Real Analysis
Lecture Notes

Set Theory

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1 Ordered Sets

We begin with introducing the notion of a partially ordered set. Suppose S is partially ordered. Given some binary relation R defined over elements $a, b \in S$, the set S must have a partial order that satisfies the following

(i) $aRa \quad \forall\, a \in S$ (reflexivity)

(ii) $aRb \wedge bRa \implies a = b$ (symmetry)

(iii) $aRb \wedge bRc \implies aRb$ (transitivity)

Example: Consider the finite set $X = \{1, 2, 3, 4\}$, and consider its power set $\mathcal{P}(X) = S = \{\text{all subsets of } X\}$. Then, for elements $A, B \in S$, does the relation '$\subseteq$' (where $\subseteq$ is defined in the typical way) satisfy a partial order over S?

To verify that $\subseteq$ is indeed a partial order we simply go through our three criteria defined above. For this particular example we can confirm our criteria via explicitly outlining the elements of our set S (this may not be feasible in sufficiently large sets, or possible in infinite sets). So, we find that[1]

$$\begin{aligned} S &= \mathcal{P}(X) \\ &= \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \\ &\qquad \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \\ &\qquad \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\Big\} \end{aligned}$$

Now, going through our criteria:

(i) $A \overset{?}{\subseteq} A \quad \forall A \in S$. Using the definition of $\subseteq$, we should find that the reflexivity criteria is easily satisfied.

[1]Note that if a finite set X has n elements then $\mathcal{P}(X)$ will contain 2^n elements.

(ii) $A \subseteq B \wedge B \subseteq A \overset{?}{\Longrightarrow} A = B$. Once again, our finite set permits us to manually verify this condition. We see that the only two elements A, B which satisfy the symmetry condition are when A and B are chosen such that $A = B$.

(iii) $A \subseteq B \wedge B \subseteq C \overset{?}{\Longrightarrow} A \subseteq C$. Perhaps this is a bit less trivial to verify, but once again we should quickly see that this is indeed the case for all elements $A, B, C \in S$, and so the transitivity of our relation $\subseteq$ holds.

Thus, $\subseteq$ is indeed a partial order, and so the set $S = \mathcal{P}(A)$ under $\subseteq$ forms a partially ordered set. It turns out that we call this relation $\subseteq$ the *containment relation*.

However, using the above example, despite being a partially ordered set we have elements who cannot be so easily compared/ordered under $\subseteq$. Namely,

$$\{1,2\} \not\subseteq \{1,3\}$$
$$\{2,3\} \not\subseteq \{1,3,4\}$$

It is not immediately clear how we should deal with these cases under a partial order. For this reason we choose to introduce the notion of a <u>total order</u> and <u>totally ordered sets</u>.

Consider the closed interval $S = [0,1]$ and the relation $a \leq b$ (as typically defined). If we were willing to do so, we could verify that $\leq$ is indeed a partial order on S. However, it is also true that *all* $a, b \in S$ satisfies either

$$a \leq b \quad \text{or} \quad b \leq a \quad \forall\, a, b \in S$$

Hence, if S is a <u>totally ordered set</u>, then for some binary relation R over a set S, we require the following four criteria:

(i) $\forall\, a \in S,\ aRa$ (reflexivity)

(ii) $aRb \wedge bRa \implies a = b$ (symmetry)

(iii) $aRb \wedge bRc \implies aRb$ (transitivity)

(iv) $\forall\, a, b \in S,\ aRb$ or bRa (comparability)

Consider the finite set of integers $\{1,2,3,4\}$ and the set of nonnegative real numbers $[0,\infty)$ under the usual order.[2] We may verify that both these sets are indeed totally ordered, and yet there turns out to be (at least one) material distinction between the two.

We pose the question: Does any nonempty subset of S have a "smallest"[3] element? (ed. For some reason I prefer calling it a *least element*) Clearly this is trivially true for $S = \{1,2,3,4\}$: If $1 \in S' \subset S$ then 1 is our least element. If $1 \notin S'$ then, if $2 \in S'$ then 2 is our least element, and so on... Consider now our second set $S = [0,\infty)$. If we select the

[2]Typically, the "usual order" will mean $\leq$ if the elements are numbers or $\subseteq$ if the elements are sets.
[3]This means what you think it means: For some total order $\leq$, we find $\forall\, s \in S,\ \exists\, t \in S,\ t \leq s$

subset $(0,1) \subseteq [0,\infty)$ we may note that there is no $x \in (0,1)$ such that for all $y \in (0,1)$ we find $x \leq y$ since, as an open interval, we may descend arbitrarily close to the left endpoint.

For the above reason we introduce the notion of a well ordered set. We say that a well ordered set is a totally ordered set such that all nonempty subsets has a least element. From this definition it should be clear that the set of naturals $\mathbb{N}$ is well ordered but the set of real numbers $\mathbb{R}$ is not, since we may construct open subsets without a least element (as illustrated above).

