Mathematical & Computational Finance I
Lecture Notes

Utility Maximization & CAPM

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1 Utility Maximization

One way to use the tools that we have developed in the previous lectures is in utility maximization. We use the risk-neutral measure & risk-neutral pricing to price derivatives which permits us to hedge risk. However, risk-neutral pricing works best for "complete" financial markets where derivative securities can be hedged by replication. It turns out that most/many markets are incomplete because it is either impossible to short the underlying asset, or even completely impossible to trade in (i.e. insurance with people's lives, some commodity markets, etc...). For such incomplete markets a unique risk-neutral measure may not exist and so we should approach the problem differently.

An alternative approach is to maximize the utility of a potential investment by the investor. We will first consider utility maximization in simplified complete markets, despite having use in all market types.

Definition 1. A utility function is a non-decreasing and concave function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. Recall that a function is concave if

$$U(\alpha x + (1-\alpha)y) \geq \alpha U(x) + (1-\alpha)U(y) \quad \forall_{x,y\in\mathbb{R}},\ \alpha \in (0,1)$$

We require this definition for a utility function since we expect that marginal utility should be decreasing because we observe that \$1 means less to an investor with high wealth than an investor with lower wealth.

Example: Log-utility

$$U(x) = \begin{cases} \ln x & \text{if } x > 0 \\ -\infty & \text{if } x \leq 0 \end{cases}$$

Example: Hyperbolic Absolute Risk Aversion (HARA) utility

The hyperbolic absolute risk aversion (HARA) class of utility functions is given by[1]

$$U_p(x) = \begin{cases} \frac{1}{p}(x-c)^p & \text{if } x > c \\ 0 & \text{if } 0 < p < 1 \text{ and } x = c \\ -\infty & \text{if } p < 0 \text{ and } x = c \\ -\infty & \text{if } x < c \end{cases}$$

for $p < 1, p \neq 0, c \in \mathbb{R}$. With $p \neq 0$ and $c = 0$ we often refer to

$$U_p(x) = \frac{x^p}{p}$$

as a *power utility* function. If we take the limit $p \to 0$ we recover the log-utility function if we modify U slightly to $U(x) = \frac{x^p - 1}{p}$.

Definition 2. The index of absolute risk aversion for a utility function $U(x)$ is

$$-\frac{U''(x)}{U'(x)}$$

For the HARA utility function $U_p(x)$ we have

$$-\frac{U''(x)}{U'(x)} = \frac{1-p}{x-c} \quad \text{for } x > c$$

The concavity of a utility function gives us a measure of the trade-off between risk and return for the particular agent. The special case of $p = 1$ corresponds to risk-neutrality of an investor since we find a linear utility function.

Example: Consider an investment with the following payoffs

$$X = \begin{cases} 1 & \text{if } \omega_1 = H \\ 99 & \text{if } \omega_1 = T \end{cases}$$

where $\mathbb{P}(\omega_1 = H) = \frac{1}{2}$. We easily find that $\mathbb{E}[X] = 50$, and using Jensen's inequality

$$\phi(\mathbb{E}[X]) \leq \mathbb{E}[\phi(X)] \quad \text{for convex function } \phi$$

consider $U(x)$ as our convex function (since U is concave) we find

$$\begin{aligned} -U(\mathbb{E}[X]) &\leq \mathbb{E}[-U(X)] \\ -U(\mathbb{E}[X]) &\leq -\mathbb{E}[U(X)] \\ \mathbb{E}[U(X)] &\leq U(\mathbb{E}[X]) \end{aligned}$$

Thus, we see that the utility of a guaranteed \$50 exceeds the expected utility of the random amount given by X.

[1] We interpret c as the baseline of cash we require. Typically we consider $c = 0, p \neq 0$.

1.1 Problem A

Consider the N-period binomial asset pricing model with $0 < d < 1 + r < u$ and an initial endowment X_0. Using an adapted portfolio process $\{\Delta_n\}_{n=0}^{N-1}$ along with a stock & bank account we wish to maximize our terminal utility generated by the wealth at time N. Our goal is to solve the following

> **Problem A:** Find an adapted portfolio process $\{\Delta_n\}_{n=0}^{N-1}$ that maximizes
>
> $$\mathbb{E}[U(X_N)]$$
>
> subject to the wealth equation
>
> $$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$
>
> for $n = 0, 1, ..., N-1$.

