Mathematical & Computational Finance I
Lecture Notes

Utility Maximization & CAPM

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1 Utility Maximization

Example: For $U(x) = 1 - e^{-ax}$, for $a > 0$, find the optimal solution to the portfolio optimization problem/utility maximization problem.

We know that the utility maximization problem is equivalent to Problem C in the previous notes. So, consider a parameterization of the sample space

$$\Omega = \left\{\omega^1, ..., \omega^M\right\}$$

where each ω^m corresponds to a possible coin toss sequence $\omega^m = \omega_1\omega_2 \cdots \omega_N$ and define (for brevity)

$$\begin{aligned}
\zeta_m &= \zeta_N(\omega^m) \\
p_m &= \mathbb{P}(\omega^m) \\
x_m &= X_N(\omega^m)
\end{aligned}$$

Recall that Problem C specified in the previous notes states: Given X_0, find a vector $(x_1, ..., x_M)$ that maximizes

$$\mathbb{E}[U(X_N)] = \sum_{m=1}^{M} p_m U(x_m)$$

subject to the constraint $\mathbb{E}[\zeta_N X_N] = \sum_{m=1}^{M} p_m \zeta_m x_m = X_0$. Let f be the objective function, λ the Lagrange multiplier, and g the constraint, then the Lagrangian L for this problem is

$$\begin{aligned}
L &= f - \lambda g \\
L &= \mathbb{E}[U(X_N)] - \lambda(\mathbb{E}[\zeta_N X_N] - X_0) \\
L(x_1, ..., x_m, \lambda) &= \sum_{m=1}^{M} p_m U(x_m) - \lambda \left(\sum_{m=1}^{M} p_m \zeta_m x_m - X_0 \right)
\end{aligned}$$

Then

$$
\begin{aligned}
&\nabla L = 0 \\
\iff &\frac{\partial L}{\partial x_m} = p_m a e^{-a x_m} - \lambda p_m \zeta_m \\
\implies &e^{-a x_m} = \frac{\lambda}{a}\zeta_m
\end{aligned}
$$

Solving for $x_m = X_N(\omega^m)$ gives us

$$
\begin{aligned}
x_m &= -\frac{1}{a}\ln\left(\frac{\lambda}{a}\zeta_m\right) \\
&= -\frac{1}{a}\ln\left(\frac{\lambda}{a}\right) - \frac{1}{a}\ln\zeta_m
\end{aligned}
$$

and substituting this solution for x_m into the constraint $\mathbb{E}[\zeta_N X_N] = X_0$

$$
\begin{aligned}
X_0 &= \sum_{m=1}^{M} p_m \zeta_m x_m \\
&= \sum_{m=1}^{M} p_m \zeta_m \left[-\frac{1}{a}\ln\left(\frac{\lambda}{a}\right) - \frac{1}{a}\ln\zeta_m\right] \\
&= \left[-\frac{1}{a}\ln\left(\frac{\lambda}{a}\right)\right]\sum_{m=1}^{M} p_m\zeta_m - \frac{1}{a}\sum_{m=1}^{M} p_m\zeta_m\ln\zeta_m \\
&= \left[-\frac{1}{a}\ln\left(\frac{\lambda}{a}\right)\right]\mathbb{E}[\zeta] - \frac{1}{a}\mathbb{E}[\zeta\ln\zeta] \\
&= \left[-\frac{1}{a}\ln\left(\frac{\lambda}{a}\right)\right]\mathbb{E}\left[\frac{Z}{(1+r)^N}\right] - \frac{1}{a}\mathbb{E}\left[\frac{Z}{(1+r)^N}\ln\left(\frac{Z}{(1+r)^N}\right)\right] \\
&= \left[-\frac{1}{a}\ln\left(\frac{\lambda}{a}\right)\right]\frac{\mathbb{E}[Z_n]}{(1+r)^N} - \frac{1}{a}\mathbb{E}\left[\frac{Z}{(1+r)^N}\ln\left(\frac{Z}{(1+r)^N}\right)\right] \\
&= \left[-\frac{1}{a}\ln\left(\frac{\lambda}{a}\right)\right]\frac{Z_0}{(1+r)^N} - \frac{1}{a}\mathbb{E}\left[\frac{Z}{(1+r)^N}\ln\left(\frac{Z}{(1+r)^N}\right)\right] \\
&= \left[-\frac{1}{a}\ln\left(\frac{\lambda}{a}\right)\right]\frac{1}{(1+r)^N} - \frac{1}{a}\frac{1}{(1+r)^N}\mathbb{E}\left[Z\ln\left(\frac{Z}{(1+r)^N}\right)\right] \\
\iff -\frac{1}{a}\ln\left(\frac{\lambda}{a}\right) &= X_0(1+r)^N + \frac{1}{a}\mathbb{E}\left[Z\ln\left(\frac{Z}{(1+r)^N}\right)\right]
\end{aligned}
$$

