Mathematical & Computational Finance I
Lecture Notes

Probability Theory on Coin Toss Space

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1 Probability Theory

1.1 Conditional Expectation

Suppose X is a random variable on the coin toss sample space

$$\Omega = \{\omega : \omega = \omega_1 \cdots \omega_n,\ \omega_i \in \{H, T\}\}$$

where $\mathbb{P}(\omega_i = H) = p$. For

$$X(\omega) = X(\omega_1 \cdots \omega_N)$$

we can estimate X (or some function of X) using only the first n coin tosses for $n \leq N$. We think of the observations of the coin tosses up until time $n \leq N$ as the information available at time n.

Definition 1. For $1 \leq n \leq N$ let $\omega_1 \cdots \omega_n$ be given. There are 2^{N-n} possible continuations of $\omega_{n+1} \cdots \omega_N$. Write

$$\begin{aligned}
\#H(\omega_{n+1} \cdots \omega_N) &= \text{number of heads} \\
\#T(\omega_{n+1} \cdots \omega_N) &= \text{number of tails}
\end{aligned}$$

Define the random variable, for $X = X(\omega_1 \cdots \omega_n \cdots \omega_N)$,

$$\begin{aligned}
\mathbb{E}_n[X](\omega_1 \cdots \omega_n) &= \sum_{\omega_{n+1} \cdots \omega_N} \mathbb{P}(\omega_{n+1} \cdots \omega_N) X(\omega_1 \cdots \omega_N) \\
&= \sum_{\omega_{n+1} \cdots \omega_N} p^{\#H(\omega_{n+1} \cdots \omega_N)} (1-p)^{\#T(\omega_{n+1} \cdots \omega_N)} X(\omega_1 \cdots \omega_N)
\end{aligned}$$

That is, we sum over all possible continuations of the coin toss sequence. We say that $\mathbb{E}_n[X]$ is the <u>conditional expectation</u> of X based on the information at time n.

Definition 2. Let $X = X(\omega_1 \cdots \omega_N)$ be a random variable depending on the first N coin tosses. In the two extreme cases $n = 0$ and $n = N$ we define the conditional expectation as

$$\begin{aligned}
\mathbb{E}_0[X] &= \mathbb{E}[X] \quad \text{(no information ordinary expectation)} \\
\mathbb{E}_N[X] &= X \quad \text{(full information)}
\end{aligned}$$

1.1.1 Properties of Conditional Expectation

Let N be a positive integer and X, Y be random varaibles depending on the first N coin tosses. Let $0 \leq n \leq N$ be given. Then

1. For constants $c_1, c_2 \in \mathbb{R}$

$$\mathbb{E}_n[c_1 X + c_2 Y] = c_1 \mathbb{E}_n[X] + c_2 \mathbb{E}_n[Y] \quad \text{linearity}$$

2. If X only depends on the first n coin tosses

$$\mathbb{E}_n[XY] = X\mathbb{E}_n[Y] \quad \text{taking out what is known/adaptedness of } X$$

3. If $0 \leq n \leq m \leq N$ then

$$\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X] \quad \text{tower property}$$

   in particular, $\mathbb{E}[\mathbb{E}_m[X]] = \mathbb{E}[X]$.

4. If X only depends on $\omega_{n+1} \cdots \omega_N$ then

$$\mathbb{E}_n[X] = \mathbb{E}[X] \quad \text{independence}$$

5. If ϕ is a convex function then

$$\phi(\mathbb{E}_n[X]) \leq \mathbb{E}_n[\phi(X)] \quad \text{conditional Jensen's inequality}$$

