Mathematical & Computational Finance I
Lecture Notes

Probability Theory on Coin Toss Space

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1 Markov Processes

Recall that last time wee looked at the example of a simple lookback option. We had shown that the joint $\{S_n, M_n\}_{n=0}^N$ process, with

$$M_n = \max_{0 \le k \le n} S_k$$

with time N payoff

$$V_N = M_N - S_N$$

is Markovian. We had shown that $V_N = \nu(s, m)$ exists by the Markov property, and by the Independence Lemma

$$\nu_n(s,m) = \frac{1}{1+r}\left[\tilde{p}\nu_{n+1}(us, \max\{m, us\}) + \tilde{q}\nu_{n+1}(ds, m)\right] \quad n = 0, 1, ..., N-1$$

In general, if we have $V_N = \nu_N(S_N, M_N)$ then the multiple time step Markov property of $\{S_N, M_N\}$ and the risk-neutral pricing formula gives us

$$V_n = \nu_n(S_n, M_n) = \tilde{\mathbb{E}}_n\left[\frac{\nu_N(S_N, M_N)}{(1+r)^{N-n}}\right]$$

Note that the Markov property doesn't actually give us what the function is and in order to make a computable algorithm we rely on the Independence Lemma.

Example: Derive the general recursive algorithm for calculating the time n price of a derivative security whose payoff depends on the asset price at time N and its maximum price to time N.

Solution: Suppose that for time time $n \in \{0, ..., N-1\}$ we have computed the function ν_{n+1}

$$V_{n+1} = \nu_{n+1}(S_{n+1}, M_{n+1})$$

then, write

$$\begin{aligned}
V_n &= \frac{1}{1+r}\tilde{\mathbb{E}}_n[V_{n+1}] \quad \text{(from risk-neutral pricing)} \\
&= \frac{1}{1+r}\tilde{\mathbb{E}}_n\left[\nu_{n+1}(S_{n+1}, M_{n+1})\right] \\
&= \frac{1}{1+r}\tilde{\mathbb{E}}_n\left[\nu_{n+1}\left(S_n\frac{S_{n+1}}{S_n}, \max\left\{M_n, S_n\frac{S_{n+1}}{S_n}\right\}\right)\right]
\end{aligned}$$

However, we have that $\frac{S_{n+1}}{S_n}$ is independent of the first n coin tosses with the rest of the terms adapted to the first n tosses. Therefore, by the independence Lemma

$$V_n = \nu_n(S_n, M_n)$$

where $\nu_n(s,m)$ is the ordinary expectation

$$\begin{aligned}
\nu_n(s,m) &= \tilde{\mathbb{E}}\left[\nu_{n+1}\left(s\frac{S_{n+1}}{S_n}, \max\left\{m, s\frac{S_{n+1}}{S_n}\right\}\right)\right] \\
&= \frac{1}{1+r}[\tilde{p}\nu_{n+1}(us, \max\{m, us\}) + \tilde{q}\nu_{n+1}(ds, \max\{m, ds\})]
\end{aligned}$$

Note that $M_n \geq S_n$ so we only need to know the function $\nu_n(s,m)$ when $m \geq s$. However, when $m \geq s$ and if $d \leq 1$ we have $\max\{m, ds\} = m$, so

$$\nu_n(s,m) = \frac{1}{1+r}[\tilde{p}\nu_{n+1}(us, \max\{m, us\}) + \tilde{q}\nu_{n+1}(ds, m)]$$

for $m \geq s > 0$ and $N = 0, ..., N-1$.

Theorem 1. Let $\{X_n\}_{n=0}^N$ be a Markov process under $\tilde{\mathbb{P}}$. Suppose that for some function $\nu_N(x)$ the payoff at time N of a derivative security is $\nu_N(X_N)$. Then, for each $n = \{0, ..., N\}$ the price V_n of the derivative security is some function ν_n of X_n

$$V_n = \nu_n(X_n) \quad n = 0, 1, ..., N$$

Furthermore, there exists a recursive algorithm for computing ν_n whose exact formula depends on the underlying Markovian process $\{X_n\}_{n=0}^N$.[1]

Example: Consider an N-period binomial model. An *Asian option* has a payoff based on the average of the stock price, i.e.,

$$V_N = f\left(\frac{1}{1+N}\sum_{n=0}^{N} S_n\right)$$

where f is determined by the details of the derivative contract

[1] I think we interpret this as that we are guaranteed a function of the time n process X_n of a security whose time N payoff is a function of the same process X_N.

