Mathematical & Computational Finance I
Lecture Notes

Probability Theory on Coin Toss Space

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1 Martingales

Let V_N (some random variable) be a derivative security payoff at time N depending on the first N coin tosses. From Chapter 1 we know that there exists some initial wealth X_0 and a replicating portfolio process $\{\Delta_0, \Delta_1, ..., \Delta_{N-1}\}$ that generates a wealth process $\{X_1, X_2, ..., X_N\}$ satisfying

$$X_N(\omega_1 \cdots \omega_N) = V_N(\omega_1 \cdots \omega_N)$$

for all sequences $\omega_1 \cdots \omega_N \in \Omega$. By the theorem presented last lecture we have that the discounted wealth process $\left\{\frac{X_n}{(1+r)^n}\right\}_{n=0}^{N}$ is a $\tilde{\mathbb{P}}$-martingale. That is, the discounted wealth process is adapted and

$$\tilde{\mathbb{E}}_n\left[\frac{X_N}{(1+r)^N}\right] = \frac{X_n}{(1+r)^n}$$

but we have that $X_N = V_N$, so

$$\begin{aligned}\frac{X_n}{(1+r)^n} &= \tilde{\mathbb{E}}_n\left[\frac{X_N}{(1+r)^N}\right] \quad \text{(by the martingale property)}\\ &= \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^N}\right] \quad \text{(since } X_N = V_N\text{)}\end{aligned}$$

Definition 1. Consider an N-period binomial asset pricing model with $0 < d < 1 + r < u$ and let V_N be a derivative security payoff at time N (random variable) depending on the first N coin tosses. Let $\Delta_0, ..., \Delta_{N-1}$ be the replicating portfolio process and $X_0, ..., X_N$ be the corresponding wealth process for hedging V_N.

For $0 \leq n \leq N$ <u>the price of the derivative security at time n</u> is X_n which we denote by V_n.

The rationale for defining the time n price to be $V_n = X_n$ is that if we start at node $\omega_1 \cdots \omega_n$ then we can replicate the payoff at time N using initial capital $X_n(\omega_1 \cdots \omega_n)$. If this were not the case, i.e. V_n would be any other price at time n, then there would be arbitrage.

However, we have that

$$\tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^N}\right] = \frac{V_n}{(1+r)^n}$$

and so by the definition above and this equation we may conclude that the discounted derivative price is a $\tilde{\mathbb{P}}$-martingale. Multiplying this equation by $(1+r)^n$ yields

$$V_n = \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right]$$

which gives us the time n derivative security price without having to set up the hedging portfolio, wealth process, or having to consider the backwards induction process.

1.1 Risk Neutral Pricing Formula

Theorem 1. Consider an N-period binomial asset pricing model with $0 < d < 1 + r < u$ and risk neutral probability measure $\tilde{\mathbb{P}}$. Let V_N be a derivative security payoff at time N (a random variable) depending on the first N coin tosses. Then, the price of the derivative security at time $n \in \{0, 1, ..., N\}$ is given by

$$V_n = \tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right] \quad \textbf{risk-neutral pricing formula}$$

and the discounted security price process

$$\left\{\frac{V_n}{(1+r)^n}\right\}_{n=0}^{N}$$

is a $\tilde{\mathbb{P}}$-martingale.

Proof. This was exercise 2.8 in the book. □

Using this risk neutral pricing formula together with properties of conditional expectation and martingales allows us to say some interesting things about various derivative securities.

Example: Consider a binomial asset pricing model with $S_0 = 5100, u = 1.053, d = 0.965, r = 0.0033$. A European gap call option has payoff

$$V_N = \begin{cases} S_N - K_1 & \text{if } S_N > K_2 \\ 0 & \text{if } S_N \leq K_2 \end{cases}$$

at time $N = 3$.

(i) For $K_1 = 5355$ and $K_2 = 5500$ find the price of the European gap call option price at all nodes of the binomial tree using the risk neutral pricing formula.

(ii) In the general case of an N-period binomial asset pricing model with $0 < d < 1+r < u$ is there a general relationship between the price at time zero of a European gap call option when $K_2 \geq K_1$ and the price at time zero of a (vanilla) European call option on the same asset with strike price K_1 and expiry date N?

