Mathematical & Computational Finance I
Lecture Notes

Black-Derman-Toy Model Calibration

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1 Black-Derman-Toy Model Calibration

Our ongoing goal is how to apply the methods developed in the binomial model in order to specify an interest rate process. We then apply such a process to price fixed-income derivatives. In the real-world we need a tree that is capable of pricing zero-coupon bonds & options correctly and in a market consistent manner. The process of matching the model to fit real-world data is known as calibration: We wish to calibrate our model to observations in the market by modifying model parameters.

In order to calibrate the BDT model we require

(i) Time-zero zero-coupon bond prices

(ii) Bond yield volatilities

Consider a derivative security paying \$1 at time m if the binomial term structure goes through node (m, j), and \$0 otherwise. Recall that we call such a derivative an Arrow-Debreu security.

Denote $A(n, i, m, j)$ to be the Arrow-Debreu security price at node (n, i) which pays \$1 if the process goes through (m, j). At the node $(m-1, i)$, we may apply the martingale property and risk-neutral pricing formula to yield

$$A(m-1, i, m, j) = \begin{cases} \frac{\tilde{p}(m-1,j-1)}{1+r(n-1,j-1)} & \text{if } i = j-1 \quad \text{(discounted probability of heads)} \\ \frac{1-\tilde{p}(m-1,j)}{1+r(m-1,j)} & \text{if } i = j \quad \text{(discounted probability of tails)} \\ 0 & \text{else} \end{cases}$$

We define[1] $A(0, 0, 0, 0) = 1$.

[1] Maybe we don't have to *define* it. It may be derivable from the definitions, but at the moment I don't feel like checking.

We may use a portfolio of simple Arrow-Debreu securities in order to price a zero-coupon bond. The price of a zero-coupon bond at node (n, i) with maturity at some future time m is given by

$$P(n, i; m) = \sum_{j=0}^{m} A(n, i, m, j)$$

Otherwise, we can show that there is arbitrage. From Jamshidian's Forward Induction formula we are able to calculate

$$A(n, i, m+1, j) \quad j = 0, 1, ..., m+1$$

given the previous maturity Arrow-Debreu price

$$A(n, i, m, j) \quad j = 0, 1, ..., m$$

The statement of Jamshidian's Forward Induction formula is[2]

$$A(n, i, m+1, j) = \begin{cases} \frac{1-\tilde{p}(m,j)}{1+r(m,j)} A(n, i, m, j) & \text{if } j = 0 \\ \frac{1-\tilde{p}(m,j)}{1+r(m,j)} A(n, i, m, j) + \frac{\tilde{p}(m,j-1)}{1+r(m,j-1)} A(n, i, m, j-1) & \text{if } 0 < j < m \\ \frac{\tilde{p}(m,j-1)}{1+r(m,j-1)} A(n, i, m, j-1) & \text{if } j = m \end{cases}$$

In the BDT model we typically assume that the conditional transition probabilities of an up step $\tilde{p}(m, j) = \frac{1}{2}$. In order to calibrate the model we require bond yield volatilities $\sigma_y(t)$ as well as observed bond prices at time 0. In the BDT model we also require that the short rates[3] satisfy

$$\sigma_t(t) = \frac{1}{2} \log \left(\frac{r(t, n+1)}{r(t, n)} \right)$$

Then, after doing some trivial manipulations, we find

$$r(t, n+1) = r(t, n) e^{2\sigma_r(t)} = r(t, n)\sigma(t) \tag{1}$$

Yield rates $Y(n, j; m)$ must satisfy[4]

$$P(n, j; m) = [1 + Y(n, j; m)]^{-(m-n)}$$

and the current yield volatility for a bond at maturity t is

$$\sigma_y(t) = \frac{1}{2} \log \left(\frac{Y(1, 1; t)}{Y(1, 0; t)} \right)$$

Hence

$$Y(1, 1; t) = Y(1, 0; t) e^{2\sigma_y(t)} \quad t = 2, 3, ...$$

[2] A simpler example was left as an exercise in an assignment.

[3] I forget what is the definition of a "short rate": What makes it special?

