MATH 680: Assignment 3

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Section 1: Subgradients and Proximal Operators

Question 1.1

1.1.(i)

Recall that a *subgradient* of f at point $x \in \mathbb{R}^n$ is defined as a vector $g \in \mathbb{R}^n$ satisfying the inequality

$$f(y) \geq f(x) + g^T(y - x), \quad \forall\, y.$$

The *subdifferential* of f at x is the set of all subgradients at x

$$\partial f(x) = \{g \in \mathbb{R}^n \ : \ g \text{ is a subgradient of } f \text{ at } x\}.$$

Let $g_1, g_2 \in \partial f(x)$ be two subgradients of f at x so that

$$\begin{aligned} f(y) &\geq f(x) + g_1^T(y - x) \\ f(y) &\geq f(x) + g_2^T(y - x). \end{aligned}$$

Let $\lambda \in [0, 1]$ and consider the linear combination of the above two inequalities, yielding

$$\begin{aligned} \lambda f(y) + (1 - \lambda) f(y) &\geq \lambda \left[f(x) + g_1^T(y - x) \right] + (1 - \lambda) \left[f(x) + g_2^T(y - x) \right] \\ \iff f(y) &\geq f(x) + \left[\lambda g_1^T + (1 - \lambda) g_2^T \right] (y - x) \\ &= f(x) + \left[\lambda g_1 + (1 - \lambda) g_2 \right]^T (y - x). \end{aligned}$$

That is, vector $\lambda g_1 + (1 - \lambda) g_2$ is a valid subgradient of f at x since it satisfies the subgradient inequality. Therefore,

$$g_1, g_2 \in \partial f(x) \implies \lambda g_1 + (1 - \lambda) g_2 \in \partial f(x), \quad \lambda \in [0, 1]$$

which informs us that $\partial f(x)$ is indeed a convex set for all $x \in \text{dom}(f)$. To show that $\partial f(x)$ is a closed set we first note that for fixed $y \in \text{dom}(f)$ the set

$$H_y = \left\{ g \mid f(y) \geq f(x) + g^T(y - x) \right\} = \left\{ g \mid f(y) - f(x) \geq g^T(y - x) \right\}$$

defines a halfspace $\left\{ z \mid b \geq a^T z \right\}$. It's easy to see that the complement $H_y^c = \left\{ g \mid f(y) - f(x) < g^T(y - x) \right\}$ is an open set since, for $a_x < b_x$, $a_x, b_x \in \mathbb{R}$,

$$\forall\, x \in H_y^c, \ \exists\, (a_x, b_x) \subset H_y^c.$$

Therefore, each H_y must be a closed set. Next, note that we may express $\partial f(x)$ as the intersection of all halfspaces H_y over all $y \in \text{dom}(f)$, i.e.,

$$
\begin{aligned}
\partial f(x) &= \left\{g \mid f(y) \geq f(x) - g^T(y-x),\ \forall\, y \in \operatorname{dom}(f)\right\} \\
&= \bigcap_{y \in \operatorname{dom}(f)} \left\{g \mid f(y) \geq f(x) - g^T(y-x)\right\}.
\end{aligned}
$$

Recall that a (potentially uncountable) intersection of closed sets is closed. Therefore, $\partial f(x)$ is indeed a closed set, as desired.

1.1.(ii)

Note that f is differentiable for all $x \neq 0$. Therefore, the subgradient of f at x is simply the gradient given by

$$
\nabla f = \frac{x}{\|x\|_2}.
$$

However, at $x = 0$, we apply the definition of the subgradient

$$
\begin{aligned}
\partial f(x)\Big]_{x=0} &= \left\{z \mid f(y) \geq f(0) + z^T(y-0),\ \forall\, y \in \operatorname{dom}(f)\right\} \\
&= \left\{z \mid \|y\|_2 \geq z^T y,\ \forall\, y \in \operatorname{dom}(f)\right\} \\
&= \left\{z \mid 1 \geq \|z\|_2\right\}.
\end{aligned}
$$

Thus,

$$
\partial f(x) = \begin{cases} \frac{x}{\|x\|_2} & \text{if } x \neq 0 \\ \{z \mid \|z\|_2 \leq 1\} & \text{if } x = 0, \end{cases}
$$

as desired.

