Real Analysis
Lecture Notes

Metric Spaces

November 16 2016
Last update: December 4, 2017

1 Compactness

We continue our work on expanding the properties of compact sets and its relationships to other definitions. In particular, we have found so far that

$$\begin{aligned} X \text{ is closed-bounded} &\implies X \text{ is compact set} \overset{\text{def}}{\iff} \text{All open covers of } X \text{ have a finite subcover} \\ &\implies \text{Infinite sequences from } X \text{ has a cluster point} \\ &\overset{\text{def}}{\iff} X \text{ has the Bolzano-Weierstrass property} \\ &\iff X \text{ is sequentially compact} \\ &\overset{\text{def}}{\iff} \text{Every infinite sequence has a convergent subsequence} \end{aligned}$$

We then reintroduced some definitions and properties of continuous functions and bounded sets. We now move on to a new definition:

Definition: ***(Totally bounded space).*** A metric space X is <u>totally bounded</u> if for any $\epsilon > 0$ there are only finitely many points $x_1, x_2, ..., x_n \in X$ satisfying

$$X \subset \bigcup_{i=1}^{n} S_{x_i,\epsilon}$$

That is, X is <u>totally bounded</u> if X is the union of finitely many open spheres of radius ϵ.

Example: Consider the bounded set $[2, 3]$. Note that $[2, 3]$ is compact by the Heine-Borel Theorem since it is closed and bounded. Therefore, for all open covers there is a corresponding finite subcover. Thus, for the open cover $\{S_{x,\epsilon}\}_{x\in[2,3]}$ we may produce the finite subcover $\{S_{x_i,\epsilon}\}_{i=1}^{n}$ so that

$$X \subset \bigcup_{i=1}^{n} S_{x_i,\epsilon}$$

and so $[2, 3]$ is totally compact by definition.

Example: In general, for all sets $[a, b]$ we find that $[a, b]$ is compact by the Heine-Borel Theorem. For all $x \in [a, b]$ let O_x be the open interval

$$O_x = (x - \epsilon, x + \epsilon)$$

then the family of O_x given by $\{O_x \ : \ x \in [a, b]\}$ covers our interval $[a, b]$. However, $[a, b]$ is compact. Therefore, all open covers of $[a, b]$ have some finite subcover $\{O_{x_1}, O_{x,2}, ...\}$ so that

$$[a, b] \subset \bigcup_{i=1}^{n} O_{x_i} = \bigcup_{i=1}^{n} (x_i - \epsilon, x_i + \epsilon)$$

and so our bounded set $[a, b]$ is totally compact.

Proposition: ***Compactness $\implies$ totally bounded.***

Proof. By the same argument as our example above we find that a metric space X is compact $\implies$ X is totally bounded. □

Example: Let $X = \mathbb{R}$ under the discrete metric

$$\rho_d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{else} \end{cases}$$

We have found that such a metric space is bounded since the distance between points is always less than $M = 2$. However, is this $X = \mathbb{R}$ totally bounded? Take $\epsilon = \frac{1}{2}$ and note that under ρ_d the sphere defined by $S_{x, \frac{1}{2}}$ is

$$S_{x, \frac{1}{2}} = \{x\}$$

Therefore any finite union of such spheres $S_{x_i, \frac{1}{2}}$ will be

$$\bigcup_{i=1}^{n} S_{x_i, \frac{1}{2}} = \{x_i\}_{i=1}^{n} \neq \mathbb{R}$$

That is, since n was arbitrary, we have shown that no finite set of open spheres on $\mathbb{R}$ in the discrete metric will be able to cover $\mathbb{R}$, and so $\mathbb{R}$ under ρ_d is *not* totally bounded.

This example gives us a sense of the distinction between *boundedness*, which $\mathbb{R}$ under ρ_d clearly is, and *total boundedness*, which $\mathbb{R}$ under ρ_d has shown to not be. Under the standard Euclidean metric $\rho(x, y) = |x - y|$ there is no such distinction in $\mathbb{R}$, but we have just seen at least one metric in which such a difference emerges.

