. The Matrix/Vector routines exploits this structure information to speed up computations.

Block Matrices and vectors support the operators +,-,*, clone etc ...

1.4 Complex numbers and Vectors

MPCMath implement complex numbers using the complex class

```
complex a = new complex(1.0, 3.0);
complex b = complex.Exp(a);
complex c = a + b;
console.Show("a", a);
console.Show("b", b);
console.Show("c", c);

complex i = complex.Sqrt(-1.0);
console.WriteLine();
console.Show("i", i);
```

with the output:

```
a  1,0000 + i* 3,0000

b  -2,6911 + i* 0,3836

c  -1,6911 + i* 3,3836


i  0,0000 + i* 1,0000
```

Complex objects supports Add, Sub, Mul, Div, RV(real value), IV (imaginary value), Mod (Modulus), Arg (Argument) and the operators +, -, * and /

Complex vectors are implemented using the CVector class. Presently CVector implements a limited set of functionalities, which will be increased when needed. *MPCMath* does not support Complex Matrices presently.

1.5 Transfer Functions

Transfer function used in linear control applications are implemented using the TransferFunction class. The following code shows examples of definitions of transfer functions:

```
double gain = 1.0;
double delay = 0.0;

// Impulse
TransferFunction Impulse =
    new TransferFunction(delay);

//Step function
TransferFunction Step =
    new TransferFunction(gain, delay);
// Pole

double tau = 10.0;
TransferFunction Pole =
    new TransferFunction(gain, delay, tau);
// second order Oscillating
tau = 10.0;
double sigma = 0.1;
TransferFunction OscillationgSecondOrder =
    new TransferFunction(gain, delay, tau, sigma);
TransferFunction.Show("OscillationgSecondOrder"
    , OscillationgSecondOrder);

// tranfer function from poles and zeroes
int np = 5;
CVector roots = new CVector(np);
for (int pos = 0; pos < np; pos++)
{
    roots[pos] = -pos;
}

int nr = 4;
CVector zeroes = new CVector(nr);
for (int pos = 0; pos < nr; pos++)
{
    zeroes[pos] = pos;
}

gain = 5.0;
delay = 2.0;
TransferFunction G =
    new TransferFunction(gain, delay, roots, zeroes);
```

```
            TransferFunction.Show("G", G);
```

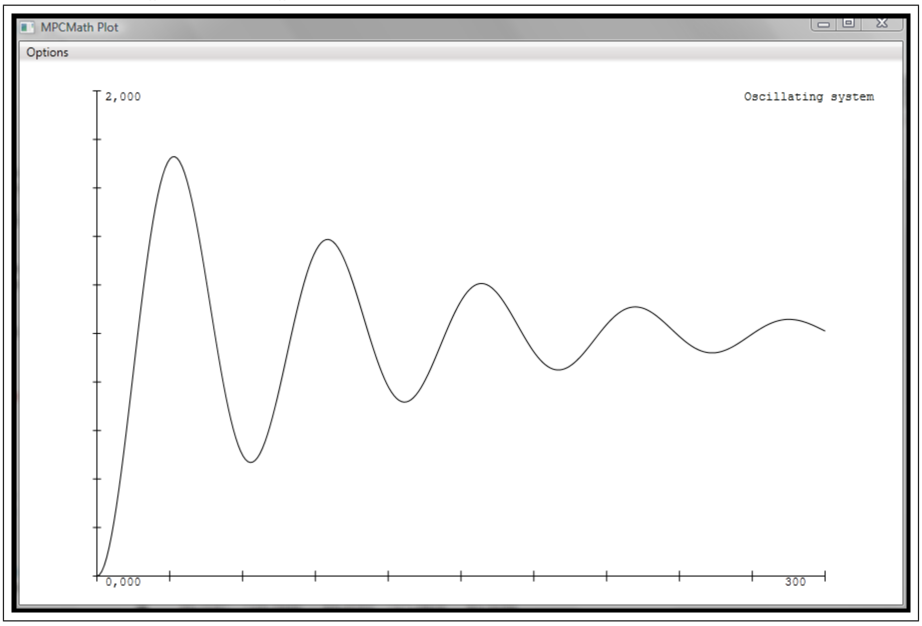
with the output:

```
OscillationgSecondOrder TransferFunction
    R
    0,0100
    P
    0,0100    0,0200         1
    Gain      0,0100
    Roots Complex List
   -0,0100 i    0,0995
   -0,0100 i   -0,0995

G TransferFunction
    Delay     2,0000 seconds
    R
  -30,0000   55,0000  -30,0000    5,0000
    P
   24,0000   50,0000   35,0000   10,0000         1
    Gain      5,0000
    Zeroes Complex List
    3,0000 i         0
    2,0000 i         0
         1 i         0

    Roots Complex List
        -1 i         0
   -2,0000 i         0
   -3,0000 i         0
   -4,0000 i         0
```

Transfer functions supports the +,-,* and / operators. Time responses are obtained using the Value function.

```
gain = 5.0;
delay = 2.0;
TransferFunction G =
    new TransferFunction(gain, delay, roots, zeroes);
TransferFunction.Show("G", G);

TransferFunction GR = OscillationgSecondOrder * Step;
int steps = 300;

Vector resp = new Vector(steps);
for (int step = 0; step < steps; step++)
{
    double time = step;
    resp[step] = GR.Value(time);
}

console.Plot(
    new PlotSeries("Oscillating system",
     0.0, 2.0, resp));
```

with the plotted output



1.6 ARX Models

ARX models describes linear plant dynamics as

$$A(q)y(t) = B(q)u(t) + \epsilon(t)$$
$$A(q) = \sum_{i=0}^{n_y} a_i$$
$$B(q) = \sum_{i=0}^{n_u} b_i$$

q is the time shift operator $q_1 y(t) = y(t-1)$. , and ϵ is the noise. The $E\Delta ARX$ model integrates the noise using the filter

$$\eta_k = \frac{1 - \alpha q^{-1}}{1 - q^{-1}} e_k$$

where α is in the range 0.0 to 1.0. The $E\Delta ARX$ integrates the noise in order to remove offsets in the closed control loop, see [Huusom et al., 2010]. The following code generates an ARX object *plant* and a $E\Delta ARX$ object *model*

```
// Arx model
Vector Aq = new Vector(1.0, -1.558, 0.5769);
Vector Bq = new Vector(0.0, 0.2094, 0.1744);
int delayCount = 25;
double dt = 2.0;

ARXModel plant =
    new ARXModel(delayCount, Aq, Bq, dt);
ARXModel.Show("plant", plant);

// Arx model with noise model
double alfa = 0.7;
ARXModel model =
    new ARXModel(delayCount, Aq, Bq, dt, alfa);
ARXModel.Show("model", model);
```

```
plant ARX Model

    Delay 25 steps
    A Vector
         1   -1,5580    0,5769
    B Vector
         0    0,2094    0,1744
model ARX Model

    Delay 25 steps
```

```
    A Vector
         1   -2,5580    2,1349   -0,5769
    B Vector
         0    0,2094   -0,0350   -0,1744
```

ARX and *E∆ARX* objects can be directly created from transfer functions. The transfer function

$$G(s) = \frac{20}{(50s+1)(4s+1)}e^{-50s}$$

can be realized as *ARX* and *E∆ARX* objects by

```
gain = 20.0;
delay = 50.0;
dt = 2.0;
Vector taus = new Vector(50.0, 4.0);
TransferFunction Furnace =
    TransferFunction.TauForm(gain, delay, taus);
TransferFunction.Show("Furnace", Furnace);
plant = new ArxModel(Furnace, dt);
ArxModel.Show("plant", plant);
```

```
Furnace TransferFunction
    Delay    50,0000 seconds
    R
    0,1000
    P
    0,0050    0,2700         1
    Gain      0,1000
    Roots Complex List
   -0,0200 i         0
   -0,2500 i         0

plant Arx Model

    Delay 25 steps
    A Vector
         1   -1,5673    0,5827
    B Vector
         0    0,1681    0,1405
```

The time responses from the *ARX* model can be calculated using the NextState and Observed functions

```
x= new Vector(plant.Dimension);
double u = 100.0;
double eps;
double epsVariance = 0.1;
double epsMean = 0.0;
resp = new Vector(steps);
for (int step = 0; step < steps; step++)
{
    eps = MPCMathLib.WhiteGaussianNoise(
        epsVariance,epsMean);
    x = plant.NextState(x, u, eps);
    resp[step] = plant.Observed(x, eps);
}
```

The state space matrices for the ARX and $E\Delta ARX$ objects are given in the ASP, BSP,KSP and C properties. The code sequence

```
for (int step = 0; step < steps; step++)
{
    eps = MPCMathLib.WhiteGaussianNoise(
        epsVariance, epsMean);
    x = plant.ASP * x + plant.BSP * u
        + plant.KSP * eps;
    resp[step] = plant.C * x + eps;
}
```

gives the same results obtained using NextState and Observed routines.

1.7 State Space Models

The state space model is used to describe linear multiple input multiple output, MIMO systems.

$$\begin{aligned} X_{k+1} &= AX_k + B * U_k + \epsilon_k \\ Y_k &= CX_k + \eta_k \end{aligned}$$

Where ϵ_k is the process noise and η_k the measurement noise.

The StateSpace object can be created directly from a set of A,B,C and K matrices, in this case with matrices with random numbers:

```
int nx = 5;
int ny = 2;
int nu = 3;

A = Matrix.Random(nx, nx, -100.0, 100.0);
B = Matrix.Random(nx, nu, -100.0, 100.0);
K = Matrix.Random(nx, ny, -100.0, 100.0);
C = Matrix.Random(ny, nx, -100.0, 100.0);

StateSpaceModel spmodel =
    new StateSpaceModel(A, B, C, K);
StateSpaceModel.Show("spmodel", spmodel);
```

where nx are the number of states, nu the number of manipulated variables and ny the number of observed variables.

StateSpaceModel objects can also be created from an ARX model

```
spmodel = new StateSpaceModel(plant);
```

where plant is the ARX model from the previous section. As the ARX model is a Single Input Single Output, SISO, state space model, this conversion is rather trivial. It more interesting creating State space models object from an arrays of transfer functions or ARX functions.

A plant can be described by the following set of transfer functions, with 2 observed variable and 2 manipulated variables. In this case a model for a cement mill derived from experimental data.

$$G(s) = \begin{bmatrix} \frac{0.62}{(45s+1)(8s+1)} e^{-5s} & \frac{0.29(8s+1)}{(2s+1)(38s+1)} e^{-1.5s} \\ \frac{-15}{(60s+1)} e^{-5s} & \frac{5}{(14s+1)(s+1)} e^{-0.1s} \end{bmatrix}$$

These four transfer functions are inserted in an array of transfer functions with the code

```
// Cement mill MIMO system
Double T = 1.0;
int ny = 2;
int nu = 2;

Matrix<TransferFunction> CMModel =
    new Matrix<TransferFunction>
        (MatrixForm.General, ny, nu);

CMModel[0, 0] = TransferFunction.TauForm
```

```
    (0.62, 5.0, new Vector(45.0, 8.0));
CMModel[1, 0] = TransferFunction.TauForm
    (-15.0, 5.0, new Vector(60.0));
CMModel[0, 1] = TransferFunction.TauForm
    (0.29, 1.5, new Vector(2.0, 38.0)
    , new Vector(8.0));
CMModel[1, 1] = TransferFunction.TauForm
    (5.00, 0.1, new Vector(14.0, 1.0));
```

and the state space model objects *Plant* and *Model* are generated

```
// Noise integration factor
Vector alfa = new Vector(0.7, 0.7);

// generate state space model of Cement Mill
StateSpaceModel Plant =
    new StateSpaceModel(CMModel, T);
StateSpaceModel Model =
    new StateSpaceModel(CMModel, T, alfa);

StateSpaceModel.Show("Plant", Plant);
StateSpaceModel.Show("Model", Model);
```

The StateSpaceModel object Model is created via $E\Delta ARX$ objects, with integration of noise, leading to offset free control.

```
Plant   StateSpace model  nx 14 ny 2 nu 2
     Delay  0


     A Matrix.Form General[14,14]
   1,8605        1        0        0        0        0        0   0,0008        0        0        0        0        0        0
  -0,8631        0        0        0        0        0        0   0,0008        0        0        0        0        0        0
        0        0   1,5806        1        0        0        0        0        0        0        0        0   0,0253        0
        0        0  -0,5908        0        0        0        0        0        0        0        0        0  -0,0223        0
        0        0        0        0   0,9835        0        0  -0,2479        0        0        0        0        0        0
        0        0        0        0        0   1,2989        1        0        0        0        0        0        0        0
        0        0        0        0        0  -0,3425        0        0        0        0        0        0        0        0
        0        0        0        0        0        0        0        0        1        0        0        0        0        0
        0        0        0        0        0        0        0        0        0        1        0        0        0        0
        0        0        0        0        0        0        0        0        0        0        1        0        0        0
        0        0        0        0        0        0        0        0        0        0        0        1        0        0
        0        0        0        0        0        0        0        0        0        0        0        0        0        0
        0        0        0        0        0        0        0        0        0        0        0        0        0        1
        0        0        0        0        0        0        0        0        0        0        0        0        0        0

     B Matrix.Form General[14,2]
        0        0
        0        0
        0        0
        0        0
        0        0
        0   0,1281
        0   0,0898
        0        0
        0        0
        0        0
        0        0
        1        0
        0        0
        0        1

     K Matrix.Form General[14,2]
   1,8605        0
  -0,8631        0
   1,5806        0
  -0,5908        0
        0   0,9835
        0   1,2989
        0  -0,3425
        0        0
        0        0
        0        0
        0        0
        0        0
        0        0
        0        0

     C Matrix.Form General[2,14]
        1        0        1        0        0        0        0        0        0        0        0        0        0        0
        0        0        0        0        1        1        0        0        0        0        0        0        0        0
```

In order to integrate the state space models the following objects are created:

```
// Define states of observed Variabes
Vector XPlant = new Vector(Plant.Dimension);
Vector XModel = new Vector(Model.Dimension);
Vector YPlant = new Vector(ny);
Vector YModel = new Vector(ny);
Vector UC = new Vector(nu);
Vector innovation;
Vector Zero = new Vector(ny);

// Measurement noise and process noise
Vector epsilonVar = new Vector(ny);
Vector xiVar = new Vector(nu);
if (noise)
{
    epsilonVar = new Vector(0.05, 10.0);
    xiVar = new Vector(0.0001, 0.1);
}

Vector xiMean = new Vector(0.0, 0.0);
Vector epsilonMean = new Vector(ny);
```

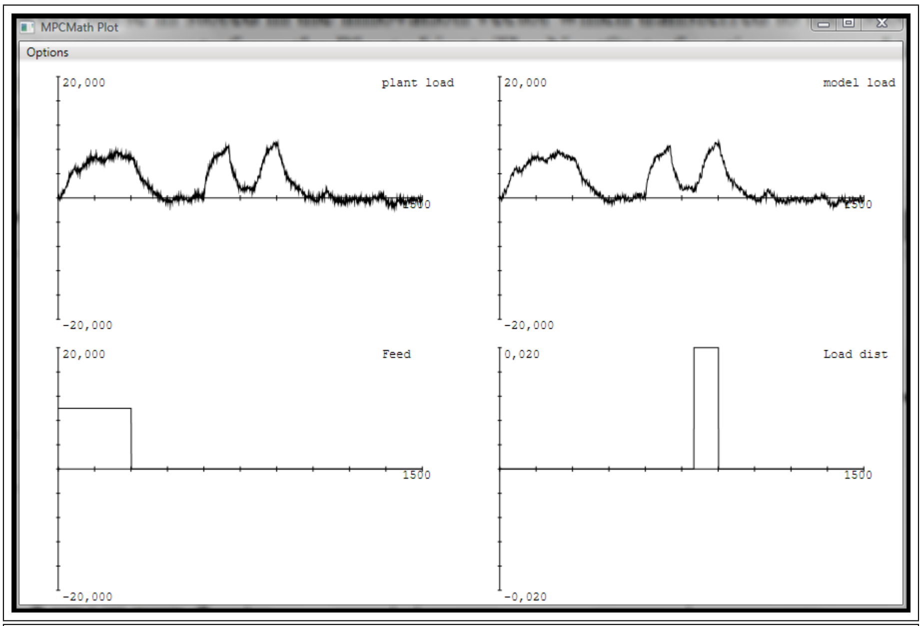
The simulation of the plant dynamics and tracking the plant with the model is done by the following code

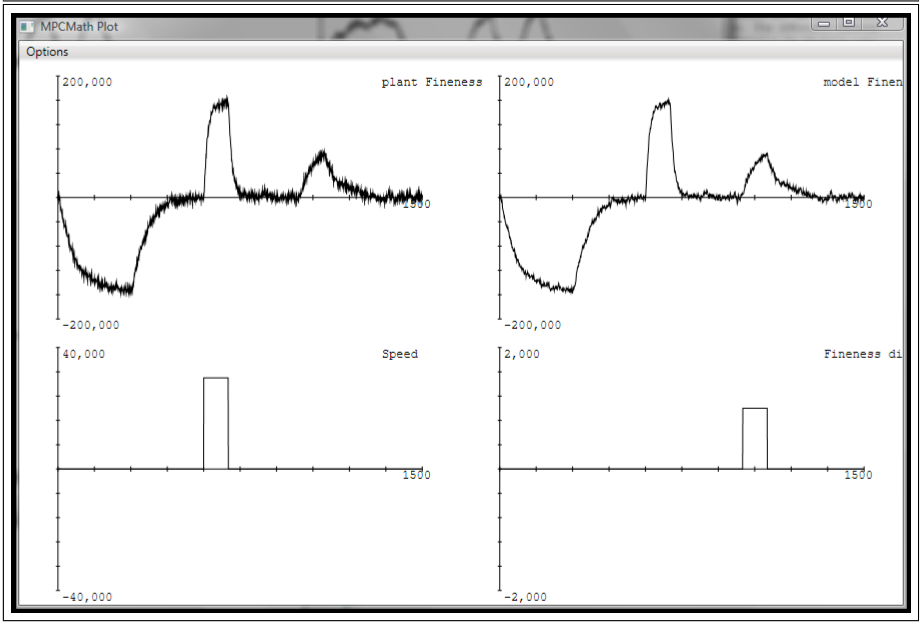
```
    steps = 1500;

    // Tracking simulation
    for (int step = 0; step < steps; step++)
    {
        // Exerxice manipulated vars
        // feed
        UC[0] = SetVal(step, 0, 300, 10.0);
        // Separator speed
        UC[1] = SetVal(step, 600, 700, 30.0);

        // umeasured disturbances
        // Load
        xiMean[0] = SetVal(step, 800, 900, 0.02);
        // Fineness
        xiMean[1] = SetVal(step, 1000, 1100, 1.0);

        // Measurement and process noise
        eps = MPCMathLib.WhiteGaussianNoise
            (epsilonVar, epsilonMean);
        xi = MPCMathLib.WhiteGaussianNoise
```

```
        (xiVar, xiMean);

    // Plant measurement with measurement noise
    YPlant = Plant.Observed(XPlant) + eps;
    YModel = Model.Observed(XModel);

    // Update Plant and Model state
    innov = YPlant - YModel;
    XModel = Model.NextState(XModel, UC, innov);
    XPlant = Plant.NextState(XPlant, UC, xi);

    // collect plot indformation
    ypload[step] = YPlant[0];
    ymload[step] = YModel[0];
    ypfineness[step] = YPlant[1];
    ymfineness[step] = YModel[1];
    Feed[step] = UC[0];
    Speed[step] = UC[1];
    dLoad[step] = xiMean[0];
    dFineness[step] = xiMean[1];
}

console.Plot(sypload, symload,
    sFeed, sdLoad);
console.Plot(sypfineness, symfineness,
    sSpeed, sdFineness);
```

The manipulated variable feed, $UC[0]$, is increased to 10.0 from step 0 to 300. The separator Speed, $UC[1]$, is increased to 100.0 from step 600 to 700, and the unmeasured disturbances are active from step 800 to 900 and step 100 to 1100 respectively.

The simulated measurement noise and process noise are calculated and stored in vectors eps and xi. The Yplant and Ymodel are the observed responses from the plant and the Model respectively. The differences between these in stored in the innovation vector which transferred to the Model in order to track the plant measurements from the Plant object. The NextState functions are used to update the Plant object (with process noise xi) and the Model object (with the innovation).

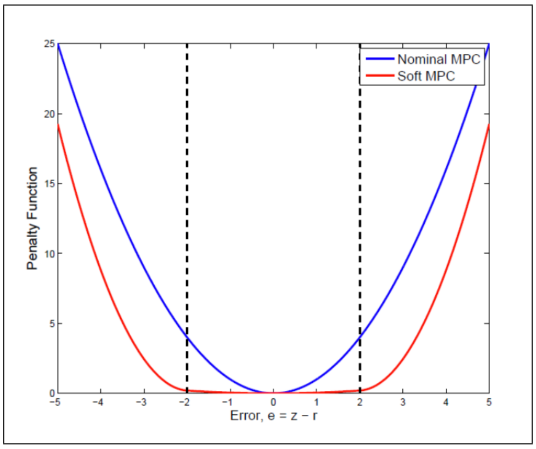




The Model objects act as a Kalman filter tracking the Plant state. Due the the noise integration, the Model is able to track the unmeasured disturbances without offsets.

1.8 Model Predictive Control

Model predictive control can be implemented using the MiMoMPC class. The MiMoMPC class implements both conventional MPC and Soft Constrained MPC. The conventional MPC has a quadratic penalty function and the Soft Constrained MPC has a dead band zone around the set point where the penalty for not reaching the exact set point is low.



The SoftConstrained MPC minimizes the problem

$$
\begin{aligned}
\min_{\{z,u,\eta\}} \phi = \frac{1}{2} \sum_{k=0}^{N-1} \|z_{k+1} - r_{k+1}\|_{Q_z}^2 + \|\Delta u_k\|_S^2 \\
+ \sum_{k=1}^{N} \frac{1}{2} \|\eta_k\|_{S_\eta}^2 + s_\eta' \eta_k
\end{aligned}
$$

subject to the constraints

$$
\begin{aligned}
& z_k = b_k + \sum_{i=1}^{n} H_i u_{k-i} && k = 1, \dots N \\
& u_{\min} \le u_k \le u_{\max} && k = 0, \dots N-1 \\
& \Delta u_{\min} \le \Delta u_k \le \Delta u_{\max} && k = 0, \dots N-1 \\
& z_k \le z_{\max,k} + \eta_k && k = 1, \dots N \\
& z_k \ge z_{\min,k} - \eta_k && k = 1, \dots N \\
& \eta_k \ge 0 && k = 1, \dots N
\end{aligned}
$$

where zk is the plant response, r_k the set-point, ΔU_k the movement of the manipulated variables. The Q_z, S and S_η are weighing matrices for reference error penalty, S movement of manipulated variables penalty and S_η the penalty for coming outside the dead-band zone. Setting S? to zero result in a conventional MPC controller. The first constraint for z_k is the linear process model. For further details consult [?Prasath and Jørgensen, 2010]

```
// MPC contoller data
int history = 100;   // history for operator display
int horizon = 50;    // optimization horion
Vector theta = null;
Vector my =    null;
```

```
        Vector ymax =  null;
        Vector ymin =  null;
        if (SoftConstraint)
        {
            // soft constrained MPC

            // reference penalty
            theta = new Vector(0.1, 0.2);
            // soft constraint penalty
            my = new Vector(1000.0, 500.0);
            // soft constraint max values
            ymax = new Vector(2.0,  5.0);
            // soft constraint min values
            ymin = new Vector(-2.0, -5.0);
        }
        else
        {
            // conventional MPC

            // reference penalty
            theta = new Vector(1000.0, 500.0);
        }

        // manipulated vars movement penalty
        Vector rho = new Vector(100.0, 200.0);
        // Hard constraint U max value
        Vector umax = new Vector( 15.0, 30.0);
        // Hard constraint U min value
        Vector umin = new Vector(-30.0, -30.0);
```

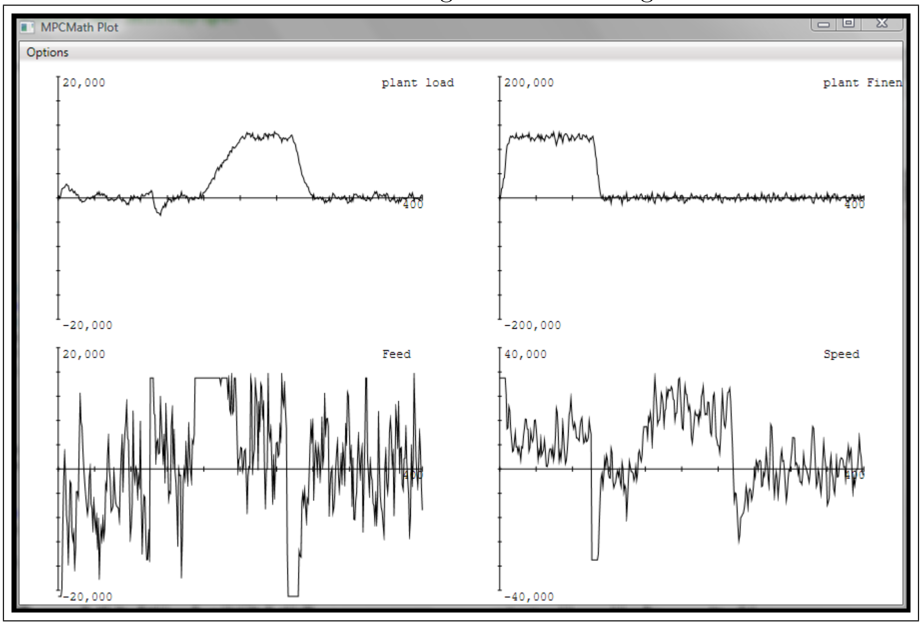
The actual MPC controller and the control loop is defined by the code below (Model and Plant objects are the StateSpace models of the Cement mill described in the previous section).

```
    // Setup MiMo MPC
    MiMoMPC mpc = new MiMoMPC(Model, history, horizon,
        theta, my, ymin, ymax, rho, umin, umax);

    // Get references to operator information
    Matrix ypopr = mpc.YPlant;
    Matrix ymopr = mpc.YModel;
    Matrix yropr = mpc.YRef;
    Matrix uopr = mpc.U;

    // MPC control
    for (int step = 0; step < steps; step++)
    {
```

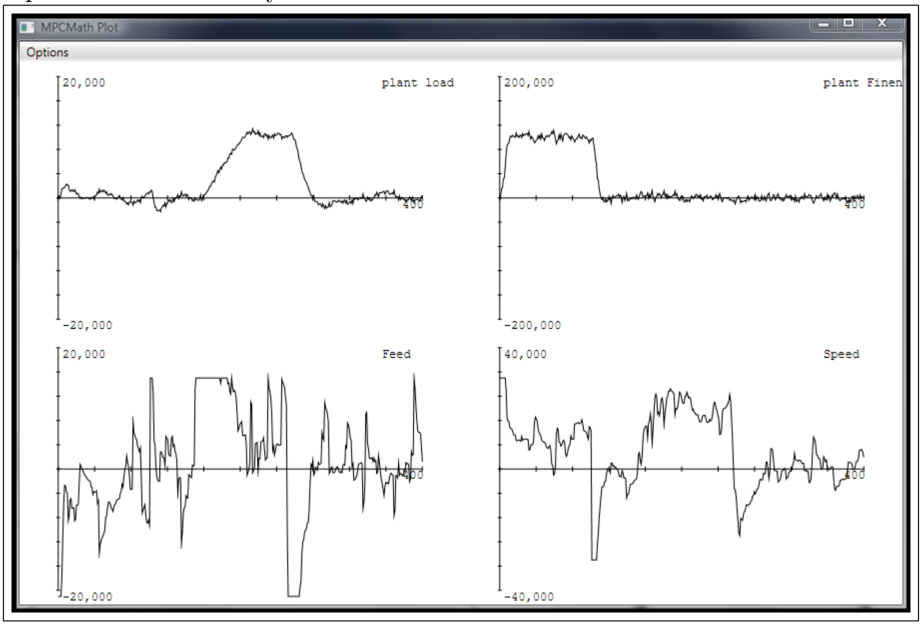
```
// Manipulate setpoints
// Fineness setpoint
YRef[1] = SetVal(step, 0, 100, 100.0);
// elevator load setpoint
YRef[0] = SetVal(step, 150, 250, 10.0);
mpc.SetRef(YRef);

// Measurement and process noise
eps = MPCMathLib.WhiteGaussianNoise
    (epsilonVar, epsilonMean);
xi = MPCMathLib.WhiteGaussianNoise
    (xiVar, xiMean);

// Plant measurement with measurement noise
YPlant = Plant.Observed(XPlant) +  eps;

// MPC controller
UC = mpc.Next(YPlant);

// send control signal and process noise to plant
XPlant = Plant.NextState(XPlant, UC, xi);

console.WriteLine("step " + step.ToString()
 + " iterations "
    + mpc.Solver.Iterations.ToString());
    }
```

The set point for the Fineness, $YRef[2]$, is set to 100.0 for step 0 to 100, and the setpoint for the Elevator load, $YRef[0]$,is set to 100.0 for step 150 to 250.

The conventional MPC controller gives the following result



a good control of elevator load and the fineness are achieved, but the manipulated variables very active. The SoftConstrained MPC results in



The achieved control is as good the conventional MPC, but the manipulated variables are much better. The SoftConstrained MPC is much more robust for model errors than the conventional MPC, [Prasath and Jørgensen, 2010]

The MPC problems are solved using the QuadraticProblemSolver object. This object find the minimum of the object function:

```
//  Minimize    X'G*X + g'*x
            //  s.t.        A*X = b
            //              G*X >= d

            QPSolver solver = new QPSolver(G, g, A, b, C, d);

            solver.Solve();

            console.Show("iterations", solver.Iterations);
            console.Show("My", solver.My); // dual gap
            BVector.Show("x", solver.X); // solution
            // equality lagrange variables
            BVector.Show("y", solver.Y);
            // slack variables
            BVector.Show("s", solver.S);
             // inequality lagrange variables
            BVector.Show("z", solver.Z);
```

The object uses a Primal-Dual Interior-Point algorithm to find the minimum. Its a general routine, which exploit the structure of G, A, C to speed up calculations especially for problems with long time horizons. The QPSolver solves the KKT system by a call to a KKTSolver defined by an interface. This makes it possible to develop KKT solver that exploit structures in the system matrices

```
solver = new QPSolver(new SoftConstraintKKTSolver
 (G, g, A, b, C, d));
```

The complete source code for this example is given in the *MPCMath* download package from http://www.2-control.dk in the TestMiMoMPC project.

1.9 Intel Math Kernel Library (BLAS and LAPACK)

Many of *MPCMath* 's functions are based on Intel Math Kernel Library [1], a highly optimized implementation of the public domain computing packages BLAS (Basic Linear Algebra Subroutines) and LAPCK (Linear Algebra Package). This gives *MPCMath* a boost in performance resulting more than an order of magnitude.

The critical cpu consuming calculations in the MPC controllers and the QPSolver are the matrix multiplications and The Cholesky factorizations.

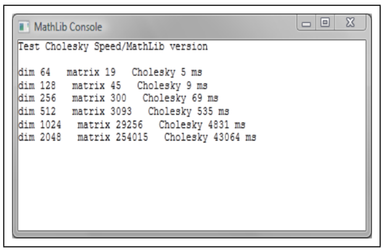
The speed of these operations can be measured with the following code

```
console.WriteLine("Test Cholesky Speed/\MPCMath");
console.WriteLine();

int dim = 32;
for (int test = 0; test < 7; test++)
{

    dim = 2 * dim;
    Matrix A = Matrix.Random
     (dim, dim, -100.0, 100.0);
    Matrix AT = Matrix.Transpose(A);
    Vector r = Vector.Random
     (dim, -1000.0, 1000.0);

    // Matrix multiplication
    DateTime start = DateTime.Now;
    A = AT * A;

    TimeSpan tm = DateTime.Now - start;
```

```
        // Cholesky factorizatio + solve
        start = DateTime.Now;
        CholeskyEquationSolver solver =
         new CholeskyEquationSolver(A);
        Vector x = solver.Solve(r);

        TimeSpan tc = DateTime.Now - start;

        console.WriteLine("dim " + dim.ToString()
         + "   matrix "
            + tm.TotalMilliseconds.ToString()
            + "   Cholesky "
            + tc.TotalMilliseconds.ToString());
    }
```

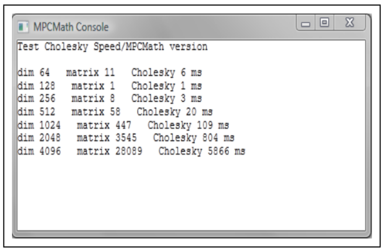
which measures calculation time for multiplication of random square matrices and subsequent Cholesky factorization.

The results obtained with the best possible C# code is



where multiplication of two 2048x 2048 matrices takes 254015 ms or 4.2 minutes (MathLib was a predecessor to *MPCMath* written entirely in C#).

With *MPCMath* using Intel MKL the results is



where multiplication of two 2048x 2048 matrices takes 3545 ms, a factor 70 improvement !

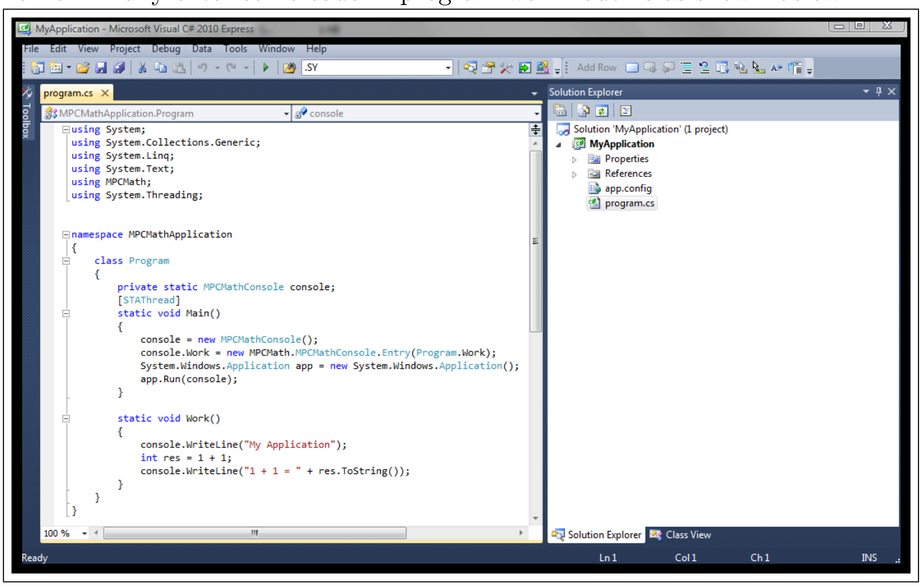
Chapter 2

Console functions

The MPCMathConsole is the development and debugging tool for *MPCMath* applications. The MPCMathConsole is normally created from the MPCMath template application delivered together with the *MPCMath* Package. Using MPCMathCosole requires a valid Development or a Demo license. It not available using a Runtime license.

