Assignment 2

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MACF 401 - Mathematical & Computational Finance I

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Part I

Solution 2.4.i: We see that $M_0 = 0$ is constant and so it is measurable/adapted. For $n \geq 1$ we have M_n depending only on the first n coin tosses by its construction. Therefore we have that M_n is adapted, satisfying the first condition.

Now, checking the martingale property of M_n:

$$\begin{aligned}
\mathbb{E}_n[M_{n+1}] &= \mathbb{E}_n\left[\sum_{j=1}^{n+1} X_j\right] \\
&= \mathbb{E}_n\left[\sum_{j=1}^{n} X_j + X_{n+1}\right] \\
&= \mathbb{E}_n\left[\sum_{j=1}^{n} X_j\right] + \mathbb{E}_n\left[X_{n+1}\right] \quad \text{(by linearity)} \\
&= \mathbb{E}_n[M_n] + \mathbb{E}_n\left[X_{n+1}\right] \quad \text{(by definition of } M_n\text{)} \\
&= M_n + \mathbb{E}_n\left[X_{n+1}\right] \quad \text{(since } M_n \text{ is known at the } n\text{th toss)} \\
&= M_n + \mathbb{E}[X_{n+1}] \quad \text{(since } X_{n+1} \text{ is independent of the first } n \text{ tosses)} \\
&= M_n + [p \cdot (1) + q \cdot (-1)] \quad \text{(by definition of ordinary expectation)} \\
&= M_n + \left[\frac{1}{2} - \frac{1}{2}\right] \\
&= M_n
\end{aligned}$$

Therefore, since M_n is both adapted and satisfies the martingale property we have that M_n is a martingale, as desired.

Solution 2.4.ii: We have that S_n is a function of an adapted process

$$f(S_n) = S_n = e^{\sigma M_n}\left(\frac{2}{e^{\sigma} + e^{-\sigma}}\right)^n$$

By the construction of S_n it does not introduce dependency on any coin tosses beyond n (in fact, f does not introduce any further dependency on any coin toss beyond those used in M_n). Hence, S_n depends only on the first n coin tosses. Therefore we have that S_n is adapted.

Confirming the martingale property of S_n:

$$\begin{aligned}
\mathbb{E}_n\left[S_{n+1}\right] &= \mathbb{E}_n\left[e^{\sigma M_{n+1}}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}\right] \\
&= \mathbb{E}_n\left[e^{\sigma(M_n+X_{n+1})}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}\right] \quad \text{(from part (i))} \\
&= \mathbb{E}_n\left[e^{\sigma M_n}e^{\sigma X_{n+1}}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}\right] \\
&= e^{\sigma M_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}\mathbb{E}_n\left[e^{\sigma X_{n+1}}\right] \quad \text{(taking out what is known)} \\
&= e^{\sigma M_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n}\mathbb{E}\left[e^{\sigma X_{n+1}}\right] \quad \text{(since } X_{n+1} \text{ is independent of the first } n \text{ tosses)} \\
&= e^{\sigma M_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}\left[p\cdot e^{\sigma\cdot(1)}+q\cdot e^{\sigma\cdot(-1)}\right] \\
&= e^{\sigma M_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}\left[\frac{1}{2}e^{\sigma}+\frac{1}{2}e^{-\sigma}\right] \\
&= e^{\sigma M_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}\frac{e^{\sigma}+e^{-\sigma}}{2} \\
&= e^{\sigma M_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n} \\
&= S_n
\end{aligned}$$

Therefore, since S_n is both adapted and satisfies the martingale property we have that S_n is a martingale, as desired.

