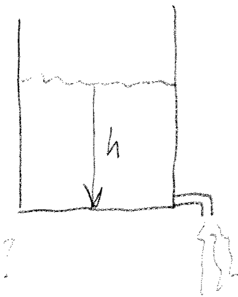
INTRODUCTION – WHAT IS DIFF. EQ. ALL ABOUT?

PROBLEM: A TANK DRAINS AT A RATE PROPORTIONAL TO THE HEAD (h) IN THE TANK

IF $\frac{1}{2}$ OF THE INITIAL HEAD DRAINS IN 120 SEC, HOW LONG DOES IT TAKE TO DRAIN 99% OF THE HEAD?



THE D.E. MODEL OF THE TANK →
$$\frac{dh(t)}{dt} = -k\,h(t) \qquad k = \text{A CONSTANT}$$

$$dh(t) = -k\,h(t)\,dt$$

$$\frac{1}{h}\,dh(t) = -k\,dt$$

$$\int \frac{1}{h(t)}\,dh(t) = \int -k\,dt$$

$$\ln h(t) = -kt + C \qquad C = \text{A CONSTANT}$$

$$h(t) = e^{(-kt+C)}$$

THIS EQ. SOLVES THE D.E. →
$$h(t) = h_0 e^{-kt} \qquad h_0 = e^{C} \quad \text{OR} \quad \frac{h}{h_0} = e^{-kt}$$

$$\frac{1}{2} = e^{-k(120)}$$

$$\ln \tfrac{1}{2} = -k(120) \implies k = 0.005776$$

$$h = h_0 e^{-0.005776t}$$

$$0.01 = e^{-0.005776t}$$

$$\ln 0.01 = -0.005776t$$

$$t = \frac{\ln 0.01}{-0.005776} = 797 \sim 800 \text{ SEC} = 13 \text{ MIN } 17 \text{ SEC}$$

PROOF OF SOLUTION

$$h(t) = h_0 e^{-kt}$$

$$\frac{dh(t)}{dt} = -kh_0 e^{-kt}$$

$$-kh(t) = -kh_0 e^{-kt}$$

THESE ARE EQUAL

$$\therefore \frac{dh(t)}{dt} = -kh(t)$$

IT TURNS OUT THAT MODELING SYSTEMS IN TERMS OF RATES OF CHANGE OF STORED SUBSTANCES IS VERY IMPORTANT.

FROM A MATH POINT-OF-VIEW, NOTE THAT THE SOLUTION OF A D.E. IS AN <u>EQUATION</u>

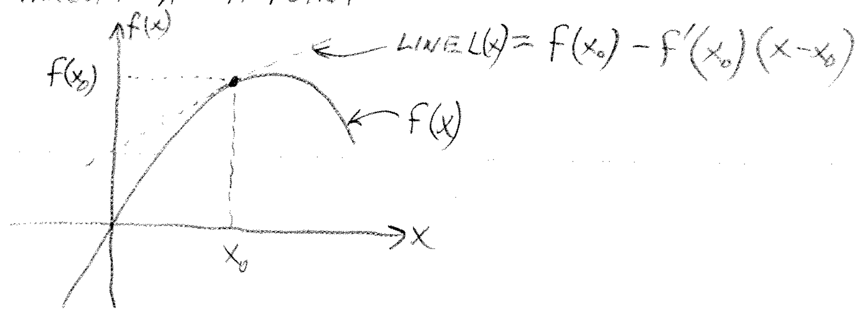
WHAT IS A DERIVATIVE? SEVERAL DEFN's, ALL USEFUL

1) THE INSTANTANTANIOUS RATE OF CHANGE OF A FUNCTION

eg $v = x' = \frac{dx}{dt}$ VELOCITY IS RATE OF CHANGE OF POSITION

$a = v' = x''$

2) SLOPE OF A TANGENT AT A POINT



3) BEST LINEAR APPROX. TO A FUNCTION

LET $f(x) = L(x) + R(x)$ WHERE $L(x)$ IS AS ABOVE

$R(x) =$ ERROR, $f(x) - L(x)$

THEN WE KNOW $\lim_{x \to x_0} R(x) = 0$

AND MORE POWERFULLY: $\lim_{x \to x_0} \frac{R(x)}{x - x_0} = 0$ (PROOF VIA TAYLOR'S THM)

4) LIMIT OF DIFFERENCE QUOTIENTS (BASIC DEFN)

$$\frac{df(x)}{dx} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

5) TABLE OF FORMULII

eg IF $f(x) = x^n$ THEN $\frac{df(x)}{dx} = n x^{n-1}$

IF $f(x) = \sin x$ THEN $\frac{df(x)}{dx} = \cos x$

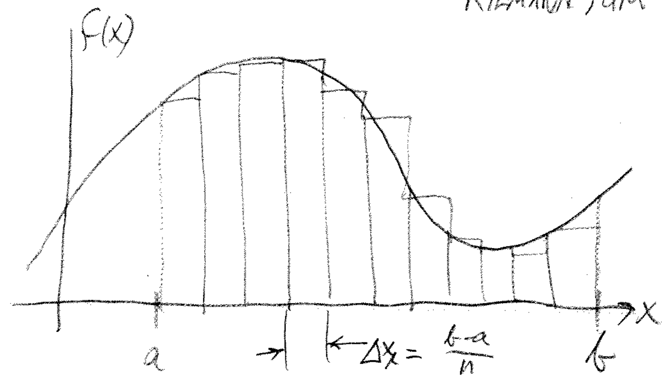
etc.

NOTE: THESE TABLES ARE USUALLY GIVEN INSTEAD AS INTEGRAL TABLES

$\int n x^{n-1} dx = x^n + C$ $\int \cos x \, dx = \sin x + C$ etc.

WHAT IS INTEGRATION?

1) THE AREA UNDER THE CURVE (DEFINITE INTEGRAL), $\int_a^b f(x)\,dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$ ← RIEMANN SUM



$x_i = a + i\Delta x_i$

2) THE ANTIDERIVATIVE (INDEFINITE INTEGRAL)

$f'(x) = g(x)$ IFF $\int g(x)\,dx = f(x) + C$ (FOR ANY LIMITS)

FUNDAMENTAL THEOREM OF CALC. $\int_a^b g(x)\,dx = f(b) - f(a)$

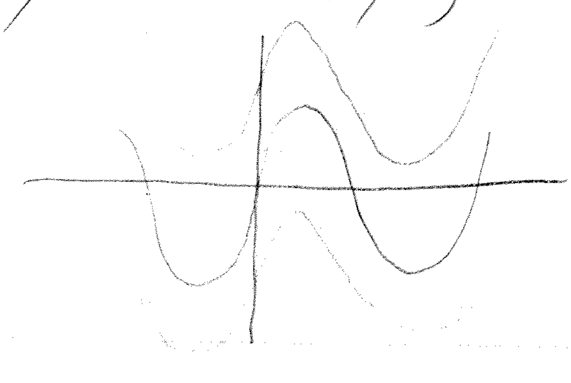
3) A SET OF TABLES — CHECK YOUR CALC BOOK

SOLUTION OF D.E. VIA INTEGRATION

SUPPOSE $y' = f(t)$ THIS IS A FIRST-ORDER D.E. (ONLY 1st DERIV. USED)

THEN $y = \int y'(t)\,dt = \int f(t)\,dt$

e.g. SUPPOSE $y' = \cos t$ THEN $y = \int \cos t\,dt = \underbrace{\sin t + C}$ IS THE GENERAL SOLUTION TO THE D.E.



