Real Analysis
Lecture Notes

Metric Spaces

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1 Separable Metric Spaces

1.1 Quick Review of Last Lecture

The last few lectures we've been working in metric spaces (X, ρ) where a metric ρ must satisfy, for $x, y, z \in X$,

$$\begin{aligned} \rho(x,y) &\geq 0 \\ \rho(x,y) &= \rho(y,x) \\ \rho(x,y) &= 0 \iff x = y \\ \rho(x,y) + \rho(y,z) &\geq \rho(x,z) \end{aligned}$$

and in the previous lecture we introduced the notion of an <u>open sphere</u> $S_{x,\delta}$ in a metric space (X, ρ) defined as:

Definition: ***Open sphere***: An <u>open sphere</u> $S_{x,\delta}$ in a metric space (X, ρ) is the set

$$S_{x,\delta} = \{y \in X \ : \ \rho(x,y) < \delta\}$$

The notion of an <u>open set</u> in a metric space $O \subset X$ defined as:

Definition: ***Open set***: A set $O \subset X$ in a metric space (X, ρ) is said to be <u>open</u> if

$$\forall\, x \in O, \ \exists\, \delta > 0 \text{ such that } S_{x,\delta} \subset O$$

where O is a union of open sets of the form $S_{x,\delta}$. That is, an open set can be expressed as the union of open spheres. From these definitions we proved the following lemma:

Lemma: If $S_{z,\epsilon}$ and $S_{x,\delta}$ are open spheres such that

$$0 < \epsilon < \delta - \rho(x,z)$$

then

$$S_{z,\epsilon} \subset S_{x,\delta}$$

That is, since $0 < \epsilon < \delta - \rho(x,z)$ we have

$$\begin{aligned} 0 &< \epsilon \\ \rho(x,z) &< \delta \\ \epsilon + \rho(x,z) &< \delta \end{aligned}$$

So $S_{z,\epsilon}$ is some open sphere centered at z with radius ϵ such that the sum of ϵ with the distance between x and z given by $\rho(x,z)$ must be less than δ. So, since this radius bound above δ and the distance $\rho(x,z) < \delta$ we have that $S_{z,\epsilon}$ must be some open sphere located within $S_{x,\delta}$, which is precisely the desired result

$$S_{z,\epsilon} \subset S_{x,\delta}$$

Proof. The more formal proof of this was that with for some $t \in S_{z,\epsilon}$ we require this t to also be an element of $S_{x,\delta}$. Thus, for $t \in S_{z,\epsilon}$ we find

$$\rho(z,t) < \epsilon$$

by definition of $t \in S_{z,\epsilon}$. However, $0 < \epsilon < \delta - \rho(x,z)$ gives us

$$0 < \epsilon + \rho(x,z) < \delta$$

but by the definition of a metric ρ we have

$$0 < \rho(x,t) < \rho(x,z) + \rho(z,t)$$

thus

$$0 < \rho(x,t) < \rho(x,z) + \rho(z,t) < \rho(x,z) + \epsilon < \delta$$

That is,

$$\rho(x,t) < \delta$$

which is precisely the requirement for

$$t \in S_{x,\delta}$$

as desired. □

A natural question consequence of the definitions of an open sphere $S_{x,\delta}$ and an open set O in a metric space is to consider *closed* sets in a metric space. To do so we first gave the definition of a point of closure of some set $E \subset X$ in a metric space (X,ρ):

Definition: *Point of closure.* A point $x \in X$ in a metric space (X,ρ) is called a point of closure of a set $E \subset X$, denoted by cl (E) or $\overline{E}$, if

$$\forall\, \delta > 0,\ \exists\, y \in E,\ \rho(x,y) < \delta$$

which is equivalent to

$$\forall\, \delta > 0,\ S_{x,\delta} \cap E \neq \emptyset$$

where the above $y \in E$ is found in the intersection $y \in S_{x,\delta} \cap E$. For example, if we consider the set $E = (0,1)$ and $x = 0$ we see that for all $\delta > 0$ we are guaranteed that a small open sphere centered at $x = 0$ is guaranteed to have a nonempty intersection with $E = (0,1)$. Then, with the definition of a point of closure we may introduce the definition of a closed set:

Definition: ***Closed set.*** A set F is closed if $\overline{F} = F$. That is, the points of closure of F are contained within F and the points of F are contained within $\overline{F}$.

