Real Analysis
Lecture Notes

The Real Number System

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1 Open and Closed Sets of $\mathbb{R}$ (con't 4)

1.1 Heine-Borel Theorem (con't)

Last class we started the Heine-Borel Theorem and managed to prove the first case, that closed-bounded intervals of real numbers, $F = [a, b]$, $a < b$, are indeed compact sets. We now extend this result to the case where F is closed and bounded, but not necessarily a connected interval.

Theorem: *(Heine-Borel Theorem (con't)*

Proof. ***(Case 2)***: F is closed and bounded, but not of the form $[a, b]$ (i.e., F is not necessarily a connected set).

Take F to be a closed and bounded set of real numbers. Since F is bounded there exists real numbers a and b such that $F \subset [a, b]$. Consider an open cover of F given by

$$F \subset \bigcup_{i \in I} O_i$$

Since F is closed its complement F^c must be open. Thus,

$$\{O_i\}_{i \in I} \cup \{F^c\}$$

forms an open covering. In fact, since

$$\mathbb{R} = F^c \cup F \subset F^c \cup \{O_i\}_{i \in I}$$

we see that this open cover $\{O_i\}_{i \in I} \cup \{F^c\}$ forms a cover for all of $\mathbb{R}$, and so it must also form an open cover for $[a, b]$. Since $[a, b]$ is a closed bounded interval we may conclude that $\{O_i\}_{i \in I} \cup \{F^c\}$ has a finite subcover such that

$$[a, b] \subset \left(\bigcup_{i=1}^{n} O_i \right) \cup F^c \subset \left(\bigcup_{i \in I} O_i \right) \cup F^c$$

and so this finite cover must also cover F. However, F^c clearly plays no role in covering F. Thus,

$$F \subset \bigcup_{i=1}^{n} O_i \subset \bigcup_{i \in I} O_i$$

That is, F is compact, as desired. $\square$

2 Borel Sets

As we have seen, we typically denote open sets by O and closed sets by F. We also consider the following notations

$$\begin{array}{ll} F_\sigma & \text{a countable union of closed sets} \\ G_\delta & \text{a countable intersection of open sets} \end{array}$$

where F is from the French *fermé*, σ from the French *somme*, G from the German *gebiet* (area, neighbourhood), and δ from the German *durchschnitt* (intersect). What is the relationship between F_σ sets and G_δ sets? Note

$$\begin{aligned}
\mathbb{R} \setminus F_\sigma &= \mathbb{R} \setminus \left(\bigcup_{n=1}^{\infty} F_n \right) \\
&= \mathbb{R} \cap \left(\bigcup_{n=1}^{\infty} F_n \right)^c \\
&= \mathbb{R} \cap \left(\bigcap_{n=1}^{\infty} F_n^c \right) \\
&= \bigcap_{n=1}^{\infty} \mathbb{R} \cap F_n^c \\
&= \bigcap_{n=1}^{\infty} \mathbb{R} \setminus F_n
\end{aligned}$$

and since each F_n is closed we have that each $\mathbb{R} \setminus F_n$ is open. Thus,

$$\begin{aligned}
\mathbb{R} \setminus F_\sigma &= \bigcap_{n=1}^{\infty} \mathbb{R} \setminus F_n \\
&= \bigcap_{n=1}^{\infty} G_n \\
&= G_\delta
\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{R} \setminus G_\delta &= \mathbb{R} \setminus \left(\bigcap_{n=1}^{\infty} O_n \right) \\ &= \bigcup_{n=1}^{\infty} \mathbb{R} \setminus O_n \\ &= F_\sigma\end{aligned}$$

Therefore, if you know what your F_σ set is then you know the corresponding G_δ, and vice-versa. In a previous assignment we had shown that the set of real numbers $\mathbb{R}$ is both an open and a closed set. Thus, $\mathbb{R}$ is both a F_σ and a G_δ.

What about the rationals $\mathbb{Q}$? Well we may write Q as

$$\begin{aligned}\mathbb{Q} &= \bigcup_{q \in \mathbb{Q}} \{q\} \\ \implies \mathbb{Q} &= F_\sigma\end{aligned}$$

since $\mathbb{Q}$ is countable. In fact, any countable set A is a F_σ set since we may write A as

$$\begin{aligned}A &= \bigcup_{a \in \mathbb{Q}} \{a\} \\ \implies A &= F_\sigma\end{aligned}$$

On the other hand, the set of irrationals $\mathbb{R} \setminus \mathbb{Q}$ can be written as

$$\begin{aligned}\mathbb{R} \setminus \mathbb{Q} &= \mathbb{R} \setminus F_\sigma \quad \text{since } \mathbb{Q} = F_\sigma \\ &= G_\delta \quad \text{(shown above)}\end{aligned}$$

It's not obvious but we can prove that, unlike $\mathbb{R}$, $\mathbb{Q}$ is **not** a G_δ and that $\mathbb{R} \setminus \mathbb{Q}$ is **not** a F_σ,[1] and so we have some examples of sets that are one but not the other.

