

Assignment 4

David Fleischer – 27101449

MACF 401 - Mathematical & Computational Finance I

Due: April 4 2016

Last update: December 4, 2017

Part I

Solution 5.2:

(i) We have

$$\begin{aligned}f(\sigma) &= pe^\sigma + q^{-\sigma} \\&= pe^\sigma + (1-p)e^{-\sigma} \\&= pe^\sigma + e^{-\sigma} - pe^{-\sigma} \\&= p(e^\sigma - e^{-\sigma}) + e^{-\sigma}\end{aligned}$$

However, note that $\forall x \in \mathbb{R}$ we have $(x-1)^2 > 0$, hence

$$\begin{aligned}(x-1)^2 &= x^2 - 2x + 1 \geq 0 \\ \implies x^2 + 1 &\geq 2x \\ \implies \frac{x^2 + 1}{x} &\geq 2 \\ \implies x + x^{-1} &\geq 2\end{aligned}$$

Thus $(e^\sigma - e^{-\sigma}) \geq 2$ and since $p > \frac{1}{2}$ we find

$$p(e^\sigma - e^{-\sigma}) \geq 1$$

This, together with the final term $e^{-\sigma} > 0$ gives us that

$$f(\sigma) = p(e^\sigma - e^{-\sigma}) + e^{-\sigma} > 1$$

as desired.

(ii) By definition we have that $M_n = \sum_{j=1}^n X_j$ depends on only the first n coin tosses $\omega_1 \cdots \omega_n$. Note that $\left(\frac{1}{f(\sigma)}\right)^n$ is deterministic we find that $S_n = e^{\sigma M_n} \left(\frac{1}{f(\sigma)}\right)^n$ is adapted since

a measurable function of an adapted process is itself adapted.

Now, to confirm the martingale property:

$$\begin{aligned}
\mathbb{E}_n [S_{n+1}] &= \mathbb{E}_n \left[e^{\sigma M_{n+1}} \left(\frac{1}{f(\sigma)} \right)^{n+1} \right] \\
&= \mathbb{E}_n \left[e^{\sigma(M_n + X_{n+1})} \left(\frac{1}{f(\sigma)} \right)^{n+1} \right] \quad (\text{by definition of } M_{n+1}) \\
&= \mathbb{E}_n \left[e^{\sigma M_n} \left(\frac{1}{f(\sigma)} \right)^n \left(\frac{1}{f(\sigma)} \right) e^{\sigma X_{n+1}} \right] \\
&= \mathbb{E}_n \left[\frac{S_n}{f(\sigma)} e^{\sigma X_{n+1}} \right] \quad (\text{by definition of } S_n) \\
&= \frac{S_n}{f(\sigma)} \mathbb{E}_n [e^{\sigma X_{n+1}}] \quad (\text{adaptedness of } \frac{S_n}{f(\sigma)} \text{ to the first } n \text{ coin tosses}) \\
&= \frac{S_n}{f(\sigma)} (pe^{\sigma \cdot (1)} + qe^{\sigma \cdot (-1)}) \quad (\text{by the independence lemma}) \\
&= \frac{S_n}{f(\sigma)} f(\sigma) \\
&= S_n
\end{aligned}$$

Therefore, since S_n is both adapted and satisfies the martingale property we have that S_n is indeed a martingale, as desired.

(iii) Applying the Optional Sampling Theorem we find that the stopped process $S_{n \wedge \tau_1}$ must be a martingale. So

$$\begin{aligned}
\mathbb{E}_0 [S_{n \wedge \tau_1}] &= \mathbb{E}_0 \left[e^{\sigma M_{n \wedge \tau_1}} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} \right] \\
&= S_{0 \wedge \tau_1} \quad (\text{by the martingale property}) \\
&= S_0 \quad (\text{since } \tau_1 > 0) \\
&= e^{\sigma M_0} \left(\frac{1}{f(\sigma)} \right)^0 \\
&= 1
\end{aligned}$$

Note

$$\lim_{n \rightarrow \infty} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} = \begin{cases} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} & \text{if } \tau_1 < \infty \\ 0 & \text{if } \tau_1 = \infty \end{cases}$$