TECHNIQUES OF INTEGRATION:

1) CHANGE OF VARIABLE AKA INTEGRATION BY SUBSTITUTION

e.g. $y'(x) = \frac{x}{30-x^2}$ OR $\frac{dy}{dx} = \frac{x}{30-x^2}$ OR $dy = \frac{x\,dx}{30-x^2}$

LET $u(x) = 30 - x^2$ THEN $du = -2x\,dx$

THUS $dy = \frac{-\frac{1}{2}du}{u}$

$$\int dy = -\frac{1}{2}\int \frac{1}{u}du$$

$$y = -\frac{1}{2}\ln|u| + C$$

$$y = -\frac{1}{2}\ln(30-x^2) + C$$

WORKS WHEN THE FUNCTION TO BE INTEGRATED CAN BE VISUALIZED AS THE PRODUCT OF A POWER OF A DIFFERENTIABLE FUNCTION ($u(x) = 30-x^2$ IN THE ABOVE EX.) AND ITS DERIVATIVE ($-2x$ IN ABOVE)

2) INTEGRATION BY PARTS $\int u\,dv = uv - \int v\,du + C$

OR $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

DO Ex 11, p14

INITIAL VALUE PROBLEMS

SUPPOSE A D.E. HAS A GENERAL SOLUTION. SOMETIMES THERE IS ENOUGH INFO. TO CHOOSE A PARTICULAR ONE OF THE GENERAL SOLUTIONS. THESE ARE CALLED INITIAL VALUE PROBLEMS

e.g. SUPPOSE $y'' = 9e^{-3x}$ AND WE WANT $y(0)=1$ AND $y'(0)=2$

INTEGRATING: $y' = -3e^{-3x} + C_0$, C_0 = A CONSTANT

$y = e^{-3x} + C_0 x + C_1$, C_1 = A CONSTANT

FROM $y' = -3e^{-3x} + C_0$ GET $y'(0) = -3e^{-0} + C_0$ AND WE KNOW $y'(0) = 2$

$\therefore\ 2 = -3 + C_0 \Rightarrow C_0 = 5$

FROM $y = e^{-3x} + C_0 x + C_1$ GET $y(0) = 1 + 5(0) + C_1$ AND KNOW $y(0) = 1$

$1 = 1 + C_1 \Rightarrow C_1 = 0$

$$\boxed{\therefore\ y = e^{-3x} + 5x}$$

NOTE THAT $y'' = 9e^{-3x}$ IS A 2nd ORDER D.E.

WE THEN NEED TWO I.C. FOR THE PARTIC. SOLN, ONE NEEDED FOR EACH INTEGRATION.

THESE TWO I.C. NEED NOT BE GIVEN IN THE FORM SHOWN ABOVE

e.g $y(0) = 1$ AND $y(1) = e^{-3} + 5$

GET TO $y = e^{-3x} + C_0 x + C_1$, NOW SET UP TWO EQ. IN 2. UNKNOWNS

$$y(0) = 1 = 1 + (0)C_0 + C_1 \Rightarrow C_1 = 0$$
$$y(1) = e^{-3} + 5 = e^{-3} + (1)C_0 + C_1 \Rightarrow C_0 = 5$$

CH 2 FIRST-ORDER EQUATIONS

START SIMPLE – WORK TOWARD MORE COMPLICATED... (2nd ORDER)

PATTERN OF CHAPTER: DEF'NS
OBSERVATIONS → DIRECTION FIELD
SEPERABLE → APPLICATION: MODELS OF MOTION
LINEAR → APPLICATION: MIXING PROBLEMS
EXACT D.E.
EXISTANCE & UNIQUENESS OF SOLUTIONS
DEPENDENCE OF SOLS ON I.C.
STABILITY

DEFINITIONS

ORDINARY D.E. VS. PARTIAL DE.

ORDINARY IF THE UNKNOWN FUNCTION IS A FUNCTION OF ONE VARIABLE

eg $\frac{dy}{dt} = y - t$ IS ORDINARY ← THIS COURSE

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$ IS PARTIAL ← GRAD SCHOOL

NORMAL FORM $y' = f(t,y)$ ← FIRST ORDER

$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$ ← nth ORDER

eg GIVEN: $y' + 3y = t + e^{-2t}$

$y' = (t + e^{-2t}) + 3y$ IS "NORMAL FORM"

GIVEN: $x^3 y''' + x^3 y'' - 2x^3 y' + 2x^3 y = 2x^4$ (y A FUNCTION OF x)

$y''' = 2x - 2y + 2y' + y''$ IS NORMAL FORM

PARTICULAR SOLUTION A FUNCTION THAT SOLVES THE DE AND SATISFIES AN I.C.

GENERAL SOLUTION A FUNCTION THAT SOLVES THE DE. AND INCLUDES n UNKNOWN CONSTANTS TO BE USED TO SATISFY I.C.

eg. $y' = t - 2ty$

TRY $y = \frac{1}{2} + ce^{-t^2}$ AS A GENERAL SOLUTION

THEN $y' = -2cte^{-t^2}$ $\quad 2ty = t + 2tce^{-t^2}$

SO $t - 2ty = t - \left(t + 2tce^{-t^2}\right) = -2tce^{-t} = y'$

$\therefore y = \frac{1}{2} + ce^{-t^2}$ IS THE GENERAL SOLUTION TO $y' = t - 2ty$

BUT SUPPOSE WE WANT $y(0) = 0$

$0 = \frac{1}{2} + ce^{-0^2} \Rightarrow c = -\frac{1}{2}$

$y = \frac{1}{2} - \frac{1}{2}e^{-t^2}$ IS A PARTICULAR SOL$^{\underline{n}}$ OF $y' = t - 2ty$

<u>INTERVAL OF EXISTANCE</u> VALID RANGE FOR THE IND. VAR. OF THE SOL$^{\underline{n}}$

e.g $y' = y^2$ HAS GENERAL SOLUTION $y = -\frac{1}{t-c}$

SUPPOSE $y(0) = 1$ THEN $y = -\frac{1}{t-1}$ $(c = 1)$

BUT NOW $t = 1$ GIVES A DIV. BY ZERO ERROR

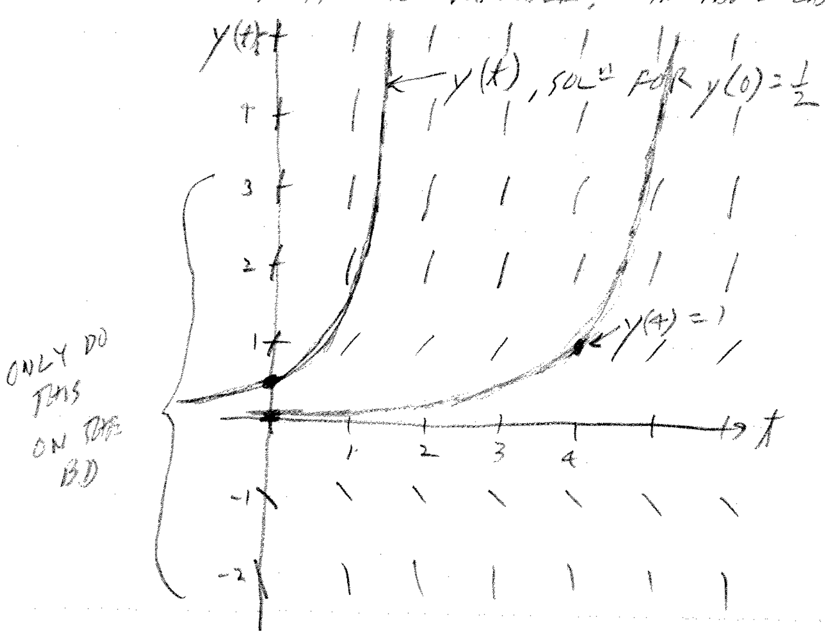
$\therefore$ SPECIFY $t < 1$ OR $t > 1$

BUT $t = 0$ IS ON THE SOL$^{\underline{n}}$ $\therefore$ $\underline{t < 1}$ IS THE INT. OF EXIST.

<u>DIRECTION FIELD</u> – GEOMETRIC INTERPRETATION OF A D.E.

PUT THE D.E. IN NORMAL FORM e.g. $y' = y$

THEN PLOT SLOPES VS. THE IND. VARIABLE, IN ABOVE CASE, NO DEPENDENCE ON t



GIVEN AN I. C.

e.g $y(0) = \frac{1}{2}$

THE SOL$^{\underline{n}}$ MUST BE TANGENT TO THE DIR. FIELD

READ TEXT FOR MORE EXAMPLES

2.2 SEPARABLE EQUATIONS

EXAMPLE: GIVEN D.E. $e^y y' - t - t^3 = 0$

FIND THE GEN'L SOLUTION.

