# Real Analysis
# Lecture Notes

The Real Number System

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# 1 Open and Closed Sets of $\mathbb{R}$ (con't)

## 1.1 Open Sets

Last class we introduced the notion of an open set:

**Definition** (*Open sets*): A set $O \subset \mathbb{R}$ said to be an open set if

$$\forall\, x \in O,\ \exists\, \delta > 0 \quad |x - y| < \delta \implies y \in O$$

That is, $O$ is an open set if, for all points $x \in O$, we remain in $O$ if we get sufficiently close to $x$, i.e. an open set is a set for which all points have a small ball surrounding them that remains enclosed in $O$. An alternate, but equivalent, definition an open set is given as follows: A set $O \subset \mathbb{R}$ is said to be an open set if

$$\forall\, x \in O,\ \exists\, (a_x, b_x) \text{ with } x \in (a_x, b_x),\ \text{such that } (a_x, b_x) \subset O$$

for the interval $(a_x, b_x)$ (depending on a given $x$) defined as $\{r \in \mathbb{R}\ :\ a_x < r < b_x\}$. That is, we say that $O$ is an *open* set if for every point $x \in O$ there is a small ball around $x$ that remains fully enclosed by $O$.

With this definition of an open set we consider the (fairly long and intensive) following result:

**Proposition**: Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals of the form $(a, b)$ (i.e. open and connected),[1] where $(a, b)$ is defined by

$$(a, b) = \{x \in \mathbb{R}\ :\ a < x < b\}$$

That is any open set $O$ of $\mathbb{R}$ can be expressed as

$$O = \bigcup_{n=1}^{\infty} (a_n, b_n), \quad a_n < b_n$$

[1] The intervals $(-\infty, b)$, $(a, \infty)$, and $(-\infty, \infty)$ count as open intervals.

where all open intervals $(a_k, b_k)$ are disjoint, meaning

$$(a_n, b_n) \cap (a_m, b_m) = \emptyset, \quad \text{if } m \neq n$$

*Proof.* Our proof will make use of the properties of $\mathbb{R}$ that we have introduced in previous lectures. In particular, we will rely on the completeness of the reals. Now, take $O$ an open set in $\mathbb{R}$, $O \neq \emptyset$. Let $x \in O$ be arbitrary and fixed. Since $O$ is an open set then our $x$ satisfies

$$\exists\, \delta > 0, \text{ such that } (x - \delta, x + \delta) \subset O$$

therefore

$$\exists\, y > x, \text{ such that } (x, y) \subset O$$
$$\exists\, z < x, \text{ such that } (z, x) \subset O$$

where the candidate for $y$ is $y = x + \delta$ so that $(x, y) = (x, x + \delta) \subset O$ and the candidate for $z$ is $z = x - \delta$ so that $(z, x) = (x - \delta, x) \subset O$. Therefore, the sets given by

$$\{y \in \mathbb{R} \;:\; x < y \text{ and } (x, y) \subset O\}$$
$$\{z \in \mathbb{R} \;:\; z < x \text{ and } (z, x) \subset O\}$$

are *not* empty since they at least contains $x+\delta$ and $x-\delta$, respectively. Lets assume that the set $\{y \in \mathbb{R} \;:\; x < y \text{ and } (x, y) \subset O\}$ has some upper bound and if it has an upper bound, then by the completeness of $\mathbb{R}$, the set must have a supremum. Call this supremum $b$. So,

$$b = \sup\{y \in \mathbb{R} \;:\; x < y, (x, y) \subset O\}$$

Similarly, by the same logic let $a$ be the infinitum of the set $\{z \in \mathbb{R} \;:\; z < x \text{ and } (z, x) \subset O\}$ so that

$$a = \inf\{z \in \mathbb{R} \;:\; (z, x) \subset O\}$$

with this pair of $a$ and $b$ let us construct the open interval $I_x = (a, b)$ which we can see contain $(x - \delta, x + \delta)$ by construction. Lets establish some properties of the interval $I_x$ for $x \in O$ and $O$ is an open set.

