B-5-1

B–5–1. A thermometer requires 1 min to indicate 98% of the response to a step input. Assuming the thermometer to be a first-order system, find the time constant.

If the thermometer is placed in a bath, the temperature of which is changing linearly at a rate of 10°/min, how much error does the thermometer show?

$\frac{5}{5}$

FIRST ORDER SYSTEM $\longrightarrow$ $T(s) = \frac{1}{Ts+1}$ (EQ 5-1)

$$c(t) = \left(1 - e^{-t/T}\right)u(t) \quad \text{(EQ. 5-3)}$$

$$0.98 = 1 - e^{-60/T} \quad (60 \text{ sec} = 1 \text{ min})$$

$$-0.02 = -e^{-60/T}$$

$$\ln(0.02) = -\frac{60}{T}$$

$$T = -\frac{60}{\ln(0.02)} = \frac{-60}{-3.912} = 15.34 \text{ s}$$

STEADY-STATE ERROR FOR A UNIT RAMP:

THIS IS A TYPE ONE SYSTEM, SEE FIG. 5-1 p 221.

$e_{ss} = \frac{1}{K_v}$ WHERE $K_v = \lim_{s \to 0} sG(s)$ (see text p 290)

$$K_v = \lim_{s \to 0} \frac{s}{Ts+1} = \frac{1}{T} = \frac{1}{15.34\, s} \quad (s = \text{seconds} = \text{units})$$

BUT THE INPUT IS NOT A UNIT RAMP $\quad r(t) = \left(\frac{10^\circ}{60\,s}\right)t = \frac{1}{6}t$
IT IS 1/6 THE SLOPE OF A UNIT RAMP.
∴ THE ERROR WILL BE $\frac{1}{6}$ OF THAT OF A UNIT RAMP

$$e_{ss} = \frac{1}{6K_v} = \frac{15.34\,s}{6(s/°)} = 2.556^\circ$$

$$\boxed{e_{ss} = 2.556^\circ}$$

THERE ARE OTHER WAYS TO SOLVE THIS PROBLEM

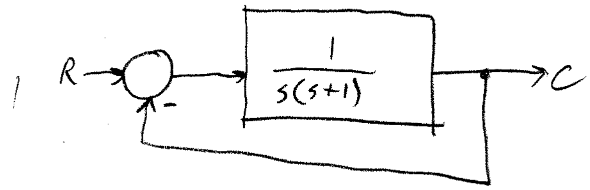
B-5-2

B–5–2. Consider the unit-step response of a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{1}{s(s+1)}$$

Obtain the rise time, peak time, maximum overshoot, and settling time.

UNITY GAIN NEG. FEEDBACK:



THE CLOSED LOOP TRANSFER FUNCTION IS

$$\frac{C}{R} = \frac{\frac{1}{s(s+1)}}{1 + \frac{1}{s(s+1)}} = \frac{1}{s^2+s+1} = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$$

WHERE $\omega_n = 1 \quad \zeta = \frac{1}{2}$

RISE TIME $t_r = \frac{\pi - \beta}{\omega_d}$ (TEXT p 231 EQ. 5-19)

WHERE $\omega_d = \omega_n\sqrt{1-\zeta^2}$ TEXT p 231

$\beta = \tan^{-1}\left(\frac{\omega_d}{\alpha}\right)$ TEXT PAGE 231 FIG. AT BOTTOM OF PAGE

$\alpha = |\text{REAL PART OF ROOT}|$ OF $= \zeta\omega_n$

$\omega_d = \sqrt{1-\frac{1}{2}^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$\beta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \cos^{-1}\zeta$ (CLASS NOTES)

6/6

$$t_r = \frac{\pi - \cos^{-1}\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = 2.418 \text{ sec}$$

PEAK TIME $t_p = \frac{\pi}{\omega_d} = \frac{3.14159}{\left(\frac{\sqrt{3}}{2}\right)} = 3.628$ sec

MAX OVERSHOOT $M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} = e^{\frac{-\left(\frac{1}{2}\right)}{\sqrt{3}/2}\pi} = e^{-\pi/\sqrt{3}} = 0.1630$

$M_p = 0.1630$ OR 16.30%

SETTLING TIME $t_s = \frac{4}{\zeta\omega_n} = \frac{4}{\left(\frac{1}{2}\right)(1)} = 8$ sec (2% CRITERION)

$$\boxed{t_r = 2.418 \text{ sec} \quad t_p = 3.628 \text{ sec} \quad M_p = 0.1630 \text{ OR } 16.3\% \quad t_s = 8 \text{ sec}}$$

B-5-4

B–5–4. Figure 5–79 is a block diagram of a space-vehicle attitude-control system. Assuming the time constant T of the controller to be 3 sec and the ratio K/J to be $\frac{2}{9}$ rad^2/sec^2, find the damping ratio of the system.

