# Assignment 1

*David Fleischer – 260396047*

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## Question 1

From our definitions of $\tilde{X}$ and $\tilde{Y}$

$$\begin{aligned}\tilde{X} &= X_{-1} - \mathbf{1}_n \bar{x}^T \\ \tilde{Y} &= Y - \mathbf{1}_n^T \bar{Y},\end{aligned}$$

we find

$$\begin{aligned}
\hat{\beta}_{-1} &= \underset{\beta \in \mathbb{R}^{p-1}}{\arg\min} \, \|\tilde{Y} - \tilde{X}\beta\|_2^2 \\
&= \underset{\beta \in \mathbb{R}^{p-1}}{\arg\min} \, \|Y - \mathbf{1}_n \bar{Y} - \left(X_{-1} - \mathbf{1}_n \bar{x}^T\right) \beta_{-1}\|_2^2 \\
&= \underset{\beta \in \mathbb{R}^{p-1}}{\arg\min} \, \|Y - X_{-1}\beta_{-1} - \mathbf{1}_n \left(\bar{Y} - \bar{x}^T \beta_{-1}\right)\|_2^2 \\
&= \underset{\beta \in \mathbb{R}^{p-1}}{\arg\min} \, \|Y - X_{-1}\beta_{-1} - \mathbf{1}_n \beta_1\|_2^2 \quad \text{(by definition of } \beta_1 \text{ above)} \\
&= \underset{\beta \in \mathbb{R}^{p-1}}{\arg\min} \, \|Y - [\mathbf{1}_n, \, X_{-1}] \, [\beta_1, \, \beta_{-1}]\|_2^2 \\
&\equiv \underset{\beta \in \mathbb{R}^{p-1}}{\arg\min} \, \|Y - X\beta\|_2^2.
\end{aligned}$$

Therefore, if $\hat{\beta} = \left(\hat{\beta}_1, \, \hat{\beta}_{-1}^T\right)^T \in \mathbb{R}^p$ and

$$\hat{\beta}_1 = \bar{Y} - \bar{x}^T \hat{\beta}_{-1},$$

then $\hat{\beta}$ also solves the uncentered problem

$$\hat{\beta} \equiv \left(\hat{\beta}_1, \, \hat{\beta}_{-1}^T\right)^T = \underset{\beta \in \mathbb{R}^p}{\arg\min} \, \|Y - X\beta\|_2^2,$$

as desired.

## Question 2

### (a)

Define our objective function $f : \mathbb{R}^p \to \mathbb{R}$ by

$$
\begin{aligned}
f(\beta) &= \|\tilde{Y} - \tilde{X}\beta\|_2^2 + \lambda\|\beta\|_2^2 \\
&= \left(\tilde{Y} - \tilde{X}\beta\right)^T \left(\tilde{Y} - \tilde{X}\beta\right)^T + \lambda\beta^T\beta \\
&= \tilde{Y}^T\tilde{Y} - \tilde{Y}^T\tilde{X}\beta - \beta^T\tilde{X}^T\tilde{Y} + \beta^T\tilde{X}^T\tilde{X}\beta + \lambda\beta^T\beta \\
&= \tilde{Y}^T\tilde{Y} - 2\beta^T\tilde{X}^T\tilde{Y} + \beta^T\tilde{X}^T\tilde{X}\beta + \lambda\beta^T\beta.
\end{aligned}
$$

Therefore, by taking the gradient we find

$$\nabla f(\beta) = -2\tilde{X}^T\tilde{Y} + 2\tilde{X}^T\tilde{X}\beta + 2\lambda\beta,$$

as desired.

## (b)

The Hessian $\nabla^2 f(\beta)$ is given by

$$\nabla^2 f(\beta) = 2\tilde{X}^T\tilde{X} + 2\lambda\mathbb{I}_{p-1},$$

where $\mathbb{I}_{p-1}$ is the $(p-1)\times(p-1)$ identity matrix. Note that $2\tilde{X}^T\tilde{X} \in \mathbb{S}_+^{p-1}$ (positive semi-definite) and, for $\lambda > 0$, we have $2\lambda\mathbb{I}_{p-1} \in \mathbb{S}_{++}^{p-1}$ (positive definite). Therefore, for all nonzero vectors $v \in \mathbb{R}^{p-1}$,

$$
\begin{aligned}
v^T\nabla^2 f(\beta)v &= v^T\left(2\tilde{X}^T\tilde{X} + 2\lambda\mathbb{I}_{p-1}\right)v \\
&= 2v^T\tilde{X}^T\tilde{X}v + 2\lambda v^T\mathbb{I}_{p-1}v \\
&= 2\left(\underbrace{\|\tilde{X}v\|_2^2}_{\geq 0} + \underbrace{\lambda\|v\|_2^2}_{>0 \text{ when } \lambda>0}\right) \\
&> 0.
\end{aligned}
$$

Hence,

$$\nabla^2 f(\beta) = 2\tilde{X}^T\tilde{X} + 2\lambda\mathbb{I}_{p-1} \in \mathbb{S}_{++}^{p-1},$$

and so $f$ must be strictly convex in $\beta$.

## (c)

Suppose a strictly convex function $f$ is globally minimized at distinct points $x$ and $y$. By strict convexity

$$\forall\, t \in (0,1) \quad f(tx + (1-t)y) < tf(x) + (1-t)f(y).$$

Since $f$ is minimized at both $x$ and $y$ we have $f(x) = f(y)$, so

$$f(tx + (1-t)y) < tf(x) + (1-t)f(x) = f(x).$$

However, this implies that the point $z = tx + (1-t)y$ yields a value of $f$ even *smaller* than at $x$, contradicting our assumption that $x$ is a global minimizer. Therefore, strict convexity implies that the global minimizer must be unique, and so for $\lambda > 0$, we are guaranteed that the above solution will be the unique solution to our penalized least squares problem.

### (d)

To write our function computing the ridge coefficients we first set $\nabla f(\beta) = 0$

$$\hat{\beta}_{-1}^{(\lambda)} = \left(\tilde{X}^T \tilde{X} + \lambda \mathbb{I}_{p-1}\right)^{-1} \tilde{X}^T \tilde{Y}.$$

For the purpose of computational efficiency we make use of the singular value decomposition of $\tilde{X}$

$$\tilde{X} = UDV^T,$$

for $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{(p-1)\times(p-1)}$ both orthogonal matrices, $U^TU = \mathbb{I}_n$, $V^TV = \mathbb{I}_{p-1}$, and $D \in \mathbb{R}^{n \times (p-1)}$ a diagonal matrix with entries $\{d_j\}_{j=1}^{\min(n,\,p-1)}$ along the main diagonal and zero elsewhere. Hence,

$$\begin{aligned}
\hat{\beta}_{-1}^{(\lambda)} &= \left(\tilde{X}^T \tilde{X} + \lambda \mathbb{I}_{p-1}\right)^{-1} \tilde{X}^T \tilde{Y} \\
&= \left(\left(UDV^T\right)^T UDV^T + \lambda VV^T\right)^{-1} \left(UDV^T\right)^T \tilde{Y} \\
&= \left(VD^TU^TUDV^T + \lambda VV^T\right)^{-1} VD^TU^T\tilde{Y} \\
&= \left(V\left(D^TD + \lambda \mathbb{I}_{p-1}\right)V^T\right)^{-1} VD^TU^T\tilde{Y} \\
&= V\left(D^TD + \lambda \mathbb{I}_{p-1}\right)^{-1} V^TVD^TU^T\tilde{Y} \\
&= V\left(D^TD + \lambda \mathbb{I}_{p-1}\right)^{-1} D^TU^T\tilde{Y}.
\end{aligned}$$

