Real Analysis
Lecture Notes

The Real Number System

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1 The Real Number System (con't)

Last class we defined groups, rings, and fields. We said that a set G with binary operation $\oplus$ defined for elements $g \in G$ is a group if it satisfies

$$\left.\begin{array}{l}\left.\begin{array}{l}(1)\ \ g_1 \oplus g_2 \in G\ (\oplus\text{-closure}).\\[6pt] (2)\ \ (g_1 \oplus g_2) \oplus g_3 = g_1 \oplus (g_2 \oplus g_3)\ (\text{associativity under } \oplus).\end{array}\right\}\text{Semigroup}\\[6pt] (3)\ \ \exists\, z \in G \text{ such that } z \text{ is unique and } g_1 \oplus z = z \oplus g_1 = g_1\ (\text{identity under } \oplus).\end{array}\right\}\text{Monoid}$$

(4) $\forall\, g \in G,\ \exists\, h \in G$ such that h_1 is unique and $g_1 \oplus h_1 = h_1 \oplus g_1 = z$ (inverse under $\oplus$).

If our group $(G, \oplus)$ also satisfies

(5) $g_1 \oplus g_2 = g_2 \oplus g_1$ (commutativity under $\oplus$).

then we say that our group is an *Abelian group*. We said that a ring R is an Abelian group with respect to $\oplus$ (i.e. it satisfies points (1)-(5))[1] if it is has a second binary operation $\odot$, defined for elements $r \in R$, satisfying

$$\left.\begin{array}{l}(6)\ \ r_1 \odot r_2 \in R\ (\odot\text{-closure}).\\[6pt] (7)\ \ \exists\, e \in R \text{ such that } r_1 \odot e = e \odot r_1 = r_1\ (\odot\text{-identity}).\\[6pt] (8)\ \ (r_1 \odot r_2) \odot r_3 = r_1 \odot (r_2 \odot r_3)\ (\text{associativity under } \odot).\end{array}\right\}\text{Monoid under } \odot$$

(9) $r_1 \odot (r_2 \oplus r_3) = r_1 \odot r_2 \oplus r_1 \odot r_3$ (left distributivity).

(10) $(r_1 \oplus r_2) \odot r_3 = r_1 \odot r_3 \oplus r_2 \odot r_3$ (right distributivity).

[1] The Abelian group R has $\oplus$-identity 0, and $\oplus$-inverse $-r$.

Within the algebraic structure of a ring with may distinguish between *non-commutative* and *commutative* rings which satisfy

(11) $r_1 \oplus r_2 = r_2 \odot r_1$ (commutativity under $\odot$).

Finally, we said that a field $(\mathbb{F}, \oplus, \odot)$ is a commutative ring with $\odot$-identity e in which all nonzero elements have a multiplicative inverse.[2] That is, if x is a nonzero element of F, then

(12) $\forall\, x \in F,\ \exists\, x^{-1}$ such that $x \odot x^{-1} = e$ ($\odot$-inverse).

2 Axioms of $\mathbb{R}$

2.1 Axioms of Order

On-top of a field we can consider the notion of an *ordered field.* We said that a set $\mathbb{F}$ is an ordered field if, noting that we can "break up" $\mathbb{F}$ into 3 parts (positives, zero, and negatives), there exists a subset $P \subset \mathbb{F}$ such that

(1) $x, y \in P \implies x \oplus y \in P$ (additive closure).

(2) $x, y \in P \implies x \odot y \in P$ (multiplicative closure).

(3) $x \in P \implies -x \notin P$.

(4) If $x \in \mathbb{F}$ then exactly one of the following hold:

  (i) $x = 0$.

  (ii) $x \in P$.

  (iii) $x \notin P$.

and if $(\mathbb{F}, \odot, \oplus)$ also satisfies the field axioms (not introduced), then we say that $\mathbb{F}$ is an ordered field. From this we may work out that the real numbers $\mathbb{R}$ equipped with standard addition $+$ and multiplication $\cdot$ is indeed an ordered field.

2.2 Axiom of Completeness

To construct the real numbers from the bottom-up we will require the notion of <u>completeness</u> in addition to the Axioms of Order listed above.

