Portfolio Rebalancing

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1. We consider the problem of rebalancing a portfolio of assets over multiple periods. We let $h_t \in \mathbb{R}^n$ denote the vector of our dollar value holdings in n positions. We will work with the portfolio weight vector, defined as $w_t = h_t/(\mathbf{1}^T h_t)$, where we assume that $\mathbf{1}^T h_t > 0$, *i.e.*, the total portfolio value is positive.

   The *target portfolio weight vector* w^* is defined as the solution of the problem,

$$\begin{array}{ll} \text{maximize} & \mu^T w - \dfrac{\gamma}{2} w^T \Sigma w \\ \text{subject to} & \mathbf{1}^T w = 1 \end{array}$$

   where $w \in \mathbb{R}^n$ is the variable, μ is the mean return, $\Sigma \in \mathbb{S}^n_{++}$ is the return covariance, and $\gamma > 0$ is the risk aversion parameter. The data $\mu, \Sigma,$ and γ are given. In words, the target weights maximize the risk-adjusted expected return.

   At the beginning of each period t we are allowed to rebalance the portfolio by buying and selling assets. We call the post-trade portfolio weights $\tilde{w}_t$. They are found by solving the (rebalancing) problem

$$\begin{array}{ll} \text{maximize} & \mu^T w - \dfrac{\gamma}{2} w^T \Sigma w - \kappa^T |w - w_t| \\ \text{subject to} & \mathbf{1}^T w = 1 \end{array}$$

   with variable $w \in \mathbb{R}^n$, where $\kappa \in \mathbb{R}^n_+$ is the vector of (so-called linear) transaction costs for the assets. (For example, these could be model bid/ask spread.) Thus we choose the post-trade weights to maximize the risk-adjusted expected return, minus the transaction costs associated with rebalancing the portfolio. Note the pre-trade weight vector w_t is known at the time we solve the problem. If we have $\tilde{w}_t = w_t$, it means that no rebalancing is done at the beginning of period t; we simply hold our current portfolio.

   After holding the rebalanced portfolio over the investment period, the dollar value of our portfolio becomes $h_{t+1} = \mathbf{diag}(r_t)\tilde{h}_t$, where $r_t \in \mathbb{R}^n_{++}$ is the (random) vector of asset returns over period t, and $\tilde{h}_t$ is the post-trade portfolio given in dollar values. The next weight vector is then given by

$$w_{t+1} = \frac{\mathbf{diag}(r_t)\tilde{w}_t}{r_t^T \tilde{w}_t}$$

The standard model is that r_t are IID random variables with mean and covariance μ and Σ, but this is not relevant in this problem.

Starting from $w_1 = w^*$, compute a sequence of portfolio weights $\tilde{w}_t$ for $t = 1, \ldots, T$. For each t, find $\tilde{w}_t$ by solving the rebalancing problem (with w_t a known constant); then generate a vector of return r_t to compute w_{t+1}.

Report the fraction of periods in which the no-trade condition holds and the fraction of periods in which the solution has only zero (or negligible) trades. Carry this out for two values of κ and comment.