By the definition of $M_{n \wedge \tau_1}$ we note that

$$M_{n \wedge \tau_1} \leq 1$$

Hence

$$0 \leq e^{\sigma M_{n \wedge \tau_1}} \leq e^{\sigma}$$

Now, considering first $\tau_1 < \infty$

$$\begin{aligned} \lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} &= \lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \lim_{n \rightarrow \infty} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} \\ &= e^{\sigma} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \end{aligned}$$

and for the case of $\tau_1 = \infty$ we use our result that $e^{\sigma M_n}$ is bound above and below, so

$$\begin{aligned} \lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} &= \lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \lim_{n \rightarrow \infty} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} \\ &= \lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \cdot 0 \\ &= 0 \end{aligned}$$

Therefore, we may combine both cases as

$$\lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} = \mathbf{1}_{\{\tau_1 < \infty\}} e^{\sigma} \left(\frac{1}{f(\sigma)} \right)^{\tau_1}$$

Now we wish to take the limit of the expectation of the stopped process $S_{n \wedge \tau_1}$ as $n \rightarrow \infty$. However, we have already shown that $\mathbb{E}_0[S_{n \wedge \tau_1}] = S_{0 \wedge \tau_1} = S_0 = 1$. So

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}_0[S_{n \wedge \tau_1}] &= \lim_{n \rightarrow \infty} 1 \\ &= 1 \end{aligned}$$

Thus

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \mathbb{E}_0[S_{n \wedge \tau_1}] \\ &= \mathbb{E}_0 \left[\lim_{n \rightarrow \infty} S_{n \wedge \tau_1} \right] \quad (\text{Dominated Convergence}) \\ &= \mathbb{E}_0 \left[\mathbf{1}_{\{\tau_1 < \infty\}} e^{\sigma} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\ \implies e^{-\sigma} &= \mathbb{E}_0 \left[\mathbf{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \end{aligned}$$

and taking the limit as $\sigma \downarrow 0$

$$\begin{aligned} \lim_{\sigma \downarrow 0} e^{-\sigma} &= \lim_{\sigma \downarrow 0} \mathbb{E}_0 \left[\mathbf{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\ \implies 1 &= \lim_{\sigma \downarrow 0} \mathbb{E}_0 \left[\mathbf{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\ &= \mathbb{E}_0 \left[\lim_{\sigma \downarrow 0} \mathbf{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \quad (\text{Dominated Convergence}) \\ &= \mathbb{E}_0 \left[\mathbf{1}_{\{\tau_1 < \infty\}} \lim_{\sigma \downarrow 0} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\ &= \mathbb{E}_0 [\mathbf{1}_{\{\tau_1 < \infty\}}] \\ &= \mathbb{P}(\{\tau_1 < \infty\}) \end{aligned}$$

as desired.

(iv) Let $\alpha \in (0, 1)$. We will first solve for the σ satisfying

$$\begin{aligned}
\alpha &= \frac{1}{f(\sigma)} \\
&= \frac{1}{pe^\sigma + qe^{-\sigma}} \\
\implies \alpha (pe^\sigma + qe^{-\sigma}) &= 1 \\
\implies \alpha pe^\sigma + \alpha qe^{-\sigma} &= 1 \\
\implies \alpha p + \alpha q (e^{-\sigma})^2 &= e^{-\sigma} \\
\implies \alpha q (e^{-\sigma})^2 - e^{-\sigma} + \alpha p &= 0 \\
\implies e^{-\sigma} &= \frac{1 \pm \sqrt{1 - 4\alpha^2 pq}}{2\alpha pq}
\end{aligned}$$

We require $\sigma > 0$ so then $0 < e^{-\sigma} < 1$. For this purpose we take the negative root

$$e^{-\sigma} = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

and from (iii) we find

$$\begin{aligned}
e^{-\sigma} &= \mathbb{E}_0 \left[\mathbf{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\
\implies \mathbb{E}_0 \left[\mathbf{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] &= \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}
\end{aligned}$$

Since we initially performed the substitution $\alpha = \frac{1}{f(\sigma)}$ we may write

$$\mathbb{E} [\mathbf{1}_{\{\tau_1 < \infty\}} \alpha^{\tau_1}] = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