1.2 Martingales

In the binomial asset pricing model with $\tilde{p} = \frac{1+r-d}{u-d}$ we saw that

$$V_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)]$$
$$S_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r}[\tilde{p}S_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}S_{n+1}(\omega_1 \cdots \omega_n T)]$$

for $n = 0, 1, ..., N-1$. These equations can be rewritten using the definition of conditional expectation under the risk-neutral probability measure $\tilde{\mathbb{P}}$ as

$$V_n(\omega_1 \cdots \omega_n) = \tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{1+r}\right](\omega_1 \cdots \omega_n)$$
$$S_n(\omega_1 \cdots \omega_n) = \tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{1+r}\right](\omega_1 \cdots \omega_n)$$

where $\tilde{\mathbb{E}}_n[\cdot]$ denotes the conditional expectation at time n with respect to the risk-neutral probability measure. Dividing both sides if the second equation by $(1+r)^n$ gives us

$$\frac{S_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right]$$

Define the discounted asset price process

$$\bar{S}_n := \frac{S_n}{(1+r)^n}$$

Then, the discounted asset price process satisfies

$$\bar{S}_n = \tilde{\mathbb{E}}_n[\bar{S}_{n+1}]$$

We say that this relationship between a process and its conditional expectation is called the martingale property. For the discounted asset price the martingale property is a consequence of using the risk-neutral (martingale) probability measure $\tilde{\mathbb{P}}$ (i.e. this relationship would not hold under the real-world measure).

The martingale property of the discounted asset price tells us that the best estimate for $\bar{S}_{n+1}$ given the information at time n is $\bar{S}_n$ (critically, only under $\tilde{\mathbb{P}}$). In general, if X, Y are random variables on a probability space $(\Omega, \mathbb{P})$ we have[1]

$$\mathbb{E}_{\mathbb{P}}[(Y - g(X))^2] \geq \mathbb{E}_{\mathbb{P}}[(Y - \mathbb{E}[Y|X])^2]$$

This implies that $\mathbb{E}_{\mathbb{P}}[Y|X]$ is the best estimate, in mean-square terms, of Y which is itself a function of X. In our notation, if $X_i = \omega_i,\ i = 1, ..., n$ are the outcomes of the first n tosses.

$$\tilde{\mathbb{E}}[(\bar{S}_{n+1} - g(\omega_1, ..., \omega_n))] \geq \tilde{\mathbb{E}}[(\bar{S}_{n+1} - \tilde{\mathbb{E}}_n[\bar{S}_{n+1}])^2] = \tilde{\mathbb{E}}_n[(\bar{S}_{n+1} - \bar{S}_n)^2]$$

Definition 3. Consider the binomial asset pricing model and let $M_0, M_1, ..., M_N$ be a sequence of random variables. The sequence $\{M_n\}_{n=0}^N$ is called an adapted stochastic process if

1. M_0 is a constant (deterministic) and
2. For each $n \in \{1, ..., N\}$ the random variable M_n depends only on the first n coin tosses.

Adapted processes are extremely important in financial modelling of asset pricing, portfolio process, and other quantities. An adapted process does not "look into the future" or use "future information" to determine its current value today.

Definition 4. Consider the binomial asset pricing model with a given probability measure $\mathbb{P}$ (can be real-world, risk-neutral, or something else). A stochastic process $\{M_n\}_{n=0}^N$ is a $\mathbb{P}$-martingale if

1. $\{M_n\}_{n=0}^N$ is adapted
2. $M_n = \mathbb{E}_n^{\mathbb{P}}[M_{n+1}]$ for $n = 0, ..., N-1$

where $\mathbb{P}$ can be the risk neutral, real world, or some other measure.

[1] I don't really know how to interpret this is in terms of martingales.

Note that both properties are necessarily required to demonstrated that a process is a martingale. Although the first criteria is sometimes trivial it is important to mention (we'll lose a mark on an exam!).

We may relax the equality in the martingale property to inequalities in order to define two important classes of processes.

Definition 5. Consider the binomial asset pricing model with a given probability measure $\mathbb{P}$. A stochastic process $\{M_n\}_{n=0}^N$ is a $\mathbb{P}$-submartingale if

1. $\{M_n\}_{n=0}^N$ is adapted
2. $M_n \leq \mathbb{E}_n^{\mathbb{P}}[M_{n+1}]$ for $n = 0, ..., N-1$

Definition 6. Consider the binomial asset pricing model with a given probability measure $\mathbb{P}$. A stochastic process $\{M_n\}_{n=0}^N$ is a $\mathbb{P}$-supermartingale if

1. $\{M_n\}_{n=0}^N$ is adapted
2. $M_n \geq \mathbb{E}_n^{\mathbb{P}}[M_{n+1}]$ for $n = 0, ..., N-1$

A process is a martingale if it is both a submartingale and a supermartingale.