Note that $D^TD + \lambda \mathbb{I}_{p-1}$ is a diagonal $(p-1)\times(p-1)$ matrix with entries $d_j^2 + \lambda$, $j = 1, ..., p-1$, and so the inverse $\left(D^TD + \lambda \mathbb{I}_{p-1}\right)^{-1}$ must also be diagonal with entries $\left(d_j^2 + \lambda\right)^{-1}$, $j = 1, ..., p-1$. We exploit this to avoid performing a matrix inversion in our function. For brevity, let

$$D^* = \left(D^TD + \lambda I_{p-1}\right)^{-1} D^T,$$

so that

$$\hat{\beta}^{(\lambda)} = VD^*U^T\tilde{Y}.$$

We present a function written in `R` performing such calculations below.

```r
ridge_coef <- function(X, y, lam) {
  Xm1 <- X[,-1] # remove leading column of 1's marking the intercept

  ytilde <- y - mean(y) # center response
  xbar <- colMeans(Xm1) # find predictor means
  Xtilde <- Xm1 - tcrossprod(rep(1, nrow(Xm1)), xbar) # center each predictor according to its mean

  # compute the SVD on the centered design matrix
  Xtilde_svd <- svd(Xtilde)
```

```
  U <- Xtilde_svd$u
  d <- Xtilde_svd$d
  V <- Xtilde_svd$v

  # compute the inverse (D^T D + lambda I_{p-1})^{-1} D^T
  Dstar <- diag(d/(d^2 + lam))

  # compute ridge coefficients
  b <- V %*% (Dstar %*% crossprod(U, ytilde)) # slopes
  b1 <- mean(y) - crossprod(xbar, b) # intercept
  list(b1 = b1, b = b)
}
```

Note the choice to use `V %*% (Dstar %*% crossprod(U, ytilde))` to compute the matrix product $VD^*U^T\tilde{Y}$ as opposed to (the perhaps more intuitive) `V %*% Dstar %*% t(U) %*% ytilde`. Such a choice is empirically justified in an appendix.

### (e)

We first take the expectation of $\hat{\beta}_{-1}^{(\lambda)}$

$$\begin{aligned}
\mathbb{E}\left[\hat{\beta}_{-1}^{(\lambda)}\right] &= \mathbb{E}\left[\left(\tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1}\right)^{-1}\tilde{X}^T\tilde{Y}\right] \\
&= \left(\tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1}\right)^{-1}\tilde{X}^T\mathbb{E}\left[\tilde{Y}\right] \\
&= \left(\tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1}\right)^{-1}\tilde{X}^T\tilde{X}\beta_{-1}
\end{aligned}$$

If $p >> n$ then using the SVD on $\tilde{X}$ may yield some speed improvements, that is, with $\tilde{X} = UDV^T$ as above, we find

$$\begin{aligned}
\mathbb{E}\left[\hat{\beta}_{-1}^{(\lambda)}\right] &= \left(\tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1}\right)^{-1}\tilde{X}^T\tilde{X}\beta_{-1} \\
&= V\left(D^TD + \lambda\mathbb{I}_{p-1}\right)^{-1}D^TDV^T\beta_{-1} \\
&= VD^*V^T\beta_{-1}
\end{aligned}$$

where $D^*$ is a diagonal $\min(n, p-1) \times \min(n, p-1)$ matrix with diagonal entries $\left\{\frac{d_j^2}{d_j^2+\lambda}\right\}_{j=1}^{\min(n,p-1)}$ and zero elsewhere. [1]

We next compute the variance of our centered ridge estimates

$$\begin{aligned}
\mathrm{Var}\left(\hat{\beta}_{-1}^{(\lambda)}\right) &= \mathrm{Var}\left(\left(\tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1}\right)^{-1}\tilde{X}^T\tilde{Y}\right) \\
&= \left(\tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1}\right)^{-1}\tilde{X}^T\mathrm{Var}\left(\tilde{Y}\right)\left(\left(\tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1}\right)^{-1}\tilde{X}^T\right)^T \\
&= \left(\tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1}\right)^{-1}\tilde{X}^T\mathrm{Var}\left(\tilde{Y}\right)\tilde{X}\left(\tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1}\right)^{-1} \\
&= \sigma_*^2\left(\tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1}\right)^{-1}\tilde{X}^T\tilde{X}\left(\tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1}\right)^{-1}
\end{aligned}$$

[1] Benchmarks are provided in an appendix for the cases of large $n$, large $p$, and $n \approx p$.

as desired. We once again may be interested in applying the SVD on $\tilde{X}$ as we had done before. Such a decomposition gives us a more concise solution

$$\operatorname{Var}\left(\hat{\beta}_{-1}^{(\lambda)}\right) = V D^{**} V^T$$

where $D^{**}$ is a diagonal $\min(n, p-1) \times \min(n, p-1)$ matrix with diagonal entries $\left\{\frac{d_j^2}{\left(d_j^2+\lambda\right)^2}\right\}_{j=1}^{\min(n,p-1)}$ and zero elsewhere.

We now wish to perform a simulation study to estimate our theoretical values $\mathbb{E}\left[\hat{\beta}_{-1}^{(\lambda)}\right]$ and $\operatorname{Var}\left(\hat{\beta}_{-1}^{(\lambda)}\right)$. For readability we first define functions computing the theoretical mean and variance according to our above expressions.

```
ridge_coef_params <- function(X, lam, beta, sigma) {
  n <- nrow(X); p <- ncol(X)
  betam1 <- beta[-1] # remove intercept term
  Xm1 <- X[,-1] # remove leading column of 1's in our design matrix

  xbar <- colMeans(Xm1) # find prector means
  Xtilde <- sweep(Xm1, 2, xbar) # center each predictor according to its mean

  if (n >= p) {
    I <- diag(p - 1)
    inv <- solve(crossprod(Xtilde) + lam * I)

    b <- solve(crossprod(Xtilde) + lam * I) %*% (crossprod(Xtilde) %*% betam1)
    vcv <- sigma^2 * inv %*% crossprod(Xtilde) %*% inv
    list(b = b, vcv = vcv)

  } else {
    # compute SVD on the centered design matrix
    Xtilde_svd <- svd(Xtilde)
    d <- Xtilde_svd$d
    V <- Xtilde_svd$v

    Dstar <- diag(d^2/(d^2 + lam))
    Dstar2 <- diag(d^2/(d^2 + lam)^2)

    b <- V %*% (Dstar %*% crossprod(V, betam1))
    vcv <- V %*% tcrossprod(Dstar2, V)
    list(b = b, vcv = vcv)
  }
}
```

