Real Analysis
Lecture Notes

The Real Number System

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1 Open and Closed Sets of $\mathbb{R}$ (con't 2)

1.1 Lindelöf Condition (con't)

Last class we introduced the definition of a Lindelöf collection: A collection C of sets is said to be Lindelöf if every open cover has a countable subcover. We continue this topic with a quick example of a countable subcover.

Example: *(Example of the Lindelöf condition)* Consider the covering of the reals

$$\{(a, b) \ : \ a < b, \ a, b \in \mathbb{R}\}$$

That is, consider the covering of $\mathbb{R}$ given by the collection of all open intervals (a, b) for real numbers $a < b$. Hopefully it is clear that we have uncountably many choices for a and b. However, we also have the countable subcollection

$$\{(-n, n) \ : \ n \in \mathbb{N}\}$$

1.2 Closed Sets in $\mathbb{R}$

Definition: *(Point of closure)* Take $E \subset \mathbb{R}$. We say that a point x is a point of closure if

$$\forall\, \delta > 0, \ (x - \delta, x + \delta) \cap E \neq \emptyset$$

or equivalently, x is a point of closure if

$$\forall\, \delta > 0, \ \exists\, y \in E \text{ such that } |x - y| < \delta$$

Example: *(Example using points of closure)* Let $E = [1, 2)$. We find that 2 is a point of closure since

$$\forall\, \delta > 0 \ (2 - \delta, 2 + \delta) \cap [1, 2) \neq \emptyset$$

since the points in E given by $(2-\delta, 2)$ will overlap with $(2-\delta, 2+\delta)$. Likewise, 1 is a point of sure by the same argument

$$\forall\, \delta > 0\ (1-\delta, 1+\delta) \cap [1,2) \neq \emptyset$$

since the points in E given by $[1, 1+\delta)$ will overlap with $(1-\delta, 1+\delta)$. In fact have that the set $E = [1,2)$ has points of closure in $[1,2]$. In general, each point $x \in E$ of a set $E \subset \mathbb{R}$ is trivially a point of closure of E.[1] We denote the set of points of closure by $\overline{E}$. Thus,

$$E \subset \overline{E}$$

Example: Let $E = \mathbb{Q}$. What are the points of closure for E? Clearly $\pi \notin \mathbb{Q}$, but by the density of $\mathbb{Q}$ in $\mathbb{R}$ we are guaranteed some rational number $q \in (\pi - \delta, \pi + \delta)$ for all $\delta > 0$. That is,

$$\forall\, \delta > 0,\ \exists\, q \in \mathbb{Q} \text{ such that } q \in (\pi - \delta, \pi + \delta)$$

Therefore, the intersection $(\pi - \delta, \pi + \delta) \cap E \neq \emptyset$ since $(\pi - \delta, \pi + \delta)$ contains at least one rational number. Hence, by definition, π is indeed a point of closure of $\mathbb{Q}$. In fact, by this argument, take arbitrary $r \in \mathbb{R}$, then

$$\forall\, r \in \mathbb{R},\ \forall\, \delta > 0,\ \exists\, q \in \mathbb{Q} \text{ such that } q \in (r - \delta, r + \delta)$$

by the density of $\mathbb{Q}$ in $\mathbb{R}$. Therefore, the intersection $(r-\delta, r+\delta) \cap \mathbb{Q} \neq \emptyset$ since it contains at least one rational point $q \in \mathbb{Q}$ from the density of $\mathbb{Q}$. Since $r \in \mathbb{R}$ was arbitrary we conclude that $\mathbb{R}$ is the set of points of closure of $\mathbb{Q}$.

Definition: *(Set of points of closure)* Let $E \subset \mathbb{R}$. We denote by $\overline{E}$ to be the set of points of closure of E. For example,

$$\begin{aligned}
\overline{[0,1]} &= [0,1] \\
\overline{[0,1)} &= [0,1] \\
\overline{(0,1)} &= [0,1] \\
\overline{\mathbb{Q}} &= \mathbb{R} \\
\overline{\mathbb{R} \setminus \mathbb{Q}} &= \mathbb{R} \\
\overline{\emptyset} &= \emptyset
\end{aligned}$$

Proposition:

(a) If $A \subset B$ then $\overline{A} \subset \overline{B}$.

(b) If $A \subset B$ then $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

[1] To show this note that for all $x \in A$ the condition $\forall\, \delta > 0\ (x-\delta, x+\delta) \cap A \neq \emptyset$ is trivially satisfied since this intersection will always contain at least x.