TECHNIQUE

① $\begin{cases} \text{SEPARATE ALL } y \text{ STUFF TO ONE SIDE, EVERYTHING ELSE OTHER SIDE} \\ \text{ALSO, USE } \frac{dy}{dt} \text{ NOTATION RATHER THAN } y' \end{cases}$

$$e^y \frac{dy}{dt} - t - t^3 = 0$$

$$e^y\, dy = (t + t^3)\, dt$$

② THEN INTEGRATE BOTH SIDES

$$\int e^y\, dy = \int (t + t^3)\, dt$$ ← MORE ON THIS STEP LATER — THIS IS A SHORTHAND VERSION

$$e^y = \tfrac{1}{2}t^2 + \tfrac{1}{4}t^4 + C$$

③ NOW SOLVE FOR $y(t)$

$$\ln e^y = \ln\left(\tfrac{1}{2}t^2 + \tfrac{1}{4}t^4 + C\right)$$

$$y(t) = \ln\left(\tfrac{1}{2}t^2 + \tfrac{1}{4}t^4 + C\right)$$

EXAMPLE: $y' = -\frac{t}{y}$

$$\frac{dy}{dt} = -\frac{t}{y}$$

$$y\, dy = -t\, dt$$

$$\int y\, dy = -\int t\, dt$$

$$\tfrac{1}{2}y^2 = -\tfrac{1}{2}t^2 + C_0$$

$$y^2 = C - t^2$$ ← EQ^{n} OF A CIRCLE, RADIUS $\sqrt{C}$

DIVIDE BY ZERO ISSUES

SUPPOSE THE D.E. IS $y' = ty^2$ AND WE WANT $y(0) = 0$

$$\frac{dy}{dt} = ty^2$$

$$\int \frac{1}{y^2}\,dy = \int t\,dt \qquad \text{IF } y \neq 0$$

$$-\frac{1}{y} = \frac{1}{2}t^2 + C$$

$$y = \frac{-2}{t^2 + 2C}$$ BUT NO FINITE VALUE OF C GIVES $y(0) = 0$

BUT IF $y = 0$ THEN $\frac{dy}{dt} = 0 \Rightarrow y(t) = C$ IS A SOL$^{\underline{n}}$, BUT IT MUST GO THRU $y(t) = 0$ FOR SOME $t \Rightarrow C = 0$

$\therefore y(0) = 0$ IS A SOLUTION OF THE D.E.

CONCLUSION: A GEN'L SOLUTION DOES NOT NECESSARILY GIVE ALL PARTICULAR SOLUTIONS WHEN YOU VARY THE CONSTANTS.

HANDLING INITIAL CONDITIONS VIA DEFINITE INTEGRATION

eg $40°$ CAN OF FAST-CO POP PLACED IN A $70°$ ROOM
AFTER 10 MIN, POP IS $50°$
WHAT IS TEMP OF POP VS. TIME

NEWTON'S LAW OF COOLING (WARMING) $\frac{dT}{dt} = -k(T - A)$

$$\int_{T(0)}^{T(t)} \frac{1}{T-A}\,dT = \int_0^t -k\,d\lambda \quad \leftarrow \text{DUMMY VARIABLE}$$

$$\ln|T - A|\Big|_{T(0)}^{T(t)} = -k\lambda\Big|_0^t$$

$$\ln\frac{T(t) - A}{T(0) - A} = -kt$$

$$\frac{T(t) - A}{T(0) - A} = e^{-kt}$$

$$T(t) = [T(0) - A]e^{-kt} + A$$

NOW PLUG IN TO FIND k:

$$T(10\text{ MIN}) = [T(0) - A]e^{-k(10\text{ MIN})}$$

$$50 = (40 - 70)e^{-10k} + 70$$

$$k = 0.0405$$

IMPLICITLY DEFINED SOLUTIONS

RECALL THE METHOD OF SEPARATION OF VARIABLES HAS 3 STEPS

1. SEPARATE, DEP. VAR ON ONE SIDE, IND. VAR ON OTHER, USE $\frac{dy}{dt}$ NOT y'
2. INTEGRATE BOTH SIDES
3. SOLVE FOR THE DEP. VAR.

THIS THIRD STEP IS NOT ALWAYS EASY OR POSSIBLE

EXAMPLE $\quad y' = \dfrac{e^x}{1+y}$ AND $y(0)=1$

(1) $\int (1+y)\,dy = \int e^x\,dx$

(2) $y + \frac{1}{2}y^2 = e^x + C$

(3) $y = ?$ NOT SO EASY THIS TIME, BUT RECALL THE QUADRATIC FORMULA

$\frac{1}{2}y^2 + y - (e^x + C) = 0$

$y^2 + 2y - 2(e^x + C) = 0$

$$y = \frac{-2 \pm \sqrt{4 - 4(-2)(e^x+C)}}{2} = -1 \pm \sqrt{1 + 2(e^x + C)}$$

SINCE $y(0) = 1$

$1 = -1 \pm \sqrt{1 + 2(e^0 + C)}$

$2 = \pm\sqrt{1 + 2(1 + C)}$ $\quad\therefore$ CHOOSE $+$

$4 = 1 + 2(1 + C)$

$\frac{3}{2} = 1 + C$

$C = \frac{1}{2}$

SUPPOSE INSTEAD $y(0) = -4$

$-4 = -1 \pm \sqrt{1 + 2(e^0 + C)}$ $\quad\therefore$ CHOOSE $-$

$-4 = -1 - \sqrt{1 + 2(e^0 + C)}$

$-3 = -\sqrt{1 + 2(e^0 + C)}$

$9 = 1 + 2(1 + C)$

$8 = 2(1 + C)$

$4 = 1 + C$

$C = 3$

INTERVAL OF EXISTANCE:
ANY $x \in \mathbb{R}$ IS OK IN THE SOL$^{\text{N}}$

OBSERVE IN THE D.E. $y \neq -1$ (DIV BY ZERO)
CHECK: IN THE SOL$^{\text{N}}$ $y \neq -1$ FOR ALL x
$\therefore$ THE INTERVAL OF EXISTANCE IS $-\infty < x < \infty$

INTERVAL OF EXISTANCE:
ANY x IS OK IN SOL$^{\text{N}}$
BUT $y \neq -1$ IN THE D.E.
NO x CAN MAKE $y = -1$
$\therefore$ $-\infty < x < \infty$

IN THE PREVIOUS EXAMPLE WE FOUND AN <u>EXPLICIT</u> <u>SOL^{n}</u>

defn AN EXPLICIT SOLUTION IS A SINGLE FUNCTION OF THE INDEPENDENT VARIABLE. IF A SOLUTION IS NOT EXPLICIT THEN IT IS <u>IMPLICIT</u>

e.g. $y = f(x)$

$$y(x) = \underbrace{-1 \pm \sqrt{1+2(e^x+c)}}$$

← NO "y" VARIABLES ON THIS SIDE

eg $y(x) = e^x + c - \frac{1}{2}y^2(x)$ MEANS SAME AS ABOVE, BUT THIS IS IMPLICIT

↑ CAN'T SOLVE FOR $y^2(x)$ BECAUSE DON'T YET KNOW $y(x)$

IN THE ABOVE CASE WE SOLVED THE IMPLICIT SOL^{n} FOR THE EXPLICIT SOL^{n}

<u>EXAMPLE</u> $x' = \dfrac{2tx}{1+x}$

$$\int \frac{1+x}{x}\,dx = \int 2t\,dt$$

$$\ln|x| + x = t^2 + c$$

NOW WE ARE STUCK

TRY $e^{(\ln|x|+x)} = e^{(t^2+c)}$ GOES NOWHERE

AT THIS POINT, TURN TO A COMPUTER FOR NUMERICAL SOLUTIONS. — READ ABOUT THIS EXAMPLE IN THE TEXT.

