Real Analysis
Lecture Notes

Metric Spaces

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1 Baire Categories

Recall our notion of *nowhere dense sets*:

Definition: ***(Nowhere dense sets).*** We say that a subset $E \subset X$ is nowhere dense in X if the intersection

$$X \setminus \overline{E}$$

is dense.

Definition: ***(First category/meagre).*** We say that a set is of the first category/meagre if it is a countable union of nowhere dense sets

$$E = \bigcup_{n}^{\infty} E_n, \quad E_n \text{ nowhere dense in } X$$

Clearly being nowhere dense implies the set is of the first category since we may construct the trivial union $E \cup E \cup E \cup \cdots$.

We can show that $\mathbb{Q}$ is of the first category since for $q \in \mathbb{Q}$ the sets $\{q\}$ are dense in $\mathbb{R}$ and $\mathbb{Q}$ is the countable union

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}, \quad \{q\} \text{ nowhere dense in } \mathbb{R}$$

Definition: ***(Second category).*** A set is of the second category if it is not of the first category.

Definition: ***(Residual/comgeare).*** A set is said to be residual if its complement is of the first category/meagre.

Note that being *residual/comeagre* is not the same as being of the second category since we could, in principle, have some first category set whose complement is also first category.

We now take a moment to restate the Baire Category Theorem in terms of the above definitions:

Baire Category Theorem: Let X be a complete metric space of real numbers. Then no nonempty open set is of the first category. That is, there is no open subset of $\mathbb{R}$ that is a countable union of nowhere dense sets.

Proof. In order to build a contradiction suppose that it is possible for us to represent an open subset U as a countable union of nowhere dense sets (i.e. it is of the first category). Then

$$U = \bigcup_{n=1}^{\infty} E_n, \quad E_n \text{ nowhere dense}$$

Since E_n is nowhere dense we have that

$$O_n = X \setminus \overline{E}_n$$

is dense. Furthermore, O_n is an open set since $\overline{E}_n$ is closed by definition. By the Baire Category Theorem we have that

$$\bigcap_{n=1}^{\infty} O_n$$

is dense since it is a countable union of dense open sets (*in a complete metric space*). Since $\bigcap_{n=1}^{\infty} O_n$ is dense we must have

$$U \cap \left(\bigcap_{n=1}^{\infty} O_n \right) \neq \emptyset$$

since U is a nonempty open set (this is the definition of a dense set).

Now, take some $x \in U \cap (\bigcap_{n=1}^{\infty} O_n)$ so that $x \in U$ and $x \in \bigcap_{n=1}^{\infty} O_n$. Under this definition of our point x we must have that

$$\begin{aligned}
x \notin X \setminus \left(\bigcap_{n=1}^{\infty} O_n \right) &= \bigcup_{n=1}^{\infty} (X \setminus O_n) \\
&= \bigcup_{n=1}^{\infty} \left(X \setminus \left(X \setminus \overline{E}_n\right)\right) \\
&= \bigcup_{n=1}^{\infty} \left(X \cap \left(X \cap \overline{E}_n^c\right)^c\right) \\
&= \bigcup_{n=1}^{\infty} \left(X \cap \left(X^c \cup \overline{E}_n^{c^c}\right)\right) \\
&= \bigcup_{n=1}^{\infty} \left(X \cap \left(X^c \cup \overline{E}_n\right)\right) \\
&= \bigcup_{n=1}^{\infty} \left((X \cap X^c) \cup \left(X \cap \overline{E}_n\right)\right) \\
&= \bigcup_{n=1}^{\infty} \left(\emptyset \cup \overline{E}_n\right) \\
&= \bigcup_{n=1}^{\infty} \overline{E}_n
\end{aligned}$$

with $\overline{E}_n$ the closure of our nowhere dense sets E_n. Hence,

$$x \notin \bigcup_{n=1}^{\infty} \overline{E}_n$$

but

$$U = \bigcup_{n=1}^{\infty} E_n$$

and

$$x \in U$$

by assumption. Contradiction! Therefore, we must conclude that it is not possible to represent an open subset U as a countable union of nowhere dense sets. That is, U is not of the first category, and since U was arbitrary we conclude that non nonempty open set is of the first category, as desired. $\square$

Proposition: If $O \subset X$ is some open set then $\overline{O} \setminus O$ is *always* nowhere dense in X. That is, the boundary of closure of an open set is nowhere dense.

