Mathematical & Computational Finance I
Lecture Notes

Binomial No-Arbitrage Pricing Model (continued)

January 14 2016
Last update: December 4, 2017

1 One-Period Binomial Model

Note that $\tilde{p}$ is not the physical/real world probability governing the evolution of the risky asset process S. There should be a real world probability of heads p such that[1]

$$(1+r)S_0 < pS_1(H) + (1-p)S_1(T)$$

with the expected return in the real world satisfying

$$\mathbb{E}\left[\frac{S_1 - S_0}{S_0}\right] > r$$

where $\mathbb{E}[\cdot]$ denotes expectation under the real world probability. In the real world the investor is compensated for the risk of holding S rather than investing \$$S_0$ in the bank account.

2 Multiperiod Binomial Model

We may generalize the one-period binomial model to accept multiple time periods, where S_i^n denotes the asset price at the i^{th} tier after n heads. In general,

$$S_i^n = d^{i-n}u^n S_0 \quad 0 \leq n \leq i$$

[1]That is, the expected return for S_1 exceeds the risk free rate.

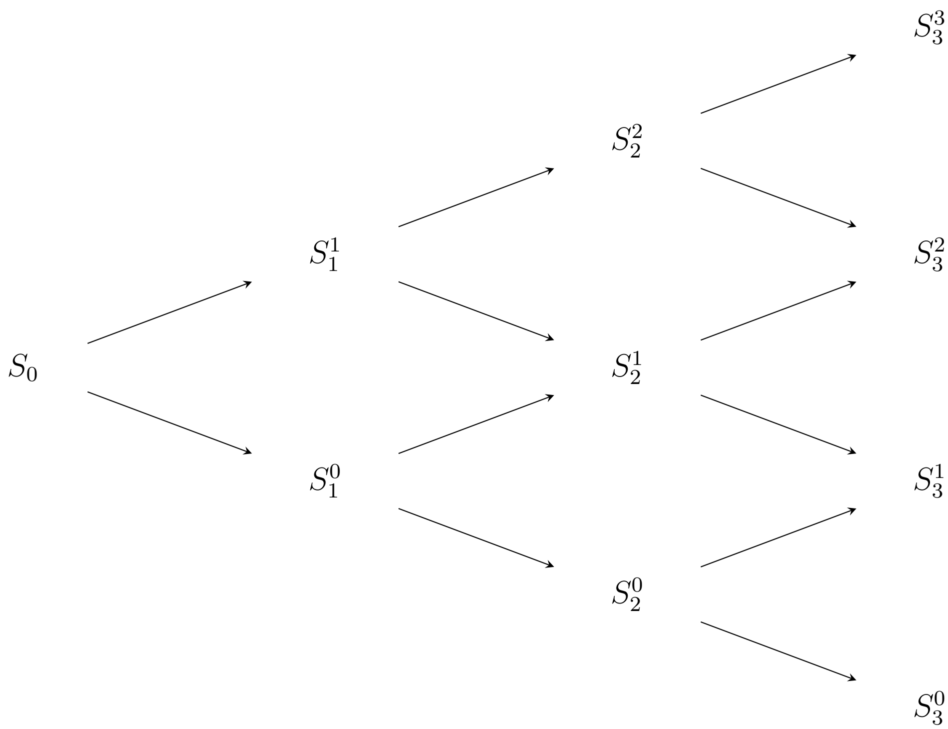

Figure 1: Recombining tree for $M = 3$ steps.

We suppose that the evolution of the risky asset is governed by sequences of coin tosses. The same space Ω for the multiperiod binomial model contains all coin toss sequences (e.g. $HTTHTTHHTT\cdots$). Let $\omega_i \in \{H, T\}$ be the outcome of the i^{th} coin toss so that we may write a sequence $\omega \in \Omega$ as $\omega = \omega_1\omega_2\cdots$. Then, write the value of the stock price at time n depending on the first n coin tosses as $S_n(\omega_1\cdots\omega_n)$. The value of S_n depends only on the number of heads and tails in the first n coin tosses and not the particular order, e.g.

$$S_3(TTH) = S_3(THT) = S_3(HTT) = S_3^1$$

We can consider the prices of derivative securities by constructing a replicating portfolio as in the one time period case. Consider the derivative security with time $t = 2$ payoff

$$V_2 = (S_2 - K)^+$$

Suppose some agent sells the option at time zero for V_0 and constructs a portfolio consisting of

Δ_0 shares of the underlying stock

$(V_0 - \Delta_0 S_0)$ invested in the riskless bank account

At time $t = 1$ the value of the portfolio is

$$X_1 = \Delta_0 S_1 + (1 + r)(V_0 - \Delta_0 S_0)$$

The value of the portfolio at time $t = 1$ depends on the outcome of the first coin toss ω_1

$$X_1(H) = \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0)$$
$$X_1(T) = \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0)$$

At time $t = 1$ we rebalance the portfolio to Δ_1 shares invested in the stock noting that Δ_1 may depend on the outcome of the first coin toss, with the remaining wealth, $X_1 - \Delta_1 S_1$, in the bank account.