[4] Think of this as the time value of money/discounting of a bond P back from final period m to initial period n. $Y(n, j; m)$ is the rate satisfying this definition (think YTM).

We use the term structure for initial yields $Y(0,t)$ and volatilities in order to fit the model.

Example: Suppose that at time 0 we have the data of time-zero bond yields/volatilities expiring at various future dates t:

Maturity t	Yield $Y(0,0;t)$	Yield Vol. $\sigma_y(t)$
1	0.06	-
2	0.07	0.19
3	0.08	0.18
4	0.09	0.17
5	0.10	0.16

Note that[5] $r(0,0) = Y(0,0;1)$ so that the time-zero Arrow-Debreu security price for a process reaching $(1,0)$ and $(1,1)$ are

$$\begin{aligned}
A(0,0,1,0) &= \frac{1-\tilde{p}(0,0)}{1+r(0,0)} A(0,0,0,0) \quad \text{(Jamshidian case } j=0\text{)} \\
&= \frac{1-\frac{1}{2}}{1+r(0,0)} A(0,0,0,0) \quad \left(\tilde{p} = \frac{1}{2} \text{ by assumption}\right) \\
&= \frac{\frac{1}{2}}{1+r(0,0)} = \frac{1}{2 \cdot 1.06} = 0.4717 \\
A(0,0,1,1) &= \frac{1-\tilde{p}(0,0)}{1+r(0,0)} A(0,0,0,0) \quad \text{(Jamshidian case } j=m\text{)} \\
&= A(0,0,1,0) = 0.4717
\end{aligned}$$

Now, at the end of the first period (time $t=1$) we must find the appropriate discount rates $r(1,0)$ and $r(1,1)$. To do so we must match observed bond yields maturing at time 2

[5]Since $r(n,j) = Y(n,j;n+1)$.

with the risk-neutral pricing formula such that

$$
\begin{aligned}
\frac{1}{Y(0,0;2)^2} = \frac{1}{1.07^2} = P(0,2) &= \tilde{\mathbb{E}}_0\left[\frac{1}{(1+r_0)(1+r_1)}\right] \\
&= \frac{1}{(1+r_0)}\left[\frac{1}{2}\cdot\frac{1}{(1+r(1,0))} + \frac{1}{2}\cdot\frac{1}{(1+r(1,1))}\right] \\
&= \frac{1}{2(1+r(0,0)}\left[\frac{1}{(1+r(1,0))} + \frac{1}{(1+r(1,1))}\right] \\
&= A(0,0,1,0)\left[\frac{1}{(1+r(1,0))} + \frac{1}{(1+r(1,1))}\right] \quad \text{(as well as } A(0,0,1,1)) \\
&= \frac{A(0,0,1,0)}{1+r(1,0)} + \frac{A(0,0,1,0)}{1+r(1,1)} \\
&= \frac{A(0,0,1,0)}{1+r(1,0)} + \frac{A(0,0,1,1)}{1+r(1,1)}
\end{aligned}
$$

Since $r(n,j) = Y(n,j;n+1)$ we find that the one-period rates at nodes $(1,0)$ and $(1,1)$ are given by

$$
\begin{aligned}
r(1,0) &= Y(1,0;2) \\
r(1,1) &= Y(1,1;2) = Y(1,0;2)e^{2\sigma_y(2)} = r(1,0)e^{2(0.19)}
\end{aligned}
$$

Substituting this relationship between $r(1,1)$ and $r(1,0)$ in our equation above gives us

$$
\begin{aligned}
\frac{1}{1.07^2} &= \frac{A(0,0,1,0)}{1+r(1,0)} + \frac{A(0,0,1,1)}{1+r(1,1)} \\
&= \frac{0.4717}{1+r(1,0)} + \frac{0.4717}{1+r(1,0)e^{2(0.19)}}
\end{aligned}
$$

We can solve this numerical/using the quadratic formula (after some rearrangement) to yield

$$
\begin{aligned}
& r(1,0) \approx 0.6522784 \\
\implies & r(1,1) = r(1,0)e^{2(0.19)} \approx 0.09538167
\end{aligned}
$$