Solution: In principle we would build a tree and recursively apply backwards induction to price the option. Instead lets practice the risk neutral pricing formula that we have just stated.

Part (i): We have the asset price tree

$$
\begin{tikzcd}
 & & & S_3^3 = 5954.636973 \\
 & & S_2^2 = 5654.9259 \arrow[ur] \arrow[dr] & \\
 & S_1^1 = 5370.3 \arrow[ur] \arrow[dr] & & S_3^2 = 5457.003494 \\
S_0 = 5100 \arrow[ur] \arrow[dr] & & S_2^1 = 5182.3395 \arrow[ur] \arrow[dr] & \\
 & S_1^0 = 4921.5 \arrow[ur] \arrow[dr] & & S_3^1 = 5000.957618 \\
 & & S_2^0 = 4749.2475 \arrow[ur] \arrow[dr] & \\
 & & & S_3^0 = 4583.023838
\end{tikzcd}
$$

Figure 1: Asset price tree S

We have corresponding risk neutral probabilities

$$
\begin{aligned}
\tilde{p} &= \frac{(1+r)-d}{u-d} \\
&= \frac{(1+0.0033)-0.965}{1.053-0.965} \\
&= \frac{\frac{10033}{10000}-\frac{965}{1000}}{0.088} \\
&= \frac{\frac{383}{10000}}{\frac{88}{1000}} = \frac{\frac{383}{10000}}{\frac{880}{10000}} \\
&= \frac{383}{880} = 0.4352\overline{27}
\end{aligned}
$$

and

$$
\tilde{q} = 1 - \tilde{p} = \frac{497}{880} = 0.5647\overline{27}
$$

First we find the time $N = 3$ payoffs are

$$\begin{aligned}
V_3(HHH) &= (S_3(HHH) - K_1) \cdot \mathbb{1}_{\{S_3(HHH) > K_2\}} \\
&= 5954.636973 - 5355 \\
&= 599.636973 \\
V_3(HHT) &= V_3(HTH) = V_3(THH) = (S_3(HHT) - K_1) \cdot \mathbb{1}_{\{S_3(HHT) > K_2\}} \\
&= 0 \\
V_3(HTT) &= V_3(THT) = V_3(TTH) = (S_3(HTT) - K_1) \cdot \mathbb{1}_{\{S_3(HTT) > K_2\}} \\
&= 0 \\
V_3(TTT) &= (S_3(TTT) - K_1) \cdot \mathbb{1}_{\{S_3(TTT) > K_2\}} \\
&= 0
\end{aligned}$$

and the risk neutral pricing formula

$$V_n(\omega_1 \cdots \omega_n) = \tilde{\mathbb{E}}_n \left[\frac{V_N}{(1+r)^{N-n}} \right] (\omega_1 \cdots \omega_n) \quad n = 0, 1, ..., N$$

we find the time $n = 0$ price

$$\begin{aligned}
V_0(\omega_1\omega_2\omega_3) &= \tilde{\mathbb{E}}_0 \left[\frac{V_3}{(1+r)^3} \right] (\omega_1\omega_2\omega_3) \\
&= \mathbb{E} \left[\frac{V_3}{(1+r)^3} \right] (\omega_1\omega_2\omega_3) \\
&= \sum_{\omega_1\omega_2\omega_3 \in \Omega} \tilde{\mathbb{P}}(\omega_1\omega_2\omega_3) \cdot \frac{V_3(\omega_1\omega_2\omega_3)}{(1+r)^3} \\
&= \tilde{\mathbb{P}}(HHH) \cdot \frac{599.636973}{1.0033^3} + 0 + 0 + 0 \\
&= \left(\frac{383}{880} \right)^3 \cdot \frac{599.636973}{1.0033^3} \\
&\approx 48.9490505
\end{aligned}$$

