# Swing-up Control of an Inverted Pendulum Comparing LQR and Pole Placement

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***Abstract*—In this paper the objective is to swing a pendulum up from the downward position using only the controls on the cart. Two methods of linear control are proposed and compared and a nonlinear control of energy shaping is utilized for control when the pendulum is in the nonlinear region. Actuator dynamics are included in the model and the voltage is limited to 12V. First the dynamics of the system are derived, then the system is linearized so it can be used for the two linear control methods and the nonlinear energy-based method is derived using Lyapunov's function. The energy-based control is integrated into the controller and energy + LQR control is compared to energy + pole placement control. used to swing the pendulum up, and finally the pendulum reaches equilibrium using a linear control model. Pole placement as well as Linear Quadratic Regulator (LQR) are compared. Both controllers are able to stabilize the system, but vary slightly in their performance.**

***Index Terms*—inverted pendulum, cart-pole, swing up, LQR, pole placement, energy control, Lyapunov**

## I. Introduction

In recent decades, underactuated systems have been used in key research topics in the field of nonlinear control. Underactuated systems are systems that have fewer actuators than degrees of freedom. Consequently, the desired control objectives must be achieved using fewer actuators. The need to control underactuated systems is common and also has important practical applications. Examples of such applications include spacecrafts, underwater vehicles, aircrafts, robots, satellites and so forth. Underactuated systems have several advantages such as minimizing the number of actuators, the reduction of cost and complexity, further driving interest into analyzing these systems.

Much of the work done in the control of underactuated systems has been done using relatively simple, low-dimensional models. The benefit of using such models is to avoid introducing unnecessary complexities to the model, but still capturing the essence of the problem. For the purposes of control, inverted pendulum mechanisms are benchmark examples of underactuated systems. There is the need to control both the position of the cart as well as the angle of the pendulum with only one control input. These underactuated, nonlinear, unstable systems are commonly used to verify various control techniques. Some of these control techniques include PID control, adaptive control, fuzzy control and neural network control. Of course, these techniques vary in complexity and performance, but here LQR and pole placement were chosen to be compared. Since these models are only valid near the equilibrium point, an energy-based method is used to get the pendulum from the downwards position near the equilibrium point, which adds another dimension to the problem. Some studies compare linear control methods where the pendulum doesn't vary much from equilibrium. It is quite common in studies for controlling the inverted pendulums to assume the system as ideal and not consider actuator dynamics.

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## II. Background

I chose to work with the cart-pole system as it is a classic inverted pendulum system of interest. This requires first deriving the equations of motion for the nonlinear system using the Euler-Lagrange method. Actuator dynamics were included as well. The system was simulated in MATLAB and the ode45 solver was used to solve the system dynamics. Then control techniques were needed to have the pendulum swing up from the downward position to the upward position. MATLAB functions such as place() and lqr() were utilized for the linear control techniques. This project was set up so both linear and non linear control techniques could be tested. A bit of tuning was required for the control parameters. The goal of this project was to show the full control (using a total of three control methods) on a relatively simple but versatile model which could then be expanded on further if desired.

## III. System Modelling

### *A. Cart-pole system*

The cart-pole system is comprised of a cart on wheels with an inverted pendulum attached to it. We will assume that the pole has neither mass nor inertia and that friction is negligible. The actuator dynamics for the wheels are included with a 12V limitation on the output voltage.

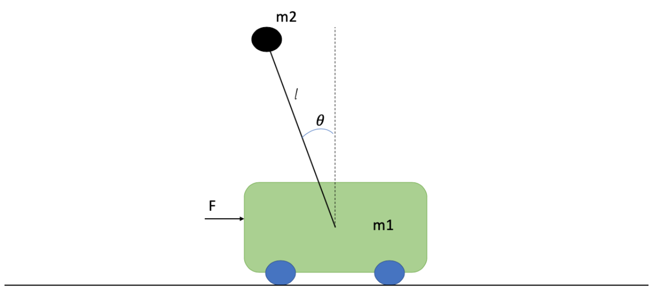


Fig. 1. Cart-pole system

We can mathematically model the nonlinear system using the Euler-Lagrange equation. The first step is finding the kinetic and potential energies of the system. The potential energy (V) depends only on the vertical position of the pendulum while the kinetic energy (K) depends on the kinetics energies of the pole and the cart. The Lagrangian, L, of the system is defined as the kinetic - potential energy.