We may now perform our simulation.

```
set.seed(124)

# set parameters
nsims <- 1e3
n <- 25
p <- 7
lam <- 4
beta_star <- 1:p
```

```r
sigma_star <- 1

# generate fixed design matrix
X <- cbind(1, matrix(rnorm(n * (p - 1)), nrow = n))

# compute theoretical mean and variance
par_true <- ridge_coef_params(X, lam, beta_star, sigma_star)
b_true <- as.vector(par_true$b)
vcv_true <- par_true$vcv

# simulate ridge coefficients nsims times
# outputs a matrix with rows corresponding to coefficients
# and columns correspond to simulation number
b_hat <- replicate(nsims, {
  y <- X %*% beta_star + rnorm(n, 0, sigma_star)
  as.vector(ridge_coef(X, y, lam)$b)
})

# estimate variance of b1, ..., b_p estimates
vcv_hat <- var(t(b_hat))

# print estimated fused ridge coefficients vs. expected values
b <- rbind(rowMeans(b_hat), b_true)
rownames(b) <- c("b_hat", "b_true")
round(b, 4)
```

```
##          [,1]   [,2]   [,3]   [,4]   [,5]   [,6]
## b_hat  0.7861 1.6595 3.2916 3.8786 4.2007 6.3650
## b_true 0.7797 1.6636 3.2936 3.8779 4.2025 6.3689
```

```r
# print absolute error between estimated and true fused ridge variances
round(abs(vcv_true - vcv_hat), 4)
```

```
##        [,1]   [,2]   [,3]   [,4]   [,5]   [,6]
## [1,] 0.0010 0.0008 0.0013 0.0012 0.0008 0.0009
## [2,] 0.0008 0.0008 0.0009 0.0017 0.0011 0.0003
## [3,] 0.0013 0.0009 0.0012 0.0006 0.0015 0.0015
## [4,] 0.0012 0.0017 0.0006 0.0014 0.0005 0.0001
## [5,] 0.0008 0.0011 0.0015 0.0005 0.0007 0.0012
## [6,] 0.0009 0.0003 0.0015 0.0001 0.0012 0.0013
```

We see that the empirical sample estimates are very close to their theoretical values, as expected.

## Question 3

Prior to writing our cross-validation function we create some helper functions for the sake of readability

```r
ridge_cv_lam <- function(X, y, lam, K) {
  # Helper function for ridge_cv()
  # perform K-fold cross-validation on the ridge regression
  # estimation problem over a single tuning parameter lam
  n <- nrow(X)

  if (K > n) {
```

```
    stop(paste0("K > ", n, "."))
  } else if (K < 2) {
    stop("K < 2.")
  }

  # groups to cross-validate over
  folds <- cut(1:n, breaks = K, labels = F)
  # get indices of training subset
  train_idxs <- lapply(1:K, function(i) !(folds %in% i))

  cv_err <- sapply(train_idxs, function(tis) {
    # train our model, extract fitted coefficients
    b_train <- unlist(ridge_coef(X[tis,], y[tis], lam))

    # find observations needed for testing fits
    test_idx <- !((1:n) %in% tis)
    # fit data
    yhat <- X[test_idx,] %*% b_train
    # compute test error
    sum((y[test_idx] - yhat)^2)
  })
  # weighted average (according to group size, some groups may have
  # +/- 1 member depending on whether sizes divided unevenly) of
  # cross validation error for a fixed lambda
  sum((cv_err * table(folds)))/n
}
```

Then, our cross-validation function is as follows:

```
ridge_cv <- function(X, y, lam.vec, K) {
  # perform K-fold cross-validation on the ridge regression
  # estimation problem over tuning parameters given in lam.vec
  n <- nrow(X); p <- ncol(X)

  cv.error <- sapply(lam.vec, function(l) ridge_cv_lam(X, y, l, K))

  # extract best tuning parameter and corresponding coefficient estimates
  best.lam <- lam.vec[cv.error == min(cv.error)]
  best.fit <- ridge_coef(X, y, best.lam)
  b1 <- best.fit$b1
  b <- best.fit$b

  list(b1 = b1, b = b, best.lam = best.lam, cv.error = cv.error)
}
```

# Question 4

For this problem we first set some global libraries/functions

```
library(doParallel)

rmvn <- function(n, p, mu = 0, S = diag(p)) {
  # generates n (potentially correlated) p-dimensional normal deviates
```

```
  # given mean vector mu and variance-covariance matrix S
  # NOTE: S must be a positive-semidefinite matrix
  Z <- matrix(rnorm(n * p), nrow = n, ncol = p) # generate iid normal deviates
  C <- chol(S)
  mu + Z %*% C # compute our correlated deviates
}
loss1 <- function(beta, b) sum((b - beta)^2)
loss2 <- function(X, beta, b) sum((X %*% (beta - b))^2)
```

and global parameters which remain constant across (a)-(d)

```
set.seed(124)

# global parameters
nsims <- 10
n <- 20
Ks <- c(5, 10, n)
lams <- 10^seq(-8, 8, 0.5)
sigma_star <- sqrt(1/2)

# empty data structure to store our results
coef_list <- vector(mode = 'list', length = length(Ks) + 1)
names(coef_list) <- c("OLS", "K5", "K10", "Kn")
```

### (a)

```
# set parameters
p <- 50
theta <- 0.5

# generate data
beta_star <- rnorm(p, 0, sigma_star)
SIGMA <- outer(1:(p - 1), 1:(p - 1), FUN = function(a, b) theta^abs(a - b))
X <- cbind(1, rmvn(n, p - 1, 0, SIGMA))

# simulation
pt <- proc.time()
registerDoParallel(cores = 4)

sim <- foreach(1:nsims, .combine = cbind) %dopar% {
  y <- X %*% beta_star + rnorm(n, 0, sigma_star)

  ols_fit <- ridge_coef(X, y, 0)
  coef_list[[1]] <- c(ols_fit$b1, ols_fit$b)

  coef_list[2:(length(Ks) + 1)] <- sapply(Ks, function(k) {
    rcv <- ridge_cv(X, y, lam.vec = lams, K = k)
    list(coefs = c(rcv$b1, rcv$b))
  })

  l1 <- sapply(coef_list, function(b) loss1(beta_star, b))
  l2 <- sapply(coef_list, function(b) loss2(X, beta_star, b))
  list(l1, l2)
```

```r
}
sim_loss <- lapply(1:nrow(sim), function(i) sapply(sim[i,], function(s) s))
names(sim_loss) <- c("Loss 1", "Loss 2")

sim_means <- t(sapply(sim_loss, function(s) rowMeans(s)))
sim_se <- t(
  sapply(sim_loss, function(s) apply(s, 1, function(x) sd(x)/sqrt(length(x)))))
proc.time() - pt
```

```
##    user  system elapsed
##   3.989   0.271   4.141
```

```r
# report results
round(sim_means, 4)
```

```
##           OLS     K5    K10     Kn
## Loss 1 8.0595 7.3142 7.3236 7.3243
## Loss 2 9.0958 8.7293 8.6791 8.6926
```

```r
round(sim_se, 4)
```

```
##           OLS     K5    K10     Kn
## Loss 1 0.2319 0.0701 0.0634 0.0580
## Loss 2 1.1475 1.1482 1.1195 1.0692
```

