TE4006

Principles of Communications

Class Notes

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Chapter 1

Introduction

The subject of *digital communications* is the study of the transmission of digital information from a *source* over a *waveform* channel. A *digital source* is discrete in time and discrete in value, it is a sequence of numbers from a finite or *countable* set, e.g., a computer terminal, a digital voltmeter.

Many physical sources are analog, i.e., *continuous* in time and continuous in value, e.g., voltage from a microphone, TV signal. The signal of these sources can be converted to digital form using an analog to digital converter (ADC or A/D).

The *waveform channel* accepts a waveform and produces a waveform as shown in Figure 1.1. The output, $Y(t)$ in general is not equal to the input, i.e., $Y(t) \neq X(t)$, instead, it is an attenuated, noise added, filtered and distorted version of the input waveform $X(t)$. The waveform channel can be any transmission medium used such as twisted pair wire, coaxial cable, fiber optic, atmosphere, vacuum, etc.

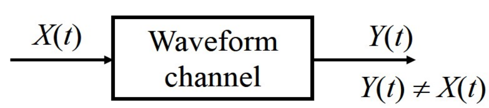

Figure 1.1: The concept of waveform channel.

1.1 General Model of a Digital Communications System

The general block diagram of a digital communication system is shown in Figure 1.2. M_i, $i = 1, 2, \ldots$, is the set of symbols being transmitted. The input signal to the End User block is the set of symbols $\widehat{M}_i$, $i = 1, 2, \ldots$, where $\widehat{M}_i$ is an **estimate** of M_i based on the received waveform $Y(t)$.

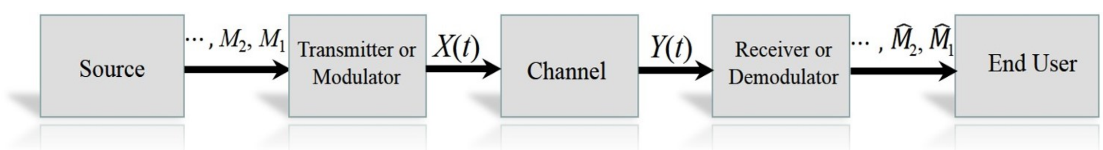

Figure 1.2: The concept of a digital communication system.

Chapter 2

The Concept of Signals

A signal is a representation of a message or information. It carries information in some format and is represented mathematically as a function. Signals can be produced by many different methods, for example an audio signal coming from a microphone, or an image coming from a camera, temperature or pressure from a sensor, etc. Different signals can represent the same information conveying the same message, see Figure 2.1.

Figure 2.1: Different signals conveying the same information.

A signal needs to have an emitter or transmitter and an interpreter or receiver, otherwise it would be information that does not make any sense. A signal describes a physically realizable process and its variability and special features such as amplitude, frequency, phase, range, location, power received measured in a space (see Figure 2.2,) etc.

Definition 1 Signal. *A signal is a measurement or observation that contains information describing some phenomenon such as the time history of an economic process, chemical signals of fragrances, variation of light intensity, among many more.*

The representation of a signal can be in continuous and discrete time, and can take forms that can be described mathematically or graphically. Then a different definition of a signal, and perhaps a formal one would be the following

Definition 2 Signal. *A signal is a real or complex valued* **function** *of one or more real variables such as time (continuous or discrete) or space.*

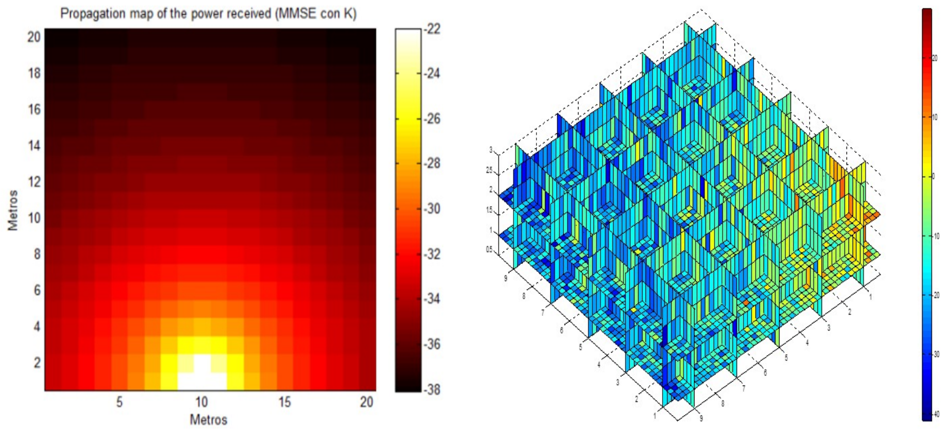

Figure 2.2: Different signals conveying the same information.

A signal may depend on one variable that would make it a one-dimensional signal, e.g., speech vs. time, temperature vs. time, etc. Dependence on two or more variables gives a *multidimensional* signal, e.g., an image, power received at certain coordinates (see Figure 2.2), etc. As one can see from definition 2, a signal is defined by the concept of a function.

A function is a rule that *maps* a *domain* set A into a *range* set or *image* set B, we denote this as $f : A \longrightarrow B$, and f is a *rule* that assigns, to each element of set A an element of set B, see [4]. The rule defines that for every element $a \in A$, there exists an element $f(a) \in B$ called the image of a under the function f. For any subset $D \subset A$, $f(D)$ will be the set of all image points of D under the function f, i.e., the image of D under f. In contrast, for a set $E \subset B$, denote $f^{-1}(E)$ the set of all points of A whose images under the function f line in E, i.e., the *preimage* or *inverse image* of E under f. Functions can be, see [4],

- *Injective or one-to-one*: If for every pair of elements $a_1, a_2 \in A$, $a_1 \neq a_2$, their images are different, in other words, $f(a_1) \neq f(a_2)$. Injectivity depends on the rule of the function.

- *Surjective or f maps A onto B*: If for every element of the range set $b \in B$, $b = f(a)$ for at least one element $a \in A$. Surjectivity depends on the rule and on the range set B.

- *Bijective or one-to-one correspondence*: If the function f is both injective and surjective. A bijective function f has an *inverse* f^{-1}, where for each $b \in B$, $f^{-1}(b)$ is that unique element $a \in A$ for which $f(a) = b$.

In general, the sets involved in the mapping of a function can be specified or can be understood from the context of the analysis, for example, in the signal analysis part, it will be understood that the domain set A will be related to time in any of its forms.

2.1 Signal Classification

Signals can be classified in different ways. In the following, a list of contrasting characteristics is presented.

1. *Continuous vs. Discrete*: A signal $s(t)$ is continuous if at any point t_0 of its domain, it satisfies that $lim_{t \to t_0^-} s(t) = lim_{t \to t_0^+} s(t)$. A signal is discrete if its range or image takes only some countable set of values.

2. *Periodic vs. aperiodic*: A signal $s(t)$ is *periodic* if for a positive constant number T_0 called the *period*, the following is satisfied

$$s\left(t\right) = s\left(t + kT_0\right), \quad \forall t \in \mathbb{R}, \;\; T_0 > 0, \;\; k = 0, \pm 1, \pm 2, \ldots . \tag{2.1}$$

   A convention is that a periodic signal always has *infinite* duration, i.e., $-\infty < t < \infty$. An aperiodic signal does not satisfy Equation (2.1) for any value of T_0. Also, an aperiodic signal can be of infinite or finite duration. See that a signal with duration from $t = 0$ up to $t = \infty$ will have infinite duration, but it will not be periodic. Figure 2.3 shows two examples of periodic signals, i.e., the cosine function and the train of rectangular pulses. Figure 2.4 shows two examples of aperiodic signals, i.e., the truncated cosine signal or cosine pulse and the rectangular pulse.

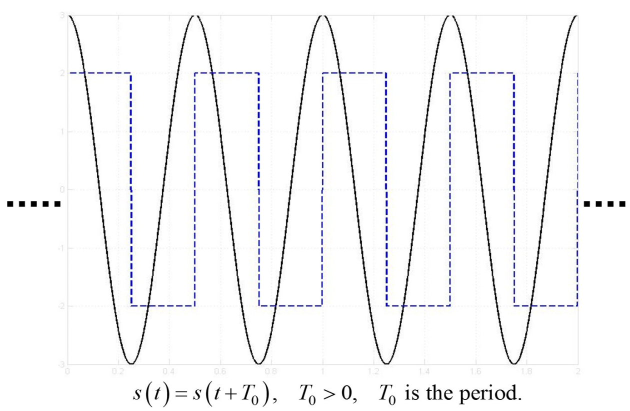

Figure 2.3: Periodic signals.

   Note that the truncated cosine signal can be seen as the product of the aperiodic rectangular pulse signal in the figure and the periodic cosine signal of Figure 2.3. See that any periodic signal has a fundamental frequency f_0 and a period $T_0 = 1/f_0$.

3. *Deterministic vs. Random*: A *deterministic* signal can be represented by explicit mathematical rules, e.g., completely specified time functions such as $s(t) = cos(2\pi f t), t \in \mathbb{R}$. A *random* signal can take a random value for any element of its domain set, and can be modeled probabilistically with distributions and density functions.

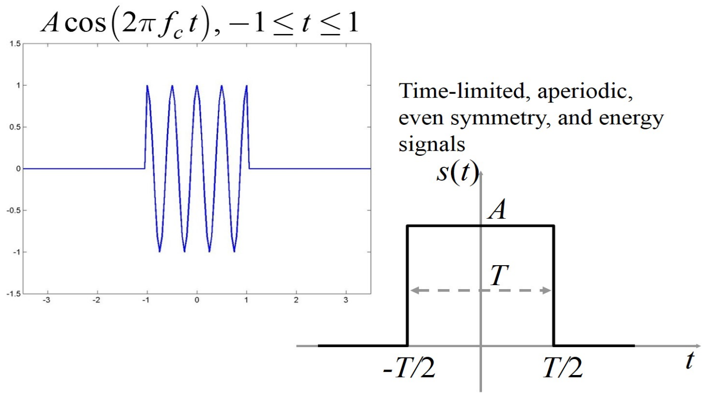

Figure 2.4: Aperiodic signals.

4. *Time limited vs. Band limited*: The signals in Figure 2.4 can be seen or classified as *time-limited* signals since their duration or existence is for a limited set of values (*interval*) of the domain or time variable, outside this interval the signal is zero or non-existent. A *band-limited* signal is zero everywhere in the frequency domain, except for a limited set or range of values of frequency.

5. *Even vs. Odd symmetry*: A signal $s(t)$ has even symmetry if it satisfies

$$s(t) = s(-t), \quad \forall t, \tag{2.2}$$

on the other hand, a signal $s(t)$ has odd symmetry if it satisfies

$$s(t) = -s(-t), \quad \forall t. \tag{2.3}$$

A signal can be the combination of an even and an odd symmetric signals, then the even and odd parts of a signal can be used to form the signal in the following way

$$s(t) = s_{even}(t) + s_{odd}(t), \tag{2.4}$$

and we can also see that

$$s_{even}(t) = \frac{1}{2}\left[s(t) + s(-t)\right], \tag{2.5}$$

and that

$$s_{odd}(t) = \frac{1}{2}\left[s(t) - s(-t)\right], \tag{2.6}$$

Figure 2.5 shows a cosine function. The cosine function has even symmetry. See how the signal on the right hand side of the vertical axis is a mirrored version of the part of the signal on the left hand side of the vertical axis. In contrast, the sine function is an odd symmetry signal, try to show that it satisfies Equation (2.3) and plot such function to see how an odd symmetry signal looks like.

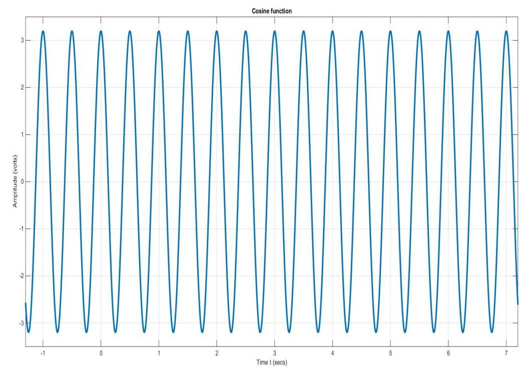

Figure 2.5: Cosine function.

2.2 Important Signals and some Properties

In this section, we present some of the most important signals because they represent relevant processes in communication systems. Also, some properties of functions will be introduced with the examples.

2.2.1 Cosine Signal

Figure 2.6 shows two representations of a cosine signal. The signal is deterministic and it is defined by $s(t) = 3\cos(2\pi t)$. Note that it is a cosine of peak amplitude of 3 V and frequency $f = 1$ Hz, i.e., see that the argument of the cosine function is $2\pi f t = 2\pi t$, hence $f = 1$. Recall that frequency f will be defined in Hz, and that it is related to $\omega = 2\pi f$. In the same figure, we can see that there is a *time-shifted* version of $s(t)$. The shift is to the right hand side of the original signal by an amount of 1/4 seconds so that the *time-shifted* signal is $s(t-\frac{1}{4}) = 3\cos\left[2\pi(t-\frac{1}{4})\right]$. Note that the same signal can be represented by another function that depends on the shift of the phase of the original signal, the phase of the original signal is zero. In the same figure, we can also see that the original signal $s(t)$ has **even** symmetry, and that the shifted version has **odd** symmetry.

Mathematically, the periodic cosine signal $s(t)$ of amplitude A and fundamental frequency f_0 is represented by any of the following equations

$$s(t) = A\cos(2\pi f_0 t), \qquad t \in \mathbb{R}, \tag{2.7}$$

or

$$s(t) = A\cos(2\pi f_0 t), \qquad -\infty < t < \infty, \tag{2.8}$$

or for any positive or negative integer number n, i.e., $n \in \mathbb{Z}$, we have

$$s(t) = \begin{cases} A\cos(2\pi f_0 t), & |t - nT_0| < \frac{T_0}{2}, \\ 0, & \text{elsewhere}. \end{cases} \tag{2.9}$$

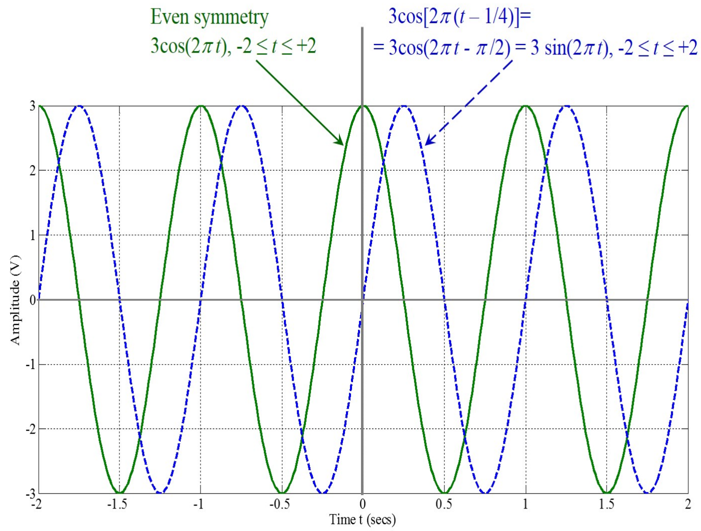

Figure 2.6: Different representations of a cosine signal.

Equation (2.9) is called the *Compact Generalized Form* (CGF) notation of signal $s(t)$. Also for any positive or negative integer number n, i.e., $n \in \mathbb{Z}$, we have

$$s(t) = \begin{cases} A\cos(2\pi f_0 t), & nT_0 - \frac{T_0}{2} < t < nT_0 + \frac{T_0}{2}, \\ 0, & \text{elsewhere} \end{cases} \tag{2.10}$$

or

$$s(t) = A\cos(2\pi f_0 t), \qquad t \in (-\infty,\ \infty). \tag{2.11}$$

See that all these equations are different mathematical expressions of the same signal. See also that the intervals of the time domain indicated in equations (2.9) and (2.10) are the same because

$$\begin{aligned} |t - nT_0| &< \frac{T_0}{2}, \\ -\frac{T_0}{2} < \quad t - nT_0 &< \frac{T_0}{2} \\ nT_0 - \frac{T_0}{2} < \qquad t \qquad &< nT_0 + \frac{T_0}{2}. \end{aligned} \tag{2.12}$$

Time-Shifting Property: *A representation $s(t-\tau)$ of any function or signal $s(t)$ implies that $s(t)$ is time-shifted to the right τ time units for $\tau > 0$. For $s(t+\tau)$, the signal $s(t)$ is time-shifted to the left τ time units for $\tau > 0$.*

2.2.2 Rectangular Pulse

Figure 4.7 shows a rectangular pulse of amplitude $A = 1$, pulse duration $T = 0.02$ secs which is centered at the origin. The signal is represented by a large pi symbol (or product symbol), i.e., mathematically, the rectangular pulse signal $s(t)$ of amplitude A, centered at the origin, and with time duration T is represented by the following equation

$$s(t) = A \prod \left(\frac{t}{T} \right), \tag{2.13}$$

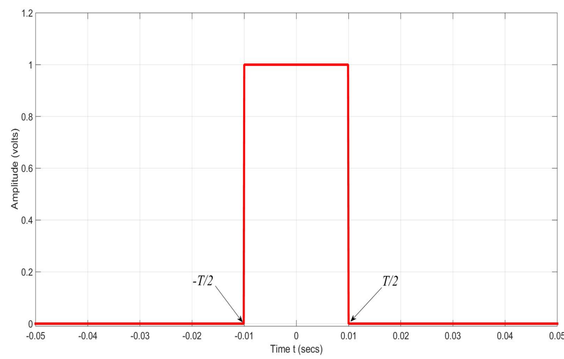

Figure 2.7: Rectangular pulse signal.

The same rectangular pulse signal can be represented mathematically by any of the following equations

$$s(t) = \begin{cases} A, & |t| < \frac{T}{2}, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.14}$$

or

$$s(t) = \begin{cases} A, & -\frac{T}{2} < t < \frac{T}{2}, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.15}$$

Time-Shifted Version. Assume that $g(t)$ is a scaled and time-shifted version of the rectangular pulse signal $s(t)$ of amplitude A, with time duration T. Assume that the time-shift is to the right of the origin by τ secs. The pulse can be seen in Figure 2.8 where the original ($s(t)$ centered at the origin) and the time-shifted version ($g(t)$ scaled to 0.7 of amplitude) are shown.

The time-shifted version $g(t)$ of the rectangular pulse $s(t)$ of Equation (2.13) can be represented by any of the following mathematical expressions

$$g(t) = s(t - \tau) = A\Pi \left(\frac{t - \tau}{T} \right), \tag{2.16}$$

or

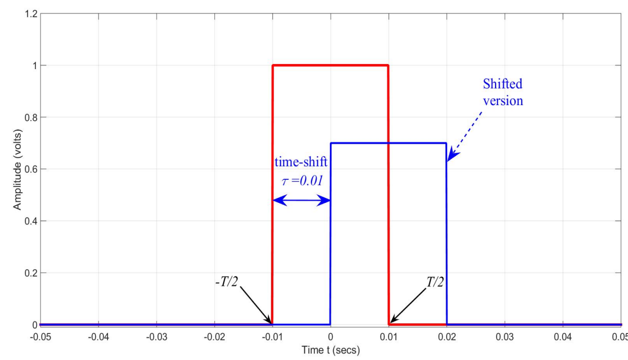

Figure 2.8: Rectangular pulse signal and time-shifted version.

$$g(t) = \begin{cases} A, & |t-\tau| < \frac{T}{2}, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.17}$$

or

$$g(t) = \begin{cases} A, & \tau - \frac{T}{2} < t < \tau + \frac{T}{2}, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.18}$$

2.2.3 Rectangular Pulse Train

Figure 2.9 shows a rectangular pulse train signal of amplitude $A = 2$, pulse duration $T = 1$ secs which is centered at the origin with period $T_0 = 4$ secs, i.e., fundamental frequency of $f_0 = 1/T_0 = 1/4$ Hz. **Duty cycle** is the quotient (or ratio) of the rectangular pulse duration T to the period of the signal T_0, and is generally denoted by $d = T/T_0$. In the case of Figure 2.9, the duty cycle is $d = 1/4$.

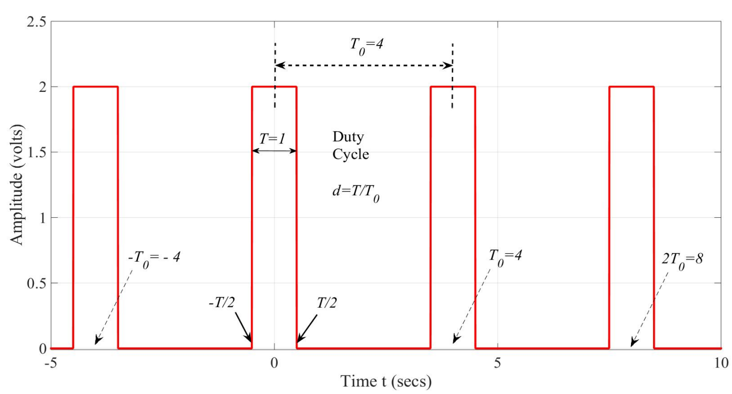

Figure 2.9: Rectangular pulse train signal.

Mathematically, the rectangular pulse train signal $s(t)$ of amplitude A, pulse duration $T < T_0$, duty cycle $d = T/T_0$, and fundamental frequency f_0 is represented as a sum of individual rectangular pulses centered (shifted) at integer multiples of the period

$$s(t) = \sum_{n=-\infty}^{\infty} A \prod \left(\frac{t - nT_0}{T} \right) \tag{2.19}$$

Also, the rectangular pulse train signal can be represented for any positive or negative integer number n, i.e., $n \in \mathbb{Z}$, as an expression for each period as

$$s(t) = \begin{cases} A, & |t - nT_0| < \frac{T}{2}, \\ 0, & \text{elsewhere} \end{cases} \tag{2.20}$$

or also for any positive or negative integer number n, i.e., $n \in \mathbb{Z}$, we have for each period

$$s(t) = \begin{cases} A, & nT_0 - \frac{T}{2} < t < nT_0 + \frac{T}{2}, \\ 0, & \text{elsewhere} \end{cases} \tag{2.21}$$

Remark: Note that the rectangular pulse train with duty cycle of $d = 1/2$ is the square wave, i.e., the time duration of each individual pulse is half of the period of the signal.

2.2.4 Truncated Cosine Signal

Figure 2.4 in previous section shows a truncated cosine signal of duration $T = 2$ and amplitude $A = 1$. A general representation of a truncated cosine pulse is shown in Figure 2.10 with amplitude A, duration T and frequency f_0. See that the figure shows a pulse centered at the origin.

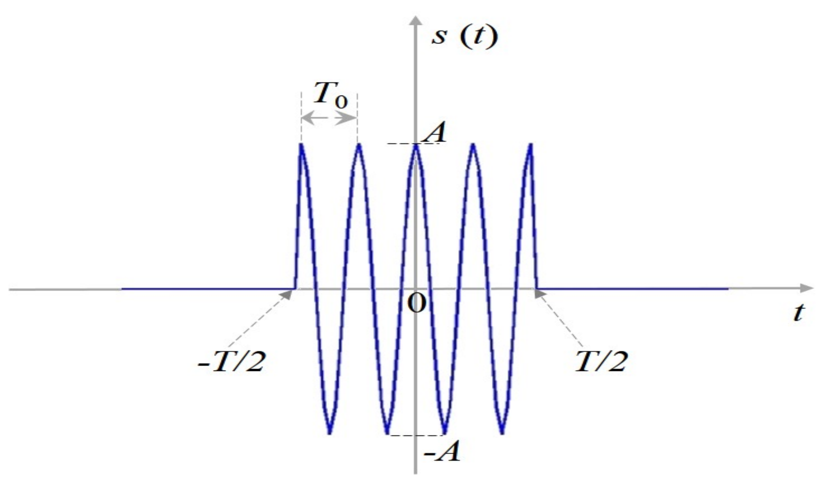

Figure 2.10: Truncated cosine pulse signal.

Mathematically, the truncated cosine signal $s(t)$ of amplitude A, duration T, fundamental frequency f_0 and centered at the origin, is represented by any of the following equations

$$s(t) = A\cos(2\pi f_0 t), \qquad -\frac{T_0}{2} < t < \frac{T_0}{2}, \tag{2.22}$$

or

$$s(t) = \begin{cases} A\cos(2\pi f_0 t), & |t| < \frac{T_0}{2}, \\ 0, & \text{elsewhere}, \end{cases} \tag{2.23}$$

or as the following product

$$s(t) = A\cos(2\pi f_0 t) \prod \left(\frac{t}{T}\right). \tag{2.24}$$

2.2.5 The Unit Step Function

The unit step function is shown in Figure 2.11 and is mathematically defined as

$$u(t) = \begin{cases} 1, & t > 0, \\ \frac{1}{2}, & t = 0, \\ 0, & t < 0. \end{cases} \tag{2.25}$$

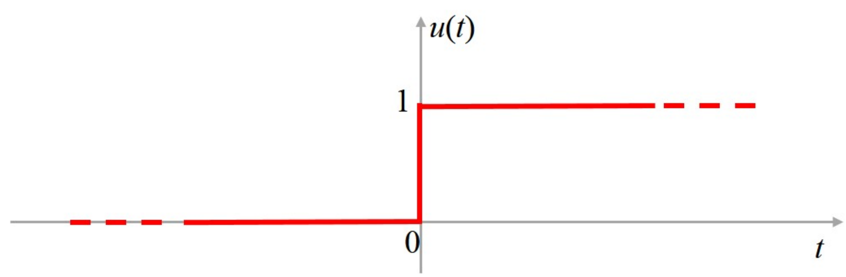

Figure 2.11: Unit step signal.

2.2.6 The Sign Function

The sign function is shown in Figure 2.12. Note that it is a function that indicates the positive or negative value of its argument, i.e., $sgn(t)$ indicates with a value of 1 when $t > 0$, similarly for the value -1. the function is mathematically defined as

$$sgn(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases} \tag{2.26}$$

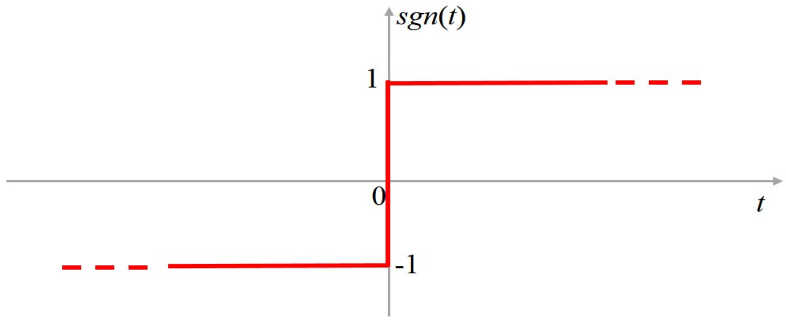

Figure 2.12: The sign signal.

2.2.7 The Sinc Function

The sinc or cardinal sine function is defined as

$$s(t) = sinc(t) = \frac{\sin(\pi t)}{\pi t}. \tag{2.27}$$

You can see in Figure 2.13 the shape of the sinc function. Note that when $t = 0$, Equation (2.27) is undetermined, i.e., $\frac{0}{0}$. The value of the sinc function at $t = 0$ must be obtained using L'Hopital's rule and the result is $s(0) = 1$, try to demonstrate this.

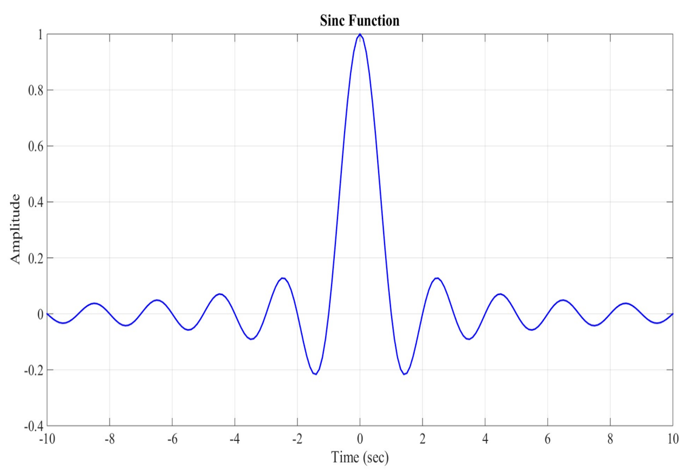

Figure 2.13: Sinc function $sinc(t)$.

2.2.8 The Dirac's Delta Function

Dirac's delta function $\delta(t)$ is an impulse centered at the origin with infinite amplitude that satisfies that its area is unitary. This is also why it is called a unitary impulse.

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1. \tag{2.28}$$

The representation in a sketch of $A\delta(t)$ is an impulse of amplitude A centered at the origin, even though it does have an infinite amplitude as said before.

An important property of the delta function is that of *Sifting* property, which is determined by the following expression

$$\int_a^b \delta(t-k)f(t)\,dt = f(k), \quad if\ k \in [a,b]. \tag{2.29}$$

2.2.9 The Triangular Pulse Function

The triangular pulse function is defined as

$$A\bigwedge\left(\frac{t}{T}\right) = \begin{cases} A - A\frac{|t|}{T} & |t| \leq T, \\ 0, & \text{elsewhere.} \end{cases}$$

Figure 2.14 shows the triangular pulse function with $A = 2$ and $T = 3$. Note that for the triangular pulse T is not the duration of the pulse as in the rectangular pulse.

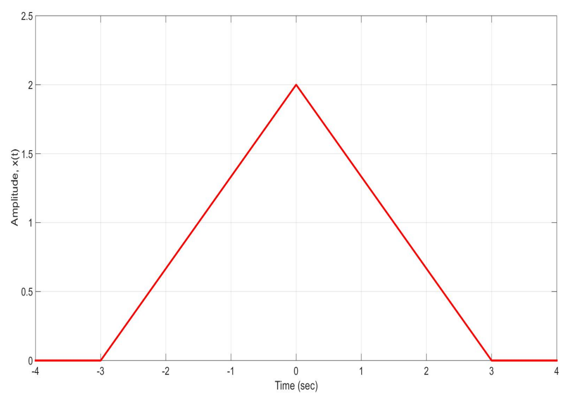

Figure 2.14: Triangular pulse function with $T = 3$ and $A = 2$.

2.3 Time Averages

The time averages of a signal are parameters used for design of systems. These time averages are obtained by the use of a linear operator and the type of signal (periodic signal, aperiodic power signal and aperiodic energy signal). The time averages are the **mean** value, the **mean squared** value and the **root mean square (rms)** value.

Before proceeding to the definitions of the time averages, we have that all time-limited signals are **energy** signals, periodic signals are **power** signals since they have infinite duration, and that aperiodic signals of infinite duration are also power signals. All **physically** realizable signals (implementable, i.e., signals that exist only for positive times $t \geq 0$) are **energy** signals since they have *finite* duration, i.e., they are time-limited signals. See that all time-limited signals are not necessarily physically realizable signals because time-limited signals can exist for $t < 0$ and we are only able to physically realize signals for $t \geq 0$.

2.3.1 Mean Value

The mean value of a signal represents the Direct Current (DC) component of the signal. If one could see the frequency components of the signal, the mean value would be situated at zero frequency. The mean value is obtained as follows

1. **Periodic signal** $x_p(t)$ with period T_0. Recall that this signal exists for all the time, i.e., $t \in \mathbb{R}$.

$$\langle x_p(t) \rangle = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_p(t) dt. \tag{2.30}$$

As an example, assume that we are given the periodic signal $x_p(t) = A\cos(2\pi f_0 t)$, as in Equation (2.7), then the mean value of this periodic signal is

$$\begin{aligned} \langle x_p(t) \rangle &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_p(t) dt \\ &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A\cos(2\pi f_0 t) dt \\ &= \frac{A}{2\pi f_0 T_0} \sin(2\pi f_0 t) \Big]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \\ &= 0. \end{aligned} \tag{2.31}$$

The last equality comes when substituting the limits of integration, because we have that the argument of the sine function gives $2\pi f_0 t = 2\pi f_0 T_0/2 = \pi$, and $\sin$ of π is always zero.

2. **Aperiodic power signal** $x(t)$. Recall that this signal has infinite duration, but it could be for all $t \in \mathbb{R}$ or for example it could be for $t \geq 0$. Define an interval of length T, then

$$\langle x(t) \rangle = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) dt. \tag{2.32}$$

3. **Aperiodic energy signal** $x(t)$. Recall that this signal exists only for a finite duration of time, for example T. The integral must be calculated only on the interval where the signal exists, i.e.,

$$\langle x(t)\rangle = \int\limits_{T} x(t)dt. \tag{2.33}$$

As an example, assume that we are given the aperiodic signal $x(t) = A\cos(2\pi f_0 t)$, for $|t| < T/2$, i.e., a truncated cosine pulse as that shown in Figure 2.10, then the mean value of this aperiodic signal is

$$\begin{aligned}
\langle x(t)\rangle &= \int\limits_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt \\
&= \int\limits_{-\frac{T}{2}}^{\frac{T}{2}} A\cos(2\pi f_0 t)dt \\
&= \frac{A}{2\pi f_0}\sin(2\pi f_0 t)\Bigg]_{-\frac{T}{2}}^{\frac{T}{2}} \\
&= 0.
\end{aligned} \tag{2.34}$$

The last equality comes when substituting the limits of integration, and with the assumption that $T = kT_0$ for some integer number $k \geq 1$. If such assumption is not true, then the result of the integral will be at most of the equivalent of the area corresponding to half a period of the cosine function (**Please demonstrate this**), which in practical ways (amplitude levels and frequencies used in communications) is much smaller than noise levels, hence it is considered to be zero anyway.

2.3.2 Mean Squared Value

The mean squared value of a signal represents the *power* or *energy* of the signal. In this case (communications and signal processing) we talk about the **normalized average total power (energy)** of a signal since we consider that the power or energy is dissipated in a 1Ω resistor. This time average is given by

1. **Periodic signal** $x_p(t)$ with period T_0. Recall that this signal exists for all the time, i.e., $t \in \mathbb{R}$.

$$\langle x_p^2(t)\rangle = \frac{1}{T_0}\int\limits_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_p^2(t)dt. \tag{2.35}$$

As an example, assume that we are given the periodic signal $x_p(t) = A\cos(2\pi f_0 t)$, as in Equation (2.7), then the mean squared value of this periodic signal is

$$\begin{aligned}
\langle x_p^2(t)\rangle &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_p^2(t)dt \\
&= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A^2\cos^2(2\pi f_0 t)dt \\
&= \frac{A^2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos^2(2\pi f_0 t)dt \\
&= \frac{A^2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left[\frac{1}{2}+\frac{1}{2}\cos(4\pi f_0 t)\right] dt \\
&= \frac{A^2}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} dt + \frac{A^2}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(4\pi f_0 t)dt \\
&= \frac{A^2}{2T_0}t\Big]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} + \frac{A^2}{4\pi f_0 T_0}\sin(4\pi f_0 t)\Big]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \\
&= \frac{A^2}{2}. \qquad (2.36)
\end{aligned}$$

For the same reason as that in Equation (2.31), the second term in the penultimate row of Equation (2.36) is zero.

2. **Aperiodic power signal** $x(t)$. Recall that this signal has infinite duration, but it could be for all $t \in \mathbb{R}$ or for example it could be for $t \geq 0$. Define an interval of length T, then

$$\langle x^2(t)\rangle = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt. \qquad (2.37)$$

3. **Aperiodic energy signal** $x(t)$. Recall that this signal exists only for a finite duration of time, for example T. The integral must be calculated only on the interval where the

signal exists, i.e.,

$$\langle x^2(t)\rangle = \int_T x^2(t)dt. \tag{2.38}$$

As an example, assume that we are given the aperiodic signal $x(t) = A\cos(2\pi f_0 t)$, for $|t| < T/2$ shown in Figure 2.10, then the mean squared value of this aperiodic signal is

$$\begin{aligned}
\langle x^2(t)\rangle &= \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f_0 t)dt \\
&= A^2\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos^2(2\pi f_0 t)dt \\
&= A^2\int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\frac{1}{2} + \frac{1}{2}\cos(4\pi f_0 t)\right] dt \\
&= \frac{A^2}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt + \frac{A^2}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(4\pi f_0 t)dt \\
&= \left.\frac{A^2}{2}t\right]_{-\frac{T}{2}}^{\frac{T}{2}} + \left.\frac{A^2}{8\pi f_0}\sin(4\pi f_0 t)\right]_{-\frac{T}{2}}^{\frac{T}{2}} \\
&= \frac{A^2 T}{2}.
\end{aligned} \tag{2.39}$$

For the same reason as that in Equation (2.34), the second term in the penultimate row of Equation (2.39) is zero. The first term represents the **Energy** of the signal and see that is equal to the product of power and duration, where power is the mean squared value obtained for the periodic cosine signal.

2.3.3 rms Value

The rms value is obtained by calculating the square root of the mean squared value, hence we have

1. **Periodic signal** $x_p(t)$ with period T_0. Recall that this signal exists for all the time, i.e., $t \in \mathbb{R}$.

$$\sqrt{\langle x_p^2(t)\rangle} = \sqrt{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_p^2(t)dt}. \tag{2.40}$$

As an example, assume that we are given the periodic signal $x_p(t) = A\cos(2\pi f_0 t)$, as in Equation (2.7), then the rms value is given by $\frac{A}{\sqrt{2}}$.

2. **Aperiodic power signal** $x(t)$. Recall that this signal has infinite duration, but it could be for all $t \in \mathbb{R}$ or for example it could be for $t \geq 0$. Define an interval of length T, then

$$\sqrt{\langle x^2(t)\rangle} = \sqrt{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt}. \tag{2.41}$$

3. **Aperiodic energy signal** $x(t)$. Recall that this signal exists only for a finite duration of time, for example T. The integral must be calculated only on the interval where the signal exists, i.e.,

$$\sqrt{\langle x^2(t)\rangle} = \sqrt{\int_T x^2(t)dt}. \tag{2.42}$$

2.4 Problems

Problem 3 *A voltage signal is applied to a coaxial cable with resistive load of 75 Ω. The voltage signal is given by*

$$v(t) = \begin{cases} 7.3\sin(2\pi f_0 t), & \left|t - \frac{T_0}{4} - nT_0\right| < \frac{T_0}{4}, \\ 0, & elsewhere, \end{cases} \tag{2.43}$$

where n is any nonnegative integer. Do the following

1. *Sketch the voltage and current waveforms. Show the most important values of time, amplitude and current.*
2. *Obtain the DC values or mean values for the voltage and the current signals.*
3. *Find the rms values for the voltage and the current signals.*
4. *Find the total average power dissipated in the resistive load.*

Chapter 3

Fourier Series

The analysis of every digital communications system considers the transmitter, the receiver, and the channel. The transmitter and receiver know the signals that are being sent and received.

The problem of the geometric interpretation of signals consist of finding a set of functions $\{\varphi_n(t)\}_{n=-\infty}^{\infty}$ defined in an interval $[a,b]$, such that the following conditions are satisfied:

1. The functions are orthogonal, i.e.

$$\int_a^b \varphi_n(t)\,\varphi_m^*(t)\,dt = \begin{cases} 0, & n \neq m, \\ \|\,\varphi_n(t)\,\|^2, & n = m. \end{cases} \tag{3.1}$$

2. Every signal $f(t)$ defined in the same interval $[a,b]$, can be expressed as a linear combination of $\{\varphi_n(t)\}_{n=-\infty}^{\infty}$, in other words

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \varphi_n(t)\,,\;\; C_n \in \mathbb{C}. \tag{3.2}$$

3.1 Orthogonal Signal Expansion

Assume that the set of functions $\{\varphi_n(t)\}_{n=-\infty}^{\infty}$ are orthogonal and complex on an interval $[a,b]$, then if $f(t)$ is also defined on the same interval $[a,b]$, it is possible to have coefficients C_k, $k=-\infty,\ldots,-1,0,1,\ldots,\infty$, such that

$$f(t) = \cdots + C_{-1}\varphi_{-1}(t) + C_0\varphi_0(t) + C_1\varphi_1(t) + C_2\varphi_2(t) + \cdots + C_n\varphi_n(t) + \cdots \tag{3.3}$$

Assume that the representation through an infinite series as that in Equation (3.3) is possible, then multiply both sides of such equation by $\varphi_n^*(t)$ and integrate on the interval $[a,b]$ to obtain

$$\begin{aligned}
\int_a^b f(t)\varphi_n^*(t)\,dt &= \cdots + \int_a^b C_{-1}\varphi_{-1}(t)\,\varphi_n^*(t)\,dt \\
&+ \int_a^b C_0\varphi_0(t)\,\varphi_n^*(t)\,dt \\
&+ \int_a^b C_1\varphi_1(t)\,\varphi_n^*(t)\,dt \\
&+ \int_a^b C_2\varphi_2(t)\,\varphi_n^*(t)\,dt \\
&+ \cdots \\
&+ \int_a^b C_n\varphi_n(t)\,\varphi_n^*(t)\,dt + \cdots
\end{aligned} \tag{3.4}$$

Now, applying the orthogonality conditions of $\{\varphi_n(t)\}_{n=-\infty}^{\infty}$ given in Equation (3.1), we get that

$$\begin{aligned}
\int_a^b f(t)\varphi_n^*(t)\,dt &= \int_a^b C_n\varphi_n(t)\,\varphi_n^*(t)\,dt \\
&= \int_a^b C_n|\varphi_n(t)|^2 dt \\
&= C_n\langle\varphi_n(t),\ \varphi_n(t)\rangle \\
&= C_n\|\ \varphi_n(t)\ \|^2
\end{aligned} \tag{3.5}$$

Therefore the coefficients C_k of the series in Equation (3.3) will be given by

$$C_n = \frac{\int_a^b f(t)\varphi_n^*(t)\,dt}{\int_a^b|\varphi_n(t)|^2 dt} = \frac{\langle f(t),\ \varphi_n(t)\rangle}{\|\ \varphi_n(t)\ \|^2}, \quad \forall n. \tag{3.6}$$

Then, the signal $f(t)$ represented by the series (a linear combination) will be given by substituting (3.6) in (3.2) to obtain the *Generalized Fourier Series (GFS)* as follows

$$f(t) = \sum_{n=-\infty}^{\infty} C_n\varphi_n(t) = \sum_{n=-\infty}^{\infty}\frac{\langle f(t),\ \varphi_n(t)\rangle}{\|\ \varphi_n(t)\ \|^2}\varphi_n(t) \tag{3.7}$$

3.2 Complex Exponential Fourier Series

The infinite series in Equation (3.7) is called the **Generalized Fourier Series (GFS)**. The GFS can be used to define other signal representations. For example, we can choose the set of periodic functions with period T_0 to be

$$\{\varphi_n(t)\}_{n=-\infty}^{\infty} = \left\{e^{j2\pi f_0 tn}\right\}_{n=-\infty}^{\infty}, \tag{3.8}$$

Recall that for a periodic signal with period T_0, its fundamental frequency f_0 is related to its period as $f_0 = 1/T_0$, in other words we have that the product of period and frequency is $f_0T_0 = 1$

and we can see that the set contains orthogonal functions since

$$\begin{aligned}
\int_{-T_0/2}^{T_0/2} \varphi_n(t)\,\varphi_m^*(t)\,dt &= \int_{-T_0/2}^{T_0/2} e^{j2\pi f_0 tn} e^{-j2\pi f_0 tm} dt \\
&= \int_{-T_0/2}^{T_0/2} e^{j2\pi f_0 t(n-m)} dt \\
&= \frac{1}{j2\pi f_0(n-m)} e^{j2\pi f_0 t(n-m)} \Bigg|_{-T_0/2}^{T_0/2} \\
&= \frac{1}{j2\pi f_0(n-m)} \left[e^{j2\pi f_0 \frac{T_0}{2}(n-m)} - e^{-j2\pi f_0 \frac{T_0}{2}(n-m)} \right] \\
&= \frac{1}{j2\pi f_0(n-m)} \left[e^{j\pi(n-m)} - e^{-j\pi(n-m)} \right] \\
&= \frac{1}{\pi f_0(n-m)} \sin\left[\pi(n-m)\right]. \qquad (3.9)
\end{aligned}$$

Recall that in Equation (3.9), the term $(n-m)$ will be an integer number since n and m are integers, therefore, we can see that for $n \neq m$ we have the sine function of an integer number $k = n - m$ times π, which is always zero for any integer number k. We can also realize that for $n = m$ we have an indetermination that needs to be solved using L'Hopital's rule. The application of this rule gives that for $n = m$ Equation (3.9) gives the result of $1/f_0$ which is the same as T_0. In summary, we get the following result for this set of orthogonal functions

$$\int_{-T_0/2}^{T_0/2} \varphi_n(t)\,\varphi_m^*(t)\,dt = \int_{-T_0/2}^{T_0/2} e^{j2\pi f_0 tn} e^{-j2\pi f_0 tn} dt = \begin{cases} 0, & n \neq m, \\ T_0, & n = m. \end{cases} \qquad (3.10)$$

Now, using the set of orthogonal functions $\{\varphi_n(t)\}_{n=-\infty}^{\infty} = \left\{e^{j2\pi f_0 tn}\right\}_{n=-\infty}^{\infty}$, and the results in (3.10) in the GFS of Equation (3.7), we get that the **Complex Exponential Fourier Series (CEFS)** of a periodic signal $f(t)$ is given by

$$\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} C_n \varphi_n(t) \\
&= \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_0 tn} \\
&= \sum_{n=-\infty}^{\infty} \frac{\langle f(t),\ \varphi_n(t)\rangle}{\|\ \varphi_n(t)\ \|^2} e^{j2\pi f_0 tn} \\
&= \sum_{n=-\infty}^{\infty} \frac{\langle f(t),\ e^{j2\pi f_0 tn}\rangle}{T_0} e^{j2\pi f_0 tn}, \qquad (3.11)
\end{aligned}$$

where the coefficients are given by

$$\begin{aligned} C_n &= \frac{\langle f(t),\ \varphi_n(t)\rangle}{\|\ \varphi_n(t)\ \|^2} \\ &= \frac{\int_{-T_0/2}^{T_0/2} f(t)\varphi_n^*(t)\,dt}{\int_{-T_0/2}^{T_0/2} |\varphi_n(t)|^2 dt} \\ &= \frac{\int_{-T_0/2}^{T_0/2} f(t)e^{-j2\pi f_0 tn} dt}{T_0} \\ &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)e^{-j2\pi f_0 tn} dt, \quad \forall n. \end{aligned} \tag{3.12}$$

Note that we can obtain from Equation (3.12) the definition of the zero coefficient that is given by

$$C_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)dt, \tag{3.13}$$

see that C_0 is exactly the same as the equation for the mean value of a periodic signal $f(t)$, i.e., $C_0 = \langle f(t)\rangle$, see Equation (2.40). Since the coefficient C_0 represents the component of signal $f(t)$ at harmonic zero, then it is the DC component of signal $f(t)$.

The conclusion is that if we have a periodic signal $f(t)$ with fundamental frequency f_0, and we want to obtain the CEFS representation of this signal, then we need to get the coefficients of the series and the series, i.e., we need to get the following

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)e^{-j2\pi n f_0 t} dt, \quad f(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}, \ \ n = 0, \pm 1, \pm 2, \ldots \tag{3.14}$$

Then one can obtain the frequency representation of $f(t)$ as a function of nf_0 as a discrete spectrum. In general since C_n are complex, one needs to obtain the magnitude spectrum $|C_n|$ as a function of nf_0. One can also obtain the magnitude square $|C_n|^2$ which is related to power of the signal.

3.2.1 Example of CEFS

Suppose that we want to calculate the CEFS of the rectangular pulse train signal $f(t)$ as that shown in Figure 3.1.

Then, first we must get the coefficients C_n for which we use the definition shown in Equa-

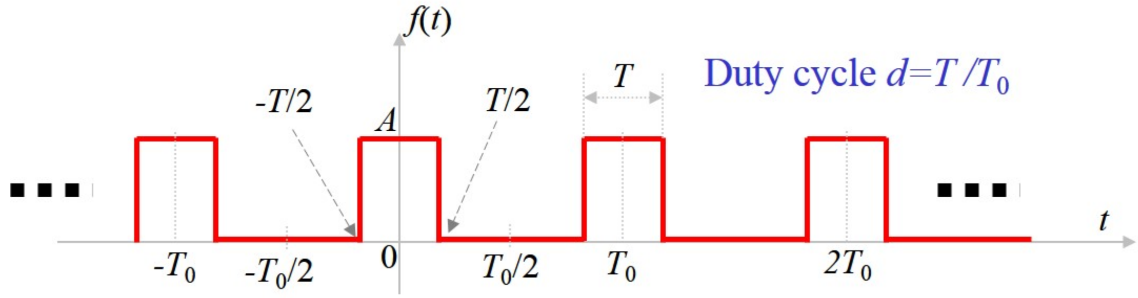

Figure 3.1: Periodic rectangular pulse train signal.

tion (3.12), and integrate to obtain the following

$$
\begin{aligned}
C_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)e^{-j2\pi n f_0 t}dt \\
&= \frac{1}{T_0}\int_{-T/2}^{T/2} Ae^{-j2\pi n f_0 t}dt \\
&= \frac{A}{T_0}\int_{-T/2}^{T/2} e^{-j2\pi n f_0 t}dt \\
&= \frac{A}{-j2\pi n f_0 T_0}e^{-j2\pi n f_0 t}\Bigg]_{-T/2}^{T/2} \\
&= \frac{A}{-j2\pi n f_0 T_0}\left(e^{-j2\pi n f_0 \frac{T}{2}} - e^{j2\pi n f_0 \frac{T}{2}}\right) \\
&= \frac{A}{\pi n f_0 T_0}\frac{1}{2j}\left(e^{j\pi n f_0 T} - e^{-j\pi n f_0 T}\right) \\
&= \frac{A}{\pi n f_0 T_0}\sin(\pi n f_0 T) \\
&= \frac{A}{\pi n f_0 T_0}\frac{T}{T}\sin(\pi n f_0 T) \\
&= \frac{AT}{T_0}\, sinc(n f_0 T) \\
&= Ad\, sinc(nd).
\end{aligned}
\tag{3.15}
$$

In the last row of Equation (3.15), we have used the definition of the duty cycle $d = T/T_0$ and the definition of the $sinc$ function.