Example: On a finite probability space $(\Omega, \mathbb{P})$ suppose that $\{X_n\}_{n=0}^N$ and $\{Y_n\}_{n=0}^N$ are $\mathbb{P}$-supermartingales. For $n = 0, ...N$ define $Z_n = (aX_n + bY_n)$ for real numbers $a, b \geq 0$. Prove that Z_n is a supermartingale under $\mathbb{P}$.

Solution:

(i) For every $n = 0, 1, ..., N$ we have that $Z_n = (aX_n + bY_n)$ depends only on the first n coin tosses since X_n and Y_n depend only on the first n coin tosses. That is, $\{Z_n\}$ is an adapted process.

(ii) For any $n \in \{0, ..., N-1\}$, since $\{X_n\}$ and $\{Y_n\}$ are supermartingales, we have that $aX_n \geq \mathbb{E}_n[aX_{n+1}]$ and $bY_n \geq \mathbb{E}_n[bY_{n+1}]$. From linearity of conditional expectation we find

$$\mathbb{E}_n[Z_{n+1}] = \mathbb{E}_n[aX_{n+1} + bY_{n+1}] = \mathbb{E}_n[aX_{n+1}] + \mathbb{E}_n[bY_{n+1}] \leq aX_n + bY_n = Z_n$$

Therefore, $\{Z_n\}$ satisfies the definition of a $\mathbb{P}$-supermartingale.

Example: On a finite probability space $(\Omega, \mathbb{P})$ suppose that $\{X_n\}_{n=0}^N$ and $\{Y_n\}_{n=0}^N$ are $\mathbb{P}$-supermartingales. For $n = 0, ..., N$ define $Z_n = (X_n \wedge Y_n)$ for real numbers a and b, where the notation $x \wedge y$ denotes the minimum of x and y. Prove that Z_n is a supermartingale under $\mathbb{P}$.

Solution:

(i) $Z_n = (X_n \wedge Y_n)$ depends only on the first n coin tosses since X_n and Y_n depend only on the first n coin tosses. That is, $\{Z_n\}$ is an adapted process.

(ii) For any $n \in \{0, ..., N-1\}$

$$\begin{aligned}
\mathbb{E}_n[Z_{n+1}] &= \mathbb{E}_n[X_{n+1} \wedge Y_{n+1})] \\
&\leq \mathbb{E}_n[X_{n+1}] \quad (\text{since } Z_{n+1} = \min\{X_{n+1}, Y_{n+1}\} \leq X_{n+1}) \\
&\leq X_n
\end{aligned}$$

Similarly, we find that

$$\mathbb{E}_n[Z_{n+1}] \leq Y_n$$

Hence

$$\mathbb{E}_n[Z_{n+1}] \leq (X_{n+1} \wedge Y_{n+1}) = Z_n$$

since

$$z \leq x \quad \text{and} \quad z \leq y \implies z \leq (x \wedge y)$$

Therefore, $\{Z_n\}$ is a $\mathbb{P}$-supermartingale by definition.

Example: On a finite probability space $(\Omega, \mathbb{P})$ suppose that $\{X_n\}_{n=0}^N$ is a $\mathbb{P}$-submartingale. Prove that $(X_n - a)^+$ is a submartingale for any constant a.

Solution: Define $Z_n = (X_n - a)^+ = \max\{X_n - a, 0\}$ for $n = 0, ..., N$.

(i) Clearly Z_n depends only on the first n coin tosses since X_n only depends on the first n coin tosses and a is a constant. Therefore, $\{Z_n\}$ is an adapted process.

(ii) For any $n \in \{0, ..., N-1\}$ since by definition $Z_{n+1} = \max\{X_{n+1} - a, 0\} \geq X_{n+1} - a$, thus

$$\begin{aligned}
\mathbb{E}_n[Z_{n+1}] &\geq \mathbb{E}_n[X_{n+1} - a] \\
&= \mathbb{E}_n[X_{n+1}] - a \\
&\geq X_n - a \quad (\text{by definition of the submartingale } X_n)
\end{aligned}$$

Also, $Z_{n+1} \geq 0$ which implies that

$$\mathbb{E}_n[Z_{n+1}] \geq 0$$

Hence, combining both results

$$\mathbb{E}_n[Z_{n+1}] \geq \max\{X_n - a, 0\} = (X_n - a)^+ = Z_n$$

Therefore, by definition, $\{Z_n\}$ is a $\mathbb{P}$-submartingale.

