Real Analysis
Lecture Notes

Metric Spaces

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1 Separable Metric Spaces

Last time we were talking about the Sorgenfry line, denoted by $\mathbb{S}$. The Sorgenfry line was defined to be a topology on $\mathbb{R}$ such that intervals of the form $[a, b)$ were said to be open. Furthermore, we had shown the result that $\mathbb{S}$ has a countable discrete subset.[1]

Finally, we ended last class with an incomplete proof that any compact subset of the Sorgenfry line $\mathcal{S}$ is countable. This was not true for $\mathbb{R}$ since, by the Heine-Borel theorem, the interval $[1, 2]$ is compact since it is a closed bounded set. We present in this class a more complete proof of this statement.

Claim: ***Any compact subset of*** $\mathbb{S}$ ***is countable.*** Let $\mathcal{F}$ be the family of intervals of the form

$$\mathcal{F} = \bigcup \left(-\infty, x - \frac{1}{n}\right), [x, \infty), \quad n \in \mathbb{N}$$

We should be able to see that, over all $n \in \mathbb{N}$, the family $\mathcal{F}$ covers the real line $\mathbb{R}$. Therefore $\mathcal{F}$ must cover all subsets of $\mathbb{R}$. Now, suppose we take a compact subset $C \subset \mathbb{R}$. For this compact set C take some point $x \in C$.

Our compact subset $C \subset \mathbb{R}$ must be covered by $\mathcal{F}$ since this family covers all subsets of $\mathbb{R}$. However, since C is compact, there must be a finite subcover of $\mathcal{F}$ that covers C. What does this finite subset of $\mathcal{F}$ look like?

Suppose our finite subcover contains the interval $[x, \infty)$. Then, in order to remain finite, $\mathcal{F}$ may only contain finitely many intervals of the form

$$\left(-\infty, \frac{1}{n_1}\right), \left(-\infty, \frac{1}{n_2}\right), ..., \left(-\infty, \frac{1}{n_k}\right)$$

[1] Did we show this? I only see that we stated $\mathbb{S}$ has a countable dense subset $\mathbb{Q}$ such that $[a, b) \cap \mathbb{Q} \neq \emptyset$ so $\mathbb{S}$ must be separable, and that any compact[2]subset of $\mathbb{S}$ is countable.

[2] Compactness: All open covers have a finite subcover.

An immediate consequence is that there must be some largest $n_p = \max\{n_1, n_2, ..., n_k\}$. For this largest n_p we see that

$$\frac{1}{n_p} \leq \frac{1}{n_i}, \quad i = 1, 2, ..., k$$
$$\implies x - \frac{1}{n_p} \geq x - \frac{1}{n_i}$$
$$\implies \left(-\infty, x - \frac{1}{n_i}\right) \subset \left(-\infty, x - \frac{1}{n_p}\right)$$

Thus, C is covered by the union

$$C \subset \left(-\infty, x - \frac{1}{n_p}\right) \cup [x, \infty)$$

Now, pick some number a_x so that

$$a_x \in \left(x - \frac{1}{n_p}, x\right)$$

and consider the interval

$$(a_x, x] \cap C$$

Clearly, $x \in C$ by assumption, so $x \in (a_x, x] \cap C$, and since C was covered by

$$C \subset \left(-\infty, x - \frac{1}{n_p}\right) \cup [x, \infty)$$

Therefore, since $C \cap \left[x - \frac{1}{n_p}, x\right) = \emptyset$,

$$(a_x, x] \cap C = \{x\}$$

Suppose we repeat this process and argument for some different $x' \in C$, $x' \neq x$. Then

$$(a_{x'}, x'] \cap C = \{x'\}$$

but $x \leq a_{x'}$,[3] hence

$$(a_x, x] \cap (a_{x'}, x'] = \emptyset$$

Therefore, the intervals $(a_x, x]$ for $x \in C$ are pairwise disjoint. However, $a_x \in \left(x - \frac{1}{n_p}, x\right) \implies a_x < x$, hence

$$\exists\, q_x \in \mathbb{Q},\ a_x < q_x < x$$

Can two different x's have the same rational q_x? No! Since our intervals $(a_x, x]$ are pairwise disjoint we know that each q_x must uniquely identify a single interval. Since the rationals $\mathbb{Q}$ are countable we may conclude that there are only countably many intervals of the form $(a_x, x]$.

