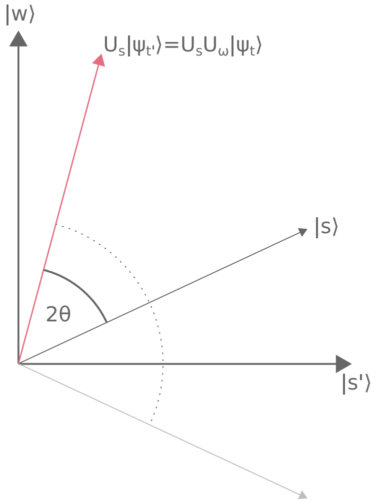
Grover's Algorithm

Hans Hohenfeld – June, 19th 2023



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Searching and Finding. . .

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1	Alice	0123/45678
2	Anna	9876/654321
3	Arthur	999/654456
⋮	⋮	

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What is **Mike**'s phone number?

No.	Name	Phone No.
1	Alice	0123/45678
2	Anna	9876/654321
3	Arthur	999/654456
⋮	⋮	
725	**Mike**	**555/789789**
⋮	⋮	

What is **Mike**'s phone number?

Assumptions

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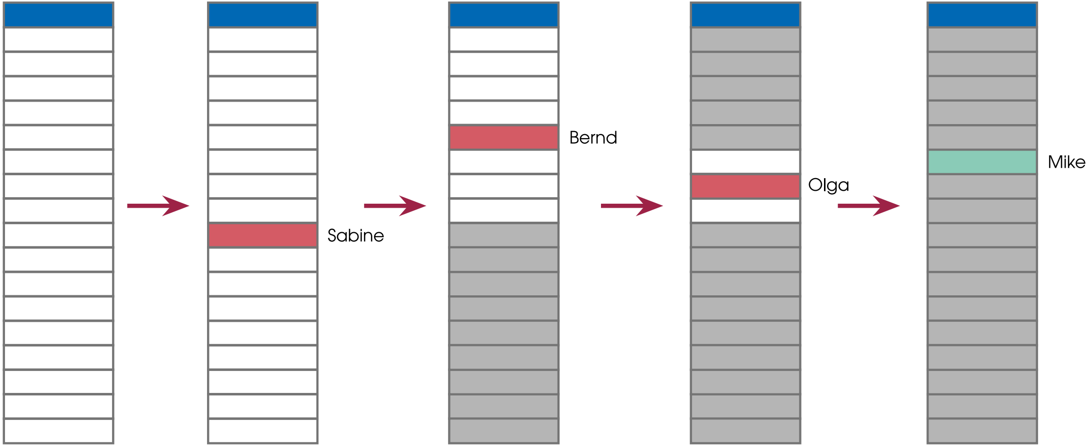
- The list is **sorted**.
- We know the list's **length**.
- We can **read** every individual line.
- Nothing else.

How do we find **Mike**'s phone number?

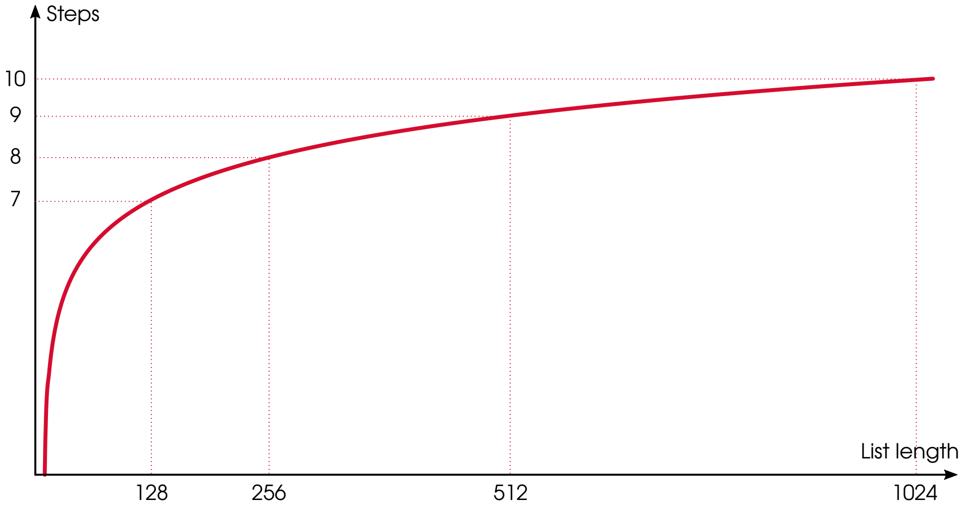
Binary Search



Time Complexity

Informally **time complexity** is a measure for how the computational effort in terms of time grows, relative to the problem size.

