Real Analysis
Lecture Notes

Metric Spaces

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1 Compactness

Last class we expanded on the definition of compactness. We now elaborate further by fleshing out a few more relationships relating to compact sets.
Definition: ***(Sequentially compact).*** Set X is said to be sequentially compact if every infinite sequence (x_n) from X has a convergent subsequence.

Lemma: A metric space has the Bolzano-Weierstrass property if and only if it is sequentially compact.

Proof. ($\Longleftarrow$) Assume X is sequentially compact so that every sequence (x_n) from X has a convergent subsequence. The definition of the Bolzano-Weierstrass Property requires every infinite sequence (x_n) to have some cluster point. This is clearly satisfied since (x_n) has a convergent subsequence $(x_{n_k}) \to x$ by assumption, which is precisely the definition for x to be a cluster point. Thus, X has the Bolzano-Weierstrass property, as desired.

($\Longrightarrow$) Assume that X has the Bolzano-Weierstrass property. Let the sequence

$$(x_n) = (x_1, x_2, x_3, ...)$$

be some infinite sequence from X. Since X has the Bolzano-Weierstrass property this sequence must have some cluster point x for which a subsequence of (x_n) converges to x.

Consider the open sphere $S_{x,1}$ in X centered at x with radius $\epsilon = 1$ Since x is a cluster point of (x_n) and our sphere $S_{x,1}$ is open, we must have some $x_{n_1} \in S_{x,1}$ since $S_{x,1}$ is open and we may get arbitrarily close to x by elements of X.

Next, consider the open sphere $S_{x,\frac{1}{2}}$ centered at x with radius $\epsilon = \frac{1}{2}$. Since x is a cluster point of (x_n) there must exist some $x_{n_2} \in S_{x,\frac{1}{2}}$ from (x_n), such that $n_1 < n_2$, since we may get arbitrarily close to x.

In general, for some $k \in \mathbb{N}$, take the open sphere $S_{x,\frac{1}{k}}$ centered at x with radius $\epsilon = \frac{1}{k}$. We may find some $x_{n_k} \in S_{x,\frac{1}{k}}$ with $n_{k-1} < n_k$ since $S_{x,\frac{1}{k}}$ is open and we may get arbitrarily close to x with elements of X.

By this construction we generate the subsequence

$$(x_{n_k}) = (x_{n_1}, x_{n_2}, x_{n_3}, ...) \longrightarrow x$$

of (x_n) which converges to x by taking $\epsilon = \frac{1}{k}$ since

$$\rho\left(x_{n_k}, x\right) < \frac{1}{k} \quad \forall\, k \in \mathbb{N}$$

since $x_{n_k} \in S_{x,\frac{1}{k}}$. Thus, with X a metric space with the Bolzano-Weierstrass property then some infinite sequence (x_n) has a convergent subsequence, and so X is sequentially compact.

Thus,

$$\text{A metric space } X \text{ has the Bolzano-Weierstrass property} \iff X \text{ is sequentially compact}$$

as desired. $\square$

Aside: Recall that we originally defined compactness by saying that every open cover of a compact set X has a finite subcover. Thus, to fill out our relationships surrounding compactness we have:

$$\begin{aligned}
X \text{ is closed-bounded} &\implies X \text{ is compact set} \overset{\text{def}}{\iff} \text{All open covers of } X \text{ have a finite subcover} \\
&\implies \text{Infinite sequences from } X \text{ has a cluster point} \\
&\overset{\text{def}}{\iff} X \text{ has the Bolzano-Weierstrass property} \\
&\iff X \text{ is sequentially compact} \\
&\overset{\text{def}}{\iff} \text{Every infinite sequence has a convergent subsequence}
\end{aligned}$$

Result: In some metric space X a continuous function $f : X \to \mathbb{R}$ maps convergent sequences to convergent sequences.

Proof. Let f be some continuous function $f : X \to \mathbb{R}$ and consider the sequence (y_n) so that

$$(y_1, y_2, y_3, ...) \longrightarrow y$$

Applying f to elements of (y_n) yields the sequence

$$(f(y_1), f(y_2), f(y_3), ...)$$

We wish to show that this sequence $(f(y_n))$ converges to $f(y)$. To do so let $\epsilon = \frac{1}{n}$ be rational and fixed. We will show that

$$|f(y) - f(y_m)| < \frac{1}{n} \quad \forall\, n \geq N$$

for some $N \in \mathbb{N}$. Note that the interval $S_{f(y),\frac{1}{n}}$ given by

$$\left(f(y) - \frac{1}{n}, f(y) + \frac{1}{n}\right)$$

Clearly such a set is open. However, we have shown that for some open set O the pre-image of O under a function function f given by $A = f^{-1}(O)$ must also be open. Thus,

$$A = f^{-1}\left(f(y) - \frac{1}{n}, f(y) + \frac{1}{n}\right)$$

is an open set. By definition of the preimage of such an open interval we see that $y \in A$. Therefore, using y as our center we may find some $\delta > 0$ so that

$$(y - \delta, y + \delta) \subset A$$

Therefore, by the Archimedean property we see that, for $y_m \in (y - \delta, y + \delta)$

$$\exists\, N \in \mathbb{N},\ |y_m - y| < \delta \quad \text{if } m \geq N$$

Hence, for such a sequence $(y_n) \longrightarrow y$ we may find some $N \in \mathbb{N}$ and this y_m for $m \geq N$ so that

$$\forall\, m \geq N,\ |f(y) - f(y_m)| < \frac{1}{n}$$

as desired. □

Claim: Let X be some sequentially compact space so that every infinite sequence has a convergent subsequence and let $f : X \to \mathbb{R}$ be a continuous function. We claim that if X is a *bounded/finite set*, so that there is some min and max of X, then f must be bounded. That is,

A continuous function on a bounded and sequentially compact set is bounded.