(A) Define

$$Y_n = \sum_{k=0}^{n} S_k$$

and use the Independence Lemma to show that the two-dimensional process $\{S_n, Y_n\},\ n = 0, 1, ..., N$ is Markovian.

Solution:

(i) Adapted? Note that S_n and $Y_n = \sum_{k=0}^{n} S_k$ depend only on the first n coin tosses, therefore $\{S_n, Y_n\}_{n=0}^{N}$ is adapted.

(ii) Markov property? Let h be an arbitrary function $h(s, y)$. We must show that there exists a function g such that

$$\mathbb{E}_n[h(S_{n+1}, Y_{n+1})] = g(S_n, Y_n) \quad 0 \le n \le N$$

Write

$$\begin{aligned} S_{n+1} &= S_n \frac{S_{n+1}}{S_n} \\ Y_{n+1} &= Y_n + S_{n+1} \\ &= Y_n + S_n \frac{S_{n+1}}{S_n} \end{aligned}$$

then we have that $\frac{S_{n+1}}{S_n}$ is independent of the first n coin tosses with the remaining terms adapted to the first n coin tosses. So

$$g(S_n, Y_n) = \mathbb{E}_n[h(S_{n+1}, Y_{n+1})] = \mathbb{E}_n \left[h\left(S_n \frac{S_{n+1}}{S_n}, Y_n + S_n \frac{S_{n+1}}{S_n} \right) \right]$$

Now, using the Independence Lemma to find g as the ordinary expectation

$$\begin{aligned} g(s, y) &= \mathbb{E} \left[h\left(S_n \frac{S_{n+1}}{S_n}, Y_n + S_n \frac{S_{n+1}}{S_n} \right) \right] \\ &= ph(us, y + us) + qh(ds, y + ds) \end{aligned}$$

Therefore, the process $\{S_n, Y_n\}_{n=0}^{N}$ is Markovian by definition. We note that this holds under $\tilde{\mathbb{P}}$ and arbitrary measure $\mathbb{P}$.

(B) According to Theorem 2.5.8 the price V_n of the Asian option at time n is some function ν_n of S_n and Y_n, i.e.,

$$V_n = \nu_n(S_n, Y_n) \quad n = 0, 1, ..., N$$

Give a formula for $\nu_N(s, y)$ and prove an algorithm for computing $\nu_n(s, y)$ in terms of ν_{n+1} in terms of ν_{n+1}.

Solution: With terminal condition

$$V_N = f\left(\frac{1}{N+1}Y_N\right)$$

and assuming we have some function $\nu_{n+1}(s, y)$ such that $V_{n+1} = \nu_{n+1}(S_{n+1}, Y_{n+1})$ for $0 \leq n \leq N-1$ then

$$\begin{aligned}
V_n &= \tilde{\mathbb{E}}_n\left[\frac{1}{1+r}V_{n+1}\right] \quad \text{(from risk-neutral pricing)} \\
&= \tilde{\mathbb{E}}_n\left[\frac{1}{1+r}\nu_{n+1}(V_{n+1}, Y_{n+1})\right] \quad \text{(by definition of } V_{n+1}\text{)} \\
&= \nu_n(S_n, Y_n) \quad \text{(by the Markov property from part } \mathbf{(A)}\text{)}
\end{aligned}$$

where by the Independence Lemma we had that

$$\nu_n(s, y) = \frac{1}{1+r}[\tilde{p}\nu_{n+1}(us, y + us) + \tilde{q}\nu_{n+1}(ds, y + ds)]$$

1.1 Examples & Exercises

Example: Dividend-paying stock.

