Mathematical & Computational Finance I
Lecture Notes

Generalizing American Derivative Securities

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1 General American Derivative Securities

In the previous notes we had proven the following propositions: Given the American derivative security price process defined by

$$V_n = \max_{\tau \in \mathcal{S}_n} \tilde{\mathbb{E}}_n \left[\mathbb{1}_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right]$$

has the following properties

(i) $V_n \geq \max(G_n, 0)$

(ii) $\left\{ \frac{V_n}{(1+r)^n} \right\}_{n=0}^{N}$ is a $\tilde{\mathbb{P}}$-supermartingale

(iii) If Y_n is any other process satisfying (i) and (ii) then

$$Y_n \geq V_n \quad \forall\, n = 0, 1, ..., N$$

Property (i) gives us that if the short party construct a hedging portfolio then all possible can be hedged regardless of when the long party chooses to exercise.

Property (ii) gives us that a short party with initial capital V_0 is able to construct a hedging portfolio with time n value given by V_n.

Combining (i) and (ii) together give us that the initial price V_0 is an acceptable price for the short party (seller).

Finally, with property (iii) we have that V_0 is the least price the short party (seller) is willing to accept for such a derivative and therefore is defined to be the fair price for the long party (buyer).

1.1 Computational Considerations

We saw last time that performing the maximization over all $\tau \in \mathcal{S}_n$ can be computationally intensive/problematic. We can show that we are able to generalize the path-independent American (recursive) pricing algorithm to path-dependent American derivative securities.

Theorem 1. Recursive Pricing Algorithm for Path-Dependent American Derivative Securities. Let V_n be the American derivative security price process defined by

$$V_n = \max_{\tau \in \mathcal{S}_n} \tilde{\mathbb{E}}_n \left[\mathbb{1}_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right]$$

then V_n satisfies

American pricing algorithm 2:

$$\begin{aligned} V_N(\omega_1 \cdots \omega_N) &= \max\{G_N(\omega_1 \cdots \omega_N), 0\} \\ V_n(\omega_1 \cdots \omega_n) &= \max\left\{ G_n(\omega_1 \cdots \omega_n), \frac{1}{1+r} \left[\tilde{p} V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q} V_{n+1}(\omega_1 \cdots \omega_n T)\right] \right\} \end{aligned}$$

for $n = N-1, N-2, ..., 1, 0$.

Proof. First, we wish to show that V_n defined by the algorithm satisfies both properties (i) and (ii) of our first result above. Then we must show that this V_n is indeed the smallest process satisfying (i) and (ii) implying that V_n must be the value corresponding to the stopping time τ for which this $\tau \in \mathcal{S}_n$ maximizes the conditional expectation. So, confirming property (i) we seek to show that

$$V_n \geq \max(G_n, 0) \quad \forall\, n = 0, 1, ..., N$$

By the algorithm above we have that, at time $n = N$,

$$V_N = \max(G_N, 0) \implies V_N \geq \max(G_N, 0)$$

We proceed inductively (backwards in time). Suppose for some $n \in \{N-1, N-2, ..., 1, 0\}$ we have that

$$V_{n+1} \geq \max(G_{n+1}, 0)$$

Then, by our algorithm we have

$$\begin{aligned} V_n(\omega_1 \cdots \omega_n) &= \max\left\{ G_n(\omega_1 \cdots \omega_n), \frac{1}{1+r} \Big[\tilde{p} V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q} V_{n+1}(\omega_1 \cdots \omega_n T)\Big] \right\} \\ &\geq \max\left\{ G_n(\omega_1 \cdots \omega_n), \frac{1}{1+r} \Big[\tilde{p} \cdot \max\left\{G_{n+1}(\omega_1 \cdots \omega_n H), 0\right\} + \right. \\ &\qquad\qquad \left. \tilde{q} \cdot \max\left\{G_{n+1}(\omega_1 \cdots \omega_n T), 0\right\} \Big] \right\} \\ &\qquad\qquad\qquad \text{(by the inductive hypothesis)} \\ &\geq \max\left\{ G_n(\omega_1 \cdots \omega_n), \frac{1}{1+r} \Big[\tilde{p} \cdot 0 + \tilde{q} \cdot 0\Big] \right\} \quad \text{(since we consider the lower bound)} \\ &\geq \max\left\{G_n(\omega_1 \cdots \omega_n), 0\right\} \end{aligned}$$