WHY SEPARATION OF VARIABLES WORKS

RECALL THIS EXAMPLE $e^y y' - t - t^3 = 0$ OR IN NORMAL FORM $y' = \frac{-t-t^3}{e^y}$

$$e^y \frac{dy}{dt} = t^3 + t$$

I SAID THIS WAS A SHORTCUT

$$e^y dy = (t^3+t)dt$$

$$\int e^y dy = \int (t^3+t)\,dt$$

$$e^y = \frac{1}{4}t^4 + \frac{1}{2}t^2 + c$$

etc...

HERE'S JUSTIFICATION FOR THE SHORTCUT

SUPPOSE THE D.E. HAS THE NORMAL FORM $y' = \frac{g(t)}{h(y)}$

THIS IS ACTUALLY REQUIRED IF THE EQUATION IS TO BE SEPARABLE
(SEPARATION OF VARIABLES DOES NOT WORK ON EVERY 1ST ORDER D.E.)

THEN $\frac{dy}{dt} = \frac{g(t)}{h(y)} \longrightarrow$ SHORTCUT $\longrightarrow h(y)dy = g(t)dt \longrightarrow 0 = 0$

ZERO IN THE LIMIT

SO THIS LOOKS MEANINGLESS, OF COURSE $0=0$ FOR ANY $h(y), g(t)$
BUT IT WORKS WHY?

LEGITIMATE STEPS:

RATIO IS $\neq 0$ IN THE LIMIT (IN GEN'L)

$$\frac{dy}{dt} = \frac{g(t)}{h(y)} \longrightarrow h(y)\frac{dy}{dt} = g(t) \longrightarrow h(y)y' = g(t) \rightarrow h(y(t))y'(t) = g(t)$$

NOW INTEGRATE BOTH SIDES WRT t

RATIO $\neq 0$ IN GEN'L

$$\int h(y(t))y'(t)\,dt = \int g(t)\,dt \qquad y'(t) = \frac{dy}{dt}$$

NOW CHANGE THE VARIABLE OF INTEGRATION TO y. NOTE $dy = y'(t)dt$

$$\int h(y)dy = \int g(t)dt$$, JUST WHAT THE SHORTCUT GIVES

2.3 MODELS OF MOTION (LINEAR)

(1) NEWTONIAN PHYSICS IN A VACUUM: $F = ma$

$$a = \frac{dv}{dt} = \frac{dx^2}{dt^2}$$

ON THE SURFACE OF THE EARTH $a = -g = -9.8\ m/s^2$ (NEG TO INDICATE DOWNWARD)

$$F = -mg \quad \text{AND} \quad F = ma$$

$$\therefore \quad a = -g$$

$$\frac{dv}{dt} = -g$$

INITIAL VELOCITY, LET $C_1 = V_0$

$$\frac{dx^2}{dt^2} = -g \Rightarrow \frac{dx}{dt} = -gt + C_1$$

$$\frac{dx}{dt} = -gt + v_0$$

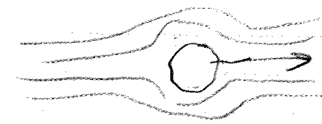
INITIAL POSITION, $C_2 = X_0$

$$x(t) = -\frac{1}{2}gt^2 + v_0 t + C_2$$

$$x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$$

(2) VISCOUS FLOW — ADD THE EFFECT OF AIR RESISTANCE

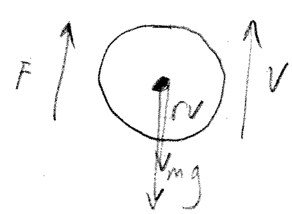
VISCOUS — LAMINAR AIR FLOW, NO TURBULANCE
(SMALL OBJECTS AND/OR LOW SPEEDS)



THE RESISTANCE DUE TO AIRFLOW IS A FORCE PROPORTIONAL TO VELOCITY

$$R(v) = -rV$$

CONSIDERED WITH EFECT OF GRAVITY,



$$F = -mg - rv$$

$$ma = -mg - rv$$

$$m\frac{dv}{dt} = -mg - rv$$

$$\frac{dv}{dt} = -g - \frac{r}{m}V$$

$$\frac{dv}{g + \frac{r}{m}V} = -dt$$

$$\frac{\frac{m}{r}dv}{\frac{mg}{r} + v} = -dt$$

$$\int \frac{dv}{\frac{mg}{r} + v} = -\frac{r}{m}\int dt$$

$$\ln\left|\frac{mg}{r} + v\right| = -\frac{r}{m}t + C_1$$

$$\frac{mg}{r} + v = e^{(-rt/m + C_1)}$$

$$v = Ce^{-rt/m} - \frac{mg}{r}$$

NOTE, AS $t \to \infty$ $\quad v \to -\frac{mg}{r}$

TERMINAL VELOCITY

③ TURBULANT FLOW $R(v) = -r\,V^2\,\mathrm{sign}(v) = -r|V|v$

NOW $m\frac{dv}{dt} = -mg - r|V|V$

$$\frac{dv}{dt} = -g - \frac{r}{m}|V|V$$

NOW RESTRICT THE APPLICATION TO FALLING OBJECTS SO THAT $v < 0$

$$\frac{dv}{dt} = -g + \frac{r}{m}V^2$$

THIS IS SEPERABLE, BUT EASIER IF WE SCALE THE VARIABLES SO THAT

$$\frac{dw}{ds} = -1 + w^2 \qquad (\text{MAKE COEFF} = 1)$$

TO DO THIS LET $V = \alpha w$ AND $t = \beta s$

OR $w = \frac{V}{\alpha}$ $\quad s = \frac{t}{\beta}$

NOW $\frac{dv}{dt} = \frac{d(\alpha w)}{d(\beta s)} = \frac{\alpha}{\beta}\frac{dw}{ds}$

SUBSTITUTING THE NEW VARIABLES INTO ORIGINAL D.E. GIVES

$$\frac{\alpha}{\beta}\frac{dw}{ds} = -g + \frac{r}{m}(\alpha w)^2 = -g + \frac{r}{m}\alpha^2 w^2$$

$$\frac{dw}{ds} = -\frac{\beta}{\alpha}g + \alpha\beta\frac{r}{m}w^2$$

NOW CHOOSE α, β SUCH THAT $-\frac{\beta}{\alpha}g = -1$ AND $\alpha\beta\frac{r}{m} = 1$

$$\alpha = \beta g$$

$$(\beta g)\beta\frac{r}{m} = 1$$

$$\beta^2 = \frac{m}{rg}$$

$$\beta = \sqrt{\frac{m}{rg}}$$

$$\alpha = \sqrt{\frac{m}{rg}}\,g = \sqrt{\frac{mg}{r}}$$

NOW SOLVE $\frac{dw}{ds} = -1 + w^2$

$$\frac{dw}{1-w^2} = -ds$$

PARTIAL FRACTIONS: NOTE $1-w^2 = (1+w)(1-w)$

$$\int\frac{1}{2}\left[\frac{dw}{1+w} + \frac{dw}{1-w}\right] = -\int ds \qquad \frac{1}{2}\ln\left|\frac{1+w}{1-w}\right| = -s + C_1 \Rightarrow \frac{1+w}{1-w} = Ce^{-2s}$$

SOLVING FOR w GIVES

$$1+w=(1-w)Ce^{-2s}$$

$$1+w=Ce^{-2s}-wCe^{-2s}$$

$$1-Ce^{-2s}=-w(1+Ce^{-2s})$$

$$w=\frac{Ce^{-2s}-1}{Ce^{-2s}+1}$$

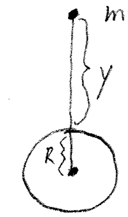
NOW SUBST ORIGINAL VARIABLES IN

$$\frac{v}{\alpha}=\frac{Ce^{-2t/\beta}-1}{Ce^{-2t/\beta}+1}$$

$$v=\sqrt{\frac{mg}{r}}\left(\frac{Ce^{-2t\sqrt{\frac{rg}{m}}}-1}{Ce^{-2t\sqrt{\frac{rg}{m}}}+1}\right)$$