Consider a binomial asset pricing model except that after each movement in the stock price a dividend is paid and the value of the stock is reduced according to the dividend. Define

$$Y_{n+1}(\omega_1 \cdots \omega_n \omega_{n+1}) = \begin{cases} u & \text{if } \omega_{n+1} = H \\ d & \text{if } \omega_{n+1} = T \end{cases}$$

Consider a random variable $A_{n+1}(\omega_1 \cdots \omega_n \omega_{n+1}) \in (0, 1)$ and the dividend paid at time $n+1$ is $A_{n+1}Y_{n+1}S_n$. After the dividend is paid the stock price at time $n+1$ is[2]

$$S_{n+1} = (1 - A_{n+1})Y_{n+1}S_n$$

An agent who begins with initial capital X_0 and at each time n takes position Δ_n shares in the risky asset, where Δ_n depends only on the first n coin tosses, has a portfolio value governed by the wealth equation

$$\begin{aligned}
X_{n+1} &= \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n) + \Delta_n A_{n+1} Y_{n+1} S_n \\
&= \Delta_n(1 - A_{n+1})Y_{n+1}S_n + (1+r)(X_n - \Delta_n S_n) + \Delta_n A_{n+1} Y_{n+1} S_n \\
&= \Delta_n Y_{n+1} S_n + (1+r)(X_n - \Delta_n S_n)
\end{aligned}$$

(A) Show that the discounted wealth process is a martingale under the risk-neutral measure.

Solution:

[2] We can think of $Y_{n+1}S_n$ is the price that the stock *would have been* had a dividend not been paid and A_{n+1} as the proportion of the assets paid as the dividend.

(i) Adapted? We proceed by induction. First, $P(0)\ :\ X_0$ is constant and so its adapted. Now, assume $P(n)\ :\ X_n$ is holds, that is, X_n depends on the first n coin tosses $\omega_1 \cdots \omega_n$. Then, we have that from the equation above X_{n+1} depends only on the first $n+1$ coin tosses since Δ_n and S_n depend on the first n coin tosses and Y_{n+1} depends only on the $(n+1)^{\text{th}}$ coin toss. Hence, $P(n+1)\ : X_{n+1}$ is adapted, holds. Therefore, by induction, we have that X_n is adapted, as desired.

(ii) Martingale property? We have

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \tilde{\mathbb{E}}_n\left[\frac{\Delta_n Y_{n+1} S_n + (1+r)(X_n - \Delta_n S_n)}{(1+r)^{n+1}}\right] \quad \text{(by definition)}\\
&= \tilde{\mathbb{E}}_n\left[\frac{\Delta_n Y_{n+1} S_n}{(1+r)^{n+1}}\right] + \tilde{\mathbb{E}}_n\left[\frac{(1+r)(X_n - \Delta_n S_n)}{(1+r)^{n+1}}\right] \quad \text{(linearity)}\\
&= \tilde{\mathbb{E}}_n\left[\frac{\Delta_n Y_{n+1} S_n}{(1+r)^{n+1}}\right] + \tilde{\mathbb{E}}_n\left[\frac{X_n}{(1+r)^{n}}\right] - \tilde{\mathbb{E}}_n\left[\frac{\Delta_n S_n}{(1+r)^{n}}\right] \quad \text{(linearity again)}\\
&= \frac{\Delta_n S_n}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n\left[Y_{n+1}\right] + \frac{X_n}{(1+r)^n} - \frac{\Delta_n S_n}{(1+r)^n} \quad \text{(adaptedness)}
\end{aligned}$$

but Y_{n+1} is independent of the first n coin tosses, so $\tilde{E}_n[Y_{n+1}] = \tilde{E}[Y_{n+1}]$, hence