## (b)

```r
# set parameters
p <- 50
theta <- 0.9

# generate data
beta_star <- rnorm(p, 0, sigma_star)
SIGMA <- outer(1:(p - 1), 1:(p - 1), FUN = function(a, b) theta^abs(a - b))
X <- cbind(1, rmvn(n, p - 1, 0, SIGMA))

# simulation
pt <- proc.time()
registerDoParallel(cores = 4)

sim <- foreach(1:nsims, .combine = cbind) %dopar% {
  y <- X %*% beta_star + rnorm(n, 0, sigma_star)

  ols_fit <- ridge_coef(X, y, 0)
  coef_list[[1]] <- c(ols_fit$b1, ols_fit$b)

  coef_list[2:(length(Ks) + 1)] <- sapply(Ks, function(k) {
    rcv <- ridge_cv(X, y, lam.vec = lams, K = k)
    list(coefs = c(rcv$b1, rcv$b))
  })

  l1 <- sapply(coef_list, function(b) loss1(beta_star, b))
  l2 <- sapply(coef_list, function(b) loss2(X, beta_star, b))
  list(l1, l2)
```

```r
}
sim_loss <- lapply(1:nrow(sim), function(i) sapply(sim[i,], function(s) s))
names(sim_loss) <- c("Loss 1", "Loss 2")

sim_means <- t(sapply(sim_loss, function(s) rowMeans(s)))
sim_se <- t(
  sapply(sim_loss, function(s) apply(s, 1, function(x) sd(x)/sqrt(length(x)))))
proc.time() - pt
```

```
##    user  system elapsed
##  11.186   0.595   3.085
```

```r
# report results
round(sim_means, 4)
```

```
##            OLS      K5     K10      Kn
## Loss 1 19.9893 20.7538 20.9312 20.9286
## Loss 2  9.4544  9.0138  9.2703  9.2594
```

```r
round(sim_se, 4)
```

```
##           OLS     K5    K10     Kn
## Loss 1 0.4374 0.2308 0.3672 0.3577
## Loss 2 1.4962 1.1203 1.3889 1.3923
```

### (c)

```r
# set parameters
p <- 200
theta <- 0.5

# generate data
beta_star <- rnorm(p, 0, sigma_star)
SIGMA <- outer(1:(p - 1), 1:(p - 1), FUN = function(a, b) theta^abs(a - b))
X <- cbind(1, rmvn(n, p - 1, 0, SIGMA))

# simulation
pt <- proc.time()
registerDoParallel(cores = 4)

sim <- foreach(1:nsims, .combine = cbind) %dopar% {
  y <- X %*% beta_star + rnorm(n, 0, sigma_star)

  ols_fit <- ridge_coef(X, y, 0)
  coef_list[[1]] <- c(ols_fit$b1, ols_fit$b)

  coef_list[2:(length(Ks) + 1)] <- sapply(Ks, function(k) {
    rcv <- ridge_cv(X, y, lam.vec = lams, K = k)
    list(coefs = c(rcv$b1, rcv$b))
  })

  l1 <- sapply(coef_list, function(b) loss1(beta_star, b))
  l2 <- sapply(coef_list, function(b) loss2(X, beta_star, b))
  list(l1, l2)
```

```r
}
sim_loss <- lapply(1:nrow(sim), function(i) sapply(sim[i,], function(s) s))
names(sim_loss) <- c("Loss 1", "Loss 2")

sim_means <- t(sapply(sim_loss, function(s) rowMeans(s)))
sim_se <- t(
  sapply(sim_loss, function(s) apply(s, 1, function(x) sd(x)/sqrt(length(x)))))
proc.time() - pt
```

```
##    user  system elapsed
##  13.728   0.487   5.606
```

```r
# report results
round(sim_means, 4)
```

```
##             OLS       K5      K10       Kn
## Loss 1 100.4377 102.8296 102.8296 102.8295
## Loss 2  10.3012  10.3012  10.3012  10.3014
```

```r
round(sim_se, 4)
```

```
##           OLS     K5    K10     Kn
## Loss 1 0.5149 0.5679 0.5679 0.5679
## Loss 2 0.7063 0.7063 0.7063 0.7063
```

### (d)

```r
# set parameters
p <- 200
theta <- 0.9

# generate data
beta_star <- rnorm(p, 0, sigma_star)
SIGMA <- outer(1:(p - 1), 1:(p - 1), FUN = function(a, b) theta^abs(a - b))
X <- cbind(1, rmvn(n, p - 1, 0, SIGMA))

# simulation
pt <- proc.time()
registerDoParallel(cores = 4)

sim <- foreach(1:nsims, .combine = cbind) %dopar% {
  y <- X %*% beta_star + rnorm(n, 0, sigma_star)

  ols_fit <- ridge_coef(X, y, 0)
  coef_list[[1]] <- c(ols_fit$b1, ols_fit$b)

  coef_list[2:(length(Ks) + 1)] <- sapply(Ks, function(k) {
    rcv <- ridge_cv(X, y, lam.vec = lams, K = k)
    list(coefs = c(rcv$b1, rcv$b))
  })

  l1 <- sapply(coef_list, function(b) loss1(beta_star, b))
  l2 <- sapply(coef_list, function(b) loss2(X, beta_star, b))
  list(l1, l2)
```

```r
}
sim_loss <- lapply(1:nrow(sim), function(i) sapply(sim[i,], function(s) s))
names(sim_loss) <- c("Loss 1", "Loss 2")

sim_means <- t(sapply(sim_loss, function(s) rowMeans(s)))
sim_se <- t(
  sapply(sim_loss, function(s) apply(s, 1, function(x) sd(x)/sqrt(length(x)))))
proc.time() - pt
```

```
##    user  system elapsed
##  15.861   0.545   6.164
```

```r
# report results
round(sim_means, 4)
```

```
##             OLS       K5      K10       Kn
## Loss 1 107.5053 105.8957 105.9118 105.9118
## Loss 2   9.7830   9.5445   9.7830   9.7830
```

```r
round(sim_se, 4)
```

```
##           OLS     K5    K10     Kn
## Loss 1 0.2474 0.1260 0.1306 0.1306
## Loss 2 1.0191 0.9498 1.0191 1.0191
```

# Question 5

## (a)

Taking the gradient of our objective function $g$ with respect to coefficient vector $\beta$ yields

$$\begin{aligned}\nabla_\beta g(\beta, \sigma^2) &= \nabla_\beta \left( \frac{n}{2} \log \sigma^2 + \frac{1}{2\sigma^2} \|\tilde{Y} - \tilde{X}\beta\|_2^2 + \frac{\lambda}{2} \|\beta\|_2^2 \right) \\ &= \frac{1}{\sigma^2} \left( -\tilde{X}^T \tilde{Y} + \tilde{X}^T \tilde{X} \beta \right) + \lambda \beta,\end{aligned}$$

while the gradient of $g$ with respect to $\sigma^2$ is given by

$$\begin{aligned}\nabla_{\sigma^2} g(\beta, \sigma^2) &= \nabla_\beta \left( \frac{n}{2} \log \sigma^2 + \frac{1}{2\sigma^2} \|\tilde{Y} - \tilde{X}\beta\|_2^2 + \frac{\lambda}{2} \|\beta\|_2^2 \right) \\ &= \frac{n}{2\sigma^2} - \frac{1}{2\sigma^4} \|\tilde{Y} - \tilde{X}\beta\|_2^2.\end{aligned}$$

as desired.