**We first claim that $\boldsymbol{I_x \subset O}$**: Take $w \in I_x = (a, b)$ and assume $x < w$. Note that since $w \in (a, b)$ we have $w < b$ and so $w$ cannot be an upper bound for

$$\{y \in \mathbb{R} \;:\; x < y \text{ and } (x, y) \subset O\}$$

because it is *false*[2] that $y \leq w$ for all such $y$ by the definition of $b$ the supremum. Therefore,

$$\exists\, y \in \mathbb{R}, \text{ such that } (x, y) \subset O \text{ and } w < y$$

Now, since $w < y$, and since have assumed $x < w$, we have that $w \in (x, y)$. But the interval $(x, y) \subset O$. Therefore, we conclude that

$$w \in O$$

[2] I need to think about this one a bit more...

and

$$w \in I_x \subset O$$

Similarly, if we had chosen $w \in I_x$ such that $a < w < x$, then for our $z$ defined above we have some $z$ such that $z < w < x$ by the definition of the infimum. So $(z, x) \subset O$, and

$$w \in I_x = (a, b) \subset O$$

**Next, we claim that $\boldsymbol{b, a \notin O}$**: Starting with $b$, suppose $b \in O$. By the definition of the open set $O$

$$\exists\, \delta > 0, \text{ such that } (b - \delta, b + \delta) \subset O$$

So $b - \delta < b \implies b - \delta$ cannot be an upper bound for the set $\{y \in \mathbb{R} \;:\; (x, y) \subset O\}$ since there is some $y > b - \delta$ for which $(x, y)$ is still in $O$. If $(x, y) \subset O$ and $(b-\delta, b+\delta) \subset O$ then the union of these sets is open and

$$(x, y) \cup (b - \delta, b + \delta) \subset O$$

So, $(x, b+\delta) \subset O$. But $b$ was the supremum of the right endpoints, $b = \sup\{y \;:\; (x, y) \subset O\}$, hence

$$b + \delta \leq b \quad \text{Contradiction!}$$

Therefore, we conclude that $b \notin O$. Similarly, by a symmetric argument we may conclude that $a \notin O$.

**Recap**:[3] We've been trying to show that all open intervals in $\mathbb{R}$ can be expressed as a countable union of disjoint open sets. As $x$ ranges over our open set $O$, the intervals $I_x$ are in $O$ (from our work above), and so $O$ is the union of all these intervals $I_x$ as we go through all $x \in O$. That is,

$$\bigcup_{x \in O} I_x = O$$

Now, consider two such sets in $O$, say $(a, b)$, $a < b$, and $(c, d)$, $c < d$. We want to show that if these intervals are *not* disjoint then they *must* be identical:

**We now claim that if $\boldsymbol{(a, b) \cap (c, d) \neq \emptyset}$ then $\boldsymbol{(a, b) = (c, d)}$**: Suppose $(a, b) \cap (c, d) \neq \emptyset$. What would happen if $c \geq b$? If $b \leq c$ then these intervals would be disjoint since we would have at least the point $b = c$ which would "divide" the two intervals into disjoint sets since $b = c \notin (a, b)$ and $b = c \notin (c, d)$. Therefore our intersection would be $\emptyset$, contradicting our assumption, and so $c < b$.

What if we have $a \geq d$? If $a \geq d$ then once again our intervals would be disjointly divided by at least one point $a = d$ since $a = d \notin (c, d)$ and $a = d \notin (a, b)$. So the intersection is once again empty, contradicting our assumption, and so $a < d$.

That is, we have found so far that $a < d$ and $c < b$.

[3] I'm having trouble with this part...

We know that $a, b, c, d \notin O$. Since $c \notin O$ then $c \notin (a, b)$. Since $c < b$, together with $c \notin (a, b)$ then we must have $c \leq a$. Similarly, since $a \notin O$ we get $a \notin (c, d)$ and since $a < d$, then we must have $a \leq c$. Taking these two inequalities together yields

$$\begin{aligned} c \leq a \leq c \\ \implies c = 0 \end{aligned}$$

A symmetric argument yields $b = d$, and so we conclude that $I_x$ are indeed either disjoint or identical. Note that for each disjoint $I_x$ we have a unique rational $q \in \mathbb{Q}$ such that $q \in I_x$ that may uniquely identity an open interval (by the density of $\mathbb{Q}$ in $\mathbb{R}$). Therefore, we get a 1-1 map from $I_x$ onto a countable subset of $q \in \mathbb{Q}$. So, we are finally able to conclude that every open set $O$ in $\mathbb{R}$ is composed of a countable union of disjoint open intervals, as desired. $\square$

**Example**: Consider the union

$$(0, 1) \cup (1, 2) \cup (2, 3) \cup \cdots \cup (k - 1, k) \cup (k, k + 1) \cup \cdots$$

Clearly this is a countable union of disjoint open sets. Therefore, by our above theorem we may conclude that sets of this form are open sets.

