Real Analysis
Lecture Notes

Set Theory

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1 Countability & Countable Sets

Last class we spoke about some countable finite sets. We now consider countably infinite sets. Let S be some countably infinite set. Such sets S are said to be countably infinite if they are not finite and if such sets are are an image of the naturals $\mathbb{N}$. That is, a countably infinite set is one that is an image of some infinite sequence $\{x_1, x_2, x_3, ...\} = \{f(1), f(2), f(3), ...\}$.

We can show that this definition is equivalent to the statement that a countably infinite set is a set for which there exists a bijective mapping with $\mathbb{N}$:

Proof. Let some infinite set S be an image of the sequence $\{x_1, x_2, x_3, ...\}$. Essentially, we now must show that there exists some bijection between the sequence $\{x_1, x_2, x_3, ...\}$ and $\mathbb{N}$. Define f as follows:

$$
\begin{aligned}
&(1) \quad \text{Map } x_1 \xrightarrow{f} 1. \\
&(2) \quad \text{If } x_2 \neq x_1, \text{ map } x_2 \xrightarrow{f} 2. \\
&(3) \quad \text{If } x_3 \neq x_2, x_3 \neq x_1, \text{ map } x_3 \xrightarrow{f} 3. \\
&\quad\ \vdots \\
&(k) \quad \text{If } x_k \neq x_i,\ \forall\, 1 \leq i \leq k-1, \text{ map } x_k \xrightarrow{f} k. \\
&\quad\ \vdots
\end{aligned}
$$

In general, we find that x_{k+1} will map to the smallest value $f(x_{k+1}) = m$ such that $x_m \neq x_i$ for all preceding $i \leq f(n)$. Since S is an infinite set, there will always be such a smallest $m \in \mathbb{N}$ to satisfy $f(n+1)$ (this actually uses the *well-ordering principle* for $\mathbb{N}$). We see that this mapping

$$x_{f(k)} \to k$$

is bijective since we can retrieve arbitrary k given $x_{f(k)}$ and vice-versa.[1] Therefore, our statements are indeed equivalent, as desired. $\square$

Proposition: Every subset of a countably finite set is countable.

Proof. Let $S = \{x_1, x_2, x_3, ...\}$ be countably infinite. Take some subset $A \subset S$, $A \neq \emptyset$ (if $A = \emptyset$ then A is countable by definition).

Let $x \in A$ be some fixed arbitrary element of A. Now, define the sequence $\{y_1, y_2, y_3, ...\}$ such that $y_n = x_n$ if $x_n \in A$ and $y_n = x$ if $x_n \notin A$. Therefore, all $y_n \in A$ by construction. By this construction we have successfully placed the set A in the range of our sequence $\{y_1, y_2, ...\}$. This is precisely our definition of countability above: An infinite set which is the image of some infinite sequence. Therefore, since A was arbitrary, we have that every subset of a countably infinite set must be countable, as desired. $\square$

Proposition: Let A be a countable set. The set of all finite sequences from A is countable.

Proof. Since A is countable it has some bijective correspondence with some subset of $\mathbb{N}$. From this, we see that it is sufficient to show that the set S set of finite sequences of $\mathbb{N}$ is countable.

Note that from the Fundamental Theorem of Arithmetic we have that every $n > 1$ can be expressed as a *unique* product of primes. That is, for some $n \in \mathbb{N}$, $n > 1$, we have

$$n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}, \quad p_m \geq 2, k_i \geq 0$$

where $p_1, p_2, ..., p_m$ are the first m primes. For example, we may decompose the following integers into their product of primes as follows:

$$\begin{aligned} 40 &= 2^3 \cdot 3^0 \cdot 5^1 \\ 24 &= 2^3 \cdot 3^1 \\ 9 &= 2^0 \cdot 3^2 \end{aligned}$$

Therefore, we define $f : \mathbb{N} \to \{\text{finite sequences from } \mathbb{N} \cup \{0\}\}$ using this prime power decomposition. As an illustrative example, our above integers would map to the finite sequences

$$\begin{aligned} f(40) &= (3, 1) \\ f(24) &= (3, 0, 1) \\ f(9) &= (0, 2) \end{aligned}$$

[1] A more rigorous proof would go through the definitions if surjectivity and injectivity... maybe I'll include that later.

