Mathematical & Computational Finance I
Lecture Notes

Interest-Rate-Dependent Assets

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1 Binomial Models for Interest Rates (con't)

Recall that we had defined the following

(i) R_n: The, potentially stochastic, interest rate over the n^{th} to $(n+1)^{\text{th}}$ period as a function of $\omega_1 \cdots \omega_n$.

(ii) D_n: The discount process given by

$$D_n = \frac{1}{(1+R_0)(1+R_1)\cdots(1+R_{n-1})} = \frac{1}{\prod_{j=0}^{n}(1+R_j)}$$

as a function of the first $\omega_1 \cdots \omega_{n-1}$ coin tosses.

(iii) $B_{n,m}$: The time n price of a zero-coupon bond maturity at time m given by

$$D_n B_{n,m} = \tilde{\mathbb{E}}_n[D_m]$$

(iv) $\Delta_{n,m}$: The number of m-maturity zero-coupon bonds held between periods n and $n+1$, as a function of the first n coin tosses $\omega_1 \cdots \omega_n$.

Using these quantities we finished last time with the proof demonstrating that the wealth process given by

$$X_{n+1} = 1 \cdot \Delta_{n,n+1} + \sum_{m=n+2}^{N} \Delta_{n,m} B_{n+1,m} + (1+R_n)\left[X_n - \sum_{m=n+1}^{N} \Delta_{n,m} B_{n,m}\right]$$

was a discounted $\tilde{\mathbb{P}}$-martingale. That is, we had proven

$$\tilde{\mathbb{E}}_n\left[D_{n+1}X_{n+1}\right] = D_n X_n$$

Since $D_n X_n$ is a $\tilde{\mathbb{P}}$-martingale we have

$$\tilde{\mathbb{E}}_0[D_n X_n] = D_0 X_0 = X_0$$

Since arbitrage is not *ipso facto* precluded from our model, suppose now that arbitrage is possible. Then, by trading in zero-coupon bonds and the bank account there exists a portfolio process sequence $\{\Delta_{n,m}\}$ satisfying

$$\begin{aligned} X_0 &= 0 \\ X_n &\geq 0 \quad \forall\, \omega_1 \cdots \omega_n \in \Omega \\ \tilde{\mathbb{P}}(X_n > 0) &> 0 \end{aligned}$$

That is, under our arbitrage assumption, with zero initial capital we have a strictly positive probability of achieving positive time n wealth. However, note that we have

$$X_0 = \tilde{\mathbb{E}}_0[D_n X_n] > 0$$

which contradicts our arbitrage assumption![1] Therefore, using risk-neutral pricing to define our zero-coupon bond price gives us a model free of arbitrage, which is nice.

By applying the risk-neutral pricing formula for $0 \leq n \leq m \leq N$ we can show that the time n price of a derivative contract paying V_m at time m, with V_m depending on only the first m coin tosses $\omega_1 \cdots \omega_n \cdots \omega_m$, is given by

$$V_n = \frac{1}{D_n} \tilde{\mathbb{E}}_n[D_m V_m]$$

From this quantity we will demonstrate that for our derivative security there exists a hedging portfolio, validating our use of risk-neutral pricing. If it is possible to construct a hedging portfolio for a short position of the derivative then the time n value must be V_n by the arbitrage-free property of the model.

Investing at time n in a zero-coupon bond maturing at time $n+1$ is identical to investing in the bank account since a \$1 investment in the bond at time n will yield $\frac{1}{B_{n,n+1}}$ bond units paying $\frac{1}{B_{n,n+1}}$ at time $n+1$, so

$$\begin{aligned} \frac{1}{B_{n,n+1}} &= \frac{1}{\tilde{\mathbb{E}}_n\left[\frac{D_{n+1}}{D_n}\right]} \\ &= \frac{1}{\tilde{\mathbb{E}}_n\left[\frac{1}{1+R_n}\right]} \\ &= \frac{1}{\frac{1}{1+R_n}} \\ &= 1 + R_n \end{aligned}$$

which is identical to the amount an agent would receive by investing \$1 in the bank account at time n. Therefore, either the bond or the bank account (whichever a particular agent prefers) is a redundant security.

[1] I'm not crystal clear on how this is true.

