Mathematical & Computational Finance I
Lecture Notes

Interest-Rate-Dependent Assets

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1 Binomial Models for Interest Rates (con't, again)

Last time we spoke about fixed income derivatives. We ended with discussing swaps and the no-arbitrage price of swaps and the corresponding swap rate: The rate K making the swap price equal 0 at contract initiation. We were also able to decompose a swap into caps and floors. That is, we found that

$$Swap_m + Cap_m = Floor_m$$

in addition to introducing the notion of options on the future spot rate.

We consider now some examples:

Example: *(Example 6.3.9)* Assume the probabilities computed from the previous example as well as the corresponding interest rate process

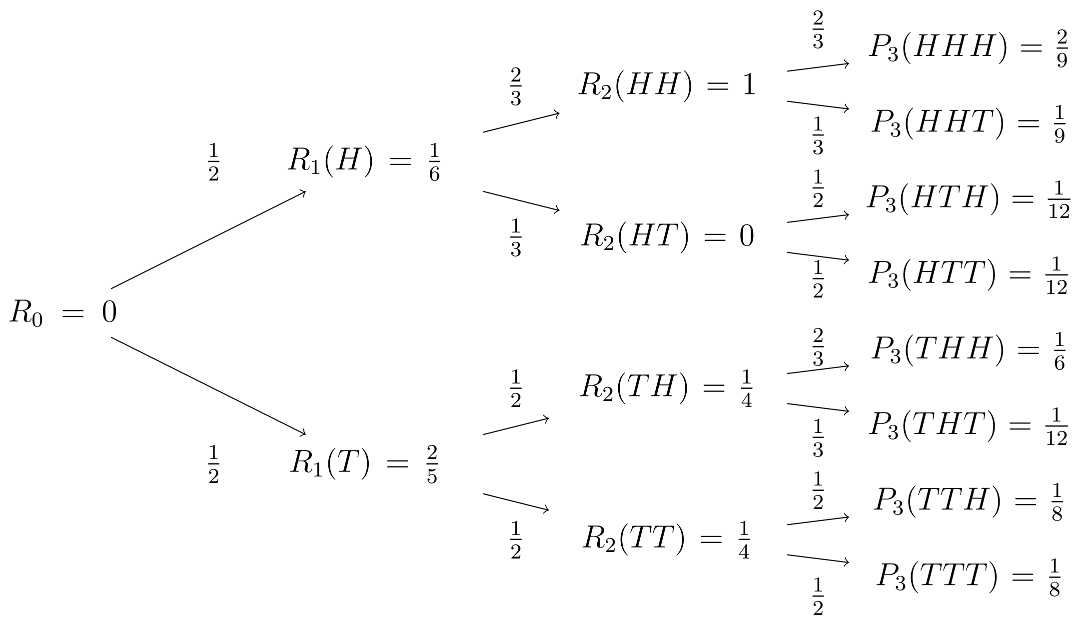

Figure 1: Interest rate process with conditional transition probabilities annotated above each branch.

Note that with

$$D_n = \frac{1}{\prod_{i=0}^{n-1}(1+R_i)}$$

we find

$$\begin{aligned} D_1 &= \frac{1}{(1+R_0)} \\ D_2 &= \frac{1}{(1+R_0)(1+R_1)} \\ D_3 &= \frac{1}{(1+R_0)(1+R_1)(1+R_2)} \end{aligned}$$

For later use we compute the following quantities depending on the outcome of the coin toss sequence $\omega_1\omega_2 \in \Omega$. Using the tree above we find

$\omega_1\omega_2$	$\frac{1}{(1+R_0)}$	$\frac{1}{(1+R_2)}$	$\frac{1}{(1+R_2)}$	D_1	D_2	D_3	$\tilde{\mathbb{P}}(\omega_1\omega_2)$
HH	1	$\frac{6}{7}$	$\frac{1}{2}$	1	$\frac{6}{7}$	$\frac{3}{7}$	$\frac{1}{3}$
HT	1	$\frac{6}{7}$	1	1	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{1}{6}$
TH	1	$\frac{5}{7}$	$\frac{4}{5}$	1	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{1}{4}$
TT	1	$\frac{5}{7}$	$\frac{4}{5}$	1	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{1}{4}$

