

MATH 680: Assignment 2

David Fleischer – 260396047

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Question 1

Proof. Let $C = \{x \in \mathbb{R}^n : Ax \leq b\}$ be our set of interest. Let $x, y \in C$, and let $t \in [0, 1]$ be an arbitrary real-valued scalar. Then,

$$\begin{aligned} A(tx + (1-t)y) &= tAx + (1-t)Ay \\ &\leq tb + (1-tb) \\ &= b. \end{aligned}$$

Thus,

$$x, y \in C \implies tx + (1-t)y \in C, \quad \text{for all } 0 \leq t \leq 1.$$

That is, C is a convex set, as desired. \square

Question 2

2.1

Proof. Recall that a (continuous, twice differentiable) function $f(z)$, $z \in C$, is convex on C if and only if its Hessian is positive semidefinite for all z on the interior of C ,

$$\nabla^2 f(z) \in \mathbb{S}_+^n,$$

and strongly convex with parameter $m > 0$ if and only if

$$\nabla^2 f(z) - m\mathbb{I}_n \in \mathbb{S}_+^n.$$

Furthermore, a matrix $M \in \mathbb{R}^{n \times n}$ is positive semidefinite if and only if all eigenvalues of M are nonnegative. Since f is nondifferentiable along $x = 0, y = 0$ we first apply the differentiable approximation f_ϵ

$$f_\epsilon(x, y) = \sqrt{x^2 + y^2 + \epsilon} + a(x^2 + y^2) \xrightarrow{\epsilon \rightarrow 0} |xy| + a(x^2 + y^2) = f(x, y)$$

Now, f_ϵ admits gradient

$$\nabla f_\epsilon(x, y) = \left(2ax + \frac{xy^2}{\sqrt{x^2y^2 + \epsilon}}, 2ay + \frac{x^2y}{\sqrt{x^2y^2 + \epsilon}} \right),$$

and Hessian

$$\nabla^2 f_\epsilon(x, y) = \begin{bmatrix} -\frac{x^2y^4}{(x^2y^2 + \epsilon)^{3/2}} + \frac{y^2}{\sqrt{x^2y^2 + \epsilon}} + 2a & \frac{2xy}{\sqrt{x^2y^2 + \epsilon}} - \frac{x^3y^3}{(x^2y^2 + \epsilon)^{3/2}} \\ \frac{2xy}{\sqrt{x^2y^2 + \epsilon}} - \frac{x^3y^3}{(x^2y^2 + \epsilon)^{3/2}} & -\frac{y^2x^4}{(x^2y^2 + \epsilon)^{3/2}} + \frac{x^2}{\sqrt{x^2y^2 + \epsilon}} + 2a \end{bmatrix}.$$

We find $\nabla^2 f_\epsilon(x, y)$ has eigenvalues

$$\lambda_{\epsilon,1} = \frac{x^2 \left(4ay^2 \sqrt{x^2 y^2 + \epsilon} + \epsilon \right) + 4a\epsilon \sqrt{x^2 y^2 + \epsilon} - \sqrt{4x^6 y^6 + x^4 \epsilon (16y^4 + \epsilon) + 14x^2 y^2 \epsilon^2 + y^4 \epsilon^2 + y^2 \epsilon}}{2(x^2 y^2 + \epsilon)^{3/2}}$$

$$\lambda_{\epsilon,2} = \frac{x^2 \left(4ay^2 \sqrt{x^2 y^2 + \epsilon} + \epsilon \right) + 4a\epsilon \sqrt{x^2 y^2 + \epsilon} + \sqrt{4x^6 y^6 + x^4 \epsilon (16y^4 + \epsilon) + 14x^2 y^2 \epsilon^2 + y^4 \epsilon^2 + y^2 \epsilon}}{2(x^2 y^2 + \epsilon)^{3/2}}.$$

Taking the limits of $\lambda_{\epsilon,1}$ and $\lambda_{\epsilon,2}$ as $\epsilon \rightarrow 0$,

$$\begin{aligned} \lambda_1 &= \lim_{\epsilon \rightarrow 0} \lambda_{\epsilon,1} = \frac{4ax^2 y^2 \sqrt{x^2 y^2} - 2\sqrt{x^6 y^6}}{2(x^2 y^2)^{3/2}} \\ &= 2a - \frac{(x^2 y^2)^{3/2}}{\sqrt{x^6 y^6}} \\ &= 2a - 1, \\ \lambda_2 &= \lim_{\epsilon \rightarrow 0} \lambda_{\epsilon,2} = \frac{4ax^2 y^2 \sqrt{x^2 y^2} + 2\sqrt{x^6 y^6}}{2(x^2 y^2)^{3/2}} \\ &= 2a + \frac{(x^2 y^2)^{3/2}}{\sqrt{x^6 y^6}} \\ &= 2a + 1. \end{aligned}$$