1.1.(iii)

Let $p, q > 0$ be conjugates so that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we can express the p-norm through the q-norm via the relationship

$$
\|x\|_p = \max_{\|z\|_q \leq 1} z^T x.
$$

To prove Holder's inequality we define vectors z and w such that

$$
z = \frac{x}{\|x\|_p} \quad \text{and} \quad w = \frac{y}{\|y\|_q}.
$$

Hence, by Young's inequality,

$$
\sum_k |z_k w_k| \leq \sum_k \left(\frac{|z_k|^p}{p} + \frac{|w_k|^q}{q} \right).
$$

However, by construction we find that both z and w have unit length

$$
\|z\|_p^p = 1 \quad \text{and} \quad \|w\|_q^q = 1.
$$

Thus,

$$\sum_k |z_k w_k| = \sum_k \left(\frac{|z_k|^p}{p} + \frac{|w_k|^q}{q} \right) = \frac{1}{p} + \frac{1}{q} = 1$$

so

$$\sum_k |z_k w_k| \leq 1.$$

That is,

$$\begin{aligned} &\sum_k \left| \frac{x_k}{\|x\|_p} \cdot \frac{y_k}{\|y\|_q} \right| \leq 1 \\ &\iff \frac{1}{\|x\|_p \|y\|_q} \sum_k |x_k y_k| \leq 1 \\ &\iff x^T y \leq \|x^T y\|_1 \leq \|x\|_p \|y\|_q, \end{aligned}$$

as desired.

1.1.(iv)

We wish to show that $g \in \partial f(x) \iff g = \underset{\|z\|_q \leq 1}{\arg\max}\, z^T x$. First, let $g \in \partial f(x)$, then

$$f(y) \geq f(x) + g^T(y - x) \iff \|y\|_p \geq \|x\|_p + g^T(y - x)$$

Taking $y = 0$

$$0 \geq \|x\|_p - g^T x \iff g^T x \geq \|x\|_p.$$

Taking $y = 2x$

$$\|2x\|_p = 2\|x\|_p \geq \|x\|_p + g^T x \iff g^T x \leq \|x\|_p.$$

Applying both inequalities we find

$$g^T x = \|x\|_p \iff g^T x = \max_{\|z\|_q \leq 1} z^T x \iff g = \underset{\|z\|_q \leq 1}{\arg\max}\, z^T x.$$

Next, suppose $g = \underset{\|z\|_q \leq 1}{\arg\max}\, z^T x$. Then, $\|g\|_q \leq 1$ and

$$g^T x = \|x\|_p.$$

However, recall that $\partial f(x)$ is defined as the set of vectors z satisfying $\|z\|_q \leq 1$ and $z^T x = \|x\|_p$. Therefore,

$$g \in \partial f(x) = \left\{ z \mid \|z\|_q \leq 1 \text{ and } z^T x = \|x\|_p \right\},$$

as desired.

Question 1.2

1.2.(i)

If $h(z) = \frac{1}{2} z^T A z + b^T z + c$, $A \in \mathbb{S}^n_+$ then our proximal operator is the minimizier

$$\text{prox}_{h,t}(x) = \arg\min_z \left\{ \frac{1}{2}\|z - x\|_2^2 + t\left(\frac{1}{2} z^T A z + b^T z + c\right) \right\}.$$

Since the proximal objective is continuous with respect to z, we may simply take the gradient of our objective to obtain

$$\begin{aligned} \frac{\partial}{\partial z}\left[\frac{1}{2}(z-x)^T(z-x) + t\left(z^T A z + b^T z + c\right)\right] &= \frac{\partial}{\partial z}\left[\frac{1}{2} z^T z - z^T x + \frac{1}{2} x^T x + t\left(z^T A z + b^T z + c\right)\right] \\ &= z - x + t z^T A + tb \end{aligned}$$

Setting this quantity to zero

$$0 = z - x + tAz + tb \implies z = (\mathbb{I} + tA)^{-1}(x - tb).$$

Therefore,

$$\text{prox}_{h,t}(x) = (\mathbb{I} + tA)^{-1}(x - tb),$$

as desired.