With the coefficients C_n of the CEFS for the rectangular pulse train, one can show the discrete spectrum of the signal. Recall that since the signal is periodic, then its spectrum will be discrete. Also, since the signal has even symmetry, its spectrum is purely real.

Figure 3.2 shows the amplitude spectrum of a rectangular pulse train with amplitude $A = 2$ and duty cycle of $d = 1/5$. See that the figure shows the shape of the amplitude spectrum of a rectangular pulse train as a function of the integer numbers n. The figure does not show frequency, but any fundamental frequency f_0 can work with the figure since n is the integer factor multiplying the fundamental frequency f_0. In other words, the shape of the amplitude spectrum is the same for all rectangular pulse trains with any frequency as long as its duty cycle d and its amplitude A are the same. The horizontal axis can be changed from n to nf_0, once one wants to consider a fundamental frequency f_0. From the figure also see that since the duty cycle is $d = 1/5$, components 5, 10, 15, 20, etc. will be zero.

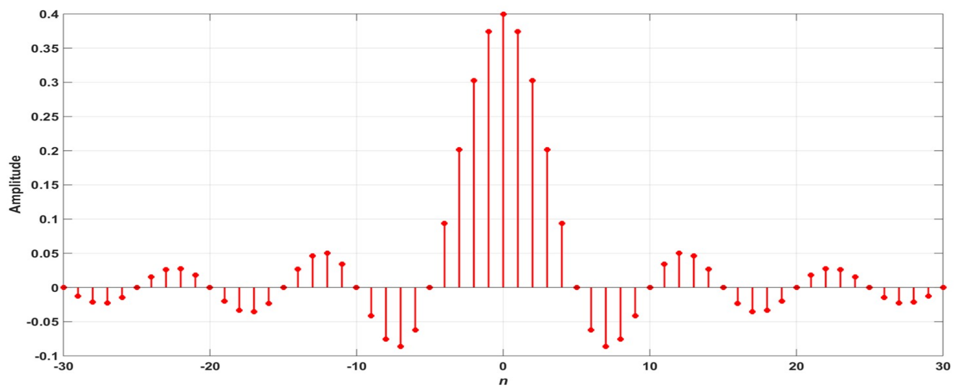

Figure 3.2: Amplitude spectrum (coefficients C_n) of rectangular pulse train with $A = 2$, $d = 1/5$.

Figure 3.3 shows the magnitude spectrum for the same rectangular pulse train considered for Figure 3.2. See that since we are showing magnitude, all components are non-negative. The basic frequency content of the signal is the same, i.e., frequency components at the same points of n appear in both figures.

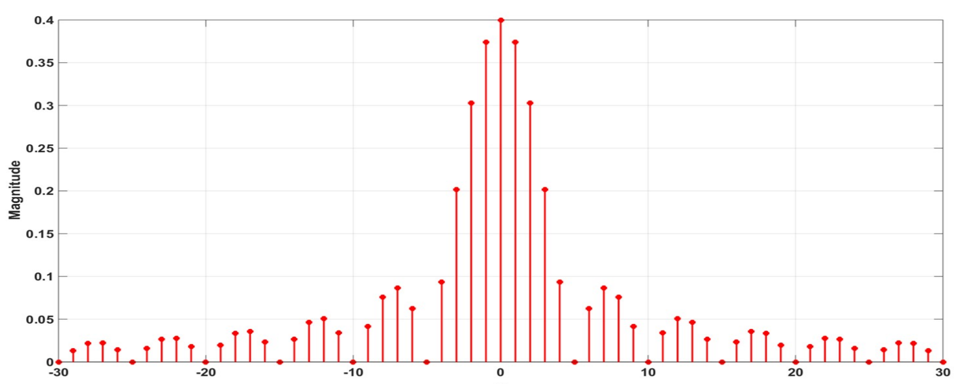

Figure 3.3: Magnitude spectrum ($|C_n|$) of rectangular pulse train with $A = 2$, $d = 1/5$.

Now that we have obtained the coefficients for the rectangular pulse train signal, we can express the signal as a linear combination of the complex exponential functions, i.e., we can show what the CEFS for the rectangular pulse train is. In order to do this, we substitute the expression in (3.15) for C_n in Equation (3.2) to get

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} C_n \varphi_n(t) \\ &= \sum_{n=-\infty}^{\infty} Ad\; sinc(nd) \varphi_n(t) \\ &= \sum_{n=-\infty}^{\infty} Ad\; sinc(nd) e^{j2\pi n f_0 t}. \end{aligned} \tag{3.16}$$

Now, as an illustration of what is of interest in the CEFS of the signal, look at some of the terms in Equation (3.16), say those terms corresponding to indexes $n = \pm 2, \pm 1$ and $n = 0$, and take a look into the superposition of such terms which is given by

$$C_{-2} e^{-j4\pi f_0 t} + C_{-1} e^{-j2\pi f_0 t} + C_0 + C_1 e^{j2\pi f_0 t} + C_2 e^{j4\pi f_0 t}. \tag{3.17}$$

Now, substituting the result obtained in (3.15) of the coefficients of the CEFS C_n, we get

$$\begin{aligned} Ad\; sinc(-2d) e^{-j4\pi f_0 t} &+ Ad\; sinc(-d) e^{-j2\pi f_0 t} + Ad\; sinc(0) \\ &+ Ad\; sinc(d) e^{j2\pi f_0 t} + Ad\; sinc(2d) e^{j4\pi f_0 t}. \end{aligned} \tag{3.18}$$

Now, see that for the first term with $Ad\; sinc(-2d)$ we have

$$Ad\; sinc(-2d) = Ad\, \frac{\sin(-2d\pi)}{-2d\pi} = Ad\, \frac{-\sin(2d\pi)}{-2d\pi} = Ad\, \frac{\sin(2d\pi)}{2d\pi} = Ad\; sinc(2d), \tag{3.19}$$

similarly for the term with $Ad\; sinc(-d)$ we have

$$Ad\; sinc(-d) = Ad\; sinc(d) \tag{3.20}$$

Therefore, in Equation (3.18) we end up with

$$\begin{aligned} Ad\; sinc(2d) e^{-j4\pi f_0 t} &+ Ad\; sinc(d) e^{-j2\pi f_0 t} + Ad\; sinc(0) \\ &+ Ad\; sinc(d) e^{j2\pi f_0 t} + Ad\; sinc(2d) e^{j4\pi f_0 t}, \end{aligned} \tag{3.21}$$

and we can associate terms to get the following expression from (3.21)

$$Ad\; sinc(2d) \left(e^{-j4\pi f_0 t} + e^{j4\pi f_0 t}\right) + 2Ad\; sinc(d) \left(e^{-j2\pi f_0 t} + e^{j2\pi f_0 t}\right) + Ad\; sinc(0). \tag{3.22}$$

Now, using trigonometric identity $2\cos(\alpha) = e^{\alpha} + e^{-\alpha}$ in (3.22) we get

$$2Ad\; sinc(2d)\cos(4\pi f_0 t) + Ad\; sinc(d)\cos(2\pi f_0 t) + Ad\; sinc(0). \tag{3.23}$$

The same procedure that we just introduced for those terms corresponding to indexes $n = \pm 2, \pm 1$ and $n = 0$ in the CEFS, can be extended for any number of terms. What is important here is that in general for this example, we will have that the CEFS of the rectangular pulse train given in Equation (3.16) is an infinite sum of cosine functions and the term $Ad\ sinc(0)$.

With this last statement, we can see that by adding several terms of the CEFS we will have an approximation to the true periodic rectangular pulse train, and if we want to represent the true signal, we would need to add an infinite number of cosines.

To see the effect of adding a number of those cosine functions obtained throughout this method in the CEFS of the rectangular pulse train given in Equation (3.16), we generate figures 3.4, 3.5 and 3.6. These figures are the superposition of 3, 11 and 201 terms of the CEFS, respectively. The big arrow in the figures indicates the direction of superposition of the terms. See how the rectangular pulse train signal is approximated as the number of terms of the CEFS increases.

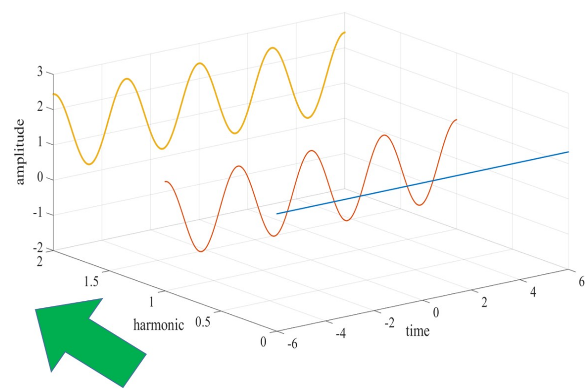

Figure 3.4: CEFS with three terms, C_{-1}, C_0, C_1.

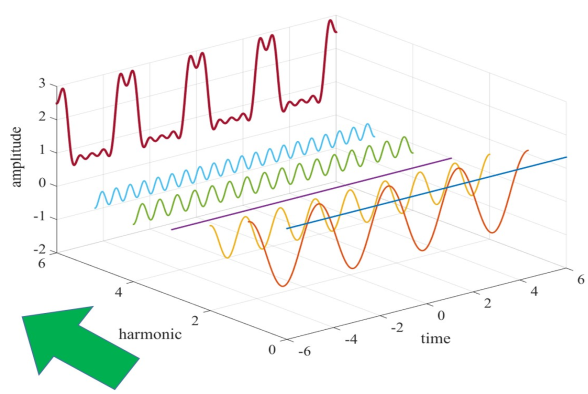

Figure 3.5: CEFS with eleven terms, $C_{-5}, \ldots, C_0, \ldots, C_5$.

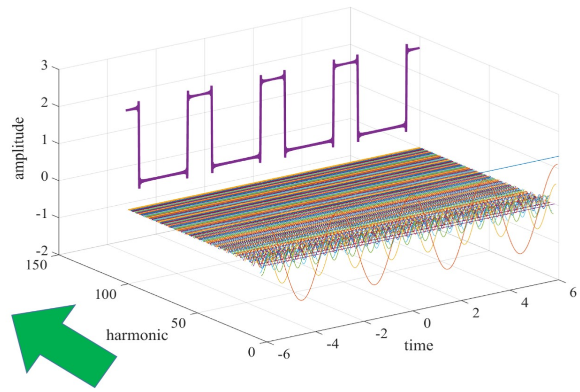

Figure 3.6: CEFS with 201 terms, $C_{-100}, \ldots, C_0, \ldots, C_{100}$.

3.3 Trigonometric Fourier Series (TFS)

Consider the definition of the CEFS for a periodic signal $s(t)$, with period T_0 or fundamental frequency f_0, given by

$$s(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_0 tn}, \tag{3.24}$$

With the definition of the CEFS in Equation (3.24), we can separate the summation into two parts to obtain an alternate expression

$$\begin{aligned} s(t) &= \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_0 tn} \\ &= \sum_{n=-\infty}^{-1} C_n e^{j2\pi f_0 tn} + C_0 + \sum_{n=1}^{\infty} C_n e^{j2\pi f_0 tn} \\ &= \sum_{n=1}^{\infty} C_{-n} e^{-j2\pi f_0 tn} + C_0 + \sum_{n=1}^{\infty} C_n e^{j2\pi f_0 tn} \\ &= C_0 + \sum_{n=1}^{\infty} \left\{ C_n e^{j2\pi f_0 tn} + C_{-n} e^{-j2\pi f_0 tn} \right\}. \end{aligned} \tag{3.25}$$

Now, in Equation (3.25) we can use the trigonometric identities of the complex exponentials in terms of the sine and cosine functions, i.e., the trigonometric identities $e^{j2\pi f_0 tn} =$

$\cos(2\pi f_0 tn) + j\sin(2\pi f_0 tn)$ and $e^{-j2\pi f_0 tn} = \cos(2\pi f_0 tn) - j\sin(2\pi f_0 tn)$, to obtain

$$\begin{aligned}
s(t) &= C_0 + \sum_{n=1}^{\infty} \left\{ C_n \left[\cos(2\pi f_0 tn) + j\sin(2\pi f_0 tn) \right] \right. \\
&\qquad \left. + C_{-n} \left[\cos(2\pi f_0 tn) - j\sin(2\pi f_0 tn) \right] \right\} \\
&= C_0 + \sum_{n=1}^{\infty} \left\{ \left[C_n + C_{-n} \right] \cos(2\pi f_0 tn) + j \left[C_n - C_{-n} \right] \sin(2\pi f_0 tn) \right\} \\
&= C_0 + \sum_{n=1}^{\infty} \left\{ A_n \cos(2\pi f_0 tn) + B_n \sin(2\pi f_0 tn) \right\}. \qquad (3.26)
\end{aligned}$$

The last row of Equation (3.26) is known as the **Trigonometric Fourier Series** representation of the periodic signal $s(t)$. In Equation (3.26), we use the following substitution

$$A_n = C_n + C_{-n}, \qquad B_n = j\left[C_n - C_{-n} \right]. \qquad (3.27)$$

Note that the coefficients A_n are associated to the real part of the CEFS, whereas the coefficients B_n are associated to the complex part of the CEFS. This also means that the coefficients A_n will be different from zero whenever the signal $s(t)$ has even symmetry or no symmetry. In contrast, the coefficients B_n will be different from zero whenever the signal $s(t)$ has odd symmetry o no symmetry. In other words, if the signal $s(t)$ has even symmetry, the coefficients B_n will be zero, and whenever $s(t)$ has odd symmetry, the coefficients A_n will be zero.

The coefficients A_n and B_n from Equation (3.27) can be obtained using the definition integral of the coefficients C_n of the CEFS, this definition is given in Equation (3.12). Once we use Equation (3.12) in Equation (3.27), we get for the coefficients A_n the following

$$\begin{aligned}
A_n &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t) e^{-j2\pi f_0 tn} dt + \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t) e^{j2\pi f_0 tn} dt \\
&= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t) \left\{ e^{-j2\pi f_0 tn} + e^{j2\pi f_0 tn} \right\} dt \\
&= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} s(t) \cos(2\pi f_0 tn) dt, \quad n = 1, 2, \ldots, \qquad (3.28)
\end{aligned}$$

and for the coefficients B_n in Equation(3.27) we obtain

$$
\begin{aligned}
B_n &= j\left[\frac{1}{T_0}\int\limits_{-T_0/2}^{T_0/2} s(t)e^{-j2\pi f_0 tn}dt - \frac{1}{T_0}\int\limits_{-T_0/2}^{T_0/2} s(t)e^{j2\pi f_0 tn}dt\right] \\
&= j\left[\frac{1}{T_0}\int\limits_{-T_0/2}^{T_0/2} s(t)\left\{e^{-j2\pi f_0 tn} - e^{j2\pi f_0 tn}\right\}dt\right] \\
&= -j\left[\frac{1}{T_0}\int\limits_{-T_0/2}^{T_0/2} s(t)\left\{-e^{-j2\pi f_0 tn} + e^{j2\pi f_0 tn}\right\}dt\right] \\
&= -j\left[\frac{2}{2T_0}\int\limits_{-T_0/2}^{T_0/2} s(t)\left\{e^{j2\pi f_0 tn} - e^{-j2\pi f_0 tn}\right\}dt\right] \\
&= \left[\frac{2}{j2T_0}\int\limits_{-T_0/2}^{T_0/2} s(t)\left\{e^{j2\pi f_0 tn} - e^{-j2\pi f_0 tn}\right\}dt\right] \\
&= \frac{2}{T_0}\int\limits_{-T_0/2}^{T_0/2} s(t)\sin(2\pi f_0 tn)dt, \quad n = 1, 2, \ldots, \qquad (3.29)
\end{aligned}
$$

3.4 Examples of Fourier Series

In this section the triangular waveform and a shifted version of it is analyzed using the CEFS.

3.4.1 Shifted Triangular Waveform

Consider the triangular waveform of Figure 3.7, and solve the following:

1. **How would you classify the signal?** The signal is continuous, with odd symmetry, periodic with period $T_0 = \pi$, and deterministic. Since it is periodic, it exists for all times, i.e., $t \in \Re$, which makes it a *power* signal because it has infinite duration.

2. **Find the Complex Exponential Fourier Series (CEFS) representation of the waveform in Figure 3.7. Provide the general expression for the coefficients C_n.**

   (a) First, define function $f(t)$. To do that, you need to see that it is made out of two straight lines, one with positive slope with value of $\frac{2}{\pi/2} = \frac{4}{\pi}$ crossing through the origin (y-intercept is zero), and another line with the "same" slope value but

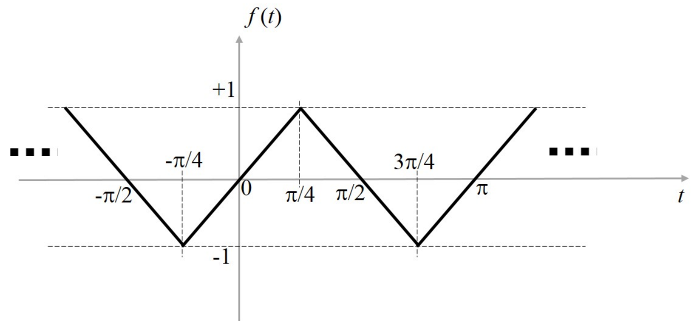

Figure 3.7: Triangular Waveform.

negative $-\frac{4}{\pi}$ and crossing the vertical axis at 2 (y-intercept is two, you need to see that this is true!). Thus, the function that defines the signal is given by

$$f(t) = \begin{cases} \frac{4}{\pi}t, & -\frac{\pi}{4} < t < \frac{\pi}{4}, \\ 2 - \frac{4}{\pi}t, & \frac{\pi}{4} \le t \le \frac{3\pi}{4}. \end{cases} \tag{3.30}$$

(b) The second step is to obtain the information from the signal such as period and fundamental frequency, which in this problem we have from Figure 3.7 the following

$$T_0 = \pi, \quad f_0 = \frac{1}{T_0} = \frac{1}{\pi}, \text{ then we have } 2\pi f_0 n = 2n. \tag{3.31}$$

(c) The third step is to obtain the complex exponential Fourier series coefficients, i.e., calculate C_n. To do this, we have to consider the definition of these coefficients and work the integrals from there on substituting the functions obtained in the first step and shown in Equation (3.30). So we start the process from the definition of the coefficients and keep working on it using Equation (3.30) and Equation (3.31) as follows.

$$\begin{aligned} C_n &= \frac{1}{T_0}\int_{T_0} f(t)e^{-j2\pi f_0 nt}\,dt \\ &= \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4}{\pi}te^{-j2nt}\,dt + \frac{1}{\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2e^{-j2nt}\,dt - \frac{1}{\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{4}{\pi}te^{-j2nt}\,dt \end{aligned} \tag{3.32}$$

Now, we work with Equation (3.32) by solving the first and the third integrals (we solve the integrals by parts or we can use a table of integrals which is what we have here). From tables of integrals, we have

$$\int te^{at}\,dt = \left(\frac{t}{a} - \frac{1}{a^2}\right)e^{at}. \tag{3.33}$$

For the specific case of the integrals to solve in Equation (3.32), we have that the parameter a in Equation (3.33) must take the value $a = -j2n$. Then using this formula, we solve the integrals in (3.32) to obtain the following

$$
\begin{aligned}
C_n &= \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4}{\pi} t e^{-j2nt}\, dt + \frac{1}{\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2e^{-j2nt}\, dt - \frac{1}{\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{4}{\pi} t e^{-j2nt}\, dt \\
&= \frac{4}{\pi^2}\left(\frac{t}{-j2n} - \frac{1}{(-j2n)^2}\right) e^{-j2nt}\Bigg]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&+ \frac{2}{\pi}\left(\frac{1}{-j2n}\right) e^{-j2nt}\Bigg]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\
&- \frac{4}{\pi^2}\left(\frac{t}{-j2n} - \frac{1}{(-j2n)^2}\right) e^{-j2nt}\Bigg]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}. \qquad (3.34)
\end{aligned}
$$

We can see that Equation (3.34) can be written as follows

$$
\begin{aligned}
C_n &= \frac{4}{\pi^2}\left(-\frac{t}{j2n} + \frac{1}{4n^2}\right) e^{-j2nt}\Bigg]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&+ \frac{2}{\pi}\left(\frac{1}{-j2n}\right) e^{-j2nt}\Bigg]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\
&- \frac{4}{\pi^2}\left(-\frac{t}{j2n} + \frac{1}{4n^2}\right) e^{-j2nt}\Bigg]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}. \qquad (3.35)
\end{aligned}
$$

Now, we substitute the initial and final values of each of the three terms in Equation (3.35) to get

$$
\begin{aligned}
C_n &= \frac{4}{\pi^2}\left[\left(-\frac{\pi}{j8n} + \frac{1}{4n^2}\right) e^{-jn\frac{\pi}{2}} - \left(\frac{\pi}{j8n} + \frac{1}{4n^2}\right) e^{jn\frac{\pi}{2}}\right] \\
&- \frac{1}{jn\pi} e^{-jn\frac{3\pi}{2}} + \frac{1}{jn\pi} e^{-jn\frac{\pi}{2}} \\
&- \frac{4}{\pi^2}\left[\left(-\frac{3\pi}{j8n} + \frac{1}{4n^2}\right) e^{-jn\frac{3\pi}{2}} - \left(-\frac{\pi}{j8n} + \frac{1}{4n^2}\right) e^{-jn\frac{\pi}{2}}\right]. \qquad (3.36)
\end{aligned}
$$

Now expand each of the terms in brackets shown in Equation (3.36) to obtain the following

$$
\begin{aligned}
C_n &= \frac{4}{\pi^2}\left[-\frac{\pi}{j8n} e^{-jn\frac{\pi}{2}} + \frac{1}{4n^2} e^{-jn\frac{\pi}{2}} - \frac{\pi}{j8n} e^{jn\frac{\pi}{2}} - \frac{1}{4n^2} e^{jn\frac{\pi}{2}}\right] \\
&- \frac{1}{jn\pi} e^{-jn\frac{3\pi}{2}} + \frac{1}{jn\pi} e^{-jn\frac{\pi}{2}} \\
&- \frac{4}{\pi^2}\left[-\frac{3\pi}{j8n} e^{-jn\frac{3\pi}{2}} + \frac{1}{4n^2} e^{-jn\frac{3\pi}{2}} + \frac{\pi}{j8n} e^{-jn\frac{\pi}{2}} - \frac{1}{4n^2} e^{-jn\frac{\pi}{2}}\right] \qquad (3.37)
\end{aligned}
$$

Now, we keep working on Equation (3.37) to expand each of the terms by multiplying the constants $4/\pi^2$ that are outside the brackets with each of the terms inside the brackets to obtain the following.

$$\begin{aligned} C_n &= -\frac{1}{j2n\pi}e^{-jn\frac{\pi}{2}} + \frac{1}{n^2\pi^2}e^{-jn\frac{\pi}{2}} - \frac{1}{j2n\pi}e^{jn\frac{\pi}{2}} - \frac{1}{n^2\pi^2}e^{jn\frac{\pi}{2}} \\ &\quad -\frac{1}{jn\pi}e^{-jn\frac{3\pi}{2}} + \frac{1}{jn\pi}e^{-jn\frac{\pi}{2}} \\ &\quad +\frac{3}{j2n\pi}e^{-jn\frac{3\pi}{2}} - \frac{1}{n^2\pi^2}e^{-jn\frac{3\pi}{2}} - \frac{1}{j2n\pi}e^{-jn\frac{\pi}{2}} + \frac{1}{n^2\pi^2}e^{-jn\frac{\pi}{2}}. \end{aligned} \tag{3.38}$$

From Equation (3.38), we can see that the first term and the ninth term add up and eliminate the sixth term. Then, see that the second term and the tenth term can be put together. Also, we can put together the fifth and seventh terms to obtain

$$\begin{aligned} C_n &= \frac{2}{n^2\pi^2}e^{-jn\frac{\pi}{2}} - \frac{1}{j2n\pi}e^{jn\frac{\pi}{2}} - \frac{1}{n^2\pi^2}e^{jn\frac{\pi}{2}} \\ &\quad +\frac{1}{jn\pi}e^{-jn\frac{3\pi}{2}}\left(\frac{3}{2}-1\right) - \frac{1}{n^2\pi^2}e^{-jn\frac{3\pi}{2}} \\ &= \frac{2}{n^2\pi^2}e^{-jn\frac{\pi}{2}} - \frac{1}{j2n\pi}e^{jn\frac{\pi}{2}} - \frac{1}{n^2\pi^2}e^{jn\frac{\pi}{2}} \\ &\quad +\frac{1}{j2n\pi}e^{-jn\frac{3\pi}{2}} - \frac{1}{n^2\pi^2}e^{-jn\frac{3\pi}{2}}. \end{aligned} \tag{3.39}$$

Now, in Equation (3.39), we can put together the first, third and fifth terms. We can do the same with the second and fourth terms, then we obtain

$$\begin{aligned} C_n &= \frac{1}{n^2\pi^2}e^{-jn\frac{\pi}{2}}\left(2 - e^{jn\pi} - e^{-jn\pi}\right) \\ &\quad +\frac{1}{j2n\pi}e^{jn\frac{\pi}{2}}\left(e^{-j2n\pi} - 1\right). \end{aligned} \tag{3.40}$$

Now, recall that we have the following identities

$$\begin{aligned} e^{jn\pi} &= \cos(n\pi) + j\sin(n\pi) = \cos(n\pi) = (-1)^n \\ e^{-jn\pi} &= \cos(n\pi) - j\sin(n\pi) = \cos(n\pi) = (-1)^n \\ e^{-j2n\pi} &= \cos(2n\pi) - j\sin(2n\pi) = \cos(2n\pi) = 1. \end{aligned} \tag{3.41}$$

Now we use the identities in Equation (3.41) and substitute them into Equation

(3.40) to obtain

$$\begin{aligned} C_n &= \frac{1}{n^2\pi^2} e^{-jn\frac{\pi}{2}} \left(2 - (-1)^n - (-1)^n\right) \\ &\quad + \frac{1}{j2n\pi} e^{jn\frac{\pi}{2}} \left(1 - 1\right) \\ &= \frac{1}{n^2\pi^2} e^{-jn\frac{\pi}{2}} \left(2 - 2(-1)^n\right) \\ &= \frac{2}{n^2\pi^2} e^{-jn\frac{\pi}{2}} \left(1 - (-1)^n\right) \\ &= \frac{2}{n^2\pi^2} \left[1 - (-1)^n\right] e^{-jn\frac{\pi}{2}}. \end{aligned} \tag{3.42}$$

See that when $n = 0$ in Equation (3.42), we have an indetermination, i.e., $\frac{0}{0}$, hence we need to treat the coefficient C_0 separately. So, Equation (3.42) is only valid for $n = \pm 1, \pm 2, \ldots$. **Verify the following result!!!**.

$$\begin{aligned} C_0 &= \frac{1}{T_0} \int_{T_0} f(t)\, dt \\ &= \frac{1}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4}{\pi} t\, dt + \frac{1}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2\, dt + \frac{1}{\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{4}{\pi} t\, dt = 0. \end{aligned} \tag{3.43}$$

Now, with the result in equations (3.42) and (3.43), we can write the CEFS of the signal in Equation (3.30), shown in Figure 3.7. This CEFS is given by

$$\begin{aligned} f(t) &= C_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n e^{j2\pi f_0 n t} \\ &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{2}{n^2\pi^2} \left[1 - (-1)^n\right] e^{-jn\frac{\pi}{2}} e^{j2\pi f_0 n t}. \end{aligned} \tag{3.44}$$

3. **Provide the expression for the magnitude** $|C_n|$ **and the phase** $\angle C_n$. The magnitude of the coefficients C_n, i.e., $|C_n|$, gives what is called the **magnitude spectrum**, i.e., the information of the frequency content of the signal. For $n = 0$, we have that C_0 gives the **DC component** or mean value of the signal because from the definition of the coefficients C_n in Equation (3.12), we have that substituting $n = 0$ gives the expression for the mean value of a signal (since $C_0 = \frac{1}{T_0} \int_{T_0} f(t)\, dt$, which is the same as the time average of *mean value* of the signal). The DC component of the signal will always be related to the value of the signal given at zero frequency. The remaining components (coefficients C_n for $n \neq 0$) will provide the frequency content at points nf_0, i.e., integer multiples of the **fundamental frequency** f_0.

(a) **Magnitude**. The coefficients C_n are given by the last row of Equation (3.42), and their magnitude is given by

$$\begin{aligned} |C_n| &= \left| \frac{2}{n^2\pi^2} \left[1 - (-1)^n\right] e^{-jn\frac{\pi}{2}} \right| \\ &= \left| \frac{2}{n^2\pi^2} \left[1 - (-1)^n\right] \right| \left| e^{-jn\frac{\pi}{2}} \right| \\ &= \left| \frac{2}{n^2\pi^2} \left[1 - (-1)^n\right] \right| \end{aligned} \tag{3.45}$$

(b) **Phase**. The phase of the coefficients is obtained by considering the last row of Equation (3.42) together with the identity: $e^{-jn\frac{\pi}{2}} = \cos(n\pi/2) - j\sin(n\pi/2)$. In this identity, see that whenever n is even or zero, i.e., $n = 0, \pm 2, \pm 4, \ldots$ the cosine function takes a value of $+1$ or -1 and the sine function is always zero, but see that when n is even, the whole coefficient C_n is zero because $1 - (-1)^n = 0$. On the other hand, see that when n is odd, i.e., $n = \pm 1, \pm 3, \ldots$, the cosine function takes the value zero and the sine function takes the value of $+1$ or -1, and at the same time we have $1 - (-1)^n = 2$, hence making C_n a purely complex coefficient for n odd, and zero for n even. Therefore, we have that the coefficients C_n are purely complex (**you should also see that this result is consistent with the symmetry of the signal, which is odd symmetry**) and that take a positive and negative values depending on the positive and negative values of n. In conclusion, the phase will be $\pi/2$ for $n > 0$ and odd, and the phase will be $-\pi/2$ for $n < 0$ and odd.

4. **Plot the discrete amplitude spectrum**. You should obtain the plots using Matlab. The plot for the magnitude is shown in Figure 3.8.

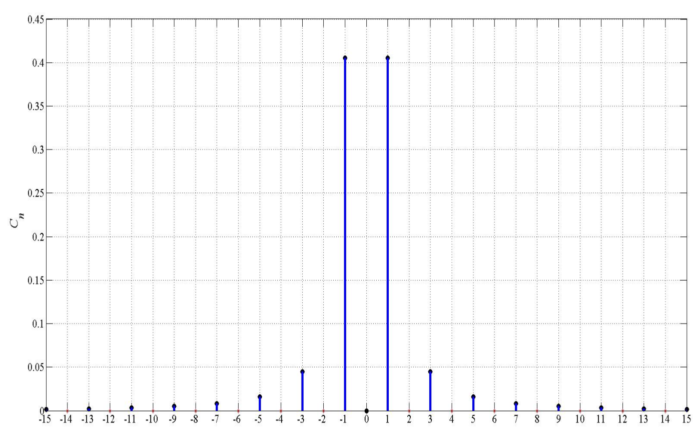

Figure 3.8: Discrete Magnitude spectrum of coefficients C_n.

3.4.2 Triangular Waveform

Answer the same questions as in previous example, but for the waveform shown in Figure 3.9. Comment the differences that the coefficients C_n have between the two representations.

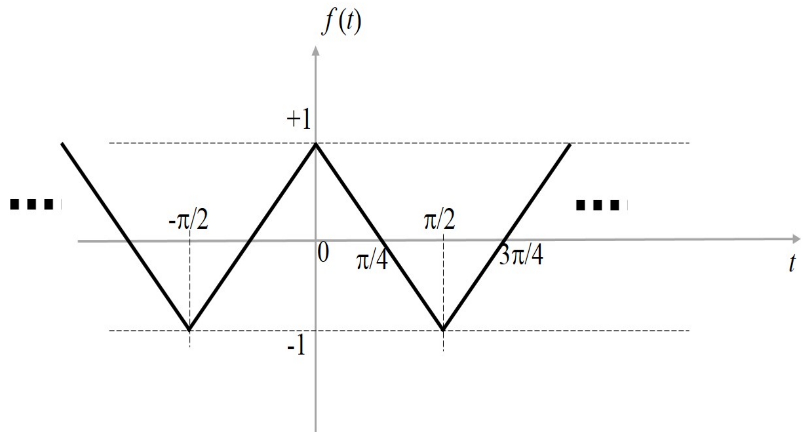

Figure 3.9: Shifted Triangular Waveform.

1. We follow the same steps as those for the first example. Just for this time, consider that the function given in Figure 3.9 is called $g(t)$. Then we have that the function is given by

$$g(t) = \begin{cases} \frac{4}{\pi}t + 1, & -\frac{\pi}{2} < t < 0, \\ 1 - \frac{4}{\pi}t, & 0 \leq t \leq \frac{\pi}{2}. \end{cases} \tag{3.46}$$

2. See that the function $g(t)$ in Equation (3.46) can be obtained by substituting in Equation (3.30) t by $t + \frac{\pi}{4}$. The second step is to obtain the information from the signal such as period and fundamental frequency, which in this problem we have the same information as that for Problem 1, in other words,

$$T_0 = \pi, \quad f_0 = \frac{1}{T_0} = \frac{1}{\pi}, \text{ then we have } 2\pi f_0 n = 2n. \tag{3.47}$$

3. The third step is to obtain the complex exponential Fourier series coefficients, i.e., calculate C_n. We start the process from the definition of the coefficients and keep working

on it using Equation (3.46) and Equation (3.47) as follows.

$$\begin{aligned}
C_n &= \frac{1}{T_0}\int_{T_0} f(t)e^{-j2\pi f_0 nt}\,dt \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{0}\left(\frac{4}{\pi}t+1\right)e^{-j2nt}\,dt + \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}}\left(1-\frac{4}{\pi}t\right)e^{-j2nt}\,dt \\
&= \frac{4}{\pi^2}\int_{-\frac{\pi}{2}}^{0} te^{-j2nt}\,dt + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{0} e^{-j2nt}\,dt \\
&\quad + \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} e^{-j2nt}\,dt - \frac{4}{\pi^2}\int_{0}^{\frac{\pi}{2}} te^{-j2nt}\,dt. \qquad (3.48)
\end{aligned}$$

Now, we work with Equation (3.48) by solving the first and fourth integrals using Equation (3.55). **You must complete all the steps to get what it is shown here**. Then we have (after solving and substituting the initial and final values)

$$\begin{aligned}
C_n &= \frac{4}{\pi^2}\left[\frac{1}{4n^2} - \frac{\pi}{j4n}e^{jn\pi} - \frac{1}{4n^2}e^{jn\pi}\right] \\
&\quad - \frac{1}{j2n\pi} + \frac{1}{j2n\pi}e^{jn\pi} - \frac{1}{j2n\pi}e^{-jn\pi} + \frac{1}{j2n\pi} \\
&\quad - \frac{4}{\pi^2}\left[-\frac{1}{4n^2} - \frac{\pi}{j4n}e^{-jn\pi} + \frac{1}{4n^2}e^{-jn\pi}\right]. \qquad (3.49)
\end{aligned}$$

By regrouping and eliminating terms in Equation (3.49) we obtain

$$\begin{aligned}
C_n &= \frac{2}{n^2\pi^2} - \frac{1}{n^2\pi^2}\left(e^{jn\pi}+e^{-jn\pi}\right) + \frac{1}{jn\pi}e^{jn\pi}\left(\frac{1}{2}-1\right) \\
&\quad + \frac{1}{jn\pi}e^{-jn\pi}\left(1-\frac{1}{2}\right) \\
&= \frac{2}{n^2\pi^2} - \frac{2}{n^2\pi^2}\cos(n\pi) - \frac{1}{j2n\pi}e^{jn\pi} + \frac{1}{j2n\pi}e^{-jn\pi} \\
&= \frac{2}{n^2\pi^2} - \frac{2}{n^2\pi^2}\cos(n\pi) - \frac{1}{n\pi}\frac{1}{2j}\left(e^{jn\pi}-e^{-jn\pi}\right) \\
&= \frac{2}{n^2\pi^2} - \frac{2}{n^2\pi^2}\cos(n\pi) - \frac{1}{n\pi}\sin(n\pi) \\
&= \frac{2}{n^2\pi^2}\left(1-\cos(n\pi)\right) \\
&= \frac{2}{n^2\pi^2}\left[1-(-1)^n\right]. \qquad (3.50)
\end{aligned}$$

Note that this result gives coefficients C_n that are purely real because the signal has even symmetry. With the result in Equation (3.50) just obtained, you can see the similarity to Equation (3.42) where you can find the extra factor $e^{-jn\frac{\pi}{2}}$. So you can see that with respect to the previous example, we have in this example, a time shift of $g(t)$ by $\pi/4$ time units to the right to obtain $f(t)$ (see functions in figures 3.7 and 3.9), in other words, $f(t) = g(t - \frac{\pi}{4})$. Whenever you have a time shift of k time units to the right of a function, the coefficients C_n will be affected by the multiplication of a factor

$$e^{-j2\pi f_0 nk} \tag{3.51}$$

In Example 1 we have a signal $f(t)$ with a time shift of $k = \pi/4$ time units to the right of the signal of problem 2, then (recall that in this problem $f_0 = 1/\pi$) we have

$$e^{-j2\pi f_0 nk} = e^{-j2\pi \frac{1}{\pi} n \frac{\pi}{4}} = e^{-jn\frac{\pi}{2}}. \tag{3.52}$$

Therefore, the factor obtained in Equation (3.52) is multiplied by the coefficient C_n of Equation (3.50) to get the coefficients of problem 1.

3.5 CEFS of Cosine Functions

In this section, a cosine function is analyzed using CEFS, the purpose is to conclude that in order to obtain the CEFS of cosine functions, we only need to use trigonometric identities.

3.5.1 Three Different Methods

Consider the periodic function $f(t)$ with period T_0 given by

$$f(t) = \cos(2\pi f_0 t). \tag{3.53}$$

We are interested in obtaining the CEFS representation of the signal. The CEFS of the signal in (3.53) can be obtained by three methods. We show those three methods in the following, beginning with the most elaborate.

1. **Method 1: Using the cosine function.**

   In order to obtain the CEFS of function $f(t)$ in Equation (3.53), first determine its period, which in this case is T_0. Then calculate the CEFS coefficients C_n which will be given by

$$\begin{aligned} C_n &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{-j2\pi f_0 nt} dt \\ &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \cos(2\pi f_0 t) e^{-j2\pi f_0 nt} dt. \end{aligned} \tag{3.54}$$

   As one can see, Equation (3.54) is an integral that has the following form

$$\int e^{\alpha\tau} \cos(\beta\tau) d\tau = \frac{1}{\alpha^2 + \beta^2} e^{\alpha\tau} \left[\alpha \cos(\beta\tau) + \beta \sin(\beta\tau)\right]. \tag{3.55}$$

To use the solution of the integral, we just see that we have for our example the following

$$\begin{aligned} \alpha &= -j2\pi f_0 n, \quad \alpha^2 = -4\pi^2 f_0^2 n^2, \\ \beta &= 2\pi f_0, \quad\ \ \beta^2 = 4\pi^2 f_0^2. \end{aligned} \tag{3.56}$$

Using the parameters defined in (3.56), we get

$$\begin{aligned} C_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \cos(2\pi f_0 t) e^{-j2\pi f_0 n t} dt \\ &= \frac{e^{-j2\pi f_0 n t}}{4\pi^2 f_0^2(1-n^2)T_0}\left\{-j2\pi f_0 n\cos(2\pi f_0 t) + 2\pi f_0 \sin(2\pi f_0 t)\right\}\Big]_{-T_0/2}^{T_0/2} \\ &= \frac{e^{-j2\pi f_0 n t}}{2\pi f_0(1-n^2)T_0}\left\{\sin(2\pi f_0 t) - jn\cos(2\pi f_0 t)\right\}\Big]_{-T_0/2}^{T_0/2} \\ &= \frac{e^{-j2\pi f_0 n t}}{2\pi(1-n^2)}\left\{\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j} - jn\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right\}\Bigg]_{-T_0/2}^{T_0/2} \\ &= \frac{e^{-j2\pi f_0 n t}}{2\pi(1-n^2)}\left\{\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j} + n\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2j}\right\}\Bigg]_{-T_0/2}^{T_0/2} \\ &= \frac{e^{-j2\pi f_0 n t}}{j4\pi(1-n^2)}\left\{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t} + ne^{j2\pi f_0 t} + ne^{-j2\pi f_0 t}\right\}\Big]_{-T_0/2}^{T_0/2} \\ &= \frac{1}{j4\pi(1-n^2)}\left\{(1+n)e^{j2\pi f_0 t(1-n)} - (1-n)e^{-j2\pi f_0 t(1+n)}\right\}\Big]_{-T_0/2}^{T_0/2} \\ &= \frac{1}{j4\pi}\left\{\frac{e^{j2\pi f_0 t(1-n)}}{1-n} - \frac{e^{-j2\pi f_0 t(1+n)}}{1+n}\right\}\Bigg]_{-T_0/2}^{T_0/2} \\ &= \frac{1}{j4\pi}\left\{\frac{e^{j2\pi f_0 \frac{T_0}{2}(1-n)}}{1-n} - \frac{e^{-j2\pi f_0 \frac{T_0}{2}(1-n)}}{1-n} - \frac{e^{-j2\pi f_0 \frac{T_0}{2}(1+n)}}{1+n} + \frac{e^{j2\pi f_0 \frac{T_0}{2}(1+n)}}{1+n}\right\} \\ &= \frac{1}{j4\pi}\left\{\frac{e^{j\pi(1-n)}}{1-n} - \frac{e^{-j\pi(1-n)}}{1-n} - \frac{e^{-j\pi(1+n)}}{1+n} + \frac{e^{j\pi(1+n)}}{1+n}\right\} \\ &= \frac{1}{2j\ 2\pi}\left\{\frac{e^{j\pi(1-n)} - e^{-j\pi(1-n)}}{1-n} + \frac{e^{j\pi(1+n)} - e^{-j\pi(1+n)}}{1+n}\right\} \\ &= \frac{1}{2\pi}\left\{\frac{\sin(\pi(1-n))}{1-n} + \frac{\sin(\pi(1+n))}{1+n}\right\}. \end{aligned} \tag{3.57}$$

Now, see that there are two critical values of n in Equation (3.57) which are $n = +1$ and $n = -1$. These are critical values because they make the denominator zero in one of the terms in the equation. For $n = +1$ we obtain C_1 using L'Hopital's rule you must show that $C_1 = 1/2$. In a similar way, see that $C_{-1} = 1/2$. **Show** that $C_n = 0$ for $n \neq \pm 1$. Therefore we have

$$C_0 = 0,\;\; C_1 = \frac{1}{2},\;\; C_{-1} = \frac{1}{2},\;\; C_m = 0,\;\; m = \pm 2, \pm 3, \ldots. \tag{3.58}$$

The cosine signal has only two coefficients, namely C_1 and C_{-1} both with value of 1/2. Therefore, the CEFS of the function $f(t) = \cos(2\pi f_0 t)$ is given by

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_0 t n} \\ &= C_1 e^{j2\pi f_0 t} + C_{-1} e^{-j2\pi f_0 t} \\ &= \frac{1}{2} e^{j2\pi f_0 t} + \frac{1}{2} e^{-j2\pi f_0 t} \\ &= \cos(2\pi f_0 t). \end{aligned} \tag{3.59}$$

In conclusion, **we only need to use the trigonometric identity of the cosine function**.

2. **Method 2: Using exponentials instead of cosine function.** In this part, we also compute the CEFS coefficients, but instead of using the integral in Equation (3.55), we first substitute the trigonometric identity of the cosine to solve for the coefficients C_n as follows

$$\begin{aligned} C_n &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \cos(2\pi f_0 t) e^{-j2\pi f_0 n t} dt \\ &= \frac{1}{2T_0} \int_{-T_0/2}^{T_0/2} \left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t} \right] e^{-j2\pi f_0 n t} dt \\ &= \frac{1}{2T_0} \int_{-T_0/2}^{T_0/2} e^{j2\pi f_0 t} e^{-j2\pi f_0 n t} dt + \frac{1}{2T_0} \int_{-T_0/2}^{T_0/2} e^{-j2\pi f_0 t} e^{-j2\pi f_0 n t} dt \\ &= \frac{1}{2T_0} \int_{-T_0/2}^{T_0/2} e^{j2\pi f_0 t(1-n)} dt + \frac{1}{2T_0} \int_{-T_0/2}^{T_0/2} e^{-j2\pi f_0 t(1+n)} dt \\ &= \left. \frac{1}{j4\pi(1-n)} e^{j2\pi f_0 t(1-n)} \right]_{-T_0/2}^{T_0/2} - \left. \frac{1}{j4\pi(1+n)} e^{-j2\pi f_0 t(1+n)} \right]_{-T_0/2}^{T_0/2} \\ &= \left. \frac{1}{j4\pi} \left\{ \frac{e^{j2\pi f_0 t(1-n)}}{1-n} - \frac{e^{-j2\pi f_0 t(1+n)}}{1+n} \right\} \right]_{-T_0/2}^{T_0/2}. \end{aligned} \tag{3.60}$$

The last row of Equation (3.60) is exactly the same as the eighth row in Equation (3.57), hence the same procedure and result as that will be obtained.

3. **Method 3: Using the trigonometric identity of the cosine function.**

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_0 tn} \\ &= C_1 e^{j2\pi f_0 t} + C_{-1} e^{-j2\pi f_0 t} \\ &= \frac{1}{2} e^{j2\pi f_0 t} + \frac{1}{2} e^{-j2\pi f_0 t} \\ &= \cos(2\pi f_0 t). \end{aligned} \tag{3.61}$$

3.5.2 A Combination of Cosine Functions

Obtain the CEFS representation of the following signal

$$g(t) = \cos^2(2\pi 200 t) - \sin(2\pi 400 t). \tag{3.62}$$

For this problem, recall to use the trigonometric identities for the cosine and sine functions. First use the trigonometric identity $\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$ to get

$$g(t) = \frac{1}{2} + \frac{1}{2}\cos(2\pi 400 t) - \sin(2\pi 400 t). \tag{3.63}$$

We can see in Equation (3.63) that the fundamental frequency of the signal is $f_0 = 400$ Hz, then its period would be $T_0 = 1/400$ sec.

