Real Analysis
Lecture Notes

Metric Spaces

November 21 2016
Last update: December 4, 2017

1 Compactness

Recall that last time we introduce the notion of a *Lebesgue number*:

Definition: ***(Lebesgue number).*** Let $\mathcal{U}$ be some open covering of X, $\mathcal{U} = \{U_i\}_{i\in I}$. We say that $\epsilon > 0$ is a Lebesgue number of $\mathcal{U}$ if

$$\forall\, \delta < \epsilon,\ \forall\, x \in X,\ \exists\, O \in \mathcal{U},\ S_{x,\delta} \subset O$$

Proposition: If X is a sequentially compact space and $\mathcal{U}$ is an open covering of X, then $\mathcal{U}$ has a *Lebesgue number*.

Proof. Suppose $X \in \mathcal{U}$. That is, X is one of the open sets $O \in \{U_i\}_{i\in I} = \mathcal{U}$. Then we see that $\delta = 2$ satisfies our definition of the Lebesgue number of X since for $\delta = 2$ we have

$$\forall\, x \in X,\ \exists\, O \in \mathcal{U},\ S_{x,2} \subset O$$

and $S_{x,2} \subset O = X \in \mathcal{U}$ with

$$S_{x,2} = \{y \in X \ : \ \rho(x,y) < 2\}$$

So if $X \in \mathcal{U}$, X trivially has a Lebesgue number.

Assume now that $X \notin \mathcal{U}$ so that X may only be the nontrivial union of open sets

$$X \subset \bigcup_i U_i$$

for each $U_i \in \mathcal{U}$. Define the function $\phi : X \to \mathbb{R}$ such that

$$\phi(r) = \sup\{r \ : \ \exists\, O \in \mathcal{U},\ S_{x,r} \subset O\}$$

That is, ϕ finds the largest radius such that the open spheres $S_{x,r}$ are still enclosed within open sets of the covering $\mathcal{U}$. We claim that

$$0 < \phi(r) < \infty$$

Since X is sequentially compact by assumption we have that

$$\begin{aligned} X \text{ is sequentially compact } &\implies X \text{ is totally bounded} \\ &\implies X \text{ is bounded.} \end{aligned}$$

Let $M \in \mathbb{R}$, $M < \infty$ be our bounding constant so that $\forall\, x \in X,\ -M < x < M$. Thus,

$$\forall\, y \in X,\ \rho(x, y) < M$$

If $\phi(x) = \infty$ then there is some finite radius $r \in R$, $r < \phi(x)$, with $r > M$ such that

$$S_{x,r} \subset O \in \mathcal{U}$$

but

$$X \subset S_{x,r} \subset O$$

and

$$O \subset X$$

by assumption that X is not composed of a single set $O \in \mathcal{U}$. Thus,

$$X \subset O \subset X$$

so

$$X = O$$

which is solved by case 1.

Thus, in either case $X \in \mathcal{U}$ or $X \notin \mathcal{U}$ we have that X has a Lebesgue number δ. That is, if X is a sequentially compact set then a covering $\mathcal{U}$ of X has a Lebesgue number, as desired.

We now consider the general case without setting $\delta = 2$.
Claim: With the above definition of $\phi(x) = \sup\{r\ :\ \exists\, O \in \mathcal{U},\ S_{x,r} \subset O\}$ we claim that

$$\forall\, x, y \in X,\ \phi(y) \geq \phi(x) - \rho(x, y)$$

Proof. If $\phi(x) \leq \rho(x, y)$ then our claim trivially holds since $\phi(x) - \rho(x, y) \leq 0$ and $\phi(y) > 0$ by definition.

So, we need only consider the case $\phi(x) > \rho(x, y)$. That is, suppose the distance between any two points in $x, y \in X$ is *less than* the maximal radius that the open sphere $S_{x,r}$ may be to still remain fully closed by at least one open sets $O \in \mathcal{U}$.