Completeness Axiom: Any nonempty set of real numbers with an upper bound has a least upper bound (supremum) in the set.

[2]That is, all elements which are not the $\oplus$-identity 0.

Example: *(Example of an incomplete ordered field)* Is the set of rationals $\mathbb{Q}$ an ordered field? We see that this set is clearly an ordered set since it may inherit its order from $\mathbb{R}$, and we can show that $\mathbb{Q}$ is a field satisfying the necessary requirements.

To test for completeness of $\mathbb{Q}$ consider the subset $\{q \in \mathbb{Q} \ : \ q^2 < 2\}$. Clearly this set is bound above by some number finite number, and so we may conclude that it has some upper bound. Is there a least upper bound? In $\mathbb{R}$ we would get that this least upper bound is $r = \sqrt{2}$. However, any $q < \sqrt{2}$ won't be an upper bound since

$$\exists\, q' \in \mathbb{Q} \text{ such that } q < q' < \sqrt{2}$$

Similarly, any $q > \sqrt{2}$ won't be a *least* upper bound since

$$\exists\, q'' \in \mathbb{Q} \text{ some other upper bound such that } q'' < q$$

and since $r = \sqrt{2}$ is provably irrational, we conclude that the rationals are not complete! That is, the rationals are *incomplete* since there is no rational number which acts as the supremum for $\{q \in \mathbb{Q} \ : \ q^2 < 2\}$ For this reason we are forced to expand our set to include the cases which cannot be "reached" by rational elements $q \in \mathbb{Q}$.

For this reason we introduce the set of real numbers $\mathbb{R}$ which satisfy essentially three conditions

(1) $\mathbb{R}$ is a field.

(2) $\mathbb{R}$ is ordered.

(3) $\mathbb{R}$ is complete.

3 The Naturals and Rationals as a Subset of $\mathbb{R}$

3.1 Archimedean Property of $\mathbb{N}$

Example: *(A consequence of completeness)* The reals have what we call the "Archimedean property". Given a real number x there is an integer n larger than x.

Proof. If $x \leq 0$ then let $n = 1$ and we're done. So, without loss of generality, suppose $x > 0$. Let S be the set of integers k such that $k \leq x$.

Is this set S empty? No! Since $x > 0$ we have that S contains at least elements of $\{0, -1, -2, ...\}$. By construction of S, we have that S is bound above by x. Therefore, by completeness of $\mathbb{R}$ there is a least upper bound for $S \in \mathbb{R}$. Call this least upper bound $y \in \mathbb{R}$.

Consider now $y - \frac{1}{2} < y$ so that $y - \frac{1}{2}$ is not a least upper bound. The only way $y - \frac{1}{2}$ may fail to satisfy the definition of a least upper bound for S is if some $k \in S$ is greater than

$y - \frac{1}{2}$. That is, since $y - \frac{1}{2}$ is not a least upper bound then there is some $k \in S$ such that $k > y - \frac{1}{2}$. So,

$$\begin{aligned} y - \frac{1}{2} &< k \\ \implies y + \frac{1}{2} &< k + 1 \\ \implies y < y + \frac{1}{2} &< k + 1 \\ \implies k + 1 &\notin S \\ \implies k + 1 &\not\leq x \quad \text{(by definition of } S\text{)} \\ \implies k + 1 &> x \end{aligned}$$

as desired. □

It is noteworthy that there exists ordered fields containing $\mathbb{R}$ (properly containing) that are *not* Archimedean (nonstandard models of $\mathbb{R}$). It is not hard to construct nonstandard models that fail to be Archimedean. In such sets we require elements that are so large that they cannot be reached by "adding 1". Similarly, if we take the reciprocal of these elements we get infinitesimally small quantities that remain > 0.

Using the Archimedean property proven above we wish to prove the following corollary:

Corollary: Between any two reals there is a rational. That is, if $x < y$ are real numbers then there is a rational q with $x < q < y$.