1.2.(ii)

Taking $h(z) = -\sum_{i=1}^n \log z_i$, $z \in \mathbb{R}^n_{++}$, we seek to solve the proximal operator

$$\text{prox}_{h,t}(x) = \arg\min_z \left\{ \frac{1}{2}\|z - x\|_2^2 - t\sum_{i=1}^n \log z_i \right\}.$$

Noting that the objective is once again continuous (on $\mathbb{R}_{++}$), we take the gradient with respect to each z_i

$$\frac{\partial}{\partial z_i}\left[\frac{1}{2}\|z - x\|_2^2 - t\sum_{i=1}^n \log z_i\right] = z_i - x_i - \frac{t}{z_i}.$$

Setting this equal to zero yields

$$0 = z_i - x_i - \frac{t}{z_i} \iff z_i = \frac{1}{2}\left(x_i - \sqrt{x_i^2 - 4t}\right).$$

Thus, for $i = 1, ..., n$, we find the i^{th} component of the proximal operator to be

$$\left[\text{prox}_{h,t}(x)\right]_i = \frac{1}{2}\left(x_i - \sqrt{x_i^2 - 4t}\right),$$

as desired.

1.2.(iii)

Consider the proximal operator

$$\operatorname{prox}_{h,t}(x) = \arg\min_z \left\{ \frac{1}{2}\|z - x\|_2^2 + t\|z\|_2 \right\}.$$

Recall that we had found the subgradient of $\|z\|_2$ to be

$$\partial h(z) = \begin{cases} \frac{z}{\|z\|_2} & \text{if } z \neq 0 \\ \{g \mid 1 \geq \|g\|_2\} & \text{if } z = 0. \end{cases}$$

Omitting the point $z = 0$ we take the gradient of our proximal objective function and set it to zero,

$$\frac{z - x}{t} + \frac{z}{\|z\|_2} = 0.$$

To solve this we consider the map to polar coordinates $x \mapsto (r_x, \theta_x)$ where

$$\begin{aligned} r_x &= \|x\|_2 \\ \theta_x &= \tan^{-1}\left(\frac{x_1}{x_2}\right). \end{aligned}$$

Note that both terms of the above gradient $\frac{z}{\|z\|_2}$ and $x - z$ must have the same angle such that the angle of $\frac{z}{\|z\|_2}$ and z must be equation to either the positive or negative angle of x. This informs us that $z = ax$, for any $a \in \mathbb{R}$. Substituting this expression for z into our gradient yields

$$\frac{a - 1}{t} r_x + \operatorname{sign}(a) = 0$$

and so

$$a = \begin{cases} \frac{r_x - t}{r_x} & \text{if } r_x > t \\ 0 & \text{else.} \end{cases} \tag{1}$$

Now, if $z = 0$, we see that $r_x \leq t$ and $\frac{1}{t}x \in \{\|x\|_2 \leq 1\}$. Therefore, we conclude that

$$\operatorname{prox}_{h,t}(x) = \begin{cases} x\frac{\|x\|_2 - t}{\|x\|_2} & \text{if } ||x||_2 > t \\ 0 & \text{else,} \end{cases} \tag{2}$$

as desired.

1.2.(iv)

Finally, consider $h(z) = t\|z\|_0$ in the proximal operator

$$\operatorname{prox}_{h,t}(x) = \arg\min_z \left\{ \frac{1}{2}\|z - x\|_2^2 + t\|z\|_0 \right\},$$

where $\|z\|_0$ denotes the sum of indicators

$$h(z) = \|z\|_0 = \sum_i \mathbb{I}_{\{z_i \neq 0\}}.$$

Note that,

$$t \cdot \mathbb{I}_{\{z_i \neq 0\}} = \begin{cases} t, & z_i \neq 0 \\ 0, & z_i = 0. \end{cases}$$