Proof. To show that $\overline{O} \setminus O$ is nowhere dense in X we must show that $X \setminus \overline{E} = X \setminus \overline{(\overline{O} \setminus O)}$ is dense. However,

$$\begin{aligned}
\overline{O} \setminus \overline{O} &= O \cap O^c \\
&= \overline{O} \cap (X \setminus O)
\end{aligned}$$

is a *closed* set since $\overline{O}$ is closed by definition and $(X \setminus O)$ is closed since O is open. Thus,

$$\begin{aligned}
\overline{\overline{O} \setminus O} &= \overline{\overline{O} \cap (X \setminus O)} \\
&= \overline{O} \cap (X \setminus O) \\
&= \overline{O} \setminus O
\end{aligned}$$

So

$$\begin{aligned}
X \setminus \overline{\left(\overline{O} \setminus O\right)} &= X \setminus \left(\overline{O} \setminus O\right) \\
&= X \cap \left(\overline{O} \setminus O\right)^c \\
&= X \cap \left(\overline{O} \cap O^c\right)^c \\
&= X \cap \left(\overline{O}^c \cup O\right) \\
&= \left(X \cap \overline{O}^c\right) \cup (X \cap O) \\
&= \left(X \setminus \overline{O}\right) \cup (X \setminus O^c) \\
&= X
\end{aligned}$$

and so $X \setminus \overline{\left(\overline{O} \setminus O\right)}$ is dense in X since its closure is X. Thus, $\overline{O} \setminus O$ is nowhere dense in X. That is, the boundary of closure of an open set is always nowhere dense in X, as desired. $\square$

Definition: ***(Interior set).*** If E is some closed set in X, $E \subset X$, then we say that E^0 is its <u>interior</u> given by

$$X \setminus \overline{(X \setminus E)}$$

That is, the interior E^0 are all the "*interior*" points of E excluding the boundary (if it is closed).

Proposition: If F is a closed set in X and F^0 its interior then $F \setminus F^0$ is *nowhere dense* in X.

Proof. Since F is closed and F^0 is open (a union of open sets is open) we have that

$$F \setminus F^0 = F \cap \left(F^0\right)^c$$

is closed because $\left(F^0\right)^c$ is closed an an intersection of closed sets is closed. Thus,

$$\begin{aligned}
X \setminus \overline{(F \setminus F^0)} &= X \setminus \left(F \setminus F^0\right) \\
&= X \cap \left(F \setminus F^0\right)^c \\
&= X \cap \left(F \cap \left(F^0\right)^c\right)^c \\
&= X \cap \left(F^c \cup F^0\right) \\
&= (X \cap F^c) \cup \left(X \cap F^0\right) \\
&= X \setminus F \cup X \setminus \left(F^0\right)^c \\
&= X
\end{aligned}$$

Thus, $X \setminus \overline{(F \setminus F^0)}$ is dense in X and so $\overline{(F \setminus F^0)}$ is *nowhere dense* in X. Another way to see this is by letting V be some nonempty open set in X. To show that $X \setminus \overline{(F \setminus F^0)}$ is dense in X we require

$$V \cap \left(X \setminus \overline{(F \setminus F^0)} \right) \overset{?}{\neq} \emptyset$$

However,

$$\begin{aligned} V \cap \left(X \setminus \overline{(F \setminus F^0)} \right) &= V \cap \left(X \setminus \left(F \setminus F^0 \right) \right) \\ &= V \cap X \\ &\neq \emptyset \end{aligned}$$

Therefore, in both cases we have shown that if F is a closed set in X and F^0 its interior, the complement $F \setminus F^0 = F \cap \left(F^0\right)^c$ is *nowhere dense* in X. That is, the complement of a dense open set (given by the interior of a closed set F^0) is always *nowhere dense*, as desired. $\square$

Example: The interior set E^0 is the set difference $X \setminus \overline{(X \setminus E)}$ is all points inside the boundary of closure around E. We can describe this interior set E^0 as the union of all open subsets of X that lie within E. For example, let

$$E = [2, 3]$$

then its interior E^0 is given by

$$E^0 = [2, 3]^0 = (2, 3)$$

Additionally, if $X = \mathbb{R}$ then

$$\begin{aligned} (2, 3]^0 &= (2, 3) \\ (2, 3)^0 &= (2, 3) \\ \mathbb{R}^0 &= \mathbb{R} \\ \emptyset^0 &= \emptyset \\ \mathbb{Q}^0 &= \emptyset \end{aligned}$$

However, if $X = \mathbb{Q}$ then we may note that $\mathbb{Q}^0 = \mathbb{Q}$.

