Mathematical & Computational Finance I
Lecture Notes

State Prices/Change of Measure

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1 Change of Measure

In the binomial asset pricing model we have two measure: $\mathbb{P}$, the real-world measure, and $\tilde{\mathbb{P}}$, the risk-neutral measure. We used $\tilde{\mathbb{P}}$ as a way to arrive to "correct" derivative prices, but what if we wish to consider $\mathbb{P}$ in the real world? How we do interpret the results we find in the risk-neutral measure for results relative to the real-world measure?

Definition 1. Consider a finite sample space Ω on which two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are defined. Assume that $\tilde{\mathbb{P}}(\omega) > 0$ and $\tilde{\mathbb{P}}(\omega) > 0$ for all $\omega \in \Omega$. The random variable

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\tilde{\mathbb{P}}(\omega)}, \quad \omega \in \Omega$$

is the Radon-Nikodým derivative of $\tilde{\mathbb{P}}(\omega)$ with respect to $\tilde{\mathbb{P}}(\omega)$. We sometimes use the notation

$$Z(\omega) = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}(\omega)$$

The Radon-Nikodým derivative has the following important properties:

Theorem 1. Let $\tilde{\mathbb{P}}(\omega)$ and $\mathbb{P}(\omega)$ be probability measures on a finite sample space Ω. Assume that $\mathbb{P}(\omega) > 0$ and $\tilde{\mathbb{P}}(\omega) > 0$ for all $\omega \in \Omega$. Then the Radon-Nikodým derivative $Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ satisfies

(i) $\mathbb{P}(Z > 0) = 1$

(ii) $\mathbb{E}[Z] = 1$

(iii) For any random variable Y

$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$$

Proof. (i) We use the definitions. Assume that $\tilde{\mathbb{P}}(\omega) > 0$, so $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$ so

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} > 0 \quad \forall_{\omega\in\Omega}$$

Hence

$$\mathbb{P}(Z(\omega) > 0) = \sum_{\{\omega: Z(\omega)>0\}} \mathbb{P}(\omega) = \sum_{\omega\in\Omega} \mathbb{P}(\omega) = 1$$

□

Proof. (ii)

$$\begin{aligned}
\mathbb{E}[Z] &= \sum_{\omega\in\Omega} Z(\omega)\mathbb{P}(\omega) \\
&= \sum_{\omega\in\Omega} \mathbb{P}(\omega)\frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} \\
&= \sum_{\omega\in\Omega} \tilde{\mathbb{P}}(\omega) = 1
\end{aligned}$$

□

Proof. (iii)

$$\begin{aligned}
\tilde{\mathbb{E}}[Z] &= \sum_{\omega\in\Omega} Y(\omega)\tilde{\mathbb{P}}(\omega) \\
&= \sum_{\omega\in\Omega} Y(\omega)\tilde{\mathbb{P}}(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(\omega)} \\
&= \sum_{\omega\in\Omega} Y(\omega)Z(\omega)\mathbb{P}(\omega) \\
&= \mathbb{E}[ZY]
\end{aligned}$$

□

Example: Consider the binomial model with $N = 3, u = 2, d = \frac{1}{2}$, and $r = \frac{1}{4}$. If we have $p = \frac{2}{3}$ and $q = \frac{1}{3}$ then we have the tree

$$
\begin{tikzcd}
 & & & S_3(HHH) = 32 \\
 & & S_2(HH) = 8 \arrow[ur] \arrow[dr] & \\
 & S_1(H) = 8 \arrow[ur] \arrow[dr] & & \begin{matrix} S_3(HHT) = \\ S_3(HTH) = \\ S_3(THH) = 8 \end{matrix} \\
S_0 = 4 \arrow[ur] \arrow[dr] & & \begin{matrix} S_2(HT) = \\ S_2(TH) = 4 \end{matrix} \arrow[ur] \arrow[dr] & \\
 & S_1(T) = 2 \arrow[ur] \arrow[dr] & & \begin{matrix} S_3(HTT) = \\ S_3(THT) = \\ S_3(TTH) = 2 \end{matrix} \\
 & & S_2(TT) = 1 \arrow[ur] \arrow[dr] & \\
 & & & S_3(TTT) = \frac{1}{2}
\end{tikzcd}
$$