Consider the open interval $(a, b) = \{x \ : \ a < x < b, \ a, b \in \mathbb{R}\}$ and the nested closed interval $\left[a + \frac{1}{n}, b - \frac{1}{n}\right]$ for $n \in \mathbb{N}$. As $n \to \infty$ we should see that we are expanding are closed set towards (a, b). Consider the countable union

$$\bigcup_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

This union cannot contain a or b since for any $n \in \mathbb{N}$ we have $a < a + \frac{1}{n}$ and $b - \frac{1}{n} < b$. However, for any $x \in (a, b)$ there exists some $N \in \mathbb{N}$ sufficiently large such that

$$x \in \bigcup_{n=1}^{N} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

[1] I think this is known as the "Baire-Category Theorem".

Therefore,

$$\bigcup_{n=1}^{\infty}\left[a+\frac{1}{n}, b-\frac{1}{n}\right]=(a,b)$$

and so by definition the open interval (a,b) is a F_σ. Recall that every open set is at most a countable union of disjoint open intervals $I_x=(a_x,b_x)$.[2] Thus, every open set is a countable union of F_σ's, and so is itself a F_σ. That is, every open set in $\mathbb{R}$ is a F_σ set.

Taking the complement of this we symmetric result we get (since the complement of an open interval is closed and vice-versa): Every closed set in $\mathbb{R}$ is a G_δ. However, there turns out to be some F_σ sets that are not G_δ sets and G_δ sets that are not F_σ sets.

2.1 Borel Hierarchy

Consider countably many closed sets F and take their union F_σ. If we take a countable union of countably many such F_σ it remains a countable union. That is,

$$(F_\sigma)_\sigma = F_{\sigma\sigma} = F_\sigma$$

However, it is less immediately obvious what the countable intersection of countably many F_σ would be. That is, we can consider collections of the form

$$(F_\sigma)_\delta = F_{\sigma\delta}$$

Iterating on this process we can consider the collections

$$\begin{aligned}((F_\sigma)_\delta)_\sigma &= F_{\sigma\delta\sigma} \quad \text{or}\\ ((G_\delta)_\sigma)_\delta &= G_{\delta\sigma\delta}\end{aligned}$$

and so on. Doing this we find an easy way to construct bigger and bigger families from subsets of $\mathbb{R}$. That is, we are taking any closed sets and taking all countable unions and countable intersections in all the ways we can.

Let $f:\mathbb{R}\to\mathbb{R}$. It turns out that the set of points of $\mathbb{R}$ where f is continuous is a G_δ (countable intersection of open sets, e.g. the irrationals but *not* the rationals). So, we can find functions $f:\mathbb{R}\to\mathbb{R}$ that is continuous at every irrational point but discontinuous at every rational point $q\in\mathbb{Q}$. However, the converse is not true! There is no function $f:\mathbb{R}\to\mathbb{R}$ that is continuous at every rational point $q\in\mathbb{Q}$ and discontinuous at every irrational point $r\in\mathbb{R}\setminus\mathbb{Q}$.

[2] This comes from the fact that because we're over the reals we have some rational q such that $a_x \le q \le b_x$ so form a bijection from $\mathbb{Q}$ to $\{I_x\}$.

3 Continuous Functions

Definition: *(Continuous at a point)* Let $E \subset \mathbb{R}$ and consider $f : E \to \mathbb{R}$. We say that f is continuous at a point $x \in \mathbb{E}$ if

$$\forall \epsilon > 0,\ \exists \delta > 0,\ f(x-\delta, x+\delta) \subset (f(x)-\epsilon, f(x)+\epsilon)$$

or equivalently,

$$\forall \epsilon > 0,\ \exists \delta > 0,\ |x-y| < \delta \implies |f(x)-f(y)| < \epsilon$$

We can show that this second definition in indeed equivalent by manipulating the implication as follows

$$\begin{aligned}
|x-y| < \delta &\implies |f(x)-f(y)| < \epsilon \\
-\delta < x-y < \delta &\implies -\epsilon < f(x)-f(y) < \epsilon \\
y-\delta < x < y+\delta &\implies f(y)-\epsilon < f(x) < f(y) < \epsilon \\
x \in (y-\delta, y+\delta) &\implies f(x) \in (f(y)-\epsilon, f(y)+\epsilon)
\end{aligned}$$

Thus, we are guaranteed to map the interval $(y-\delta, y+\delta)$ within $(f(y)-\epsilon, f(y)+\epsilon)$. That is,

$$f(y-\delta, y+\delta) \subset (f(y)-\epsilon, f(y)+\epsilon)$$

as desired.