Proof. If $y > x$ then $y - x > 0$ and by the field axioms

$$\begin{aligned} &(y - x)^{-1} \in \mathbb{R} \quad \text{and} \\ &(y - x)^{-1} > 0 \end{aligned}$$

By the Archimedean property, there exists an natural number $q \in \mathbb{N}$ such that

$$\begin{aligned} &q > (y - x)^{-1}, \quad q > 0 \\ \implies &\frac{1}{q} < y - x \end{aligned}$$

Now, consider <u>**Case 1**</u>: $0 \leq x$. Then

$$0 \leq x < y$$

and since y is positive, then we get

$$y \cdot q > 0$$

We ask now whether there exists natural numbers $n \in \mathbb{N}$ such that

$$\begin{aligned} y &\stackrel{?}{\leq} \frac{n}{q} \\ y \cdot q &\stackrel{?}{\leq} n \end{aligned}$$

However, we are guaranteed such an $n \in \mathbb{N}$ exists by the Archimedean property for $y \cdot q \in \mathbb{R}$! That is, we are guaranteed that there exists an integer n larger than $y \cdot q$, and since $y \cdot q$ is positive we are guaranteed that this n is a natural number.

By induction/well ordering of $\mathbb{N}$ there exists a smallest such n satisfying our inequality $y \leq \frac{n}{q}$. Call this least natural number by p so that

$$\begin{aligned} y \cdot q &\leq p \quad \text{but} \\ y \cdot q &\nleq p - 1 \quad \text{by construction of } p-1\text{, so} \\ y &\leq \frac{p}{q} \quad \text{since } p, q > 0\text{, and} \\ y &\nleq \frac{p-1}{q} \end{aligned}$$

Therefore,

$$\frac{p-1}{q} < y \leq \frac{p}{q}$$

We would like to show that that this rational number $\frac{p-1}{q}$ is greater than x. To so write the following

$$\begin{aligned} & x = y - (y - x) \\ \implies & x \leq \frac{p}{q} - (y - x) \end{aligned}$$

but we had that $\frac{1}{q} < y - x$, so

$$\begin{aligned} & x \leq \frac{p}{q} - (y - x) < \frac{p}{q} - \frac{1}{q} \\ \implies & x < \frac{p-1}{q} \\ \implies & x < \frac{p-1}{q} < y \end{aligned}$$

as desired. We now consider **<u>Case 2</u>**: $0 > x \implies -x > 0$. By the Archimedean principle we are guaranteed that there exists some $n \in \mathbb{N}$ such that

$$\begin{aligned} & -x < n \\ \implies & n + x > 0 \end{aligned}$$

and since $x < y$ we have that

$$0 < n + x < n + y$$

Therefore, by Case 1 we have that there exists some $r \in \mathbb{Q}$ such that

$$\begin{aligned} & x + n < r < n + y \\ \implies & x < r - n < y \end{aligned}$$

Since $r \in \mathbb{Q}$ and $n \in \mathbb{N}$ we find that $r - n \in \mathbb{Q}$, and so we have successfully found some rational number $r - n$ satisfying

$$x < r - n < y$$

as desired. $\square$

We skipped two sections on "The Extended Real Numbers" and "Sequences of Real Numbers".

4 Open and Closed Sets of $\mathbb{R}$

4.1 Open Sets

Definition (*Open sets*): We will define what it means to be an open set on $\mathbb{R}$. A subset O of $\mathbb{R}$ is called open if

$$\forall\, x \in O,\ \exists\, (a_x, b_x) \text{ such that } (a_x, b_x) \subset O, \text{ and}$$
$$(a_x, b_x) = \{r \in \mathbb{R} \ :\ a_x < r < b_x\}$$

That is, we say that O is an *open* set if for every element $x \in O$ there is an open interval (depending on x) $I_x = (a_x, b_x)$ such that this interval is a subset of O, $x \in I_x \subset O$. Note that our typical notion of a "closed" interval, say $[0, 1]$ fails this definition since we may find some $x \in [0, 1]$, namely $x = 0$ and $x = 1$, where we are *unable* to form an open interval (a_x, b_x) containing x that is also in $[0, 1]$. For example, if we take the left endpoint $x = 0$ then we are forced to create some open interval containing 0. That is, our open interval will look something like

$$\{r \ :\ a < 0 < b\}$$

However, any $a < 0$ we choose will not be in $[0, 1]$! So we see that such an interval cannot be open.