$$V = m_2 glcos(\theta)$$

$$K = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2[(\dot{x} - l\dot{\theta}cos(\theta))^2 + (l\dot{\theta}sin(\theta))^2]$$

$$L = K - V$$

$$L = \frac{1}{2}(m_1+m_2)\dot{x}^2 - m_2 l\dot{x}\dot{\theta}cos(\theta) + \frac{1}{2}m_2 l^2\dot{\theta}^2 - m_2 glcos(\theta)$$

The equations of motion can then be derived using the Euler-Lagrange equations. Where the state vector, q, is defined as q = $\begin{bmatrix} x \\ \theta \end{bmatrix}$. and F = $\begin{bmatrix} F \\ 0 \end{bmatrix}$ since there is only a force component in the x direction.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = F_i$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = F$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

After performing the calculations, the following equations can be obtained.

$$(m_1+m_2)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta = F$$

$$m_2 l\ddot{x}\cos\theta - m_2 l^2\ddot{\theta} + m_2 gl\sin\theta = 0$$

Rearranging the terms, the dynamic equations of the under-actuated system can be represented as follows.

$$M(q)\ddot{q} + C(q,\dot{q}) + G(q) + B_{damp}\dot{q} = Bu$$

$$M(q) = \begin{bmatrix} m_1 + m_2 + J_m & -m_2 lcos(\theta) \\ -m_2 lcos(\theta) & m_2 l^2 \end{bmatrix}$$

$$C(q,\dot{q}) = \begin{bmatrix} m_2 l\dot{\theta}^2 sin(\theta) \\ 0 \end{bmatrix} B = \begin{bmatrix} \frac{k_T}{R} \\ 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} 0 \\ -m_2 glsin(\theta) \\ 0 \end{bmatrix} B_{damp} = \begin{bmatrix} B_m + \frac{k_V k_T}{R} & 0 \\ 0 & 0 \end{bmatrix}$$

M(q) is the inertia matrix, C(q,) is the Coriolis-centripital, G(q) is the gravitation torques vector, $B_{damp}$ is the dampening term multiplied by  and B is the matrix that multiplies the input torques (which would just be the force on the cart). The input, u, to the system is the input voltage being supplied to the wheels which is why is is being multiplied by a $\frac{k_T}{R}$. Here the actuator dynamics are accounted for, and the values of the system parameters can be seen in the appendix. Once grouping the terms, the equations of motion can be found using.

$$\ddot{q} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} = (Bu - C(q,\dot{q}) - G(q) - B_{damp}\dot{q})/M(q)$$

## IV. Controllers

Before implementing the pole placement and LQR controller, a linearized model of the system must be developed. This system expresses the system as linear around a fixed point which we will specify as the states as x = [$x$; $\dot{x}$; $\theta$; $\dot{\theta}$] the input as u and x* = [0;0;0;0] u* = 0. This is done using the Taylor expansion. The equation is seen below.

$$\dot{x} = A(x - x^*) + B(u - u^*)$$

$$A = \frac{\partial f}{\partial s}\Big|_{x=x^*} B = \frac{\partial f}{\partial u}\Big|_{u=u^*}$$

Once complete, we can express the full system in linearized form using the equation where $A_{lin}$ and $B_{lin}$ can be found in the appendix.

$$\dot{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = A_{lin}x + B_{lin}u$$

This system is controllable if it is possible to construct and unconstrained control signal that will take an initial state to any final state in a finite amount of time. To ensure that this is valid and that the system is controllable, we check the rank of the controllability matrix of the system. If this rank is 4 (which is equal to the number of states), then the system is controllable. An important observation here is that underactuated systems can be controllable. The system cannot follow arbitrary trajectories, but they can arrive to set point in space.