Definition: *(Continuous functions)* Let $E \subset \mathbb{R}$ and consider $f : E \to \mathbb{R}$. We say that f is continuous if f is continuous at each $x \in E$.

Proposition: Suppose $f : \mathbb{R} \to \mathbb{R}$. For all open subsets $O \subset \mathbb{R}$ the function f is continuous on $\mathbb{R}$ if and only if the inverse imagine $f^{-1}(O)$ is open on $\mathbb{R}$.[3]

Proof. ($\impliedby$) Suppose $f^{-1}(O)$ is open in $\mathbb{R}$ for all open sets $O \subset \mathbb{R}$. Take $x \in \mathbb{R}$ and let $\epsilon > 0$ be fixed. Consider the the interval

$$I = (f(x)-\epsilon, f(x)+\epsilon)$$

Since I is open by assumption the inverse image $f^{-1}(I)$ is open by hypothesis. By the openness of $f^{-1}(I)$ we have, for $x \in f^{-1}(I)$,

$$\exists \delta > 0,\ (x-\delta, x+\delta) \subset f^{-1}(I)$$

Now, by definition of f^{-1} we have for $x \in f^{-1}(I)$

$$f(x) \in I$$

hence

$$\begin{aligned}
f(x-\delta, x+\delta) &\subset I \\
f(x-\delta, x+\delta) &\subset (f(x)-\epsilon, f(x)+\epsilon)
\end{aligned}$$

[3]Note that we are now interested in when the inverse function f^{-1} is "well-behaved", which is perhaps surprising.

which is precisely the definition of the continuity of f, as desired.

($\implies$) Suppose f is continuous and let O be open for all $O \subset \mathbb{R}$. Consider the point $x \in f^{-1}(O)$. By definition of f^{-1} we have that $f(x) \in O$ for open O, and since O is open we find

$$\exists\, \delta > 0,\ (f(x) - \epsilon, f(x) + \epsilon) \subset O$$

Additionally, since f is continuous we find

$$\exists\, \delta > 0,\ f(x - \delta, x + \delta) \subset (f(x) - \epsilon, f(x) + \epsilon)$$

Putting these two lines together yields

$$f(x - \delta, x + \delta) \subset (f(x) - \epsilon, f(x) + \epsilon) \subset O$$

so

$$f(x - \delta, x + \delta) \subset O$$

Thus

$$(x - \delta, x + \delta) \subset f^{-1}(O)$$

which is precisely the definition of $f^{-1}(O)$ being open, as desired. □

The following results (Propositions 17, 18, and 19) were not done in class but we were asked to review them independently before the next class.

Proposition 17: Extreme Value Theorem. A continuous real valued function on a closed and bounded set is bounded and attains its maximum and minimum.

Proposition 17: Let $f : F \to \mathbb{R}$ be continuous for F a closed and bounded subset of $\mathbb{R}$. Then, (1) f is bounded on F and (2) f assumes its maximum and minimum on F. That is,

$$\exists\, x_1, x_2 \in F,\ \forall\, x \in F,\ f(x_1) \leq f(x) \leq f(x_2)$$

Proof. We will first show that f is bounded on F. Let f be continuous on every point $x \in F$ Since f is continuous on F, for all $x \in F$, we have that f satisfies

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

where

$$|x - y| < \delta \iff y - \delta < x < y + \delta$$

That is, for $x \in (y - \delta, y + \delta) \cap F$ we have

$$\begin{aligned} |f(x) - f(y)| &< \epsilon \\ \Big||f(x)| - |f(y)|\Big| &< |f(x) - f(y)| < \epsilon \quad \text{(reverse triangle inequality)} \\ |f(y)| - \epsilon &< |f(x)| < |f(y)| + \epsilon \\ \implies |f(x)| &< |f(y)| + \epsilon \end{aligned}$$

and so f is bounded for $x \in (y - \delta, y + \delta)$. Now, note that the collection

$$\{(y - \delta, y + \delta)\}_{y \in F}$$

forms an open cover of F. However, since F is closed and bounded we may apply the Heine-Borel Theorem to conclude that there is a finite subcover

$$F \subset \bigcup_{i=1}^{n} (y_i - \delta, y_i + \delta) \subset \bigcup_{y \in F} (y - \delta, y + \delta)$$

