Real Analysis
Lecture Notes

Metric Spaces

November 28 2016
Last update: December 4, 2017

1 Baire Categories

Baire Category Theorem: Let X be a complete metric space and take a countable family of dense open sets $\{O_i\}_{i=1}^{\infty}$ from X. The intersection

$$\bigcap_{i=1}^{\infty} O_i$$

is dense.

Proof. Take a nonempty open set U of X. Since we claim that the countable intersection $\bigcap_{i=1}^{\infty} O_i$ is dense in X we wish to prove that

$$U \cap \left(\bigcap_{i=1}^{\infty} O_i \right) \neq \emptyset$$

It turns out that this nonemptyness will emerge as a consequence of completeness. We rewrite this intersection as

$$U \cap \left(\bigcap_{i=1}^{\infty} O_i \right) = \bigcap_{i=1}^{\infty} (U \cap O_i)$$

and the intersection of each $(U \cap O_i) \neq \emptyset$ since O_i is each O_i is dense in X so that $\overline{O_i} = X$.

Take $x_1 \in U \cap O_1$. Since U and each O_i is open we have that each $(U \cap O_i)$ is open and so we can form the open spheres

$$S_{x_1,r_1} \equiv S_1 \subset U \cap O_1$$

Thus, we remain within our open subset U by remaining within the open sphere S_1 since $S_1 \subset U \cap O_1$ and so $S_1 \subset U$.

Now, pick O_2 in our intersection. We have that O_2 is dense and open in X. Therefore, much like O_1,

$$O_2 \cap S_1 \neq \emptyset$$

and $O_2 \cap S_1$ is open since both sets are open. Take $x_2 \in O_2 \cap S_1$ and note that since this intersection is open we may take some open sphere S_2 such that

$$S_2 = S_{x_2,r_2} \subset O_2 \cap S_1$$

From these points x_1 and x_2 we have the distance $\rho(x_1, x_2) < r_1 - r_2$ since $x_1 \in S_1 \equiv S_{x_1,r_1}$ and $x_2 \in S_2 \equiv S_{x_2,r_2}$ (draw a diagram).

In addition to this construction of S_1 and S_2 let us insist that r_2 is bound above by

$$r_2 < \frac{1}{2} r_1$$

so that our sequence of radii $r_n \to 0$.

Claim: We claim that the closure of S_2 is a subset of S_1, $\overline{S_2} \subset S_1$. To show this let $y \in \overline{S}_2 \setminus S_2$ so that y is a bound along the closed boundary of $\overline{S}_2$. We find that

$$\begin{aligned}
\rho(y, x_2) &= r_2 \\
\rho(y, x_1) &\leq \rho(y, x_2) + \rho(x_2, x_1) \\
&< r_2 + (r_1 - r_2) \\
&= r_1 \\
\implies \rho(y, x_1) &< r_1
\end{aligned}$$

Therefore,

$$\begin{aligned}
&y \in \overline{S}_2 \setminus S_2 \\
\implies &y \in S_1 \\
\implies &\overline{S}_2 \setminus S_2 \subset S_1 \\
\implies &\overline{S}_2 \subset S_1
\end{aligned}$$

Now, take $O_3 \cap S_2$ and $x_3 \in O_3 \cap S_2$ and take the sphere S_3

$$S_3 = S_{x_3,r_3}$$

such that

$$r_3 < \frac{1}{2} r_2 < \frac{1}{4} r_1$$

Then, once again, we have that $\overline{S}_3 \subset S_2$ by the same argument as above.

Repeating this process inductively we get

$$r_n < \frac{1}{2(n-1)} r_1 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

Thus, our radii of open spheres $S_n \subset O_n \cap S_{n-1}$ vanish as $n \to \infty$. By this construction we find our sequence of centers $(x_n) = (x_1, x_2, ...)$ is Cauchy since the radii of the open spheres around these points $r_n \to 0$ and so these points must be getting arbitrarily close together.

Therefore, by the assumption of the completeness of X we have that $(x_n) \longrightarrow x \in X$.

Now, let N be some fixed natural number sufficiently large so that

$$x_n \in S_{N+1} \quad \text{for } n = N+1, N+2$$

and since $(x_n) \to x$ we have that x must also be a limit point of this subsequence $(x_{N+1}, x_{N+2}, ...)$. However, by construction of $x_n \in S_{N+1}$ we have

$$\{x_{N+1}, x_{N+2}, x_{N+3}, ...\} \subset S_{N+1}$$

Therefore

$$x \in \overline{S}_{N+1}$$

but

$$\overline{S}_{N+1} \subset S_N \subset O_N$$

hence

$$x \in O_N \quad \forall\, N$$

$$\implies x \in \bigcap_{i=1}^{N} O_i$$

and so

$$x \in U \cap \left(\bigcap_{i=1}^{\infty} O_i \right)$$

$$\implies U \cap \left(\bigcap_{i=1}^{\infty} O_i \right) \neq \emptyset$$

Therefore, since U was an arbitrary open subset of X we have that the countable intersection $\bigcap_{i=1}^{\infty} O_i$ is dense in X, as desired. $\square$

Lemma: Let A, B, and C be metric spaces such that $A \subset B \subset C$. Suppose that A is dense in C. Then, subspace A is also dense in B.