Now you can apply the trigonometric identities of the sine and cosine functions in terms of the exponentials in Equation (3.63) to obtain the CEFS.

$$\begin{aligned} g(t) &= \frac{1}{2} + \frac{1}{4}\left(e^{j2\pi 400 t} + e^{-j2\pi 400 t}\right) - \frac{1}{2j}\left(e^{j2\pi 400 t} - e^{-j2\pi 400 t}\right) \\ &= \frac{1}{2} + \frac{1}{4} e^{j2\pi 400 t} + \frac{1}{4} e^{-j2\pi 400 t} - \frac{1}{2j} e^{j2\pi 400 t} + \frac{1}{2j} e^{-j2\pi 400 t} \\ &= \frac{1}{2} + \frac{1}{4} e^{j2\pi 400 t} + \frac{1}{4} e^{-j2\pi 400 t} + j\frac{1}{2} e^{j2\pi 400 t} - j\frac{1}{2} e^{-j2\pi 400 t}. \end{aligned} \tag{3.64}$$

Then the coefficients of this CEFS are

$$C_0 = \frac{1}{2},\ \ C_1 = \frac{1}{4} + j\frac{1}{2},\ \ C_{-1} = \frac{1}{4} - j\frac{1}{2}. \tag{3.65}$$

Chapter 4

Fourier Transform

The Fourier Transform (FT) is a mathematical tool that allows us to analyze signals in the frequency domain. The FT is applied in general for the analysis of *aperiodic* signals that could be of infinite (power signals) or finite duration (energy signals). However, there is a way to apply the Fourier transform to periodic signals, this is explained in Section 4.4. Consider an aperiodic signal $x(t)$, its Fourier transform is defined to be

$$\begin{aligned} X(f) &= \mathcal{F}\left\{x(t)\right\} \\ &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt. \end{aligned} \tag{4.1}$$

If we have information about a Fourier transform of a signal $X(f)$, i.e., the signal in the frequency domain, the inverse Fourier Transform of $X(f)$ results in a time function $x(t)$, and this inverse Fourier transform is defined as

$$\begin{aligned} x(t) &= \mathcal{F}^{-1}\left\{X(f)\right\} \\ &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df. \end{aligned} \tag{4.2}$$

The Fourier transform has several properties that could be used to obtain the transform of signals. In this document, you can find some of the properties and applications of them as needed throughout the examples. One of the properties is the **time-shift** property. Consider a signal $s(t)$ with Fourier transform $S(f)$, now if we need to obtain the Fourier transform of the shifted version $s(t - t_0)$, we can define first the following signal $x(t) = s(t - t_0)$ and apply to it the FT definition in Equation (4.1) to obtain

$$\begin{aligned} X(f) &= \mathcal{F}\{x(t)\} \\ &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} s(t-t_0)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} s(z)e^{-j2\pi f(z+t_0)}dz \\ &= \int_{-\infty}^{\infty} s(z)e^{-j2\pi fz}e^{-j2\pi ft_0}dz \\ &= e^{-j2\pi ft_0}\int_{-\infty}^{\infty} s(z)e^{-j2\pi fz}dz \\ &= e^{-j2\pi ft_0}S(f). \end{aligned} \tag{4.3}$$

Equation (4.3) tells us that the Fourier transform of a shifted version of a signal $s(t)$ by t_0 seconds is given by the product of the Fourier transform of the signal (un-shifted version) $s(t)$ multiplied by the complex exponential $e^{-j2\pi ft_0}$.

4.1 Example of FT of a Signal

In this section we present a signal that consists of an exponential signal that is multiplied by a time-shifted version of a unit step function. The time shift is of k units. Consider the definition of the Fourier transform and the signal $x(t)$ given by

$$x(t) = u(t-k)e^{-at}, \tag{4.4}$$

where k is the time-shift, e.g., $k = 7$, and a is the parameter in the exponential indicating how fast it decays. In this case, the problem will be solved using delay k in the equations and in the graphs it will take a number as indicated.

1. Sketch the signal $x(t)$.

   Figure 4.1 contains a representation of signal $x(t)$ for $k = 2$ and $a = 2$. We also include the exponential e^{-at} and the unit step signal $u(t-2)$. The resulting signal $x(t)$ is marked in red color in the figure.

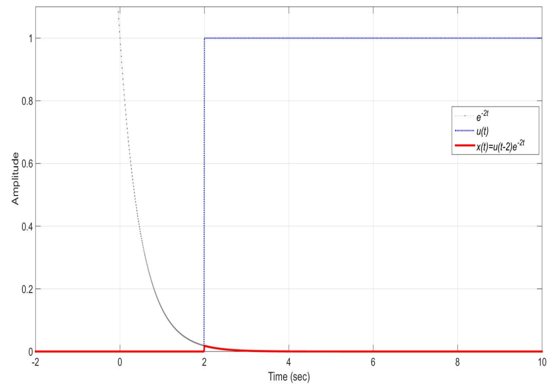

Figure 4.1: Signal $x(t)$ with $a = 2$ and $k = 2$.

2. Obtain the Fourier transform of $x(t)$, i.e., $X(f)$.

   From the sketch shown in Figure 4.1, we can see that the time where the signal exists is limited to the interval from k up to infinity. We can also see that the exponential is the only part that we need to consider since the unit step function $u(t-k)$ is only affecting the interval where the signal exists, then we have that the Fourier transform is given by

$$\begin{aligned} X(f) &= \int_{k}^{\infty} x(t)e^{-j2\pi ft}dt \\ &= \int_{k}^{\infty} e^{-at}e^{-j2\pi ft}dt \\ &= \int_{k}^{\infty} e^{-t(a+j2\pi f)}dt \\ &= \frac{1}{-(a+j2\pi f)}e^{-t(a+j2\pi f)}\Big|_{k}^{\infty} \\ &= \frac{1}{-(a+j2\pi f)}\left(0 - e^{-k(a+j2\pi f)}\right) \\ &= \frac{e^{-ka}}{a+j2\pi f}e^{-j2\pi fk}. \end{aligned} \tag{4.5}$$

   Note that $\frac{1}{a+j2\pi f}$ is the Fourier transform of an exponential signal e^{-at}, also note that since we are considering the initial time to be k, the amplitude of the exponential is e^{-ka}. Also, since the signal is considering a time-delay of k units, we have the complex exponential term $e^{-j2\pi fk}$.

4.2 The Rectangular Pulse Signal

The rectangular pulse signal is one of the most important signals in the systems and communications area. It is an energy signal, hence it is an aperiodic-even signal, and it can represent information by relating its characteristics to bits or symbols to be transmitted. Its principal characteristics are the pulse width or time duration, and its amplitude. In this section, we analyze in the frequency domain the rectangular pulse signal by obtaining its Fourier transform. We start with the rectangular pulse signal that is centered at the origin and that it has duration T. Recall that this signal is mathematically represented by the following equation

$$s(t) = A \prod \left(\frac{t}{T} \right). \tag{4.6}$$

To show an example of a realization of this rectangular pulse signal, Figure 11.3 shows a rectangular pulse signal with amplitude $A = 3$ and pulse width of duration $T = 10$. Note that the signal has its center at the origin, and the pulse width of $T = 10$ can be seen by recognizing that the pulse starts at time $-T/2 = -5$ and ends at time $+T/2 = 5$ as shown in the figure

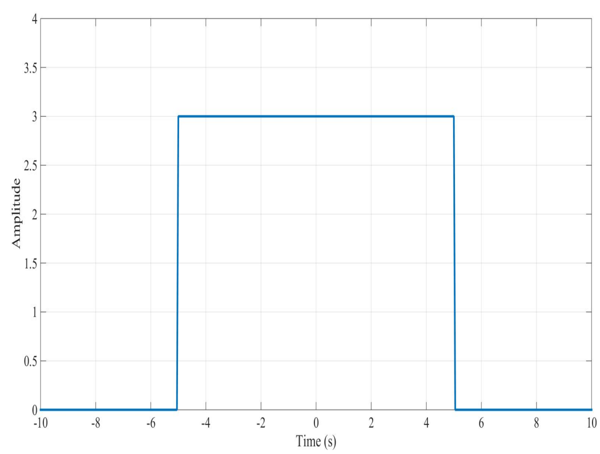

Figure 4.2: Rectangular pulse signal $A = 3$, $T = 10$.

In the following, we obtain the Fourier transform of this rectangular pulse signal. We use the definition of the Fourier transform. The integral in the definition runs from $-\infty$ up to $+\infty$, but these integration limits will be changed since the rectangular pulse signal only exists (i.e., is different from zero) for the time interval from $-T/2$ up to $+T/2$. The Fourier transform of signal $s(t)$ is given by

$$\begin{aligned}
X(f) &= \mathcal{F}\{x(t)\} \\
&= \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} A\prod\left(\frac{t}{T}\right)e^{-j2\pi ft}dt \\
&= \int_{-T/2}^{T/2} A\,e^{-j2\pi ft}dt \\
&= A\int_{-T/2}^{T/2} e^{-j2\pi ft}dt \\
&= \frac{A}{-j2\pi f}e^{-j2\pi ft}\Big|_{-T/2}^{T/2} \\
&= \frac{A}{-j2\pi f}\left\{e^{-j2\pi fT/2}-e^{j2\pi fT/2}\right\} \\
&= \frac{A}{j2\pi f}\left\{e^{j\pi fT}-e^{-j\pi fT}\right\} \\
&= \frac{A}{\pi f}\sin(\pi fT) \\
&= \frac{AT}{\pi fT}\sin(\pi fT) \\
&= AT\;sinc(fT). \qquad (4.7)
\end{aligned}$$

Recall that in general, a Fourier transform must be considered a complex function with real and imaginary parts. In the particular example of the rectangular pulse signal that is centered at the origin, we can see that it is an even signal, hence the Fourier transform will only consists of the real part. If the signal had odd symmetry, then the Fourier transform would be purely imaginary, and if the signal had no symmetry, then the Fourier transform would have both real and imaginary parts.

With the Fourier transform, we can obtain other characteristics of the signal, for example, in order to plot the signal, we obtain the magnitude of the Fourier transform (recall the FT is complex). Now, the magnitude of the Fourier transform of a rectangular pulse with $A = 3$ and $T = 2$ is shown in Figure 11.4. Note in the figure how the magnitude of the sinc function touches the horizontal axis in integer multiples of the inverse of the duration, i.e., k/T for $k = 0, \pm1, \pm2, \ldots$. The continuous line in the figure is the Fourier transform obtained by using the command *fft* in MATLAB, whereas the red dots correspond to the evaluation of the sinc function in Equation (4.7).

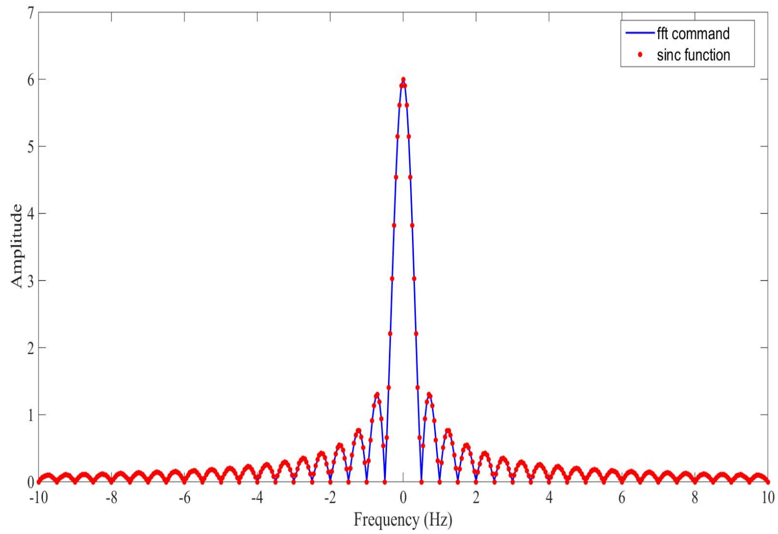

Figure 4.3: Fourier transfrom of a rectangular pulse signal $A = 3$, $T = 2$.

4.3 The Triangular Pulse Signal

Consider the signal $x(t)$ defined as

$$x(t) = \begin{cases} \frac{A}{T}t + A, & -T \leq t < 0, \\ A - \frac{A}{T}t, & 0 \leq t \leq T, \\ 0, & elsewhere. \end{cases} \tag{4.8}$$

Here, we also use the general form of a triangular pulse of amplitude A and limited in time from $-T$ to T.

1. Sketch the signal $x(t)$.

2. Obtain the Fourier transform of $x(t)$, i.e., $X(f)$.

   Considering the triangular pulse in Figure 4.4, we can see that the signal is time limited to the interval $[-T,\ T]$, then we proceed applying the definition of the Fourier transform

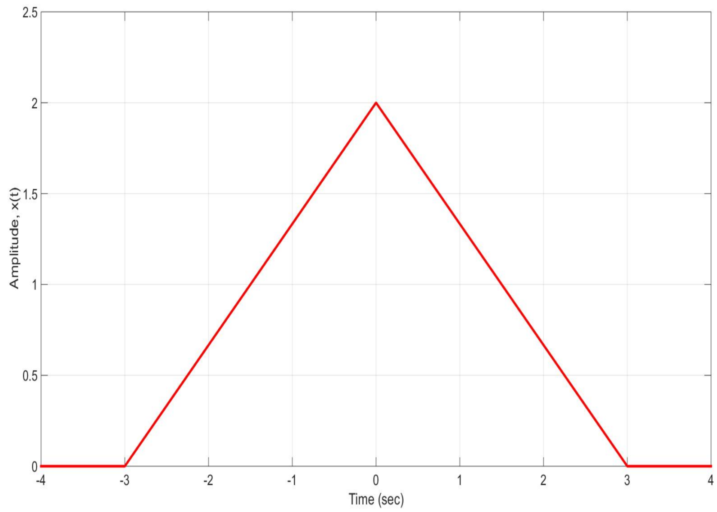

Figure 4.4: Triangular pulse for $A = 2$ and $T = 3$.

as follows

$$\begin{aligned} X(f) &= \int_{-T}^{T} x(t) e^{-j2\pi f t} dt \\ &= \int_{-T}^{0} x(t) e^{-j2\pi f t} dt + \int_{0}^{T} x(t) e^{-j2\pi f t} dt \\ &= \int_{-T}^{0} \left(\frac{A}{T} t + A \right) e^{-j2\pi f t} dt + \int_{0}^{T} \left(A - \frac{A}{T} t \right) e^{-j2\pi f t} dt. \end{aligned} \tag{4.9}$$

In this case, we are going to need the following integral

$$\int_{a}^{b} t e^{ct} dt = \left(\frac{t}{c} - \frac{1}{c^2} \right) e^{ct} \Big]_{a}^{b} \tag{4.10}$$

Note that in our case, $c = -j2\pi f$ and that $c^2 = -4\pi^2 f^2$. Thus we have the following

$$\begin{aligned}
X(f) &= \int_{-T}^{0}\left(\frac{A}{T}t + A\right)e^{-j2\pi ft}dt + \int_{0}^{T}\left(A - \frac{A}{T}t\right)e^{-j2\pi ft}dt \\
&= \frac{A}{T}\int_{-T}^{0} te^{-j2\pi ft}dt + A\int_{-T}^{0} e^{-j2\pi ft}dt + A\int_{0}^{T} e^{-j2\pi ft}dt - \frac{A}{T}\int_{0}^{T} te^{-j2\pi ft}dt \\
&= \frac{A}{T}\left(\frac{t}{-j2\pi f} - \frac{1}{-4\pi^2 f^2}\right)e^{-j2\pi ft}\Bigg]_{-T}^{0} \\
&\quad + \frac{A}{-j2\pi f}e^{-j2\pi ft}\Bigg]_{-T}^{0} \\
&\quad + \frac{A}{-j2\pi f}e^{-j2\pi ft}\Bigg]_{0}^{T} \\
&\quad - \frac{A}{T}\left(\frac{t}{-j2\pi f} - \frac{1}{-4\pi^2 f^2}\right)e^{-j2\pi ft}\Bigg]_{0}^{T} \\
&= \frac{A}{T}\left(\frac{1}{4\pi^2 f^2} - \left(\frac{-T}{-j2\pi f} + \frac{1}{4\pi^2 f^2}\right)e^{j2\pi fT}\right) \\
&\quad - \frac{A}{j2\pi f}\left(1 - e^{j2\pi fT}\right) \\
&\quad - \frac{A}{j2\pi f}\left(e^{-j2\pi fT} - 1\right) \\
&\quad - \frac{A}{T}\left(\left(\frac{T}{-j2\pi f} + \frac{1}{4\pi^2 f^2}\right)e^{-j2\pi fT} - \frac{1}{4\pi^2 f^2}\right) \\
&= \frac{2A}{4\pi^2 f^2 T} + \frac{A}{j2\pi f}\left(e^{j2\pi fT} - e^{-j2\pi fT}\right) \\
&\quad - \frac{A}{T}\left(\frac{T}{j2\pi f} + \frac{1}{4\pi^2 f^2}\right)e^{j2\pi fT} - \frac{A}{T}\left(\frac{-T}{j2\pi f} + \frac{1}{4\pi^2 f^2}\right)e^{-j2\pi fT} \\
&= \frac{2A}{4\pi^2 f^2 T} + \frac{A}{\pi f}\sin(2\pi fT) - \frac{A}{j2\pi f}e^{j2\pi fT} + \frac{A}{j2\pi f}e^{-j2\pi fT} \\
&\quad - \frac{A}{4\pi^2 f^2 T}e^{j2\pi fT} - \frac{A}{4\pi^2 f^2 T}e^{-j2\pi fT} \\
&= \frac{2A}{4\pi^2 f^2 T} + \frac{A}{\pi f}\sin(2\pi fT) - \frac{A}{j2\pi f}\left(e^{j2\pi fT} - e^{-j2\pi fT}\right) \\
&\quad - \frac{A}{4\pi^2 f^2 T}\left(e^{j2\pi fT} + e^{-j2\pi fT}\right) \\
&= \frac{2A}{4\pi^2 f^2 T} + \frac{A}{\pi f}\sin(2\pi fT) - \frac{A}{\pi f}\sin(2\pi fT) - \frac{A}{2\pi^2 f^2 T}\cos(2\pi fT) \\
&= \frac{A}{2\pi^2 f^2 T} - \frac{A}{2\pi^2 f^2 T}\cos(2\pi fT).
\end{aligned} \tag{4.11}$$

In Equation (4.11), we can see that

$$X(f) = \frac{A}{2\pi^2 f^2 T}\Bigg\{1 - \cos(2\pi f T)\Bigg\}, \tag{4.12}$$

and recall that there is a trigonometric identity as follows $\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos(\alpha)}{2}$ which can be used in Equation (4.12) to get

$$\begin{aligned}
X(f) &= \frac{A}{2\pi^2 f^2 T}\Bigg\{1 - \cos(2\pi f T)\Bigg\} \\
&= \frac{A}{2\pi^2 f^2 T} 2\sin^2(\pi f T) \\
&= \frac{A}{\pi^2 f^2 T}\frac{T}{T}\sin^2(\pi f T) \\
&= \frac{AT}{\pi^2 f^2 T^2}\sin^2(\pi f T) \\
&= AT\frac{\sin(\pi f T)}{\pi f T}\frac{\sin(\pi f T)}{\pi f T} \\
&= AT sinc(fT)\, sinc(fT) \\
&= AT sinc^2(fT).
\end{aligned} \tag{4.13}$$

Figure 4.5 shows the Fourier transform $X(f)$ of the triangular pulse signal $x(t)$. Figure 4.6 shows the same figure but with logarithmic vertical axis.

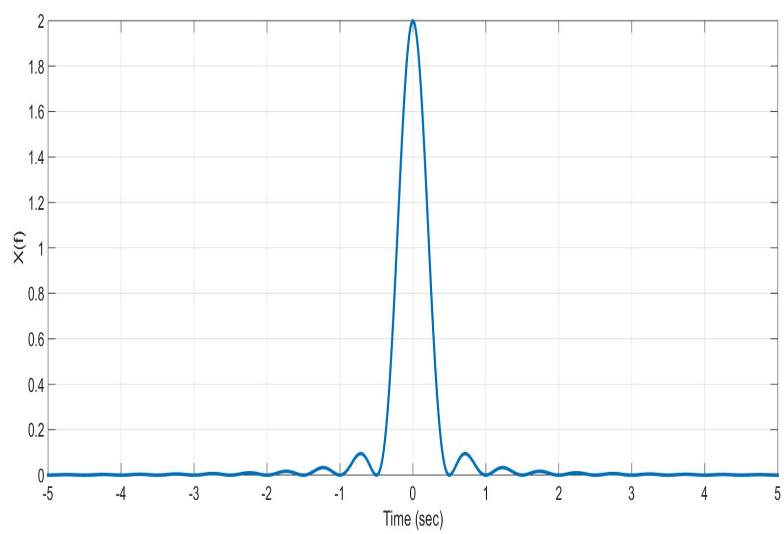

Figure 4.5: $X(f)$ for $T = 2$ and $A = 1$.

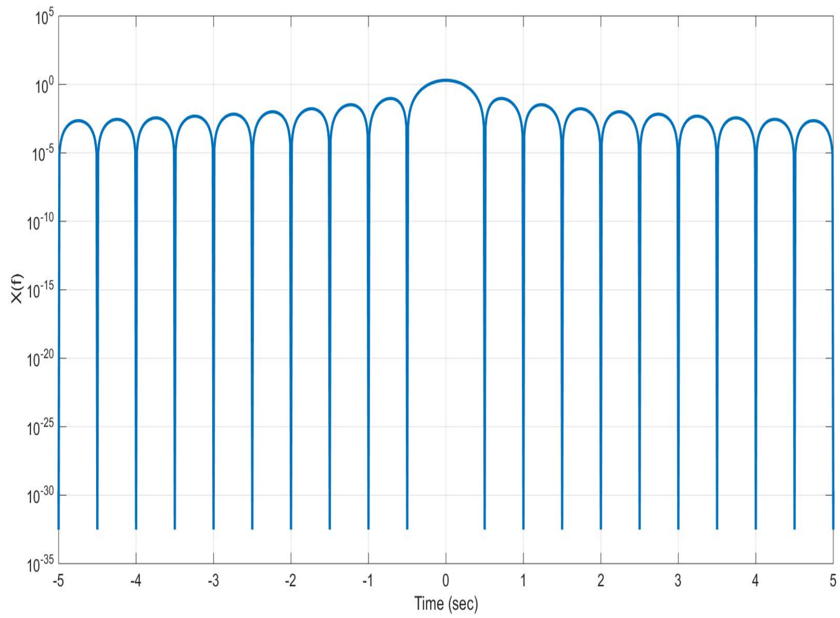

Figure 4.6: $X(f)$ for $T = 2$ and $A = 1$.

4.4 Fourier Transform of Periodic Signals

Even if the FT is said to apply to aperiodic signals, we have that it can be applied to periodic signals as well. This makes the FT the most powerful tool for signal analysis.

In order to obtain the Fourier transform of periodic signal, we need to follow a four-step procedure as described in the following paragraphs.

Consider a periodic signal $x_p(t)$ with period T_0, its Fourier transform is given by the following four-step method.

1. **Step 1: Truncation.** Take the periodic signal $x_p(t)$ and truncate one period, consider always as the best option to truncate the period centered at the origin, i.e., from $-T_0/2$ to $T_0/2$. This truncated signal can be represented mathematically by the product of the original periodic signal $x_p(t)$ and a unitary amplitude rectangular pulse signal centered at the origin and of duration T_0, in other words, you need to obtain the truncated signal $x(t)$ as
$$x(t) = x_p(t) \prod \left(\frac{t}{T_0} \right). \tag{4.14}$$

2. **Step 2: Fourier Transform.** Now that we have a truncated version of the periodic signal, we can see that it is aperiodic and time-limited, hence we can obtain the Fourier transform of the truncated signal $x(t)$ by using the definition in Equation (4.1). In other words, obtain $X(f)$.

3. **Step 3: "Sampling".** Although sampling is always considered to be a time-domain process, we obtain from the Fourier transform $X(f)$ some samples in this step. The samples are those values of $X(f)$ that are located in integer multiples of the fundamental frequency $f_0 = 1/T_0$. In the Fourier transform just obtained, change every instance of the variable f by nf_0 where $f_0 = 1/T_0$ to obtain $X(nf_0)$.

4. **Step 4: Scale.** The last step in this procedure is to scale the Fourier transform $X(nf_0)$ just obtained by multiplying it by the constant $1/T_0$. The Fourier transform of the periodic signal $x_p(t)$ will be finally given by a scaled version of $X(nf_0)$ as follows

$$X_p(f) = \frac{1}{T_0} X(nf_0) = C_n. \tag{4.15}$$

In the examples, you can verify that this Fourier transform obtained by the four-step procedure gives exactly the CEFS coefficients C_n.

4.5 FT of a Rectangular Pulse Train Signal

Consider the periodic signal given by a rectangular pulse train with period T_0, amplitude A and the duration of each rectangular pulse being T where $T \leq T_0/2$. The signal can be represented mathematically by

$$x_p(t) = \sum_{n=-\infty}^{\infty} A \prod \left(\frac{t - nT_0}{T} \right). \tag{4.16}$$

See that the signal in Equation (4.16) has even symmetry, and hence the rectangular pulse corresponding to $n = 0$ is centered at the origin.

Now we can apply the four-step method

1. **Step 1: Truncation.** Consider the case of $n = 0$ to obtain

$$x(t) = A \prod \left(\frac{t}{T} \right). \tag{4.17}$$

2. **Step 2: Fourier Transform.** The FT was just obtained in the previous example given in Equation (4.7), hence we get

$$X(f) = AT\ sinc(fT). \tag{4.18}$$

3. **Step 3: "Sampling".**

$$X(nf_0) = AT\ sinc(nf_0T). \tag{4.19}$$

4. **Step 4: Scale.**

$$\begin{aligned} X_p(f) &= \frac{AT}{T_0}\, sinc\left(n\frac{T}{T_0}\right) \\ &= Ad\; sinc(nd). \end{aligned} \tag{4.20}$$

In Equation (4.20) d is the *duty cycle* and is given by $d = T/T_0$. We generally are interested in duty cycles of the form $1/2, 1/3, 1/4, \ldots$.

Check by yourself that the Fourier transform of a periodic signal gives the CEFS coefficients C_n, i.e., Equation (4.20) is exactly the same as the coefficients of the CEFS C_n for a rectangular pulse train.

4.6 FT and CEFS of a Cosine Signal

In this section, we obtain the CEFS and the FT of a periodic cosine function by using the four steps just described in Section 4.4. We start first with the calculation of the CEFS for the periodic signal and the analysis of its series, then (in point 8 of the example) we use the four step method to obtain the FT. At the end of the example we conclude that the CEFS coefficients and the FT are strongly related to one another.

Consider the *periodic* signal $g_p(t) = A\cos(2\pi f_0 t + \phi)$ with period $T_0 = 1/f_0$. An example of a realization of this signal is shown in Figure 4.7 for the specific case of frequency $f_0 = 2$ Hz, phase shift of $\phi = \pi/4$, and amplitude $A = 3$V. Note that since the phase shift is positive, the signal is shifted to the left of the vertical axis as shown in the figure. Also, this shift produces a signal that has no symmetry, therefore, you should expect to get that the CEFS coefficients C_n be complex with real and imaginary parts different from zero.

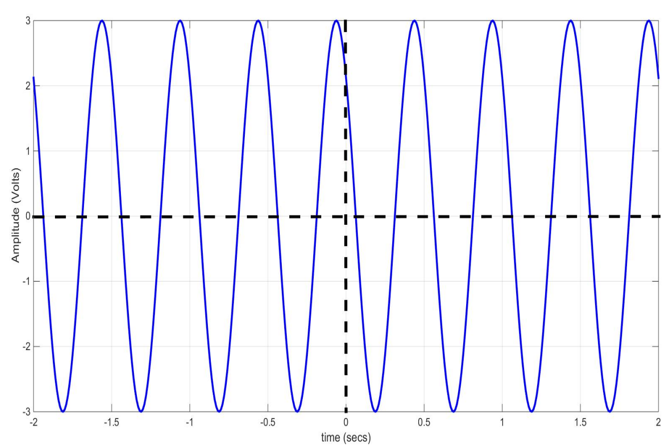

Figure 4.7: Signal $g_p(t)$ for $A = 3$, $\phi = \pi/4$, and $f_0 = 2$Hz.

1. Obtain the coefficients c_n of the CEFS. You need to use the definition of the coefficients with the integral and show all your work.

$$\begin{aligned}
C_n &= \frac{1}{T_0}\int\limits_{-T_0/2}^{T_0/2} g_p(t)e^{-j2\pi f_0 nt}dt \\
&= \frac{1}{T_0}\int\limits_{-T_0/2}^{T_0/2} A\cos(2\pi f_0 t+\phi)e^{-j2\pi f_0 nt}dt \\
&= \frac{A}{2T_0}\int\limits_{-T_0/2}^{T_0/2} \left[e^{j2\pi f_0 t+j\phi}+e^{-j2\pi f_0 t-j\phi}\right]e^{-j2\pi f_0 nt}dt \\
&= \frac{A}{2T_0}\int\limits_{-T_0/2}^{T_0/2} \left[e^{j\phi}e^{j2\pi f_0 t}+e^{-j\phi}e^{-j2\pi f_0 t}\right]e^{-j2\pi f_0 nt}dt \\
&= \frac{A}{2T_0}\int\limits_{-T_0/2}^{T_0/2} \left[e^{j\phi}e^{-j2\pi f_0(n-1)t}+e^{-j\phi}e^{-j2\pi f_0(n+1)t}\right]dt \\
&= \frac{A}{2T_0}\left\{e^{j\phi}\frac{1}{-j2\pi f_0(n-1)}e^{-j2\pi f_0(n-1)t}\right. \\
&\qquad \left.\left. +e^{-j\phi}\frac{1}{-j2\pi f_0(n+1)}e^{-j2\pi f_0(n+1)t}\right\}\right]_{-T_0/2}^{T_0/2} \\
&= \frac{A}{2T_0}\left\{\frac{-e^{j\phi}}{j2\pi f_0(n-1)}\left[e^{-j\pi(n-1)}-e^{j\pi(n-1)}\right]\right. \\
&\qquad \left. -\frac{e^{-j\phi}}{j2\pi f_0(n+1)}\left[e^{-j\pi(n+1)}-e^{j\pi(n+1)}\right]\right\} \\
&= \frac{A}{2T_0}\left\{\frac{e^{j\phi}}{\pi f_0(n-1)}\sin(\pi(n-1))+\frac{e^{-j\phi}}{\pi f_0(n+1)}\sin(\pi(n+1))\right\} \\
&= \frac{Ae^{j\phi}}{2}\frac{\sin(\pi(n-1))}{\pi(n-1)}+\frac{Ae^{-j\phi}}{2}\frac{\sin(\pi(n+1))}{\pi(n+1)} \\
&= \begin{cases} \frac{Ae^{j\phi}}{2}, & n=1, \\ \frac{Ae^{-j\phi}}{2} & n=-1, \\ 0, & n\neq\pm 1. \end{cases} \qquad (4.21)
\end{aligned}$$

2. Provide an expression of the CEFS for $g_p(t)$

$$\begin{aligned} g_p(t) &= \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_0 nt} \\ &= \frac{A}{2} e^{j\phi} e^{j2\pi f_0 t} + \frac{A}{2} e^{-j\phi} e^{-j2\pi f_0 t} \\ &= \frac{A}{2} e^{j(2\pi f_0 t + \phi)} + \frac{A}{2} e^{-j(2\pi f_0 t + \phi)} \\ &= A\cos(2\pi f_0 t + \phi). \end{aligned} \tag{4.22}$$

   Recall that the CEFS of a cosine function is obtained using the trigonometric identity.

3. Give an expression for the magnitude of the coefficients, i.e., $|C_n|$

$$|C_{+1}| = \frac{A}{2}, \quad |C_{-1}| = \frac{A}{2}. \tag{4.23}$$

   This can also be expressed as $|C_n| = \frac{A}{2}\delta(n-1) + \frac{A}{2}\delta(n+1)$.

4. Give an expression for the phase of the coefficients, i.e., $\angle C_n$. Since $C_{-1} = \frac{Ae^{-j\phi}}{2} = \frac{A}{2}\cos(\phi) - j\frac{A}{2}\sin(\phi)$ and $C_{+1} = \frac{Ae^{j\phi}}{2} = \frac{A}{2}\cos(\phi) + j\frac{A}{2}\sin(\phi)$, we have that the phase of these is given by

$$\begin{aligned} \angle C_{+1} &= \tan^{-1}\left\{\frac{\frac{A}{2}\sin(\phi)}{\frac{A}{2}\cos(\phi)}\right\} = \tan^{-1}(\tan(\phi)) = \phi, \\ \angle C_{-1} &= \tan^{-1}\left\{\frac{-\frac{A}{2}\sin(\phi)}{\frac{A}{2}\cos(\phi)}\right\} = \tan^{-1}(-\tan(\phi)) = -\phi. \end{aligned} \tag{4.24}$$

5. **Plot** the magnitude spectrum of the coefficients, $|C_n|$ Figure 4.8 shows a plot of the magnitude of the coefficients when $A = 3$.

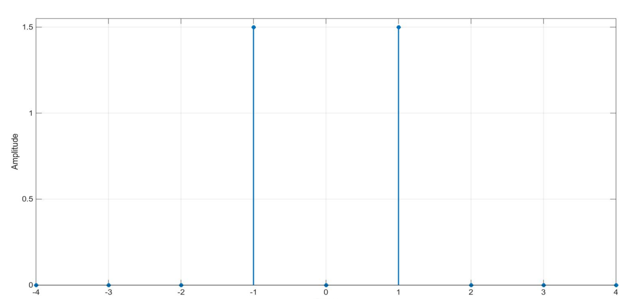

Figure 4.8: Magnitude of coefficients C_n for $A = 3$.

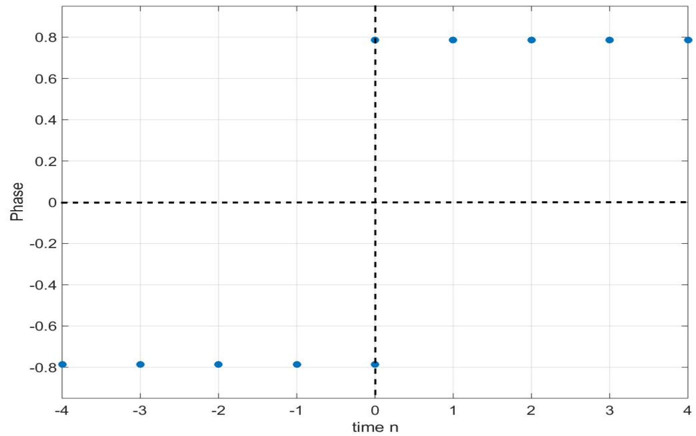

Figure 4.9: Phase of coefficients C_n for $A = 3$.

6. **Plot** the phase spectrum of the coefficients, $|C_n|$ Figure 4.9 shows a plot of the phase of the coefficients when $\phi = \pi/4$.

7. Obtain the coefficients C_0, A_n and B_n of the TFS. From the solution of the coefficients C_n, we have that $C_0 = 0$ and that we only have coefficients for $n = 1$ and $n = -1$. Then we can apply the identities $A_n = C_n + C_{-n}$ and $B_n = j(C_n - C_{-n})$ to obtain

$$A_1 = \frac{Ae^{j\phi}}{2} + \frac{Ae^{-j\phi}}{2} = A\cos(\phi). \tag{4.25}$$

and

$$B_1 = j\left(\frac{Ae^{j\phi}}{2} - \frac{Ae^{-j\phi}}{2}\right) = -A\sin(\phi). \tag{4.26}$$

8. Obtain the Fourier transform of $g_p(t)$ using the method seen in class given by

   (a) **Truncate** $g_p(t)$, give a sketch or an expression of this signal, call it $x(t)$. We provide an expression of this signal which is

$$x(t) = A\cos(2\pi f_0 t + \phi)\prod\left(\frac{t}{T_0}\right). \tag{4.27}$$

(b) Obtain the **Fourier transform** $X(f)$ of $x(t)$

$$\begin{aligned}
X(f) &= \int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi ft}dt \\
&= \int_{-T_0/2}^{T_0/2} A\cos(2\pi f_0 t + \phi)e^{-j2\pi ft}dt \\
&= \frac{A}{2}\int_{-T_0/2}^{T_0/2} \left[e^{j2\pi f_0 t + j\phi} + e^{-j2\pi f_0 t - j\phi}\right] e^{-j2\pi ft}dt \\
&= \frac{A}{2}\int_{-T_0/2}^{T_0/2} \left[e^{j\phi}e^{j2\pi f_0 t} + e^{-j\phi}e^{-j2\pi f_0 t}\right] e^{-j2\pi ft}dt \\
&= \frac{A}{2}\int_{-T_0/2}^{T_0/2} \left[e^{j\phi}e^{-j2\pi(f-f_0)t} + e^{-j\phi}e^{-j2\pi(f+f_0)t}\right] dt \\
&= \frac{A}{2}\left\{e^{j\phi}\frac{1}{-j2\pi(f-f_0)}e^{-j2\pi(f-f_0)t}\right. \\
&\qquad \left.\left. + e^{-j\phi}\frac{1}{-j2\pi(f+f_0)}e^{-j2\pi(f+f_0)t}\right\}\right]_{-T_0/2}^{T_0/2} \\
&= \frac{A}{2}\left\{\frac{-e^{j\phi}}{j2\pi(f-f_0)}\left[e^{-j\pi(f-f_0)T_0} - e^{j\pi(f-f_0)T_0}\right]\right. \\
&\qquad \left. - \frac{e^{-j\phi}}{j2\pi(f+f_0)}\left[e^{-j\pi(f+f_0)T_0} - e^{j\pi(f+f_0)T_0}\right]\right\} \\
&= \frac{A}{2}\left\{\frac{e^{j\phi}}{\pi(f-f_0)}\sin(\pi(f-f_0)T_0) + \frac{e^{-j\phi}}{\pi(f+f_0)}\sin(\pi(f+f_0)T_0)\right\} \\
&= e^{j\phi}\frac{AT_0}{2}sinc((f-f_0)T_0) + e^{-j\phi}\frac{AT_0}{2}sinc((f+f_0)T_0). \qquad (4.28)
\end{aligned}$$

(c) **"Sample"** $X(f)$ by changing f for nf_0, i.e., give $X(nf_0)$. To do this we use the expression $(f \pm f_0) = (nf_0 \pm f_0) = (n \pm 1)f_0$ and the identity $f_0T_0 = 1$ to get

$$\begin{aligned}
X(nf_0) &= e^{j\phi}\frac{AT_0}{2}sinc((nf_0 - f_0)T_0) + e^{-j\phi}\frac{AT_0}{2}sinc((nf_0 + f_0)T_0) \\
&= e^{j\phi}\frac{AT_0}{2}sinc((n-1)) + e^{-j\phi}\frac{AT_0}{2}sinc((n+1)) \\
&= e^{j\phi}\frac{AT_0}{2}\frac{\sin(\pi(n-1))}{\pi(n-1)} + e^{-j\phi}\frac{AT_0}{2}\frac{\sin(\pi(n+1))}{\pi(n+1)}. \qquad (4.29)
\end{aligned}$$

The last row of Equation (4.29) is close to the penultimate row in Equation (4.21) with he only difference of a scaling factor of T_0.

(d) **Scale** your result, i.e., provide $\frac{1}{T_0}X(nf_0)$

$$G_p(f) = \frac{1}{T_0}X(nf_0) = e^{j\phi}\frac{A}{2}\delta(f - f_0) + e^{-j\phi}\frac{A}{2}\delta(f + f_0). \tag{4.30}$$

(e) Compare this result to the coefficient C_n obtained in part (a), conclude. Comparison of equations (4.29) and (4.21) shows that the scaling of $X(nf_0)$ gives the same result as that of the coefficients of the CEFS C_n.

9. Provide the transform pair of $g_p(t)$

$$g_p(t) = A\cos(2\pi f_0 t + \phi) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad G_p(f) = e^{j\phi}\frac{A}{2}\delta(f - f_0) + e^{-j\phi}\frac{A}{2}\delta(f + f_0). \tag{4.31}$$

4.7 Example of FT and CEFS of a Periodic Signal

Define a periodic signal $x_p(t)$ with period $T_0 = 2\pi$. One period of the signal is given by

$$x_p(t) = \begin{cases} t, & 0 \le t \le \pi, \\ 0, & -\pi < t < 0. \end{cases} \tag{4.32}$$

1. Sketch the signal with three periods. Figure 4.10 shows three periods of the signal.

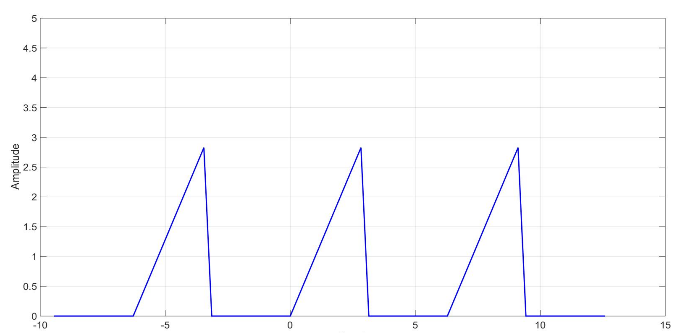

Figure 4.10: Three periods of signal $x_p(t)$.

2. Obtain the CEFS coefficients C_n and show that they are given by

$$C_n = \begin{cases} \frac{1}{2\pi n^2}\left[(-1)^n - 1\right] + j\frac{1}{2n}(-1)^n, & n = \pm 1, \ \pm 2, \ \dots, \\ \frac{\pi}{4}, & n = 0. \end{cases} \tag{4.33}$$

First note that $T_0 = 2\pi$, then we start with the definition of the CEFS coefficients C_n and work by integrating by parts to obtain them as follows

$$\begin{aligned}
C_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_p(t)e^{-j2\pi f_0 nt}dt \\
&= \frac{1}{2\pi}\int_0^{\pi} te^{-j2\pi f_0 nt}dt \\
&= \frac{1}{2\pi}\int_0^{\pi} te^{-jnt}dt \\
&= \left.\frac{-1}{2\pi}\frac{t}{jn}e^{-jnt}\right]_0^{\pi} + \frac{1}{2\pi}\frac{1}{jn}\int_0^{\pi} e^{-jnt}dt \\
&= \frac{-1}{j2\pi n}\pi e^{-jn\pi} + \left.\frac{1}{j2\pi n}\left(\frac{-1}{jn}\right)e^{-jnt}\right]_0^{\pi} \\
&= \frac{j}{2n}e^{-jn\pi} + \frac{1}{2\pi n^2}\left(e^{-jn\pi} - 1\right) \\
&= \frac{1}{2\pi n^2}\left[(-1)^n - 1\right] + j\frac{1}{2n}(-1)^n. \qquad (4.34)
\end{aligned}$$

The coefficients in Equation (4.34) are valid for negative and positive $n \neq 0$. To obtain C_0 we also apply the definition of the coefficients to obtain the following

$$\begin{aligned}
C_0 &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_p(t)dt \\
&= \frac{1}{2\pi}\int_0^{\pi} tdt \\
&= \left.\frac{1}{2\pi}\frac{t^2}{2}\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left(\frac{\pi^2}{2} - 0\right) \\
&= \frac{\pi}{4}. \qquad (4.35)
\end{aligned}$$

3. Provide the magnitude $|C_n|$. To calculate this, we use the magnitude square first, i.e., $|C_n|^2$. See that $|C_0| = \pi/4$. We also use the fact that $(-1)^{2n} = ((-1)^2)^n = 1^n = 1$. Then for $n \neq 0$ we have

$$\begin{aligned}
|C_n|^2 &= \left\{\frac{1}{2\pi n^2}\left[(-1)^n - 1\right]\right\}^2 + \left\{\frac{1}{2n}(-1)^n\right\}^2 \\
&= \frac{1}{4\pi^2 n^4}\left[(-1)^{2n} - 2(-1)^n + 1\right] + \frac{1}{4n^2}(-1)^{2n} \\
&= \frac{1}{4\pi^2 n^4}\left[1 - 2(-1)^n + 1\right] + \frac{1}{4n^2} \\
&= \frac{1}{4\pi^2 n^4}\left[2 - 2(-1)^n\right] + \frac{1}{4n^2} \\
&= \frac{1}{2\pi^2 n^4}\left[1 - (-1)^n\right] + \frac{1}{4n^2}, \qquad |n| > 0.
\end{aligned} \tag{4.36}$$

4. Give terms $n = 0, \pm 1, \pm 2$ of the CEFS.

   The CEFS would be given by

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_0 n t} \tag{4.37}$$

   For $n = 0$ we have $C_0 = \pi/4$, for $n = 1$, we have C_1 from Equation (4.34) and so on. We show the first term for $n = 0$, the second term for $n = 1$, the third term for $n = -1$, the fourth term for $n = 2$ and the fifth term for $n = -2$.

$$\begin{aligned}
x_p(t) &= \frac{1}{4} \\
&= \left(-\frac{1}{\pi} - j\frac{1}{2}\right) e^{jt} \\
&= \left(-\frac{1}{\pi} + j\frac{1}{2}\right) e^{-jt} \\
&= j\frac{1}{4}e^{j2t} - j\frac{1}{4}e^{-j2t}.
\end{aligned} \tag{4.38}$$

5. Give the coefficients C_0, A_n and B_n of the TFS.

   $C_0 = \pi/4$. Use identities $A_n = C_n + C_{-n}$ and $B_n = j(C_n - C_{-n})$ to obtain the coefficients for the TFS.

$$\begin{aligned}
A_n &= \frac{1}{n^2\pi^2}\left[(-1)^n - 1\right], \\
B_n &= -\frac{1}{n\pi}(-1)^n.
\end{aligned} \tag{4.39}$$

6. Give terms $n = 0, 1, 2, 3$ of the TFS. Similar to the CEFS terms.

7. Give the Fourier transform of the truncated signal. Show that it is given by

$$X(f) = \frac{1}{4\pi^2 f^2}\left(e^{-j2\pi^2 f} - 1\right) + j\frac{1}{2f}e^{-j2\pi^2 f}. \tag{4.40}$$

Consider the truncated signal as $x(t) = x_p(t)\prod\left(\frac{t-\pi}{2\pi}\right)$. Using the definition and integrating by parts we get

$$\begin{aligned} X(f) &= \int_0^\pi t e^{-j2\pi f t} dt \\ &= -\frac{t}{j2\pi f}e^{-j2\pi f t}\Big]_0^\pi + \frac{1}{j2\pi f}\int_0^\pi e^{-j2\pi f t} dt \\ &= -\frac{1}{j2f}e^{-j2\pi^2 f} + \frac{1}{j2\pi f}\left(-\frac{1}{j2\pi f}\right)e^{-j2\pi f t}\Big]_0^\pi \\ &= j\frac{1}{2f}e^{-j2\pi^2 f} + \frac{1}{4\pi^2 f^2}\left(e^{-j2\pi^2 f} - 1\right). \end{aligned} \tag{4.41}$$

Chapter 5

Review of Probability Theory and Random Processes

5.1 Probability

5.1.1 Gaussian Random Variable

A Gaussian random variable X with mean m and standard deviation σ is denoted as $X \sim \mathcal{N}(m, \sigma)$ and has probability density function (pdf) given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}, \tag{5.1}$$

its expectation, average or mean value is given by

$$E(x) = \int_{-\infty}^{\infty} x f_X(x) dx = m, \tag{5.2}$$

and its variance is

$$\text{var}(X) = E\left[(X-m)^2\right] = \int_{-\infty}^{\infty} (x-m)^2 f_X(x) dx = \sigma^2. \tag{5.3}$$

The cumulative distribution function (cdf) of the random variable X is given by

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(\alpha) d\alpha \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(\alpha-m)^2}{2\sigma^2}} d\alpha. \end{aligned} \tag{5.4}$$

In (5.4), we can use *normalization*, i.e., change of variable $z = \frac{(\alpha - m)}{\sigma}$ to obtain

$$\begin{aligned} F_X(x) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\frac{(x-m)}{\sigma}} e^{-\frac{z^2}{2}} \sigma dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{(x-m)}{\sigma}} e^{-\frac{z^2}{2}} dz \\ &= \Phi_X\left(\frac{x-m}{\sigma}\right). \end{aligned} \tag{5.5}$$

To evaluate the probabilities of a Gaussian random variable, the $Q()$ function is used. $Q()$ is defined as

$$Q(x) = 1 - \Phi(x) = P(X > x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{z^2}{2}} dz, \tag{5.6}$$

it corresponds to the area of the tail from x to ∞, and it has the property that

$$Q(-x) = 1 - Q(x). \tag{5.7}$$

Other functions used in the evaluation of probabilities are the error function and the complementary error function defined as

$$erf(x) = \frac{2}{\pi} \int_{-\infty}^{x} e^{-z^2} dz, \qquad erfc(x) = 1 - erf(x) = \frac{2}{\pi} \int_{x}^{\infty} e^{-z^2} dz, \tag{5.8}$$

and both are related by

$$Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right). \tag{5.9}$$

As a simple example, try to demonstrate that if $X_1 \sim \mathcal{N}(0,1)$ and $X_2 = 2X_1$, then $X_2 \sim \mathcal{N}(0,4)$.

Fact: if $Y = aX + b$ where a, b are constants and $X \sim \mathcal{N}(m, \sigma)$ is a Gaussian random variable, then Y is a Gaussian random variable because Y is a linear function of X. This does not occur if the function is not linear, e.g., if $Z = X^2$, then Z is not Gaussian. Also, we have that $E(Y) = am + b$ and $\text{var}(Y) = a^2\sigma^2$.

In general, for an arbitrary random variable X, if we define $Y = aX + b$, then $E(Y) = aE(X) + b$ and $\text{var}(Y) = a^2\text{var}(X)$.