Now, for time $t = 2$ we use the Jamshidian Forward Induction formula to compute

$$
\begin{aligned}
A(0,0,2,0) &= \frac{A(0,0,1,0)}{2(1+r(1,0))} = 0.221564266 \\
A(0,0,2,1) &= \frac{A(0,0,1,0)}{2(1+r(1,0))} + \frac{A(0,0,1,1)}{2(1+r(1,1))} = 0.437093829 \\
A(0,0,2,2) &= \frac{A(0,0,1,1)}{2(1+r(1,1))} = 0.21559562
\end{aligned}
$$

Then, to compute the time-1 Arrow-Debreu securities expiring at time-2 we use risk-neutral pricing

$$A(1,0,2,0) = \frac{1}{2}\cdot\frac{1}{1+r(1,0)} = 0.469716245 \underbrace{= A(1,0,2,1)}_{\text{since } \tilde{p}=\tilde{q}}$$

$$A(1,1,2,1) = \frac{1}{2}\cdot\frac{1}{1+r(1,1)} = 0.456922672 = A(1,1,2,2)$$

$$A(1,0,2,2) = A(1,1,2,0) = 0 \quad \text{(there is no way to reach these nodes in one step)}$$

We seek the one-period time-two interest rates $r(2,0), r(2,1), r(2,2)$, where

$$r(2,n+1) = r(2,n)e^{2\sigma_t(t)} = r(2,n)\sigma(t)$$

Again matching the observed prices with the risk-neutral pricing formula

$$\begin{aligned}\frac{1}{Y(0,0;3)^3} = \frac{1}{1.08^2} = P(0,3) &= \tilde{\mathbb{E}}\left[\frac{1}{(1+r_0)(1+r_1)(1+r_2)}\right] \\ &= \frac{1}{(1+r_0)}\tilde{\mathbb{E}}\left[\frac{1}{(1+r_1)}\tilde{\mathbb{E}}\left[\frac{1}{(1+r_2)}\;\middle|\;\omega_1\right]\right]\end{aligned}$$

Note that the internal expectation is

$$\tilde{\mathbb{E}}\left[\frac{1}{(1+r_2)}\;\middle|\;\omega_1\right] = \begin{cases} \frac{1}{2}\frac{1}{1+r(2,2)} + \frac{1}{2}\frac{1}{1+r(2,1)} & \text{if } \omega_1 = H \\ \frac{1}{2}\frac{1}{1+r(2,1)} + \frac{1}{2}\frac{1}{1+r(2,0)} & \text{if } \omega_1 = T \end{cases}$$

Thus

$$\begin{aligned}\tilde{\mathbb{E}}\left[\frac{1}{(1+r_1)}\tilde{\mathbb{E}}\left[\frac{1}{(1+r_2)}\;\middle|\;\omega_1\right]\right] &= \frac{1}{2}\frac{1}{1+r(1,1)}\left\{\frac{1}{2}\frac{1}{1+r(2,2)} + \frac{1}{2}\frac{1}{1+r(2,1)}\right\} + \\ &\qquad \frac{1}{2}\frac{1}{1+r(1,0)}\left\{\frac{1}{2}\frac{1}{1+r(2,1)} + \frac{1}{2}\frac{1}{1+r(2,0)}\right\} \\ &= \frac{1}{4}\left(\frac{1}{(1+r(1,0)(1+r(2,0))}\right) + \\ &\qquad \frac{1}{4}\left(\frac{1}{1+r(1,0)} + \frac{1}{1+r(1,1)}\right)\frac{1}{1+r(2,1)} + \\ &\qquad \frac{1}{4}\left(\frac{1}{(1+r(1,1))(1+r(2,2))}\right)\end{aligned}$$

