Mathematical & Computational Finance I
Lecture Notes

Backwards Induction & Probability Theory on Coin Toss Space

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1 Exercise 1.8: Asian Call Option

to do...

2 Backwards Induction

Definition 1. The general recursive procedure for finding the price of a European derivative security at time zero, denoted V_0, is called <u>backwards induction</u>.

For an N-period binomial model with $0 < d < 1 + r < u$ let V_N be a random variable (the payoff) depending on the first N coin tosses $\omega_1 \cdots \omega_N$. Define recursively, backwards in time, the sequence of random variables $V_{N-1}, V_{N-2}, ..., V_0$ by

$$V_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)]$$

where

$$\tilde{p} = \frac{1+r-d}{u-d} \quad \tilde{q} = 1 - \tilde{p}$$

Theorem 1. Replication. Define

$$\Delta_n(\omega_1 \cdots \omega_n) = \frac{V_{n+1}(\omega_1 \cdots \omega_n H) - V_{n+1}(\omega_1 \cdots \omega_n T)}{S_{n+1}(\omega_1 \cdots \omega_n H) - S_{n+1}(\omega_1 \cdots \omega_n T)}$$

if we set $X_0 = V_0$ and define recursively, forwards in time, the values $X_1, ..., X_N$ by

$$X_N(\omega_1 \cdots \omega_N) = V_N(\omega_1 \cdots \omega_N) \quad \forall\ \omega_1 \cdots \omega_N \in \Omega$$

Proof. We will proceed by induction on n. At time 0 we have that $P(0) : X_0 = V_0$ is trivially true by construction. Therefore $P(0)$ holds.

Assume now that our inductive hypothesis $P(n)$ holds

$$P(n) : X_n(\omega_1 \cdots \omega_n) = V_n(\omega_1 \cdots \omega_n) \; \forall \omega_1 \cdots \omega_n \in \Omega$$

Now, let $\omega_1 \cdots \omega_n$ be an arbitrary and fixed sequence of coin tosses so that our inductive hypothesis $P(n)$ holds. For the next coin toss ω_{n+1} we wish to show that $P(n+1)$ holds, that is

$$P(n+1) : X_{n+1}(\omega_1 \cdots \omega_n \omega_{n+1} \stackrel{?}{=} V_{n+1}(\omega_1 \cdots \omega_n \omega_{n+1})$$

but note that from the wealth equation we have[1]

$$X_{n+1}(\omega_1 \cdots \omega_n \omega_{n+1}) = \Delta_n S_{n+1} + (1+r)[X_n - \Delta_n S_n]$$

Now, consider the case of the $n+1^{\text{th}}$ coin toss to be H, then

$$\begin{aligned}
X_{n+1}(H) &= \Delta_n S_{n+1}(H) + (1+r)[X_n - \Delta_n S_n] \\
X_{n+1}(H) &= \Delta_n u S_n + (1+r)[X_n - \Delta_n S_n] \\
&= \Delta_n u S_n + (1+r)X_n - (1+r)\Delta_n S_n \\
&= (1+r)X_n + \Delta_n S_n[u - (1+r)] \\
&= (1+r)V_n + \Delta_n S_n[u - (1+r)] \quad \text{(by the inductive hypothesis)}
\end{aligned}$$

but by definition

$$\begin{aligned}
\Delta_n &= \frac{V_{n+1}(\omega_1 \cdots \omega_n H) - V_{n+1}(\omega_1 \cdots \omega_n T)}{S_{n+1}(\omega_1 \cdots \omega_n H) - S_{n+1}(\omega_1 \cdots \omega_n T)} \\
&= \frac{V_{n+1}(\omega_1 \cdots \omega_n H) - V_{n+1}(\omega_1 \cdots \omega_n T)}{uS_n(\omega_1 \cdots \omega_n) - dS_n(\omega_1 \cdots \omega_n)} \\
&= \frac{V_{n+1}(\omega_1 \cdots \omega_n H) - V_{n+1}(\omega_1 \cdots \omega_n T)}{(u-d)S_n(\omega_1 \cdots \omega_n)}
\end{aligned}$$

So,

$$\begin{aligned}
X_{n+1}(H) &= (1+r)V_n + \Delta_n S_n[u - (1+r)] \\
&= (1+r)V_n + \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} S_n[u - (1+r)] \\
&= (1+r)V_n + [V_{n+1}(H) - V_{n+1}(T)] \left(\frac{u - (1+r)}{u-d} \right)
\end{aligned}$$

but

$$\tilde{q} = \frac{u - (1+r)}{u-d}$$

[1]For brevity we have dropped the functional $(\omega_1 \cdots \omega_n)$.

