Mathematical & Computational Finance I
Lecture Notes

Assignment 3 – DRAFT

Due: February 11 2016
Last update: December 4, 2017

Solution 3.1.i: Assume that $\tilde{\mathbb{P}}(\omega) > 0$, so $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$, then

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} > 0 \implies \frac{1}{Z(\omega)} > 0 \quad \forall_{\omega \in \Omega}$$

Therefore,

$$\begin{aligned}
\tilde{\mathbb{P}}\left(\frac{1}{Z(\omega)} > 0\right) &= \sum_{\left\{\omega \,:\, \frac{1}{Z(\omega)} > 0\right\}} \tilde{\mathbb{P}}(\omega) \\
&= \sum_{\{\omega \,:\, Z(\omega) > 0\}} \tilde{\mathbb{P}}(\omega) \\
&= \sum_{\omega \in \Omega} \tilde{\mathbb{P}}(\omega) \\
&= 1
\end{aligned}$$

as desired.

Solution 3.1.ii:

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\frac{1}{Z}\right] &= \sum_{\omega \in \Omega} \frac{1}{Z(\omega)} \tilde{\mathbb{P}}(\omega) \\
&= \sum_{\omega \in \Omega} \frac{\mathbb{P}(\omega)}{\tilde{\mathbb{P}}(\omega)} \tilde{\mathbb{P}}(\omega) \\
&= \sum_{\omega \in \Omega} \mathbb{P}(\omega) \\
&= 1
\end{aligned}$$

Solution 3.1.iii:

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{\omega\in\Omega} Y(\omega)\mathbb{P}(\omega) \\
&= \sum_{\omega\in\Omega} Y(\omega)\frac{\tilde{\mathbb{P}}(\omega)}{\tilde{\mathbb{P}}(\omega)}\mathbb{P}(\omega) \\
&= \sum_{\omega\in\Omega} Y(\omega)\frac{\mathbb{P}(\omega)}{\tilde{\mathbb{P}}(\omega)}\tilde{\mathbb{P}}(\omega) \\
&= \sum_{\omega\in\Omega} Y(\omega)\frac{1}{Z(\omega)}\tilde{\mathbb{P}}(\omega) \\
&= \tilde{\mathbb{E}}\left[Y\frac{1}{Z}\right]
\end{aligned}$$

as desired.

Solution 3.4.i: We have the state price density defined by

$$\zeta_n(\omega) = \frac{Z_n(\omega)}{(1+r)^n}$$

Hence

$$\begin{aligned}
\zeta_3(HHH) &= \frac{1}{\left(1+\frac{1}{4}\right)^3}\cdot\frac{27}{64} = \frac{64}{125}\cdot\frac{27}{64} = \frac{27}{125} \\
\zeta_3(HHT) = \zeta_3(HTH) = \zeta_3(THH) &= \frac{1}{\left(1+\frac{1}{4}\right)^3}\cdot\frac{27}{32} = \frac{64}{125}\cdot\frac{27}{32} = \frac{54}{125} \\
\zeta_3(HTT) = \zeta_3(THT) = \zeta_3(TTH) &= \frac{1}{\left(1+\frac{1}{4}\right)^3}\cdot\frac{27}{16} = \frac{64}{125}\cdot\frac{27}{16} = \frac{108}{125} \\
\zeta_3(TTT) &= \frac{1}{\left(1+\frac{1}{4}\right)^3}\cdot\frac{27}{8} = \frac{64}{125}\cdot\frac{27}{8} = \frac{216}{125}
\end{aligned}$$

