# Real Analysis
# Lecture Notes

Metric Spaces

November 23 2016
Last update: December 4, 2017

## 1 Compactness

We wish to work up to the following result: A metric space is compact if and only if it is totally bounded and complete (where *complete* will be defined shortly). Note that this is similar to the Heine-Borel Theorem stating that a closed-bounded set is compact. However, in our case the added restrictions permit us to now state the result in both directions.

**Definition:** ***Complete metric space.*** We say that a metric space is complete if every Cauchy sequence from $X$ is convergent.

Recall our previous result: If $X$ is a metric space then the following statements are equivalent

1. $X$ is compact.
2. $X$ satisfies the Bolzano-Weierstrass property.
3. $X$ is sequentially compact (all sequences have a convergent subsequence).
4. $X$ is totally bounded (+ complete, to be shown)

**Example:** Consider $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{R}$ under the discrete metric $\rho_d$

$$\rho_d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{else} \end{cases}$$

Then

1. $N$ is not bounded and so it cannot be totally bounded. However, $\mathbb{N}$ is complete since any Cauchy sequence from $\mathbb{N}$ will produce $0 < \epsilon$ so that our Cauchy sequence $(x_1, x_2, ...)$ will eventually repeat (and thus, be convergent) when $|x_n - x|$ gets small enough.
2. Likewise, $\mathbb{R}$ is not bounded, and so not totally bounded, but complete.

3. $\mathbb{Q}$ is not bounded, and so not totally bounded. However, unlike $\mathbb{N}$ and $\mathbb{R}$, $\mathbb{Q}$ is *not* complete (recall the completeness axiom). Cauchy sequence from $\mathbb{Q}$ might *not* be convergent since the sequence $(x_n) \to \sqrt{2}$ will never converge to $\sqrt{2}$ in $\mathbb{Q}$ for all $\epsilon > 0$.

4. $(0,1)$ is totally bounded, but is *not* complete since we may have the Cauchy sequence $(x_n) \to 0$ or $(x_n) \to 1$, converging outside of our space.

5. $\mathbb{R}$ under $\rho_d$ is complete since if $|x_m - x_n| < 1$ then $x_m = x_n$ and so $|x_m - x_n| = |x_m - x_m| = 0 < \epsilon$. However, $\mathbb{R}$ under the discrete metric is not totally bounded since the open cover
$$\left\{ S_{x,\epsilon=\frac{1}{2}} \right\}_{x\in\mathbb{R}}$$
will not be able to cover $\mathbb{R}$ with only finitely many $S_{x,\epsilon}$.

**Theorem:** ***A metric space is compact if and only if it is totally bounded and complete.***

*Proof.* ($\Longrightarrow$) Suppose some metric space $X$ is compact. We immediately get that our space $X$ must be totally bounded since if $X$ is compact all covers have a finite subcover. Hence, the cover given by
$$X \subset \bigcup_{x\in X} S_{x,\epsilon}$$
has some finite subcover
$$X \subset \bigcup_{i=1}^{n} S_{x_i,\epsilon}$$
satisfying the definition of total boundedness. We now seek to show that all Cauchy sequences are convergent. Let $(x_n)$ be some Cauchy sequence from $X$. We wish to prove that $(x_n) \to x$ for some $x \in X$.

Since $X$ is compact we are given that $X$ is *sequentially compact*. Therefore, all sequences from $X$ have a convergent subsequence. Therefore, our sequence $(x_n)$ has a subsequence $(x_{n_k})$ such that $(x_{n_k}) \to x$ for some $x \in X$.

However, since $(x_n)$ is a Cauchy sequence it must also converge to the same limit as its subsequences. We can see this by letting $(x_{n_k})$ be our subsequence and $x$ its limit. Then,
$$\forall\, \epsilon > 0,\ \exists, K \in \mathbb{N},\ \forall\, k \geq K,\ \ |x_{n_k} - x| < \epsilon$$
and if the sequence is Cauchy then for sufficiently large $N$ and $n, m \geq N$
$$|x_n - x_m| < \epsilon$$
Thus
$$\forall\, \epsilon > 0,\ \exists, N \in \mathbb{N},\ \forall\, n \geq N,\ \ |x_n - x| < \epsilon$$
in particular, we may let $N = n_K$, and where $\forall\, n \geq N$ holds since for all $n, m \geq N$, $|x_n - x_m| < \epsilon$, i.e. all points beyond $N$ are at least as close as each other.