In Problem A the expectation is calculated with respect to the real-world measure $\mathbb{P}$. If $\tilde{\mathbb{P}}$ was used then the stock and bank account with both have expected return equal to r, hence we would only end up investing in the riskless bank account.

Example: Consider the 2-period binomial model with $u = 2, d = \frac{1}{2}, S_0 = 4, r = \frac{1}{4}$ and assume the following probabilities:

$$\begin{aligned} \mathbb{P}(HH) &= \frac{4}{9} \\ \mathbb{P}(HT) = \mathbb{P}(TH) &= \frac{2}{9} \\ \mathbb{P}(TT) &= \frac{1}{9} \end{aligned}$$

Now, assume that we have the utility function $U(x) = \ln x$ and assume that we begin with initial wealth $X_0 = 4$ and must choose values $\Delta_0, \Delta_1(H), \Delta_1(T)$ to maximize the expected utility $\mathbb{E}[\ln(X_2)]$. At time $N = 1$ our wealth is

$$\begin{aligned}
X_1(H) &= \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) \quad \text{(from the wealth equation)}\\
&= \Delta_0 (uS_0) + (1+r)(X_0 - \Delta_0 S_0)\\
&= \Delta_0 \cdot (2 \cdot 4) + \left(1 + \frac{1}{4}\right)(4 - \Delta_0 \cdot 4)\\
&= 8\Delta_0 + \frac{5}{4}(4 - 4\Delta_0)\\
&= 3\Delta_0 + 5\\
X_1(T) &= \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0)\\
&= \Delta_0 (dS_0) + (1+r)(X_0 - \Delta_0 S_0)\\
&= \Delta_0 \cdot \left(\frac{1}{2} \cdot 4\right) + \left(1 + \frac{1}{4}\right)(4 - \Delta_0 \cdot 4)\\
&= 2\Delta_0 + \frac{5}{4}(4 - 4\Delta_0)\\
&= -3\Delta_0 + 5
\end{aligned}$$

and at time $N = 2$ we find

$$\begin{aligned}
X_2(HH) &= \Delta_1(H) S_2(HH) + (1+r)(X_1(H) - \Delta_1(H) S_1(H))\\
&= \Delta_1(H)(uS_1(H)) + (1+r)(X_1(H) - \Delta_1(H) S_1(H))\\
&= \Delta_1(H) \cdot (2 \cdot 8) + \left(1 + \frac{1}{4}\right)([3\Delta_0 + 5] - \Delta_1(H) \cdot 8)\\
&= 16\Delta_1(H) + \frac{5}{4}(3\Delta_0 - 8\Delta_1(H) + 5)\\
&= 6\Delta_1(H) + \frac{15}{4}\Delta_0 + \frac{25}{4}
\end{aligned}$$

similarly, we may compute

$$\begin{aligned}
X_2(HT) &= -6\Delta_1(H) + \frac{15}{4}\Delta_0 + \frac{25}{4}\\
X_2(TH) &= \frac{3}{2}\Delta_1(T) - \frac{15}{4}\Delta_0 + \frac{25}{4}\\
X_2(TT) &= -\frac{3}{2}\Delta_1(T) - \frac{15}{4}\Delta_0 + \frac{25}{4}
\end{aligned}$$

Then, our objective function to maximize, $\mathbb{E}[\ln X_2]$, becomes

$$\begin{aligned}
\mathbb{E}[\ln X_2] &= \sum_{\omega_1\omega_2 \in \Omega} \ln\left(X_2(\omega_1\omega_2)\right)\mathbb{P}(\omega_1\omega_2)\\
&= \ln(X_2(HH))\mathbb{P}(HH) + \ln(X_2(HT))\mathbb{P}(HT) +\\
&\qquad \ln(X_2(TH))\mathbb{P}(TH) + \ln(X_2(TT))\mathbb{P}(TT)\\
&= \frac{4}{9}\ln\left[6\Delta_1(H) + \frac{15}{4}\Delta_0 + \frac{25}{4}\right] + \frac{2}{9}\ln\left[-6\Delta_1(H) + \frac{15}{4}\Delta_0 + \frac{25}{4}\right]\\
&\qquad + \frac{2}{9}\ln\left[\frac{3}{2}\Delta_1(T) - \frac{15}{4}\Delta_0 + \frac{25}{4}\right] \frac{1}{9}\ln\left[-\frac{3}{2}\Delta_1(T) - \frac{15}{4}\Delta_0 + \frac{25}{4}\right]
\end{aligned}$$