An alternate definition of openness is given in the book: A set O of real numbers is called open if for each $x \in O$, there is some $\delta > 0$ such that each y satisfying $|x - y| < \delta$ must be an element of O, $y \in O$. That is,

$$\forall\, x \in O,\ \exists\, \delta > 0 \quad |x - y| < \delta \implies y \in O$$

We check again that our set $[0, 1]$ *fails* this criteria. Taking $\delta > 0$, we see that at the left endpoint $x = 0$, if we consider all the y satisfying the inequality

$$\begin{aligned} |x - y| &< \delta \\ |0 - y| &< \delta \\ |y| &< \delta \\ \implies -\delta < y &< \delta \end{aligned}$$

However, for **all** $\delta > 0$, we see that there are some y satisfying $y < \delta$ which cannot be in our set $y \notin [0,1]$, namely we have some negative $y < 0$. That is, we cannot find some $\delta > 0$ which permits all y satisfying the inequality above to be wholly included in our set $[0,1]$.

Now, consider the analogous *open* set $(0,1)$. Note that we cannot take the left endpoint of our close set $x = 0$, and so we choose some sufficiently small $x = \epsilon > 0$. Consider all y satisfying

$$\begin{aligned} |x - y| &< \delta \\ |\epsilon - y| &< \delta \\ \implies -\delta < \epsilon - y &< \delta \\ \epsilon - \delta < y &< \delta + \epsilon \end{aligned}$$

Since our definition only requires some $\delta > 0$ to exist, we see that a sufficiently small $\delta > 0$ will permit y to be sufficiently close to x so that y may be an element of $(0,1)$, $y \in (0,1)$.

Example: *(Is all of $\mathbb{R}$ an open set?)* Take $x \in \mathbb{R}$. Can we find an interval (a connected interval),[3]

$$(a_x, b_x) \quad \text{such that} \quad (a_x, b_x) \in \mathbb{R}$$

Clearly at any point $x \in \mathbb{R}$, we can find *at least one* subset (a_x, b_x) where (a_x, b_x) is a subset of our original set, which happens to be $\mathbb{R}$. This is pretty trivial since we can take any arbitrary interval, say

$$(x - 5, x + 12)$$

and find that this interval is obviously a subset of $\mathbb{R}$.

Example: *(Is the empty set $\emptyset$ open?)* Take any $x \in \emptyset$ (there are no $x \in \emptyset$!). For any such x, we satisfy our definition of openness. This statement is said to be *vacuously* true since all $x \in \emptyset$ satisfy because there are no $x \in \emptyset$.

Example *(Non-connected intervals)* Consider a non-connected interval, for example

$$O = (1,2) \cup (4,5)$$

First, take any $x \in (1,2) \subset O$. Note that x is in a connected open "sub-interval" in O. Likewise, if we take $x \in (4,5) \subset O$ then we see that this x is also in a connected open "sub-interval" in O. From this we can see that any $x \in O$ must be in either the open set $(1,2)$ or $(4,5)$. That is, any $x \in O$ satisfies the definition of openness, and so O is indeed open.

Claim: The intersection of two open sets O_1, O_2 is an open set $O_1 \cap O_2$.

[3]I don't think we've officially defined connectedness. As far as I understand, a set X is connected if X *cannot* be represented as the union of two or more nonempty open subsets. For example, $(0,1)$ is a (open) connected set since any union of *open* subsets, say $(0, 0.5) \cup (0.5, 1)$, will leave out at least one element in our original interval $(0,1)$.