Definition 1. Let $0 \leq m \leq N$ be given. A coupon bond can be modelled as a sequence of constant (i.e. non-random) quantities

$$C_0, C_1, ..., C_m$$

We say that, for $0 \leq n \leq m-1$, the amount C_n is the coupon payment made at time n. The final payment C_m is the payment that includes the principle as well as the interest accrued over periods $m-1$ to m. For a \$1 face value zero-coupon bond we have that

$$C_0 = C_1 = \cdots = C_{m-1} = 0 \quad \text{and} \quad C_m = 1$$

We can view a coupon bond as a portfolio of zero-coupon bonds consisting of

C_0 paid in cash at time $n = 0$ (typically $C_0 = 0$)

C_1 face-value zero-coupon bond maturing at time $n = 1$

C_2 face-value zero-coupon bond maturing at time $n = 2$

$\vdots$

C_m face-value zero-coupon bond maturing at time $n = m$

Therefore, the rational time zero price of a coupon bond is given by

$$\sum_{k=0}^{m} C_k B_{0,k} = \tilde{\mathbb{E}}_0 \left[\sum_{k=0}^{m} D_k C_k \right]$$

and for any period $n \leq m$, before the payment C_n is made but after all payments $C_0, ..., C_{n-1}$, the price is given by

$$\sum_{k=n}^{m} C_k B_{n,k} = \tilde{\mathbb{E}}_n \left[\sum_{k=n}^{m} \frac{D_k}{D_n} C_k \right]$$

which is similar to our result for the price of a sequence of cash-flows, for a *deterministic* rate r, given by

$$V_n = \tilde{\mathbb{E}}_n \left[\sum_{k=n}^{N} \frac{C_k}{(1+r)^{k-n}} \right], \quad n = 0, 1, ..., N$$

discussed in Chapter 2.

1.1 Fixed-Income Derivatives

Suppose that we have a risky asset whose time n price is given by S_n in addition to stochastic interests rate dictated by the binomial model. Assume that S_n depends only on the coin tosses up to n, $\omega_1 \cdots \omega_n$. Since we have the risk-neutral measure $\tilde{\mathbb{P}}$ we require that the discounted asset price must be a $\tilde{\mathbb{P}}$-martingale satisfying

$$D_n S_n = \tilde{\mathbb{E}}_n[D_{n+1} S_{n+1}], \quad n = 0, 1, ..., N-1$$

1.1.1 Forward Prices

Definition 2. A forward contract is an agreement to pay a specified delivery price K at a specified future time m, for $0 \leq m \leq N$, for the asset whose time m price is S_m.

The m-forward price of S at time n denote $Fwd_{n,m}$, for $0 \leq n \leq m$, is the delivery price K that gives the forward contract a no-arbitrage value of zero at time n.

Theorem 1. Consider a risky asset price process $\{S_n\}_{n=0}^{N}$ in the binomial interest rate model. Assume that zero-coupon bonds of all maturities are sufficiently liquid and can be traded. For $0 \leq n \leq m \leq N$, the m-forward price of S at time n is

$$Fwd_{n,m} = \frac{S_n}{B_{n,m}}$$

Proof. Suppose that at time n an agent

(i) Shorts the forward contract with delivery price K at time m (i.e. the agent has an obligation to deliver one unit of S at time m in exchange for K). If K is chosen such that the forward contract has a no-arbitrage value of zero at time n then the shorted forward position generates no cash-flow at time n.

(ii) Shorts $\frac{S_n}{B_{n,m}}$ units of a zero-coupon bond maturing at time m. Then, the agent receives $\frac{S_n}{B_{n,m}} \cdot B_{n,m} = S_n$ in cash at time n.

(iii) The agent uses these proceeds to buy one share of the asset at time n so that the net gain/loss at time n is zero (to cover the shorted forward position).

Then, at time m we have that

(i) The agent must deliver the one purchased share in exchange for K units in cash (i.e. cover the shorted forward position).

(ii) Deliver the $\frac{S_n}{B_{n,m}}$ in the m-maturity bond (i.e. cover the shorted bond position).