To look for a maximum we should take partial derivatives with respect to $\Delta_0, \Delta_1(H), \Delta_1(T)$

$$
\begin{aligned}
\frac{\partial}{\partial \Delta_0}\mathbb{E}[\ln X_2] &= \frac{4}{9}\frac{14}{4}\frac{1}{X_2(HH)} + \frac{2}{9}\frac{15}{4}\frac{1}{X_2(HT)} - \\
&\quad \frac{2}{9}\frac{15}{4}\frac{1}{X_2(TH)} - \frac{1}{9}\frac{15}{4}\frac{1}{X_2(TT)} \\
&= \frac{5}{3X_2(HH)} + \frac{5}{6X_2(HT)} - \frac{5}{6X_2(TH)} - \frac{15}{36X_2(TT)} \\
&= \frac{5}{12}\left[\frac{4}{X_2(HH)} + \frac{2}{X_2(HT)} - \frac{2}{X_2(TH)} - \frac{1}{X_2(TT)}\right] \\
\frac{\partial}{\partial \Delta_1(H)}\mathbb{E}[\ln X_2] &= \frac{4}{3}\left[\frac{2}{X_2(HH)} - \frac{1}{X_2(HT)}\right] \\
\frac{\partial}{\partial \Delta_1(T)}\mathbb{E}[\ln X_2] &= \frac{1}{6}\left[\frac{2}{X_2(TH)} - \frac{1}{X_2(TT)}\right]
\end{aligned}
$$

and setting these equal to zero gives us

$$
\begin{aligned}
\frac{4}{X_2(HH)} + \frac{2}{X_2(HT)} &= \frac{2}{X_2(TH)} + \frac{1}{X_2(TT)} \quad \left(\text{from } \frac{\partial}{\partial \Delta_0}\mathbb{E}[\ln X_2]\right) \\
\frac{2}{X_2(HH)} &= \frac{1}{X_2(HT)} \quad \left(\text{from } \frac{\partial}{\partial \Delta_1(H)}\mathbb{E}[\ln X_2]\right) \\
\frac{2}{X_2(TH)} &= \frac{1}{X_2(TT)} \quad \left(\text{from } \frac{\partial}{\partial \Delta_1(T)}\mathbb{E}[\ln X_2]\right)
\end{aligned}
$$

The last two equations in this system give us

$$
\begin{aligned}
2X_2(HT) &= X_2(HH) \\
2X_2(TT) &= X_2(TH)
\end{aligned}
$$

and substituting this into the first equation in the system

$$
\begin{aligned}
\frac{4}{X_2(HH)} + \frac{2}{X_2(HT)} &= \frac{2}{X_2(TH)} + \frac{1}{X_2(TT)} \\
\implies \frac{4}{2X_2(HT)} + \frac{2}{X_2(HT)} &= \frac{2}{2X_2(TT)} + \frac{1}{X_2(TT)} \\
\frac{4}{X_2(HT)} &= \frac{2}{X_2(TT)} \\
4X_2(TT) &= 2X_2(HT) \\
2X_2(TT) &= X_2(HT)
\end{aligned}
$$

So, we have

$$
\begin{cases}
X_2(HH) = 2X_2(HT) \\
X_2(TH) = 2X_2(TT) \\
X_2(HT) = 2X_2(TT)
\end{cases}
$$