We calculate our time-zero zero-coupon bond prices using

$$
\begin{aligned}
B_{0,n} &= \tilde{\mathbb{E}}_0\left[D_n\right] \\
B_{0,1} &= \tilde{\mathbb{E}}_0\left[D_1\right] \\
&= \tilde{\mathbb{E}}_0\left[\frac{1}{1+R_0}\right] = \frac{1}{1+R_0} = 1 \\
B_{0,2} &= \tilde{\mathbb{E}}_0\left[D_2\right] \\
&= \sum_{\omega_1} D_2(\omega_1)\tilde{\mathbb{P}}(\omega_1) = \frac{1}{2}\cdot\frac{1}{(1+0)(1+\frac{2}{5})} + \frac{1}{2}\cdot\frac{1}{(1+0)(1+\frac{1}{6})} = \frac{11}{14} \\
B_{0,3} &= \tilde{\mathbb{E}}_0\left[D_3\right] \\
&= \sum_{\omega_1\omega_2} D_2(\omega_1\omega_2)\tilde{\mathbb{P}}(\omega_1\omega_2) = \frac{1}{3}\cdot\frac{3}{7} + \frac{1}{6}\cdot\frac{6}{7} + \frac{1}{4}\cdot\frac{4}{7} + \frac{1}{4}\cdot\frac{4}{7} = \frac{4}{7}
\end{aligned}
$$

and the time-one zero-coupon bond prices are

$$
\begin{aligned}
B_{n,m} &= \tilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}\right] = \frac{1}{D_n}\tilde{\mathbb{E}}_n\left[D_m\right] \\
B_{1,1} &= \tilde{\mathbb{E}}_1\left[\frac{D_1}{D_1}\right] = 1 \\
B_{1,2}(H) &= \tilde{\mathbb{E}}_1\left[\frac{D_2}{D_1}\right](H) = \frac{1}{D_1(H)}\tilde{\mathbb{E}}_1\left[D_2\right](H) = \frac{1}{D_1(H)}D_2(H) \quad \text{(predictability)} \\
&= \frac{1}{\frac{1}{(1+R_0)}}\frac{1}{(1+R_0)(1+R_1(H))} = \frac{6}{7} \\
B_{1,2}(T) &= \frac{5}{7} \quad \text{(similar computation to } B_{1,2}(H)\text{)} \\
B_{1,3}(H) &= \frac{1}{D_2(H)}\sum_{\omega_2} D_3(H\omega_2)\tilde{\mathbb{P}}(H\omega_2) = \frac{4}{7} \\
B_{1,3}(T) &= \frac{1}{D_2(T)}\sum_{\omega_2} D_3(T\omega_2)\tilde{\mathbb{P}}(T\omega_2) = \frac{4}{7}
\end{aligned}
$$

the time-two zero-coupon bond prices

$$
\begin{aligned}
B_{2,2} &= \tilde{\mathbb{E}}_2\left[\frac{D_2}{D_2}\right] = 1 \\
B_{2,3}(HH) &= \frac{1}{D_2(HH)}\tilde{\mathbb{E}}_2\left[D_3\right](HH) = \frac{D_3(HH)}{D_2(HH)} = \frac{1}{2} \quad \text{(predictability)} \\
B_{2,3}(HT) &= 1 \\
B_{2,3}(TH) &= \frac{4}{5} \\
B_{2,3}(TT) &= \frac{4}{5}
\end{aligned}
$$

Consider now a three-period interest rate cap with cap $K = \frac{1}{3}$. The payoff from the caplets (which compose the cap) are

$\omega_1\omega_2$	R_0	$(R_0 - \frac{1}{3})^+$	R_1	$(R_1 - \frac{1}{3})^+$	R_2	$(R_2 - \frac{1}{3})^+$
HH	0	0	$\frac{1}{6}$	0	1	$\frac{2}{3}$
HT	0	0	$\frac{1}{6}$	0	0	0
TH	0	0	$\frac{2}{5}$	$\frac{1}{15}$	$\frac{1}{4}$	0
TT	0	0	$\frac{2}{5}$	$\frac{1}{15}$	$\frac{1}{4}$	0