NOTE; AS $t\to\infty$ $v\to-\sqrt{\frac{mg}{r}}$

PG 45 Ex 14



ONE GREAT DISCOVERY IN SCIENCE IS $|F| = \dfrac{GMm}{r^2}$

WHERE $G = 6.6726 \times 10^{-11}\ Nm^2/Kg^2$

SUPPOSE AN OBJECT WITH MASS m IS LAUNCHED FROM EARTH'S SURFACE WITH INITIAL VELOCITY V_0. LET y REPRESENT ITS POSITION ABOVE THE EARTH'S SURFACE

a) IF AIR RESISTANCE IS IGNORED, USE THE IDEA $a = \dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = \dfrac{dv}{dx}v$

TO SHOW THAT $v\dfrac{dv}{dy} = -\dfrac{GM}{(R+y)^2}$

$$F = ma \quad \text{AND} \quad F = \frac{GMm}{r^2}$$

THUS $ma = \dfrac{GMm}{r^2}$ NOW LET $a = \dfrac{dv}{dt} = \dfrac{dv}{dy}\cdot\dfrac{dy}{dt} = \dfrac{dv}{dy}v$ SINCE $v = dy/dt$

$$a = \frac{GM}{(R+y)^2} \qquad \text{ALSO} \quad r = R+y$$

$$v\frac{dv}{dy} = \frac{GM}{(R+y)^2} \qquad \text{Q.E.D.}$$

b) ASSUMING THAT $y(0) = 0$ (LAUNCHED FROM EARTH'S SURFACE) AND $v(0) = V_0$, SOLVE THE D.E. ABOVE TO SHOW THAT

$$v^2 = V_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right)$$

$$\int_{V_0}^{v} s\,ds = \int_0^y \frac{GM}{(R+u)^2}du$$

(u IS A DUMMY VARIABLE IN PLACE OF WHAT WAS y SIMILAR s REPLACES v)

WHERE $u = R+y$ $du = dy$

$$\frac{1}{2}s^2\Big|_{V_0}^{v} = \frac{-GM}{R+u}\Big|_0^y$$

$$\frac{1}{2}v^2 - \frac{1}{2}V_0^2 = -\frac{GM}{R+y} - \frac{GM}{R}$$

$$v^2 = V_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right) \qquad \text{Q.E.D.}$$

PG 45 EX 14 CONTINUED

c) SHOW THAT THE MAXIMUM HEIGHT IS $y = \dfrac{v_0^2 R}{\frac{2GM}{R} - v_0^2}$

SOLVE PREVIOUS FOR y GIVEN v = 0

$$0 = v_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right)$$

$$0 = \frac{v_0^2}{2GM} - \frac{1}{R} + \frac{1}{R+y}$$

$$\frac{1}{R+y} = \frac{1}{R} - \frac{v_0^2}{2GM}$$

$$R+y = \frac{1}{\frac{1}{R} - \frac{v_0^2}{2GM}}$$

$$y = -R + \frac{1}{\frac{1}{R} - \frac{v_0^2}{2GM}} = -R + \frac{2GM}{\frac{2GM}{R} - v_0^2} = \frac{-R\left(\cancel{\frac{2GM}{R}} - v_0^2\right)}{\left(\frac{2GM}{R} - v_0^2\right)} + \frac{\cancel{2GM}}{\left(\frac{2GM}{R} - v_0^2\right)}$$

$$y = \frac{v_0^2}{\frac{2GM}{R} - v_0^2} \quad \text{Q.E.D.}$$

d) SHOW THAT IF $v_0 = \sqrt{\frac{2GM}{R}}$ IS MINIMUM ESCAPE VELOCITY $(y \to \infty)$

IN THE ABOVE EQ. NOTE THE DENOMINATOR.

IF $v_0 < \sqrt{\frac{2GM}{R}}$ THEN $v_0^2 < \frac{2GM}{R}$ AND y IS FINITE

AS DENOM. GOES TO ZERO, $y \to \infty$

IF DENOM. = ZERO $y \to \infty$ THIS HAPPENS WHEN $v_0^2 = \frac{2GM}{R}$

$\therefore v_0 = \sqrt{\frac{2GM}{R}}$ IS MIN. ESCAPE VELOCITY

2.4 LINEAR EQUATIONS (FIRST ORDER)

defn A D.E IS LINEAR IFF THE UNKNOWN FUNCTION AND ITS DERIVATIVES APPEARE ALONE IN THE D.E. OR MULTIPLIED BY A CONSTANT OR A FUNCTION OF ONLY THE INDEPENDENT VARIABLE.

$$x' + a(t)x = f(t)$$

CALLED THE "FORCING FUNCTION"

OR n^{TH} ORDER $\quad x^{(n)} + a_{n-1}(t)x^{(n-1)} + \dots + a_1(t)x' + a_0(t)x = f(t)$

defn A. D.E. IS HOMOGENIOUS IFF THE FORCING FUNCTION ($f(t)$ ABOVE) $=0$

MORE GENERALLY, IFF ZERO IS A SOL^{n}

eg $\quad y' = 5y$, y IS A FUNCTION OF t

SUPPOSE $y(t) = 0 \rightarrow$ SOLVES THE D.E. $\therefore$ HOMOGENIOUS

eg $\quad y' = 5y + 1$

SUPPOSE $y(t) = 0 \rightarrow$ DOES NOT SOLVE THE D.E. $\therefore$ INHOMOGENIOUS

SOLUTION OF HOMOGENIOUS EQ^{n} (NOTHING NEW HERE - USE SEPARATION)

$$x' = a(t)x$$

$$\frac{dx}{dt} = a(t)x$$

$$\frac{1}{x}dx = a(t)dt$$

$$\int \frac{1}{x}dx = \int a(t)dt$$

$$\ln|x| = \int a(t)dt + C$$

$$|x| = e^{\int a(t)dt + c} = Ae^{\int a(t)dt} \quad \text{WHERE } A = e^{C}$$

LET $A > 0$, DUMP $|\ |$, THEN FIND IT WORKS FOR ANY A

EXAMPLE $\quad y' = 5y$, y A FUNCT. OF t

$$y = Ae^{\int 5dt} = Ae^{5t}$$

TRY IT: $\quad y' = 5Ae^{5t} = 5y \quad \therefore$ OK

SOL$^{\underline{N}}$ OF INHOMOGENIOUS EQ$^{\underline{N}}$ — USE AN <u>INTEGRATING FACTOR</u> TO ENABLE SEPARATION

AN EXAMPLE FIRST, THEN GENERALIZE

SUPPOSE $T' = -k(T-A)$ (NEWTON'S LAW OF COOLING)

$T' + kT = kA$

[COMPARE TO GEN'L FORM FOR 1ST ORDER LINEAR D.E.