In principle we could expand these terms to solve a system of three wealth equations in three unknowns $\Delta_0, \Delta_1(H), \Delta_1(T)$, though this would be quite tedious. However, we know that under the risk-neutral measure $\tilde{\mathbb{P}}$ the discounted portfolio value process $\left\{\frac{X_n}{(1+r)^n}\right\}_{n=0}^{N}$ is a martingale and that

$$\tilde{\mathbb{E}}\left[\frac{X_n}{(1+r)^n}\right] = X_0$$

so, with $\tilde{p} = \frac{1}{2}$, we find

$$\begin{aligned}
X_0 = 4 &= \tilde{\mathbb{E}}\left[\frac{X_2}{(1+r)^2}\right] \\
&= \left(\frac{4}{5}\right)^2 \left[\frac{1}{4}X_2(HH) + \frac{1}{4}X_2(HT) + \frac{1}{4}X_2(TH) + \frac{1}{4}X_2(TT)\right]
\end{aligned}$$

However, we found above that

$$\begin{cases} X_2(HH) = 2X_2(HT) = 2(2X_2(TT)) \\ X_2(TH) = 2X_2(TT) \\ X_2(HT) = 2X_2(TT) \end{cases}$$

So,

$$\begin{aligned}
4 &= \left(\frac{4}{5}\right)^2 \left[\frac{1}{4}(4X_2(TT)) + \frac{1}{4}(2X_2(TT) + \frac{1}{4}(2X_2(TT)) + \frac{1}{4}X_2(TT)\right] \\
&= \frac{16}{25}\left[X_2(TT) + \frac{1}{2}X_2(TT) + \frac{1}{2}X_2(TT) + \frac{1}{4}X_2(TT)\right] \\
&= \frac{16}{25}\frac{9}{4}X_2(TT) \\
&= \frac{36}{25}X_2(TT) \\
\implies \frac{25}{9} &= X_2(TT) \\
&\implies \begin{cases} X_2(HH) = \frac{100}{9} \\ X_2(HT) = \frac{50}{9} \\ X_2(TH) = \frac{50}{9} \\ X_2(TT) = \frac{25}{9} \end{cases}
\end{aligned}$$

Treating these terminal portfolio values as the payoff of a derivative security we may find the replicated portfolio process satisfying

$$\begin{aligned}
\Delta_1(H) &= \frac{X_2(HH) - X_2(HT)}{S_2(HH) - S_2(HT)} = \frac{25}{54} \\
\Delta_1(T) &= \frac{X_2(TH) - X_2(TT)}{S_2(TH) - S_2(TT)} = \frac{25}{27}
\end{aligned}$$

Using the martingale property again

$$\tilde{\mathbb{E}}_1\left[\frac{X_2}{(1+r)^2}\right] = \frac{X_1}{(1+r)}$$
$$\implies \tilde{\mathbb{E}}_1\left[\frac{X_2}{(1+r)}\right] = X_1$$

Hence

$$X_1(H) = \frac{1}{1+\frac{1}{4}}\left[\frac{1}{2}X_2(HH) + \frac{1}{2}X_2(HT)\right] = \frac{20}{3}$$
$$X_1(T) = \frac{1}{1+\frac{1}{4}}\left[\frac{1}{2}X_2(HT) + \frac{1}{2}X_2(TT)\right] = \frac{10}{3}$$

and once again finding the corresponding time zero portfolio process

$$\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)} = \frac{5}{9}$$

1.2 Problem B

We may generalize the previous process in Problem A to form a new utility maximization problem:

Problem B: Given X_0 find a random variable X_N (without regard to the portfolio process) that maximizes

$$\mathbb{E}[U(X_N)]$$

subject to

$$\tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] = X_0$$

That is, subject to the constraint that X_n must be a discounted martingale.

Lemma 1. Suppose $\Delta_0^*, \Delta_1^*, ..., \Delta_{N-1}^*$ is an optimal portfolio process for Problem A and X_N^* is the corresponding time N optimal terminal wealth. Then X_N^* is optimal for Problem B.