We can express this indicator as the sum $t \cdot \mathbb{I}(z_i) = t \cdot \mathbb{J}(z_i) + t$ for $\mathbb{J}$ given by

$$t \cdot \mathbb{J}(z_i) = \begin{cases} 0, & z_i \neq 0 \\ -t, & z_i = 0. \end{cases}$$

Section 2: Properties of Proximal Mappings and Subgradients

2.(b)

We wish to show that, for $\forall x, y \in \mathbb{R}$, $u \in \partial f(x)$, and $v \in \partial f(y)$,

$$(x - y)^T (u - v) \geq 0.$$

To see this, first note that if $u \in \partial f(x)$ then by definition

$$f(y) \geq f(x) + u^T (y - x).$$

It follows that (result from Stanford's notes)

$$f(y) \leq f(x) \implies u^T (y - x) \leq 0$$

Similarly, if $v \in \partial f(y)$ then

$$f(x) \geq f(y) + v^T (x - y),$$

so

$$f(x) \leq f(y) \implies v^T (x - y) \leq 0.$$

Therefore, putting these two inequalities together,

$$\begin{aligned} &u^T (y - x) + v^T (x - y) \leq 0 \\ \implies &(x - y)^T (v - u) \leq 0 \\ \implies &(x - y)^T (u - v) \geq 0, \end{aligned}$$

as desired.

2.(d)

We wish to show

$$\text{prox}_t(x) = u \iff h(y) \geq h(u) + \frac{1}{t}(x-u)^T(y-u), \quad \forall y.$$

First, recall that

$$\text{prox}_t(x) = \arg\min_u \left\{ \frac{1}{2t}||x-u||_2^2 + h(u) \right\}.$$

If h is closed and convex, then the proximal mapping exists and is unique for all x. That is, it is closed, bounded, and strongly convex. It follows, from these optimality conditions that

$$\begin{aligned} u = \text{prox}_t(x) &\iff x - u \in \partial h(u) \\ &\iff h(y) \geq h(u) + \frac{1}{t}(x-u)^T(y-u) \end{aligned}$$

as desired.

2.(e)

We will now show that the prox_t mapping is non-expansive, that is,

$$||\text{prox}_t(x) - \text{prox}_t(y)||_2 \leq ||x-y||_2, \quad \forall x, y.$$

We first denote

$$u = \text{prox}_t(x) \quad \text{and} \quad v = \text{prox}_t(y).$$

Then, by definition,

$$x - u \in \partial f(u) \quad \text{and} \quad y - v \in \partial f(v),$$

where ∂f is monotone. This leads to

$$\langle x - u - (y - v), u - v \rangle \geq 0$$

$$\implies \langle x - y, u - v \rangle \geq ||u - v||_2^2.$$

In other words,

$$||\text{prox}_t(x) - \text{prox}_t(y)||_2 \leq ||x-y||_2, \quad \forall x, y.$$

and so prox_t is non-expansive, as desired.

Section 3: Properties of Lasso

Question 3.1

First, note that the Lagrangian of the Lasso problem is

$$\widehat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2n}\|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda\|\beta\|_1 \right\}$$

for centered response vector $\mathbf{y} \in \mathbb{R}^n$ and centered design matrix $\mathbf{X} \in \mathbb{R}^{n \times p}$. The solution $\widehat{\beta}_j$, $j = 1, ..., p$, to the above minimization problem must satisfy the subgradient condition

$$0 = -\frac{1}{n}\langle X_j, \mathbf{y} - X_j\widehat{\beta}_j\rangle + \lambda s_j,$$

where X_j denotes the j^{th} column/predictor of $\mathbf{X}$ and s_j is

$$s_j = \text{sign}\left(\widehat{\beta}_j\right).$$

Therefore, for $\widehat{\beta}_j = 0$, $j = 1, ..., p$, we find that λ must satisfy

$$0 = -\frac{1}{n}\langle X_j, \mathbf{y}\rangle + \lambda s_j.$$

and so, for $\widehat{\beta}_j \equiv 0$, we find

$$\lambda = \left|\frac{1}{n}\langle X_j, \mathbf{y}\rangle\right|.$$

Hence, for all $\widehat{\beta}_j \equiv 0$ we must set

$$\lambda_{\max} = \max_j \left|\frac{1}{n}\langle X_j, \mathbf{y}\rangle\right|,$$

as desired.