## 1.2 Compact Sets

*"What does finiteness mean in topology?"* We know that *finite* sets are "nice". That is, (1) finite sets are *bounded*; (2) In $\mathbb{R}$, finite sets contain their sup and inf and so it makes sense to consider a max and min; (3) When performing some process on finite sets we are guaranteed that this process will end. So, it makes sense to introduce some definition of "small" sets.[4] That is, since finite sets are so nice to work with it makes sense to define precisely what we mean so that we can work with these objects in a rigorous framework. To do this we will need to begin with the following definitions:

**Definition** *(Open cover of a set)*: An open cover of a subset $E$ in some space $X$ is a collection of *open sets* $\{O_i\}$ whose *union* "covers" (i.e. wholly contains) $E$, where $i \in I$ is some (potentially uncountable!) indexing. That is, $\{O_i\}$ is an open cover of $E \subset X$ if

$$E \subset \bigcup_{i \in I} O_i$$

**Definition** *(Subcover of a cover)*: A subcover is a subcollection of $\{O_i\}_{i \in I}$. That is, $\{O_{i_n}\}_{i_n \in I}$ is a subcover of $\{O_i\}_{i \in I}$ if each $i_n$ are specific indices found in $I$. In the future we will usually want to consider finite subcovers, but this definition doesn't necessarily require this (i.e. the subcover may still be uncountable).

[4] I think this is where the term "compact" comes from – Using the real-world definition of "compact" we can imagine that if something is "compact" then it is containable, in some sense.

**Example**: The interval $\left[\frac{1}{2}, 1\right)$ has a cover $\{O_n\}_{n=3}^{\infty}$ where each $O_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$.[5] Note that our intervals $O_n$ look like

$$\begin{aligned} n = 3 &\mapsto \left(\frac{1}{3}, \frac{2}{3}\right) \\ n = 4 &\mapsto \left(\frac{1}{4}, \frac{3}{4}\right) \\ n = 5 &\mapsto \left(\frac{1}{5}, \frac{4}{5}\right) \\ &\vdots \\ n = m &\mapsto \left(\frac{1}{m}, 1 - \frac{1}{m}\right), \quad m = 3, 4, ... \\ &\vdots \end{aligned}$$

By construction we see that all $O_n$ are open sets, satisfying our openness criteria. Clearly $\frac{1}{2}$ is covered by all our sets $O_n$. Also, $\frac{3}{4}$ is covered, not by $O_4$, but instead by $O_5$ since $O_4 = \left(\frac{1}{4}, \frac{3}{4}\right)$ is open and does not contain $\frac{3}{4}$. In fact, any point $x \in \left[\frac{1}{2}, 1\right)$ will *eventually* be covered by some set $O_i \in \{O_n\}_{n=3}^{\infty}$. That is, any point $x \in \left[\frac{1}{2}, 1\right)$ will be covered by the some interval $O_k = \left(\frac{1}{k}, 1 - \frac{1}{k}\right)$ for sufficiently large $k$.[6]

However, we could have considered lots of covers! For example, is the collection of intervals $\{(0, 2)\}$ an open cover? Well our collection of "intervals" is actually just the single interval $(0, 2)$ which is open by construction. Furthermore, any $x \in \left[\frac{1}{2}, 1\right)$ will be contained by some open interval in the collection $\{(0, 2)\}$, namely the interval $(0, 2)$.

We could also consider an *uncountably large* cover. We could construct the cover $\{O_x\}_{x \in \left[\frac{1}{2}, 1\right)}$ such that

$$O_x = (x - \delta, x + \delta), \quad \delta > 0$$

Thus, our cover becomes $\{(x - \delta, x + \delta)\}_{x \in \left[\frac{1}{2}, 1\right)}$, which consists of open sets by construction. Hopefully it is clear why this cover is indeed uncountable (there are an uncountable number of $x$ in the interval $\left[\frac{1}{2}, 1\right)$), and trivially every $x$ is contained by some open interval in $\{O_x\}$, namely the interval $O_x = (x - \delta, x + \delta)$.