Proposition: ***Total boundedness $\implies$ boundedness.***

Proof. For boundedness we require $\sup \rho(x,y) \le M$ for all $x, y \in X$. Let $\epsilon = 1$. Then, assuming X is totally bounded, there exists finitely many points $x_1, x_2, ..., x_n$ such that

$$X \subset \bigcup_{i=1}^{n} S_{x_i,1}$$

Now, take any two points $y_1, y_2 \in X$ such that

$$\begin{aligned} y_1 &\in S_{x_1,1} \\ y_2 &\in S_{x_2,1} \end{aligned}$$

By the definition of a metric we have the inequality

$$\begin{aligned} \rho(y_1, y_2) &\le \rho(y_1, x_{n_1}) + \rho(x_{n_1}, x_{n_2}) + \rho(x_{n_2}, y_2) \\ &= 1 + \rho(x_{n_1}, x_{n_2}) + 1 \\ &= \rho(x_{n_1}, x_{n_2}) + 2 \end{aligned}$$

Although we don't know the distance $\rho(x_{n_1}, x_{n_2})$ we do know that there are only finitely many such points. Thus, take the largest such distance

$$\sup \left\{ \rho(x_{n_i}, x_{n_j}) \right\}_{i,j=1,2,...,n} = M$$

Since X is totally bounded there are only finitely many spheres covering X, and so this largest distance between points M is itself finite. Therefore,

$$\begin{aligned} \rho(y_1, y_2) &\le \rho(y_1, x_{n_1}) + \rho(x_{n_1}, x_{n_2}) + \rho(x_{n_2}, y_2) \\ &= \rho(x_{n_1}, x_{n_2}) + 2 \\ &= M + 2 \end{aligned}$$

That is, $M + 2$ is some bound between arbitrary points $y_1, y_2 \in X$, and so X is indeed bounded. Hence

$$\text{A totally bounded metric space } X \implies X \text{ is bounded,}$$

as desired. □

Proposition: *Sequentially compact $\implies$ totally bounded.*

Proof. Suppose *not*. That is, suppose that there exists some $\epsilon > 0$ so that X is *not* covered by finitely many spheres $S_{x,\epsilon}$.

Pick any point $x_1 \in X$. Note that under our contradictory assumption we have that $S_{x_1,\epsilon} \subsetneq X$. Since there is a point x_2 outside of $S_{x_1,\epsilon}$ take the union

$$S_{x_1,\epsilon} \cup S_{x_2,\epsilon} \subsetneq X$$

Similarly, there exists some $x_3 \in X$ so that

$$\rho(x_1, x_3) \geq \epsilon$$
$$\rho(x_2, x_3) \geq \epsilon$$

with union

$$S_{x_1,\epsilon} \cup S_{x_2,\epsilon} \cup S_{x_3,\epsilon} \subsetneq X$$

Continue defining points x_n so that we produce the sequence. However, recall that if X is sequentially compact then all subsequences have convergent subsequences. Therefore our sequence $(x_1, x_2, x_3, ...)$ has some convergent subsequence (x_{n_k}) in which the distance

$$\rho(x_i, x_j) < \frac{\epsilon}{2}$$

However, we have defined x_n to satisfy

$$\rho(x_i, x_j) \geq \epsilon$$

Contradiction! Therefore, is X is totally bounded. That is,

If a metric space X sequentially compact then it must be totally bounded,

as desired. $\square$

Example: Consider now some metric space X and $\mathcal{U}$ an open cover of X so that

$$\mathcal{U} = \{O_i\}$$

and

$$X \subset \bigcup_{i \in I} O_i$$

Consider some open set O_x containing a point x and produce the open sphere $S_{x,\delta}$ so that

$$S_{x,\epsilon} \subset O_x$$

Note that every point x has will have its own δ so that $S_{x,\delta} \subset O_x$. In general, as we approach the boundaries of X we will find this radius $\delta \to 0$.