Proof. Assume that $\Delta_0^*, ..., \Delta_{N-1}^*$ is the optimal wealth process for Problem A and that X_N^* is the corresponding terminal wealth, that is, X_N^* is the wealth generated by X_0. We have that

$$X_{n+1}^* = \Delta_n^* S_{n+1} + (1+r)(X_n^* - \Delta_n^* S_n) \quad n = 0, ..., N-1$$

We must first show that the portfolio process generating X_N^* is feasible, that is, the portfolio process satisfies the constraint

$$\tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] = X_0$$

Note that since X_N^* is generated by an adapted portfolio process $\Delta_0^*, ..., \Delta_{N-1}^*$ which solves Problem A then we have that it is a discounted martingale[2], that is,

$$\tilde{\mathbb{E}}\left[\frac{X_N^*}{(1+r)^N}\right] = X_0$$

Now that we have that this process is feasible we must show that it is the optimal solution for Problem B. Assume instead that we have better solution. Let X_N be any other random variable satisfying the martingale property

$$\tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] = X_0$$

Consider X_N to be equal to the payoff of some derivative security. Then, by the risk-neutral pricing theorem we have that the time zero price must be X_0. As such, we can calculate the corresponding portfolio process

$$\Delta_0, \Delta_1, ..., \Delta_{N-1}$$

However, since X_N^* was the optimal solution for Problem A we have that[3]

$$\mathbb{E}[U(X_N)] \leq \mathbb{E}[U(X_N^*)]$$

since $\mathbb{E}[U(X)]$ was the constraint for Problem A. Therefore, we conclude that X_N^* must be optimal for Problem B, as desired. $\square$

Lemma 2. Suppose X_N^* is optimal for Problem B. Then there exists a portfolio process $\Delta_0^*, \Delta_1^*, ..., \Delta_{N-1}^*$ that starts with initial wealth X_0 generating time N terminal value X_N^* that is optimal for Problem A.

Proof. If X_N^* is optimal for Problem B then we should consider a derivative security with time N payoff $V_N = X_N^*$. By going recursively backwards in time, we have

$$V_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)] \quad n = N-1, N-2, ..., 0$$

Then,[4]

$$\Delta_n^*(\omega_1 \cdots \omega_n) = \frac{V_{n+1}(\omega_1 \cdots \omega_n H) - V_{n+1}(\omega_1 \cdots \omega_n T)}{S_{n+1}(\omega_1 \cdots \omega_n H) - S_{n+1}(\omega_1 \cdots \omega_n T)} \quad n = 0, 1, ..., N-1$$

and we obtain the portfolio process $\Delta_0^*, \Delta_1^*, ..., \Delta_N^*$. From the **replication theorem for the multiperiod binomial model**[5], starting with initial wealth $V_0 = X_0$, the portfolio process

[2]It's obviously adapted, but I'm not sure how I see it necessarily satisfies the martingale property. Was this a consequence of having $\Delta_0^*, ..., \Delta_{N-1}^*$ solved Problem A?

[3]Was this because both wealth processes were generated by the same initial wealth X_0?

[4]How can we get to this step? I'm sure this is discussed in Chapter 1.

[5]See Theorem 1.2.2 from Chapter 1

$\Delta_0^*, ..., \Delta_{N-1}^*$ will achieve time N wealth X_N^*, that is, X_N^* is feasible for Problem A.

Now, let $\Delta_0, \Delta_1 ..., \Delta_{N-1}$ be any other portfolio process starting with initial wealth X_0, investing according to the wealth equation, leading to time N terminal wealth X_N (i.e. feasible & optimal for Problem A). Under the risk neutral measure $\tilde{\mathbb{P}}$ we have that the discounted process $\frac{X_n}{(1+r)^n}$ is a martingale, hence, X_N satisfies

$$\tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] = X_0$$

Therefore, X_N is a feasible solution to Problem B. However, we were given that X_N^* is optimal for Problem B, so

$$\mathbb{E}[U(X_N)] \leq \mathbb{E}[U(X_N^*)]$$

Therefore, $\Delta_0^*, \Delta_1^*, ..., \Delta_{N-1}^*$ is optimal for Problem A, as desired. $\square$