Noting that

$$\mathbb{E} [\alpha^{\tau_1}] = \mathbb{E} [\mathbf{1}_{\{\tau_1 = \infty\}} \alpha^{\tau_1}] + \mathbb{E} [\mathbf{1}_{\{\tau_1 < \infty\}} \alpha^{\tau_1}]$$

and since $\alpha \in (0, 1)$ we find $\mathbf{1}_{\{\tau_1 = \infty\}} \alpha^{\tau_1} = 0$. Therefore, we may conclude with

$$\mathbb{E} [\alpha^{\tau_1}] = \mathbb{E} [\mathbf{1}_{\{\tau_1 < \infty\}} \alpha^{\tau_1}] = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

as desired.

(v) By the Dominated Convergence Theorem write

$$\begin{aligned}
\mathbb{E} [\tau_1 \alpha^{\tau_1 - 1}] &= \frac{\partial}{\partial \alpha} \mathbb{E} [\alpha^{\tau_1}] \\
&= \frac{\partial}{\partial \alpha} \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q} \quad (\text{from (iv)}) \\
&= \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha^2 q \sqrt{1 - 4\alpha^2 pq}}
\end{aligned}$$

Since we had defined $\alpha \in (0, 1)$ we must take the limit as $\alpha \uparrow 1$

$$\begin{aligned}
\lim_{\alpha \uparrow 1} \mathbb{E} [\tau_1 \alpha^{\tau_1 - 1}] &= \lim_{\alpha \uparrow 1} \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha^2 q \sqrt{1 - 4\alpha^2 pq}} \\
&= \frac{1 - \sqrt{1 - 4pq}}{2q \sqrt{1 - 4pq}} \\
&= \frac{1 - \sqrt{1 - 4p(1 - p)}}{2(1 - p) \sqrt{1 - 4p(1 - p)}} \\
&= \frac{1 - \sqrt{1 - 4p + 4p^2}}{2(1 - p) \sqrt{1 - 4p + 4p^2}} \\
&= \frac{1 - \sqrt{(1 - 2p)^2}}{2(1 - p) \sqrt{(1 - 2p)^2}} \\
&= \frac{1 - |1 - 2p|}{2(1 - p) |1 - 2p|} \\
&= \frac{1 - (2p - 1)}{2(1 - p)(2p - 1)} \quad \text{since } 1 - 2p < 0 \text{ for } \frac{1}{2} < p < 1 \\
&= \frac{2 - 2p}{2(1 - p)(2p - 1)} \\
&= \frac{1}{2p - 1} = \frac{1}{p + p - 1} = \frac{1}{p - q} \quad (\text{I think the final expression is the nicest})
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{p - q} &= \lim_{\alpha \uparrow 1} \mathbb{E} [\tau_1 \alpha^{\tau_1 - 1}] \\
&= \mathbb{E} \left[\lim_{\alpha \uparrow 1} \tau_1 \alpha^{\tau_1 - 1} \right] \quad (\text{Dominated Convergence}) \\
&= \mathbb{E} [\tau_1]
\end{aligned}$$

as desired.

Solution 5.3:

(i) With the substitution $x = e^{\sigma_0}$ we solve

$$\begin{aligned}
1 &= px + \frac{q}{x} \\
&= \frac{px^2 + q}{x} \\
\implies x &= px^2 + q \\
\implies px^2 - x + q &= 0 \\
\implies x &= \frac{1 \pm \sqrt{1 - 4pq}}{2p} \\
&= \frac{1 \pm \sqrt{1 - 4p(1 - p)}}{2p} \\
&= \frac{1 \pm \sqrt{(1 - 2p)^2}}{2p} \\
&= \frac{1 \pm (1 - 2p)}{2p}
\end{aligned}$$

Clearly we must have $x = e^{\sigma} > 0$ for all $\sigma \in \mathbb{R}$, so we are required to take the positive term. This yields

$$\begin{aligned}
x &= \frac{1 + 1 - 2p}{2p} \\
&= \frac{1 - p}{p} \\
&= \frac{q}{p} \\
\implies e^{\sigma_0} &= \frac{q}{p} \\
\implies \sigma_0 &= \log \frac{q}{p}
\end{aligned}$$