2.1 Creating a new *MPCMath* application

Copy the folder C:2-controlMPCMathTemplate to a new folder and rename the new folder the name of your application i.e. "MyApplication". Open the new folder and double click the solution file MPCMathTemplate.sln.In the Solution explorer window rename the solution and the project file to your applications name. Finally enter some code in program work routine as shown below



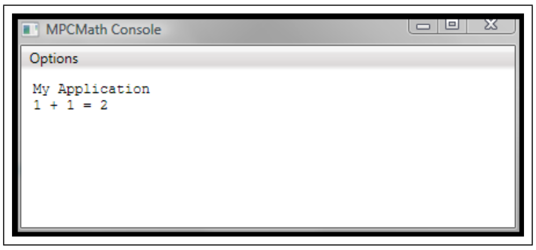
The normal Main program for the application, creates the MPCMathConsole, set the entry of the Program.Work routine, and then starts the created MPCMathConsole by calling *app.Run*(*console*);. The MPCMath console runs in a new parallel tread, execution the code specified in the *Program.Work* routine. Separating the MPCMathConsole main tread from the work tread, enables print out of information during program execution.

```
private static MPCMathConsole console;
[STAThread]
static void Main()
{
    console = new MPCMathConsole();
    console.Work =
     new MPCMath.MPCMathConsole.Entry(Program.Work);
    System.Windows.Application
     app = new System.Windows.Application();
    app.Run(console);
}
```

the work routine is:

```
static void Work()
{
    console.WriteLine("My Application");
    int res = 1 + 1;
    console.WriteLine("1 + 1 = " + res.ToString());
}
```

Running the application shows The MPCMathConsole



2.2 Displaying and plotting variables

Text output to the console is done using the Show and Writeline routines, some exaples are

```
// Show routines
console.WriteLine("output a line to console");
```

```
// output a blank line
console.WriteLine();

// output a bool var
bool Ok = true;
console.Show("Ok", Ok);

// output an array of integers
int[] intArr = { 1, 4, 5, 6, 7 };
console.Show("Integer array", intArr);

// output double
double pi = Math.PI;
console.Show("pi short", pi);

MPCMathConsole.Digits = 8;
console.Show("pi", pi);
```

with the following output

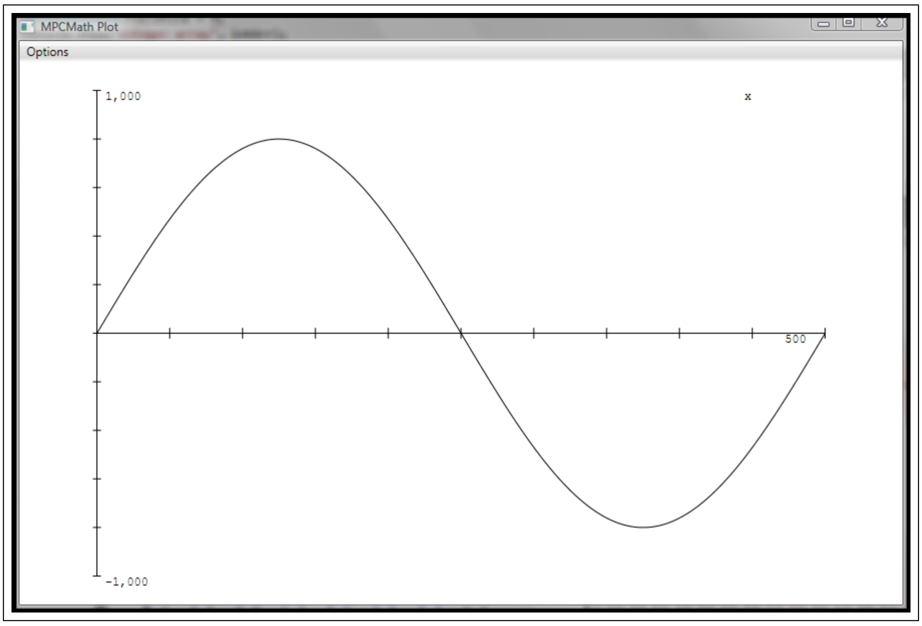
```
output a line to console

Ok True

Integer array
         1         4         5         6         7

pi short 3,1416

pi 3,14159265

Integer array
  1  4  5  6  7
```

Plots are generated using the Plot routine. The following code defines a vector and fills it with a sinus curve. The *PlotSeries* defines the label, min, max plot values (to be found automatically, and finally set x as the vector holding the values to be plotted. The call to *console.Plot* creates a new window with the plot.

```
int steps = 500;
Vector x = new Vector(steps);
for (int step = 0; step < steps; step++)
{
   x[step] = 0.8 * Math.Sin(2.0 * pi * step /(steps - 1));
}
```

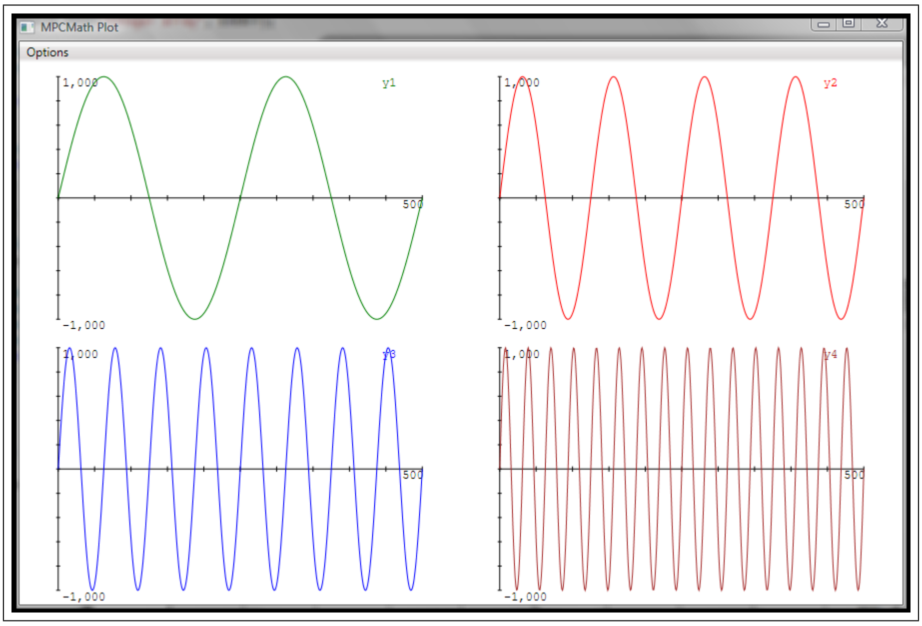
```
PlotSeries xs = new PlotSeries("x", -1.0, 1.0, x);
console.Plot(xs);
```

The resulting plot is



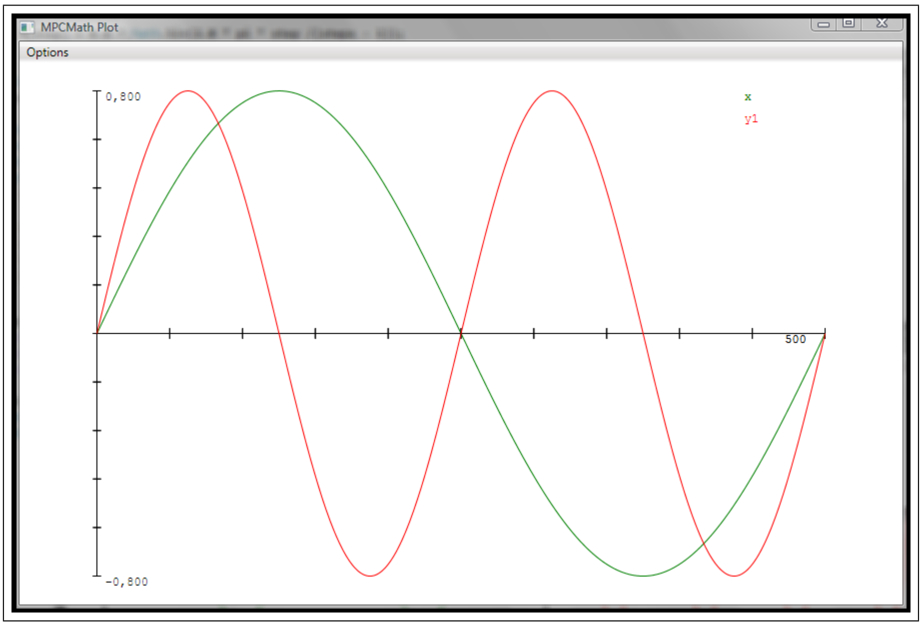
Plot can handle of to four plot series

```
Vector y1 = new Vector(steps);
Vector y2 = new Vector(steps);
Vector y3 = new Vector(steps);
Vector y4 = new Vector(steps);
for (int step = 0; step < steps; step++)
{
   y1[step] = Math.Sin(4.0 * pi * step / (steps - 1));
   y2[step] = Math.Sin(8.0 * pi * step / (steps - 1));
   y3[step] = Math.Sin(16.0 * pi * step / (steps - 1));
   y4[step] = Math.Sin(32.0 * pi * step / (steps - 1));
}

console.Plot(
    new PlotSeries("y1", 0.0, 0.0, SeriesColor.Green, y1),
    new PlotSeries("y2", 0.0, 0.0, SeriesColor.Red, y2),
    new PlotSeries("y3", 0.0, 0.0, SeriesColor.Blue, y3),
    new PlotSeries("y4", 0.0, 0.0, SeriesColor.Brown, y4));
```



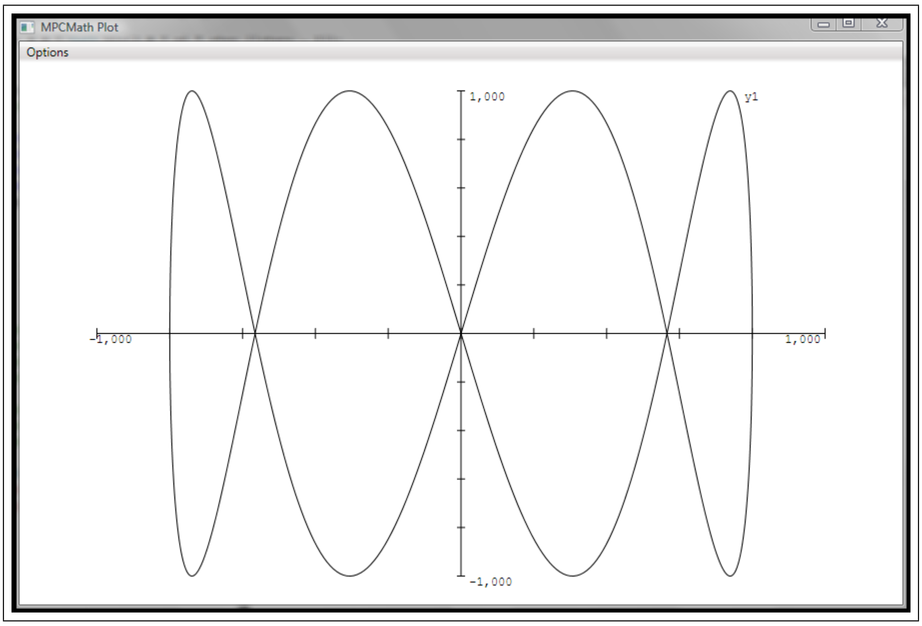
Using the Plot class several plot series can be shown in one coordinate system.

```
console.Plot(new Plot(
    new PlotSeries("x", 0.0, 0.0, SeriesColor.Green, x),
    new PlotSeries("y1", 0.0, 0.0, SeriesColor.Red, y1)));
```

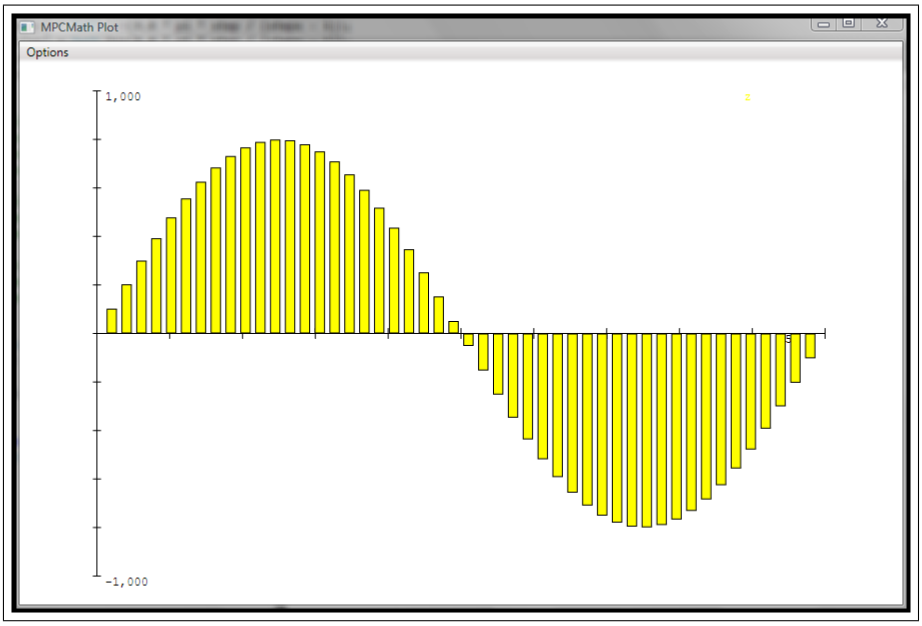


The Plot class also allows one to specify the plot type using the PlotType enumeration

```
console.Plot(new Plot(PlotType.XY,
    new PlotSeries("x", -1.0, 1.0,  x),
    new PlotSeries("y1", -1.0, 1.0, y2)));
```



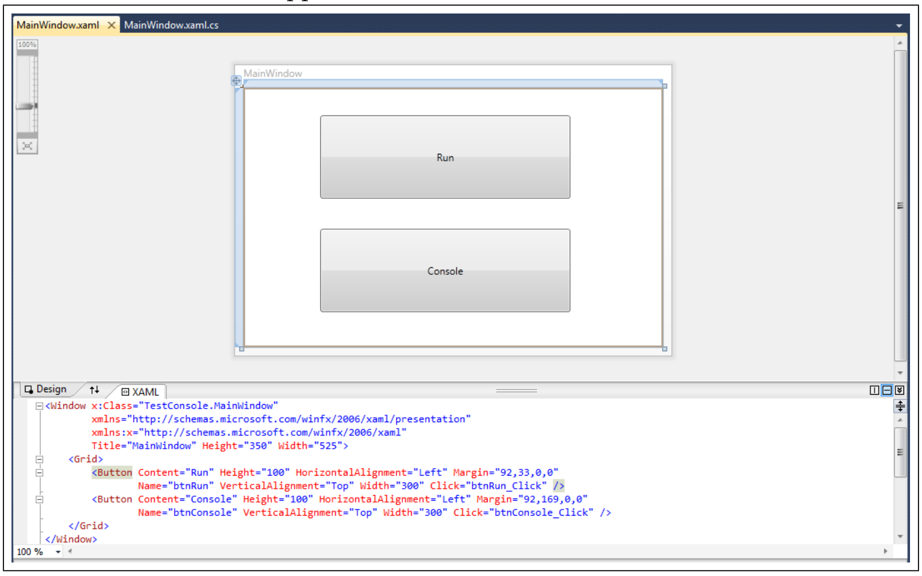

```
steps = 50;
Vector z = new Vector(steps);
for (int step = 0; step < steps; step++)
{
    z[step] = 0.8 * Math.Sin(2.0 * pi * step / (steps - 1));
}

console.Plot(new Plot(PlotType.TBar,
    new PlotSeries("z", -1.0, 1.0,SeriesColor.Yellow, z)));
```



2.3 Using Console for test of GUI programs

It can be very convenient to use the Console as a part of a GUI program. Here console can be used to display debug information or detailed information about program operation.

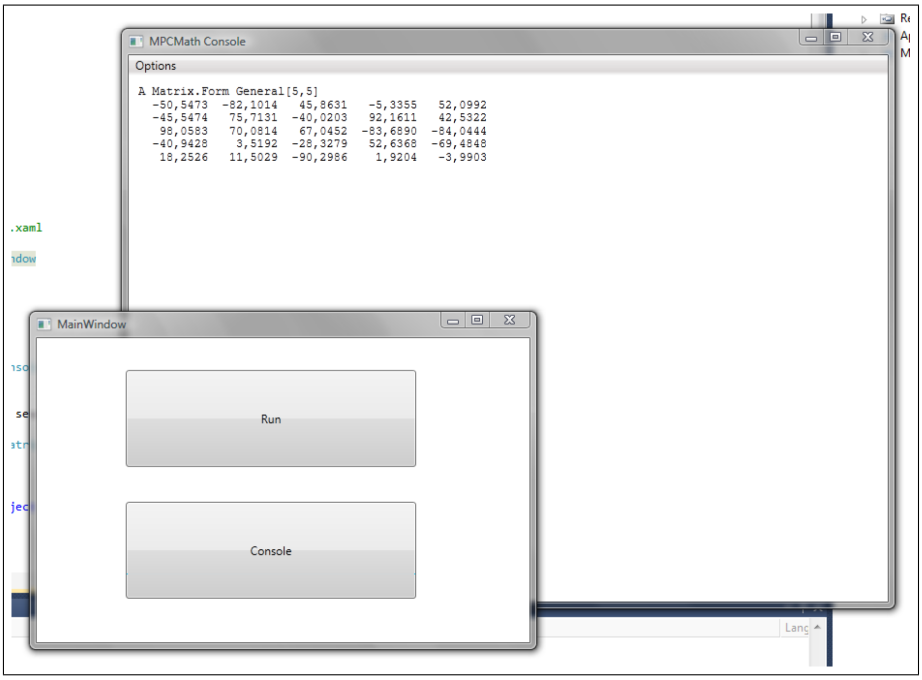
Define a new WPF application and with two buttons *Run* and *Console*.




```
using System.Windows.Media.Imaging;
```

```
using System.Windows.Navigation;
using System.Windows.Shapes;
using MPCMath;

namespace TestConsole
{
    /// <summary>
    /// Interaction logic for MainWindow.xaml
    /// </summary>
    public partial class MainWindow : Window
    {
        private MPCMathConsole console;

        public MainWindow()
        {
            InitializeComponent();
            this.console = new MPCMathConsole(true);
        }

        private void btnRun_Click(object sender, RoutedEventArgs e)
        {
            MPCMath.Matrix A = MPCMath.Matrix.Random(5, 5, -100.0, 100.0);
            MPCMath.Matrix.Show("A", A);
        }

        private void btnConsole_Click(object sender, RoutedEventArgs e)
        {
            this.console.Show();
        }
    }
}
```

The constructor creates the MPCMathConsole and stores a reference to in the private variable console. The *Run* button creates some output on the console. The *Run* makes the console visible. The console can be hidden by selecting the menu item *Hide*.



2.4 MPCMathConsole

Class MPCMathConsole

MPCMath console class for test and debugging of MPCMath programs. Running MPCMath Console requires a valid Development or Demo license

```
class MPCMathConsole : MPCMathWindow
```

Constructors

```
MPCMathConsole() : this(false)
```

Default constructor, without Hide option in console window

```
MPCMathConsole(bool EnableHide) : base(EnableHide)
```

Constructor
Parameters
EnableHide   Enable Hide option in console window

Properties

Console

Get reference to MPCMath console

```
static MPCMathConsole Console {get;}
```

Digits

Digits for doubles

```
static int Digits {set; get;}
```

FieldSize

Field size of vector and matrix diplay

```
static int FieldSize {set; get;}
```

NumbersPerLine

Numbers per Line

```
static int NumbersPerLine {set; get;}
```

Trace

Trace console output to file ”console.txt” in bin/debug or bin/release directory

```
static bool Trace {set; get;}
```

Work

Callback entry point defining work thread

```
Entry Work {set;}
```

Methods

Abort

Abort execution of Work thread. Usefull for debugging code generating unhandled Exceptions, which prevents reading the output on the MPCMath console. The routine aborts the current work thread.

```
void Abort()
```

Abort

Abort execution of Work thread. Usefull for debugging code generating unhandled Exceptions, which prevents reading the output on the MPCMath console. The routine aborts the current work thread.

```
void Abort(string text)
```

Parameters

text	Text string outputted to console

AssertEqual

Assert equal for bool variable Show actual and expected if they are not equal

```
void AssertEqual(string text, bool actual, bool expected)
```

Parameters

text	variable desription
actual	actual value
expected	expected value

AssertEqual

Assert equal for integers Show actual and expected if they are not equal

```
void AssertEqual(string text, int actual, int expected)
```

Parameters

text	variable desription
actual	actual value
expected	expected value

AssertEqual

Assert equal for integer array Show actual and expected if they are not equal

```
void AssertEqual(string text, int[] actual, int[] expected)
```

Parameters

text	variable desription
actual	actual value
expected	expected value

AssertEqual

Assert equal for double Show actual and expected if they are not equal

```
void AssertEqual(string text, double actual, double expected)
```

Parameters

text	variable desription
actual	actual value
expected	expected value

AssertEqual

Assert equal for complex Show actual and expected if they are not equal

```
void AssertEqual(string text, complex actual, complex expected)
```

Parameters

text	variable desription
actual	actual value
expected	expected value

Clear

Clear Console window

```
void Clear()
```

DFormat

Format double variable

```
static string DFormat(double a)
```

Parameters

a	double variable
returns	Formatted output string

DFormat

Format double variable

```
static string DFormat(int Fieldsize, double a)
```

Parameters

Fieldsize	Field size
a	double variable
returns	Formatted output string

Entry

Delegate specifing Work routine format

```
delegate void Entry()
```

Plot

Plot series in new window, each series have a separate coordinate system. Up to four series

```
void Plot(params PlotSeries[] Series)
```

Parameters

Series	Series definition

Plot

Plot series i new or existing window

```
void Plot(string PlotID, params PlotSeries[] Series)
```

Parameters

PlotID	Window identifiction string
Series	Series definition

Plot

Plot plots in new window. Each plot can have a number of series. Up to four plots in the window

```
void Plot(params Plot[] Plots)
```

Parameters

Plots

Plot

Plot plots in new or existing window.

```
void Plot(string PlotID, params Plot[] Plots)
```

Parameters

PlotID	window identification
Plots	Plot definitions

Print

Print console window

```
void Print()
```

Print

Print text

```
void Print(string text)
```

Show

Show Console Window

```
void Show()
```

Show

Show bool

```
void Show(string text, bool a)
```

Parameters

text	Variable description
a	bool variable

Show

Show int, integer

```
void Show(string text, int r)
```

Parameters

text	Variable description
r	integer variable

Show

Show int[], integer array

```
void Show(string txt, int[] a)
```

Parameters

txt	Variable description
a	integer array

Show

Show double

```
void Show(string text, double r)
```

Parameters

text	Variable description
r	double variable

Show

Show complex

```
void Show(String text, complex c)
```

Parameters

text	Variable description
c	complex variable

Show

Show complex List

```
void Show(string txt, List<complex> a)
```

Parameters

txt	Variable description
a	list of complex variables

Show

Show double[], double array

```
void Show(string txt, double[] a)
```

Parameters

txt	Variable description
a	double array

WriteLine

Write empty line to console

```
void WriteLine()
```

WriteLine

Write line to console

```
void WriteLine(string txt)
```

Parameters

txt	text string

2.5 PlotSeries

Class PlotSeries

Class for definition of plot series.

```
[Serializable]
class PlotSeries
```

Constructors

`PlotSeries()`

Default constructor for serilization

```
PlotSeries(string Text, double Min, double Max, Vector Values):
this(Text, Min, Max, SeriesColor.Black, Values)
```

Constructor. If both Min = 0.0 and Max = 0.0 uses the NiceMin and NiceMax routines to find suitable Min and Max values

Parameters

Text	Series desciption
Min	Minimum plot value
Max	Maximum plot value
Values	Vector with series values

`PlotSeries(string Text, double Min, double Max, SeriesColor Color, Vector Values)`

Parameters

Text	Series desciption
Min	Minimum plot value
Max	Maximum plot value
Color	series color
Values	Vector with series values

Properties

Color

Series color

`SeriesColor Color {set; get;}`

Max

Maximum plot value

`double Max {set; get;}`

Min

Minimum plot value

`double Min {set; get;}`

Text

Series desciption

`string Text {set; get;}`

Values

Plot values

```
Vector Values {set; get;}
```

Methods

Equal

Equal Plotseries objects

```
static bool Equal(PlotSeries a, PlotSeries b)
```

Parameters

a	Plotseries object a
b	Plotseries object a
returns	true if equal

NiceMax

Nice Maximum value for plot (example x = 0.75464, NiceMax(x) = 0.8)

```
static double NiceMax(double x)
```

Parameters

x	value

NiceMin

Nice Minimum value for plot

```
static double NiceMin(double x)
```

Parameters

x

ScalePlot

Scale Plot

```
void ScalePlot()
```

Enumeration SeriesColor

Series Colors

```
enum SeriesColor
```

Fields
- Black
- Blue
- Brown
- Cyan
- DarkBlue
- DarkGreen
- DarkRed
- Gray
- Green
- LightGray
- Magenta
- Orange
- Red
- Yellow

2.6 Plot

Class **Plot**

Plot definition class

```
[Serializable]
class Plot
```

Constructors

`Plot()`

Constructor

`Plot(params PlotSeries[] Series)`

Constructor for default plot with PlotType.TY . Up to four series can be specified
Parameters
- Series

`Plot(PlotType Type)`

Constructor for plot plot type, series must be set using property Series
Parameters
- Type Plot type

`Plot(PlotType Type, params PlotSeries[] Series)`

Constructor for up to four series.
Parameters
- Type Plot type
- Series Plot Series

```
Plot(PlotType Type, double XMin, double XMax)
```

Constructor fro a plot type, series must be set separately

Parameters

Type	Plot type
XMin	Min x-axis value
XMax	Max x-axis value

```
Plot(PlotType Type, double XMin, double XMax, params PlotSeries[] Series)
```

Constructor a given plottype with up to fopur series

Parameters

Type	Plot type
XMin	Min x-axis value
XMax	Max x-axis value
Series	Plot Series

Properties

PlotSeries

Plot Series

```
PlotSeries[] PlotSeries {set; get;}
```

Type

Plot Type

```
PlotType Type {set; get;}
```

XMax

X axis maximum value

```
double XMax {set; get;}
```

XMin

X axis minimum value

```
double XMin {set; get;}
```

Enumeration PlotType

Plot type

```
enum PlotType
```

Fields

MPC	MPC Operator plot
TBar	TY with bars (only one series)
TY	TY plots Time for x-axis , series for Y axix
XY	Y versus X Plot

2.7 MPCMathWindow.xaml

Class MPCMathWindow

MPCMath Console window (MPCMathWindow.xaml). This is mainly for internal use

```
partial class MPCMathWindow : Window
```

Constructors

```
MPCMathWindow(bool EnableHide)
```

Constructor for MPCMathWindow
Parameters

EnableHide	Enable Hide field in menu

Properties

Text

Get window text

```
string Text {get;}
```

Methods

AddLine

Add line to text area

```
void AddLine(string Line)
```

Parameters

Line	Text line

Clear

Clear Text area

```
void Clear()
```

Chapter 3

Vectors and Matrices

Linear algebra is handled using Vector and Matrix objects.

Vector and Matrix objects supports operator over loads, documented as

operator

```
static Vector operator *(Matrix a, Vector x)
```

Parameters

a	a matrix
x	x vector
returns	result vector

means that the compiler accepts the $A * x$ statement in the code below:

```
int dim = 5;
Vector x = Vector.Random(dim,-100.0, 100.0);
Matrix A = Matrix.Random(dim,dim, -100.0,100.0);
Vector res = A * x;
```

The individual elements of a vector can be read or set using $[pos], [row, col,]$ statements as below:

```
double velm = x[3];
double aelm = A[0,4];
x[2] = 1.3 * x[3] / A[4, 4] + velm;
```

The Vector class is build on the base class $VBase < double >$ and implements the interface $ICommon < Vector >$. VBase provides basic vector functionalities and ICommon allows the Vector object to be included as elements in Block and Generic Vectors and Matrices.

The Matrix class is build on the base class $MBase < double >$ and implements the interface $ICommon < Matrix >$. MBase provides basic matrix functionalities and ICommon allows the Matrix object to be included as elements in Block and Generic Vectors and Matrices.

VBase, Mbase and ICommon are documented in appendix C.

Matrices are created as null matrices as a default. The different MatrixForms can be created using special constructors. Examples are:

```
// Null matrix with 5 rows and 5 columns
Matrix B = new Matrix(dim);

// Constant diagonal matrix with 5 rows and 5 columns
Matrix C = new Matrix(MatrixForm.ConstantDiagonal, 5);

// Diagonal matrix with 6 rows and 7 columns
Matrix D = new Matrix(MatrixForm.Diagonal, 6,7);

// General matrix with 10 rows and 20 columns
Matrix e = new Matrix(MatrixForm.General, 10, 20);
```

Addressing non diagonal element in a diagonal matrix, $C[3,4]$, creates an indexing error. The best way to locate the sinner is to look in the call stack until you find your call. Forgetting to specify the MatrixForm also generates a lot of indexing errors.

The form of a Matrix is read or changed using the Form property.

```
// Upgrade C matrix to a general matrix
C.Form = MatrixForm.General;
```

Upgrades to a Matrix preserves data. Attempts to degrade the status of a Matrix are ignored.

3.1 Sparse Matrices

MPCMath implements sparse matrices using the PARDISO storage scheme described in appendix G. The sparse matrices are used by the SparseSolver 8.7 and the sparse format are supported for matrix / vector and matrix/ matrix multiplications.

3.2 Vector

Class Vector

Vector class

```
[Serializable]
class Vector : VBase<double>, ICommon<Vector>
```

Constructors

```
Vector()
: base(0)
```

Default constructor for serialization

```
Vector(int n)
: base(n)
```

Constructor
Parameters
n Dimension

```
Vector(int n, double val)
: base(n, val)
```

vector with equal values
Parameters
n Dimension
val Value

```
Vector(params double[] vals)
: base(vals)
```

Constructor from array of doubles. Vector dimension equal to array length
Parameters
vals Array of doubles

Methods

Add

Add Vector, y = y + x

```
void Add(Vector x)
```

Parameters
x x vector

Add

Add constant to vector y[i] = y[i] + a

```
void Add(double a)
```

Parameters
a constant

AssertEqual

assert equal, Compare two vectors and output value if not equal

```
static void AssertEqual(string text, Vector actual, Vector expected)
```

Parameters

text	Text string
actual	Vector with actual values
expected	Vector with expected values

Axpy

Add Vector (y = a*x + y)

```
void Axpy(double a, Vector x)
```

Parameters

a	a factor
x	x vector

Covr

Covariance vector cov[k]

```
static Vector Covr(int kFirst, int kLast, Vector a, Vector b)
```

Parameters

kFirst	first k
kLast	end k
a	Input vector
b	Input vector
returns	Vector of covariances

DivElements

Divide Vector elements, res[i] = y[i]/x[i]

```
static Vector DivElements(Vector y, Vector x)
```

Parameters

y	y vector
x	z vector
returns	result vector

Equal

Equal, compare two vectors. (Error limit given in MPCMathLib.MaxError)

```
static bool Equal(Vector y, Vector x)
```

Parameters

y	Vector y
x	Vector x
returns	true if equal

Gemv

Gemv, y := alpha * op(a) * x + beta * y

```
void Gemv(Matrix a, Vector x, MatrixOp opr, double alpha, double beta)
```

Parameters

a	Matrix a
x	Vector x
opr	MatrixOp N,T
alpha	constant
beta	constant

Get

Get subvector

```
Vector Get(int pos, int dim )
```

Parameters

pos	Initial position
dim	Dimension of subvector
returns	Subvector

Get

Get subvector with increments

```
Vector Get(int pos, int incr, int dim)
```

Parameters

pos	Initial position
incr	Increment
dim	Dimension of subvector
returns	Subvector

Max

Get Maximum value

```
double Max()
```

Parameters

returns	Maximum value of vector elements

Mean

Get Mean value

```
double Mean()
```

Parameters

returns	Mean value of vector elements

Min

Get Minimm value

```
double Min()
```

Parameters

returns	Minimum value of vector elements

MulElements

Multiply Vecor elements, res[i] = y[i]*x[i]

```
static Vector MulElements(Vector y, Vector x)
```

Parameters

y	y vector
x	x vector
returns	res vector

Norm

Euclidian norm of Vector

```
double  Norm()
```

Parameters

returns	Norm of vector elements

Normalize

Normalize vector (Making Norm = 1.0)

```
void Normalize()
```

NormInf

Infinity Norm of vector

```
double NormInf()
```

operator

Clone vector operatos, res = +x

```
static Vector operator +(Vector x)
```

Parameters

x	Input vector
returns	Clone of x

operator

Add vector operator, res = y + x

```
static Vector operator +(Vector y, Vector x)
```

Parameters

y	y vector
x	

operator

Negate vector, res = -x

```
static Vector operator -(Vector x)
```

Parameters

x	x vector

operator

Sub vector operator, res = y - x

```
static Vector operator -(Vector y, Vector x)
```

Parameters

y	y vector
x	x vector
returns	y - x

operator

Multiply operator res = y'*x

```
static double operator *(Vector y, Vector x)
```

Parameters

y	y vector
x	x vector
returns	Inner product of vectors

operator

Matrix * vector operator, res = A * x

`static Vector operator *(Matrix a, Vector x)`

Parameters

a	a matrix
x	x vector
returns	result vector

operator

Vector' * Matrix operator, res = x'*A

`static Vector operator *(Vector x, Matrix a)`

Parameters

x	x vector
a	a matrix
returns	res vector

operator

Multiply scalar and Vector operator, res[i] = alpha* x[i]

`static Vector operator *(double alpha, Vector x)`

Parameters

alpha	constant
x	x vector
returns	res vector

operator

Multiply scalar and Vector operator, res[i] = x[i]* alpha

`static Vector operator *(Vector x, double alpha)`

Parameters

x	x vector
alpha	constant
returns	res vector

operator

Divide vector with scalar, res[i] = x[i]/alpha

`static Vector operator /(Vector x, double alpha)`

Parameters

x	x vector
alpha	constant
returns	res vector

operator

Add scalar to vector, res[i] = x[i] + alpha

```
static Vector operator +(Vector x, double alpha)
```

Parameters

x	x vector
alpha	constant
returns	res vector

operator

Add scalar to vector, res[i] = alpha + x[i]

```
static Vector operator +(double alpha, Vector x)
```

Parameters

alpha	constant
x	x vector
returns	res vector

operator

Subtract scalar from vector res[i] = x[i] - alpha

```
static Vector operator -(Vector x, double alpha)
```

Parameters

x	x vector
alpha	constant
returns	res vector

operator

Subtract scalar from vector res[i] = alpha - x[i]

```
static Vector operator -(double alpha, Vector x)
```

Parameters

alpha	constant
x	x vector
returns	res vector

ProperZeroes

Convert small values to proper xeroes

```
void ProperZeroes()
```

Random

Random vector

```
static Vector Random(int n, double min, double max)
```

Parameters

n	Dimension
min	Minimum value of random elements
max	Minimum value of random elements
returns	Vector with random values

Random

Random vector

```
static Vector Random(Vector XMin, Vector XMax)
```

Parameters

XMin	Minimum values
XMax	Maximum values
returns	Vector with random values

Set

Set, insert subvector

```
void Set(int pos, Vector a)
```

Parameters

pos	Start position
a	Subvector

Set

Set subvector with increments

```
void Set(int pos, int incr, Vector a)
```

Parameters

pos	Start position
incr	Increment
a	Subvector

Show

Show vector on console

```
static void Show(string txt, Vector a)
```

Parameters

txt	Text string
a	Vector

Show

Show Vector

```
void Show(string txt)
```

Parameters
  txt   Text string

Sort

Sort elements in vector descenting

```
void Sort()
```

Sub

Sub Vector, y = y - x

```
void Sub(Vector x)
```

Parameters
  x   x vector

Sub

Subtract constant from vector y[i] = y[i] - a

```
void Sub(double a)
```

Parameters
  a   constant

Sum

Get Sum of elements

```
double Sum()
```

Parameters
  returns   Sum of all element in vector

Variance

Get variance

```
double Variance()
```

Parameters
  returns   Variance of vector elements

3.3 MatrixForm

Enumeration MatrixForm

Matrix Form Enumerator

```
enum MatrixForm
```

Fields

ConstantDiagonal	Matrix with constant diagonal elemenets
Diagonal	Diagonal matrix
General	General matrix
Null	Null matrix
Sparse	Sparse matrix in PARDISO format

Enumeration MatrixOp

Operator Enumerator, for BLAS, LAPACK routines

```
enum MatrixOp
```

Fields

C	Conjungate Transpose (complex)
N	No tranpose
T	Transpose

Enumeration UPLO

Upper or Lower part of Matrix for BLAS, LAPACK routines

```
enum UPLO
```

Fields

Lower	Lower triangular
Upper	Upper triangular

Enumeration Side

Side of matrix, for BLAS, LAPACK routines

```
enum Side
```

Fields

Left	Left side
Right	Right side

Enumeration **Diagonal**

Type of diagonal matrix, for BLAS, LAPACK routines

```
enum Diagonal
```

Fields

NonUnit	Unit matrix
Unit	Non unit matrix

Enumeration **SparseMatrixType**

Sparse Matrix type enumeration.

```
enum SparseMatrixType
```

Fields

RealUnsymmetrical	Real and unsymmetrical matrix
StructurallySymmetrical	Real and structurally symmetrical matrix
SymmetricalIndefinite	Real and symmetric indfinite matrix
SymmetricalPositiveDefinit	Real and symmetrical positive definite matrix

3.4 Matrix

Class **Matrix**

Matrix class

```
[Serializable]
class Matrix : MBase<double> , ICommon<Matrix>
```

Constructors

```
Matrix()
: base(0)
```

Constructor default for serialization

```
Matrix(int n) : base(n)
```

Constructor square matrix (Form = Null)
Parameters

n	Dimenson

```
Matrix(MatrixForm Form, int n) : base(Form, n)
```

Constructor square matrix
Parameters

Form	Matrix form
n	Dimenson

```
Matrix(int n, int m) : base(n, m)
```

Constructor n x m matrix, (Form = Null)
Parameters

n	Rows
m	Columns

```
Matrix(MatrixForm Form, int n, int m): base(Form, n, m)
```

