# CH 3 MODELING & APPLICATIONS

(LATER CALL ATTENTION TO ROY CLOUSER'S PAPER IN JUNE 2003 PRO-REGE
READ FROM 1ST PARAGRAPH ON P 7 ONWARD TO THE END — FROM "PERHAPS THIS...")
RE-READ PAGE 1, PAGE 2 UP TO "MECHANICS" P 104
HOMEWORK QUESTION: SHOULD DIFF-EQ. BE TAUGHT DIFFERENTLY AT A CHRISTIAN COLLEGE AS COMPARED TO A STATE UNIVERSITY. (YES OR NO AND DEFEND YOUR CHOICE.)

1. NO, 1+1=2 EVERYWHERE "2. GOD CREATED IT" 3. MATH POINTS TO AND EXPLAINS PART OF GOD'S NATURE. A PROPER STUDY OF MATH MAKES GOD MORE REAL TO US

## 3.1 POPULATION GROWTH

MALTHUSIAN MODEL

ASSUMPTION — UNLIMITED RESOURCES, NO POLLUTION

$P(t)$ = POPULATION, A FUNCTION OF TIME

THERE IS A CERTAIN BIRTH RATE $b$ SUCH THAT IN A TIME $\Delta t$ THERE WILL BE $b\,P(t)\,\Delta t$ BIRTHS (OR CELL DIVISIONS)

DEATH RATE $d$ SUCH THAT $d\,P(t)\,\Delta t$ (OR $d\,P(t)\,dt$)

THEN $$P'(t) = (b-d)\,P(t)$$

REPRODUCTIVE RATE $r = b-d$

$$P'(t) = r\,P(t)$$ ← CALLED THE MALTHUSIAN MODEL

SEPERABLE
$$\frac{1}{P}\,dP = r\,dt$$
$$\ln|P| = rt + A$$
$$P = Ce^{rt} \qquad C = e^{A}$$

OR AN INITIAL VALUE PROBLEM, $P(t) = P_0 e^{rt}$ WHERE $P(t_0) = P_0$

EVALUATING $r$ AND $P_0$ TO FIT OBSERVED DATA:

Ex 8

eg

| DAY | # OF FLIES OBSERVED | # FLIES PREDICTED ← 1ST DAY | # FLIES PREDICTED ← 5TH DAY |
|---|---|---|---|
| 0 | 10 | 10 | 10 |
| 1 | 14 | 14 | 13 |
| 2 | 19 | 20 | 17 |
| 3 | 24 | 27 | 22 |
| 4 | 28 | 38 | 29 |
| 5 | 39 | 54 | 38 |
| 6 | 55 | 75 | 50 |
| 7 | 72 | 105 | 65 |
| 8 | 85 | 148 | 85 |
| 9 | 123 | 207 | 111 |
| 10 | 136 | 289 | 144 |

$14 = 10e^{r}$

$\ln 1.4 = r = 0.33647$

$38 = 10e^{5r}$

$\ln 3.8 = 5r \quad r = 0.267$

A MORE SOPHISTICATED METHOD IS LINEAR REGRESSION

LOGISTIC MODEL

THE ASSUMPTION OF UNLIMITED RESOURCES MIGHT NOT BE REALISTIC

SUPPOSE THE DEATH RATE INCREASES WITH THE POPULATION DUE TO LACK OF RESOURCES —

INCREASES W/ P DECLINES w/ P

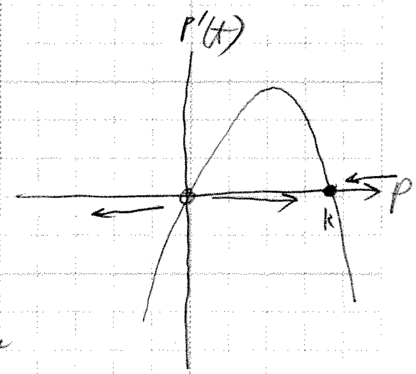
DEATH RATE : $d + aP$ BIRTH RATE $b - cP$