Then, the time-zero price for the caplet, using the risk-neutral pricing formula, is given by

$$\begin{aligned}
\tilde{\mathbb{E}}_0\left[D_1\left(R_0 - \frac{1}{3}\right)^+\right] &= 0 \\
\tilde{\mathbb{E}}_0\left[D_2\left(R_1 - \frac{1}{3}\right)^+\right] &= \sum_{\omega_1} D_2(\omega_1)\left(R_1(\omega_1) - \frac{1}{3}\right)^+ \tilde{\mathbb{P}}(\omega_1) \\
&= \frac{6}{7}\left(\frac{1}{6} - \frac{1}{3}\right)^+ \frac{1}{2} + \frac{5}{7}\left(\frac{2}{5} - \frac{1}{3}\right)^+ \frac{1}{2} = \frac{1}{42} \\
\tilde{\mathbb{E}}_0\left[D_3\left(R_2 - \frac{1}{3}\right)^+\right] &= \sum_{\omega_1\omega_2} D_3(\omega_1\omega_2)\left(R_2(\omega_1\omega_2) - \frac{1}{3}\right)^+ \tilde{\mathbb{P}}(\omega_1\omega_2) \\
&= \frac{2}{21}
\end{aligned}$$

Therefore, applying our risk-neutral pricing formula for an m-period cap

$$\mathrm{Cap}_m = \tilde{\mathbb{E}}_0\left[\sum_{n=1}^{m} D_n\left(R_{n-1} - K\right)^+\right] \quad \text{(the cap is the sum of caplets)}$$

we have

$$\mathrm{Cap}_3 = 0 + \frac{1}{42} + \frac{2}{21} = \frac{5}{42}$$

as desired.

Example: *(Exercise 6.4)* Consider the same data as in Example 6.3.9 above. We wish to hedge a short position in the caplet paying $\left(R_2 - \frac{1}{3}\right)^+$ at time $n = 3$. From Example 6.3.9 we have that the time $n = 3$ payoff of the caplets are

$$\begin{aligned}
V_3(HH) &= \frac{2}{3} \\
V_3(HT) &= 0 \\
V_3(TH) &= 0 \\
V_3(TT) &= 0
\end{aligned}$$

Since $V_3(\omega_1\omega_2)$ depends on only the first two coin tosses, we find that the time-two price of the caplet can be determined by simply discounting the payoffs:

$$\begin{aligned}
V_2(HH) &= \frac{1}{1+R_2(HH)}V_3(HH) = \frac{1}{3} \\
V_2(HT) &= 0 \\
V_2(TH) &= 0 \\
V_2(TT) &= 0
\end{aligned}$$

To determine the time-one prices $V_1(H)$ and $V_1(T)$ we compute the risk-neutral expectation

$$\begin{aligned}
V_1(H) &= \tilde{\mathbb{E}}_1\left[\frac{V_2}{1+R_1}\right](H) \\
&= \frac{1}{1+\frac{1}{6}}\left[\frac{2}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot 0\right] = \frac{4}{21} \\
V_1(T) &= \tilde{\mathbb{E}}_1\left[\frac{V_2}{1+R_1}\right](T) = 0
\end{aligned}$$

We wish to now show that beginning with initial wealth $X_0 = \frac{2}{21}$ and by investing in strictly the money market & maturity-two bond we may achieve a portfolio value $X_1 = V_1$, regardless of the outcome of ω_1. Note that we have no need for time-three maturity bonds since this would, in principle, the ability to hedge over times two to three. However, the interest rate at time 2 is already known and given by R_2! That is, since our model ends at time 3 we have no interest rate risk over the final period.

To do so we buy Δ_0 units of the time-two maturity bond. Doing so gives us time-one portfolio value

$$X_1 = \Delta_0 B_{1,2} + (1+R_0)[X_0 - \Delta_0 B_{0,2}]$$

Since we require $X_1(\omega) = V_1(\omega)$ for all $\omega \in \Omega$

$$\begin{cases} \frac{4}{21} = V_1(H) = X_1(H) = \Delta_0 B_{1,2}(H) + (1+R_0)[X_0 - \Delta_0 B_{0,2}] \\ 0 = V_1(T) = X_1(T) = \Delta_0 B_{1,2}(T) + (1+R_0)[X_0 - \Delta_0 B_{0,2}] \end{cases}$$

$$\implies \begin{cases} \frac{4}{21} = \Delta_0\frac{6}{7} + \left[\frac{2}{21} - \Delta_0\frac{11}{14}\right] \\ 0 = \Delta_0\frac{5}{7} + \left[\frac{2}{21} - \Delta_0\frac{11}{14}\right] \end{cases}$$

$$\implies \begin{cases} \frac{4}{21} = \frac{\Delta_0}{14} + \frac{2}{21} \\ 0 = -\frac{\Delta_0}{14} + \frac{2}{21} \end{cases}$$