We ended the lecture with the definition of a separable metric space. Before doing so it will be useful to reintroduce the definition of a dense set:

Definition: ***Dense set.*** A set $A \subset X$ in a topological space X is said to be dense if every point $x \in X$ is *either* an element of A, $x \in A$ *or* a limit point of A, where a limit point x satisfies

$$(x - \delta, x + \delta) \cap A = S_{x,\delta} \cap A \neq \emptyset$$

That is, A is dense in X if all $x \in X$ we satisfy either

$$\begin{aligned} & x \in A \quad \text{or} \\ \exists\, y \in A \quad \text{such that} \quad & y \in S_{x,\delta} \cap A \end{aligned}$$

For example, the set of rational numbers $\mathbb{Q}$ is clearly dense in $\mathbb{R}$ since for arbitrary $r \in \mathbb{R}$ we have that either $r \in \mathbb{Q}$ or

$$\exists\, q \in \mathbb{Q} \text{ such that } q \in S_{r,\delta} \cap \mathbb{Q}$$

We can also see that the second criteria may be satisfied by the sequence $\{q_n \,:\, q_n \in \mathbb{Q}\}_{n=1}^{\infty}$ such that $q_n \to r$. On the other hand $\mathbb{N}$ is clearly not dense in $\mathbb{Q}$ since for some $r \in \mathbb{R}$ we can build a sufficiently small open sphere $S_{x,\delta}$, $\delta > 0$, such that the intersection $S_{x,\delta} \cap \mathbb{N} = \emptyset$. Finally, we may give the definition of a separable metric space:

Definition: ***Separable metric space.*** A metric space (X, ρ) is said to be separable if it contains a countable and dense subset $D \subset X$ such that the closure of D, denoted $\overline{D}$, is $\overline{D} = X$.

1.2 Elaborating on Separability

We first require the notion of a base B of a topological space X:

Definition: ***Base of a topological space.*** A family $B = \{O_i\}_{i \in I}$ of open sets is said to be a base of a topological space X if B

1. The family B covers X.

2. Any open set $O \subset X$ is a union of sets in the base

$$O = \bigcup_{i \in J \subset I} O_i$$

Theorem: ***Equivalent statements to separability.*** Let (X, ρ) be a metric space. The following are equivalent:

1. X is separable.

2. X has a countable base $\{O_i\}$. That is, for all open sets $O \subset X$ we have

$$O = \bigcup_{n=1}^{\infty} O_i$$

This is *not* true in general. Bases for an arbitrary topological space may not be separable. Fortunately for us, this turns out to be true for metric spaces.

Proof. ((1) $\implies$ (2)). Since X is separable we have that X must have a countable dense subset. Let $D = \{x_n\}_{n=1}^{\infty} \subset X$, $\overline{D} = X$, be such a subset. Consider all open spheres centered at each x_n of rational radius $q \in \mathbb{Q}$, i.e. consider the open spheres

$$\{S_{x_n,q}\}$$

Since $x_n \in \{x_n\}_{n=1}^{\infty} = D \subset X$ is countable and $q \in \mathbb{Q}$ is countable we see that the family of open spheres $\{S_{x_n,q}\}$ must be countable. We claim that this family $\{S_{x_n,q}\}$ forms a countable base for X. That is, we wish to show that if O is any open set and $y \in O \subset X$ then

$$y \in O_i \subset O$$

Consider an open set $O \subset X$ (nonempty) and take $y \in O$. Since O is open we must have some open sphere $S_{x,\delta}$ contained in O:

$$\exists\, \delta > 0,\ S_{y,\delta} \subset O$$

Without loss of generality (since $\mathbb{Q}$ is dense in $\mathbb{R}$ so we may take δ even smaller such that δ is now rational) take $\delta \in \mathbb{Q}$. Since D is dense in X we have that $\overline{D}$ and so we may choose $y \in O \subset X$ such that $y \in \overline{D} = X$. That is, since y is taken to be a point of closure of D

$$\forall\, \delta > 0,\ \exists\, z \in D, \quad \rho(z, y) < \frac{\delta}{2}, \quad \text{or equivalently}$$
$$\forall\, \delta > 0, \quad S_{y,\delta} \cap D \neq \emptyset$$

Using the first definition we note that since $D = \{x_n\}$ is countable we may restate this as

$$\forall\, \delta > 0,\ \exists\, x_n \in D \quad \text{such that} \quad \rho(y, x_n) < \frac{\delta}{2}$$

That is, we have some $x_n \in D$ such that $x_n \in S_{y,\delta}$ and the distance between x_n and y is bound above by

$$\rho(x_n, y) = \rho(y, x_n) = \frac{\delta}{2}$$

By our earlier result stating for $0 < \epsilon < \delta - \rho(x, z)$, $S_{z,\epsilon} \subset S_{x,\delta}$, with $\epsilon = \frac{\delta}{2}$ with

$$0 < \frac{\delta}{2} < \delta - \rho(x_n, y)$$

we have

$$y \in S_{x_n, \frac{\delta}{2}} \subset S_{y,\delta}$$

hence

$$\begin{aligned} &y \in y \in S_{x_n, \frac{\delta}{2}} \subset S_{y,\delta} \subset O \\ \implies &y \in S_{x_n, \frac{\delta}{2}} \subset O \end{aligned}$$

and so the open spheres $\left\{ S_{x_n, \frac{\delta}{2}} \right\}$ form our countable base of X, as desired.