Solution 2.5.i: We reasons which will become clear through the solution, instead consider $2I_n$. We will show that $2I_n = M_n^2 - n$. So,

$$\begin{aligned}
2I_n &= 2\sum_{j=0}^{n-1} M_j(M_{j+1}-M_j) \quad \text{(by definition)} \\
&= 2\sum_{j=0}^{n-1}\left(M_jM_{j+1}-M_j^2\right) \\
&= 2\sum_{j=0}^{n-1} M_jM_{j+1} - 2\sum_{j=0}^{n-1} M_j^2
\end{aligned}$$

but

$$
\begin{aligned}
M_n^2 &= \sum_{j=0}^{n-1} M_{j+1}^2 - \sum_{j=0}^{n-1} M_j^2 \\
\implies \sum_{j=0}^{n-1} M_j^2 &= \sum_{j=0}^{n-1} M_{j+1}^2 - M_n^2
\end{aligned}
$$

so

$$
\begin{aligned}
2I_n &= 2\sum_{j=0}^{n-1} M_j M_{j+1} - 2\sum_{j=0}^{n-1} M_j^2 \\
&= 2\sum_{j=0}^{n-1} M_j M_{j+1} - \sum_{j=0}^{n-1} M_j^2 - \left(\sum_{j=0}^{n-1} M_{j+1}^2 - M_n^2\right) \\
&= 2\sum_{j=0}^{n-1} M_j M_{j+1} - \sum_{j=0}^{n-1} M_j^2 - \sum_{j=0}^{n-1} M_{j+1}^2 + M_n^2 \\
&= M_n^2 + \sum_{j=0}^{n-1} \left(2M_j M_{j+1} - M_j^2 - M_{j+1}^2\right) \\
&= M_n^2 - \sum_{j=0}^{n-1} \left(M_{j+1} - M_j\right)^2 \\
&= M_n^2 - \sum_{j=0}^{n-1} X_{j+1}^2 \\
&= M_n^2 - \sum_{j=0}^{n-1} 1 \\
&= M_n^2 - n
\end{aligned}
$$

Hence

$$
\begin{aligned}
2I_n &= M_n^2 - n \\
\implies I_n &= \frac{1}{2} M_n^2 - \frac{1}{2} n
\end{aligned}
$$

as desired.

Solution 2.5.ii:

$$\begin{aligned}
\mathbb{E}_n\left[f(I_{n+1})\right] &= \mathbb{E}_n\left[f\left(\sum_{j=0}^{n} M_j(M_{j+1}-M_j)\right)\right] \quad \text{(by definition)} \\
&= \mathbb{E}_n\left[f\left(\sum_{j=0}^{n-1} M_j(M_{j+1}-M_j) + M_n(M_{n+1}-M_n)\right)\right] \\
&= \mathbb{E}_n\left[f(I_n + M_n X_{n+1})\right]
\end{aligned}$$

However, I_n and M_n are adapted to the first n coin tosses and X_{n+1} is independent of the first n tosses. Therefore, our conditional expectation may be rewritten as the ordinary expectation

$$\begin{aligned}
\mathbb{E}_n\left[f(I_{n+1})\right] &= \mathbb{E}_n\left[f(I_n + M_n X_{n+1})\right] \\
&= \mathbb{E}\left[f(I_n + M_n X_{n+1})\right] \quad \text{(by the Independence Lemma)} \\
&= p \cdot f(I_n + M_n \cdot (1)) + q \cdot f(I_n + M_n \cdot (-1)) \\
&= \frac{1}{2} f(I_n + M_n) + \frac{1}{2} f(I_n - M_n)
\end{aligned}$$

but we're offended by the presence of M_n in our solution, so, from part (i),

$$\begin{aligned}
I_n &= \frac{1}{2} M_n^2 - \frac{1}{2} n \\
\Longrightarrow \quad M_n &= \sqrt{2I_n + n}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}_n\left[f(I_{n+1})\right] &= \frac{1}{2} f(I_n + M_n) + \frac{1}{2} f(I_n - M_n) \\
&= \frac{1}{2} f(I_n + \sqrt{2I_n + n}) + \frac{1}{2} f(I_n - \sqrt{2I_n + n})
\end{aligned}$$

Therefore

$$\mathbb{E}_n[f(I_{n+1})] = g(I_n)$$

where

$$g(i) = \frac{1}{2} f(i + \sqrt{2i + n}) + \frac{1}{2} f(i - \sqrt{2i + n})$$

as desired.