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \frac{\Delta_n S_n}{(1+r)^{n+1}}\tilde{\mathbb{E}}\left[Y_{n+1}\right] + \frac{X_n}{(1+r)^n} - \frac{\Delta_n S_n}{(1+r)^n}\\
&= \frac{\Delta_n S_n}{(1+r)^{n+1}}\left(\tilde{\mathbb{E}}\left[Y_{n+1}\right] - 1\right) + \frac{X_n}{(1+r)^n}\\
&= \frac{\Delta_n S_n}{(1+r)^{n+1}}\left(\frac{\tilde{p}u + \tilde{q}d}{1+r} - 1\right) + \frac{X_n}{(1+r)^n}
\end{aligned}$$

but

$$\begin{aligned}
\frac{\tilde{p}u + \tilde{q}d}{1+r} - 1 &= \frac{\tilde{p}u + \tilde{q}d - (1+r)}{1+r}\\
&= \frac{\frac{(1+r)-d}{u-d}u + \frac{u-(1+r)}{u-d}d - (1+r)}{1+r}\\
&= \frac{\frac{(1+r)u - du + ud - (1+r)d}{u-d} - (1+r)}{1+r}\\
&= \frac{\frac{u-d}{u-d}(1+r) - (1+r)}{1+r}\\
&= 0
\end{aligned}$$

so

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \frac{\Delta_n S_n}{(1+r)^{n+1}}\left(\frac{\tilde{p}u + \tilde{q}d}{1+r} - 1\right) + \frac{X_n}{(1+r)^n}\\
&= \frac{X_n}{(1+r)^n}
\end{aligned}$$

as desired.

(B) Show that the risk-neutral pricing formula still applies.

Solution: We define recursively, backwards in time,

$$V_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)] \quad n = 0, ..., N-1$$

Then, with $X_0 = V_0$ constant and X_{n+1} defined by

$$X_{n+1} = \Delta_n Y_{n+1} S_n + (1+r)(X_n - \Delta_n S_n)$$

and Δ_n defined the usual way we claim that

$$X_n(\omega_1 \cdots \omega_n) = V_n(\omega_1 \cdots \omega_n) \quad n = 0, ..., N, \ \forall_{\omega \in \Omega}$$

We proceed inductively. We begin with $P(0) \ : \ X_0 = V_0$ is true by definition. Now, assume that $P(n) \ : \ X_n = V_n$ holds for $0 \leq n \leq N-1$ for arbitrary fixed $\omega_1 \cdots \omega_n$. For $P(n+1)$ begin with $\omega_{n+1} = H$

$$\begin{aligned}
X_{n+1}(H) &= \Delta_n Y_{n+1}(H) S_n + (1+r)(X_n - \Delta_n S_n) \\
&= \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} u S_n + (1+r)\left[X_n - \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} S_n\right] \quad \text{(definition of } \Delta_n\text{)} \\
&= \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} u S_n + (1+r)\left[V_n - \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} S_n\right] \quad \text{(inductive hypothesis)} \\
&= \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} u S_n + (1+r)V_n - (1+r)\frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} S_n \\
&= \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} S_n(u - (1+r)) + (1+r)V_n \\
&= (V_{n+1}(H) - V_{n+1}(T))\frac{u-(1+r)}{u-d} + (1+r)V_n \\
&= \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T) + (1+r)V_n \\
&= \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T) + (1+r)\frac{\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)}{1+r} \quad \text{(from risk-neutral pricing)} \\
&= \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T) + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= (\tilde{p} + \tilde{q})V_{n+1}(H) \\
&= V_{n+1}(H)
\end{aligned}$$

Similarly, we can show that $X_{n+1}(\omega_1 \cdots \omega_n T) = V_{n+1}(\omega_1 \cdots \omega_n T)$. Therefore, we have that $P(n) \ : \ X_n = V_n$ holds for all $0 \leq n \leq N$ by induction. Finally, define

$$V'_n = \tilde{\mathbb{E}}_n \left[\frac{V_N}{(1+r)^{N-n}}\right] \quad n = 0, 1, ..., N-1$$