Therefore,

$$
\begin{aligned}
\frac{1}{1.08^2} &= \frac{1}{(1+r_0)}\tilde{\mathbb{E}}\left[\frac{1}{(1+r_1)}\tilde{\mathbb{E}}\left[\frac{1}{(1+r_2)}\;\middle|\;\omega_1\right]\right] \\
&= \frac{1}{(1+r_0)}\Bigg[\frac{1}{4}\left(\frac{1}{(1+r(1,0)(1+r(2,0))}\right)+ \\
&\qquad \frac{1}{4}\left(\frac{1}{1+r(1,0)}+\frac{1}{1+r(1,1)}\right)\frac{1}{1+r(2,1)}+ \\
&\qquad \frac{1}{4}\left(\frac{1}{(1+r(1,1))(1+r(2,2))}\right)\Bigg] \\
&= \frac{1}{2(1+r_0)}\Bigg[\frac{1}{2}\left(\frac{1}{(1+r(1,0)(1+r(2,0))}\right)+ \\
&\qquad \frac{1}{2}\left(\frac{1}{1+r(1,0)}+\frac{1}{1+r(1,1)}\right)\frac{1}{1+r(2,1)}+ \\
&\qquad \frac{1}{2}\left(\frac{1}{(1+r(1,1))(1+r(2,2))}\right)\Bigg] \\
&= \Bigg[\frac{1}{2}\left(\frac{A(0,0,1,0)}{(1+r(1,0)(1+r(2,0))}\right)+ \\
&\qquad \frac{1}{2}\left(\frac{A(0,0,1,0)}{1+r(1,0)}+\frac{A(0,0,1,1)}{1+r(1,1)}\right)\frac{1}{1+r(2,1)}+ \\
&\qquad \frac{1}{2}\left(\frac{A(0,0,1,1)}{(1+r(1,1))(1+r(2,2))}\right)\Bigg] \\
&= \frac{A(0,0,2,0)}{1+r(2,0)}+\frac{A(0,0,2,1)}{1+r(2,1)}+\frac{A(0,0,2,2)}{1+r(2,2)}
\end{aligned}
$$

and using the relationship

$$r(2,n+1) = r(2,n)e^{2\sigma_r(2)} = r(2,n)\sigma(2)$$

gives us the equation in terms of $r(2,0)$

$$\frac{1}{1.08^2} = \frac{A(0,0,2,0)}{1+r(2,0)}+\frac{A(0,0,2,1)}{1+r(2,0)\sigma(2)}+\frac{A(0,0,2,2)}{1+r(2,0)[\sigma(2)]^2}$$

Note that we express the volatility parameters in terms of σ and not σ_r as is given in the table. The volatility parameter $\sigma(2)$ is still unknown to us and so our equation is still intractable with two unknowns. We use the yield volatilities $\sigma_y(2)$ to obtain a separate equation in these unknowns.

To build our second equation we start with finding out intermediate bond yields for maturity at time $t = 3$. These prices should be consistent with the initial bond prices & the short rate.

Note

$$\begin{aligned} P(1,m;3) &= [1+Y(1,m;3)]^{-2} \\ \implies Y(1,m;3) &= P(1,m;3)^{-\frac{1}{2}} - 1 \end{aligned}$$

Then, from our relationship $r(t,n+1) = r(t,n)e^{2\sigma_r(t)} = r(t,n)\sigma(t)$, with the rate $Y(1,1;3)$, we find

$$Y(1,1;3) = P(1,1;3)^{-\frac{1}{2}} - 1 = \left[P(1,0;3)^{-\frac{1}{2}} - 1\right] e^{2\sigma_y(3)}$$

From risk-neutral pricing we have that we can also express $P(1,0;3)$ by

$$\begin{aligned} P(1,0;3) &= \tilde{\mathbb{E}}_1\left[\frac{1}{(1+r_1)(1+r_2)}\right](T) \\ &= \frac{1}{(1+r(1,0))}\left[\frac{1}{2}\frac{1}{(1+r(2,0))} + \frac{1}{2}\frac{1}{(1+r(2,1))}\right] \\ &= \frac{1}{2(1+r(1,0))}\left[\frac{1}{(1+r(2,0))} + \frac{1}{(1+r(2,1))}\right] \\ &= \frac{A(1,0,2,0)}{1+r(2,0)} + \frac{A(1,0,2,1)}{1+r(2,1)} \\ &= \frac{A(1,0,2,0)}{1+r(2,0)} + \frac{A(1,0,2,1)}{1+r(2,0)\sigma(2)} \end{aligned}$$

