RAM Model

Memory

Infinite sequence of cells, contains w bits. Every cell has an address starting at 1

CPU

32 registers of width w bits. **Operations**

Set value to register (constant or from other register). Take two integers from other registers and store the result of; $a + b$, $a - b$, $a \cdot b$, a/b. Take two registers and compare them; $a < b$ $a = b$, $a > b$. Read and write from memory.

Definitions

An algorithm is a set of atomic operations. It's cost is the number of atomic operations. A word is a sequence of w bits

Definitions

Worst-case

Worst-case cost of an algorithm is the longest possible running time of input size n

Random

RANDOM(x, y) returns an integer between x and y chosen uniformly at random

Data Structure

Data Structure describes how data is stored in memory.

Dictionary Search

let n be register 1, and v be register 2
register $left \to 1, right \to 1$
while $left \leq right$
register $mid \to (left + right)/2$
if the memory cell at address $mid = v$ then
return yes
else if memory cell at address $mid > v$ then
$right = mid - 1$
else
$left = mid + 1$
return no

Worst-case time: $f_2(n) = 2 + 6\log_2 n$

Function Comparison

Big-O

We say that $f(n)$ grows asymptotically no faster than $g(n)$ if there is a constant $c_1 > 0$ such that $f(n) \leq c_1 \cdot g(n)$ and holds for all n at least a constant c_2. This is denoted by $f(n) = O(g(n))$.
$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ for some constant c

Example

$1000\log_2 n = O(n)$,
$n \neq O(10000\log_2 n)$
$\log_{b_1} n = O(\log_{b_2} n)$ for any constants $b_1 > 1$ and $b_2 > 1$. Therefore $f(n) = 2 + 6\log_2 n$ can be represented; $f(n) = O(\log n)$

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Big-Ω

If $g(n) = O(f(n))$, then $f(n) = \Omega(g(n))$ to indicate that $f(n)$ grows asymptotically no slower than $g(n)$. We say that $f(n)$ grows asymptotically no slower than $g(n)$ if $c_1 > 0$ such $f(n) \geq c_1 \cdot g(n)$ for $n > c_2$; denoted by $f(n) = \Omega(g(n))$

Big-Θ

If $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, then $f(n) = \Theta(g(n))$ to indicate that $f(n)$ grows asymptotically as fast as $g(n)$

Sort

Merge Sort

Divide the array into two parts, sort the individual arrays then combine the arrays together. $f(n) = O(n \log n)$.

This is the fastest sorting time possible (apart from $O(n \log \log n)$

Counting Sort

A set S of n integers and every integer is in the range [1, U]. (all integers are distinct)

Step 1: Let A be the array storing S. Create array B of length U. Set B to zero.

Step 2: For $i \in [1, n]$; Set x to A[i], Set B[x] = 1

Step 3: Clear A, For $x \in [1, U]$; If B[x] = 0 continue, otherwise append x to A

Analysis

Step 1 and 3 take O(U) time, while Step 2 O(n) time. Therefore running time is O(n + U) = O(U).

Data

LinkedList

Every node stores pointers to its succeeding and preceding nodes (if they exist). The first node is called the head and last called the tail. The space required for a linkedlist is $O(n)$ memory cells. Starting at the head node, the time to enumerate over all the integers is $O(n)$. Time for assertion and deletion is equal to $O(1)$

Stack

The stack has two operations; Push (Inserts a new element into the stack), Pop (Removes the most recently inserted element from the stack and returns it. Since a stack is just a linkedlist, push and pop use $O(1)$ time.

Queue

The queue has two operations; En-queue (Inserts a new element into the queue), De-queue (Removes the least recently used element from the queue and returns it). Since a queue is just a linkedlist, push and pop use $O(1)$ time.<Paste>

Dynamic Arrays

Naive Algorithm

insert(e): Increase n by 1, initial an array A' of length n, copy all n-1 of A to A', Set A'[n]=e, Destroy A.

This takes $O(n^2)$ time to do n insertions.

A Better Algorithm

insert(e): Append e to A and increase n by 1. If A is full; Create A' of length 2n, Copy A to A', Destroy A and replace with A'
This takes $O(n)$ time to do n insertions.

Hashing

The main idea of hashing is to divide the dataset S into a number m of disjoint subsets such that only one subset needs to be searched to answer any query.