From this we get the sequence/process

$$V_0', \frac{V_1'}{1+r}, \cdots, \frac{V_{N-1}'}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N}$$

is a $\tilde{\mathbb{P}}$-martingale since

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{V_{n+1}'}{(1+r)^{n+1}}\right] &= \tilde{\mathbb{E}}_n\left[\frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_{n+1}\left[\frac{V_N'}{(1+r)^{N-(n+1)}}\right]\right] && \text{(risk-neutral pricing)} \\
&= \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^N}\right] && \text{(tower property)} \\
&= \frac{1}{(1+r)^n}\tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right] \\
&= \frac{V_n'}{(1+r)^n}
\end{aligned}$$

and by Assignment 2, Exercise 2.8 we have that $X_n = V_n = V_n'$ for all n. Therefore we have that the risk-neutral pricing formula still holds, as desired.

(C) Show that the discounted stock price is *not* a martingale under the risk-neutral measure. However, if $A_{n+1} = a \in (0,1)$ regardless of the value of n and the outcome of the coin tosses then $\frac{S_n}{(1-a)^n(1+r)^n}$ is a martingale under the risk-neutral measure.

Solution: Note that for $n = 0, 1, ..., N-1$ we have

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] &= \tilde{\mathbb{E}}_n\left[\frac{(1-A_{n+1})Y_{n+1}S_n}{(1+r)^{n+1}}\right] \quad \text{(by definition)} \\
&= \frac{S_n}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n[(1-A_{n+1})Y_{n+1}] \quad \text{(adaptedness)} \\
&= \frac{S_n}{(1+r)^{n+1}}\tilde{\mathbb{E}}[(1-A_{n+1})Y_{n+1}] \quad \text{(Independence Lemma)} \\
&= \frac{S_n}{(1+r)^{n+1}}[\tilde{p}(1-A_{n+1}(H))u + \tilde{q}(1-A_{n+1})(T))d] \\
&= \frac{S_n}{(1+r)^{n+1}}[\tilde{p}u - \tilde{p}A_{n+1}(H)u + \tilde{q}d - \tilde{q}A_{n+1}(T)d] \\
&= \frac{S_n}{(1+r)^{n+1}}[\tilde{p}u + \tilde{q}d] - \frac{S_n}{(1+r)^{n+1}}[\tilde{p}A_{n+1}(H)u + \tilde{q}A_{n+1}(T)d] \\
&= \frac{S_n}{(1+r)^n}\frac{\tilde{p}u + \tilde{q}d}{1+r} - \frac{S_n}{(1+r)^{n+1}}[\tilde{p}A_{n+1}(H)u + \tilde{q}A_{n+1}(T)d] \\
&= \frac{S_n}{(1+r)^n}\frac{\frac{(1+r)-d}{u-d}u + \frac{u-(1+r)}{u-d}d}{1+r} - \frac{S_n}{(1+r)^{n+1}}[\tilde{p}A_{n+1}(H)u + \tilde{q}A_{n+1}(T)d] \\
&= \frac{S_n}{(1+r)^n}\frac{\frac{(1+r)u-du+ud-(1+r)d}{u-d}}{1+r} - \frac{S_n}{(1+r)^{n+1}}[\tilde{p}A_{n+1}(H)u + \tilde{q}A_{n+1}(T)d] \\
&= \frac{S_n}{(1+r)^n} - \frac{S_n}{(1+r)^{n+1}}[\tilde{p}A_{n+1}(H)u + \tilde{q}A_{n+1}(T)d]
\end{aligned}$$

but

$$\frac{S_n}{(1+r)^n} - \frac{S_n}{(1+r)^{n+1}}[\tilde{p}A_{n+1}(H)u + \tilde{q}A_{n+1}(T)d] < \frac{S_n}{(1+r)^n}$$

since $\frac{S_n}{(1+r)^{n+1}}[\tilde{p}A_{n+1}(H)u + \tilde{q}A_{n+1}(T)d] > 0$. That is, the discounted asset price with dividends is no longer a $\tilde{\mathbb{P}}$-martingale. However, if we consider the process $\frac{S_n}{(1-a)^n(1+r)^n}$ with $A_{n+1} = a \in (0, 1$ for all n we have

