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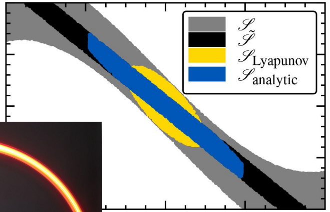
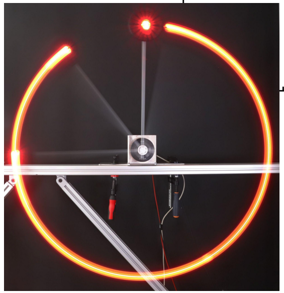




Analytic Estimation of Region of Attraction of an LQR Controller for Torque Limited Simple Pendulum

contributors:


Lukas Gross


Lasse Maywald


Shivesh Kumar


Frank Kirchner


Christoph Lüth

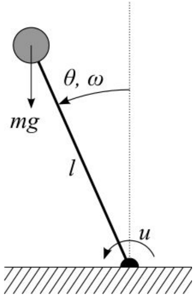
contact: lukas.gross@dfki.de

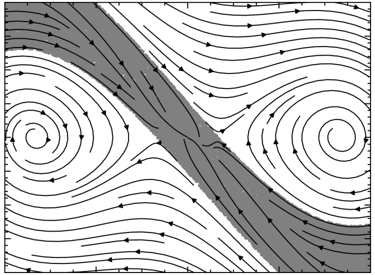
This work was done within the project VeryHuman funded by the Federal Ministry of Education and Resarch of Germany.

VeryHuman

Federal Ministry of Education and Research

Overview





Why a torque limited pendulum, why care about the ROA?

$$\dot{x} = \mathbf{A}x + \mathbf{B}u$$

LQR IVP

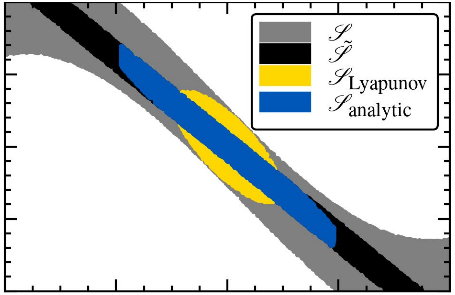
$$u(x) = -\mathbf{k}x \qquad x(t)$$

$$u(t)$$

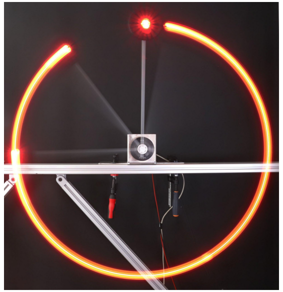
Can we come up with some analytic approximation?

$$\{\gamma \,|\, mgl\,|\sin(\theta) - \theta| \leq \bar{u}\}$$
$$\cap \{\gamma \,|\, |u_{\mathrm{lin},\gamma}(0)| \leq \bar{u}\}$$
$$\cap \{\gamma \,|\, t^*_\gamma > 0 \implies |u_{\mathrm{lin},\gamma}(t^*_\gamma)| \leq \bar{u}\}$$

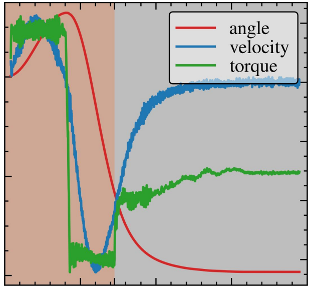
Turns out, the answer is yes!



How does it compare to other methods in simulation…



… and in experiment?

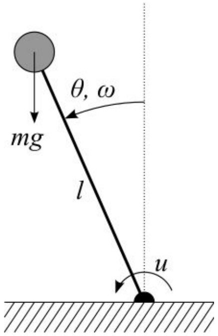


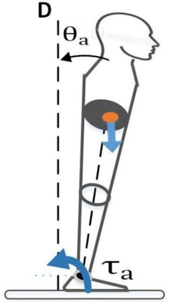
And more experiments!