Time complexity of Binary Search



Time Complexity

More formally:

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The computational effort to find an element in a sorted list of length n does not grow faster than $\log(n)$ (up to constant factors).

Or even more formally...

Time Complexity

Binary Search: Time Complexity

Let $T_{\mathrm{BS}} : \mathbb{N} \to \mathbb{N}$ be the **time step** function of Binary Search. It holds:

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We write:

$$T_{\text{BS}} \in O(\log(n))$$

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We write:

$$T_{\text{BS}} \in O(\log(n))$$

We say: Binary Search has (time) complexity $O(\log(n))$.

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Big-O	Name	Problem

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$O(1)$	Constant	Median of a sorted list

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$O(c^n)$	Exponential	Chess, Go, ...

We change the problem a bit. . .

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Who has the phone number **1234/999888**?

This type of problem is called

Unstructured Search.

For reasons that become more clear later, we divide the problem between two instances.

The Seeker

- Has access to the list . . .

The Seeker

- Has access to the list ...
- Can read arbitrary entries in constant time ...

The Seeker

- Has access to the list . . .
- Can read arbitrary entries in constant time . . .
- Decides which entry to read next . . .

The Oracle

- Knows the answer ...

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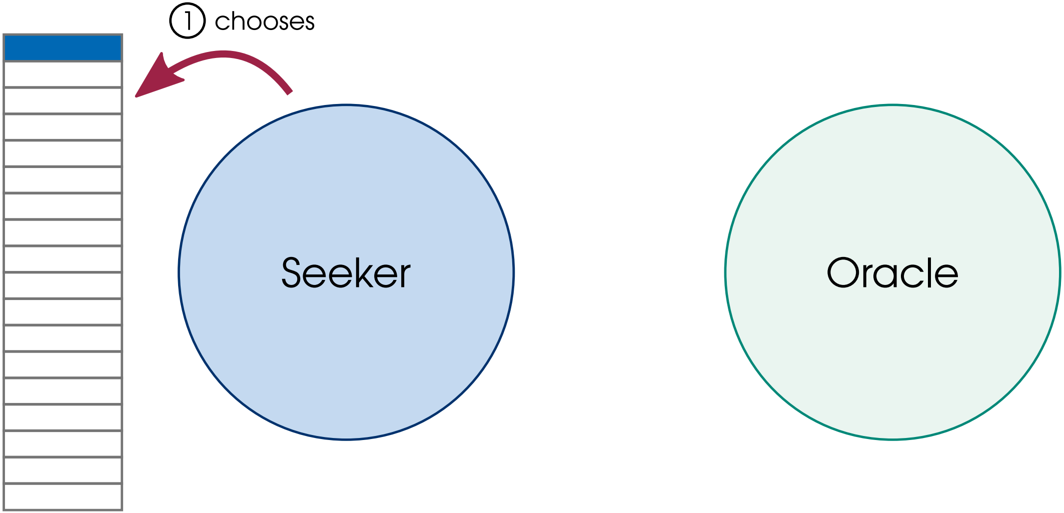
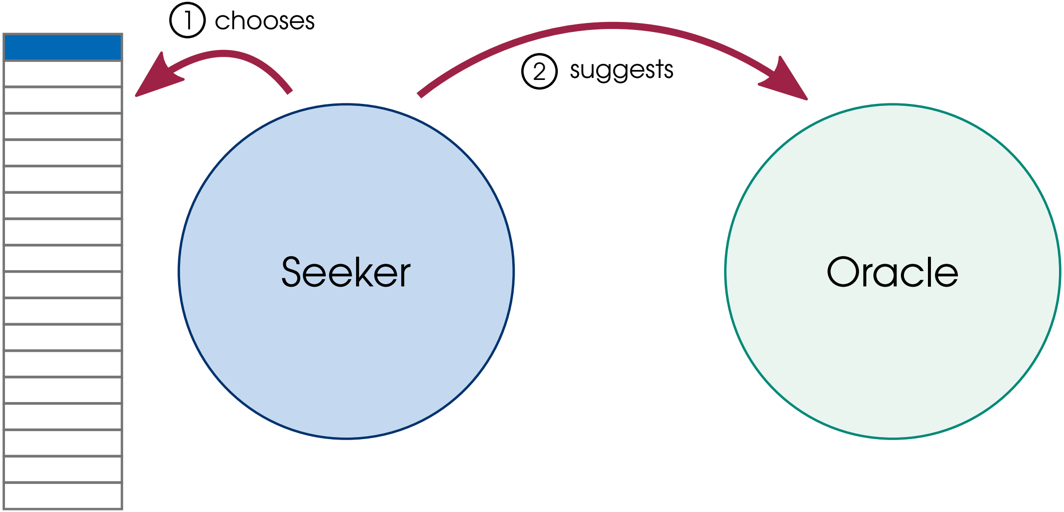
The Oracle