Question 3.2

3.2.(a)

Suppose solutions $\widehat{\beta}$, $\widehat{\gamma}$ have common optimum c^* such that

$$\mathbf{X}\widehat{\beta} \neq \mathbf{X}\widehat{\gamma}.$$

Recall that the squared-loss function $f(a) = \|y - a\|_2^2$ is strictly convex, and that the ℓ_1 norm is convex, implying that the lasso minimization problem must also be strictly convex. Therefore, the solution set $\mathcal{B}$ to the lasso problem must also be convex. Thus, by convexity of $\mathcal{B}$,

$$\alpha\widehat{\beta} + (1-\alpha)\widehat{\gamma} \in \mathcal{B}$$

for $0 < \alpha < 1$. It follows that

$$\begin{aligned}\frac{1}{2}\|\mathbf{y} - \mathbf{X}\left[\alpha\widehat{\beta} + (1-\alpha)\widehat{\gamma}\right]\|_2^2 + \lambda\|\alpha\widehat{\beta} + (1-\alpha)\widehat{\gamma}\|_1 &< \alpha\left(\frac{1}{2}\|\mathbf{y} - \mathbf{X}\widehat{\beta}\|_2^2 + \lambda\|\widehat{\beta}\|_1\right) + (1-\alpha)\left(\frac{1}{2}\|\mathbf{y} - \mathbf{X}\widehat{\gamma}\|_2^2 + \lambda\|\widehat{\gamma}\|_1\right)\\ &= \alpha c^* + (1-\alpha)c^*\\ &= c^*.\end{aligned}$$

This implies that the solution of $\alpha\widehat{\beta} + (1-\alpha)\widehat{\gamma}$ attains a new optima $c^{\text{new}} < c^*$, which is a contradiction. Therefore, we must conclude

$$\mathbf{X}\widehat{\beta} = \mathbf{X}\widehat{\gamma},$$

as desired.

3.2.(b)

The statement $\|\widehat{\beta}\|_1 = \|\widehat{\gamma}\|_1$, for $\lambda > 0$, is directly implied by the above proof. Specifically, since $\mathbf{X}\widehat{\beta} = \mathbf{X}\widehat{\gamma}$, we must have that both solutions must have the same squared residuals

$$\|\mathbf{y} - \mathbf{X}\widehat{\beta}\|_2^2 = \|\mathbf{y} - \mathbf{X}\widehat{\gamma}\|_2^2,$$

and since both Lagrangian loss functions attain the same optimum c^* we find that the penalty terms must also be equal

$$\lambda\|\widehat{\beta}\|_1 = \lambda\|\widehat{\gamma}\|_1,$$

as desired.

Section 4: Convergence Rates for Proximal Gradient Descent

Question 4.(a)

We wish to show that

$$s = G_t(x^{(i-1)}) - \nabla g(x^{(i-1)})$$

is a subgradient of h evaluated at $x^{(i)}$. Note that h is convex, but not necessarily differentiable, and recall that from Question 2.(d) we had shown that

$$\text{prox}_t(x) = u \iff h(y) \geq h(u) + \frac{1}{t}(x-u)^T(y-u), \quad \forall y.$$

In this case,

$$\begin{aligned}
x^{(i)} &= \text{prox}_{t,h}(x^{(i-1)} - t\nabla g(x^{(i-1)})) \\
\iff h(y) &\geq h(x^{(i)}) + \frac{1}{t}(x^{(i-1)} - x^{(i)} - t\nabla g(x^{(i-1)}))^T(y - x^{(i)}) \\
&= h(x^{(i)}) + (G_t(x^{(i-1)}) - \nabla g(x^{(i-1)}))^T(y - x^{(i)}) \\
&= h(x^{(i)}) + s^T(y - x^{(i)}).
\end{aligned}$$

That is, s precisely satisfies the definition of a subgradient, $s \in \partial h(x^{(i)})$, as desired.