(iii) Therefore, the net cash-flow at time m is

$$K - \frac{S_n}{B_{n,m}}$$

Note that if $K \neq \frac{S_n}{B_{n,m}}$ there is an arbitrage opportunity.[2] Therefore, in order to satisfy no-arbitrage we must have

$$Fwd_{n,m} = K = \frac{S_n}{B_{n,m}}$$

[2] If $K > \frac{S_n}{B_{n,m}}$ then we have found arbitrage. If $K < \frac{S_n}{B_{n,m}}$ then go long on the forward contract, long the m-maturity bond, and short the asset.

as desired.[3] We call such positions undertaken in the proof *static hedges* since the agent does not need (or should need) to trade/rebalance between time n when the hedge is initiated and time m when the contracts expire.

□

Example: *(Exercise 6.2)* Verify that the discounted value of the static hedging portfolio constructed immediately above is a $\tilde{\mathbb{P}}$-martingale.

Proof. First we must identity what our portfolio actually is. At time n we short the forward contract with delivery date m and delivery price K with time n value zero. Our hedged position is

(i) Short $\frac{S_n}{B_{n,m}}$ units of the zero-coupon bond.

(ii) Long one unit of the underlying asset for S_n.

Therefore, at time $k = n, n+1, ..., m$ the value of this hedging portfolio is given by

$$X_k = S_k - \frac{S_n}{B_{n,m}} B_{k,m}$$

Adaptedness: First note that $\frac{S_n}{B_{n,m}}$ remains constant through $k = n, n+1, ..., m$ and depends on the first n coin tosses $\omega_1 \cdots \omega_n$, and so it is adapted. Next, by definition we have that S_k and $B_{k,m}$ depend on the first $k = n, n+1, ..., m$ coin tosses $\omega_1 \cdots \omega_k$. Therefore, our portfolio X_k is adapted.

Martingale property: For $k = n, ..., m-1$ we have the conditional expectation with respect to time k of the discounted portfolio is

$$\begin{aligned}
\tilde{\mathbb{E}}_k \left[D_{k+1} X_{k+1} \right] &= \tilde{\mathbb{E}}_k \left[D_{k+1} \left(S_{k+1} - \frac{S_n}{B_{n,m}} B_{k+1,m} \right) \right] \quad \text{(by definition)} \\
&= \tilde{\mathbb{E}}_k \left[D_{k+1} S_{k+1} \right] - \frac{S_n}{B_{n,m}} \tilde{\mathbb{E}}_k \left[D_{k+1} B_{k+1,m} \right] \quad \text{(linearity \& adaptedness)} \\
&= D_k S_k - \frac{S_n}{B_{n,m}} \tilde{\mathbb{E}}_k \left[D_{k+1} B_{k+1,m} \right] \quad (D_k S_k \text{ is a } \tilde{\mathbb{P}}\text{-martingale}) \\
&= D_k S_k - \frac{S_n}{B_{n,m}} D_k B_{k,m} \quad (D_k B_{k,m} \text{ is a } \tilde{\mathbb{P}}\text{-martingale}) \\
&= D_k \left(S_k - \frac{S_n}{B_{n,m}} B_{k,m} \right) \\
&= D_k X_k \quad \text{(by definition)}
\end{aligned}$$

which satisfies the martingale property for $D_k X_k$. Therefore, the discounted hedging portfolio process $D_k X_k$ is a $\tilde{\mathbb{P}}$-martingale, as desired. □

[3] I don't quite see this final step.

So, thankfully, we have confirmed that the discounted value of the static hedging portfolio is indeed a $\tilde{\mathbb{P}}$-martingale, as any discounted asset value should be in our model. This permits us to apply the risk-neutral pricing formula when need be. Hence, the payoff (to a short party) of a forward contract will be $S_m - K$ at time m. Therefore, we wish to solve

$$\begin{aligned}
\tilde{\mathbb{E}}_n[D_m(S_m-K)] &= \tilde{\mathbb{E}}_n[D_mS_m] - K\tilde{\mathbb{E}}_n[D_m] \quad \text{(linearity \& adaptedness)} \\
&= \tilde{\mathbb{E}}_n[D_mS_m] - K\tilde{\mathbb{E}}_n\left[\frac{D_n}{D_n}D_m\right] \\
&= \tilde{\mathbb{E}}_n[D_mS_m] - KD_n\tilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}\right] \quad \text{(adaptedness)} \\
&= D_nS_n - KD_n\tilde{\mathbb{E}}_n\left[\frac{D_m}{D_n}\right] \quad \text{(martingale property of } D_nS_n) \\
&= D_nS_n - KD_nB_{n,m} \quad \text{(definition of } B_{n,m}) \\
&= D_n\left(S_n - KB_{n,m}\right)
\end{aligned}$$