- Knows the answer ...
- Gets entries suggested by the seeker and answers with **yes/no** ...
- Takes constant time for an answer ...
- Is never wrong ...


1 chooses
Seeker
Oracle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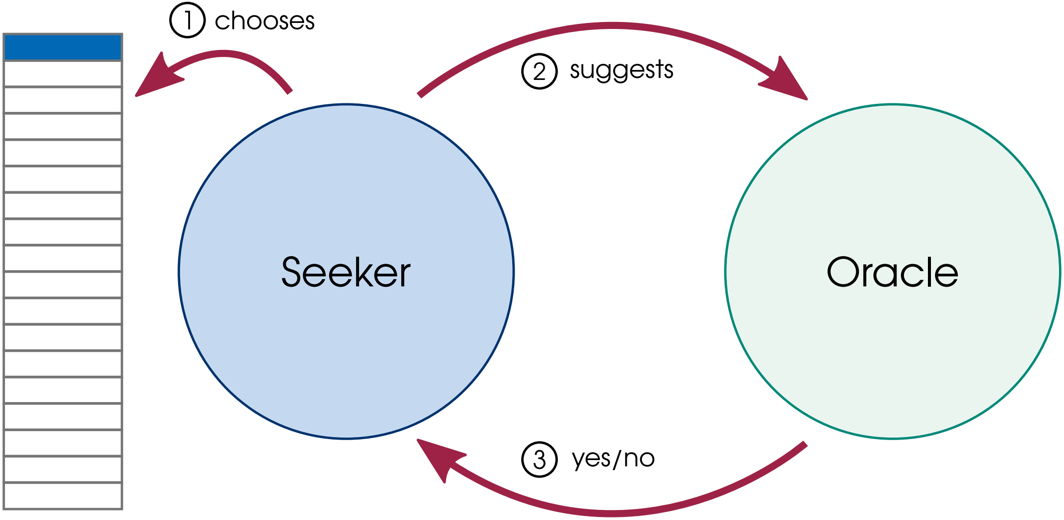
1 chooses
2 suggests
3 yes/no
Seeker
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How often does the Seeker need to query the Oracle?

Linear Search

- The only reasonable strategy is to query the Oracle for one entry after the other, starting from the first.

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- The only reasonable strategy is to query the Oracle for one entry after the other, starting from the first.
- For a list of length n this requires n queries in the worst case.
- On average, the search is successful after $\frac{n}{2}$ queries.
- **Important:** The Oracle does not change anything with regards to complexity.

Linear Search

Lineare Suche has (query/time) complexity of $O(n)$.

Questions?

Searching and Finding . . .

Searching and Finding . . . on Quantum Computers

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- Implemented for 3 qubits on a programmable quantum computer in 2017 [3].

Problem Statement

Given n qubits with $N = 2^n$ basis states

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$$
\begin{gathered}
|0\ldots0\rangle = |0\rangle \\
|0\ldots1\rangle = |1\rangle \\
\ldots \\
|1\ldots1\rangle = |N-1\rangle
\end{gathered}
$$

Problem Statement

One of the basis states is the one we are searching for. We will call this state:

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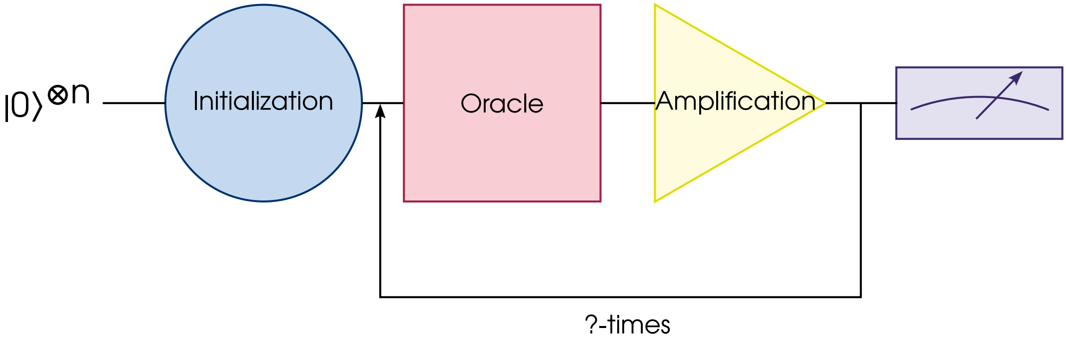
$$|\omega\rangle$$

Let's first look at a schematic of the algorithm...



