Real Analysis
Lecture Notes

The Real Number System

September 28 2016
Last update: October 15, 2016

1 Open and Closed Sets of $\mathbb{R}$ (con't 3)

1.1 Closed Sets in $\mathbb{R}$ (con't)

Recall from last class a few definitions relating to closed sets:

Definition: *(Point of closure)* A point $x \in E \subset \mathbb{R}$ is said to be a point of closure of E if

$$\forall\, \delta > 0,\ (x - \delta, x + \delta) \cap E \neq \emptyset$$

That is, x is a point of closure of E if an arbitrarily small open ball around x has some overlap with E, i.e. x is in some sense "deep enough" within the set E. We also presented an alternative, but equivalent, definition for a point of closure: A point $x \in E \subset \mathbb{R}$ is said to be a point of closure of E if

$$\forall\, \delta > 0,\ \exists\, y \in E \text{ such that } |x - y| < \delta$$

That is, x is a point of closure of E if we can get arbitrarily close to x while still remaining in E. Notice that this definition requires us to be able to get arbitrarily close to x while staying in E, i.e. we are still interested in whether x is "deep enough" within E.

Definition: *(Set of points of closure)* For a set $E \subset \mathbb{R}$, we denote $\overline{E}$ to be the set of points of closure of E. That is, $\overline{E}$ contains all $x \in E$ such that x is a point of closure.

Definition: *(Closed sets)* A set F is said to be closed if $F = \overline{F}$. That is, F is closed if the set of points of closure of F is identical to the original set F.

Some typical examples of closed sets include

$$\begin{aligned} [a, b], &\quad [1, 2] \cup [3, 4] \\ [a, \infty), &\quad (-\infty, a] \\ \mathbb{R}, &\quad \emptyset \end{aligned}$$

1.1.1 Properties of Closed Sets

Proposition: *(A union of closed sets is closed)* Let F_1 and F_2 be closed sets. Then the union $F_1 \cup F_2$ forms a closed set.

Proof. Consider the set of points of closure $\overline{F}_1$ and $\overline{F}_2$,

$$\begin{aligned}\overline{F_1 \cup F_2} &= \overline{F}_1 \cup \overline{F}_2 \quad \text{(from last class)} \\ &= F_1 \cup F_2 \quad \text{(since } F_1 \text{ and } F_2 \text{ are closed)}\end{aligned}$$

as desired. □

Corollary: *(Arbitrary intersections of closed sets are closed)* Let $\{F_i\}$ be a collection of closed sets. Then, the intersection

$$\bigcap_{i \in I} F_i$$

is closed for arbitrary indexing $i \in I$.

Proof. Consider the intersection $\bigcap_i F_i$ and let $x \in \overline{\bigcap_i F_i}$. Then, by assumption of x in this set we have

$$\forall\, \delta > 0,\ (x - \delta, x + \delta) \cap \left(\bigcap_i F_i \right) \neq \emptyset$$

so

$$\begin{aligned} & y \in \bigcap_i F_i \\ \implies & y \in F_i \end{aligned}$$

Hence, x is a point of closure for all sets F_i. That is

$$\begin{aligned} & x \in \overline{F}_i \\ \implies & x \in \bigcap_i \overline{F}_i \end{aligned}$$

However, since F_i is closed we have that $F_i = \overline{F}_i$. Thus

$$\begin{aligned} & x \in \bigcap_i F_i \\ \implies & \overline{\bigcap_i F_i} \subset \bigcap_i F_i \end{aligned}$$

That is, for $x \in \overline{\bigcap_i F_i}$ we have found that we must have $x \in \bigcap_i F_i$. Since $F \subset \overline{F}$ is always true for any arbitrary set,[1] it is sufficient to show $\overline{\bigcap F} \subset \bigcap F$, as we have done. □

Proposition: *(The complement of a open set is closed and the complement of a closed set is open)* A subset of the reals is open if and only if its complement is closed.

[1] I think I want to prove this at some point...