Constructor n x m matrix
Parameters

Form	Matrix form
n	Rows
m	Columns

```
Matrix(int n, double val) : base(n, val)
```

Constructor ConstantDiagonal Matrix (Form = ConstantDiagonal)
Parameters

n	Dimension
val	value

```
Matrix(int n, int m, double[] vals): base(n, m, vals)
```

Constructor general matrix from array of values
Parameters

n	Rows
m	Columns
vals	Value array

```
Matrix(Vector a) : base(a.values)
```

Constructor Diagonal matrix (Form = Diagonal)
Parameters

a	Vector of diagonal values

```
Matrix(params double[] vals) : base(vals)
```

Constructor Diagonal matrix (Form = Diagonal)
Parameters

vals	array of values

Properties

Density

Matrix density (0.0 - 1.0) Null -¿ 0.0 General -¿ 1.0

```
double Density {get;}
```

MatrixForm

Matrix form (Null, ConstantDiagonal, Diagonal, Sparse or General) Moving from Null towards General preserves data Moving from General to Null, ConstantDiagonal and Diagonal are ignored. Moving from General to Sparse is executed

```
override MatrixForm Form {set; get;}
```

SMType

Sparse Matrix type

```
SparseMatrixType SMType {set; get;}
```

Methods

Add

Add, B = B + A

```
void Add(Matrix a)
```

Parameters

a	Matrix to be added

AddScaled

Add scaled this = alpha * a + this;

```
void AddScaled(double alpha, Matrix a)
```

Parameters

alpha	Constant
a	Input matrix

AddScaled

res = alpha* this

```
Matrix AddScaled(double alpha)
```

Parameters

a
txt
alpha

AssertEqual

Assert equal, Show actual and expected matrices are not equal

```
static bool AssertEqual(string text, Matrix actual, Matrix expected)
```

Parameters

text	text string
actual	actual matrix
expected	expected matrix
returns	true if equal

EigenValues

Eigenvalues

```
static CVector EigenValues(Matrix a)
```

Parameters

a	Input matrix
returns	Complex vector with eigenvalues

Equal

Equal matrices, test whether two Matrices are equal

```
static bool Equal(Matrix a, Matrix b)
```

Parameters

a	Input matrix a
b	Input Matrix b
returns	true if equal

Exp

Exponential function

```
static Matrix Exp(Matrix A)
```

Parameters

A

Gemm

General Matrix Multiplication this = alpha*Op(a)*Op(b) + beta * this

```
void Gemm(Matrix a, Matrix b, MatrixOp opra,
MatrixOp oprb, double alpha, double beta)
```

Parameters

a	input matrix a
b	input matrix b
opra	MatrixOp direct or transposed
oprb	MatrixOp direct or transposed
alpha	constant
beta	constant

Ger

Rank1 update of Matrix a := alpha * x * y' + a

```
void Ger(Vector x, Vector y, double alpha)
```

Parameters

x	input vector x
y	input vector y
alpha	constant

Get

Get Submatrix

```
Matrix Get(int row, int n, int col, int m)
```

Parameters

row	Row position
n	Rows
col	Column position
m	Columns
returns	Submatrix

GetColumn

Get GetColumn

```
Vector GetColumn(int col)
```

Parameters

col	Column
returns	Column vector

GetColumn

Get GetColumn

```
Vector GetColumn(int col, int indx, int length)
```

Parameters

col	Column
indx	First row
length	Number of rows

GetRow

Get Row

```
Vector GetRow(int row)
```

Parameters

row	Row
returns	row vector

GetRow

Get Row

```
Vector GetRow(int row, int indx, int length)
```

Parameters

row	Row
indx	First Column position
length	Nomber of columns

Invert

Invert Matrix

```
static Matrix Invert(Matrix a)
```

Parameters

a

Mul

outer product A = x*y'

```
static Matrix Mul(Vector x, Vector y)
```

Parameters

x	input vector x
y	input vector y
returns	outer product matrix

MulAdd

Multiply Add r = r + a*b

```
void MulAdd(Matrix a, Matrix b)
```

Parameters

a	Input matrix a
b	input matrix b

MulTransposed

Multiply tranpose res = a'*b

```
static Matrix MulTransposed(Matrix a, Matrix b)
```

Parameters

a	Input matrix a
b	Input matrix b
returns	a'*b

Norm

Frobenius Norm of matrix

```
double Norm()
```

Norm1

Norm 1 of Matrix

```
double Norm1()
```

NormInf

Norm Infinity of Matrix

```
double NormInf()
```

operator

Clone Matrix, A = +B

```
static Matrix operator +(Matrix a)
```

Parameters

a	Input matrix
returns	clone of input matrix

operator

Negate Matrix, A = -B

```
static Matrix operator -(Matrix a)
```

Parameters

a	Input matrix
returns	Negated input matrix

operator

Add binary operator, Res = A + B

```
static Matrix operator +(Matrix a, Matrix b)
```

Parameters

a	A matrix
b	B matrix
returns	Sum of input matrices

operator

Sub binary operator, Res = A - B

```
static Matrix operator -(Matrix a, Matrix b)
```

Parameters

a	A matrix
b	B matrix

operator

Matrix multiplication operator, Res = A * B

```
static Matrix operator *(Matrix a, Matrix b)
```

Parameters

a	Input matrix a
b	Input matrix b
returns	Product

operator

Multiply matrix with constant, Res = alpha * A

```
static Matrix operator *(double alpha, Matrix a)
```

Parameters

alpha	constant
a	Input matrix
returns	Result matrix

operator

Multiply matrix with constant, Res = A * alpha

```
static Matrix operator *(Matrix a, double alpha)
```

Parameters

a	Input matrix
alpha	constant
returns	Result matrix

operator

Divide matrix with constant, Res = A/alpha

```
static Matrix operator /(Matrix a, double alpha)
```

Parameters

a	Input matrix
alpha	constant
returns	Result matrix

PseudoInvert

Pseudo Invert

```
static Matrix PseudoInvert(Matrix a)
```

Parameters

a	Input matrix
returns	Result matrix

Random

Random Matrix

```
static Matrix Random(int n, int m, double min, double max)
```

Parameters

n	Number of Rows
m	Number of Columns
min	Minimum value
max	Maximum value
returns	Random Matrix

Set

Set sub matrix

```
void Set(int row, int col, Matrix a)
```

Parameters

row	Row position
col	Column position
a	sub matrix

SetColumn

Set Column

```
void SetColumn(int col, Vector Vals)
```

Parameters

col	Column
Vals	Vector of values

SetRow

Set Row

```
void SetRow(int row, Vector Vals)
```

Parameters

row	Row
Vals	Vector of values

Show

Show matrix

```
static void Show(string text, Matrix a)
```

Parameters

text	text string
a	Matrix a

Sub

Sub B = B - A

```
void Sub(Matrix a)
```

Parameters

a	Matrix to be subtracted

SVD

SVD Dcomposition A = U*S*VT

```
static void SVD(Matrix a, out Matrix U, out Vector S, out Matrix VT)
```

Parameters

a	Input matrix A
U	Left side othogonal matrix
S	Singular Values vector
VT	Rigth side orthogonal matrix

ToSparse

Copy Matrix to Sparse Matrix. equivalent to Sparse(SparseMatrixType.RealUnsymmetrical, UPLO.Lower, A)

```
static Matrix ToSparse(Matrix A)
```

Parameters

A	Input matrix
returns	Sparse matrix

ToSparse

Copy Matrix to Sparse Matrix. equivalent to Sparse(type, UPLO.Lower, A)

```
static Matrix ToSparse(SparseMatrixType type, Matrix A)
```

Parameters

type	Sparse matrix Type
A	Input matrix
returns	Sparse matrix

ToSparse

Copy Matrix to Sparse Matrix

```
static Matrix ToSparse(SparseMatrixType type, UPLO UpLo, Matrix A)
```

Parameters

type	Sparse matrix Type
UpLo	Position for values in symmetric values
A	Input matrix
returns	Sparse matrix

Transpose

Transpose matrix

```
static Matrix Transpose(Matrix a)
```

Parameters

a	Input matrix
returns	Transposed input matrix

Trmm

Matrix product with triangular matrix Side = Left this := alpha * Opr(A) * this side = Right this : alpha * this * Opr(A)

```
void Trmm(Side side, UPLO uplo, MatrixOp opr, double alpha, Matrix A)
```

Parameters

side	Position af A matrix
uplo	Upper or lower matrix
opr	Transpose operator
alpha	constant
A	input matrix

UnityMatrix

Unity matrix

```
static Matrix UnityMatrix(int n)
```

Parameters

n	Dimension
returns	Unity matrix

3.5 CVector

Class CVector

Complex vector

```
[Serializable]
class CVector : VBase<complex>
```

Constructors

```
CVector()
: base(0)
```

Constructor for serialization

```
CVector(int n) : base(n)
```

Constructor
Parameters

n	Dimension

```
CVector(int n, complex val) : base(n, val)
```

Constructor, with equal elements
Parameters

n	Dimension
val	complex value

```
CVector(params complex[] vals)
: base(vals)
```

Constructor, form array of complex values
Parameters
vals

```
CVector(List<complex> a) : base(a.Count)
```

Constructor from List of complex values
Parameters

a	List of Complex values

```
CVector(Vector rv, Vector iv)
: base(rv.Dimension)
```

Constructor from real and imaginary values
Parameters

rv	real parts
iv	imaginary parts

Methods

Arg

Get Argument

```
Vector Arg()
```

Parameters

returns	Vector of argument

AssertEqual

Assert equal, Show actual and expected if no equal

```
static bool AssertEqual(string text, CVector actual, CVector expected)
```

Parameters

text	text string
actual	actual value
expected	expected values

Equal

Equal

```
static bool Equal(CVector a, CVector b)
```

Parameters

a	Vector a
b	Vector b
returns	true if equal

EqualRoots

Equal Roots

```
static bool EqualRoots(CVector a, CVector b)
```

Parameters

a	Vector a
b	Vector b
returns	true if equal

IV

Get imaginary values

```
Vector IV()
```

Parameters
 returns Vector of Imaginary values

Mod

Get moduli

```
Vector Mod()
```

Parameters
 returns Vector of Moduli

operator

Clone vector, res = + x

```
static CVector operator +(CVector x)
```

Parameters
 x Input vector

RV

Get real values

```
Vector RV()
```

Parameters
 returns Vector of real values

Show

Show Complex Vector

```
static void Show(string txt, CVector a)
```

Parameters
 txt text string
 a complex vector

Sort

Sort elements in vector descenting according 1 according to real part, 2 according to imag part

```
void Sort()
```

Chapter 4

Block Vectors and Matrices

A block vector is a vector who's elements consist of vectors. A Block matrix is a matrix who's elements consists of matrices. Block vectors and block matrices are very convenient for two reasons. It simplifies the programming task, making it easier to configure matrices for Optimization tasks. MPC controllers are often described in literature using block matrices. Secondly *MPCMath* stores information about the structure of the block matrix and its sub matrices making it possible to exploit structures automatically during program execution. The MatrixForm enumeration is described in C.2.

There are some rules for block vectors and block matrices that must be obeyed. All sub matrices in a block matrix row, must have an equal number rows. All sub matrices in a block matrix column must have the same number of columns. The structure of a block matrix or block vector is recorded in a *Structure* C.1 object.

For a block vector the *Structure* object defines the number of sub vectors and the dimension of each sub vector. The code below defines a a block vector X with three sub vectors. The dimension of the sub vectors are 3, 3 and 1. Three random sub vectors are stored in X and finally a sub vector, $xsub$, is retrieved from X

```
Structure strc = new Structure(3);
strc[0] = 3;
strc[1] = 2;
strc[2] = 1;

BVector X = new BVector(strc);
Structure.Show("x structure", X.Structure);

X[0] = Vector.Random(3, -100.0, 100.0);
X[1] = Vector.Random(2, -100.0, 100.0);
X[2] = Vector.Random(1, -100.0, 100.0);

BVector.Show("x", X);
```

```
Vector xsub = X[1];
Vector.Show("xsub", xsub);
```

The MPCMathConole output is

```
x structure
         3         2         1

x Vector
  -60,3688  -48,1193  -98,3050  -22,4367  -49,2921   82,2724
xsub Vector
  -22,4367  -49,2921
```

The following code defines a block matrix A with random sub matrices. It displays the structure of A and finally demonstrates a multiplication between a block matrix and a block vector.

```
Structure rowStr = Structure.Values(1, 2, 3, 1);
Structure colStr = Structure.Values(3, 2, 1);
BMatrix A = new BMatrix(MatrixForm.General, rowStr, colStr);

A[0, 0] = Matrix.Random(1, 3, -100.0, 100.0);
A[1, 1] = Matrix.UnityMatrix(2);
A[2, 1] = Matrix.Random(3, 2, -100.0, 100.0);

BMatrix.ShowStructure("A structure", A);
BMatrix.Show("A", A);

BVector R = A * X;
BVector.Show("R", R);
```

The MPCMathConole output is

```
A structure Matrix.Form General
General matrix[4][3]

General          Null             Null
Null             ConstantDiagonal Null
Null             General          Null
Null             Null             Null

A Matrix.Form General[7,6]
```

```
  -23,0554  -85,4168   79,6318         0         0         0
         0         0         0         1         0         0
         0         0         0         0         1         0
         0         0         0  -27,7048  -23,4950         0
         0         0         0   19,7783   26,8926         0
         0         0         0   34,7124   74,2146         0
         0         0         0         0         0         0

R Vector
 -1705,0789   -2,0829   90,7942 -2075,5061 2400,4981 6665,9496         0
```

Block vectors and block matrices can be created from vector dimensions or numbers of rows and columns. Then the block matrix builds the structure information, when the sub matrix object are stored in the matrix. In general its safer, and prevents a lot of indexing errors, if you one carefully plan the structure of the task and use *Structure* objects to define the block vectors and matrices.

4.1 Structure

Class Structure

Structure class

```
[Serializable]
class Structure
```

Constructors

`Structure()`

Default constructor for serialization

`Structure(int n)`

Constructor for basic structure
Parameters
n Dimension

`Structure(int n, int val)`

Structure with equal substructures
Parameters
n Dimension
val Substructure dimensions

Properties

Dimension

Dimension

```
int Dimension {get;}
```

FlatDimension

Flat dimension

```
int FlatDimension {get;}
```

Indexer

Get or set value

```
int this[int p] {set; get;}
```

Methods

AssertEqual

assert equal, Compare two Structures and output value if not equal

```
static void AssertEqual(string text, Structure actual, Structure expected)
```

Parameters

text	Text string
actual	Vector with actual values
expected	Vector with expected values

Equal

Test if two structrures are equal

```
static bool Equal(Structure a, Structure b)
```

Parameters

a	Input a
b	Input b
returns	true if equal

operator

Merge structures (warning a + b != b + a)

```
static Structure operator +(Structure a, Structure b)
```

Parameters

a	Input a
b	Input b
returns	a+b

Show

Show Structure

```
static void Show(string txt, Structure a)
```

Parameters

txt	text string
a	Structure object

Values

Construct new Structure from array of values

```
static Structure Values(params int[] vals)
```

Parameters

vals	array of dimensions
returns	Structure

4.2 BVector

Class BVector

Block Vector class

```
[Serializable]
class BVector : Vector<Vector>
```

Constructors

```
BVector()
: base()
```

Constructor default

```
BVector(int n) : base(n)
```

Constructor

Parameters

n	Dimension

```
BVector(int n, Vector val) : base(n,val)
```

Constructor vector with equal values
Parameters

n	Dimension
value	Value

```
BVector(params Vector[] vals)
: base(vals)
```

Generate vector from array of Vectors
Parameters

vals	Vector array

```
BVector(Vector<Vector> a)
: base(a.Dimension)
```

Constructor BVector from $Vector < Vector >$
Parameters
a

```
BVector(Structure s)
```

Construct BVector from structure
Parameters

s	Structure

```
BVector(Structure s, double val)
```

Construct BVector from structure with equal values
Parameters

s	Structue
val	value

```
BVector(Vector a)
: base(1)
```

Constructor, Wrap Vector a in block vector
Parameters
a

Methods

DivElements

Divide Vector elements, z[i] = y[i]/x[i]

```
static BVector DivElements(BVector y, BVector x)
```

Parameters

y	Input y
x	Input X
returns	y[i]/x[i]

Flatten

Flatten BVector to plain Vector

```
static Vector Flatten(BVector a)
```

Parameters

a	Input BVector
returns	Vector

Get

Get subvector

```
new BVector Get(int pos, int dim)
```

Parameters

pos	Start position
dim	Dimension

Get

Get subvectorwith increments

```
new BVector Get(int pos,int incr, int dim)
```

Parameters

pos	Start position
incr	Position incremenst
dim	Dimension

Max

Maximum value

```
double Max()
```

Parameters

returns	Maximum value of all elements in member vectors

Min

Minimum value

```
double Min()
```

Parameters

returns	Minimum value of all elements in member vectors

MulElements

Multiply Vecor elements, z[i] = y[i]*x[i]

```
static BVector MulElements(BVector y, BVector x)
```

Parameters

y	Input y
x	Input X
returns	y[i]*x[i]

Norm

Euclidian norm of Vector

```
double Norm()
```

Parameters

returns	Norm of vector elements

operator

Clone, unary + operator

```
static BVector operator +(BVector a)
```

Parameters

a	input a
returns	+a

operator

Negate, unary -

```
static BVector operator -(BVector a)
```

Parameters

a	Input a
returns	-a

operator

Binary add operator

```
static BVector operator +(BVector a, BVector b)
```

Parameters

a	Input a
b	Input b
returns	a+b

operator

Binary Sub

```
static BVector operator -(BVector a, BVector b)
```

Parameters

a	Input a
b	Input b
returns	a-b

operator

Multiply constant and BVector

```
static BVector operator *(double alpha, BVector a)
```

Parameters

alpha	constant
a	Input a
returns	alpha*a

operator

Multiply BVector and constant

```
static BVector operator *(BVector a, double alpha)
```

Parameters

a	Input a
alpha	Constant
returns	alpha*a

operator

Divide BVector and constant

```
static BVector operator /(BVector a, double alpha)
```

Parameters

alpha	Constant
a	Input a
returns	a/alpha

operator

Multiply , inner product

```
static double operator *(BVector a, BVector b)
```

Parameters

a	Input a
b	Input b
returns	a'*b

operator

Matrix * vector operator

```
static BVector operator *(BMatrix a, BVector x)
```

Parameters

a	Input BMatix a
x	Input BVector x
returns	a*x

operator

Vector * matrix operator

```
static BVector operator *(BVector x, BMatrix a)
```

Parameters

x	BVector x
a	BMatrix a
returns	x'*a

Random

Random BVector

```
static BVector Random(Structure strc, double min, double max)
```

Parameters

strc	Structure
min	Minimum value
max	Maximum value

Restructure

Restructure Vector from plain Vector

```
static BVector Restructure(Structure strct, Vector a)
```

Parameters

strct	Structure
a	Input vecto
returns	Block vector

Show

Show

```
static void Show(string text, BVector a)
```

Parameters

text	text string
a	BVector object

Show

Show first elements of each subvector

```
static void Show(string text, BVector a, int dim)
```

Parameters

text	text string
a	BVector object
dim	Number of elemenets to show from each subvector

4.3 BMatrix

Class BMatrix

Blok Matrix class

```
[Serializable]
class BMatrix : Matrix<Matrix>
```

Constructors

```
BMatrix()
: base()
```

Constructor default

```
BMatrix(int n)
: base(n)
```

Constructor square matrix as Null matrix
Parameters

n	Dimension

```
BMatrix(MatrixForm Form, int n)
: base(Form, n)
```

Constructor square matrix
Parameters

Form	Matrix form
n	Dimension

```
BMatrix(int n, int m)
: base(n, m)
```

Constructor n x m matrix as Null matrix
Parameters

n	Rows
m	Columns

```
BMatrix(MatrixForm Form, int n, int m)
: base(Form, n, m)
```

Constructor n x m matrix
Parameters

Form	Matrix Form
n	Rows
m	Columns

```
BMatrix(int n, Matrix val)
: base(n, val)
```

Constructor ConstantDiagonal Matrix
Parameters

n	Dimension
val	Value

```
BMatrix(BVector a)
: base(a.Dimension)
```

Constructor Diagonal matrix, from BVector
Parameters

a	Diagonal elements

```
BMatrix(params Matrix[] vals)
: base(vals)
```

Constructor Diagonal matrix
Parameters

vals	Diagonal sub matrices

```
BMatrix(Structure Structure) : base(Structure.Dimension)
```

Constructor for square matrix as null matrix
Parameters

Structure	Structure for rows and Columns

```
BMatrix(MatrixForm form, Structure Structure)
: base(form, Structure.Dimension)
```

Constructor for square matrix
Parameters

form	Matrix Form
Structure	Structure for rows and Columns

```
BMatrix(Structure RowStructure, Structure ColStructure)
: base(RowStructure.Dimension, ColStructure.Dimension)
```

Constructor for general matrix as Null matrix
Parameters

RowStructure	Row Structure
ColStructure	Column Structure

```
BMatrix(MatrixForm form, Structure RowStructure, Structure ColStructure)
: base(RowStructure.Dimension, ColStructure.Dimension)
```

Constructor for general matrix
Parameters

form	Matrix Form
RowStructure	Row Structure
ColStructure	Column Structure

```
BMatrix(Matrix<Matrix> a)
: base(a.form, a.rows, a.columns)
```

Constructor BMatrix from generic Matrix¡Matrix¿ Obs Values and structures are copied by reference, not cloned
Parameters
a Input a

```
BMatrix(Matrix a)
: base(1)
```

Constructor , Wrap matrix a in Block Matrix
Parameters
a Input matrix

Methods

Allocate

Allocate Matrix element. If null element a new matrix is allocated. If element exist matrix form is changed:

```
void Allocate(int row, int col)
```

Parameters
row
col
form

Allocate

Allocate Matrix element. (Empty elements of a Block matrix are nor allocated by the constructor).

```
void Allocate(int row, int col, MatrixForm form)
```

Parameters
row
col
form

Flatten

Flatten BMatrix

```
static Matrix Flatten(BMatrix a)
```

Parameters

a	Input Bmatrix
returns	Plain Matrix

Get

Get submatrix

```
new BMatrix Get(int row, int rows, int col, int cols)
```

Parameters

row	Row position
rows	Number of Rows
col	Column position
cols	Number of Columns
returns	Submatrix

MulTransposed

Multiply transposed a'*b

```
static BMatrix MulTransposed(BMatrix a, BMatrix b)
```

Parameters

a	Input a
b	Input b
returns	a'*b

operator

Clone, + operator

```
static BMatrix operator +(BMatrix a)
```

Parameters

a	Input a
returns	+a

operator

Negate, - operator

```
static BMatrix operator -(BMatrix a)
```

Parameters

a	Input a
returns	-a

operator

Binary add

```
static BMatrix operator +(BMatrix a, BMatrix b)
```

Parameters

a	Input a
b	Input b
returns	a+b

operator

Binary sub

```
static BMatrix operator -(BMatrix a, BMatrix b)
```

Parameters

a	Input a
b	Input b
returns	a-b

operator

Multiply

```
static BMatrix operator *(BMatrix a, BMatrix b)
```

Parameters

a	Input a
b	Input b
returns	a*b

operator

Multiply with constant

```
static BMatrix operator *(double alpha, BMatrix a)
```

Parameters

alpha	constant
a	Input a
returns	alpha*a

operator

Multiply with constant

```
static BMatrix operator *(BMatrix a, double alpha)
```

Parameters

a	Input a
alpha	Constant

operator

Divede by constant

```
static BMatrix operator /(BMatrix a, double alpha)
```

Parameters

a	Input a
alpha	constant
returns	a/alpha

OuterProduct

Ouher Product of two BVectors

```
static BMatrix OuterProduct(BVector a, BVector b)
```

Parameters

a	Input a
b	Input b
returns	BMatrix

Random

Random Block Matrix

```
static BMatrix Random(Structure rowStrc, Structure colStrc, double min, double max)
```

Parameters

rowStrc	Row structure
colStrc	Column structure
min	Minimum value
max	Maximum value
returns	Random BMatrix

Restructure

Restructure Matrix

```
static BMatrix Restructure(Structure rowStr,Structure colStr, Matrix a)
```

Parameters

rowStr	Row Structure
colStr	Columns structure
a	Input matrix
returns	BMatrix

Show

Show

```
static void Show(string text, BMatrix a)
```

Parameters

text	Text string
a	BMatrix object

ShowStructure

Show matrix structure

```
static void ShowStructure(string text, BMatrix A)
```

Parameters

text	text string
A	BMatrix object

ToSparse

Copy Block Matrix to sparse matrix Equivalent to ToSparse(SparseMatrixType.RealUnsymmetrical, UPLO.Lower, A);

```
static Matrix ToSparse(BMatrix A)
```

Parameters

A	Input matrix
returns	Sparse matrix

ToSparse

Copy Block Matrix to sparse matrix Equivalent to ToSparse(type, UPLO.Lower, A);

```
static Matrix ToSparse(SparseMatrixType type, BMatrix A)
```

Parameters

type	Sparse matrix Type
A	Input matrix
returns	Sparse matrix

ToSparse

Copy Block Matrix to sparse matrix

```
static Matrix ToSparse(SparseMatrixType type, UPLO UpLo, BMatrix A)
```

Parameters

type	Sparse matrix Type
UpLo	Position for values in symmetric values
A	Input matrix
returns	Sparse matrix

Transpose

Transpose

```
static BMatrix Transpose(BMatrix a)
```

Parameters

a	Input a
returns	a'

Chapter 5

Generic Vectors and Matrices

The main part of the block vector 4.2 and block matrix 4.3 functionality is build using the generic Vector and generic matrices classes.

Generic vector and matrices are populated with objects that implements the interface *ICommon* 5.1 and a default new T() constructor. In *MPCMath* this is the case for the Classes complex **??**, Vector 3.2, Matrix 3.4. For the classes ArxModel 10.1, LinearFilter 9.1, TransferFunction 6.1 there is a limited implementation of ICommon, that does not support calculations.

The following code demonstrates generic vector and matrices for complex objects.

```
int dim = 4;
Vector<complex> x = new Vector<complex>(dim);
Matrix<complex> A =
    new Matrix<complex>(MatrixForm.General, dim, dim);

// fill in some crazy numbers
for (int row = 0; row < dim; row++)
{
    x[row] = complex.Sqrt(-row);
    for (int col = 0; col < dim; col++)
    {
        A[row, col] = new complex(row, col);
    }
}

Vector<complex>.Show("x", x);
Matrix<complex>.Show("A", A);

// Matrix calculation
Vector<complex> Res = A * x;
```

```
Vector<complex>.Show("Res", Res);
```

The console output is

Test Generic Matrix x MPCMath.Vector‘1[MPCMath.complex] [0] 0,0000 + i* 0,0000

[1] 0,0000 + i* 1,0000

[2] 0,0000 + i* 1,4142

[3] 0,0000 + i* 1,7321

A Matrix.Form General Size [4,4] Pos [0,0] 0,0000 + i* 0,0000

Pos [0,1] 0,0000 + i* 1,0000

Pos [0,2] 0,0000 + i* 2,0000

Pos [0,3] 0,0000 + i* 3,0000

Pos [1,0] 1,0000 + i* 0,0000

Pos [1,1] 1,0000 + i* 1,0000

Pos [1,2] 1,0000 + i* 2,0000

Pos [1,3] 1,0000 + i* 3,0000

Pos [2,0] 2,0000 + i* 0,0000

Pos [2,1] 2,0000 + i* 1,0000

Pos [2,2] 2,0000 + i* 2,0000

Pos [2,3] 2,0000 + i* 3,0000

Pos [3,0] 3,0000 + i* 0,0000

Pos [3,1] 3,0000 + i* 1,0000

Pos [3,2] 3,0000 + i* 2,0000

Pos [3,3] 3,0000 + i* 3,0000

Res MPCMath.Vector‘1[MPCMath.complex] [0] -9,0246 + i* 0,0000

[1] -9,0246 + i* 4,1463

[2] -9,0246 + i* 8,2925

[3] -9,0246 + i* 12,4388

The generic Show routine cannot produce compact printouts.

5.1 ICommon

Interface **ICommon**

Common functions for elements in Generic Vectors and Matrices

```
interface ICommon<T>
```

Properties

Columns

Columns

```
int Columns {get;}
```

Form

Object form

```
MatrixForm Form {get;}
```

Rows

Rows

```
int Rows {get;}
```

Methods

AddScaled

Add/Clone +, alpha * this

```
T AddScaled(double alpha)
```

Parameters

alpha	constant

AddScaled

Add this + alpha * a

```
T AddScaled(double alpha, T a)
```

Parameters

alpha	constant
a	object

Equal

Equal

```
bool Equal(T a)
```

Parameters

a	Compared to
returns	true if equal

MulAdd

MulAdd this = this + a x b

```
void MulAdd(T a, T b)
```

Parameters

a	object a
b	object b
returns	Result object

Restructure

Restructure Object

```
void Restructure(int Rows, int Cols)
```

Parameters
form
Rows
Cols

Show

Show object

```
void Show(string txt)
```

Parameters

txt	text string
a	object

TMulAdd

Transpose a and MulAdd this = this + a' x b

```
void TMulAdd(T a, T b)
```

Parameters

a	object a
b	object b

Transpose

Transpose

```
T Transpose()
```

Parameters

returns	Transposed object

Enumeration **ObjectType**

Object type

```
enum ObjectType
```

Fields
  Element
  Matrix
  Vector

5.2 TVector

Class **Vector**

Generic Vector class

```
[Serializable]
class Vector<T> : VBase<T>, ICommon<VBase<T>> where T : ICommon<T>, new()
```

Constructors

```
Vector(int n)
: base(n)
```

Constructor
Parameters
  n

```
Vector(int n, T val)
: base(n, val)
```

vector with equal values
Parameters
  n
  val

```
Vector(params T[] vals): base(vals)
```

Constructor
Parameters
  vals

```
Vector(Structure s):base(s.Dimension)
```

Construct Vector from structure
Parameters
  s

Properties

T

Set or get matrix value

```
override T this[int pos] {set; get;}
```

Methods

AssertEqual

assert equal, Show if not equal

```
static void AssertEqual(string text, Vector<T> actual, Vector<T> expected)
```

Parameters
text
actual
expected

Equal

Equal

```
static bool Equal(Vector<T> y, Vector<T> x)
```

Parameters

y	Vector y
x	Vector x
returns	true if equal

Get

Get subvector

```
Vector<T> Get(int pos, int dim )
```

Parameters

pos	Position
dim	Dimension
returns	Subvector

Get

Get subvector with increment

```
Vector<T> Get(int pos, int incr, int dim)
```

Parameters

pos	Position
incr	Increment
dim	Dimension
returns	Subvector

operator

Unary add/clone operator +

```
static Vector<T> operator +(Vector<T> a)
```

Parameters

a	Input a
returns	+a

operator

Unary Sub/clone operator -

```
static Vector<T> operator -(Vector<T> a)
```

Parameters

a	Input a
returns	-a

operator

Add operator +

```
static Vector<T> operator +(Vector<T> a, Vector<T> b)
```

Parameters

a	Input a
b	Input b
returns	a+b

operator

Vector * Vector , inner product

```
static Vector<T> operator *(Vector<T> a, Vector<T> b)
```

Parameters

a	Input a
b	Input b
returns	a+b

operator

Vector * Matrix operator

```
static Vector<T> operator *(Matrix<T> a, Vector<T> x)
```

Parameters

a	Input a
x	Input b
returns	a*x

operator

Matrix * Vector operator

```
static Vector<T> operator *(Vector<T> x, Matrix<T> a)
```

Parameters

a	Input a
x	Input b
returns	x'*a

operator

Scalar vector multiply

```
static Vector<T> operator *(double alpha, Vector<T>a)
```

Parameters

alpha	constant
a	Input a
returns	alpha*a

operator

Scalar vector multiply

```
static Vector<T> operator *(Vector<T> a, double alpha)
```

Parameters

a	Input a
alpha	constant
returns	alpha*a

operator

Vector Scalar divide

```
static Vector<T> operator /(Vector<T> a, double alpha)
```

Parameters

alpha	constant
a	Input a

operator

Sub operator -

```
static Vector<T> operator -(Vector<T> a, Vector<T> b)
```

Parameters

a	Input a
b	Input b
returns	a-b

Set

Set subvector

```
void Set(int pos , Vector<T>a)
```

Parameters

pos	position
a	sub vector

Set

Set subvector with increments

```
void Set(int pos, int incr ,Vector<T> a)
```

Parameters

pos	position
incr	increment
a	sub vector

Show

Show

```
static void Show(string txt, Vector<T> a)
```

Parameters

txt	text string
a	Object

5.3 TMatrix

Class Matrix

Generic Matrix class

```
[Serializable]
class Matrix<T>: MBase<T>, ICommon<Matrix<T>>  where T : ICommon<T>, new()
```

Constructors

```
Matrix(int n)
: base(n)
```

Constructor square matrix, null form

Parameters

n	Dimension

```
Matrix(MatrixForm Form, int n)
: base(Form, n)
```

Constructor square matrix
Parameters

Form	Matrix Form
n	Dimension

```
Matrix(int n, int m)
: base(n, m)
```

Constructor n x m matrix
Parameters
n
m

```
Matrix(MatrixForm Form, int n, int m)
: base(Form, n, m)
```

Constructor n x m matrix
Parameters

n	Rows
m	Columns

```
Matrix(int n, T val)
: base(n, val)
```

Constructor ConstantDiagonal Matrix
Parameters

n	Dimension
val	value of all diagonal elements

```
Matrix(Vector<T> a)
: base(a.values)
```

Constructor Diagonal matrix
Parameters

a	Vector vith diagonal values

```
Matrix(params T[] vals)
: base(vals)
```

Constructor Diagonal matrix
Parameters

vals	Array of diagonal values

```
Matrix(Structure Structure) : base(Structure.Dimension)
```

Constructor for square matrix Null matrix
Parameters

Structure	Structure

```
Matrix(MatrixForm form, Structure Structure)
: base(form, Structure.Dimension)
```

Constructor for square matrix
Parameters

form	Matrix Form
Structure	Structure

```
Matrix(Structure RowStructure, Structure ColStructure)
: base(RowStructure.Dimension, ColStructure.Dimension)
```

Constructor, null matrix
Parameters

Structure	Structure

```
Matrix(MatrixForm form, Structure RowStructure, Structure ColStructure)
: base(RowStructure.Dimension, ColStructure.Dimension)
```

Constructor
Parameters

form	Matrix Form
RowStructure	Row structure
ColStructure	Column structure

Properties

T

Set or get matrix value

```
override T this[int row, int col] {set; get;}
```

Methods

Add

Add r = r + a

```
void Add(Matrix<T> a)
```

Parameters

a	Input a

AddScaled

Add scaled r = alpha * a + r;

```
void AddScaled(double alpha, Matrix<T> a)
```

Parameters

alpha	constant
a	Input a

AssertEqual

Assert equal, Show if a not equal to b

```
static bool AssertEqual(string text, Matrix<T> actual, Matrix<T> expected)
```

Parameters
text
actual
expected

Equal

Equal generic Matrix

```
static bool Equal(Matrix<T> a, Matrix<T> b)
```

Parameters

a	Input a
b	Input b
returns	true if a equal to b

Get

Get submatrix

```
Matrix<T> Get(int row, int col, int rows, int cols)
```

Parameters

row	Row position
col	Column position
rows	Number of rows
cols	Number of columns
returns	Sub matrix

MulAdd

MulAdd r = rs + a*b

```
void MulAdd(Matrix<T> a, Matrix<T> b)
```

Parameters

a	Input a
b	Input b

MulTransposed

Multiply Transposed res = a'*b

```
static Matrix<T> MulTransposed(Matrix<T> a, Matrix<T> b)
```

Parameters

a	Input a
b	INput b
returns	a'*b

operator

Clone Matrix

```
static Matrix<T> operator +(Matrix<T> a)
```

Parameters

a	Input a
returns	-a

operator

Negate Matrix

```
static Matrix<T> operator -(Matrix<T> a)
```

Parameters

a	Input a
returns	-a

operator

Add binary operator

```
static Matrix<T> operator +(Matrix<T> a, Matrix<T> b)
```

Parameters

a	Input a
b	Input b
returns	a+b

operator

Sub binary operator

```
static Matrix<T> operator -(Matrix<T> a, Matrix<T> b)
```

Parameters

a	Input a
b	Input b
returns	a+b

operator

Matrix constant multiplikation

```
static Matrix<T> operator *(Matrix<T> a, double alpha)
```

Parameters

a	Input a
alpha	constant
returns	alpha*a

operator

Constrant matrix multiplikation

```
static Matrix<T> operator *(double alpha, Matrix<T> a)
```

Parameters

alpha	constant
a	Input a
returns	alpha*a

operator

Matrix constant division

```
static Matrix<T> operator /(Matrix<T> a, double alpha)
```

Parameters

a	Input a
alpha	constant
returns	a/alpha

operator

Matrix multiply operator

```
static Matrix<T> operator *(Matrix<T> a, Matrix<T> b)
```

Parameters

a	Input a
b	Input b
returns	a*b

Set

Set submatrix

```
void Set(int row, int col, Matrix<T> a)
```

Parameters

row	Row Position
col	Column position
a	Sub matrix

Show

Show matrix

```
static void Show(string text, Matrix<T> a)
```

Parameters

text
A

Sub

Sub, r = r - a

```
void Sub(Matrix<T> a)
```

Parameters

a	Input a

TMulAdd

TMulAdd r = r + a'*b

```
void TMulAdd(Matrix<T> a, Matrix<T> b)
```

Parameters

a	Input a
b	Input b

Transpose

Transpose

```
static Matrix<T> Transpose(Matrix<T> a)
```

Parameters

a	Input a
returns	a'

Chapter 6

Transfer Functions

Transfer Functions are the classical way to describe the dynamics of a linear system. Creating transfer functions using delay and time constants are an intuitive way to specify process dynamic. The transfer function can be used directly or the can be used to as input to Linear Filter, which are the discrete form of transfer functions.