$|0\rangle^{\otimes n}$
Initialization
Oracle
Amplification
?-times

Oracle and **Amplitude Amplification** are executed several times

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How often will be the result of the following discussion.

First the details of each component. . .

Initialization

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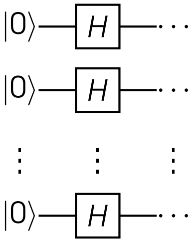

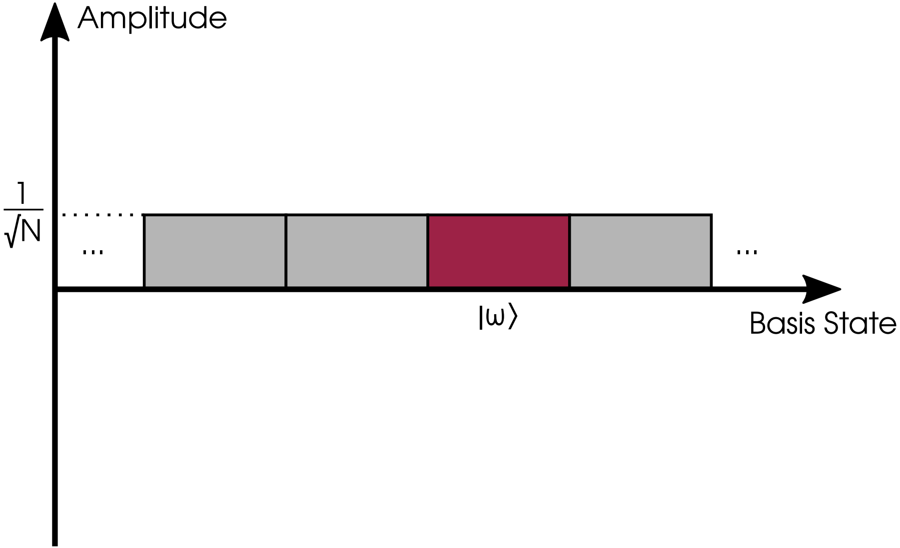
Amplitude
$\frac{1}{\sqrt{N}}$
...
...
$|\omega\rangle$
Basis State

After initializing the system, we have an equal probability of $\frac{1}{N}$ to measure either of the N basis states.

Oracel

The oracle U_ω marks the state $|\omega\rangle$ we are searching for. After applying the oracle, we want for each basis state:

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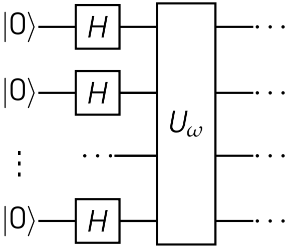

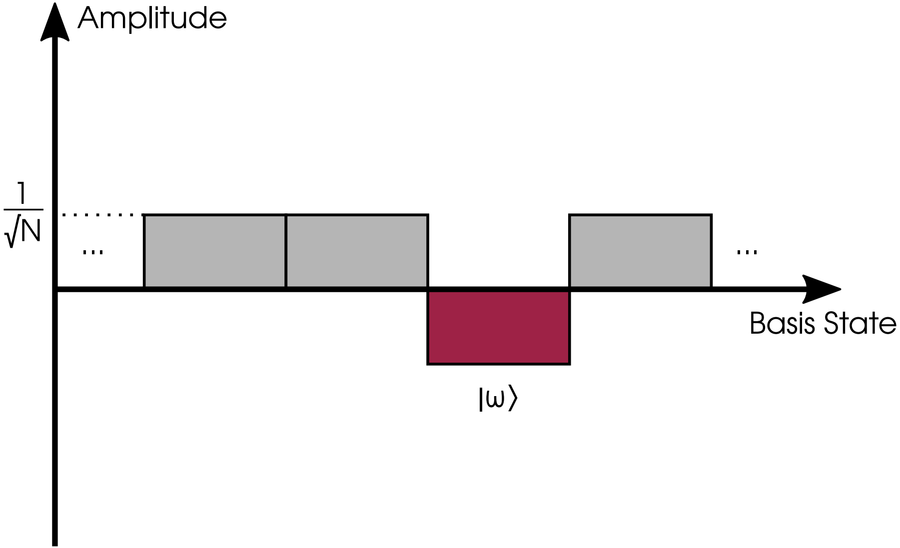
Amplitude
$\frac{1}{\sqrt{N}}$
...
...
Basis State
$|\omega\rangle$

Probabilities of measuring either basis state remain unchanged by this.
The oracle only inverted the phase of the state we are searching for!