(a) *Proof.* Let $x \in \overline{A}$. That is, x is a point of closure of A. Take $\delta > 0$, then

$$(x-\delta, x+\delta) \cap A \neq \emptyset \implies (x-\delta, x+\delta) \cap B \neq \emptyset$$

since $A \subset B$. More formally,

$$\exists\, y \in \{(x-\delta, x+\delta) \cap A\}$$

Thus, we have some $y \in A$. Since $A \subset B$ we also find that $y \in B$. Therefore,

$$\exists\, y \in \{(x-\delta, x+\delta) \cap B\}$$

That is, $(x-\delta, x+\delta) \cap B \neq \emptyset$, so x is a point of closure of B, $x \in \overline{B}$. Since $x \in \overline{A}$ was arbitrary we find

$$\overline{A} \subset \overline{B}$$

as desired. □

(b) *Proof.* Take $A \subset A \cup B$ and $B \subset A \cup B$. From part (a) we have that

$$\overline{A} \subset \overline{A \cup B}$$
$$\overline{B} \subset \overline{A \cup B}$$

So $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$ which completes the first direction. We proceed using proof by contrapositive. Instead of showing $x \in \overline{A \cup B} \implies x \in \overline{A} \cup \overline{B}$ we will show $x \notin \overline{A} \cup \overline{B} \implies x \notin \overline{A \cup B}$

Now, suppose $x \notin \overline{A} \cup \overline{B}$ so that $x \notin \overline{A}, x \notin \overline{B}$. Then

$$\exists\, \delta_1 > 0 \text{ such that } (x-\delta_1, x+\delta_1) \cap A = \emptyset$$
$$\exists\, \delta_2 > 0 \text{ such that } (x-\delta_2, x+\delta_2) \cap B = \emptyset$$

Taking the minimum $\delta^* = \min\{\delta_1, \delta_2\}$ we still satisfy both equalities with respect to A and B and so

$$(x-\delta^*, x+\delta^*) \cap (A \cup B) = \emptyset$$

But this is precisely the definition of *not* being a point of closure of $(A \cup B)$! Therefore, if $x \notin \overline{A} \cup \overline{B}$ then $x \notin \overline{A \cup B}$. Taking the contrapositive statement we get

$$x \in \overline{A \cup B} \implies x \in \overline{A} \cup \overline{B}$$

which is precisely $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Thus, taking both directions we get

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

as desired. □

Definition: (*What does is mean to be a **closed** set?*) A set F is[2] <u>closed</u> if $F = \overline{F}$.

Proposition: $\overline{\overline{E}} = \overline{E}$. That is, the set of points of closure of $\overline{E}$ is precisely $\overline{E}$. Equivalently, $\overline{E}$ is a closed set.

Proof. Let $x \in \overline{\overline{E}}$. Take $\delta > 0$ and consider $\frac{\delta}{2} > 0$. Since x is a point of closure of $\overline{E}$ we must have some y in the intersection of $\left(x - \frac{\delta}{2}, x + \frac{2}{\delta}\right)$ and $\overline{E}$. That is,

$$\exists\, y \in \left(\overline{E} \cap \left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right)\right)$$

hence, we have some y satisfying

$$|x - y| < \frac{\delta}{2}$$

Since $y \in \left(\overline{E} \cap \left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right)\right)$ we have that $y \in \overline{E}$. Since y is a point of closure of E we must have some z in the intersection of $\left(y - \frac{\delta}{2}, y + \frac{2}{\delta}\right)$ and E. That is,

$$\begin{aligned}
\left(y - \frac{\delta}{2}, y + \frac{\delta}{2}\right) \cap E \neq \emptyset \\
\exists\, z \in \left(E \cap \left(y - \frac{\delta}{2}, y + \frac{\delta}{2}\right)\right) \\
|y - z| < \frac{\delta}{2}
\end{aligned}$$

Thus,

$$\begin{aligned}
|x - y| &< \frac{\delta}{2} \\
|y - z| &< \frac{\delta}{2} \\
\implies |x - y| + |y - z| &= \frac{\delta}{2} + \frac{\delta}{2} = \delta, \quad \text{but} \\
|x - y| + |y - z| &\geq |x - y + y - z| \quad \text{(triangle inequality)} \\
&= |x - z| \\
\implies |x - z| &\leq |x - y| + |y - z| < \delta \\
\implies |x - z| &< \delta
\end{aligned}$$

That is, for $x \in \overline{\overline{E}}$ there is some $z \in E$ such that

$$|x - z| < \delta$$

or equivalently,

$$\exists\, z \in E \quad \text{such that } (x - \delta, x + \delta) \cap E \neq \emptyset$$

So $x \in \overline{\overline{E}}$ is a point of closure of E, so $x \in \overline{E}$, as desired.

$\square$

[2]The notation F for a closed set is from the French *fermé*.