and since $A(1,0,2,2) = A(1,1,2,0) = 0$ we can generalize this as

$$P(1,0;3) = \sum_{j=0}^{2} \frac{A(1,0,2,j)}{1+r(2,0)\sigma(2)^j}$$

Similarly

$$\begin{aligned} P(1,1;3) &= \tilde{\mathbb{E}}_1\left[\frac{1}{(1+r_1)(1+r_2)}\right](H) \\ &= \frac{1}{(1+r(1,1))}\left[\frac{1}{2}\frac{1}{(1+r(2,1))} + \frac{1}{2}\frac{1}{(1+r(2,2))}\right] \\ &= \frac{A(1,1,2,1)}{1+r(2,0)} + \frac{A(1,1,2,2)}{1+r(2,1)} \\ &= \frac{A(1,1,2,1)}{1+r(2,0)} + \frac{A(1,1,2,2)}{1+r(2,0)\sigma(2)} \\ &= \sum_{j=0}^{2} \frac{A(1,1,2,j)}{1+r(2,0)\sigma(2)^j} \quad \text{(since } A(1,1,2,0)=0) \end{aligned}$$

Substituting these expressions for $P(1,0;3)$ and $P(1,1;3)$ into

$$P(1,1;3)^{-\frac{1}{2}} - 1 = \left[P(1,0;3)^{-\frac{1}{2}} - 1\right] e^{2\sigma_y(3)}$$

gives

$$\left(\sum_{j=0}^{2}\frac{A(1,1,2,j)}{1+r(2,0)\sigma(2)^j}\right)^{-\frac{1}{2}}-1=\left[\left(\sum_{j=0}^{2}\frac{A(1,0,2,j)}{1+r(2,0)\sigma(2)^j}\right)^{-\frac{1}{2}}-1\right]e^{2\sigma_y(3)}$$

We now have two equations in two unknowns

$$\frac{1}{1.08^2}=\frac{A(0,0,2,0)}{1+r(2,0)}+\frac{A(0,0,2,1)}{1+r(2,0)\sigma(2)}+\frac{A(0,0,2,2)}{1+r(2,0)[\sigma(2)]^2}$$
$$\left(\sum_{j=0}^{2}\frac{A(1,1,2,j)}{1+r(2,0)\sigma(2)^j}\right)^{-\frac{1}{2}}-1=\left[\left(\sum_{j=0}^{2}\frac{A(1,0,2,j)}{1+r(2,0)\sigma(2)^j}\right)^{-\frac{1}{2}}-1\right]e^{2\sigma_y(3)}$$

which may be used to solve numerically for $r(2,0)$ and $\sigma(2)$. We find

$$\begin{aligned}\sigma(2)&=0.1724846\\ r(2,0)&=0.06949409\\ r(2,1)&=0.09981177\\ r(2,2)&=0.13853693\end{aligned}$$

We can use these (including $r(0,0), r(1,0), r(1,1)$) short rates r to price other derivatives, such as an option expiring at time two with underlying asset equal to the time-3 maturity zero-coupon bond.
In general, for time-t calibration we can use the equations

$$\begin{aligned}P(0,t+1)&=\sum_{j=0}^{t}\frac{A(0,0,t,j)}{1+r(t,0)\sigma(t)^j}\\ Y(1,1;t+1)&=Y(1,0;t+1)e^{2\sigma_y(t+1)}\end{aligned}$$

to match initial yield rates with yield volatilities, such that

$$Y(1,m;t+1)=\left(\sum_{j=0}^{t}\frac{A(1,m,t,j)}{1+r(t,0)\sigma(t)^j}\right)^{-\frac{1}{t}}-1,\quad m=0,1.$$

Note that we have constructed our tree under the risk-neutral measure $\tilde{\mathbb{P}}$. We can implement this procedure recursively forwards in time to continue matching the interest rate tree to initial yields & bond volatilities by solving the system of two equations in two unknowns at each step.