Figure 1: Asset price tree S_n

We can calculate the probabilities for a particular node under $\mathbb{P}$

$$
\begin{tikzcd}
 & & & p_3(HHH) = \frac{8}{27} \\
 & & p_2(HH) = \frac{4}{9} \arrow[ur] \arrow[dr] & \\
 & p_1(H) = \frac{2}{3} \arrow[ur] \arrow[dr] & & \begin{matrix} p_3(HHT) = \\ p_3(HTH) = \\ p_3(THH) = \frac{4}{27} \end{matrix} \\
\mathbb{P} \arrow[ur] \arrow[dr] & & \begin{matrix} p_2(HT) = \\ p_2(TH) = \frac{2}{9} \end{matrix} \arrow[ur] \arrow[dr] & \\
 & p_1(T) = \frac{1}{3} \arrow[ur] \arrow[dr] & & \begin{matrix} p_3(HTT) = \\ p_3(THT) = \\ p_3(TTH) = \frac{2}{27} \end{matrix} \\
 & & p_2(TT) = \frac{1}{9} \arrow[ur] \arrow[dr] & \\
 & & & p_3(TTT) = \frac{1}{27}
\end{tikzcd}
$$

Figure 2: Real world probabilities $\mathbb{P}$

Similarly, under $\tilde{\mathbb{P}}$ we have $\tilde{p} = \frac{1}{2}$, so

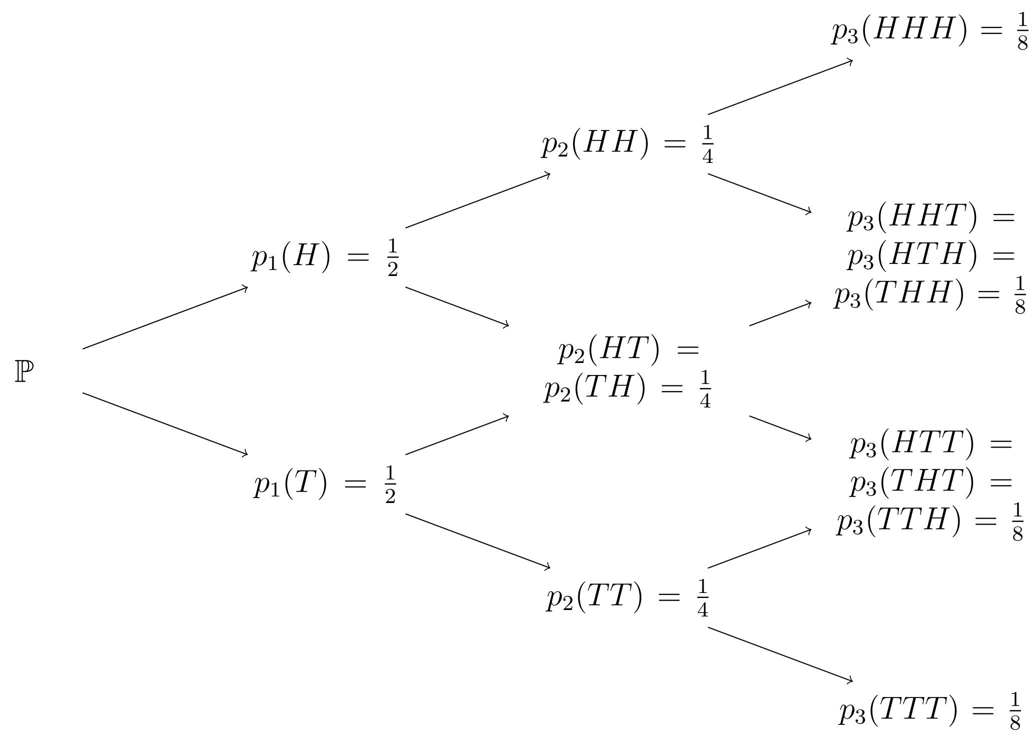

Figure 3: Real world probabilities $\tilde{\mathbb{P}}$

Then, from the definition of the Radon-Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ is

$$
\begin{aligned}
Z(HHH) &= \frac{\tilde{\mathbb{P}}(HHH)}{\mathbb{P}(HHH)} = \frac{\frac{1}{8}}{\frac{8}{27}} = \frac{27}{64} \\
Z(HHT) &= Z(HTH) = Z(THH) = \frac{\frac{1}{8}}{\frac{4}{27}} = \frac{27}{32} \\
Z(HTT) &= Z(THT) = Z(THH) = \frac{\frac{1}{8}}{\frac{2}{27}} = \frac{27}{16} \\
Z(TTT) &= \frac{\frac{1}{8}}{\frac{1}{27}} = \frac{27}{8}
\end{aligned}
$$