Since the choice of $n \in \{N-1, N-2, ..., 1, 0\}$ was arbitrary we have that

$$V_n \geq \max(G_n, 0) \quad \forall\, n = 0, 1, ..., N$$

satisfying (i), as desired. Now, we must show that this process defined by the algorithm above is a discounted $\tilde{\mathbb{P}}$-supermartingale. From the algorithm we have

$$\begin{aligned}
V_n(\omega_1 \cdots \omega_n) &= \max\left\{G_n(\omega_1 \cdots \omega_n), \frac{1}{1+r}\left[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)\right]\right\} \\
&\geq \frac{1}{(1+r)}\Big[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)\Big] \\
&= \tilde{E}_n\left[\frac{1}{1+r}V_{n+1}\right](\omega_1 \cdots \omega_n) \\
\implies \frac{V_n(\omega_1 \cdots \omega_n)}{(1+r)^n} &\geq \tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right](\omega_1 \cdots \omega_n)
\end{aligned}$$

satisfying the supermartingale property, and since V_n as defined by the algorithm above clearly depends on only the first n coin tosses we may conclude that V_n is indeed a supermartingale, as desired. Finally, we must show that this process is the "smallest" process satisfying (i) and (ii). First, by the algorithm we have that

$$V_N(\omega_1 \cdots \omega_N) = \max\{G_N(\omega_1 \cdots \omega_N), 0\}$$

Then it is clear that V_N is indeed the smallest at time N such that

$$V_N \geq \max\{G_N, 0\}$$

We once again proceed inductively (backwards). Assume that for some $n \in \{N-1, ..., 1, 0\}$ that V_{n+1} is as small as possible while satisfying (i) and (ii). Then, (ii) (supermartingale) gives us that

$$\begin{aligned}
V_n(\omega_1 \cdots \omega_n) &\geq \tilde{\mathbb{E}}_n\left[\frac{1}{1+r}V_{n+1}\right](\omega_1 \cdots \omega_n) \\
&= \frac{1}{1+r}\Big[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)\Big]
\end{aligned}$$

and by (i) we have

$$V_n(\omega_1 \cdots \omega_n) \geq G_n(\omega_1 \cdots \omega_n)$$

Hence, combining the above two inequalities,

$$V_n(\omega_1 \cdots \omega_n) \geq \max\left\{G_n(\omega_1 \cdots \omega_n), \frac{1}{1+r}\Big[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)\Big]\right\}$$

for $n = N-1, ..., 0$. However, the American pricing algorithm above gives us that

$$V_n(\omega_1 \cdots \omega_n) = \max\left\{G_n(\omega_1 \cdots \omega_n), \frac{1}{1+r}\Big[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)\Big]\right\}$$

Hence V_n is as small as possible, for arbitrary $n \in \{0, 1, ..., N\}$, as desired. $\square$

This result gives us an alternative algorithm to the maximization over all $\tau \in \mathcal{S}_n$ by using an algorithm that is essentially identical to the algorithm for the path-independent American option.

1.2 Replication

1.2.1 Short Position

It remains to show exactly how we can replicate/hedge the payoff of a path-dependent American derivative security, as well as finding the optimal exercise strategy for the long party.

Theorem 2. Replication of Path-Dependent American Derivative Securities. Consider an N-period binomial asset pricing model with $0 < d < 1 + r < u$. For each $n = 0, 1, ..., N$ let G_n be an adapted random variable depending on the first n coin tosses. Let V_n be the American derivative security price process defined by

$$V_n = \max_{\tau \in \mathcal{S}_n} \tilde{\mathbb{E}}_n \left[\mathbb{1}_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau - n}} G_\tau \right]$$

and define Δ_n and C_n such that

$$\begin{aligned}
\Delta_n(\omega_1 \cdots \omega_n) &= \frac{V_{n+1}(\omega_1 \cdots \omega_n H) - V_{n+1}(\omega_1 \cdots \omega_n T)}{S_{n+1}(\omega_1 \cdots \omega_n H) - S_{n+1}(\omega_1 \cdots \omega_n T)} \\
C_n(\omega_1 \cdots \omega_n) &= V_n(\omega_1 \cdots \omega_n) - \frac{1}{1+r}\Big[\tilde{p} V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q} V_{n+1}(\omega_1 \cdots \omega_n T)\Big]
\end{aligned}$$

for $n = 0, 1, ..., N - 1$. Then

1. $C_n \geq 0$ for all n.