The cumulative distribution function (cdf) and pdf of the Gaussian random variable X with $m = 3$ and different values of σ is shown in Figure 5.1

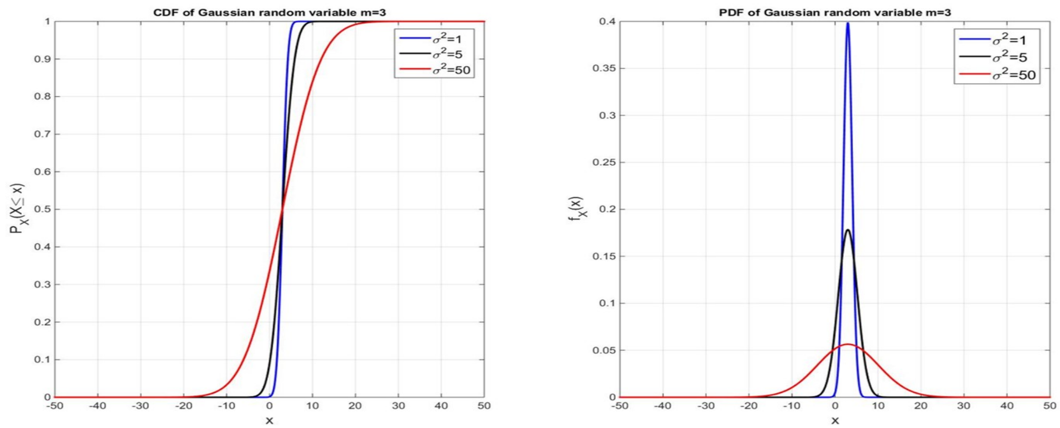

Figure 5.1: CDF and pdf of a Gaussian random variable with $m = 3$.

5.1.2 Gaussian Random Vectors

$\mathbf{X} = (X_1, X_2, \ldots, X_n)$ are *jointly* Gaussian if the joint pdf is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\,|\Lambda|^{1/2}} exp\left\{-\frac{1}{2}\left[(\mathbf{x}-\mathbf{m})^T\Lambda^{-1}(\mathbf{x}-\mathbf{m})\right]\right\}, \tag{5.10}$$

where Λ is the **covariance** matrix, which is an array of size nxn, $|\Lambda|$ is the absolute value of the determinant of Λ, var$(X_i) = \lambda_{ii}$ and the elements of Λ are λ_{ij}. $\mathbf{x}$ and $\mathbf{m}$ are nx1 vectors. The elements of $\mathbf{m}$ are $m_i = E(X_i)$.

Note that for $i \neq j$, we have λ_{ij} =cov(X_i, X_j). Also we have λ_{ii} =var(X_i) =cov$(X_i, X_i) = \sigma^2_{X_i}$. Recall that we have the definition

$$\text{cov}(X_i, X_j) = E\left[(X_i - E(X_i))(X_j - E(X_j))\right] = E(X_iX_j) - E(X_i)E(X_j), \tag{5.11}$$

and that we have cov(X_i, X_j) =cov(X_j, X_i).

The covariance matrix Λ is given as

$$\Lambda = \left[\begin{array}{ccccc} \sigma^2_{X_1} & \text{cov}(X_1, X_2) & \text{cov}(X_1, X_3) & \cdots & \text{cov}(X_1, X_n) \\ \text{cov}(X_2, X_1) & \sigma^2_{X_2} & \text{cov}(X_2, X_3) & \cdots & \text{cov}(X_2, X_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \text{cov}(X_n, X_1) & \text{cov}(X_n, X_2) & \text{cov}(X_n, X_3) & \cdots & \sigma^2_{X_n} \end{array}\right]. \tag{5.12}$$

Each component X_i is a Gaussian random variable with mean $m_i = E(X_i)$ and var$(X_i) = \lambda_{ii} = \sigma^2_{X_i}$.

If $X_1, X_2, \ldots, X_n$ are **uncorrelated**, then cov$(X_i, X_j) = 0$ for $i \neq j$ and

$$\Lambda = \text{diag}\left[\sigma^2_{X_1}, \sigma^2_{X_2}, \ldots, \sigma^2_{X_n}\right], \tag{5.13}$$

and

$$\Lambda^{-1} = \text{diag}\left[\frac{1}{\sigma^2_{X_1}}, \frac{1}{\sigma^2_{X_2}}, \ldots, \frac{1}{\sigma^2_{X_n}}\right], \tag{5.14}$$

and as a consequence of this diagonal matrix, we have that the pdf in (5.10) takes the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\left(\prod_{i=1}^{n}\sigma_{X_i}^2\right)^{1/2}} exp\left[-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i - m_i)^2}{\sigma_{X_i}^2}\right]. \tag{5.15}$$

Equation (5.15) can be rewritten as follows

$$f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n}\left\{\frac{1}{(2\pi)^{1/2}\sigma_{X_i}} e^{-\frac{(x_i-m_i)^2}{2\sigma_{X_i}^2}}\right\} = \prod_{i=1}^{n} f_{X_i}(x_i), \tag{5.16}$$

where

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}\sigma_{X_i}} e^{-\frac{(x_i-m_i)^2}{2\sigma_{X_i}^2}}. \tag{5.17}$$

Equation (5.16) implies that $X_1, X_2, \ldots, X_n$ are **independent**.

Fact: If $X_1, X_2, \ldots, X_n$ are **uncorrelated** Gaussian random variables, then they are **independent**. This only applies to Gaussian random variables.

For general random variables **independence** implies **uncorrelation**, but not the other way around, i.e.,

$$\begin{array}{ccc} \text{independent} & \text{implies} & \text{uncorrelated} \\ \text{uncorrelated} & \text{does not imply} & \text{independent,} \end{array} \tag{5.18}$$

but if the random variables are Gaussian, then **independence** $\Longleftrightarrow$ **uncorrelation**.

If $\mathbf{X} = [X_1, X_2, \ldots, X_n]^T$, $\mathbf{X} \sim \mathcal{N}(\mathbf{m_X}, \Lambda_{\mathbf{X}})$ is a Gaussian random vector and we define $\mathbf{Y} = A\mathbf{X} + B$ for matrices A of size mxn and B of size mx1, then $\mathbf{Y} = [Y_1, Y_2, \ldots, Y_m]^T$ is a Gaussian random vector because it is a linear function of $\mathbf{X}$ and

$$\mathbf{m_Y} = AE(\mathbf{X}) + B = A\mathbf{m_X} + B, \tag{5.19}$$

and

$$\Lambda_{\mathbf{Y}} = E[(\mathbf{Y} - \mathbf{m}_Y)(\mathbf{Y} - \mathbf{m}_Y)^T], \tag{5.20}$$

where $\mathbf{Y} - \mathbf{m}_Y$ is a vector of size mx1. We also have that

$$\begin{aligned} \Lambda_{\mathbf{Y}} &= E[(\mathbf{Y} - \mathbf{m}_Y)(\mathbf{Y} - \mathbf{m}_Y)^T] \\ &= E\left\{[A\mathbf{X} + B - (A\mathbf{m_X} + B)]\,[A\mathbf{X} + B - (A\mathbf{m_X} + B)]^T\right\} \\ &= E\left\{A(\mathbf{X} - \mathbf{m_X})[A(\mathbf{X} - \mathbf{m_X})]^T\right\} \\ &= E\left\{A(\mathbf{X} - \mathbf{m_X})(\mathbf{X} - \mathbf{m_X})^T A^T\right\} \\ &= AE\left\{(\mathbf{X} - \mathbf{m_X})(\mathbf{X} - \mathbf{m_X})^T\right\} A^T \\ &= A\Lambda_{\mathbf{X}}A^T. \end{aligned} \tag{5.21}$$

Equation (5.21) is true for any random vectors $\mathbf{X}$ and $\mathbf{Y}$.

Fact: A linear transformation of a Gaussian random vector is Gaussian.

5.2 Random Processes

A random process is an indexed family (collection) of random variables. Let T be a subset of $\mathbb{R}$, then for every $t \in T$, we have a random variable X_t, i.e., the random process is defined as $\{X_t; t \in T\}$. T is associated in general to time. When T is a countable set, then the random process is a **Discrete Time** random process, commonly called **Stochastic Process**.

Example: A Bernoulli random process $\{X_k; k \geq 1\}$, where X_i are independent and identically distributed (iid) random variables with

$$\begin{aligned} P(X_i = 0) &= p \\ P(X_i = 1) &= 1 - p, \end{aligned} \tag{5.22}$$

and for example we can have

$$P(X_1 = 0, X_{10} = 1, X_{15} = 0, X_{20} = 1) = p^2(1-p)^2. \tag{5.23}$$

Each such outcome of the random process is called a **Sample Function or Path** or a **Realization** of the process.

When the set T is **uncountable**, then the random process is **Continuous Time**, e.g., $T = \mathbb{R}$, then $\{X_t\}$ is a random process.

5.2.1 Characterization of Random Processes

For a complete characterization of a random process, we need to know the joint distribution of every finite collection of random variables that form the process. A partial characterization is what we really deal with. Given a random process $\{X_t\}$, the **Partial Characterization** can be obtained, for example, by obtaining the mean function

$$m_X(t) = E(X_t). \tag{5.24}$$

We can also obtain the n-th moments or the n-th central moments. There are other alternatives for the characterization of a random process such as

- **Autocorrelation Function.** It is defined as

$$R_X(t, s) = E(X_t X_s). \tag{5.25}$$

- **Covariance Function.** It is defined as

$$K_X(t, s) = R_X(t, s) - m_X(t) m_X(s). \tag{5.26}$$

Example: Let the random process be defined as

$$X_t = \cos(2\pi f_0 t + \Theta), \tag{5.27}$$

where f_0 is a constant and Θ is a uniformly distributed random variable in the interval $[0, 2\pi)$, i.e., $\Theta \sim U[0, 2\pi)$. The pdf of the random variable is

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta < 2\pi, \\ 0, & \text{elsewhere.} \end{cases} \tag{5.28}$$

the mean value is given by

$$\begin{aligned} m_X(t) &= E(X_t) \\ &= E\left[\cos(2\pi f_0 t + \Theta)\right] \\ &= \int_{-\infty}^{\infty} \cos(2\pi f_0 t + \theta) f_\Theta(\theta) d\theta \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \cos(2\pi f_0 t + \theta) d\theta \\ &= 0. \end{aligned} \tag{5.29}$$

The autocorrelation is obtained as follows

$$\begin{aligned} R_X(t,s) &= E\left[\cos(2\pi f_0 t + \Theta)\cos(2\pi f_0 s + \Theta)\right] \\ &= E\left\{\frac{1}{2}\cos[2\pi f_0(t-s)] + \frac{1}{2}\cos[2\pi f_0(t+s) + 2\Theta]\right\} \\ &= \frac{1}{2}\cos[2\pi f_0(t-s)] + \frac{1}{2}E\left\{\cos[2\pi f_0(t+s) + 2\Theta]\right\} \\ &= \frac{1}{2}\cos[2\pi f_0(t-s)] + 0 \\ &= \frac{1}{2}\cos[2\pi f_0(t-s)] \end{aligned} \tag{5.30}$$

As you can see in this example, the mean function is constant, and the autocorrelation function depends only on the time difference $(t - s)$. Any random process that satisfies these two conditions is said to be a **Wide Sense Stationary** (WSS) random process.

5.2.2 Properties of Random Processes

Stationarity

A random process is **stationary** if for any n, and $t_1, t_2, \ldots t_n$, and for any τ, the random vectors

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_n}), \tag{5.31}$$

and

$$(X_{t_1+\tau}, X_{t_2+\tau}, \ldots, X_{t_n+\tau}), \tag{5.32}$$

have the same distribution function.

[To do: *show that previous example is not stationary*].

Wide Sense Stationary (WSS)

The random process $\{X_t\}$ is WSS if

1. $m_X(t) = E(X_t)$ is independent of t, i.e., it is a constant.

2. $R_X(t_1, t_2) = E(X_{t_1} X_{t_2})$ depends **only** on the time difference $t_2 - t_1$.

Fact: Stationarity implies WSS, but WSS does not imply stationarity in general.
For a WSS random process

1. $m_X(t) = m$.

2. $R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau) = E(X_t X_{t+\tau})$. with $\tau = t_2 - t_1$.

Properties of the **autocorrelation function**

1. $R_X(\tau) = R_X(-\tau)$. The autocorrelation function is an even function.

2. $|R_X(\tau)| = R_X(0)$. From Schwarz inequality. The maximum value of the autocorrelation function is at the origin.

3. $R_X(0) = E(X_t^2)$ for any t. This is the **total average normalized** power of the random process (*signal*). This is the same as the mean squared value of Section 2.3.2.

Two random processes $\{X_t\}$ and $\{Y_t\}$ are jointly WSS if

1. $\{X_t\}$ is WSS

2. $\{Y_t\}$ is WSS, and

3. $R_{XY}(t, s) = E(X_t Y_s)$ depends only on $(s - t)$. This is called **Cross-Correlation** function.

5.2.3 Power Spectral Density (PSD)

The power spectral density tells how power is spread over frequency. Let us denote the PSD of process $\{X_t\}$ by $S_X(f)$, we want this to be a density, then

1. Total average power is given by $\int\limits_{-\infty}^{\infty} S_X(f) df$, and

2. $S_X(f) \geq 0$ for all f.

The definition of the PSD is the Fourier transform of the autocorrelation function, which is given by

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau, \tag{5.33}$$

note that the process $\{X_t\}$ MUST be WSS in order to have a Fourier transform.

The inverse is

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df, \tag{5.34}$$

and

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f) df. \tag{5.35}$$

Given a function $g(\tau)$, how can we say if $g(\tau)$ is a candidate for an autocorrelation function of some random process? See if it satisfies the three properties of the autocorrelation function. If it does, then a fourth criterion would be

$$\mathcal{F}\{g(\tau)\} = S(f) \geq 0, \qquad \forall f. \tag{5.36}$$

Properties of PSD

1. $S_X(f) \geq 0, \quad \forall f.$
2. $R_X(0) = \int_{-\infty}^{\infty} S_X(f) df$. Total average power.
3. Total average power in the bandwidth $[W_1, W_2]$

$$\int_{-W_2}^{-W_1} S_X(f) df + \int_{W_1}^{W_2} S_X(f) df. \tag{5.37}$$

4. $S_X(f) = S_X(-f)$, $S_X(f)$ is a real even function.

5.2.4 Gaussian Random Process

$\{X_t\}$ is a Gaussian random process (GRP) if for any n, any $t_1, t_2, \ldots, t_n$, the random variables $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ are jointly Gaussian.

A GRP is completely characterized by its mean $m_X(t)$ and autocorrelation function $R_X(t_1, t_2)$. To write joint pdf of $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ we need

$$m_X(t) = \begin{bmatrix} E(X_{t_1}) \\ E(X_{t_2}) \\ \vdots \\ E(X_{t_n}) \end{bmatrix} = \begin{bmatrix} m_X(t_1) \\ m_X(t_2) \\ \vdots \\ m_X(t_n) \end{bmatrix}, \tag{5.38}$$

and we also need the covariance matrix Λ_X =cov(X_{t_i}, X_{t_j}), where

$$\text{cov}(X_{t_i}, X_{t_j}) = R_X(t_i, t_j) - m_X(t_i)m_X(t_j). \tag{5.39}$$

If a GRP is WSS, then it is also **Strictly Stationary**. If the mean value is a constant, i.e., if $m_X(t) = m$, $\forall t$, then

$$\text{cov}(X_{t_i}, X_{t_j}) = R_X(t_j - t_i) - m^2. \tag{5.40}$$

5.2.5 White Gaussian Noise

$\{N(t)\}$ is a GRP with mean zero and autocorrelation function given by

$$R_N(\tau) = \frac{N_0}{2}\delta(\tau), \tag{5.41}$$

where $\delta(\tau)$ is the Dirac delta function. Its spectral density is the Fourier transform of the autocorrelation function which is

$$S_N(f) = \frac{N_0}{2}, \tag{5.42}$$

and the power of the signal $N(t)$ is given by

$$P = \int_{-\infty}^{\infty} S_N(f)df = R_N(0) = \infty, \tag{5.43}$$

therefore, $N(t)$ is a signal that is not physically realizable.

5.3 Linear Time Invariant (LTI) Systems

Figure 5.2 shows a typical linear time invariant system, with input $X(t)$ a random process, output $Y(t)$, *impulse response* $h(t)$ and frequency response $H(f)$, sometimes called transfer function. The output is given by the convolution integral in any of its two formats, i.e., recall that convolution is commutative.

$$Y(t) = X(t) * Y(t) = \int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} X(t-\tau)h(\tau)d\tau. \tag{5.44}$$

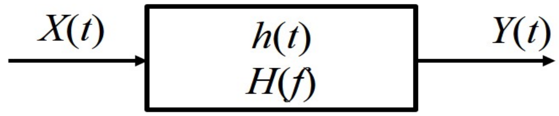

Figure 5.2: LTI System with input $X(t)$ and output $Y(t)$.

Now, assume that $X(t)$ is a WSS random process with mean value m_X and autocorrelation function $R_X(\tau)$ with τ the time difference, then the mean value of the output process will be given by

$$\begin{aligned} E\left[Y(t)\right] &= E\left[X(t) * Y(t)\right] \\ &= E\left[\int\limits_{-\infty}^{\infty} X(t-\tau)h(\tau)d\tau\right] \\ &= \int\limits_{-\infty}^{\infty} E\left[X(t-\tau)\right]h(\tau)d\tau \\ &= \int\limits_{-\infty}^{\infty} m_X h(\tau)d\tau \\ &= m_X \int\limits_{-\infty}^{\infty} h(\tau)d\tau \\ &= m_X H(0) \\ &= m_Y, \end{aligned} \tag{5.45}$$

hence m_Y is constant and does not depend on t. So as a conclusion, whenever a WSS process is at the input of a LTI system, the output will be also WSS, although at this point we still need to show that the autocorrelation of $Y(t)$ depends only on the time difference. Note that $H(0) = \int\limits_{-\infty}^{\infty} h(\tau)d\tau$.

Now we calculate the autocorrelation of $Y(t)$ as follows

$$\begin{aligned} R_Y(t_1, t_2) &= E\left[Y(t_1) * Y(t_2)\right] \\ &= E\left[\int_{-\infty}^{\infty} X(t_1 - \tau)h(\tau)d\tau \int_{-\infty}^{\infty} X(t_2 - \nu)h(\nu)d\nu\right] \\ &= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t_1 - \tau)X(t_2 - \nu)h(\tau)h(\nu)d\tau d\nu\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[X(t_1 - \tau)X(t_2 - \nu)\right] h(\tau)h(\nu)d\tau d\nu \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t_2 - t_1 - \nu + \tau)h(\tau)h(\nu)d\tau d\nu, \end{aligned} \tag{5.46}$$

hence the autocorrelation of $Y(t)$ depends only on the time difference $(t_2 - t_1)$, therefore $Y(t)$ is WSS, and defining $\eta = (t_2 - t_1)$, we get

$$R_Y(\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\eta - \nu + \tau)h(\tau)h(\nu)d\tau d\nu, \tag{5.47}$$

therefore, we have that whenever a WSS process $X(t)$ is at the input of an LTI system, the output will also be WSS.

When $h(t)$ is real, we have that $H(-f) = H^*(f)$, then the spectral density of the output is given by

$$\begin{aligned} S_Y(f) &= S_X(f)H(f)H(-f) \\ &= S_X(f)H(f)H^*(f) \\ &= S_X(f)|H(f)|^2. \end{aligned} \tag{5.48}$$

Part II

Digital Communication Systems

Chapter 6

Model of the Communication System

The block diagram of a digital communication system is shown in Figure 6.1, where symbols M_k belong to a stochastic process $\{M_k, k \geq 0\}$.

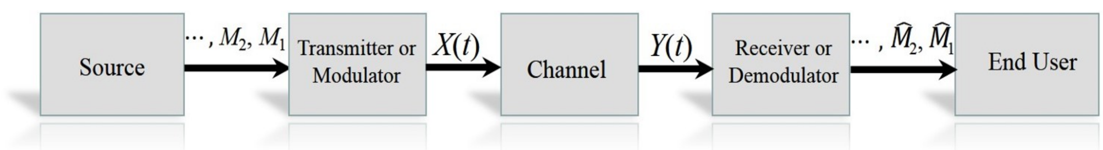

Figure 6.1: The concept of a digital communication system.

The source is a discrete time finite valued random process where $\{M_k\}$ are iid **source symbols**. The values that $\{M_k\}$ takes for any k, come from a sample space that is known as the **source alphabet** Ω_M, i.e.

$$\Omega_M = \{m_1, m_2, \ldots, m_q\}, \quad M_i \in \Omega_M. \tag{6.1}$$

The pmf of the alphabet is known, i.e., $p_M(m_i) = P(M_k = m_i)$, $\forall k$ is known, and $\sum_{i=1}^{q} p_M(m_i) = 1$. Each element of the source alphabet m_i is known as a **source letter**. The rate of the source is R_s, i.e., every time $T = 1/R_s$ seconds, one symbol is emitted by the source.

6.1 Channel

The channel in a communication system is considered to be Additive White Gaussian Noise (AWGN) as shown in Figure 6.2. The output of the channel is the input signal $X(t)$ with an additive signal which is noise $N(t)$. The noise process $\{N(t)\}$ is independent of the input signal $X(t)$. Since $\{N(t)\}$ is AWGN, we have that

$$E[N(t)] = 0, \quad S_N(f) = \frac{N_0}{2}, \ \forall f. \tag{6.2}$$

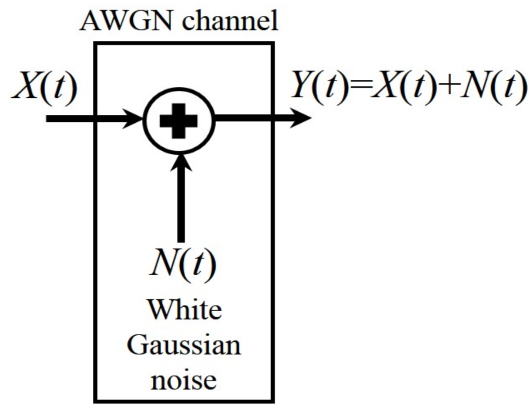

Figure 6.2: The concept of an Additive White Gaussian Noise (AWGN) channel.

6.2 Modulator

The modulator in the communication system is what is known as a *letter by letter* modulator. It assigns signals to source letters, i.e., for each letter m_i in the alphabet Ω_M, there is a signal $s_i(t)$. The **signal set** specifies completely the modulator and is given by $\mathbb{S} = \{s_1(t), s_2(t), \ldots, s_q(t)\}$. All signals must be distinct. Note that the number of signals in $\mathbb{S}$ is the same as the number of letters in Ω_M.

6.3 Demodulator

The demodulator is the system in the receiver that will detect and deliver the estimated symbol based on the received signal $Y(t)$. It receives $Y(t) = X(t) + N(t)$, and it must produce an estimate of the source output symbol. We call these estimates $\widehat{M}_i$, $i = 1, 2, \ldots$. The optimal demodulator is designed using *decision theory*.

6.4 Major Goals

The major goals of a digital communication system are:

1. Given the modulator $\mathbb{S}$, find the demodulator that has the lowest probability of error $P(\mathcal{E})$, or the maximum probability of correct $P(\mathcal{C})$, i.e.,

$$P(\mathcal{E}) = P(M_i \neq \widehat{M_i}), \tag{6.3}$$

$$P(\mathcal{C}) = P(M_i = \widehat{M_i}) = 1 - P(\mathcal{E}). \tag{6.4}$$

2. For a given source and channel, design a modulator such that when used with the best demodulator gives a low error probability we want. It will have low power, small bandwidth and low complexity.

6.5 Examples of Signal Sets

The signal set $\mathbb{S}$ can have several different options such as the following examples:

1. BPSK: For $q = 2$ we can have

$$s_1(t) = A\cos(2\pi f_0 t), \quad s_2(t) = -A\cos(2\pi f_0 t), \quad 0 \le t \le T. \tag{6.5}$$

2. mASK

$$s_i(t) = A_i \cos(2\pi f_0 t), \;\; i = 1, 2, \ldots, q, \quad 0 \le t \le T. \tag{6.6}$$

3. QPSK: For $q = 4$

$$s_i(t) = A\cos\left(2\pi f_0 t + (i-1)\frac{\pi}{2}\right), \;\; i = 1, 2, 3, 4, \quad 0 \le t \le T. \tag{6.7}$$

4. BFSK: For $q = 2$ we have

$$s_1(t) = A\cos(2\pi f_0 t), \quad s_2(t) = A\cos(2\pi f_1 t), \quad 0 \le t \le T. \tag{6.8}$$

Chapter 7

Decision and Estimation Theory

Suppose we have two random variables M and R. If M (e.g., a message transmitted) is the outcome of some random experiment we would like to know, but we can not observe it, and if R (e.g., the signal received) is the outcome of some random experiment that we can observe, and if M and R are not independent, then it is reasonable to expect that R might profitably be used to make an estimate of the unobserved value of M, i.e., based on the observation of R, we would like to guess the outcome of M.

The study of rules for doing just this is the topic of decision and estimation theory. This theory tells us how to find a decision (or estimation) rule which, for any $R = r$ that might occur, indicates a good guess for M. That is, an estimation rule is a function which assigns an estimate $\widehat{m}$ to every potential observed value of $R = r$. In particular the theory can provide us with the best such rule.

In order to characterize the best rule, we must know the relationship between M and R, i.e., their joint distribution. In addition, we must specify how to measure the *goodness* of an estimation rule. Indeed for different measures of goodness different rules turn out to be best. In discrete processes, estimation and decision are the same.

Suppose we have two random variables (r.v.) M and R, where M is discrete with alphabet $\Omega_M = \{m_1, m_2, \ldots, m_q\}$, while R can be discrete or continuous and has alphabet Ω_R. The joint distribution of M and R is given by $p_{MR}(m, r)$. If R is discrete, then $p_{MR}(m, r)$ is the probability mass function (pmf), while if R is continuous, $p_{MR}(m, r)$ is mixed. We want to know the value of M, but we can only observe the outcome of the r.v. R and must estimate M from the observation of R. When the r.v. M is discrete, the estimation rule is commonly called a decision rule rather than an estimation rule.

We need a **decision rule** i.e., a function $g : \Omega_R \longrightarrow \Omega_M$. g is a function that maps outcomes from Ω_R into Ω_M. If the outcome of R is r, then our estimate of M is $\widehat{M} = g(r)$.

7.1 Decision Regions

We need to have a good decision rule (the best we can). In order to achieve this, we can define a criterion based on decision regions. A good way to picture a decision rule is via its **decision**

regions. For $i = 1, 2, \ldots, q$, let $I_i \triangleq \{r : g(r) = m_i\}$, i.e., I_i is called the i^{th} decision region, $I_i \subset \Omega_R$. If $r \in I_j$, then $g(r) = m_j$ is our estimate of M, see Figure 7.1. The collection of decision regions $\{I_1, I_2, \ldots, I_q\}$ forms a partition of Ω_R, that is

$$I_i \cap I_j = \emptyset, \quad \forall \quad i \neq j, \quad \text{and} \quad \bigcup_{i=1}^{q} I_i = \Omega_R. \tag{7.1}$$

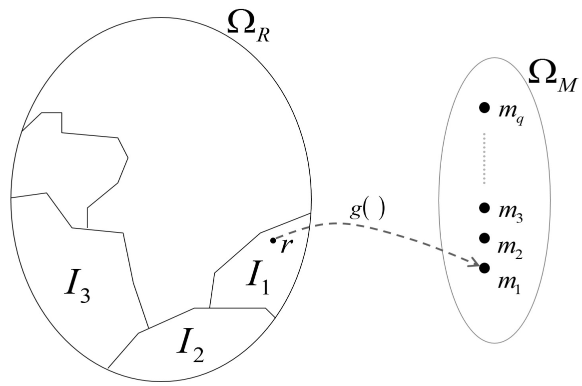

Figure 7.1: Mapping of decision regions into estimates.

Our criterion of goodness can be the probability of error, then this is called **Minimum Probability of Error Decision Rule**, which is valid only for M discrete.

7.2 Minimum Probability of Error Decision Rule

Let us agree that the best decision rule is one that minimizes the probability that it is in error. For a rule $g(r)$ this is defined as

$$\text{Probability of error } = P(\mathcal{E}) \triangleq P(M \neq g(R)) = P(\{(m,r) : g(r) \neq m\}). \tag{7.2}$$

Recall that $I_1, I_2, \ldots, I_q$ are called decision regions, also that $P(\mathcal{C}) = 1 - P(\mathcal{E})$. With this, we can write

$$\begin{aligned} P(\mathcal{E}) &= P(\{(m,r) : m \neq g(r)\}) \\ &= \sum_{m_i} \int\limits_{\{r:g(r)\neq m_i\}} p_{MR}(m,r)\, dr \\ &= \sum_{m_i} \int\limits_{I_i^c} p_{MR}(m,r)\, dr. \end{aligned} \tag{7.3}$$

Equivalently, we can write this in terms of the probability of being correct, i.e.,

$$\begin{aligned}
P(\mathcal{C}) &= P(\{(m,r) : m = g(r)\}) \\
&= \sum_{m_i} \int_{\{r:g(r)=m_i\}} p_{MR}(m,r)\,dr \\
&= \sum_{m_i} \int_{I_i} p_{MR}(m,r)\,dr. \qquad (7.4)
\end{aligned}$$

In order to obtain the probability of error in (7.3) or that of correct in (7.4), we need to condition to find the joint density needed, i.e, $p_{MR}(m,r)$. One can condition on two events, $\{M = m_i\}$ or $\{R = r\}$. In this case, we need to condition on the latter, since the evidence that the receiver has is that of the received signal R and not of the signal being transmitted M. With this in mind, and assuming that R is continuous, we get

$$\begin{aligned}
P(\mathcal{C}) &= P(M = g(R)) \\
&= \int_{\Omega_R} P(M = g(R)|R = r)p_R(r)dr \\
&= \int_{\Omega_R} P(M = g(r)|R = r)p_R(r)dr. \qquad (7.5)
\end{aligned}$$

The best decision rule is that which maximizes the probability of correct. Now, since all the terms in the expression are non-negative, to maximize $P(\mathcal{C})$, we can maximize $P(\mathcal{C})$ by choosing (for each given r) $g(r)$ to be that $m \in \Omega_M$ for which $P(M = g(r)|R = r) = P(M = m|R = r)$ is maximum. For example, suppose that $R = \alpha$, and that we have

$$\begin{aligned}
P(M = m_1|R = \alpha) &= 0.1 \\
P(M = m_2|R = \alpha) &= 0.5 \\
P(M = m_3|R = \alpha) &< 0.5 \\
&\vdots \\
P(M = m_q|R = \alpha) &< 0.5, \qquad (7.6)
\end{aligned}$$

then choose $g(\alpha) = m_2$ because

$$P(M = m_2|R = \alpha) > P(M = m_i|R = \alpha), \quad \forall i \neq 2. \qquad (7.7)$$

Therefore, to maximize $P(\mathcal{C})$, for each $r \in \Omega_R$, let $g(r)$ be that value in Ω_M, say $\widehat{m}$, for which $P(M = \widehat{m}|R = r) = p_{M|R}(\widehat{m}|r)$ is largest.

7.2.1 Decision Rule

Based on what has been discussed in previous paragraphs, the decision rule would be the following:

Maximum A Posteriori Rule (MAP)
Given $R = r$, let

$$g(r) = m_i, \tag{7.8}$$

IFF (if and only if)

$$p_{M|R}(m_i|r) \geq p_{M|R}(m_j|r), \quad \forall j \neq i. \tag{7.9}$$

Break off ties arbitrarily, say that

$$p_{M|R}(m_k|r) = p_{M|R}(m_i|r) > p_{M|R}(m_j|r), \;\; \forall j \neq k, i, \tag{7.10}$$

then choose $g(r) = m_i$ arbitrarily. The probability of correct or the probability of error will be the same for both choices.

Now, we will have some other useful expressions for the probability of error and probability of correct. For these expressions, we use the law of total probability

$$\begin{aligned} P(\mathcal{E}) &= P(M \neq g(R)) \\ &= \sum_{i=1}^{q} P(g(R) \neq M | M = m_i) P(M = m_i) \\ &= \sum_{i=1}^{q} P(g(R) \neq m_i | M = m_i) p_M(m_i) \\ &= \sum_{i=1}^{q} P(\mathcal{E}|m_i)\, p_M(m_i). \end{aligned} \tag{7.11}$$

From (7.11), we can see that

$$\begin{aligned} P(\mathcal{E}|m_i) &= P(g(R) \neq m_i | M = m_i) \\ &= \int_{\{r: g(r) \neq m_i\}} p_{R|M}(r|m_i) dr \\ &= \int_{I_i^c} p_{R|M}(r|m_i) dr, \end{aligned} \tag{7.12}$$

then

$$P(\mathcal{E}) = \sum_{i=1}^{q} P(\mathcal{E}|m_i)\, p_M(m_i), \tag{7.13}$$

and

$$P(\mathcal{C}) = \sum_{i=1}^{q} P(\mathcal{C}|m_i)\, p_M(m_i), \tag{7.14}$$

where

$$
\begin{aligned}
P(\mathcal{C}|m_i) &= P(g(R) = m_i | M = m_i) \\
&= \int_{\{r:g(r)=m_i\}} p_{R|M}(r|m_i)dr \\
&= \int_{I_i} p_{R|M}(r|m_i)dr.
\end{aligned}
\tag{7.15}
$$

7.2.2 Example

Let $\Omega_M = \{m_1, m_2\}$, $p_M(m_1) = 1/4$ and

$$
p_{R|M}(r|m_1) = \begin{cases} e^{-r}, & r \geq 0, \\ 0, & r < 0, \end{cases} \tag{7.16}
$$

and

$$
p_{R|M}(r|m_2) = \begin{cases} 2e^{-2r}, & r \geq 0, \\ 0, & r < 0. \end{cases} \tag{7.17}
$$

Find the best decision rule and $P(\mathcal{E})$.

Solution.

The best decision rule would be

$$
g(r) = m_i, \tag{7.18}
$$

IFF (if and only if)

$$
p_{M|R}(m_i|r) \geq p_{M|R}(m_j|r), \quad \forall j \neq i, \tag{7.19}
$$

then as we can see, we need to get $p_{M|R}(m_i|r)$. We can obtain this conditional probability using Bayes rule, i.e.,

$$
p_{M|R}(m_i|r) = \frac{p_{R|M}(r|m_i)p_M(m_i)}{p_R(r)}. \tag{7.20}
$$

With (7.20), we can substitute it into (7.19) to get the best decision rule as

$$
g(r) = m_i, \tag{7.21}
$$

IFF (if and only if)

$$
\frac{p_{R|M}(r|m_i)p_M(m_i)}{p_R(r)} \geq \frac{p_{R|M}(r|m_j)p_M(m_j)}{p_R(r)}, \quad \forall j \neq i. \tag{7.22}
$$

Therefore, we get from (7.22) the following

$$
g(r) = m_i, \tag{7.23}
$$

IFF (if and only if)

$$p_{R|M}(r|m_i)p_M(m_i) \geq p_{R|M}(r|m_j)p_M(m_j), \quad \forall j \neq i, \tag{7.24}$$

or

$$g(r) = m_1, \tag{7.25}$$

IFF (if and only if)

$$\begin{aligned} p_{R|M}(r|m_1)p_M(m_1) &\geq p_{R|M}(r|m_2)p_M(m_2), \\ e^{-r}\frac{1}{4} &\geq 2e^{-2r}\frac{3}{4}. \end{aligned} \tag{7.26}$$

Thus

$$g(r) = m_1, \quad \text{IFF} \quad r \geq \ln 6, \tag{7.27}$$

$I_1 = \{r : r \geq \ln 6\}$ and $I_2 = \{r : r < \ln 6\} = I_1^c$,

$$g(r) = \begin{cases} m_1, & \text{if } r \geq \ln 6, \\ m_2, & \text{if } r < \ln 6. \end{cases} \tag{7.28}$$

This can be seen in Figure 7.2, where decision regions I_1 and I_2 are shown together with the threshold of $\ln(6)$ as obtained by the decision rule.

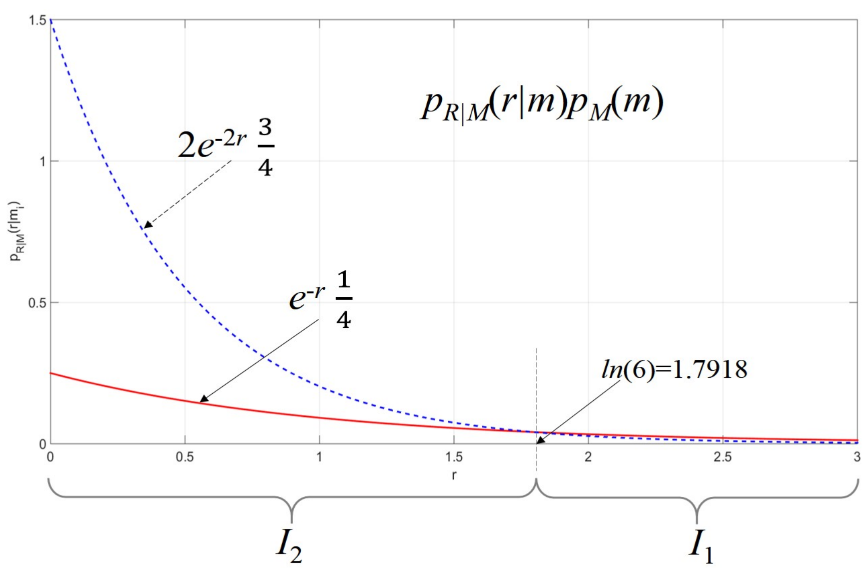

Figure 7.2: Decision regions for example.

Now, the probability of correct can be calculated using the law of total probability and considering that the sample space is partitioned in two by the events $\{M = m_1\}$ and $\{M = m_2\}$. This is given by

$$P(\mathcal{C}) = P(\mathcal{C}|m_1)\,p_M(m_1) + P(\mathcal{C}|m_2)\,p_M(m_2), \tag{7.29}$$

then we need to get $P\left(\mathcal{C}|m_i\right)$ for $i = 1, 2$.

$$\begin{aligned} P\left(\mathcal{C}|m_1\right) &= P(g(R) = m_1|M = m_1) \\ &= \int_{I_1} p_{R|M}(r|m_1)dr \\ &= \int_{\ln 6}^{\infty} e^{-r}dr \\ &= -e^{-r}\Big]_{\ln 6}^{\infty} \\ &= \frac{1}{6}. \end{aligned} \tag{7.30}$$

Recall that $e^{\ln x} = x^{\ln e} = x$, also $a^{\log_c b} = b^{\log_c a}$ and $k\log(x) = \log(x^k)$. Similarly, we get for m_2

$$\begin{aligned} P\left(\mathcal{C}|m_2\right) &= P(g(R) = m_2|M = m_2) \\ &= \int_{I_2} p_{R|M}(r|m_2)dr \\ &= \int_{0}^{\ln 6} 2e^{-2r}dr \\ &= -e^{-2r}\Big]_{0}^{\ln 6} \\ &= \frac{35}{36}, \end{aligned} \tag{7.31}$$

then we get

$$\begin{aligned} P\left(\mathcal{C}\right) &= P\left(\mathcal{C}|m_1\right)p_M(m_1) + P\left(\mathcal{C}|m_2\right)p_M(m_2) \\ &= \frac{1}{6} * \frac{1}{4} + \frac{35}{36} * \frac{3}{4} \\ &= \frac{1}{24} + \frac{35}{48} \\ &= \frac{37}{48}. \end{aligned} \tag{7.32}$$

We can also perform the analysis with the probability of error as follows

$$P\left(\mathcal{E}\right) = P\left(\mathcal{E}|m_1\right)p_M(m_1) + P\left(\mathcal{E}|m_2\right)p_M(m_2), \tag{7.33}$$

then we need to get $P(\mathcal{E}|m_i)$ for $i = 1, 2$.

$$
\begin{aligned}
P(\mathcal{E}|m_1) &= P(g(R) \neq m_1 | M = m_1) \\
&= \int_{I_1^c} p_{R|M}(r|m_1) dr \\
&= \int_0^{\ln 6} e^{-r} dr \\
&= -e^{-r}\Big]_0^{\ln 6} \\
&= \frac{5}{6}.
\end{aligned} \tag{7.34}
$$

Similarly, we get for m_2

$$
\begin{aligned}
P(\mathcal{E}|m_2) &= P(g(R) \neq m_2 | M = m_2) \\
&= \int_{I_2^c} p_{R|M}(r|m_2) dr \\
&= \int_{\ln 6}^{\infty} 2e^{-2r} dr \\
&= -e^{-2r}\Big]_{\ln 6}^{\infty} \\
&= \frac{1}{36},
\end{aligned} \tag{7.35}
$$

then we get

$$
\begin{aligned}
P(\mathcal{E}) &= P(\mathcal{E}|m_1) p_M(m_1) + P(\mathcal{E}|m_2) p_M(m_2) \\
&= \frac{5}{6} * \frac{1}{4} + \frac{1}{36} * \frac{3}{4} \\
&= \frac{5}{24} + \frac{1}{48} \\
&= \frac{11}{48} \\
&= 1 - P(\mathcal{C}).
\end{aligned} \tag{7.36}
$$

Maximum Likelihood (ML) Decision Rule.
This is a special case of the MAP rule. The MAP rule is given by:
Given $R = r$, let

$$g(r) = m_i, \tag{7.37}$$

IFF (if and only if)

$$
\begin{aligned}
p_{M|R}(m_i|r) &\geq p_{M|R}(m_j|r), \quad \forall j \neq i \\
\frac{p_{R|M}(r|m_i) p_M(m_i)}{p_R(r)} &\geq \frac{p_{R|M}(r|m_j) p_M(m_j)}{p_R(r)}, \quad \forall j \neq i \\
p_{R|M}(r|m_i) p_M(m_i) &\geq p_{R|M}(r|m_j) p_M(m_j), \quad \forall j \neq i.
\end{aligned} \tag{7.38}
$$

Now, suppose that $p_M(m_i) = \frac{1}{q}$ for all $i = 1, 2, \ldots, q$, i.e., all the values of M are equally likely. Then the MAP rule simplifies to what is called the **Maximum Likelihood (ML)** Decision Rule, which is

Given $R = r$, let

$$g(r) = m_i, \tag{7.39}$$

IFF (if and only if)

$$p_{R|M}(r|m_i) \geq p_{R|M}(r|m_j), \quad \forall j \neq i. \tag{7.40}$$

$p_{R|M}(r|m)$ is the likelihood of m.

7.2.3 Another Example

Consider the system shown in Figure 7.3, where the input M is a discrete amplitude in volts and takes values from the set $\Omega_M = \{m_1, m_2\}$ with $m_2 > m_1$ and a-priori probabilities $p_M(m_1) = p$, $p_M(m_2) = 1 - p$.

The channel has noise N added where $N \sim \mathcal{N}(0, \sigma^2)$ is a Gaussian random variable with zero mean and variance σ^2. M and N are independent.

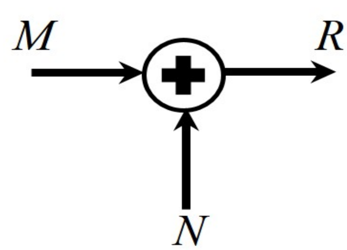

Figure 7.3: System for example.

Find the best decision rule for M based on R, and the probability of error.

Solution:

First note that the noise N has pdf

$$f_N(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x)^2}{2\sigma^2}}. \tag{7.41}$$

Second, we need to get $p_{R|M}(r|m)$. We start by trying to get the conditional cdf, i.e.,

$$\begin{aligned}
F_{R|M}(r|m) &= P(R \leq r|M = m) \\
&= P(M + N \leq r|M = m) \\
&= P(N + m \leq r|M = m) \\
&= P(N \leq r - m|M = m) \quad \text{then by independence} \\
&= P(N \leq r - m) \\
&= F_N(r - m).
\end{aligned} \tag{7.42}$$

From the cdf in Equation (7.42), we can get the pdf by taking the derivative with respect to r as follows

$$\begin{aligned}
f_{R|M}(r|m) &= p_{R|M}(r|m) \\
&= \frac{d}{dr} F_{R|M}(r|m) \\
&= \frac{d}{dr} F_N(r - m) \\
&= f_N(r - m) \\
&= p_N(r - m) \\
&= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r-m)^2}{2\sigma^2}}.
\end{aligned} \tag{7.43}$$

From the pdf in Equation (7.43), we can see that the random variable R is also Gaussian with the mean m and variance σ^2.

Now, we can obtain the decision rule.

Decision rule.

$$g(r) = m_1 \quad \textbf{IFF} \quad p_{M|R}(m_1|r) \geq p_{M|R}(m_2|r). \tag{7.44}$$

Equivalently, using Baye's rule, we can say that

$$g(r) = m_1 \quad \textbf{IFF} \quad p_{R|M}(r|m_1)p_M(m_1) \geq p_{R|M}(r|m_2)p_M(m_2). \tag{7.45}$$

Now, using the a-priori probabilities together with Equation (7.43), we get

$$\begin{aligned}
p_N(r-m_1)p &\geq p_N(r-m_2)(1-p) \\
\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(r-m_1)^2}{2\sigma^2}}p &\geq \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(r-m_2)^2}{2\sigma^2}}(1-p) \\
\frac{p}{1-p} &\geq e^{\frac{(r-m_1)^2}{2\sigma^2}}e^{-\frac{(r-m_2)^2}{2\sigma^2}} \\
\ln\left(\frac{p}{1-p}\right) &\geq \frac{(r-m_1)^2}{2\sigma^2}-\frac{(r-m_2)^2}{2\sigma^2} \\
2\sigma^2\ln\left(\frac{p}{1-p}\right) &\geq r^2-2rm_1+m_1^2-r^2+2rm_2-m_2^2 \\
2\sigma^2\ln\left(\frac{p}{1-p}\right) &\geq 2r(m_2-m_1)+m_1^2-m_2^2. \qquad (7.46)
\end{aligned}$$

Now, from Equation (7.46) we can obtain an inequality in terms of r to get the decision rule as follows

$$\begin{aligned}
g(r) &= m_1 \quad \text{IFF} \\
r &\leq \frac{\sigma^2}{m_2-m_1}\ln\left(\frac{p}{1-p}\right)+\frac{m_1+m_2}{2}=\mathbf{Th}. \qquad (7.47)
\end{aligned}$$

The decision rule, together with the pdfs is shown in Figure 7.4.

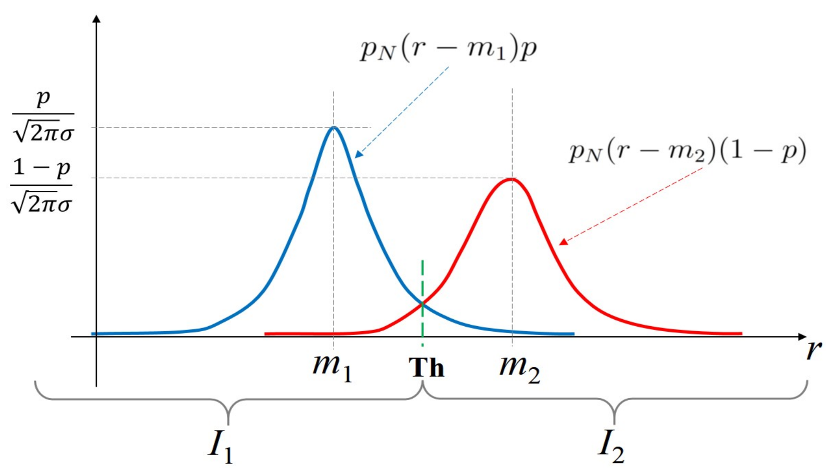

Figure 7.4: Decision regions for example, $p > (1-p)$ is assumed.

Now, to find the probability of error, we use the law of total probability with the sample space Ω_M as follows

$$P(\mathcal{E})=\sum_{i=1}^{2}P(\mathcal{E}|m_i)\,p_M(m_i), \qquad (7.48)$$

then

$$
\begin{aligned}
P(\mathcal{E}|m_1) &= P\left(R \in I_1^c | M = m_1\right) \\
&= \int_{I_1^c} p_{R|M}(r|m_1)dr \\
&= \int_{\mathbf{Th}}^{\infty} p_{R|M}(r|m_1)dr \\
&= \int_{\mathbf{Th}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r-m_1)^2}{2\sigma^2}} dr, \quad \text{change to } \; z = \frac{r-m_1}{\sigma}, \\
&= \int_{\frac{\mathbf{Th}-m_1}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2}} \sigma dz \\
&= \int_{\frac{\mathbf{Th}-m_1}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= Q\left(\frac{\mathbf{Th}-m_1}{\sigma}\right),
\end{aligned}
\tag{7.49}
$$

and similarly, we can get

$$
\begin{aligned}
P(\mathcal{E}|m_2) &= P\left(R \in I_2^c | M = m_2\right) \\
&= \int_{I_2^c} p_{R|M}(r|m_2)dr \\
&= \int_{-\infty}^{\mathbf{Th}} p_{R|M}(r|m_1)dr \\
&= \int_{-\infty}^{\mathbf{Th}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r-m_2)^2}{2\sigma^2}} dr, \quad \text{change to } \; z = \frac{r-m_2}{\sigma}, \\
&= \int_{-\infty}^{\frac{\mathbf{Th}-m_2}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= 1 - \int_{\frac{\mathbf{Th}-m_2}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= 1 - Q\left(\frac{\mathbf{Th}-m_2}{\sigma}\right) = Q\left(\frac{m_2-\mathbf{Th}}{\sigma}\right).
\end{aligned}
\tag{7.50}
$$

Then the probability of error for the system is

$$P(\mathcal{E}) = pQ\left(\frac{\mathbf{Th} - m_1}{\sigma}\right) + (1-p)Q\left(\frac{m_2 - \mathbf{Th}}{\sigma}\right). \quad (7.51)$$

As a special case, we can consider that $p = 1/2$, then $\mathbf{Th} = \frac{m_1+m_2}{2}$ and we get

$$P(\mathcal{E}) = Q\left(\frac{m_2 - m_1}{2\sigma}\right). \quad (7.52)$$

One can see that the $Q()$ function in Equation (7.52) will decrease in value when $m_2 - m_1$ increases, or when σ decreases. This will make probability of error decrease, i.e., the larger the difference in values of the symbols m_1 and m_2, the smaller the probability of error.