Fact: The Radon-Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ can be calculated as[1]

$$
Z(\omega_1 \cdots \omega_N) = \frac{\tilde{\mathbb{P}}(\omega_1 \cdots \omega_N)}{\mathbb{P}(\omega_1 \cdots \omega_N)} = \left(\frac{\tilde{p}}{p}\right)^{\#H(\omega_1 \cdots \omega_N)} \left(\frac{\tilde{q}}{q}\right)^{\#T(\omega_1 \cdots \omega_N)}
$$

[1] I think this is only true in our binomial model. It we weren't in a coin toss space then this wouldn't necessarily hold.

Now, suppose that V_3 is the payoff of a derivative security (in particular the lookback option example used in the text). By the risk-neutral pricing formula we have

$$
\begin{aligned}
V_0 &= \tilde{\mathbb{E}}\left[\frac{V_3}{(1+r)^3}\right] \\
&= \frac{1}{(1+r)^3}\sum_{\omega\in\Omega} V_3(\omega)\tilde{\mathbb{P}}(\omega) \\
&= \frac{1}{(1+r)^3}\sum_{\omega\in\Omega} V_3(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(\omega)}\tilde{\mathbb{P}}(\omega) \\
&= \frac{1}{(1+r)^3}\sum_{\omega\in\Omega} V_3(\omega)Z(\omega)\mathbb{P}(\omega) \\
&= \mathbb{E}\left[Z\cdot\frac{V_3}{(1+r)^3}\right]
\end{aligned}
$$

which is essentially an application of the Theorem $\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$ above. This formula weights the payoff in each state of the world using the Radon-Nikodým derivative/random variable Z and calculates the expectation using the corresponding real-world probabilities under $\mathbb{P}$.

Definition 2. In the N-period binomial model with real-world probability measure $\mathbb{P}$ and risk-neutral probability measure $\tilde{\mathbb{P}}$ suppose that $\tilde{\mathbb{P}}(\omega) > 0$ and $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$. The <u>state-price density</u> is the random variable

$$\zeta(\omega) = \frac{Z(\omega)}{(1+r)^N}$$

where Z is the Radon-Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$. The random variable

$$\zeta(\omega)\mathbb{P}(\omega)$$

is called the <u>state-price corresponding to ω</u>.[2]

Definition 3. Let $\overline{\omega} = \overline{\omega_1}\cdots\overline{\omega_N}$ be a particular coin toss sequence in the N-period binomial asset pricing model. The derivative security with payoff

$$V_N(\omega) = \begin{cases} 1 & \text{if } \omega = \overline{\omega} \\ 0 & \text{else} \end{cases}$$

is called the <u>Arrow-Debreu security</u> associated with state $\overline{\omega}$.

[2] The state price density $\zeta(\omega)$ includes the discount factor $(1+r)^{-N}$ and a measure of risk $Z(\overline{\omega})$.

By the risk-neutral pricing formula

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\frac{V_N}{(1+r)^N}\right] &= \tilde{\mathbb{E}}\left[\frac{\mathbb{1}_{\{\overline{\omega}\}}}{(1+r)^N}\right] \\
&= \frac{\tilde{\mathbb{P}}(\overline{\omega})}{(1+r)^N} \\
&= \frac{Z(\overline{\omega})\mathbb{P}(\overline{\omega})}{(1+r)^N} \\
&= \zeta(\overline{\omega})\mathbb{P}(\overline{\omega})
\end{aligned}$$