At time $n = 1$ we find the prices

$$\begin{aligned}
V_1(H\omega_1\omega_2) &= \tilde{\mathbb{E}}_1\left[\frac{V_N}{(1+r)^{N-1}}\right](H\omega_1\omega_2) \\
&= \sum_{H\omega_2\omega_3\in\Omega} \tilde{\mathbb{P}}(\omega_1\omega_2\omega_3|\omega_1 = H)\cdot\frac{V_N}{(1+r)^{N-1}} \\
&= \tilde{\mathbb{P}}(HHH|H)\frac{V_3(HHH)}{(1.0033^2)} + 0 + 0 \\
&= \left(\frac{383}{880}\right)^2 \frac{599.636973}{1.0033^2} \\
&\approx 112.838936 \\
V_1(T\omega_2\omega_3) &= \tilde{\mathbb{E}}_1\left[\frac{V_N}{(1+r)^{N-1}}\right](T\omega_1\omega_2) \\
&= \sum_{T\omega_2\omega_3\in\Omega} \tilde{\mathbb{P}}(\omega_1\omega_2\omega_3|\omega_1 = T)\cdot\frac{V_N}{(1+r)^{N-1}} \\
&= 0
\end{aligned}$$

We may repeat this process to find the time $n = 2$ prices

$$\begin{aligned}
V_2(HH\omega_3) &\approx 260.11968 \\
V_2(HT\omega_3) &= 0 \\
V_2(TH\omega_3) &= 0 \\
V_2(TT\omega_3) &= 0
\end{aligned}$$

Part (ii): Using the risk neutral pricing formula we may write the time zero gap call price

$$V_0^{gc} = \tilde{\mathbb{E}}\left[\frac{(S_N - K_1)}{(1+r)^N}\mathbb{1}_{\{S_N > K_2\}}\right]$$

However, since we assume that $K_1 \leq K_2$, we have that[1] $\{S_N > K_2\} \subseteq \{S_N > K_1\}$, so

$$\mathbb{1}_{\{S_N > K_2\}} \leq \mathbb{1}_{\{S_N > K_1\}}$$

Therefore

$$V_0^{gc} = \tilde{\mathbb{E}}\left[\frac{(S_N - K_1)}{(1+r)^N}\mathbb{1}_{\{S_N > K_2\}}\right] \leq \tilde{\mathbb{E}}\left[\frac{(S_N - K_1)}{(1+r)^N}\mathbb{1}_{\{S_N > K_1\}}\right] = V_0^{ec}$$

where V_0^{ec} is the time zero price of a vanilla European call option. Note that if $\{S_N > K_2\} \subset \{S_N > K_1\}$ (proper subset) then our inequality above becomes a strict inequality.

[1]For brevity we denote $\{S_N > K_i\} := \{\omega_1 \cdots \omega_N \in \Omega \mid S_N(\omega_1 \cdots \omega_N) > K_i\}$

An alternative answer: For $K_1 \leq K_2$,

$$\begin{aligned}
V_0^{ec} &= (S_N - K_1)^+ \\
&= \tilde{\mathbb{E}}\left[\frac{(S_N - K_1)}{(1+r)^N}\mathbb{1}_{\{S_N > K_1\}}\right] \\
&= \tilde{\mathbb{E}}\left[\frac{(S_N - K_1)}{(1+r)^N}\mathbb{1}_{\{S_N > K_2\}\cup\{K_1 < S_N \leq K_2\}}\right] \\
&= \tilde{\mathbb{E}}\left[\frac{(S_N - K_1)}{(1+r)^N}\mathbb{1}_{\{S_N > K_2\}}\right] + \tilde{\mathbb{E}}\left[\frac{(S_N - K_1)}{(1+r)^N}\mathbb{1}_{\{K_1 < S_N \leq K_2\}}\right] \\
&= V_0^{gc} + \tilde{\mathbb{E}}\left[\frac{(S_N - K_1)}{(1+r)^N}\mathbb{1}_{\{K_1 < S_N \leq K_2\}}\right]
\end{aligned}$$

where we may analyse further if we were so inclined to do so.[2]