The use of the TransferFunction class can be illustrated by a simple example. The example shows the closed loop response for a PI controller controlling a Plant describes as a second order systems.

The *Plant* is described by the transfer function

$$Plant(s) = \frac{1}{(\tau s)^2 + 2\tau\sigma s + 1)}$$

The PI *Controller* is defined by

$$Controller(s) = P(1 + \frac{1}{\tau_i s})$$

where P is the controller gain and τ_i the integration time.

```
double gain = 1.0;
double delay = 0.0;
double tau = 10.0;
double sigma = 0.5;
// second order process
TransferFunction Plant = new TransferFunction(gain, delay, tau, sigma);
double P = 3.0;
double TI = 25.0;
// Integrator
TransferFunction Integrator = new TransferFunction(1.0 / TI, 0.0);
TransferFunction Controller = P * (1.0 + Integrator);
```

creates the *Plant* and *controller* objects.

The closed loop dynamic G_{cl} is defined by

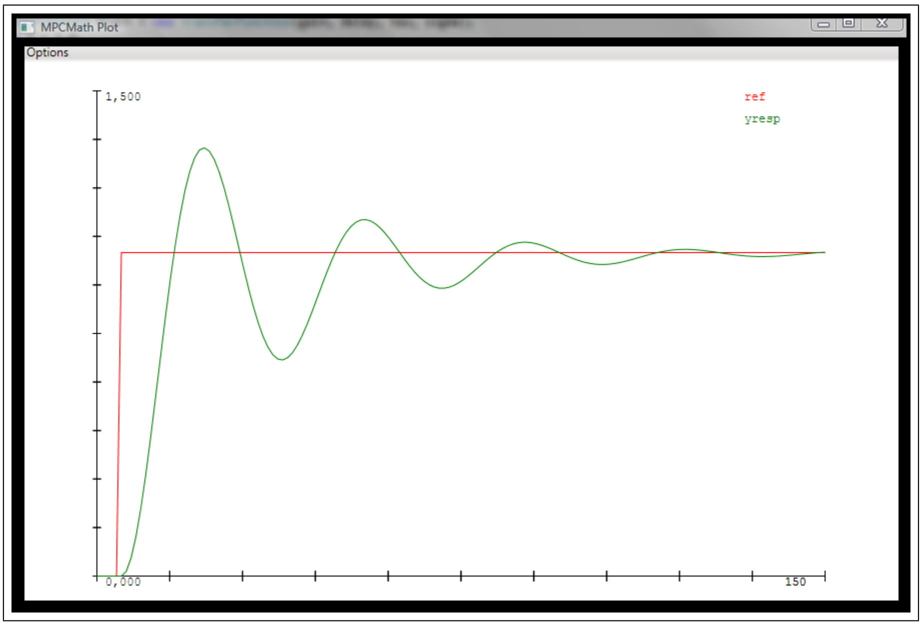
$$G_{cl}(s) = \frac{Controller(s)Plant(s)}{1 + Controller(s)Plant(s)}$$

```
 // Closed loop dynamics
TransferFunction Gcl = Controller * Plant / (1.0 + Controller * Plant);

// reference signal, step after 5 time units
TransferFunction Ref = new TransferFunction(1.0, 5.0);
// Closed loop response
TransferFunction Y = Gcl * Ref;
```

creates the transfer function of the closed loop dynamic *Gcl* and the a reference signal Ref (a step delayed 5 seconds) and the response signal Y.

Plotting of the Ref and Y signal are performed by the code

```
int steps = 150;
Vector yresp = new Vector(steps);
Vector Reference = new Vector(steps);
for (int step = 0; step < steps; step++)
{
    Reference[step] = Ref.Value(step);
    yresp[step] = Y.Value(step);
}

double max = 1.5;
console.Plot(new Plot(
    new PlotSeries("ref", 0.0, max, SeriesColor.Red, Reference),
    new PlotSeries("yresp", 0.0, max,SeriesColor.Green, yresp)));
    }
}
```



6.1 TransferFunction

Class TransferFunction

Linear Transfer Function $G(S) = e^{-\theta S}\frac{R(S)}{P(S)}$

```
[Serializable]
class TransferFunction : ICommon<TransferFunction>
```

Constructors

```
TransferFunction()
: this(0.0, 0.0)
```

Default constructor

```
TransferFunction(double Gain, double Delay, CVector Roots)
```

TransferFunction Constructor $G(s) = \gamma e^{-\theta s}\frac{1}{\Pi(s-p_i)}$

Parameters

Gain	Gain γ
Delay	Delay θ
Roots	Roots p_1

```
TransferFunction(double Gain, double Delay, CVector Roots, CVector Zeroes)
```

TransferFunction Constructor $G(s) = \gamma e^{-\theta s} \frac{\Pi(s-r_i)}{\Pi(s-p_i)}$
Parameters

Gain	Gain γ
Delay	Delay θ
Roots	Roots p_i
Zeroes	Zeroes r_i

```
TransferFunction(double Delay, Polynomial P)
```

TransferFunction constructor $G(s) = e^{-\theta s} \frac{1}{P(s)}$
Parameters

Delay	Delay θ
P	Root polynomial $P(s)$

```
TransferFunction(double Delay, Polynomial P, Polynomial R)
```

TransferFunction constructor $G(s) = e^{-\theta s} \frac{R(s)}{P(s)}$
Parameters

Delay	Delay θ
P	Root polynomial $P(s)$
R	Zero polynomial $R(s)$

```
TransferFunction(double Delay)
```

Constructor Transfer function Impulse $G(s) = e^{-\theta s}$
Parameters

Delay	Delay θ

```
TransferFunction(double Gain, double Delay)
```

Constructor Transfer function, Step $G(s) = \gamma e^{-\theta s}/s$
Parameters

Gain	Gain γ
Delay	Delay θ

```
TransferFunction(double Gain, double Delay, double Tau)
```

constructor Transfer function, one pole $G(s) = \gamma e^{-\theta s}/(\tau s + 1)$
Parameters

Gain	Gain γ
Delay	Delay θ
Tau	time constant $\tau(\tau! = 0)$

```
TransferFunction(double Gain, double Delay, double Tau, double Sigma)
```

constructor Transfer function, two poles $G(s) = \gamma e^{-\theta s}/((\tau s)^2 + 2\sigma\tau s + 1)$
Parameters

Gain	Gain γ
Delay	Delay θ
Tau	Time constant $\tau(\tau! = 0)$
Sigma	Damping σ

Properties

Delay

Transfer function Delay

```
double Delay {set; get;}
```

Gain

Transfer function Gain

```
double Gain {set; get;}
```

P

Denominator Polynomial $G(s) = e^{-\theta s}\frac{R(s)}{P(s)}$

```
Polynomial P {get;}
```

R

Numerator Polynomial $G(s) = e^{-\theta s}\frac{R(s)}{P(s)}$

```
Polynomial R {get;}
```

Roots

Transfer function Roots

```
CVector Roots {get;}
```

Zeroes

Transfer function Zeroes

```
CVector Zeroes {get;}
```

Methods

Add

Add transfer functions

```
static TransferFunction Add(TransferFunction a, TransferFunction b)
```

Parameters

a	input a
b	input b
returns	a+b

AssertEqual

Assert equal, Show transfer function if not equal

```
static void AssertEqual(string text,
TransferFunction actual, TransferFunction expected)
```

Parameters

text	text string
actual	actual transfer function
expected	expected transfer function

Clone

Clone Transfer function

```
static TransferFunction Clone(TransferFunction a)
```

Parameters

a	input a
returns	Clone TransferFunction

Div

Divide transferfunctions

```
static TransferFunction Div(TransferFunction a, TransferFunction b)
```

Parameters

a	input a
b	input b
returns	a/b

Equal

Equal function

```
static bool Equal(TransferFunction a, TransferFunction b)
```

Parameters

a	Input a
b	Input b
returns	true if a equal b

Mul

Multiply transferfunctions

```
static TransferFunction Mul(TransferFunction a, TransferFunction b)
```

Parameters

a	input a
b	input b
returns	a*b

Mul

Multiply Transferfunction with constant

```
static TransferFunction Mul(double a, TransferFunction b)
```

Parameters

a	constant
b	input b
returns	a*b

operator

TransferFunction clone operator

```
static TransferFunction operator +(TransferFunction a)
```

Parameters

a	input TransferFunction
returns	Cloned TransferFunction

operator

TransferFunction Negate operator

```
static TransferFunction operator -(TransferFunction a)
```

Parameters

a	Input TransferFunction
returns	Negated TransferFunction

operator

Transfer function Add operator MPCMath requires the two Transferfunction a b to have equal delay !

```
static TransferFunction operator +(TransferFunction a, TransferFunction b)
```

Parameters

a	input a
b	input b
returns	a+b

operator

Transfer function Add operator MPCMath requires the two Transferfunction b to have equal delay = 0!

```
static TransferFunction operator +(double a, TransferFunction b)
```

Parameters

a	constant
b	TransferFunction
returns	TransferFunction

operator

Transfer function Add operator MPCMath requires the two Transferfunction a to have equal delay = 0!

```
static TransferFunction operator +(TransferFunction a, double b)
```

Parameters

a	TransferFunction
b	constant
returns	TransferFunction

operator

Transferfunction Sub

```
static TransferFunction operator -(TransferFunction a, TransferFunction b)
```

Parameters

a	input a
b	input b
returns	a-b

operator

Multiply transferfunctions operator

```
static TransferFunction operator *(TransferFunction a, TransferFunction b)
```

Parameters

a	input a
b	input b
returns	a*b

operator

Multiply TransferFunction with Constant operator

```
static TransferFunction operator *(double a, TransferFunction b)
```

Parameters

a	constant
b	Input b
returns	a*b

operator

Multiply TransferFunction with Constant operator

```
static TransferFunction operator *(TransferFunction a, double b)
```

Parameters

a	input a
b	constant
returns	a*b

operator

Divide TransferFunction with constant operator

```
static TransferFunction operator /(TransferFunction a, double b)
```

Parameters

a	input a
b	constant
returns	a/b

operator

Divide transferfunctions operator

```
static TransferFunction operator /(TransferFunction a, TransferFunction b)
```

Parameters

a	input a
b	input b
returns	a/b

Show

Show Transfer function

```
static void Show(string txt, TransferFunction a)
```

Parameters

txt	text string
a	TransferFunction

Sub

Transferfunction Sub

```
static TransferFunction Sub(TransferFunction a, TransferFunction b)
```

Parameters

a	input a
b	input b
returns	a-b

TauForm

"Constructor" for Tau form $G(s) = \gamma e^{-\theta s} \frac{1}{\Pi(\tau_i s+1)}$

```
static TransferFunction TauForm(double Gain, double Delay, Vector Taus)
```

Parameters

Gain	Gain γ
Delay	Delay θ
Tau	Time constants τ
returns	Transfer function

TauForm

"Constructor" for Tau form $G(s) = \gamma e^{-\theta s} \frac{\Pi(z_i s+1)}{\Pi(\tau_i s+1)}$

```
static TransferFunction TauForm(double Gain, double Delay, Vector Taus, Vector Zeroes
```

Parameters

Gain	Gain γ
Delay	Delay θ
Tau	time constants τ
zeroes	
returns	Transfer function

Value

Time response of Transfer function

```
double Value(double t)
```

Parameters

t	time
returns	f(t)

Chapter 7

State Space Models

7.1 StateSpaceModel

Class StateSpaceModel

State Space model class X+ = A*X + B*U + K * eps + off Y = C*X + eps

```
[Serializable]
class StateSpaceModel
```

Constructors

```
StateSpaceModel()
```

Constructor for serialization

```
StateSpaceModel(Matrix A, Matrix B, Matrix C, Matrix K)
```

Constructor from system matrices
Parameters
A System matrix A
B System matrix B
D System matrix C
K System matrix K

```
StateSpaceModel(ARXModel arxModel)
```

Constructor ArxModel to StateSpace Nodel
Parameters
arxModel ARX model

```
StateSpaceModel(TransferFunction G, double T):
this(new ARXModel(G, T))
```

Constructor TransferFunction to StateSpace Model
Parameters
G Transfer function
T Sample time

```
StateSpaceModel(Matrix<TransferFunction> model, double T)
: this(model, T, new Vector(model.rows, 1.0))
```

Constructor MiMo TransferFunction to StateSpace model
Parameters

model	Generic matrix of TransferFunction objects
T	Sample time

```
StateSpaceModel(Matrix<TransferFunction> model, double T, Vector Alfa)
```

Constructor MiMo TransferFunction to StateSpace Models
Parameters

model	Generic matrix of TransferFunction objects
T	Sample tim
alpha	Noise Integrator Coificients

```
StateSpaceModel(Matrix<ARXModel> model)
```

Constructor MiMo Arx model to StateSpace
Parameters

model	Generic matrix of ArxModel objects

Properties

A

System Matrix A

```
Matrix A {set; get;}
```

B

System Matrix B

```
Matrix B {set; get;}
```

C

System Matrix C

```
Matrix C {set; get;}
```

Created

Created (Administrative field)

```
DateTime Created {set; get;}
```

Delay

Minimum delay for all manipulated vars

```
int Delay {get;}
```

Delays

Delay chains for U 17.3

```
DelayChain[] Delays {set; get;}
```

Description

Description (Administrative field)

```
string Description {set; get;}
```

Dimension

States

```
int Dimension {set; get;}
```

K

System Matrix K

```
Matrix K {set; get;}
```

NU

Manipulated vars

```
int NU {set; get;}
```

NY

Observed vars

```
int NY {set; get;}
```

Off

Offset (X+ = A*X + B*U + K * Eps + Off)

```
Vector Off {get;}
```

US

Operating point for inputs (Administrative field)

```
Vector US {set; get;}
```

XS

Operating point for State

```
Vector XS {set; get;}
```

Methods

AssertEqual

Assert equal, and show objects if not equal

```
static void AssertEqual(string text, StateSpaceModel actual, StateSpaceModel expected
```

Parameters

text	text string
actual	actual statespace model
expected	expected statespace model

Equal

Equal compare two StateSpace models

```
static bool Equal(StateSpaceModel actual, StateSpaceModel expected)
```

Parameters

actual	actual statespace model
expected	expected statespace model

NextState

Next state X+ = A*X + B*U + K*Eps

```
Vector NextState(Vector X, Vector U, Vector Eps)
```

Parameters

X	State vector
U	Manipulated vector
Eps	Noise
returns	Next state

Observed

Observed Y = C*X

```
Vector Observed(Vector X)
```

Parameters
X
returns Observed vector

Reset

Reset delay chains

```
void Reset()
```

SetKalmanGain

Set Kalman Gain from noise spectrum

```
void SetKalmanGain(Vector q, Vector r)
```

Parameters
q Process Noise
r Measurement Noise

Show

Show State Space model

```
static void Show(string txt, StateSpaceModel model)
```

Parameters
txt
model

Chapter 8

Equation solvers

Equation solvers are used to solve linear equations $A * x = r$. The factorization of the coefficient matrix is performed during the creation of the solver object, making multiple solutions equations with equal coefficient matrices very efficient.

```
int dim = 5;
Matrix A = Matrix.Random(dim, dim, -100.0, 100.0);

ISolver solver = new LinearEquationSolver(A);

Vector r = Vector.Random(dim, -100.0, 100.0);
Vector x = solver.Solve(r);
Vector.AssertEqual("r", A * x, r);

Matrix  R = Matrix.Random(dim, 3, -100.0, 100.0);
Matrix  X = solver.Solve(R);
Matrix.AssertEqual("R", A * X, R);
```

In this example the $Vector.AssertEqual$ and $Matrix.AssertEqual$ tests whether the correct solutions to the equations has been obtained.

The *LinearEquationSolver* performs the $A = P * L * U$ factorizations, and can be used to invert matrices.

The equation solvers implements the interface *ISolver* facilitating easy switch between solvers.

The *SparseEquationSolver* is using Intel's PARDISO solver.

8.1 ISolver

Interface ISolver

Interface for Equation solvers

```
interface ISolver
```

Methods

Solve

Solve A*x = r

```
Vector Solve(Vector r)
```

Parameters

r	Vector r
returns	Vector x

Solve

Solve A*X = R

```
Matrix Solve(Matrix R)
```

Parameters

R	Matrix R
returns	Matrix X

8.2 LinearEquationSolver

Class LinearEquationSolver

Linear equation solver, PLU factorization. Matrix determinant. Solve A*x = r and A*X = R

```
class LinearEquationSolver : ISolver
```

Constructors

```
LinearEquationSolver(Matrix A)
```

Constructor
Parameters

A	Coifficient matrix

Properties

L

Lover factor in A = PLU factiorization

```
Matrix L {get;}
```

U

Upper factor in A = PLU factiorization

```
Matrix U {get;}
```

Methods

Determinant

Determinant

```
double Determinant()
```

Parameters
returns determinant of A

Invert

Invert matrix

```
Matrix Invert()
```

Parameters
returns A^{-1}

Pivot

```
void Pivot(Matrix LU)
```

Parameters
LU L*U

Solve

Solve A*x = r

```
Vector Solve(Vector r)
```

Parameters
r vector r
returns vector x

Solve

Solve A*X = R

```
Matrix Solve(Matrix R)
```

Parameters
r Matrix R
returns Matrix X

8.3 LeastSquareEquationSolver

Class LeastSquareEquationSolver

Least Square Equation Solver. QR Factorization

```
class LeastSquareEquationSolver : ISolver
```

Constructors

```
LeastSquareEquationSolver(Matrix A)
```

Constructor
Parameters

A	Coifficient matrix

Properties

Q

Q matrix, A = Q*R Q orthogonal

```
Matrix Q {get;}
```

R

R matrix , A = Q*R , R upper triangular

```
Matrix R {get;}
```

R1

R1 matrix R upper triangular
$A = Q \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$

```
Matrix R1 {get;}
```

Methods

Solve

Solve A*x = r

```
Vector Solve(Vector r)
```

Parameters

r	vector r
returns	vector x

Solve

Solve A*X = R

```
Matrix Solve(Matrix R)
```

Parameters

R	Matrix R
returns	Matrix X

8.4 CholeskyEquationSolver

Class CholeskyEquationSolver

Cholesky equations solver

```
class CholeskyEquationSolver : ISolver
```

Constructors

```
CholeskyEquationSolver(Matrix a)
```

Constructor Set Equations and do Cholesky decomposition
Parameters

a	Coificient matrix

Properties

LTR

Factor lower triangular

```
Matrix LTR {get;}
```

Methods

Solve

Solve A*x = r

```
Vector Solve(Vector r)
```

Parameters

r	vector r
returns	vector x

Solve

Solve A*X = R

```
Matrix Solve(Matrix R)
```

Parameters

R	Matrix R
returns	Matrix X

8.5 SymmetricEquationSolver

Class SymmetricEquationSolver

Symmetrical Equation solver class

```
class SymmetricEquationSolver : ISolver
```

Constructors

```
SymmetricEquationSolver(Matrix A)
```

Constructor
Parameters

A	Symmetrical coifficient matrix

Methods

Invert

Invert matrix

```
Matrix Invert()
```

Solve

Solve A*x = r

```
Vector Solve(Vector r)
```

Parameters

r	vector r
returns	vector x

Solve

Solve A*X = R

```
Matrix Solve(Matrix R)
```

Parameters

R	Matrix R
returns	Matrix X

8.6 SymmetricalBlockEquationSolver

Class SymmetricalBlockEquationSolver

Symmetrical Block Equation Solver, exploiting diagonal elements in diagonal of block matrix.

```
class SymmetricalBlockEquationSolver
```

Constructors

```
SymmetricalBlockEquationSolver(BMatrix A)
```

Constructor
Parameters
A Symmetrical block matrix with values stored in lower triangle

Methods

Solve

Solve A * x = r

```
BVector Solve(BVector r)
```

Parameters
r coificient block vector

8.7 SparseEquationSolver

Class SparseEquationSolver

Sparse Linear Equation Solver, (PARDISO, Parallel Direct Sparse Solver) Solve A*x = r and A*X = R

```
class SparseEquationSolver : ISolver
```

Constructors

```
SparseEquationSolver(Matrix A)
: this(SparseMatrixType.RealUnsymmetrical, UPLO.Upper, A)
```

Constructor for unsymmetrical matrix
Parameters
A Coificient matrix

```
SparseEquationSolver(SparseMatrixType type, UPLO UpLo, Matrix A)
```

Constructor
Parameters

type	Sparse matrix type
UpLo	Upper or lower triangular for symmetric matrices
A	Coificient matrix

```
SparseEquationSolver(BMatrix A)
: this(SparseMatrixType.RealUnsymmetrical, UPLO.Upper, A)
```

Constructor for Block Matrix/Vector equations
Parameters

A	Coificient matrix

```
SparseEquationSolver(SparseMatrixType type, UPLO UpLo, BMatrix A)
```

Constructor for Block Matrix/Vector equations
Parameters

type	Sparse matrix type
UpLo	Upper or lower triangular for symmetric matrices
A	Coificient matrix

Methods

Solve

Solve equations

```
Vector Solve(Vector r)
```

Parameters

r	right hand side coifficients
returns	solution vector

Solve

Solve equations

```
Matrix Solve(Matrix  R)
```

Parameters

R	right hand side coifficients
returns	solution matrix

Solve

Solve equations

```
BVector Solve(BVector r)
```

Parameters

r	right hand side coifficients
returns	solution vector

Solve

Solve equations

```
BMatrix Solve(BMatrix R)
```

Parameters

R	right hand side coifficients
returns	solution matrix

8.8 Banded matrix storage

The equation solvers *BandEquationSolver* 8.9 and *SymmetricBandEquationSolver* 8.10 stores the data in a compact format as follows. The m by n band matrix A with $ku = 1$ non-zero super-diagonals and $kl = 2$ sub-diagonals is stored in a matrix AB with $m = kl + 1 + ku$ rows and n columns. Columns of the matrix is stored in the corresponding columns in the AB matrix and diagonals of the matrix are stored in the rows of the array. Thus a_{ij} is stored in $ab[ku + 1 + i - j, j]$ for $max(1, j - ku, \leq 1 \leq min(n, j + kl)$. The elements marked NaN inAB are ignored.

```
A Matrix.Form General[8,8]
   -78,7643   65,9043          0          0          0          0          0          0
   -94,1250   -2,6987     9,3032          0          0          0          0          0
   -26,2801  -61,2631    61,8430    18,1162          0          0          0          0
          0  -21,6376   -80,8072   -79,1933    42,3467          0          0          0
          0         0   -30,0282    41,7612    26,4536    -3,5851          0          0
          0         0          0    76,4280    35,7596   -72,9231   -81,3391          0
          0         0          0          0   -83,8892    34,8921   -34,3952    39,6899
          0         0          0          0          0    75,5007   -84,5224    10,9481

AB Matrix.Form General[4,8]
        NaN   65,9043     9,3032    18,1162    42,3467    -3,5851   -81,3391    39,6899
   -78,7643   -2,6987    61,8430   -79,1933    26,4536   -72,9231   -34,3952    10,9481
   -94,1250  -61,2631   -80,8072    41,7612    35,7596    34,8921   -84,5224        NaN
   -26,2801  -21,6376   -30,0282    76,4280   -83,8892    75,5007        NaN        NaN
```

Symmetrical banded matrix A with n rows and columns and $kd = 1$ non-zero sub and super diagonals is stored in a compact matrix AB with $1 + kd$ rows and n columns. The first row is the diagonal of A and the following rows are the sub-diagonals of A. The elements marked NaN inAB are ignored.

```
A Matrix.Form General[8,8]
    62,0381   20,6993          0          0          0          0          0          0
    20,6993   50,3402   -40,3750          0          0          0          0          0
```

```
        0   -40,3750    42,2135   -13,3814          0          0          0
        0          0   -13,3814   103,5437   -14,6392          0          0
        0          0          0   -14,6392    65,9977   -60,5258          0
        0          0          0          0   -60,5258    76,3448    15,1430
        0          0          0          0          0    15,1430    83,1616    61,1
        0          0          0          0          0          0    61,1746    65,3

AB Matrix.Form General[2,8]
    62,0381    50,3402    42,2135   103,5437    65,9977    76,3448    83,1616    65,3
    20,6993   -40,3750   -13,3814   -14,6392   -60,5258    15,1430    61,1746
```

8.9 BandEquationSolver

Class BandEquationSolver

Band Equation solver class

```
class BandEquationSolver : ISolver
```

Constructors

```
BandEquationSolver(int Kl, int Ku, Matrix AB)
```

Constructor
Parameters

Kl	Number of sub-diagonals
Ku	Number of Super-diagonals
AB	Coefficient matrix for banded matrix

Methods

Solve

Solve A*x = r

```
Vector Solve(Vector r)
```

Parameters

r	vector r
returns	vector x

Solve

Solve A*X = R

```
Matrix Solve(Matrix R)
```

Parameters

R	Matrix R
returns	Matrix X

8.10 SymmetricBandEquationSolver

Class SymmetricBandEquationSolver

Symmetrical positive definite band matrix Equation solver class

```
class SymmetricBandEquationSolver : ISolver
```

Constructors

```
SymmetricBandEquationSolver(int Kd, Matrix A)
```

Constructor

Parameters

Kd	number of super- and sup- diagonals
A	Coifficent matrix

Methods

Solve

Solve A*x = r

```
Vector Solve(Vector r)
```

Parameters

r	vector r
returns	vector x

Solve

Solve A*X = R

```
Matrix Solve(Matrix R)
```

Parameters

R	Matrix R
returns	Matrix X

Chapter 9

Linear Filter

The linear filter or the discrete transfer function is defined by

$$F(q) = q^{-k}R(q)/P(q) \tag{9.1}$$

where q is the shift operator, $u(t+1) = qu(t)$ and $q^{-1}u(t) = u(t-1)$

The numerator and denominator polynomials are defined by

$$R(q) = \sum_{i=0}^{nz} r_i q^{-i} \quad P(q) = \sum_{i=0}^{np} p_i q^{-i} \tag{9.2}$$

The following example illustrates the use of linear filters. We have an ARX process modelled as

$$\begin{aligned} Y_{Plant} &= Y_{Det} + Y_{Stoc} && (9.3)\\ &= G(q)u(t) + H(q)e(t) && (9.4)\\ &= q^{-k}\frac{R(q)}{P(q)}u(t) + \frac{1}{P(q)}e(t) && (9.5) \end{aligned}$$

where the input signal $u(t)$ is a pseudo random binary signal, and $e(t)$ is white noise.

The process is a second order oscillating process defined by the transfer function *Plant*. The LinearFilter $G(q)$ describing the deterministic part of Y_{Plant} is derived directly from *Plant* and the sample time *DT*. The LinearFilter $H(q)$ is derived from $G(q)$ denominator polynomial.

The input signal $u(t)$ is generated using the PBRS 9.1routine. The deterministic part comes from $Y_{Det} = G(q)u(t)$. The stochastic part $Y_{Stoc} = H(q)e(t)$ where the white noise series is generated using the *Niid* 9.1 routine. The plant response comes from $Y_{Plant} = Y_{Det} + Y_{Stoc}$

Having an input and an output series a LinearFilter $\hat{G}(q)$ can be estimated using the Regress 9.1 routine. An estimate $\hat{Y}(t)$ of the deterministic response is calculated The estimated deterministic response is calculated from $\hat{Y}(t) = \hat{G}(q)u(t)$

The code is

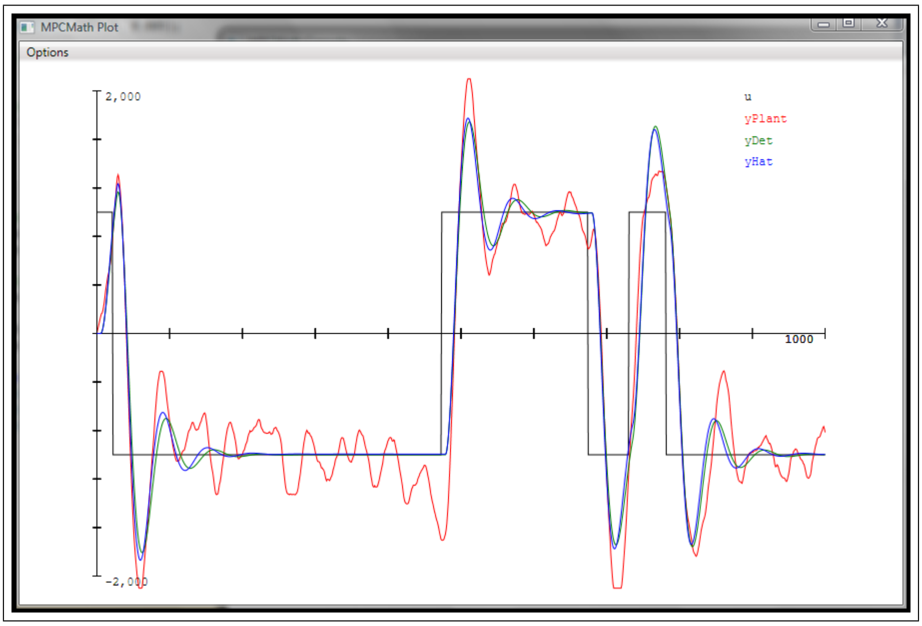
```
console.WriteLine("Test Linear Filter");

int N = 1000;
double mean = 0.0;
double var = 1.0e-4;

double gain = 1.0;
int delay = 5;
double tau = 10.0;
double sigma = 0.3;
double DT = 1.0;

// second order system
TransferFunction Plant = new TransferFunction(gain, delay, tau, sigma);

// model for deterministic part
LinearFilter G = new LinearFilter(Plant, DT);
// model for stochastic part
LinearFilter H = new LinearFilter(new Polynomial(1.0), G.P);

LinearFilter.Show("G", G);
LinearFilter.Show("H", H);
console.Show("H.Gain", H.Gain);

// Pseudo Random Binary Signal
Vector u = LinearFilter.PBRS(N, 0.005);

// Deterministic response
Vector yDet = G * u;

// Stochatic resonse
Vector yStoch = H * LinearFilter.Niid(N, mean, var);
Vector yPlant = yDet + yStoch;

int np = 3;
int nz = 2;
int k = 6;
// Estimate model form input signal u and output signal u
LinearFilter GHat = LinearFilter.Regress(np, yPlant, k, nz, u);
LinearFilter.Show("GHat", GHat);

// Estimated deterministic response
Vector yHat = GHat * u;

double max = 2.0;
double min = -max;
console.Plot(
    new Plot(new PlotSeries("u", min, max, u),
    new PlotSeries("yPlant", min, max, SeriesColor.Red, yPlant),
    new PlotSeries("yDet", min, max, SeriesColor.Green, yDet),
```

```
new PlotSeries("yHat", min, max, SeriesColor.Blue, yHat)));
```

with the following output

```
Test Linear Filter
G ShiftFunction K = 6
     Z
    0,0049     0,0048
     P
         1    -1,9321     0,9418
H ShiftFunction
     Z
         1
     P
         1    -1,9321     0,9418
H.Gain 103,1159
GHat ShiftFunction K = 6
     Z
    0,0023     0,0087
     P
         1    -1,9298     0,9409
```

end the plotted time series



9.1 LinearFilter

Class LinearFilter

LinearFilter (Discrete Transfer function)

```
[Serializable]
class LinearFilter : ICommon<LinearFilter>
```

Constructors

`LinearFilter() : this (new Polynomial(1.0))`

Default constructor, F(q) = 1

`LinearFilter(int K, Polynomial R, Polynomial P)`

Constructor $F(q) = q^{-k}R(q)/P(q)$
Parameters

k	Delay
R	R Polynomial
P	P Polynomial

`LinearFilter(Polynomial R, Polynomial P)`

Constructor $F(q) = R(q)/P(q)$
Parameters

R	R Polynomial
P	P Polynomial

`LinearFilter(Polynomial R)`

Constructor $F(q) = R(q)$
Parameters

R	R Polynomial

`LinearFilter(params double[] vals)`

Constructor $F(q) = R(q)$
Parameters

vals	R coificient array

`LinearFilter(TransferFunction TF, double T)`

Constructor LinearFilter from TransferFunction
Parameters

TF	TransferFunction object
T	Sample time

`LinearFilter(int K, params double[] vals)`

Constructor $F(q) = q^{-k}R(q)$
Parameters

K	Delay
vals	R coificient array

Properties

Gain

Gain (stationary)

```
double Gain {get;}
```

InitialValue

Inital value for input time series, used by LinearFilter object for $y(t)$ where $t < 0$

```
double InitialValue {set; get;}
```

K

Delay

```
int K {set; get;}
```

P

Denominator Polynomial

```
Polynomial P {set; get;}
```

R

Numerator Polynomial

```
Polynomial R {set; get;}
```

Roots

Roots

```
CVector Roots {get;}
```

Zeroes

Zeroes

```
CVector Zeroes {get;}
```

Methods

AssertEqual

Assert Equal, show if not equal

```
static bool AssertEqual(string text, LinearFilter actual, LinearFilter expected)
```

Parameters

text	text string
actual	actual filter object
expected	expected filter obkject
returns	true if equal

Equal

Test if Equal

```
static bool Equal(LinearFilter A, LinearFilter B)
```

Parameters

A	A Filter object
B	B Filter object
returns	true if equal

Invert

Invert LinearFilter

```
LinearFilter Invert()
```

Parameters

returns	F^{-1}

Invertible

Check if LinearFilter is invertible. If $v = H(q)e(t)$ implies that $e(t) = H^{-1}(q)v(t)$

```
bool Invertible()
```

Parameters

returns	true if invertible

Niid

Normal independent identical distributed process

```
static Vector Niid(int N, double Mean, double Variance)
```

Parameters

N	Series lenght
Mean	Series mean
Variance	Series variance

operator

Filter time series

```
static Vector operator *(LinearFilter F, Vector u)
```

Parameters

F	Filter object
u	Input time series
returns	$F(q)u(t)$

operator

Filter time series

```
static Vector operator *(Vector u, LinearFilter H)
```

Parameters

u	Input time series
F	Filter object
returns	$F(q)u(t)$

operator

Operator + , clone

```
static LinearFilter operator +(LinearFilter F)
```

Parameters

F	Filter object
returns	$+F$

operator

Operator -

```
static LinearFilter operator -(LinearFilter F)
```

Parameters

F	Filter object
returns	$-F$

operator

Multiply by constant

```
static LinearFilter operator *(double fak, LinearFilter F)
```

Parameters

fak	constant
F	Filter object
returns	$fak * F$

operator

Multiply by constant

```
static LinearFilter operator *(LinearFilter F, double fak)
```

Parameters

F	Filter object
fak	constant
returns	$fak * F$

operator

Divide by constant

```
static LinearFilter operator /(LinearFilter F, double fak)
```

Parameters

F	Filter object
fak	constant
returns	F/fak

operator

Add Linear Filters

```
static LinearFilter operator +(LinearFilter A, LinearFilter B)
```

Parameters

A	A Filter object
B	B Filter object
returns	$A + B$

operator

Subtract Linear Filters

```
static LinearFilter operator -(LinearFilter A, LinearFilter B)
```

Parameters

A	A Filter object
B	B Filter object
returns	$A - B$

operator

Add constant and LinearFilter

```
static LinearFilter operator +(double alpha, LinearFilter A)
```

Parameters

alpha	constant
A	A Filter object
returns	$A + alpha$

operator

Add LinearFilter and constant

`static LinearFilter operator +(LinearFilter A, double alpha)`

Parameters

A	A Filter object
alpha	constant
returns	$alpha + A$

operator

Subtract constant and LinearFilter

`static LinearFilter operator -(double alpha, LinearFilter A)`

Parameters

alpha	constant
A	A Filter object
returns	$A - alpha$

operator

Subtract LinearFilter and constant

`static LinearFilter operator -(LinearFilter A, double alpha)`

Parameters

A	A Filter object
alpha	constant
returns	$A - alpha$

operator

Multiply operator

`static LinearFilter operator *(LinearFilter A, LinearFilter B)`

Parameters

A	A Filter object
B	B Filter object
returns	$A * B$

operator

Divide operator

`static LinearFilter operator /(LinearFilter A, LinearFilter B)`

Parameters

A	A Filter object
B	B Filter object
returns	A/B

operator

Operator divide constant with linear filter

```
static LinearFilter operator /(double alpha, LinearFilter A)
```

Parameters

alpha	constant
A	A Filter object
returns	$alpha/A$

PBRS

Pseudo Binary Random Signal series. Produces a PRBS series u of length N, where $u[i]$ switches between -1 and 1 with probality *prob*

```
static Vector PBRS(int N, double prob)
```

Parameters

N	Series length
prob	Probability for switch
returns	u series

Regress

Find Shift Function $F(q) = 1/P(q)$ where $P(q)y(t) = e(t)$, $P(q) = \sum_{i=0}^{np} p_i q^{-i}$ and and $e(t) = N_{iid}(0.0, 1.0)$

```
static LinearFilter Regress(int np, Vector y)
```

Parameters

np	Number of poles
y	Output series
returns	$F(q) = 1/P(q)$

Regress

Find Shift Function $F(q) = q^{-k}R(q)/P(q)$ where $R(q) = \sum_{i=0}^{nz} r_i q^{-i}$, $P(q) = \sum_{i=0}^{np} p_i q^{-i}$ and $y(t) = F(q)u(t)$

```
static LinearFilter Regress(int np, Vector y, int k, int nz, Vector u)
```

Parameters

np	Number of poles
y	Output series
k	Delay
nz	Number of poles
u	Output series

Regress

Find Shift Function $F(q) = q^{-k}R(q)/P(q)$ where $R(q) = \sum_{i=0}^{nz} r_i q^{-i}$, $P(q) = \sum_{i=0}^{np} p_i q^{-i}$ and $y(t) = F(q)u(t)$

```
static LinearFilter Regress(Norm norm, double gamma,
int np, Vector y, int k, int nz, Vector u)
```

Parameters

norm	Regression norm
gamma	Huber norm parameter
np	Number of poles
y	Output series
k	Delay
nz	Number of poles
u	Output series

Show

Show LinearFilter

```
static void Show(string txt, LinearFilter A)
```

Parameters

txt	text string
A	A Filter object

Chapter 10

ARX models

The ARX model describes the process

$$A(q)y(t) = B(q)u(t) + e(t) \tag{10.1}$$

q is the time shift operator, and the polynomials are

$$A(q) = 1 + \sum_{j=1}^{ny} a_j q^{-j} \tag{10.2}$$

$$B(q) = \sum_{j=1}^{nu} B_j q^{-j} \tag{10.3}$$

The $Extended\Delta ARX$ [Huusom et al., 2010] model is

$$A(q)y(t) = B(q)u(t) + \frac{1 - \alpha q_{-1}}{1 - q_{-1}} e(t) \tag{10.4}$$

see description given in Introduction 1.6.