LET $r_0 = b - d$ AND $\frac{r_0}{K} = c + a$

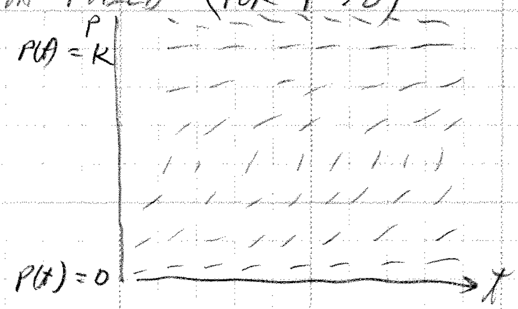
NOW $P'(t) = r_0\left(1 - \frac{P(t)}{K}\right)P(t)$

NOTE, THIS IS AUTONOMOUS AND SEPERABLE

GRAPH OF RIGHT-HAND-SIDE




CONSIDER THE DIRECTION FIELD (FOR $P > 0$)




EQUILIBRIUM SOLUTIONS AT

$P(t) = 0$ (UNSTABLE)

$P(t) = K$ (STABLE)

SOLUTION — THE DE IS AUTONOMOUS ∴ SEPERABLE

$$\frac{P'(t)}{P(t)} = r_0\left(1 - \frac{P(t)}{K}\right) = r_0\Big($$

$$\frac{KP'(t)}{P(t)} = r_0\left(K - P(t)\right)$$

$$\int \frac{K}{P(t)(K - P(t))}\,dP = \int r_0\,dt$$

NOW DO PARTIAL FRACTION EXPANSION

$$\frac{K}{P(K-P)} = \frac{a}{P} + \frac{b}{K-P} = \frac{a(K-P)}{P(K-P)} + \frac{bP}{P(K-P)} \quad \therefore a = 1,\ b = 1$$

$$\int \frac{1}{P(t)}\,dP + \int \frac{1}{K - P(t)}\,dP = \int r_0\,dt$$

$$\ln|P(t)| - \ln|P(t) - K| = r_0 t + C$$

$$\ln\left|\frac{P(t)}{K - P(t)}\right| = r_0 t + C$$

$$** \quad \frac{P(t)}{K - P(t)} = Ae^{r_0 t} \qquad A = e^C$$

$A > 0$, $K \geq P(t)$ DROP ABS VALUE

NOW SOLVE FOR $P(t)$

$$P(t) = [K - P(t)]Ae^{r_0 t}$$

$$P(t)\left[1 + Ae^{r_0 t}\right] = KAe^{r_0 t}$$

$$P(t) = \frac{KAe^{r_0 t}}{1 + Ae^{r_0 t}}$$

APPLY INITIAL CONDITIONS, AT $t_0$ POPULATION IS $P_0$ OR $P(t_0) = P_0$

THEN FROM ** $\frac{P_0}{K-P_0} = Ae^{r_0 t_0} \Rightarrow A = \frac{P_0 e^{-r_0 t_0}}{K-P_0}$ ***

SUBST BACK INTO THE SOLUTION

$$P(t) = \frac{K\left(\frac{P_0 e^{-r_0 t_0}}{K-P_0}\right)e^{r_0 t}}{1 + \left(\frac{P_0 e^{-r_0 t_0}}{K-P_0}\right)e^{r_0 t}} = \frac{K P_0 e^{r_0(t-t_0)}}{(K-P_0) + P_0 e^{r_0(t-t_0)}}$$

$$\boxed{P(t) = \frac{K P_0}{(K-P_0)e^{-r_0(t-t_0)} + P_0}}$$

NOTE THAT AS $t \to \infty$ $e^{-r_0(t-t_0)} \to 0$ $P(t) \to K$

## DERIVING $K$, $P_0$, $r_0$ FROM OBSERVED DATA

IF $P_0$, $K$ ARE KNOWN AND $t_0 = 0$ USE *** TO FIND $A$

$$A = \frac{P_0}{K-P_0}$$

THEN FROM ** $\frac{P(t)}{K-P(t)} = \frac{P_0}{K-P_0}e^{r_0 t}$

NOW, GIVEN ONE DATA POINT, $P(t)$ AT $t$, FIND $r_0$

$$e^{r_0 t} = \frac{\left(\frac{P(t)}{K-P(t)}\right)}{\left(\frac{P_0}{K-P_0}\right)} \Rightarrow r_0 = \frac{1}{t}\ln\left(\frac{\left(\frac{P(t)}{K-P(t)}\right)}{\left(\frac{P_0}{K-P_0}\right)}\right)$$