hence

$$
\begin{aligned}
X_{n+1}(H) &= (1+r)V_n + [V_{n+1}(H) - V_{n+1}(T)]\left(\frac{u-(1+r)}{u-d}\right) \\
&= (1+r)V_n + [V_{n+1}(H) - V_{n+1}(T)]\tilde{q}
\end{aligned}
$$

but

$$(1+r)V_n = \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)$$

so

$$
\begin{aligned}
X_{n+1}(H) &= (1+r)V_n + [V_{n+1}(H) - V_{n+1}(T)]\tilde{q} \\
&= \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) + [V_{n+1}(H) - V_{n+1}(T)]\tilde{q} \\
&= \tilde{p}V_{n+1}(H) + V_{n+1}(H)\tilde{q} \\
&= (\tilde{p}+\tilde{q})V_{n+1}(H) \\
&= V_{n+1}(H)
\end{aligned}
$$

which completes the case of the $n+1^{\text{th}}$ coin toss as heads. A nearly identical procedure will confirm that

$$X_{n+1}(T) = V_{n+1}(T)$$

Thus

$$P(n+1) : X_{n+1}(\omega_1 \cdots \omega_n \omega_{n+1}) = V_{n+1}(\omega_1 \cdots \omega_n \omega_{n+1})$$

holds, as desired. Hence, by induction $P(n)$ holds for all $0 \leq n \leq N$, for all sequences $\omega_1 \cdots \omega_n$. □

The above replication theorem gives us that, by no-arbitrage, that the price of the European derivative security at time n must be $V_n(\omega_1 \cdots \omega_n)$.

3 Probability Theory on Coin Toss Space

Definition 2. A finite probability space consists of

1. A finite set Ω called the sample space and
2. A function $\mathbb{P} : \Omega \to [0,1]$ called a probability measure such that

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$$

Definition 3. An event is a subset Ω. We define the probability of an event $A \subset \Omega$ to be

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega)$$

The above definitions lead to all the usual properties of probability that we should be familiar from introductory courses (STAT 249, 250, 349, ...), e.g.

1. $\mathbb{P}(\Omega) = 1$
2. If $A, B \subseteq \Omega$ such that $A \cap B = \emptyset$ then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

3. etc...

Definition 4. Let $(\Omega, \mathbb{P})$ be a finite probability space. A random variable X is a function

$$\begin{aligned} X: \ \Omega &\to \mathbb{R} \\ \omega &\to X(\omega) \end{aligned}$$

Sometimes it is useful to include $\pm\infty$ as values that our random variables may take, so, using the extended real numbers

$$X : \Omega \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$$

Definition 5. The distribution of a random variable is a specification of the probabilities that a random variable takes certain values

$$F(y) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \le y\}) \quad \forall\, y \in \mathbb{R}$$

or

$$f(y) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = y\}) \quad \forall\, y \in \text{Range}(X)$$

Definition 6. Let X be a random variable defined on the finite probability space $(\Omega, \mathbb{P})$. The expectation of X is

$$\mathbb{E}_{\mathbb{P}}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega)$$

and the variance of X is

$$\text{Var}[X] = \mathbb{E}_{\mathbb{P}}[(X - \mathbb{E}_{\mathbb{P}}[X])^2]$$

The usual properties of expectation and variance hold:

1. Linearity: For $a, b \in \mathbb{R}$ and random variables X, Y

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$$

2.

$$\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}^2[X]$$

Definition 7. A function $\phi : \mathbb{R} \to \mathbb{R}$ is convex if, for all $t \in (0, 1)$ and all $x, y \in \mathbb{R}$, that is, for all linear combinations of X and Y

$$\phi(tx + (1 - t)y) \le t\phi(x) + (1 - t)\phi(y)$$

Theorem 2. Jensen's Inequality. Let X be a random variable on a finite probability space $(\Omega, \mathbb{P})$ and let ϕ be convex. Then

$$\phi(\mathbb{E}[X]) \leq \mathbb{E}[\phi(X)]$$

Proof. Let $x \in \mathbb{R}$ be arbitrary. Then, there exists some supporting line l through $(x, \phi(x))$ such that the graph of ϕ is above l.

Therefore, $\phi(y) \geq \phi(x) + \lambda(y - x)$ where $\lambda = \text{slope}(l)$.

Let $x = \mathbb{E}[x]$, then for all y

$$\phi(y) \geq \phi(\mathbb{E}[X]) + \lambda(y - \mathbb{E}[X])$$

Note that λ depends on $x = \mathbb{E}[X]$ but not y. Let $y = X$ to find λ, so

$$\phi(X) \geq \phi(\mathbb{E}[X]) + \lambda(X - \mathbb{E}[X])$$

Take the expectation of both sides to yield

$$\begin{aligned}
\mathbb{E}[\phi(X)] &\geq \mathbb{E}[\phi(\mathbb{E}[X]) + \lambda(X - \mathbb{E}[X])] \\
&\geq \mathbb{E}[\phi(\mathbb{E}[X])] + \lambda(\mathbb{E}[X] - E[E[X]]) \\
\implies \mathbb{E}[\phi(X)] &\geq \phi(E[X])
\end{aligned}$$

as desired. $\square$