10.1 ARXModel

Class ARXModel

ARX model and Extended Delta ARX Model

```
[Serializable]
class ARXModel : ICommon<ARXModel>
```

Constructors

```
ARXModel() : this(0, new Vector(1, 1.0), new Vector(1), 1.0)
```

Default constructor for serialization

```
ARXModel(int Delay, Vector A, Vector B, double T)
```

Constructor ARX Model
Parameters

Delay	Delay
A	A polynomial
B	B polynomial
T	Sample time

```
ARXModel(int Delay, Vector A, Vector B, double T, double Alfa)
```

Constructor Extended Delta ARX Model
Parameters

Delay	Delay
A	A polynomial
B	B polynomial
T	Sample time
Alfa	Alfa coifficient

```
ARXModel(TransferFunction TF, double T)
: this(0, new Vector(1, 1.0), new Vector(1), 1.0)
```

Constructor ARX model
Parameters

TF	System transfer function
T	Sample time

```
ARXModel(TransferFunction TF, double T, double Alfa)
```

Constructor Extended Delta ARX Model
Parameters

TF	System transfer function
T	Sample time
Alfa	Alfa coifficient

Properties

A

ARX coefficients A, A(q-1)y(t) = (q-delay)*B(q-1)*u(t)

```
Vector A {get;}
```

Alfa

Extended Delta ARX model alpha coefficient

```
double Alfa {get;}
```

ASP

State space matrix ASP
X+ = ASP*X + BSP*U + KSP* eps
Y = C*X + eps

```
Matrix ASP {get;}
```

B

ARX coefficients B, A(q-1)y(t) = (q-delay)*B(q-1)*u(t)

```
Vector B {get;}
```

BSP

State space matrix BSP
X+ = ASP*X + BSP*U + KSP* eps
Y = C*X + eps

```
Vector BSP {get;}
```

C

State space vector C
X+ = ASP*X + BSP*U + KSP* eps
Y = C*X + eps

```
Vector C {get;}
```

Delay

Delay states

```
int Delay {get;}
```

Dimension

State dimension

```
int Dimension {get;}
```

ExtendedDeltaARX

Extended Delta ARX model

```
bool ExtendedDeltaARX {get;}
```

Gain

ARX function gain

```
double Gain {get;}
```

KSP

State space matrix KSP
X+ = ASP*X + BSP*U + KSP* eps
Y = C*X + eps

```
Vector KSP {get;}
```

T

Sampling time (in seconds);

```
double T {get;}
```

Methods

AssertEqual

Assert equal and show function if nor equal

```
static void AssertEqual(string txt, ARXModel a, ARXModel b)
```

Parameters

txt	text string
a	Actual ARX model
b	Expected ARX model

NextState

Next state

```
Vector NextState(Vector X, double u, double eps)
```

Parameters

X	state vector
u	Manipulated variables
eps	Noise vector
returns	Next state x

Observed

Observed values

```
double Observed(Vector X)
```

Parameters

X	state vector
returns	Y vector

Reset

Reset u delay chain

```
void Reset()
```

Show

Show ARX Model

```
static void Show(string txt, ARXModel a)
```

Parameters

txt	text string
a	ARXModel object

Chapter 11

Function and Model interfaces

MPCMath provides the $IFun$ and $IConFun$ interfaces as general interfaces to unconstrained and Constrained functions.

The function interface IFunt is useful for fitting function parameters, as described in section Least Square Fitting 12.6

The IModel interfaces is general interface to plant objects.

11.1 IFun

Interface IFun

Interface for function $\mathbb{R}_n \Rightarrow \mathbb{R}$

```
interface IFun
```

Properties

N

Number of variables in x

```
int N {get;}
```

Methods

DDx

Hessian matrix

```
Matrix DDx(Vector x)
```

Parameters

x	state
returns	Hessian matrix. null if Hessian undefined or x infeasible

Dx

Jacobian vector

```
Vector Dx(Vector x)
```

Parameters

x	state
returns	Jacobian vector or null if x infeasible

Value

Function value

```
double Value(Vector x)
```

Parameters

x	State
returns	function value or double. MaxValue if x infeasible

11.2 IConFun

Interface IConFun

Interface for Constrained functions $\mathbb{R}_n \Rightarrow \mathbb{R}$. Equality constraints $Ce(X) = 0$ and Inequality Constraints $Ci(x) \geq 0$

```
interface IConFun : IFun
```

Properties

Ne

Number of Equality conditions

```
int Ne {get;}
```

Ni

Number of Inequality conditions

```
int Ni {get;}
```

Methods

DDxe

Hessians of Equality Constraints

```
BMatrix DDxe(Vector x)
```

Parameters

x	state
returns	Hessian of constraints. null if Hessian undefined or x infeasible

DDxi

Hessians of Inequality Constraints

```
BMatrix DDxi(Vector x)
```

Parameters

x	state
returns	Hessian of constraints. null if Hessian undefined or x infeasible

Dxe

Gradients of Equality constraints

```
Matrix Dxe(Vector x)
```

Parameters

x	state
returns	Gradients of constraints

Dxi

Gradients of Inequality constraints

```
Matrix Dxi(Vector x)
```

Parameters

x	state
returns	Gradients of constraints

Vale

Equality Constraint Values

```
Vector Vale(Vector x)
```

Parameters

x	State
returns	Constraint values

Vali

Inequality Constraint Values $Vali \geq 0$

```
Vector Vali(Vector x)
```

Parameters

x	State
returns	Constraint values

11.3 IFunt

Interface IFunt

Interface for function $\mathbb{R}_n \Rightarrow \mathbb{R}$

```
interface IFunt
```

Properties

N

Number of variables in x

```
int N {get;}
```

Methods

DDx

Hessian

```
Matrix DDx(Vector x, double t)
```

Parameters

x	
t	
returns	Hessian matrix. null if Hessian undefined or x infeasible

Dx

Jacobian

```
Vector Dx(Vector x, double t)
```

Parameters

x	parameters
t	t
returns	Jacobian or null if infeasible

Value

Function value, (return infinity if x infeasible)

```
double Value(Vector x, double t)
```

Parameters

x	parameters
t	t
returns	function value or infinity if infeasible

11.4 IModel

Interface **IModel**

Process model interface

```
interface IModel
```

Properties

Dimension

Number of states

```
int Dimension {get;}
```

NU

Number of manipulated vars

```
int NU {get;}
```

NY

Number of Observed vars

```
int NY {get;}
```

Parameters

Model parameters, useful for iterative optimization programs

```
Vector Parameters {set; get;}
```

Sparse

Sparse matrices supported

```
bool Sparse {get;}
```

T

Sample time for Next and LinearModel functions

```
double T {get;}
```

Methods

Derivative

Derivative function dx/dt

```
Vector Derivative(Vector X, Vector U)
```

Parameters

X	Present state
U	Manipulated variables
returns	Time derivative of state X or null

Ju

Jacobian with respect to INputs

```
Matrix Ju(Vector X, Vector U)
```

Parameters

X	State vector
U	Manipulated vars
returns	Jacobian or Null

Jx

Jacobian with respect to outputs

```
Matrix Jx(Vector X, Vector U)
```

Parameters

X	State vector
U	Manipulated vars
returns	Jacobian or Null

LinearModel

Linear State Space Model

```
StateSpaceModel LinearModel(Vector XS, Vector US)
```

Parameters

XS	Stationary State
XY	
returns	StateSpaceModel

Next

Next step function

```
Vector Next(Vector X, Vector U, Vector W)
```

Parameters

X	of state X
U	Manipulated variables
W	Disturbance/Process Noise
returns	Next value of state X

Observed

Observed output

```
Vector Observed(Vector X, Vector U)
```

Parameters

X	of state X
U	Manipulated variables

Step

Iteration step with sensitivites

```
void Step(Vector X0, Vector U0, out Vector X,
out Matrix A, out Matrix B, out Matrix c)
```

Parameters

X0	Initial state
U0	Manipulated variables
X	End state variables
A	Sensitivity dX/dXOi
B	Sensitivity dX/dU0i
C	Sensitivity dY/dX

11.5 Infeasible State exceptions

Programs implementing Models using the IModel interface can can notify the calling program if it has entered into an infeasible state. For the U-Loop reactor example below, the states are Chemical concentrations, which must be positive. The ODESolver 15.3 and the SteadyState 16.4 routines uses these exceptions to back track if error parameter set to 1, other wise the exception is passed on the calling program.

```
/// <summary>
/// Test whether input are feasible
/// </summary>
/// <param name="X"></param>
/// <param name="U"></param>
private void Feasible(Vector X, Vector U)
{
    X.ProperZeroes();
    for (int p = 0; p < X.Dimension; p++)
    {
        if (X[p] < 0.0)
        {
            throw new MPCMathException("ULoop reactor", "Infeasible state", 1);
        }
    }
}
```

In the example above `"ULoop reactor"` can be replaced by your own identifying text.

Chapter 12

Unconstrained and constrained minimization

Unconstrained minimization defined by

$$\min_{x \in \mathbb{R}^n} \quad \theta = fun(x) \tag{12.1a}$$

and equality constrained minimization problems

$$\min_{x \in \mathbb{R}^n} \quad \theta = fun(x) \tag{12.2a}$$
$$s.t. \qquad Ax = b \tag{12.2b}$$

are solved using the Newton method [Boyd and Vanderberghe, 2004]. The function to be minimized must implement the IFun 11.1 interface.

12.1 NewtonMethod

Class NewtonMethod

Newtons Method

```
class NewtonMethod
```

Constructors

`NewtonMethod(IFun function)`

Constructor for unconstrained solver
Parameters
 function function definition

`NewtonMethod(IFun function, Matrix A, Vector B)`

Constructor for equality constrained solver
Parameters
function   function definition

Properties

Alfa

Back tracking parameter $(0.0 < Alfa < 0.5)$

```
double Alfa {set; get;}
```

Beta

Back tracking parameter $(0.0 < Beta < 1.0)$

```
double Beta {set; get;}
```

Epsilon

Error margin

```
double Epsilon {set; get;}
```

Lambda

Newton decrement

```
double Lambda {get;}
```

MaxBacktrack

Maximum number of backtrack steps

```
long MaxBacktrack {set; get;}
```

MaxIterationSteps

Maximum number of iteration steps

```
long MaxIterationSteps {set; get;}
```

Mu

Dual variables

```
Vector Mu {get;}
```

Steps

number of Newton steps

```
long Steps {get;}
```

X

Postion of optimum

```
Vector X {get;}
```

Methods

Minimize

Minimize function

```
double Minimize(Vector xo)
```

Parameters

xo	Start position of x
returns	Minimum value

12.2 Extended linear-Quadratic Optimal Control problem

The extended Linear-Quadratic Optimal Control Problem is a special equality constrained minimization problem, which are encountered in solvers for MPC and LQR controllers. This problem can be solved extremely cpu efficient using the RiccatiSolver class 12.8. For a detailed description of the mathematics behind the Extended Linear-Quadratic Optimal Control problem and the the Riccati series a description can be found in [Jørgensen, 2004]

The RiccatiSolver class solves the problem

$$\min_{\{x_{k+1},u_k\}_{k=0}^{N-1}} \phi = \sum_{k=0}^{N-1} l_k(x_k,u_k) + l_N(x_N) \tag{12.3a}$$

subject to

$$\boldsymbol{x}_{k+1} = A_k\boldsymbol{x}_k + B_k u_k + b_k \quad k = 0,\dots,N-1 \tag{12.3b}$$

with the stage costs

$$l_k(x_k,u_k) = \frac{1}{2}x_k' Q_k x_k + x_k' M_k u_k + \frac{1}{2}u_k' R_k u_k + q_k' x_k + r_k' u_k + f_k \tag{12.4a}$$

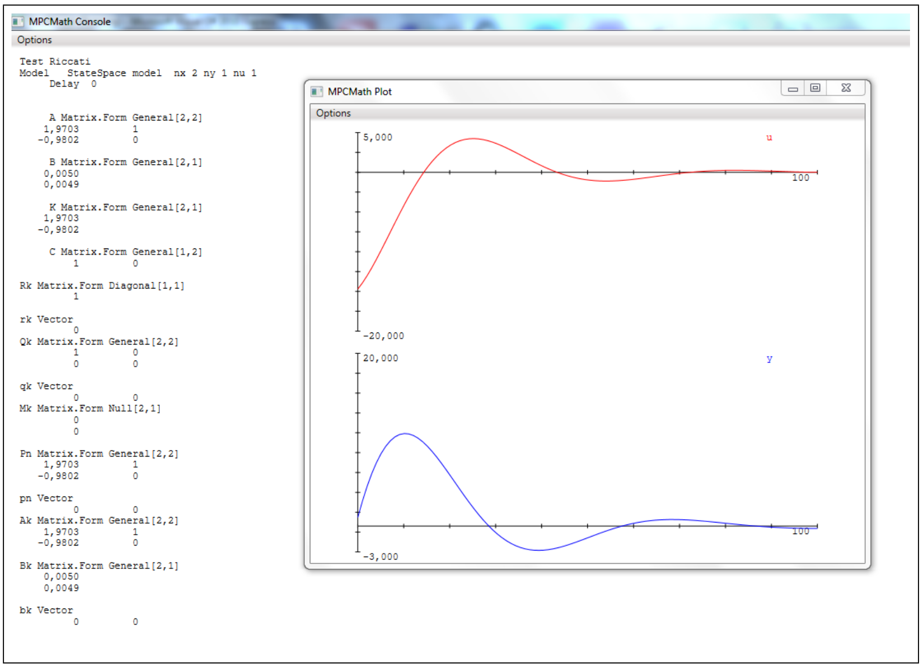
$$l_N(x_N) = \frac{1}{2}x_N' Q_N x_N + p_N' x_N + \gamma_k \tag{12.4b}$$

The initial state x_0 is a parameter and not a decision variable.
The following code example illustrates the use of the Riccati Solver class

```
console.WriteLine("Test Riccati");

int N = 100;

Double T = 1.0;
// Second order system
double gain = 1.0;
double delay = 0.0;
double tau = 10.0;
double sigma = 0.1;
TransferFunction G = new TransferFunction(gain, delay,tau, sigma);
StateSpaceModel Plant = new StateSpaceModel(G, T);
StateSpaceModel.Show("Model", Plant);
int nx = Plant.Dimension;
int nu = Plant.NU;
int ny = Plant.NY;

// initial condition
Vector X0 = new Vector(nx, 1.0);

Matrix Ak = Plant.A;
Matrix Bk = Plant.B;
Vector bk = Plant.Off;

Vector theta = new Vector(ny, 1.0);
Vector rho = new Vector(nu, 1.0);

// define stage penalties
Matrix Qk = Matrix.MulTransposed(Plant.C, new Matrix(theta)) * Plant.C;
Vector qk = new Vector(nx);
Matrix Rk = new Matrix(rho);
Vector rk = new Vector(nu);
Matrix Mk = new Matrix(nx, nu);
// end condition penalties
Matrix PN = Ak;
Vector pN = qk;

Matrix.Show("Rk", Rk);
Vector.Show("rk", rk);
Matrix.Show("Qk", Qk);
Vector.Show("qk", qk);
Matrix.Show("Mk", Mk);
Matrix.Show("Pn", PN);
Vector.Show("pn", pN);
```

```
Matrix.Show("Ak", Ak);
Matrix.Show("Bk", Bk);
Vector.Show("bk", bk);

// store stage penalites as diagonal elements in Block matrices
BMatrix Q = new BMatrix(N, Qk);
BMatrix R = new BMatrix(N, Rk);
BMatrix M = new BMatrix(N, Mk);
BMatrix A = new BMatrix(N, Ak);
BMatrix B = new BMatrix(N, Bk);

RiccatiSolver ricattiSolver = new RiccatiSolver(PN, Q, M, R, A, B);

BVector q = new BVector(N, qk);
BVector r = new BVector(N, rk);
BVector b = new BVector(N, bk);

BVector U;
BVector X;

ricattiSolver.Solve(X0, pN, q, r, b, out U, out X);

// Plot results

Vector y = new Vector(N);
Vector u = new Vector(N);
for(int k = 0; k < N; k++)
{
    y[k] = X[k][0];
    u[k] = U[k][0];
}

console.Plot(new PlotSeries("u", 0.0, 0.0, SeriesColor.Red, u),
    new PlotSeries("y", 0.0, 0.0, SeriesColor.Blue, y));
```

The code produces the following output



12.3 Function Minimization

The function minimizer $FunMin$ finds a local minima of a non-linear function, using a quasi Newton method (BFGS).

$$\min_{x \in \mathbb{R}^n} \quad fun(x) \tag{12.5a}$$

The function to be minimized must implement the $IFun$ interface , 11.1, with the property N and methods $Value$ and Dx. $FunMin$ does not use the Hessian function DDx.

The constrained function minimizer $ConFunMin$ 12.5find a locals minima of fun(x) subject to equality and inequality constraints

$$\min_{x \in \mathbb{R}^n} \quad fun(x) \tag{12.6a}$$

$$s.t. \quad C_e(x) = 0 \tag{12.6b}$$

$$C_i(x) \geq 0 \tag{12.6c}$$

Where C_e is a vector of n_e equality conditions and C_i is a vector of n_i inequality conditions.

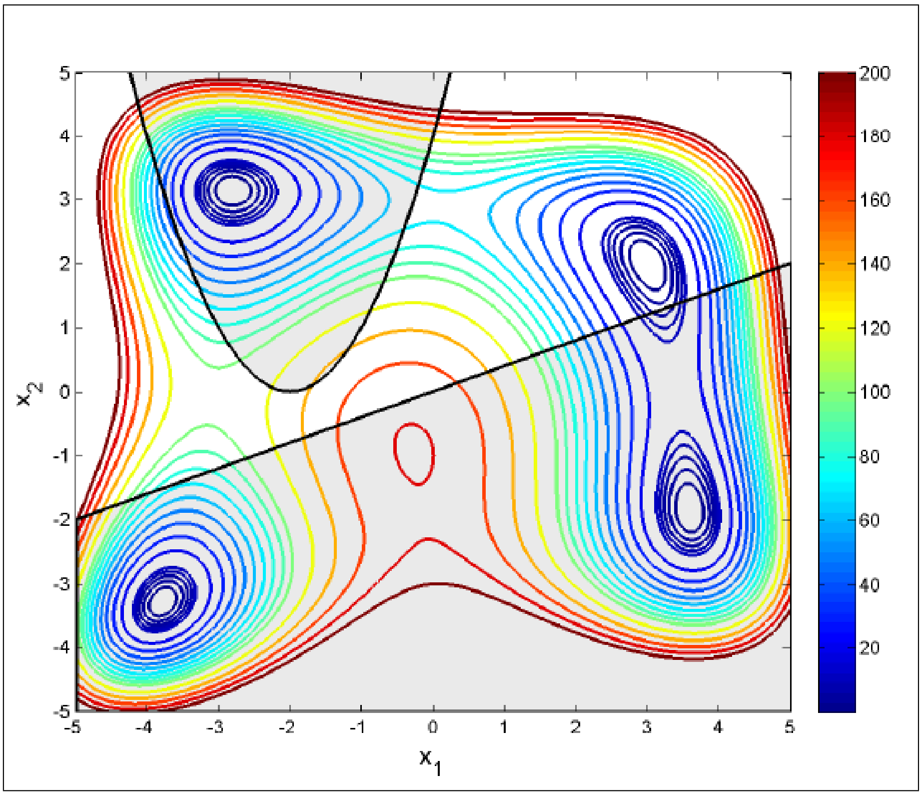
The function to be minimized must implement the $IConFun$ interface , 11.2, with the properties N,Ne,Ni, and methods $value$, Dx, $Vale$, Dxe, $Vali$ and Dxi. $ConFunMin$ does not require the Hessian functions DDx, $DDxe$, $DDxi$.

Use of $FunMin$ and $ConFunMin$ are illustrated on a simple non-linear function:

$$fun(x_1, x_2) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 - 7)^2 \quad (12.7a)$$

$$s.t. \quad C_1(x_1, x_2) = (x_1 + 2)^2 - x_2 \geq 0 \quad (12.7b)$$

$$C_2(x_1, x_2) = -4x_1 + 10x_2 \geq 0 \quad (12.7c)$$



The implementation of fun with the $IConFun$ interface is shown in appendix D

The use of $FunMin$ is demonstrated is the code example below:

```
console.WriteLine("Test Constrained Minimizer");

IConFun fun = new TestConFun();

Vector X0 = new Vector(3.5, 1.5);
Vector XMin = new Vector(-5.0, -5.0);
Vector XMax = new Vector( 5.0,  5.0);

Vector.Show("X0", X0);
Vector.Show("fun.Dx", fun.Dx(X0));
Matrix.Show("fun.DDx", fun.DDx(X0));
```

```
double val;
Vector X;

// Unconstrained minimization

FunMin min = new FunMin(fun);
val = min.Minimize(X0, XMin, XMax);
console.WriteLine("Iterations " + min.Iterations +
    " Minimum value " + val);

X = min.X;
Vector.Show("X*", X);
Vector.Show("fun.Dx", fun.Dx(X));
Matrix.Show("fun.DDx", fun.DDx(X));
```

Notice that the interface *IConFun* inherits the interface *IFun* , making the function *TestConFun* usable as argument requiring an *IFun* interface.

The resulting output on the console is:

```
X0 Vector
         3,5000         1,5000
fun.Dx Vector
        36,0000        -2,0000
fun.DDx Matrix.Form General[2,2]
        31,0000        20,0000
        20,0000        15,0000

Iterations 32 Minimum value 3,09858265117597E-18
X* Vector
         3,0000         2,0000
fun.Dx Vector
         0,0000         0,0000
fun.DDx Matrix.Form General[2,2]
        34,0000        20,0000
        20,0000        34,0000
```

The *Consistent* routine can be used to check the coding of the function to be optimized. *Consistent* checksthe approximation:

$$\tilde{f}(x) \simeq f(x) + \nabla_x f(x)^{'} \Delta x \tag{12.8}$$

The use of *Consistent* is demonstrated in the code below

```
FunMin min = new FunMin(fun);

if (min.Consistent(X0))
{
```

```
        val = min.Minimize(X0, XMin, XMax);
        console.WriteLine("Iterations " + min.Iterations +
            " Minimum value " + val);

        X = min.X;
        Vector.Show("X*", X);
        Vector.Show("fun.Dx", fun.Dx(X));
        Matrix.Show("fun.DDx", fun.DDx(X));
    }
    else
    {
        console.WriteLine("Inconsistent function");
    }
```

The constrained minimizer $ConFunMin$ is based on the Augmented Lagrange multiplier method. The Lagrange function for the optimization problem, 12.6, is

$$L(x, \lambda_e, \lambda_i, z) = fun(x) - \lambda_e C_e(x) - \lambda_i (C_i(x) - z) \tag{12.9a}$$

$$s.t. \quad z \geq 0 \tag{12.9b}$$

where z is a vector of slack variables. This Lagrange function is augmented to the Lagrange function

$$\begin{aligned} L(x, \lambda_e, \lambda_i, z) = {} & fun(x) - \lambda_e C_e(x) - \lambda_i (C_i(x) - z) \\ & + \frac{1}{2}\mu (C_e(x)^{'} C_e(x) + (C_i(x) - z)^{'} (C_i(x) - z)) \end{aligned} \tag{12.10}$$

The square term $\frac{1}{2}\mu(C_e(x)^{'} C_e(x) + (C_i(x) - z)^{'}(C_i(x) - z))$ gives a penalty on constraint violations. $ConFunMin$ fixes a set of $(\lambda_e, \lambda_i, \mu, z)_0$ and then uses the unconstrained minimizer $FunMin$ to minimize the augmented Lagrange function over x. If the constraint violations are unacceptable, the value of μ is increased, and a new minimization is performed. $ConFunMin$ continues to iterate on the parameter set $(\lambda_e, \lambda_i, \mu, z)_k$ until an optimal solutions is found. For further details see [Nocedal and J.Wright, 2006] or [K.Madsen et al., 2004]

The following code demonstrate how to locate all four minima's in the simple problem 12.7

```
IConFun fun = new TestConFun();
Vector XMin = new Vector(-5.0, -5.0);
Vector XMax = new Vector( 5.0,  5.0);

double val;
Vector X;
```

```
// Constrained Minimization

console.WriteLine("Constrained minimizer ConFunMin");
ConFunMin lmin = new ConFunMin(fun);

int ntry = 50;
List<Vector> minimas = new List<Vector>();
for (int i = 0; i < ntry; i++)
{
    Vector X0 = Vector.Random(XMin, XMax);
    val = lmin.Minimize(X0, XMin, XMax);
    X = lmin.X;

    if (lmin.Ok)
    {
        bool insert = true;
        for (int p = 0; p < minimas.Count; p++)
        {
            Vector Xminima = minimas[p];
            if ((X - Xminima).Norm() < 1.0e-2)
            {
                insert = false;
                break;
            }
        }

        if (insert)
        {
            minimas.Add(X);
        }
    }
}

console.WriteLine("Search end, found " + minimas.Count + " minimas");
for (int p = 0; p < minimas.Count; p++)
{
    X = minimas[p];
    val = fun.Value(X);
    console.Show("val", val);
    Vector.Show("X", X);
}
```

The program starts minimization for *ntry* random start points. Found minimas are collected in the list *minims*, if it is not already in the list.

The console output is;

Test Constrained Minimizer Constrained minimizer ConFunMin Search end, found 4 minimas val 65,4261 X Vector -0,2983 2,8956 val 72,8556 X Vector -3,5485 -1,4194 val 0,0000 X Vector 3,0000 2,0000 val 35,9298 X Vector -3,6546 2,7377

12.4 FunMin

Class FunMin

Unconstrained function minimizer. Quasi Newton method (BFGS) .

```
class FunMin
```

Constructors

```
FunMin(IFun Fun)
```

Constructor
Parameters
Fun Function to be minimized

Properties

BackTracks

Total number of backtracks in minimization

```
int BackTracks {get;}
```

C1

Wolfe condition constant C1

```
double C1 {set; get;}
```

C2

Wolfe condition constant C2

```
double C2 {set; get;}
```

EpsGrad

Gradient residual limit

```
double EpsGrad {set; get;}
```

EpsStep

Step size residual limit

```
double EpsStep {set; get;}
```

Iterations

Number of Iterations

```
int Iterations {get;}
```

MaxBacktracks

Maximum number af backtracks in each line search

```
int MaxBacktracks {set; get;}
```

MaxIterations

Maximum number of iteration steps

```
int MaxIterations {set; get;}
```

Ok

Minimize or Consistent result

```
bool Ok {get;}
```

ResGrad

Gradient residual

```
double ResGrad {get;}
```

ResStep

Step size residual

```
double ResStep {get;}
```

Tau

Initial Steepest descent gain

```
double Tau {set; get;}
```

Trace

Trace iterations

```
bool Trace {set; get;}
```

TraceLevel

Trace level

```
int TraceLevel {set; get;}
```

W

Inverse Hessian BFGS approximation;

```
Matrix W {get;}
```

WarmStart

Warm start, (keep BFGS Hesssian estimate between calls)

```
bool WarmStart {set; get;}
```

X

Argmin X

```
Vector X {get;}
```

Methods

Consistent

Check consistence of function to be minimized

```
bool Consistent(Vector x)
```

Parameters

x	X operation point
returns	true if consistent

Consistent

Check consistence of function to be minimized

```
bool Consistent(Vector x, Vector dx)
```

Parameters

x	X operation point
dx	variations of x
returns	true if consistent

Minimize

Minimize function, find local minimum

```
double Minimize(Vector X0)
```

Parameters

X0	Inital X
returns	minimum value

Minimize

Minimize function, find local minimum, with box constraints on variables

```
double Minimize(Vector X0, Vector XMin, Vector XMax)
```

Parameters

X0	Initial X guess
XMin	Min values for X, null if no check wanted
XMax	Max values for X, null if no check wanted
returns	minimum value at local minimum

12.5 ConFunMin

Class ConFunMin

Constrained Minimization of Function using Augmented Lagrangian method

```
class ConFunMin : IFun
```

Constructors

`ConFunMin(IConFun Fun)`

Constructor
Parameters
Fun

Properties

EpsGradL

Lagrange iteration residual limit

```
double EpsGradL {set; get;}
```

EpsGradX

X iterations limit

```
double EpsGradX {set; get;}
```

Iterations

Number of Iterations

```
int Iterations {get;}
```

LambdaE

Equality Lagrange multipliers

```
Vector LambdaE {set; get;}
```

LambdaI

Inequality Lagrange multipliers

```
Vector LambdaI {set; get;}
```

MaxIterations

Maximum number of iteration steps

```
int MaxIterations {set; get;}
```

Mu

Penalty of squared constraints

```
double Mu {set; get;}
```

Mu0

Initial penalty of squared constraints

```
double Mu0 {set; get;}
```

Ok

Minimize or Consistent result

```
bool Ok {get;}
```

ResGradL

Lagrange iteration residual

```
double ResGradL {get;}
```

ResGradX

X iteration residual

```
double ResGradX {get;}
```

Trace

Trace iterations

```
bool Trace {set; get;}
```

X

Argmin X

```
Vector X {get;}
```

XIterations

Total number of iterations i XMinimizer

```
int XIterations {get;}
```

XMinimizer

Reference to Minimizer for augmented Lagrangian function

```
FunMin XMinimizer {get;}
```

Methods

Consistent

Check consistence of function to be minimized

```
bool Consistent(Vector x)
```

Parameters

x	X operation point
returns	true if consistent

Consistent

Check consistence of function to be minimized

```
bool Consistent(Vector x, Vector dx)
```

Parameters

x	X operation point
dx	variations of x
returns	true if consistent

Minimize

Minimize Constrained function

```
double Minimize(Vector X0)
```

Parameters

X0	Inital X
returns	minimum value

Minimize

```
double Minimize(Vector X0, Vector XMin, Vector XMax)
```

Parameters

X0	Initial X guess
XMin	Min values for X, null if no check wanted
XMax	Max values for X, null if no check wanted
returns	minimum value

12.6 Least Square Fitting

The class *LeastSquareFit* 12.7 is useful for fitting the parameters x of a given function to a set of experimental results.

An example is

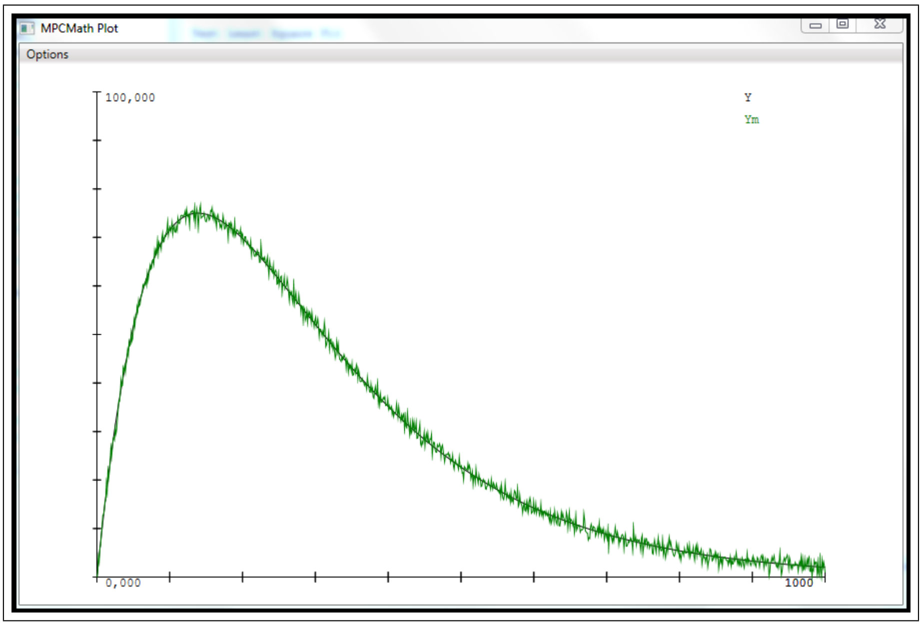
$$fun(t,x) = x_2 e^{X_0 t} + x_3 e^{x_1 t} + \varepsilon(t) \tag{12.11}$$

where ε is measurement noise. The function fun must implement the $IFunt$ interface as shown in the code example below:

```
/// <summary>
/// test function for parameter fitting, Least square fit
/// </summary>
public class Function : IFunt
{
    /// <summary>
    /// X dimension
    /// </summary>
    int IFunt.N
    {
        get
        {
            return 4;
        }
    }
```

```
    /// <summary>
    /// Function value
    /// </summary>
    /// <param name="x"></param>
    /// <param name="t"></param>
    /// <returns></returns>
    double IFunt.Value(Vector x, double t)
    {
        return x[2] * Math.Exp(x[0] * t) + x[3] * Math.Exp(x[1] * t); ;
    }

    /// <summary>
    /// Jacobian
    /// </summary>
    /// <param name="x"></param>
    /// <param name="t"></param>
    /// <returns></returns>
    Vector IFunt.Dx(Vector x, double t)
    {
        Vector dx = new Vector(4);
        dx[2] = Math.Exp(x[0] * t);
        dx[3] = Math.Exp(x[1] * t);
        dx[0] = t * x[2] * dx[2];
        dx[1] = t * x[3] * dx[3];
        return dx;
    }

    /// <summary>
    /// Hessian (not implemented)
    /// </summary>
    /// <param name="x"></param>
    /// <param name="t"></param>
    /// <returns></returns>
    Matrix IFunt.DDx(Vector x, double t)
    {
        return null;
    }
}
```

Plot of the input data with and without noise.