Solution 3.4.ii: We first compute the payoffs V_3 of our Asian call option

$$
\begin{aligned}
V_3(HHH) &= \left(\frac{1}{4}60 - 4\right)^+ = 11 \\
V_3(HHT) &= \left(\frac{1}{4}36 - 4\right)^+ = 5 \\
V_3(HTH) &= \left(\frac{1}{4}24 - 4\right)^+ = 2 \\
V_3(HTT) &= \left(\frac{1}{4}18 - 4\right)^+ = 0.5 \\
V_3(THH) &= \left(\frac{1}{4}18 - 4\right)^+ = 0.5 \\
V_3(THT) &= \left(\frac{1}{4}12 - 4\right)^+ = 0 \\
V_3(TTH) &= \left(\frac{1}{4}9 - 4\right)^+ = 0 \\
V_3(TTT) &= \left(\frac{1}{4}7.5 - 4\right)^+ = 0
\end{aligned}
$$

and from Theorem 3.2.7 we have

$$V_n = \frac{1}{\zeta_n}\mathbb{E}_n[\zeta_N V_N]$$

so

$$
\begin{aligned}
V_0 &= \frac{1}{\zeta_0}\mathbb{E}_0[\zeta_3 V_3] \\
&= \frac{1}{Z_0}\mathbb{E}[\zeta_3 V_3]
\end{aligned}
$$

but $Z_0 = 1$, hence

$$
\begin{aligned}
V_0 &= \mathbb{E}[\zeta_3 V_3] \\
&= \zeta_3(HHH)V_3(HHH)\mathbb{P}(HHH) + \zeta_3(HHT)V_3(HHT)\mathbb{P}(HHT) + \\
&\qquad \zeta_3(HTH)V_3(HTH)\mathbb{P}(HTH) + \zeta_3(HTT)V_3(HTT)\mathbb{P}(HTT) + \\
&\qquad \zeta_3(THH)V_3(THH)\mathbb{P}(THH) + \zeta_3(THT)V_3(THT)\mathbb{P}(THT) + \\
&\qquad \zeta_3(TTH)V_3(TTH)\mathbb{P}(TTH) + \zeta_3(TTT)V_3(TTT)\mathbb{P}(TTT) \\
&= \left[\frac{27}{125}\cdot 11 \cdot \frac{8}{27}\right] + \left[\frac{54}{125}\cdot 5 \cdot \frac{4}{27}\right] + \left[\frac{54}{125}\cdot 2 \cdot \frac{4}{27}\right] + \left[\frac{108}{125}\cdot 0.5 \cdot \frac{2}{27}\right] + \left[\frac{54}{125}\cdot 0.5 \cdot \frac{4}{27}\right] + \\
&\qquad \left[\frac{108}{125}\cdot 0 \cdot \frac{2}{27}\right] + \left[\frac{108}{125}\cdot 0 \cdot \frac{2}{27}\right] + \left[\frac{216}{125}\cdot 0 \cdot \frac{1}{27}\right] \\
&= \frac{152}{125} \\
&= 1.216
\end{aligned}
$$

as desired.

Solution 3.4.iii: We have

$$\begin{aligned}
\zeta_2(TH) = \zeta_2(HT) &= \frac{Z_2(HT)}{\left(1+\frac{1}{4}\right)^2} \\
&= \frac{\frac{9}{8}}{\frac{25}{16}} \quad \left(Z_2(HT) = \frac{9}{8} \text{ from Fig. 3.2.1}\right) \\
&= \frac{18}{25} \\
&= 0.72
\end{aligned}$$

as desired.

Solution 3.4.iv: We have

$$\begin{aligned}
V_2(HT) &= \frac{1}{\zeta_2(HT)}\mathbb{E}_2[\zeta_3 V_3](HT) \\
&= \frac{25}{18}\mathbb{E}_2[\zeta_3 V_3](HT) \\
&= \frac{25}{18}\left[\zeta_3(HTH)V_3(HTH)\mathbb{P}(H) + \zeta_3(HTT)V_3(HTT)\mathbb{P}(T)\right] \\
&= \frac{25}{18}\left[\left[\frac{54}{125}\cdot 2\cdot\frac{2}{3}\right] + \left[\frac{108}{125}\cdot 0.5\cdot\frac{1}{3}\right]\right] \\
&= 1
\end{aligned}$$