Under this definition we note that the set $S = \{..., -3, -2, -1, 0, 1\}$ is not well ordered since, although we may find a greatest element, if we select the subset $\{..., -3, -2\}$ we see that there is no least element.

Now, suppose T is totally ordered by the total order $\leq$. Let $T_1 \subseteq T$, $T_1 \neq \emptyset$. If T_1 inherits a total order from its totally ordered parent T then it's easy to see that it remains totally ordered itself:

Proof. For $a, b, c \in T_1$ we have that since $T_1 \subseteq T$, $a, b, c \in T$. Since T is totally ordered we have that $\leq$ satisfies the four criteria of total ordered above with respect to a, b, c. However, since a, b, c were chosen to be arbitrary elements of T_1, we may conclude that T_1 is indeed totally ordered, as desired. $\square$

Suppose T is now a well ordered set. It is perhaps less clear whether $T_1 \subseteq T$ is well ordered. However, it turns out that the subset T_1 does indeed remain well ordered:

Proof. Suppose T is well ordered and let $T_1 \subseteq T$. If T is well ordered then T is totally ordered and all nonempty subsets of T contain a least element. Since we have shown that $T_1 \subseteq T$ must be totally ordered, it is sufficient to show that all nonempty subsets of T_1 contain a least element. However, all subsets of T_1 must be subsets of T since the elements of T_1 are wholly contained by T. Since the nonempty subsets of T have a least element, we may conclude that the nonempty subsets of T_1 must have a least element, as desired. $\square$

It turns out that being well ordered on $\mathbb{N}$ is identical to the principle of mathematical induction.[4] That is, if a given set is well ordered then we may use mathematical induction to prove some statement is true for all elements of our set. We quickly summarize mathematical induction: If we have given statements indexed by $n \in \mathbb{N}$, say $P(1), P(2), ..., P(n), ...$ we know that all statements $P(n), n \in \mathbb{N}$ hold if

1. $P(1)$ is true.

2. $P(k) \implies P(k+1)$.

[4] This was given as an assignment problem in class.

2 Sequences

A finite sequence from X is a function from a set of positive integers $\{1, 2, ..., n\}$ into X. We may denote this by

$$f\{1, 2, ..., n\} \longrightarrow X \quad \text{or}$$
$$\{x_1, x_2, ..., x_n\} \quad n \in \mathbb{N}$$

A countably infinite sequence from X is a function $f : \mathbb{N} \to X$ where $\mathbb{N}$ has the natural order. A countably infinite set is a set which is in the range of a countably infinite sequence.

Example: Is the set $S = \{-1, -2, -3, ...\}$ countably infinite? Yes! Define $f : \mathbb{N} \to S$ such that

$$\begin{aligned} f(1) &= -1 \\ f(2) &= -2 \\ f(3) &= -3 \\ &\vdots \\ f(n) &= -n \\ &\vdots \end{aligned}$$

We should note that this function f is surjective (onto) since all elements of S are mapped to by f from some element of $\mathbb{N}$, and so we have found a function $f : \mathbb{N} \to S$ to satisfy the definition of a countably infinite set.

Example: Consider the set $S = \{0, 2, 4, 8, ...\}$. We may define $f : \mathbb{N} \to S$ by $f(n) = 2n - 2$ to show that S is indeed countably infinite.

So, we have defined a countably infinite set S to be a set that may be mapped from by $\mathbb{N}$ by $f : \mathbb{N} \to S$. However, it is sometimes inconvenient to construct a mapping from $\mathbb{N}$ to S. For this reason we introduce the following result: If we map a set S to a second set T by $g : S \to T$ we immediately see that the composition $(g \circ f)(n) = g(f(n))$, $g \circ f : \mathbb{N} \to T$, will map $\mathbb{N}$ onto T. Therefore, if S is countable and if there exists some map from S onto T, then T must also be countable. Hence, we may prove that some set T is countable by showing that there exists some surjective mapping from S onto T.

2.1 Subsequences

For some sequence from the set X, say $\{x_1, x_2, x_3...\}$, we define a subsequence of a sequence to be the the sequence $\{x_{n_1}, x_{n_2}, x_{n_3}, ...\}$ with indices $n_1, n_2, n_3, ... \in \mathbb{N}$. Alternatively, if

$$f\{1, 2, ..., \}$$

is some sequence, then the set

$$h\{1, 2, ...\}$$

is a subsequence of $f\{1, 2, ...\}$ if there exists a monotone function $g : \mathbb{N} \to \mathbb{N}$ such that

$$h = g \circ f$$

That is, our subsequence $\{x_{n_1}, x_{n_2}, ...\}$ can be given by

$$\begin{aligned} x_{n_1} &= f(n_1) = g(f(n_1)) = h(1) \\ x_{n_2} &= f(n_2) = g(f(n_2)) = h(2) \\ &\vdots \end{aligned}$$

3 Unions, Intersection, and Complementation

It will be useful to recall DeMorgan's Laws:

(1) $(A \cap B)^c = A^c \cup B^c$

(2) $(A \cup B)^c = A^c \cap B^c$

(ed. Essentially everything else was omitted from this section of the text?)