7.3 Sufficient Statistics

As discussed previously, we are often encountered with the problem of having to estimate something (a random variable M) from the observation of something else (another random variable R). It turns out however that in many cases the data provided by the observation is more than necessary. In other words the observation contains information that is useless as far as the estimation process is concerned. In such situations we often process the observed data by removing the unnecessary information before implementing an estimation rule. Obviously, in such processing we do not want to discard any useful information.

In the following, we will present this processing by a function. An important question then is what type of functions can be used on the observation so that while removing unnecessary data, they do not remove any useful information. In the following, these notions will be made clear rigorously.

Definition 1. Let f be a given function. $Z = f(R)$ is called a **sufficient statistic** for decisions about M based on R if M and R are conditionally independent given Z, i.e.,

$$p_{MR|Z}(m, r|z) = p_{M|Z}(m|z)p_{R|Z}(r|z), \quad \text{for all } m, r, z. \quad (7.53)$$

Recall that the last result comes from the definition of the conditional probability, i.e., we know that if A and B are two independent events, we have $P(A|B) = P(A, B)/P(B) = P(A)P(B)/P(B) = P(A)$, then

$$\begin{aligned} p_{MR|Z}(m, r|z) &= \frac{p_{MRZ}(m, r, z)}{p_Z(z)} \\ &= \frac{p_{MRZ}(m, r, z)}{p_Z(z)} \frac{p_{MZ}(m, z)}{p_{MZ}(m, z)} \\ &= \frac{p_{MZ}(m, z)}{p_Z(z)} \frac{p_{MRZ}(m, r, z)}{p_{MZ}(m, z)} \\ &= p_{M|Z}(m|z)p_{R|MZ}(r|m, z) \\ &= p_{M|Z}(m|z)p_{R|Z}(r|z). \end{aligned} \quad (7.54)$$

Note

1. If M and R are independent, it does not imply that M and R are conditionally independent given Z.
2. If M and R are conditionally independent given Z, it does not imply that M and R are independent

Definition 2. An alternative definition of sufficient statistics would be: $Z = f(R)$ is a sufficient statistic for M based on R if

$$p_{R|MZ}(r|m,z) = p_{R|Z}(r|z), \quad \text{for all } \ m,r,z \quad \text{such that } \ p_{M|Z}(m|z) > 0, \tag{7.55}$$

or equivalently

$$p_{M|RZ}(m|r,z) = p_{M|Z}(m|z), \quad \text{for all } \ m,r,z \quad \text{such that } \ p_{R|Z}(r|z) > 0, \tag{7.56}$$

The following result provides a justification for the definition of sufficient statistic.

Claim: If Z is sufficient statistic for decision about M based on R, then the best decision rule based on Z ($g'(z)$ to estimate $\widehat{\widehat{M}}$) is as good as the best decision rule based on R ($g(r)$ to estimate $\widehat{M}$), see Figure 7.5.

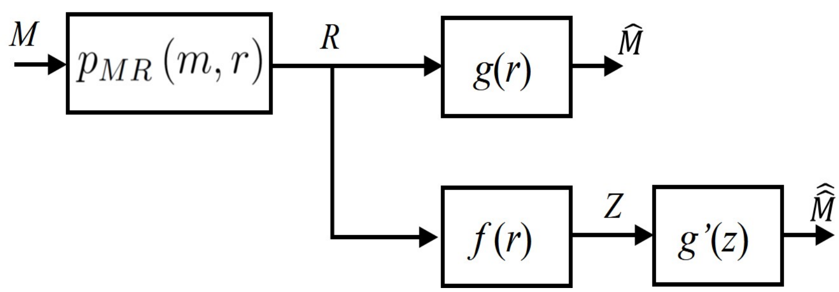

Figure 7.5: Simplifying process in sufficient statistics.

Proof. To prove this we show the following:

1. The best estimate based on Z and R is as good as the best estimate based on Z. This is obvious because we can just ignore R.
2. The best estimate based on Z is as good as the best estimate based on Z and R.

1 and 2 prove our claim. Then we prove 2.
The best rule based on Z and R is, say g_1, where

$$\begin{aligned} g_1(z,r) = \quad & m_i \quad \text{IFF } \ (z,r) \in I_i \\ I_i \quad & = \left\{(z,r) : p_{M|ZR}(m_i|z,r) \geq p_{M|ZR}(m_j|z,r) \ \text{ for all } \ j \neq i\right\}, \end{aligned} \tag{7.57}$$

or

$$g_1(z,r) = m_i \qquad \text{IFF} \\ p_{M|ZR}(m_i|z,r) \geq p_{M|ZR}(m_j|z,r) \;\text{ for all }\; j \neq i. \tag{7.58}$$

Now, say that the best rule based on Z is g_2, i.e.,

$$g_2(z) = m_i \qquad \text{IFF} \\ p_{M|Z}(m_i|z) \geq p_{M|Z}(m_j|z) \;\text{ for all }\; j \neq i. \tag{7.59}$$

Now, show that $P_{g_1}(\mathcal{E}) = P_{g_2}(\mathcal{E})$, or $P_{g_1}(\mathcal{C}) = P_{g_2}(\mathcal{C})$. Suppose R is discrete. The case R is continuous is similar.

$$\begin{aligned} P_{g_1}(\mathcal{C}) &= \sum_{\Omega_Z \Omega_R} p_{M|ZR}(g_1(z,r)|z,r) p_{ZR}(z,r) \\ &= \sum_{\Omega_Z \Omega_R} \left[\max_i p_{M|ZR}(m_i|z,r)\right] p_{ZR}(z,r). \end{aligned} \tag{7.60}$$

Now, Z is sufficient statistic, therefore

- $p_{M|ZR}(m_i|z,r) = p_{M|Z}(m_i|z)$ for all m, r, z such that $p_{R|Z}(r|z) > 0$.
- If $p_{R|Z}(r|z) = 0$, then $p_{RZ}(r,z) = 0$.

Thus, by the two previous points,

$$\begin{aligned} P_{g_1}(\mathcal{C}) &= \sum_{\Omega_Z \Omega_R} \left[\max_i p_{M|ZR}(m_i|z,r)\right] p_{ZR}(z,r) \\ &= \sum_{\Omega_Z \Omega_R} \left[\max_i p_{M|Z}(m_i|z)\right] p_{ZR}(z,r) \\ &= \sum_{\Omega_Z} \left[\max_i p_{M|Z}(m_i|z)\right] \sum_{\Omega_R} p_{ZR}(z,r) \\ &= \sum_{\Omega_Z} \left[\max_i p_{M|Z}(m_i|z)\right] p_Z(z) \\ &= P_{g_2}(\mathcal{C}). \end{aligned} \tag{7.61}$$

Hence, using (Z, R) is the same as using Z.

7.3.1 Examples

Example 1. Let $Z = f(R)$ where f is a one to one function. Claim: Z is a sufficient statistic for M based on R, see Figure 7.6. We have to show that $p_{R|MZ}(r|m,z) = p_{R|Z}(r|z)$.

$$p_{R|MZ}(r|m,z) = \begin{cases} 1, & \text{if } \; r = f^{-1}(z), \\ 0, & \text{if } \; r \neq f^{-1}(z). \end{cases} \tag{7.62}$$

Also,

$$p_{R|Z}(r|z) = \begin{cases} 1, & \text{if } r = f^{-1}(z), \\ 0, & \text{if } r \neq f^{-1}(z). \end{cases} \tag{7.63}$$

This proves the claim.

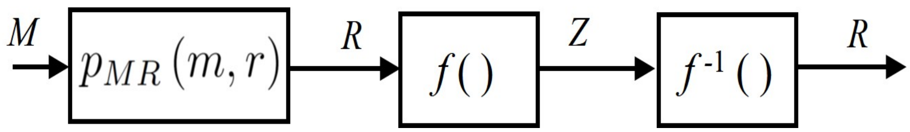

Figure 7.6: Block diagram for example

Example 2. Consider the block diagram in Figure 7.7, where N_1, N_2 and M are independent. Let $Z = f(R_1, R_2) = R_1$. Claim: Z is a sufficient statistic for M based on (R_1, R_2), i.e., we have to show that $p_{R_1R_2|MZ}(r_1, r_2|m, z) = p_{R_1R_2|Z}(r_1, r_2|z)$

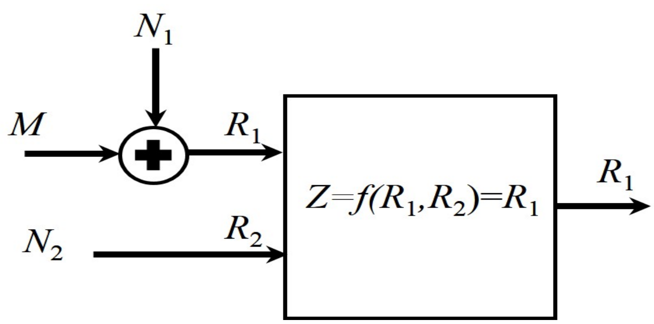

Figure 7.7: Block diagram for example

We have

$$p_{R_1R_2|MZ}(r_1, r_2|m, z) = p_{R_2|MZ}(r_2|m, z)p_{R_1|R_2MZ}(r_1|r_2, m, z). \tag{7.64}$$

Now

$$p_{R_2|MZ}(r_2|m, z) = p_{N_2}(r_2), \tag{7.65}$$

and

$$p_{R_1|R_2MZ}(r_1|r_2, m, z) = \begin{cases} 1, & \text{if } r_1 = z, \\ 0, & \text{if } r_1 \neq z. \end{cases} \tag{7.66}$$

Therefore

$$p_{R_1R_2|MZ}(r_1, r_2|m, z) = \begin{cases} p_{N_2}(r_2), & \text{if } r_1 = z, \\ 0, & \text{if } r_1 \neq z, \end{cases} \tag{7.67}$$

on the other hand,

$$p_{R_1R_2|Z}(r_1, r_2|z) = p_{R_2|R_1Z}(r_2|r_1, z)p_{R_1|Z}(r_1|z). \tag{7.68}$$

Now,

$$p_{R_1|Z}(r_1|z) = \begin{cases} 1, & \text{if } r_1 = z, \\ 0, & \text{if } r_1 \neq z, \end{cases} \tag{7.69}$$

and

$$p_{R_2|R_1Z}(r_2|r_1, z) = p_{N_2}(r_2), \tag{7.70}$$

therefore

$$p_{R_1R_2|Z}(r_1, r_2|z) = \begin{cases} p_{N_2}(r_2), & \text{if } r_1 = z, \\ 0, & \text{if } r_1 \neq z. \end{cases} \tag{7.71}$$

This proves the claim. In this example, we might think of N_2 as irrelevant or unnecessary data that can be ignored without compromising the optimality of the decision rule.

Chapter 8

Optimum Receivers for a Given Modulator

Consider the system shown in Figure 8.1, where the source produces symbols or messages M_i that take values from an alphabet set, i.e., $M_i \in \Omega_M = \{m_1, m_2, \ldots, m_q\}$, with pmf given (known) by $p_M(m_i)$, $i = 1, 2, \ldots, q$. M_i are iid random variables. The signal alphabet set is $\mathcal{S} = \{s_1(t), s_2(t), \ldots, s_q(t)\}$,where $s_i(t) = 0$ for $t \notin [0, T]$, and it is assumed that the source produces one symbol every T seconds. $N(t)$ is white Gaussian noise with mean zero and power spectral density $S_N(f) = N_0/2$ for all f, i.e., the channel is AWGN. Noise in the interval $[0, T]$ is independent of noise in any other interval, e.g., $[T, 2T]$. One can do better if M_i are not independent, and $N(t)$ is correlated on the intervals.

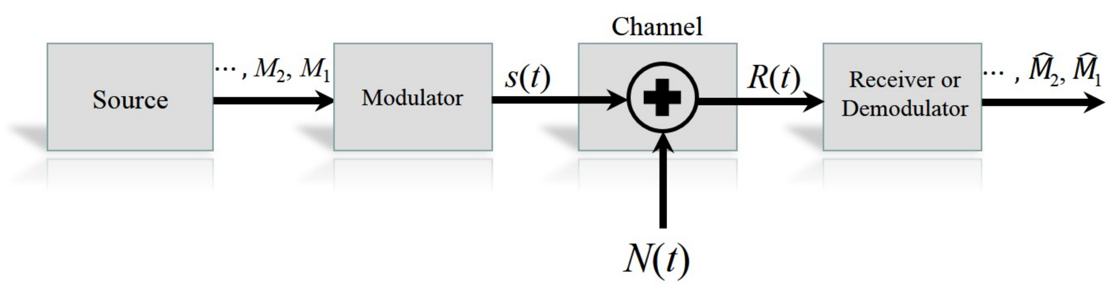

Figure 8.1: Block diagram for optimum receivers

The goal of the demodulator is to make a decision for M based on $R(t)$ for $0 \le t \le T$. We wish to make $P(\mathcal{E})$ small, where

$$P(\mathcal{E}) = P(\widehat{M_k} \neq M_k). \tag{8.1}$$

$\{R(t)\}$ and $\{M_k\}$ are random processes, once you observe them, they become random variables (one sample).

The best decision rule for the detection problem in the system of Figure 8.1 is to make a decision about M when we observe $R(t)$ for $0 \le t \le T$, i.e., $r(t)$. This decision rule would be

$$\widehat{M} = m_i, \qquad \text{IFF} \\ p_{M|R(t)}(m_i|r(t)) \geq p_{M|R(t)}(m_j|r(t)), \;\; \forall j \neq i. \tag{8.2}$$

In Equation (8.2) we can see that we are conditioning on $R(t)$ which is a random process, it is not easy to calculate. Hence, we look for sufficient statistics for M based on $R(t)$, a simplifying procedure.

8.1 Receiver Fundamentals

In this Section, a review of the fundamentals of digital communications is presented. The intention of this section is not to elaborate thoroughly on each of the topics, but to show their application directly to the context of interest.

8.1.1 Fundamental block diagram

The basic digital communication system can be divided in functional blocks, where each block comprises a set of operations carried out on the signal or information generated by a source or user. It generally consists of a transmitter, a receiver and a channel or transmission medium. Figure 8.2 shows the fundamental block diagram of a digital communication system where the information is generated by the source in a binary form. The block identified as source can be the actual information source together with an analog to digital converter such that a stream of bits is generated. Bits are delivered to a block denoted processing, where information is processed digitally to produce symbols. There are several processing tasks that can be included in such block, for example, source and channel coding, compression, encryption, filtering, etc.

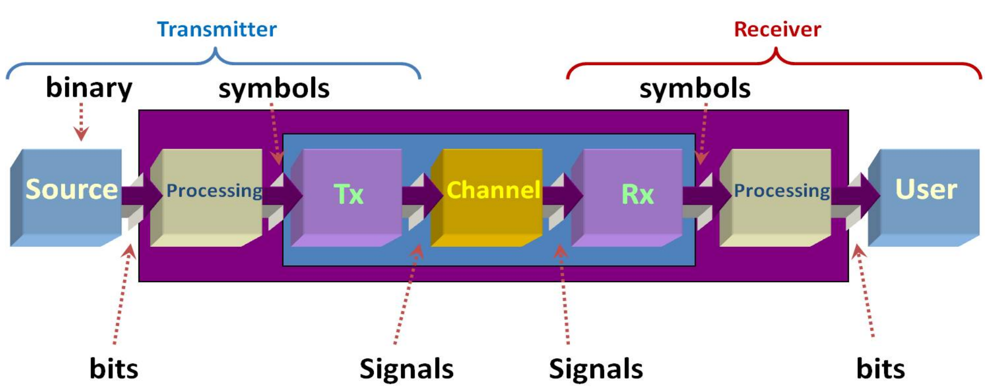

Figure 8.2: Digital communications block diagram

The Tx block consists of the functions that transmit the information through a signal representation suitable for the transmission medium used. Functions that can be included in the Tx block are modulation, spreading, pulse shaping, filtering among others. Signals are generated and transmitted through the channel which at least adds noise to the signal. In the receiver, the complementary operations of those carried out by the transmitter, are performed with the objective of delivering the bits estimated from the signal received.

Figure 8.2 also shows two background rectangles, the larger one includes those blocks for which the input is a stream of bits and the output is the stream of estimated bits. The set of blocks within this large rectangle form the binary channel. The small rectangle has as input symbols and the output is a stream of symbols.

In some cases, the bits estimated are not equal to those from the source, case in which the system produces an error. Since errors are measured in terms of bits, the performance measure is the Bit Error Rate (BER) which is the average number of bits that are not estimated correctly at the receiver divided by the total number of bits transmitted.

Figure 8.3 shows a simplified version of the block diagram of Figure 8.2, but the simplification will help to obtain performance equations for the system. Also, this model is usually considered when channel effects such as noise and fadings need to be included.

Figure 8.3: Simplified block diagram

Another important part of the block model in Figure 8.3 is that it consists of a transmitter that produces signals that are well known in advanced. These signals form a set which is called the *signal alphabet*, which is usually a discrete finite set, for example, a signal alphabet of q signals is given by

$$\Omega_M = \{s_1(t), s_2(t), \ldots, s_q(t)\}. \tag{8.3}$$

The signal alphabet is the set of signals used by the transmitter and the receiver. The signals in Equation (8.3) are time limited signals. The receiver will get a signal from the channel and will compare it to the known signals in the alphabet and will make a decision to determine which signal was actually transmitted. As in the binary case, the receiver can have errors caused by the channel effects. The receiver will get messages made out of signals from the alphabet. The signals of the alphabet are produced by the source with certain frequency, which defines a probability for each signal to be generated by the source. These probabilities are known as the *a-priori* probabilities and form a pmf, in other words, the source generates a random variable M that takes values from the set Ω_M, and that has pmf given by

$$p_M(s_i(t)) = P[M = s_i(t)] = p_i, \quad i = 1, 2, \ldots, q, \quad \sum_i p_i = 1. \tag{8.4}$$

8.1.2 Energy and Power Signals.

Signals are classified in several ways, for example, the signals can be deterministic / random, continuous / discrete, time-limited / band-limited, periodic / aperiodic. Another classification is the energy / power signals. Every physically realizable signal is an energy signal, which means that every time-limited signal is an energy signal. The classification of signals together with the time averages can be found in [6] and [5].

Even though energy signals are the ones generated by sources, many times power signals are used to study and design the systems, for example, a power signal mostly used is

$$g(t) = A\cos(2\pi f_c t), \quad t \in \mathbb{R}. \tag{8.5}$$

The signal in (8.5) is periodic with period $T_c = 1/f_c$, and exists for $t \in \mathbb{R}$, which means that it is not physically realizable. A signal that is time-limited is the rectangular pulse of amplitude A, duration T, and centered at T/2, denoted as

$$x(t) = A \prod\left(\frac{t - \frac{T}{2}}{T}\right). \tag{8.6}$$

An energy signal can be produced by the multiplication of the signals in (8.5) and (8.6) to obtain

$$s(t) = g(t)x(t) = A\cos(2\pi f_c t), \quad 0 \leq t \leq T. \tag{8.7}$$

8.2 Signal Alphabet and Matched Filter Receiver

As mentioned previously, the source in the communication system is characterized by a set of signals called signal alphabet which has a set of known a-priori probabilities. With this signal set, the receiver is designed by using the concept of *Matched filter* which consists of a system with parallel filters matched to each of the signals of the alphabet whose impulse response for the i-th signal is given by, see [5],

$$h_i(t) = Ks_i(T - t), \ \ K \neq 0, \ \ T > 0, \ \ i = 1, 2, \ldots, q. \tag{8.8}$$

The matched filter is the system that maximizes the signal to noise ratio at the output for the signal to which it is matched.

In (8.8), K is a constant different from zero, and it is usually set to one, and T is the duration of the set of energy signals and it is chosen such that each filter in the receiver is a causal system. T is usually chosen to be the maximum duration of the signals in the set. Figure 8.4 shows an alphabet of energy signals consisting of three signals with individual durations of $T_1 = 1$, $T_2 = 3$ and $T_3 = 2$. To choose the value of the duration T, you need to determine the maximum time at which a signal exists ($t_{\max} = 3$ in this example) and the minimum time at which they exist ($t_{\min} = 0$), and the duration of your signal set will be from $t_{\min}$ to $t_{\max}$, i.e., $T = 3$. The figure also shows the corresponding matched filters for each signal of the alphabet applying Equation (8.8).

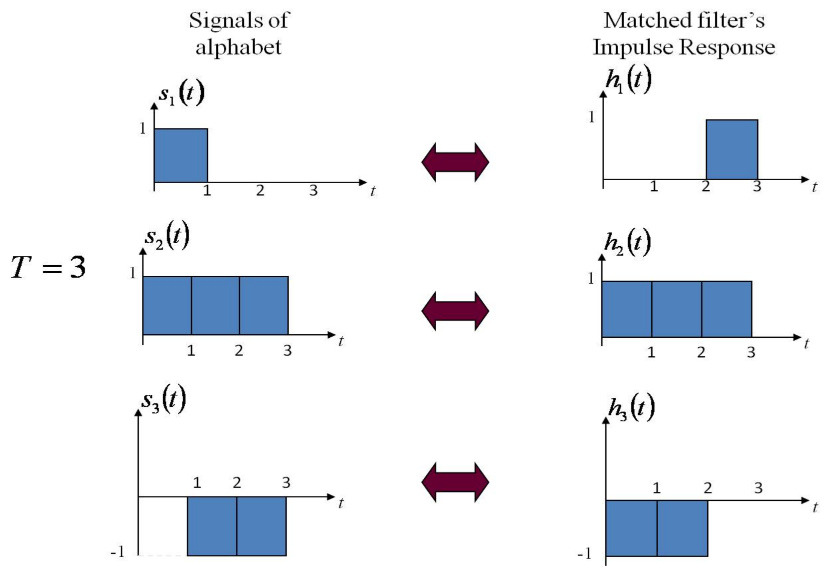

Figure 8.4: Signal alphabet and matched filters

The receiver for the signal alphabet in Figure 8.4 is shown in Figure 8.5, where three parallel branches are used to identify the signal that was transmitted.

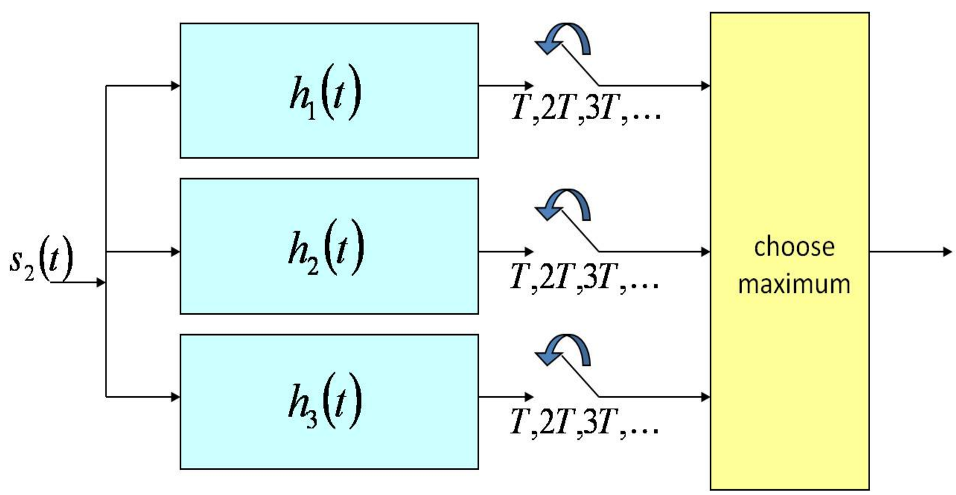

Figure 8.5: Receiver with Matched filters

The output of the matched filters in Figure 8.5 can be obtained by convolution, and for example in the absence of noise in the channel for the transmitted signal 2 of the alphabet, the output can be shown to be as those responses in Figure 8.6. From the convolutions, it can be seen that the output at time $T = 3$ is convenient to use it in a decision making process. In this case since signal 2 is the one being received, it can be seen in Figure 8.6 that at time $T = 3$ the output of the second filter produces the maximum value, and this is the of the decision block in the receiver of Figure 8.5.

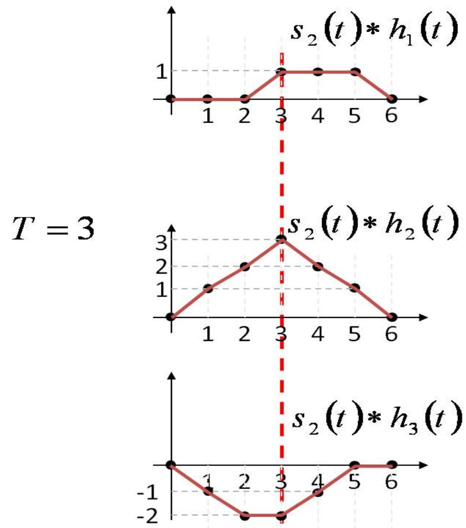

Figure 8.6: Output of filters by convolution

In general, matched filters are not implemented from those impulse responses in Figure 8.4. The implementation of the receiver is based on the process of convolution which for the branch of the signal that is being transmitted after the sample at T seconds is given by

$$
\begin{aligned}
s_2(t) * h_2(t)|_{t=T} &= \left. \int_0^t s_2(\tau) h_2(t-\tau) d\tau \right|_{t=T} \\
&= \left. \int_0^t s_2(\tau) s_2(T-t+\tau) d\tau \right|_{t=T} \\
&= \int_0^T s_2(\tau) s_2(\tau) d\tau \\
&= \int_0^T s_2^2(\tau) d\tau \\
&= E_2 \\
&= \langle s_2^2(t) \rangle. \qquad (8.9)
\end{aligned}
$$

In Equation (8.9), the energy of signal 2, E_2, is obtained. As seen from (8.9), the same result would have been obtained if the incoming signal is multiplied by signal 2 and then integrating the product and sample it at T seconds. And since the receiver knows as well the signal alphabet, then the same operation of multiplication and integration can be repeated in each of the branches giving a receiver constructed with correlators where each branch is comparing the received signal with each signal in the alphabet, the decision block will be in charge of providing the final decision as to which signal the received signal is most similar to. The receiver with correlators can be seen in Figure 8.7.

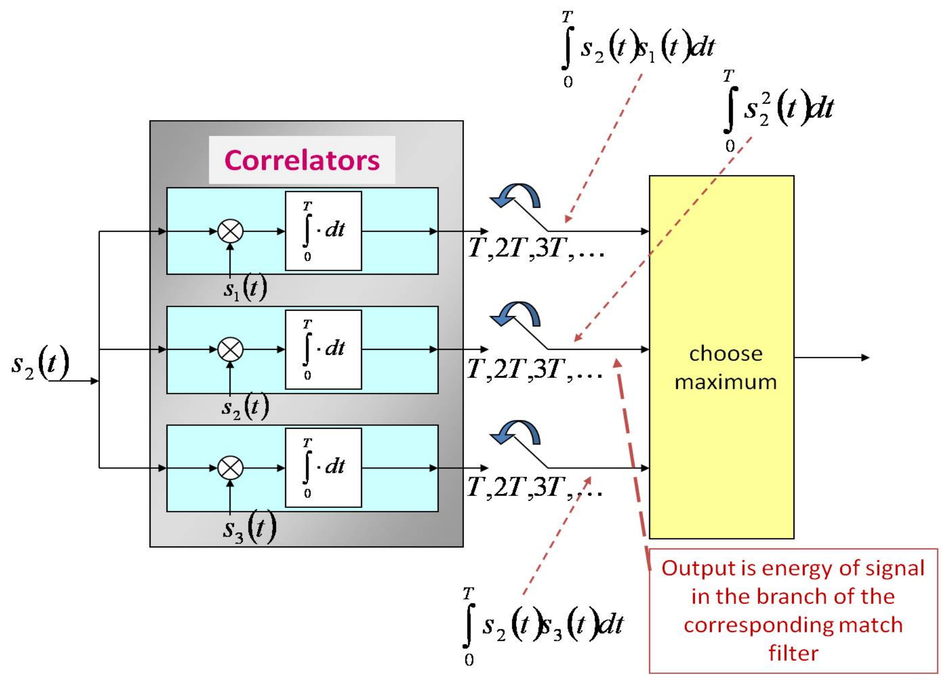

Figure 8.7: Correlator receiver

An important characteristic of the receivers in figures 8.5 and 8.7 is that they have as many branches as signals in the alphabet of the system. Since the number of signals can be large, for example for an actual system such as WiFi, we have modulators with 64 signals available; hence the number of branches would be unmanageable. The optimum receiver reduces the number of branches and changes the decision block, as it will be seen.

8.3 A Geometrical Representation of Signals

We are going to map the random process to a vector space through a finite set of functions. And every function (signal) will be represented as a linear combination of such signals.

Consider the communication system shown in Figure 8.8.

The source is an iid process $\{M_n\}$ with alphabet or signal set $\Omega_M = \{m_1, m_2, \ldots, m_q\}$ and probability mass function $p_M(m_i)$. We assume that the source emits symbols at the rate of one symbol very T seconds. The source rate is $R = 1/T$ source symbols per second.

The modulator is specified by the signal set $\mathcal{S} = \{s_1(t), s_2(t), \ldots, s_q(t)\}$,where $s_i(t) = 0$ for $t \notin [0, T]$. The channel is AWGN with noise process $N(t)$ with power spectral density $S_N(f) = N_0/2$ for all f.

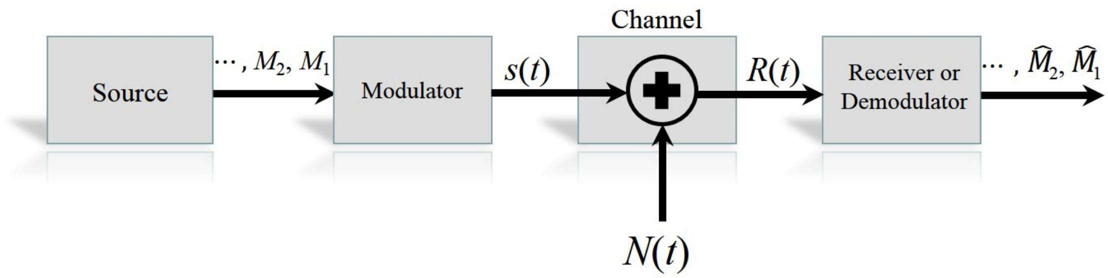

Figure 8.8: Block diagram for optimum receivers

Recall that the objective of the demodulator is to make a decision about M based on the received signal $R(t)$ with $0 \leq t \leq T$. We wish our demodulator to have small error probability, i.e., small value of

$$P(\mathcal{E}) = P(\widehat{M}_k \neq M_k). \tag{8.10}$$

We observe $R(t)$ for $0 \leq t \leq T$ and want to make a decision about M. According to the MAP rule, we let

$$\begin{aligned} g(R(t)) = m_i, \quad & \text{IFF} \\ p_{M|R(t)}(m_i|r(t)), \quad & 0 \leq t \leq T \text{ is largest among all messages } m_j \neq m_i. \end{aligned} \tag{8.11}$$

We cannot evaluate this probability because $R(t)$ is a waveform, i.e., a random process. Therefore, we look for a sufficient statistic for decisions about M based on $R(t)$, a simplifying procedure.

Suppose that we are given a set of functions $\{\varphi_1(t), \varphi_2(t), \ldots, \varphi_L(t)\}$ such that

- **Duration**. $\varphi_i(t) = 0$ if $t \notin [0, T)$.
- **Finite Energy**. $\int_0^T \varphi_i^2(t)dt < \infty$.
- Each signal $s_k(t)$ in $\mathcal{S}$ can be expressed as a linear combination of $\varphi_i(t)$, i.e., for each $s_k(t) \in \mathcal{S}$,

$$s_k(t) = \sum_{i=1}^{L} a_{ki}\varphi_i(t), \quad a_{ki} \in \mathbb{R}. \tag{8.12}$$

Then, given $R(t)$, for $0 \leq t \leq T$, compute the statistic

$$\mathbf{R} = (R_1, R_2, \ldots, R_L), \tag{8.13}$$

where

$$R_i = \int_0^T R(t)\varphi_i(t)dt, \quad i = 1, 2, \ldots, L. \tag{8.14}$$

Claim: $\mathbf{R}$ is a sufficient statistic for decisions about M based on $R(t)$, for $0 \leq t \leq T$. This is the same as saying that the best decision based on $\mathbf{R}$ is the same as the best decision based on $R(t)$. This will be proved later.

Now, we use $\mathbf{R}$ to make a decision about M, i.e., to estimate M. The best decision rule for M based on $\mathbf{R}$ is: Given $\mathbf{R} = \mathbf{r}$, let

$$\begin{aligned} & g(\mathbf{r}) = m_i, \quad \text{IFF} \\ & p_{M|\mathbf{R}}(m_i|\mathbf{r}), \qquad \text{is largest among all messages } m_j \neq m_i, \end{aligned} \tag{8.15}$$

or equivalently (using Bayes rule), iff

$$p_M(m_i)p_{\mathbf{R}|M}(\mathbf{r}|m_i), \text{ is largest among all messages } m_j \neq m_i. \tag{8.16}$$

For this, we need to compute $p_{\mathbf{R}|M}(\mathbf{r}|m_i)$. This is done as follows. Given that $M = m_i$, then $s_i(t)$ was transmitted, and the received signal is given by

$$R(t) = s_i(t) + N(t), \tag{8.17}$$

see that in (8.17) $s_i(t)$ is a deterministic waveform, and $N(t)$ is a random process. Now, we can get vector $\mathbf{R} = (R_1, R_2, \ldots, R_L)$ by substituting (8.17) in (8.14) as follows

$$\begin{aligned} R_j &= \int_0^T R(t)\varphi_j(t)dt \\ &= \underbrace{\int_0^T s_i(t)\varphi_j(t)dt}_{s_{ij} \text{ a real number}} + \underbrace{\int_0^T N(t)\varphi_j(t)dt}_{N_j \text{ a random variable}}. \end{aligned} \tag{8.18}$$

Therefore, $\mathbf{R} = \mathbf{s}_i + \mathbf{N}$, where $\mathbf{s}_i = (s_{i1}, s_{i2}, \ldots, s_{iL})$ and $\mathbf{N} = (N_1, N_2, \ldots, N_L)$, and where

$$s_{ij} = \int_0^T s_i(t)\varphi_j(t)dt \quad \text{and} \quad N_j = \int_0^T N(t)\varphi_j(t)dt. \tag{8.19}$$

Now, we need to find $p_{\mathbf{R}|M}(\mathbf{r}|m_i)$. For this we first evaluate the conditional distribution function $F_{\mathbf{R}|M}(\mathbf{r}|m_i)$ as follows

$$\begin{aligned} F_{\mathbf{R}|M}(\mathbf{r}|m_i) &= P\left(R_1 \leq r_1, R_2 \leq r_2, \ldots, R_L \leq r_L | M = m_i\right) \\ &= P\left(N_1 + s_{i1} \leq r_1, N_2 + s_{i2} \leq r_2, \ldots, N_L + s_{iL} \leq r_L | M = m_i\right) \\ &= P\left(N_1 \leq r_1 - s_{i1}, N_2 \leq r_2 - s_{i2}, \ldots, N_L \leq r_L - s_{iL}\right) \\ &= F_{\mathbf{N}}(\mathbf{r} - \mathbf{s}_i). \end{aligned} \tag{8.20}$$

For Equation (8.20) we use the independence of the noise process and the source symbol process. Also, Equation (8.20) implies that

$$p_{\mathbf{R}|M}(\mathbf{r}|m_i) = \frac{d}{d\mathbf{r}} F_{\mathbf{R}|M}(\mathbf{r}|m_i) = p_{\mathbf{N}}(\mathbf{r} - \mathbf{s}_i). \tag{8.21}$$

In (8.21) we need to find $p_{\mathbf{N}}(\mathbf{n})$. For this, note that

$$N_j = \int_0^T \underbrace{N(t)}_{\text{WGRP}} \varphi_j(t) dt, \quad j = 1, 2, \ldots, L. \tag{8.22}$$

Now, $N(t)$ is a white Gaussian random process (WGRP), and the random vector is obtained by performing a linear operation on $\{N(t)\}$ which is the integral, then N_j is a Gaussian random variable for all j, and then $\mathbf{N} = (N_1, N_2, \ldots, N_L)$ is a Gaussian random vector. In other words N_j is obtained from a linear operation of a Gaussian random process hence $N_1, N_2, \ldots, N_L$ are jointly Gaussian. Since every Gaussian random variable or vector is completely defined by its mean and variance, then we need to obtain the mean vector and the covariance matrix to get the probability density function. We proceed as follows

$$\begin{aligned} E(N_j) &= E\left[\int_0^T N(t)\varphi_j(t)dt\right] \\ &= \int_0^T E[N(t)]\,\varphi_j(t)dt \quad \text{recall that } E[N(t)] = 0, \\ &= 0, \end{aligned} \tag{8.23}$$

and the covariance is obtained as $\text{cov}(N_i, N_j) = E(N_iN_j) - E(N_i)E(N_j) = E(N_iN_j)$, then we have

$$\begin{aligned} \text{cov}(N_i, N_j) &= E(N_iN_j) \\ &= E\left[\int_0^T N(t)\varphi_i(t)dt \int_0^T N(\tau)\varphi_j(\tau)d\tau\right] \\ &= \int_0^T \int_0^T \underbrace{E[N(t)N(\tau)]}_{\text{autocorrelation } R_N(t-\tau)} \varphi_i(t)\varphi_j(\tau)dtd\tau \\ &= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-\tau)\varphi_i(t)\varphi_j(\tau)dtd\tau \\ &= \frac{N_0}{2}\int_0^T \varphi_i(t)\underbrace{\int_0^T \delta(t-\tau)\varphi_j(\tau)d\tau}_{\text{sifting property } \varphi_j(t)} dt \\ &= \frac{N_0}{2}\int_0^T \varphi_i(t)\varphi_j(t)dt. \end{aligned} \tag{8.24}$$

In Equation (8.24), we can see that everything becomes simpler if we choose the functions $\{\varphi_j(t)\}_{j=1}^{L}$ so that the following conditions are satisfied

1. Finite duration T, i.e., $\varphi_j(t)$ is an energy signal
2. We can get linear combinations of the function $\varphi_j(t)$ for all j.

3. **Orthonormal** functions $\varphi_j(t)$ for all j, i.e.,

$$\int_0^T \varphi_i(t)\varphi_j(t)dt = \begin{cases} 1, & \text{if } i = j, \quad (i) \\ 0, & \text{if } i \neq j, \quad (ii) \end{cases} \tag{8.25}$$

In Equation (8.25), condition (i) means that function $\varphi_i(t)$ has unit energy (normalized energy), condition (ii) means that $\varphi_i(t)$ and $\varphi_j(t)$ are orthogonal. If functions $\{\varphi_j(t)\}_{j=1}^L$ satisfy these conditions, then we say that $\{\varphi_j(t)\}_{j=1}^L$ is a set of **Orthonormal** functions.

Once we consider these conditions, then we get the covariance to be

$$\text{cov}(N_i, N_j) = \begin{cases} \frac{N_0}{2}, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{8.26}$$

In Equation (8.26), the value zero when $i \neq j$ implies that we have N_i and N_j *uncorrelated*, and since N_i and N_j are Gaussian random variables, then they are also *independent*. Hence

$$\text{var}(N_i) = \sigma_{N_i}^2 = \frac{N_0}{2}, \tag{8.27}$$

and

$$\begin{aligned} p_{\mathbf{N}}(\mathbf{n}) &= p_{N_1,N_2,\ldots,N_L}(n_1, n_2, \ldots, n_L) \\ &= \prod_{i=1}^L p_{N_i}(n_i) \\ &= \prod_{i=1}^L \frac{1}{\sqrt{2\pi\sigma_{N_i}^2}} e^{-\frac{n_i^2}{2\sigma_{N_i}^2}} \\ &= \prod_{i=1}^L \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n_i^2}{N_0}} \\ &= \frac{1}{(\pi N_0)^{L/2}} e^{-\frac{\sum_{i=1}^L n_i^2}{N_0}}. \end{aligned} \tag{8.28}$$

Now, from (8.21) we need to get $p_{\mathbf{N}}(\mathbf{r} - \mathbf{s}_i)$ hence from (8.28) we get

$$p_{\mathbf{N}}(\mathbf{r} - \mathbf{s}_i) = \frac{1}{(\pi N_0)^{L/2}} e^{-\frac{\sum_{j=1}^L (r_j - s_{ij})^2}{N_0}}. \tag{8.29}$$

Therefore, now we can get the best decision rule which is now given by

$$\begin{aligned} &g(\mathbf{r}) = m_i, \ \text{IFF} \\ &p_M(m_i)p_{\mathbf{R}|M}(\mathbf{r}|m_i), \qquad \text{is largest among all messages } m_j \neq m_i. \end{aligned}$$

$$p_M(m_i)\frac{1}{(\pi N_0)^{L/2}} \quad \exp\left[-\frac{\sum_{j=1}^L (r_j - s_{ij})^2}{N_0}\right] \geq \text{ all other messages } m_k \neq m_i. \tag{8.30}$$

We can use a shorthand notation in (8.30) by defining for any vector $\mathbf{x} = (x_1, x_2, \ldots, x_L)$ its norm as

$$\|\mathbf{x}\|^2 = \sum_{i=1}^{L} x_i^2, \tag{8.31}$$

then we get as the decision rule the following

$$\begin{aligned} g(\mathbf{r}) &= m_i, \text{ IFF} \\ & p_M(m_i)\frac{1}{(\pi N_0)^{L/2}}\exp\left[-\frac{\|\mathbf{r}-\mathbf{s}_i\|^2}{N_0}\right] \text{ Largest among all } m_k \neq m_i. \end{aligned} \tag{8.32}$$

The term $\|\mathbf{r}-\mathbf{s}_i\|^2$ is the Euclidean distance between vectors $\mathbf{r}$ and $\mathbf{s}_i$. Now, in Equation (8.32) note that the terms $\frac{1}{(\pi N_0)^{L/2}}$ will cancel each other out and also that we can use the ln to get

$$\begin{aligned} g(\mathbf{r}) &= m_i, \text{ IFF} \\ & \ln p_M(m_i) - \frac{\|\mathbf{r}-\mathbf{s}_i\|^2}{N_0} \text{ is largest,} \end{aligned} \tag{8.33}$$

or

$$\frac{\|\mathbf{r}-\mathbf{s}_i\|^2}{N_0} - \ln p_M(m_i) \text{ is smallest.} \tag{8.34}$$

We can also change the decision rule to be

$$\begin{aligned} g(\mathbf{r}) &= m_i, \text{ IFF} \\ & \frac{\|\mathbf{r}-\mathbf{s}_i\|^2}{N_0} - \ln p_M(m_i) \leq \frac{\|\mathbf{r}-\mathbf{s}_j\|^2}{N_0} - \ln p_M(m_j), \quad \forall j \neq i \\ & \|\mathbf{r}-\mathbf{s}_i\|^2 - N_0 \ln p_M(m_i) \leq \|\mathbf{r}-\mathbf{s}_j\|^2 - N_0 \ln p_M(m_j), \quad \forall j \neq i \end{aligned} \tag{8.35}$$

Define

$$U_i = \|\mathbf{r}-\mathbf{s}_i\|^2 - N_0 \ln p_M(m_i), \tag{8.36}$$

then a receiver is given (knows) signals $\mathbf{s}_i$ for all $i = 1, 2, \ldots, q$, pmf $p_M(m_i)$ and functions $\{\varphi_j(t)\}_{j=1}^{L}$. First the receiver gets vector $\mathbf{s}_i$ and then calculates vector $\mathbf{R} = (R_1, R_2, \ldots, R_L)$ by using

$$s_{ij} = \int_0^T s_i(t)\varphi_j(t)dt, \qquad R_j = \int_0^T R(t)\varphi_j(t)dt. \tag{8.37}$$

Then with this information the receiver calculates u_i for all i and compares them according to (8.35) to choose m_i. The receiver associated to this problem can be seen in Figure 8.9.

The following questions remain to be solved

1. How to find functions $\{\varphi_j(t)\}_{j=1}^{L}$ that satisfy the **orthonormal** properties

$$\int_0^T \varphi_i(t)\varphi_j(t)dt = \begin{cases} 1, & \text{if } i = j, \ (i) \\ 0, & \text{if } i \neq j, \ (ii) \end{cases} \tag{8.38}$$

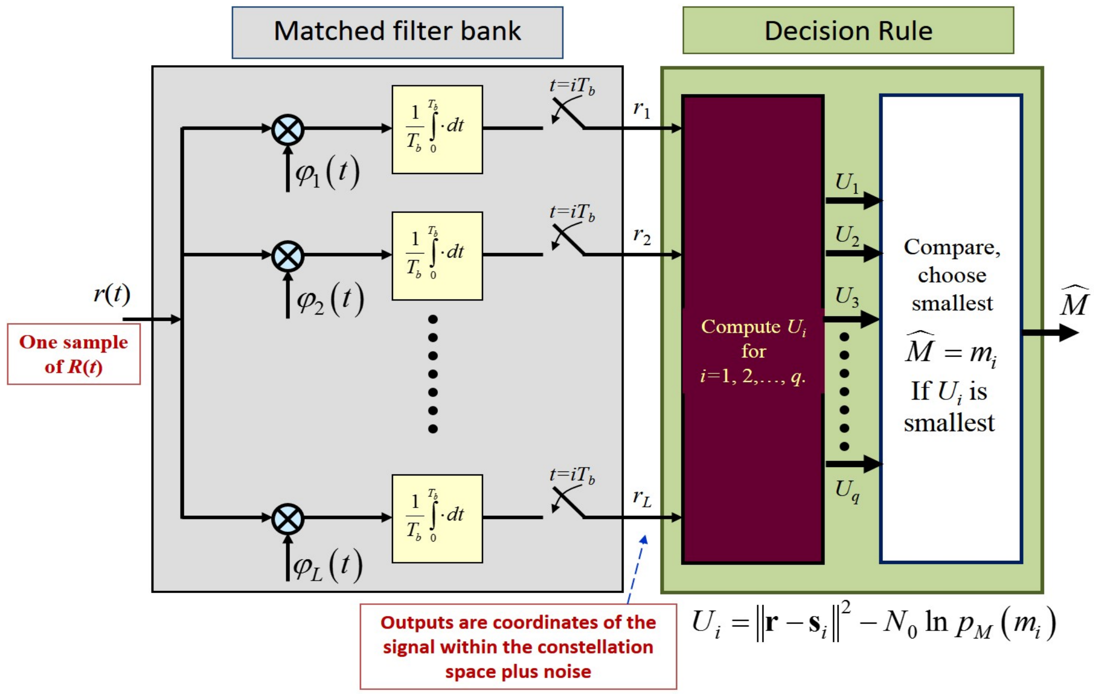

Figure 8.9: Block diagram for receiver

and that every signal in the alphabet, i.e., $s_i(t) \in \mathcal{S}$ for all i can be expressed as a linear combination of $\varphi_j(t)$ for $j = 1, \ldots, L$, in other words

$$s_i(t) = \sum_{j=1}^{L} s_{ij}\varphi_j(t), \tag{8.39}$$

where $s_{ij} \in \mathbb{R}$ and

$$s_{ij} = \int_0^T s_i(t)\varphi_j(t)dt. \quad \mathbf{s}_i = (s_{i1}, s_{i2}, \ldots, s_{iL}). \tag{8.40}$$

Note that s_{ij} by the integral used to obtain it, is the *correlation* between waveform $s_i(t)$ and the function $\varphi_j(t)$.