These are all examples of covers. A natural question to ask is (especially when considering infinite covers): Given a cover of $E$, do we need all the sets $O_i$ to cover $E$? Using our previous example of the interval $\left[\frac{1}{2}, 1\right)$ and its cover $\{O_n\}_{n=3}^{\infty} = \left\{\left(\frac{1}{n}, 1 - \frac{1}{n}\right)\right\}$, we should notice that since $\left(\frac{1}{3}, 1 - \frac{1}{3}\right) \subset \left(\frac{1}{4}, 1 - \frac{1}{4}\right)$, we can throw away the first open set at $n = 3$ and still cover our interval. In fact, we can throw away any finite number of open sets from $\{O_n\}_{n=3}^{\infty}$ since eventually there will be some $k$ sufficiently large such that all removed intervals will

[5] We start at $n = 3$ because if we started at $n = 1$ we'd get the interval $(1, 0)$ which either doesn't make sense, or is $(0, 1)$ which makes our example trivial. If we started at $n = 2$ then we'd get the interval $\left(\frac{1}{2}, \frac{1}{2}\right)$ which isn't particularly interesting.

[6] I think this relies on the Archimedean property.

be wholly contained by $\left(\frac{1}{k}, 1-\frac{1}{k}\right)$.

What about the cover $\{O_n\} = \{(0,2)\}$? Well, we can consider the trivial subcover, the cover itself, but if we remove any open set in the collection (i.e. we remove the only open set $(0,2)$ in $\{O_n\}$) then we will not longer be able to cover our original interval $\left[\frac{1}{2}, 1\right)$.

What about our uncountable covering $\{O_x\}_{x\in\left[\frac{1}{2},1\right)} = \{(x-\delta, x+\delta)\}_{x\in\left[\frac{1}{2},1\right)}, \delta > 0$? Well, lets consider the case where $\delta = \frac{1}{10}$. We should be able to see that we can *completely* cover $\left[\frac{1}{2}, 1\right)$ by taking the collection of open sets

$$\begin{aligned} x = \frac{1}{2} &\mapsto \left(\frac{1}{2} - \frac{1}{10}, \frac{1}{2} + \frac{1}{10}\right) = \left(\frac{4}{10}, \frac{6}{10}\right) \\ x = \frac{6}{10} &\mapsto \left(\frac{6}{10} - \frac{1}{10}, \frac{6}{10} + \frac{1}{10}\right) = \left(\frac{5}{10}, \frac{7}{10}\right) \\ x = \frac{7}{10} &\mapsto \left(\frac{7}{10} - \frac{1}{10}, \frac{7}{10} + \frac{1}{10}\right) = \left(\frac{6}{10}, \frac{8}{10}\right) \\ x = \frac{8}{10} &\mapsto \left(\frac{8}{10} - \frac{1}{10}, \frac{8}{10} + \frac{1}{10}\right) = \left(\frac{7}{10}, \frac{9}{10}\right) \\ x = \frac{9}{10} &\mapsto \left(\frac{9}{10} - \frac{1}{10}, \frac{9}{10} + \frac{1}{10}\right) = \left(\frac{8}{10}, 1\right) \end{aligned}$$

so that their union is a cover for $\left[\frac{1}{2}, 1\right)$

$$\left[\frac{1}{2}, 1\right) \subset \left(\frac{4}{10}, \frac{6}{10}\right) \cup \left(\frac{5}{10}, \frac{7}{10}\right) \cup \left(\frac{6}{10}, \frac{8}{10}\right) \cup \left(\frac{7}{10}, \frac{9}{10}\right) \cup \left(\frac{8}{10}, 1\right)$$

That is, we have found not only a subcover from a uncountable cover, but we have found a finite subcover containing 5 open sets from an uncountable set.

On the topic of finite subcovers, lets go back to our cover $\{O_n\}_{n=3}^{\infty} = \left\{\left(\frac{1}{n}, 1-\frac{1}{n}\right)\right\}$ of $\left[\frac{1}{2}, 1\right)$. We said that we could construct many subcovers by removing any finite number of open intervals contained in the cover. However, we *cannot* construct a finite subcover from this particular cover since stopping at any finite $n$ yielding the interval $\left(\frac{1}{n}, 1-\frac{1}{n}\right)$ will still leave some points in $\left[\frac{1}{2}, 1\right)$ uncovered. That is, we require infinitely many elements in our cover in order to successfully construct a cover for $\left[\frac{1}{2}, 1\right)$.

We are now in a good position to define what a compact set is.