4 Algebras of Sets

Let X be a set. An algebra $\mathcal{A}$ of subsets of X is a collection of sets closed under intersection, union, set difference,[5] and complementation as follows: For $A, B \in \mathcal{A}$, then

(a) $A \cap B \in \mathcal{A}$

(b) $A \cup B \in \mathcal{A}$

(c) $B \setminus A \in \mathcal{A}$

(d) $A^c \in \mathcal{A}$

Strictly speaking, we only need $A \cup B \in \mathcal{A}$ and $A^c \in \mathcal{A}$ in order to derive the remaining two properties. For example, $A \cap B = (A^c \cup B^c)^c$ and so since $A, B \in \mathcal{A}$, and since $\mathcal{A}$ is closed under complementation, we find $A^c, B^c \in \mathcal{A}$. Since $\mathcal{A}$ is closed under unions we find $A^c \cup B^c \in \mathcal{A}$. Finally once again realizing $\mathcal{A}$ is closed under complementation we find $(A^c \cup B^c)^c \in \mathcal{A}$. We can perform similar manipulations to show that the explicit statement of $B \setminus A = B \cap A^c \in \mathcal{A}$ is not necessary.

Example: Let $X = \mathbb{N}$ and consider the collection $\mathcal{A} = \{\emptyset, \mathbb{N}\}$. We can verify that our closure criteria over our two elements to conclude that $\mathcal{A}$ is indeed an algebra.

[5] For sets X and Y we say that "set subtraction is given by $X \setminus Y = X \cap Y^c$.

Example: Let $X = \mathbb{N}$ and $\mathcal{B} = \{\text{all subsets of } \mathbb{N}\}$. This collection can too be shown as a valid algebra.

Example: *(Boolean algebra on $\mathbb{N}$)* Take $X = \mathbb{N}$ and $\mathcal{C} = \{\text{all subsets of } X \text{ which are either finite or have finite complements (cofinite)}\}$.[6] We start with determining what type of elements belong to $\mathcal{C}$. Listing some examples we find:

$$\begin{aligned} \{2,3\} &\in \mathcal{C} \quad \text{(finite)} \\ \mathbb{N} \setminus \{5\} &\in \mathcal{C} \quad \text{(cofinite)} \\ \mathbb{N} &\in \mathcal{C} \quad \text{(cofinite)} \\ \emptyset &\in \mathcal{C} \quad \text{(finite)} \end{aligned}$$

From these examples it should be immediately obvious that $\mathcal{C}$ is indeed closed under complementation.[7]

What about intersection? If sets A and B are finite then, by definition, $A, B \in \mathcal{C}$ and clearly $A \cap B$ must be finite, so $A \cap B \in \mathcal{C}$. Similarly, for cofinite B, if A is finite then $A \cap B$ must be finite, and so $A \cap B \in \mathcal{C}$. Finally, if both A and B are cofinite we must consider $A \cap B$. Note

$$\begin{aligned} A \cap B &= [(A \cap B)^c]^c \\ &= [A^c \cup B^c]^c \end{aligned}$$

but since A and B are cofinite, A^c and B^c must be finite, so $A^c \cup B^c$ must be finite, and because

$$(A \cap B)^c = ([A^c \cup B^c]^c)^c = A^c \cup B^c$$

is finite, we may conclude that, by definition, $A \cap B$ is cofinite.

What about unions? If $A, B \in \mathcal{C}$ are both finite, then clearly $A \cup B$ is finite and so the union must be finite: $A \cup B \in \mathcal{C}$. If A is finite and B is cofinite, then

$$\begin{aligned} & B \subseteq A \cup B \\ \implies & B^c \supseteq (A \cup B)^c \quad \text{(complementation reverses containment}^8\text{)} \end{aligned}$$

Since B is cofinite, B^c must be finite. Thus, from $(A \cup B)^c \subseteq B^c$ we have that the complement of $A \cup B$ is finite, and so $A \cup B$ is cofinite. Hence, $A \cup B \in \mathcal{C}$. Finally, if both A and B are both cofinite then we should note that

$$\begin{aligned} A \cup B &= [(A \cup B)^c]^c \\ &= [A^c \cap B^c]^c \\ \implies (A \cup B)^c &= A^c \cap B^c \end{aligned}$$

[6] An example of a set which is neither finite nor cofinite is the set of even naturals over the set of naturals. Its complement is the set of odd naturals which is clearly infinite.

[7] I would like to more rigorously prove this, but this is essentially as was stated in class.

[8] Draw a diagram if you don't see this.

Since A and B are cofinite, A^c and B^c must be finite, and clearly the intersection $A^c \cap B^c$ must also be finite. Therefore, by definition, the union $A \cup B \in \mathcal{C}$, as desired.

Are all subsets of $\mathbb{N}$ finite or cofinite? No! As a counterexample, consider $A = \{\text{all primes}\}$. We know that this is an infinite set with complement $A^c = \{\text{all commposites}\}$, which is itself infinite set.