Real Analysis
Lecture Notes

Metric Spaces

November 2 2016
Last update: December 4, 2017

1 Relative Topologies

Definition: ***Closed and open sets in a subspace.*** Let S and X be metric spaces such that $S \subset X$ and A some set $A \subset X$. We say that $\underline{A \text{ is closed in subspace } S}$ and $\underline{A \text{ is open in subspace } S}$ if

$$A = F \cap S \quad \text{for some closed set } F \subset X$$

and

$$A = O \cap S \quad \text{for some closed set } O \subset X$$

respectively.

Proposition: ***A closed set in a subspace is the intersection of the subspace with a closed set in the superspace.*** Let X be some metric space and $E \subset S \subset X$. The closure of E relative to S is the closure of E relative to X intersected with S.

Proof. A point of closure $x \in E \subset X$ is a point of X in which its neighbourhood in X intersects with E. That is, $x \in E$ is a point of closure of E if a small ball $S_{x,\epsilon}$ centered at x contains a point in $E \subset X$. Similarly, a point of closure $x \in E \subset S$ is a point of S in which its neighbourhood in S intersects with E.

Clearly the closure of E in S, denoted by $\mathrm{cl}_S(E) = \overline{E}_S$, is the closure of X intersected with S since $S \subset X$. That is,

$$\begin{aligned} &\overline{E} \quad \text{closed in } X \\ \implies &\overline{E} \cap S \quad \text{closed in } S \end{aligned}$$

Suppose A is closed in S. Then the closure of A in S must be the set A itself, $\overline{A} \cap S = A$, by the definition of a closed set: $\overline{F} = F$.

However, since this closure is $\overline{A} \cap S$ where $\overline{A}$ is closed in X, and since A was an arbitrary subset $A \subset S \subset X$, we may conclude that every closed set in S is the intersection over S of some closed set in X. □

Proposition: ***A closed set in a superspace intersected with a subspace is closed in the subspace.*** This is the the converse proposition: If F is a closed set in X then the intersection $\overline{F} \cap S$ closed.

Proof. Note that we have the intersection

$$S \cap \overline{(S \cap F)} \subset S$$

Now, clearly $S \cap F \subset S$ and so, by our earlier results on set closures,

$$\overline{(S \cap F)} \subset \overline{S}$$

and by the definition of intersection we must also have $(S \cap F) \subset F$ and so

$$\overline{(S \cap F)} \subset \overline{F}$$

Therefore

$$\overline{(S \cap F)} \subset \overline{S} \cap \overline{F}$$

but

$$\begin{aligned} S \cap \overline{(S \cap F)} &\subset S \cap \left(\overline{S} \cap \overline{F}\right) \\ &= S \cap \overline{S} \cap \overline{F} \\ &= \left(S \cap \overline{S}\right) \cap \overline{F} \\ &= S \cap \overline{F} \\ &= S \cap F \quad \text{(since } F \text{ is closed)} \end{aligned}$$

Hence, if F is closed in X and $S \subset X$, then

$$S \cap \overline{(S \cap F)} \subset S \cap F$$

but since $F \subset S$, we also note that

$$\begin{aligned} & F \subset S \cap F \\ \implies & \overline{F} \subset \overline{S \cap F} \\ \implies & F \subset \overline{S \cap F} \quad (\text{since } \overline{F} = F) \\ \implies & S \cap F \subset S \cap \overline{S \cap F} \end{aligned}$$

and since both

$$\begin{aligned} S \cap \overline{(S \cap F)} &\subset S \cap F \\ S \cap F &\subset S \cap \overline{S \cap F} \end{aligned}$$

we may conclude that

$$S \cap F = S \cap \overline{S \cap F}$$

for $\overline{(S \cap F)}$ the closure of $S \cap F = S \cap \overline{F}$ in X. Therefore, then $S \cap F = S \cap \overline{F}$ is indeed closed in S since it satisfies the definition of closure:

Set A is closed in $S \subset X$ if $A = F' \cap S$ for some closed set $F' \subset X$

with $A = S \cap F = S \cap \overline{F}$ and $F' = \overline{S \cap F}$ clearly closed in X, as desired. $\square$