Solution 2.8.i: From the "multistep-ahead" version of the martingale property (equation 2.4.3) we have, for $0 \leq n \leq m \leq N$,

$$M_n = \mathbb{E}_n[M_m]$$

So, since M_n and M'_n are martingales, with $m = n$, for $0 \leq n \leq N$

$$\begin{aligned}\tilde{\mathbb{E}}_n[M_N] &= M_n \\ \tilde{\mathbb{E}}_n[M'_N] &= M'_n\end{aligned}$$

Hence, if $M_N = M'_N$

$$\begin{aligned}&\tilde{\mathbb{E}}_n[M_N] = \tilde{\mathbb{E}}_n[M'_N] \\ \implies &M_n = M'_n\end{aligned}$$

for arbitrary $0 \leq n \leq N$, as desired.

Solution 2.8.ii: Let V_N be the time N payoff of a derivative security. Define the process $\{V_n\}_{n=0}^N$ to be the derivative security price process such that

$$V_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r}\left[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)\right] \quad \star$$

By the construction of V_n from equation (1.2.16) we see that V_n is only dependent on the first n coin tosses $(\omega_1 \cdots \omega_n)$, hence V_n is adapted. Now, confirming the martingale property

$$\begin{aligned}\tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right] &= \frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n\left[V_{n+1}\right] \quad \text{(taking out what is known)} \\ &= \frac{1}{(1+r)^{n+1}}\left[\tilde{p}V(H) + \tilde{q}V(T)\right] \\ &= \frac{1}{(1+r)^{n+1}}(1+r)V_n \quad (\text{from } \star) \\ &= \frac{V_n}{(1+r)^n}\end{aligned}$$

Hence

$$V_0, \frac{V_1}{1+r}, \cdots, \frac{V_{N-1}}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N}$$

is a martingale under $\tilde{\mathbb{P}}$, as desired.

Solution 2.8.iii: Once again, by the same argument as in (ii), we see that V'_n depends only on the first n tosses. That is, V'_n is adapted. Now, for $0 \leq n \leq m \leq N$,

$$\begin{aligned}\tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right] &= \tilde{\mathbb{E}}_n\left[\frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_{n+1}\left[\frac{V_N}{(1+r)^{N-(n+1)}}\right]\right] \\ &= \frac{1}{(1+r)^n}\tilde{\mathbb{E}}_n\left[\tilde{\mathbb{E}}_{n+1}\left[\frac{V_N}{(1+r)^{N-n}}\right]\right] \quad \text{(taking out what is known)} \\ &= \frac{1}{(1+r)^n}\tilde{\mathbb{E}}_n\left[\frac{V_N}{(1+r)^{N-n}}\right] \quad \text{(by the tower property)} \\ &= \frac{V'_n}{(1+r)^n} \quad \text{(by definition)}\end{aligned}$$

Hence

$$V_0', \frac{V_1'}{1+r)}, \cdots, \frac{V_{N-1}'}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N}$$

is a martingale under $\tilde{\mathbb{P}}$, as desired.

Solution 2.8.iv: We have from parts (ii) and (iii) that

$$\begin{cases} V_0, \frac{V_1}{1+r}, \cdots, \frac{V_{N-1}}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N} \\ V_0', \frac{V_1'}{1+r)}, \cdots, \frac{V_{N-1}'}{(1+r)^{N-1}}, \frac{V_N}{(1+r)^N} \end{cases}$$

are martingales under $\tilde{\mathbb{P}}$. Additionally, we have that at time N both processes have the same terminal value V_N. Therefore, by part (i) we must have that $V_n = V_n'$ for arbitrary $0 \leq n \leq N$, as desired.