Since F is covered by $\{(y_i - \delta, y_i + \delta)\}_{i=1}^{n}$ we have that each $x \in F$ has some open interval such that $x \in (y_k - \delta, y_k + \delta)$. Thus,

$$\begin{gathered} \forall\, x \in F,\ \exists\, k \in \{1, ..., n\} \text{ such that } |f(y_k)| - \epsilon < f(x) < |f(y_k)| + \epsilon \\ \implies \forall\, x \in F,\ \exists\, k \in \{1, ..., n\} \text{ such that } |f(x)| < |f(y_k)| + \epsilon \end{gathered}$$

At this point we see that this k is not necessarily fixed for every $x \in F$. A simple solution for this is to find the maximum over all k, letting

$$|f(y_M)| = \max\{|f(y_1)|, ..., |f(y_n)|\}$$

where it was meaningful to consider the maximum since we have only a finite number of elements k. Clearly, by construction we have that $|f(x)| < |f(y_M)| + \epsilon$ for all $x \in F$. So, letting

$$M = |f(y_M)| + \epsilon$$

we find

$$\forall\, x \in F,\ \exists\, M > 0 \quad |f(x)| < M$$

Therefore f is indeed bounded by M on F. Next, we wish to show that f attains its extrema on F. We must show that there is some $x^* \in F$ such that $f(x^*) = \sup_{x \in F} f(x)$. Since f is bounded we have that this supremum

$$m = \sup_{x \in F} f(x)$$

is exists and is finite. Let $n \in N$. If m is supremum of f on F. Since m is the *least upper bound*, $m - \frac{1}{n}$ cannot be an upper bound for f on F. Therefore, for $n \in \mathbb{N}$

$$\exists\, x_n \in F \text{ such that } m - \frac{1}{n} < f(x_n)$$

Therefore, we have defined a sequence (x_n) such that

$$m - \frac{1}{n} < f(x_n) < m$$

and as $n \to \infty$

$$m - \frac{1}{n} \to m < f(x_n) < m$$

so we get that this sequence (x_n) is constructs a convergent sequence $(f(x_n)) \to m$. Furthermore, since F is bounded we have that our sequence (x_n) must be bounded and so by the Bolanzo-Weierstrauss Theorem[4] we may conclude that there is a convergent subsequence (x_{k_i}) such that this subsequence is convergent to some value

$$(x_{n_i}) \to x^*$$

and that this subsequence maintains the convergence

$$(f(x_{n_i})) \to m$$

Now, since F is closed we must have that $(x_{n_i}) \to x^* \in F$. Finally, since f is continuous on F we must have that f is continuous at x^* so that $f(x^*) = m^*$. However, we already know that

$$(f(x_{n_i})) \to m$$

Therefore

$$m^* = f(x^*) = m$$

as desired. An essentially identical proof can be done with the minimum by considering $\inf_{x \in F} f(x) = m$, the sequence $m + \frac{1}{n} > m$ and $m + \frac{1}{m} \to m$ as $n \to \infty$, and defining the sequence $(x_n) \in F$, bounded on F by closure, such that $m < f(x_n) < m + \frac{1}{n}$ since $m + \frac{1}{n}$ is not a *greatest lower bound* of f on F. $\square$

Proposition 18: Let f be a real-valued function defined in $(-\infty, \infty)$. Then f is continuous *if and only if* for each open set O of real numbers $f^{-1}(O)$ is an open set, where

$$f^{-1}(O) = \{x \ : \ f(x) \in O\}$$

Proof. ($\implies$) Suppose $f : (-\infty, \infty) \to \mathbb{R}$ is continuous and let $O \subset \mathbb{R}$ be an arbitrary open set. Since f is continuous

$$\forall \epsilon > 0, \ \exists, \delta > 0, \ f(x-\delta, x+\delta) \subset (f(x) - \epsilon, f(x) + \epsilon)$$

Applying f^{-1}

$$\begin{aligned} f^{-1}(f(x-\delta, x+\delta)) &\subset f^{-1}(f(x) - \epsilon, f(x) + \epsilon) \\ \implies (x-\delta, x+\delta) &\subset f^{-1}(f(x) - \epsilon, f(x) + \epsilon) \end{aligned}$$

However, since O is arbitrary we may let $O = (f(x) - \epsilon, f(x) + \epsilon)$ since this set is indeed continuous for $\epsilon > 0$. Thus,

$$\exists \, \delta > 0, \ (x - \delta, x + \delta) \subset f^{-1}(O)$$

and since $(x - \delta, x + \delta)$ is open for $\delta > 0$ we have there exists some small ball $(x - \delta, x + \delta)$ such that this ball is fully enclosed by $f^{-1}(O)$. That is, $f^{-1}(O)$ satisfies the definition of an open set, as desired.