1.2.1 Martingales Over Multiple Time Steps

The martingale property holds over multiple time steps:

Theorem 1. If $\{M_n\}_{n=0}^{N}$ is a $\mathbb{P}$-martingale then

$$M_n = \mathbb{E}_n[M_m]$$

for all $0 \leq n \leq m \leq N$.

Proof.

$$\begin{aligned}\mathbb{E}_n[M_m] &= \mathbb{E}_n[\mathbb{E}_{m-1}[M_m]] \quad \text{(applying the tower property)} \\ &= \mathbb{E}_n[M_{m-1}] \quad \text{(martingale property over one time step)}\end{aligned}$$

Since the one-step martingale property holds we suspect that we may proceed inductively on m. Define $P(m): \ \mathbb{E}_n[M_m] = M_n$. For $P(1)$ we see that

$$P(0): \ \mathbb{E}_n[M_{n+1}] = M_n$$

is true by definition of the one-step martingale property. Now, assume that our inductive hypothesis holds, that is,

$$P(k): \ \mathbb{E}_n[M_{n+k}] = M_n$$

For $P(k+1)$ we find

$$\begin{aligned}P(k+1): \ \mathbb{E}_n[M_{n+k+1}] &= \mathbb{E}_n[\mathbb{E}_{n+1}[M_{n+k+1}]] \quad \text{(tower property)} \\ &= \mathbb{E}_n[M_{n+k}] \quad \text{(martingale property)} \\ &= M_n \quad \text{(by our inductive hypothesis)}\end{aligned}$$

Therefore $P(k+1)$ holds. We may conclude that $P(m)$ is true for arbitrary m such that $0 \leq n \leq m \leq N$, as desired. □

Corollary 1. If $\{M_n\}_{n=0}^{N}$ is a $\mathbb{P}$-martingale then

$$M_0 = \mathbb{E}[M_n]$$

for every $n = 0, ..., N$.

Proof. Using the previous result with $n = 0$ and $m = n$ we find

$$\mathbb{E}[M_n] \equiv \mathbb{E}_0[M_n] = M_0$$

as desired. □

In general, we do not expect risky asset processes to be martingales with respect to the real-world measure. That is, the real-world/physical probabilities under $\mathbb{P}$ should not be such that S_n is a martingale. If it were the case that S_n was a martingale with respect to the real-world measure then there would be no incentive for risk-taking: The one-period expected return is zero! In the real world we hope that the expected asset prices should rise faster than the bank account to appropriately compensate investors for the extra risk.

Theorem 2. Consider the general binomial asset-pricing model with $0 < d < 1 + r < u$. Let the risk-neutral probabilities be given by

$$\tilde{p} = \frac{(1+r) - d}{u - d}, \quad \tilde{q} = \frac{u - (1+r)}{u - d}$$

Then, the discounted asset price process $\left\{ \frac{S_n}{(1+r)^n} \right\}_{n=0}^{N}$ is a $\tilde{\mathbb{P}}$-martingale.

Proof. First, we see that $\frac{S_n}{(1+r)^n}$ depends only on the first n coin tosses by construction of the binomial mode. Therefore the discounted price process is adapted.

Taking the expectation with respect to the risk neutral measure

$$\begin{aligned}
\tilde{\mathbb{E}}_n \left[\frac{S_{n+1}}{(1+r)^{n+1}} \right] &= \frac{1}{(1+r)^n} \frac{1}{1+r} \tilde{\mathbb{E}}_n [\tilde{p} S_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q} S_{n+1}(\omega_1 \cdots \omega_n T)] \\
&= \frac{1}{(1+r)^n} \frac{1}{1+r} (\tilde{p} u S_n + \tilde{q} d S_n) \\
&= \frac{S_n}{(1+r)^n} \frac{1}{1+r} (\tilde{p} u + \tilde{q} d)
\end{aligned}$$