eg GIVEN $P_0 = 1$ AT $t_0 = 0$, $K = 100$ AND $P(1) = 2$

$$r_0 = \frac{1}{1}\ln\frac{\left(\frac{2}{100-2}\right)}{\left(\frac{1}{100-1}\right)} \cong 0.7033$$

$$\therefore P(t) = \frac{100}{99e^{-0.7033t} + 1}$$

## 3.3 TIME VALUE OF MONEY (PERSONAL FINANCE)

CONSIDER BORROWING $P$ DOLLARS AT A SIMPLE INTEREST RATE OF $i$ FOR $n$ PERIODS

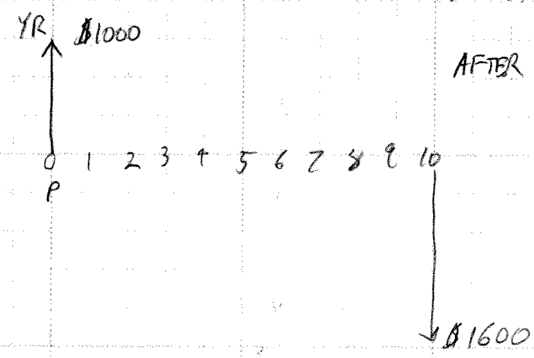
$P$ = THE PRESENT VALUE (THE AMOUNT BORROWED IN THIS CASE)

SIMPLE INTEREST { INTEREST IS PAYED ONLY ON INITIAL PRINCIPAL, THEREFORE $I = Pi$ WHERE $I$ IS THE TOTAL AMOUNT OF INTEREST PAYED IN A STATED PERIOD (NO INTEREST ON INTEREST)

$n$ = PERIOD OF THE LOAN IN SO MANY UNITS – eg YEARS, MONTHS etc

$S$ = FUTURE VALUE A

e.g. BORROW \$1000 FOR 10 YEARS AT 6% PER YEAR SIMPLE INTEREST, WITH ONE BALLOON PAYMENT AT THE END




AFTER 10 YRS, MAKE ONE PAYMENT

$S = P + nIP = 1000 + 10(0.06)1000 = 1600$

ANOTHER WAY TO LOOK AT THIS – FROM THE POINT OF VIEW OF THE BANK: \$1000 INVESTED TODAY HAS A FUTURE VALUE OF $S = P + nIP$

i.e. SAVINGS ACCOUNTS AND LOANS ARE THE SAME – JUST A DIFFERENT PERSPECTIVE.

COMPOUND INTEREST INTEREST IS ADDED TO PRINCIPLE PERIODICALLY

FIRST PERIOD $S_1 = P + Pi = P(1+i)$ NEXT PD USE $S_1$ AS PRINCIPLE

2nd PERIOD $S_2 = S_1 + S_1 i = P(1+i) + P(1+i)i = P(1+i)(1+i) = P(1+i)^2$

3rd PERIOD $S_3 = S_2 + S_2 i = P(1+i)^3$

⋮

nth PERIOD $S_n = P(1+i)^n$ $(1+i)^n$ IS THE "SINGLE PAYMENT COMPOUND AMOUNT FACTOR"

EXAMPLE: BORROW \$1000 AT 6% NOMINAL INTEREST COMPOUNDED ANNUALLY FOR 10 YRS WITH ONE BALLOON PAYMENT AT THE END $(P = S_0)$

| n | S |
|---|---|
| 0 | 1000 |
| 1 | 1060 |
| 2 | 1124 |
| 3 | 1191 |
| 4 | 1262 |
| 5 | 1338 |
| 6 | 1419 |
| 7 | 1504 |
| 8 | 1594 |
| 9 | 1689 |
| 10 | 1791 |