[Eßer et al., 2020]

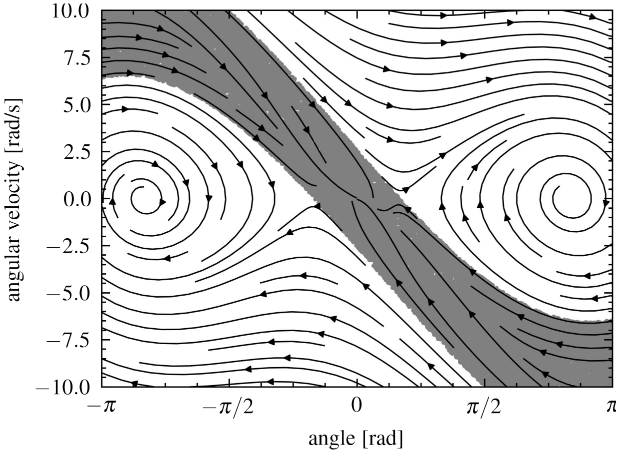
Challenges ahead?

Why a Torque Limited Pendulum?
Why Care About Region of Attraction?





[Fotuhi et al., 2020]



Torque Limited Pendulum:

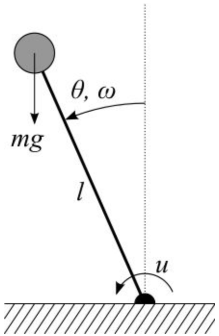
- easy to understand toy problem to study control
- nonlinear and underactuated, if torque is limited enough
- good approximation for a human standing upright (torque limitation due to finite foot length)

Region of Attraction (ROA):

- phase space region within which a dynamic system converges towards a fixed point
- for an actively stabilized system, the ROA is the region within which the stabilizer is viable

➔ **in safety critical scenarios, we must know, when the stabilizer fails!**

LQR Controlled Pendulum



$$\ddot{\theta} = \frac{1}{ml^2}\left(mgl\sin(\theta) - b\dot{\theta} + u\right)$$
$$\simeq \frac{1}{ml^2}\left(mgl\theta - b\dot{\theta} + u\right),$$

ODE and its linear approximation

set of 1st order ODEs

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{pmatrix}}_{\mathbf{A}} \mathbf{x} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix}}_{\mathbf{B}} u$$

$$\int_0^\infty \left(\mathbf{x}^T \mathbf{Q}\mathbf{x} + uRu\right) dt$$

quadratic cost

$$u(\mathbf{x}) = -\mathbf{K}\mathbf{x}$$

Linear Quadratic Regulator (LQR)

$$\ddot{\theta} = -\frac{1}{ml^2}\left(mgl\sin(\theta) - K_0\theta - (b + K_1)\dot{\theta}\right)$$
$$\simeq -\frac{1}{ml^2}\left((mgl - K_0)\theta - (b + K_1)\dot{\theta}\right).$$

closed loop dynamics

Effect of Torque Limitation on the Region of Attraction

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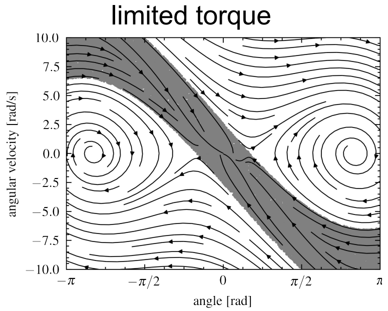

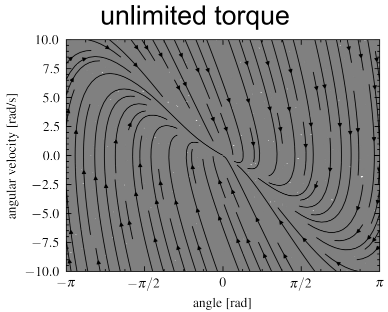

- Starting Point: The set of all points, for which all phase space trajectories starting at those points will go to zero as time goes to infinity:

$$\mathscr{S} = \left\{\gamma \,\middle|\, \mathbf{x}_\gamma(t \to \infty) \to 0\right\}$$