but

$$\begin{aligned}
\tilde{p} u + \tilde{q} d &= \left(\frac{(1+r) - d}{u - d} \right) u + \left(\frac{u - (1+r)}{u - d} \right) d \\
&= \frac{1}{u - d} [(1+r)u - du + ud - (1+r)d] \\
&= \frac{1}{u - d} [(1+r)u - (1+r)d] \\
&= \frac{u - d}{u - d} (1+r) \\
&= 1 + r
\end{aligned}$$

so

$$\begin{aligned}
\tilde{\mathbb{E}}_n \left[\frac{S_{n+1}}{(1+r)^{n+1}} \right] &= \frac{S_n}{(1+r)^n} \frac{1}{1+r} (\tilde{p} u + \tilde{q} d) \\
&= \frac{S_n}{(1+r)^n} \frac{1}{1+r} (1+r) \\
&= \frac{S_n}{(1+r)^n}
\end{aligned}$$

as desired. $\square$

1.2.2 Portfolio Processes

In the binomial model with N coin tosses suppose at each time n the investor

1. Takes a position Δ_n shares of stock.
2. Holds the position until time $n+1$.
3. Based on the outcome of the $(n+1)^{\text{th}}$ coin toss takes a new position at time $(n+1)$ of Δ_{n+1} shares.
4. Portfolio is rebalanced by either (a) Liquidating shares and investing the proceeds in the bank account or (b) Reducing the bank account (borrowing more if necessary) and buying additional shares.

$\implies$ The portfolio process $\{\Delta_0, \Delta_1, ..., \Delta_{N-1}\}$ is adapted.

If the investor has initial wealth X_0 at time 0 then the wealth at time n is

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

for $n = 0, ..., N-1$. We call this equation the wealth equation. Note that X_n depends only on the first n coin tosses so the wealth process $\{X_n\}_{n=0}^N$ is adapted.

Theorem 3. Consider the binomial model with N periods and $0 < d < 1+r < u$. Let $\Delta_0, ..., \Delta_{N-1}$ be an adapted portfolio process and X_0 be constant. Then the wealth process $\{X_n\}_{n=0}^N$ define by the wealth equation is a discounted martingale. That is,

$$\frac{X_n}{(1+r)^n} = \tilde{\mathbb{E}}_n \left[\frac{X_{n+1}}{(1+r)^{n+1}} \right]$$

for $n = 0, ...N-1$.

Proof. From the wealth equation we have

$$\begin{aligned}
\tilde{\mathbb{E}}_n \left[\frac{X_{n+1}}{(1+r)^{n+1}} \right] &= \tilde{\mathbb{E}}_n \left[\frac{\Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)}{(1+r)^{n+1}} \right] \\
&= \tilde{\mathbb{E}}_n \left[\frac{1}{(1+r)^{n+1}} \Delta_n S_{n+1} \right] + \tilde{\mathbb{E}}_n \left[\frac{1}{(1+r)^n} (X_n - \Delta_n S_n) \right] \\
&= \Delta_n \tilde{\mathbb{E}}_n \left[\frac{S_{n+1}}{(1+r)^{n+1}} \right] + \frac{X_n}{(1+r)^n} - \Delta_n \frac{S_n}{(1+r)^n} \quad \text{(taking out what is known)} \\
&= \Delta_n \frac{S_n}{(1+r)^n} + \frac{X_n}{(1+r)^n} - \Delta_n \frac{S_n}{(1+r)^n} \quad \left(\text{since } \frac{S_{n+1}}{(1+r)^{n+1}} \text{ is a martingale}\right) \\
&= \frac{X_n}{(1+r)^n}
\end{aligned}$$

as desired. □

Corollary 2. An immediate consequence of the previous theorem is that, under the conditions of the previous theorem,

$$\tilde{\mathbb{E}} \left[\frac{X_n}{(1+r)^n} \right] = X_0$$

for $n = 0, ..., N$.

This corollary implies that there can be no arbitrage in the binomial model![2]

[2] I'm not quite sure I see this...

2 Fundamental Theorem of Asset Pricing

Theorem 4. First Fundamental Theorem of Asset Pricing. If there exists a risk neutral probability measure then arbitrage is not possible.