NOTE $1791 = 1000\,(1.06)^{10}$

SAME AS ABOVE EXCEPT COMPOUNDED MONTHLY $r = 6\%/YR$ NOTE $i = \frac{r}{\text{NOMINAL RATE PD.}}$

NOTE: $r$ = NOMINAL INTEREST RATE, $i = \frac{r}{p}$

$p$ = # PERIODS OF COMPOUNDING PERIODS IN THE INTERVAL OF THE NOMINAL RATE

$i = \frac{r}{12\ \text{month/YR}} = \frac{1}{2}\%/\text{MONTH}$

AFTER 10 yrs = 120 months

$$S_{120} = 1000\,(1.005)^{120} = 1819$$

SAME AS ABOVE EXCEPT COMPOUNDED DAILY $r = 6\%\ YR$ $i = \frac{r}{365} = 0.0164\%/\text{DAY}$

$$S_{3650} = 1000\,(1.000164)^{3650} = 1822.03$$

SAME AS ABOVE EXCEPT COMPOUNDED CONTINUOUSLY

LET $p$ = NUMBER OF COMPOUNDING PERIODS/ YEAR, $i = \frac{r}{p}$

$$S = P \lim_{p \to \infty} \left(1 + \frac{r}{p}\right)^{pn}$$

RECALL FROM CALCULUS $\lim_{x \to 0} (1+x)^{1/x} = e$

LET $x = \frac{r}{p}$ THEN $pn = \left(\frac{r}{x}\right)n = \frac{1}{x} rn$

$$S = P \lim_{x \to 0} (1+x)^{\frac{rn}{x}} = Pe^{rn}$$

$$S = Pe^{rn}$$

EXAMPLE – SAME AS ABOVE EXCEPT COMPOUND CONTINUOUSLY

$$S = 1000e^{(0.06)10}$$

$$S = 1822.12$$

A D.E. MODELING APPROACH TO CONTINUOUS COMPOUNDING

LET $P(t)$ BE THE LOAN BALANCE ($P = P(0)$ $S = P(\text{FINAL PERIOD})$)

THE D.E. MODEL OF THIS LOAN IS

$$\boxed{P'(t) = rP}$$

UNITS \$/PD UNITS $PD^{-1}$ UNITS \$

$$\boxed{P(t) = P_0 e^{rt}}$$

SOLUTION: $\int \frac{1}{P}\,dP = \int r\,dt$

$\ln P = rt + C$

$P(t) = Ae^{rt}$

OBVIOUSLY $A = P_0 = P(0)$

SAME RESULT AS PREVIOUSLY w/ MUCH LESS FUSS

PERIODIC PAYMENTS INTO A SAVINGS ACCOUNT

$R$ = PERIODIC PAYMENT

$S_n$ = VALUE (BALANCE) AT END OF PERIOD $n$

$r$ = NOMINAL INTEREST RATE

$i = \frac{r}{p}$, $p$ = # COMPOUNDING PERIODS IN $r$

$n$ = NUMBER OF PERIODS OF COMPOUNDING

AT END OF 1ST PD. $S_1 = R$

2nd PD $S_2 = S_1 + S_1 i + R = R + Ri + R = R(1+i) + R$

3rd PD $S_3 = S_2(1+i) + R = R(1+i)^2 + R(1+i) + R$

$\vdots$

nTH PD $S_n = R\left[(1+i)^{n-1} + (1+i)^{n-2} + \cdots + 1\right]$

MULT. BY $(1+i)$ $(1+i)S_n = R\left[(1+i)^n + (1+i)^{n-1} + (1+i)^{n-2} + \ldots (1+i)\right]$

SUBTRACT $(1+i)S_n - S_n = R\left[(1+i)^n - 1\right]$

$$iS_n = R\left[(1+i)^n - 1\right]$$

$$S_n = \frac{R\left[(1+i)^n - 1\right]}{i}$$

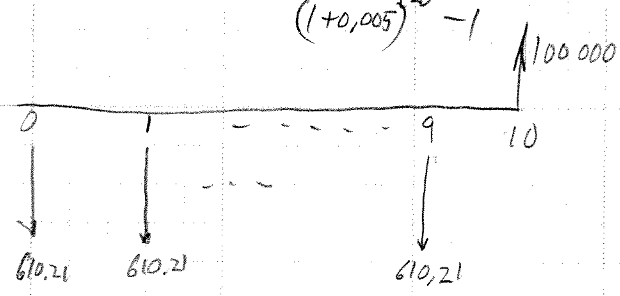
OR SOLVE FOR $R$

$$R = \frac{Si}{(1+i)^n - 1}$$

EXAMPLE HOW MUCH SHOULD I SAVE PER MONTH TO ACCUMULATE \$100000 AFTER 10 YEARS IF INTEREST IS NOMINALLY 6%/YR COMPOUNDED MONTHLY?