Oracle

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- U_ω is a diagonal matrix.
- All but one entry on its diagonal are 1.
- In the line with index corresponding to the searched basis state, the entry is -1.

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Let $|N-1\rangle$ be the searched basis state. The corresponding oracle is then:

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$$U_\omega = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix}$$

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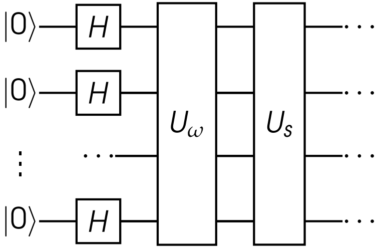
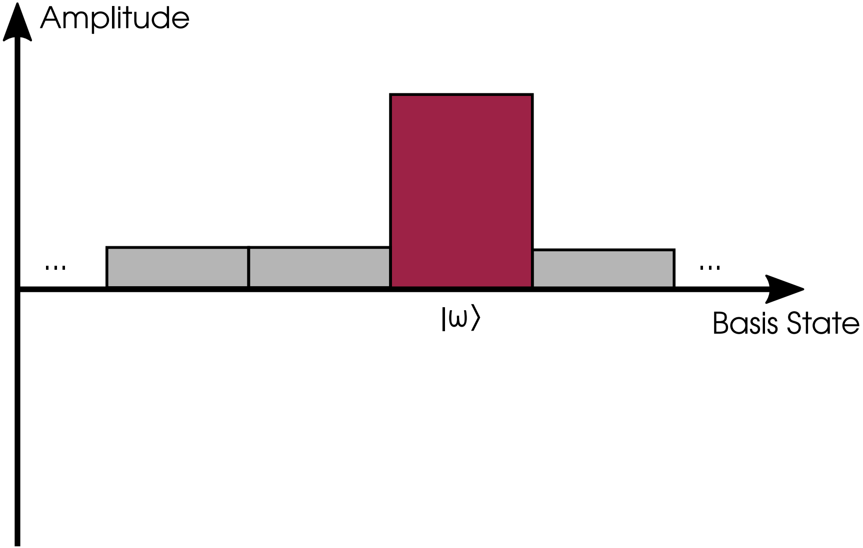
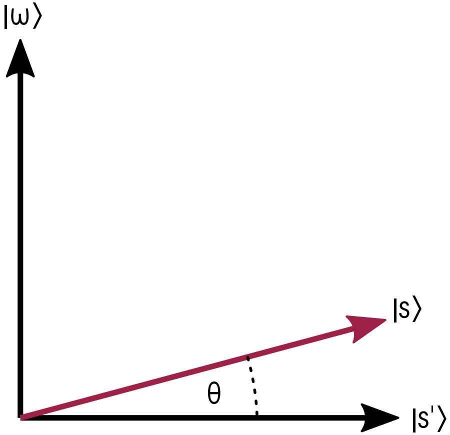
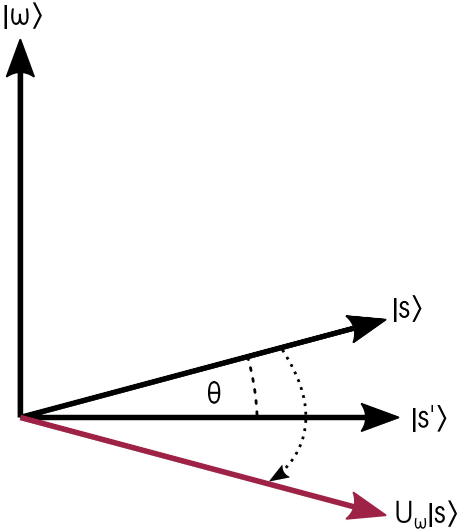
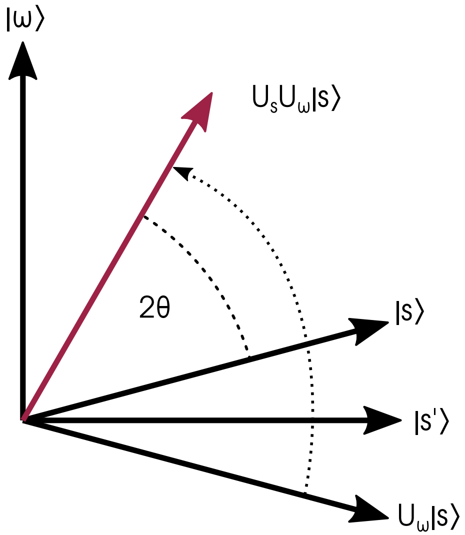