The program below illustrates the use of *LeastSquareFit* to estimate the four parameters of the function

```
console.WriteLine("Test Least Square Fit");
double tau1 = 0.005;
double tau2 = 0.01;
double gain = 300.0;
double var = 1.0;
Vector X = new Vector(-tau1, -tau2, gain , -gain);
Vector.Show("X", X);

IFunt fun = new Function();

int m = 1000;
Vector T = new Vector(m);
Vector Y = new Vector(m);
Vector Ym = new Vector(m);
for (int i = 0; i < m; i++)
{
    double time = (1000.0 * i)/m;
    T[i] = time;
    Y[i] = fun.Value(X, time);
    Ym[i] = Y[i] + MPCMathLib.WhiteGaussianNoise(0.0, var);
}

console.Plot(new Plot(
    new PlotSeries("Y", 0.0, 100.0, Y),
    new PlotSeries("Ym", 0.0, 100.0, SeriesColor.Green, Ym)));
```

```
        LeastSquareFit lsqfit = new LeastSquareFit(T, Ym, fun);
        Vector X0 = new Vector(0.0, 0.0,  100.0, -100.0);
        Vector Xhat = lsqfit.Fit(X0);

        console.WriteLine("Iterations " + lsqfit.Iterations);
        Vector.Show("X", X);
        Vector.Show("Xhat", Xhat);

        // constrain the function parameters

        Vector XMin = new Vector(-10.0, -10.0, 0.0, -500.0);
        Vector XMax = new Vector(0.0, 0.0, 290.0, -310.0);

        Xhat = lsqfit.Fit(X0, XMin, XMax);
        console.WriteLine();
        console.WriteLine("Constrained result");
        Vector.Show("XMin", XMin);
        Vector.Show("XMax", XMax);
        console.WriteLine("Iterations " + lsqfit.Iterations);
        Vector.Show("X", X);
        Vector.Show("Xhat", Xhat);
    }
```

The console output is

```
Test Least Square Fit
X Vector
        -0,0050        -0,0100       300,0000      -300,0000
Iterations 50
X Vector
        -0,0050        -0,0100       300,0000      -300,0000
Xhat Vector
        -0,0050        -0,0098       311,1983      -311,1504

Constrained result
XMin Vector
       -10,0000       -10,0000              0      -500,0000
XMax Vector
              0              0       290,0000      -310,0000
Iterations 56
X Vector
        -0,0050        -0,0100       300,0000      -300,0000
Xhat Vector
        -0,0050        -0,0098       290,0000      -311,1504
```

The *LeastSquareFit* uses the Levenberg-Marquardt method as described in [Nocedal and J.Wright, 2006] and [Madsen et al., 2004]

12.7 LeastSquareFit

Class LeastSquareFit

```
class LeastSquareFit
```

Constructors

```
LeastSquareFit(Vector T, Vector Y, IFunt Fun)
```

Constructor Least Square Fit
Parameters

T	t values
Y	y values
X0	Initial guess for parameter vector X
Fun	Approximation function

Properties

EpsGrad

Gradient residual limit

```
double EpsGrad {set; get;}
```

EpsStep

Step size residual limit

```
double EpsStep {set; get;}
```

Iterations

Number of Iterations

```
int Iterations {get;}
```

MaxIterations

Maximum number of iteration steps

```
int MaxIterations {set; get;}
```

ResGrad

Gradient residual

```
double ResGrad {get;}
```

ResStep

Step size residual

```
double ResStep {get;}
```

Tau

Tau, initial mu factor. Tau equal 0.0 , start with Newton direction. Tau equal infinity, start with steepest descent direction.

```
double Tau {set; get;}
```

Trace

Trace iterations

```
bool Trace {set; get;}
```

Methods

Fit

Fit parameter

```
Vector Fit(Vector X0)
```

Parameters

X0	Initial parameter guess
returns	Local Minimizer parameters

Fit

Fit parameter, constrained

```
Vector Fit(Vector X0, Vector XMin, Vector XMax)
```

Parameters

X0	Initial parameter guess
XMin	Min values for X, null if no check wanted
XMax	Max values for X, null if no check wanted
returns	Local Minimizer parameters

12.8 RiccatiSolver

Class RiccatiSolver

Riccati solver for extended linear-quadratic optimal control problem

```
class RiccatiSolver
```

Constructors

```
RiccatiSolver(Matrix Pn, BMatrix Q, BMatrix M, BMatrix R, BMatrix A, BMatrix B)
```

Constructor with Factorization
Parameters

N	Horizon
Pn	Penalty on final state
Q	State penalties
M	cross term penalties. (null is legal)
R	Penalties on inputs
A	Linear state equation matrices
B	Linear state equation matrices

Properties

K0

Initial Gain

```
Matrix K0 {get;}
```

p0

Initail cost

```
Vector p0 {get;}
```

P0

Initial cost P0

```
Matrix P0 {get;}
```

Methods

Solve

Solve Riccati series

```
void Solve(Vector X0, Vector pn, BVector q, BVector r,
BVector b, out BVector U, out BVector X)
```

Parameters

X0	Initial state
pn	linear penalty on final state
q	linear penalty on state
r	linear penalty on inputs
b	linear state space term
U	Optimal inputs
X	Optimal plant state

Solve

Solve Riccati series

```
void Solve(Vector X0, Vector pn, BVector q, BVector r,
BVector b, out BVector U, out BVector X, out BVector Lambda)
```

Parameters

X0	Initial state
pn	linear penalty on final state
q	linear penalty on state
r	linear penalty on inputs
b	linear state space term
U	Optimal inputs
X	Optimal plant state
Lambda	Dual variable

Chapter 13

QP and LP solvers

The *QPSolver* class is used to solve quadratic and linear optimization tasks. *QPSolver* implements a Primal-Dual Interior-Point Algorithm including Mehrotra's modifications. For quadratic problems the problem

$$\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & \frac{1}{2}x'Gx + g'x && (13.1a)\\
s.t. \quad & Ax = b && (13.1b)\\
& Cx \geq d && (13.1c)
\end{aligned}$$

is solved. For linear problems the problem

$$\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & g'x && (13.2a)\\
s.t. \quad & Ax = b && (13.2b)\\
& Cx \geq d && (13.2c)
\end{aligned}$$

is solved.

An quadratic problem example:

$$\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & \frac{1}{2}(x_1-1)^2 + \frac{1}{2}(x_2-2)^2 + \frac{1}{2}(x_3-1)^2 && (13.3a)\\
s.t. \quad & x_1 = x_2 + 0.5 && (13.3b)\\
& -1 \leq x_1 \leq 1 && (13.3c)\\
& -1 \leq x_2 \leq 1 && (13.3d)\\
& -1 \leq x_3 \leq 1 && (13.3e)
\end{aligned}$$

reformulaions of (13.3) to the standard form (13.1) gives

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13.4}$$

$$g' = \begin{pmatrix} -1 & -2 & -1 \end{pmatrix} \tag{13.5}$$

$$A = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \tag{13.6}$$

$$b' = \begin{pmatrix} 0.5 \end{pmatrix} \tag{13.7}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{13.8}$$

$$d' = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix} \tag{13.9}$$

The *MPCMath* code for this problem is

```
static void Work()
{
    console.WriteLine("Test QP solver");

    int nx = 3;
    int ny = 1;
    int nz = 3;
    Structure sx = new Structure(1, nx);
    Structure sy = new Structure(1, ny);
    Structure sz = new Structure(2, nz);
    BMatrix G = new BMatrix(MatrixForm.General, sx);
    BVector g = new BVector(sx);
    BMatrix A = new BMatrix(MatrixForm.General, sy, sx);
    BVector b = new BVector(sy);
    BMatrix C = new BMatrix(MatrixForm.General, sz, sx);
    BVector d = new BVector(sz);

    Matrix I = Matrix.UnityMatrix(nx);
    G[0, 0] = I;
    g[0] = new Vector(-1, -2, -1);

    A[0, 0] = new Matrix(MatrixForm.General, ny, nx);
    A[0, 0][0, 0] = 1.0;
    A[0, 0][0, 1] = -1.0;
    b[0] = new Vector(1, 0.5);

    C[0, 0] = I;
    C[1, 0] = -I;
    d[0] = new Vector(-1, -1, -1);
    d[1] = new Vector(-1, -1, -1);
```

```
            Structure.Show("sx", sx);
            Structure.Show("sy", sy);
            Structure.Show("sz", sz);
            BMatrix.Show("G", G);
            BVector.Show("g", g);
            BMatrix.Show("A", A);
            BVector.Show("b", b);
            BMatrix.Show("C", C);
            BVector.Show("d", d);

            QPSolver solver = new QPSolver(G, g, A, b, C, d);
            solver.Solve();
            BVector.Show("x optimal", solver.X);
            console.WriteLine();
            console.WriteLine("KKT conditions");
            console.Show("Dual gap", solver.My);
            BVector.Show("A*X - b", A * solver.X - b);
            BVector.Show("C*X - d", C * solver.X - d);
            BVector.Show("S*E", BVector.MulElements(solver.S , solver.Z));
        }
    }
```

with the output

```
Test QP solver
sx
         3

sy
         1

sz
         3         3

G Matrix.Form General[3,3]
         1         0         0
         0         1         0
         0         0         1

g Vector
        -1   -2,0000        -1
A Matrix.Form General[1,3]
         1        -1         0

b Vector
    0,5000
C Matrix.Form General[6,3]
         1         0         0
```

```
        0         1         0
        0         0         1
       -1         0         0
        0        -1         0
        0         0        -1

d Vector
       -1        -1        -1        -1        -1        -1
x optimal Vector
    1,0000    0,5000    0,9934

KKT conditions
Dual gap 0,0000
A*X - b Vector
        0
C*X - d Vector
    2,0000    1,5000    1,9934    0,0000    0,5000    0,0066
S*E Vector
    0,0000    0,0000    0,0000    0,0000    0,0000    0,0000
```

The interior point algorithm requires that the initial value of X strictly fulfils the inequality conditions $Cx > d$. The equality condition $Ax = b$ don't have to be fulfilled. For most MPC problems $X = 0$ fulfils this requirement.

13.1 Quadratic Optimization

Quadratic program

$$\min_{x\in\mathbb{R}^n} \quad \frac{1}{2}x'Gx + g'x \tag{13.10a}$$

$$s.t. \quad Ax = b \tag{13.10b}$$

$$Cx \geq d \tag{13.10c}$$

Lagrange function

$$L(x,y,z) = \frac{1}{2}x'Gx + g'x - y'(Ax - b) - z'(Cx - d) \tag{13.11}$$

Optimality Conditions

$$\nabla_x L(x,y,z) = Gx + g - A'y - C'z = 0 \tag{13.12a}$$

$$\nabla_y L(x,y,z) = -(Ax - b) = 0 \tag{13.12b}$$

$$\nabla_z L(x,y,z) = -(Cx - d) \leq 0 \tag{13.12c}$$

$$z \geq 0 \tag{13.12d}$$

$$(Cx - d)_i z_i = 0 \quad i = 1,2,\ldots,m_c \tag{13.12e}$$

Slack variables

$$s \triangleq Cx - d \geq 0 \tag{13.13}$$

implies

$$
\begin{aligned}
-Cx + s + d &= 0 && (13.14a)\\
s &\geq 0 && (13.14b)
\end{aligned}
$$

Optimality Conditions

$$
\begin{aligned}
&r_L = Gx + g - A'y - C'z = 0 && (13.15a)\\
&r_A = -Ax + b = 0 && (13.15b)\\
&r_C = -Cx + s + d = 0 && (13.15c)\\
&z \geq 0 && (13.15d)\\
&s \geq 0 && (13.15e)\\
&s_i z_i = 0 \quad i = 1, 2, \ldots, m_c && (13.15f)
\end{aligned}
$$

Notation

$$
S = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_{m_c} \end{bmatrix} \quad z = \begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_{m_c} \end{bmatrix} \quad e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{13.16}
$$

The complementarity conditions

$$
s_i z_i = 0 \quad i = 1, 2, \ldots, m_c \tag{13.17}
$$

can be expressed as

$$
SZe = 0 \tag{13.18}
$$

The optimality conditions can be expressed as

$$
\begin{aligned}
&r_L = Gx + g - A'y - C'z = 0 && (13.19a)\\
&r_A = -Ax + b = 0 && (13.19b)\\
&r_C = -Cx + s + d = 0 && (13.19c)\\
&r_{SZ} = SZe = 0 && (13.19d)\\
&s \geq 0, \quad z \geq 0 && (13.19e)
\end{aligned}
$$

which is the same as

$$
\begin{aligned}
&F(x, y, z, s) = \begin{bmatrix} r_L \\ r_A \\ r_C \\ r_{SZ} \end{bmatrix} = \begin{bmatrix} Gx + g - A'y - C'z \\ -Ax + b \\ -Cx + s + d \\ SZe \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} && (13.20a)\\
&(z, s) \geq 0 && (13.20b)
\end{aligned}
$$

Solve $F(x, y, z, s) = 0$ such that $(z, s) \geq 0$ using Newton's method.

13.2 Matrix Factorization

Matrix factorizations are used by *QPSolver* to speed up calculation. The KKT equations

$$\begin{bmatrix} G & -A' & -C' & 0 \\ -A & 0 & 0 & 0 \\ -C & 0 & 0 & I \\ 0 & 0 & S & Z \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_L \\ -r_A \\ -r_C \\ -\bar{r}_{SZ} \end{bmatrix} \tag{13.21}$$

can be factorized as follows. The fourth equation gives

$$S\Delta z + Z\Delta s = -\bar{r}_{SZ} \quad \Rightarrow \quad \Delta s = -Z^{-1}\bar{r}_{SZ} - Z^{-1}S\Delta z \tag{13.22}$$

Substitution in the third equation gives

$$-r_C = -C'\Delta x + \Delta s = -C'\Delta x - Z^{-1}S\Delta z - Z^{-1}\bar{r}_{SZ} \tag{13.23}$$

and

$$-C'\Delta x - Z^{-1}S\Delta z = -r_C + Z^{-1}\bar{r}_{SZ} \tag{13.24}$$

Augmented form

$$\begin{bmatrix} G & -A' & -C' \\ -A & 0 & 0 \\ -C & 0 & -Z^{-1}S \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} -r_L \\ -r_A \\ -r_C + Z^{-1}\bar{r}_{SZ} \end{bmatrix} \tag{13.25}$$

13.3 KKT solvers

QPSolver solves the KKT equations (13.20) by calling a KKTSolver routine. The KKT solver determines the type of optimization problem quadratic, linear , second order cone problems. The KKT solver must implement the interface *IKKTSolver* making it possible to switch between KKT solvers and making it possible to produce custom KKT solvers exploiting matrix structures.

The relevant KKTSolver is passed to *QPSolver* using a constructor. The two following calls are equivalent (as *KKTSolver* is the default KKT solver for *QPSolver*)

```
QPSolver solver = new QPSolver(G, g, A, b, C, d);
QPSolver solver = new QPSolver(new KKTSolver(G, g, A, b, C, d));
```

The *IKKTSolver* defines three methods used by *QPSolver*.The KKT solves the augmented KKT equations (13.25).

- Calculate the residuals r_L, r_A and r_C.
- Factorize

$$\begin{bmatrix} G & -A' & -C' \\ -A & 0 & 0 \\ -C & 0 & -Z^{-1}S \end{bmatrix} \tag{13.26}$$

- Calculate Newton search directions Δx, Δy and Δz

The default KKTSolver perform a further factorization of (13.25) to the form:

$$\begin{bmatrix} G + C'\left(S^{-1}Z\right)C & -A' \\ -A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -r_L + C\left(S^{-1}Z\right)\left(r_C - Z^{-1}\bar{r}_{SZ}\right) \\ -r_A \end{bmatrix} \tag{13.27}$$

The main computational effort for QPSolver and the associated KKTSolver is the computation of the term $G+C'\left(S^{-1}Z\right)C$ and the factorization of (13.27) using a SymmetricalBlockEquationSolver 8.6.

MPCMath automatically exploits structures in the block matrices, but but further exploitation of marix structures or other factorization methods can be implemeneted by writing a new KKTSolver. The source code for the default KKTSolver is given in appendix F.

13.4 QPSolver

Class QPSolver

IP Solver class Primal-Dual Interior-Point Algorithm

```
class QPSolver
```

Constructors

```
QPSolver(BMatrix G, BVector g, BMatrix A, BVector b, BMatrix C, BVector d)
: this(new KKTSolver(G, g, A, b, C, d))
```

Constructor with default KKT Solver
min. X'Gx + g'x
s.t. Ax = b
Gx ≥ d

Parameters

G	G coefficients
g	g coefficients
A	A coefficients
b	b coefficients
C	C coefficients
d	d coefficients

```
QPSolver(IKKTSolver Solver)
```

Solver with specific KKT solver
Parameters

Solver	KKT solver

Properties

Disabled

Disabled inequality constrains. Included = 0, disabled = 1;

```
BVector Disabled {get;}
```

ErrorLimit

Error limit

```
double ErrorLimit {set; get;}
```

Iterations

Iterations in last solve call

```
int Iterations {get;}
```

MaxIterations

Maximum iterations

```
int MaxIterations {set; get;}
```

Mu

Duality gap

```
double Mu {get;}
```

S

Slack variables for (Cx - d) reference

```
BVector S {set; get;}
```

Time

Execution Time for last Solve call

```
TimeSpan Time {get;}
```

Trace

Trace iterations

```
bool Trace {set; get;}
```

TraceLevel

Trace level 0 trace residual 1 trace residual and x 2 trace residual and x,y,z,s 3 trace residual, x,y,z,s and detailed info

```
int TraceLevel {set; get;}
```

X

Solution vector x

```
BVector X {set; get;}
```

Y

Laggrange multiplier for (Ax - b) reference

```
BVector Y {set; get;}
```

Z

Lagrange multiplier for (CX-d) reference

```
BVector Z {set; get;}
```

Methods

Solve

Solve

```
void Solve()
```

13.5 IKKTSolver

Interface IKKTSolver

KKT Solver interface for solving KKT equation system in augmented form, se equation (13.25)

```
interface IKKTSolver
```

Properties

d

```
BVector d {get;}
```

Strcx

X structure

```
Structure Strcx {get;}
```

Strcy

Y structure

```
Structure Strcy {get;}
```

Strcz

Z structure

```
Structure Strcz {get;}
```

Methods

Factorize

Factorize KKT system

```
void Factorize(BVector SInvZ)
```

Parameters

SInvZ	$S^{-1}Z$ diagonal vector

Residuals

Calculate residulas

```
void Residuals(BVector x, BVector y, BVector z, BVector s,
out BVector rL, out BVector rA, out BVector rC, out BVector rSZ)
```

Parameters

x	state variables
y	Lagrange multipliers for equality conditions
z	Lagrange multipliers for inequality conditions
s	Slack variables
rL	residuals
rA	residuals
rC	residuals
rSZ	residuals

Solve

Find Newton search directions

```
void Solve(BVector rL, BVector rA, BVector rC,
out BVector DX, out BVector DY, out BVector DZ)
```

Parameters

rL	residual
rA	residual
rC	residual
DX	X search direction
DY	Y search direction
DZ	Z search direction

13.6 KKTSolver

Class KKTSolver

Quadratic Program KKT solver, default KKT solver for QPSolver. Q must be positive semidefinit. Solves Linear problems by setting Q = null.

```
class KKTSolver : IKKTSolver
```

Constructors

```
KKTSolver(BMatrix G, BVector g, BMatrix A,
BVector b, BMatrix C, BVector d)
```

Constructor for KKT solver
Parameters

G	G coefficients
g	g coefficients
A	A coefficients
b	b coefficients
C	C coefficients
d	d coefficients

13.7 LPKKTSolver

Class LPKKTSolver

Linear problem KKT solver for problems defined in equation (13.2)

```
class LPKKTSolver : IKKTSolver
```

Constructors

`LPKKTSolver(BVector g, BMatrix A, BVector b, BMatrix C, BVector d)`

Constructor, Linear Programming KKTSolver
Parameters

g	g coefficients
A	A coefficients
b	b coefficients
C	C coefficients
d	d coefficients

Chapter 14

Model Predictive Control, MPC

This chapter describes how to set up MPC controller using *MPCMath* . The implementation of MPC is based on the MiMoMPC object. Source code for MiMoMPC is available for the *MPCMath* user, making it possible to tailor the MPC controllers specifications and functionality to the users requirement

The MiMoMPC class implements both conventional MPC and Soft Constrained MPC [Prasath and Jørgensen, 2010], [Prasath et al., 2010] . The conventional MPC has a quadratic penalty function and the Soft Constrained MPC has a dead band zone around the set point where the penalty for not reaching the exact set point is low.

The plant is assumed to be a linear state space system in innovation form.

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k + K\epsilon_k \tag{14.1a}$$

$$\boldsymbol{y}_k = C\boldsymbol{x}_k + \epsilon_k \tag{14.1b}$$

where $\boldsymbol{x}_k$ is the plant state vector, u_k the manipulated variables. ϵ_k noise and $\boldsymbol{y}_k$ is the observed plant output. Knowing the present plant state $\boldsymbol{x}_k$ and the measured plant output $\boldsymbol{y}_k$ the noise ϵ_k can be estimated from

$$\epsilon_k = y_k - CAx_k \tag{14.2}$$

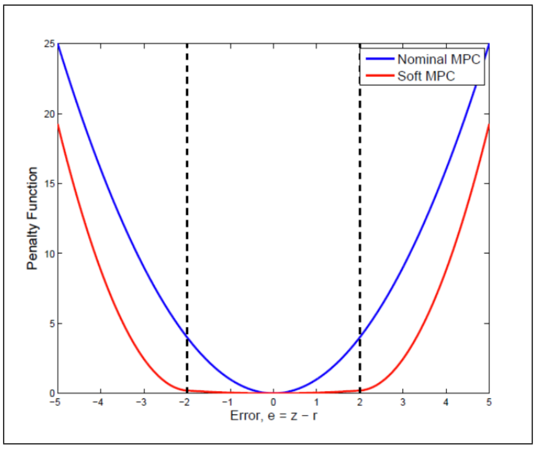
The MPC control problem is formulated as minimization of

$$\min_{\{y,u,\eta\}} \phi = \frac{1}{2}\sum_{k=0}^{N-1}(\|y_{k+1} - r_{k+1}\|_{Q_y}^2 + \|\Delta u_k\|_{S_u}^2) + \frac{1}{2}\sum_{k=1}^{N}\|\eta_k\|_{S_\eta}^2 \tag{14.3a}$$

subject to the constraints

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k + K\epsilon_k \qquad k = 0, \dots, N-1 \tag{14.3b}$$
$$\boldsymbol{y}_k = C\boldsymbol{x}_k + \epsilon_k \qquad k = 0, \dots, N \tag{14.3c}$$
$$u_{\min} \leq u_k \leq u_{\max} \qquad k = 0, \dots, N-1 \tag{14.3d}$$
$$\Delta u_{\min} \leq \Delta u_k \leq \Delta u_{\max} \qquad k = 0, \dots, N-1 \tag{14.3e}$$
$$y_k \leq y_{\max} + \eta_k \qquad k = 1, \dots, N \tag{14.3f}$$
$$y_k \geq y_{\min} - \eta_k \qquad k = 1, \dots, N \tag{14.3g}$$
$$\eta_k \geq 0 \qquad k = 1, \dots, N \tag{14.3h}$$

In which $\Delta u_k = u_k - u_{k-1}$. The term $\|y_{k+1} - r_{k+1}\|^2_{Q_y}$ in (14.3a) penalizes the deviation between plant output and the reference r_k. The term $\|\Delta u_k\|^2_{S_u}$ penalizes the movements of the manipulated variables. $y_{\min}$ and $y_{\max}$ are the soft constraint limits on the controlled variables. η_k is the violation of soft constraints. $\|\eta_k\|^2_{S_\eta}$ is the penalty for violation of the soft constraints.



The state space description of the plant (14.1) can be rearranged to

$$y_k = CA^k x_0 + CA^{k-1}K\epsilon_0 + \sum_{i=1}^{k-1} CA^{k-i-1}Bu_i \qquad 1 \leq k \leq N \tag{14.4}$$

where the predicted future value of y_{k+i} ia a function of x_k , ε_k and the future values of the manipulated variables u_k.

Defining $y_{free,k} = CA^k x_0 + CA^{k-1}K\epsilon_0$ and $H_i = CA^{i-1}B$ (The Markov parameters) gives

$$y_k = y_{free,k} + \sum_{j=1}^{i} H_i u_{j-1} \qquad 1 \leq i < N \tag{14.5}$$

Define the vectors Y,Y_{free}, R and η

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \quad Y_{free} = \begin{bmatrix} y_{free,1} \\ y_{free,2} \\ \vdots \\ y_{free,N} \end{bmatrix} \quad R = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \quad U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{bmatrix} \tag{14.6}$$

and the matrix Γ

$$\Gamma = \begin{bmatrix} H1 & 0 & \dots & 0 & 0 \\ H_2 & H_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{N-1} & H_{N-2} & & H_1 & 0 \\ H_N & H_{N-1} & \dots & H_2 & H_1 \end{bmatrix} \tag{14.7}$$

Then the prediction from (14.5) can be expressed as

$$Y = Y_{free} + \Gamma U \tag{14.8}$$

Define the matrix Λ and vector I_0 by
Λ and I_0 by

$$\Lambda = \begin{bmatrix} I & 0 & 0 & \dots & 0 & 0 \\ -I & I & 0 & \dots & 0 & 0 \\ 0 & -I & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & \dots & -I & I \end{bmatrix} \quad I_0 = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14.9}$$

Define $\mathcal{Q}_y$, $\mathcal{S}_u$ and $\mathcal{S}_\eta$

$$\mathcal{Q}_y = \begin{bmatrix} Q_y & & & \\ & Q_y & & \\ & & \ddots & \\ & & & Q_y \end{bmatrix} \mathcal{S}_u = \begin{bmatrix} S_u & & & \\ & S_u & & \\ & & \ddots & \\ & & & S_u \end{bmatrix} \mathcal{S}_\eta = \begin{bmatrix} S_\eta & & & \\ & S_\eta & & \\ & & \ddots & \\ & & & S_\eta \end{bmatrix} \tag{14.10}$$

Then the objective function (14.3) may be expressed as

$$\begin{aligned} \phi &= \frac{1}{2} \sum_{k=0}^{N-1} \|y_{k+1} - r_{k+1}\|_{Q_y}^2 + \|\Delta u_k\|_{S_u}^2 + \frac{1}{2} \|\eta_{k+1}\|_{S_\eta}^2 \\ &= \frac{1}{2} \|Y - R\|_{\mathcal{Q}_y}^2 + \frac{1}{2} \|\Lambda U - I_0 u_{-1}\|_{\mathcal{S}_u}^2 + \frac{1}{2} \|\eta\|_{\mathcal{S}_\eta}^2 \\ &= \frac{1}{2} \|Y_{free} + \Gamma U - R\|_{\mathcal{Q}_y}^2 + \frac{1}{2} \|\Lambda U - I_0 u_{-1}\|_{\mathcal{S}_u}^2 + \frac{1}{2} \|\eta\|_{\mathcal{S}_\eta}^2 \\ &= \frac{1}{2} U' \left(\Gamma' \mathcal{Q}_y \Gamma + \Lambda' \mathcal{S}_u \Lambda\right) U + \left(\Gamma' \mathcal{Q}_z (Y_{free} - R) - \Lambda' \mathcal{S}_u I_0 u_{-1}\right)' U \\ &\qquad + \left(\frac{1}{2} \|Y_{free} - R\|_{\mathcal{Q}_y}^2 + \frac{1}{2} \|I_0 u_{-1}\|_{\mathcal{S}_u}^2\right) + \frac{1}{2} \eta' \mathcal{S}_\eta \eta \\ &= \frac{1}{2} U' G U + g' U + \rho + \frac{1}{2} \eta' \mathcal{S}_\eta \eta \\ &= \frac{1}{2} x' \bar{G} x + \bar{g}' x + \rho \end{aligned} \tag{14.11}$$

with

$$G = \Gamma' \mathcal{Q}_y \Gamma + \Lambda' \mathcal{S}_u \Lambda \tag{14.12a}$$
$$g = \Gamma' \mathcal{Q}_y (c - R) - \Lambda' \mathcal{S}_u I_0 u_{-1} \tag{14.12b}$$
$$\rho = \frac{1}{2} \left\| Y_{free} - R \right\|_{\mathcal{Q}_y}^2 + \frac{1}{2} \left\| u_{-1} \right\|_{S_u}^2 \tag{14.12c}$$
$$x = \begin{bmatrix} U \\ \eta \end{bmatrix} \; \bar{G} = \begin{bmatrix} G & 0 \\ 0 & \mathcal{S}_\eta \end{bmatrix} \; \bar{g} = \begin{bmatrix} g \\ s_\eta \end{bmatrix} \tag{14.12d}$$

Consequently, we may solve MPC regulator problem (14.3) by solution of the following convex quadratic program

$$\min_x \quad \psi = \frac{1}{2} x' \bar{G} x + \bar{g}' x \tag{14.13a}$$
$$s.t. \quad x_{\min} \le x \le x_{\max} \tag{14.13b}$$
$$b_l \le \bar{C} x \le b_u \tag{14.13c}$$

in which

$$x_{\min} = \begin{bmatrix} U_{\min} \\ 0 \end{bmatrix} \quad x_{\max} = \begin{bmatrix} U_{\max} \\ \infty \end{bmatrix} \tag{14.14a}$$

$$b_l = \begin{bmatrix} \Delta U_{\min} \\ -\infty \\ Z_{\min} - c \end{bmatrix} C = \begin{bmatrix} \Lambda & 0 \\ \Gamma & -I \\ \Gamma & I \end{bmatrix} b_u = \begin{bmatrix} \Delta U_{\max} \\ Z_{\max} - c \\ \infty \end{bmatrix} \tag{14.14b}$$

In a model predictive controller only the first vector, u_0^*, of $U^* = \begin{bmatrix} (u_0^*)' & (u_1^*)' & \dots & (u_{N-1}^*)' \end{bmatrix}'$, is implemented on the process. At the next sample time the open-loop optimization is repeated with new information due to a new measurement.

14.1 MiMoMPC

Class MiMoMPC

Multiple Input Multiple Output MPC *MPCMath*

```
class MiMoMPC
```

Constructors

```
MiMoMPC(StateSpaceModel Model, int History, int Horizon,
Vector Theta, Vector Mu, Vector YMin, Vector YMax,
Vector Rho, Vector UMin, Vector UMax)
```

Constructor for MiMoMPC object
Parameters

Model	Controller model
History	History length
Horizon	Control horizon, N
Theta	Reference violation penalty, Q_y
Mu	Soft constraint violation penalty, S_η
YMin	Y Max soft constraint, y_{max}
YMax	Y Min soft constraint, y_{min}
Rho	U change penalty, S_u
UMin	U max value, u_{max}
UMax	U min value, u_{max}

Properties

Alfa

Error integration coifficient Alfa = 0.0 Dedadbeat elimination of stationary error Alfa = 1.0 No elimination of stationary error

```
Vector Alfa {set; get;}
```

History

Length of History part

```
int History {get;}
```

Horizon

Length of control Horizon

```
int Horizon {get;}
```

Model

reference to Arx model for MPC controller

```
StateSpaceModel Model {get;}
```

Mu

Soft Constrained penalty, Q_η

```
Vector Mu {set; get;}
```

Rho

Delta U penalty, S_u

```
Vector Rho {set; get;}
```

SoftConStrained

MPC controller soft constrained

```
bool SoftConStrained {get;}
```

Solver

Reference to QP solver

```
QPSolver Solver {get;}
```

Theta

Delta Y penalty, Q_y

```
Vector Theta {set; get;}
```

U

Reference to U response

```
Matrix U {get;}
```

UMax

U maximum value, U_{max}

```
Vector UMax {set; get;}
```

UMin

U minimum value, U_{min}

```
Vector UMin {set; get;}
```

X

Reference to state

```
Vector X {get;}
```

YFree

Reference to YFree response

```
Matrix YFree {get;}
```

YMax

Soft Constrained Uppper Limit, y_{max}

```
Vector YMax {set; get;}
```

YMin

Soft Constrained Lower Limit, y_{min}

```
Vector YMin {set; get;}
```

YModel

Reference to YModel response

```
Matrix YModel {get;}
```

YPlant

Reference to YPlant response

```
Matrix YPlant {get;}
```

YRef

Reference to YRef

```
Matrix YRef {get;}
```

Methods

Next

Next control step

```
Vector Next(Vector YPlant)
```

Parameters

YPlant	Plant value
returns	U value

NextY

Next expected Y value

```
Vector NextY()
```

SetRef

Set future reference

```
void SetRef(Matrix Ref)
```

Parameters
  Ref   future reference Matrix

SetRef

Set future reference

```
void SetRef(Vector Ref)
```

Parameters
  Ref   future reference value

14.2 LinearModel

Class LinearModel

Linear Model dx/dt = A * X + B * U + W
Y = C * X + V

```
class LinearModel : IModel
```

Constructors

```
LinearModel(Matrix A, Matrix B, Matrix C, double T)
: this(A, B, C, T, T)
```

Constructor
Parameters

A	System matrix A
B	System matrix A
C	System matrix A
T	Step size for NextSate

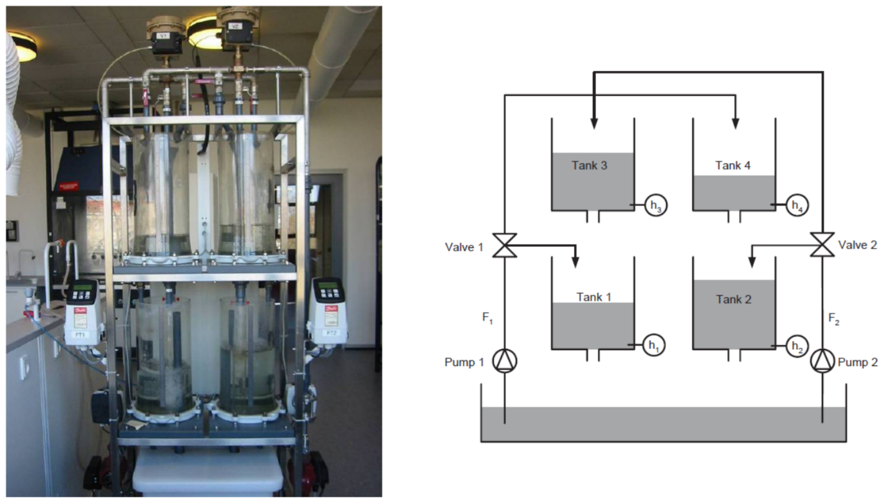
```
LinearModel(Matrix A, Matrix B, Matrix C, double T, double DT)
```

Constructor
Parameters

A	System matrix A
B	System matrix A
C	System matrix A
T	Step size for NextSate
DT	Integration step size

14.3 Four Tank Process

The four tank process is an simple multi variable process used to demonstrate MIMO process dynamic in education . The four tank system was introduced by Johanson [2000] as a benchmark for control design.



The Controlled variables are the water levels in the four tanks, H1, H2, H3 and H4. The manipulated variables are the two inflows, F1 and F2.

The parameters in the IModel interface are

Parameters[0]	cross sectional area of outlet a1 [cm2]
Parameters[1]	cross sectional area of outlet a2 [cm2]
Parameters[2]	cross sectional area of outlet a3 [cm2]
Parameters[3]	cross sectional area of outlet a4 [cm2]
Parameters[4]	cross sectional area of inlet A1 [cm2]
Parameters[5]	cross sectional area of inlet A2 [cm2]
Parameters[6]	cross sectional area of inlet A3 [cm2]
Parameters[7]	cross sectional area of inlet A4 [cm2]
Parameters[8]	valve position valve $\gamma 1$
Parameters[9]	valve position valve $\gamma 2$

14.4 FourTankProcess

Class FourTankProcess

Four Tank Process model

```
class FourTankProcess : IModel
```

Constructors

```
FourTankProcess(double T)
: this(T, false)
```

Constructor

Parameters

T Step size

`FourTankProcess(double T, bool Extended)`

Constructor
Parameters

T	Step size
Extended	Extended model

14.5 VanDerPol

Class VanDerPol

Van der Pol's problem $y"(t) = \mu(1 - y(t)^2)y'(t) - y(t)$

```
class VanDerPol : IModel
```

Constructors

`VanDerPol(double Mu, double T)`

Construtor
Parameters

Mu	μ parameter
T	Step size

Implementation of Van der Pol equation is shown in appendix E

Chapter 15

ODE solvers

15.1 Introduction

Ordinary Differential Equations can be solved using the ODESolver class. ODESolver implements a number of Explicit Runge-Kutta methods where the solutions can be found without iterations, and a number of ESDIRK methods, where the solution in each integration step requires a number of iteration steps. The simpler Explicit Runge-Kutta methods are suitable for differential equation system, where the eigenvalues have an equal order of magnitude. The ESDIRK methods are recommended for stiff,differential equation systems where there is a large difference between the systems eigenvalues. The test example below shows how to integrate the Van der Pol equation, where the parameter μ determines the systems stiffness . $\mu = 0.0$ gives a non - stiff sinus curve, values of $\mu = 100.0$ gives a very stiff problem.