Furthermore, if we set $X_0 = V_0$ and define recursively

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)[X_n - C_n - \Delta_n S_n]$$

then we get

(ii) $X_n(\omega_1 \cdots \omega_n) = V_n(\omega_1 \cdots \omega_n)$ for all $n = 0, 1, ..., N$ and all $\omega_1 \cdots \omega_n \in \Omega$.

(iii) In particular, $X_n \geq G_n$.

Proof. ***(Proof of (i))*** In the last result we had shown that V_n as defined by the recursive algorithm is a supermartingale. That is

$$V_n(\omega_1 \cdots \omega_n) \geq \tilde{\mathbb{E}}_n \left[\frac{V_{n+1}}{1+r} \right] (\omega_1 \cdots \omega_n)$$

and by the definition of C_n we have

$$\begin{aligned}
C_n(\omega_1 \cdots \omega_n) &= V_n(\omega_1 \cdots \omega_n) - \frac{1}{1+r}\Big[\tilde{p} V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q} V_{n+1}(\omega_1 \cdots \omega_n T)\Big] \\
&= V_n(\omega_1 \cdots \omega_n) - \tilde{\mathbb{E}}_n \left[\frac{V_{n+1}}{1+r} \right] (\omega_1 \cdots \omega_n) \\
&\geq V_n(\omega_1 \cdots \omega_n) - V_n(\omega_1 \cdots \omega_n) \\
&= 0 \\
\implies C_n &\geq 0
\end{aligned}$$

as desired. ∎

Proof. ***(Proof of (ii))*** We will prove this by induction on n. For $n = 0$ we have that $V_0 = X_0$ is trivially true by definition. Now assume that

$$X_n(\omega_1 \cdots \omega_n) = V_n(\omega_1 \cdots \omega_n) \quad \forall\, n = 0, 1, ..., N,\ \forall\, \omega_1 \cdots \omega_n \in \Omega$$

Then, by the definition of C_n we have that

$$C_n(\omega_1 \cdots \omega_n) = V_n(\omega_1 \cdots \omega_n) - \frac{1}{1+r}\Big[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)\Big]$$

$$\implies V_n(\omega_1 \cdots \omega_n) - C_n(\omega_1 \cdots \omega_n) = \frac{1}{1+r}\Big[\tilde{p}V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \cdots \omega_n T)\Big]$$

Assume $\omega_{n+1} = H$. Using the wealth equation for American derivatives we have

$$\begin{aligned}
X_{n+1}(H) &= \Delta_n S_{n+1}(H) + (1+r)[X_n - C_n - \Delta_n S_n] \\
&= \Delta_n S_{n+1}(H) + (1+r)[V_n - C_n - \Delta_n S_n] \quad \text{(by the inductive hypothesis)} \\
&= \Delta_n S_{n+1}(H) + (1+r)\left[\frac{1}{1+r}\left(\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)\right) - \Delta_n S_n\right] \\
&= \Delta_n S_{n+1}(H) + [\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) - (1+r)\Delta_n S_n] \\
&= \Delta_n S_{n+1}(H) - (1+r)\Delta_n S_n + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= \Delta_n u S_n - (1+r)\Delta_n S_n + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= \Delta_n S_n(u - (1+r)) + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} S_n(u - (1+r)) + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= \Big[V_{n+1}(H) - V_{n+1}(T)\Big]\frac{u - (1+r)}{u-d} + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)
\end{aligned}$$

and recalling that

$$\tilde{p} = \frac{(1+r) - d}{u-d}$$
$$\tilde{q} = \frac{u - (1+r)}{u-d}$$

we find

$$\begin{aligned}
X_{n+1}(H) &= \Big[V_{n+1}(H) - V_{n+1}(T)\Big]\tilde{q} + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T) + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(H) \\
&= V_{n+1}(H) \\
\implies X_{n+1}(H) &= V_{n+1}(H)
\end{aligned}$$

We can show an equivalent result holds for $\omega_{n+1} = T$. Therefore, since the choice for n was arbitrary, we have that by induction

$$X_n(\omega_1 \cdots \omega_n) = V_n(\omega_1 \cdots \omega_n)$$

∎

Proof. ***(Proof of (iii))*** Finally, we seek to confirm that

$$X_n \geq G_n$$

Since we have just shown that

$$X_n(\omega_1 \cdots \omega_n) = V_n(\omega_1 \cdots \omega_n)$$

we may use this result together with the result $V_n \geq \max(G_n, 0) \geq G_n$ to give

$$X_n \geq G_n$$

as desired. □

Therefore, by $(i), (ii)$, and (iii) we have that we may replicated a general American derivative security, and thereby hedge our **short** position.