Solving this system we find

$$\Delta_0 = 14\cdot\frac{2}{21} = \frac{4}{3}$$

We can verify $\Delta_0 = \frac{4}{3}$ given by

$$\begin{aligned}\Delta_0 &= \frac{V_1(H) - V_1(T)}{B_{1,2}(H) - B_{1,2}(T)} \\ &= \frac{\frac{4}{21} - 0}{\frac{6}{7} - \frac{5}{7}} \\ &= \frac{\frac{4}{21}}{\frac{1}{7}} \\ &= \frac{4}{3}\end{aligned}$$

Applying the wealth equation again we can find the portfolio process $\Delta_1(\omega)$ such that $X_2 = V_2$ for all $\omega \in \Omega$ with initial wealth $X_1(H)$ and $X_1(T)$ using bonds $B_{2,3}$ and $B_{1,3}$ so that

$$V_2 = X_2 = \Delta_1 B_{2,3} + (1 + R_1)[X_1 - \Delta_1 B_{1,3}]$$

Now, we had that

$$V_2(HH) = \frac{1}{3}$$

so then

$$\begin{aligned}\frac{1}{3} = V_2(HH) = X_2(HH) &= \Delta_1(H)B_{2,3}(HH) + (1 + R_1(H))[X_1(H) - \Delta_1(H)B_{1,3}(H)] \\ &= \Delta_1(H)\frac{1}{2} + \frac{7}{6}\left[\frac{4}{21} - \Delta_1(H)\frac{4}{7}\right] \\ &= -\frac{1}{6}\Delta_1(H) + \frac{2}{9} \\ \implies \Delta_1(H) &= -\frac{2}{3}\end{aligned}$$

Similarly, we may verify that from setting the wealth equation $0 = V_2(HT) = X_2(HT)$ we find that $\Delta_1(H) = -\frac{2}{3}$, as is consistent with our model.

If $\omega_1 = T$ then we should consider $V_2(TH) = X_2(TH)$ and $V_2(TT) = X_2(TT)$. However, since we had found $X_2(TH) = 0 = X_2(TT)$ then we find that

$$\Delta_1(T) = 0$$

and so we are still fully hedged if $\omega_1 = T$ without trading additional units of the asset.

2 Forward Measures

Let V_m denote the time-m payoff of some interest rate dependent derivative security. From our risk-neutral pricing formula we have that the time-n price of such a derivative is

$$V_n = \frac{1}{D_n}\tilde{\mathbb{E}}_n\left[D_m V_m\right], \quad n = 0, 1, ..., m$$

However, in order for us to make use of this formula we require the joint condition distribution (conditional with respect to the information up to n) of D_m and V_m under $\tilde{\mathbb{P}}$. We make use of the **change-of-measure** technique, generalized from our earlier work, to simplify this risk-neutral formula. Recall we had defined our change-of-measure variable Z such that

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$$

so that we may transform a random variable Y from the real-world measure to the risk-neutral, satisfying

$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$$

We now introduce the following definition:

Definition 1. Let $1 \leq m \leq N$ be fixed and define

$$Z_{m,m} = \frac{D_m}{B_{0,m}}$$

and define the <u>m-forward measure</u> $\tilde{\mathbb{P}}^m$ by

$$\tilde{\mathbb{P}}^m(\omega) = Z_{m,m}(\omega)\tilde{\mathbb{P}}(\omega), \quad \forall\, \omega \in \Omega$$

To quickly convince ourselves that $\tilde{\mathbb{P}}^m$ is indeed a probability measure we note

$$\begin{aligned}
\tilde{\mathbb{P}}^m(\Omega) &= \sum_{\omega\in\Omega} \tilde{\mathbb{P}}^m(\omega) \\
&= \sum_{\omega\in\Omega} Z_{m,m}(\omega)\tilde{\mathbb{P}}(\omega) \\
&= \tilde{\mathbb{E}}_0\left[Z_{m,m}\right] \\
&= \tilde{\mathbb{E}}_0\left[\frac{D_m}{B_{0,m}}\right] \\
&= \frac{1}{B_{0,m}}\tilde{\mathbb{E}}_0[D_m] \quad \text{(adaptedness of } B_{0,m}\text{)} \\
&= 1
\end{aligned}$$

with all other properties just as straightforwardly confirmable. We wish to also generalize the *Radon-Nikodým* process to the discussion of stochastic interest rates:

Definition 2. Let $1 \leq m \leq N$ be fixed. With

$$Z_{m,m} = \frac{D_m}{B_{0,m}}$$

define the <u>Radon-Nikodým process</u>

$$Z_{n,m} = \tilde{\mathbb{E}}_n[Z_{m,m}], \quad n = 0, 1, ..., m$$

From this definition we find

$$\begin{aligned}
Z_{n,m} &= \tilde{\mathbb{E}}_n[Z_{m,m}] \\
&= \tilde{\mathbb{E}}_n\left[\frac{D_m}{B_{0,m}}\right] \quad \text{(by definition)} \\
&= \tilde{\mathbb{E}}_n\left[\frac{D_n}{D_n}\frac{D_m}{B_{0,m}}\right] \\
&= \frac{D_n}{B_{0,m}}\tilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}\right] \quad \text{(taking out what is known)} \\
&= \frac{D_n}{B_{0,m}}B_{n,m} \quad \text{(definition of } B_{n,m})
\end{aligned}$$

Proposition 1. Let $\tilde{\mathbb{E}}_n^m[\cdot]$ denote[1] the conditional expectation at time n with respect to the m-forward measure $\tilde{\mathbb{P}}^m$. If V_m is a random variable depending on only the first m coin tosses we have

$$\begin{aligned}
\tilde{\mathbb{E}}_0[Z_{m,m}V_m] &= \sum_{\omega\in\Omega} Z_{m,m}(\omega)V_m(\omega)\tilde{\mathbb{P}}(\omega) \\
&= \sum_{\omega\in\Omega} V_m(\omega)\tilde{\mathbb{P}}^m(\omega) \\
&= \tilde{\mathbb{E}}_0^m[V_m]
\end{aligned}$$

and, in generality we have

$$\frac{1}{Z_{n,m}}\tilde{\mathbb{E}}_n[Z_{m,m}V_m] = \tilde{\mathbb{E}}_n^m[V_m], \quad n = 0, 1, ..., m$$

Since from our earlier lemma,[2] for $0 \le n \le m \le N$, if Y is a random variable depending on only $\omega_1 \cdots \omega_m$, then

$$\tilde{\mathbb{E}}_n[Y] = \frac{1}{Z_n}\mathbb{E}_n[Z_m Y]$$

Hence with $Y = V_m$, $Z_n = Z_{n,m}$, $Z_m = Z_{m,m}$, and $\tilde{\mathbb{E}}_n = \tilde{\mathbb{E}}_n^m$, we find establish our identity, as desired.

We are now in a position to present a new formulation of the General Bayes' Theorem discussed in previous sections. We have

Theorem 1. General Bayes' Theorem II Let m be fixed such that $1 \le m \le N$ and let $\tilde{\mathbb{P}}^m$ be the m-forward measure. If $0 \le n \le m$ and V_m depends only on the first m coin tosses, then

$$\tilde{\mathbb{E}}_n^m[V_m] = \frac{1}{D_n B_{n,m}}\tilde{\mathbb{E}}_n[D_m V_m]$$

[1] Perhaps a more informative notation would be $\tilde{\mathbb{E}}_n^{\tilde{\mathbb{P}}^m}[\cdot]$, but we abandon this for brevity.

[2] Lemma 3.2.6 in Shreve's Stochastic Calculus for Finance I.

Proof. We have

$$
\begin{aligned}
\tilde{\mathbb{E}}^m_n[V_m] &= \frac{1}{Z_{n,m}}\tilde{\mathbb{E}}_n[Z_{m,m}V_m] \quad \text{(from the above proposition)}\\
&= \frac{B_{0,m}}{D_nB_{n,m}}\tilde{\mathbb{E}}_n\left[\frac{D_m \overbrace{B_{m,m}}^{=1}}{B_{0,m}}V_m\right] \quad \text{(by definition of } Z_{n,m} \text{ and } Z_{m,m})\\
&= \frac{1}{B_{n,m}}\tilde{\mathbb{E}}_n\left[\frac{B_{0,m}}{D_n}\cdot\frac{D_m}{B_{0,m}}V_m\right] \quad \text{(“taking in” what is known)}\\
&= \frac{1}{D_nB_{n,m}}\tilde{\mathbb{E}}_n\left[D_mV_m\right]
\end{aligned}
$$

