Real Analysis
Lecture Notes

Metric Spaces

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1 Connected Sets

We begin by restating the last few definitions and examples relating to connected sets that we introduced in the previous notes:

Definition: A metric space X is called connected if there does not exist any nonempty open subsets A and B in X such that

$$A \cap B = \emptyset$$
$$A \cup B = X$$

Example: Is $\mathbb{R}$ connected? The only open A and B for $A \cup B$ to construct $\mathbb{R}$ and satisfy $A \cap B = \emptyset$ is

$$\mathbb{R} \cap \emptyset = \emptyset$$
$$\mathbb{R} \cup \emptyset = \mathbb{R}$$

but clearly this involved the empty set. Thus, $\mathbb{R}$ is indeed connected.

Example: Consider the punctured real line $\mathbb{R} \setminus \{2\} = X$. Then

$$X = (-\infty, 2) \cup (2, \infty)$$

Clearly we can construct X by $A = (-\infty, 2)$ and $B = (2, \infty)$ with $A \cap B = \emptyset$. Therefore, X is *not* connected. Similarly,

$$X = [2, 3] \cup [4, 5]$$

is not connected since we cannot find open A or B to construct X without either being empty.

Interestingly, the puncture plane $\mathbb{R}^2 \setminus \{(0, 0)\}$ *is connected*. We can think of this geometrically as the whole plane without the origin remains connected. However, by a similar argument, the real plane without the x-axis is not connected.

We can show that $\mathbb{Q}$ is not connected by the following argument

$$\begin{aligned}
(-\infty, \pi] \text{ is closed in } \mathbb{R} &\text{ and } [\pi, \infty) \text{ is closed in } \mathbb{R} \\
\text{Let } A = (-\infty, \pi] \cap \mathbb{Q} \text{ is closed in } \mathbb{Q} \\
\text{Let } B = [\pi, \infty) \cap \mathbb{Q} \text{ is closed in } \mathbb{Q} \\
A \neq \emptyset &\text{ and } B \neq \emptyset \\
A \cup B &= \mathbb{Q} \\
A \cap B &= \emptyset \\
\text{Both } A \text{ and } B \text{ are } &\textit{clopen} \text{ sets} \\
\mathbb{Q} \text{ is } &\textit{not} \text{ connected}
\end{aligned}$$

by a similar argument we can show that the irrationals $\mathbb{R} \setminus \mathbb{Q}$.

Example: We say that $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

is the *topologists sine curve*. It turns out that the topologists sine curve defined by f is connected. We can show this by letting T denote the set of points defined by f and considering the set

$$A = \left\{ \left(x, \sin \frac{1}{x} \right) \in \mathbb{R}^2 \ : \ x \in \mathbb{R}^+ \right\}$$

and

$$B = \left\{ \left(x, \sin \frac{1}{x} \right) \in \mathbb{R}^2 \ : \ x \in \mathbb{R}^- \right\}$$

Then

$$T = \overline{A \cup B} = \overline{A} \cup \overline{B}$$

(It isn't difficult to show that both A and B are connected and so we just need that a) if A is connected and $A \subset C \subset \overline{A}$ then C is connected, and b) if A and B are connected and $A \cap B \neq \emptyset$ then $A \cup B$ is connected).

Proposition: Suppose $f : X \overset{\text{onto}}{\to} Y$ is onto and continuous. If X is connected then Y is connected.

Proof. Suppose Y is *not* connected. Then there exists open sets A and B such that

$$\begin{aligned} A \cup B &= Y \\ A \cap B &= \emptyset \end{aligned}$$

Since f is onto and continuous we have that, because A and B are open

$$\begin{aligned} f^{-1}(A) &\text{ is open} \\ f^{-1}(B) &\text{ is open} \end{aligned}$$

Is $f^{-1}(A) \cup f^{-1}(B) = X$? Take $x \in X$ so that $f(x) \in Y$. Without loss of generality assume $f(x) \in A \subset Y$. Then, for all x,

$$x \in f^{-1}(A)$$

and so, together with the case for $f(x) \in B \subset Y$,

$$f^{-1}(A) \cup f^{-1}(B) = X$$

Is $f^{-1}(A) \cap f^{-1}(B) = \emptyset$? Assume not so that there is at least one point x such that

$$x \in f^{-1}(A) \cap f^{-1}(B)$$

Then $f(x) \in A$ and $f(x) \in B$. However, $A \cap B = \emptyset$ by assumption that Y is not connected. Contradiction! Therefore

$$f^{-1}(A) \cap f^{-1}(B) = \emptyset$$

Without loss of generality, is $f^{-1}(A) \neq \emptyset$? Take some $a \in A$. Since f is onto we do indeed have some x such that $x = f^{-1}(a)$ with $f(x) = a \in A$