We say that Problem A is the *primal problem* and Problem B is the corresponding *dual problem*. By transforming Problem A we end up with the dual Problem B that ends up being easier in practice to solve. Typically by doing such a transformation the dual problem ends up being in a different space to the primal problem. In this case we had Problem A:

$$\max_{\Delta_0, \Delta_1, ..., \Delta_{N-1}} \mathbb{E}[U(X)]$$

with corresponding dual problem:

$$\max_{x \in \mathbb{X}} \mathbb{E}[U(x)], \quad \mathbb{X} = \left\{x \ : \ \tilde{\mathbb{E}}\left[\frac{x}{(1+r)^N}\right] = X_0\right\}$$

In Problem B it is inconvenient for our computations that the objective function is under the real-world measure while the constraint is under the risk-neutral measure. We may reformulate Problem B using the change of measure techniques developed earlier to write

$$X_0 = \tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] = \mathbb{E}\left[\frac{Z_N \cdot X_N}{(1+r)^N}\right]$$

where Z is the Radon-Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$. Alternatively, we may use the state-price density $\zeta_n = \frac{Z_n}{(1+r)^n}$ to write our Problem B maximization problem as

$$\begin{aligned} \max \quad & \mathbb{E}[U(X_N)] \\ \text{s.t.} \quad & \mathbb{E}[\zeta_N X_N] = X_0 \end{aligned}$$

<u>Example</u>: Now, the question remains how to solve Problem B with respect to $\mathbb{P}$? Consider the Radon-Nikodým derivative

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$$

We have, using the values from our previous examples in the book[6]

$$\begin{aligned}
Z(HH) &= \frac{\tilde{\mathbb{P}}(HH)}{\mathbb{P}(HH)} = \frac{9}{16} \\
Z(HT) = Z(TH) &= \frac{9}{8} \\
Z(TT) &= \frac{9}{4}
\end{aligned}$$

and corresponding state-price densities

$$\begin{aligned}
\zeta_1 := \zeta(HH) &= \frac{Z(HH)}{(1+r)^2} = \frac{9}{25} \\
\zeta_2 := \zeta_2(HT) &= \frac{18}{25} \\
\zeta_3 := \zeta(TH) &= \frac{18}{25} \\
\zeta_4 := \zeta_2(TT) &= \frac{36}{25}
\end{aligned}$$

Now, recall that we had the change of measure result

$$\begin{aligned}
X_0 &= \tilde{\mathbb{E}}\left[\frac{X_N}{(1+r)^N}\right] \\
&= \mathbb{E}\left[\frac{X_N Z_N}{(1+r)^N}\right] \\
&= \mathbb{E}\left[\zeta_N X_N\right]
\end{aligned}$$

and so with

$$\begin{aligned}
x_1 &:= X_2(HH),\ p_1 := \mathbb{P}(HH) \\
x_2 &:= X_2(HT),\ p_2 := \mathbb{P}(HT) \\
x_3 &:= X_2(TH),\ p_3 := \mathbb{P}(TH) \\
x_4 &:= X_2(TT),\ p_4 := \mathbb{P}(TT)
\end{aligned}$$

we may compute Problem B by rewriting it as

$$\begin{aligned}
\max_{(x_1,x_2,x_3,x_4)} \quad & \mathbb{E}[U(X)] = \sum_{m=1}^{4} p_m U(x_m) = \sum_{m=1}^{4} p_m \ln x_m \\
\text{subject to} \quad & X_0 = \mathbb{E}[\zeta X] = \sum_{m=1}^{4} p_m \zeta_m x_m
\end{aligned}$$

[6] I forget which one exactly...

How do we solve such constrained optimization problems? We use Lagrange multipliers! With the utility function $U(x) = lnx$ we have the objective function

$$\begin{aligned}
f(x_1, x_2, x_3, x_4) &= \mathbb{E}[\ln(X)] \\
&= p_1 \ln x_1 + p_2 \ln x_2 + p_3 \ln x_3 + p_4 \ln x_4 \\
&= \frac{4}{9} \ln x_1 + \frac{2}{9} \ln x_2 + \frac{2}{9} \ln x_3 + \frac{1}{9} \ln x_4
\end{aligned}$$

subject to the constraint

$$\begin{aligned}
g(x_1, x_2, x_3, x_4) &= \mathbb{E}[\zeta X] \\
&= p_1\zeta_1 x_1 + p_2\zeta_2 x_2 + p_3\zeta_3 x_3 + p_4\zeta_4 x_4 \\
&= \frac{4}{9}\frac{9}{25} x_1 + \frac{2}{9}\frac{18}{25} x_2 + \frac{2}{9}\frac{18}{25} x_3 + \frac{1}{9}\frac{36}{25} x_4 \\
&= \frac{4}{25} x_1 + \frac{4}{25} x_2 + \frac{4}{25} x_3 + \frac{4}{25} x_4 \\
&= X_0
\end{aligned}$$