2. Show that $\mathbf{R}$ is a sufficient statistic

3. Simplify the receiver shown in Figure 8.9

4. Compute $P(\mathcal{E})$

5. Compute $P(\mathcal{E})$ for different signal sets, each with its own optimum receiver

Note that the following two equations form a **Fourier Series Pair**

$$s_i(t) = \sum_{j=1}^{L} s_{ij}\varphi_j(t), \qquad s_{ij} = \int_0^T s_i(t)\varphi_j(t)dt. \tag{8.41}$$

With the analysis just used, we can see that the system can be seen in block diagram as that in Figure 8.10

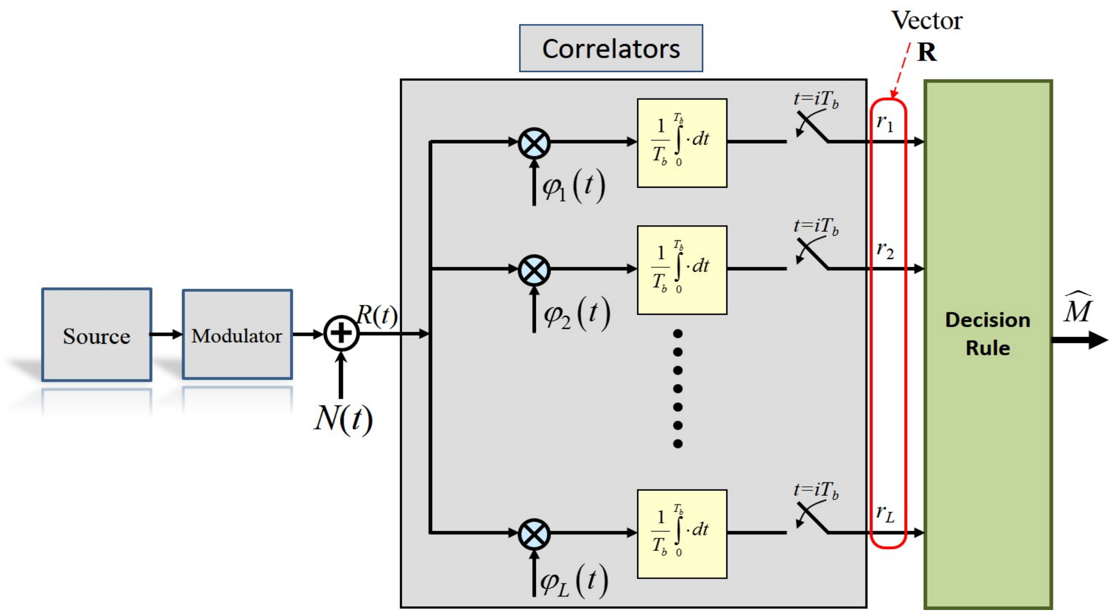

Figure 8.10: Block diagram for receiver

Also, with the analysis carried on, we basically changed the use of $s(t)$, $R(t)$, and $N(t)$ by their vector counterparts $\mathbf{R}$, $\mathbf{s}$ and $\mathbf{N}$, respectively, which makes the system have a different look where we consider a vector channel version as shown in Figure 8.11. See that the figure has a block for a *vector modulator* that takes symbols generated by the source and produces a signal in vector form. The vector channel has as input a vector and produces at the output a vector which is the noisy and distorted version of the input vector signal.

8.4 Example 1.

Assume that you are given the signal alphabet as that shown in Figure 8.12. Note that all the signals are energy signals as discussed. Also note that the duration of the set of signals will be $T = 2$. See that there is a dependency of the first and the third signals, i.e., $s_1(t) = -s_3(t)$.

Figure 8.13 shows a given set of orthonormal signals that will be used to *vectorize* the system in order to consider it as that with the vector channel concept. Note that we can verify the properties of orthonormality of these two signals, i.e., unitary energy and orthogonality.

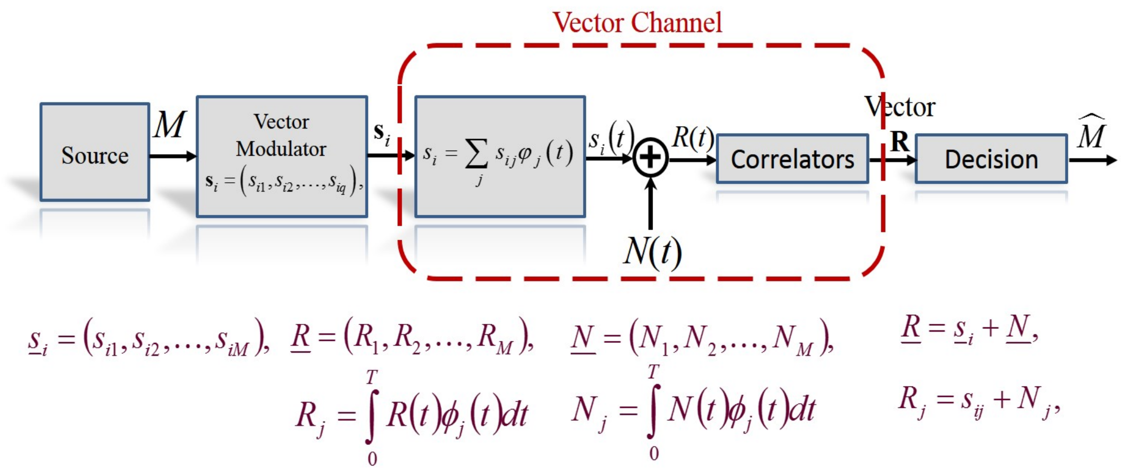

Figure 8.11: Block diagram for system with vector channel concept

The property of orthogonality can be seen easily since both signals occupy different time intervals, so they do not overlap and their product will be zero due to this condition.

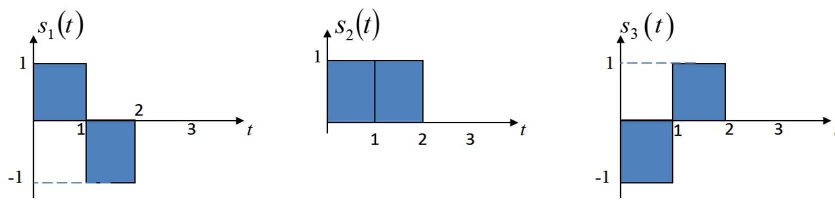

Figure 8.12: Signal set or alphabet for example 1.

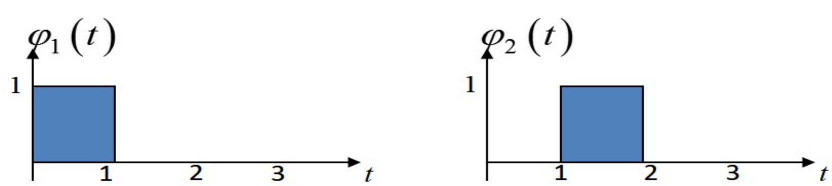

Figure 8.13: Orthonormal function set for example 1.

The first point to address in this example is to find a linear combination of functions $\varphi_j(t)$ for $j = 1, 2$, that gives each and every signal of the alphabet, i.e., we can see that

$$\begin{aligned} s_1(t) &= \varphi_1(t) - \varphi_2(t), \\ s_2(t) &= \varphi_1(t) + \varphi_2(t), \\ s_3(t) &= -\varphi_1(t) + \varphi_2(t). \end{aligned} \tag{8.42}$$

See that in Equation (8.42) the signals are represented by linear combinations of the orthonormal functions which are somehow easy to see. Later we will see how to get the orthonormal signals, and as a consequence how to determine the linear combinations.

Now, from the set of equations in (8.42), we can take the coefficients that are multiplying the orthonormal functions to determine the components of the vectors for each signal, i.e.

$$\begin{aligned} s_1(t) &= \varphi_1(t) - \varphi_2(t), \quad \Rightarrow \quad \mathbf{s}_1 = (1, -1), \\ s_2(t) &= \varphi_1(t) + \varphi_2(t), \quad \Rightarrow \quad \mathbf{s}_2 = (1, 1), \\ s_3(t) &= -\varphi_1(t) + \varphi_2(t), \quad \Rightarrow \quad \mathbf{s}_3 = (-1, 1). \end{aligned} \tag{8.43}$$

The receiver will consists of a system with two branches, one for each orthonormal function, and it will apply the MAP rule as shown in Figure 8.14. See that the last block shows the computation of U_i and the decision making process to get $\widehat{M}$. One can draw the receiver with two blocks at the end to emphasize the computation of U_i and the decision rule. See that for this example, we can consider that all symbols are equally likely, then $p_M(m_i) = 1/3$ for all i. Also, consider that the Euclidean distance is $\|\mathbf{r} - \mathbf{s}_i\|^2 = (r_1 - s_{i1})^2 + (r_2 - s_{i2})^2$, and recall that

$$U_i = \|\mathbf{r} - \mathbf{s}_i\|^2 - N_0 \ln p_M(m_i). \tag{8.44}$$

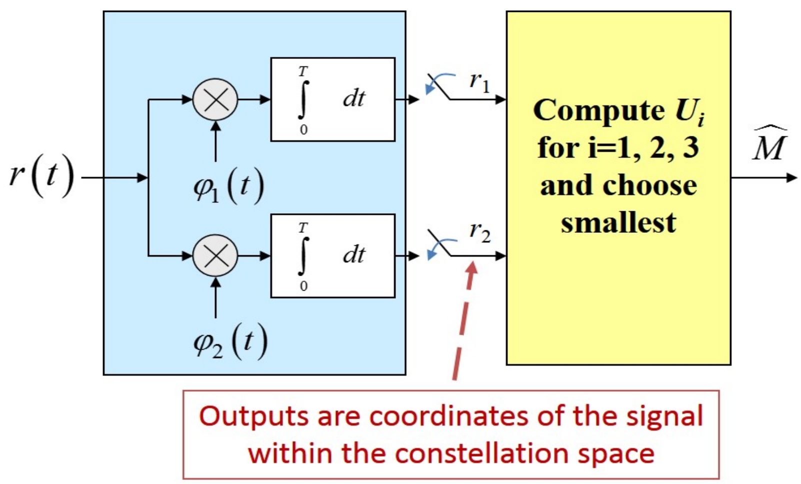

Figure 8.14: Block diagram of receiver for example 1.

See that in the case of equally likely messages, the term $N_0 \ln p_M(m_i)$ from all U_i's is canceled, and the decision rule will only be determined by the comparison of the Euclidean distances $\|\mathbf{r} - \mathbf{s}_i\|^2$, so the MAP rule becomes ML rule as follows

$$\begin{aligned} g(\mathbf{r}) &= m_i, \text{ IFF} \\ &\quad \|\mathbf{r} - \mathbf{s}_i\|^2 \leq \|\mathbf{r} - \mathbf{s}_j\|^2, \; \forall j \neq i \end{aligned} \tag{8.45}$$

With the vectors shown in (8.43), and the ML decision rule in (8.45), we can show a two-dimensional space where the signals exist called *signal constellation*, as shown in Figure

8.15. The same figure shows the decision regions I_i for all i. See that the end block of the receiver, has as input the vector $\mathbf{r} = (r_1, r_2)$ which is used to denote the dimensions of the constellation in this example. Later we will see that such dimensions are defined by the orthonormal functions.

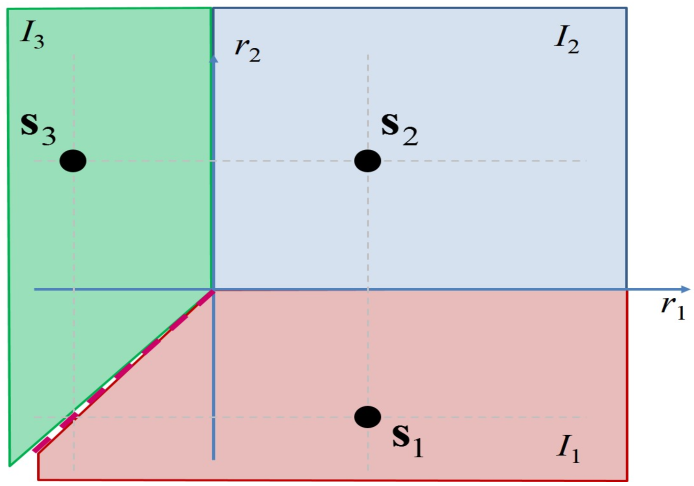

Figure 8.15: Signal constellation for example 1.

Note that if $M = m_i$, then the signal transmitted is $s_i(t)$ and the transmitted vector is $\mathbf{s}_i$. The received vector will be

$$\mathbf{R} = \mathbf{s}_i + \mathbf{N}, \quad \text{where } \mathbf{N} = (N_1, N_2). \tag{8.46}$$

If $\mathbf{s}_i$ is the transmitted vector, and the noise vector $\mathbf{N} = \mathbf{n}$ is added, then $\mathbf{r} = \mathbf{s}_i + \mathbf{n}$ is the received vector. Then, if $\mathbf{r}$ is still inside the decision region I_i, the receiver makes a correct decision, otherwise a wrong decision is made.

Remark:

The density function of the noise vector $\mathbf{N}$ is given by

$$\begin{aligned} p_{\mathbf{N}}(\mathbf{n}) &= \frac{1}{(\pi N_0)^{L/2}} \exp\left\{ -\frac{\sum_{i=1}^{L} n_i^2}{N_0} \right\} \\ &= \frac{1}{(\pi N_0)^{L/2}} \exp\left\{ -\frac{\|\mathbf{n}\|^2}{N_0} \right\}. \end{aligned} \tag{8.47}$$

We see that the density function $p_{\mathbf{N}}(\mathbf{n})$ depends on the length of the noise vector and not on its direction. This is why we say that noise is a **spherically symmetric** or **circularly symmetric** Gaussian random variable.

8.5 Problems.

Due on Friday October 13, 2017.

1. Consider the block diagram shown in Figure 8.16, where M is a binary random variable with $p_M(+1) = p_M(-1) = 1/2$, and where N_1 and N_2 are two independent zero mean Gaussian random variables with variances $\sigma_{N_1}^2 = 1$ and $\sigma_{N_2}^2 = 2$, respectively. M is independent of N_1 and N_2.

   (a) Find the best decision rule (minimum probability of error decision rule) for M based on (R_1, R_2). See example in Section 7.2.2.

   (b) Draw the decision regions for the decision rule obtained in the previous part (a).

   (c) Find the probability of error for the decision rule of part (a).

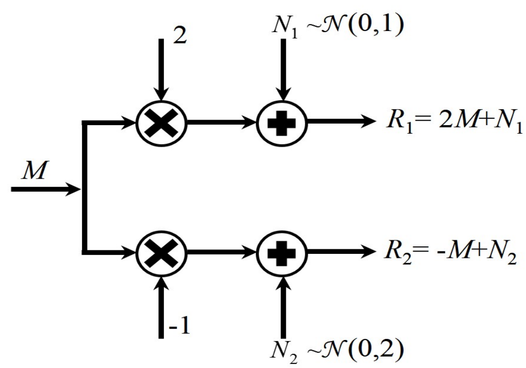

Figure 8.16: Block diagram for problem 1.

2. Read Chapter 4 of Wozencraft and Jacobs and solve problem 4.1 (page 273)

3. Solve problem 4.3 (page 274)

4. Solve problem 4.8 (page 278)

Part III

Signal Representation

Chapter 9

Geometric Interpretation of Signals

The analysis of every digital communications system considers the transmitter, the receiver, and the channel. The transmitter and receiver know the signals that are being sent and received. In general, a transmitter usually consist of a system such as a modulator, which is specified by a set of signals that is called the *signal alphabet*. Define the signal alphabet as

$$\Omega = \{s_1(t), s_2(t), \ldots, s_q(t)\}. \tag{9.1}$$

In Equation (9.1), each of the elements of the alphabet is an energy signal with duration T seconds, in other words,

$$\begin{aligned} s_i(t) &= 0, \quad t \notin [0, T], \quad i = 1, 2, \ldots, q; \\ E_i &= \int_0^T s_i^2(t)\,dt < \infty. \end{aligned} \tag{9.2}$$

The problem of the geometric interpretation of signals consist of finding a set of functions $\{\varphi_1(t), \varphi_2(t), \ldots, \varphi_M(t)\}$ such that the following conditions are satisfied:

1. The functions are orthonormal
2. Every signal in the alphabet can be expressed as a linear combination of $\{\varphi_i(t)\}_{i=1}^{M}$, in other words

$$s_k(t) = \sum_{j=1}^{M} s_{kj}\varphi_j(t), \; s_{kj} \in \mathbb{R}, \; k = 1, 2, \ldots, q; \; j = 1, 2, \ldots, M. \tag{9.3}$$

Then we have that

$$\mathbf{s}_k = [s_{k1}, s_{k2}, \ldots, s_{kM}], \tag{9.4}$$

and there exists a one-to-one transform from signals to vectors, in other words, $s_k(t) \longrightarrow \mathbf{s}_k$, in fact, signals themselves have vector like properties.

9.1 Vector Space

Define the following set of signals

$$\Omega_X = \left\{ x(t) : x(t) = 0 \text{ for } t \notin [0,T], \text{ and } \int_0^T x^2(t)\, dt < \infty \right\}. \tag{9.5}$$

The set defined in (9.5) satisfies the following:

Claim 4 Ω_X *is a vector space over the field* $(\mathbb{R}, \cdot, +)$*, where* $\mathbb{R} = (-\infty, \infty)$.

In order to have a *space*, we first need a *field*, and we also need other conditions that we summarize in three subsections as follows.

9.1.1 Field $(\mathbb{R}, \cdot, +)$

A field is a set with two operations, and these operations must satisfy certain conditions as follows

1. **Commutative Group.** A commutative group is a set, e.g., $\mathbb{R}$, and an operation that satisfies the commutativity property with elements of the set and such operation, i.e., for all $a, b, c \in \mathbb{R}$, we have
$$a \cdot b = b \cdot a \in \mathbb{R}.$$
   (a) **Group.** A group does not necessarily satisfy the commutativity property, but it satisfies the following conditions.
      i. *Associativity*, $a \cdot (b \cdot c) = (a \cdot b) \cdot c \in \mathbb{R}$.
      ii. *Unit element*. There exists an element in the set, called *the unit element* such that $a \cdot 1 = a$.
      iii. *Inverse element*. Every element in the set must have an inverse, i.e., $a \cdot (a^{-1}) = 1$, $a^{-1} \in \mathbb{R}$.

The groups $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot)$ are called **Abelian groups**, and together form a field. The set of integers is NOT a field because the integers do not have an inverse element.

9.1.2 The operation addition

In order to consider the operation addition, first define an operation $\oplus$, and then, we can define the addition of elements of the set Ω_X with the operation $\oplus$ as follows

1. $\forall x(t), y(t) \in \Omega_X, \Longrightarrow (x(t) \oplus y(t)) \in \Omega_X$
2. Commutativity $x(t) \oplus y(t) = y(t) \oplus x(t)$

3. Associativity $\forall x(t), y(t), z(t) \in \Omega_X, \Longrightarrow x(t) \oplus (y(t) \oplus z(t)) = (x(t) \oplus y(t)) \oplus z(t)$

4. Zero element $\forall x(t) \in \Omega_X, \exists\, o(t) \in \Omega_X$, such that $x(t) \oplus o(t) = x(t)$.

5. Negative functions, i.e., there exists a negative function $y(t) \in \Omega_X$ of $x(t)$ such that $x(t) \oplus y(t) = o(t)$.

The question now is how to define the operation $\oplus$, so that conditions 1 through 5, just stated, are satisfied. For example, we can make the following definition $x(t) \oplus y(t) = \max\{x(t), y(t)\}$, but this definition does not satisfy the zero element condition for $x(t) < o(t)$.

The operation that does satisfy the five conditions is the one known as **pointwise addition of functions,**

$$(x + y)(t) = x(t) + y(t). \tag{9.6}$$

9.1.3 Scalar multiplication

Elements of the field are called *scalars*. For $a \in \mathbb{R}$ and $x(t) \in \Omega_X$ define the scalar multiplication as point-wise multiplication, then it must satisfy

1. For all $a, b \in \mathbb{R}$ and all $x(t) \in \Omega_X$, $ax(t) \in \Omega_X$

2. $a(x(t) + y(t)) = ax(t) + ay(t) \in \Omega_X$

3. $a(bx(t)) = (ab)x(t) \in \Omega_X$, we have that $a \cdot (b \cdot x(t))$ and $(ab) \cdot x(t)$ are scalar multiplications, $(a \cdot b)$ is a multiplication on the field.

4. Unit scalar, $1 \cdot x(t) = x(t)$.

Ω_X is usually shown by $L^2[0,T]$, the set of square integrable functions on the interval $[0,T]$. For example, the function $x(t) = 1/\sqrt{t} \notin L^2[0,1]$. Also $x(t) = 1/t^2 \notin L^2[0,1]$.

Definition 5 *Suppose $\Omega_Y \subset \Omega_X$, then Ω_Y is called a vector subspace of Ω_X if Ω_Y is itself a vector space.*

9.2 Geometric Concepts for $L^2[0,T]$.

A function $x(t) \in L^2[0,T]$ satisfies the following

$$\left\{ \int_0^T x^2(t)\, dt \right\}^{\frac{1}{2}} < \infty. \tag{9.7}$$

We can say that the set of functions Ω_X, is equivalent to the set of functions $L^2[0,T]$, and both are a special case of the space of functions $L^p[0,T]$, where the functions satisfy for $p=2$ the following

$$\left\{ \int_0^T x^p(t)\,dt \right\}^{\frac{1}{p}} < \infty. \tag{9.8}$$

The following are the characteristics or properties of the squared integrable function space $L^2[0,T]$.

1. **Inner product:** the inner product or **correlation** (measure of the similarity of two signals) of two functions $x(t), y(t) \in L^2[0,T]$ is defined as

$$\langle x(t), y(t)\rangle \triangleq \int_0^T x(t)\,y(t)\,dt. \tag{9.9}$$

2. **Norm or length:** the norm or length of a function $x(t) \in L^2[0,T]$ is denoted as $\|x(t)\|$, and is defined based on the inner product. The **energy** E of the function or signal is given by $\|x(t)\|^2$, and is obtained as follows

$$\langle x(t), x(t)\rangle = \int_0^T x^2(t)\,dt = \|x(t)\|^2 = E. \tag{9.10}$$

3. **Distance:** the distance is the separation between any pair of functions and for $x(t)$, $y(t) \in L^2[0,T]$ is defined as follows

$$d\left[x(t), y(t)\right] = \|x(t) - y(t)\|. \tag{9.11}$$

4. **Angle:** the angle formed by two functions $x(t), y(t) \in L^2[0,T]$, denoted as $\angle x(t), y(t)$, is defined through the cosine function as follows

$$\cos\left[\angle x(t), y(t)\right] = \frac{\langle x(t), y(t)\rangle}{\|x(t)\|\,\|y(t)\|}. \tag{9.12}$$

By Schwartz inequality, we have the following

$$\left[\int_0^T x(t)\,y(t)\,dt\right]^2 \leq \left[\int_0^T x^2(t)\,dt \int_0^T y^2(t)\,dt\right], \tag{9.13}$$

In conclusion, the Schwartz inequality in (9.13) states that

$$|\langle x(t), x(t)\rangle|^2 \leq \|x(t)\|^2\,\|y(t)\|^2. \tag{9.14}$$

Two functions $x(t), y(t) \in L^2[0,T]$, are **orthogonal**, i.e., form a right angle between them ($\angle x(t), y(t) = \pi/2$) if

$$\langle x(t), x(t) \rangle = 0 \tag{9.15}$$

The set of functions Ω_X, usually shown by $L^2[0,T]$, is called an **inner product space**, it is also called a **Hilbert space** if it has an additional property, i.e., if every Cauchy sequence has a limit. We say that $\{P_n\}$ is a Cauchy sequence if $|P_{n+1} - P_n| \longrightarrow 0$ as $n \longrightarrow \infty$. Also, it is important that If every Cauchy sequence has a limit, then the space is a **complete space**. A Hilbert space is a complete space.

9.3 Bases for a Vector Space

A **Basis** for a vector space $\mathcal{V}$ is a set of vectors that is **linearly independent** and **spans** the space V. For vectors $\mathbf{V}_i \in \mathcal{V}$, $i = 1, 2, 3, \ldots$, we have the following

1. A set of vectors $\mathbf{V}_i$, $i = 1, 2, 3, \ldots$, is linearly independent if

$$a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + \cdots = 0, \implies a_1 = a_2 = a_3 = \cdots = 0.$$

2. A set of vectors $\mathbf{V}_i$, $i = 1, 2, 3, \ldots$, spans $\mathcal{V}$ if every vector in $\mathcal{V}$ can be written as a linear combination of $\mathbf{V}_i$, $i = 1, 2, 3, \ldots$,

Facts about bases.

1. There are bases for **all** vector spaces, in fact many bases (i.e., not necessarily unique).
2. All the bases have the same number of vectors
3. The number of vectors or elements in the bases is called the **dimension** of the vector space.

The following is an example of a set of functions that forms a basis of $L^2[0,T]$,

$$\begin{aligned}
\varphi_0(t) &= \frac{1}{\sqrt{T}},\ 0 \le t \le T, \qquad (9.16)\\
\varphi_1(t) &= \sqrt{\frac{2}{T}}\cos\left(2\pi\frac{t}{T}\right),\ 0 \le t \le T,\\
\varphi_2(t) &= \sqrt{\frac{2}{T}}\sin\left(2\pi\frac{t}{T}\right),\ 0 \le t \le T,\\
\varphi_3(t) &= \sqrt{\frac{2}{T}}\cos\left(2\pi\frac{2t}{T}\right),\ 0 \le t \le T,\\
\varphi_4(t) &= \sqrt{\frac{2}{T}}\sin\left(2\pi\frac{2t}{T}\right),\ 0 \le t \le T.
\end{aligned}$$

The set of functions $\{\varphi_i(t)\}$ $i = 0, 1, 2, 3, 4;$ in Equation (9.16) forms a basis, i.e., the functions are linearly independent which means that

$$\sum_{i=0}^{4} a_i \varphi_i(t) = 0, \implies a_i = 0, \ \forall i.$$

In general, a basis of $L^2[0, T]$ is given by

$$\begin{aligned}
\varphi_0(t) &= \frac{1}{\sqrt{T}}, \qquad (9.17)\\
\varphi_{2k-1}(t) &= \sqrt{\frac{2}{T}} \cos\left(2\pi \frac{kt}{T}\right),\\
\varphi_{2k}(t) &= \sqrt{\frac{2}{T}} \sin\left(2\pi \frac{kt}{T}\right), \ 0 \le t \le T, \ k = 1, 2, 3, \ldots
\end{aligned}$$

Assume that there is a basis $\{\varphi_i(t)\}$, $i = 0, 1, 2, 3, \ldots,$ then given any function $x(t) \in L^2[0, T]$, we can write the given function as a linear combination of the basis functions as follows

$$x(t) = \sum_{i=0}^{\infty} x_i \varphi_i(t), \ x_i \in \mathbb{R}. \qquad (9.18)$$

The set of functions $\{\varphi_i(t)\}_{i=0}^{\infty}$ is an **orthonormal** basis, and the coefficients in the linear combination in (9.18) satisfy the following

$$x_i = \int_0^T x(t) \varphi_i(t) \, dt = \langle x(t), \varphi_i(t) \rangle, \ i = 1, 2, \ldots \qquad (9.19)$$

The dimension of $L^2[0, T]$ is the cardinality of the set of functions $\{\varphi_i(t)\}_{i=0}^{\infty}$, i.e., the number of functions $\varphi_i(t)$, which is infinity, but there are other basis for $L^2[0, T]$. Recall that equations (9.18) and (9.19) form a Fourier series pair.

A set of functions $\{\varphi_i(t)\}_{i=0}^{\infty}$ is **orthonormal** if it satisfies the following

$$\langle \varphi_i(t), \varphi_j(t) \rangle = \int_0^T \varphi_i(t) \varphi_j(t) \, dt = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases} \qquad (9.20)$$

9.4 Bases for a Communication System

Now recall the transmitter-receiver system where we had a signal alphabet given by

$$\Omega = \{s_1(t), s_2(t), \ldots, s_q(t)\} \qquad (9.21)$$

and that each of the elements of the alphabet is an energy signal with duration T seconds, in other words,

$$\begin{aligned} s_i(t) &= 0, \;\; t \notin [0,T], \; i = 1,2,\ldots,q; \qquad (9.22)\\ E_i &= \int_0^T s_i^2(t)\, dt < \infty. \end{aligned}$$

From the definition in Equation (9.21), and the properties in Equation (9.22), it can be concluded that

$$\Omega \subset L^2[0,T].$$

The important question is if Ω is a subspace of $L^2[0,T]$. In general the answer is NO since it is not known if for example $s_i(t) + s_j(t) \in \Omega$. Therefore, we cannot talk about a basis of Ω, because Ω is *not a vector space*. Hence, we need to find a set of functions $\{\varphi_i(t)\}_{i=0}^{L}$ such that they are orthonormal and span Ω, in other words, for each k

$$s_k(t) = \sum_{j=1}^{L} s_{kj}\varphi_j(t).$$

Let $\sum$ be the set of all linear combinations of $s_1(t), s_2(t), \ldots, s_q(t)$, in other words

$$\sum = \left\{ x(t) : x(t) = \sum_{i=1}^{q} a_i s_i(t), \text{ for some } a_1, a_2, \ldots, a_q \in \mathbb{R} \right\}. \qquad (9.23)$$

The set of signals in the alphabet of (9.21) satisfies the condition $\Omega \subset \sum$, also the set $\sum$ is a subspace of Ω_X, which was defined in Equation (9.5). We can calculate the energy of the signals that are linear combinations in $\sum$, as follows

$$\begin{aligned} \int_0^T x_i^2(t)\, dt &= \int_0^T \left[\sum_{i=1}^{q} a_i s_i(t)\right]^2 dt \qquad (9.24)\\ &= \int_0^T \left[\sum_{i=1}^{q} a_i s_i(t)\right]\left[\sum_{j=1}^{q} a_j s_j(t)\right] dt \\ &= \int_0^T \sum_{i=1}^{q}\sum_{j=1}^{q} a_i a_j s_i(t)\, s_j(t)\, dt \\ &= \sum_{i=1}^{q}\sum_{j=1}^{q} a_i a_j \int_0^T s_i(t)\, s_j(t)\, dt \\ &= \sum_{i=1}^{q}\sum_{j=1}^{q} a_i a_j \left\langle s_i(t), s_j(t)\right\rangle. \end{aligned}$$

In Equation (9.24), the last equality is obtained by using the definition of the inner product in Equation (9.9). Now, we can apply the Schwartz inequality in Equation (9.13) or in Equation (9.14) to obtain the following

$$|\langle s_i(t), s_j(t)\rangle|^2 \le \int_0^T s_i^2(t)\,dt \int_0^T s_j^2(t)\,dt < \infty. \tag{9.25}$$

With Equation (9.25), it can be concluded that the functions $x(t) \in \sum$ satisfy the finite energy criterion

$$\int_0^T x_i^2(t)\,dt < \infty.$$

Therefore it is concluded that

$$\Sigma \subset L^2[0, T] \tag{9.26}$$

Now, to show that Σ is a subspace, it is needed the *closure under addition,* i.e., for $x(t), y(t) \in \Sigma$, we have $x(t) + y(t) \in \Sigma$, together with the *closure under scalar multiplication*, i.e., for $a \in \mathbb{R}$, and $x(t) \in \Sigma$, we have $ax(t) \in \Sigma$. Consider $a_i, b_i \in \mathbb{R}$, $i = 1, 2, \ldots, q$, and $x(t), y(t) \in \Sigma$, then we have for the closure under addition the following

$$\begin{aligned} x(t) + y(t) &= \sum_{i=1}^{q} a_i s_i(t) + \sum_{i=1}^{q} b_i s_i(t) \\ &= \sum_{i=1}^{q} (a_i + b_i)\, s_i(t) \\ &= \sum_{i=1}^{q} c_i s_i(t) \in \Sigma;\ c_i = a_i + b_i \in \mathbb{R}. \end{aligned} \tag{9.27}$$

Hence, Σ has the property of being closed under addition since the addition of two functions from the set is also a linear combination of the signal alphabet which is also an element of the set Σ. Closure under scalar multiplication can be shown similarly. Now, we can obtain a basis for Σ, and in fact, we need an orthonormal basis for the set Σ. In order to do this, the application of the Gram-Schmidt procedure helps to obtain such orthonormal basis.

Chapter 10

Gram-Schmidt Orthogonalization Procedure

The geometric interpretation of signals helps us to understand that we can work with spaces of functions or spaces of vectors that are equivalent. Also, the interpretation helps us to see that a basis is needed in order to obtain linear combinations of signals that form a space. Hence, any signal of an alphabet can be represented by a linear combination of the basis functions, and the basis functions have the orthonormal property in (9.20). In this section, we find the procedure to obtain such basis functions when an signal alphabet is given.

10.1 G-S Procedure

We are given a signal alphabet $\Omega = \{s_1(t), s_2(t), \ldots, s_q(t)\}$, and we need to find a set of functions that form a basis and that represent by linear combinations the signals in the alphabet. The problem reduces to find functions $\{\varphi_i(t)\}_{i=0}^{L}$, such that they are *orthonormal* and *span* the set Ω, i.e., an orthonormal basis for Σ. Before starting with the procedure, recall the definition of the norm of a signal $x(t)$ given by

$$\|x(t)\| = \sqrt{\int_0^T x^2(t)\,dt}. \tag{10.1}$$

The procedure is carried out with the signals in the alphabet set. The basis will consist of an orthonormal set of functions that span Σ and Ω as well. The signals in the alphabet can be ordered and each signal can be taken in the procedure as ordered in the set. We follow the order of the signals as indicated by the subindex in the alphabet set. The procedure is defined by the following steps:

Step 1. Obtain the first basis $\varphi_1(t)$ function as a normalized version of the first signal. The function does not need to satisfy any orthogonal condition since the signal space created has only the first basis function.

$$\varphi_1(t) = \frac{s_1(t)}{\|s_1(t)\|}, \tag{10.2}$$

it can be verified the normality condition of (10.2) as follows

$$\begin{aligned}
\int_0^T \varphi_1^2(t)\,dt &= \int_0^T \frac{s_1^2(t)}{\|s_1(t)\|^2}dt \\
&= \frac{1}{\|s_1(t)\|^2}\int_0^T s_1^2(t)\,dt \\
&= \frac{\|s_1(t)\|^2}{\|s_1(t)\|^2} \\
&= 1.
\end{aligned}$$

Step 2. Define an auxiliary function as follows

$$g_2(t) = s_2(t) - \langle s_2(t), \varphi_1(t)\rangle \varphi_1(t). \tag{10.3}$$

The auxiliary function in (10.3) provides a projection of the signal $s_2(t)$ on the direction defined by $\varphi_1(t)$, and then this projection is subtracted from the signal $s_2(t)$ of the alphabet. The auxiliary function $g_2(t)$ is orthogonal to $\varphi_1(t)$, as seen in Figure 10.1 for vectors. Recall that through the use of the geometric interpretation, signals and vectors are equivalent. The second orthonormal basis function is given by

$$\varphi_2(t) = \frac{g_2(t)}{\|g_2(t)\|}. \tag{10.4}$$

Step 3. In this step, we take the third signal from the alphabet, and obtain the auxiliary function that will be orthogonal to the two previous basis functions. This is achieved by subtracting the projections of the third signal on the directions indicated by the two previous basis functions. The auxiliary function is given by

$$g_3(t) = s_3(t) - \langle s_3(t), \varphi_1(t)\rangle \varphi_1(t) - \langle s_3(t), \varphi_2(t)\rangle \varphi_2(t). \tag{10.5}$$

Then the third orthonormal basis function is obtained by normalizing the auxiliary function of Equation (10.5) to get

$$\varphi_3(t) = \frac{g_3(t)}{\|g_3(t)\|}. \tag{10.6}$$

The procedure follows through all the steps by taking each and every signal in the alphabet, so the procedure will consist of q steps. In general, the $k-$th step is the following:

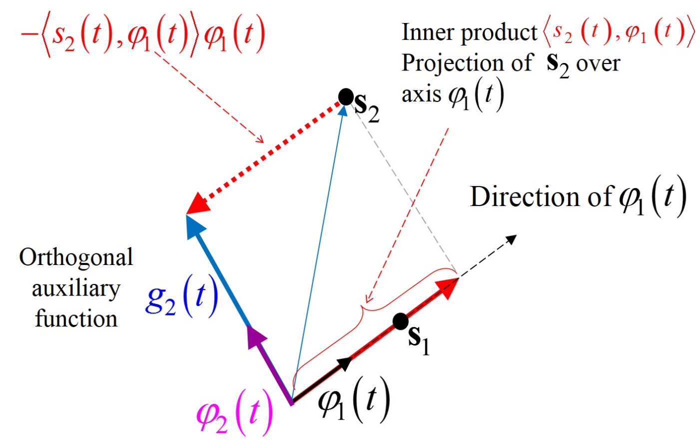

Figure 10.1: Orthogonality of auxiliary function.

Step k. Take the $k-$th signal in the alphabet, $k = 2, 3, \ldots, q$, and obtain the auxiliary function

$$g_k(t) = s_k(t) - \sum_{i=1}^{k-1} \langle s_k(t), \varphi_i(t)\rangle \varphi_i(t). \tag{10.7}$$

The $k-$th orthonormal signal will be given by

$$\varphi_k(t) = \frac{g_k(t)}{\|g_k(t)\|}. \tag{10.8}$$

Recall that **ALL** the signals in the alphabet must be taken in the procedure. In some of the steps, say step k, the auxiliary function $g_k(t)$ can be zero, meaning that the signal $s_k(t)$ being analyzed in such step can be represented as a linear combination of the basis functions $\varphi_1(t), \ldots, \varphi_{k-1}(t)$, in other words, the $k-$th signal is a linear combination of other signals in the alphabet.

In the case of having a zero auxiliary function, it can be concluded that there is no orthonormal basis function for that particular step. Even in this condition of a zero auxiliary function, the procedure must continue with the next signal in the alphabet in another step of the procedure. At the end of the procedure, the orthonormal basis functions found will be $\varphi_1(t), \ldots, \varphi_L(t)$, where $L \leq q$, meaning that the space being spanned has an $L-$dimensional space. If the signals of the alphabet are taken in different order, then the orthonormal set of functions will most likely be different, but the dimension (number of functions) will be the same.

10.2 Vector Space Representation

After applying the Gram-Schmidt orthogonalization procedure to a signal alphabet $\Omega = \{s_1(t), s_2(t), \ldots$ we obtain the set of basis function that span the space, i.e., a set of functions such that represents the signals through linear combinations. Now, since the basis functions $\varphi_1(t), \ldots, \varphi_L(t)$, where $L \leq q$, are orthonormal, they also are what is called *unitary functions or unitary vectors.* Hence, every linear combination of the basis functions will be a vector in the space generated by them.

To see this vector representation of the signals in the alphabet, from Equation (10.2) we get the linear combination that represents the first signal of the alphabet as follows

$$\begin{aligned} s_1(t) &= \|s_1(t)\| \varphi_1(t) \\ &= \|s_1(t)\| \varphi_1(t) + 0\varphi_2(t) + 0\varphi_3(t) + \cdots + 0\varphi_L(t). \end{aligned} \tag{10.9}$$

From equations (10.3) and (10.4), we get the linear combination for the second signal of the alphabet as follows

$$\begin{aligned} s_2(t) &= g_2(t) + \langle s_2(t), \varphi_1(t)\rangle \varphi_1(t) \\ &= \|g_2(t)\| \varphi_2(t) + \langle s_2(t), \varphi_1(t)\rangle \varphi_1(t) \\ &= \langle s_2(t), \varphi_1(t)\rangle \varphi_1(t) + \|g_2(t)\| \varphi_2(t) + 0\varphi_3(t) + \cdots + 0\varphi_L(t). \end{aligned} \tag{10.10}$$

And in general, from equations (10.7) and (10.8), the linear combination that represents the $k-$th signal is given by

$$\begin{aligned} s_k(t) &= g_k(t) + \sum_{i=1}^{k-1} \langle s_k(t), \varphi_i(t)\rangle \varphi_i(t) \\ &= \|g_k(t)\| \varphi_k(t) + \sum_{i=1}^{k-1} \langle s_k(t), \varphi_i(t)\rangle \varphi_i(t) + 0\varphi_k(t) + \cdots + 0\varphi_L(t) \\ &= \sum_{i=1}^{k-1} \langle s_k(t), \varphi_i(t)\rangle \varphi_i(t) + \|g_k(t)\| \varphi_k(t) + 0\varphi_{k+1}(t) + \cdots + 0\varphi_L(t). \end{aligned} \tag{10.11}$$

Since the basis functions represent unitary vectors in a vector space, from equations (10.9), (10.10) and (10.11), we get the vector representations of the signals in the alphabet in an $L-$dimensional vector space as follows

$$s_1(t) \longrightarrow \mathbf{s}_1 = (s_{11}, s_{12}, s_{13}, s_{14}, \ldots, s_{1L}) = \underbrace{(\|s_1(t)\|, 0, 0, 0, \ldots, 0)}_{L \text{ elements}}, \tag{10.12}$$

$$s_2(t) \longrightarrow \mathbf{s}_2 = (s_{21}, s_{22}, s_{23}, s_{24}, \ldots, s_{2L}) = \underbrace{(\langle s_2(t), \varphi_1(t)\rangle, \|g_2(t)\|, 0, 0, \ldots, 0)}_{L \text{ elements}}, \tag{10.13}$$

$$\begin{aligned} s_k(t) \longrightarrow \mathbf{s}_k &= (s_{k1}, s_{k2}, s_{k3}, s_{k4}, \ldots, s_{kL}) \\ &= \underbrace{(\langle s_k(t), \varphi_1(t)\rangle, \langle s_k(t), \varphi_2(t)\rangle, \ldots, \langle s_k(t), \varphi_{k-1}(t)\rangle, \|g_k(t)\|, 0, \ldots)}_{L \text{ elements}} \end{aligned} \tag{10.14}$$

The vectors in equations () - (), define in the $L-$dimensional vector space, what is known as a **signal constellation**. In summary, we can say that any set of q energy signals $\Omega =$

$\{s_1(t), s_2(t), \ldots, s_q(t)\}$ can be represented as linear combinations of L orthonormal basis functions where $L \leq q$, in other words, we have the representation also known as the **synthesis equation** as follows

$$s_i(t) = \sum_{j=1}^{L} s_{ij}\varphi_j(t),\ 0 \leq t \leq T,\ i = 1, 2, \ldots, q. \tag{10.15}$$

The **analysis equation** is given by the inner products used to obtain the coefficients of the linear combination, i.e.,

$$s_{ij} = \langle s_i(t), \varphi_j(t)\rangle = \int_0^T s_i(t)\varphi_j(t)\,dt,\ j = 1, 2, \ldots, L;\ \ i = 1, 2, \ldots, q. \tag{10.16}$$

In equations (10.15) and (10.16), the set of orthonormal basis functions $\varphi_1(t), \ldots, \varphi_L(t)$ satisfy the orthonormality conditions

$$\langle \varphi_i(t), \varphi_j(t)\rangle = \int_0^T \varphi_i(t)\varphi_j(t)\,dt = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{10.17}$$

In Figure 10.2, we can see the representation in a block diagram of the synthesis and analysis equations. These block diagram representation is used as the base for every digital communications system, specially the analysis equation which is used for the receivers.

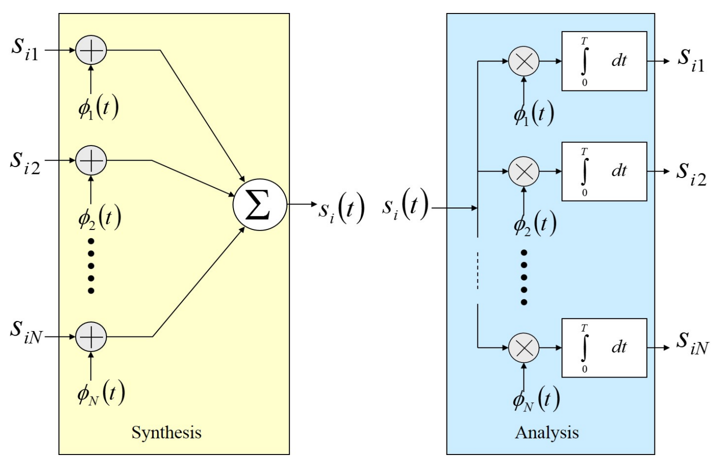

Figure 10.2: Synthesis and analysis block diagrams for an $N-$dimensional vector space.

The vector space representation of the signal alphabet provides an equivalence between the functions representing the signals and the vectors representing the signals, we can see this by comparing the properties of a signal $x(t) \in \Sigma$, or $x(t) \in L^2[0, T]$, and those of $\mathbf{x} = (x_1, x_2, \ldots, x_L)$. Suppose that we have signals $x(t), y(t) \in \Sigma$, and that $\varphi_1(t), \ldots, \varphi_L(t)$ is an orthonormal basis of Σ, then, there is a representation of the signals $x(t), y(t)$ as linear combinations of the basis functions as follows

$$x(t) = \sum_{j=1}^{L} x_j \varphi_j(t), \quad y(t) = \sum_{j=1}^{L} y_j \varphi_j(t). \tag{10.18}$$

The inner product of these two signals and considering equations (10.17) and (10.18), we obtain the following equivalence

$$\begin{aligned}
\langle x(t), y(t) \rangle &= \int_0^T x(t)\, y(t)\, dt \\
&= \int_0^T \left\{ \sum_{i=1}^{L} x_i \varphi_i(t) \sum_{j=1}^{L} y_j \varphi_j(t) \right\} dt \\
&= \int_0^T \sum_{i=1}^{L} \sum_{j=1}^{L} x_i y_j \varphi_i(t)\, \varphi_j(t)\, dt \\
&= \sum_{i=1}^{L} \sum_{j=1}^{L} x_i y_j \int_0^T \varphi_i(t)\, \varphi_j(t)\, dt \\
&= \sum_{i=1}^{L} x_i y_i \\
&= \langle \mathbf{x}, \mathbf{y} \rangle .
\end{aligned} \tag{10.19}$$

As seen in Equation (10.19), the inner product does not depend on the basis functions.

Also, the norm of a vector $\mathbf{x} = (x_1, \ldots, x_L)$ and the norm of a signal are related as follows

$$\begin{aligned}
\|\mathbf{x}\| &= \left\{ \sum_{i=0}^{L} x_i^2 \right\}^{1/2} \qquad (10.20) \\
&= \left\{ \sum_{i=0}^{L} x_i x_i \right\}^{1/2} \\
&= \left\{ \sum_{i=1}^{L} \sum_{j=1}^{L} x_i x_j \int_0^T \varphi_i(t)\, \varphi_j(t)\, dt \right\}^{1/2} \\
&= \left\{ \int_0^T \sum_{i=1}^{L} \sum_{j=1}^{L} x_i x_j \varphi_i(t)\, \varphi_j(t)\, dt \right\}^{1/2} \\
&= \left\{ \int_0^T \sum_{i=1}^{L} x_i \varphi_i(t) \sum_{j=1}^{L} x_j \varphi_j(t)\, dt \right\}^{1/2} \\
&= \left\{ \int_0^T x(t)\, x(t)\, dt \right\}^{1/2} \\
&= \left\{ \int_0^T x^2(t)\, dt \right\}^{1/2} \\
&= \|x(t)\|
\end{aligned}$$

10.3 Digital Communication Problem

Consider the system shown in Figure 10.3 with signal alphabet $\mathcal{S} = \{s_1(t), s_2(t), \ldots, s_q(t)\}$. For such system what we have so far is that

1. We can find an orthonormal basis $\{\varphi_j(t)\}_{j=1}^{L}$ for the signal set $\mathcal{S}$.

2. The received signal $R(t)$ is processed as a vector $\mathbf{R} = (R_1, R_2, \ldots, R_L)$, where $R_i = \int_0^T R(t)\varphi_i(t)\, dt$, in an L-dimensional space spanned by the orthonormal basis.

3. We use $\mathbf{R}$ to make a decision for M.

We claim that $\mathbf{R}$ is sufficient statistic for decisions about M based on $R(t)$, $0 \le t \le T$.

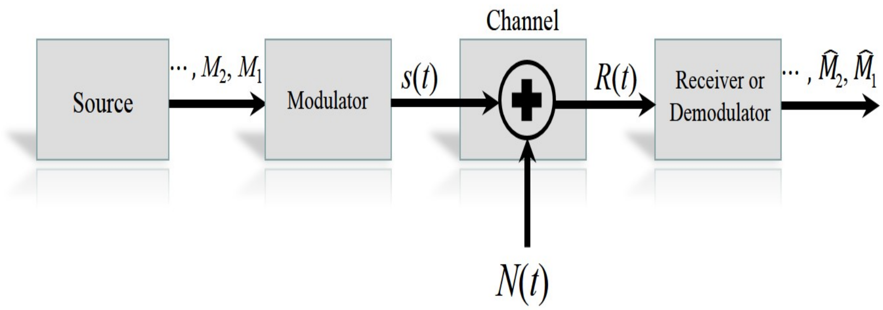

Figure 10.3: Block diagram of a digital communication system.