## (b)

We first consider the objective function in terms of $\beta$. We find the Hessian with respect to $\beta$

$$
\begin{aligned}
\nabla^2_\beta g\left(\beta, \sigma^2\right) &= \nabla^2_\beta \left( \frac{n}{2} \log \sigma^2 + \frac{1}{2\sigma^2} \|\tilde{Y} - \tilde{X}\beta\|_2^2 + \frac{\lambda}{2}\|\beta\|_2^2 \right) \\
&= \nabla_\beta \left( \frac{1}{\sigma^2} \tilde{X}^T \left(-\tilde{Y} + \tilde{X}\beta\right) + \lambda\beta \right) \\
&= \tilde{X}^T\tilde{X} + \lambda \mathbb{I}_{p-1}.
\end{aligned}
$$

The symmetric matrix $\tilde{X}^T\tilde{X}$ is always positive semi-definite, and for $\lambda \geq 0$, $\lambda\mathbb{I}_{p-1}$ will also be positive semi-definite (and strictly positive definite when $\lambda > 0$). Thus, the Hessian with respect to $\beta$ must be positive semi-definite

$$
\nabla^2_\beta g\left(\beta, \sigma^2\right) = \tilde{X}^T\tilde{X} + \lambda\mathbb{I}_{p-1} \in \mathbb{S}_+^{p-1},
$$

and so our objective function $g\left(\beta, \sigma^2\right)$ is convex in $\beta$. Now, considering the Hessian with respect to $\sigma^2$,

$$
\begin{aligned}
\nabla^2_{\sigma^2} g\left(\beta, \sigma^2\right) &= \nabla^2_{\sigma^2} \left( \frac{n}{2} \log \sigma^2 + \frac{1}{2\sigma^2} \|\tilde{Y} - \tilde{X}\beta\|_2^2 + \frac{\lambda}{2}\|\beta\|_2^2 \right) \\
&= \nabla_{\sigma^2} \left( \frac{n}{2\sigma^2} - \frac{1}{2\sigma^4} \|\tilde{Y} - \tilde{X}\beta\|_2^2 \right) \\
&= -\frac{n}{2\sigma^4} + \frac{1}{\sigma^6}\|\tilde{Y} - \tilde{X}\beta\|_2^2.
\end{aligned}
$$

For $g$ to be convex in $\sigma^2$ we require $\nabla^2_{\sigma^2} g\left(\beta, \sigma^2\right) \geq 0$. However, such a condition is equivalent to

$$
n \geq \frac{2}{\sigma^2}\|\tilde{Y} - \tilde{X}\beta\|_2^2.
$$

As a counterexample consider the following data

```r
set.seed(124)
n <- 20
p <- 100
beta <- rep(0.1, p)
sigma <- sqrt(2)

Xtilde <- matrix(rnorm(n * p), nrow = n)
eps <- rnorm(n, 0, sigma^2)
ytilde <- Xtilde %*% beta + eps

rhs <- as.numeric(2/sigma^2 * crossprod(ytilde - Xtilde %*% beta))
rhs
```

```
## [1] 55.03599
```

```r
n >= rhs
```

```
## [1] FALSE
```

and so it is not the case that $\nabla^2_{\sigma^2} g\left(\beta, \sigma^2\right)$ is (always) nonnegative, implying that our objective function $g\left(\beta, \sigma^2\right)$ is *not* convex in $\sigma^2$.

## (c)

Let $\bar{\beta}$ be a solution to our maximum likelihood ridge estimation problem such that, for $\lambda > 0$, we have

$$\tilde{Y} - \tilde{X}\bar{\beta} = 0.$$

Since $\bar{\beta}$ is a solution it must satisfy our first order condition

$$\nabla_\beta g(\beta, \sigma^2) = \frac{1}{\sigma^2}\left(-\tilde{X}^T\tilde{Y} + \tilde{X}^T\tilde{X}\beta\right) + \lambda\beta = 0 \iff \frac{1}{\sigma^2}\left(\tilde{X}^T\left(-\tilde{Y} + \tilde{X}\beta\right)\right) + \lambda\beta = 0.$$

Thus, for such a solution $\bar{\beta}$ and $\lambda > 0$,

$$\begin{aligned} 0 &= \frac{1}{\sigma^2}\left(\tilde{X}^T\left(-\tilde{Y} + \tilde{X}\bar{\beta}\right)\right) + \lambda\bar{\beta} \\ &= \frac{1}{\sigma^2}\left(\tilde{X}^T\left(-\tilde{Y} + \tilde{Y}\right)\right) + \lambda\bar{\beta} \\ &= \lambda\bar{\beta} \\ \iff \bar{\beta} &= 0. \end{aligned}$$

Similarly, using our second first order condition $\nabla_{\sigma^2} g(\beta, \sigma^2) = 0$, at $\beta = \bar{\beta}$,

$$\begin{aligned} \nabla_{\sigma^2} g(\beta, \sigma^2) &= \frac{n}{2\sigma^2} - \frac{1}{2\sigma^4}\|\tilde{Y} - \tilde{X}\beta\|_2^2 \\ &= \frac{n}{2\sigma^2} - \frac{1}{2\sigma^4}\|\tilde{Y} - \tilde{X}\bar{\beta}\|_2^2 \\ &= \frac{n}{2\sigma^2} - \frac{1}{2\sigma^4}\|\tilde{Y} - \tilde{Y}\|_2^2 \\ &= \frac{n}{2\sigma^2} = 0 \end{aligned}$$

This conditions implies that either $n = 0$ or $\sigma^2 \to \infty$. Thus, no such global minimizer could exist.

## (d)

Solving our first order conditions

$$\begin{aligned} \frac{1}{\sigma^2}\left(\tilde{X}^T\left(-\tilde{Y} + \tilde{X}\bar{\beta}\right)\right) + \lambda\bar{\beta} &= 0 \\ \frac{n}{2\sigma^2} - \frac{1}{2\sigma^4}\|\tilde{Y} - \tilde{X}\beta\|_2^2 &= 0, \end{aligned}$$

we find the maximum likelihood estimate $\hat{\beta}^{(\lambda,\, ML)}$ to be

$$\hat{\beta}^{(\lambda,\, ML)} = \left(\tilde{X}^T\tilde{X} + \sigma^2\lambda\mathbb{I}_{p-1}\right)^{-1}\tilde{X}^T\tilde{Y}.$$

and the maximum likelihood estimate $\hat{\sigma}^{2\,(\lambda,\, ML)}$ to be

$$\hat{\sigma}^{2\,(\lambda,\,ML)} = \frac{1}{n}\|\tilde{Y} - \tilde{X}\hat{\beta}^{(\lambda,\,ML)}\|_2^2$$

To compute such estimates we may use the following algorithm: Consider some fixed data set $\mathcal{D} = \{X, Y\}$ and a fixed tuning parameter $\lambda$.

(1) Center the data: Center each predictor by its mean $X \mapsto \tilde{X}$, center the response vector by its mean $Y \mapsto \tilde{Y}$.

(2) Have some initial proposal for the estimate $\hat{\sigma}_0^{2\,(\lambda,\,ML)} \in \mathbb{R}^+$.

(3) Compute an initial proposal for $\hat{\beta}_0^{(\lambda,\,ML)}$ based on $\hat{\sigma}_0^{2\,(\lambda,\,ML)}$.

(4) Update our variance estimate $\hat{\sigma}_i^{2\,(\lambda,\,ML)}$ using the previous estimate of $\hat{\beta}_{i-1}^{(\lambda,\,ML)}$.

(5) Update our coefficient estimate $\hat{\beta}_i^{(\lambda,\,ML)}$ using the new estimate of $\hat{\sigma}_i^{2\,(\lambda,\,ML)}$.