**Definition**: *(Compact set)* A subset $A$ of the reals is called compact if every open cover of $A$ contains a finite subcover. That is, we call a subset $A \subset \mathbb{R}$ compact if whenever

$$A \subset \bigcup_{i\in I} O_i, \quad O_i \text{ open sets}$$

then there is a finite subset $J$ of $I$ such that

$$A \subset \bigcup_{i \in J} O_i$$

From our previous work it is easy to see that $\left[\frac{1}{2}, 1\right)$ is *not* compact since we had found a cover of $\left[\frac{1}{2}, 1\right)$ that did not contain a finite subcover of $A$.

**Example**: *($\mathbb{R}$ is not compact)* In order to show that $\mathbb{R}$ is not compact we must demonstrate that no finite collection of open sets can cover $\mathbb{R}$.

Suppose we try to cover $\mathbb{R}$ by the collection of sets

$$\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$$

Clear $\mathbb{R}$ is the union of these connected open sets. However, we cannot form a union of a *finite* number of open sets from this collect, i.e.

$$\nexists\, (-n_1, n_1) \cup (-n_2, n_2) \cup \cdots \cup (-n_k, n_k)$$

such that

$$\mathbb{R} = \bigcup_{i \in \{1, \ldots, k\} \text{ finite}} (-n_i, n_i)$$

and so $\mathbb{R}$ fails our definition of compactness.

**Example**: *(Is $\{2, 7\}$ compact?)* Yes! Any cover of $\{2, 7\}$ must contain either one or two open sets containing $\{2, 7\}$. Removing all open sets from the cover, we are left with at most two open sets. That is, we are left with a finite subcover for all covers of $\{2, 7\}$, and so $\{2, 7\}$ is compact. In fact, from this example it should be fairly easy to see that any finite subset of $\mathbb{R}$ is compact.

## 1.3 Lindelöf Condition

We wish to generalize the notion of compactness to consider the case of a countably infinite number of subcovers for every cover of a given set. This generalization is the Lindelöf Condition and is as follows:

**Lindelöf Condition**: If $U$ is a collection of open sets $U = \{O_i\} = \bigcup O_i$ of $\mathbb{R}$, then $U$ is a union of <u>countably many open sets $O_i$</u>. That is, if $U = \bigcup O_i$ is a set of real numbers then

$$U = \bigcup O_i = \bigcup_{i=1}^{\infty} O_i$$

Notice that this is a weakening the definition of compactness to include countable unions in $\mathbb{R}$. So one may think of the Lindelöf condition as the weakening of compactness.

*Proof.* Let $U$ be a collection of potentially an uncountable number of open sets $\{O_i\}$. Let $x \in U$ be an arbitrary real number. By construction of $U$, since $x \in U$, we must have at least one open set $O_i$ in our (potentially uncountable) collection $\{O_i\}$ such that $x \in O_i$. Since $O_i$ is open there exists an open interval $I_x = (a, b)$ so that $x \in (a_x, b_x) = I_x \subset O_i$, for $a_x < b_x$ real numbers. Recall that between any two reals there is a rational and so we may pick rational numbers $p_x, q_x$ such that $a_x < p_x < x < q_x < b$ and construct the open interval with rational endpoints $(p_x, q_x)$ such that

$$x \in (p_x, q_x) \subset (a_x, b_x) = I_x \subset O_i \subset U$$

We have shown before that the collection of all intervals with rational endpoints $(p_x, q_x)$ is countable. However, we can construct $U$ by the union

$$U = \bigcup_{x \in U} (p_x, q_x)$$

For each interval in the collection of intervals with rational endpoints $\{(p_x, q_x)\}$ select any single set $O_i$ in the (potentially uncountable) original collection $\{O_i\}$ that contains $(p_x, q_x)$, $(p_x, q_x) \subset O_i$. This gives us our bijection $(p_x, q_x) \leftrightarrow O_i$ and so we may conclude that since

$$U = \bigcup_{x \in U} (p_x, q_x)$$

then since $(p_x, q_x) \subset O_i$, we have, for countably many $O_i$

$$U = \bigcup_{i=1}^{\infty} O_i$$

as desired. $\square$

Critically, notice that we're presenting the Lindelöf condition is the context of subsets of $\mathbb{R}$. It may not be the case that any arbitrary collection of sets is Lindelöf. In general, a collection of sets $C$ may or may not have the Lindelöf property:

> A collection $C$ of sets is said to be Lindelöf if every open cover has a countable subcover.

In relation to this definition we see the obvious weakening of compactness:

> A collection $C$ of sets is said to be compact if every open cover has a finite subcover.