1.2 Cash-Flow Valuation

Recall that we may find the net present value (NPV) of some series of cash flows C at time n by

$$NPV_n(C) = \sum_{k=n}^{N} \frac{C_k}{(1+r)^{k-n}}$$

Theorem 2. Cash-Flow Valuation. Consider an N-period binomial asset pricing model with $0 < d < 1+r < u$ and risk neutral probability measure $\tilde{\mathbb{P}}$. Let $C_0, C_1, ..., C_N$ be a sequence of random variables such that each C_n depends only on $\omega_1 \cdots \omega_n$. The price at time n of the derivative security that makes payments $C_n, ..., C_N$ at time $n, ..., N$ is given by

$$V_n = \tilde{\mathbb{E}}_n\left[\sum_{k=n}^{N} \frac{C_k}{(1+r)^{k-n}}\right] \quad n = 0, ..., N$$

and the price process $\{V_n\}_{n=0}^N$ satisfies

$$C_n(\omega_1 \cdots \omega_n) = V_n(\omega_1 \cdots \omega_n) - \frac{1}{1+r}\left[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)\right]$$

Now, define

$$\Delta_n(\omega_1 \cdots \omega_n) = \frac{V_{n+1}(\omega_1 \cdots \omega_n H) - V_{n+1}(\omega_1 \cdots \omega_n T)}{S_{n+1}(\omega_1 \cdots \omega_n H) - S_{n+1}(\omega_1 \cdots \omega_n T)} \quad n = 0, 1, ..., N-1$$

If we set $X_0 = V_0$ and define recursively forwards in time the portfolio values $X_1, ..., X_N$ by

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n)$$

then

$$X_n(\omega_1 \cdots \omega_n) = V_n(\omega_1 \cdots \omega_n) \quad n = 0, ..., N, \ \omega_1 \cdots \omega_n \in \Omega$$

[2] We see that a European call option with strike K_1 is equivalent to a gap call option with strike K_1 and gap price K_2 as well as an up & out barrier option with barrier K_2 and a singleton monitoring point at expiry.

We say that V_n is the net present value at time n of the payments $C_n, ..., C_N$. Note that C_n depends only on $\omega_1 \cdots \omega_n$, so for $n = 0, ..., N-1$

$$\begin{aligned}
V_n &= \tilde{\mathbb{E}}_n \left[\sum_{k=n}^{N} \frac{C_k}{(1+r)^{k-n}} \right] \quad \text{(by the definition)} \\
&= C_n + \frac{1}{1+r} \tilde{\mathbb{E}}_n \left[\sum_{k=n+1}^{N} \frac{C_k}{(1+r)^{k-(n+1)}} \right] \quad \text{(taking out what is known)} \\
&= C_n + \frac{1}{1+r} \tilde{\mathbb{E}}_n \left[\tilde{\mathbb{E}}_{n+1} \left[\sum_{k=n+1}^{N} \frac{C_k}{(1+r)^{k-(n+1)}} \right] \right] \quad \text{(tower property)} \\
&= C_n + \frac{1}{1+r} \tilde{\mathbb{E}}_n \left[V_{n+1} \right] \quad \text{(by the definition of } NPV\text{)}
\end{aligned}$$

and for $n = N$ we have $V_N = C_N$. Now, consider an agent with a short position on the cash flows (i.e. makes payment C_n at time n – if C_n is negative the short party will have received C_n). The agent invests in a risky asset and the risk free bank account so that at time n

1. Just before making a payment C_n the value of the portfolio is $X_n = V_n$
2. Then the agent makes payment C_n
3. The agent rebalances the portfolio and takes Δ_n units in the stock

The time $n+1$ value of the portfolio (prior to making the time $n+1$ payment) is

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n)$$

If $X_0 = V_0$ and the agent chooses Δ_n according to

$$\Delta_n(\omega_1 \cdots \omega_n) = \frac{V_{n+1}(\omega_1 \cdots \omega_n H) - V_{n+1}(\omega_1 \cdots \omega_n T)}{S_{n+1}(\omega_1 \cdots \omega_n H) - S_{n+1}(\omega_1 \cdots \omega_n T)} \quad n = 0, 1, ..., N-1$$

then we claim that

$$X_n(\omega_1 \cdots \omega_n) = V_n(\omega_1 \cdots \omega_n)$$

and

$$X_N(\omega_1 \cdots \omega_N) = V_N(\omega_1 \cdots \omega_N) = C_N(\omega_1 \cdots \omega_N)$$