$r = \frac{6\%}{12\ \frac{\text{months}}{\text{YR}}} = 0.005$ $S = 100\,000$ $n = (10\text{ yrs})(12\ \text{months}/\text{yr}) = 120$

$$R = \frac{(100\,000)(0.005)}{(1+0.005)^{120} - 1} = \$610.21/\text{month}$$ (NOTE $610.21 \times 120 = 73\,225$)



NOW WE HAVE A FORMULA FOR A BALLOON PAYMENT ON A LOAN OR THE FUTURE VALUE OF A SAVED AMOUNT!

$$S_n = P(1+i)^n$$

AND FOR THE FUTURE VALUE OF PERIODIC PAYMENTS

$$S_n = \frac{R\left[(1+i)^n - 1\right]}{i}$$

ADDING THEM TOGETHER GIVES THE $\boxed{\text{TVM FORMULA}}$

$$S_n = P(1+i)^n + \frac{R\left[(1+i)^n - 1\right]}{i}$$

EXAMPLE CAR LOAN: 20 000 LOAN AMOUNT
6% NOMINAL ANNUAL INTEREST COMPOUNDED ANNUALLY
5 YEARS

WHAT MUST THE PAYMENT BE?

PERIODS WILL BE MONTHS (COMPOUNDED MONTHLY)

$$i = \frac{6\%}{12} = 0.005$$

$$n = 12 \times 5 = 60 \text{ MONTHS}$$

$S_{60} = 0$, LOAN BALANCE IS ZERO AFTER 6 months

$P = 20\,000$ POSITIVE = MONEY FROM BANK → CUSTOMER (R WILL BE NEG.)
NEG = " " CUSTOMER → BANK

YOU CHOOSE THE CONVENTION

$$0 = (20000)(1+0.005)^{60} + \frac{R\left[(1+0.005)^{60} - 1\right]}{0.005} \Rightarrow R = -386.66$$

NOTE $386.66 \times 60 = 23199.60$

THE TVM FORMULA HAS 5 VARIABLES — IF YOU KNOW ANY 4, SOLVE FOR THE 5TH

- $P$ = PRESENT (INITIAL) BALANCE
- $R$ = REGULAR PAYMENT
- $S_n$ = FUTURE VALUE AT END OF PERIOD $n$
- $n$ = # OF PERIODS
- $i$ = INTEREST RATE (PER PERIOD)

AN APPROXIMATE APPROACH TO THE TVM FORMULA — ASSUME CONTINUOUS COMPOUNDING AND CONTINUOUS PAYMENTS

$P(t)$ = VALUE A TIME $t$

$$P'(t) = rP(t) + R$$

$r$ = NOMINAL INTEREST RATE
$R$ = PAYMENT RATE

$$P'(t) - rP(t) = R$$

LINEAR, $u(t) = e^{-\int r\,dt} = e^{-rt}$

$$\left(e^{-rt}P(t)\right)' = Re^{-rt}$$

$$e^{-rt}P(t) = -\frac{R}{r}e^{-rt} + C$$

$$P(t) = -\frac{R}{r} + Ce^{rt}$$

OBVIOUSLY $P(0) = -\frac{R}{r} + C$

$C = P(0) + R/r$ LET $P_0 = P(0)$

$$\boxed{P(t) = -\frac{R}{r} + \left(P_0 + \frac{R}{r}\right)e^{rt} = P_0 e^{rt} + \frac{R}{r}\left(e^{rt} - 1\right)}$$ ← APPROX MODEL TVM FORMULA

eg CAR LOAN \$20000 LOAN AMOUNT

6% NOMINAL ANNUAL INTEREST $i = \frac{r}{12} = 0.005$

5 YEARS $t = 5 \text{ YRS} \times 12 \text{ months/yr} = 60$

SOLVE FOR $R$

$$P(t) - P_0 e^{rt} = \frac{R}{r}\left(e^{rt} - 1\right)$$

$$\frac{R}{r} = \frac{P(t) - P_0 e^{rt}}{e^{rt} - 1}$$

$$R = \frac{P(t) - P_0 e^{rt}}{e^{rt} - 1}\, r = \frac{0 - 20000\, e^{(0.005)(60)}}{e^{(0.005)(60)} - 1}(0.005) = 385.83$$

↗ OFF BY 0.83