Amplitude
...
...
|ω⟩
Basis State

A geometric view on the algorithm…


|ω⟩
|s⟩
θ
|s'⟩


|ω⟩
|s⟩
θ
|s'⟩
U_ω|s⟩



$|\omega\rangle$
$U_sU_\omega|s\rangle$
2θ
$|s\rangle$
$|s'\rangle$
$U_\omega|s\rangle$

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The amplitude amplification step mirrors $U_\omega \left|s\right\rangle$ on the vector $\left|s\right\rangle$.The operator to achieve this is given by:

$$U_s = 2\left|s\right\rangle\left\langle s\right| - I.$$

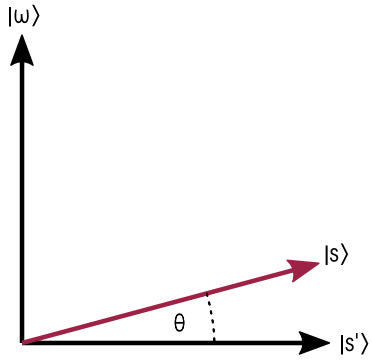
Grover's Algorithm: Complexity

How often do we have to apply the oracle and amplitude amplification?

Looking at θ gives an intuition.

Grover's Algorithm: Complexity

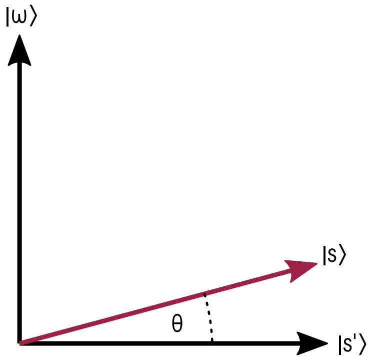
We have:



Grover's Algorithm: Complexity

We have:

$$|s\rangle = \sin\theta\,|\omega\rangle + \cos\theta\,|s'\rangle$$



Grover's Algorithm: Complexity

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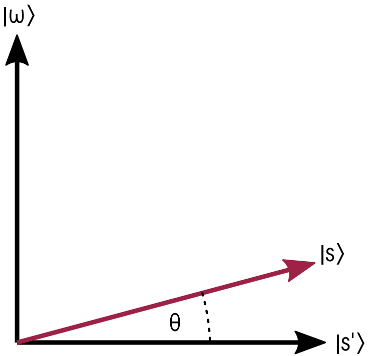
with

$$\theta = \arcsin\langle s|\omega\rangle = \frac{1}{\sqrt{N}}$$



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$$\frac{1}{\sqrt{N}}.$$

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The amplitude of the searched basis state increases linearly in

$$\frac{1}{\sqrt{N}}.$$

Hence, after approximately $\sqrt{N}$ steps, the probability of measuring the correct state is very close to 100%.

Grover's Algorithm: Complexity

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Grover's Algorithm: Complexity

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- Unstructured search, classical (Linear search): $O(n)$.

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- Unstructured search, quantum (Grover): $O(\sqrt{n})$.

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- Unstructured search, classical (Linear search): $O(n)$.
- Unstructured search, quantum (Grover): $O(\sqrt{n})$.

A quantum computer can solve this problem **quadratically faster**.

Questions?

Thank you.

Images

- Slide 29: Based on [3].
- Slides 34, 37, 42, 44-46 and title slide: Based on [1].

References

[1] Amira Abbas u. a. *Learn Quantum Computation Using Qiskit*. 2020. URL: http://community.qiskit.org/textbook.

[2] Charles H. Bennett u. a. „Strengths and Weaknesses of Quantum Computing". In: *SIAM Journal on Computing* 26.5 (1997), S. 1510–1523. DOI: 10.1137/S0097539796300933.

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[5] Michael A. Nielsen und Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge Univ Press, 9. Dez. 2010.

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