However, every x generates a unique interval $(a_x, x]$! Therefore, there are only countably many points $x \in C$. That is, our set C, a compact subset of $\mathbb{R}$ is countable, and since C was arbitrary we may conclude that any compact subset of $\mathbb{S}$ is indeed countable, as desired.

[3] Where does this inequality come from? Is this an assumption/criteria for selecting x' or is this a consequence of something else?

2 Topological Bases

Definition: ***(Base for a topology)*** We say that a base for a topology (X, ρ) is a family of open sets B such that every open set is the union of open sets in B. That is, for any open set $O \subset X$, a base B composed of open sets B_i, $B = \{B_i\}_{i \in I}$, satisfies

$$O = \bigcup_{i \in I} B_i$$

Example: Consider B a base for the topology on the Sorgenfry line $\mathbb{S}$. Take point $x \in \mathbb{R}$. Then set $[x, x+1)$ is open in $\mathbb{S}$ is open by definition. Therefore, under our base B we have

$$\exists\, B_x \in B \text{ such that } x \in B_x \subset [x, x+1)$$

Take $y \neq x$. Without loss of generality let $x < y$. From $[y, y+1)$ we get an open set B_y such that

$$y \in B_y \subset [y, y+1)$$

Since $x < y$ we know that $x \notin B_y \subset [y, y+1)$. So, $x \in B_x, y \in B_y$ and since $x \neq y$ we have $B_x \neq B_y$. This gives us that a mapping from $\mathbb{R} \to B$ is one-to-one. Therefore, the image of $\mathbb{R}$ must have cardinality greater than to $\mathbb{R}$ itself. However, we know that $\mathbb{R}$ is uncountable. Hence,

$$\text{card } B \geq \text{card } \mathbb{R}$$

Therefore, B must be uncountable. However, we found last time that if X is a metric space then the following are equivalent:

1. X is separable.
2. X has a countable base.

but for the Sorgenfry $\mathbb{S}$ we have that

1. $\mathbb{S}$ is separable.
2. Any base for $\mathbb{S}$ is uncountable.

Therefore, we must conclude that $\mathbb{S}$ is not *metrizable*. That is, there is no metric on $\mathbb{R}$ for which the open sets generated by the metric are exactly the open sets in the Sorgenfry line.

Definition: ***(Equivalent metric spaces)*** Let X be some space with metrics ρ and ρ'. We say that metrics ρ and ρ' are equivalent if they both give rise to the same open sets.

[BOOK DEFINITION OF EQUIVALENT METRIC SPACES]

Example: Consider $X = \mathbb{R}$ and

$$\rho(x, y) = |y - x|$$
$$\rho(x, y) = \frac{1}{2}|y - x|$$

Clearly any open set defined by the metric spaces (X, ρ) can also be found in (X, ρ').

Definition: ***(Bounded metrics)*** A metric ρ is said to be <u>bounded</u> in space X if, for $M \in \mathbb{R}$ finite,

$$\forall\, x, y \in X,\ \ \rho(x, y) \leq M$$

An immediate example is given by the discrete metric

$$\rho_d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

then, for all $x, y \in X$ we find $\rho_d(x, y) \leq 1$.

Proposition: ***Every metric is equivalent to some bounded metric***. In particular, if (X, ρ) is some metric space then ρ will be equivalent to

$$\frac{\rho(x, y)}{1 + \rho(x, y)} = \rho'(x, y)$$

which will be shown to be bound by $\rho'(x, y) \leq 1$ and give rise to the same open sets generated by ρ.

Proof.

1. Is ρ' bounded? Clearly $\rho' \geq 0$ by definition, and as $\rho \to \infty$ we see that

   $$\rho' = \frac{\rho}{1 + \rho} \to 1 \implies \rho' \in [0, 1)$$

   Hence, $\rho'(x, y) \leq 1$ for all $x, y \in X$.

2. Given that ρ is a metric, is ρ' also a metric?

   (a) Clearly $\rho'(x, y) \geq 0$.

   (b) If $\rho' = \frac{\rho}{1+\rho} = 0$ then $\rho = 0 \iff x = y$. Thus, $\rho' = 0 \iff x = y$.