Proof. ($\implies$) It is sufficient to show that $\overline{O^c} \subset O^c$ since $O^c \subset \overline{O^c}$ is true for all sets. Let O be some open set and take $x \in O$. By definition of an open set we have that we are guaranteed a sufficiently small open interval around $x \in O$ that remains in O. That is,

$$\exists\, \delta > 0,\ (x - \delta, x + \delta) \subset O$$

Can x be a point of closure of $O^c = \mathbb{R} \setminus O$? No! Note that

$$(x - \delta, x + \delta) \cap O^c = \emptyset$$

since this interval is fully enclosed by O. Thus, if $x \in O$ then $x \notin \overline{O^c}$. That is,

$$\begin{aligned} x \in O &\implies x \notin \overline{O^c} \\ \iff x \in \overline{O^c} &\implies x \notin O \\ \iff x \in \overline{O^c} &\implies x \in O \end{aligned}$$

Hence, $\overline{O^c} \subset O^c$, and so O^c is a closed set.

($\impliedby$) Let F be some closed set. To show that F^c is open we must show that each point $x \in F^c$ has a small interval around it such that this interval remains fully enclosed in F^c. That is, we must show that for all $x \in F^c$

$$\exists\, \delta > 0,\ (x - \delta, x + \delta) \subset F^c$$

So, let $x \in F^c$. Can this x be a point of closure for F? No! Our point x cannot be a point of closure of F since $x \in F^c \iff x \notin F$, and since $F = \overline{F}$, we have $x \notin \overline{F}$. Thus, $x \in F^c \implies x \notin \overline{F}$. Thus, since x is *not* a point of closure of F we have

$$\exists\, \delta > 0,\ (x - \delta, x + \delta) \cap F = \emptyset$$

That is, there is a sufficiently small ball around $x \in F^c$ which doesn't intersect with F. Therefore, if this small ball doesn't intersect F at any point, it must be fully enclosed within $\mathbb{R} \setminus F = F^c$, which is precisely the definition of an open set. Hence if F is closed then F^c is open, as desired.

□

Example: *(Is $\mathbb{N}$ closed?)* Is the set of natural numbers $\mathbb{N} = \{1, 2, ...\}$ closed? Is $\mathbb{N}$ compact?[2]

Consider the open intervals of width $\frac{2}{3}$ around each $n \in \mathbb{N}$, i.e. intervals of the form

$$\left(1 - \frac{1}{3}, 1 + \frac{1}{3}\right),\ \left(2 - \frac{1}{3}, 2 + \frac{1}{3}\right),\ \left(3 - \frac{1}{3}, 3 + \frac{1}{3}\right), ...$$

In general, denote these open intervals by

$$O_n = \left(n - \frac{1}{3}, n + \frac{1}{3}\right)$$

[2]Compactness: All open covers have a finite subcover.

Note that each O_n contains a single point in $\mathbb{N}$, namely n. Therefore

$$O_n \cap \mathbb{N} = \{n\}$$

Hence

$$\mathbb{N} \subset O_1 \cup O_2 \cup \cdots = \bigcup_{i=1}^{\infty} O_i$$

That is, we have constructed a countably infinite cover of $\mathbb{N}$. However, removing any single O_k from the cover yields

$$\mathbb{N} \cap \bigcup_{i=1, i\neq k}^{\infty} O_i = \mathbb{N} \setminus \{k\}$$

and so $\mathbb{N}$ is no longer covered. Therefore, $\mathbb{N}$ cannot be compact. I don't have the proof of whether $\mathbb{N}$ is closed or not for this in my notes, but the intuition is obvious since $\mathbb{R} \setminus \mathbb{N} = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup \cdots$ is an open set (a union of open sets is open).[3]. If this isn't enough, show that $(-\infty, 1)$ is open and all $(n-1, n)$ is open. Therefore, the complement $(\mathbb{R} \setminus \mathbb{N})^c = \mathbb{N}$ must be closed.

1.2 Heine-Borel Theorem

This work on open, closed, and compact sets has essentially been for us to work up to this point. Finally, we are able to state an important result:

Theorem: *(Heine-Borel Theorem)* If F is a closed bounded subset of $\mathbb{R}$, then F is compact, i.e. every open cover of closed bounded sets F have a finite subcover.

Proof. Let F be a closed bounded subset of real numbers. We consider the following case:

(Case 1): $F = [a, b],\ a < b,\ a, b \in \mathbb{R}$.