(6) Repeat steps (5)-(6) until some convergence criteria is met, say $\|\hat{\sigma}_i^{2\,(\lambda,\,ML)} - \hat{\sigma}_{i-1}^{2\,(\lambda,\,ML)}\|$, is small.

## (e)

Our function is as follows

```r
ridge_coef_mle <- function(X, y, lam, tol = 1e-16) {
  Xm1 <- X[,-1] # remove leading column of 1's marking the intercept

  ytilde <- y - mean(y) # center response
  xbar <- colMeans(Xm1) # find predictor means
  Xtilde <- sweep(Xm1, 2, xbar) # center each predictor according to its mean

  # compute the SVD on the centered design matrix
  Xtilde_svd <- svd(Xtilde)
  U <- Xtilde_svd$u
  d <- Xtilde_svd$d
  V <- Xtilde_svd$v

  ## generate some initial guess for sigma and beta
  sig0 <- rexp(1)
  Dstar <- diag(d/(d^2 + sig0^2 * lam))
  b0 <- V %*% (Dstar %*% crossprod(U, ytilde))

  i <- 1
  repeat {
    # update sigma and beta
    sig_new <- sqrt(1/n * crossprod(ytilde - Xtilde %*% b0))
    Dstar <- diag(d/(d^2 + sig_new^2 * lam))
    b_new <- V %*% (Dstar %*% crossprod(U, ytilde))

    if (abs(sig_new^2 - sig0^2) < tol)
      break

    sig0 <- sig_new
    b0 <- b_new
    i <- i + 1
  }
```

```r
  list(niter = i, sigma = as.numeric(sig_new), b = b_new)
}
grad_mle <- function(X, y, lam, b, s) {
  n <- nrow(X)
  Xm1 <- X[,-1] # remove leading column of 1's marking the intercept
  ytilde <- y - mean(y) # center response
  xbar <- colMeans(Xm1) # find predictor means
  Xtilde <- sweep(Xm1, 2, xbar) # center each predictor according to its mean

  gb <- 1/s^2 * crossprod(Xtilde, Xtilde %*% b - ytilde) + lam * b
  gs <- n/(2 * s^2)  - 1/(2 * s^4) * crossprod(ytilde - Xtilde %*% b)
  c(grad_b = gb, grad_s = gs)
}
```

### (f)

```r
set.seed(124)
n <- 100
p <- 5
lam <- 1
beta_star <- (-1)^(1:p) * rep(5, p)
sigma_star <- sqrt(1/2)

X <- cbind(1, matrix(rnorm(n * (p - 1)), nrow = n))
y <- X %*% beta_star + rnorm(n, 0, sigma_star)

rcm <- ridge_coef_mle(X, y, lam)
rcm
```

```
## $niter
## [1] 9
##
## $sigma
## [1] 0.6559084
##
## $b
##           [,1]
## [1,]  4.976904
## [2,] -5.000078
## [3,]  4.888082
## [4,] -5.017066
```

```r
grad_mle(X, y, lam, rcm$b, rcm$sigma)
```

```
##       grad_b1       grad_b2       grad_b3       grad_b4        grad_s
##  5.178080e-13 -1.419309e-12  4.849454e-13 -9.281464e-13  1.421085e-14
```

as desired.

# Question 6

## (a)

Consider our objective function

$$f(\beta) = \frac{1}{2}\|\tilde{Y} - \tilde{X}\beta\|_2^2 + \frac{\lambda_1}{2}\|\beta\|_2^2 + \frac{\lambda_2}{2}\sum_{j=2}^{p}(\beta_j - \beta_{j-1})^2$$

To show convexity we wish to show $\nabla^2 f(\beta) \in \mathbb{S}_+^{p-1}$. However, it's not immediately obvious how to take such a gradient with our fused sum terms $(b_j - \beta_{j-1})^2$. One way to get around this is to define vector $B \in \mathbb{R}^{p-1}$ given by

$$B = \begin{bmatrix} \beta_2 - \beta_1 \\ \vdots \\ \beta_p - \beta_{p-1} \end{bmatrix}$$

Then

$$\sum_{j=2}^{p}(\beta_j - \beta_{j-1})^2 = B^T B$$

In order to achieve our task of expressing the fused sum in terms of the vector $\beta$ we must next decompose $B$ into a product of $\beta$ and some matrix. To this end we define matrix $A \in \mathbb{R}^{(p-2)\times(p-1)}$ with entries -1 along the main diagonal and 1 along the upper diagonal, i.e.,

$$A = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

Then

$$\begin{aligned} \sum_{j=2}^{p}(\beta_j - \beta_{j-1})^2 &= B^T B \\ &= \beta^T A^T A \beta \\ &\equiv \|A\beta\|_2^2 \end{aligned}$$

Therefore, our objective function can be expressed as

$$\begin{aligned} f(\beta) &= \frac{1}{2}\|\tilde{Y} - \tilde{X}\beta\|_2^2 + \frac{\lambda_1}{2}\|\beta\|_2^2 + \frac{\lambda_2}{2}\|A\beta\|_2^2 \\ &\equiv \frac{1}{2}\tilde{Y}^T\tilde{Y} - \beta^T\tilde{X}^T\tilde{Y} + \frac{1}{2}\beta^T\tilde{X}^T\tilde{X}\beta + \frac{\lambda_1}{2}\beta^T\beta + \frac{\lambda_2}{2}\beta^T A^T A\beta \end{aligned}$$

Hence

$$\nabla f(\beta) = -\tilde{X}^T\tilde{Y} + \tilde{X}^T\tilde{X}\beta + \lambda_1\beta + \lambda_2 A^T A\beta$$

admitting the Hessian

$$\nabla^2 f(\beta) = \tilde{X}^T\tilde{X} + \lambda_1\mathbb{I}_{p-1} + \lambda_2 A^T A$$

Recalling that a matrix multiplied with its transpose must always be positive semi-definite, we find $\tilde{X}^T X$ and $A^T A$ must be positive semi-definite. Thus, since $\lambda_1 > 0$, we find that our sum $\tilde{X}^T\tilde{X} + \lambda_1\mathbb{I}_{p-1} + \lambda_2 A^T A = \nabla^2 f(\beta)$ is positive semi-definite, and so $f(\beta)$ must be strictly convex, as desired.

## (b)

We first solve for $\hat{\beta}_{-1}^{(\lambda_1,\,\lambda_2)}$ in (a) by setting $\nabla f(\beta) = 0$

$$\begin{aligned}
0 &= -\tilde{X}^T\tilde{Y} + \tilde{X}^T\tilde{X}\beta + \lambda_1\beta + \lambda_2 A^T A\beta \\
\tilde{X}^T\tilde{Y} &= \left(\tilde{X}^T\tilde{X} + \lambda_1\mathbb{I}_{p-1} + \lambda_2 A^T A\right)\beta \\
\implies \hat{\beta}_{-1}^{(\lambda_1,\,\lambda_2)} &= M\tilde{X}^T\tilde{Y}
\end{aligned}$$

where we have set $M = \left(\tilde{X}^T\tilde{X} + \lambda_1\mathbb{I}_{p-1} + \lambda_2 A^T A\right)^{-1}$ for brevity. Therefore