However, X is connected so it is a *contradiction* to state that

$$f^{-1}(A) \neq \emptyset \quad \text{and} \quad f^{-1}(B) \neq \emptyset$$

and

$$f^{-1}(A) \cap f^{-1}(B) = \emptyset$$

and

$$f^{-1}(A) \cup f^{-1}(B) = X$$

Thus, we have both X is connected and X is not connected. Contradiction! Therefore, we must conclude that Y is connected. That is, if $f : X \to Y$ is a continuous onto function and X is some connected set, then Y must also be connected, as desired. $\square$

Recall the Intermediate Value Theorem:

Theorem: Intermediate Value Theorem. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < f(b)$. Then there exists some $c \in [a, b]$ such that $f(a) \leq z \leq f(b)$ with $f(c) = z$.

Now, the Intermediate Value Theorem for connected spaces:

Theorem: Intermediate Value Theorem for Connected Spaces. Let X be connected and consider a continuous function $f : X \to \mathbb{R}$. Take $x, y \in X$ and $c \in \mathbb{R}$ such that $f(x) < c < f(y)$. Then $\exists\, z \in X$ such that $f(z) = c$.

Proof. Suppose the result is false. That is, c is not in the image of f for some $z \in X$. Consider the open sets

$$(-\infty, c) \text{ and } (c, \infty)$$

Let

$$A = f^{-1}(-\infty, c) \quad \text{(open in } X\text{)}$$
$$B = f^{-1}(c, \infty) \quad \text{(open in } X\text{)}$$

Therefore

$$A \cap B \neq \emptyset$$

We have that $A \neq \emptyset$ since $x \in A$ and $B \neq \emptyset$ since $y \in B$ by assumption. Furthermore

$$A \cup B = X$$

since c is *not in the range of f by assumption*. Thus, X is not connected. Contradiction! Therefore, c is in the range of f for some $z \in X$, as desired. □

Lemma: If a set of real numbers is bounded above by a positive real then it is bounded above by a positive integer.

Proof. This is an obvious application of the Archimedean principle. □

Lemma: Let $f : X \to \mathbb{R}$ be continuous. Then $|f| : X \to \mathbb{R}$ is also continuous.

Proof. We have done this proof before when discussing continuity. □

Note that for $[0, t]$, $t > 0$ a closed set we have that $|f|^{-1}[0, t]$ must be closed in X since it is continuous. Therefore

$$\{x \in X \ : \ |f(x)| \leq t\}$$

is closed in X.

Theorem: Uniform Boundedness Principle. Let $\mathcal{F}$ be a family of real-valued continuous functions on a complete metric space X. Suppose that for each $x \, inX$ there is some number M_x such that $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$. Then, there is a nonempty open set $O \subset X$ and a constant M such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and all $x \in O$.

Proof. For each integer m, let $E_{m,f} = \{x \in X \ : \ |f(x)| \leq m\}$, and

$$E_m = \bigcap_{f \in \mathcal{F}} E_{m,f}$$

Since each f is continuous, $E_{m,f}$ is closed, and so E_m must also be closed.

Now, for each $x \in X$ there is an integer m such that $|f(x)| \leq m$ for all $f \in \mathcal{F}$. That is, there is an integer m such that $x \in E_m$. Therefore

$$X = \bigcup_{m=1}^{\infty} E_m$$

If all E_m were nowhere dense then X is of the first category. However, X is an open set and so some E_m is *not* nowhere dense. Since this non-nowhere dense E_m is closed, it must have an open set within it, say

$$O \subset E_m$$

However, for every $x \in O$ we have defined

$$O \subset \bigcap_{f \in \mathcal{F}} E_{m,f} = \bigcap_{f \in \mathcal{F}} \{x \in X \ : \ |f(x)| \leq m\}$$

That is, for every $x \in O$ we have $|f(x)| \leq m$ for all $f \in \mathcal{F}$. Thus, there is a nonempty open set $O \subset X$ and a constant M such that

$$\forall\, f \in \mathcal{F}, \ \forall\, x \in O, \ |f(x)| \leq M$$

as desired.

□