Note that under this construction we don't have a definition for $f(1)$ since we were forced to limit ourselves to $n > 1$. However, since this is only finitely many values we can just "throw it into the trash" to some finite sequence, say

$$f(1) = (0)$$

Is this f surjective? Do we get **all** finite sequences from $\mathbb{N} \cup \{0\}$? Yes! We can show this more rigorously, but we should be able to immediately see that any sequence $(n_1, n_2, ..., n_k)$ will uniquely define some $n \in \mathbb{N}$.

Is this f injective? Yes! By the Fundamental Theorem of Arithmetic we have that all $n \in \mathbb{N}$ have some unique sequence $(n_1, n_2, ..., n_k)$ given by f.

So, the image of f is countably infinite and contains all finite sequences defined using elements $n \in \mathbb{N}$. Therefore, our set S is indeed in the range of f, and so it must be countably infinite itself, as desired. $\square$

We have shown before that the set of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ is countable. From this we can show that the set of rationals $\mathbb{Q} = \left\{ \frac{z_1}{z_2} \; : \; z_1, z_2 \in \mathbb{Z}, z_2 \neq 0 \right\}$ is also countable: Note that some rational number $q \in \mathbb{Q}$ is determined by a *pair* of integers (z_1, z_2), $z_2 \neq 0$. Note that these pairs of integers are clearly a finite sequence formed from the countable set $\mathbb{Z}$. Therefore, by the above proposition we may conclude that $\mathbb{Q}$ is indeed countable.

Proposition: A countable union of countable sets is countable.

Proof. Let $\mathcal{A} = \bigcup_{n=1}^{\infty} A_n$, $A_n \neq \emptyset$ for all n. Our sets A_n look something like $A_n = \{x_{n,m}\}_{m=1}^{\infty}$. Now, note that (n, m) is a finite sequence made from elements in $\mathbb{N}$, and so the set of $\{(n, m) \; : \; n, m \in \mathbb{N}\}$ must be countable. Therefore, the mapping

$$(n, m) \to x_{n,m}$$

is a mapping of ordered pairs from $\mathbb{N}$ onto the elements $x_{n,m} \in A_n$. That is, the mapping from (n, m) to $x_{n,m}$ is a mapping onto the countable union $\mathcal{A}$. Since we have determined that the set of ordered pairs (n, m) must be countable, we conclude that this union $\mathcal{A}$ must also be countable since we have shown the necessary mapping exists. $\square$

We say that the cardinality of countably infinite sets is given by the cardinal number $\aleph_0$ (aleph-null). That is,

$$\text{card } \mathbb{N} = \text{card } \mathbb{Z} = \text{card } \mathbb{Q} = \text{card } \Big(\underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{\text{countably many times}} \Big) = \aleph_0$$

Proposition: We claim that there exists a set S of all countable sequences from $\{0, 1\}$ that is *not* countable.

Proof. Suppose S is countable. Then, we may express S by

$$S = \{s_1, s_2, s_3, ...\}$$

We may list the elements of S as follows:

$$\begin{aligned} s_1 &= (s_{1,1},\ s_{1,2},\ s_{1,3},\ ...) \\ s_2 &= (s_{2,1},\ s_{2,2},\ s_{2,3},\ ...) \\ s_3 &= (s_{3,1},\ s_{3,2},\ s_{3,3},\ ...) \\ &\vdots \end{aligned}$$

where all $s_{n,m} \in \{0, 1\}$. Now, construct the sequence s such that the k^{th} element in s is given by $s_{k,k} + 1 \mod 2$ so that s differs from s_1 in the first position, s differs from s_2 in the second position, s differs from s_3 in the third position, and so on. For example, if S is given by the sequences

$$\begin{aligned} s_1 &= (\underline{0}, 1, 0, 0, ...) \\ s_2 &= (0, \underline{0}, 1, 0, ...) \\ s_3 &= (1, 1, \underline{1}, 1, ...) \\ s_4 &= (0, 0, 0, \underline{0}, ...) \\ &\vdots \end{aligned}$$