Question 4.(b)

We wish to derive the following inequality

$$f\left(x^{(i)}\right) \leq f(z) + G_t\left(x^{(i-1)}\right)^T\left(x^{(i-1)} - z\right) - \frac{t}{2}\left\|G_t\left(x^{(i-1)}\right)\right\|_2^2.$$

Recall that our objective function f can be decomposed as

$$f(x) = g(x) + h(x),$$

for g convex and differentiable, with ∇g being Lipschitz, and h convex. Therefore, f must also be convex. Now, by (A4),

$$g\left(x^{(i)}\right) \leq g\left(x^{(i-1)}\right) - t\nabla g\left(x^{(i-1)}\right)^T G_t\left(x^{(i-1)}\right) + \frac{t}{2}\left\|G_t\left(x^{(i-1)}\right)\right\|_2^2.$$

Furthermore, since ∇g is Lipschitz,

$$\begin{aligned}\left\|\nabla g(x^{(i-1)}) - g(x^{(i)})\right\|_2^2 &\leq L \left\|x^{(i-1)} - x^{(i)}\right\|_2^2 \\ &= \frac{1}{t}\left\|x^{(i-1)} - x^{(i)}\right\|_2^2 \\ &= \left\|G_t(x^{(i-1)})\right\|_2^2.\end{aligned}$$

On the other hand, s is a subgradient of h at $x^{(i)}$, $s \in \partial h\left(x^{(i)}\right)$, so

$$\begin{aligned}&h\left(x^{(i-1)}\right) \geq h\left(x^{(i)}\right) + s^T\left(x^{(i-1)} - x^{(i)}\right) \\ &\iff h\left(x^{(i)}\right) < h\left(x^{(i-1)}\right) + G_t\left(x^{(i-1)}\right) - \nabla g\left(x^{(i-1)}\right)^T\left(x^{(i)} - x^{(i-1)}\right).\end{aligned}$$

Rearranging the above expressions, it follows that

$$\begin{aligned}f\left(x^{(i)}\right) &= g\left(x^{(i)}\right) + h\left(x^{(i)}\right) \\ &\leq h\left(x^{(i-1)}\right) + g\left(x^{(i-1)}\right) + G_t\left(x^{(i-1)}\right) - \nabla g\left(x^{(i-1)}\right)^T\left(x^{(i)} - x^{(i-1)}\right) - \\ &\qquad\qquad t\nabla g\left(x^{(i-1)}\right)G_t\left(x^{(i-1)}\right) + \frac{t}{2}\left\|G_t\left(x^{(i-1)}\right)\right\|_2^2 \\ &\leq f(z) + G_t\left(x^{(i-1)}\right)^T\left(x^{(i-1)} - z\right) - \frac{t}{2}\left\|G_t\left(x^{(i-1)}\right)\right\|_2^2,\end{aligned}$$

for $z \in \mathbb{R}^n$, as desired.

Question 4.(c)

We now wish to show that the sequence $\left\{f(x^{(i)})\right\}$ is nonincreasing for $i = 0, ..., k$. That is, we wish to show, for $i = 1, ..., k$,

$$f(x^{(i)}) \leq f(x^{(i-1)}).$$

We recall the inequality from the previous question,

$$f(x^{(i)}) \leq f(z) + G_t(x^{(i-1)})^T(x^{(i-1)} - z) - \frac{t}{2}||G_t(x^{(i-1)})||_2^2, \quad z \in \mathbb{R}^n.$$

If we let $z = x^{(i-1)}$, we see that

$$\begin{aligned}f(x^{(i)}) &\leq f(x^{(i-1)}) + G_t(x^{(i-1)})^T(x^{(i-1)} - x^{(i-1)}) - \frac{t}{2}\left\|G_t(x^{(i-1)})\right\|_2^2 \\ &= f(x^{(i-1)}) - \frac{t}{2}\left\|G_t(x^{(i-1)})\right\|_2^2.\end{aligned}$$

Note that $\frac{t}{2}\left\|G_t(x^{(i-1)})\right\|_2^2$ will always be positive unless $G_t(x^{(i-1)}) = 0$. This implies that

$$f(x^{(i)}) \leq f(x^{(i-1)}), \quad i = 1, ..., k,$$

as desired.