Pre-processing

Create an array of linkedlist(L) from 1 to m and an array H of length m. Store the heads of L in H, for all $x \in S$; calculate hash value ($h(x)$), insert x into $L_{h(x)}$. We will always choose $m = O(n)$, so $O(n+m) = O(n)$

Querying

Query with value v, calculate the hash value $h(v)$, Look for v in $L_h(v)$. Query time: $O(\mid L_{h(v)} \mid)$

Hash Function

Pick a prime p; $p \geq m, p \geq$ any integer k. Choose α and β uniformly random from $1, \ldots, p-1$. Therefore: $h(k) = 1 + (((\alpha k + \beta) mod\, p) mod\, m)$

Any Possible Integer

The possible integers is finite under the RAM Model. Max: $2^w - 1$. Therefore p exists between $[2^w, w^{w+1}]$.

Timing

Space: $O(n)$, Preprocessing time: $O(n)$, Query time: $O(1)$ in expectation

Week 3 – Extra

When using 'Direction 1: Constant Finding' setting c_1, always set it to match the coefficent on the LHS so that you can cancel.
When trying to get a contradiction, try and isolate an $x \cdot c_1$ on the RHS, where $x \in Z$, such that an expression that contains n is $\leq x \cdot c_1$
Make judicious use of the max function when adding functions together If $f_1(n) + f_2(n) \leq c_1 \cdot g_1(n) + c_1' \cdot g_2(n) \leqslant max\{c_1, c_1'\} \cdot (g_1(n) + g_2(n))$, for all $n \geqslant max\{c_2, c_2'\}$.

The Master Theorem

Theorem 1

$n + \frac{n}{c} + \frac{n}{c^2} + \ldots + \frac{n}{c^h} = O(n)$

Theorem 2

Let $f(n)$ be a function that returns a positive value for every integer $n > 0$. We know:

$$f(1) \leqslant c_1$$
$$f(n) \leqslant \alpha \cdot f(\lceil n/\beta \rceil) + c_2 \cdot n^\gamma \text{ for } n \geqslant 2$$

where $\alpha, \beta, \gamma, c_1$ and c_2 are positive constants. Then:

- If $log_b \alpha < \gamma$ then $f(n) = O(n^\gamma)$
- If $log_b \alpha = \gamma$ then $f(n) = O(n^\gamma \cdot log(n))$
- If $log_b \alpha > \gamma$ then $f(n) = O(n^{log_\beta(a)})$

Hierarchy

$$O(1) \leqslant O(log(n)) \leqslant O(n^c)$$
$$\leqslant O(n) \leqslant O(n^2)$$
$$\leqslant O(n^c) \leqslant O(c^n)$$

Trees

Undirected Graphs

An undirected graph is a pair of (V, E) where:

- V is a set of elements, each of which called a node.
- E is a set of pairs (u, v) such that:
  - u and v are distinct nodes;
  - If (u, v) is in E, then (v, u) is also in E – we say that there is an edge between u and v.

A node may also be called a vertex. We will refer to V as the vertex set or the node set of the graph, and E the edge set.

Paths and Cycles

Let G = (V, E) be an undirected graph. A path in G is a sequence of nodes $(v_1, v_2, \ldots, v_k)$ such that

- For every $i \in [1, k-1]$, there is an edge between v_i and v_{i+1}.

A cycle in G is a path $(v_1, v_2, \ldots, v_k)$ such that; $k \geq 4$, $v_1 = v_k$, $v_1, v_2, \ldots, v_{k-1}$ are distinct

Connected Graphs

An undirected graph G = (V, E) is connected if, for any two distinct vertices u and v, G has a path from u to v.

Trees

A tree is a connected undirected graph contains no cycles.

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Rooting a Tree

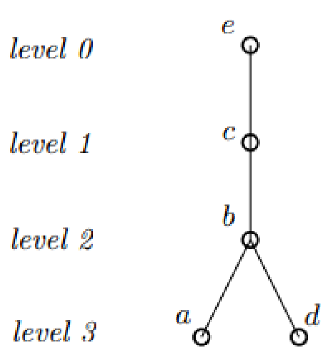
Given any tree T and an arbitrary node r, we can allocate a level to each node as follows:

- r is the root of T – this is level 0 of the tree.
- All the nodes that are 1 edge away from r constitute level 1 of the tree.
- All the nodes that are 2 edges away from r constitute level 2 of the tree.
- And so on.

The number of levels is called the height of the tree. We say that T has been rooted once a root has been designated.