Then the first four elements of s will be given by

$$s = (1, 1, 0, 1, ...)$$

Since the k^{th} digit of s will differ from from the k^{th} sequence in the k^{th} term, we see that s cannot be in our set S. That is, s is a sequence from $\{0, 1\}$ and $s \notin S$. But we had stated that S contains all sequences from $\{0, 1\}$! Contradiction: We must conclude that S cannot be a countable set (we say it is *uncountable*) whose cardinality card S is given by $2^{\aleph_0}$.

In this course we will not provide the proof for $\aleph_0 < 2^{\aleph_0}$. $\square$

Continuum Hypothesis: The "Continuum Hypothesis" states that there is no "infinity" (cardinality) strictly between the cardinality of the naturals $\aleph_0$ and $2^{\aleph_0}$ (this turns out to be the cardinality of the reals, $\mathbb{R}$).

2 Orderings

In the previous class we introduced the notion of partially ordered and totally ordered sets. A partially ordered set S satisfies:

$$\begin{array}{ll} (1) & \text{For all } a \in S,\ a \leq a. \text{ (reflexivity)} \\ (2) & \text{if } a \leq b \text{ and } b \leq a \text{ then } a = b. \text{ (antisymmetry)} \\ (3) & \text{if } a \leq b \text{ and } b \leq c \text{ then } a \leq c. \text{ (transitivity)} \end{array}$$

A totally ordered set S requires a fourth criteria:

$$(4) \quad \text{For all } a, b \in S \text{ then either } a \leq b \text{ or } b \leq a.$$

We also introduced the notion of a well-ordered set. A well-ordered set S is a totally ordered set if and only if every nonempty subset of S has a least element.

Consider a set S and let the least element of S be s_1. Now consider the difference $S \setminus \{s_1\}$. If $S \setminus \{s_1\} = \emptyset$ then $S = \{s_1\}$ and so S is finite and trivially well ordered. So, suppose $S \setminus \{s_1\} \neq \emptyset$. By the well-ordering principle $S \setminus \{s_1\}$ has a least element, and since $s_1 \notin S$ this least element cannot be s_1. We call this new least element s_2. Clearly $s_2 \neq s_1$.

If S is not a finite set then by this process we get the ordering on S

$$s_1 < s_2 < s_3 < \cdots$$

Clearly this ordering of $(s_1, s_2, s_3, ...)$ is in a bijection with $\mathbb{N}$ and so it is countable. With this in mind we introduce the following:

Well-ordering principle: Every set X can be well-ordered. That is, there exists a ordering relation $\prec$ that well orders X.

Note that this well-ordering principle is not constructive. Obviously we won't use the total order $<$ on $\mathbb{R}$ since this won't yiesld a well-order (i.e. we will have open subsets without a least element). It's not intuitively obvious what such an order on $\mathbb{R}$ would be. As a result, there are uncountable well-ordered sets.

Proposition: There exists an uncountable set X that is well-ordered by a relation $\prec$ so that:

(i) There is a largest element $\Omega \in X$.

(ii) If $x \in X$ and $x \neq \Omega$ then x has only a countable number of predecessors. That is, the set $\{y \in X \ : \ y < x, x \in X, x \neq \Omega\}$ is countable.

Recalling our construction of $S = \{s_1, s_2, s_3, ...\}$, we may picture X, with the well-order $\prec$, to look something like

$$X = \{s_1, s_2, s_3, ..., \Omega\}$$

Proof. Proof given next class. $\square$