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1-a)^{n+1}(1+r)^{n+1}}\right] &= \frac{(1-a)S_n}{(1-a)^{n+1}(1+r)^{n+1}}\tilde{\mathbb{E}}_n[Y_{n+1}] \quad \text{(since } S_{n+1} = (1-a)S_nY_{n+1}) \\
&= \frac{(1-a)S_n}{(1-a)^{n+1}(1+r)^{n+1}}\tilde{\mathbb{E}}[Y_{n+1}] \quad \text{(Independence Lemma)} \\
&= \frac{(1-a)S_n}{(1-a)^{n+1}(1+r)^{n+1}}\left[(1+r)\frac{\tilde{p}u + \tilde{q}d}{1+r}\right] \\
&= \frac{S_n}{(1-a)^n(1+r)^n}
\end{aligned}$$

Hence, the process $\frac{S_n}{(1-a)^n(1+r)^n}$ is a $\tilde{\mathbb{P}}$-martingale.

Example: Put-call parity.

Consider a stock that pays no dividends in an N-period binomial model. A European call has payoff $C_N = (S_N - K)^+$ at time N and a European put has time N payoff $P_N = (K - S_N)^+$. The price of the call and put, denoted C_n and P_n respectively, are given by the risk-neutral pricing formula.

A *forward contract* to buy one share of the stock at time N for delivery price K has payoff $F_N = S_N - K$ and its price at earlier times is denoted by F_n and given by the risk-neutral pricing formula.

(A) If at time zero we buy one forward contract and one put option and hold them until expiry, explain why the payoff we receive is the same as the payoff of the call. That is, show that

$$C_N = F_N + P_N$$

Solution: Note that

$$P_N = \begin{cases} K - S_N & \text{if } K > S_N \\ 0 & \text{if } K \leq S_N \end{cases}$$

and

$$C_N = \begin{cases} 0 & \text{if } K > S_N \\ K - S_N & \text{if } K \leq S_N \end{cases}$$

Therefore

$$\begin{aligned} F_N + P_N &= \begin{cases} (S_N - K) + (K - S_N) & \text{if } K > S_N \\ (S_N - K) + (0) & \text{if } K \leq S_N \end{cases} \\ &= \begin{cases} 0 & \text{if } K > S_N \\ S_N - K & \text{if } K \leq S_N \end{cases} \\ &= C_N \end{aligned}$$

(B) Using the risk-neutral pricing formulae for C_n, P_n and F_n and the properties of conditional expectations show that $C_n = F_n + P_n$ for $0 \leq n \leq N$.

Solution: Note

$$\begin{aligned} F_n + P_n &= \tilde{\mathbb{E}}_n\left[\frac{F_N}{(1+r)^{N-n}}\right] + \tilde{\mathbb{E}}_n\left[\frac{P_N}{(1+r)^{N-n}}\right] \quad \text{(risk-neutral pricing)} \\ &= \tilde{\mathbb{E}}_n\left[\frac{F_N + P_N}{(1+r)^{N-n}}\right] \quad \text{(linearity)} \\ &= \tilde{\mathbb{E}}_n\left[\frac{C_N}{(1+r)^{N-n}}\right] \quad \text{(from } \textbf{(A)}\text{)} \\ &= C_n \quad \text{(risk-neutral pricing)} \end{aligned}$$

(C) Using the fact that the discounted stock price is a $\tilde{\mathbb{P}}$-martingale show that $F_0 = S_0 - \frac{K}{(1+r)^N}$.