### A. State Feedback Controller by Pole Placement

The first method to consider is that of the pole placement. The aim of this feedback controller is to stabilize the unstable system which is done by selected desired closed-loop poles. Using the linearized, controllable system that was derived above, a proper gain matrix (k) can be found that assigns the closed loop poles of the system to the desired locations on the complex plane. These desired closed-loop poles that are chosen need to lie on the left-hand side of the complex plane. Moving the poles too far to the left, however, results in an aggressive control effort. Selection of these poles requires a compromise in control effort and how quickly the response is as well as how responsive it is to any noise. The state space representation of our system can be written as

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The closed loop system is then formed by feeding back the state variables through the gain matrix k.

$$u = -kx$$

It follows that the control signal, u, is determined by the instantaneous state which is known as state feedback.

$$\dot{x} = (A - Bk)\, x$$

The characteristic equation of the closed loop system can then be found.

$$|SI - (A - Bk)| = 0$$

The poles are the eigenvalues of the matrix (A-Bk). Designing this control law means selecting the feedback gain matrix, k, such that the poles are in a desirable location. Trial and error is required to obtain the desired performance. In MATLAB there is a function place(A,B,p) that will find the k matrix given A B and desired poles. The matrix of desired poles, p, as well as the gain matrix, k, was found for the system.

$$p = \begin{bmatrix} -3 \\ -4 \\ -5 \\ -6 \end{bmatrix} k = [-24.74, -25.36, 111.70, 28.58]$$

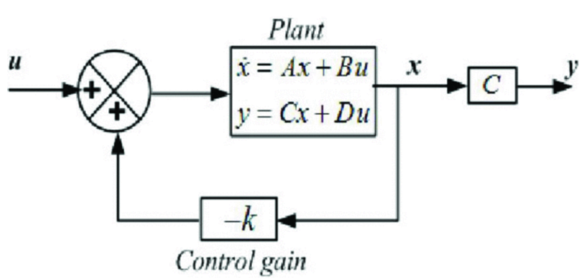


Fig. 2. State Feedback Controller

### B. Linear Quadratic Regulator LQR

The second linear controller we will investigate is the linear quadratic regulator (LQR). The control strategy is to optimally find k by minimizing a quadratic cost function, J. J is depended on the two weight matrices Q and R where Q must be a positive semi-definite symmetric matrix and R must be a positive definite matrix. It weights the state vectors by Q and the control input by R. In our case R is a single number. Each number in the diagonal of Q corresponds to its respective state (x,$\dot{x}$,$\theta$,$\dot{\theta}$) so it can be weighted to favor, perhaps, a more aggressive control on theta and a less aggressive control on position depending on desired output. Below we see the cost function J that the LQR method minimizes, and again gain matrix k being fedback into the system.

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt$$

$$u = -kx$$

The solution can be obtained using the solution to the State Dependent Algebraic Riccati Equation. And then k can be obtained.

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$
$$k = R^{-1}B^T P$$

The weight matrices Q and R, as well and the resulting gain matrix, k, was found for the system. Q favors the smallest error in theta, then a small error in position and R favors a small control input.

$$Q = \begin{bmatrix} 200 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R = 0.0001$$

$$k = [-1414.21, -1180.17, 4795.02, 966.35]$$

### C. Swing-up using Energy Control Method

To get the pendulum from the down position to swing up near the equilibrium point so it can be controlled by the linear control methods, we use and energy control technique. This technique accomplishes the task by controlling the systems energy rather than the energy directly. A way to swing the pendulum up is to give the system the energy it needs to get the pendulum to the upright position. This corresponds to the trajectory that passes through the unstable equilibrium point at the upright position. It doesn't need to control the perfectly, as one of the linear controls will take over when the pendulum gets within 30° of the equilibrium position.