Solution 2.9.i: We first compute the up & down factors at each node

$$\begin{aligned} u_0 &= \frac{S_1(H)}{S_0} = \frac{8}{4} = 2 \\ d_0 &= \frac{S_1(T)}{S_0} = \frac{2}{4} = \frac{1}{2} \\ u_1(H) &= \frac{S_2(HH)}{S_1(H)} = \frac{12}{8} = \frac{3}{2} \\ d_1(H) &= \frac{S_2(HT)}{S_1(H)} = \frac{8}{8} = 1 \\ u_1(T) &= \frac{S_2(TH)}{S_1(T)} = \frac{8}{2} = 4 \\ d_1(T) &= \frac{S_2(TT)}{S_1(T)} = \frac{2}{2} = 1 \end{aligned}$$

Thus,

$$\begin{aligned} \tilde{p}_0 &= \frac{(1+r_0) - d_0}{u_0 - d_0} = \frac{(1+\frac{1}{4}) - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{\frac{3}{4}}{\frac{3}{2}} = \frac{6}{12} = \frac{1}{2} \\ \implies \tilde{q}_0 &= 1 - \tilde{p}_0 = \frac{1}{2} \\ \tilde{p}_1(H) &= \frac{(1+r_1(H)) - d_1(H)}{u_1(H) - d_1(H)} = \frac{1+\frac{1}{4}-1}{\frac{3}{2}-1} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \\ \implies \tilde{q}_1(H) &= 1 - \tilde{p}_1(H) = \frac{1}{2} \\ \tilde{p}_1(T) &= \frac{(1+r_1(T)) - d_1(T)}{u_1(T) - d_1(T)} = \frac{1+\frac{1}{2}-1}{4-1} = \frac{\frac{1}{2}}{3} = \frac{1}{6} \\ \implies \tilde{q}_1(T) &= 1 - \tilde{p}_1(T) = \frac{5}{6} \end{aligned}$$

Therefore, by the independence of the coin tosses,

$$\begin{aligned}
\tilde{\mathbb{P}}(HH) &= \tilde{p}_0\tilde{p}_1(H) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}\\
\tilde{\mathbb{P}}(HT) &= \tilde{p}_0\tilde{q}_1(H) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}\\
\tilde{\mathbb{P}}(TH) &= \tilde{q}_0\tilde{p}_1(T) = \frac{1}{2}\cdot\frac{1}{6} = \frac{1}{12}\\
\tilde{\mathbb{P}}(TT) &= \tilde{q}_0\tilde{p}_1(T) = \frac{1}{2}\cdot\frac{5}{6} = \frac{5}{12}
\end{aligned}$$

as desired.

Solution 2.9.ii: Going backwards through our tree

$$\begin{aligned}
V_1(H) &= \frac{1}{1+r_1(H)}[\tilde{p}_1(H)V_2(HH) + \tilde{q}_1(H)V_2(HT)]\\
&= \frac{4}{5}\left[\frac{1}{2}\cdot 5 + \frac{1}{2}\cdot 1\right]\\
&= \frac{12}{5} = 2.4\\
V_1(T) &= \frac{1}{1+r_1(T)}[\tilde{p}_1(T)V_2(TH) + \tilde{q}_1(T)V_2(TT)]\\
&= \frac{2}{3}\left[\frac{1}{6}\cdot 1 + \frac{5}{6}\cdot 0\right]\\
&= \frac{1}{9} = 0.111...
\end{aligned}$$

Hence

$$\begin{aligned}
V_0 &= \frac{1}{1+r_0}[\tilde{p}_0V_1(H) + \tilde{q}_0V_1(T)]\\
&= \frac{4}{5}\left[\frac{1}{2}\cdot\frac{12}{5} + \frac{1}{2}\cdot\frac{1}{9}\right]\\
&= \frac{226}{225} = 1.00444...
\end{aligned}$$