Solution: Note

$$\begin{aligned}
F_0 &= C_0 - P_0 \quad \text{(from } \mathbf{(B)}\text{)} \\
&= \tilde{\mathbb{E}}_0\left[\frac{C_N}{(1+r)^N}\right] - \tilde{\mathbb{E}}_0\left[\frac{P_N}{(1+r)^N}\right] \quad \text{(risk-neutral pricing)} \\
&= \tilde{\mathbb{E}}_0\left[\frac{C_N - P_N}{(1+r)^N}\right] \quad \text{(linearity)} \\
&= \tilde{\mathbb{E}}_0\left[\frac{S_N - K}{(1+r)^N}\right] \quad \text{(since } C_N - P_N = F_N = S_N - K\text{)} \\
&= \tilde{\mathbb{E}}_0\left[\frac{S_N}{(1+r)^N}\right] - \frac{K}{(1+r)^N} \quad \text{(linearity)} \\
&= S_0 - \frac{K}{(1+r)^N} \quad \text{(martingale property)}
\end{aligned}$$

(D) Suppose we begin at time zero with F_0, buy one share of the risk asset, borrowing the necessary money to do so, and make no further trades. Show that at time N we have the portfolio valued at F_N.[3]

Solution: Consider the following portfolio: At time $t = 0$

1. Borrow $\frac{K}{(1+r)^N}$ from the bank account
2. Use the cash
$$F_0 + \frac{K}{(1+r)^N} = \left(S_0 - \frac{K}{(1+r)^N}\right) + S_0 = S_0$$
to buy one share of the stock.

Then, at time $t = N$

1. We owe K on the loan
2. The stock position is worth S_N
3. The net value of the portfolio is then
$$X_N = S_N - K = F_N$$

(E) The **forward price** of the stock at time zero is define to be the value of K that causes the forward contract to have price zero at time zero. The forward price in this model is $(1+r)^N S_0$.

[3]This is called **static replication** of the forward contract. If you sell the forward contract for F_0 at time zero you can use this static replication to hedge the short position in the forward contract.)

Show that, at time zero, the price of a call struck at the forward price is the same as the price of a put stuck at the forward price. This fact is known as **put-call parity**.

Solution: Let $F(0, S_0)$, the forward price, be such that $K = F(0, S_0)$ the price of the forward contract at time zero is $F_0 = 0$. That is,

$$0 = F_0 = S_0 - \frac{F(0, S_0)}{(1+r)^N} \quad (\text{since } F_n = S_n - \frac{K}{(1+r)^{N-n}})$$

implying that $S_0 = \frac{F(0,S_0)}{(1+r)^N}$. Solving for $F(0, S_0)$ gives us

$$F(0, S_0) = (1+r)^N S_0$$

as desired. Now, at time zero the prices of a call and put option struck at $K = F(0, S_0) = (1+r)^N S_0)$ is

$$\begin{aligned} C_0 &= \tilde{\mathbb{E}}\left[\frac{(S_N - (1+r)^N S_0)^+}{(1+r)^N}\right] \\ P_0 &= \tilde{\mathbb{E}}\left[\frac{((1+r)^N S_0 - S_N)^+}{(1+r)^N}\right] \end{aligned}$$

Then

$$\begin{aligned} C_0 - P_0 &= \tilde{\mathbb{E}}\left[\frac{(S_N - (1+r)^N S_0)^+}{(1+r)^N}\right] - \tilde{\mathbb{E}}\left[\frac{((1+r)^N S_0 - S_N)^+}{(1+r)^N}\right] \\ &= \tilde{\mathbb{E}}\left[\frac{(S_N - (1+r)^N S_0)^+ - ((1+r)^N S_0 - S_N)^+}{(1+r)^N}\right] \quad (\text{linearity}) \\ &= \tilde{\mathbb{E}}\left[\frac{S_N - (1+r)^N S_0}{(1+r)^N}\right] \quad (\text{since } C_N - P_N = S_N - K) \\ &= \tilde{\mathbb{E}}\left[\frac{S_N}{(1+r)^N}\right] - S_0 \quad (\text{linearity}) \\ &= S_0 - S_0 \quad (\text{martingale property of } S_n) \\ &= 0 \end{aligned}$$

So, we have that $C_0 = P_0$ when $K = (1+r)^N S_0$.

(F) If we choose $K = (1+r)^N S_0$ we have $C_0 = P_0$. Do we have $C_n = P_n$ for every n?