$$E = \frac{1}{2} J\dot{\theta}^2 + m_2 gl(cos(\theta) - 1) = 0$$

To perform the energy control, we take the derivative of E with respect to time to find

$$\frac{d}{dt} E = -m_2 ul\dot{\theta} cos(\theta)$$

This allows us to understand how the energy is influenced by accelerations around the pivot. The control technique is relatively straight forward in nature, and we can see how controllability is lost when $\dot{\theta} = 0$ or $cos(\theta) = 0$, or when $\theta = \pm 90°$. When the pendulum is directly horizontal or switching directions there is no controllability, and controllability is maximized when the pendulum is swinging quickly and the pendulum close the straight downward or upward position. We want to increase the energy, so u should be positive when $\dot{\theta}cos(\theta)$ is negative. Using Lyapunov function $V = (E - E_o)^2/2$ and the control law $u = (E - E_o)\dot{\theta}cos\theta$ and $\Delta E = E - E_o$ we find

$$\frac{d}{dt}V = -mlk(\Delta E\dot{\theta}cos(\theta))^2$$

$$\Delta E = \frac{2}{3}m_2 l^2\dot{\theta}^2 - m_2 gl(cos(\theta) - 1)$$

$$u = k\Delta E\dot{\theta}\cos(\theta), k = 0.2$$

The Lyapunov function decreases for everywhere besides the two conditions where the system is uncontrollable as discussed above. This isn't an issue as the system won't stay in these states for every long. This control law will drive the function towards the desired energy $E_o$. The factor of k can be tuned for the desired response.

## V. Results

Here we see how the stabilization of the cart is able to be acheived by first using the energy control method and then either the pole placement or the LQR controller. We see the responses for the energy control plus LQR and then energy control plus pole placement method.

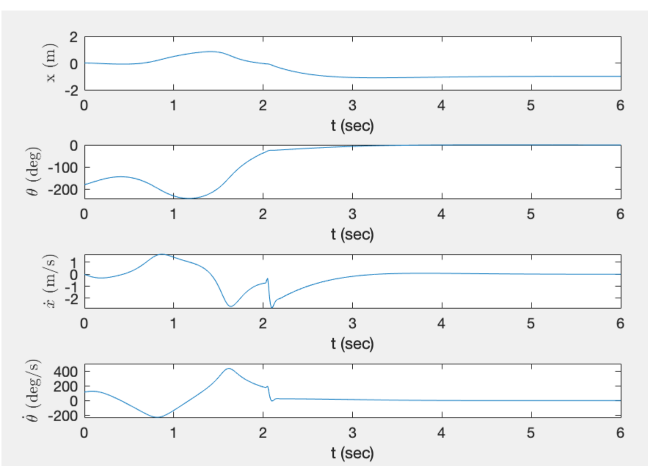


Fig. 3. LQR response

While the system reaches the desired position more quickly with the LQR response, when comparing it with the pole placement response we see how there is a trade off. The movements are more jagged, that is, we see quick changes in $\dot{x}$ and $\dot{\theta}$ for the LQR method more than we see it in the pole placement method. This could be a drawback as it could be hard on the system.

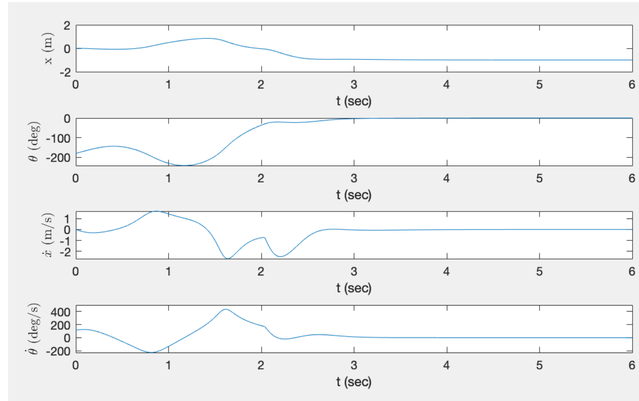


Fig. 4. Pole placement response

Also wee see the voltage input responses for both control methods.