Proof. Let $O \subset B$ be some nonempty open set in B. To show that A is dense in B we must show that

$$O \cap A \overset{?}{\neq} \emptyset$$

Since $B \subset C$ we have some open set in C such that

$$V \cap B = O$$

Since $O \neq \emptyset$ we must have $V \neq \emptyset$ and so

$$A \cap V \neq \emptyset$$

Thus

$$\begin{aligned} A \cap V &= (A \cap V) \cap B \\ &= A \cap (V \cap B) \\ &= A \cap O \\ \implies A \cap O &\neq \emptyset \end{aligned}$$

and so A is dense in B, as desired. $\square$

Claim: The set of irrationals $\mathbb{R} \setminus \mathbb{Q}$ has the Baire Category property.

Note that we have found a Cauchy sequence (x_n) from $\mathbb{R} \setminus \mathbb{Q}$ that does not converge in $\mathbb{R} \setminus \mathbb{Q}$, in particular

$$\left(\frac{\pi}{n}\right) \longrightarrow 0$$

and so $\mathbb{R} \setminus \mathbb{Q}$ is *not* complete. Thus, if our claim is true we see that the Baire Category property *does not imply* completeness.

Proof. Suppose $O_1, O_2, ...$ are dense and open in $\mathbb{R} \setminus \mathbb{Q}$. We can find an open set $V_1 \subset \mathbb{R}$ such that

$$V_1 \cap (\mathbb{R} \setminus \mathbb{Q}) = O_1$$

Is V_1 dense in $\mathbb{R}$?

Suppose $P_1 \neq \emptyset$ is open in $\mathbb{R}$ and $P \cap V_1 = \emptyset$. Then

$$\begin{aligned} &P \cap (\mathbb{R} \setminus \mathbb{Q}) \quad \text{is open in } \mathbb{R} \setminus \mathbb{Q}, \text{ and} \\ &P \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset \end{aligned}$$

since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$ (and so an intersection of an open set with a dense set is nonempty).

Since O_1 is dense in $\mathbb{R} \setminus \mathbb{Q}$ by assumption we have (since $P \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$)

$$\begin{aligned} &(P \cap (\mathbb{R} \setminus \mathbb{Q})) \cap O_1 \neq \emptyset \\ &(P \cap (\mathbb{R} \setminus \mathbb{Q})) \cap O_1 \subset V_1 \cap (\mathbb{R} \setminus \mathbb{Q}) \end{aligned}$$

Therefore, the countable intersection

$$\begin{aligned} \left(\bigcap_{n=1}^{\infty} \underbrace{V_n}_{\text{dense and open in } \mathbb{R}} \right) \bigcap_{q \in \mathbb{Q}} \underbrace{(\mathbb{R} \setminus \{q\})}_{\text{open and dense in } \mathbb{R}} &= \bigcap_{n=1}^{\infty} V_n \cap (\mathbb{R} \setminus \mathbb{Q}) \\ &= \bigcap_{n=1}^{\infty} (V_n \cap (\mathbb{R} \setminus \mathbb{Q})) \\ &= \bigcap_{n=1}^{\infty} O_n \end{aligned}$$

Since each O_n is dense in $\mathbb{R}$ we have by our lemma each O_n is dense in $\mathbb{R} \setminus \mathbb{Q}$. Thus, the Baire Category property is satisfied for $\mathbb{R} \setminus \mathbb{Q}$ since each O_n are dense. Therefore, we have found a metric space satisfying the Baire Category property that is not complete. $\square$

Example: Is $[0, 1]$ countable? Since $[0, 1]$ is closed and bounded we have that it must be compact. Since $[0, 1]$ is compact we then have that it is totally bounded, and since total boundedness $\implies$ completeness we see that $[0, 1]$ is complete.

Now, suppose $[0, 1]$ is countable. Take $r \in [0, 1]$ and look at the intersection

$$[0, 1] \setminus \{r\}$$

Since $\{r\}$ is closed we have $[0, 1] \setminus \{r\}$ is dense and open. However, the intersection over all elements of $[0, 1]$ is

$$\bigcap_{r \in [0,1]} [0, 1] \setminus \{r\} = \emptyset$$

If $[0, 1]$ *were* countable then this intersection would have been a countable intersection of open sets. Therefore, since $[0, 1]$ is also complete, we may use the Baire Category theorem to conclude that this countable union of dense open sets $[0, 1] \setminus \{r\}$ would itself be dense. However, $\emptyset$ is clearly not dense in $[0, 1]$ and so we must conclude that $[0, 1]$ is, in fact, uncountable.

Example: Consider the set of natural numbers $\mathbb{N}$. We know that $\mathbb{N}$ is complete since any Cauchy sequence will eventually have ϵ so small that $|x_n - x_m| < \epsilon$ implies $x_n = x_m$. Furthermore, $\mathbb{N}$ is countable, but the countable intersection from this complete space

$$\bigcap_{n \in \mathbb{N}} \mathbb{N} \setminus \{n\} = \emptyset$$

This *does not* violate the Baire Category Theorem since the intersections $\mathbb{N} \setminus \{n\}$ is not dense because the closure $\overline{\mathbb{N} \setminus \{n\}} \neq \mathbb{N}$.