Since $q > p$ we have that $\frac{q}{p} > 1 \implies \sigma_0 = \log \frac{q}{p} > 1 > 0$, which satisfies our positivity criteria. That is, we have found $\sigma_0 > 0$ such that $f(\sigma_0) = 1$ since

$$\begin{aligned}
f(\sigma_0) &= pe^{\sigma_0} + qe^{-\sigma_0} \\
&= pe^{\log \frac{q}{p}} + qe^{-\log \frac{q}{p}} \\
&= p \cdot \frac{q}{p} + q \cdot \frac{p}{q} \\
&= q + p = 1
\end{aligned}$$

We wish now to confirm that $f(\sigma) > 1$ for all $\sigma > \sigma_0 = \log \frac{q}{p}$. To this end, again let

$x = e^\sigma$ and calculate

$$\begin{aligned}
1 &\leq px + \frac{q}{x} \\
1 &\leq \frac{px^2 + q}{x} \\
x &\leq px^2 + q \\
0 &\leq px^2 - x + q \\
0 &\leq (x-1)(px - q)
\end{aligned}$$

which is satisfied when $x \geq \frac{q}{p} \implies e^\sigma \geq \frac{q}{p} \implies \sigma \geq \log \frac{q}{p} = \sigma_0$, as desired.

(ii) We go through the same process as in (5.2). Defining the process $S_n = e^{\sigma M_n} \left(\frac{1}{f(\sigma)} \right)^n$ we see by the same argument that S_n is an adapted process since it is a measurable function of M_n , which is itself a function of the first n coin tosses. To confirm the martingale property we have

Now, to confirm the martingale property:

$$\begin{aligned}
\mathbb{E}_n [S_{n+1}] &= \mathbb{E}_n \left[e^{\sigma M_{n+1}} \left(\frac{1}{f(\sigma)} \right)^{n+1} \right] \\
&= \mathbb{E}_n \left[e^{\sigma(M_n + X_{n+1})} \left(\frac{1}{f(\sigma)} \right)^{n+1} \right] \quad (\text{by definition of } M_{n+1}) \\
&= \mathbb{E}_n \left[e^{\sigma M_n} \left(\frac{1}{f(\sigma)} \right)^n \left(\frac{1}{f(\sigma)} \right) e^{\sigma X_{n+1}} \right] \\
&= \mathbb{E}_n \left[\frac{S_n}{f(\sigma)} e^{\sigma X_{n+1}} \right] \quad (\text{by definition of } S_n) \\
&= \frac{S_n}{f(\sigma)} \mathbb{E}_n [e^{\sigma X_{n+1}}] \quad (\text{adaptedness of } \frac{S_n}{f(\sigma)} \text{ to the first } n \text{ coin tosses}) \\
&= \frac{S_n}{f(\sigma)} (pe^{\sigma \cdot (1)} + qe^{\sigma \cdot (-1)}) \quad (\text{by the independence lemma}) \\
&= \frac{S_n}{f(\sigma)} f(\sigma) \\
&= S_n
\end{aligned}$$

So S_n remains a martingale for the case of $0 < p < \frac{1}{2}$. Now, again applying the Optional Sampling Theorem we have that the stopped process $S_{n \wedge \tau_1}$ must be a martingale. Hence

$$\begin{aligned}
\mathbb{E}_0 [S_{n \wedge \tau_1}] &= \mathbb{E}_0 \left[e^{\sigma M_{n \wedge \tau_1}} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} \right] \\
&= S_{0 \wedge \tau_1} \quad (\text{by the martingale property}) \\
&= S_0 \quad (\text{since } \tau_1 > 0) \\
&= e^{\sigma M_0} \left(\frac{1}{f(\sigma)} \right)^0 \\
&= 1
\end{aligned}$$