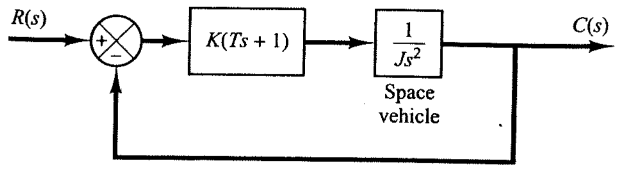


Figure 5–79
Space-vehicle attitude-control system.

$$\frac{C}{R} = \frac{K(Ts+1)\frac{1}{Js^2}}{1 + K(Ts+1)\frac{1}{Js^2}} = \frac{KTs + K}{Js^2 + KTs + K} = \frac{\left(\frac{KT}{J}\right)s + \left(\frac{K}{J}\right)}{s^2 + \left(\frac{KT}{J}\right)s + \left(\frac{K}{J}\right)}$$

$$\omega_n^2 = \frac{K}{J} = \frac{2}{9} \qquad 2\zeta\omega_n = \frac{KT}{J}$$

$$\omega_n = \frac{\sqrt{2}}{3}$$

4/4

$$2\zeta\frac{\sqrt{2}}{3} = \frac{2}{9}(3)$$

$$\zeta\frac{\sqrt{2}}{3} = \frac{1}{3}$$

$$\boxed{\zeta = \frac{1}{\sqrt{2}} = 0.707}$$

B-5-5

B–5–5. Consider the system shown in Figure 5–80. The system is initially at rest. Suppose that the cart is set into motion by an impulsive force whose strength is unity. Can it be stopped by another such impulsive force?

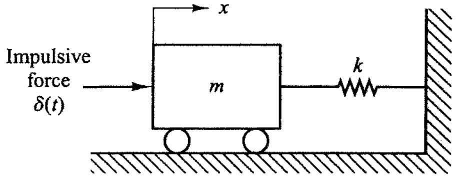


Figure 5–80
Mechanical system.

THE ASSIGNMENT IS TO PROVE THAT IT CAN BE STOPPED BY ADDING A DELAYED IMPULSIVE FORCE TO THE INPUT.

6/6

$\Sigma f = m\ddot{x}$

I WILL FIRST FIND THE IMPULSE RESPONSE

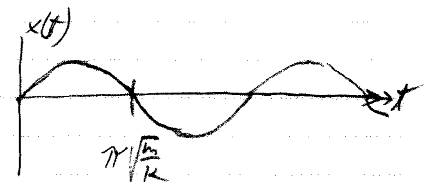
$\delta(t) - kx = m\ddot{x}$

$m\ddot{x} + kx = \delta(t)$

$X(s)\left(ms^2 + k\right) = 1$

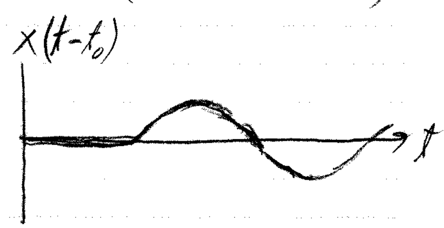
$X(s) = \dfrac{1}{ms^2+k} = \dfrac{\frac{1}{m}}{s^2+\frac{k}{m}} = \left(\dfrac{1}{\sqrt{mk}}\right)\dfrac{\sqrt{\frac{k}{m}}}{s^2+\frac{k}{m}}$

$x(t) = \dfrac{1}{\sqrt{mk}}\sin\left(\sqrt{\frac{k}{m}}\,t\right)u(t)$



SINCE THE SYSTEM IS TIME INVARIANT THE RESPONSE TO $\delta(t-t_0)$ IS $x(t-t_0)$

LET $t_0 = \pi\sqrt{\frac{m}{k}}$



$x\left(t-\pi\sqrt{\frac{m}{k}}\right) = \dfrac{1}{\sqrt{mk}}\sin\left[\sqrt{\frac{k}{m}}\left(t-\pi\sqrt{\frac{m}{k}}\right)\right]u\left(t-\pi\sqrt{\frac{m}{k}}\right)$

TRIG ID: $\sin(\theta-\pi) = -\sin\theta$

$x\left(t-\pi\sqrt{\frac{m}{k}}\right) = \dfrac{-1}{\sqrt{mk}}\sin\left(\sqrt{\frac{k}{m}}\,t\right)u\left(t-\pi\sqrt{\frac{m}{k}}\right)$

BY SUPERPOSITION THE RESPONSE TO $\delta(t)+\delta\left(t-\pi\sqrt{\frac{m}{k}}\right)$ IS $x(t)-x\left(t-\pi\sqrt{\frac{m}{k}}\right)$
AND FOR $t > \pi\sqrt{\frac{m}{k}}$ BOTH $u(t)$ AND $u\left(t-\pi\sqrt{m/k}\right) = 1$

$\therefore$ FOR $t > \pi\sqrt{\frac{m}{k}}$ THE TOTAL RESPONSE $= \dfrac{1}{\sqrt{mk}}\sin\sqrt{\frac{k}{m}}\,t - \dfrac{1}{\sqrt{mk}}\sin\sqrt{\frac{k}{m}}\,t = 0$

CERTAIN OTHER DELAYS, $\pi\sqrt{\frac{m}{k}} + \ell\pi$, ℓ AN INTEGER, ALSO WORK

Q.E.D.