$$\begin{aligned}
\mathbb{E}\left[\hat{\beta}_{-1}^{(\lambda_1,\,\lambda_2)}\right] &= \mathbb{E}\left[M\tilde{X}^T\tilde{Y}\right] \\
&= M\tilde{X}^T\mathbb{E}\left[\tilde{Y}\right] \\
&= M\tilde{X}^T\beta_{*,\,-1}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}\left(\hat{\beta}_{-1}^{(\lambda_1,\,\lambda_2)}\right) &= \mathrm{Var}\left(M\tilde{X}^T Y\right) \\
&= M\tilde{X}^T\mathrm{Var}\left(\tilde{Y}\right)\tilde{X}M^T \\
&= \sigma_*^2 M\tilde{X}^T\tilde{X}M^T
\end{aligned}$$

as desired. We now perform our fused ridge simulation study to test the theoretical values with some empirical estimates. We first define our fused ridge coefficient estimation function (as well as functions permitting us to easily compute the theoretical means and variances of the fused ridge problem)

```r
fused_ridge_coef <- function(X, y, lam1, lam2) {
  n <- nrow(X); p <- ncol(X)
  Xm1 <- X[,-1] # remove leading column of 1's marking the intercept

  ytilde <- y - mean(y) # center response
  xbar <- colMeans(Xm1) # find predictor means
  Xtilde <- sweep(Xm1, 2, xbar) # center each predictor according to its mean

  I <- diag(p - 1)
  UD <- cbind(rep(0, p - 2), diag(p - 2)) # upper diagonal matrix
```

```r
  J <- -1 * cbind(diag(p - 2), rep(0, p - 2)) # diag (p - 2)*(p - 1) matrix
  A <- J + UD

  M <- solve(crossprod(Xtilde) + lam1 * I + lam2 * crossprod(A))
  b <- M %*% crossprod(Xtilde, y)
  b0 <- mean(y) - crossprod(xbar, b)
  return(list(b0 = b0, b = b))
}
fused_ridge_coef_params <- function(X, lam1, lam2, beta, sigma) {
  # omits intercept term b0
  # returns theoretical means and variances for the fused ridge problem
  n <- nrow(X); p <- ncol(X)
  Xm1 <- X[,-1] # remove leading column of 1's marking the intercept
  betam1 <- beta[-1] # remove intercept term

  xbar <- colMeans(Xm1) # find predictor means
  Xtilde <- sweep(Xm1, 2, xbar) # center each predictor according to its mean

  I <- diag(p - 1)
  UD <- cbind(rep(0, p - 2), diag(p - 2)) # upper diagonal matrix
  J <- -1 * cbind(diag(p - 2), rep(0, p - 2)) # diag (p - 2)*(p - 1) matrix
  A <- J + UD

  M <- solve(crossprod(Xtilde) + lam1 * I + lam2 * crossprod(A))
  b <- M %*% crossprod(Xtilde, (Xtilde %*% betam1))

  vcv <- matrix(0, nrow = p - 1, ncol = p - 1)
  if (n > p) { # when n > p this matrix multiplication routine is quicker
    vcv <- sigma^2 * M %*% tcrossprod(crossprod(Xtilde), M)
  } else { # when p > n this matrix multiplication routine is quicker
   vcv <- sigma^2 * tcrossprod(M, Xtilde) %*% tcrossprod(Xtilde, M)
  }

  return (list(b = b, vcv = vcv))
}
```

We now simulate some data to test our estimates:

```r
set.seed(124)

# set parameters
nsims <- 1e4
n <- 1e2
p <- 5
lam1 <- 1
lam2 <- 1
sigma_star <- 1
beta_star <- rnorm(p)

# generate (fixed) design matrix
X <- cbind(rep(1, n), matrix(rnorm(n * (p - 1)), nrow = n, ncol = p - 1))

# compute expected parameter values
par_true <- fused_ridge_coef_params(X, lam1, lam2, beta_star, sigma_star)
```

```r
b_true <- as.vector(par_true$b)
vcv_true <- par_true$vcv

# simulate our fused ridge coefficients nsims times
# outputs a matrix with rows corresponding to coefficients
# and columns correspond to simulation number
pt <- proc.time()
b_hat <- replicate(nsims, {
  y <- X %*% beta_star + rnorm(n, 0, sigma_star) # generate response
  return (as.vector(fused_ridge_coef(X, y, lam1, lam2)$b))
})
proc.time() - pt
```

```
##    user  system elapsed
##   2.184   0.066   2.696
```

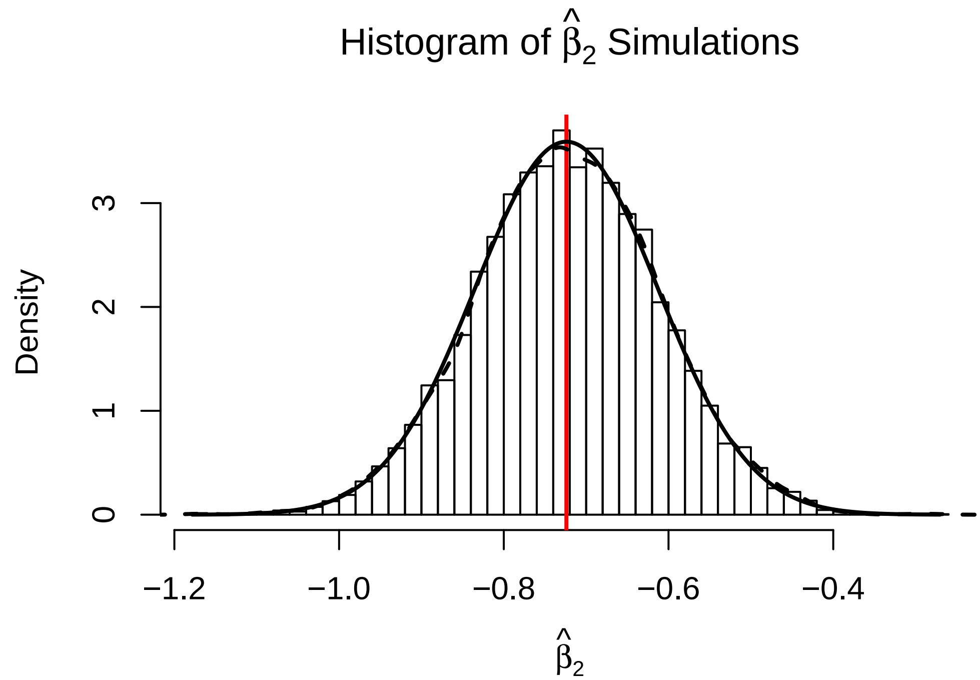
```r
# estimate variance of b2, ..., b_p estimates
vcv_hat <- var(t(b_hat))

# print estimated fused ridge coefficients vs. expected values
b <- rbind(rowMeans(b_hat), b_true)
rownames(b) <- c("b_hat", "b_true")
round(b, 4)
```

```
##          [,1]    [,2]   [,3]   [,4]
## b_hat  0.0316 -0.7226 0.2226 1.3899
## b_true 0.0313 -0.7240 0.2235 1.3920
```

```r
# print absolute error between estimated and true fused ridge variances
round(abs(vcv_true - vcv_hat), 4)
```

```
##       [,1]  [,2]  [,3]  [,4]
## [1,] 2e-04 1e-04 1e-04 1e-04
## [2,] 1e-04 1e-04 1e-04 2e-04
## [3,] 1e-04 1e-04 0e+00 1e-04
## [4,] 1e-04 2e-04 1e-04 3e-04
```

As a case study, we may look at the simulations of $\hat{\beta}_2^{(\lambda_1,\lambda_2)}$ and compare it with it's theoretical distribution. Note that the estimates $\hat{\beta}^{(\lambda_1,\lambda_2)} = M\tilde{X}^T\tilde{Y}$ are normally distributed because they are a linear combination of $\tilde{Y} \sim \mathcal{N}(\tilde{X}\beta, \sigma^2)$ (when our noise terms $\epsilon \sim \mathcal{N}(0, \sigma^2)$). We visualize the histogram of the $\hat{\beta}_2^{(\lambda_1,\lambda_2)}$ simulations with its empirical and theoretical densities overlaid (dashed, solid), along with its expected value (vertical line) below.