Since we require the time n price of this contract to be zero we must have

$$\begin{aligned}
0 &= S_n - KB_{n,m} \\
\implies K &= \frac{S_n}{B_{n,m}}
\end{aligned}$$

which is consistent to our result from risk-neutral pricing above.

1.1.2 Forward Interest Rates

In additional to forward prices we may also consider forward rates between periods. Let $0 \le n \le m \le N-1$. At time n perform the following actions

(i) Short one unit of the m-maturity zero-coupon bond to receive $B_{n,m}$.

(ii) Use the proceeds to purchase $\frac{B_{n,m}}{B_{n,m+1}}$ units of the zero-coupon bond maturity at time $m+1$ for a cost of $\frac{B_{n,m}}{B_{n,m+1}} \cdot B_{n,m+1} = B_{n,m}$.

and at time m

(i) Invest \$1 in order to cover the short position in the m-maturity bond.[4]

and finally, at time $m+1$

(i) Receive $\frac{B_{n,m}}{B_{n,m+1}}$ from the long position in the $(m+1)$-maturity zero-coupon bond.

[4]This doesn't make sense to me. Why would we invest now when the m-maturity bond is already expiring presently and so we must deliver \$1 now.

We write $F_{n,m}$ (not to be confused with $Fwd_{n,m}$) for the forward interest rate earned between times m and $m+1$ starting with the actions taken at time n. The \$1 invested at time m will accumulate to

$$\$1 \cdot (1 + F_{n,m}) = \frac{B_{n,m}}{B_{n,m+1}}$$

Therefore, with $\frac{B_{n,m}}{B_{n,m+1}}$ denoting the current value of the invested \$1 and -1 denoting the principle we must repay/deliver,

$$\begin{aligned} F_{n,m} &= \frac{B_{n,m}}{B_{n,m+1}} - 1 \\ &= \frac{B_{n,m} - B_{n,m+1}}{B_{n,m+1}} \end{aligned}$$

is the rate (decided/locked in at time n) at which we may invest \$1 between time m and $m+1$ for a single period. Note that we have

$$\begin{aligned} F_{m,m} &= \frac{B_{m,m}}{B_{m,m+1}} - 1 \\ &= \frac{1}{\frac{1}{1+R_m}} - 1 \\ &= 1 + R_m - 1 \\ &= R_m \end{aligned}$$

so the one-period rate that may be locked in at time m is identical to R_m, the *spot rate* at time m.

Definition 3. Let $0 \leq n \leq m \leq N-1$. The one-period forward interest rate at time n for investing at time m is given by

$$F_{n,m} = \frac{B_{n,m}}{B_{n,m+1}} - 1$$

Theorem 2. Let $0 \leq n \leq m \leq N-1$ be given. The no-arbitrage time n price of a contract paying R_m at time $m+1$ is

$$B_{n,m+1} F_{n,m} = B_{n,m} - B_{n,m+1}$$

Proof. At time n perform the following actions

(i) Short a bond paying R_m at time $m+1$ with value to be determined.

(ii) Long a m-maturity zero-coupon bond worth $B_{n,m}$.

(iii) Short a $(m+1)$-maturity zero-coupon bond worth $B_{n,m+1}$.

Notice that this will require some initial capital in order to set up since

$$B_{n,m} - B_{n,m+1} > 0$$

At time m we have

(i) Receive \$1 from the m-maturity zero-coupon bond.

(ii) Invest the \$1 in the bank account for rate R_m

(iii) Net cash flow = 0.

and at time $m+1$ we have

(i) The bank account is now worth $1+R_m$.

(ii) Pay R_m to the long counterparty of the initial shorted contract.

(iii) Pay the remaining \$1 to the cover the short position from the $m+1$-maturity zero-coupon bond.

(iv) Net cash flow = 0.