- conservative approach to account for torque limit: do not trust LQR-controller outside of limits

$$\tilde{\mathscr{S}} = \left\{\gamma \,\middle|\, \forall t > 0 : |u_\gamma(t)| \leq \bar{u}\right\} \cap \mathscr{S}_{\text{unlim}}$$

- In this case RoA for unlimited LQR is huge

$$\tilde{\mathscr{S}} = \left\{\gamma \,\middle|\, \forall t > 0 : \boxed{|u_\gamma(t)|} \leq \bar{u}\right\}$$

➔ **we want to know the time evolution of the applied torques**

Time Evolution of the Torque

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$$\ddot{\theta} \simeq -\frac{1}{ml^2}\left((mgl - K_0)\theta - (b + K_1)\dot{\theta}\right)$$

solve IVP

$$\mathbf{x}(t) = \begin{pmatrix} 1 & 1 \\ \kappa_0 & \kappa_1 \end{pmatrix} \begin{pmatrix} C_0 e^{\kappa_0 t} \\ C_1 e^{\kappa_1 t} \end{pmatrix}$$

insert into $u(\mathbf{x}) = -\mathbf{K}\mathbf{x}$

$$u_{\text{lin},\mathbf{x}(0)}(t) = -(K_0 + K_1\kappa_0)C_0 e^{\kappa_0 t} - (K_0 + K_1\kappa_1)C_1 e^{\kappa_1 t}$$

find extremum

$$t^* = -\frac{\ln\left(-\frac{A_0}{A_1}\right)}{\sqrt{D}}, \quad A_i = -(K_0 + K_1\kappa_i)\,\kappa_i C_i$$

insert into $\tilde{\mathscr{S}} = \{\gamma \,|\, \forall t > 0 : |u_\gamma(t)| \leq \bar{u}\}$

$$\boxed{\begin{aligned} \tilde{\mathscr{S}} \simeq \quad & \{\gamma \,|\, |u_{\text{lin},\gamma}(0)| \leq \bar{u}\} \\ & \cap \{\gamma \,|\, t^*_\gamma > 0 \implies |u_{\text{lin},\gamma}(t^*_\gamma)| \leq \bar{u}\} \end{aligned}}$$

Additional Heuristic Needed

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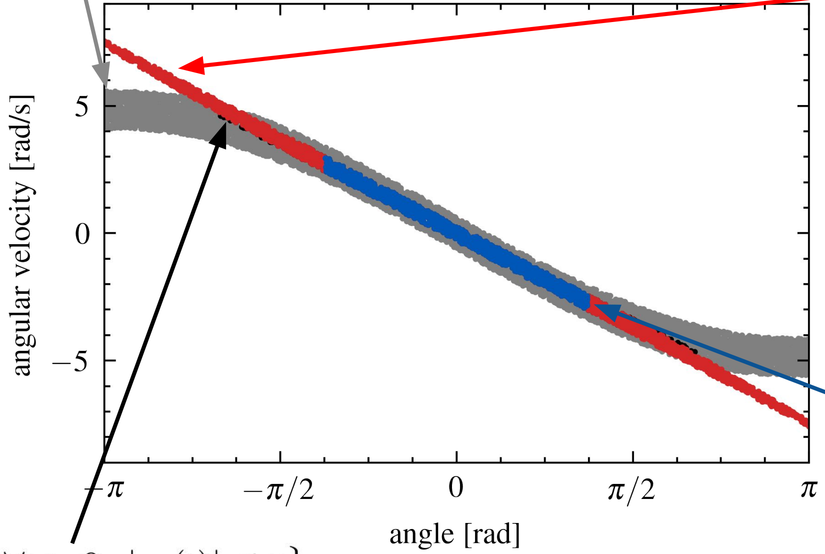
actual ROA

$$\mathscr{S} = \{\gamma \,|\, \mathbf{x}_\gamma(t \to \infty) \to 0\}$$

linear approximation

$$\tilde{\mathscr{S}} \simeq \{\gamma \,|\, |u_{\text{lin},\gamma}(0)| \leq \bar{u}\} \cap \{\gamma \,|\, t^*_\gamma > 0 \implies |u_{\text{lin},\gamma}(t^*_\gamma)| \leq \bar{u}\}$$