Suppose now that $\sigma > \sigma_0 = \log \frac{q}{p}$ from part (i). Then we have that $f(\sigma) > 1$. Therefore, $0 < \frac{1}{f(\sigma)} < 1$. From this fact we find

$$\lim_{n \rightarrow \infty} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} = \begin{cases} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} & \text{if } \tau_1 < \infty \\ 0 & \text{if } \tau_1 = \infty \end{cases}$$

By the definition of $M_{n \wedge \tau_1}$ we note that

$$M_{n \wedge \tau_1} \leq 1$$

Hence

$$0 \leq e^{\sigma M_{n \wedge \tau_1}} \leq e^\sigma$$

Now, considering first $\tau_1 < \infty$

$$\begin{aligned} \lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} &= \lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \lim_{n \rightarrow \infty} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} \\ &= e^\sigma \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \end{aligned}$$

and for the case of $\tau_1 = \infty$ we use our result that $e^{\sigma M_n}$ is bound above and below, so

$$\begin{aligned} \lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} &= \lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \lim_{n \rightarrow \infty} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} \\ &= \lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \cdot 0 \\ &= 0 \end{aligned}$$

Therefore, we may combine both cases as

$$\lim_{n \rightarrow \infty} e^{\sigma M_{n \wedge \tau_1}} \left(\frac{1}{f(\sigma)} \right)^{n \wedge \tau_1} = \mathbf{1}_{\{\tau_1 < \infty\}} e^\sigma \left(\frac{1}{f(\sigma)} \right)^{\tau_1}$$

Now we wish to take the limit of the expectation of the stopped process $S_{n \wedge \tau_1}$ as $n \rightarrow \infty$. However, we have already shown that $\mathbb{E}_0[S_{n \wedge \tau_1}] = S_{0 \wedge \tau_1} = S_0 = 1$. So

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}_0[S_{n \wedge \tau_1}] &= \lim_{n \rightarrow \infty} 1 \\ &= 1 \end{aligned}$$

Thus

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \mathbb{E}_0[S_{n \wedge \tau_1}] \\ &= \mathbb{E}_0 \left[\lim_{n \rightarrow \infty} S_{n \wedge \tau_1} \right] \quad (\text{Dominated Convergence}) \\ &= \mathbb{E}_0 \left[\mathbf{1}_{\{\tau_1 < \infty\}} e^\sigma \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\ \implies e^{-\sigma} &= \mathbb{E}_0 \left[\mathbf{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \end{aligned}$$

Instead of taking taking the limit as $\sigma \downarrow 0$ as we did in **(5.2)** we now consider the limit as $\sigma \downarrow \sigma_0$ since we have restricted σ such that $\sigma > \sigma_0$. Taking this limit

$$\begin{aligned}
\lim_{\sigma \downarrow \sigma_0} e^{-\sigma} &= \lim_{\sigma \downarrow \sigma_0} \mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\
\implies e^{-\sigma_0} &= \lim_{\sigma \downarrow \sigma_0} \mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\
&= \mathbb{E}_0 \left[\lim_{\sigma \downarrow \sigma_0} \mathbb{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \quad (\text{Dominated Convergence}) \\
&= \mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 < \infty\}} \lim_{\sigma \downarrow \sigma_0} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\
&= \mathbb{E}_0 [\mathbb{1}_{\{\tau_1 < \infty\}}] \\
&= \mathbb{P}(\{\tau_1 < \infty\})
\end{aligned}$$

We may simplify this as

$$\begin{aligned}
\mathbb{P}(\{\tau_1 < \infty\}) &= e^{-\sigma_0} \\
&= e^{-\log \frac{q}{p}} \\
&= \frac{p}{q}
\end{aligned}$$

which is a permissible probability for this event since our definition of p and q satisfying $0 < p < \frac{1}{2}$ and $q = 1 - p$ implies that $0 < \frac{p}{q} < 1$.