[4] Every bounded sequence has a convergent subsequence.

($\Longleftarrow$) Suppose that $f^{-1}(O)$ is open for all open sets $O \subset \mathbb{R}$. Since $f^{-1}(O)$ is open we have that there is a small ball centered at each $x \in f^{-1}(O)$ such that this ball is fully enclosed by $f^{-1}(O)$. That is,

$$\exists\, \delta > 0,\ (x - \delta, x + \delta) \subset f^{-1}(O)$$

Thus

$$f(x - \delta, x + \delta) \subset f(f^{-1}(O)) = O$$

Since O is any open set $O \subset \mathbb{R}$, let $O = (f(x) - \epsilon, f(x) + \epsilon)$ for all $\epsilon > 0$. Thus,

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0,\ f(x - \delta, x + \delta) \subset (f(x) - \epsilon, f(x) + \epsilon)$$

which is precisely the definition of continuity, as desired. $\square$

Proposition 19: Intermediate Value Theorem. Let f be a continuous real-valued function defined on a closed interval $[a, b]$ and suppose that $f(a) \leq \gamma \leq f(b)$ (or vice-versa). Then, there exists a point $c \in [a, b]$ such that $f(c) = \gamma$. That is, for any point in $[f(a), f(b)]$ there is some point in the domain $[a, b]$ for which f is in the range $[f(a), f(b)]$.

Proof. If $\gamma = f(a)$ or $\gamma = f(b)$ then the proof is trivial. So, without loss of generality, suppose $f(a) < \gamma < f(b)$. Since f is continuous on $[a, b]$, for all points in $x \in [a, b]$ we satisfy

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

where this y which satisfies $|x - y| < \delta$ is within $[a, b]$. So, letting $y = c$ we have for all $x \in [a, b]$, whenever $c - \delta < x < c + \delta$

$$f(c) - \epsilon < f(x) < f(c) + \epsilon$$

or equivalently

$$x \in (c - \delta, c + \delta) \implies f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$$

Now, construct c as follows. For the interval [a,b] consider the subset

$$\{x \in [a, b]\ :\ f(x) < \gamma\}$$

Clearly this set is nonempty since it contains at least a since we supposed that $f(a) < \gamma < f(b)$. Hence, since this set is bound above by b we have that the supremum of this set exists. That is, let c satisfy

$$c = \sup\{x \in [a, b]\ :\ f(x) < \gamma\}$$

Clearly $c \in [a, b]$ since this set was shown to be bound below by a and above by b. Now, for $\delta > 0$ satisfying continuity at $c \in [a, b]$ we have some x^* in the half interval $x^* \in (c - \delta, c]$ such that

$$f(c) - \epsilon < f(x^*) < \gamma$$

where $f(x^*) < \gamma$ follows since $x^* \leq c$ and c was the supremum of the values for which $f(x) < \gamma$. So, choose some other point x^{**} in the other half interval $[c, c + \delta)$. By definition this x^{**} will not satisfy $f(x) < \gamma$ since c was the supremum of these satisfying x. Therefore

$$f(c) + \epsilon > f(x^{**}) \geq \gamma$$

Now, we may rearrange these two inequalities to yield

$$\begin{aligned} f(c) &< f(x^*) + \epsilon < \gamma + \epsilon \\ f(c) &> f(x^{**}) - \epsilon \geq \gamma - \epsilon \end{aligned}$$

Thus,

$$\begin{aligned} f(c) &< \gamma + \epsilon \\ f(c) &> \gamma - \epsilon \\ \implies \gamma - \epsilon < f(c) &< \gamma + \epsilon \end{aligned}$$

for all $\epsilon > 0$. Therefore, we must conclude that $f(c) = \gamma$, as desired. □

Definition: (Uniform continuity) A function f on a set E is uniformly continuous on E if

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x, y \in E,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

That is, $\delta > 0$ remains constant across all points $x, y \in E$.[5]

[5] However, δ can (and probability will) depend on ϵ.