Putting our two propositions together we get the following result:
Theorem: Let $S \subset X$ for spaces S and X and A some subset of X. Then,

$$\text{A set } A \text{ is closed in } S \iff A = F \cap S \text{ for } F \text{ some closed set in } X$$

Proof. ($\implies$) Our proof follows immediately from our two propositions above. □

Theorem: ***(The analogous Theorem for open sets).*** Let $S \subset X$ for spaces S and X and A some subset of X. Then,

$$\text{A set } A \text{ is open in } S \iff A = O \cap S \text{ for } O \text{ some open set in } X$$

Proof. We consider now open sets in our subspace. Suppose that A is now *open* in S. Then, using our results on the complements of open sets, the intersection

$$S \setminus A = S \cap A^c$$

must be closed in S. Therefore, from the above Theorem on closed sets, there exists a closed set F in X such that

$$\begin{aligned} S \setminus A &= F \cap S \\ \iff S \cap A^c &= F \cap S \\ \implies A^c &= F \\ \implies X \setminus A^c &= X \setminus F \\ \iff X \cap (A^c)^c &= X \setminus F \\ \iff X \cap A &= X \setminus F \\ \iff A &= X \setminus F \end{aligned}$$

Since F is closed in X then $X \setminus F = X \cap F^c$ is open in X. Hence

$$S \cap A = (X \setminus F) \cap S$$

That is, if A is some open set in S then $A = (X \setminus F) \cap S$ for some open set $(X \setminus F)$, for some open set $(X \setminus F)$ in X, or more succinctly:

$$\text{If } A \text{ is some open set in } S \implies A = O \cap S \text{ for } O = (X \setminus F) \text{ some open set in } X.$$

which completes the first direction.

($\impliedby$) Conversely, take J some open set $J \subset X$. Then the intersection

$$X \setminus J = X \cap J^c$$

is *closed* in X. Now, let A be given by

$$A = (X \setminus J) \cap S = (X \cap J^c) \cap S$$

and from our second proposition regarding closed sets in a superspace we may conclude that A is closed in S. However,

$$\begin{aligned} A &= (X \setminus J) \cap S \\ &= (X \cap J^c) \cap S \\ &= X \cap J^c \cap S \\ &= X \cap S \cap J^c \\ &= (X \cap S) \cap J^c \\ &= S \cap J^c \\ &= S \setminus J \end{aligned}$$

and so if A is closed in S then the complement $S \setminus A = S \cap A^c$ must be open in A, but

$$\begin{aligned} A &= S \setminus J \\ \implies S \setminus A &= S \setminus (S \setminus J) \\ &= S \cap (S \setminus J)^c \\ &= S \cap (S \cap J^c)^c \\ &= S \cap (S^c \cup (J^c)^c) \\ &= S \cap (S^c \cup J) \\ &= S \cap S^c \cup S \cap J \\ &= \emptyset \cup S \cap J \\ &= S \cap J \\ A &= S \cap J \end{aligned}$$

That is, if $A = (X \setminus J) \cap S$ for some closed set $(X \setminus J)$ then $A = (X \setminus J) \cap S$ is closed in S and so $S \setminus A = S \cap J$ is open in S. Putting this all together:

$$\text{If } A = O \cap S \text{ for some open set } O = J \text{ in } X \implies A \text{ is open in } S.$$

completing the converse direction.
Placing these two conclusions together gives us the result

$$\text{A set } A \text{ is open in } S \iff A = O \cap S \text{ for } O \text{ some open set in } X$$

which completes our analogous Theorem for relative topologies with respect to open sets, as desired. $\square$

Recall that a metric space is separable by definition

$$\iff \text{it has a countable dense subset}$$

However, we have also proven that a metric space is separable

$$\iff \text{there is a countable } \textit{base} \text{ for the topology}$$

where a *base* is a family of open sets $\mathcal{O} = \{O_i\}_{i \in I}$ such that every open set in the metric space can be generated by a union of base elements.

