Real Analysis
Lecture Notes

Metric Spaces

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1 Distance in Metric Spaces

What is a reasonable way of defining distance in some space? Suppose we are given some set A and points $x \notin A$ and $a \in A$. Clearly, under the definition of a metric we have

$$\rho(x, a) \geq 0$$

However, what is a good way of defining the distance from x to the entire set A? We typically evaluate this as

$$\rho(x, A) = \inf_{a \in A} \rho(x, a)$$

Suppose now that A is a circle given by

$$A = \{(x, y) \ : \ x^2 + y^2 \leq 4\}$$

Then, under our definition for $\rho(x, A)$ we have

$$\rho((\text{origin}), A) = \rho((0, 0), A) = 0$$

Suppose $A = [1, 2) \subset \mathbb{R}$. We find that

$$\rho(3, A) = 1$$
$$\rho(2, A) = 0$$

Note that $\rho(2, A) = 0$ despite the fact that $\rho(2, a \in A) > 0$ since our distance $\rho(2, A)$ is given by the infimum $\inf_{a \in A} \rho(2, a)$.

Recall the sup metric between functions given by

$$\rho(f, g) = \sup_{x \in D} |f(x) - g(x)|$$

and consider the open "*sphere*" $B_{\sin x, \frac{1}{2}, \text{sup metric}}$. This sphere defines a cloud around the function $\sin x$ with open boundaries given by $\sin x \pm \frac{1}{2}$. In particular, any function within these bounds will lie within the open sphere $B_{\sin x, \frac{1}{2}, \sup \text{metric}}$.

Example: Let X be a metric space and let $A \subset X$, $A \neq \emptyset$. Let d be a metric on X and define the distance from set A by

$$d(x, A) = \inf_{a \in A} d(x, a) = f(x)$$

Claim: *f is continuous.* That is, $f : A \to \mathbb{R}^+$ given by $f(x) = d(x, A) = \inf_{a \in A} d(x, a)$ is continuous.

Proof. For all $a \in A$ and $\forall\, x, y \in X$ the distance $f(y)$ is bound above by

$$f(y) = d(x, A) = \in_{a \in A} d(x, a) \leq d(y, a)$$

by the definition of the infimum. Using the triangle inequality gives us

$$\begin{aligned} f(y) &\leq d(y, a) \\ &\leq d(y, x) + d(x, a) \end{aligned}$$

Taking the infimum over A of both sides yields

$$\begin{aligned} \inf_{a \in A} f(y) &= \inf_{a \in A} d(a, A) \\ &= \inf_{a \in A} \left[\inf_{a \in A} d(x, a) \right] \\ &= \inf_{a \in A} d(x, a) \\ &= f(y) \\ \inf_{a \in A} d(y, x) &= d(y, x) \\ \inf_{a \in A} d(x, a) &= d(x, A) \\ &= f(x) \end{aligned}$$

Therefore, we may express our

$$f(y) \leq d(y, x) + d(x, a)$$

as

$$\begin{aligned} & f(y) \leq d(y, x) + f(x) \\ \implies & f(y) - f(x) \leq d(y, x) \\ \iff & f(x) - f(y) \leq d(x, y) \\ \implies & |f(x) - f(y)| \leq d(x, y) \end{aligned}$$

Hence, $\forall\, \epsilon > 0$, take the distance between points x and y to be bound above by $\delta = \epsilon$ so that $d(x, y) < \delta = \epsilon$. That is,

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0, |x - y| < \delta \implies |f(x) - f(y)| \leq d(x, y) < \delta = \epsilon$$

which is precisely the definition of continuity, as desired. □

Proposition: *A subspace of a metric space is a metric space.* Consider some metric space X with metric d and some subspace $S \subset X$. Is the topology on S given by (S, d) also a metric space? Yes!

Proof. Recall the basic properties of a metric d, for all $x, y, z \in X$:

1. $d(x, y) \geq 0$
2. $d(x, y) = d(y, x)$
3. $d(x, y) = 0 \iff x = y$
4. $d(x, z) \leq d(x, y) + d(y, z)$

Therefore, if we take points $x, y, z \in S \subset X$ we see that all four properties hold under the metric space (X, d) by assumption. So, since x, y, z were arbitrary points from S we have that all four properties of a metric space (S, d) must be satisfied by inheritance from X. $\square$

Although we have just shown that $S \subset X$ inherits its metric from its superspace space X, potential ambiguities arise if we consider subtopologies (E, d) and (S, d) such that $E \subset S \subset X$. That is, when considering subspaces $E \subset S \subset X$ a natural question to ask is: How can we relate being *close in* X with being *closed in* S?

Example: Let $X = \mathbb{R}$ and let subspaces S and E be given by

$$S = (0, 1)$$
$$E = \left(0, \frac{1}{2}\right)$$

so that $E \subset S \subset X$. Clearly, the closure of E in $\mathbb{R}$ is

$$\mathrm{cl}(E) = \overline{E} = \left[0, \frac{1}{2}\right]$$

However, the closure of E in S must be

$$\mathrm{cl}_S(E) = \overline{E}_S = \left(0, \frac{1}{2}\right]$$

Proposition: Let X be a metric space and E, S be subspaces $E \subset S \subset X$. The *closure of E relative to S* is

$$\mathrm{cl}_S(E) = \overline{E}_S = \overline{E} \cap S$$

where $\overline{E}$ is the *closure of E relative to the common parent space X*. We say that subspace $A \subset S$ is *closed* in S if

$$A = S \cap F$$

for some closed set F which is closed in X. Analogously, subspace $A \subset S$ is *open* in S if

$$A = S \cap O$$

for some open set O which is open in X.
Let A be any subset of X, $A \subset X$ and S a subspace of X, $S \subset X$. To generate a subset of S we may perform the intersection $A \cap S$. That is, if $A \subset X$ then

$$A \cap S \subset S$$

If we want to produce open sets in S then suppose sets $O_i \subset X$ are open in X so that

$$O_i \cap S \subset S$$

In the next set of notes we will prove that by considering the intersection $O_i \cap S$ we will see that S inherits the relative topology from that defined in X.

Example: Let $X = \mathbb{R}$ and $S = \mathbb{Q}$, so that we are considering the subspace $\mathbb{Q} \subset \mathbb{R}$. What happens if we consider the intersection with the intervals $[e, \pi]$ and (e, π)? Using our previous proposition we must conclude that these intersections

$$\begin{array}{rl} [e, \pi] \cap \mathbb{Q} & \text{is } \textit{closed} \text{ in } \mathbb{Q}, \text{ but} \\ (e, \pi) \cap \mathbb{Q} & \text{is } \textit{open} \text{ in } \mathbb{Q}! \end{array}$$

since the intersection with an open set is open and the intersection with a closed set is closed. However, since $e \notin \mathbb{Q}$ and $\pi notin \mathbb{Q}$ we find

$$[e, \pi] \cap \mathbb{Q} = (e, \pi)$$

Therefore, we must conclude that , unlike the set of reals $\mathbb{R}$, the set of rationals $\mathbb{Q}$ contains sets that are *both closed and open*!