Histogram of $\hat{\beta}_2$ Simulations
Density
0
1
2
3
−1.2
−1.0
−0.8
−0.6
−0.4
$\hat{\beta}_2$

# Appendix

## Computing $\mathbb{E}\left[\hat{\beta}^{(\lambda)}\right]$

Consider the case of $n >> p$

```r
library(microbenchmark)
set.seed(124)

#===== Large n case =====#
# parameters
n <- 1e2
p <- 1e1
lam <- 1

# generate data
beta <- rnorm(p)
X <- matrix(rnorm(n * p), nrow = n)
I <- diag(p)

# define functions
f1 <- function() solve(crossprod(X) + lam * I) %*% (crossprod(X) %*% beta)
f2 <- function() {
  X_svd <- svd(X)
  V <- X_svd$v
  d <- X_svd$d
  Dstar <- diag(d^2/(d^2 + lam))
  V %*% (Dstar %*% crossprod(V, beta))
}

# test speed
microbenchmark(f1(), f2(), times = 1e3, unit = "us")
```

```
## Unit: microseconds
##   expr     min       lq      mean   median       uq      max neval
##   f1()  40.572  48.0635  79.43704  55.4970  83.3045 4564.463  1000
##   f2() 135.893 145.7960 213.36906 160.1835 244.4215 3070.065  1000
```

and the case for $p >> n$

```r
#===== Large p case =====#
# parameters
n <- 1e1
p <- 1e2
lam <- 1

# generate data
beta <- rnorm(p)
X <- matrix(rnorm(n * p), nrow = n)
I <- diag(p)

# define functions
f1 <- function() solve(crossprod(X) + lam * I) %*% (crossprod(X) %*% beta)
f2 <- function() {
  X_svd <- svd(X)
```

```r
  V <- X_svd$v
  d <- X_svd$d
  Dstar <- diag(d^2/(d^2 + lam))
  V %*% (Dstar %*% crossprod(V, beta))
}

# test speed
microbenchmark(f1(), f2(), times = 1e3, unit = "us")
```

```
## Unit: microseconds
##  expr      min        lq      mean    median        uq       max neval
##  f1() 2521.519 2912.1900 3964.0194 3334.0515 4185.9735 45830.058  1000
##  f2()  144.999  189.0555  348.1425  250.9225  347.9005  7483.617  1000
```

and $n \approx p$

```r
#===== n ~ p case =====#
# parameters
n <- 1e2
p <- 1e2
lam <- 1

# generate data
beta <- rnorm(p)
X <- matrix(rnorm(n * p), nrow = n)
I <- diag(p)

# define functions
f1 <- function() solve(crossprod(X) + lam * I) %*% (crossprod(X) %*% beta)
f2 <- function() {
  X_svd <- svd(X)
  V <- X_svd$v
  d <- X_svd$d
  Dstar <- diag(d^2/(d^2 + lam))
  V %*% (Dstar %*% crossprod(V, beta))
}

# test speed
microbenchmark(f1(), f2(), times = 1e3, unit = "us")
```

```
## Unit: microseconds
##  expr      min      lq     mean   median       uq      max neval
##  f1() 3297.170 3526.36 4467.752 3987.380 4662.688 63293.94  1000
##  f2() 6355.631 6870.40 8826.251 7641.675 9029.426 92591.84  1000
```

## Matrix Multiplication Timing

Consider the following matrix multiplication benchmarks (for the cases of $n >> p$ and $p >> n$).

```r
set.seed(124)
#===== Large n case =====#

# set parameters
n <- 1e3
p <- 1e2
```

```r
lam <- 1

# generate data
X <- matrix(rnorm(n * p), nrow = n)
beta <- rnorm(p)
eps <- rnorm(n)
y <- X %*% beta + eps

ytilde <- y - mean(y)
xbar <- colMeans(X)
Xtilde <- sweep(X, 2, xbar)

# compute decomposition
Xtilde_svd <- svd(Xtilde)
U <- Xtilde_svd$u
d <- Xtilde_svd$d
V <- Xtilde_svd$v
Dstar <- diag(d/(d^2 + lam))

# define multiplication functions
f1 <- function() V %*% Dstar %*% t(U) %*% ytilde
f2 <- function() V %*% Dstar %*% (t(U) %*% ytilde)
f3 <- function() V %*% (Dstar %*% (t(U) %*% ytilde))
f4 <- function() V %*% (Dstar %*% crossprod(U, ytilde))
f5 <- function() V %*% crossprod(Dstar, crossprod(U, ytilde))

# test speed
microbenchmark(f1(), f2(), f3(), f4(), f5(), times = 100, unit = "us")
```

```
## Unit: microseconds
##   expr      min        lq       mean    median         uq       max neval
##   f1() 8539.360 9370.2700 10694.9810 9881.4655 10768.1475 56889.227   100
##   f2() 1105.282 1246.9100  1592.0975 1375.7585  1619.7890  3953.049   100
##   f3()  373.493  456.1785  1131.2226  568.3030   757.7270 41895.561   100
##   f4()  130.554  148.4855   184.9980  158.8245   178.5810   921.319   100
##   f5()  126.529  136.6590   163.0916  149.5500   161.0145  1053.371   100
```

```r
#===== Large p case =====#
set.seed(124)

# set parameters
n <- 1e2
p <- 1e3
lam <- 1

# generate data
X <- matrix(rnorm(n * p), nrow = n)
beta <- rnorm(p)
eps <- rnorm(n)
y <- X %*% beta + eps

# define multiplication functions
f1 <- function() V %*% Dstar %*% t(U) %*% ytilde
f2 <- function() V %*% Dstar %*% (t(U) %*% ytilde)
```

```r
f3 <- function() V %*% (Dstar %*% (t(U) %*% ytilde))
f4 <- function() V %*% (Dstar %*% crossprod(U, ytilde))
f5 <- function() V %*% crossprod(Dstar, crossprod(U, ytilde))

# test speed
microbenchmark(f1(), f2(), f3(), f4(), f5(), times = 100, unit = "us")
```

```
## Unit: microseconds
##  expr      min        lq       mean    median         uq       max neval
##  f1() 8723.032 9411.6130 10615.3308 9959.4255 10547.3115 52695.971   100
##  f2() 1111.522 1238.8020  1522.7300 1335.4380  1530.9570  3822.139   100
##  f3()  373.276  428.6810   674.6204  503.7100   615.6695  2140.836   100
##  f4()  130.146  138.8240   170.0265  150.9685   166.5490   836.547   100
##  f5()  126.331  130.9575   155.3439  146.5125   160.5565   469.328   100
```