Claim: ***Any subspace of a metric space is separable.***

Proof. Let X be a separable metric space. By definition X must have a countable base $\mathcal{O} = \{O_i\}_{i=1}^{\infty}$. Let S be a subspace of X so that $S \subset X$. It is sufficient to show that

$$\mathcal{B} = \{\{O_1 \cap S\}, \{O_2 \cap S\}, ...\}$$

forms a base for S since this would be the desired countable base to yield separability. Take J some open set in S. From our earlier result we know that J can be written as the intersection of some open set O in X so that

$$J = O \cap S$$

However, since the family $\{O_i\}_{i=1}^{\infty}$ is a base for X we have that our open set $O \subset X$ must be expressible as a union of base elements

$$O = \bigcup_{k \in K} O_k, \quad K \subset \mathbb{N}$$

Thus

$$\begin{aligned} J &= O \cap S \\ &= \left(\bigcup_{k \in K} O_k \right) \cap S \\ &= \bigcup_{k \in K} (O_k \cap S) \end{aligned}$$

Therefore, since J was an arbitrary open set in S, we have that the desired family of sets $\mathcal{B}$ may be given by the union of all possible $O_i \cap S$. That is, the family

$$\mathcal{B} = \{\{O_i \cap S\}\}_{i=1}^{\infty}$$

is countable since $\mathcal{O} = \{O_i\}_{i=1}^{\infty}$ is countable. Since S has a countable base $\mathcal{B}$ for its topology we may conclude that S is indeed separable, as desired. $\square$

Corollary: The set of real numbers $\mathbb{R}$ has as its base $\mathbb{Q}$, and since $\mathbb{Q}$ is countable and dense, $\mathbb{R}$ is a separable metric space. Furthermore, since the set of *irrationals*, $\mathbb{R} \setminus \mathbb{Q}$, is a subspace of $\mathbb{R}$ we have that $\mathbb{R} \setminus \mathbb{Q}$ is separable (i.e. it has a countable dense subset). Similarly, since the set of natural numbers $\mathbb{N}$ is a subspace of $\mathbb{R}$ we are giving that $\mathbb{N}$ is separable.[1]

[1] Separability of the natural numbers $\mathbb{N}$ is obvious from the fact that it's a countable set, and so it clearly has a countable dense subset (dense subset with respect to $\mathbb{N}$).

Example: *(Example of open and closed sets not surviving the ambient topology)*. Let us revisit the example introduced in the previous set of notes. Consider the metric spaces $X = \mathbb{R}$ and $S = \mathbb{Q}$, and the intervals (e, π) and $[e, \pi]$. Clearly the intervals

$$(e, \pi)$$
$$[e, \pi]$$

are open and closed in $\mathbb{R}$ by construction. If we consider the intersections

$$(e, \pi) \cap \mathbb{Q}$$

we see that $(e, \pi) \cap \mathbb{Q}$ is *open* in $\mathbb{Q}$ since (e, π) is open in $\mathbb{R}$ (i.e. a direct application of our earlier Theorem for open sets). On the other hand, the intersection

$$[e, \pi] \cap \mathbb{Q}$$

is *closed* in $\mathbb{Q}$ by the analogous Theorem for closed sets. However,

$$(e, \pi) \cap \mathbb{Q} = [e, \pi] \cap \mathbb{Q}$$

since neither e nor π in $[e, \pi]$ are elements of $\mathbb{Q}$. Therefore, $\mathbb{Q}$ contains sets which are *both open and closed*! Such sets which are both open and closed are said to be clopen sets.

We have been working with open and closed sets and their interaction with metric spaces and their subspaces. We found that open/closed sets may not survive as open/closed into the subspace. For this reason we wish to create a notion similar to *openness* and *closedness* which remains unaffected when considering subspaces of metric spaces.

We suggest now that the "next best thing" to a closed set will be a *compact set*.

Result: Let X be a metric space such that $A \subset S \subset X$.

Statement 1: A is a *compact* subset of X. That is, if A is covered by the open cover $\mathcal{P} = \{P_i\}_{i \in I}$, for open sets $P_i \subset X$, then A is covered by a finite number of such P_i to form the finite subcover $\{P_1, P_2, ..., P_n\}$ so that $A \subset \bigcup_{i=1}^{n} P_i$.

Statement 2: A is a *compact* subset of S. That is, if A is covered by the open cover $\mathcal{J} = \{J_i\}_{i \in I}$, for open sets $J_i \subset S$, then A is covered by a finite number of such J_i to form the finite subcover $\{J_1, J_2, ..., J_n\}$ so that $A \subset \bigcup_{i=1}^{n} J_i$.