Example: Consider the subspace of $\mathbb{R}$ given by

$$X = \left\{ 2, 1\frac{1}{2}, 1\frac{1}{3}, 1\frac{1}{4}, ... \right\} = \left\{ 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, ..., 1 \right\}$$

In this definition for our set X we can find an open set $E \subset \mathbb{R}$ whose intersection $X \cap E = \left\{1\frac{1}{2}\right\} = \left\{\frac{3}{2}\right\}$. For some reason that I don't see we find that this point $\left\{1\frac{1}{2}\right\}$ is an open set. Clearly it's also a closed set. In particular, letting E be the open interval $\left(1\frac{2}{5}, 1\frac{3}{5}\right)$

$$X \cap \left(1\frac{2}{5}, 1\frac{3}{5} \right) = \left\{ 1\frac{1}{2} \right\}$$

All points in our set are open except the point $x = 1$. To see that $x = 1$ is not an open point we note that if we take an open interval around 1 given by $S_{x=1,\epsilon}$ then we capture an infinite number of terms in our sequence (i.e. we capture the whole sequence except only finitely many). Thus, we have the set X which has a *countable* number of open points with an additional non-open point added onto it.

Clearly X is closed and bounded above by 2 and below by 1 and so it is a compact set. This set is the *one-point compactification of* $\mathbb{N}$ (for some reason I don't understand).

Claim: If F is some closed set of the first category (i.e. a countable union of *nowhere dense sets*) in some complete metric space X (i.e. every Cauchy sequence from X is convergent), then F is nowhere dense.

Proof. Let F be some closed set of the first category so that

$$F = \bigcup_{n=1}^{\infty} F_n, \quad F_n \text{ nowhere dense}$$

To show that F is nowhere dense in X we seek to show that $X \setminus \overline{F}$ is dense. However, $\overline{F} = F$ so

$$\begin{aligned}
X \setminus \overline{F} &= X \setminus F \\
&= X \setminus \left(\bigcup_{n=1}^{\infty} F_n \right) \\
&= X \cap \left(\bigcup_{n=1}^{\infty} F_n \right)^c \\
&= X \cap \left(\bigcap_{n=1}^{\infty} F_n^c \right) \\
&= \bigcap_{n=1}^{\infty} (X \cap F_n^c) \\
&= \bigcap_{n=1}^{\infty} (X \setminus F_n)
\end{aligned}$$

However, we have that X is *complete*, $(X \setminus F_n)$ is *open*, and we have a *countable intersection*. Therefore, by the Baire Category Theorem

$$X \setminus \overline{F} = \bigcap_{n=1}^{\infty} (X \setminus F_n)$$

is dense in X. Thus, F is *nowhere dense* in X. That is, a closed set of the first category in a complete metric space is nowhere dense, as desired. □

Quickly recall the definition of a *residual/comeagre set*:

Definition: ***(Residual/comgeare).*** A set is said to be <u>residual</u> if its complement is of the first category/meagre.

Claim: A subset E of a complete metric space X is residual *if and only if* E contains a dense G_δ. That is, if E is a subset of a complete metric space X then $E^c = X \setminus E$ is of the first category (countable union of nowhere dense sets) *if and only if* E contains a dense countable intersection of open sets.

Proof. *(Unproven?)* □

Proposition: Let $\{F_n\}$ be some countable collection of closed sets such that

$$X = \bigcup_{n=1}^{\infty} F_n$$

Then

$$O = \bigcup_{n=1}^{\infty} F_n^0$$

is a residual and open set. That is, O is an open set whose complement is a countable union of nowhere dense sets.

Proof. To prove that O is residual we must show that $X \setminus O$ is of the first category, and so we must prove that $X \setminus O$ is a countable union of nowhere dense sets. Recall that

$$F_n^0 = X \setminus \overline{(X \setminus F_n)}$$

Now, let E_n be given by the closed boundary of F_n

$$E_n = F_n \setminus F_n^0 = F_n \cap \left(F_n^0\right)^c$$

We find that E_n is *nowhere dense* in X by our earlier proposition. Therefore, the countable union of these nowhere dense E_n is

$$E = \bigcup_{n=1}^{\infty} E_n$$

is of the first category/meagre. However, with $O = \bigcup_{n=1}^{\infty} F_n^0$ we find

$$\begin{aligned} X \setminus O &= X \setminus \left(\bigcup_{n=1}^{\infty} F_n^0 \right) \\ &= \left(\bigcup_{n=1}^{\infty} F_n \right) \setminus \left(\bigcup_{n=1}^{\infty} F_n^0 \right) \\ &\subset \bigcup_{n=1}^{\infty} E_n \end{aligned}$$