Therefore, if $X$ is compact then $X$ is totally bounded, since the cover of open spheres has a finite subcover, and complete, since every Cauchy sequence is convergent, which completes the first direction

$(\Longleftarrow)$ Suppose some metric space $X$ is totally bounded and complete. To show that $X$ is complete it is sufficient to show that $X$ is sequentially compact. That is, it is sufficient to show that every sequence $(x_n)$ has a convergent subsequence. However, if we must show that every sequence $(x_n)$ has a convergent subsequence then it is sufficient to show that every sequence $(x_n)$ has a Cauchy subsequence $(x_{n_k})$.

As an aside note that if some set $B$ is an infinite set and we conver it by finitely many sets $A_1, ..., A_n$ we must have that at least one of the $A_i$ is an infinite set.

Now, consider the sequence $(x_n)$ and let $r = 1$ be a radius for a set of open spheres in $X$. Since $X$ is totally bounded $X$ must be covered by finitely many such spheres with radius $r = 1$. However, by our aside, at least one of our spheres must be an infinite set containing infinitely many such $x_i$. Label this infinite set by

$$S_{\epsilon=1}$$

Pick one point from this spheres $x_{n_1} \in \{x_i\} = S_1$. Clearly the intersection

$$S_1 \setminus \{x_i\}$$

still has infinitely many elements left within it.

Let $\epsilon = \frac{1}{2}$. Since $X$ is totally bounded $X$ is covered by finitely many spheres of radius $\frac{1}{2}$. However, these finitely many spheres which cover $X$ must also cover the intersection $S_1 \setminus \{x_i\}$. Therefore, there must exist some sphere $S_2$ with radius $\epsilon = \frac{1}{2}$ so that

$$S_2 \cap (S_1 \setminus \{x_i\})$$

is still infinite. From this intersection pick the next point $x_{n_2}$

$$x_{n_2} \in S_2 \cap (S_1 \setminus \{x_i\})$$

Consider the distance $\rho(x_{n_1}, x_{n_2})$. Since both points are within $S_1$ by definition we must have a worst-case upper bound

$$\rho(x_{n_1}, x_{n_2}) < 2$$

With the same argument, let $r = \frac{1}{3}$ and construct $S_3$ a sphere of radius $\frac{1}{3}$ so that

$$S_3 \cap S_2 \cap S_1$$

has infinitely many points *beyond* the finite sequence

$$\{x_1, x_2, ..., x_{n_1}, ..., x_{n_2}\}$$

Pick one of these points beyond $x_{n_2}$ and call it $x_{n_3}$. Since $x_{n_3}$ and $x_{n_2}$ are both in $S_2$ with radius $\frac{1}{2}$ we have the worst-case upper bound of their distance given by

$$\rho(x_{n_1}, x_{n_2}) < 1$$

Continue inductively this manner.

From this process we have constructed the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ which is a subsequence of $\{x_n\}_{n=1}^{\infty}$ with distances

$$\rho(x_{n_k}, x_{n_l}) < \frac{2}{N}, \quad \text{if } k, l \geq N$$

Thus $\rho(x_{n_k}, x_{n_l}) \to 0$ for sufficiently large $N$, and so our subsequence $(x_{n_k})$. Therefore, if $X$ is some totally bounded and complete metric space we have that every sequence from $X$ has a Cauchy subsequence, and so every sequence from $X$ has a convergent subsequence, and so $X$ is sequentially compact, and so $X$ is complete, as desired. □

**Aside:** In general, if $X$ is some metric space it can be *completed* via Cauchy sequences, and if $X$ is a totally bounded metric space its completion will also be totally bounded.

**Baire Category Theorem:** Let $X$ be a complete metric space and take a countable family of dense open sets $\{O_i\}_{i=1}^{\infty}$ from $X$. The intersection

$$\bigcap_{i=1}^{\infty} O_i$$

is dense.

