Mathematical & Computational Finance I
Lecture Notes

Binomial No-Arbitrage Pricing Model

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1 One-Period Binomial Model

Let S_0 be the price of one share of stock at time $t = 0$. Assume that $S_0 > 0$ and that after one time period that the price can go up by a factor of u or down by a factor of d. That is, S_0 can either go up to $S_1^u = uS_0$ or down to $S_1^d = dS_0$. We may think of the price at time step 1 as depending on the outcome of a coin toss (either H or T). So,

$$\begin{aligned} S_1(H) &= S_1^u = uS_0 \\ S_1(T) &= S_1^d = dS_0 \end{aligned}$$

Assume now that the probability of heads is given by p and tails is $q = 1 - p$, then[1]

$$\begin{aligned} u &= \frac{S_1(H)}{S_0} \\ d &= \frac{S_1(T)}{S_0} \end{aligned}$$

Suppose that there is a constant interest rate r over a time period and that \$1 invested in the risk-free bank account at time zero will grow to $\$1 \cdot (1 + r)$ after one time period.

Definition 1. The <u>one-period binomial model</u> consists of

1. An initial stock price S_0
2. A risk-free bank account which pays interest at a rate of r per period
3. A finite probability space $\Omega = \{H, T\}$ and a probability measure $\mathbb{P}$ specified by $\mathbb{P}(H) = p$ and $\mathbb{P}(T) = 1 - p$

[1]For now we assume (without proof) that $d < u$. This will be later proven to be the case in a no-arbitrage world.

4. A random variable S_1 taking values

$$S_1(\omega) = \begin{cases} uS_0 & \text{if } \omega = H \\ dS_0 & \text{if } \omega = T \end{cases}$$

representing the price of the stock after one time period

We assume that we can borrow from the bank account at the same rate: Borrow \$1 at time zero then repay $\$1 \cdot (1+r)$ after one time period. We require $r \geq -1$ (usually specifying $r \geq 0$).

1.1 Arbitrage

In an efficient market there should be no way to earn a riskless profit.

Definition 2. A trading strategy which

1. Requires no initial investment
2. Has zero probability of losing money
3. Has a positive probability of profit

is called an arbitrage strategy.

Real markets sometimes present arbitrage opportunities but these are expected to quickly disappear in a sufficiently efficient market.

Axiom: *There are no arbitrage strategies in the one-period binomial model.*

Theorem 1. There are no arbitrage strategies in the one period binomial model if and only if

$$0 < d < 1 + r < u$$

Proof. ($\Longrightarrow$) *If there are no arbitrage strategies then* $0 < d < 1 + r < u$.

Assume that $d \geq 1 + r$. That is, it is always preferable to buy the stock since the riskless rate will be less than the down factor. Now, at time $t = 0$, borrow S_0 from the bank account and use it to buy 1 unit of S. At time $t = 1$ we will repay the loan for $(1 + r) \cdot S_0$. We finance this repayment by selling the stock for S_1.

If $\omega = T$ then $S_1^d = dS_0$. Thus, the net profit is $dS_0 - (1 + r)S_0 = S_0(d - (1 + r)) \geq 0$. Where the ≥ 0 follows from the hypothesis.

If $\omega = H$ then $S_1^u = uS_0$. Thus, the net profit is $uS_0 - (1 + r)S_0 = S_0(u - (1 + r)) > S_0(u - (1 + r)) \geq 0$.

So, there is no possibility of a loss in both states of ω. Contradicts our axiom! Therefore we have that $d < 1 + r$.

Assume now that $u \leq 1+r$. That is, it is always preferable to invest in the bank account than to buy the risky stock. At time $t = 0$ short sell 1 unit of the stock to receive $\$S_0$. Invest the short proceeds into the bank account. At time $t = 1$ this will accumulate to $(1 + r)S_0$. Return the shorted stock by buying 1 unit in the market for S_1.

If $\omega = H$ then $S_1 = S_1^u = uS_0$. Thus, the net profit is $(1+r)S_0 - uS_0 = S_0((1+r) - u) \geq 0$ by hypothesis.

If $\omega = T$ then $S_1 = S_1^d = dS_0$. Thus, the net profit is $(1+r)S_0 - dS_0 = S_0((1+r) - d) > S_0((1 + r) - u) \geq 0$.

Once again we find that we are guaranteed a net profit. Contradiction! Therefore we have that $u > 1+r$. So, from the first and second parts we conclude that $0 < d < 1+r < u$, as desired.

($\Longleftarrow$) *Converse left as an exercise (Exercise 1.1)* $\square$

Definition 3. A European call option is a contract which gives the owner (long party) the right to buy one share of the stock after one time period for strike price K. If $S_1^d < K < S_1^u$ then

If $S \downarrow$ ($\omega = T$) the holder will not exercise the option since he could buy the stock on the market for $S_1^d < K$.

If $S \uparrow$ ($\omega = H$) the holder will exercise the option and buy the stock from the short party for $K < S_1^u$. He can then immediately sell the stock and realize a profit of $S_1^u - K > 0$.

The payoff of such an option after one time period is

$$\max(S_1 - K, 0) \equiv (S_1 - K)^+$$

1.2 Arbitrage Pricing Theory

How much is an option worth at time zero *before* we know the price of the stock after one time period? Arbitrage pricing theory answers this question by replicating the payoff of the option (in both states $\omega = H, T$ of the world) with a portfolio composed of the stock and the bank account.