We should also note that we only permit trading at these discrete time nodes in our multiperiod tree. At time $t = 2$ the value of the investor's wealth/portfolio should be

$$X_2 = \Delta_1 S_2 + (1+r)(X_1 - \Delta_1 S_1)$$

We wish to have $X_2 = V_2$ in all possible states (regardless of the outcome of the two coin tosses)

$$\begin{aligned}
V_2(HH) &= \Delta_1(H) S_2(HH) + (1+r)[X_1(H) - \Delta_1 S_1(H)] \\
V_2(HT) &= \Delta_1(H) S_2(HT) + (1+r)[X_1(H) - \Delta_1 S_1(H)] \\
V_2(TH) &= \Delta_1(T) S_2(TH) + (1+r)[X_1(T) - \Delta_1 S_1(T)] \\
V_2(TT) &= \Delta_1(T) S_2(TT) + (1+r)[X_1(T) - \Delta_1 S_1(T)]
\end{aligned}$$

Including the two equations for $X_1(H)$ and $X_1(T)$ we find that we have six equations for the time-one and time-two wealth in six unknowns $(V_0, \Delta_0, \Delta_1(H), \Delta_1(T), X_1(H), X_1(T))$. Subtracting $V_2(TH) - V_2(TT)$ gives

$$\begin{aligned}
V_2(TH) - V_2(TT) &= \Delta_1(T) S_2(TH) - \Delta_1(T) S_1(TT) \\
&= \Delta_1(T) \left[S_2(TH) - S_1(TT) \right] \\
\implies \Delta_1(T) &= \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_1(TT)}
\end{aligned}$$

is the Δ-hedge ratio at time $t = 1$ for the note T. Substituting our value for $\Delta_1(T)$ into either equation for $V_2(TH)$ or $V_2(TT)$ yields

$$X_1(T) = \frac{1}{1+r}[\tilde{p} V_2(TH) + \tilde{q} V_2(TT)]$$

where, as before,

$$\tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = 1 - \tilde{p}$$

At time $t - 1$ if the first coin toss was a T then the price of the option is

$$V_1(T) = \frac{1}{1+r}[\tilde{p} V_2(TH) + \tilde{q} V_2(TT)] \quad (\textbf{risk-neutral valuation formula})$$

otherwise there would be arbitrage (consider the one period sub-tree remaining). We can also verify that

$$S_1(T) = \frac{1}{1+r}[\tilde{p} S_2(TH) + \tilde{q} S_2(TT)] \quad (\textbf{martingale property})$$

Similarly, if we subtract $V_2(HH) - V_2(HT)$, we find

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)}$$

Likewise, by substituting $\Delta_1(H)$ into $V_2(HH)$ yields us that $X_1(H) = V_1(H)$ is the price of the option at time $t = 1$ if the first coin toss is a H satisfying

$$V_1(H) = \frac{1}{1+r}[\tilde{p}V_2(HH) + \tilde{q}V_2(HT)]$$

Once again, we require that at time step $t = 1$ we require $X_1(\omega_1) = V_1(\omega_1)$ to avoid arbitrage. Substituting $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$ into our previous formulas for $X_1(\omega)$ yields

$$\begin{aligned} V_1(H) &= \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) \\ V_1(T) &= \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0) \end{aligned}$$

which we find to be identical to the one-period case, hence

$$\begin{aligned} V_1(H) - V_1(T) &= \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0) - \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0) \\ &= \Delta_0(S_1(H) - S_1(T)) \\ \implies \Delta_0 &= \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} \end{aligned}$$

This method defines a recursive procedure for finding V_0 by proceeding backwards in time through the nodes of the binomial tree. Essentially, we solve a series of one-period examples backwards through the nodes. The stochastic processes

$$\{\Delta_0, \Delta_1\}, \quad \{X_0, X_1, X_2\}, \quad \{V_0, V_1, V_2\}$$

define the replication problem. These processes are clearly composed of random variables since they depend on the outcome of the coin tosses. If we begin with initial wealth $X_0 = V_0$ and specify $\Delta_0, \Delta_1(\omega)$ we can compute the value of the portfolio

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n) \quad (\textbf{wealth equation})$$

that replicates the derivative. The value of the derivative at time zero must be the value of the replicating portfolio X_0, otherwise we would find arbitrage.