**Example:** Let $X = \mathbb{Q}$ and take $q \in \mathbb{Q}$. Is the intersection $\mathbb{Q} \setminus \{q\}$ dense in $\mathbb{Q}$? Since $\{q\}$ is closed we must have that

$$\mathbb{Q} \setminus \{q\} \quad \text{is open}$$

and

$$\bigcap_{q \in \mathbb{Q}} (\mathbb{Q} \setminus \{q\}) = \emptyset$$

but $\emptyset$ is trivially *not* dense in $\mathbb{Q}$! This is fine and doesn't contradict the Baire Category Theorem since $\mathbb{Q}$ is not a complete metric space.

**Corollary:** ***(Consequence of the Baire Category Theorem).*** We can show that $\mathbb{Q}$ is not a countable intersection of open sets. That is, $\mathbb{Q}$ is not a $G_\delta$.

*Proof.* Suppose that $\mathbb{Q}$ is in fact a $G_\delta$ set so that

$$\mathbb{Q} = \bigcap_{i=1}^{\infty} O_i$$

for $O_i$ open sets in $\mathbb{R}$. Since $\mathbb{Q} \subset O_i$ by definition, and $\mathbb{Q}$ is dense in $\mathbb{R}$, we have that each $O_i$ must also be dense in $\mathbb{R}$.

Now, consider the intersection

$$\left(\underbrace{\bigcap_{i=1}^{\infty} O_i}_{=\mathbb{Q}}\right) \bigcap_{q \in Q} (\mathbb{R} \setminus \{q\}) = \emptyset$$

However, this is a countable intersection of dense open sets in a complete space $\mathbb{R}$! Therefore, by the Baire Category Theorem this intersection *must be dense*, but the empty set $\emptyset$ is not dense! Therefore, $\mathbb{Q}$ is not a $G_\delta$. $\square$

It turns out that the set of irrationals $\mathbb{R} \setminus \mathbb{Q}$ is a $G_\delta$ since

$$\begin{aligned} \mathbb{Q} &= \bigcup_{q \in \mathbb{Q}} \{q\} \\ \implies \mathbb{R} \setminus \mathbb{Q} &= \mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} \{q\} \\ &= \bigcap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{q\}) \end{aligned}$$

and this forms an intersection countably many open sets since there are countably many $q \in \mathbb{Q}$ and the singleton set $\{q\}$ is closed and so $\mathbb{R} \setminus \{q\}$ is open.
Note that we began by writing $\mathbb{Q}$ as the countable union

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$

so although $\mathbb{Q}$ is not a $G_\delta$, we have that $\mathbb{Q}$ is a $F_\sigma$.

**Example:** Is the set of *irrationals* $\mathbb{R} \setminus \mathbb{Q}$ complete? (does every Cauchy sequence of irrationals converge? No!

*Proof.* Consider the sequence $\left(2 + \frac{1}{n\pi}\right)_{n=1}^{\infty}$ taken from $\mathbb{R} \setminus \mathbb{Q}$. It is easy to see that our sequence does indeed converge and

$$\left(2 + \frac{1}{n\pi}\right)_{n=1}^{\infty} \longrightarrow 2$$

Thus, we have found at least one Cauchy sequence taken from $\mathbb{R} \setminus \mathbb{Q}$ that does not converge within $\mathbb{R} \setminus \mathbb{Q}$ and so $\mathbb{R} \setminus \mathbb{Q}$ cannot be complete, as desired. $\square$

**Definition:** ***(Nowhere dense sets).*** A subset $E$ of $X$, $E \subset X$, is said to be <u>nowhere dense</u> in $X$ if the intersection

$$X \setminus \text{cl}(E) \equiv X \setminus \overline{E} = X \cap \overline{E}^c$$

is dense in $X$. Alternatively, we can given an equivalent definition: A subset $E$ of $X$, $E \subset X$, is said to be <u>nowhere dense</u> in $X$ if there is no nonempty open set $V$ that is a subset of the closure of $E$, $V \not\subseteq \overline{E}$.

That is, $E$ is nowhere dense in $X$ if there is *no neighbourhood* in $X$ in which $E$ is dense. We can think of this as $E \subset X$ is nowhere dense in $X$ if it is *not dense* in any open subset of $X$.