$x'(t) + a(t)x(t) = f(t)$ ABOVE $x = T$, $a(t) = k$ $f(t) = kA$

BACK TO $T' + kT = kA$

MULT BY e^{kt} ⟵ CALLED AN INTEGRATING FACTOR

$$T'e^{kt} + kTe^{kt} = kAe^{kt}$$

NOTE: L.H.S. $= \left[Te^{kt}\right]' = T'e^{kt} + kTe^{kt}$

$$\therefore \frac{d}{dt}\left[Te^{kt}\right] = kAe^{kt}$$

SEPARATE etc (JUST DO ANTI-DERIVATIVE ON L.H.S)

$$Te^{kt} = \int kAe^{kt}\,dt$$

$$Te^{kt} = Ae^{kt} + C$$

$$T = A + Ce^{-kt}$$

NOW GENERALIZE THIS IDEA

START W/ A GEN'L 1ST ORDER LINEAR D.E.

$$x'(t) - a(t)x(t) = f(t)$$

MULTIPLY THRU BY THE INTEGRATING FACTOR $u(t)$

$$u(t)x'(t) - u(t)a(t)x(t) = u(t)f(t)$$

ASSUME WE KNOW THE INTEGRATING FACTOR SUCH THAT

$$\left[u(t)\,x(t)\right]' = u(t)x'(t) - u(t)a(t)x(t) \quad \leftarrow \text{NOTE}^{*}$$

THEN $$\left[u(t)x(t)\right]' = u(t)f(t)$$

$$u(t)x(t) = \int u(t)f(t)\,dt + C$$

$$x(t) = \frac{1}{u(t)}\int u(t)f(t)\,dt + \frac{C}{u(t)}$$

THE KEY IS TO FIND THE INTEGRATING FACTOR

GO BACK TO THE EQ^{N} OF NOTE

$$\left[u(t)x(t)\right]' = u(t)x'(t) - u(t)a(t)x(t)$$

CHAIN RULE $$u'(t)x(t) + u(t)x'(t) = u(t)x'(t) - u(t)a(t)x(t)$$

$$u'(t)x(t) = -u(t)a(t)x(t)$$

$$u'(t) = -u(t)a(t) \quad \leftarrow \text{NOTE: LINEAR HOMOGENIOUS DE.}$$

$$u(t) = Ae^{-\int a(t)\,dt} \quad (\text{LET } A = 1 \text{ FOR CONVENIENCE})$$

$$u(t) = e^{-\int a(t)\,dt}$$ IS THE NEEDED INTEGRATING FACTOR

USE THIS FORMULA TO FIND INTEGRATING FACTORS

EXAMPLE P 55 Ex 3

$$y' + \frac{2}{x}y = \frac{\cos x}{x^2}$$ IDENTIFY $a(x) = \frac{2}{x}$

INTEGRATING FACTOR $u(x) = e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = e^{-\ln x^2} = -x^2$

SIGN NOT IMPORTANT

$\therefore$ MULTIPLY THE DE BY x^2

$$x^2y' + 2xy = \cos x$$

NOW USE WHAT THE INTEGRATING FACTOR DID TO NOTE THAT

$$x^2y' + 2xy = (x^2y)'$$

$$(x^2y)' = \cos x$$

$$x^2y = \int \cos x\, dx + C$$

$$x^2y = \sin x + C$$

$$\boxed{y = \frac{\sin x + C}{x^2}}$$

EXAMPLE (from Rabenstein, p 26)

$$(x+1)y' - y = x$$

NOTE: NOT IN THE FORM $y' + ay = f$ SO PUT IT IN THAT FORM

$$y' - \frac{1}{x+1}y = \frac{x}{x+1}$$

ONLY NEED ONE I. F. CHOOSE C_1 SO THAT $e^{C_1} = 1$

INTEGRATING FACTOR: $u(x) = e^{-\int \frac{1}{x+1}dx} = e^{-\ln|x+1| + C_1} = \frac{e^{C_1}}{|x+1|} = \frac{1}{x+1}, \ x > -1$

$$\frac{1}{x+1}y' - \frac{1}{(x+1)^2}y = \frac{x}{(x+1)^2}$$

NOW OBSERVE THAT THE L.H.S. IS $\left(\frac{1}{x+1}y\right)' = \frac{1}{x+1}y' - \frac{1}{(x+1)^2}y$

$$\left(\frac{1}{x+1}y\right)' = \frac{x}{(x+1)^2}$$

$$\frac{1}{x+1}y = \int \frac{x}{(x+1)^2}dx = \int \frac{1}{x+1} - \frac{1}{(x+1)^2}dx$$ (PARTIAL FRACTION EXPANSION w/ REPEATED ROOTS)

NOTE $\frac{x}{(x+1)^2} = \frac{a}{x+1} + \frac{b}{(x+1)^2} = \frac{a(x+1)+b}{(x+1)^2} \quad a = 1, \ b = -1$

$$\frac{1}{x+1}y = \ln|x+1| + \frac{1}{x+1} + C$$

$$y = 1 + (x+1)(C + \ln|x+1|)$$

THE METHOD OF VARIATION OF PARAMETERS

AN ALTERNATIVE TO USING AN INTEGRATING FACTOR

CONSIDER AGAIN $y'(t) - a(t)y(t) = f(t)$

WE KNOW A SOLUTION OF THE HOMOGENIOUS EQ. $y_h'(t) = a(t)y_h(t)$ IS

$$y_h(t) = e^{\int a(t)\,dt} \qquad \text{(BY SEPARATION OF VARIABLES)}$$

NOW ASSUME THERE IS A GENERAL SOL^{N} TO THE INHOMOGENIOUS EQ., $y(t)$

DEFINE $v(t) = \dfrac{y(t)}{y_h(t)}$ (OK: $y_h(t) \neq 0$ SINCE $e^x \neq 0$ FOR ALL x)

THEN $y(t) = v(t)y_h(t) \longrightarrow$ SUBST THIS INTO THE D.E. AND SOLVE FOR $v(t)$

$$\left(v(t)y_h(t)\right)' - a(t)\left(v(t)y_h(t)\right) = f(t)$$

$$v'(t)y_h(t) + v(t)y_h'(t) - a(t)v(t)y_h(t) = f(t) \quad \text{AND RECALL } y_h'(t) = a(t)y_h(t)$$

$$v'(t)y_h(t) + v(t)\left[\cancel{y_h'(t) - a(t)y_h(t)}^{\,0}\right] = f(t)$$

$$v'(t)y_h(t) = f(t)$$

$$v'(t) = \frac{f(t)}{y_h(t)} \implies v(t) = \int \frac{f(t)}{y_h(t)}\,dt$$

$$y(t) = y_h(t)\int \frac{f(t)}{y_h(t)}\,dt$$

(NOTE THE INTEGRATING FACTOR IS $u(t) = \dfrac{1}{y_h(t)}$)

SO THE MATH IS EQUIVALENT TO USING AN INTEGRATING FACTOR
BUT THE STEPS ALONG THE WAY LOOK DIFFERENT

WE WILL SEE THAT THE VARIATIONS OF PARAMETERS IS PARTICULARLY
EASY IN SOME CASES

EXAMPLE

$$y' - 2y = t^2 e^{2t}$$

NOTE $a(t) =$ A CONSTANT — 1^{ST} ORDER LINEAR W/ CONSTANT COEFFICIENTS

(NOT NEEDED FOR VARIATION OF PARAMETERS BUT MAKES GOOD EXAMPLE)

HOMOG. EQ. IS $y_h' = 2y_h \Rightarrow \frac{1}{y_h}\frac{dy_h}{dt} = 2 \quad \ln|y_h| = 2t \quad y_h = Ae^{2t}$ (A > 0, BUT TRY A < 0 → WORKS TOO)

JUST LET A = 1

NOW ASSUME $y(t) = v(t)y_h(t) = v(t)e^{2t}$

NOTE $y'(t) = 2v(t)e^{2t} + v'(t)e^{2t}$

NOW SUBST INTO ORIGINAL D.E.

$$\left(2v(t)e^{2t} + v'(t)e^{2t}\right) - 2\left(v(t)e^{2t}\right) = t^2 e^{2t}$$

$$v'(t) = t^2$$

$$v(t) = \tfrac{1}{3}t^3$$

$$\boxed{\therefore\ y(t) = \tfrac{1}{3}t^3 e^{2t}}$$

CHECK

$$y'(t) = \tfrac{2}{3}t^3 e^{2t} + t^2 e^{2t}$$

SUBST INTO ORIGINAL D.E.

$$\left(\tfrac{2}{3}t^3 e^{2t} + t^2 e^{2t}\right) - 2\left(\tfrac{1}{3}t^3 e^{2t}\right) \stackrel{?}{=} t^2 e^{2t}$$

YES Q.E.D.

EXAMPLE

$$y' + \frac{1}{t}y = 3\cos 2t \quad , \quad t>0$$

VARIATION OF PARAMETERS ALSO WORKS W/ NON-CONSTANT COEF.