Hence, we have successfully hedged our short positions in the contract using initial capital

$$B_{n,m} - B_{n,m+1} > 0$$

Therefore, the no-arbitrage price of the initial shorted contract must be

$$B_{n,m} - B_{n,m+1} > 0$$

Therefore, by our result

$$\begin{aligned} F_{n,m} &= \frac{B_{n,m}}{B_{n,m+1}} - 1 \\ \implies B_{n,m+1}F_{n,m} &= B_{n,m} - B_{n,m+1} \end{aligned}$$

We have that

$$B_{n,m+1}F_{n,m} = B_{n,m} - B_{n,m+1} > 0$$

is the rational no-arbitrage price of this contract paying R_m at time $m+1$, as desired. $\square$

If we consider such a forward contract initiated at time n for delivery of "asset" R_m at time $m+1$ the forward price $Fwd_{n,m+1}$ at time n, using the forward rate $F_{n,m}$, is given by

$$\begin{aligned} Fwd_{n,m+1} &= \frac{S_n}{B_{n,m+1}} \\ &= \frac{B_{n,m+1}F_{n,m}}{B_{n,m+1}} \\ &= F_{n,m} \end{aligned}$$

since, by our theorem, the value of the underlying asset at time n is

$$S_n = B_{n,m+1}F_{n,m}$$

1.1.3 Interest Rate Swap

Once again, consider the following portfolio. At time n:

(i) Short a forward contract on the interest rate process R: Agree to receive (fixed) $F_{n,m}$ at time $m+1$ in exchange for a payment of (random) R_m at time $m+1$.

(ii) Such a contract should have time n value of zero to enter.

At time m

(i) Invest \$1 at rate R_m.

and at time $m+1$

(i) Receive $1 + R_m$ from the investment at time m.

(ii) Pay R_m to the long party of the contract.

(iii) Receive $F_{n,m}$ from the long party of the contract.

(iv) Net cash received $= 1 + F_{n,m}$.

After subtracting the \$1 injected into the portfolio at time m we find that we have effectively locked-in an interest rate $F_{n,m}$ over the period m to $m+1$ at time n. This transaction leads to the definition of an interest rate swap:

Definition 4. Let m be given with $1 \leq m \leq N$. An m-period interest rate swap is a contract that makes payments $S_1, ..., S_m$ at time $1, ..., m$ where

$$S_n = K - R_{n-1}, \quad n = 1, ..., m$$

The m-period swap rate SR_m is the value of K that makes the time zero no-arbitrage price of the interest rate swap equal to zero.

We say that the long party **("receive fixed")** of the swap contract

1. Receives a constant payment at each period n: Receive fixed interest K on national principal of \$1.

2. Pays a variable rate **("pays floating")** of R_{n-1} at each time n: Receive variable (stochastic) interest R_{n-1} on notional principal of \$1.

If the long party in such a contract already has some loan on which fixed interest payments must be paid then the long swap position will convert[5] such a fixed rate loan into a variable rate loan. Conversely, a short swap position converts a variable rate loan to a fixed rate loan (paying fixed & receiving floating).

[5] Assuming the appropriate parameters match.

Theorem 3. The time zero no-arbitrage price of the m-period interest rate swap is given by

$$\begin{aligned} Swap_m &= \sum_{n=1}^{m} \underbrace{B_{0,n}}_{\text{discounting}} \underbrace{(K - F_{0,n-1})}_{\text{risk-neutral forecast}} \\ &= K \sum_{n=1}^{m} B_{0,n} - (1 - B_{0,m}) \end{aligned}$$

and the m-period swap rate (fixed rate) is

$$\begin{aligned} SR_m &= \frac{\sum_{n=1}^{m} B_{0,n} F_{0,n-1}}{\sum_{n=1}^{m} B_{0,n}} \\ &= \frac{1 - B_{0,m}}{\sum_{n=1}^{m} B_{0,n}} \end{aligned}$$