Solution 2.9.iii: Recalling our formula for Δ_0 we have

$$\begin{aligned}
\Delta_0 &= \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}\\
&= \frac{\frac{12}{5} - \frac{1}{9}}{8-2}\\
&= \frac{103}{270} = 0.3\overline{814}
\end{aligned}$$

Solution 2.9.iv: Once again

$$\begin{aligned}\Delta_1 &= \frac{V_1(HH) - V_1(HT)}{S_1(HH) - S_1(HT)} \\ &= \frac{5-1}{12-8} \\ &= 1\end{aligned}$$

Part II

Solution 1.a: We first note that $J_k(\omega_1 \cdots \omega_k)$ is a function of the first k coin tosses, namely, the value J_k takes will depend on the kth coin toss. Hence, J_k is adapted since it does not rely on information beyond time k. Now, we have that Y_k is a function of deterministic variables k, p and random variables J_i. We have already determined that J_i is adapted and so, since Y_k introduces no further dependency on any coin tosses, we may state that Y_k is in fact adapted.

Now, to confirm that the martingale property holds

$$\begin{aligned}\mathbb{E}_k\left[Y_{k+1}\right] &= \mathbb{E}_k\left[\sum_{i=0}^{k+1} J_i - (k+1)p\right] \quad \text{(by definition)} \\ &= \mathbb{E}_k\left[\sum_{i=0}^{k} J_i + J_{k+1} - kp - p\right] \\ &= \mathbb{E}_k\left[Y_k + J_{k+1} - p\right] \\ &= \mathbb{E}_k\left[Y_k - p\right] + \mathbb{E}_k\left[J_{k+1}\right] \quad \text{(linearity)} \\ &= Y_k - p + \mathbb{E}_k\left[J_{k+1}\right] \quad \text{(since } Y_k \text{ and } p \text{ are known at } k\text{)}\end{aligned}$$

but we have that J_{k+1} is independent of the first k coin tosses since its value is uniquely determined by the $(k+1)$th coin toss, hence

$$\begin{aligned}\mathbb{E}_k\left[Y_{k+1}\right] &= Y_k - p + \mathbb{E}_k\left[J_{k+1}\right] \\ &= Y_k - p + p \cdot (1) + (1-p) \cdot (0) \\ &= Y_k - p + p \\ &= Y_k\end{aligned}$$

Therefore, since Y_k is adapted and satisfies the martingale property we may conclude that Y_k is in fact a martingale, as desired.

Solution 1.b: From part (a) we have that Y_k is adapted. Now, to confirm that the Markov property holds

$$\begin{aligned}
\mathbb{E}_k\left[f(Y_{k+1})\right] &= \mathbb{E}_k\left[f\left(\sum_{i=1}^{k+1} J_i - (k+1)p\right)\right] \\
&= \mathbb{E}_k\left[f\left(\sum_{i=1}^{k} J_i + J_{k+1} - kp - p\right)\right] \\
&= \mathbb{E}_k\left[f\left(Y_k - J_{k+1} - p\right)\right]
\end{aligned}$$

We now note that Y_k is adapted to the first k coin tosses, p some constant $0 < p < 1$, and J_{k+1} is independent of the first k coin tosses. That is, we have that Y_k and p is measurable with respect to the information up to k and J_{k+1} is independent of this information, therefore, by the Independence Lemma we have that

$$\mathbb{E}_k\left[f\left(Y_k - J_{k+1} - p\right)\right] = g(Y_k)$$

such that g is the ordinary expectation

$$\begin{aligned}
g(y) &= \mathbb{E}\left[f\left(y - J_{k+1} - p\right)\right] \\
&= p \cdot f(y + (1) - p) + (1 - p) \cdot f(y + (0) - p) \\
&= pf(y + 1 - p) + (1 - p)f(y - p)
\end{aligned}$$

Since we have that Y_k is adapted and have successfully found a function g for arbitrary f such that

$$\mathbb{E}_k\left[f(Y_{k+1})\right] = g(Y_k)$$

we may conclude that the process $\{Y_k\}_{k=0}^{N}$ is Markovian, as desired.