B-5-6

B–5–6. Obtain the unit-impulse response and the unit-step response of a unity-feedback system whose open-loop transfer function is

$$G(s) = \frac{2s+1}{s^2}$$

$$\frac{C}{R} = \frac{\frac{2s+1}{s^2}}{1+\frac{2s+1}{s^2}} = \frac{2s+1}{s^2+2s+1} = \frac{2s+1}{(s+1)^2} = \frac{A}{(s+1)^2} + \frac{B}{s+1}$$

COVER-UP: LET $s=-1$ $A=-1$

$2s+1 = -1 + B(s+1)$ FROM THE s TERM $2 = B$

$$\frac{C}{R} = \frac{2}{s+1} - \frac{1}{(s+1)^2}$$

IMPULSE RESPONSE: $R(s)=1$ $H(s) = \frac{2}{s+1} - \frac{1}{(s+1)^2}$ ($H(s) = C(s)$ WHEN $R(s)=1$)

$$h(t) = \left(2e^{-t} - te^{-t}\right)u(t)$$

IMPULSE RESPONSE → $\boxed{h(t) = e^{-t}(2-t)u(t)}$

5/5

STEP RESPONSE, $R(s) = \frac{1}{s}$

$$C(s) = \frac{2s+1}{(s+1)^2 s} = \frac{A}{(s+1)^2} + \frac{B}{s+1} + \frac{D}{s}$$

COVER UP: LET $s=-1$ $A = \frac{2(-1)-1}{(-1)} = 1$, LET $s=0$ $D=1$

$2s+1 = As + B(s+1)s + D(s+1)^2$

$2s+1 = As + B(s^2+s) + D(s^2+2s+1)$ FROM THE s TERM $2 = (A+B+2D)$

$2 = 1+B+2$

$B=-1$

$$C(s) = \frac{1}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{s}$$

$\boxed{c(t) = \left(1 + te^{-t} - e^{-t}\right)u(t)}$ ← STEP RESPONSE

B-5-10

B–5–10. Referring to the system shown in Figure 5–84, determine the values of K and k such that the system has a damping ratio ζ of 0.7 and an undamped natural frequency ω_n of 4 rad/sec.

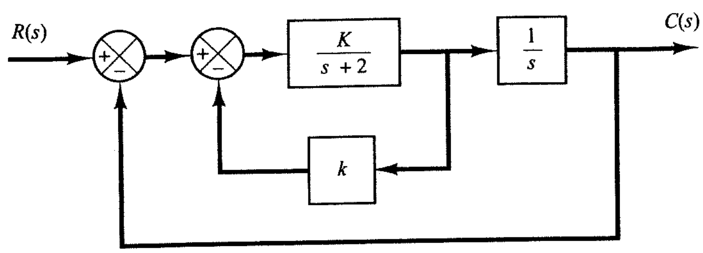


Figure 5–84
Closed-loop system.

MASON'S GAIN RULE: $P_1 = \dfrac{K}{s(s+2)}$ $\quad L_1 = \dfrac{-Kk}{(s+2)}$ $\quad L_2 = \dfrac{-K}{s(s+2)}$

$\Delta = 1 - L_1 - L_2 \qquad \Delta_1 = 1$

$$H(s) = \frac{P_1 \Delta_1}{\Delta} = \frac{\frac{K}{s(s+2)}}{1 + \frac{Kk}{(s+2)} + \frac{K}{s(s+2)}} = \frac{K}{s(s+2) + K(ks+1)}$$

$$H(s) = \frac{K}{s^2 + 2s + Kks + K} = \frac{K}{s^2 + (Kk+2)s + K} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\dfrac{5}{5}$

$\omega_n^2 = K \Rightarrow K = 4^2 = 16$

$2\zeta\omega_n = Kk + 2 \Rightarrow 2(0.7)(4) = 16k + 2$

$5.6 = 16k + 2$

$3.6 = 16k$

$k = \dfrac{3.6}{16} = 0.225$

$$\boxed{\begin{array}{l} K = 16 \\ k = 0.225 \end{array}}$$