That is, we must solve

$$\begin{aligned}
\max\ & f \\
\text{s.t. } & g = X_0
\end{aligned}$$

Let λ be a Lagrange multiplier. We wish to find all values $x_1, x_2, x_3, x_4, \lambda$ such that

$$\begin{aligned}
\text{grad } f = \nabla f &= \lambda \nabla g \\
\text{and } g &= X_0
\end{aligned}$$

That is, we solve

$$0 = \frac{\partial f}{\partial x_i} = \lambda \frac{\partial g}{\partial x_i} \quad i = 1, 2, 3, 4$$

yielding

$$\begin{aligned}
\frac{4}{9x_1} &= \lambda \frac{4}{25} \implies x_1 = \frac{25}{9\lambda} \\
\frac{2}{9x_2} &= \lambda \frac{4}{25} \implies x_2 = \frac{25}{18\lambda} \\
\frac{2}{9x_3} &= \lambda \frac{4}{25} \implies x_3 = \frac{25}{18\lambda} \\
\frac{1}{9x_4} &= \lambda \frac{4}{25} \implies x_4 = \frac{25}{36\lambda}
\end{aligned}$$

Plugging in these values for x_1, x_2, x_3, x_4 into our constraint $g = X_0$ we find

$$\begin{aligned}
\frac{4}{25}x_1 + \frac{4}{25}x_2 + \frac{4}{25}x_3 + \frac{4}{25}x_4 &= X_0 = 4 \\
\frac{4}{25}\frac{25}{9\lambda} + \frac{4}{25}\frac{25}{18\lambda} + \frac{4}{25}\frac{25}{18\lambda} + \frac{4}{25}\frac{25}{36\lambda} &= 4 \\
\frac{4}{9\lambda} + \frac{2}{9\lambda} + \frac{2}{9\lambda} + \frac{1}{9\lambda} &= 4 \\
\frac{1}{\lambda} &= 4 \\
\implies \lambda &= \frac{1}{4}
\end{aligned}$$

Hence, plugging in our value for λ into $\frac{\partial f}{\partial x_i} = 0$

$$\begin{aligned}
x_1 = X_2(HH) &= \frac{25}{9\lambda} = \frac{25}{9} \cdot 4 = \frac{100}{9} \\
x_2 = X_2(HT) &= \frac{25}{18\lambda} = \frac{25}{18} \cdot 4 = \frac{50}{9} \\
x_3 = X_2(TH) &= \frac{25}{18\lambda} = \frac{25}{18} \cdot 4 = \frac{50}{9} \\
x_4 = X_2(TT) &= \frac{25}{36\lambda} = \frac{25}{36} \cdot 4 = \frac{25}{9}
\end{aligned}$$

With these values for $X_2(HH), X_2(HT), X_2(TH), X_2(TT)$ we may compute $\Delta_1(H), \Delta_1(T), \Delta_0$ by using our formula from risk-neutral valuation & the wealth equation

$$\begin{aligned}
\Delta_1(H) &= \frac{X_2(HH) - X_2(HT)}{S_2(HH) - S_2(HT)} \\
\Delta_1(T) &= \frac{X_2(TH) - X_2(TT)}{S_2(TH) - S_2(TT)} \\
X_1(H) &= \frac{1}{1+r}\left[\tilde{\mathbb{P}}(H)X_2(HH) + \tilde{\mathbb{P}}(T)X_2(HT)\right] \\
X_1(T) &= \frac{1}{1+r}\left[\tilde{\mathbb{P}}(H)X_2(TH) + \tilde{\mathbb{P}}(T)X_2(TT)\right] \\
\Delta_0 &= \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)}
\end{aligned}$$

which would complete the problem.