1.2.2 Long Position

We now wish to consider the perspective of a long party in an American derivative contract. Fix $n \leq N$ and suppose that the American derivative security has not yet been exercised. Let $\tau^* \in \mathcal{S}_n$ be the optimal exercise satisfying

$$V_n = \tilde{\mathbb{E}}_n \left[\mathbb{1}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^* - n}} G_{\tau^*} \right]$$

For $k = n, n+1, ..., N$ define

$$C_k = \mathbb{1}_{\{\tau^* = k\}} G_k$$

If the long party exercises according to the optimal stopping time τ^* then the long party will receive cash flows

$$C_n, C_{n+1}, ..., C_N$$

at times $n, n+1, ..., N$, and we find that at most one of the $C_k > 0$. We find that if the option is exercised at or before time N then the C_k corresponding to the time at which the derivative is exercised will be the only nonzero cash flow. However, along different paths $\omega_1 \cdots \omega_N$ the payment C_k may come at different times. So, the value of this cash-flow stream at time n is given by

$$\begin{aligned} V_n &= \tilde{\mathbb{E}}_n \left[\sum_{k=n}^{N} \mathbb{1}_{\{\tau^* = k\}} \frac{1}{(1+r)^{k-n}} G_k \right] \\ &= \tilde{\mathbb{E}}_n \left[\sum_{k=n}^{N} \frac{C_k}{(1+r)^{k-n}} \right] \end{aligned}$$

This is essentially just a present value calculation, as was discussed in the later portion of Chapter 2, of the cash flows received at times $n, n+1, ..., N$. Once the long party selects the exercise strategy according to τ^* then the conditional expectation of the cash flows above is precisely the contract given by the derivative. Therefore V_n must be an acceptable price for the long position in this contract.

1.2.3 Optimal Exercise

It remains to show precisely how the long party of an American derivative security may find the optimal exercise strategy. We will consider the problem of selecting an optimal exercise strategy/time given by $\tau^* \in \mathcal{S}_0$ at time $n = 0$.

Theorem 3. Optimal Exercise. With

$$V_0 = \max_{\tau \in \mathcal{S}_0} \tilde{\mathbb{E}} \left[\mathbb{1}_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau}} G_{\tau} \right]$$

and

$$\tau^* = \min \{n : V_n = G_n\}$$

then

$$V_0 = \tilde{\mathbb{E}} \left[\mathbb{1}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*} \right]$$

That is, the τ^* maximizing our conditional expectation is given by the first time for which the risk-neutral price of the option is equal to its intrinsic value. If no such time exists (i.e. minimizing for the first time $V_n = G_n$ over the empty set $\emptyset$) then $\tau^* = \infty$.

Proof. Consider the stopped process

$$\left\{ \frac{V_{n\wedge\tau^*}}{(1+r)^{n\wedge\tau^*}} \right\}_{n=0}^{N}$$

Clearly our process is adapted since V_n is adapted, τ^* is adapted, and stopping a process does not introduce any dependency on further coin tosses. Now, fix the first n tosses $\omega_1 \cdots \omega_n$ and suppose we have not yet exercised the derivative (i.e. $\tau^* \geq n+1$). Then

$$\tau^* \geq n+1 \implies V_n(\omega_1 \cdots \omega_n) > G_n(\omega_1 \cdots \omega_n)$$

must be true by the definition of the optimal/maximizing stopping time τ^*. By the recursive American algorithm we have

$$\begin{aligned} V_{n\wedge\tau^*}(\omega_1 \cdots \omega_n) &= V_n(\omega_1 \cdots \omega_n) \\ &= \max \left\{ G_n(\omega_1 \cdots \omega_n), \frac{1}{1+r} \left[\tilde{p} V_{n+1}(\omega_1 \cdots \omega_n H) + \tilde{q} V_{n+1}(\omega_1 \cdots \omega_n T) \right] \right\} \end{aligned}$$