Example: Cover the set of naturals $\mathbb{N}$ by the open covers $\left\{\left(n - \frac{1}{n}, n + \frac{1}{n}\right)\right\}$ where $\delta = \frac{1}{n}$. Clearly as $n \to \infty$ we will find $\delta \to 0$. However, since with $\delta = 0$ we have the covers $\{n\}$, which still covers $\mathbb{N}$.

Definition: ***(Lebesgue Number).*** Let $\mathcal{U}$ be some covering of X with $\mathcal{U} = \{U_i\}$. The value $\epsilon > 0$ is called a Lebesgue number of the covering if, for all $\delta < \epsilon$,

$$\forall x \in X,\ \exists O \in \mathcal{U},\ S_{x,\delta} \subset O$$

Example: Let $X = \mathbb{R}$ under the discrete metric ρ_d

$$\rho_d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{else} \end{cases}$$

Let $\mathcal{U}$ be an open covering of X. Does $\mathcal{U}$ have a Lebesgue number? We may be able to see that $\epsilon = 1$ works as a Lebesgue number since for all $\delta < \epsilon = 1$ we find that $\rho(x, y) < \delta < \epsilon = 1 \implies x = y \iff \rho_d(x, y) = 0$, so with our open spheres $S_{x,\delta}$

$$S_{x,\delta} = \{x \ : \ \rho_d(x, y) < \delta\}$$

we see that for this $\epsilon = 1$

$$\forall x \in X, \ \exists O \in \mathcal{U}, \ S_{x,\delta} \subset O$$

and so $\epsilon = 1$ is indeed a Lebesgue number for $X = \mathbb{R}$ under the discrete metric.

Proposition: ***If X is a sequentially compact metric space and $\mathcal{U}$ some open covering of X then $\mathcal{U}$ has a Lebesgue number.***

Proof. *(Proof later)* □

Borel-Lebesgue Theorem: Let X be a metric space. The following are equivalent:

1. X is compact.
2. X has the Bolzano-Weierstrass property.
3. X is sequentially compact.

Proof. We have already proven that $1 \implies 2 \implies 3$. All that remains is to prove that $3 \implies 1$. That is, all we must prove is that if X is sequentially compact then X is compact.

Assume that X is sequentially compact. Take an open covering $\mathcal{U}$ of X. We have two tools under our disposal: (a) From an earlier result, X is totally bounded and (b) From our above Theorem, X has a Lebesgue number.

In particular, let $\epsilon > 0$ be the Lebesgue number of the covering $\mathcal{U}$. Take $0 < \delta < \epsilon$. Since X is sequentially compact we have that X is totally bounded from our previous work. Therefore, by total boundedness of X, there exists finitely many open covers $\{S_{x_i,\delta}\}_{i=1}^n$ that cover X.

Since $\delta < \epsilon$ and ϵ is the Lebesgue number of $\mathcal{U}$, each of these open spheres in X must lie within an open set U_i from $\mathcal{U}$. That is,

$$\begin{gathered} S_{x_1,\delta} \subset O_1 \\ S_{x_2,\delta} \subset O_2 \\ \vdots \\ S_{x_n,\delta} \subset O_n \end{gathered}$$

Since $S_{x_i,\delta} \subset O_i$ and $X \subset \bigcup_{i=1}^n S_{x_i,\delta}$ by total boundedness, we conclude that

$$X \bigcup_{i=1}^{n} O_i \quad O_i \in \mathcal{U}$$

and so, for any open cover $\mathcal{U}$ there is a finite subcover $\{O_i\}_{i=1}^n \subset \mathcal{U}$. That is, if X is sequentially compact then X is compact, as desired. $\square$