Lemma: Let $\{\psi_j(t)\}_{j=1}^{\infty}$ be an orthonormal basis for $L^2[0,T]$. Let $\tilde{\mathbf{R}} = (\tilde{R}_1, \tilde{R}_2, \ldots)$ where

$$\tilde{R}_i = \int_0^T R(t)\psi_i(t)\,dt, \tag{10.21}$$

then $\tilde{\mathbf{R}}$ is sufficient statistic for M based on $R(t)$, $0 \le t \le T$,

$$R(t) = \sum_{i=1}^{\infty} \tilde{R}_i \psi_i(t)\,. \tag{10.22}$$

See that if the signal received has finite energy, i.e., $P\left(\int_0^T R^2(t)dt < \infty\right) = 1$, then the mapping $R(t) \longrightarrow \tilde{\mathbf{R}}$ is one to one, and this implies that $\tilde{\mathbf{R}}$ is sufficient statistic.

First, form the orthonormal basis $\{\psi_j(t)\}_{j=1}^{\infty}$ as follows.

- Suppose that $\{\theta_j(t)\}_{j=1}^{\infty}$ is an orthonormal basis for $L^2[0,T]$.

- Apply the Gram Schmidt procedure to the following sequence

$$\varphi_1(t), \varphi_2(t), \ldots, \varphi_L(t), \theta_1(t), \theta_2(t), \ldots \tag{10.23}$$

  the orthonormal functions $\{\varphi_j(t)\}_{j=1}^{L}$ are taken from point (1) at the beginning of this section.

- Then define $\psi_j(t) = \varphi_j(t)$ for $j = 1, \ldots, L$, and $\psi_j(t) = \theta_{j-L}(t)$ for $j = L+1, \ldots$.

Then we have $\tilde{\mathbf{R}} = (\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_L, \tilde{R}_{L+1}, \ldots)$ where

$$\tilde{R}_i = \int_0^T R(t)\psi_i(t)\,dt = R_i, \quad 1 \le i \le L, \tag{10.24}$$

and for $i > L$ we have

$$\tilde{R}_i = \int_0^T R(t)\psi_i(t)\,dt. \tag{10.25}$$

Suppose that $M = m_k$, then $\mathbf{s}_k$ is transmitted so $R(t) = s_k(t) + N(t)$, then we have in the correlator receiver the following

$$\begin{aligned} \tilde{R}_i &= \int_0^T s_k(t)\psi_i(t)\,dt + \int_0^T N(t)\psi_i(t)\,dt \\ &= \int_0^T \left[\sum_{j=1}^{L} s_{kj}\varphi_j(t)\right]\psi_i(t)\,dt + \int_0^T N(t)\psi_i(t)\,dt \\ &= \sum_{j=1}^{L} s_{kj}\int_0^T \varphi_j(t)\,\psi_i(t)\,dt + \int_0^T N(t)\psi_i(t)\,dt. \end{aligned} \tag{10.26}$$

In the last row of Equation (10.26), the first integral is zero for $i \neq j$ since $\varphi_j(t)$ and $\psi_i(t)$ are orthogonal for $j \leq L$ and $\varphi_j(t) = 0$ for $j > L$ because the way the basis was constructed, hence for $i > L$

$$\tilde{R}_i = \int_0^T N(t)\psi_i(t)\,dt = N_i. \tag{10.27}$$

Now, for $i \leq L$ and $i = j$, we have that $\int_0^T \varphi_j(t)\,\psi_i(t)\,dt = 1$ then

$$\tilde{R}_i = s_{ki} + N_i = R_i. \tag{10.28}$$

Note that N_i and N_j are independent, also recall that $\mathbf{R} = (\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_L) = \mathbf{s}_{+}\mathbf{N}$. In Figure 10.4 we can see the system with input $\mathbf{R}$ of L elements and another input $\mathbf{N}' = (R_{\tilde{L+1}}, R_{\tilde{L+2}}, \ldots,)$. Note that for $i > L$ from Equation (10.27) we have that $\mathbf{N}' = (N_{L+1}, N_{L+2}, \ldots,)$, and that $\mathbf{N}$ and $\mathbf{N}'$ are independent. The output in Figure 10.4 says that it is sufficient statistics based on $\tilde{R}$, and that $\tilde{R}$ is sufficient statistic based on $R(t)$.

If $\mathbf{s} \in \mathbb{R}^2$, this implies that we have orthonormal functions $\varphi_1(t)$ and $\varphi_2(t)$, then $\mathbf{N}'$ is the noise in other dimensions.

Example 6 *Gram-Schmidt Orthogonalization.*

Consider the signal alphabet shown in Figure 10.5 and obtain the orthonormal basis with the Gram-Schmidt orthogonalization procedure.

In this case, we are going to take the signals in the order they are to apply the orthogonalization procedure. Before starting, note that the alphabet or energy signal set has a duration of

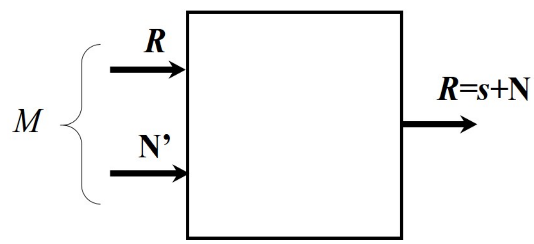

Figure 10.4: Sufficient statistics.

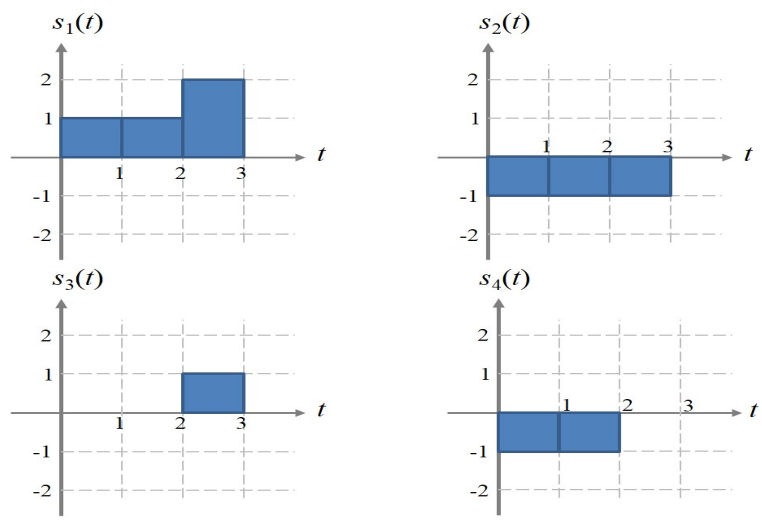

Figure 10.5: Signal alphabet for example.

$T = 3$. The first step is to obtain the signal energy of $s_1(t)$ which is

$$||s_1(t)||^2 = \int_0^T s_1^2(t)\, dt = 6. \tag{10.29}$$

Now we can apply the GS procedure to obtain the following first basis function

$$\varphi_1(t) = \frac{s_1(t)}{||s_1(t)||}. \tag{10.30}$$

The first basis function $\varphi_1(t)$ is shown in Figure 10.6

Next step is to obtain the auxiliary function $g_2(t)$ which is given by

$$g_2(t) = s_2(t) - \langle s_2(t), \varphi_1(t)\rangle \varphi_1(t). \tag{10.31}$$

Recall that the inner product must give a scalar. The inner product needed is obtained as follows

$$\begin{aligned} \langle s_2(t), \varphi_1(t)\rangle &= \int_0^T s_2(t)\varphi_1(t)\, dt \\ &= -\frac{4}{\sqrt{6}}, \end{aligned} \tag{10.32}$$

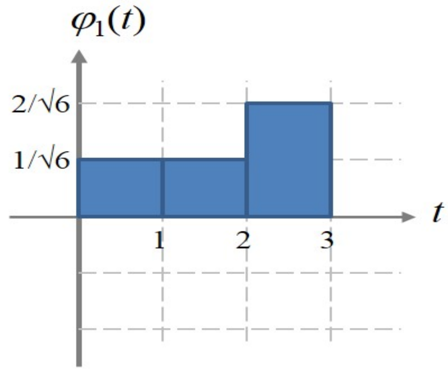

Figure 10.6: Orthonormal signal $\varphi_1(t)$ for example.

thus, the function is obtained by solving $g_2\left(t\right) = s_2\left(t\right) + \frac{4}{\sqrt{6}}\varphi_1\left(t\right)$ as shown in Figure 10.7.

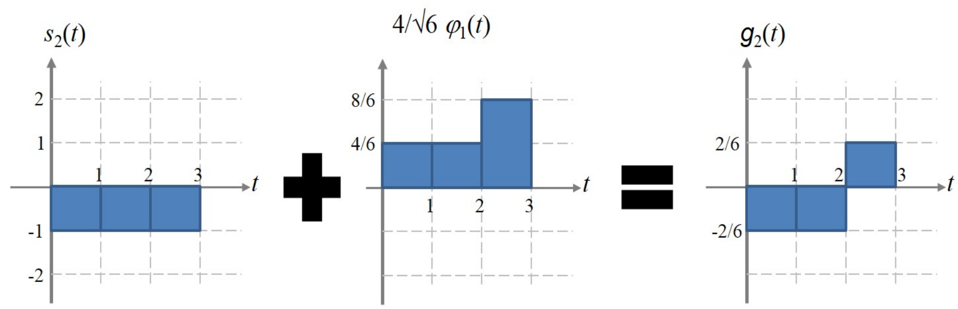

Figure 10.7: Function $g_2(t)$ for example.

Now we can apply the GS procedure to obtain the second basis function as follows

$$\varphi_2(t) = \frac{g_2(t)}{||g_2(t)||}, \tag{10.33}$$

where the norm of $g_2(t)$ is given by

$$||g_2(t)||^2 = \int_0^T g_2^2(t)\, dt = \frac{12}{36} = \frac{1}{3}. \tag{10.34}$$

Therefore, the second basis function is as that shown in Figure 10.8.

We must continue the procedure to obtain the function $g_3(t)$ as follows

$$g_3\left(t\right) = s_3\left(t\right) - \left\langle s_3\left(t\right), \varphi_1\left(t\right)\right\rangle \varphi_1\left(t\right) - \left\langle s_3\left(t\right), \varphi_2\left(t\right)\right\rangle \varphi_2\left(t\right). \tag{10.35}$$

The inner products needed are obtained as follows

$$\begin{aligned} \left\langle s_3\left(t\right), \varphi_1\left(t\right)\right\rangle &= \int_0^T s_3(t)\varphi_1(t)\, dt \\ &= \frac{2}{\sqrt{6}}, \end{aligned} \tag{10.36}$$

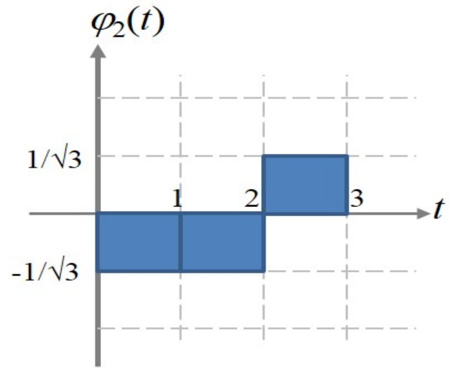

Figure 10.8: Orthonormal signal $\varphi_2(t)$ for example.

$$\begin{aligned} \langle s_3(t), \varphi_2(t) \rangle &= \int_0^T s_3(t)\varphi_2(t)\, dt \\ &= \frac{1}{\sqrt{3}}, \end{aligned} \tag{10.37}$$

Once you compute the function $g_3(t)$ in Equation (10.35), you can demonstrate that gives

$$g_3(t) = 0. \tag{10.38}$$

Hence, we need to continue with the procedure and analyze the last signal of the alphabet by obtaining the function $g_4(t)$ which is

$$g_4(t) = s_4(t) - \sum_{i=1}^{4-1} \langle s_4(t), \varphi_i(t) \rangle \varphi_i(t). \tag{10.39}$$

The inner products needed are obtained as follows

$$\begin{aligned} \langle s_4(t), \varphi_1(t) \rangle &= \int_0^T s_4(t)\varphi_1(t)\, dt \\ &= -\frac{2}{\sqrt{6}}, \end{aligned} \tag{10.40}$$

$$\begin{aligned} \langle s_4(t), \varphi_2(t) \rangle &= \int_0^T s_4(t)\varphi_2(t)\, dt \\ &= \frac{2}{\sqrt{3}}, \end{aligned} \tag{10.41}$$

Now, we need to calculate the function $g_3(t)$ as follows

$$g_4(t) = s_4(t) + \frac{2}{\sqrt{6}}\varphi_1(t) - \frac{2}{\sqrt{3}}\varphi_2(t). \tag{10.42}$$

If you carry on the calculations, you will also see that you get $g_3(t) = 0$. Therefore, the orthonormal basis function set is given by $\{\varphi_1(t), \varphi_2(t)\}$ which are shown in figures 10.6 and 10.8, respectively. Thus, the signal space or constellation of our alphabet will be a two-dimensional space. Now we must obtain the signal representation of the alphabet by finding the linear combinations. These linear combinations are given by

$$\begin{aligned}
s_1(t) &= \sqrt{6}\varphi_1(t), \\
s_2(t) &= -\frac{4}{\sqrt{6}}\varphi_1(t) + \frac{1}{\sqrt{3}}\varphi_2(t), \\
s_3(t) &= \frac{2}{\sqrt{6}}\varphi_1(t) + \frac{1}{\sqrt{3}}\varphi_2(t), \\
s_4(t) &= -\frac{2}{\sqrt{6}}\varphi_1(t) + \frac{2}{\sqrt{3}}\varphi_2(t).
\end{aligned} \tag{10.43}$$

With the linear combinations in Equation (10.43), we can obtain the vector representations of the signals by considering the coefficients in (10.43) as the coordinates for each signal vector, this is as follows

$$\begin{aligned}
\mathbf{s}_1 &= (\sqrt{6}, 0), \\
\mathbf{s}_2 &= (-\frac{4}{\sqrt{6}}, \frac{1}{\sqrt{3}}), \\
\mathbf{s}_3 &= (\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{3}}), \\
\mathbf{s}_4 &= (-\frac{2}{\sqrt{6}}, \frac{2}{\sqrt{3}}).
\end{aligned} \tag{10.44}$$

The constellation of the alphabet is shown in Figure 10.9, where we also have decision regions shown.

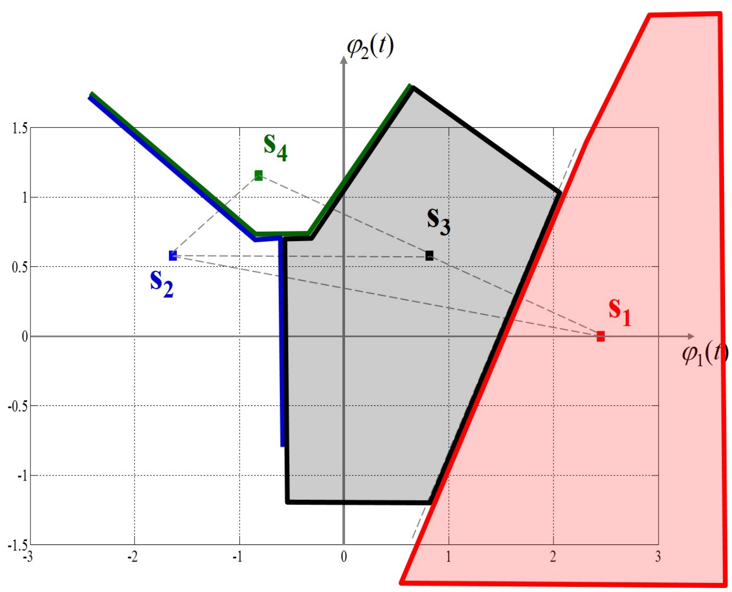

Figure 10.9: Orthonormal signal $\varphi_2(t)$ for example.

Chapter 11

Optimum Receivers

Given a modulator with signal set $\mathcal{S} = \{s_i(t)\}_{i=1}^{q}$, we want to find the optimal receiver.

First, find an orthonormal spanning set $\{\varphi_j(t)\}_{j=1}^{L}$, with $L \leq q$. Recall that there are many orthonormal spanning sets, but the performance is the same no matter which one you choose, however the receiver designs will be different and the vector representation (constellation) will be different.

In Figure 11.1, we have one set of signals with two different orthonormal basis which give different vectors.

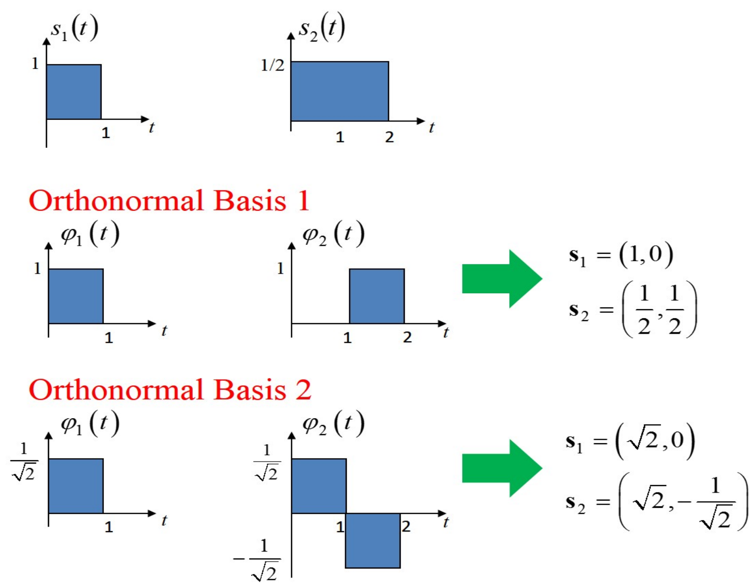

Figure 11.1: Orthonormal signal $\varphi_i(t)$, $i = 1, 2$, for example.

Although vector representations are different, norms, distances between signals, inner products are the same.

In Figure 11.2, one can see the receiver with L multipliers and L integrators. Recall that the inputs to the decision block are random variables, i.e., we have the random vector

$\mathbf{r} = (r_1, r_2, \ldots, r_L)$.

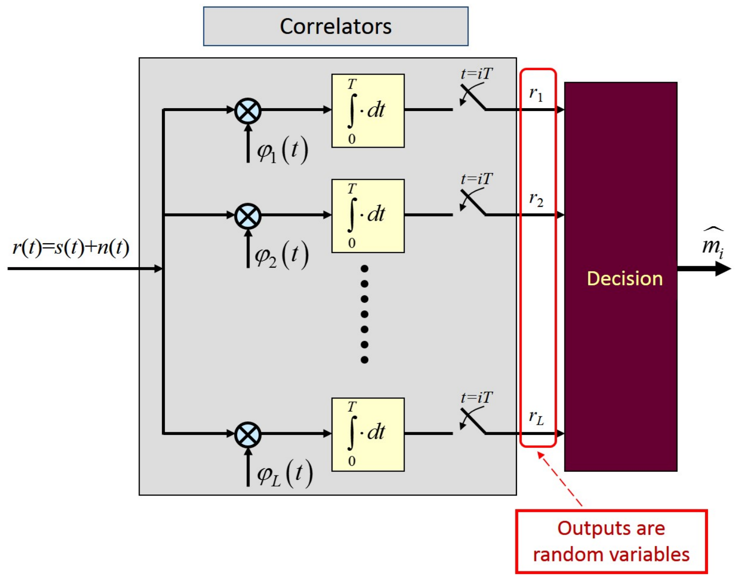

Figure 11.2: Optimum receiver with L multipliers and integrators.

From the notions on sufficient statistic, we have not lost optimality, then for the decision block use the optimum vector decision rule on $\mathbf{r} = (r_1, r_2, \ldots, r_L)$, which is

$$\boxed{\begin{array}{l} g(\mathbf{r}) = m_i, \quad \text{IFF} \\ \qquad ||\mathbf{r} - \mathbf{s}_i||^2 \quad -N_0 \ln p_M(m_i) \quad \text{IS SMALLEST} \,. \end{array}} \tag{11.1}$$

Now, expanding the decision rule, we have

$$||\mathbf{r} - \mathbf{s}_i||^2 = \sum_{j=1}^{L} (r_j - s_{ij})^2 = \sum_{j=1}^{L} r_j^2 - 2\sum_{j=1}^{L} r_j s_{ij} + \sum_{j=1}^{L} s_{ij}^2. \tag{11.2}$$

On the right hand side of Equation (11.2), the first term does not depend on i, i.e., it is independent of what $g(\mathbf{r})$ chooses. Therefore, to minimize $||\mathbf{r} - \mathbf{s}_i||^2$, we can use the remaining terms in (11.2). Note that in the second term on the right hand side of (11.2), we have the following inner product

$$\sum_{j=1}^{L} r_j s_{ij} = \langle \mathbf{r}, \mathbf{s}_i \rangle. \tag{11.3}$$

Also, note that the third term on the right hand side of (11.2) is the signal energy, i.e.,

$$\sum_{j=1}^{L} s_{ij}^2 = E_i = ||\mathbf{s}_i||^2. \tag{11.4}$$

Then, we can rewrite the decision rule as

$$\boxed{\begin{array}{ll} g(\mathbf{r}) = m_i, & \text{IFF} \\ & E_i - 2\langle \mathbf{r}, \mathbf{s}_i \rangle \quad -N_0 \ln p_M(m_i) \quad \text{IS SMALLEST}\ . \end{array}} \tag{11.5}$$

E_i and $N_0 \ln p_M(m_i)$ are known, so we have to calculate in *real-time* only $\langle \mathbf{r}, \mathbf{s}_i \rangle$. For ease of presentation, let

$$c_i = -\frac{1}{2}E_i + \frac{N_0}{2}\ln p_M(m_i), \tag{11.6}$$

then the decision rule is given by

$$\boxed{\begin{array}{ll} g(\mathbf{r}) = m_i, & \text{IFF} \\ & \langle \mathbf{r}, \mathbf{s}_i \rangle \quad +c_i \quad \text{IS LARGEST}\ . \end{array}} \tag{11.7}$$

The receiver according to the decision rule shown in (11.7) is shown in Figure 11.3. Recall that in the decision rule blocks, the receiver knows the L-dimensional constellation $\mathbf{s}_i$ for all $i = 1, 2, \ldots, q$, the energy of signal i, E_i, the a-priori probabilities $p_M(m_i)$ and the noise N_0. The first block in the decision rule computes the inner products $\langle \mathbf{r}, \mathbf{s}_i \rangle$, for all $i = 1, 2, \ldots, q$ when the input vector is obtained, i.e., $\mathbf{r} = (r_1, r_2, \ldots, r_L)$.

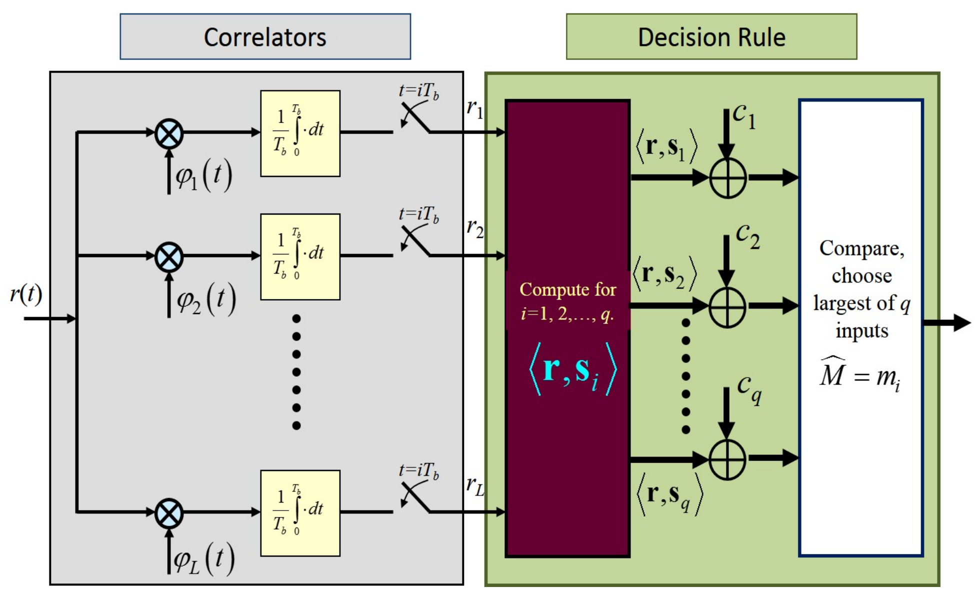

Figure 11.3: Optimum receiver with L multipliers and L integrators.

A different implementation of the receiver with the same decision rule but with q multipliers and integrators is shown in Figure 11.4.

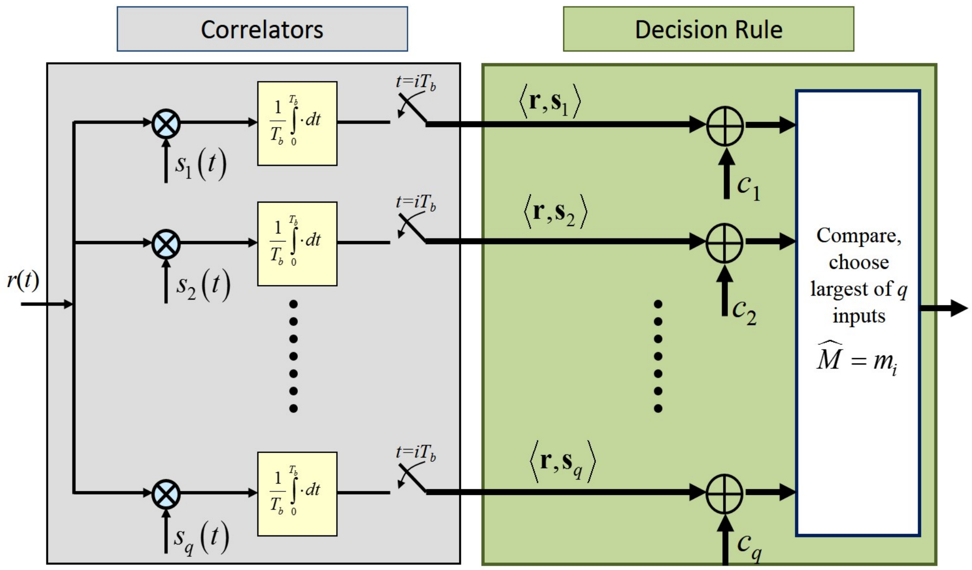

Figure 11.4: Optimum receiver with q multipliers and q integrators.

11.1 Signal Set Geometries

In this section, we have some examples of signal geometries (constellations) that are common to represent modulators.

11.1.1 Antipodal Signal Set

The signal alphabet is

$$\mathcal{S}_{BPSK} = \{s_0(t), s_1(t)\}, \quad t \in [0, T]. \tag{11.8}$$

We also have the condition of antipodal signals which is

$$s_1(t) = -s_0(t), \tag{11.9}$$

BPSK Modulation

Binary Phase Shift Keying (BPSK) is a binary modulation that changes phases between different symbols (bits). The receiver must be able to detect phase changes of π radians. Example of a modulator that uses this set of signals is **BPSK**, where we can define the signals as follows

$$\begin{aligned} s_0(t) &= A\cos(2\pi f_c t + \theta), \quad 0 \le t \le T, \\ s_1(t) &= A\cos(2\pi f_c t + \theta + \pi), \quad 0 \le t \le T, \end{aligned} \tag{11.10}$$

We can use the Gram-Schmidt orthogonalization procedure with the set of signals in (11.10). Note that since $s_1(t) = -s_0(t)$, there is a dependence of one signal, i.e., one of the signals can be expressed as a linear combination of the other. This dependence means that the orthogonalization procedure will give an orthonormal basis which has a dimension that is at least of one unit less than signals in the alphabet. In this case it means that we only will have one orthonormal function and both signals in the alphabet can be expressed as linear combinations of the orthonormal function.

The orthonormal signal set will be given by only one function which is

$$\varphi_1(t) = \frac{s_0(t)}{||s_0(t)||} \tag{11.11}$$

The signals in the alphabet have the same energy, i.e.,

$$||s_0(t)||^2 = ||s_1(t)||^2 = E. \tag{11.12}$$

If you get the energy of the signals using the root mean square time average, we get

$$E = \frac{A^2}{2}T, \tag{11.13}$$

with this energy, in general we know the bit duration T, since we know the bit rate R at which we transmit, and since $R = 1/T$, then T is known. We also have in general channel limitations that make us define upper limits in energy E, hence E is also known, then what we do not know is the amplitude A, then A is given by

$$A = \sqrt{\frac{2E}{T}}, \tag{11.14}$$

and we can re-write the signal set as

$$\begin{aligned} s_0(t) &= \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \theta), \quad 0 \le t \le T, \\ s_1(t) &= \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \theta + \pi), \quad 0 \le t \le T, \end{aligned} \tag{11.15}$$

or as

$$\begin{aligned} s_0(t) &= \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \theta), \quad 0 \le t \le T, \\ s_1(t) &= -\sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \theta), \quad 0 \le t \le T. \end{aligned} \tag{11.16}$$

See that the orthonormal function is given by

$$\begin{aligned} \varphi_1(t) &= \frac{s_0(t)}{||s_0(t)||} \\ &= \frac{A\cos(2\pi f_c t + \theta)}{\sqrt{\frac{A^2}{2}T}} \\ &= \sqrt{\frac{2}{T}}\cos(2\pi f_c t + \theta), \quad 0 \le t \le T. \end{aligned} \tag{11.17}$$

Therefore, the signals in the alphabet can be expressed as linear combination of the basis functions as follows

$$
\begin{aligned}
s_0(t) &= ||s_0(t)||\varphi_1(t) = \sqrt{E}\,\varphi_1(t)\,, \\
s_1(t) &= -\sqrt{E}\,\varphi_1(t)\,.
\end{aligned} \tag{11.18}
$$

With the linear combinations in (11.18), we can obtain the one-dimensional vector representations for each signal as follows $\mathbf{s}_0 = (\sqrt{E})$, and $\mathbf{s}_1 = (-\sqrt{E})$. Figure 11.5 shows the constellation of the BPSK modulation, see that the distance from the vectors to the origin are the squared root of the energy of the signal.

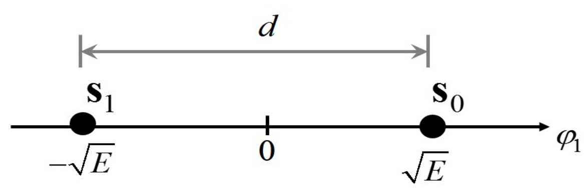

Figure 11.5: BPSK constellation.

Now, since we have only one orthonormal function for BPSK, the optimum receiver will have only one branch as shown in Figure 11.6.

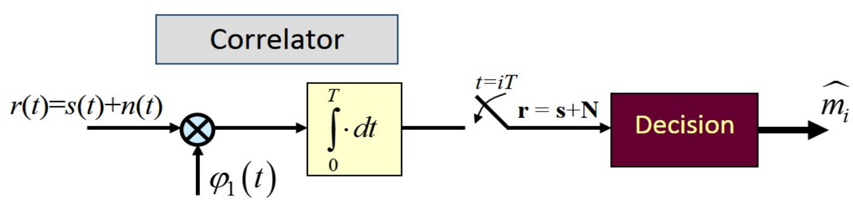

Figure 11.6: BPSK receiver.

Also, with two possible symbols being transmitted, the decision rule is

$$
\boxed{\begin{array}{c} g(\mathbf{r}) = m_0, \quad \text{IFF} \\ \|\mathbf{r}-\mathbf{s}_0\|^2 - N_0 \ln p_M(m_0) \;\; \leq \|\mathbf{r}-\mathbf{s}_1\|^2 - N_0 \ln p_M(m_1). \end{array}} \tag{11.19}
$$

We can develop the inequality in the decision rule in (11.19) taking into account that the vectors are one-dimensional to get

$$
\begin{aligned}
\|\mathbf{r}-\mathbf{s}_0\|^2 - N_0 \ln p_M(m_0) &\leq \|\mathbf{r}-\mathbf{s}_1\|^2 - N_0 \ln p_M(m_1) \\
r^2 - 2rs_0 + s_0^2 - N_0 \ln p_M(m_0) &\leq r^2 - 2rs_1 + s_1^2 - N_0 \ln p_M(m_1) \\
2rs_1 - 2rs_0 &\leq s_1^2 - s_0^2 + N_0 \ln\left(\frac{p_M(m_0)}{p_M(m_1)}\right) \\
r &\leq \frac{s_1+s_0}{2} + \frac{N_0}{2|s_1-s_0|} \ln\left(\frac{p_M(m_0)}{p_M(m_1)}\right).
\end{aligned} \tag{11.20}
$$

Note that in Equation (11.20), $|s_1 - s_0|$ is the separation distance d shown in the constellation of Figure 11.5, also that $N_0/2 = \sigma^2$ is the variance of the noise, i.e., $N \sim \mathcal{N}(0, \sigma^2)$. See that when m_0 is transmitted, $\sqrt{E}$ is the point of the constellation that is transmitted.

The right hand side of the last inequality in Equation (11.20) is a threshold, i.e., we have

$$r \leq \frac{s_1 + s_0}{2} + \frac{N_0}{2|s_1 - s_0|} \ln\left(\frac{p_M(m_0)}{p_M(m_1)}\right) = \frac{N_0}{2|s_1 - s_0|} \ln\left(\frac{p_M(m_0)}{p_M(m_1)}\right) = \mathbf{Th}, \tag{11.21}$$

see that when the constellation is symmetric as shown in Figure 11.5, the term $\frac{s_1+s_0}{2}$ will be zero, and the decision rule can be summarized as

$$\boxed{g(\mathbf{r}) = \widehat{M} = m_0, \quad \text{IFF } r \geq \mathbf{Th}, \;\; \text{otherwise } \widehat{M} = m_1, \;\; r < \mathbf{Th}.} \tag{11.22}$$

Now, to calculate the probability of error, we use the law of total probability as follows

$$P(\mathcal{E}) = P(\mathcal{E}|M = m_0)p_M(m_0) + P(\mathcal{E}|M = m_1)p_M(m_1), \tag{11.23}$$

hence, we first find the conditional error probabilities. See that $P(\mathcal{E}|M = m_0)$ is assuming that s_0 is transmitted hence the error probability will be given by the probability that $\mathbf{r} = \mathrm{s}_0 + \mathbf{N}$ is $r < \mathbf{Th}$. Also, recall that $\mathbf{r} = \mathrm{s}_0 + \mathbf{N}$ is Gaussian since s_0 is fixed and $\mathbf{N}$ is Gaussian. We can determine that $\mathbf{r}$ is Gaussian with variance σ^2 and mean value s_0. In other words, the bell-shaped Gaussian density is shifted and centered at the point for s_0. This is something similar that is described by Figure 11.7 and substituting $m_2 = s_0$ and $m_1 = s_1$.

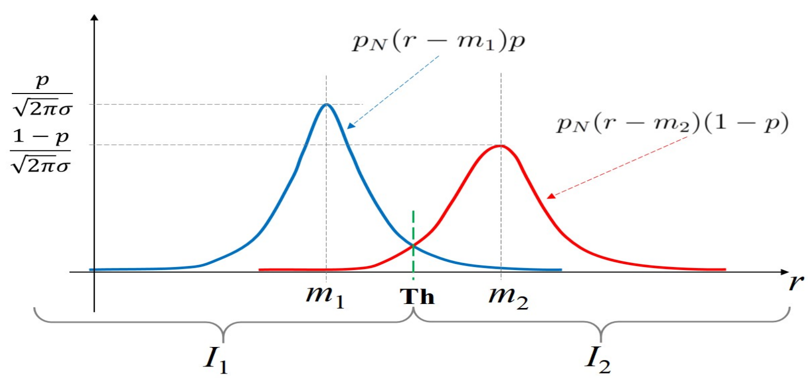

Figure 11.7: Constellation and Gaussian density functions

Hence, in Figure 11.7, the conditional error probability $P(\mathcal{E}|M = m_0)$ will be given by the area of the tail of the Gaussian density (color red) to the left of the threshold. In other words, the equivalent event is that the vector $\mathbf{r}$ results on the left hand side of the threshold $\mathbf{Th}$, so we can write

$$P(\mathcal{E}|M = m_0) = P(\mathbf{r} < \mathbf{Th}). \tag{11.24}$$

We can elaborate the calculation of the conditional error probability by considering that $N \sim \mathcal{N}(0, \sigma^2 = N_0/2)$ as follows

$$
\begin{aligned}
P(\mathcal{E}|M = m_0) &= P(\mathbf{r} < \mathbf{Th}|M = m_0) \\
&= P(s + N < \mathbf{Th}|M = m_0) \\
&= P(s_0 + N < \mathbf{Th}|M = m_0) \\
&= P(N < \mathbf{Th} - s_0) \\
&= P\left(N < \mathbf{Th} - \sqrt{E}\right) \\
&= 1 - P\left(N > \mathbf{Th} - \sqrt{E}\right) \\
&= 1 - \int\limits_{\mathbf{Th}-\sqrt{E}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx, \qquad (11.25)
\end{aligned}
$$

We can make a change of variable (normalization) in the integral of (11.25) by using the substitution $z = x/\sigma$ to get

$$
\begin{aligned}
P(\mathcal{E}|M = m_0) &= 1 - \int\limits_{\mathbf{Th}-\sqrt{E}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= 1 - \int\limits_{\frac{\mathbf{Th}-\sqrt{E}}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2}} \sigma dx \\
&= 1 - \int\limits_{\frac{\mathbf{Th}-\sqrt{E}}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dx \\
&= 1 - Q\left(\frac{\mathbf{Th} - \sqrt{E}}{\sigma}\right) \\
&= Q\left(\frac{\sqrt{E} - \mathbf{Th}}{\sigma}\right). \qquad (11.26)
\end{aligned}
$$

Similarly, we can get

$$
P(\mathcal{E}|M = m_1) = Q\left(\frac{\sqrt{E} + \mathbf{Th}}{\sigma}\right). \qquad (11.27)
$$

With (11.26) and (11.27) we can solve (11.23) as follows

$$
\begin{aligned}
P(\mathcal{E}) &= P(\mathcal{E}|M = m_0)p_M(m_0) + P(\mathcal{E}|M = m_1)p_M(m_1) \\
&= Q\left(\frac{\sqrt{E} - \mathbf{Th}}{\sigma}\right) p_M(m_0) + Q\left(\frac{\sqrt{E} + \mathbf{Th}}{\sigma}\right) p_M(m_1) \qquad (11.28)
\end{aligned}
$$

If we have equally likely messages, i.e., $p_M(m_0) = p_M(m_1) = 1/2$, then $\mathbf{Th} = 0$ and the probability of error is given by

$$P(\mathcal{E}) = Q\left(\frac{\sqrt{E}}{\sigma}\right), \tag{11.29}$$

and substituting $\sigma = \sqrt{N_0/2}$ we get

$$P(\mathcal{E}) = Q\left(\sqrt{\frac{2E}{N_0}}\right). \tag{11.30}$$

Appendix A. Complex Analysis

A.1 Complex Variable

This chapter is a review of the basic concepts of complex numbers and their operations and functions. Before proceeding to the complex number context, we recall some fundamental concepts of number theory.

Recall that numbers are organized in different sets, such as the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of rational numbers $\mathbb{Q} = \{p/q : \ p, q \in \mathbb{Z}, \ q \neq 0\}$, the set of real numbers $\mathbb{R}$, which sometimes and depending on the context, the set will contain the $-\infty$ and $+\infty$ symbols. The definitions and axioms in the following sections are based on [1].

A.1.1 The Real numbers

Definition 7 *An ordered set S, is a set with elements satisfying a relation such as $<$.*

Definition 8 *A field is a set F with two operations, addition and multiplication which satisfy the following axioms.*

- Addition
  - For any two numbers $x, y \in F$, we have $x + y \in F$.
  - *Commutative property*, i.e., $\forall x, y \in F$, $x + y = y + x$.
  - *Associativity property*, i.e., $\forall x, y, z \in F$, $(x + y) + z = x + (y + z)$.
  - *Null element*, or *additive identity*, $\forall x \in F$, there exists and element $0 \in F$ such that $x + 0 = x$.
  - *Additive inverse*, $\forall x \in F$, there is an element $-x \in F$ such that $x + (-x) = 0$.
- Multiplication
  - For any two numbers $x, y \in F$, we have $xy \in F$.
  - *Commutative property*, i.e., $\forall x, y \in F$, $xy = yx$.
  - *Associativity property*, i.e., $\forall x, y, z \in F$, $(xy)\, z = x\,(yz)$.

  - *Multiplicative identity*, $\forall x \in F$, there exists and element $1 \in F$ such that $x1 = x$.
  - *Multiplicative inverse*, $\forall x \in F$, $x \neq 0$, there is an element $1/x \in F$ such that $x(1/x) = 1$.

- *Distributive Law*, $\forall x, y, z \in F$, $x\,(y + z) = xy + xz$.

Definition 9 *An ordered field is a field F which is also an ordered set, such that*

- $\forall x, y, z \in F$, $x + y < x + z$ implies $y < z$.
- $\forall x, y \in F$, $xy > 0$ implies $x > 0$ and $y > 0$, or $x < 0$ and $y < 0$.

Definition 10 *Consider an ordered set S, and a subset $B \subset S$. The subset B is bounded above if $\exists \delta \in S$ such that $\forall x \in B$, $x \leq \delta$. δ is the upper bound of B.*

Definition 11 *Let $B \subset S$ be bounded above. Assume $\exists \alpha \in S$ such that*

- *α is an upper bound of B*
- *if $\beta < \alpha$, then β is not an upper bound of B.*

Then α is the least upper bound of the set B or the supremum of B, i.e., $\alpha = sup\ B$.

Theorem 12 *The set $\mathbb{R} = (-\infty,\ \infty)$ of real numbers together with the addition and multiplication operations is an ordered field with the least upper bound property, i.e. $(\mathbb{R}, \cdot, +)$ is a field. The set of rational numbers $\mathbb{Q}$ with the same operations is a subfield of $(\mathbb{R}, \cdot, +)$.*

Theorem 13 *For every positive $x \in \mathbb{R}$ and every integer $n > 0$, there is a unique real number y such that $y^n = x$, or $y = \sqrt[n]{x} = x^{1/n}$.*

Definition 14 *The real field $(\mathbb{R}, \cdot, +)$ together with the symbols $-\infty$ and $+\infty$ form the Extended Real Number System (ERNS).*

$\forall x \in \mathbb{R}$, some implications of the ERNS are the following

- $x/\infty = x/(-\infty) = 0$
- $x + \infty = +\infty$.
- $x + (-\infty) = -\infty$.
- if $x > 0$, $x(+\infty) = +\infty$, $x(-\infty) = -\infty$
- if $x < 0$, $x(+\infty) = -\infty$, $x(-\infty) = +\infty$

A.1.2 Complex Numbers

Recall that there are different sets of numbers such as the natural numbers, the rational numbers, the irrational numbers, the integer numbers, the real numbers and the complex numbers. It is common to see that the real numbers are extended by the ser of complex numbers, specially when solutions to equations such as $x^2 = -2$ need to be obtained. In order to work with the complex numbers, we need the definition $i^2 = -1$, sometimes instead of the symbol i,the symbol is changed to $j^2 = -1$.We denote that z is a complex number as $z \in \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. A complex number z is written in a form such as

$$z = a + jb, \tag{A.31}$$

where $a, b \in \mathbb{R}$, and a is called the real part of z, and b is called the imaginary part of z, i.e., $a = \Re\{z\}$, and $b = \Im\{z\}$. The complex conjugate of the complex number z is denoted as

$$z^* = a - jb. \tag{A.32}$$

Consider the complex number in (A.31) and another complex number $w = c + jd$. We can perform the following operations with these two numbers

1. Addition: $z + w = (a + c) + j(b + d)$ .This operation satisfies the laws of commutativity $(z + w = w + z)$, and associativity $(z + (w + x) = (z + w) + x)$, where z, w,and x are complex numbers. The conjugate of a sum is the sum of the conjugates, i.e., $(z + w)^* = z^* + w^*$.

2. Difference: $z - w = (a - c) + j(b - d)$ .This operation does not satisfy the laws of commutativity and associativity. The conjugate of a difference is the difference of the conjugates, i.e., $(z - w)^* = z^* - w^*$.

3. Multiplication: $zw = (ac - bd) + j(ad + bc)$. Multiplication also satisfies the commutative law since $zw = wz$, and the associativity law, $z(wx) = (zw)x$.The conjugate of a multiplication is the multiplication of the conjugates, i.e., $(zw)^* = z^*w^*$.

The multiplication and addition operation also satisfy the distributive law since $z(w + x) = zw + zx$. When a complex number is multiplied by its complex conjugate, the result is always a purely positive real number (when $z \neq 0$), since

$$\begin{aligned} zz^* &= (a + jb)(a - jb) \\ &= a^2 - jab + jba - j^2b^2 \\ &= a^2 + b^2. \end{aligned} \tag{A.33}$$

In Equation (A.33), the condition $j^2 = -1$ was used. Also, with the complex conjugate of z we have the following result

$$\begin{aligned} z + z^* &= (a + jb) + (a - jb) \\ &= 2a + j0 \\ &= 2\Re\{z\}, \end{aligned} \tag{A.34}$$

similarly, we have the following

$$\begin{aligned} z - z^* &= (a + jb) - (a - jb) \\ &= 0 + j2b \\ &= j2\Im\{z\}, \end{aligned} \tag{A.35}$$

A real number m can be written as a complex number y with zero imaginary part, i.e., $y = m + j0$. the product of a real number and a complex number is obtained as follows

$$\begin{aligned} mz &= yz \\ &= (m + j0)\,(a + jb) \\ &= m\,(a + jb) \\ &= am + jbm. \end{aligned} \tag{A.36}$$

For complex numbers z and w, with $w \neq 0$, the quotient z/w is obtained by multiplication and division with the complex conjugate of the denominator w^* as shown in the following expression

$$\begin{aligned} \frac{z}{w} &= \frac{a + jb}{c + jd} \\ &= \left(\frac{a + jb}{c + jd}\right)\left(\frac{c - jd}{c - jd}\right) \\ &= \frac{(ac + bd) + j\,(cb - ad)}{c^2 + d^2}. \end{aligned} \tag{A.37}$$

The square of a complex number z is given by

$$\begin{aligned} z^2 &= zz \\ &= (a + jb)\,(a + jb) \\ &= a^2 + jab + jba + j^2b^2 \\ &= \left(a^2 - b^2\right) + j2ab. \end{aligned} \tag{A.38}$$

The reciprocal of a complex number z is given by

$$\begin{aligned} \frac{1}{z} &= \frac{1}{(a + jb)} \\ &= \frac{1}{(a + jb)}\frac{(a - jb)}{(a - jb)} \\ &= \frac{a - jb}{a^2 + b^2} \\ &= \frac{z^*}{zz^*}. \end{aligned} \tag{A.39}$$

From Equation (A.39) and the previous results, suggest that every operation performed with complex numbers must be algebraically handled in order to obtain a final expression in the form $f + jg$, where the real and imaginary parts can be numbers, or rational expressions. With all the discussion carried out so far, it can be concluded that the set of complex numbers $\mathbb{C}$ together with the addition and multiplication operations, forms an ordered field. The complex numbers $0 + j0$ and $1 + j0$ are the additive and multiplicative identity elements, respectively.

Definition 15 *The modulus or absolute value of a complex number z is*

$$|z| = (zz^*)^{1/2} = \sqrt{a^2 + b^2}. \tag{A.40}$$

Some of the properties of the absolute value of complex numbers $z, w \in \mathbb{C}$ are the following

1. $|z| > 0$, unless $z = 0$.
2. $|z^*| = |z|$.
3. $|zw| = |z||w|$.
4. $|\Re\{z\}| \leq |z|$.
5. **Triangular inequality.** $|z + w| \leq |z| + |w|$.