But $V_n > G_n$ (shown above) so

$$\begin{aligned}
V_{n\wedge\tau^*}(\omega_1\cdots\omega_n) &= \max\left\{G_n(\omega_1\cdots\omega_n), \frac{1}{1+r}\Big[\tilde{p}V_{n+1}(\omega_1\cdots\omega_n H) + \tilde{q}V_{n+1}(\omega_1\cdots\omega_n T)\Big]\right\} \\
&= \frac{1}{1+r}\Big[\tilde{p}V_{n+1}(\omega_1\cdots\omega_n H) + \tilde{q}V_{n+1}(\omega_1\cdots\omega_n T)\Big] \quad \textbf{\textit{(I don't see this...)}} \\
&= \frac{1}{1+r}\Big[\tilde{p}V_{(n+1)\wedge\tau^*}(\omega_1\cdots\omega_n H) + \tilde{q}V_{(n+1)\wedge\tau^*}(\omega_1\cdots\omega_n T)\Big] \quad \text{(since } \tau^* \geq n+1\text{)} \\
&= \tilde{\mathbb{E}}\left[\frac{V_{(n+1)\wedge\tau^*}}{1+r}\right](\omega_1\cdots\omega_n) \\
\implies V_{n\wedge\tau^*}(\omega_1\cdots\omega_n) &= \tilde{\mathbb{E}}\left[\frac{V_{(n+1)\wedge\tau^*}}{1+r}\right](\omega_1\cdots\omega_n) \\
\implies \frac{V_{n\wedge\tau^*}}{(1+r)^n} &= \tilde{\mathbb{E}}\left[\frac{V_{(n+1)\wedge\tau^*}}{(1+r)^{n+1}}\right]
\end{aligned}$$

so, the stopped process $\left\{\frac{V_{n\wedge\tau^*}}{(1+r)^{n\wedge\tau^*}}\right\}_{n=0}^{N}$ is both adapted and satisfies the martingale property. Therefore, the process is indeed a $\tilde{\mathbb{P}}$-martingale. If, on the other hand, along the path $\omega_1\cdots\omega_n$ we have that

$$\tau^* \leq n$$

Then

$$\begin{aligned}
V_{n\wedge\tau^*}(\omega_1\cdots\omega_n) &= V_{\tau^*}(\omega_1\cdots\omega_{\tau^*}) \\
&= \tilde{p}V_{\tau^*}(\omega_1\cdots\omega_{\tau^*}) + \tilde{q}V_{\tau^*}(\omega_1\cdots\omega_{\tau^*}) \\
&= \tilde{p}V_{(n+1)\wedge\tau^*}(\omega_1\cdots\omega_{\tau^*}\cdots\omega_n H) + \tilde{q}V_{\tau^*}(\omega_1\cdots\omega_{(n+1)\wedge\tau^*}\cdots\omega_n T) \\
&= \tilde{\mathbb{E}}_n\left[V_{(n+1)\wedge\tau^*}\right](\omega_1\cdots\omega_n)
\end{aligned}$$

However, note that on the path $\omega_1\cdots\omega_n$, for $\tau^* \leq n$, we have

$$(1+r)^{(n+1)\wedge\tau^*} = (1+r)^{n\wedge\tau^*} \quad \left(= (1+r)^{\tau^*}\right)$$

Therefore

$$\frac{V_{n\wedge\tau^*}}{(1+r)^{n\wedge\tau^*}} = \tilde{\mathbb{E}}_n\left[\frac{V_{(n+1)\wedge\tau^*}}{(1+r)^{(n+1)\wedge\tau^*}}\right]$$

So, the stopped process is indeed a martingale for both cases of τ^*. Using this result we may state

$$\begin{aligned}
V_0 &= \tilde{\mathbb{E}}_0\left[\frac{1}{(1+r)^{N\wedge\tau^*}}V_{N\wedge\tau^*}\right] \\
&= \tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau^*\leq N\}}\frac{1}{(1+r)^{\tau^*}}G_{\tau^*}\right] + \tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau^*=\infty\}}\frac{1}{(1+r)^N}V_N\right]
\end{aligned}$$

Now, since $V_N = \max(G_N, 0)$ we have that on all paths for which $\tau^* = \infty$, that is, paths for which we **do not** exercise (paths for which the option value > intrinsic value)

$$V_N > G_N$$

However, since $V_N = \max(G_N, 0)$, this may only happen when

$$G_N < 0, \quad V_N = 0$$

Hence

$$\mathbb{1}_{\{\tau^*=\infty\}} V_N = 0$$

So then we may rewrite our above equation

$$\begin{aligned} V_0 &= \tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*}\right] + \tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau^*=\infty\}} \frac{1}{(1+r)^N} V_N\right] \\ &= \tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*}\right] \end{aligned}$$

as desired. That is, $\tau^* = \min\{n : V_N = G_n\}$ satisfies

$$V_0 = \tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*}\right]$$

$\square$