Properties of the Q-function (also see your textbook, pages 199 – 201)

1.) $Q(-x) = 1 - Q(x)$

2.) $Q(0) = 1/2$ exactly

3.) $Q(\infty) = 0$

4.) If a Gaussian random variable X has a mean of $\overline{x}$ and a standard deviation of σ then $P(X > x) = Q\left(\frac{x-\overline{x}}{\sigma}\right)$

Proof of the last property:

$$P(X > x) = \int_x^\infty f_X(x)dx = \int_x^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\overline{x})^2}{2\sigma^2}} dx$$

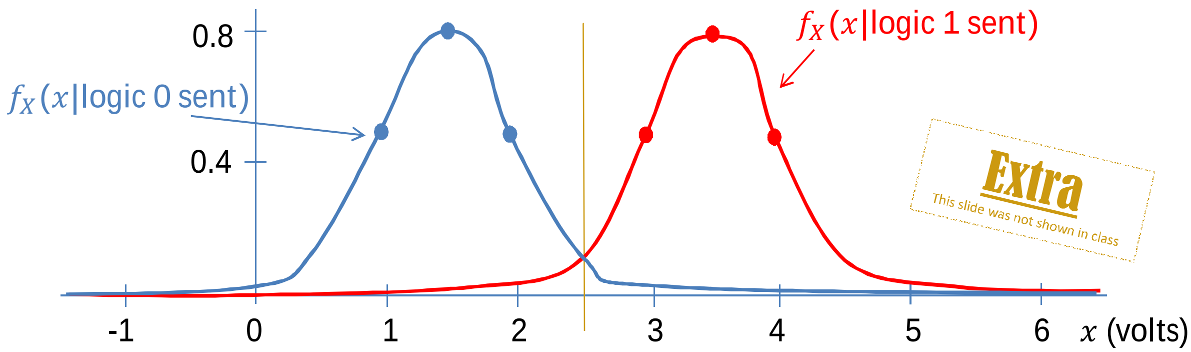
Change of variable: let $u = \frac{x-\overline{x}}{\sigma}$ then $du = \frac{dx}{\sigma}$ and $x = \sigma u + \overline{x}$

$$\int_x^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\overline{x})^2}{2\sigma^2}} dx = \int_{\frac{x-\overline{x}}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-(u)^2}{2}} du \triangleq Q\left(\frac{x-\overline{x}}{\sigma}\right)$$

Extra
This slide was not shown in class

Example: Binary data sent. Probability of sending a logic-1 is 50%, has voltage = 3.5 V
Probability of sending a logic-0 is 50%, has voltage = 1.5 V
By the time the signal gets to the receiver it has added Gaussian noise with voltage = 0.5 V rms (power = voltage squared = 1/4 W w.r.t. 1 Ω)

Change the means and the noise power.



Q: Suppose the logic threshold is at 2.5 V and a logic-1 is sent. What is the probability of an error?

$f_X(x|\text{logic 0 sent}) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\overline{x})^2}{2\sigma^2}}$ where $\overline{x} = 1.5$ and $\sigma = 0.5$ observe $\frac{1}{\sqrt{2\pi}\sigma} \cong \frac{0.4}{\sigma} = 0.8$

$f_X(x|\text{logic 1 sent}) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\overline{x})^2}{2\sigma^2}}$ where $\overline{x} = 3.5$ and $\sigma = 0.5$ observe $\frac{1}{4\sigma} = 0.5$

We can sketch the plots above based on that data.

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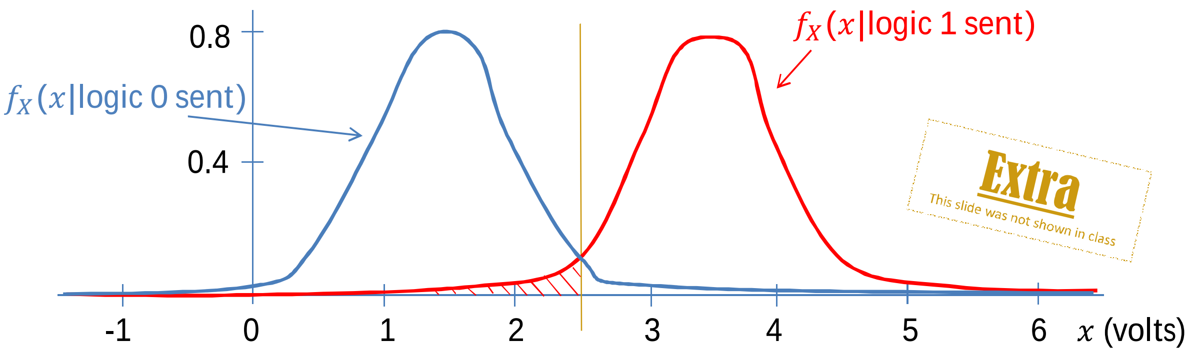

Q: Suppose the logic threshold is at 2.5 V and a logic-1 is sent. What is the probability of an error?

$f_X(x|\text{logic 1 sent}) = \frac{1}{\sqrt{2\pi}(0.5)} e^{\frac{-(x-3.5)^2}{2(0.5)^2}}$ And $P(\text{error})$ = area shaded in red above

$$P(\text{error}) = \int_{-\infty}^{2.5} f_X(x|\text{logic 1 sent})dx = \int_{-\infty}^{2.5} \frac{1}{\sqrt{2\pi}(0.5)} e^{\frac{-(x-3.5)^2}{2(0.5)^2}} dx$$

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Example: Binary data sent. Probability of sending a logic-1 is 50%, has voltage = 3.5 V
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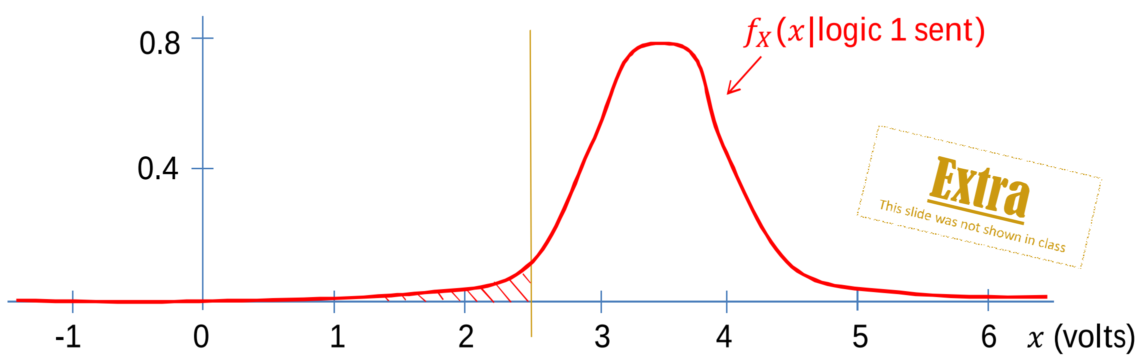
Change the means and the noise power.



$P(\text{error}) = \int_{-\infty}^{2.5} \frac{1}{\sqrt{2\pi}(0.5)} e^{\frac{-(x-3.5)^2}{2(0.5)^2}} dx$ and now do a change of variable. $u = \frac{x-\overline{x}}{\sigma}$ and $du = dx/\sigma$

$P(\text{error}) = \int_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}} e^{\frac{-(u)^2}{2}} du = 1 - Q(-2) = Q(2) = 0.023$ if $x = 2.5$ then $u = \frac{2.5-3.5}{0.5} = -2$

ANSWER