(iii) Let $\alpha \in (0, 1)$. Repeating the arguments from **(5.2.iv)** (since the preliminary steps are not affected by the requirement $0 < p < \frac{1}{2}$) we find that the σ satisfying

$$\alpha = \frac{1}{f(\sigma)}$$

yielding

$$e^{-\sigma} = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

From **(5.3.ii)** we have that

$$e^{-\sigma} = \mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right]$$

Therefore, with our substitution of $\alpha = \frac{1}{f(\sigma)}$, we may write

$$\mathbb{E}_0 [\mathbb{1}_{\{\tau_1 < \infty\}} \alpha^{\tau_1}] = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

Now, write

$$\mathbb{E}_0 [\alpha^{\tau_1}] = \mathbb{E}_0 [\mathbb{1}_{\{\tau_1 = \infty\}} \alpha^{\tau_1}] + \mathbb{E}_0 [\mathbb{1}_{\{\tau_1 < \infty\}} \alpha^{\tau_1}]$$

but if $\tau_1 = \infty$ then, recalling $\alpha \in (0, 1)$, we have $\mathbb{1}_{\{\tau_1 = \infty\}}\alpha^{\tau_1} = 0$. Hence

$$\begin{aligned}\mathbb{E}_0 [\alpha^{\tau_1}] &= \mathbb{E}_0 [\mathbb{1}_{\{\tau_1 = \infty\}}\alpha^{\tau_1}] + \mathbb{E}_0 [\mathbb{1}_{\{\tau_1 < \infty\}}\alpha^{\tau_1}] \\ &= \mathbb{E}_0 [\mathbb{1}_{\{\tau_1 < \infty\}}\alpha^{\tau_1}] \\ &= \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}\end{aligned}$$

as desired.

(iv) Applying the Dominated Convergence Theorem we write

$$\begin{aligned}\mathbb{E} [\tau_1 \alpha^{\tau_1 - 1}] &= \frac{\partial}{\partial \alpha} \mathbb{E} [\alpha^{\tau_1}] \\ &= \frac{\partial}{\partial \alpha} \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q} \quad (\text{from (5.3.iii)}) \\ &= \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha^2 q \sqrt{1 - 4\alpha^2 pq}}\end{aligned}$$

Since we had defined $\alpha \in (0, 1)$ we must take the limit as $\alpha \uparrow 1$

$$\begin{aligned}\lim_{\alpha \uparrow 1} \mathbb{E} [\tau_1 \alpha^{\tau_1 - 1}] &= \lim_{\alpha \uparrow 1} \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha^2 q \sqrt{1 - 4\alpha^2 pq}} \\ &= \frac{1 - \sqrt{1 - 4pq}}{2q \sqrt{1 - 4pq}} \\ &= \frac{1 - \sqrt{1 - 4p(1 - p)}}{2(1 - p) \sqrt{1 - 4p(1 - p)}} \\ &= \frac{1 - \sqrt{1 - 4p + 4p^2}}{2(1 - p) \sqrt{1 - 4p + 4p^2}} \\ &= \frac{1 - \sqrt{(1 - 2p)^2}}{2(1 - p) \sqrt{(1 - 2p)^2}} \\ &= \frac{1 - |1 - 2p|}{2(1 - p) |1 - 2p|}\end{aligned}$$

Now, since $0 < p < \frac{1}{2}$ we have that $1 - 2p > 0$, thus

$$\begin{aligned}\lim_{\alpha \uparrow 1} \mathbb{E} [\tau_1 \alpha^{\tau_1 - 1}] &= \frac{1 - (1 - 2p)}{2(1 - p)(1 - 2p)} \\ &= \frac{p}{(1 - p)(1 - 2p)} \\ &= \frac{p}{q(1 - p - p)} = \frac{p}{q(q - p)}\end{aligned}$$

as desired.

Solution 5.6: $S_0 = 4, K = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}$. Show that an American put expiring at time $N = 1$ has price $V_0 = 0.8$, expiring at time $N = 3$ has price $V_0 = 0.928$, and expiring at time $N = 5$ has price $V_0 = 0.96896$.