A derivative security with general payoff V_N at time N can be regarded as a portfolio of Arrow-Debreu securities since

$$\begin{aligned}
V_0 &= \tilde{\mathbb{E}}\left[\frac{V_N}{(1+r)^N}\right] \quad \text{(risk-neutral pricing)} \\
&= \mathbb{E}\left[Z \cdot \frac{V_N}{(1+r)^N}\right] \quad \text{(theorem: } \tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]\text{)} \\
&= \mathbb{E}[\zeta V_N] \quad \text{(definition of } \zeta\text{)} \\
&= \sum_{\omega\in\Omega} V_N(\omega)\zeta(\omega)\mathbb{P}(\omega) \quad \text{(definition of expectation)}
\end{aligned}$$

where $V_N(\omega)$ corresponds to the portfolio weight on a particular state ω and $\zeta(\omega)\mathbb{P}(\omega)$ corresponds to the state-price for an Arrow-Debreu security given ω.

1.1 Radon-Nikodým Process

We now consider the time n conditional expectation under $\mathbb{P}$ of the Radon-Nikodý derivative Z. This permits us to investigate prices of derivative securities at intermediate time steps in terms of the real-world measure.

Theorem 2. Let Z be a random variable on a finite probability space $(\Omega, \mathbb{P})$. Define the random variable

$$Z_n = \mathbb{E}_n[Z] \quad n = 0, 1, ..., N$$

Then the process $\{Z_n\}_{n=0}^N$ is a $\mathbb{P}$-martingale.

Proof.

(i) Adapted? By definition of conditional expectation we have that $Z_n = \mathbb{E}_n[Z]$ depends on the first n coin tosses. Hence, the process $\{Z_n\}$ is an adapted stochastic process.

(ii) Martingale property? We must show that for all n, $0 \le n \le N$, that $\mathbb{E}_n[Z_{n+1}] = Z_n$. Note

$$\begin{aligned}
\mathbb{E}_n[Z_{n+1}] &= \mathbb{E}_n[E_{n+1}[Z]] \quad \text{(by definition)} \\
&= \mathbb{E}_n[Z] \quad \text{(tower property)} \\
&= Z_n \quad \text{(by definition)}
\end{aligned}$$

$\square$

In general we may make successive estimates of a random variable by conditioning further in time. We may expect that doing so would provide more information, but this theorem tells us that, on average, that the estimates of the future state remain the same. This fits with out intuition of the definition of a martingale.

Example: Consider the binomial model with $N = 3, u = 2, d = \frac{1}{2}, r = \frac{1}{4}, p = \frac{2}{3}$, and $q = \frac{1}{3}$. We had used this example to calculate values of Z conditioned at time zero (ordinary expectation). We may use the theorem above to compute intermediate values for Z. For example, to calculate Z_2

$$
\begin{aligned}
Z_n(\omega) &= \mathbb{E}_n[Z](\omega) \\
\implies Z_2(HH) &= \mathbb{E}_2[Z](HH) \\
&= pZ(HHH) + qZ(HHT) \\
&= \frac{2}{3} \cdot \frac{27}{64} + \frac{1}{3} \cdot \frac{27}{32} \\
&= \frac{9}{16}
\end{aligned}
$$

Note that in the N-step binomial model $Z_N = \mathbb{E}_N[Z] = Z$ as Z only depends on the first N coin tosses. We could use the definition to calculate $Z_1 = \mathbb{E}_1[Z]$ but it is simpler to use the one-step martingale property of Z

$$Z_1 = \mathbb{E}_1[Z_2]$$

to go recursively backwards in time. Similarly, $Z_0 = \mathbb{E}_0[Z_1] \equiv \mathbb{E}[Z_1]$ but from part (ii) of the theorem stating $\mathbb{E}[Z] = 1$ we find that this computation is not necessary.

Definition 4. Let $\tilde{\mathbb{P}}(\omega)$ and $\mathbb{P}(\omega)$ be probability measures on a finite sample space Ω. Assume that $\mathbb{P}(\omega) > 0$ and $\tilde{\mathbb{P}}(\omega) > 0$ for all $\omega \in \Omega$. For the Radon-Nikodým derivative $Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$, the associated Radon-Nikodým process is

$$Z_n = \mathbb{E}_n[Z] \quad n = 0, 1, ..., N$$

In particular, $Z_N = Z$ and $Z_0 = 1$.