Proof. Let M be some real number so that

$$M = \sup\{f(x) \ :\ x \in X\}$$

permitting $M = +\infty$. Consider the following cases:

Case 1: $M = +\infty$. Take $x_1 \in X$ and consider the mapping $f(x_1)$. If $M = +\infty$ then we must have

$$f(x_1) + 1 < M$$

and, since $M = +\infty$, there exists some x_2 so that

$$f(x_1) + 1 < f(x_2)$$

Similarly, there exists some x_3 so that

$$f(x_2) + 1 < f(x_3)$$

Continue defining this sequence $(f(x_n))$ in such a manner. Clearly

$$(f(x_1), f(x_2), f(x_3), ...) \longrightarrow \infty = M$$

However, X is assumed to be *sequentially compact*. Therefore, (x_n) must have some convergent subsequence $(x_n) \longrightarrow x \in X$

$$(x_{n_1}, x_{n_2}, x_{n_3}, ...) \longrightarrow x$$

If f is a continuous function then we have shown that $(f(x_{n_k}))$ must also be convergent so that $(f(x_{n_k})) \to f(x)$ for finite $f(x) \in \mathbb{R}$. However, we have assumed that $(f(x_n)) \longrightarrow \infty$ is nonfinite. Contradiction! Therefore $M \neq +\infty$.

Case 2: M is finite. We have $M = \sup_{x \in X} f(x) < \infty$. We seek some x so that $f(x) = M$ to show us that f is indeed bounded by $f(x) = M$.

Take $x_1 \in X$. Clearly $f(x_1) \leq M$ since $\sup_{x \in X} f(x) = M$. If $f(x_1) = M$ then we're done.

If not, take $x_2 \in X$ such that $f(x_2) > M - \frac{1}{2}$ and $f(x_2) > f(x_1)$. If $f(x_2) = M$ then we're done.

If not, take $x_3 \in X$ such that $f(x_3) > M - \frac{1}{3}$ and $f(x_3) > f(x_2)$... etc.

Continue is this manner. If we never get some $x_n \in X$ so that $f(x_n) = M$ we construct the sequence $(x_n) = (x_1, x_2, x_3, ...)$ such that

$$(f(x_1), f(x_2), f(x_3), ...) \longrightarrow M$$

However, X is sequentially compact, and so our sequence (x_n) from X has a convergent subsequence $(x_{n_k}) \to x$. Now, since $(f(x_n)) \longrightarrow M$ we clearly have its infinite subsequence

$$(f(x_{n_1}), f(x_{n_2}), f(x_{n_3}), ...) \longrightarrow M$$

and so for this definition of M we have that f is finite and bounded above by M. Similarly, since f is a continuous function we have that $-f$ is continuous. Thus, since $-f$ is continuous on a bounded set X it must achieve its minimum at some point on X, say $-m$.

Therefore, for this continuous f on a sequentially compact X we find that f is bounded between $-m \leq f \leq M$, as desired. □

Aside: ***(Alternate definition of boundedness).*** An alternate definition for boundedness is that if X is a bounded set then for all $x, y \in X$ we find $\rho(x, y) \leq M$.

Definition: ***(Bounded subspaces).*** Let X be some metric metric and A a subspace of X so that $A \subset X$. We say that A is a bounded subspace of X if there is some finite $M \in \mathbb{R}$ such that

$$\forall\, a, b \in A,\ \ \rho(a, b) < M$$

Examples: Under this definition of a bounded subspace we see that

$$\begin{aligned} X = \mathbb{R},\ A = \mathbb{R} &\implies A \text{ is not bounded in } X \\ X = \mathbb{R},\ A = \mathbb{N} &\implies A \text{ is not bounded in } X \\ X = \mathbb{R},\ A = [1, 3] &\implies A \text{ is bounded in } X \\ X = \mathbb{R},\ A = (2, 7) &\implies A \text{ is bounded in } X \end{aligned}$$

Example: Take $Y = \mathbb{R}$ under the discrete metric $\rho_d(x, y)$ defined by

$$\rho_d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{else} \end{cases}$$

What are some bounded subsets of Y? For $A \subset Y$ to be a bounded subset we require some $M \in \mathbb{R}$ such that $\forall\, a, b \in A$ we find $\rho_d(a, b) < M$. Therefore, all subsets of Y are bounded sets! In particular, if we take $M = 2$ we may generate any subset of Y and it will be a bounded set.

Definition: ***(Totally bounded space).*** A metric space X is totally bounded if for any $\epsilon > 0$ there are only finitely many points $x_1, x_2, ..., x_n \in X$ satisfying

$$X = \bigcup_{i=1}^{n} S_{x_i, \epsilon}$$

That is, X is totally bounded if X is the union of finitely many open spheres of radius ϵ.