1.3 Problem C

We may generalize the previous example. Note that in the N-period binomial model there are $M = 2^N$ possible coin toss sequences in our sample space Ω which we may enumerate by

$$\Omega = \left\{\omega^1, \omega^2, ..., \omega^M\right\}$$

and define

$$\begin{aligned}
\zeta_m &= \zeta_N(\omega^m) \\
p_m &= \mathbb{P}(\omega^m) \\
x_m &= X_N(\omega^m)
\end{aligned}$$

then, we may reformulate Problem B as the following:

Problem C: Given X_0 find a vector $(x_1, ..., x_M)$ that maximizes

$$\mathbb{E}[U(X_N)] = \sum_{m=1}^{M} p_m U(x_m)$$

subject to

$$\mathbb{E}[\zeta_N X_N] = \sum_{m=1}^{M} p_m x_m \zeta_m = X_0$$

Theorem 1. Let $I(x) = (U')^{-1}(x)$. The solution of Problem A can be found by solving

$$\mathbb{E}\left[\frac{Z_N}{(1+r)^N} I\left(\frac{\lambda Z_N}{(1+r)^N}\right)\right] = X_0$$

for λ and then substituting this value for λ to solve

$$X_N = I\left(\frac{\lambda Z}{(1+r)^N}\right)$$

Then, with this value for X_N, the optimal portfolio process $\{\Delta_n\}_{n=0}^{N-1}$ and wealth $\{X_n\}_{n=0}^{N-1}$ can be calculated by considering the replication of a derivative security with payoff $V_N = X_N$ using our backwards induction method.

Proof. Consider the Lagrangian

$$\begin{aligned}
L(x_1, ..., x_M, \lambda) &= f(x_1, ...x_M) - \lambda(g(x_1, ..., x_M) - X_0) \\
&= \mathbb{E}[U(X_N)] - \lambda\left(\mathbb{E}[\zeta_N X_N] - X_0\right) \\
&= \sum_{m=1}^{M} p_m U(x_m) - \lambda\left(\sum_{m=1}^{M} p_m \zeta_m x_m - X_0\right)
\end{aligned}$$

Then, the Lagrange multiplier equations are

$$\begin{aligned}
&\nabla f = \lambda \nabla g \\
\implies &\frac{\partial L}{\partial x_m} = 0, \quad m = 1, ..., M
\end{aligned}$$

which is equivalent to

$$
\begin{aligned}
p_m U'(x_m) - \lambda p_m \zeta_m &= 0 \\
U'(x_m) &= \lambda \zeta_m \\
\iff U'\left(X_N\right) &= \frac{\lambda Z_N}{(1+r)^N}
\end{aligned}
$$

Note that U is strictly concave everywhere by its construction and that $U(X_N)$ is finite[7]. Then we have that U' must be a decreasing function $\implies U'$ is invertible.[8] Hence,

$$
\begin{aligned}
I(x) &:= \left(U'\right)^{-1}(x) \quad \text{exists, and} \\
X_N &= I\left(\frac{\lambda Z}{(1+r)^N}\right)
\end{aligned}
$$

Substituting this value for X_N into our constraint yields

$$
\begin{aligned}
\mathbb{E}\left[\frac{Z_N X_N}{(1+r)^N}\right] &= \sum_{m=1}^{M} p_m \zeta_m x_m \\
&= \mathbb{E}\left[\frac{Z}{(1+r)^N} I\left(\frac{\lambda Z}{(1+r)^N}\right)\right] \\
&= X_0
\end{aligned}
$$

If we can solve this equation for the Lagrange multiplier λ then we should substitute this quantity into

$$
X_N = I\left(\frac{\lambda Z}{(1+r)^N}\right)
$$

and find the corresponding portfolio process $\Delta_{N-1}, \Delta_{N-2}, ..., \Delta_0$ that generates X_N from X_0 by our usual backwards induction methods, as desired. □

[7] U may be continuous, but since X is defined on a finite probability space we have that $U(X)$ is finite.

[8] We have from Analysis that a monotonically decreasing function is invertible