The differential equations to be integrated must be implemented using the *IModel* interface 11.4. Implementation of Van der Pol equation is shown in appendix E. The ERK methods does not require implementation of the Jacobian method, but the ESDIRK methods require that the Jacobian method is implemented.

The ODESolver class can be run with fixed step length and with adaptive step length, if you specify a tolerance for the integration. The step length is adjusted based on calculation of the index

$$r = max \frac{e_i}{max(abstol_i, |x_i| reltol_i)} \quad i \in 1, \ldots, n \tag{15.1}$$

The error e_i is estimated based on comparing integration result with of one step with two integration steps with half the step length. Values of $r > 1.0$ rejects the step and decreases the step length. Values of $r \leq 1.0$ increases the step length and the step is accepted.

The program for integration of the Van der Pol equation is:

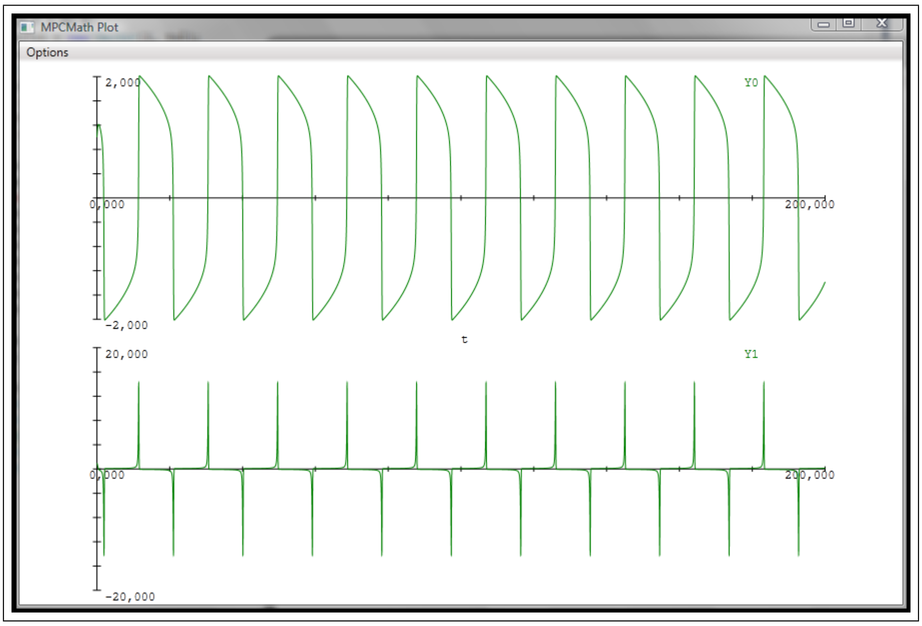
```
console.WriteLine("Test Ordinary Differential Equation solvers");

double dt = 0.01;
```

```
            double T = 200.0;
            ODEMethods method = ODEMethods.ESDIRK34;

            // Create Van der Pol model
            double mu = 10.0;
            IModel fun = new VanDerPol(mu, dt);

            double tol = 1.0e-6;
            Vector absTol = new Vector(2, tol);
            Vector relTol = new Vector(2, tol);
            // Create ODE solver
            ODESolver solver = new ODESolver(method, fun, dt, absTol, relTol);

            // integrate Van der Pol
            Vector tval = null;
            Matrix X = null;
            Vector XS = new Vector(fun.Dimension, 1.0);
            Vector US = new Vector(fun.NU);
            DateTime start = DateTime.Now;
            solver.Integrate(T, XS, US, out tval, out X);
            TimeSpan elapsed = DateTime.Now - start;

            console.WriteLine();
            console.WriteLine("Method " + method.ToString()
                + " absTol " + tol.ToString()
                + " Tolerance " + solver.Tolerance.ToString());
            console.Show("Time ms ", elapsed.TotalMilliseconds);
            console.Show("stepsize", solver.Step);
            console.Show("steps", tval.Dimension);
            console.Show("evaluations", solver.Evaluations);
            console.Show("accepted", solver.AcceptedSteps);
            console.Show("rejected", solver.RejectedSteps);
            console.Show("Factorizations", solver.Factorizations);

            // Plot results
            Vector Y0 = X.GetRow(0);
            Vector Y1 = X.GetRow(1);

            console.Plot(new Plot(PlotType.XY,
                new PlotSeries("t", 0.0, T, tval),
                new PlotSeries("Y0", 0.0, 0.0, SeriesColor.Green, Y0)),
                new Plot(PlotType.XY,
                new PlotSeries("t", 0.0, T, tval),
                new PlotSeries("Y1", 0.0, 0.0, SeriesColor.Green, Y1)));
        }
    }
```

The console output is

```
Test Ordinary Differential Equation solvers

Method ESDIRK34 absTol 1E-06 Tolerance 1E-16
Time ms  2005,0000
stepsize 0,1242
steps 4099

evaluations 208592

accepted 4098

rejected 1171

Factorizations 15848
```

The resulting plots are



15.2 ODEMethods

Enumeration **ODEMethods**

ODE solver methods enumeration

```
enum ODEMethods
```

Fields

DOPRI5	DOPRI5(4) 7 stages, order 4
ERK1	Explicit Euler
ERK2A	Explicit Trapezoidal alpha = 1;
ERK2B	Explicit Trapezoidal alpha = ;
ERK3A	Three stage ERK
ERK3B	Three stage ERK
ERK4A	Four Stage ERK
ERK4B	Four Stage ERK
ERK4C	Classical Runge Kutta (4 order, 4 tages)
ERK4D	Four Stage ERK
ESDIRK12	Explicit Singly Diagonally Implicit Runge-Kutta method 12
ESDIRK23	Explicit Singly Diagonally Implicit Runge-Kutta method 23
ESDIRK34	Explicit Singly Diagonally Implicit Runge-Kutta method 34

15.3 ODESolver

Class ODESolver

ODE solver for ordinary differential equations

```
class ODESolver
```

Constructors

```
ODESolver(ODEMethods Method, IModel Model, double Step)
: this(Method, Model, Step, null, null)
```

Constructor for fixed step size

Parameters

Method	Integration Method
Model	Differential equations
Step	Step size

```
ODESolver(ODEMethods Method, IModel Model,
double InitialStep, Vector AbsTol, Vector RelTol)
```

Constructor for adaptive step size

Parameters

Method	Integration Method
Model	Differential equations
Step	Initial step size
AbsTol	Absolute tolerance
RelTol	Relative tolerance

Properties

A

Butcher tableau A

```
Matrix A {get;}
```

AcceptedSteps

Number of Accepted steps

```
int AcceptedSteps {set; get;}
```

B

Butcher tableau B

```
Vector B {get;}
```

C

Butcher tableau C

```
Vector C {get;}
```

Evaluations

Number of evaluations of Model.Derivative

```
int Evaluations {set; get;}
```

Factorizations

Number of factorizations

```
int Factorizations {set; get;}
```

InfeasibleSteps

Number of Backtracks due to infeasible states

```
int InfeasibleSteps {set; get;}
```

Iterations

Maximum number Newton iterations used by ESDIRK step

```
int Iterations {set; get;}
```

MaxIterations

Max Iterartions for ESDIRK Newton iterations

```
int MaxIterations {set; get;}
```

NewtonErrors

Number of Newton iterations not converged

```
int NewtonErrors {set; get;}
```

RejectedSteps

Number of Rejected steps

```
int RejectedSteps {set; get;}
```

Step

Integration step size

```
double Step {set; get;}
```

Tolerance

Tolerance for ESDIRK Newton iterations

```
double Tolerance {set; get;}
```

Methods

Integrate

Integrate ODE for T time units, and return all integration steps

```
void Integrate(double T, Vector X0, Vector U, out Vector Steps, out Matrix Xnew)
```

Parameters

T	Integration time
X0	Initial state
U	Manipulated variables
Steps	Integration steps (time values)
Xnew	Integration states, stored column wise

Next

Integrate ODE for T time units

```
Vector Next(double T, Vector X0, Vector U)
```

Parameters

T	Integration time
X0	Initial state
U	Manipulated variables
returns	new State

Chapter 16

MPCMathLib

16.1 MPCMathLib-MinNorm

Class MPCMathLib

Minimize sum(norm(y - A*x))

```
partial class MPCMathLib
```

Methods

MinNorm

Minimize sum(norm(y- A*x))

```
static Vector MinNorm(Norm norm, double gamma, Matrix A, Vector y)
```

Parameters

norm	Norm
gamma	Huber norm coificient
A	Coifficient matrix
y	vector of observed values
returns	vector x

Enumeration Norm

```
[0] = Zeroes;
g[1] = Ones;
Cqp[0, 0] = +A;
Cqp[0, 1] = I;
Cqp[1, 0] = -A;
Cqp[1, 1] = I;
```

```
Cqp[2, 1] = I;
d[0] = y;
d[1] = -y;
errorlim = 1.0e-9;
break;
case Norm.NormInfinity:
Ones = new Vector(1, 1.0);
Zeroes = new Vector(cols);
I = Matrix.Random(rows, 1, 1.0, 1.0);
strx = Structure.Values(cols, 1);
stry = new Structure(0);
strz = Structure.Values(rows, rows, 1);
g = new BVector(strx);
Aqp = new BMatrix(MatrixForm.General, stry, strx);
b = new BVector(stry);
Cqp = new BMatrix(MatrixForm.General, strz, strx);
d = new BVector(strz);
Gqp = new BMatrix(strx);
g[0] = Zeroes;
g[1] = Ones;
Cqp[0, 0] = +A;
Cqp[0, 1] = I;
Cqp[1, 0] = -A;
Cqp[1, 1] = I;
Cqp[2, 1] = new Matrix(1, 1.0);
d[0] = y;
d[1] = -y;
break;
case Norm.Huber:
strx = Structure.Values(cols, rows, rows, rows);
stry = Structure.Values(rows);
strz = Structure.Values(rows, rows, rows, rows);
Gqp = new BMatrix(MatrixForm.Diagonal, strx);
g = new BVector(strx);
Aqp = new BMatrix(MatrixForm.General, stry, strx);
b = new BVector(stry);
Cqp = new BMatrix(MatrixForm.General, strz, strx);
d = new BVector(strz);
I = Matrix.UnityMatrix(rows);
Vector gammaI = new Vector(rows, gamma);
Gqp[1, 1] = I;
g[2] = +gammaI;
g[3] = +gammaI;
Aqp[0, 0] = A;
Aqp[0, 1] = I;
Aqp[0, 2] = I;
Aqp[0, 3] = -I;
b[0] = y;
Cqp[0, 1] = I;
Cqp[1, 1] = -I;
```

```
Cqp[2, 2] = I;
Cqp[3, 3] = I;
d[0] = -gammaI;
d[1] = -gammaI;
break;
}
KKTSolver kktsolver = new KKTSolver(Gqp, g, Aqp, b, Cqp, d);
qpsolver = new QPSolver(kktsolver);
qpsolver.MaxIterations = 100;
qpsolver.ErrorLimit = errorlim;
qpsolver.Solve();
x = qpsolver.X[0];
if (qpsolver.Iterations >= qpsolver.MaxIterations)
{
throw new MPCMathException(" + norm.ToString(), ");
}
}
else
{
lssolver = new LeastSquareEquationSolver(A);
x = lssolver.Solve(y);
}
return x;
}
else
{
throw new MPCMathException(", ");
}
}
}
enum Norm
```

Fields

Huber	Huber norm
Norm1	Norm 1, absolute value
Norm2	Norm2, Euclid norm
NormInfinity	Norm Infinity

16.2 MPCMathLib-RK4

Class MPCMathLib

```
partial class MPCMathLib
```

Methods

RK4

Runge Kutta 4 integrator

```
static Vector RK4(IModel model, Vector X, Vector U, double T)
```

Parameters

model	Process model
X	Present state
U	Manipulated variables
T	End time
returns	State at endtime

RK4

Runge Kutta 4 integrator

```
static Vector RK4(IModel model, Vector X, Vector U, double T, double step)
```

Parameters

model	Process model
X	Present state
U	Manipulated variables
T	End time
step	step size
returns	State at endtime

16.3 MPCMathLib-Discretize

Class MPCMathLib

```
partial class MPCMathLib
```

Methods

Discretize

Calculate disretized linear equations from Linear differential equations.
$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$
$x(k+1) = AdX(k) + BdU(k)$

```
static void Discretize(Matrix A, Matrix B, double T,
out Matrix AD, out Matrix BD)
```

Parameters

A	A matrix
B	B matrix
T	Discretization time
AD	AD matrix for discrete system
BD	BD matrix for discrete system

Discretize

Discretize linear or unlinear Plant Model around an operating point
$\frac{dx(t)}{dt} = Fun(X(t), U(t))$
$x(k+1) = AdX(k) + BdU(k)$

```
static void Discretize(IModel model, Vector XS, Vector US,
double T, out Matrix AD, out Matrix BD)
```

Parameters

model	IModel model
XS	State of operating point
US	State of manipulated var operating point
AD	AD matrix for discrete system
BD	BD matrix for discrete system

16.4 MPCMathLib-SteadyState

Class MPCMathLib

Find stationary state of Process with manipulated variable US given

```
partial class MPCMathLib
```

Properties

Iterations

Iterarions

```
static int Iterations {get;}
```

MaxIterations

Maximum number of iterations (default 100)

```
static int MaxIterations {set; get;}
```

Methods

SteadyState

Find Steady State of Process

```
static Vector SteadyState(IModel model, Vector XI, Vector US)
```

Parameters

model	Plant model
XI	Initial guess off state variables
US	Manipulated variables
returns	XS state variable

16.5 MPCMathLib-Util

Class MPCMathLib

MPCMathLib Utility routines

```
partial class MPCMathLib
```

Properties

MaxError

Maximum error for comparing doubles

```
static double MaxError {set; get;}
```

Ran

Random number generator

```
static Random Ran {get;}
```

Methods

Distribution

Density curve for a values

```
static Vector Distribution(int N, double Range, Vector a)
```

Parameters

N	Groups
Range	max variantion
a	input vector

Fact

Factorial n!

```
static int Fact(int n)
```

Parameters

n	input
returns	n!

Gauss

Gauss density function

```
static double Gauss(double x, double mean, double var)
```

Parameters

x	Input
mean	Mean
var	Variation

GaussDistribution

Gauss Distribution

```
static Vector GaussDistribution(int N, double Range, double mean, double var)
```

Parameters

N	Groups
Range	max variantion
mean	
var	

IsZero

Is Zero, test wheter $-MaxError \geq a \leq MaxError$

```
static bool IsZero(double a)
```

Parameters

a	
returns	true if —a— ¡= MaxError

KalmanGain

Kalman filter gain

```
static Matrix KalmanGain(Matrix a, Matrix c, Vector q, Vector r)
```

Parameters

a	system matrix (x+ = a*x + b*u)
c	system matrix (y = c*x)
q	Process Noise
r	Measurement Noise

KalmanGain

Kalman filter gain

```
static void KalmanGain(Matrix a, Matrix c,
Vector q, Vector r, out Matrix K, out Matrix P)
```

Parameters

a	system matrix (x+ = a*x + b*u)
c	system matrix (y = c*x)
q	Process Noise
r	Measurement Noise
K	Kalman Gain
P	Error Covariance

ProperZero

Return a proper 0.0 if —a— less than MaxError

```
static double ProperZero(double a)
```

Parameters
a

PseudoRandom

Set Pseudo random mode if seed != 0 Needed for debugging purposes, get the same random number every time the program is started

```
static void PseudoRandom(int seed)
```

Parameters
seed

WhiteGaussianNoise

White Gaussian Noise

```
static double WhiteGaussianNoise(double mean, double var)
```

Parameters

var	mean
mean	variance

WhiteGaussianNoise

White Gaussian Noise

```
static Vector WhiteGaussianNoise(Vector mean, Vector var)
```

Parameters

mean	mean
var	variance

Chapter 17

Miscellaneous functions

17.1 Complex

Class complex

Basic complex functionality

```
[Serializable]
class complex : ICommon<complex> , IComparable<complex>, IEquatable<complex>
```

Constructors

```
complex()
```

Constructor

```
complex(double RV, double IV)
```

Constructor
Parameters
RV real value
IV imaginary value

```
static implicit operator complex(double a)
```

implicit conversion from double to complex
Parameters
a input var

Properties

Arg

Angle

```
double Arg {get;}
```

Con

Complex conjugate

```
complex Con {get;}
```

IV

Imaginary value

```
double IV {set; get;}
```

Mod

Modulus

```
double Mod {get;}
```

RV

Real value

```
double RV {set; get;}
```

Methods

Equals

```
override bool Equals(object o)
```

Exp

Complex exponential

```
static complex Exp(complex a)
```

Parameters

a	input var
returns	exp(a)

GetHashCode

```
override int GetHashCode()
```

operator

Unary + operator/ clone

```
static complex operator +(complex a)
```

Parameters

a	Input complex
returns	Out complex

operator

Add complex numbers

```
static complex operator +(complex a, complex b)
```

Parameters

a	input a
b	input b
returns	a+b

operator

Add complex numbers

```
static complex operator +(double a, complex b)
```

Parameters

a	input a
b	input b
returns	a+b

operator

Add complex numbers

```
static complex operator +(complex a, double b)
```

Parameters

a	input a
b	input b
returns	a + b

operator

Unary - operator

```
static complex operator -(complex a)
```

Parameters

a	input var
returns	-a

operator

Subtract complex numbers

```
static complex operator -(complex a, complex b)
```

Parameters

a	input var
b	input var
returns	a-b

operator

Subtract complex numbers

```
static complex operator -(double a, complex b)
```

Parameters

a	input var
b	input var
returns	a-b

operator

Subtract complex numbers

```
static complex operator -(complex a, double b)
```

Parameters

a	input var
b	input var
returns	a-b

operator

Multiply complex numbers

```
static complex operator *(complex a, complex b)
```

Parameters

a	input var
b	input var
returns	a*b

operator

Multiply complex numbers

```
static complex operator *(double a, complex b)
```

Parameters

a	input var
b	input var
returns	a*b

operator

Multiply complex numbers

```
static complex operator *(complex a, double b)
```

Parameters

a	input var
b	input var
returns	a*b

operator

Divide complex numbers

```
static complex operator /(complex a, complex b)
```

Parameters

a	input var
b	input var
returns	a/b

operator

Divide complex numbers

```
static complex operator /(double a, complex b)
```

Parameters

a	input var
b	input var
returns	a/b

operator

Divide complex numbers

```
static complex operator /(complex a, double b)
```

Parameters

a	input var
b	input var
returns	a/b

operator

Equal operator

```
static bool operator ==(complex a, complex b)
```

Parameters

a	input var
b	input var
returns	true if equal

operator

Not equal operator

```
static bool operator !=(complex a, complex b)
```

Parameters

a	input var
b	input var
returns	true if not equal

Sqrt

Complex Sqrt

```
static complex Sqrt(complex a)
```

Parameters

a	input var
returns	sqrt(a)

17.2 Polynomial

Class Polynomial

Polynomial class with $p(x) = c[0] + c[1] * x + c[2] * x^2 + +c[n] * x^n$

```
[Serializable]
class Polynomial
```

Constructors

```
Polynomial():this(0)
```

Constructor fro serialization

```
Polynomial(double Gain, CVector Roots)
```

Constuctor from a vector of roots
Parameters

dim	Dimension
Gain	Gain
Roots	Roots

```
Polynomial(Vector A)
```

Constructor from vector of coifficients
Parameters

A	$Coefficientsa0 + a1 * x + a2 * x^2 + ...$

```
Polynomial(params double[] vals)
```

Constructor from an array of coifficinets
Parameters

vals	$Coefficientsa0 + a1 * x + a2 * x^2 + ...$

Properties

Coefficients

Polynomial coifficients

```
Vector Coefficients {get;}
```

Gain

Gain

```
double Gain {get;}
```

N

Polynomial dimension

```
int N {get;}
```

Roots

Polynomial roots

```
CVector Roots {get;}
```

Methods

Add

Add polynomials

```
static Polynomial Add(Polynomial a, Polynomial b)
```

Parameters

a	Input a
b	Input b
returns	a+b

AssertEqual

Assert equal show actual and expected if not equal

```
static bool AssertEqual(string text, Polynomial actual, Polynomial expected)
```

Parameters

text	text string
actual	actual value
expected	expected value

Clone

Clone Polynomial

```
static Polynomial Clone(Polynomial a)
```

Parameters

a	input value
returns	Cloned polynomial

Derivative

Derivative of polynomial

```
Polynomial Derivative()
```

Parameters

returns	Derivative polynomial

Div

Divide Polynomials, res = a/b

```
static Polynomial Div(Polynomial a, Polynomial b)
```

Parameters

a	Polynomial a
b	Polynomial b
returns	a/b

Equal

Equal Polynomial

```
static bool Equal(Polynomial a, Polynomial b)
```

Parameters

a	Input a
b	Input b
returns	true if equal

Mul

Multiply Polynomial with constant res = fak *b

```
static Polynomial Mul(double fak, Polynomial b)
```

Parameters

fak	Constant
b	Polynomila

Mul

Multiply Polynomials operator res = P * Q

```
static Polynomial Mul(Polynomial P, Polynomial Q)
```

Parameters

P	Polynomial P
Q	Polynomial Q
returns	P * Q

operator

Clone operator, res = +a

```
static Polynomial operator +(Polynomial a)
```

Parameters

a	input a
returns	Cloned polynomial

operator

Negate operator, res = - a;

```
static Polynomial operator -(Polynomial a)
```

Parameters

a	input a
returns	- a

operator

Add operator

```
static Polynomial operator +(Polynomial a, Polynomial b)
```

Parameters

a	Input a
b	Input b
returns	a+b

operator

Subtract operator

```
static Polynomial operator -(Polynomial a, Polynomial b)
```

Parameters

a	Input a
?	Input b
returns	a-b

operator

Multiply operator, res = fak*a

```
static Polynomial operator *(double fak, Polynomial a)
```

Parameters

fak	Constant
a	Polynomial a a
returns	fak*res

operator

Multiply operator, res = a*fak

```
static Polynomial operator *(Polynomial a, double fak)
```

Parameters

a	Polynomial a
fak	Constant
returns	res*fak

operator

```
static Polynomial operator /(Polynomial a, double fak)
```

Parameters

a	Polynomial a
fak	constant
returns	a/fak

operator

Multiply Polynomials operator res = P * Q

```
static Polynomial operator *(Polynomial P, Polynomial Q)
```

Parameters

P	Polynomial P
Q	Polynomial Q
returns	P * Q

operator

Div operator, res = a/b

```
static Polynomial operator /(Polynomial a, Polynomial b)
```

Parameters

a	Polynomial a
b	Polynomial b
returns	a/b

operator

Remainder operator

```
static Polynomial operator %(Polynomial a, Polynomial b)
```

Parameters

a	Polynomial a
b	Polynomial a
returns	a

Random

Generate random Polynomial

```
static Polynomial Random(int n, double min, double max)
```

Parameters

n	dimension
min	min coifficients
max	max coifficients

Rem

Remainder

```
static Polynomial Rem(Polynomial a, Polynomial b)
```

Parameters

a	Polynomial a
b	Polynomial a
returns	a

Show

Show Polynomial

```
static void Show(string txt, Polynomial a)
```

Parameters

txt	text string
a	Polynomial

Sub

Subtract

```
static Polynomial Sub(Polynomial a, Polynomial b)
```

Parameters

a	Input a
b	Input b
returns	a-b

Truncate

Truncate Polynomial to effective dimension

```
void Truncate()
```

Value

Polynomial value

```
double Value(double x)
```

Parameters

x	input x
returns	Value

Value

Polynomial value

```
complex Value(complex x)
```

Parameters

x	Complex value
returns	Complex value

17.3 DelayChain

Class DelayChain

Delay Chain for delay of manipulated variables

```
[Serializable]
class DelayChain
```

Constructors

`DelayChain()`

Constructor for serialization

`DelayChain(int Delay)`

Constructor
Parameters

Delay	Delay

`DelayChain(int Delay, double Value)`

Constructor
Parameters

Delay	Delay
Value	Initial value

Properties

Delay

Delay

```
int Delay {set; get;}
```

ResetValue

Reset value

```
double ResetValue {set; get;}
```

Methods

AssertEqual

assert equal, Compare two DelayChains and output value if not equal

```
static void AssertEqual(string text, DelayChain actual, DelayChain expected)
```

Parameters

text	Text string
actual	DelayChain with actual values
expected	DelayChain with expected values

Equal

Equal, compare two Delay chains. (Error limit given in MPCMathLib.MaxError)

```
static bool Equal(DelayChain y, DelayChain x)
```

Parameters

y	DelayChain y
x	DelayChain x
returns	true if equal

Get

Get delayed value

```
double Get()
```

Parameters

returns	u[k - delay]

Get

Get delayed value with offset

```
double Get(int Offset)
```

Parameters

Offset	Ofsett
returns	u[k + Offset - delay]

Reset

Reset DelayChain

```
void Reset()
```

Set

Set value

```
void Set(double value)
```

Parameters

value	u[k]

Show

Show DelayChain

```
static void Show(string text, DelayChain x)
```

Parameters

text	text string
x	DelayChaiin object

17.4 HuberFunction

Class HuberFunction

Huber penalty function.
$val = \sum(\phi(Y - A * X))$
$\phi(e) = e^2, |e| \leq \gamma$
$\phi(e) = \gamma * |e| - * \gamma^2, |e| > \gamma$

```
class HuberFunction : IFun
```

Constructors

```
HuberFunction(double Gamma, Matrix A, Vector Y)
```

Constructor
Parameters

Gamma	Threshhold, γ
A	Coifficient matrix
Y	Value vector

Properties

A

A in $val = \sum(\phi(Y - A * X))$

```
Matrix A {get;}
```

Gamma

Threshhold, γ

```
double Gamma {set; get;}
```

Y

y in $val = \sum(\phi(Y - A * X))$

```
Vector Y {get;}
```

17.5 Spline and Cubic Smoothing Spline functions

A spline function is a curve constructed from polynomial segments (Splines) that are subject to continuity conditions at their joints. A spline function is approximating a set of data $(x_0, y_o), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ of the function $y = y(x)$. The Spline function segments are:

$$S_i(x) = a_i * (x - x0_i)^3 + b_i * (x - x0_i) * 2 + c_i * (x + x0_i) + d_i \tag{17.1}$$

For a simple Spline function the spline function segments fulfils the conditions:

$$\begin{aligned} S_{i-1}(x_i) &= S_i(x_i) && (17.2a) \\ S'_{i-1}(x_i) &= S'_i(x_i) && (17.2b) \\ S''_{i-1}(x_i) &= S''_i(x_i) && (17.2c) \\ S_i(x_i) &= y_i && (17.2d) \end{aligned}$$

In the cubic smoothing Spline function the spline function is allowed to deviate from the data points. The cubic smoothing spline function is m minimizing the value

$$\min_{\{x\}} \quad L(x) = \lambda \sum_0^n \sigma_i (y_i - S_i(x_i))^2 + (1 - \lambda) \int_{x_0}^{x_n} S''(x)^2 \tag{17.3}$$

In the limiting case, where $\lambda = 0.0$ the spline function will become a straight line. At the other extreme , where $\lambda = 1.0$ the smoothing spline converges to the simple spline function.

17.6 Spline

Class Spline

Spline function. $S(x) = a * (x - x0)^3 + b * (x - x0) * 2 + c * (x + x0) + d$

```
class Spline : IComparable
```

Properties

A

Spline cubic coifficient

```
double A {set; get;}
```

B

Spline square coifficient

```
double B {set; get;}
```

C

Spline linear coifficient

```
double C {set; get;}
```

D

Spline constant coifficient

```
double D {set; get;}
```

H

Spline length

```
double H {set; get;}
```

Sigma

Spline data weight

```
double Sigma {set; get;}
```

X0

Spline starting point

```
double X0 {set; get;}
```

Y

Spline data value

```
double Y {set; get;}
```

Methods

Position

Is x inside Spline interval

```
int Position(double X)
```

Parameters
X
returns -1 if x below spline, 0 if inside, 1 if above

Value

Spline value

```
double Value(double X)
```

Parameters
X

17.7 CubicSpline

Class CubicSpline

Smoothing Cubic Spline

```
class CubicSpline
```

Constructors

```
CubicSpline(Vector X, Vector Y) : this(X, Y, 1.0, null)
```

Constructor for non smoothing Spline
Parameters

X	data values
Y	funtion value

```
CubicSpline(Vector X, Vector Y, double Lambda)
: this(X, Y, Lambda, null)
```

Constructor for smoothing Spline
Parameters

X	data values
Y	funtion value
Lambda	Smoothing factor

```
CubicSpline(Vector X, Vector Y, double Lambda, Vector Sigma)
```

Constructor for smoothing Spline
Parameters

X	data values
Y	funtion value
Lambda	Smoothing factor
Sigma	Weigths for data points

Properties

Lambda

Smoothing factor ($0.0 < Lambda \leq 1.0$)

```
double Lambda {set; get;}
```

Splines

Array of Spline functions

```
Spline[] Splines {get;}
```

X

Spline x points

```
Vector X {get;}
```

Y

Spline y points

```
Vector Y {get;}
```

Methods

Value

Value of smoothed spline

```
double Value(double x)
```

Parameters
x

17.8 BarrierFunction

Class BarrierFunction

Logaritmic barrier function.
Given a set of inequality conditions $B \geq Ax$ the barrier function is

$$\phi = -\sum_{i=0}^{N-1} ln(b_i - a_i x) \tag{17.4}$$

where a_i is the i row of A and N is the number of rows in A

```
class BarrierFunction : IFun
```

Constructors

```
BarrierFunction(Matrix A, Vector B)
```

Constructor
Parameters
A A matrix
B B vector

17.9 Reports

In C# it is surprisingly difficult to program a simple plain report with some values in nice columns. One way is to use Windows Presentation Foundation, WPF, function to make the columns, but in many cases its an over kill. The ReportBuilder object provides a simple solution to the problem.

The following program demonstrates the use of the report builder object

```
console.WriteLine("Test Report Builder");
console.WriteLine();

string[] names = { "Sten", "Peter", "John", "Jorgen", "Niels" };
double[] values = { 1000.00, 33234.12, 27945.28, 1677.45, 56.44 };

ReportBuilder rb = new ReportBuilder(500);

int pname = 0;       // position of name column
int pvalue = 15;     // position of value column
int fieldsize = 10;
int digits = 2;
rb.Write(pname, "Name");
rb.Write(pvalue, "value");
rb.NewLine();
for (int p = 0; p < names.Length; p++)
{
    rb.Write(pname, names[p]);
    rb.Write(pvalue, values[p], fieldsize, digits);
    rb.NewLine();
}

console.WriteLine(rb.ToString());
```

with the output

```
Test Report Builder

Name           value
Sten              1000,00
Peter            33234,12
John             27945,28
Jorgen            1677,45
Niels               56,44
```

17.10 ReportBuilder

Class ReportBuilder

Report builder

```
class ReportBuilder
```

Constructors

```
ReportBuilder(int Capacity)
```

Constructor fro Report builder
Parameters

Capacity	Capcity of report builder

Methods

NewLine

New line

```
void NewLine()
```

ToString

ToString

```
override string ToString()
```

Write

Write string

```
void Write(int Pos, string Value)
```

Parameters

Pos	position
Value	text string

Write

Write double

```
void Write(int Pos, double Value, int Size, int Digits)
```

Parameters

Pos	Position
Value	Value
Size	Field size
Digits	Digits

17.11 File I/O and serialization

Serialization refers to the term of converting or eventually transferring he state of an object into a stream (e.g. a file stream or a memory stream) The stream contains all the information needed to reconstruct , deserialize, the object for later use. This can be used to store object on disc or transferring object between applications or computers.

MPCMath objects defined as [Serializable] can be serialized using .NET's BinaryFormatter. *MPCMath* provides the BinaryIO class to save and read object from binary files and the XmlIO class to save and read from XML files. As shown below.

```
Vector x = Vector.Random(10, -100.0, 100.0);

// save and read binary file
string file = "object.txt";
BinaryIO.Save(file, x);
Vector y = (Vector)BinaryIO.Read(file);
Vector.AssertEqual("BinaryIO Vector", y, x);

// save and read XML file
string xfile = "xobject.xml";
XmlIO<Vector>.Save(xfile, x);
y = XmlIO<Vector>.Read(xfile);
Vector.AssertEqual("XmlIO Vector", y, x);
```

Some of the *MPCMath* objects marked as [Serializable] can be serialized using .NET'a SoapFormatter as ASCII streams. The SoapFormatter does not support generic classes as $BMatrix$, $BVector$.

To support XmlSerialization a number of properties with names like xvl and xvla has been defined. These routines are not for general use.

17.12 BinaryIO

Class BinaryIO

Binary Input Output routines. Saves or read a serializable object to or from binary file.

```
static class BinaryIO
```

Methods

Read

Read object from file

```
static Object Read(string FileName)
```

Parameters

FileName	File name
returns	read object

Save

Save object to disc

```
static void Save(string FileName, Object ObjGraph)
```

Parameters

FileName	File name
ObjGraph	Object to be saved

17.13 XmlIO

Class XmlIO

XML Input Output routines. Saves or read a serializable object to or from XML file.

```
static class XmlIO<T> where T : new()
```

Methods

Read

Read object from file

```
static T Read(string FileName)
```

Parameters

FileName	File name
returns	read object

Save

Save object to disc

```
static void Save(string FileName, T ObjGraph)
```

Parameters

FileName	File name
ObjGraph	Object to be saved

17.14 MPCMathLib-JacobianApprox

Class MPCMathLib

Calculate Jacobian approximantion

```
partial class MPCMathLib
```

Methods

JacobianApprox

Jacobian Approximation, , for debugging of model implemented using IModel inteface. The initial variation of X, DX is reduced until norm of error is less than eps

```
static Matrix JacobianApprox(IModel model, Vector X, Vector U, Vector DX, double eps)
```

Parameters

model	Plant model
X	Operating state
U	Manipulated variables
DX	Initial Variation of X
eps	Error limit
returns	Approximated Jacobian

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Appendix A

Installing *MPCMath*

This appendix describes how to get *MPCMath* up and running on your machine. If you get any problems or have any questions, please don't hesitate to contact me.

A.1 Prerequisites

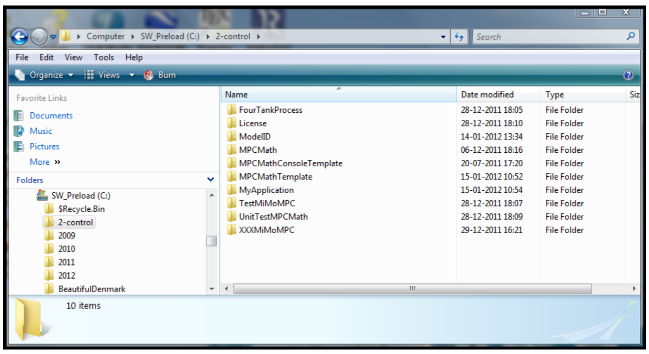
MPCMath requires that the most recent version of visual Studio is installed on your computer. A free version of Microsoft Visual C# 2010 Express can be downloaded from Microsoft

http://www.microsoft.com/express/Downloads/#2010-Visual-CS.

Remember to register Microsoft Visual C# 2010 Express, otherwise it becomes inactive after 30 days.

