Real Analysis
Lecture Notes

The Real Number System

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1 Borel Sets

Last class we briefly introduced the notion of countable unions of closed sets F_σ and countable intersections of open sets G_δ. We had shown that

$$\mathbb{R} \setminus F_\sigma = G_\delta$$
$$\mathbb{R} \setminus G_\delta = F_\sigma$$

We also found that the rationals $\mathbb{Q}$ were a F_σ since we can consider the closed sets (in $\mathbb{R}$) $\{q\}$ for $q \in \mathbb{Q}$ and construct the countable union

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$

and that the irrationals $\mathbb{R} \setminus \mathbb{Q}$ were a G_δ since

$$\begin{aligned} \mathbb{R} \setminus \mathbb{Q} &= \mathbb{R} \setminus (F_\sigma) \\ &= G_\delta \end{aligned}$$

Furthermore, we had stated without proof that, unlike $\mathbb{R}$, the set of rationals $\mathbb{Q}$ is *not* a G_δ and that $R \setminus \mathbb{Q}$ is *not* a F_σ. Another important result was that every open interval (a, b) is a F_σ since

$$\bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

in which the union considers expanding closed intervals $\left[a + \frac{1}{n}, b - \frac{1}{n}\right]$ that are inversely proportional to n. Since every open set O can be expressed as a countable union of disjoint open intervals $I_x = (a_x, b_x)$[1] Therefore, since we have shown that every open *interval* is a F_σ, then we may conclude that every open *set* is also a F_σ since it is a countable union of

[1] Countably comes from the density of $\mathbb{Q}$ in $\mathbb{R}$ since we can form a bijection on a rational $q \in (a_x, b_x)$, and since our intervals are disjoint this rational q uniquely identifies (a_x, b_x).

F_σ's. That is, every open set in $\mathbb{R}$ is a F_σ.

By considering the complement of this result we obtain the symmetric result that every close set in $\mathbb{R}$ is a G_δ, since the complement of an open set is a closed set.

We also introduced the notion of the Borel Hierarchy. If we consider the union of countably many closed sets F we know that we form a F_σ. However, we consider the union of countably many such F_σ sets we should see that it remains a countable union. Hence

$$(F_\sigma)_\sigma = F_{\sigma\sigma} = F_\sigma$$

Similarly, we find that

$$(G_\delta)_\delta$$

It's less obvious what the countable intersection of F_σ sets or the countable union of G_δ sets would be. That is, we wish to think about collections of the form

$$(F_\sigma)_\delta = F_{\sigma\delta}$$

We can iterate on this process to form the collections

$$((F_\sigma)_\delta)_\sigma = F_{\sigma\delta\sigma}$$
$$((G_\delta)_\sigma)_\delta = G_{\delta\sigma\delta}$$

and so on. This process provides us with an easy way to construct larger and larger families from subsets of $\mathbb{R}$. Finally, we also stated the following result: Let $f : \mathbb{R} \to \mathbb{R}$. The set of all points $\mathbb{R}$ where f is continuous is a G_δ (i.e. $\mathbb{R} \setminus \mathbb{Q}$). That is, we can find functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every irrational point and discontinuous at every rational point, but not vice-versa. We cannot construct a function that is continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{R} \setminus \mathbb{Q}$.

Now, we call <u>borel sets</u> of $\mathbb{R}$ to be the result of all possible iterations

$$F_{\sigma\delta\sigma\delta\ldots}$$
$$G_{\delta\sigma\delta\sigma\ldots}$$

It turns out that the probability of choosing a Borel subset of $\mathbb{R}$ (assuming uniform selection across all possible subsets of $\mathbb{R}$) is 0. However, it's difficult to actually pick anything that's not a Borel subset. That is, the cardinality of Borel subsets is less than the cardinality of all possible subsets of $\mathbb{R}$.

Example: pg. 53, # 53. Suppose f is a real-valued function defined for all $\mathbb{R}$. Prove that the set of all points at which f is continuous is a G_δ (intersection of open sets) set.

Proof. Let f be continuous at x so that $x \mapsto f(x)$. By the definition of continuity

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0,\ f(x-\delta, x+\delta) \subset (f(x)-\epsilon, f(x)+\epsilon)$$

Therefore, since f is continuous we have that $f^{-1}(O)$ is open for all open sets $O \subset \mathbb{R}$. That is,

$$f^{-1}(f(x) - \epsilon, f(x) + \epsilon)$$

is open. In particular, for all n, the inverse image

$$f^{-1}\left(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\right)$$

must be open since f is continuous. Thus, knowing that f is continuous at x becomes the question: Are all inverse images of the form

$$f^{-1}\left(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\right)$$

open in $\mathbb{R}$? Now, let f be continuous at $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Consider the open interval $\left(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\right)$ centered at $f(x)$ with radius $\frac{1}{n}$. Using continuity we have that

$$\exists\, \delta_{x,n} > 0,\ f(x - \delta_{x,n}, x + \delta_{x,n}) \subset \left(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\right)$$

Let $U_{n,x} = (x - \delta_{x,n}, x + \delta_{x,n})$ and let

$$U_n = \bigcup_{\text{all } x \text{ where } f \text{ is continuous at } x} U_{n,x}$$

Since each $U_{n,x}$ is open we have that U_n must be open since an arbitrary union of open sets is open. Now, each x is f is continuous must lie in all U_n for all n. That is, the points where f is continuous lies in

$$\bigcap_n U_n$$

which is precisely a countable intersection of open sets, and so the set of points where x is continuous is a G_δ.