HOMOG. EQ. IS $y_h' + \frac{1}{t}y_h = 0$ OR $\frac{1}{y_h}y_h' = -\frac{1}{t}$

$$\ln|y_h| = -\ln|t| + c \Rightarrow y_h = -\frac{1}{t}$$

SIGN NOT IMPORTANT

ASSUME $y(t) = v(t)/t$

NOTE $y'(t) = \frac{v'(t)}{t} - \frac{v(t)}{2t^2}$

SUBST INTO ORIGINAL D.E.

PG 55 (Ex 25)

USE FOR CLASSROOM EXAMPLE

USE THE TECHNIQUE OF Ex 22 (Pg 55). FIND THE GENERAL SOL^n

$xy' + y = x^4 y^3$ (y IS A FUNCTION OF x)

$y' + \frac{1}{x} y = x^3 y^3$

$y' = -\frac{1}{x} y + x^3 y^3$ OBSERVE $n = 3$ (COMPARE TO THE BERNOULLI EQ^n IN Ex 22)

LET $z = y^{(1-n)} = y^{-2}$ THEN $z' = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = -2y^{-3}\frac{dy}{dx} = -2y^{-3}y'$

$\therefore$ MULT THE EQ THRU BY $-2y^{-3}$

$-2y^{-3}y' = 2y^{-3}\frac{1}{x}y - 2y^{-3}x^3y^3$

$z' = 2\frac{1}{x}y^{-2} - 2x^3$

$z' = 2\frac{1}{x}z - 2x^3$

$z' - 2\frac{1}{x}z = -2x^3$ COMPARE TO $z' - a(x)z = f(x)$

IGNORE, $C_1 = 0$

AN INTEGRATING FACTOR IS $u(x) = e^{-\int a(x)dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\ln(x) + C_1} = \frac{1}{x^2}$

$\frac{1}{x^2}z' - 2\frac{1}{x^3}z = -2x$

$\left(\frac{1}{x^2}z\right)' = -2x$

$\frac{z}{x^2} = -2\int x\,dx = -2\left(x^2 + C_2\right) = -x^2 + C$ $\quad C = -2C_2$

$z = -x^4 + Cx^2$

NOW SUBSTITUTE FOR z TO PUT IN TERMS OF y

$\frac{1}{y^2} = -x^4 + Cx^2$

$y = \frac{\pm 1}{\sqrt{Cx^2 - x^4}}$

BERNOULLI EQ EXAMPLE

$y' = ry - ky^2$ IS IMPORTANT IN POPULATION DYNAMICS

COMPARE TO $y' = a(t)y + f(t)y^n$, y A FUNCTION OF t

$n = 2$ $\therefore$ $z = y^{1-n} = \frac{1}{y}$ THEN $\frac{dz}{dt} = \frac{dz}{dy}\frac{dy}{dt} = -y^{-2}y'$

SO MULTIPLY THRU BY $-y^{-2}$

$$-y^{-2}y' = -\frac{r}{y} - k$$

$$z' = -rz - k$$

$z' + rz = -k$ AN INTEGRATING FACTOR IS $e^{-\int r\,dt} = e^{-rt}$

$$e^{-rt}z' + re^{-rt}z = -ke^{-rt}$$

$$\left(e^{-rt}z\right)' = -ke^{-rt}$$

$$e^{-rt}z = -k\int e^{-rt} = \frac{k}{r}e^{-rt} + C$$

$$z = \frac{k}{r} + Ce^{rt}$$

NOW SUBST FOR z TO GET BACK IN TERMS OF y

$$\frac{1}{y} = \frac{k}{r} + Ce^{rt}$$

$$y = \frac{1}{\frac{k}{r} + Ce^{rt}}$$

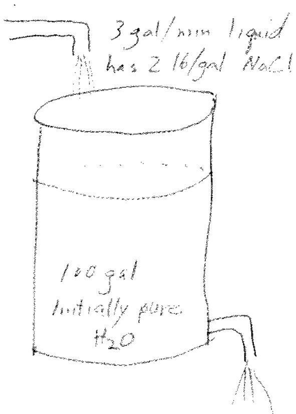
2.5 MIXING PROBLEMS — VERY PRACTICAL

A GENERAL ASSUMPTION — COMPLETE MIXING, SAY LIKE SOAP IN A WASHING MACHINE

SOME GENERAL IDEAS: LET $x(t)$ = THE AMOUNT (IN LBS eg) OF SOMETHING IN THE TANK

THEN $x' = \frac{dx}{dt}$ IS THE RATE OF CHANGE

eg



← This is a "concentration"

rate of change = (rate in) − (rate out)

rate in = (volume rate)(concentration)

$= (3\ gal/min)(2\ lb/gal) = 6\ lb/min$

rate out = (volume rate)(concentration)

$= (3\ gal/min)(c(t))$

NOW LET $x(t)$ = AMOUNT OF SALT IN TANK (lb)

$c(t)$ = CONCENTRATION OF SALT IN TANK $= \frac{x(t)}{100 gal}$

$$x' = 6 - 3\left(\frac{x}{100}\right)$$

$$x' + \frac{3}{100}x = 6 \quad \longleftarrow \text{A LINEAR D.E.}, \quad a(t) = \frac{-3}{100}$$

$$\text{INT. FACTOR} = u(t) = e^{-\int a(t)dt} = e^{3t/100}$$

$$e^{3t/100}x' + \frac{3}{100}e^{3t/100}x = 6e^{3t/100}$$

$$\left(e^{3t/100}x\right)' = 6e^{3t/100}$$

$$e^{3t/100}x = 6\int e^{3t/100}dt = 200e^{3t/100} + C$$

$$x = 200 + Ce^{-3t/100}$$

ALSO NOTE $x(0) = 0$ (PURE H_2O IN TANK AT $t=0$)

$\therefore C = -200$

$$\boxed{x = 200\left(1 - e^{-3t/100}\right)}$$

EXAMPLE 5.3

GIVEN
- 600 gal TANK HAS 300 gal PURE H_2O IN IT — TANK IS WELL MIXED
- SOLn FLOWS IN AT 3 gal/min, HAS $\frac{3}{2}$ lb/gal NaCl DISSOLVED IN IT
- TANK DRAINS AT 1 gal/min
- HOW MUCH SALT (IN lbs) WILL BE IN THE TANK WHEN FILLED TO 600 gal?

NOTE: VOLUME IS NOT CONSTANT: VOLUME INCREASES $3-1=2$ gal/min

$$V(t) = 300 + 2t \quad (t \text{ IN MINUTES}, \ V \text{ in gal})$$

WRITE THE "USUAL EQ."

$$\text{rate of change} = (\text{rate in}) - (\text{rate out})$$

LET $x(t)$ = AMOUNT OF SALT IN TANK (IN lbs)

$$\text{rate in} = \left(3 \ \tfrac{gal}{min}\right)\left(\tfrac{3}{2} \ \tfrac{lb}{gal}\right) = \tfrac{9}{2} \ \tfrac{lb}{min}$$

$$\text{rate out} = (\text{volume flow rate})(\text{concentration}) = \left(1 \ \tfrac{gal}{min}\right)\left(\tfrac{x}{V}\right)$$

$$x' = \frac{9}{2} - \frac{x}{300+2t}$$

$$x' + \frac{x}{300+2t} = \frac{9}{2} \qquad a(t) = \frac{-1}{300+2t} \qquad \int a(t)\,dt = \frac{-1}{2}\int \frac{1}{t+150}\,dt = -\frac{1}{2}\ln|t+150|$$

$$u(t) = e^{-\int a(t)dt} = e^{(\ln|t+150|)\frac{1}{2}} = (t+150)^{\frac{1}{2}} = \sqrt{t+150}$$

$t > 0$, (ABS VALUE NOT NEEDED)

$$(t+150)^{\frac{1}{2}} x' + \frac{1}{2}\frac{x}{(t+150)^{1/2}} = \frac{9}{2}(t+150)^{\frac{1}{2}}$$

$$\left[(t+150)^{\frac{1}{2}} x\right]' = \frac{9}{2}(t+150)^{\frac{1}{2}}$$

$$(t+150)^{\frac{1}{2}} x = \frac{9}{2}\int (t+150)^{\frac{1}{2}} = \frac{9}{2}\left[\frac{2}{3}(t+150)^{\frac{3}{2}} + C_1\right] = 3(t+150)^{\frac{3}{2}} + C$$

$$x(t) = 3(t+150) + \frac{C}{\sqrt{t+150}} = 450 + 3t + \frac{C}{\sqrt{t+150}} \longleftarrow \text{GEN'L SOLUTION}$$

INITIAL CONDITION: $x(0) = 0$

$$0 = 450 + 3(0) + \frac{C}{\sqrt{0+150}} \quad \therefore \ C = -450\sqrt{150} = -450\sqrt{\frac{300}{2}} = -4500\sqrt{\frac{3}{2}}$$

$$x(t) = 450 + 3t - \frac{4500\sqrt{3}}{\sqrt{2t+300}}$$

CONTINUES...