Theorem 16 Schwartz Inequality. *Let $a_1, a_2, \ldots, a_n \in \mathbb{C}$ and $b_1, b_2, \ldots, b_n \in \mathbb{C}$, then*

$$\left|\sum_{j=1}^{n} a_j b_j^*\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2 . \tag{A.41}$$

Geometric Interpretation

A complex number $z = a + jb$ is an ordered pair (a, b) of real numbers, where the first component of the ordered pair corresponds to the real part $\Re\{z\}$ and the second component of the ordered pair corresponds to the imaginary part $\Im\{z\}$. In this way, the ordered pair representing a complex number z can be associated to a point (a, b) in a coordinate plane or in rectangular coordinates. Since a point (a, b) can be interpreted as a vector with initial point the origin and end point (a, b), then a complex number can also be seen as a vector in the **complex plane** where the horizontal axis corresponds to the real part, and the vertical axis to the imaginary part. This is shown in Figure A.8

Within this context, we have a new way to represent the complex number $z = a + jb$ as $z = (a, b)$. Recall that a point (a, b) in rectangular coordinates as that shown in Figure A.8 can be expressed in terms of polar coordinates (r, θ) as follows

$$a = r\cos\theta,\ b = r\sin\theta, \tag{A.42}$$

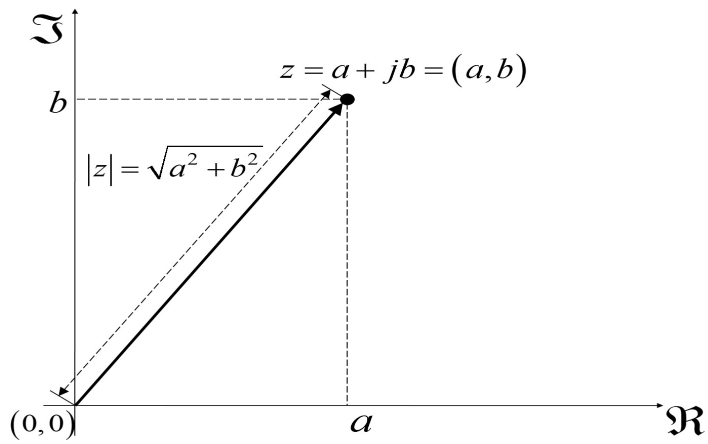

Figure A.8: Geometric interpretation of a complex number.

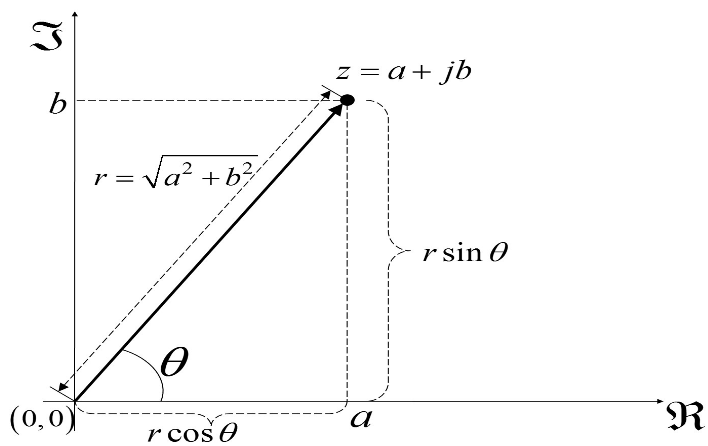

Figure A.9: Polar coordinates of a complex number.

with the relationship of

$$r = \sqrt{a^2 + b^2},\ \theta = \tan^{-1}\left(\frac{b}{a}\right), \tag{A.43}$$

the coordinate transformations can be seen in Figure A.9.

With the polar coordinate representation in (A.43), the complex number $z = a + jb$ can be represented as $z = r\cos\theta + jr\sin\theta$ or in Euler's notation as $z = re^{j\theta}$, with $r = |z|$ the magnitude of the complex number and θ its phase. As exercise, derive the following results for the multiplication and division of two complex numbers $z = a + jb$ or in polar form $z = r_z(\cos\theta_z + j\sin\theta_z)$ and $w = c + jd$ or $w = r_w(\cos\theta_w + j\sin\theta_w)$, see [3],

$$zw = r_z r_w \left[\cos(\theta_z + \theta_w) + j\sin(\theta_z + \theta_w)\right], \tag{A.44}$$

$$\frac{z}{w} = \frac{r_z}{r_w}\left[\cos(\theta_z - \theta_w) + j\sin(\theta_z - \theta_w)\right]. \tag{A.45}$$

Powers of Complex Numbers

With the results in equations (A.44) and (A.45), integer powers of complex numbers can be found, for example, with $w = z$ in (A.44), we can get

$$z^2 = r_z^2 \left[\cos\left(2\theta_z\right) + j \sin\left(2\theta_z\right)\right]. \tag{A.46}$$

To calculate z^3, we can use (A.46) with $z^3 = z^2 z$ to obtain

$$z^3 = r_z^3 \left[\cos\left(3\theta_z\right) + j \sin\left(3\theta_z\right)\right], \tag{A.47}$$

and in general, for $n > 0$ the n-th power of a complex number z is given by

$$z^n = r_z^n \left[\cos\left(n\theta_z\right) + j \sin\left(n\theta_z\right)\right] \tag{A.48}$$

In Figure A.9, the angle θ is also known as the argument of the complex number z, i.e. $\arg(z) = \theta$. We can follow the same inductive argument as that in equations (A.46) - (A.48) for the case when $n < 0$, by noting first that $1 = 1 + j0$ and that $1 = (1, 0)$ which implies that $\arg(1) = 0$, then from (A.45) we get

$$\frac{1}{z} = \frac{1}{r_z} \left[\cos\left(-\theta_z\right) + j \sin\left(-\theta_z\right)\right], \tag{A.49}$$

and then we can obtain from (A.46) and (A.49) the following

$$\frac{1}{z^2} = \frac{1}{r_z^2} \left[\cos\left(-2\theta_z\right) + j \sin\left(-2\theta_z\right)\right], \tag{A.50}$$

which in general takes us as well to Equation (A.48).

DeMoivre's Formula. For a normalized complex number $z = \cos\theta_z + j\sin\theta_z$, i.e., a complex number with $r_z = 1$, Equation (A.48) gives

$$z^n = \left[\cos\left(n\theta_z\right) + j \sin\left(n\theta_z\right)\right] = \left(\cos\theta_z + j\sin\theta_z\right)^n. \tag{A.51}$$

Roots of Complex Numbers

In order to obtain the n-th root of a complex number we can consider that we have two complex numbers in polar form, e.g., $z = r_z\left(\cos\theta_z + j\sin\theta_z\right)$ and $w = r_w\left(\cos\theta_w + j\sin\theta_w\right)$, and that we have the equality $w^n = z$, which becomes from (A.48) the following

$$r_w^n \left(\cos n\theta_w + j\sin n\theta_w\right) = r_z\left(\cos\theta_z + j\sin\theta_z\right). \tag{A.52}$$

From (A.52), it can be concluded that $r_w^n = r_z$, hence $r_w = r_z^{1/n}$, and we also have that

$$\cos n\theta_w + j\sin n\theta_w = \cos\theta_z + j\sin\theta_z. \tag{A.53}$$

From (A.53), we can get

$$\cos n\theta_w = \cos\theta_z, \quad \text{and} \quad \sin n\theta_w = \sin\theta_z. \tag{A.54}$$

Equality in expressions shown in (A.54) will be obtained for k integer as follows

$$n\theta_w = \theta_z + 2\pi k, \tag{A.55}$$

which gives

$$\theta_w = \frac{\theta_z + 2\pi k}{n}, \quad k = 0, 1, 2, \ldots, n-1. \tag{A.56}$$

Note that from Equation (A.56), we will obtain n different roots for the complex number z. Therefore, the n roots of complex number $z = r_z(\cos\theta_z + j\sin\theta_z)$, i.e., $w = z^{1/n}$ is given by

$$z^{1/n} = r_z^{1/n}\left[\cos\left(\frac{\theta_z + 2\pi k}{n}\right) + j\sin\left(\frac{\theta_z + 2\pi k}{n}\right)\right], \quad k = 0, 1, \ldots, n-1. \tag{A.57}$$

From Equation (A.57), it can be seen that the n roots of the complex number z are located on the circumference of the circle of radius r_z, at angles defined by $(\theta_z + 2\pi k)/n$, for $k = 0, 1, \ldots, n-1$. The concept of the circle defined by the complex number z is important since it defines regions on the complex plane and will be useful to define those regions where functions converge.

Complex Plane Sets

In the complex plane as that in Figure A.10, different sets can be defined, for example, the set of point indicated by the circle in the figure are all those points w with a distance to the complex number z that is less than or equal to ρ, i.e., the distance between the complex numbers $z = a + jb$ and $w = c + jd$ is given by

$$|z - w| = \sqrt{(a-c)^2 + (b-d)^2}, \tag{A.58}$$

and the circle in Figure A.10 of radius ρ, including the circumference is given by the following set of points

$$B_z(\rho) = \{w \in \mathbb{C} : \ |z - w| \leq \rho\}. \tag{A.59}$$

Different sets in the complex plane can be defined, for example, the circle in Equation (A.59) is a closed circle, the open circle or disc of radius ρ is defined by excluding the circumference to get

$$B_z(\rho) = \{w \in \mathbb{C} : \ |z - w| < \rho\}, \tag{A.60}$$

the right half plane of the complex plane including the imaginary axis is defined as

$$E = \{w \in \mathbb{C} : \ \Re\{w\} \geq 0\}. \tag{A.61}$$

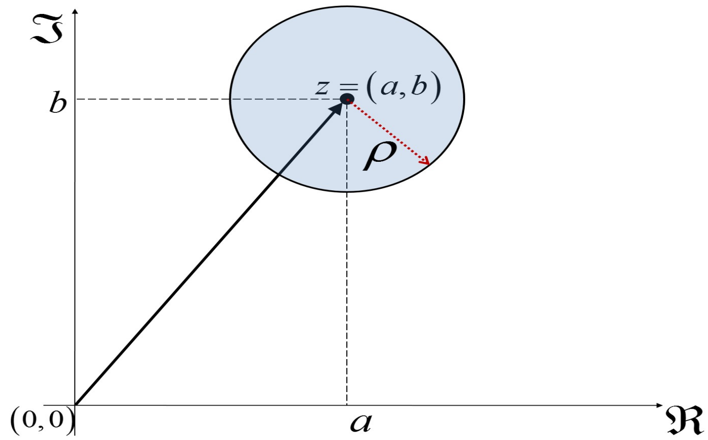

Figure A.10: Circle of radius ρ.

A ring centered at the origin in the complex plane can be defined by two radii ρ_1 and ρ_2, where $\rho_1 < \rho_2$, hence the closed set of points in the ring is defined as

$$E(\rho_1, \rho_2) = \{w \in \mathbb{C} : \ \rho_1 \leq |w| \leq \rho_2\}, \tag{A.62}$$

and the open ring set is defined as

$$E(\rho_1, \rho_2) = \{w \in \mathbb{C} : \ \rho_1 < |w| < \rho_2\}. \tag{A.63}$$

A closed ring centered at a point $z = (a, b)$, with two radii ρ_1 and ρ_2, where $\rho_1 < \rho_2$, is defined by

$$E_z(\rho_1, \rho_2) = \{w \in \mathbb{C} : \ \rho_1 \leq |z - w| \leq \rho_2\}. \tag{A.64}$$

Appendix B. Complex Functions

Recall that a function in general is a mapping from a set B to a set E that assigns to each element of B a unique element in E, i.e., $f : B \longrightarrow E$. The set B is called the domain of the function f, and the set E the range. Our interest is in functions with a complex domain set, i.e., $B \subset \mathbb{C}$, then f is a **complex function**.

Consider a complex number $z = a + jb \in B$ and a complex function such that $w = f(z)$, then since $w = c + jd$, we actually have that the real and imaginary parts of w are functions of a and b, i.e., we have

$$w = f(z) = c(a, b) + jd(a, b). \tag{A.65}$$

The mapping of a complex function can be seen in Figure A.11, where a complex number z in the domain B is mapped to a complex number w in the range E of the complex function f.

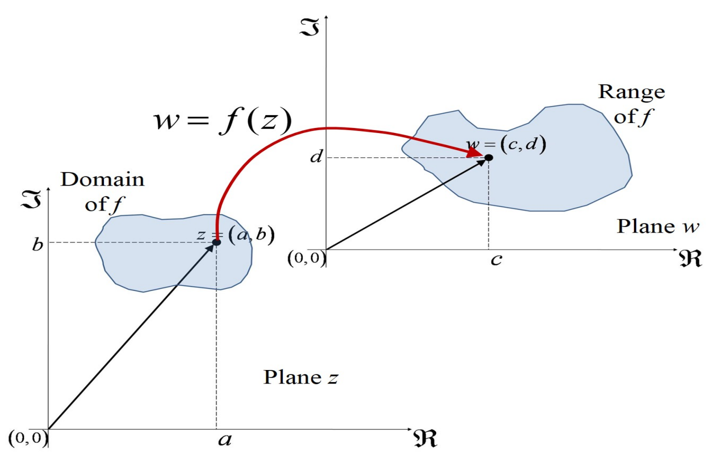

Figure A.11: Mapping of complex function.

Example 17 *Consider a complex number $z = a + jb \in \mathbb{C}$, and let the complex function f be $f(z) = z + \Re\{z\}$, then we can express the function using the standard notation as in the following equation*

$$f(z) = a + jb + a = 2a + jb, \tag{A.66}$$

then, we obtain that

$$c(a, b) = 2a, \quad d(a, b) = b. \tag{A.67}$$

Example 18 *Consider the function $f : B \longrightarrow E$, where the domain is the set $B = \{z = a + jb \in \mathbb{C} : \Re\{z\} = 1$, and the mapping is given by $w = c + jd = f(z) = z^2$. The range of the function is obtained as follows. From Equation (A.38) we can see that*

$$f(z) = \left(a^2 - b^2\right) + j2ab, \tag{A.68}$$

and then we have that

$$c(a, b) = \left(a^2 - b^2\right), \quad d(a, b) = 2ab. \tag{A.69}$$

Since $\Re\{z\} = 1$, we obtain by substituting this in (A.69) that

$$c(1, b) = \left(1 - b^2\right), \quad d(1, b) = 2b. \tag{A.70}$$

From (A.70) we can see that $b = d/2$ and that $c = 1 - d^2/4$. This last expression can be used to plot the result of the mapping as shown in Figure A.12.

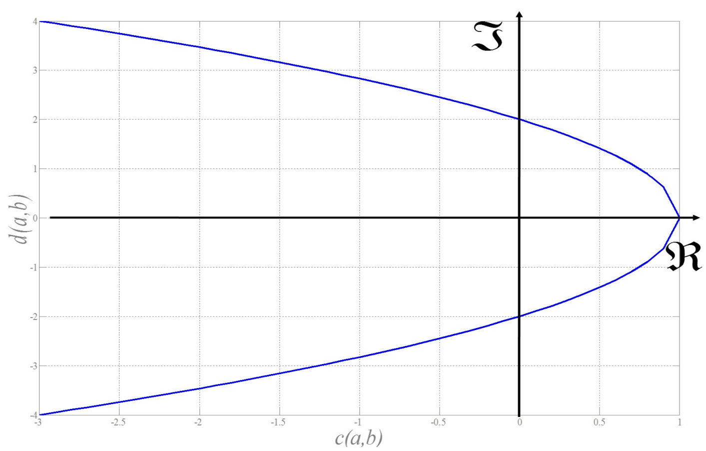

Figure A.12: Mapping of complex function $f = z^2$ with domain $B = \{z = a + jb \in \mathbb{C} : \Re\{z\} = 1$.

Definition 19 *A complex function f is continuous at a point z_0 if*

$$\lim_{z \to z_0} f(z) = f(z_0) \tag{A.71}$$

Consider the complex constant coefficients a_i, $i = 0, 1, \ldots, n$ and n a nonnegative integer, then define the polynomimal of degree n as

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0, \quad a_n \neq 0. \tag{A.72}$$

Definition 20 *A complex function $w = f(z)$ is analytical in a point z_0 if its derivative exists in such point and in a neighborhood of the point. A complex function is analytical in a domain B if it is analytical in all the points of the domain.*

A complex number c is a zero of a polynommial if and only if $z-c$ is a factor of $f(z)$. For example, we can have

$$\begin{aligned} f(z) &= z^4+5z^2+4 \\ &= \left(z^2+1\right)\left(z^2+4\right) \\ &= (z+j)(z-j)(z+j2)(z-j2). \end{aligned} \quad (A.73)$$

From the second expression in (A.73), the roots are obtained and then the last expression gives the factors of the polynomial.

Recall that a complex number $z=a+jb$ is considered in a complex domain $D \subseteq \mathbb{C}$. A complex-valued function is a function of the form

$$f(z) = u(a,b) + jv(a,b), \quad (A.74)$$

where the real part of the function is $\Re\{f(z)\} = u(a,b)$, and the imaginary part $\Im\{f(z)\} = v(a,b)$. The complex conjugate of the function $f(z)$ is $f^*(z) = u(a,b) - jv(a,b)$, and the magnitude or absolute value is given by

$$|f(z)| = \sqrt{u^2(a,b) + v^2(a,b)} = \sqrt{f(z)\, f^*(z)}. \quad (A.75)$$

The values of the complex-valued function $f(z)$ correspond to points on the complex plane, i.e., points with coordinates $(u(a,b), v(a,b))$. The function maps the complex domain D to a range or image $E \subset \mathbb{C}$, where $(u(a,b), v(a,b)) \in E$, this is denoted by $f: D \longrightarrow E$. The domain of the function can be many different things, including other number sets such as the real or the rational numbers for example $D = \mathbb{R} \subset \mathbb{C}$, or $D = \mathbb{N} \subset \mathbb{C}$.

Example 21 *Reduce the function, obtain the standard form representation, and plot the real and imaginary parts as a function of the domain. Let the domain of the complex-valued function f be the real numbers, i.e., $D = \{x : x \in (-\infty, \infty)\}$. There can also be other definitions of the domain, for example, $D = \{z = a + jb : b = 0,\ a \in (-\infty, \infty)\}$. Define the function as $f(x) = (3 + j2x)^2/2$. To solve this, develop the function as follows*

$$\begin{aligned} f(x) &= \frac{1}{2}(3+j2x)^2 \\ &= \frac{1}{2}\left(9+j12x-4x^2\right) \\ &= 4.5-2x^2+j6x. \end{aligned} \quad (A.76)$$

The real part of the function is $u(x) = (4.5 - 2x^2)$, and the imaginary part is $v(x) = 6x$. In Figure A.13, the real and imaginary parts of the function are shown.

At this moment, consider a complex -valued function whose domain are the real numbers, in other words $f: \mathbb{R} \longrightarrow E$, let the domain or independent variable be x, then the following points are satisfied

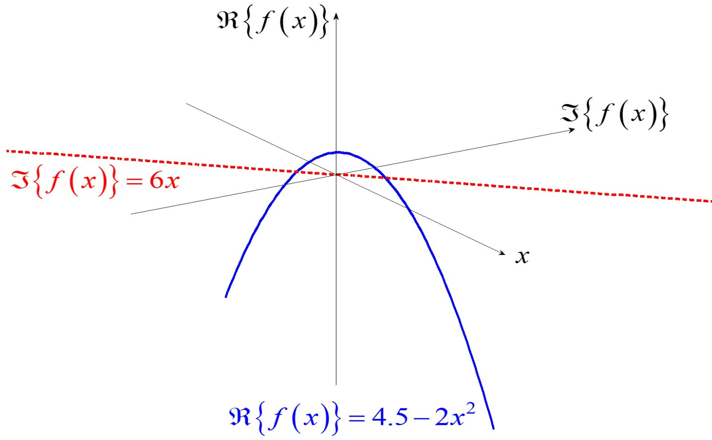

Figure A.13: Plot of the real and imaginary parts of the complex-valued function $f(x) = (3 + j2x)^2/2$.

1. The derivative of the complex-valued function exists on an interval if and only if the derivatives of the real and imaginary parts of the function exist on that interval, in other words, for $x \in [x_0, x_1]$,

$$\frac{df(x)}{dx} = \frac{du(x)}{dx} + j\frac{dv(x)}{dx}. \tag{A.77}$$

2. The complex-valued function will be differentiable if it is continuous. Then, $f(x)$ is continuous at a point x_0 if and only if $u(x)$ and $v(x)$ are continuous at the same point x_0. This is extended for intervals, i.e., the function is continuous on an interval if and only if the real and imaginary parts of the function are continuous on the interval. Continuity can also be replaced for boundedness, smooth, even, periodic, etc. as the characteristic of the function that needs to be verified.

3. The integral of the complex-valued function exists on an interval if and only if the integrals of the real and imaginary parts of the function exist on that interval, in other words, for $x \in [x_0, x_1]$,

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} u(x)\,dx + j\int_{x_0}^{x_1} v(x)\,dx. \tag{A.78}$$

4. The complex conjugate of the integral of the complex-valued function is the integral of

the complex conjugate of the function, in other words

$$\begin{aligned}\left(\int_{x_0}^{x_1} f(x)\,dx\right)^* &= \left(\int_{x_0}^{x_1} u(x)\,dx + j\int_{x_0}^{x_1} v(x)\,dx\right)^* \\ &= \int_{x_0}^{x_1} u(x)\,dx - j\int_{x_0}^{x_1} v(x)\,dx \\ &= \int_{x_0}^{x_1} \left[u(x) - jv(x)\right]dx \\ &= \int_{x_0}^{x_1} f^*(x)\,dx.\end{aligned} \tag{A.79}$$

5. Also, the integrals satisfy

$$\left|\int_{x_0}^{x_1} f(x)\,dx\right| \leq \int_{x_0}^{x_1} |f(x)|\,dx. \tag{A.80}$$

Appendix C. Sequences and Series of Functions

We briefly discuss about convergence of series and sequences of complex functions. Results and facts about these kinds of series and sequences is similar to those of series and sequences of real-valued functions.

Definition 22 *A sequence of complex numbers is a function with domain set the positive integers $n = 1, 2, 3...$, where for each n, a complex number z_n is assigned. We denote such sequence as $\{z_n\}$.*

Example 23 *Consider the sequence $z_n = \frac{1}{n^2} + j^n \frac{1}{2^n}$, $n = 1, 2, ...$ then the sequence takes the values*

$$1 + j\frac{1}{2},\ \ \frac{1}{4} - \frac{1}{4} = 0,\ \ \frac{1}{9} - j\frac{1}{8}, \ldots$$

Definition 24 *The sequence $\{z_n\}$ converges to a number L, if for any small positive number ϵ, there is an integer number N so that for $n > N$, $|z_n - L| < \epsilon$.*

In Figure A.14, we can see how the sequence of the previous example converges to $z_n = 0$ as n grows.

Theorem 25 *The sequence $\{z_n\}$ converges to a complex number L, if and only if,*

$$\Re\{z_n\} \underset{n\to\infty}{\longrightarrow} \Re\{L\}\,,\ \text{and}\ \Im\{z_n\} \underset{n\to\infty}{\longrightarrow} \Im\{L\}\,. \tag{A.81}$$

Example 26 *Consider the sequence $\{z_n\} = \left\{\frac{3+jn}{n+j}\right\}$ and obtain the convergence and the number to which it converges. One way to proceed in this example is to obtain the sequence in standard form and then reduce it algebraically, in other words,*

$$\begin{aligned} z_n &= \frac{3+jn}{n+j} \cdot \frac{n-j}{n-j} \qquad (A.82)\\ &= \frac{4n + j\left(n^2-3\right)}{n^2+1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\ &= \frac{\frac{4}{n}}{1+\frac{1}{n^2}} + j\frac{\left(1-\frac{3}{n^2}\right)}{1+\frac{1}{n^2}}. \end{aligned}$$

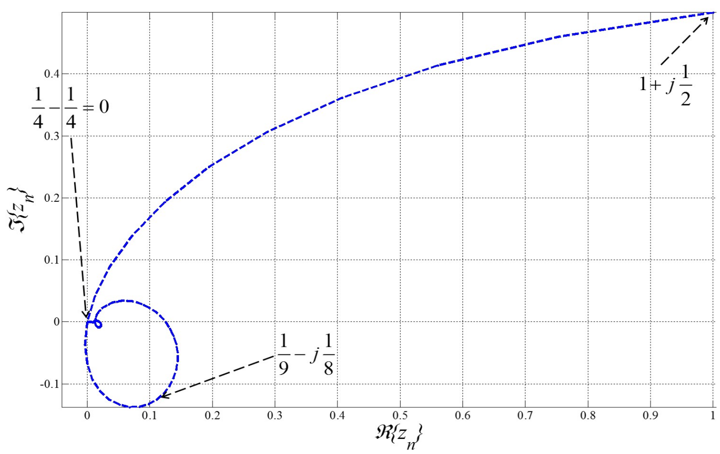

Figure A.14: Convergence of the sequence in graphical form.

From Equation (A.82), the limits can be taken separately to the real and imaginary parts of the result. this gives

$$\lim_{n\to\infty} \Re\{z_n\} = 0, \quad \lim_{n\to\infty} \Im\{z_n\} = j.$$

Hence, the sequence converges to $L = 0 + j$*.*

Definition 27 *An* **infinite series of complex numbers** *takes the form*

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots \tag{A.83}$$

Consider the partial sums defined as follows

$$S_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + z_3 + \cdots + z_{n-1} + z_n, \tag{A.84}$$

and see the sequence of partial sums $\{S_n\}$, then the infinite series in Equation (A.83) converges to a complex number L when

$$\{S_n\} \underset{n\to\infty}{\longrightarrow} L.$$

Example 28 *Geometric complex series. Consider a constant number* r*, and the infinite complex series defined as*

$$\sum_{k=1}^{\infty} rz^{k-1} = r + rz + rz^2 + \cdots \tag{A.85}$$

The n*-th partial sum of the infinite series in (A.85) is given by*

$$S_n = \sum_{k=1}^{n} rz^{k-1} = r + rz + rz^2 + \cdots + rz^{n-1}. \tag{A.86}$$

In order to obtain the convergence of (A.85) and the conditions under which it occurs, consider the following

$$zS_n = \sum_{k=1}^{n} rz^k = rz + rz^2 + \cdots + rz^{n-1} + rz^n. \tag{A.87}$$

Now, subtract (A.87) from (A.86) to obtain

$$S_n - zS_n = r - rz^n. \tag{A.88}$$

Solve for S_n in (A.88) and we get

$$S_n = \frac{r\left(1 - z^n\right)}{1 - z}. \tag{A.89}$$

In (A.89), as n goes to infinity, the term that dominates convergende is z^n, and it is known that $z^n \longrightarrow 0$ as $n \longrightarrow \infty$ when $|z| < 1$, otherwise, i.e., when $|z| \geq 1$, z^n diverges without limit. Therefore, the geometric series in (A.85) converges as $n \longrightarrow \infty$ when $|z| < 1$, and the limit is

$$\sum_{k=1}^{\infty} rz^{k-1} = \frac{r}{1 - z}. \tag{A.90}$$

It is left to the reader to show the following results

$$\sum_{k=1}^{\infty} z^{k-1} = \frac{1}{1 - z}, \tag{A.91}$$

$$\begin{aligned} \sum_{k=1}^{\infty} r\left(-z\right)^{k-1} &= r - rz + rz^2 - rz^3 + \cdots \\ &= \frac{r}{1 + z}. \end{aligned} \tag{A.92}$$

Also, see that the following relations are satisfied

$$\frac{1 - z^n}{1 - z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1}, \tag{A.93}$$

and

$$\frac{1}{1 - z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1} + \frac{z^n}{1 - z} \tag{A.94}$$

Theorem 29 *If the infinite complex series in Equation (A.83) converges, then*

$$\lim_{n \to \infty} z_n = 0. \tag{A.95}$$

The opposite, i.e., $\lim\limits_{n \to \infty} z_n \neq 0$, will make the infinite complex series diverge.

Definition 30 ***Absolute Convergence***. *An infinite complex series converges absolutely if* $\sum_{k=1}^{\infty} |z_k|$ *converges as well.*

Theorem 31 ***Ratio Test.*** *Given an infinite complex series* $\sum_{k=1}^{\infty} z_k$

1. *it converges if*
$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| < 1, \tag{A.96}$$

2. *it diverges if* $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| > 1$,*for* $n \geq n_0$ *for some fixed integer* n_0.

3. *If* $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = 1$, *the test is inconclusive.*

Theorem 32 ***Root Test***. *Given a series* $\sum_{k=1}^{\infty} z_k$, *define* $\alpha = \lim_{n\to\infty} \sqrt[n]{|z_n|}$, *then*

1. *if* $\alpha < 1$, $\sum_{k=1}^{\infty} z_k$ *converges absolutely;*

2. *if* $\alpha > 1$, $\sum_{k=1}^{\infty} z_k$ *diverges;*

3. *if* $\alpha = 1$, *there is no conclusion as if the series converges or diverges.*

Appendix D. Power Series

Definition 33 Power Series. *Consider a sequence $\{c_n\}$ of complex numbers that are the* **coefficients** *of the series, then the power series is defined as*

$$\sum_{n=0}^{\infty} c_n z^n. \tag{A.97}$$

Depending on the choice of z in (A.97), the series will converge or diverge. With every power series, there is a *circle of convergence* such that Equation (A.97) converges if z is inside that circle, and the series diverges otherwise. The following two theorems are tests for convergence of series. These theorems will be useful to determine convergence or divergence of series.

Theorem 34 Root Test. *Given a series $\sum_n a_n$, define $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, then*

1. *if $\alpha < 1$, $\sum_n a_n$ converges;*
2. *if $\alpha > 1$, $\sum_n a_n$ diverges;*
3. *if $\alpha = 1$, there is no conclusion as if the series converges or diverges.*

Theorem 35 Ratio Test. *Given a series $\sum_n a_n$,*

1. *it converges if*
$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1, \tag{A.98}$$
2. *it diverges if*
$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \geq 1, \tag{A.99}$$

for $n \geq n_0$ for some fixed integer n_0.

Appendix E. Introduction to Matlab

Matlab is a software that allows us to quickly and efficiently work with any set of vectors and matrices. All expressions in Matlab are considered to be working on matrices including scalar elements which are considered matrices of dimensions 1x1.

Matlab is formed by several elements, one of them is the command prompt which can be seen in Figure A.15 that shows the Matlab main window.

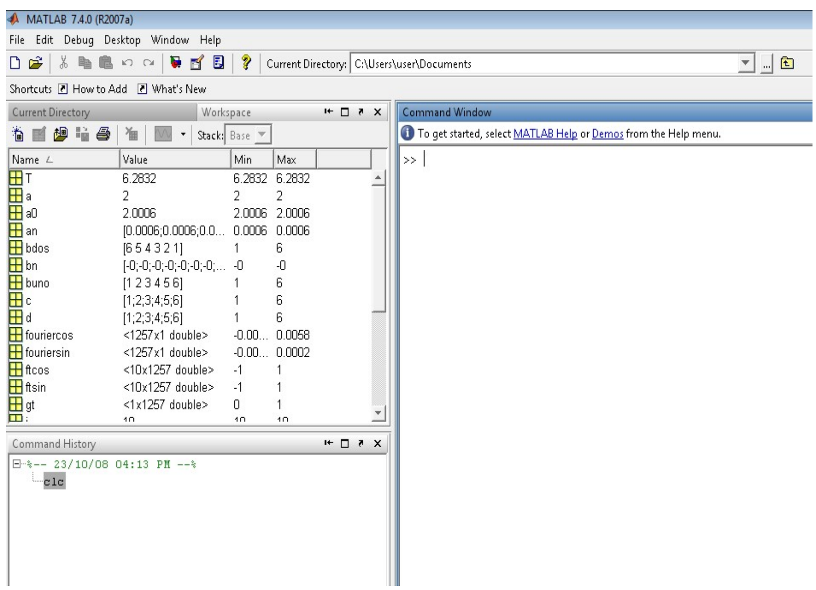

Figure A.15: Main environment of Matlab software. You can observe on the right hand side the command prompt.

To create a **scalar**, it is sufficient in the command line the assignment, for example:

>> c=6

When you write the line and click on Enter, you will see on the screen that the variable and its value will be shown. If you do not want Matlab to show on the screen a variable and its value every time you click on Enter, add a semicolon at the end, i.e.,

>> c=6;

To create a **row vector** you only need to write the elements of the vector separated by spaces or commas and all elements written within brackets as follows

>> C=[1 2 3 4 5];

or

>> C=[1,2,3,4,5];

This creates a row vector C of dimensions 1x5. If you want a **column vector**, then separate elements of your vector by semicolons as follows

>> D=[1;2;3;4;5];

This creates a column vector of dimensions 5x1. As you know, the transpose of a row vector is a column vector and vice versa, then we have that the transpose of row vector C is column vector D. Another way to write column vector D once you already defined row vector C would be with the use of transpose as follows

>> D=C';

A matrix can be created by writing each row as a row vector and rows separated by semicolons as follows

>> M= [1 2 3 4 5 6 7 8; 1 2 3 4 5 6 7 8; 1 2 3 4 5 6 7 8];

Matrix M has dimensions 3x8 (3 rows and 8 columns). If we transpose matrix M, we will get a matrix of dimensions 8x3. If you do not know the dimensions of a matrix, then you can use the command $size$ as follows

>> [m1,m2]=size(M)

This command will assign the value 3 to the variable m1 (for 3 rows) and 8 to the variable m2 (for 8 columns). The size of a vector is obtained by the following command

>> L=length(C)

This command assigns the value 5 to the variable L.

Matlab allows you to execute basic arithmetic operations with scalars, vectors and matrices. You only need to consider that each of the elements in the command has the same dimensions, otherwise an error will be produced. For example, two matrices, say A and B of the same dimensions, say mxn can be added as follows

```
>> X=A+B;
```

You can multiply matrices, also considering that the dimensions are in agreement, i.e., the number of columns of the first matrix are the same as the number of rows of the second matrix. For example for two matrices, A of dimensions mxn and B of dimensions nxq, we can use the command

```
>> Y=A*B;
```

You can multiply matrices by any scalar as follows:

```
>> Z=c*A;
```

In some cases, you need to multiply element by element, i.e., consider matrix A with elements a(1,1), a(1,2),..., a(m,n). And consider matrix B with elements b(1,1), b(1,2), ..., b(m,n). Then we can obtain a matrix W with elements w(1,1), w(1,2),...,w(m,n); where w(i,j)=a(i,j)*b(i,j) as follows

```
>> W=A.*B;
```

Note that both matrices must have the same dimensions mxn. Also, all the operations as subtraction, division can be used in the same way as it has been shown for the multiplication.

You can use powers as well on scalars or on each element of matrices, for example the command

```
>> Q=A.^3;
```

will obtain matrix Q where element q(i,j)=a(i,j)^3, i.e., the third power of a(i,j). In conclusion, the use of a dot before an operation, indicates that the operation will be performed on an element-wise base.

From a matrix, you can extract elements or rows and columns as follows:

- To extract the third column of matrix A you use the command

  ```
  >> S=A(:,3);
  ```

- To extract the fifth row of matrix A you use the command

  ```
  >> S=A(5,:);
  ```

- To extract the element (3,6) of matrix A you use the command

  $>>$ S=A(3,6);

- To extract the third, fourth and fifth columns together with the first, second and third rows of matrix A you use the command

  $>>$ S=A(1:3,3:5);

You can also generate functions, for example with the independent variable being time t. Assume you want to generate the function $f(t) = 3.2\cos(2\pi f_0 t)$ with a fundamental frequency of $f_0 = 2$ Hz, and for times between -1.3 seconds up to 7.2 seconds; then you write the following commands

```
>> t=(-1.3:0.01:7.2)';
>> f0=2;
>> f=3.2*cos(2*pi*f0*t);
```

The first command generates a column vector that contains the times from -1.3 up to 7.2 seconds in increments of 0.01 seconds. The second command defines the fundamental frequency of 10 Hz. The third command gives a column vector with each j-th element being the value of the function $f(t)$ evaluated at time $t(j)$.

You can also plot this function, see Figure A.16 so that you can see how it looks like by writing the commands

```
>> plot(t,f)
>> xlabel('Time t (secs)')
>> ylabel('Amplitude (volts)')
>> title('Cosine function')
>> grid
>> axis([-1.3 7.2 -3.5 3.5])
```

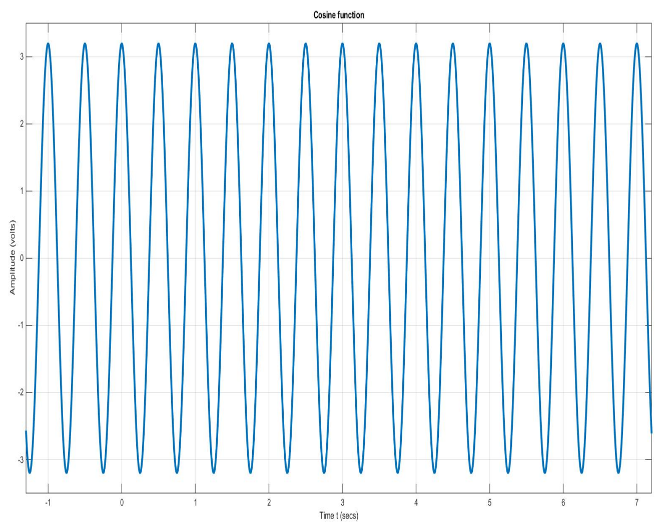

Figure A.16: Cosine function.

Appendix F. Differential Equations

In this Appendix, we will elaborate a procedure for the solution of different forms of First Order Differential Equations (FODE), usually found in simple passive circuits.

In general, and within the scope of this text, we will consider a system that has an input denoted by $x(t)$ and an output by $y(t)$. The system will be described by FODE in this appendix. The FODE can be written in such a form that the coefficient of the derivative of $y(t)$ is one, with all the terms that involve $y(t)$ are on the left hand side and all the terms with $x(t)$ are on the right hand side.

A.1 First Order Differential Equation in Standard Form

Any First Order Differential Equation in Standard Form (FODESF) that relates the signals $x(t)$ and $y(t)$ has the following form

$$\frac{dy(t)}{dt} + A(t)y(t) = Bx(t), \tag{A.100}$$

note that on the right hand side of the equation, we have the input $x(t)$ multiplied by a constant and no derivative of $x(t)$ exists. This is what defines the standard form. The second term on the left hand side can be a function of time, but it certainly cannot be a function of either $y(t)$ or $x(t)$.

In order to solve the FODESF given in Equation (A.100), we follow the procedure described in the next steps.

1. **Step 1**: In this step, we take the function or constant that multiplies $y(t)$ in Equation (A.100) and solve the integral of $A(t)$ and place it in the following exponent of the exponential function

$$e^{\int A(t)dt} \tag{A.101}$$

2. **Step 2**: In step two, we take the term that was obtained in Equation (A.101) and multiply

the whole Equation (A.100) by it on both sides of the equality as follows

$$
\begin{aligned}
e^{\int A(t)dt}\left\{\frac{dy(t)}{dt}+A(t)y(t)\right\} &= \{Bx(t)\}\, e^{\int A(t)dt}, \\
e^{\int A(t)dt}\frac{dy(t)}{dt}+A(t)e^{\int A(t)dt}y(t) &= Bx(t)e^{\int A(t)dt}, \\
e^{\int A(t)dt}\frac{dy(t)}{dt}+y(t)A(t)e^{\int A(t)dt} &= Bx(t)e^{\int A(t)dt}, \qquad \text{(A.102)}
\end{aligned}
$$

The last row of Equation (A.102) only rearranges the terms.

3. **Step 3**: In this step, first one should recall the derivative of the product of two functions, i.e., $\frac{d}{dt}f(t)g(t) = f(t)\frac{dg(t)}{dt} + g(t)\frac{df(t)}{dt}$. With this in mind, one can see that the last row of Equation (A.102) on its left hand side contains the derivative of a product of two functions, the first being $e^{\int A(t)dt}$ and the second one being $y(t)$.

   In order to see this, one should recall that

   $$\frac{de^{\int A(t)dt}}{dt} = A(t)e^{\int A(t)dt}. \qquad \text{(A.103)}$$

   Thus, we can write the last row of Equation (A.102) as follows

   $$\frac{d}{dt}\left\{e^{\int A(t)dt}y(t)\right\} = Bx(t)e^{\int A(t)dt}. \qquad \text{(A.104)}$$

4. **Step 4**: In this step, we solve Equation (A.104) for $y(t)$ by integrating both sides of the equation and getting

   $$e^{\int A(t)dt}y(t) = B\int_{-\infty}^{t} x(\tau)e^{\int A(\tau)d\tau}d\tau. \qquad \text{(A.105)}$$

   Note the change of the variable of integration on the right hand side to τ and the limits of integration indicating the accumulation of time up to t.

5. **Step 5**: In this step present the solution of the differential equation as follows

   $$y(t) = Be^{-\int A(t)dt}\int_{-\infty}^{t} x(\tau)e^{\int A(\tau)d\tau}d\tau. \qquad \text{(A.106)}$$

A.2 FODE Not in Standard Form

A FODE that relates the signals $x(t)$ and $y(t)$ and that it does not have the standard form can be one of the following equations

$$\frac{dy(t)}{dt} + A(t)y(t) = C\frac{dx(t)}{dt}, \qquad \text{(A.107)}$$

$$\frac{dy(t)}{dt} + A(t)y(t) = C\frac{dx(t)}{dt} + Bx(t). \tag{A.108}$$

Note that on the right hand side of the equation, we have the input $x(t)$ in a derivative. This is what determines that it is not in the standard form.

In the following subsections, the FODE of each of these forms in equations (A.107) and (A.108) will be solved.

A.2.1 FODE with only $\frac{dx(t)}{dt}$

In order to solve the FODE given in Equation (A.107), we follow the procedure described for the FODESF case and modify Step 3. We rewrite the differential equation here

$$\frac{dy(t)}{dt} + A(t)y(t) = C\frac{dx(t)}{dt}, \tag{A.109}$$

1. **Step 1**: In this step, we take the function or constant that multiplies $y(t)$ in Equation (A.109) and solve the integral of $A(t)$ and place it in the following exponent of the exponential function

$$e^{\int A(t)dt} \tag{A.110}$$

2. **Step 2**: In step two, we take the term that was obtained in Equation (A.110) and multiply the whole Equation (A.109) by it on both sides of the equality as follows

$$\begin{aligned} e^{\int A(t)dt}\left\{\frac{dy(t)}{dt} + A(t)y(t)\right\} &= \left\{C\frac{dx(t)}{dt}\right\}e^{\int A(t)dt}, \\ e^{\int A(t)dt}\frac{dy(t)}{dt} + A(t)e^{\int A(t)dt}y(t) &= C\frac{dx(t)}{dt}e^{\int A(t)dt}, \\ e^{\int A(t)dt}\frac{dy(t)}{dt} + y(t)A(t)e^{\int A(t)dt} &= Ce^{\int A(t)dt}\frac{dx(t)}{dt}, \end{aligned} \tag{A.111}$$

The last row of Equation (A.102) only rearranges the terms.

3. **Step 3**: In this step, first one should recall the derivative of the product of two functions, i.e., $\frac{d}{dt}f(t)g(t) = f(t)\frac{dg(t)}{dt} + g(t)\frac{df(t)}{dt}$. With this in mind, one can see that the last row of Equation (A.111) on its left hand side contains the derivative of a product of two functions, the first being $e^{\int A(t)dt}$ and the second one being $y(t)$.

In order to see this, one should recall that

$$\frac{de^{\int A(t)dt}}{dt} = A(t)e^{\int A(t)dt}. \tag{A.112}$$

Thus, we can write the last row of Equation (A.111) as follows

$$\frac{d}{dt}\left\{e^{\int A(t)dt}y(t)\right\} = Ce^{\int A(t)dt}\frac{dx(t)}{dt}. \tag{A.113}$$

Here is where the modification comes. On the right hand side of Equation (A.113), we have to complete the expression so that we get also an expression for the derivative of the product of two functions. In this case, one of the functions is $x(t)$, and the other function needs to be $e^{\int A(t)dt}$. Hence, we add and subtract the same term in order to obtain

$$\frac{d}{dt}\left\{e^{\int A(t)dt}y(t)\right\} = C\left\{e^{\int A(t)dt}\frac{dx(t)}{dt} + x(t)A(t)e^{\int A(t)dt} - x(t)A(t)e^{\int A(t)dt}\right\}. \tag{A.114}$$

Now, we can manipulate (A.114) to get the following

$$\frac{d}{dt}\left\{e^{\int A(t)dt}y(t)\right\} = C\frac{d}{dt}\left\{e^{\int A(t)dt}x(t)\right\} - Cx(t)A(t)e^{\int A(t)dt}. \tag{A.115}$$

4. **Step 4**: In this step, we solve Equation (A.115) for $y(t)$ by integrating both sides of the equation and getting

$$e^{\int A(t)dt}y(t) = Ce^{\int A(t)dt}x(t) - C\int_{-\infty}^{t} A(\tau)x(\tau)e^{\int A(\tau)d\tau}d\tau. \tag{A.116}$$

Note the change of the variable of integration on the right hand side to τ and the limits of integration indicating the accumulation of time up to t.

5. **Step 5**: In this step present the solution of the differential equation as follows

$$y(t) = Cx(t) - Ce^{-\int A(t)dt}\int_{-\infty}^{t} A(\tau)x(\tau)e^{\int A(\tau)d\tau}d\tau. \tag{A.117}$$

A.2.2 FODE with Terms including $\frac{dx(t)}{dt}$ and $x(t)$

In order to solve the FODE given in Equation (A.108), we follow the procedure described for the FODESF case and modify Step 3. We rewrite the differential equation here

$$\frac{dy(t)}{dt} + A(t)y(t) = C\frac{dx(t)}{dt} + Bx(t), \tag{A.118}$$

1. **Step 1**: In this step, we take the function or constant that multiplies $y(t)$ in Equation (A.118) and solve the integral of $A(t)$ and place it in the following exponent of the exponential function

$$e^{\int A(t)dt} \tag{A.119}$$

2. **Step 2**: In step two, we take the term that was obtained in Equation (A.119) and multiply the whole Equation (A.118) by it on both sides of the equality as follows

$$\begin{aligned} e^{\int A(t)dt}\left\{\frac{dy(t)}{dt}+A(t)y(t)\right\} &= \left\{C\frac{dx(t)}{dt}+Bx(t)\right\}e^{\int A(t)dt}, \\ e^{\int A(t)dt}\frac{dy(t)}{dt}+A(t)e^{\int A(t)dt}y(t) &= C\frac{dx(t)}{dt}e^{\int A(t)dt}+Bx(t)e^{\int A(t)dt}, \\ e^{\int A(t)dt}\frac{dy(t)}{dt}+y(t)A(t)e^{\int A(t)dt} &= Ce^{\int A(t)dt}\frac{dx(t)}{dt}+Bx(t)e^{\int A(t)dt}, \quad \text{(A.120)} \end{aligned}$$

The last row of Equation (A.120) only rearranges the terms.

3. **Step 3**: In this step, first one should recall the derivative of the product of two functions, i.e., $\frac{d}{dt}f(t)g(t) = f(t)\frac{dg(t)}{dt} + g(t)\frac{df(t)}{dt}$. With this in mind, one can see that the last row of Equation (A.120) on its left hand side contains the derivative of a product of two functions, the first being $e^{\int A(t)dt}$ and the second one being $y(t)$.

In order to see this, one should recall that

$$\frac{de^{\int A(t)dt}}{dt} = A(t)e^{\int A(t)dt}. \quad \text{(A.121)}$$

Thus, we can write the last row of Equation (A.120) as follows

$$\frac{d}{dt}\left\{e^{\int A(t)dt}y(t)\right\} = Ce^{\int A(t)dt}\frac{dx(t)}{dt} + Bx(t)e^{\int A(t)dt}. \quad \text{(A.122)}$$

Here is where the modification comes. On the right hand side of Equation (A.122), we have to complete the expression so that we get also an expression for the derivative of the product of two functions. In this case, one of the functions is $x(t)$, and the other function needs to be $e^{\int A(t)dt}$. Hence, we add and subtract the same term in order to obtain

$$\begin{aligned} \frac{d}{dt}\left\{e^{\int A(t)dt}y(t)\right\} &= C\left\{e^{\int A(t)dt}\frac{dx(t)}{dt}+x(t)A(t)e^{\int A(t)dt}-x(t)A(t)e^{\int A(t)dt}\right\} \\ &\quad +Bx(t)e^{\int A(t)dt}. \quad \text{(A.123)} \end{aligned}$$

Now, we can manipulate (A.123) to get the following

$$\begin{aligned} \frac{d}{dt}\left\{e^{\int A(t)dt}y(t)\right\} &= C\frac{d}{dt}\left\{e^{\int A(t)dt}x(t)\right\} - Cx(t)A(t)e^{\int A(t)dt} + Bx(t)e^{\int A(t)dt} \\ &= C\frac{d}{dt}\left\{e^{\int A(t)dt}x(t)\right\} - \left[CA(t)-B\right]x(t)e^{\int A(t)dt}. \quad \text{(A.124)} \end{aligned}$$

4. **Step 4**: In this step, we solve Equation (A.124) for $y(t)$ by integrating both sides of the equation and getting

$$e^{\int A(t)dt}y(t) = Ce^{\int A(t)dt}x(t) - \int_{-\infty}^{t} [CA(\tau) - B]\, x(\tau)e^{\int A(\tau)d\tau} d\tau. \tag{A.125}$$

   Note the change of the variable of integration on the right hand side to τ and the limits of integration indicating the accumulation of time up to t.

5. **Step 5**: In this step, we present the solution of the differential equation as follows

$$y(t) = Cx(t) - e^{-\int A(t)dt} \int_{-\infty}^{t} [CA(\tau) - B]\, x(\tau)e^{\int A(\tau)d\tau} d\tau. \tag{A.126}$$

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