Lemma 1. "*This lemma allows us to connect concepts from our previous work*" Assume the conditions of the above definition. Let $n \in \{0, ..., N\}$ and let Y be a random variable depending only on the first n coin tosses. Then

$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[Z_n Y]$$

Proof.

$$
\begin{aligned}
\tilde{\mathbb{E}}[Y] &= \mathbb{E}[ZY] \quad \text{(by an earlier result)} \\
&= \mathbb{E}[\mathbb{E}_n[ZY]] \quad \text{(tower property)} \\
&= \mathbb{E}[Y\mathbb{E}_n[Z]] \quad \text{(adaptedness of } Y) \\
&= \mathbb{E}[YZ_n] \quad \text{(by definition: } Z_n = \mathbb{E}_n[Z])
\end{aligned}
$$

$\square$

Now, consider a time n fixed coin toss sequence $\overline{\omega}_1 \cdots \overline{\omega}_n$ and define the random variable

$$Y(\omega_1 \cdots \omega_n \omega_{n+1} \cdots \omega_N) = \begin{cases} 1 & \text{if } \omega_1 \cdots \omega_n = \overline{\omega}_1 \cdots \overline{\omega}_n \\ 0 & \text{else} \end{cases}$$

We can think of this as an Arrow-Debreu security at an intermediate time step n such that the future coin tosses $\omega_{n+1} \cdots \omega_N$ are irrelevant to the payoff. Then

$$\begin{aligned} \tilde{\mathbb{E}}[Y] &= \tilde{\mathbb{P}}(\omega_1 \cdots \omega_n = \overline{\omega}_1 \cdots \overline{\omega}_n) \\ &= \tilde{p}^{\#H(\overline{\omega}_1 \cdots \overline{\omega}_n)} \tilde{q}^{\#T(\overline{\omega}_1 \cdots \overline{\omega}_n)} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[YZ_n] &= Z_n(\overline{\omega}_1 \cdots \overline{\omega}_n) \mathbb{P}(\omega_1 \cdots \omega_n = \overline{\omega}_1 \cdots \overline{\omega}_n) \\ &= Z_n(\overline{\omega}_1 \cdots \overline{\omega}_n) \tilde{p}^{\#H(\overline{\omega}_1 \cdots \overline{\omega}_n)} \tilde{q}^{\#T(\overline{\omega}_1 \cdots \overline{\omega}_n)} \end{aligned}$$

and since we had that $\mathbb{E}[Z_n Y] = \tilde{\mathbb{E}}[Y]$ from our above lemma, we may state

$$\begin{aligned} Z_n(\omega_1 \cdots \omega_n) \tilde{p}^{\#H(\omega_1 \cdots \omega_n)} \tilde{q}^{\#T(\omega_1 \cdots \omega_n)} &= \tilde{p}^{\#H(\omega_1 \cdots \omega_n)} \tilde{q}^{\#T(\omega_1 \cdots \omega_n)} \\ \implies Z_n(\omega_1 \cdots \omega_n) &= \left(\frac{\tilde{p}}{p}\right)^{\#H(\omega_1 \cdots \omega_n)} \left(\frac{\tilde{q}}{q}\right)^{\#T(\omega_1 \cdots \omega_n)} \end{aligned}$$

which permits us to compute intermediate time steps for Z_n fairly easily.

Theorem 3. General Bayes Theorem Let $0 \leq n \leq m \leq N$ and let Y be a random variable depending on only the first n coin tosses. Then

$$\begin{aligned} \tilde{\mathbb{E}}_n[Y] &= \frac{\mathbb{E}_n[Z_m Y]}{\mathbb{E}_n[Z_m]} \\ &= \frac{\mathbb{E}_n[Z_m Y]}{Z_n} \end{aligned}$$

Proof.