Proof. We proceed inductively. First

$$P(0) \ : \ X_0 = V_0$$

where V_0 is calculated from the risk neutral pricing formula, is true by definition. Now, fix arbitrary $\omega_1 \cdots \omega_n$, we set our inductive hypothesis as

$$P(n) \ : \ X_n(\omega_1 \cdots \omega_n) = V_n(\omega_1 \cdots \omega_n) \quad 1 \le n \le N-1$$

It is our task to show that $P(n+1)$ holds under the assumption that $P(n)$ is true. From the definition we have that

$$V_n = C_n + \tilde{\mathbb{E}}_n \left[\frac{1}{1+r} V_{n+1} \right]$$

So, by the definition of conditional expectation we have

$$V_n(\omega_1 \cdots \omega_n) - C_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r} \left[\tilde{p} V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q} V_{n+1}(\omega_1 \cdots \omega_n T) \right]$$

First consider the case $\omega_{n+1} = H$. By construction of the portfolio

$$X_{n+1}(\omega_1 \cdots \omega_n H) = X_{n+1}(H) = \Delta_n S_{n+1}(H) + (1+r) \left[X_n - C_n - \Delta_n S_n \right]$$

but

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} \quad n = 0, 1, ..., N-1$$

so

$$\begin{aligned}
X_{n+1}(H) &= \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} \cdot S_{n+1}(H) + (1+r) \left[X_n - C_n - \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} \cdot S_n \right] \\
&= \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} \left[S_{n+1}(H) - (1+r) S_n \right] + (1+r) \left[X_n - C_n \right] \\
&= \frac{V_{n+1}(H) - V_{n+1}(T)}{u S_n - d S_n} \left[u S_n - (1+r) S_n \right] + (1+r) \left[X_n - C_n \right] \\
&= \frac{V_{n+1}(H) - V_{n+1}(T)}{u - d} \left[u - (1+r) \right] + (1+r) \left[X_n - C_n \right] \\
&= \left[V_{n+1}(H) - V_{n+1}(T) \right] \left[\frac{u - (1+r)}{u - d} \right] + (1+r) \left[X_n - C_n \right]
\end{aligned}$$

Note

$$\begin{aligned}
\tilde{p} &= \frac{(1+r) - d}{u - d} \\
\implies \tilde{q} &= 1 - \tilde{p} = \frac{u-d}{u-d} - \frac{(1+r)-d}{u-d} \\
&= \frac{u - (1+r)}{u - d}
\end{aligned}$$

hence

$$\begin{aligned}
X_{n+1}(H) &= \left[V_{n+1}(H) - V_{n+1}(T) \right] \left[\frac{u - (1+r)}{u - d} \right] + (1+r) \left[X_n - C_n \right] \\
&= \left[V_{n+1}(H) - V_{n+1}(T) \right] \tilde{q} + (1+r) \left[X_n - C_n \right] \\
&= \left[V_{n+1}(H) - V_{n+1}(T) \right] \tilde{q} + (1+r) \left[V_n - C_n \right] \quad \text{(since } X_n = V_n \text{)}
\end{aligned}$$

By the inductive hypothesis we have $V_n = C_n + \tilde{\mathbb{E}}_n\left[\frac{1}{1+r}V_{n+1}\right]$ so we may simplify our result to

$$\begin{aligned}
X_{n+1}(H) &= \left[V_{n+1}(H) - V_{n+1}(T)\right]\tilde{q} + (1+r)\left[C_n + \tilde{\mathbb{E}}_n\left[\frac{1}{1+r}V_{n+1}\right] - C_n\right] \\
&= \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T) + \tilde{\mathbb{E}}_n\left[V_{n+1}\right] \\
&= \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T) + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= (\tilde{p}+\tilde{q})V_{n+1}(H) \\
&= V_{n+1}(H)
\end{aligned}$$

and so we find that

$$X_{n+1}(\omega_1\cdots\omega_n H) = V_{n+1}(\omega_1\cdots\omega_n H)$$

A similar procedure will yield

$$X_{n+1}(\omega_1\cdots\omega_n T) = V_{n+1}(\omega_1\cdots\omega_n T)$$

Therefore, by induction, we have that

$$X_n(\omega) = V_n(\omega) \quad 0 \le n \le N,\ \omega \in \Omega$$

$\square$