Now, note that the left-hand side to this equation is the only term containing the unknown λ and that this expression is close to our value for $x_m = X_N(\omega^m)$. In order to eliminate this unknown λ we may substitute the current form for the constraint into our expression for

$x_m \equiv X_N(\omega^m) = -\frac{1}{a}\ln\left(\frac{\lambda}{a}\right) - \frac{1}{a}\ln\zeta_m$. We find

$$\begin{aligned} x_m &= -\frac{1}{a}\ln\left(\frac{\lambda}{a}\right) - \frac{1}{a}\ln\zeta_m \\ &= \left(X_0(1+r)^N + \frac{1}{a}\mathbb{E}\left[Z\ln\left(\frac{Z}{(1+r)^N}\right)\right]\right) - \frac{1}{a}\ln\zeta_m \\ &= X_0(1+r)^N + \frac{1}{a}\mathbb{E}\left[Z\ln\left(\frac{Z}{(1+r)^N}\right)\right] - \frac{1}{a}\ln\zeta_m \end{aligned}$$

which gives us the terminal wealth X_N maximizing $\mathbb{E}[U(X_N)]$ subject to $\mathbb{E}[\zeta X] = X_0$. We could use backwards induction and our results in risk-neutral valuation to find the corresponding wealth process $\{X_n\}$ and portfolio process $\{\Delta_n\}$. We could stop here but instead let us look a little deeper at the result. We have

$$\begin{aligned} x_m &= X_0(1+r)^N + \frac{1}{a}\mathbb{E}\left[Z\ln\left(\frac{Z}{(1+r)^N}\right)\right] - \frac{1}{a}\ln\zeta_m \\ &= X_0(1+r)^N + \frac{1}{a}\mathbb{E}\left[Z\ln\left(\frac{Z}{(1+r)^N}\right)\right] - \frac{1}{a}\ln\left(\frac{Z_N(\omega^m)}{(1+r)^N}\right) \\ &= X_0(1+r)^N + \frac{1}{a}\tilde{\mathbb{E}}\left[\ln\left(\frac{Z}{(1+r)^N}\right)\right] - \frac{1}{a}\ln\left(\frac{Z_N}{(1+r)^N}\right) \quad (\text{since } \tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]) \\ &= X_0(1+r)^N + \left[\frac{1}{a}\tilde{\mathbb{E}}\left[\ln Z\right] - \frac{1}{a}\ln\left((1+r)^N\right)\right] - \left[\frac{1}{a}\ln Z_N - \frac{1}{a}\ln\left((1+r)^N\right)\right] \\ &= X_0(1+r)^N + \frac{1}{a}\tilde{\mathbb{E}}[\ln Z] - \frac{1}{a}\ln Z_N \end{aligned}$$

That is, the optimal attainable terminal utility is achieved at wealth

$$X_N^* = X_0(1+r)^N + \frac{1}{a}\left(\tilde{\mathbb{E}}[\ln Z] - \ln Z_N\right)$$

We may see that X_N^* is composed of a riskless bank account term accruing interest $X_0(1+r)^N$ and some risky term that is determined by our particular utility function $\frac{1}{a}\left(\tilde{\mathbb{E}}[\ln Z] - \ln Z_N\right)$. The question remains: How can we achieve this terminal wealth? We should treat X_N^* as the payoff of a derivative security at time N (i.e. set $V_N = X_N^*$). We replicate this payoff with the portfolio process

$$\Delta_{n-1}(\omega_1\cdots\omega_{n-1}) = \frac{V_n(\omega_1\cdots\omega_{n-1}H) - V_n(\omega_1\cdots\omega_{n-1}T)}{S_n(\omega_1\cdots\omega_{n-1}H) - S_n(\omega_1\cdots\omega_{n-1}T)}$$