EXAMPLE 5.3 CONTINUED

$$v(t) = 300 + 2t = 600$$
$$2t = 300$$
$$t = 150 \quad \text{WHEN TANK IS FULL (600 gal)}$$

$$x(600) = 450 + 3(150) - \frac{4500\sqrt{3}}{\sqrt{2(150)+300}} = 581.8 \text{ lb}$$

2.6 EXACT DIFFERENTIAL EQUATIONS

REVIEW: PARTIAL DERIVATIVES OF FUNCTIONS OF MORE THAN ONE VARIABLE

WE DEFINE $y' \triangleq \frac{dy}{dx} \triangleq \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$

WHAT IF WE HAVE A FUNCTION OF TWO VARIABLES?

eg $F(x,y) = (3x+y)^2$

THEN WE DEFINE <u>PARTIAL</u> DERIVATIVES

1ST HOLD x CONSTANT $\frac{\partial F}{\partial y} \triangleq \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}$ $\qquad \frac{\partial F}{\partial y} = 2(3x+y)$

SIMILARLY HOLD y CONSTANT $\frac{\partial F}{\partial x} \triangleq \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y) - f(x,y)}{\Delta x}$ $\qquad \frac{\partial F}{\partial x} = 2(3x+y)3 = 6(3x+y)$

WHAT IF x AND y CHANGE SIMULTANEOUSLY?

THEN WE DEFINE THE DIFFERENTIAL (NOT SLOPE) OF THE FUNCTION IN DIRECTION (dx, dy)

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

WHERE dx AND dy ARE UNDERSTOOD AS SMALL CHANGES IN x AND y AND dF IS THE RESULTING SMALL CHANGE IN F

THE BASIC IDEA:

NOW CONSIDER THIS DE.: $3(3x+y) + (3x+y)y' = 0$ WHERE y IS A FUNCTION OF x

$$3(3x+y) + (3x+y)\frac{dy}{dx} = 0$$

$$3(3x+y)dx + (3x+y)dy = 0$$

LOOKS LIKE A DIFFERENTIAL $\rightarrow$ $6(3x+y)dx + 2(3x+y)dy = 0$

NOW LET $F(x,y) = (3x+y)^2$ THEN

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

$dF = 0$ NOW INTEGRATE

$$F(x,y) = 0 + C$$

$(3x+y)^2 = C$ IS AN IMPLICIT SOLUTION

NOW GENERALIZE THIS PROCESS

GIVEN: A D.E. OF THE FORM $P(x,y) + Q(x,y)y' = 0$

WE CHECK TO SEE IF THERE EXISTS A FUNCTION $F(x,y)$ SUCH THAT

$$\frac{\partial F}{\partial x} = P(x,y) \quad \text{AND} \quad \frac{\partial F}{\partial y} = Q(x,y)$$

THM 6.20 THIS WILL BE THE CASE IF $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

PROOF SUPPOSE THE EQUATION IS EXACT THEN THERE EXISTS $F(x,y)$ SUCH THAT $dF = 0$

AND $\frac{\partial F}{\partial x} = P(x,y)$ $\frac{\partial F}{\partial y} = Q(x,y)$

NOW FIND MORE DERIVATIVES

$$\frac{\partial P}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} \quad \text{AND} \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial y \partial x} \quad \therefore \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

ASSUMING THE EQ DOES TURN OUT TO BE EXACT, WE FIND $F(x,y)$

THERE ARE TWO WAYS TO GO:

$$\frac{\partial F}{\partial x} = P(x,y) \qquad - \text{OR} - \qquad \frac{\partial F}{\partial y} = Q(x,y)$$

$$F(x,y) = \int P(x,y)\,dx + \phi_y(y) \qquad F(x,y) = \int Q(x,y)\,dy + \phi_x(x)$$

FOLLOW THIS ONE

DIFF WRT y

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int P(x,y)\,dx\right) + \phi_y'(y) \qquad \text{BUT ALSO} \quad \frac{\partial F}{\partial y} = Q(x,y)$$

$$Q(x,y) = \frac{\partial}{\partial y}\left(\int P(x,y)\,dx\right) + \phi_y'(y)$$

$$\phi_y'(y) = Q(x,y) - \frac{\partial}{\partial y}\int P(x,y)\,dx$$

REV ORDER OF INT, DIFF

$$\phi_y'(y) = Q(x,y) - \int \frac{\partial P}{\partial y}\,dx$$, THUS FIND $\phi_y'(y) \longrightarrow F(x,y)$

THE IMPLICIT SOL'N IS $F(x,y) = C$

EXAMPLE: $3(3x+y) + (3x+y)y' = 0$ (PRETEND WE DON'T KNOW SOL$^{\underline{n}}$)

MANIPULATE TO IDENTIFY P AND Q

$$\underbrace{3(3x+y)}_{P}dx + \underbrace{(3x+y)}_{Q}dy = 0$$

TEST FOR EXACTNESS $\frac{\partial P}{\partial y} = 3$ $\quad \frac{\partial Q}{\partial x} = 3$ EQUAL ∴ THIS IS AN EXACT D.E.

$$F(x,y) = \int P(x,y)\,dx + \phi(y)$$

$$= \int 3(3x+y)\,dx + \phi(y)$$

SET = 0 → ONLY NEED ONE SOL$^{\underline{n}}$

$$= \frac{9}{2}x^2 + 3xy + C_1 + \phi(y)$$

NOW LOOK FOR A $\phi(y)$ SUCH THAT $\frac{\partial F(x,y)}{\partial y} = Q(x,y)$

$$\frac{\partial F}{\partial y} = 3x + \phi'(y) = 3x+y$$

SET = 0

$$\therefore \phi'(y) = y \Rightarrow \phi(y) = \frac{1}{2}y^2 + C_2$$

$$\therefore F(x,y) = \frac{9}{2}x^2 + 3xy + \frac{1}{2}y^2 = \frac{1}{2}(9x^2 + 6xy + y^2) = \frac{1}{2}(3x+y)^2$$

∴ A SOLUTION IS $\frac{1}{2}(3x+y)^2 = C_3$

OR $\boxed{(3x+y)^2 = C}$ ← IMPLICIT GEN'L SOL$^{\underline{n}}$

$\boxed{y = \pm\sqrt{C} - 3x}$ ← GEN'L SOL$^{\underline{n}}$

SUMMARY IF AN EXACT D.E.

$$F(x,y) = \int P(x,y)\,dx + \phi_y(y) \quad \text{OR} \quad F(x,y) = \int Q(x,y)\,dy + \phi_x(x)$$

TO FIND $\phi_y(y)$ NOTE (THE DEF$^{\underline{n}}$ OF EXACT) $\frac{\partial F(x,y)}{\partial y} = Q(x,y)$ OR $\frac{\partial F(x,y)}{\partial x} = P(x,y)$

THIS LEADS TO A D.E. → SOLVE → GET ϕ

ASSEMBLE GEN'L IMPLICIT SOL$^{\underline{n}}$ $F(x,y) = 0$

SOLVE FOR y IF POSSIBLE