Proof. The time zero no-arbitrage rational price of the payment $\$K$ made at time n must be

$$\begin{aligned} K \cdot \tilde{\mathbb{E}}_0 \left[\frac{1}{\prod_{j=0}^{n} (1 + R_j)} \right] &= K \cdot \tilde{\mathbb{E}}_0 \left[D_n \right] \\ &= K \cdot B_{0,n} \end{aligned}$$

and by our previous theorem stating that the no-arbitrage price of a payment R_m at time $m + 1$ is given by

$$B_{n,m+1} F_{n,m} = B_{n,m} - B_{n,m+1} > 0$$

Therefore, we have that the time zero no-arbitrage price of a payment R_{n-1} made at time n is $B_{0,n} F_{0,n-1}$. Hence, the time zero price of an the payments[6] S_n at time n is

$$\begin{aligned} S_n &= \underbrace{K \cdot B_{0,n}}_{\text{time zero price of fixed}} - \underbrace{B_{0,n} F_{0,n-1}}_{\text{time zero price of floating}} \\ &= B_{0,n}(K - F_{0,n-1}) \end{aligned}$$

summing over all time periods n yields

$$Swap_m = \sum_{n=1}^{m} B_{0,n}(K - F_{0,n-1})$$

Using our result

$$F_{0,n-1} = \frac{B_{0,n-1} - B_{0,n}}{B_{0,n}}$$

[6]As defined above, $S_n = K - R_{n-1}$.

we get

$$
\begin{aligned}
Swap_m &= \sum_{n=1}^{m} B_{0,n}(K - F_{0,n-1}) \\
&= \sum_{n=1}^{m} B_{0,n}\left(K - \frac{B_{0,n-1} - B_{0,n}}{B_{0,n}}\right) \\
&= \sum_{n=1}^{m} B_{0,n}K - (\underbrace{B_{0,n-1} - B_{0,n}}_{\text{telescoping}}) \\
&= \sum_{n=1}^{m} B_{0,n}K - (\underbrace{B_{0,0}}_{=1} - B_{0,m}) \\
&= K\sum_{n=1}^{m} B_{0,n} - (1 - B_{0,m})
\end{aligned}
$$

which is the quantity for $Swap_m$ to be proven. Now, setting $Swap_m = 0$ in order to find the m-period rate we find

$$
\begin{aligned}
0 &= Swap_m \\
&= \sum_{n=1}^{m} B_{0,n}(K - F_{0,n-1}) \\
&= K\sum_{n=1}^{m} B_{0,n} - \sum_{n=1}^{m} B_{0,n}F_{0,n-1} \\
\implies K\sum_{n=1}^{m} B_{0,n} &= \sum_{n=1}^{m} B_{0,n}F_{0,n-1} \\
\implies K &= \frac{\sum_{n=1}^{m} B_{0,n}F_{0,n-1}}{\sum_{n=1}^{m} B_{0,n}} \\
&= \frac{\sum_{n=1}^{m} B_{0,n}\frac{B_{0,n-1}-B_{0,n}}{B_{0,n}}}{\sum_{n=1}^{m} B_{0,n}} \\
&= \frac{\sum_{n=1}^{m} B_{0,n-1} - B_{0,n}}{\sum_{n=1}^{m} B_{0,n}} \\
&= \frac{B_{0,0} - B_{0,m}}{\sum_{n=1}^{m} B_{0,n}} \\
&= \frac{1 - B_{0,m}}{\sum_{n=1}^{m} B_{0,n}}
\end{aligned}
$$

and if the notional principal to the swap is \$1 we have that $K = SR_m$, as desired. $\square$

The formula given in the above theorem

$$Swap_m = \sum_{n=1}^{m} B_{0,n}(K - F_{0,n-1})$$

for the no-arbitrage price of an m-period swap is, unsurprisingly, consistent with risk-neutral pricing. In particular note that