Doing backwards induction/risk-neutral valuation at each time steps we can find each $X_n = V_n$ at each $n = 1, ..., N$ and since X_n is a martingale in our model we note

$$X_n = \tilde{\mathbb{E}}_n\left[\frac{X_N^*}{(1+r)^{N-n}}\right] \quad (\text{martingale property})$$

Now,

$$
\begin{aligned}
X_n &= \tilde{\mathbb{E}}_n\left[\frac{X_N^*}{(1+r)^{N-n}}\right] \\
&= \tilde{\mathbb{E}}_n\left[\frac{\left(X_0(1+r)^N + \frac{1}{a}\left(\tilde{\mathbb{E}}[\ln Z] - \ln Z_N\right)\right)}{(1+r)^{N-n}}\right] \\
&= \frac{1}{(1+r)^{N-n}}\left[X_0(1+r)^N + \frac{1}{a}\tilde{\mathbb{E}}_n\left[\tilde{\mathbb{E}}[\ln Z]\right] - \frac{1}{a}\tilde{\mathbb{E}}_n[\ln Z]\right] \\
&= \frac{1}{(1+r)^{N-n}}\left[X_0(1+r)^N + \frac{1}{a}\tilde{\mathbb{E}}[\ln Z] - \frac{1}{a}\tilde{\mathbb{E}}_n[\ln Z]\right] \quad \text{(tower property)}
\end{aligned}
$$

So, with $V_n = X_n$ we find

$$
\begin{aligned}
\Delta_{n-1}(\omega_1\cdots\omega_{n-1}) &= \frac{X_n(\omega_1\cdots\omega_{n-1}H) - X_n(\omega_1\cdots\omega_{n-1}T)}{(u-d)S_{n-1}(\omega_1\cdots\omega_{n-1})} \\
&= \frac{\tilde{\mathbb{E}}_n[\ln Z](\omega_1\cdots\omega_{n-1}T) - \tilde{\mathbb{E}}_n[\ln Z](\omega_1\cdots\omega_{n-1}H)}{a(1+r)^{N-n}(u-d)S_{n-1}(\omega_1\cdots\omega_{n-1}}
\end{aligned}
$$

since all constant terms in $X_n = \frac{1}{(1+r)^{N-n}}\left[X_0(1+r)^N + \frac{1}{a}\tilde{\mathbb{E}}[\ln Z] - \frac{1}{a}\tilde{\mathbb{E}}_n[\ln Z]\right]$ cancel each other out in the $(\omega_1\cdots\omega_{n-1}H)$ and $(\omega_1\cdots\omega_{n-1}T)$ terms. Hence, with this utility function we have an explicit portfolio process $\{\Delta_n\}_{n=1}^N$. Note that this portfolio process is initial-wealth invariant, that is, Δ_n is not a function of our initial endowment X_0.

End of midterm material

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2 CAPM

How can we relate our work in this chapter to the capital asset pricing model? Consider the one-step binomial model with the no-arbitrage require $0 < d < 1+r < u$. We know that for some $0 \le p \le 1$

$$\mathbb{E}[S_1] = pS_1(H) + qS_1(T)$$

and under $\tilde{\mathbb{P}}$

$$S_0 = \tilde{\mathbb{E}}\left[\frac{S_1}{1+r}\right] = \frac{1}{(1+r)}\left[\tilde{p}S_1(H) + \tilde{q}S_1(T)\right]$$

Define the return of the risk asset S to be

$$
\begin{aligned}
r_s := \frac{S_1 - S_0}{S_0} &= \begin{cases} \frac{S_1(H)-S_0}{S_0} & \omega_1 = H \\ \frac{S_1(T)-S_0}{S_0} & \omega_1 = T \end{cases} \\
&= \begin{cases} u-1 & \omega_1 = H \\ d-1 & \omega_1 = T \end{cases}
\end{aligned}
$$