$$\begin{aligned} \mathbb{E}_n[Y](\omega_1 \cdots \omega_n) &= \sum_{\omega_{n+1} \cdots \omega_m} Y(\omega_1 \cdots \omega_n \omega_{n+1} \cdots \omega_m) \tilde{p}^{\#H(\omega_{n+1} \cdots \omega_m)} \tilde{q}^{\#T(\omega_{n+1} \cdots \omega_m)} \\ &= \left[\frac{p}{\tilde{p}}\right]^{\#H(\omega_1 \cdots \omega_n)} \left[\frac{q}{\tilde{q}}\right]^{\#T(\omega_1 \cdots \omega_n)} \sum_{\omega_{n+1} \cdots \omega_m} Y(\omega_1 \cdots \omega_m) \cdot \\ &\qquad \left[\frac{\tilde{p}}{p}\right]^{\#H(\omega_1 \cdots \omega_m)} \left[\frac{\tilde{q}}{q}\right]^{\#T(\omega_1 \cdots \omega_m)} \cdot \tilde{p}^{\#H(\omega_{n+1} \cdots \omega_m)} \tilde{q}^{\#T(\omega_{n+1} \cdots \omega_m)} \end{aligned}$$

but

$$Z_n(\omega_1 \cdots \omega_n) = \left[\frac{\tilde{p}}{p}\right]^{\#H(\omega_1 \cdots \omega_n)} \left[\frac{\tilde{q}}{q}\right]^{\#T(\omega_1 \cdots \omega_n)}$$

so

$$\begin{aligned}
\mathbb{E}_n[Y](\omega_1 \cdots \omega_n) &= \frac{1}{Z_n} \sum_{\omega_{n+1} \cdots \omega_m} Y(\omega_1 \cdots \omega_m) Z_m \tilde{p}^{\#H(\omega_{n+1} \cdots \omega_m)} \tilde{q}^{\#T(\omega_{n+1} \cdots \omega_m)} \\
&= \frac{1}{Z_n} \mathbb{E}_n[Z_m Y]
\end{aligned}$$

and by the tower property we can show that, with $n \leq m$,

$$Z_n = \mathbb{E}_n[Z_m] = \mathbb{E}_n[\mathbb{E}_m[Z]] = \mathbb{E}_n[Z]$$

hence

$$\tilde{\mathbb{E}}_n[Y] = \frac{\mathbb{E}_n[Z_m Y]}{Z_n} = \frac{\mathbb{E}_n[Z_m Y]}{\mathbb{E}_n[Z_m]}$$

as desired. □

The general Bayes Theorem allows us to give a version of the risk-neutral pricing formula based on state-prices and the real-world probability measure $\mathbb{P}$.

Definition 5. In the N-period binomial model with real-world probability measure $\mathbb{P}$ and risk-neutral measure $\tilde{\mathbb{P}}$ suppose that $\tilde{\mathbb{P}}(\omega) > 0$ and $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$.

The state-price density process $\{\zeta_n\}_{n=0}^N$ is given by

$$\zeta_n = \frac{Z_n}{(1+r)^n}$$

where Z_n is the Radon-Nikodým process of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$.

Theorem 4. Consider the N-period binomial model with $0 < d < 1 + r < u$, the real-world probability measure $\mathbb{P}$, and risk-neutral probability measure $\tilde{\mathbb{P}}$. Suppose that $\tilde{\mathbb{P}}(\omega) > 0$ and $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$.

Let $\{Z_n\}_{n=0}^N$ be the Radon-Nikodým process and $\{\zeta\}_{n=0}^N$ be the state-price density process, and let V_N be the payoff at time N of a derivative security which may depend on all N coin tosses. The price at time n is

$$V_n = \frac{1}{\zeta_n} \mathbb{E}_n[\zeta_N V_N]$$

Proof. By the risk-neutral pricing formula

$$\begin{aligned}
V_n &= \tilde{\mathbb{E}}_n \left[\frac{V_N}{(1+r)^{N-n}} \right] \\
&= \frac{\mathbb{E}_n \left[Z_N \cdot \frac{V_N}{(1+r)^{N-n}} \right]}{Z_n} \quad \text{(by the General Bayes Theorem with } m = N) \\
&= \frac{(1+r)^n}{Z_n} \mathbb{E}_n \left[\frac{Z_N V_N}{(1+r)^N} \right] \quad \text{(taking out what is known at time } n) \\
&= \frac{1}{\zeta_n} \mathbb{E}_n[\zeta_N V_N]
\end{aligned}$$

□