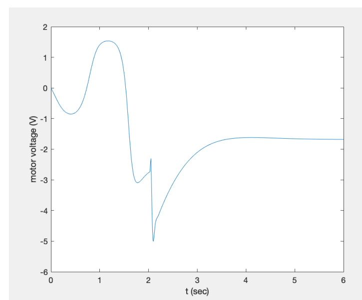


Fig. 5. LQR voltage input

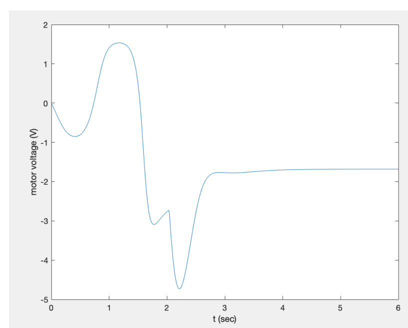


Fig. 6. Pole placement voltage input

While the voltage isn't particularly large, it is depended on constraints of the system. Here voltage is certainly below 12V, but we may wish to change this threshold. The cart also doesn't move around too much, but that is something that could be taken into consideration. Also, we do see sharp changes in $\dot{x}$ and $\dot{\theta}$ which could prove to be problematic for the real system.

## VI. Discussion

With comparing the two different linear control techniques, as well as integrating a nonlinear control technique, it is evident that different control techniques can be beneficial for different purposes. The LQR method allowed us to specify weights to favor the control effort of different states which can be particularly useful depending on the situation. The pole placement method, however, allowed direct control of the placement of the poles which could be beneficial but also more complicated if one isn't too familiar with how to go about pole placement. It ultimately lies in the control of the designer and the desired output of the system. This control is done on a nearly ideal system with actuator dynamics, but imposing more constraints on the system would be good grounds for future work. Also, introducing disturbances for a more realistic filter is important as well as filtering techniques to remove noise. Imposing other limitations in addition to voltage control could be added as well as friction and inertia of the pole.

In this study the LQR and pole-placement methods were compared and an energy control method was implemented on the cart-pole system. First the nonlinear system was modeled using Euler-Lagrange equations. Actuator dynamics were accounted for and the terms were grouped so the state space model could be found to describe the equations of motion. The system was then linearized using the Taylor expansion. This linarized system was used to find the state feedback controller using pole placement as well as an LQR controller. To account for the control needed when the pole wasn't close to it's equilibrium point, a nonlinear energy controller was utilized. This took advantage of the system's energy rather than trying to control state directly. Overall, we were successfully able to control the cart from the downwards position and saw slightly better performance with the LQR method.

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## Appendix

| Physical Quantity | Symbol | Units |
|---|---|---|
| mass of cart | $m_1$ | 1 kg |
| mass at end of pole | $m_2$ | 0.5 kg |
| length of pole | $l$ | 0.7 m |
| gravity | g | 9.81 m/s |
| speed constant | $k_V$ | 1.0 $\frac{rad}{sV}$ |
| winding resistance | $R$ | 1.25 $\Omega$ |
| torque constant | $k_T$ | 1.25 $\frac{Nm}{A}$ |
| bearing viscous friction | $B_m$ | 1x$10^{-5}$ Nmm |
| rotor inertia | $J_m$ | 7x$10^{-6}$ kgm$^2$ |

System Parameters

$$A_{lin} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(k_T k_V + B_m R}{R(J_m+m_1)} & \frac{g m_2}{J_m+m_1} & 0 \\ 0 & \frac{m_p g}{m_c} & 0 & 1 \\ 0 & \frac{-(k_T k_V + B_m R}{Rl(J_m+m_1)} & \frac{g(J_m+m_1+m_2)}{l(J_m+m_1)} & 0 \end{bmatrix}$$

$$B_{lin} = \begin{bmatrix} 0 \\ \frac{k_T}{R(J_m+m_1)} \\ 0 \\ \frac{k_T}{Rl(J_m+m_1)} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$