A.2 Installing *MPCMath*

Unzip the file "2-control 14-feb-2011.zip" to c:\creating the folder c:\2-control. (If the file you received has the extension .sip, rename it to .zip). Copy the license file MPCMath00007.lic to c:\2-control\ License. Thats all, now you should be ready to use *MPCMath* . The c:\2-control folder should look something like this



The c:\2-control folder contains the following folders:

License	Holding the license file
MPCMath	The binary *MPCMath* .dll
MPCMathConsoleTemplate	Template for Console based *MPCMath* applications. Can be used with MCMath Runtime licenses.
MPCMathWPFTemplate	Template for WPF based application, this is the normal starting point for development of a new *MPCMath* application. Using the WPF based console requires a Demo or a Development license.
TestMiMoMPC	Multiple Input Multiple Output MPC program, used in the?Introduction to *MPCMath* ? chapter.
UnitTestMPCMath	Unit test of the *MPCMath* library. It contains many examples of calls to *MPCMath* .
XXXMiMoMPC	As TestMiMoMPC with included source code for MIMOMPC object

A.3 Trouble shooting, Debugging and Exceptions

The MPCMath.dll requires that the directories "C:\2-control\MPCMath" and "C:\2-control\License" exists. Its also required that a valid license file is present in the License directory.

If the MPCMathTemplate runs properly and display the license information. Then the License information is ok. If MPCMathTemplate executes and MPCMathConsoleTemplate don't execute properly, then there is a problems with the c:\2-control\MPCMath\MPCMathMKLia32.dll which includes Intel MKL library routines. Use "DependencyWalker" to test whether your operating system is missing some sub dll. DependencyWalker is free sw which can be downloaded from the net.

If you move programs as MIMOMPC ,UnitTestMPCMath or MPCMath-Template away from the c:\2-control directory the reference til MPCMath must be redefined. Its done in the solution explorer Add references. Browse to the routine "c:\2-control\MPCMath*MPCMath* .dll"

The *MPCMath* function checks parameter values and error situations. If an error happens the function creates an MPCMathException, as show below where the vector x and y should have the same dimension.

```
static void Work()
{
    console.WriteLine("Test Exceptions");

    try
    {
        Vector x = Vector.Random(4, -100.0, 100.0);
        Vector y = Vector.Random(3, -100.0, 100.0);
        double res = x * y;
    }
    catch (MPCMathException e)
```

```
            {
                console.WriteLine(e.Message);
            }
        }
```

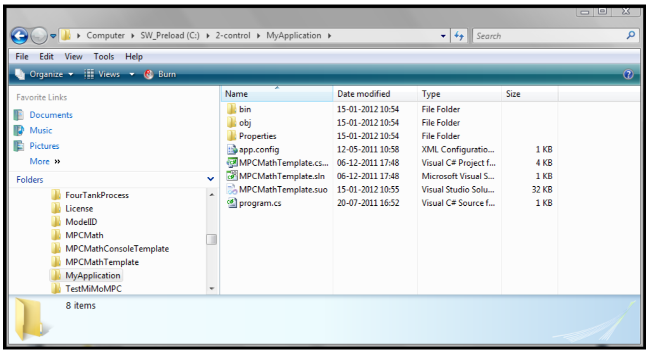
the console print out is

```
Test Exceptions
MPCMath.Vector.Mul/Illegal dimensions
```

Normally its easier debug programs without the *try/catch* construction, and use the information from the call stack to locate the sinner. In the final program its advisable to include *try/catch* construction.

A.4 MPCMathTemplate

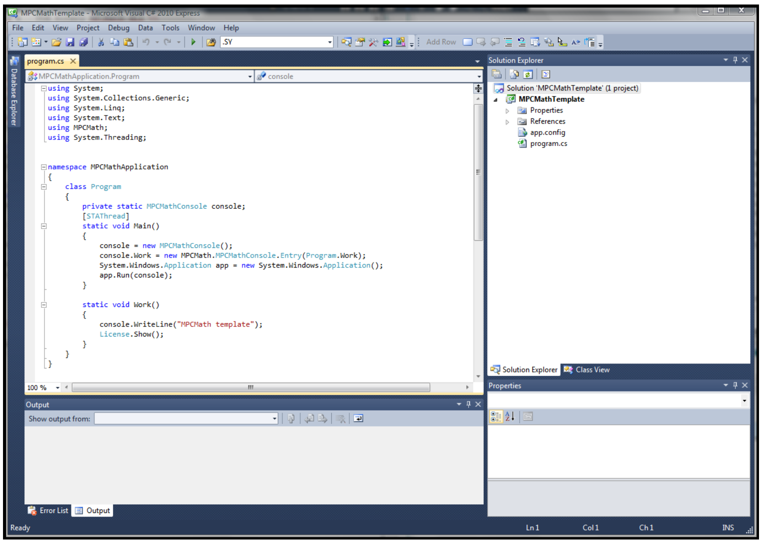
This is the starting point for development of a new *MPCMath* application. Copy the contents of MPCMathTemplate to a new folder named "MyApplication".



Double click the solution file MPCMathTemplate.sln to start Microsoft Visual C# 2010 Express. Click the menu View / Other Windows / Solution Explorer to open the Solution explorer.

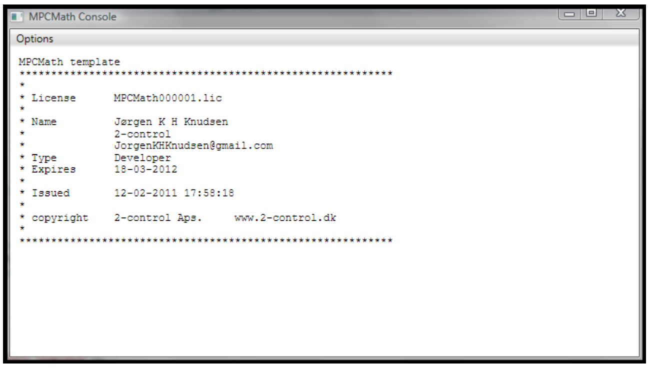
If "MyApplication" in not located the c:\2-control directory the reference til MPCMath must be redefined. Its done in the solution explorer Add references. Browse to the routine "c:\2-control\MPCMath*MPCMath* .dll"

Rename the Solution and the project to MyApplication. (Right click / Rename). Open the main program program.cs by double clicking it. Now the screen should look like this:



The first routine $staticvoidMain()$ starts the *MPCMath* console in a separate thread, this section should not be modified by you. Your code should be placed in the $staticvoidWork()$ routine.

Start the application by clicking the run button.
If everything is ok, the *MPCMath* console should look like this:

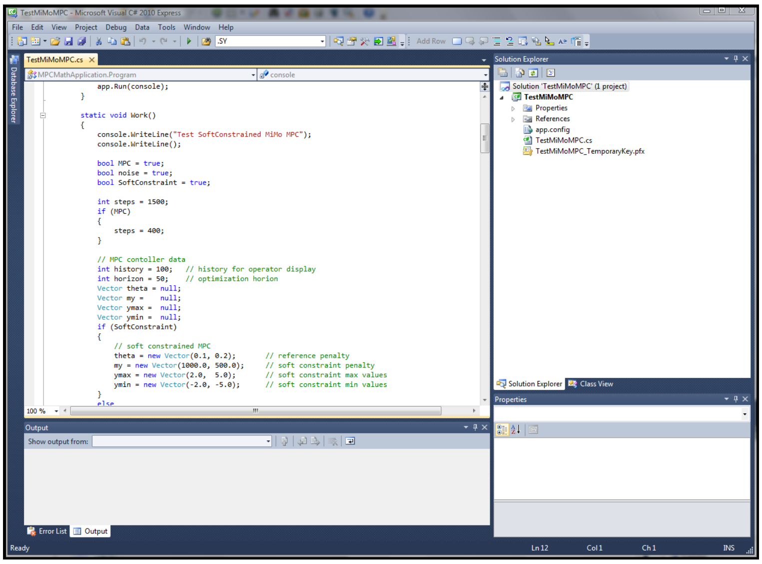


Closing the *MPCMath* console window or clicking the  button returns the system to development mode.

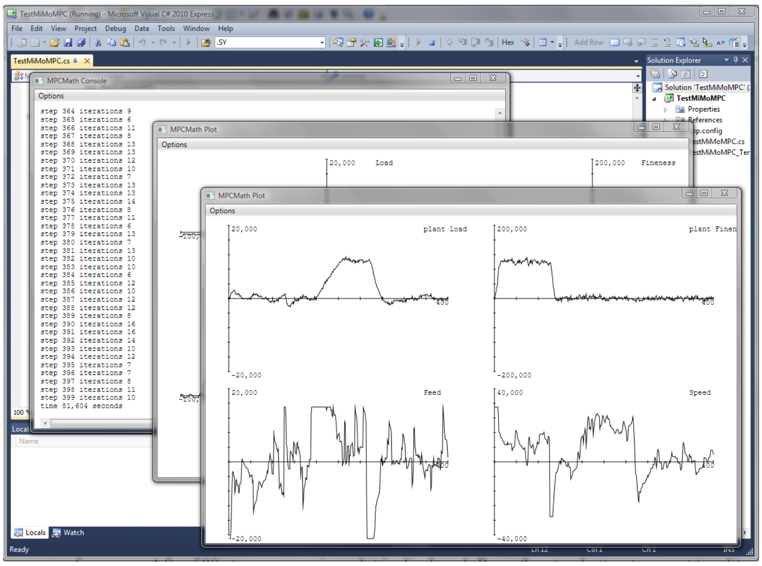
Before starting developing new code, I recommend that you study the TestMiMoMPC application and the chapter "Introduction to *MPCMath*".

A.5 TestMiMoMPC

This folder contains the Multiple Input Multiple Output MPC program, used in the "Introduction to *MPCMath*" document. Start the application by opening the solution file "TestMiMoMPC.sln".



Press the run button to start the MiMo MPC controller.



The MPC controller runs 500 steps and display an operator display with history and predicted performance. After 500 step an overview plot is displayed. Press the stop button to revert to edit mode.

Play with the application parameter and study the code.

```
bool MPC = true;                  // MPC or trackin exaple
bool noise = true;                // noise on/off
bool SoftConstraint = true;       // Conventional or softconstrained MPC
```

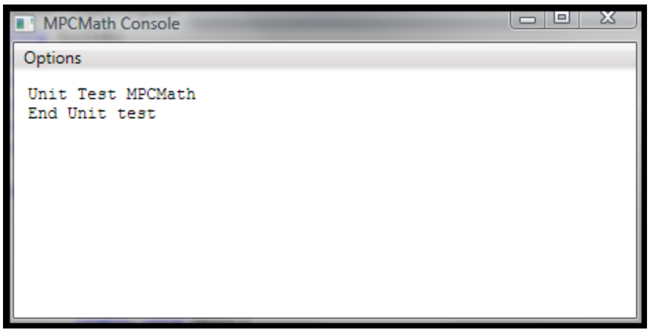
The MiMoMPC history and prediction horizon are defined by:

```
// MPC contoller data
int history = 100;    // history for operator display
int horizon = 50;     // optimization horion
```

The history value must be equal or larger the the state space model's time delay.

A.6 UnitTestMPCMath

The UnitTestMPCMath application test the functionality of *MPCMath* . Run the application. If everything is OK the output will look like this:



In the source code for the unit test presents a lot of *MPCMath* code examples.

A.7 MPCMathConsoleTemplate

This template can be used to develop or test *MPCMath* applications without *MPCMath* 's console. Typically intended for the final runtime systems.

Appendix B

Debugging tool for Console applications

B.1 Cnsl

Class Cnsl

MPCMath Cnsl class for test and debugging of MPCMath programs. Running DOS Command console. It requires a valid Demo or Developer license. Output is ignored when running under runtime licenses

```
static class Cnsl
```

Properties

Digits

Digits for doubles

```
static int Digits {set; get;}
```

FieldSize

Field size of vector and matrix diplay

```
static int FieldSize {set; get;}
```

NumbersPerLine

Numbers per Line

```
static int NumbersPerLine {set; get;}
```

Trace

Trace console output to file "console.txt" in bin/debug or bin/release directory

```
static bool Trace {set; get;}
```

Methods

AssertEqual

Assert equal for bool variable Show actual and expected if they are not equal

```
static void AssertEqual(string text, bool actual, bool expected)
```

Parameters

text	variable desription
actual	actual value
expected	expected value

AssertEqual

Assert equal for integers Show actual and expected if they are not equal

```
static void AssertEqual(string text, int actual, int expected)
```

Parameters

text	variable desription
actual	actual value
expected	expected value

AssertEqual

Assert equal for integer array Show actual and expected if they are not equal

```
static void AssertEqual(string text, int[] actual, int[] expected)
```

Parameters

text	variable desription
actual	actual value
expected	expected value

AssertEqual

Assert equal for double Show actual and expected if they are not equal

```
static void AssertEqual(string text, double actual, double expected)
```

Parameters

text	variable desription
actual	actual value
expected	expected value

AssertEqual

Assert equal for complex Show actual and expected if they are not equal

```
static void AssertEqual(string text, complex actual, complex expected)
```

Parameters

text	variable desription
actual	actual value
expected	expected value

DFormat

Format double variable

```
static string DFormat(double a)
```

Parameters

a	double variable
returns	Formatted output string

DFormat

Format double variable

```
static string DFormat(int Fieldsize, double a)
```

Parameters

Fieldsize	Field size
a	double variable
returns	Formatted output string

ReadLine

ReadLine from console

```
static void ReadLine()
```

Show

Show bool

```
static void Show(string text, bool a)
```

Parameters

text	Variable description
a	bool variable

Show

Show int, integer

```
static void Show(string text, int r)
```

Parameters

text	Variable description
r	integer variable

Show

Show int[], integer array

```
static void Show(string txt, int[] a)
```

Parameters

txt	Variable description
a	integer array

Show

Show double

```
static void Show(string text, double r)
```

Parameters

text	Variable description
r	double variable

Show

Show complex

```
static void Show(String text, complex c)
```

Parameters

text	Variable description
c	complex variable

Show

Show complex List

```
static void Show(string txt, List<complex> a)
```

Parameters

txt	Variable description
a	list of complex variables

Show

Show double[], double array

```
static void Show(string txt, double[] a)
```

Parameters

txt	Variable description
a	double array

Show

Show vector on console

```
static void Show(string txt, Vector a)
```

Parameters

txt	Text string
a	Vector

Show

Show matrix

```
static void Show(string text, Matrix a)
```

Parameters

text	text string
a	Matrix a

Show

Show ARX Model

```
static void Show(string txt, ARXModel a)
```

Parameters

txt	text string
a	ARXModel object

Show

Show Structure

```
static void Show(string txt, Structure a)
```

Parameters

txt	text string
a	Structure object

Show

Show

```
static void Show(string text, BMatrix a)
```

Parameters

text	Text string
a	BMatrix object

Show

Show

```
static void Show(string text, BVector a)
```

Parameters

text	text string
a	BVector object

Show

Show Polynomial

```
static void Show(string txt, Polynomial a)
```

Parameters

txt	text string
a	Polynomial

Show

Show Transfer function

```
static void Show(string txt, TransferFunction a)
```

Parameters

txt	text string
a	TransferFunction

Show

Show Complex Vector

```
static void Show(string txt, CVector a)
```

Parameters

txt	text string
a	complex vector

Show

Show LinearFilter

```
static void Show(string txt, LinearFilter A)
```

Parameters
txt text string
A A Filter object

Show

Show State Space model

```
static void Show(string txt, StateSpaceModel model)
```

Parameters
txt
model

ShowStructure

Show matrix structure

```
static void ShowStructure(string text, BMatrix A)
```

Parameters
text text string
A BMatrix object

WriteLine

Write empty line to console

```
static void WriteLine()
```

WriteLine

Write line to console

```
static void WriteLine(string txt)
```

Parameters
txt text string

Appendix C

Base classes for Vectors and Matrices

C.1 VBase

Class VBase

Base class for Vector holding values of type T

```
[Serializable]
abstract class VBase<T>
```

Constructors

```
VBase(int n)
```

Constructor
Parameters
n

```
VBase(int n, T val)
```

Constructor vector with equal values
Parameters

n	Dimension
value	value

```
VBase(T[] vals)
```

Generate vector from array
Parameters

vals	array of values

Properties

Dimension

Get vector Dimension

```
int Dimension {get;}
```

Structure

Get vector structure

```
Structure Structure {get;}
```

T

Set ot get vector value

```
virtual T this[int pos] {set; get;}
```

Methods

G

Get value (equivalent to res = a[pos])

```
T G(int pos)
```

Parameters

pos	position
returns	value

S

Set value (equivalent to a[pos] = val)

```
void S(int pos, T val)
```

Parameters
pos
val

C.2 MBase

Class MBase

Base class for Matrix objects type T

```
[Serializable]
abstract class MBase<T>
```

Constructors

`MBase(int n)`

Constructor for square matrix
Parameters

n	Dimension

`MBase(MatrixForm Frm, int n)`

Constructor for square matrix
Parameters

n	

`MBase(int n, int m)`

Constructor for matrix
Parameters

n	Rows
m	Columns

`MBase(MatrixForm Frm, int n, int m)`

Constructor for matrix
Parameters

n	Rows
m	Columns

`MBase(int n, int m, T[] vals)`

Constructor for general matrix
Parameters

n	Rows
m	Columns
vals	Values stored row vise

`MBase(int n, T val)`

Constructor for diagonal matrix with constant diagonal values
Parameters

n	Dimension
val	value

`MBase(T[] vals)`

Constructor for diagonal matrix from vector a[i,i] = x[i]
Parameters

vals	vector of values

Properties

Columns

Columns

```
int Columns {get;}
```

ColumnStructure

Get Column structure

```
Structure ColumnStructure {get;}
```

MatrixForm

Matrix form (Null, ConstantDiagonal, Diagonal or General) Moving from Null towards General preserves data Moving from General to Null, ConstantDiagonal and Diagonal are ignored. Moving from General to Sparse is executed

```
virtual MatrixForm Form {set; get;}
```

NZ

Get or set space for Non zero values in sparse matrix. If NZ is set to less than the actual number of of non-zero values, the value buffer is shortened to actual number of non- zero values.

```
int NZ {set; get;}
```

Rows

Rows

```
int Rows {get;}
```

RowStructure

Get Row structure

```
Structure RowStructure {get;}
```

T

Set or get matrix value , res = A[row,col]

```
virtual T this[int row, int col] {set; get;}
```

Type

Object type

```
ObjectType Type {get;}
```

Methods

G

Get value (equivalent to res = a[row,col])

```
T G(int row, int col)
```

Parameters
row Row
col

S

Set value (equivalent to a[row,col] = res)

```
void S(int row, int col, T val)
```

Parameters
row
col
val

Appendix D

Implementation of test fun using $IConFun$

Implementation of non -linear test function :

```
using System;
using System.Collections.Generic;
using System.Linq;
using System.Text;
using MPCMath;

namespace MPCMathApplication
{
    /// <summary>
    /// Non linear function with four minimas and two inequality constriants \index{IConFun}
    /// </summary>
    public class TestConFun : IConFun
    {

        // fun(x0,x1) = (x0^2 + x1 .11)^2 + (x0 + x1^2 -7)^2
        int IFun.N
        {
            get
            {
                return 2;
            }
        }

        /// <summary>
        /// Value
        /// </summary>
        /// <param name="x"></param>
        /// <returns></returns>
```

```
double IFun.Value(Vector x)
{
    double ra = x[0] * x[0] + x[1] - 11.0;
    double rb = x[0] + x[1] * x[1] - 7.0;
    double res = ra*ra + rb*rb;
    return res;
}

/// <summary>
/// Jacobian
/// </summary>
/// <param name="x"></param>
/// <returns></returns>
Vector IFun.Dx(Vector x)
{
    double ra = x[0] * x[0] + x[1] - 11.0;
    double rb = x[0] + x[1] * x[1] - 7.0;
    Vector dx = new Vector(2);
    dx[0] = 4.0 * x[0] *ra + 2.0 * rb;
    dx[1] = 2.0 *ra + 4.0 * x[1] * rb;
    return dx;
}

/// <summary>
/// Hessian
/// </summary>
/// <param name="x"></param>
/// <returns></returns>
Matrix IFun.DDx(Vector x)
{
    Matrix DDX = null;
    double ra = x[0] * x[0] + x[1] - 11.0;
    double rb = x[0] + x[1] * x[1] - 7.0;
    DDX = new Matrix(MatrixForm.General, 2);
    DDX[0, 0] = 4.0 * ra + 8.0 * x[1] * x[1] + 2.0;
    DDX[0, 1] = 4.0 * (x[0] + x[1]);
    DDX[1, 0] = DDX[0, 1];
    DDX[1, 1] = 2.0 + 4.0 * rb + 8.0 * x[1] * x[1];
    return DDX;
}


/// <summary>
/// Number of Equality Constraints
/// </summary>
int IConFun.Ne
{
    get
    {
        return 0;
```

```
    }
}

/// <summary>
/// Equality coinstraint values
/// </summary>
/// <param name="x"></param>
/// <returns></returns>
Vector IConFun.Vale(Vector x)
{
    Vector Cval = new Vector(0);
    return Cval;
}

/// <summary>
/// Equality constraint gradients
/// </summary>
/// <param name="x"></param>
/// <returns></returns>
Matrix IConFun.Dxe(Vector x)
{
    Matrix Dx = new Matrix(0, 2);
    return Dx;
}

/// <summary>
/// Equality constraint Hessians
/// </summary>
/// <param name="x"></param>
/// <returns></returns>
BMatrix IConFun.DDxe(Vector x)
{
    BMatrix DDx = null;
    DDx = new BMatrix(new Structure(1, 2));
    return DDx;
}

/// <summary>
/// Number of Inequality Constraints
/// </summary>
int IConFun.Ni
{
    get
    {
        return 2;
    }
}

/// <summary>
/// Constraint values
```

```
        /// </summary>
        /// <param name="x"></param>
        /// <returns></returns>
        Vector IConFun.Vali(Vector x)
        {
            Vector Cval = new Vector(2);
            Cval[0] = Math.Pow(x[0] + 2.0, 2.0) - x[1];
            Cval[1] = -4.0 * x[0] + 10.0 * x[1];
            return Cval;
        }

        /// <summary>
        /// Cosntraint gradients
        /// </summary>
        /// <param name="x"></param>
        /// <returns></returns>
        Matrix IConFun.Dxi(Vector x)
        {
            Matrix Dx = new Matrix(MatrixForm.General, 2, 2);
            Dx[0, 0] = 2.0 * (x[0] + 2.0);
            Dx[0, 1] = -1.0;
            Dx[1, 0] = -4.0;
            Dx[1, 1] = 10.0;
            return Dx;
        }

        /// <summary>
        /// Constraint Hessians
        /// </summary>
        /// <param name="x"></param>
        /// <returns></returns>
        BMatrix IConFun.DDxi(Vector x)
        {
            BMatrix DDx = null;
            DDx = new BMatrix(MatrixForm.Diagonal, new Structure(1, 2));
            Matrix ddx00 = new Matrix(MatrixForm.Diagonal, 2);
            ddx00[0, 0] = 2.0;
            DDx[0, 0] = ddx00;
            return DDx;
        }
    }
}
```

Appendix E

Implementaion of Van der Pol equations using *IModel*

Implementation of the Van der Pol equations using IModel is shown below:

```
using System;
using System.Collections.Generic;
using System.Linq;
using System.Text;
using MPCMath;

namespace MPCMath
{
    /// <summary>
    /// Van der Pol's problem
    /// $ y"(t) =  \mu (1- y(t)^2)y'(t) -y(t) $
    /// </summary>
    public class VanDerPol : IModel
    {
        private double mu;
        private double t;

        /// <summary>
        /// Construtor
        /// </summary>
        /// <param name="Mu">$\mu$ parameter</param>
        /// <param name="T">Step size</param>
        public VanDerPol(double Mu, double T)
        {
            this.mu = Mu;
            this.t = T;
        }
```

```
int IModel.Dimension
{
    get
    {
        return 2;
    }
}

int IModel.NU
{
    get
    {
        return 0;
    }
}

int IModel.NY
{
    get
    {
        return 1;
    }
}

Vector IModel.Derivative(Vector X, Vector U)
{
    Vector res = new Vector(2);
    res[0] = X[1];
    res[1] = this.mu*(1.0 - X[0]*X[0])* X[1] - X[0];
    return res;
}

Matrix IModel.Jacobian(Vector X, Vector U)
{
    Matrix res = new Matrix(MatrixForm.General, 2);
    res[0,1] = 1.0;
    res[1,0] = 2.0 * this.mu * X[0]*X[1] - 1.0;
    res[1,1] = this.mu*(1.0 - X[0]*X[0]);
    return res;
}

BMatrix IModel.BJacobian(Vector X, Vector U)
{
    return null;
}

Vector IModel.NextState(Vector X, Vector U, Vector W)
{
    throw new NotImplementedException();
}
```

```
        Vector IModel.Observed(Vector X, Vector U)
        {
            return +X;
        }

        StateSpaceModel IModel.LinearModel(Vector XS, Vector US)
        {
            throw new NotImplementedException();
        }

        Vector IModel.Parameters
        {
            get
            {
                return new Vector(1,this.mu);
            }
            set
            {
                this.mu = value[0];
            }
        }
    }
}
```

Appendix F

KKTSolver code

Source code listing for default KKTSolver ?? implementing interface IKKT-Solver 13.5.

```
// <code-header>
//  <file>KKTSolver.cs</file>
//  <author>Jrgen K H Knudsen</author>
//  <copyright>Copyright (c) 2-control Aps. 2011</copyright>
// </code-header> using System;
using System;
using System.Collections.Generic;
using System.Linq;
using System.Text;

namespace MPCMath
{
    // Quadratic Program KKT solver ( Q >= 0)
    // Linear Program solver ( Q = Null matrix or Q.Form == MatrixForm.Null);
    // min.   xGx + g'
    // s.t    Ax  = b
    //        Cx >= d
    //
    //      |G    -A'   -C'     |   | DX |    | -rL |
    //      |-A    0     0      | * | DY |  = | -rA |
    //      |-C    0    - ZInvS |   | DZ |    | -rC |
    //
    //      Factorized into
    //
    //      |G - C'*SinvZ*C        -A'| * |DX| = | -rL + C'* SinvZ* rc|
    //      |- A                    0 |   |DY| = | -rA                |
    //
    //      DZ = DX'* C' * SInvZ + SInvZ*rC

    /// <summary>
    /// Quadratic Program KKT solver, default KKT solver for QPSolver.
    /// Q must be positive semidefinit.
```

```
/// Solves Linear problems by setting Q = null.
/// </summary>
public class KKTSolver : IKKTSolver
{
    private static MPCMathConsole console = MPCMathConsole.Console;

    // Linear equation solver
    private SymmetricalBlockEquationSolver solver;

    private Structure strc;
    private Structure strcx;
    private Structure strcy;
    private Structure strcz;

    private BMatrix mG;
    private BVector vg;
    private BMatrix mA;
    private BVector vb;
    private BMatrix mC;
    private BVector vd;

    private BMatrix J;

    private BMatrix W;
    private BMatrix mCTSinvZ;
    /// <summary>
    /// Constructor for KKT solver
    /// </summary>
    /// <param name="G">G coefficients</param>
    /// <param name="g">g coefficients</param>
    /// <param name="A">A coefficients</param>
    /// <param name="b">b coefficients</param>
    /// <param name="C">C coefficients  </param>
    /// <param name="d">d coefficients</param>
    public KKTSolver(BMatrix G, BVector g, BMatrix A, BVector b, BMatrix C, BVect
    {
        if (G != null)
        {
            if (!Structure.Equal(G.rowStructure, G.columnStructure))
            {
                throw new MPCMathException("MPCMath.KKTSolver", "Illegal G struct
            }

            this.mG = G;
        }
        else
        {
            this.mG = new BMatrix(g.structure);
        }
```

```
if (!Structure.Equal(G.rowStructure, g.structure))
{
    throw new MPCMathException("MPCMath.KKTSolver", "Illegal g structure");
}


if (!Structure.Equal(G.rowStructure, A.columnStructure))
{
    throw new MPCMathException("MPCMath.KKTSolver", "Illegal A structure");
}

if (!Structure.Equal(A.rowStructure, b.structure))
{
    throw new MPCMathException("MPCMath.KKTSolver", "Illegal b structure");
}

if (!Structure.Equal(G.rowStructure, C.columnStructure))
{
    throw new MPCMathException("MPCMath.KKTSolver", "Illegal C structure");
}

if (!Structure.Equal(C.rowStructure, d.structure))
{
    throw new MPCMathException("MPCMath.KKTSolver", "Illegal d structure");
}

this.vg = g;
this.mA = A;
this.vb = b;
this.mC = C;
this.vd = d;

this.strcx = this.mG.rowStructure;
this.strcy = this.mA.rowStructure;
this.strcz = this.mC.rowStructure;
this.strc = new Structure(G.rows + A.rows);

for (int row = 0; row < this.strcx.Dimension; row++)
{
    this.strc[row] = this.strcx[row];
}

int p = this.strcx.Dimension;
for (int row = 0; row < A.rows; row++)
{
    this.strc[p] = A.rowStructure[row];
    p++;
}

this.J = new BMatrix(MatrixForm.General, strc);
```

```
        p = this.strcx.Dimension;
        for (int row = 0; row < this.strcy.Dimension; row++)
        {
            for (int col = 0; col < p; col++)
            {
                J[row + p, col] = -this.mA[row, col];
            }
        }
    }

    /// <summary>
    /// X Structure
    /// </summary>
    Structure IKKTSolver.Strcx
    {
        get
        {
            return this.strcx;
        }
    }

    /// <summary>
    /// Y Structure
    /// </summary>
    Structure IKKTSolver.Strcy
    {
        get
        {
            return this.strcy;
        }
    }

    /// <summary>
    /// X Structure
    /// </summary>
    Structure IKKTSolver.Strcz
    {
        get
        {
            return this.strcz;
        }
    }

    /// <summary>
    /// Slacks
    /// </summary>
    BVector IKKTSolver.d
    {
        get
        {
```

```
        return this.vd;
    }
}

/// <summary>
/// Calculate residulas
/// </summary>
/// <param name="x">state variables</param>
/// <param name="y">Lagrange multipliers for equality conditions</param>
/// <param name="z">Lagrange multipliers for inequality conditions</param>
/// <param name="s">Slack variables</param>
/// <param name="rL">residuals</param>
/// <param name="rA">residuals</param>
/// <param name="rC">residuals</param>
/// <param name="rSZ">residuals</param>
void IKKTSolver.Residuals(BVector x, BVector y,
    BVector z, BVector s,
    out BVector rL, out BVector rA,
    out BVector rC, out BVector rSZ)
{
    // rL = Gx + g - A'y -C'z
    rL = this.mG * x + this.vg - y * this.mA - z * this.mC;

    // rA = b- Ax
    rA = this.vb - this.mA * x;

    //  rC = d - Cx + s
    rC = this.vd - this.mC * x + s;

    // rSZ = SZe
    rSZ = BVector.MulElements(s, z);
}

/// <summary>
/// Factorize KKT system
/// </summary>
/// <param name="SInvZ">diagonal vector</param>
void IKKTSolver.Factorize(BVector SInvZ)
{
    this.W = new BMatrix(SInvZ);

    this.mCTSinvZ = BMatrix.MulTransposed(this.mC, W);

    //   J = | G + C'*SInvZ * C      -A' |
    //       | -A                     0  |

    BMatrix GM = this.mG + this.mCTSinvZ * this.mC;

    int dim = this.strcx.Dimension;
    for (int row = 0; row < dim; row++)
```

```
        {
            for (int col = 0; col < dim; col++)
            {
                this.J[row, col] = GM[row, col];
            }
        }

        this.solver = new SymmetricalBlockEquationSolver(J);
    }

    /// <summary>
    /// Find Newton search directions
    /// </summary>
    /// <param name="rL">residual</param>
    /// <param name="rA">residual</param>
    /// <param name="rC">residual</param>
    /// <param name="DX">X search direction</param>
    /// <param name="DY">Y search direction</param>
    /// <param name="DZ">Z search direction</param>
    void IKKTSolver.Solve(BVector rL, BVector rA, BVector rC,
        out BVector DX, out BVector DY, out BVector DZ)
    {
        BVector rX = -rL + this.mCTSinvZ * rC;
        BVector R = new BVector(this.strc);
        int nx = this.strcx.Dimension;
        for (int p = 0; p < nx; p++)
        {
            R[p] = rX[p];
        }

        for (int p = 0; p < this.strcy.Dimension; p++)
        {
            R[p + nx] = -rA[p];
        }

        BVector res = this.solver.Solve(R);

        DX = new BVector(this.strcx);
        DY = new BVector(this.strcy);

        for (int p = 0; p < nx; p++)
        {
            DX[p] = res[p];
        }

        for (int p = 0; p < this.strcy.Dimension; p++)
        {
            DY[p] = res[p + nx];
        }
```

```
            DZ = -DX * this.mCTSinvZ + this.W * rC;
            return;
        }
    }
}
```

.

Appendix G

PARDISO sparse matrix storage format

The compression of the non-zeros of a sparse matrix A into a linear array is done by walking down each column (column major format) or across each row (row major format) in order, and writing the non-zero elements to a linear array in the order that they appear in the walk.

When storing symmetric matrices, it is necessary to store only the upper triangular half of the matrix (upper triangular format) or the lower triangular half of the matrix (lower triangular format).

The Intel MKL direct sparse solver uses a row major upper triangular storage format. That is, the matrix is compressed row-by-row and for symmetric matrices only non-zeros in the upper triangular half of the matrix are stored.

The Intel MKL storage format accepted for the PARDISO software for sparse matrices consists of three arrays, which are called the values, columns, and rowIndex arrays. The following table describes the arrays in terms of the values, row, and column positions of the non-zero elements in a sparse matrix A.

values	A real or complex array that contains the non-zero entries of A. The non-zero values of A are mapped into the values array using the row major, upper triangular storage mapping described above.
columns	Element i of the integer array columns contains the number of the column in A that contained the value in values(i).
rowIndex	Element j of the integer array rowIndex gives the index into the values array that contains the first non-zero element in a row j of A.

The length of the values and columns arrays is equal to the number of non-zeros in A.

Since the rowIndex array gives the location of the first non-zero within a row, and the non-zeros are stored consecutively, then we would like to be able to compute the number of non-zeros in the i-th row as the difference of rowIndex(i) and rowIndex(i+1).

In order to have this relationship hold for the last row of A, we need to add an entry (dummy entry) to the end of rowIndex whose value is equal to the number of non-zeros in A, plus one. This makes the total length of the rowIndex array one larger than the number of rows of A.

The Intel MKL sparse storage scheme uses the Fortran programming language convention of starting array indices at 1, rather than the C programming language convention of starting at 0.

MPCMath takes care of all updating of the indexes of this storage scheme, and the indexes rowIndex and Columns are not presented for the *MPCMath* programmer. This description is provided solely to give an understanding of the sparse matrix storage mechanism.

Appendix H

MPCMath change history

Feb 25, 2012. Moved NiceMax 2.5 and NiceMin 2.5 from *Plot* 2.6 class to *PlotSeries* 2.6 class.

March 1, 2012 Support for serialization 17.11 of *MPCMath* objects. New *BinaryIO* 17.12 for saving and reading objects to disc files

March 6, 2012 IModel 11.4 property Hessian renamed to Jacobian

March 13, 2012 ODESolver included, chapter 15 . Van der Pol routines 14.5 and appendix E for demonstration of ODESolver included.

March 14, 2012 Support for serialization 17.11 of *MPCMath* objects to XML files . New *XmlIO* 17.13 for saving and reading objects to XML files

May 8, 2102 New Vector functionality ProperZeroes 3.2. IModel interface expanded with BJacobian **??**. SparseMatrixSolver 8.7 based in Intel's PARDISO routine. ODESolver 15.3 and SteadyState 16.4support for BJacobian **??** and support for Infeasible State Exception **??**. JacobianApprox 17.14 for debugging models implemented using IModel 11.4 interface.

May 12, 2012 IModel 11.4 interface, included parameter for sample time T and renamed NextStep to Next. Added administrative properties to StateSpaceModel 7.1

June 8, 2012 Sparse matrices implemented using PARDISO matrix storage. Matrix GetSubmatrix and SetSubmatrix changed to Get and Set.

June 23, 2012 Sparse matrices implemented as one of the possible Matrix Forms C.2. A quite big jobs, with a lot of testing. IModel 11.4 changed as usual. It will probably not stabilize until work with Non linear MPC is finished.

July 4, 2012 BMatrix 4.3 Get parameter sequence changed, to same form as Matrix Get routine

August 4, 2013. Minor changes to StateSpaceModel 7.1. Removed property YS, included properties XS and Off. New constructor included. Included RiccatiSolver 12.8 object. Included Cnsl class B.1 for debugging of Console based programs

December 15, 2014. Added BandEquationSolver 8.9, SymmetricBandEquationSolver 8.10, Spline 17.6, CubicSpline 17.7, IConFun 11.2, IFunt 11.3, FunMin , 12.4, ConFunMin 12.5, LeastSquareFit 12.7 and an additional version of Vector.Random 3.2

Index