   (c) $\rho'(x, y) = \rho'(y, x)$ since $\rho(x, y) = \rho(y, x)$.

   (d) Does the triangle inequality hold? Working backwards we wish to prove that

   $$\rho'(x, z) \leq \rho'(x, y) + \rho'(y, z)$$

   but $\rho' = \frac{\rho}{1+\rho}$, so

   $$\begin{aligned} \frac{\rho(x, z)}{1 + \rho(x, z)} &\leq \frac{\rho(x, y)}{1 + \rho(x, y)} + \frac{\rho(y, z)}{1 + \rho(y, z)} \\ \frac{\rho(x, z)}{1 + \rho(x, z)} &\leq \frac{\rho(x, y) + \rho(x, y)\rho(y, z) + \rho(y, z) + \rho(y, z)\rho(x, y)}{(1 + \rho(x, y))(1 + \rho(y, z))} \\ &\vdots \end{aligned}$$

Continuous this tedious algebra we find that the triangle inequality is indeed satisfied.

3. Are the metrics ρ and $\rho' = \frac{\rho}{1+\rho}$ equivalent? Recall that open sets under a metric ρ are unions of spheres of the form

$$B_{x,\epsilon,\rho} = \{y \in X \ : \ \rho(x,y) < \epsilon\}$$

Therefore, we will have equivalence between ρ and ρ' if the spheres generated by ρ are also open in the topology generated by ρ', and if the spheres generated by ρ' are open in the topology of ρ.

Before moving on to prove this result we must quickly prove the following lemma:

Lemma:

(a) $B_{x,r,\rho} \subseteq B_{x,r,\rho'}$ (B centered at x with radius r under ρ).

Proof. Let $y \in B_{x,r,\rho}$. Then

$$\begin{aligned} &\rho(x,y) < r \\ \implies &\frac{\rho(x,y)}{\rho(x,y)+1} < r \quad (\text{since } 1 \leq 1+\rho(x,y)) \\ \iff &\rho'(x,y) < r \\ \implies &y \in B_{x,r,\rho'} \\ \implies &B_{x,r,\rho} \subseteq B_{x,r,\rho'} \end{aligned}$$

□

(b) $B_{x,\frac{r}{r+1},\rho'} \subseteq B_{x,r,\rho}$.

Proof. Take $z \in B_{x,\frac{r}{r+1},\rho'}$. Then

$$\begin{aligned} &\rho'(x,z) < \frac{r}{r+1} \\ \implies &\frac{\rho(x,z)}{\rho(x,z)+1} < \frac{r}{r+1} \\ \implies &\rho(x,z)(r+1) < r(\rho(x,z)+1) \\ \implies &\rho(x,z)r + \rho(x,z) < r\rho(x,z) + r \\ \implies &\rho(x,z) < r \\ \implies &z \in B_{x,r,\rho} \\ \implies &B_{x,\frac{r}{r+1},\rho'} \subseteq B_{x,r,\rho} \end{aligned}$$

□

Now, back to our proposition at hand. Are the open sets under ρ open under ρ' and vice-versa? In particular, is $B_{x,r,\rho'}$ open in the ρ-topology? Take some point $t \in B_{x,r,\rho'}$. Since $B_{x,r,\rho'}$ is open we can find a smaller sphere lying within it:

$$\exists\, d > 0 \text{ such that } B_{t,d,\rho'} \subset B_{x,r,\rho'}$$

and using our first lemma:

$$B_{t,d,\rho} \subset B_{t,d,\rho'}$$

Therefore, every point $t \in B_{x,r,\rho'}$ lies within some open set in the ρ metric given by $B_{t,d,\rho}$. That is, every point lies within some open subset $B_{t,d,\rho}$ of $B_{x,r,\rho'}$.

Therefore, taking the union of all $t \in B_{x,r,\rho'}$ we find that the open set $B_{x,r,\rho'}$ is a union of open sets of the form $B_{t,d,\rho}$, which are open in the ρ-topology.

By the same argument we may find that $B_{x,r,\rho}$ will be open in the ρ'-topology. Hence, every metric ρ is equivalent to a bounded metric $\rho' = \frac{\rho}{1+\rho}$.

$\square$