Real Analysis
Lecture Notes

Metric Spaces

November 9 2016
Last update: December 4, 2017

1 Compactness

Goal: Last time we introduced our goal to *characterize compact metric spaces* in greater detail. We will soon make use of a fairly involved proof by contrapositive. Recall that this method of proof uses the logical identity

$$(p \implies q) \iff (\neg q \implies \neg p)$$

In particular, let

$$\begin{aligned} p &: \{O_i\} \text{ cover } X \\ q &: \text{ finitely many } \{O_i\} \text{ cover } X \end{aligned}$$

We said that a set is compact in some space of all its covers have a finite subcover. In this case we may express this definition as

$$\text{Compactness} \iff (p \implies q)$$

Negating these statements yield

$$\begin{aligned} \neg q &: \text{ finitely many } \{O_i\} \textit{ do not } \text{cover } X \\ \neg p &: \{O_i\} \textit{ do not } \text{cover } X \end{aligned}$$

Note that if $\{O_i\}$ *do not cover* X then $X \not\subseteq \{O_i\}$ which is equivalent to

$$
\begin{aligned}
&\{O_i\} \text{ do not cover } X \\
\iff\ & X \not\subseteq \bigcup_i O_i \\
\implies\ & X^c \not\supseteq \left(\bigcup_i O_i\right)^c \\
\implies\ & (X \cap X^c) \not\supseteq \left(X \cap \left(\bigcup_i O_i\right)^c\right) \\
\implies\ & \emptyset \not\supseteq X \setminus \left(\bigcup_i O_i\right) \\
\implies\ & X \setminus \left(\bigcup_i O_i\right) \neq \emptyset \\
\implies\ & X \cap \left(\bigcup_i O_i\right)^c \neq \emptyset \\
\implies\ & X \cap \left(\bigcap_i O_i^c\right) \neq \emptyset \\
\implies\ & \bigcap_i (X \cap O_i^c) \neq \emptyset \\
\implies\ & \bigcap_i (X \setminus O_i) \neq \emptyset
\end{aligned}
$$

Note that the set defined by $(X \setminus O_i)$ is a closed set since O_i is open. Therefore, we may express our definition of compactness

$$
\begin{aligned}
p &: \{O_i\} \text{ cover } X \iff X \subset \bigcup_{i \in I} O_i \\
q &: \text{ finitely many } \{O_i\} \text{ cover } X \iff X \subset \bigcup_{i=1}^{n} O_i
\end{aligned}
$$

to the contrapositive, $\neg q \implies \neg p$, by

$$
\begin{aligned}
\neg q &: \bigcap_{i=1}^{n} (X \setminus O_i) \neq \emptyset \\
\neg p &: \bigcap_{i \in I} (X \setminus O_i) \neq \emptyset
\end{aligned}
$$

for O_i open sets and $(X \setminus O_i)$ closed sets.

Therefore, we may reformulate our definition of compactness as follows: A set X is said to be compact if, for any family of *closed sets* $(X \setminus O_i)$,

$$\bigcup_{i=1}^{n} (X \setminus O_i) = \emptyset \implies \bigcup_{i \in I} (X \setminus O_i) = \emptyset$$

That is, in some compact set X, the intersection of any finite number of closed sets is empty implies that the intersection of *all closed sets* $(X \setminus O_i)$ in the family is empty.

Our original definition of compactness made use of open sets. Now, our current translation of compactness permits us to work with closed sets. We summarize this more succinctly by the *finite intersection property*:

Definition: ***(Finite intersection property).*** A family of sets $\{M_i\}_{i \in I}$ is said to have the finite intersection property if its intersection over any finite subcollection is nonempty:

$$\bigcap_{i \in J} M_i \quad J \text{ a finite subset of } I,\ J \subset I$$

Thus,

Definition: ***(Compactness).*** A set X is said to be compact if *any* family of closed subsets $\{F_i\}_{i \in I}$ with the finite intersection property itself has a nonempty intersection.

Definition: ***(Limit point).*** A point x is a limit point of sequence the (x_n) if the sequence (x_n) is a Cauchy sequence towards x. That is,

$$\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N},\ \forall\, n \geq N,\ |x - x_n| < \epsilon$$

Note that this definition of a limit point *does not* include points in alternating sequences with finite lim sup and lim inf. That is, consider the sequence (x_n) given by

$$(x_n) = \left(\frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, \frac{1}{4}, \ldots \right)$$

In particular, each element in x_n is defined by

$$x_n = \begin{cases} \frac{1}{n/2+1} & \text{if } n \text{ even} \\ \frac{(n-1)/2+1}{(n-1)/2+2} & \text{if } n \text{ odd} \end{cases} \qquad n = 1, 2, \ldots$$

so that

$$\limsup(x_n) = 1$$
$$\liminf(x_n) = 0$$

For this definition of (x_n) we see that both $x = 1$ and $x = 0$ are *similar* to limit points, but fail to satisfy the definition because it is not the case that

$$\forall\, n \geq N,\ |x - x_n| < \epsilon$$

since the distance $|x - x_n|$ will alternate between $< \epsilon$ and $< 1 - \epsilon$. For this reason we introduce a weaker notion than a limit point to describe such cases:

Definition: ***(Cluster point).*** The definition of a cluster point contrasts with that of a limit point in that its definition relies on the use of lim sup and lim inf. We say that x is a cluster point of the sequence (x_n) if

$$\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N},\ \exists\, n \geq N,\ |x - x_n| < \epsilon$$

Note the change of the final quantifier

$$\exists\, n \geq N,\ |x - x_n| < \epsilon$$

in which we may ignore any future points beyond n that do not satisfy the inequality.