Solution: By party **(B)** we have

$$\begin{aligned} C_n - P_n = F_n &= \tilde{\mathbb{E}}_n\left[\frac{S_N - (1+r)^N S_0}{(1+r)^{N-n}}\right] \\ &= \tilde{\mathbb{E}}_n\left[\frac{S_N}{(1+r)^{N-n}}\right] - (1+r)^n S_0 \\ &= S_n - (1+r)^n S_0 \end{aligned}$$

and $S_n - (1+r)^n S_0$ is not necessarily equal to 0 for all n and all coin toss sequences $\omega_1 \cdots \omega_n$ since

$$S_n(\omega_1 \cdots \omega_n) = u^{\#H(\omega_1 \cdots \omega_n)} d^{\#T(\omega_1 \cdots \omega_n)} S_0$$

and $0 < d < 1 + r < u$.

Example: Choose options

Let $1 \leq m \leq N - 1$ and $K > 0$ be given. A **chooser option** is a contract sold at time zero that gives the owner the right to receive either a call or a put option at time m. The owner of the chooser may wait until time m before choosing. The call or put chosen expires at time N with strike K.

Show that the time-zero price of a chooser option is the sum of the time-zero price of a put, expiring at time N and having strike price K, and a call expiring at time m having strike price $\frac{K}{(1+r)^{N-m}}$.

Solution: At time $t = m$ the invest will be given the right to receive the call option if the time m price $C_m > P_m$. That is, the time m price of the call expiring at N exceeds the time m price of the put expiring at N. Otherwise the investor would choose the put option.

The holder of the chooser can, at time $t = m$, sell the option that was chosen. Therefore, the time zero price of the chooser option is

$$V_0 = \tilde{\mathbb{E}}\left[\frac{\max\{C_m, P_m\}}{(1+r)^m}\right]$$

and using the put-call parity result from the previous exercise, write

$$\begin{aligned}
\max\{C_m, P_m\} &= \max\{F_m + P_m, P_m\} \\
&= P_m + \max\{F_m, 0\} \\
&= P_m + \max\left\{S_m - \frac{K}{(1+r)^{N-m}}, 0\right\}
\end{aligned}$$

Hence

$$
\begin{aligned}
V_0 &= \tilde{\mathbb{E}}\left[\frac{\max\{C_m, P_m\}}{(1+r)^m}\right] \\
&= \tilde{\mathbb{E}}\left[\frac{P_m + \max\left\{S_m - \frac{K}{(1+r)^{N-m}}, 0\right\}}{(1+r)^m}\right] \\
&= \tilde{\mathbb{E}}\left[\frac{P_m}{(1+r)^m}\right] + \tilde{\mathbb{E}}\left[\frac{\max\left\{S_m - \frac{K}{(1+r)^{N-m}}, 0\right\}}{(1+r)^m}\right] \\
&= \tilde{\mathbb{E}}\left[\frac{1}{(1+r)^m}\tilde{\mathbb{E}}_m\left[\frac{(K-S_N)^+}{(1+r)^{N-m}}\right]\right] + \tilde{\mathbb{E}}\left[\frac{\max\left\{S_m - \frac{K}{(1+r)^{N-m}}, 0\right\}}{(1+r)^m}\right] \qquad \text{(definition of } P_m\text{)} \\
&= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}_m\left[\frac{(K-S_N)^+}{(1+r)^N}\right]\right] + \tilde{\mathbb{E}}\left[\frac{\max\left\{S_m - \frac{K}{(1+r)^{N-m}}, 0\right\}}{(1+r)^m}\right] \\
&= \tilde{\mathbb{E}}\left[\frac{(K-S_N)^+}{(1+r)^N}\right] + \tilde{\mathbb{E}}\left[\frac{\max\left\{S_m - \frac{K}{(1+r)^{N-m}}, 0\right\}}{(1+r)^m}\right] \qquad \text{(tower property)} \\
&= P_0(N, K) + C_0\left(m, \frac{K}{(1+r)^{N-m}}\right)
\end{aligned}
$$

which is the sum of the price at the time zero price of a put option with expiry at time N and strike K and the time zero price of a call option with expiry at time m and strike $\frac{K}{(1+r)^{N-m}}$.