Consider the difference $r - \rho(x, y)$. Since $\phi(y)$ is the greatest such radius we must have

$$r - \rho(x, y) \leq \phi(y)$$

since $\rho(x, y) \geq 0$ by assumption. Taking the sup over all such r yields our result, for $\phi(y) = \sup_{x \in X} r$

$$\phi(y) \geq \phi(x) - \rho(x, y)$$

as desired. $\square$

Now, using this result note that if $\phi(y) \geq \phi(x) - \rho(x,y)$ then

$$\begin{aligned} & \phi(y) \geq \phi(x) - \rho(x,y) \\ \iff & \phi(x) \geq \phi(y) - \rho(x,y) \end{aligned}$$

by symmetry since x and y are arbitrary points in X. Noting that the distance $\rho(x,y)$ is identical in both inequalities we may state that

$$|\phi(x) - \phi(y)| \leq \rho(x,y)$$

So, for $\rho(x,y) < \delta = \epsilon$ we get that

$$\rho(x,y) < \delta \implies |\phi(x) - \phi(y)| < \epsilon$$

and so ϕ is a continuous function. That is, if our inequality is indeed true we get that ϕ is a continuous function. However, since X is sequentially compact by assumption, we have that all function $f : X \to \mathbb{R}$ *must* be continuous, and so ϕ *must* be a continuous function.

Since ϕ must be continuous we know that it must achieve its infimum. So, there exists some x for which $\inf_{x \in X} \phi(x)$ is attained. Let this infimum $\inf_{x \in X} \phi(x) = \epsilon$. We claim that this $\epsilon = \inf_{x \in X}$ is a Lebesgue number for our cover $\mathcal{U}$.

To see this take δ so that $0 < \delta < \epsilon$. The small value of ϕ must be ϵ by definition of $\epsilon = \inf_{x \in X} \phi(x)$ (since ϕ achieves its infimum on X). Therefore,

$$\forall\, x \in X,\ \delta < \phi(x)$$

since $\delta < \epsilon = \inf_{x \in X} \phi(x) < \phi(x)$. Hence,

$$\delta < \sup\{r \ : \ S_{x,r} \subset O \in \mathcal{U}\}$$

or more completely

$$\forall\, \delta < \epsilon,\ \forall\, x \in X,\ \exists\, O \in \mathcal{U},\ S_{x,\delta} \subset O$$

which is precisely the definition for ϵ satisfying the definition of a Lebesgue number for $\mathcal{U}$. That is, since X is compact there will always be such $\delta < \epsilon = \inf_{x \in X} \phi(x)$ to act as a Lebesgue number since $\phi(x)$ is continuous and thus it must achieve such an infimum. Thus, for a compact set X, $\epsilon = \inf_{x \in X} \phi(x)$ works as our general Lebesgue number for our cover $\mathcal{U}$ of X, as desired. $\square$

A good final question may be to prove such a result under the restriction that X is a metric space.

Definition: ***(Compactification).*** Let X be some topological space. We say that a <u>compactification</u> of X is a compact space K such that X is a *dense subset* of K.

Recall for a moment the definition of a *dense subset* X of K:

$$\forall\, O \neq \emptyset, O \text{ open in K},\ O \cap X \neq \emptyset$$

That is, X is a dense subset of K if all open subsets of K form nonempty intersections with X. Equivalently, X is dense in K if

$$\overline{X} = K$$

For example, the set of rationals $\mathbb{Q}$ is dense in $\mathbb{R}$ since any open set in $\mathbb{R}$ must contain at least one $q \in \mathbb{Q}$. However, $\mathbb{N}$ is not dense in $\mathbb{R}$ since we may construct nonempty open sets that do not contain a single natural number $n \in \mathbb{N}$, say $(2, 3) \subset \mathbb{R}$.

Example: ***(Example of a compactification).*** Consider the real interval $(0, 1)$. We have shown that such an interval is not compact since we were able to construct a covering with no finite subcover. However, we can construct the set $[0, 1]$ which will be the *two-point compactification* of $(0, 1)$ since the closure of $(0, 1)$ is exactly $[0, 1]$, and so $(0, 1)$ is dense in $[0, 1]$.