Using our above example $(x_n) = \left(\frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, \frac{1}{4}, ...\right)$ we found that

$$\begin{aligned} \limsup(x_n) &= 1 \\ \liminf(x_n) &= 0 \end{aligned}$$

so that both 1 and 0 are cluster points of (x_n). If we split (x_n) into the two subsequences

$$\begin{aligned} \left(x_{n_{2k-1}}\right) &= \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\right) \to 1 \\ \left(x_{n_{2k}}\right) &= \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right) \to 0 \end{aligned}$$

we may be able to see the result that

$$\text{cluster point} \iff \text{limit point of some subsequence}$$

Recall the Bolzano-Weierstrass property:

Definition: ***(Bolzano-Weierstrass Property).*** We say that some set X is said to have the Bolzano-Weierstrass Property if every infinite sequence $(x_n) = x_1, x_2, x_3, ...$ taken from X has a cluster point (finite lim sup or lim inf?).

Example: $X = \mathbb{R}$ *does not* have the Bolzano-Weierstrass property since the sequence $x_n = 1, 2, 3, ...$ from $\mathbb{R}$ has no cluster point.

Theorem: ***If X is compact then X has the Bolzano-Weierstrass Property.***

Proof. Let X be some compact set and take an infinite sequence (x_n) from X. We want to find a cluster point of (x_n).

Define some subsets of X by

$$
\begin{aligned}
B_1 &= \{x_1, x_2, x_3, ...\} \\
B_2 &= \{x_2, x_3, x_4, ...\} \\
B_3 &= \{x_3, x_4, x_5, ...\} \\
&\vdots \\
B_n &= \{x_n, x_{n+1}, x_{n+2}, ...\}
\end{aligned}
$$

so that

$$B_1 \supset B_2 \supset B_3 \supset \cdots \supset B_n \supset \cdots$$

Then their closures $\overline{B}_1, \overline{B}_2, \overline{B}_3, ...$ also satisfy

$$\overline{B}_1 \supset \overline{B}_2 \supset \overline{B}_3 \supset \cdots \supset \overline{B}_n \supset \cdots$$

Consider the family of these closures

$$\left\{\overline{B}_n \ : \ n = 1, 2, ...\right\}$$

Clearly our family $\left\{\overline{B}_n\right\}$ is a closed subset of X by construction, so does this family $\left\{\overline{B}_n\right\}$ have the finite intersection property? Well, for any finite intersection

$$\bigcap_i \overline{B}_{n_i}$$

there must be some greatest $m = \max(n_1, n_2, ..., n_k)$. But recall that

$$\overline{B}_1 \supset \overline{B}_2 \supset \overline{B}_3 \supset \cdots \supset \overline{B}_n \supset \cdots$$

Then for this maximal m we have

$$\overline{B}_m = \bigcap_i \overline{B}_{n_i}$$

Clearly $\overline{B}_m \neq \emptyset$ since $x_m \in B_m \in \overline{B}_m$. However, X is a compact set by assumption. Therefore, for any family of sets from X with the finite intersection property must itself has a nonempty intersection. Therefore, since $\overline{B}_m = \bigcap_i \overline{B}_{n_i}$ we have

$$\bigcap_{i=1}^{\infty} \overline{B}_i \neq \emptyset$$

Take some $x \in \bigcap_{i=1}^{\infty} \overline{B}_i$. We claim that x is a cluster point of our sequence (x_n). In particular, take any open set O of X containing x, and take any N so that $x \in \overline{B}_N$, where

$$B_N = \{x_N, x_{N+1}, x_{N+2}, ...\}$$

Since

$$\{x_1, x_2, x_3, ..., x_N, x_{N+1}, x_{N+2}, ...\}$$

has x as its cluster point, then the intersection

$$O \cap \{x_N, x_{N+1}, x_{N+2}, ...\} \neq \emptyset$$

Therefore, there exists some $n \geq N$ so that $x_n \in O$, and so x is indeed a cluster point of (x_n). That is,

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \exists n \geq N,\ |x - x_n| < \epsilon$$

and so since (x_n) was an arbitrary infinite sequence from X we may conclude that X does indeed have the Bolzano-Weierstrass property, as desired. $\square$

Up until now we have proven the following results

$$\begin{aligned} X \text{ is closed-bounded} &\implies X \text{ is compact set} \\ &\implies \text{Infinite sequences from } X \text{ has a cluster point} \\ &\iff X \text{ has the Bolzano-Weierstrass property} \end{aligned}$$