Conversely, if $x \in \bigcap_n U_n$ then the interval

$$f^{-1}\left(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\right)$$

is open for all x and all n, and since we have proven that continuity $\iff$ $f^{-1}(O)$ is open for all open sets O we may conclude that f is continuous at x, as desired. $\square$

As an application of this result (the set of points where f is continuous is a G_δ) we will prove the Baire Category Theorem (next week), which will give us that $\mathbb{Q}$ is not a G_δ. That is, there is no function which is continuous on Q but discontinuous on $\mathbb{R} \setminus \mathbb{Q}$.

Example: pg. 53, # 54. Suppose we have a sequence of functions $(f_n)_{n=1}^{\infty}$ with each $f_n : \mathbb{R} \to \mathbb{R}$ continuous. Prove that the set of points $C \subset \mathbb{R}$ where all f_n converge is a $F_{\sigma\delta}$

(a countable intersection of countable unions of closed sets), i.e. show that $C \subset \mathbb{R}$ can be expressed as

$$C = \bigcap_{n}^{\infty} \bigcup_{m_n}^{\infty} F_{m_n}$$

Proof. For example, if we had the function

$$f_n(x) = n$$

then the set of points for which $(f_n(x))$ converge is $C = \emptyset$. If we tried

$$f_n(x) = \frac{1}{n}$$

then $f_n(x) \to 0$ for all x and so $C = \mathbb{R}$. Trying

$$f_n(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ nx & x \in (0, \infty] \end{cases}$$

so that we only have convergence on the negative reals $x \in (-\infty, 0] = C$. With $f_n(x) = \frac{\sin x}{n}$ we find convergence on $\mathbb{R}$ and $f_n(x) = n \sin x$ we only find convergence at every $2k\pi$.

So, we have $\mathbb{R}$ and a subset $C \subset \mathbb{R}$ and we want to show that this C must always be a $F_{\sigma\delta}$. What does it *mean* to say that f belongs to C? We require that

$$(f_n(x)) \to f(x)$$

where convergence means (using the Cauchy criteria)

$$\forall\, \epsilon,\ \exists\, N \geq 1,\ \forall\, n, k \geq N,\ |f_k(x) - f_n(x)| \leq \frac{1}{m} < \epsilon$$

Furthermore, if (f_n) is continuous then the difference $f_k - f_n$ is continuous and the absolute value $|f_k - f_n|$ is continuous. Now, we ask the question: What is the inverse image of the ***closed*** set $\left[-\frac{1}{m}, \frac{1}{m}\right]$? Again, from our earlier result we know that continuity $\iff$ f maps closed sets to closed sets and open sets to open sets. Thus,

$$|f_k - f_n| \left(\left[-\frac{1}{m}, \frac{1}{m} \right] \right)$$

Let $F_{n,m}$ be the set

$$F_{n,m} = \left\{ x \in \mathbb{R} \ : \ |f_k(x) - f_n(x)| \leq \frac{1}{m},\ \forall\, k \geq n \right\}$$

That is, $F_{n,m}$ is the preimage $|f_k - f_n|^{-1}$ of $\left[-\frac{1}{m}, \frac{1}{m}\right]$, which we have said to be a closed set by the continuity of f for fixed m. Thus, $F_{n,m}$ is closed for each n, m.

Now, the statement $x \in C$ means that for all m there is some n sufficiently large such that $|f_k - f_n(x)| \leq \frac{1}{m}$ (for all $k \geq n$). Therefore, by the construction of these closed sets we find

$$x \in F_{n,m} \implies x \in \bigcup_{\text{all } n \text{ sufficiently large}} F_{n,m}$$

and

$$\implies x \in \bigcap_{\text{all } m} \left[\bigcup_{\text{all } n \text{ sufficiently large}} F_{n,m} \right]$$

which is precisely the definition of a $F_{\sigma\delta}$. Conversely, suppose $x \in \bigcap_m \bigcup_n F_{n,m}$ take $\epsilon > 0$ and find M such that $\frac{1}{M} < \epsilon$. Then

$$\implies x \in F_{n,M} \text{ for some } n$$
$$\implies |f_k(x) - f_n(x)| < \frac{1}{M}, \quad \forall\, k \geq n$$

which is the definition of convergence at x.

$\square$