Solution 5.7: (i) Recall that we had found

$$v(s) = \begin{cases} 4 - s & \text{if } s \leq 2 \\ \frac{4}{s} & \text{if } s \geq 4 \end{cases}$$

so with $s = 2^j$ we have

$$v(2^j) = \begin{cases} 4 - 2^j & \text{if } j \leq 1 \\ \frac{4}{2^j} & \text{if } j \geq 2 \end{cases}$$

Therefore, for $j \leq 0$, we find

$$\begin{aligned} c(2^j) &= v(2^j) - \frac{4}{5} \left[\frac{1}{2} v(2 \cdot 2^j) + \frac{1}{2} v\left(\frac{1}{2} 2^j\right) \right] \\ &= (4 - 2^j) - \frac{2}{5} [(4 - 2^{j+1}) + (4 - 2^{j-1})] \\ &= 4 - 2^j - \frac{2}{5} [8 - 2^{j-1}(4 + 1)] \\ &= 4 - 2^j - \frac{2}{5} [8 - 5 \cdot 2^{j-1}] \\ &= 4 - 2^j - \frac{16}{5} + 2^j \\ &= \frac{4}{5} \end{aligned}$$

For $j = 1$ we find

$$\begin{aligned} c(2^j) &= v(2^j) - \frac{4}{5} \left[\frac{1}{2} v(2 \cdot 2^j) + \frac{1}{2} v\left(\frac{1}{2} 2^j\right) \right] \\ &= v(2) - \frac{2}{5} [v(4) + v(1)] \\ &= (4 - 2) - \frac{2}{5} \left[\frac{4}{4} + (4 - 1) \right] \\ &= 2 - \frac{2}{5} [1 + 3] \\ &= 2 - \frac{8}{5} \\ &= \frac{2}{5} \end{aligned}$$

and for $j \geq 2$

$$\begin{aligned}
c(2^j) &= v(2^j) - \frac{4}{5} \left[\frac{1}{2} v(2 \cdot 2^j) + \frac{1}{2} v\left(\frac{1}{2} 2^j\right) \right] \\
&= \frac{4}{2^j} - \frac{2}{5} \left[\frac{4}{2^{j+1}} + \frac{4}{2^{j-1}} \right] \\
&= \frac{4}{2^j} - \frac{2}{5} \frac{4}{2^{j-1}} \left[\frac{1}{4} + 1 \right] \\
&= \frac{4}{2^j} - \frac{8}{5 \cdot 2^{j-1}} \frac{5}{4} \\
&= \frac{4}{2^j} - \frac{2}{2^{j-1}} \\
&= \frac{4}{2^j} - \frac{4}{2^j} \\
&= 0
\end{aligned}$$

(ii) For $j \leq 0$ we find

$$\begin{aligned}
\delta(2^j) &= \frac{v(2^{j+1}) - v(2^{j-1})}{2^{j+1} - 2^{j-1}} \\
&= \frac{(4 - 2^{j+1}) - (4 - 2^{j-1})}{2^{j+1} - 2^{j-1}} \\
&= \frac{-(2^{j+1} - 2^{j-1})}{2^{j+1} - 2^{j-1}} \\
&= -1
\end{aligned}$$

for $j = 1$

$$\begin{aligned}
\delta(2^j) &= \frac{v(2^{j+1}) - v(2^{j-1})}{2^{j+1} - 2^{j-1}} \\
&= \frac{v(4) - v(1)}{4 - 1} \\
&= \frac{\frac{4}{4} - (4 - 1)}{3} \\
&= -\frac{2}{3}
\end{aligned}$$

and for $j \geq 2$

$$\begin{aligned}
\delta(2^j) &= \frac{v(2^{j+1}) - v(2^{j-1})}{2^{j+1} - 2^{j-1}} \\
&= \frac{\frac{4}{2^{j+1}} - \frac{4}{2^{j-1}}}{2^{j+1} - 2^{j-1}} \\
&= \frac{4}{2^{j-1}} \cdot \frac{\frac{1}{4} - 1}{2^{j+1} - 2^{j-1}} \\
&= -\frac{4}{2^{j-1}} \cdot \frac{\frac{3}{4}}{3 \cdot 2^{j-1}} \\
&= -\frac{1}{2^{j-1} \cdot 2^{j-1}} \\
&= -\frac{1}{2^{2j-2}} \\
&= -\frac{1}{2^{2j} 2^{-2}} \\
&= -\frac{4}{2^{2j}}
\end{aligned}$$

(iii)

Solution 5.8:

- (i)
- (ii)
- (iii)
- (iv)