and for $V_2(TH)$ we have

$$\begin{aligned}
V_2(TH) &= \frac{1}{\zeta_2(TH)}\mathbb{E}_2[\zeta_3 V_3](TH) \\
&= \frac{25}{18}\mathbb{E}_2[\zeta_3 V_3](TH) \\
&= \frac{25}{18}\left[\zeta_3(THH)V_3(THH)\mathbb{P}(H) + \zeta_3(THT)V_3(THT)\mathbb{P}(T)\right] \\
&= \frac{25}{18}\left[\left[\frac{54}{125}\cdot 0.5\cdot\frac{2}{3}\right] + \left[\frac{108}{125}\cdot 0\cdot\frac{1}{3}\right]\right] \\
&= \frac{1}{5} \\
&= 0.20
\end{aligned}$$

as desired.

Solution 3.5.i: We had the risk-neutral measure

$$\tilde{\mathbb{P}}(HH) = \frac{1}{4} \quad \tilde{\mathbb{P}}(HT) = \frac{1}{4} \quad \tilde{\mathbb{P}}(TH) = \frac{1}{12} \quad \tilde{\mathbb{P}}(TT) = \frac{5}{12}$$

hence

$$
\begin{aligned}
Z(HH) &= \frac{\frac{1}{4}}{\frac{4}{9}} = \frac{9}{16} \\
Z(HT) &= \frac{\frac{1}{4}}{\frac{2}{9}} = \frac{9}{8} \\
Z(TH) &= \frac{\frac{1}{12}}{\frac{2}{9}} = \frac{9}{24} = \frac{3}{8} \\
Z(TT) &= \frac{\frac{5}{12}}{\frac{1}{9}} = \frac{45}{12} = \frac{15}{4}
\end{aligned}
$$

Solution 3.5.ii: By Theorem 3.2.1 we have

$$
Z_n = \mathbb{E}_n[Z]
$$

So, with $Z_2 = Z$ and $n = 1$ we have

$$
\begin{aligned}
Z_1(H) &= \mathbb{E}_1[Z](H) \\
&= Z_2(HH)\mathbb{P}(H|H) + Z_2(HT)\mathbb{P}(T|H) \\
&= \frac{9}{16} \cdot \frac{\mathbb{P}(HH)}{\mathbb{P}(HH) + \mathbb{P}(HT)} + \frac{9}{8} \cdot \frac{\mathbb{P}(HT)}{\mathbb{P}(HH) + \mathbb{P}(HT)} \\
&= \frac{9}{16} \cdot \frac{\frac{4}{9}}{\frac{4}{9} + \frac{2}{9}} + \frac{9}{8} \cdot \frac{\frac{2}{9}}{\frac{4}{9} + \frac{2}{9}} \\
&= \frac{3}{4} \\
Z_1(T) &= \mathbb{E}_1[Z](T) \\
&= Z_2(TH)\mathbb{P}(H|T) + Z_2(TT)\mathbb{P}(T|T) \\
&= \frac{3}{8} \cdot \frac{\mathbb{P}(TH)}{\mathbb{P}(TH) + \mathbb{P}(TT)} + \frac{15}{4} \cdot \frac{\mathbb{P}(TT)}{\mathbb{P}(TH) + \mathbb{P}(TT)} \\
&= \frac{3}{8} \cdot \frac{\frac{2}{9}}{\frac{2}{9} + \frac{1}{9}} + \frac{15}{4} \cdot \frac{\frac{1}{9}}{\frac{2}{9} + \frac{1}{9}} \\
&= \frac{3}{2} \\
Z_0 &= \mathbb{E}_0[Z_1] \\
&= Z_1(H)\mathbb{P}(\omega_1 = H) + Z_1(T)\mathbb{P}(\omega_1 = T) \\
&= \frac{3}{4}\left[\frac{4}{9} + \frac{2}{9}\right] + \frac{3}{2}\left[\frac{2}{9} + \frac{1}{9}\right] \\
&= 1
\end{aligned}
$$

as desired.

Solution 3.5.iii: We had payoffs

$$
\begin{aligned}
V_2(HH) &= 5 \\
V_2(HT) &= 1 \\
V_2(TH) &= 1 \\
V_2(TT) &= 0
\end{aligned}
$$