Now, if we take $t \in \bigcup E_n = \bigcup (F_n \setminus F_n^0)$ then

$$t \in F_n$$

for some n but

$$t \notin F_n^0$$

for all n. That is,

$$t \in \left(\bigcup F_n\right) \setminus \left(\bigcup F_n^0\right) = X \setminus O$$

and so

$$X \setminus O \supset \bigcup_{n=1}^{\infty}$$

Therefore, with both $X \setminus O \subset \bigcup_{n=1}^{\infty}$ and $X \setminus O \supset \bigcup_{n=1}^{\infty}$ we find

$$X \setminus O = \bigcup_{n=1}^{\infty}$$

for E_n *nowhere dense*. That is, $X \setminus O$ is a countable union of nowhere dense sets/of the first category. Therefore, the complement O is of the first category and so O is residual by definition. Furthermore, O is *dense* in X since

$$\begin{aligned}
\overline{O} &= \overline{\bigcup_{n=1}^{\infty} F_n^0} \\
&= \bigcup_{n=1}^{\infty} \overline{F_n^0} \\
&= \bigcup_{n=1}^{\infty} F_n \\
&= X
\end{aligned}$$

So, if $X = \bigcup F_n$ for F_n closed and $O = \bigcup F_n^0$ then $O = \bigcup_n^0$ is a residual open set (*it's complement is a countable union of nowhere dense sets*). $\square$

Example: Is there a function $f : \mathbb{R} \to \mathbb{R}$ which is unbounded on every nonempty open subset of $\mathbb{R}$? Yes! However, this function cannot be continuous:

$$f(x - \delta, x + \delta) \nsubseteq (f(x) - \epsilon, f(x) + \epsilon)$$

For example, such a function which is unbounded on every nonempty subset of real numbers is given by

$$f(x) = \begin{cases} q & \text{if } x = \frac{p}{q} \in \mathbb{Q} \ (x \text{ is rational}) \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Clearly this function is not continuous. Take $y \in \mathbb{R}$ and let

$$I = \left(y - \frac{1}{3}, y + \frac{1}{3}\right)$$

We know that I has infinitely many rationals $\frac{p}{q} \in I$. However, there are also infinitely many q such that $\frac{p}{q} \in I$ for a fixed p. Thus, on I we have that

$$f\left(\frac{p}{q}\right) = q$$

is indeed unbounded on I.

2 Connected Sets

Definition: A metric space X is called <u>connected</u> if there does not exist any nonempty open subsets A and B in X such that

$$A \cap B = \emptyset$$
$$A \cup B = X$$

Example: Is $\mathbb{R}$ connected? The only open A and B for $A \cup B$ to construct $\mathbb{R}$ and satisfy $A \cap B = \emptyset$ is

$$\mathbb{R} \cap \emptyset = \emptyset$$
$$\mathbb{R} \cup \emptyset = \mathbb{R}$$

but clearly this involved the empty set. Thus, $\mathbb{R}$ is indeed connected.

Example: Consider the punctured real line $\mathbb{R} \setminus \{2\} = X$. Then

$$X = (-\infty, 2) \cup (2, \infty)$$

Clearly we can construct X by $A = (-\infty, 2)$ and $B = (2, \infty)$ with $A \cap B = \emptyset$. Therefore, X is *not* connected. Similarly,

$$X = [2, 3] \cup [4, 5]$$

is not connected since we cannot find open A or B to construct X without either being empty.

Interestingly, the puncture plane $\mathbb{R}^2 \setminus \{(0, 0)\}$ *is connected*. We can think of this geometrically as the whole plane without the origin remains connected. However, by a similar argument, the real plane without the x-axis is not connected.

We can show that $\mathbb{Q}$ is not connected by the following argument

$$\begin{aligned}
(-\infty, \pi] \text{ is closed in } \mathbb{R} &\text{ and } [\pi, \infty) \text{ is closed in } \mathbb{R} \\
\text{Let } A = (-\infty, \pi] \cap \mathbb{Q} &\text{ is closed in } \mathbb{Q} \\
\text{Let } B = [\pi, \infty) \cap \mathbb{Q} &\text{ is closed in } \mathbb{Q} \\
A \neq \emptyset &\text{ and } B \neq \emptyset \\
A \cup B &= \mathbb{Q} \\
A \cap B &= \emptyset \\
\text{Both } A \text{ and } B &\text{ are } \textit{clopen} \text{ sets} \\
\mathbb{Q} &\text{ is } \textit{not} \text{ connected}
\end{aligned}$$

by a similar argument we can show that the irrationals $\mathbb{R} \setminus \mathbb{Q}$.

We say that $f : [0, 1] \to [0, 1]$ given by

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \in (0, 1] \\ 0 & x = 0 \end{cases}$$