Solution 2.a: We first consider the asset price tree. Since the option value is clearly path dependent we construct the following non-recombining tree for later use in determining the derivative price

$$
\begin{tikzcd}
 & & & S_3(HHH) = 133.1 \\
 & & S_2(HH) = 121 \arrow[ur] \arrow[r] & S_3(HHT) = 108.9 \\
 & S_1(H) = 110 \arrow[ur] \arrow[dr] & & S_3(HTH) = 108.9 \\
 & & S_2(HT) = 99 \arrow[ur] \arrow[r] & S_3(HTT) = 89.1 \\
S_0 = 100 \arrow[uur] \arrow[ddr] & & & S_3(THH) = 108.9 \\
 & & S_2(TH) = 99 \arrow[ur] \arrow[r] & S_3(THT) = 89.1 \\
 & S_1(T) = 90 \arrow[ur] \arrow[dr] & & S_3(TTH) = 89.1 \\
 & & S_2(TT) = 81 \arrow[ur] \arrow[r] & S_3(TTT) = 72.9
\end{tikzcd}
$$

Figure 1: Asset price process tree

Now, at time 3 we have the option payoff

$$V_3(\omega_1\omega_2\omega_3) = (S_3(\omega_1\omega_2\omega_3) - K)^+ \cdot \prod_{i=1}^{3} \mathbb{1}_{\{S_i(\omega_1\cdots\omega_i)>B\}}$$

So, with strike $K = 105$ and barrier $B = 95$

$$
\begin{aligned}
V_3(HHH) &= (133.1 - 105)^+ \cdot 1 = 28.1 \\
V_3(HHT) &= (108.9 - 105)^+ \cdot 1 = 3.9 \\
V_3(HTH) &= (108.9 - 105)^+ \cdot 1 = 3.9 \\
V_3(HTT) &= (89.1 - 105)^+ \cdot 0 = 0 \\
V_3(THH) &= (108.9 - 105)^+ \cdot 0 = 0 \\
V_3(THT) &= (89.1 - 105)^+ \cdot 0 = 0 \\
V_3(TTH) &= (89.1 - 105)^+ \cdot 0 = 0 \\
V_3(TTT) &= (72.9 - 105)^+ \cdot 0 = 0
\end{aligned}
$$

and determining the risk-neutral probabilities

$$
\begin{aligned}
&\tilde{p} = \frac{(1+r)-d}{u-d} = \frac{(1+(e^{0.05}-1))-0.9}{1.1-0.9} = \frac{e^{0.05}-0.9}{0.2} = 5e^{0.05} - 4.5 \approx 0.7563\,5548 \\
\implies &\tilde{q} = 1 - \tilde{p} = 1 - (5e^{0.05} - 4.5) = 5.5 - 5e^{0.05} \approx 0.2346\,4451
\end{aligned}
$$

From these values it is quick work to fill in the remaining nodes of the non-recombining derivative price tree