**Example:** Consider $X = \mathbb{R}$ and the singleton set $E = \{2\} \subset \mathbb{R}$. Clearly the closure of this set is

$$\overline{\{2\}} = \{2\}$$

Therefore, $X \setminus \overline{E}$ is given by

$$\mathbb{R} \setminus \{2\}$$

Then, to check if this intersection is dense in $X$ we take its closure

$$\overline{\mathbb{R} \setminus \{2\}} = \mathbb{R}$$

since $\mathbb{R} \setminus \{2\}$ is open with point 2 removed. Thus, since $\overline{\mathbb{R} \setminus \{2\}} = \mathbb{R}$ we see that $X \setminus \overline{E} = \mathbb{R} \setminus \{2\}$ is indeed dense in $X = \mathbb{R}$, and so $E = \{2\}$ is nowhere dense in $X = \mathbb{R}$.

**Example:** Let $X = \mathbb{R}$ and $E = \mathbb{N}$. We can see that the closure of $\mathbb{N}$ is $\overline{\mathbb{N}} = \mathbb{N}$. Thus, the intersection $X \setminus \overline{E}$ is

$$\mathbb{R} \setminus \overline{\mathbb{N}} = \mathbb{R} \setminus \mathbb{N}$$

and so to check if $\mathbb{R} \setminus \mathbb{N}$ is dense in $\mathbb{R}$ we again take its closure

$$\overline{\mathbb{R} \setminus \mathbb{N}} = \mathbb{R}$$

Thus, $\mathbb{R} \setminus \mathbb{N}$ is dense in $\mathbb{R}$ and so $\mathbb{N}$ is nowhere dense in $\mathbb{R}$.

**Example:** The set of rationals $\mathbb{Q}$ is *not* nowhere dense in $\mathbb{R}$ since

$$\mathbb{R} \setminus \overline{\mathbb{Q}} = \mathbb{R} \setminus \mathbb{R} = \emptyset$$

but

$$\overline{\mathbb{R} \setminus \overline{\mathbb{Q}}} = \overline{\mathbb{R} \setminus \mathbb{R}} = \overline{\emptyset} = \emptyset$$

and clearly $\emptyset$ is not dense in $\mathbb{R}$ and so $\mathbb{Q}$ is not nowhere dense in $\mathbb{R}$.

In general, no dense subset of some set will also be a nowhere dense set.

**Result:** Suppose $O$ is some dense and open set in $X = \mathbb{R}$ and $E = \mathbb{R} \setminus O$. Since $O$ is an open set we have that $E = \mathbb{R} \setminus O$ must be closed so that $\overline{E} = E$. Therefore

$$\begin{aligned}
R \setminus \overline{E} &= R \setminus E \\
&= \mathbb{R} \setminus (\mathbb{R} \setminus O) \\
&= \mathbb{R} \cap (\mathbb{R} \setminus O)^c \\
&= \mathbb{R} \cap (\mathbb{R} \cap O^c)^c \\
&= \mathbb{R} \cap (\mathbb{R}^c \cup O) \\
&= \mathbb{R} \cap (\emptyset \cup O) \\
&= \mathbb{R} \cap (O) \\
&= \mathbb{R} \cap O \\
&= O
\end{aligned}$$

However, $O$ is dense by assumption and $X \setminus \overline{E} = \mathbb{R} \setminus (\mathbb{R} \setminus O) = O$ is dense. Thus,

$$E = \mathbb{R} \setminus O$$

is nowhere dense.

**Definitions:** A subset $E$ of $X$ is called a <u>meagre set</u> or <u>of the first category</u> if it is a countable union of nowhere dense sets $E_n$

$$E = \bigcup_{n=1}^{\infty} E_n \quad E_n \text{ nowhere dense}$$

For example, since $\mathbb{Q}$ can be written as

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$

where each $\{q\}$ is nowhere dense in $\mathbb{R}$. Thus, $\mathbb{Q}$ must be a meagre set/of the first category by definition.

We say that a set is <u>of the second category</u> if it is not of the first category, i.e. it cannot be expressed as a countable union of nowhere dense sets.

A set is <u>residual</u> or <u>comeagre</u> if it is the complement of a meagre set.