which established our result, as desired. □

Corollary 1. We may continue with this result to show that $\tilde{\mathbb{E}}^m_n[V_m]$ is the time-n price of any derivative paying V_m at time m ***denominated in units of the zero-coupon bond maturing at time m.*** That is, the *forward price* of the asset as given by

$$
Fwd_{n,m} = \frac{S_n}{B_{n,m}}
$$

We say that the price of an asset denominated this way is a martingale under the m-forward measure $\tilde{\mathbb{P}}^m$.

Proof. From the Theorem above

$$
\begin{aligned}
\tilde{\mathbb{E}}^m_n[V_m] &= \frac{1}{D_nB_{n,m}}\tilde{\mathbb{E}}_n\left[D_mV_m\right]\\
&= \frac{1}{B_{n,m}}\tilde{\mathbb{E}}_n\left[\frac{V_m}{(1+R_n)\cdots(1+R_{m-1})}\right]\\
&= \frac{1}{B_{n,m}}V_n \quad \text{(risk neutral pricing)}
\end{aligned}
$$

In additional the the result, this gives us that the m-forward prices of assets are $\tilde{\mathbb{P}}^m$-martingales. □

Our application of this result will be that it is sometimes more simple to compute the m-forward expectation $\tilde{\mathbb{E}}^m_n[V_m]$ than it is to calculate the risk-neutral expectation $\tilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}V_m\right]$. In fact, we can see that we may compute the time-n price of our derivative V_n by rearranging our terms to yield

$$
V_n = B_{n,m}\tilde{\mathbb{E}}^m_n[V_m]
$$

This gives us the time-n price of a security paying V_m at time m **denominated in units of the zero-coupon bond maturing at time** m**.** We call this m-maturity zero-coupon bond the <u>numéraire</u> for the m-forward measure $\tilde{\mathbb{P}}^m$. If we know the distribution of V under $\tilde{\mathbb{P}}^m$ then we don't have to do any discounting under the expectation to arrive to our fair price V_m!

Example: *(Exercise 6.5)* Let $0 \leq m \leq N-1$ be fixed and consider the forward interest rate given by

$$F_{n,m} = \frac{B_{n,m} - B_{n,m+1}}{B_{n,m+1}}, \quad n = 0, 1, ..., m$$

Show that $F_{n,m}$, for $n = 0, 1, ..., m$, is a martingale under the $(m+1)$-forward measure $\tilde{\mathbb{P}}^{m+1}$.

Solution: Note that both $B_{n,m}$ and $B_{n,m+1}$ depend only on the first n coin tosses $\omega_1 \cdots \omega_n$. Therefore, since $B_{n,m}$ and $B_{n,m+1}$ are adapted, we find $F_{n,m}$ to be adapted.

Now, to test the martingale property under $\tilde{\mathbb{P}}^{m+1}$:

$$\begin{aligned}
\tilde{\mathbb{E}}_n^{m+1}\left[F_{n+1,m}\right] &= \frac{1}{D_n B_{n,m+1}} \tilde{\mathbb{E}}_n\left[D_{m+1} F_{n+1,m}\right] \quad \text{(from Bayes' II)} \\
&= \frac{1}{B_{n,m+1}} \tilde{\mathbb{E}}_n\left[\frac{D_{m+1}}{D_n} F_{n+1,m}\right] \quad \text{(predictability of } D_n\text{)} \\
&= \frac{1}{B_{n,m+1}} \tilde{\mathbb{E}}_n\left[\frac{D_{m+1}}{D_n}\left(\frac{B_{n+1,m} - B_{n+1,m+1}}{B_{n+1,m+1}}\right)\right] \quad \text{(definition)} \\
&= \frac{1}{B_{n,m+1}}\left(\tilde{\mathbb{E}}_n\left[\frac{D_{m+1}}{D_n}\frac{B_{n+1,m}}{B_{n+1,m+1}}\right] - \tilde{\mathbb{E}}_n\left[\frac{D_{m+1}}{D_n}\right]\right) \quad \text{(linearity)} \\
&= \frac{1}{B_{n,m+1}}\left(\tilde{\mathbb{E}}_n\left[\frac{D_{m+1}}{D_n}\frac{B_{n+1,m}}{B_{n+1,m+1}}\right] - B_{n,m+1}\right) \quad \text{(definition of } B_{n,m+1}\text{)} \\
&= \frac{1}{B_{n,m+1}}\left(\tilde{\mathbb{E}}_n\left[\frac{D_{m+1}}{D_n}\frac{\tilde{\mathbb{E}}_{n+1}\left[\frac{D_m}{D_{n+1}}\right]}{\tilde{\mathbb{E}}_{n+1}\left[\frac{D_{m+1}}{D_{n+1}}\right]}\right] - B_{n,m+1}\right) \quad \text{(definition of } B_{n+1,m} \text{ and } B_{n+1,m+1}\text{)} \\
&= \frac{1}{B_{n,m+1}}\left(\tilde{\mathbb{E}}_n\left[\frac{D_{m+1}}{D_n}\frac{\tilde{\mathbb{E}}_{n+1}\left[D_m\right]}{\tilde{\mathbb{E}}_{n+1}\left[D_{m+1}\right]}\right] - B_{n,m+1}\right) \quad \text{(predictability of } D_{n+1}\text{)}
\end{aligned}$$

However, note that

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{D_{m+1}}{D_n}\frac{\tilde{\mathbb{E}}_{n+1}\left[D_m\right]}{\tilde{\mathbb{E}}_{n+1}\left[D_{m+1}\right]}\right] &= \tilde{\mathbb{E}}_n\left[\tilde{\mathbb{E}}_{n+1}\left[\frac{D_{m+1}}{D_n}\frac{\tilde{\mathbb{E}}_{n+1}\left[D_m\right]}{\tilde{\mathbb{E}}_{n+1}\left[D_{m+1}\right]}\right]\right] \quad \text{(tower property)}\\
&= \tilde{\mathbb{E}}_n\left[\frac{1}{D_n}\tilde{\mathbb{E}}_{n+1}\left[D_{m+1}\frac{\tilde{\mathbb{E}}_{n+1}\left[D_m\right]}{\tilde{\mathbb{E}}_{n+1}\left[D_{m+1}\right]}\right]\right] \quad \text{(predictability of } D_n\text{)}\\
&= \tilde{\mathbb{E}}_n\left[\frac{\tilde{\mathbb{E}}_{n+1}\left[D_m\right]}{D_n}\tilde{\mathbb{E}}_{n+1}\left[D_{m+1}\frac{1}{\tilde{\mathbb{E}}_{n+1}\left[D_{m+1}\right]}\right]\right]\\
&= \tilde{\mathbb{E}}_n\left[\frac{\tilde{\mathbb{E}}_{n+1}\left[D_m\right]}{D_n\tilde{\mathbb{E}}_{n+1}\left[D_{m+1}\right]}\tilde{\mathbb{E}}_{n+1}\left[D_{m+1}\right]\right]\\
&= \tilde{\mathbb{E}}_n\left[\frac{\tilde{\mathbb{E}}_{n+1}\left[D_m\right]}{D_n}\right]\\
&= \tilde{\mathbb{E}}_n\left[\tilde{\mathbb{E}}_{n+1}\left[\frac{1}{D_n}D_m\right]\right] \quad \text{(predictability of } D_n\text{)}\\
&= \tilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}\right] \quad \text{(tower property)}\\
&= B_{n,m} \quad \text{(definition)}
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde{\mathbb{E}}_n^{m+1}\left[F_{n+1,m}\right] &= \frac{1}{B_{n,m+1}}\left(\tilde{\mathbb{E}}_n\left[\frac{D_{m+1}}{D_n}\frac{\tilde{\mathbb{E}}_{n+1}\left[D_m\right]}{\tilde{\mathbb{E}}_{n+1}\left[D_{m+1}\right]}\right] - B_{n,m+1}\right)\\
&= \frac{1}{B_{n,m+1}}\left(B_{n,m} - B_{n,m+1}\right)\\
&= F_{n,m}
\end{aligned}$$

Therefore, $\{F_{n,m}\}_{n=0}^{m}$ is a $\tilde{\mathbb{P}}^{m+1}$-martingale, as desired.