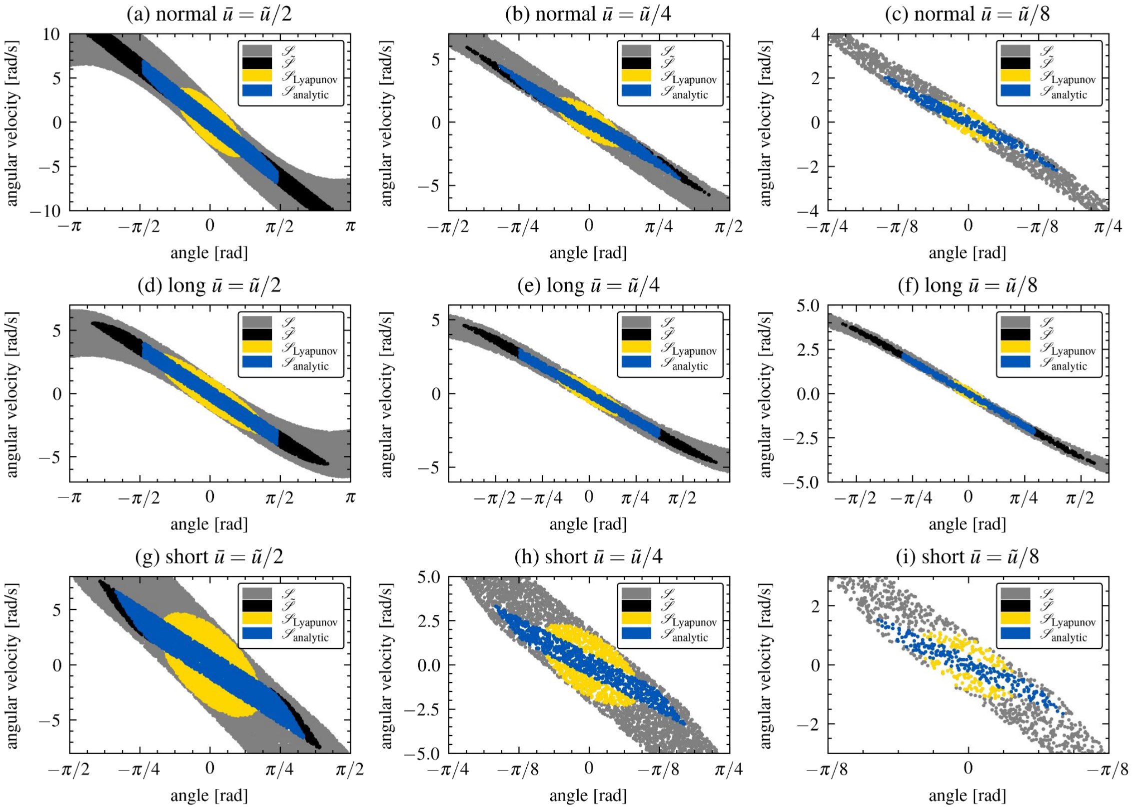
$$\tilde{\mathscr{S}} = \{\gamma \,|\, \forall t > 0 : |u_\gamma(t)| \leq \bar{u}\}$$

torque limit not violated

➔ **heuristic: ensure approximation error is less than torque limit!**

$$\{\gamma \,|\, \boxed{mgl\,|\sin(\theta) - \theta| \leq \bar{u}}\} \cap \{\gamma \,|\, |u_{\text{lin},\gamma}(0)| \leq \bar{u}\} \cap \{\gamma \,|\, t^*_\gamma > 0 \implies |u_{\text{lin},\gamma}(t^*_\gamma)| \leq \bar{u}\}$$

Numerical Experiments



- evaluated 100000 random initial values for 3 sets of pendulum parameters and 3 different torque limits
- comparison with Lyapunov-sampling method from Najafi et al., 2016 (Yellow)