In this form we see that $\nabla^2 f(x, y)$ has nonnegative eigenvalues if and only if $a \geq \frac{1}{2}$, and so f is convex for $a \geq \frac{1}{2}$. To show strong convexity, we use the result that if matrix M has eigenvalues $\{\lambda_i\}_{i=1}^n$ then $M - k\mathbb{I}_n$ has eigenvalues $\{\lambda_i - k\}_{i=1}^n$. Therefore, $\nabla^2 f(x, y) - m\mathbb{I}_2$ has eigenvalues

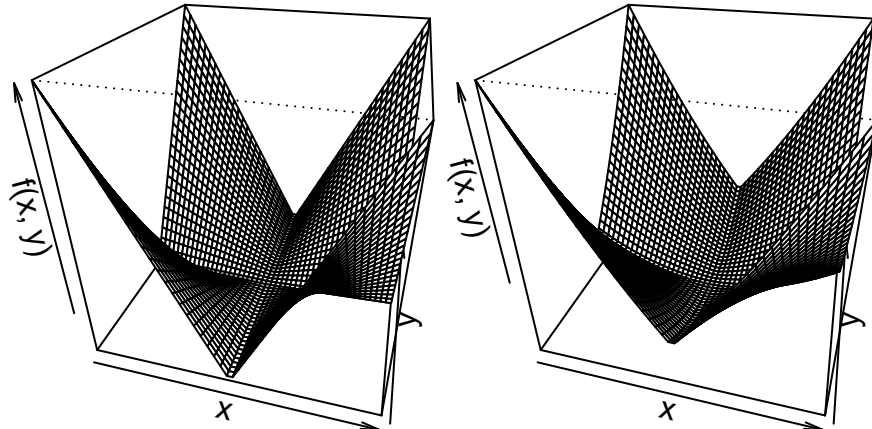
$$\begin{aligned} \lambda_{m,1} &= 2a - 1 - m \\ \lambda_{m,2} &= 2a + 1 - m. \end{aligned}$$

To ensure $\lambda_{m,1}, \lambda_{m,2}$ are nonnegative we set $a > \frac{1}{2}$ and $m \leq a$. Therefore, f is strongly convex with parameter m , $a \geq m > 0$, as desired. \square

We present below figures of f evaluated on $[-1, 1] \times [-1, 1]$ for $a \in \{0, 0.25, 0.5, 0.75\}$.

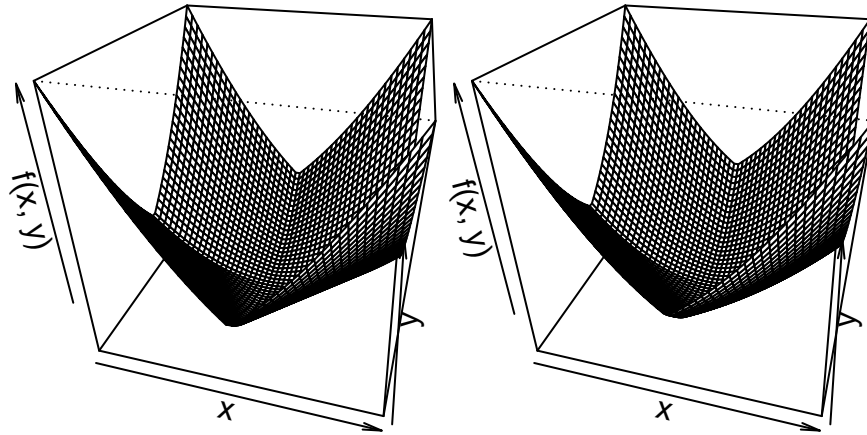
f(x, y; a), a = 0

f(x, y; a), a = 0.25



$f(x, y; a), a = 0.5$

$f(x, y; a), a = 0.75$



2.2

2.2 (a)

For $x \in \mathbb{R}_{++}^n$ we find gradient

$$\nabla f(x) = -[x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}],$$

and Hessian

$$\nabla^2 f(x) = \begin{bmatrix} x_1^{-2} & 0 & \cdots & 0 \\ 0 & x_2^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n^{-2} \end{bmatrix}$$

Since $\nabla^2 f(x)$ is diagonal we may immediately obtain its eigenvalues $\{\lambda_i\}_{i=1}^n$,

$$\lambda_i = x_i^{-2}.$$

2.2 (b)

2.3

Question 3

Question 4