Quiz 8 of Selected Topics of Mathematical Statistics Seminar

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Quiz

Let $(X_i)_{i=1}^n$ be i.i.d random variables with distribution $\mathcal{N}(\mu, 1)$, find $\lim\limits_{n \to \infty} \mathbb{P}(\sqrt{n}(\overline{X_n} - \mu) \leqslant x)$ where $\overline{X_n} = \frac{1}{n} \sum_{i=1}^n X_i$

First property of the characteristic function

Let φ_{X_i} be the characteristic function of the random variable X_i .We know that $X_i \sim \mathcal{N}(\mu, 1)$ we have :

$$\varphi_{X_i}(t) = \exp(\mu i t - \tfrac{t^2}{2})$$

let $S_n = \sum_{j=1}^{n} X_j$ we known that $\{X_i\}$ is a iid family of random variables, so with the property of the characteristic function and because $\{X_i\}$ are independents, we have:

$$\varphi_{S_n}(t) = \varphi_{\sum\limits_{j=1}^{n} X_j}(t) = \prod_{j=1}^{n} \varphi_{X_j}(t) = (\varphi_{X_1}(t))^n = \exp(in\mu t - \frac{nt^2}{2})$$

Second property of the characteristic function

We know also that for any (a, b) $\in \mathbb{R}^2$, $\varphi_{aX+b}(t) = \varphi_X(at)e^{itb}$. So with $a = \frac{1}{\sqrt{n}}$ and $b = \sqrt{n}\mu$ we can derive the characteristic function of the random variable $\sqrt{n}(\bar{X}_n - \mu) = aS_n - b$

$$\begin{aligned}
\varphi_{\sqrt{n}(\bar{X}_n-\mu)}(t) = \varphi_{aS_n-b}(t) &= \varphi_{S_n}(\frac{t}{\sqrt{n}})e^{-i\sqrt{n}\mu t} \\
&= exp(i\sqrt{n}\mu t - \frac{nt^2}{2n} - i\sqrt{n}\mu t) \\
&= e^{-\frac{t^2}{2}} = \phi(t)
\end{aligned}$$

where φ is the characteristic function of the reduced-centered normal distribution.

Conclusion

For any n $\in \mathbb{N}^*$

$$\varphi_{\sqrt{n}(\bar{X}-\mu)}(t) = \varphi(t)$$

hence for any t $\in \mathbb{R}$

$$\lim_{n \to \infty} \varphi_{\sqrt{n}(\overline{X}-\mu)}(t) = \varphi(t)$$

Thanks to theorem 17, for any t $\in \mathbb{R}$ we have :

$$\lim_{n \to \infty} \mathbb{P}(\sqrt{n}(\bar{X} - \mu) \leqslant x) = \Phi(x)$$

where Φ is the cdf of the reduced-centered normal distribution.