Consider some open cover $\{O_i\}_{i \in I}$ of $F = [a, b]$ and consider the set E such that

$$E = \{x \ : \ a \leq x \leq b \text{ and } [a, x] \text{ can be covered by a finite subcover}\}$$

By construction it should be clear that E is bound above by b. Is $E = \emptyset$? No! Our set E contains at least a single point $a \in E$ since

$$a \in \{x \ : \ a \leq x \leq b \text{ and } [a, a] = a \text{ can be covered by some open set}\}$$

Since the original open covering $\{O_i\}_{i \in I}$ covers $F = [a, b]$, there exists at least one open set, say O', where we may find $a \in O'$ such that $O' \in \{O_i\}_{i \in I}$. Thus,

$$E \neq \emptyset$$

[3]From the Sept. 19 notes

Therefore, since E nonempty and bounded above by b, we may invoke the *completeness of* $\mathbb{R}$ to conclude that our set E has some supremum $c \in \mathbb{R}$. Since b is an upper bound and c is the *least upper bound* we clearly have

$$\sup E = c \leq b$$

Since $c \in [a, b]$ we have that c must be covered by some $O \in \{O_i\}_{i \in I}$. That is,

$$\exists\, O \in \{O_i\}_{i \in I},\ c \in O$$

and since O is open we have that there must be a sufficiently small interval around c that remains enclosed by O:

$$\exists\, \delta > 0,\ (c - \delta, c + \delta) \subset O$$

If $c + \delta$ extends past our original interval $F = [a, b]$, i.e. $c + \delta \geq b \iff \delta \geq b - c$, then shrink our small open interval by replacing δ with

$$\epsilon = \frac{b - c}{2}$$

so that $(c - \epsilon, c + \epsilon) \subset O$ also becomes a subset of $F = [a, b]$. Now, $c - \epsilon$ cannot an upper bound of E since c is the *least upper bound* and so there must be since there is some other $x \in E$ such that $x > c - \epsilon$. However, by construction of E and since $x \in E$, there must be a finite cover $\{O_i\}_{i=1}^n$ which covers $[a, x]$.

Therefore, combining $(c - \epsilon, c + \epsilon) \subset O$ with this finite subcover $\{O_i\}_{i=1}^n$, we find that the interval $[a, x] \cup (c - \epsilon, c + \epsilon) = [a, c + \epsilon)$ has the finite subcover $\{O_i\}_{i=1}^n \cup \{O\}$.

However, we recall that E was defined to be the set of elements in $[a, b]$ that have a finite subcover. Thus, each point $c^* \in [c, c + \epsilon)$ is an element of E if the point $c^* \leq b$. But $\sup E = c$ and so no point of $[c, c + \epsilon)$ except c can possibly be an element of E! Therefore, we must have that $c = b$. Hence, $[a, c] = [a, b]$ can be finitely covered by $\{O_i\}_{i=1}^n \cup \{O\}$. That is, $[a, b] = F$ is compact, as desired.

(Case 2): F is closed and bounded, but not of the form $[a, b]$ (i.e., F is not necessarily a connected set). *Left for next class.* □

Corollary: The Cantor set is compact.

Proof. The Cantor set is a subset of $[0, 1]$ and so it is bounded. Additionally, we can show that the Cantor set is a closed set. Let $\mathfrak{C}$ be the Cantor set. Recall that the Cantor set is constructed by recursively removing the middle thirds of subsets from the unit interval. That is, if we take intermediate sets C_i to be

$$\begin{aligned}
C_1 &= [0, 1] \\
C_2 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
C_3 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\
&\vdots
\end{aligned}$$

we get

$$\mathfrak{C} = \bigcap_{i=1}^{\infty} C_i$$

and if we note that $C_1 \supset C_2 \supset \cdots \supset C_i \supset \cdots$ we may see that

$$\mathfrak{C} = \bigcap_{i=1}^{\infty} C_i = \left\{ \text{the set of all endpoints of the form } \frac{d}{3^k},\ d \in \{0,1,2\},\ k \in \mathbb{N} \right\}$$

However, we see that we have constructed the Cantor set $\mathfrak{C}$ from an intersection of the closed C_n (each C_n is closed since it is a finite union of closed sets). Since an intersection of closed sets is closed we may conclude that $\mathfrak{C}$ is closed. Therefore, since the Cantor set $\mathfrak{C}$ is a closed bounded set we may invoke the Heine-Borel Theorem to conclude that it is compact, as desired. □