Proof. If $O_1 \cap O_2 = \emptyset$ then $O_1 \cap O_2$ is open by vacuity. So, without loss of generality assume $O_1 \cap O_2 \neq \emptyset$. Take any $x \in O_1 \cap O_2$. We are required that for all such x we must have an interval around x given by

$$(x - \delta, x + \delta)$$

which remains in the intersection $O_1 \cap O_2$. Note that O_1 is open by assumption and $x \in O_1 \cap O_2 \implies x \in O_1$. So

$$\forall\, x \in O_1,\ \exists\, \delta_1 > 0,\ \text{such that } (x - \delta_1, x + \delta_1) \subset O_1$$

is satisfied by assumption because O_1 is open. Similarly, since O_2 is open by assumption, and $x \in O_1 \cap O_2 \implies x \in O_2$. So,

$$\forall\, x \in O_2,\ \exists\, \delta_2 > 0,\ \text{such that } (x - \delta_2, x + \delta_2) \subset O_2$$

That is, for a small movement from x given by $\delta_1 > 0$ and $\delta_2 > 0$ we remain inside our original set. If we are inside our set for both δ_1 and δ_2 then clearly we are inside the set for the *minimum* of δ_1 and δ_2. That is, if we set

$$\delta = \min\{\delta_1, \delta_2\}$$

then any $x \in O_1 \cap O_2$ will have some small open interval surrounding it given by this small movement δ, and so

$$\forall\, x \in O_2 \cap O_2,\ \exists\, \delta = \min\{\delta_1, \delta_2\} > 0,\ \text{such that } (x - \delta, x + \delta) \subset O_1 \cap O_2$$

is satisfied. Since O_1 and O_2 were arbitrary open sets we conclude that the intersection of two open sets is open, as desired.

$\square$

Corollary: Any intersection of a finite number of open intervals is open.

Proof. I think we didn't do an official proof in class. However, I think that a way that we can see this is that for any finite number of open intervals, say n open intervals, we may proceed inductively on n using the above result. That is, for a collection $\{O_1, O_2, ..., O_n\}$ of open sets, we may use the above result to conclude that

$$\begin{aligned}
&O_1 \cap O_2 \text{ is open.}\\
&(O_1 \cap O_2) \cap O_3 \equiv O_1 \cap O_2 \cap O_3 \text{ is open.}\\
&((O_1 \cap O_2) \cap O_3) \cap O_4 \equiv O_1 \cap O_2 \cap O_3 \cap O_4 \text{ is open.}\\
&\quad\vdots\\
&(...(O_1 \cap O_2) \cap ...) \cap O_n \equiv O_1 \cap O_2 \cap \cdots \cap O_n \text{ is open.}
\end{aligned}$$

$\square$

Example *(Intersections of countably infinite open sets)*: What is the intersection of the following open sets?

$$(-1,1)\cap\left(-\frac{1}{2},\frac{1}{2}\right)\cap\left(-\frac{1}{3},\frac{1}{3}\right)\cap\cdots\cap\left(-\frac{1}{k},\frac{1}{k}\right)\cap\cdots$$

Repeating this process as $k\to\infty$ we find that what we are left with at the end of this process is the set

$$\{0\}=[0]$$

Is this set open? No! For all $x\in\{0\}$, namely $x=0$, we want to be able to find at least one $\delta>0$ such that $(x-\delta,x+\delta)\in\{0\}$. However, clearly no such nonzero δ exists since any movement will take us out of the singleton set $\{0\}=[0]$. So, we have just demonstrated that although a finite number of intersections of open sets is open, a countably infinite number of intersections of open sets may not be open.

Proposition: Let O_i be open sets in $\mathbb{R}$, where $i\in I$ is some indexing, and let

$$S=\bigcup_{i\in I}O_i$$

Then we claim that S is an open set. That is, the union of any collection of open sets $\{O_i\}$ is open.

Proof. Take $x\in S$. For some index $i\in I$, we have that our x is an element of at least one O_i. Since all the O_i are open then for this x

$$\exists\,\delta>0,\ \text{such that}\ (x-\delta,x+\delta)>0$$

Since we let $x\in S$ be arbitrary we may conclude that S is indeed an open set, as desired. Note that we set no limitations on the indexing set I. Thus, we have not only proven that the union of a finite number of open sets is open, but also a union of infinite open sets is open. $\square$

Aside: A collection of subsets of a set X are called a topology on X if

1. $\emptyset$ and X are open.
2. Finite intersections of the collection are in the collection.
3. Arbitrary unions from the collection are in the collection.

We find that the "smallest" topology for a set X is the collection $\{\emptyset,X\}$ and the "biggest" is the power set $\mathcal{P}(X)$.