$$
\begin{aligned}
V_2(HH) &= \frac{1}{1+r}[\tilde{p}V_3(HHH) + \tilde{q}V_3(HHT)] \\
&= \frac{1}{1+(e^{0.05}-1)}[0.7563\,5548 \cdot (28.1) + 0.2346\,4451 \cdot (3.9)] \\
&= e^{-0.05} \cdot 22.1687\,0257 \\
&= 21.0875\,2219 \\
V_2(HT) &= \frac{1}{1+r}[\tilde{p}V_3(HTH) + \tilde{q}V_3(HTT)] \\
&= \frac{1}{1+(e^{0.05}-1)}[0.7563\,5548 \cdot (3.9) + 0.2346\,4451 \cdot (0)] \\
&= e^{-0.05} \cdot 2.9497\,8637 \\
&= 2.8059\,2359 \\
V_2(TH) &= \frac{1}{1+r}[\tilde{p}V_3(THH) + \tilde{q}V_3(THT)] \\
&= \frac{1}{1+(e^{0.05}-1)}[0.7563\,5548 \cdot (0) + 0.2346\,4451 \cdot (0)] \\
&= e^{-0.05} \cdot 0 \\
&= 0 \\
V_2(TT) &= \frac{1}{1+r}[\tilde{p}V_3(TTH) + \tilde{q}V_3(TTT)] \\
&= \frac{1}{1+(e^{0.05}-1)}[0.7563\,5547(0) \cdot +0.2346\,4451 \cdot (0)] \\
&= e^{-0.05} \cdot 0 \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
V_1(H) &= \frac{1}{1+r}[\tilde{p}V_2(HH) + \tilde{q}V_2(HT)] \\
&= \frac{1}{1+(e^{0.05}-1)}[0.7563\,5547 \cdot (21.0875\,2219) + 0.2346\,4451 \cdot (2.8059\,2359)] \\
&= e^{-0.05} \cdot 16.6080\,5753 \\
&= 15.7980\,7301 \\
V_1(T) &= \frac{1}{1+r}[\tilde{p}V_2(TH) + \tilde{q}V_2(TT)] \\
&= \frac{1}{1+(e^{0.05}-1)}[0.7563\,5547 \cdot 0 + 0.2346\,4451 \cdot 0] \\
&= e^{-0.05} \cdot 0 \\
&= 0
\end{aligned}
$$

Thus

$$\begin{aligned}
V_0 &= \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] \\
&= \frac{1}{1+(e^{0.05}-1)}[0.7563\,5547 \cdot (15.7980\,7301) + 0.2346\,4451 \cdot (0)] \\
&= e^{-0.05} \cdot 11.9489\,5909 \\
&= 11.3662\,0148
\end{aligned}$$

as desired.

Solution 2.b: We calculate our values for Δ_n, $0 \le n \le 3$, at each node of the tree using

$$\Delta_n(\omega_1 \cdots \omega_n) = \frac{S_{n+1}(\omega_1 \cdots \omega_n H) - S_{n+1}(\omega_1 \cdots \omega_n T)}{V_{n+1}(\omega_1 \cdots \omega_n H) - V_{n+1}(\omega_1 \cdots \omega_n T)}$$

Hence at time $t = 0$

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{15.7980\,7301 - 0}{110 - 90} = \frac{15.7980\,7301}{20} = 0.7899\,0365$$

at time $t = 1$ we find

$$\begin{aligned}
\Delta_1(H) &= \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{21.0875\,2219 - 2.8059\,2359}{121 - 99} = \frac{18.2815\,9860}{22} = 0.8309\,8175 \\
\Delta_1(T) &= \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} = \frac{0-0}{99-81} = \frac{0}{18} = 0
\end{aligned}$$

and finally at time $t = 2$

$$\begin{aligned}
\Delta_2(HH) &= \frac{V_3(HHH) - V_3(HHT)}{S_3(HHT) - S_3(HHT)} = \frac{28.1 - 3.9}{133.1 - 108.9} = \frac{24.2}{24.2} = 1 \\
\Delta_2(HT) &= \frac{V_3(HTH) - V_3(HTT)}{S_3(HTH) - S_3(HTT)} = \frac{3.9 - 0}{108.9 - 89.1} = \frac{3.9}{19.8} = 0.1\overline{96} \\
\Delta_2(TH) &= \frac{V_3(THH) - V_3(THT)}{S_3(THH) - S_3(THT)} = \frac{0-0}{108.9 - 89.1} = \frac{0}{19.8} = 0 \\
\Delta_2(TT) &= \frac{V_3(TTH) - V_3(TTT)}{S_3(THH) - S_3(TTT)} = \frac{0-0}{89.1 - 72.9} = \frac{0}{16.2} = 0
\end{aligned}$$

as desired.