1. *Proof of Statement 1.*

   Proof. ($2 \implies 1$) Suppose Statement 2 holds so that A is compact in S. Then A has the finite subcover of open sets $J_i \subset S$

$$A \subset \bigcup_{i=1}^{n} J_i \quad J_i \text{ open in } S$$

but

$$
\begin{aligned}
A &\subset \bigcup_{i=1}^{n} J_i \\
\implies A \cap S &\subset \left(\bigcup_{i=1}^{n} J_i\right) \cap S \\
\iff A \cap S &\subset \bigcup_{i=1}^{n} (J_i \cap S) \\
\iff A &\subset \bigcup_{i=1}^{n} (J_i \cap S) \quad (\text{since } A \subset S)
\end{aligned}
$$

Since each $(J_i \cap S) \subset X$ we have that

$$\bigcup_{i=1}^{n} (J_i \cap S) \subset X$$

so that the family of open sets

$$A \subset \mathcal{P} = \{P_i\}_{i=1}^{n} = \{J_i \cap S\}_{i=1}^{n}$$

form a finite open cover for A in X. Therefore, A is compact in X, as desired. □

2. *Proof of Statement 2.*

Proof. ($1 \implies 2$) Suppose Statement 1 holds. Take $A \subset \bigcup J_i$ for J_i open sets in S. Since each J_i are open we have from our earlier Theorems that

$$J_i = P_i \cap S \quad \text{for some } P_i \text{ open set in } X$$

and by assumption of Statement 1 we have that A is compact in X so that

$$A \subset \bigcup_{i=1}^{n} P_i$$

for P_i open sets in X. Therefore, to generate a finite subcover of S we may intersect A with S to produce the union

$$
\begin{aligned}
A \cap S &= \left(\bigcup_{i=1}^{n} P_i\right) \\
&= \bigcup_{i=1}^{n} (P_i \cap S)
\end{aligned}
$$

Letting each $J_i = P_i \cap S$ we form the finite subcover $\mathcal{J} = \{J_i\}_{i=1}^n$ such that

$$A \subset \bigcup_{i=1}^{n} J_i$$

for arbitrary compact set $A \subset X$. That is, A is covered by the finite open cover $\{J_i\}_{i=1}^n$ of open sets $J_i \subset S$, so A is compact in S, as desired. □

Goal: Our present and upcoming goal will be to *characterize compact metric spaces* in greater detail. We will soon make use of a fairly involved proof by contrapositive. Recall that this method of proof uses the logical identity

$$(p \implies q) \iff (\neg q \implies \neg p)$$

In particular, let

$$\begin{aligned} p &: \ \{O_i\} \text{ cover } X \\ q &: \ \text{finitely many } \{O_i\} \text{ cover } X \end{aligned}$$

We said that a set is compact in some space of all its covers have a finite subcover. In this case we may express this definition as

$$\text{Compactness} \iff (p \implies q)$$

Negating these statements yield

$$\begin{aligned} \neg q &: \ \text{finitely many } \{O_i\} \textit{ do not } \text{cover } X \\ \neg p &: \ \{O_i\} \textit{ do not } \text{cover } X \end{aligned}$$

Note that if $\{O_i\}$ *do not cover* X then $X \not\subseteq \{O_i\}$ which is equivalent to

$$\begin{aligned} &\{O_i\} \text{ do not cover } X \\ &\iff X \not\subseteq \{O_i\} \\ &\implies X^c \not\supseteq \{O_i\}^c \\ &\implies (X \cap X^c) \not\supseteq (X \cap \{O_i\}^c) \\ &\implies \emptyset \not\supseteq X \setminus \{O_i\} \\ &\implies X \setminus \{O_i\} \neq \emptyset \end{aligned}$$

Therefore, we can simplify our contrapositive definition of compactness to

$$\begin{aligned} \neg q &: \ X \setminus \{O_i\} \neq \emptyset \text{ for finitely many } O_i \\ \neg p &: \ X \setminus \{O_i\} \neq \emptyset \end{aligned}$$