$$\begin{aligned}
Swap_m &= \sum_{n=1}^{m} B_{0,n}(K - F_{0,n-1}) \\
&= \sum_{n=1}^{m} B_{0,n}K - B_{0,n}F_{0,n-1} \\
&= \sum_{n=1}^{m} B_{0,n}K - (B_{0,n-1} - B_{0,n}) \\
&= \sum_{n=1}^{m} \tilde{\mathbb{E}}_0[D_n]K - \Big(\tilde{\mathbb{E}}_0[D_{n-1}] - \tilde{\mathbb{E}}_0[D_n]\Big) \\
&= \sum_{n=1}^{m} \tilde{\mathbb{E}}[KD_n] - \Big(\tilde{\mathbb{E}}[D_n(1 + R_{n-1})] - \tilde{\mathbb{E}}[D_n]\Big) \\
&= \sum_{n=1}^{m} \tilde{\mathbb{E}}[KD_n] - \tilde{\mathbb{E}}[D_n(1 + R_{n-1}) - D_n] \\
&= \sum_{n=1}^{m} \tilde{\mathbb{E}}[KD_n] - \tilde{\mathbb{E}}[D_n(1 + R_n - 1)] \\
&= \sum_{n=1}^{m} \tilde{\mathbb{E}}[KD_n] - \tilde{\mathbb{E}}[D_nR_{n-1}] \\
&= \sum_{n=1}^{m} \tilde{\mathbb{E}}[KD_n - D_nR_{n-1}] \\
&= \sum_{n=1}^{m} \tilde{\mathbb{E}}[D_n(K - R_{n-1})] \\
&= \tilde{\mathbb{E}}\left[\sum_{n=1}^{m} D_n(K - R_{n-1})\right]
\end{aligned}$$

That is, we have that the time zero rational price of the m-period interest rate swap (net swap cash flows for the long/fixed position, $K - R_{n-1}$) is given by

$$Swap_m = \tilde{\mathbb{E}}_0\left[\sum_{n=1}^{m} D_n(K - R_{n-1})\right]$$

Notice that the above arguments for the no-arbitrage price of fixed income derivatives have been through the construction of hedging portfolios for the short position. In general, whenever a hedging portfolio is successfully constructed the application of risk-neutral pricing is justified.

1.1.4 Caps & Floors

The remainder of this section discusses the risk-neutral pricing of interest rate caps & floors (options on interest rates). Although we can typically construct a hedging portfolio for short positions in these securities, we do not[7] work out the assumptions that would guarantee this is possible.

Definition 5. Let m be given with $1 \leq m \leq N$. A m-period interest rate cap is a contract that makes payments $C_1, ..., C_m$ at times $1, ..., m$ where each payment is given by

$$C_n = (R_{n-1} - K)^+, \quad n = 1, ..., m$$

An m-period interest rate floor is a contract that makes payments $F_1, ..., F_m$ at times $1, ..., m$ where

$$F_n = (K - R_{n-1})^+, \quad n = 1, ..., m$$

A contract which makes a single payment C_n at only time n is called an interest rate caplet.

A contract which makes a single payment F_n at only time n is called an interest rate floorlet.

We can show that the risk-neutral price at time zero of an m period interest rate cap is

$$Cap_m = \tilde{\mathbb{E}} \left[\sum_{n=1}^{m} D_n (R_{n-1} - K)^+ \right]$$

If we have are paying a variable rate loan presently at R_{n-1} then an interest rate cap will effectively "cap" the interest rate we are obliged to pay at K: Whenever the interest owed R_{n-1} is in excess of K we receive the rate

$$(R_{n-1} - K)^+ = R_{n-1} - K$$

with a net rate owed of K at time n. Similarly, we can show that the risk-neutral price at time zero of an m period interest rate cap is

$$Floor_m = \tilde{\mathbb{E}} \left[\sum_{n=1}^{m} D_n (R_{n-1} - K)^+ \right]$$

[7] Do we not do this because it's impossible or just not within the scope of the section?

and if are receiving a variable rate loan presently at R_{n-1} then an interest rate floor will effectively "floor" the interest rate we will receive at a guaranteed minimum rate K: Whenever the interest paid R_{n-1} is less than K we receive

$$(K - R_{n-1}^{+} = K - R_{n-1}$$

with a net rate received of K at time n.

Note that we have the relationship

$$\underbrace{K - R_{n-1}}_{\text{swap}} + \underbrace{(R_{n-1} - K)^{+}}_{\text{cap}} = \underbrace{(K - R_{n-1})^{+}}_{\text{floor}}$$

That is, holding a swap and a cap is equivalent to holding a floor:

$$Swap_m + Cap_m = Floor_m$$

If we have that $K = SR_m$ then $Swap_m = 0$ and the cap and floor on the same underlying rate will have the same price

$$K = SR_m \implies Swap_m = 0 \implies Cap_m = Floor_m$$