Proposition: *A closed subset of a compact space is compact.*

Proof. Let X be some compact space and F a closed subset of X, $F \subset X$. Take any open covering $\mathcal{O} = \{O_i\}$ so that

$$F \subset \bigcup_{i \in I} O_i$$

Consider the complement $X \setminus F$. Since $F \subset \bigcup_{i \in I} O_i$ we find that our compact space X is covered by the union

$$X \subset (X \setminus F) \cup \{O_i\}_{i \in I}$$

However, X is compact. Thus, we can find finitely many such sets

$$X \subset (X \setminus F) \cup \{O_i\}_{i=1}^n$$

and since $X \setminus$ obviously does not contain F and $F \subset X$ we see that all points in F must be contained by the finite set $\{O_i\}_{i=1}^n$ so that

$$F \subset X \subset (X \setminus F) \cup \{O_i\}_{i=1}^n$$

In particular,

$$F \subset X \subset \{O_i\}_{i=1}^n$$

since $F \cap (X \setminus F) = \emptyset$. Thus, F is covered by finitely many open sets $\{O_i\}_{i=1}^n$ and so F must be compact, as desired. $\square$

Proposition: *A compact subspace of a metric space is closed and bounded.*

Proof. Let X be a metric space and K a compact subspace so that $K \subset X$. Let y be some point of closure of K. We have proven that the function $f \ : \ K \to \mathbb{R}$ given by

$$f(x) = \rho(x, y)$$

is a continuous functon on K. Computing the infimum on f we see that

$$\inf_{x \in K} f(x) = \inf_{x \in K} \rho(x, y) = 0$$

since y is a point of closure of K. However, we have shown that a continuous function on a *sequentially compact* (and so compact) metric space must assume its infinum. Therefore,

$$\exists\, z \in K, \ f(z) = 0$$

but $f(z) = \rho(z, y)$ and

$$\rho(z, y) = 0 \iff z = y$$

Therefore, f assumes its infimum at $y \in K$. Similarly, since f is continuous it must achieve its maximum on K, say at point y', since K is compact. Thus,

$$0 \leq f(x) \leq M \quad M \in \mathbb{R}$$

and so K is indeed bounded. That is, K a compact subspace of metric space X is a closed and bounded set, as desired. □

Corollary: ***All compact subspaces of real numbers are closed and bounded (reverse of the Heine-Borel Theorem).***

Proof. This is an immediate consequence of the above proposition for $X = \mathbb{R}$. □

Proposition: ***A continuous image of a compact space is compact.*** That is, if X is a compact space and $f : X \overset{\text{onto}}{\to} Y$ is a continuous surjective function then Y is compact. That is, . It turns out that an analogous result holds for Linedelof spaces. That is, if X is a Lindelof space and $f : X \overset{\text{onto}}{\to} Y$ is a continuous surjective function, then Y Lindelof.

For example, if we have some function $f : [2, 3] \to (0, 1)$ it must be the case that f is *not* continous since $(0, 1)$ has been shown to not be compact.

Proof. Let X be some compact space and Y some continuous mapping $f : X \to Y$. Let $\mathcal{O} = \{O_i\}$ be some open covering of Y such that

$$Y \subset \bigcup_{i \in I} O_i$$

We wish to show that under f, Y is covered by only finitely many such O_i. However, we already know that the preimage of a an open set under a continuous function is open. That is, if O_i is an open set and f is continuous

$$P_i = f^{-1}(O_i)$$

is open. Since f is *onto* we have that $f(x) \in O_i \implies x \in X$ and so by the definition of f^{-1} we have

$$X \subset \bigcup_{i \in I} f^{-1}(O_i)$$

So these sets $\{f^{-1}(O_i)\}_{i \in I}$ cover our compact space X, and since X is compact we can select finitely many such open sets and still cover X, i.e.

$$X \subset \bigcup_{i=1}^{n} f^{-1}(O_i)$$

Applying f to each $f^{-1}(O_i)$ yields $f\left(f^{-1}(O_i)\right) = O_i$ since f is onto. Hence,

$$\begin{aligned} f(X) &\subset \bigcup_{i \in I} f\left(f^{-1}(O_i)\right) \\ \iff Y &\subset \bigcup_{i=1}^{n} O_i \end{aligned}$$

and so Y is covered by finitely many such open sets O_i. Thus, Y is indeed compact, as desired. $\square$