PAREMETERS FOR SIMULATION

	mass [kg]	length [m]	inertia [kg m^2]
normal	0.676	0.45	0.137
long	0.174	1.744	0.531
short	1.744	0.174	0.0531

$$\tilde{u} := mlg \approx 2.98 \text{ Nm}$$

Numerical Experiments

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	$\tilde{u}/2$	$\tilde{u}/4$	$\tilde{u}/8$
normal	0.99	1.183	1.150
long	0.723	1.152	1.721
short	0.718	0.72	0.767

$T_{\text{analytic}} = 0.00136 \text{ s}$ $T_{\text{Lyapunov}} = 12.8 \text{ s}$

Relative Area:

- area of analytic estimation divided by volume of Lyapunov estimation
- depends on parameters
- estimation tends to get comparatively better with more severe torque limitation

Computation Time:

- average time to initialize classifier, given a new set of parameters
- lyapunov method relies on sampling and is thus orders of magnitude slower

Physical Experiments

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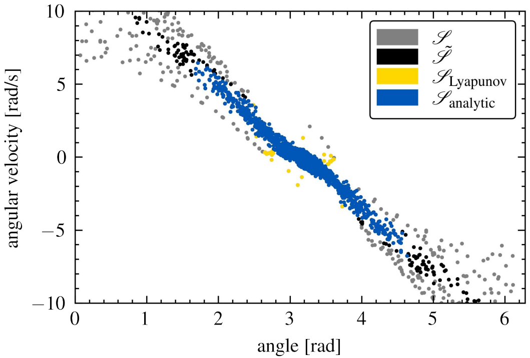
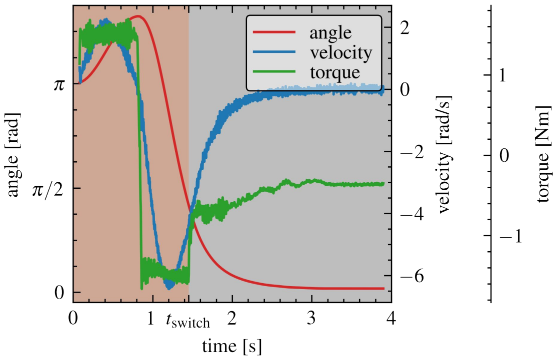




Region of Attraction:

- experiment conducted on physical actuated pendulum as described in [Wiebe et al., 2022]
- 570 runs of the LQR controlled pendulum from random initial values have been conducted
- results similar to numerical

Swing-Up:

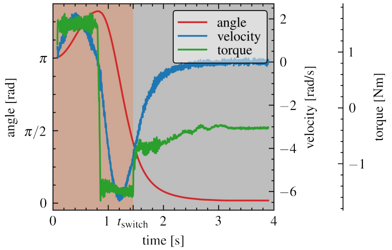
- demonstrate switching behavior based on analytic ROA estimation
- energy shaping type controller for swing up motion (brown area)
- LQR for stabilization (gray area)

Conclusion

$$
\begin{aligned}
&\{\gamma \,|\, mgl|\sin(\theta) - \theta| \leq \bar{u}\} \\
\cap &\{\gamma \,|\, |u_{\text{lin},\gamma}(0)| \leq \bar{u}\} \\
\cap &\{\gamma \,|\, t^*_\gamma > 0 \implies |u_{\text{lin},\gamma}(t^*_\gamma)| \leq \bar{u}\}
\end{aligned}
$$

	$\tilde{u}/2$	$\tilde{u}/4$	$\tilde{u}/8$
normal	0.99	1.183	1.150
long	0.723	1.152	1.721
short	0.718	0.72	0.767

$T_{\text{analytic}} = 0.00136$ s $\quad T_{\text{Lyapunov}} = 12.8$ s



- we found analytically derived conditions classifying phase space points as part of ROA
- performs similar in terms of size but is orders of magnitude faster to compute than baseline
- has been shown to work in experiment as switching condition

Outlook

- generalization towards higher dimensional systems?
  → there are methods for dealing with higher dimensional linear ODEs
- heuristic can definitely be improved
- possible use of this to formally verify a hybrid system switching between LQR and another controller

Thank you!

contact:
lukas.gross@dfki.de

source code and more information:
dfki-ric-underactuated-lab.github.io/analytic_roa_lqr_pendulum/