((2) $\implies$ (1)) Let X have a countable base $\{O_i\}$ so that for all open sets $O \subset X$ so that

$$O = \bigcup_{i=1} O_i$$

Without loss of generality all O_n are nonempty. For each n choose a point $x_n \in O_n$. Clearly the set

$$D = \{x_n\}_{n=1}^{\infty}$$

is a countable subset of X. So, to complete the proof we wish to show that D is dense in X. That is, we must show that any open sphere $S_{x,\delta}$, $x \in X$, will have a nonempty intersection

$$S_{x,\delta} \cap D \neq \emptyset$$

Let $x \in X$ be arbitrary and take any open sphere centered at x, $S_{x,\delta}$, $\delta > 0$. Since $\{O_n\}$ is a base for X we must have that every open set in X can be expressed as a union of the base elements O_n. Therefore, all open spheres $S_{x,\delta}$ can be written as

$$S_{x,\delta} = \bigcup_n O_n$$

so, there exists some n such that

$$O_n \subset S_{x,\delta}$$

Clearly this gives us that $x_n \in O_n \implies x_n \in S_{x,\delta}$. Therefore, since $x_n \in S_{x,\delta}$

$$\forall\, \delta > 0,\ S_{x,\delta} \cap \{x_n\} \neq \emptyset, \quad \{x_n\} = D$$

which is precisely the statement that $X = \overline{D}$ and so D is indeed dense in X, as desired. $\square$

1.3 Sorgenfry Line

We will change the topology on $\mathbb{R}$ to construct the *"Sorgenfry line"*. To get the usual topology on $\mathbb{R}$ we take $a, b \in \mathbb{R}$, $a < b$ and let (a, b) be the interval defined by the set

$$(a, b) = \{x \in \mathbb{R} \ : \ a < x < b\}$$

For the Sorgenfry line we now define intervals of the form $[a, b)$ to be open sets. We will denote the Sorgenfry line by $\mathbb{S}$ just as the real line is denoted by $\mathbb{R}$. This leads to the following remarks:

1. $\mathbb{S}$ has a countable dense subset, namely $\mathbb{Q}$. That is,

   $$\forall\, a < b, \ [a, b) \cap \mathbb{Q} \neq \emptyset$$

   and so the Sorgenfry line must be separable.

2. Claim: The interval (a, b) is open in $\mathbb{S}$. This will give us that all open sets in $\mathbb{S}$ are open set since all open sets can be constructed by a countable union of open intervals

   $$O = \bigcup (a, b) \quad \text{or} \quad O = \bigcup (a, b]$$

   To show this take the union

   $$\bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right), \quad a - 1 < b$$

   Note that a is never reached in this countable union since we are always a nonzero distance $\frac{1}{n}$ away. However, any point $a + \epsilon$, $\epsilon > 0$ is an element of this union since for any $\epsilon > 0$ we can make n sufficiently large such that $a + \frac{1}{n} < a + \epsilon$. By definition this countable union of open intervals must itself be open. Therefore, since a is *not* within this union we have

   $$\bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right) = (a, b) \quad \text{is open.}$$

3. Claim: The interval $[a, b)$ is also *closed* in $\mathbb{S}$. To see this consider the union

   $$[b, b + 1) \cup [b + 1, b + 2) \cup \cup [b + 2, b + 3) \cup \cdots = [b, \infty)$$

   Since this is a union of open sets we have that $[b, \infty)$ must be open. Furthermore, the union

   $$\cdots \cup [a - 3, a - 2) \cup [a - 2, a - 1) \cup [a - 1, a) = (-\infty, a)$$

   is also open by the same reasoning. Therefore, taking the union of these two open sets

   $$(-\infty, a) \cup [b, \infty) \quad \text{is open.}$$

   and taking the complement of this open set

   $$\Big((-\infty, a) \cup [b, \infty)\Big)^c = [a, b) \quad \text{is closed.}$$

   We say that such an interval $[a, b)$ under the Sorgenfry topology is *"clopen"*.

4. Claim: Any compact[1] subsets of $\mathbb{S}$ are countable. Note that this contrasts with $\mathbb{R}$ under the usual topology since $[1, 2]$ is compact by the Heine-Borel theorem.[2]

Proof. (Start of the proof) Begin with C some compact set and take $x \in C$. Consider the family of intervals of the form

$$[x, \infty) \text{ and } \left(-\infty, x - \frac{1}{n}\right), \quad n \in \mathbb{N}$$

Under $\mathbb{S}$ the interval $[x, \infty)$ was shown to be open. We should be able to see that this family of sets covers $(-\infty, \infty)$ over all $n \in \mathbb{N}$.

(And then the proof stops here...) □

[1] Compactness: All open covers have a finite subcover.

[2] All bounded closed intervals (sets?) are compact.