Ancestors and Descendants

Let u and v be two nodes in T. u is an ancestor of v is one of the following holds: u = v, u is the parent of v, u is the parent of an ancestor of v. Accordingly, we say that v is a descendant of u. In particular, if $u \neq v$, we say that u is a proper ancestor of v, and likewise, v is a proper descendant of u.



Node b is an ancestor of b, a and d.
Node c is an ancestor of c, b, a and d.
Node c is a proper ancestor of b, a, d.

Subtrees

The subtree of u is the part of T that is “at or below” u.

Internal and Leaf Nodes

In a rooted tree, a node is a leaf node if it has no children; otherwise, it is an internal node.

k-Ary and Binary

A k-Ary tree is a rooted tree where every internal node has at most k child nodes. A 2-ary tree is called a binary tree.

Full Level

Consider a binary tree with height h. Its Level $l(0 \leq l \leq h-1)$ is full if it contains 2^l nodes.

Complete Binary Tree

A binary tree of height h is complete if:

- Levels 0, 1, …, h-2 are all full
- At Level h-1, the leaf nodes are “as far left as possible”. This means that if you were to add a leaf node v at Level h-1, v would need to be on the right of all the existing leaf nodes.

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Priority Queue

A priority queue stores a set S of n integers and supports the following operations:

- Insert(e): Adds a new integer to S.
- Delete-min: Removes the smallest integer in S, and returns it.

Next we will implement a priority queue using a data structure called the binary heap to achieve the following guarantees:

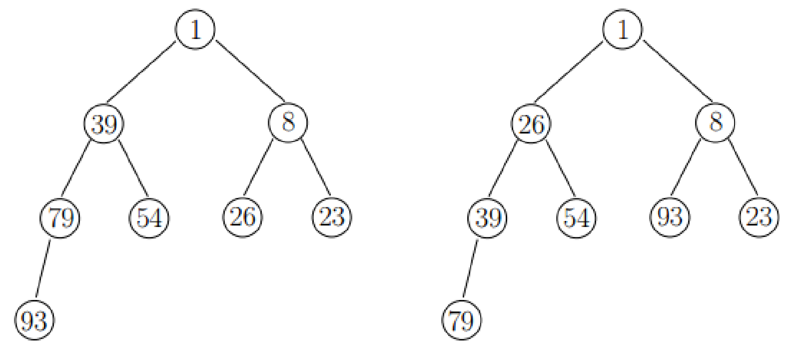
- O(n) space consumption
- O(log n) insertion time
- O(log n) delete-min time

Binary Heap

Let S be a set of n integers. A binary heap on S is a binary tree T satisfying:

- T is complete
- Every node u in T corresponds to a distinct integer in S – the integer is called the key of u (and is stored at u).
- If u is an internal node, the key of u is smaller than those of its child noeds.

Two possible binary heaps on S = {93, 39, 1, 26, 8, 23, 79, 54}:



The smallest integer of S must be the key of the root.

Insertion

We preform insert(e) on a binary heap T as follows:

1. Create a leaf node z with key e, while ensuring that T is a complete binary tree – notice that there is only one place where z can be added.
2. Set $u \to z$.
3. If u is the root, return.
4. If the key of $u >$ the key of its parent p, return.
5. Otherwise, swap the keys of u and p. Set $u \to p$, and repeat from Step 3.

Delete-Min

We perform a delete-min on a binary heap T as follows:

1. Report the key of the root.
2. Identify the rightmost leaf z at the bottom level of T.
3. Delete z, and store the key of z at the root.
4. Set $u \to$ the root.
5. If u is a leaf, return.
6. If the key of $u <$ the keys of the children of u, return.
7. Otherwise, let v be the child of u with a smaller key. Swap the keys of u and v. Set $u \to v$, and repeat from Step 5.

Dynamic Binary Heaps

Storing a Complete Binary Tree Using an Array

Let T be any complete binary tree with n nodes. Let us linearize the nodes in the following manner:

- Put nodes at a higher level before those at a lower level.
- Within the same level, order the nodes from left to right.

Let us store the linearized sequence of nodes in an array A of length n.

Property 1

Lemma: Suppose that node u of T is stored at A[i]. Then, the left child of u is stored at A[2i], and the right child at A[2i + 1].

Corollary: Suppose that node u of T is stored at A[i]. Then, the parent of u is stored at $A[\lfloor i/2 \rfloor]$.

Property 2

Lemma: The rightmost leaf node at the bottom level is stored at $A[n]$.

Performance Guarantees

- Space consumption O(n)
- Insertion: O(log n) time amortized
- Delete-min: O(log n) time amertized

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