Under this model we rely on some assumptions

1. Fractional shares can be bough or sold
2. Can borrow (and lend) at the risk free rate r
3. The purchase price of the stock is the same as its selling price (i.e. no bid-ask spread)

4. The stock price takes only two possible values after one time step

Definition 4. A derivative security is a contract that pays

$$V_1 = \begin{cases} V_1(H) = V_1^u & \text{at time 1 if } \omega = H \\ V_1(T) = V_1^d & \text{at time 1 if } \omega = T \end{cases}$$

A European call option is a particular derivative security with $V_1 = (S_1 - K)^+$. A European put option is a derivative security with $V = (K - S_1)^+$. A forward contract is a derivative security with $V = S_1 - K$.

Suppose we wish to find the time zero price, V_0 , of a derivative security in the binomial model. To do so we replicate the payoff with

Initial wealth X_0

Buying Δ_0 shares of stock at time zero

Holding a cash position of $X_0 - \Delta_0 S_0$ for one time period

After one time period the value of this position is

$$\begin{aligned} X_1 &= \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \\ &= \Delta_0 S_1 + (1+r)X_0 - (1+r)\Delta_0 S_0 \\ &= (1+r)X_0 + \Delta_0(S_1 - (1+r)S_0) \end{aligned}$$

To replicate the payoff of the derivative at time $t = 1$ with the portfolio we choose X_0 and Δ_0 such that

$$\begin{aligned} X_1(H) &= V_1(H) \\ X_1(T) &= V_1(T) \end{aligned}$$

Equivalently, if we discount back to time zero,

$$\begin{aligned} \frac{1}{1+r}X_1(H) &= \frac{1}{1+r}V_1(H) \\ \frac{1}{1+r}X_1(T) &= \frac{1}{1+r}V_1(T) \end{aligned}$$

Substituting our equation for X_1 into the above equation gives us

$$\begin{aligned} X_0 + \Delta_0\left(\frac{S_1(H)}{1+r} - S_0\right) &= \frac{1}{1+r}V_1(H) \\ X_0 + \Delta_0\left(\frac{S_1(T)}{1+r} - S_0\right) &= \frac{1}{1+r}V_1(T) \end{aligned}$$

which gives us a system of two equations in two unknowns (X_0, Δ_0). Rearranging a bit yields

$$X_0 = \frac{1}{1+r}V_1(H) - \Delta_0\left(\frac{S_1(H)}{1+r} - S_0\right)$$

and substituting into the second equation in our system

$$
\begin{aligned}
\frac{1}{1+r}V_1(H) - \Delta_0\left(\frac{S_1(H)}{1+r} - S_0\right) + \Delta_0\left(\frac{S_1(T)}{1+r} - S_0\right) &= \frac{1}{1+r}V_1(T) \\
\implies \Delta_0\left(\frac{S_1(T) - S_1(H)}{1+r}\right) &= \frac{V_1(T) - V_1(H)}{1+r} \\
\implies \Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}
\end{aligned}
$$

Substituting Δ_0 into our equation for X_0 yields

$$
X_0 = \frac{1}{1+r}V_1(H) - \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}\left(\frac{S_1(H)}{1+r} - S_0\right)
$$

Starting at time $t = 0$ the initial wealth X_0 needed to replicate the derivative payoff is given by the above equation. We find that Δ_0 is the units of the long stock position in the portfolio with the remaining $X_0 - \Delta_0 S_0$ invested in the bank account. That is, starting with X_0 invested in the replicating portfolio we find that

If the coin toss is H the portfolio will be worth $V_1(H)$

If the coin toss is T the portfolio will be worth $V_1(T)$

We could sell the derivative security at time zero and be able to pay the long counterparty the amounts $V_1(H)$ or $V_1(T)$ (for cash-settled options) depending on the outcome of the coin toss. That is, we have hedged a short position in one unit of the derivative security.

The price received at time zero for the derivative should be X_0, otherwise we would find arbitrage.

The initial price, X_0, of the derivative security given by the above formula is not particularly informative. We may simplify it by using the substitutions $S_1(H) = uS_0$ and $S_1(T) = dS_0$ by

$$
\begin{aligned}
X_0 &= \frac{V_1(H)}{1+r} - \frac{V_1(H)}{uS_0 - dS_0}\left(\frac{uS_0}{1+r} - S_0\right) + \frac{V_1(T)}{uS_0 - dS_0}\left(\frac{uS_0}{1+r} - S_0\right) \\
&= \left[1 - \frac{u + (1+r)}{u-d}\right]\frac{V_1(H)}{1+r} + \left[\frac{u - (1+r)}{u-d}\right]\frac{V_1(T)}{1+r} \\
&= \frac{1+r-d}{u-d}\frac{V_1(H)}{1+r} + \left[1 - \frac{1+r-d}{u-d}\right]\frac{V_1(T)}{1+r} \\
&= \tilde{p}\frac{V_1(H)}{1+r} + (1-\tilde{p})\frac{V_1(T)}{1+r}
\end{aligned}
$$

where $\tilde{p} = \frac{1+r-d}{u-d}$. We define a probability measure $\tilde{\mathbb{P}}$ on $\Omega = \{H, T\}$ by

$$
\begin{aligned}
\tilde{\mathbb{P}}(H) &= \tilde{p} = \frac{1+r-d}{u-d} \\
\tilde{\mathbb{P}}(T) &= 1 - \tilde{p}
\end{aligned}
$$

Then, the initial wealth required to replicate the derivative security payoff using the delta hedging strategy $\Delta_0 = \frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}$ is

$$X_0 = \tilde{p}\frac{V_1(H)}{1+r} + \tilde{q}\frac{V_1(T)}{1+r} = \tilde{\mathbb{E}}\left[\frac{V_1}{1+r}\right]$$

where $\tilde{\mathbb{E}}[\cdot]$ indicates expectation with respect to $\tilde{\mathbb{P}}$.