Lemma 1. For any $p, \tilde{p} \in (0,1)$ we have

$$\begin{aligned}\mathbb{E}[r_s] - \tilde{\mathbb{E}}[r_s] &= (p - \tilde{p}) \left[\frac{S_1(H) - S_1(T)}{S_0} \right] \\ &= (p - \tilde{p})(u - d)\end{aligned}$$

Proof. to do... □

Note that $\tilde{\mathbb{E}}[r_s] = r$ by definition, so as corollary we get

Corollary 1.

1. $\mathbb{E}[r_s] - r = (p - \tilde{p})(u - d)$
2. $\mathbb{E}[r_s] - r_s(H) = (p - 1)(u - d)$
3. $\mathbb{E}[r_s] - r_s(T) = p(u - d)$

For the following suppose that given some measure $\mathbb{P}$ we have two traded assets S^1 and S^2.

Definition 1. The <u>covariance</u> of the returns for S^1 and S^2 as

$$\begin{aligned}V^{\mathbb{P}}_{S^1,S^2} &= \mathrm{Cov}^{\mathbb{P}}(r_{S^1}, r_{S^2}) \\ &= \mathbb{E}[r_{S^1}, r_{S^2}] - \mathbb{E}[r_{S^1}]\mathbb{E}[r_{S^2}]\end{aligned}$$

Lemma 2. We can rewrite the covariance of two assets as

$$V^{\mathbb{P}}_{S^1,S^2} = p(1-p)[u_1 - d_1][u_2 - d_2]$$

Lemma 3. Assume that $\mathbb{E}[r_S] \geq r$. We can show that if $0 < p < 1$ then

$$\mathbb{E}[r_s] - r = \frac{|p - \tilde{p}|}{\sqrt{p(1-p)}} \sigma_S$$

where σ_S is a measure of volatility/risk of the asset.

Proof. to do... □

We may interpret $\mathbb{E}[r_S] - r$ as the risk premium/expected return above the risk free rate for investing in S. If σ_S is higher we would expect a higher return for our investment, and when $p = \tilde{p}$ we are insensitive to risk (i.e. risk-neutral). Define

$$\Lambda(p) = \frac{|p - \tilde{p}|}{\sqrt{p(1-p)}}$$

So we have

$$S_0 = \frac{\mathbb{E}[S_1]}{(1 + r) + \Lambda(p)\sigma_S}$$

That is, discounting must be risk adjusted if we use the expectation under the real-world measure. We can find that

$$\mathbb{E}[r_{S^1}] - r = \beta_{S^1,S^2} \left(\mathbb{E}[r_{S^2}] - r\right)$$

where

$$\begin{aligned}\beta_{S^1,S^2} &= \frac{\mathrm{Cov}^{\mathbb{P}}(S^1, S^2)}{\mathrm{Var}^{\mathbb{P}}(S^1, S^2)} \\ &= \frac{V^{\mathbb{P}}_{S^1,S^2}}{V^{\mathbb{P}}_{S^1,S^2}}\end{aligned}$$

We say that β is a "regression coefficient" for the returns of S^1 onto those of S^2. Usually we have that S^2 is a market index or some proxy for the whole market. Doing so permits us to find idiosyncratic risks associated with a particular asset S^1. Note that we can write[1]

$$S^1_0 = \frac{\mathbb{E}[S^1_1]}{(1+r) + \beta_{S^1,S^2}(\mathbb{E}[r_{S^2}] - r)}$$

where we say that S^1_0 is a "relative pricing formula" (under a subjective measure $\mathbb{P}$). This means that given some information about S^2 we can price S^1, assuming we know the correlation[2] of the returns between S^1 and S^2. We may derive more general versions of CAPM using "representative agents" and the utility maximization tools that we had previously explored. Doing so gives us an "equilibrium pricing model" which is explored in greater depth in the Shreve book but will not be discussed here.

[1]How? Where does this come from? Is this some substitution?

[2]Often we make the simplifying assumption that the correlations between assets are constant in time – this is typically a bad idea in practice.