Question 4.(d)

We will now derive the following inequality

$$f(x^{(i)}) - f(x^*) \leq \frac{1}{2t}\left(\left\|x^{(i-1)} - x^*\right\|_2^2 - \left\|x^{(i)} - x^*\right\|_2^2\right),$$

for x^* the minimizer of f (where $f(x^*)$ is assumed to be finite). Using the previous inequality, and the fact that $f(x^{(i)}) \leq f(x^*)$,

$$\begin{aligned}
& f(x^{(i)}) \leq f(x^*) + G_t(x^{(i-1)})^T(x^{(i-1)} - x^*) - \frac{t}{2}\left\|G_t(x^{(i-1)})\right\|_2^2 \\
\iff\ & f(x^{(i)}) - f(x^*) \leq G_t(x^{(i-1)})^T(x^{(i-1)} - x^*) - \frac{t}{2}\left\|G_t(x^{(i-1)})\right\|_2^2 \\
\iff\ & f(x^{(i)}) - f(x^*) \leq \frac{1}{2t}\left(2t \cdot G_t(x^{(i-1)})^T(x^{(i-1)} - x^*) - t^2\left\|G_t(x^{(i-1)})\right\|_2^2\right) \\
\iff\ & f(x^{(i)}) - f(x^*) \leq \frac{1}{2t}\left(2t \cdot G_t(x^{(i-1)})^T(x^{(i-1)} - x^*) - t^2\left\|G_t(x^{(i-1)})\right\|_2^2 - \left\|x^{(i-1)} - x^*\right\|_2^2 + \left\|x^{(i-1)} - x^*\right\|_2^2\right).
\end{aligned}$$

Note that

$$\left\|x^{(i-1)} - x^* - tG_t(x^{(i-1)})\right\|_2^2 = \left\|x^{(i-1)} - x^*\right\|_2^2 - 2t \cdot G_t(x^{(i-1)})^T(x^{(i-1)} - x^*) + t^2\left\|G_t(x^{(i-1)})\right\|_2^2.$$

Therefore

$$f(x^{(i)}) - f(x^*) \leq \frac{1}{2t}\left(\left\|x^{(i-1)} - x^*\right\|_2^2 - \left\|x^{(i-1)} - tG_t(x^{(i-1)}) - x^*\right\|_2^2\right).$$

Furthermore, we have that $G_t(x^{(i-1)}) = \frac{1}{t}(x^{(i-1)} - x^{(i)})$. Hence,

$$\begin{aligned}
f(x^{(i)}) - f(x^*) &\leq \frac{1}{2t}\left(\left\|x^{(i-1)} - x^*\right\|_2^2 - \left\|x^{(i-1)} - t\left(\frac{1}{t}(x^{(i-1)} - x^{(i)})\right) - x^*\right\|_2^2\right) \\
&= \frac{1}{2t}\left(\left\|x^{(i-1)} - x^*\right\|_2^2 - \left\|x^{(i)} - x^*\right\|_2^2\right),
\end{aligned}$$

as desired.

Question 4.(e)

We will now show

$$f(x^{(k)}) - f(x^*) \leq \frac{1}{2kt}\left\|x^{(0)} - x^*\right\|_2^2.$$

We begin with the result above, summing over all k iterations,

$$\begin{aligned}
\sum_{i=1}^{k} f(x^{(i)}) - f(x^*) &\leq \sum_{i=1}^{k} \frac{1}{2t}\left(\left\|x^{(i-1)} - x^*\right\|_2^2 - \left\|x^{(i)} - x^*\right\|_2^2\right) \\
&= \frac{1}{2t}\left(\left\|x^{(0)} - x^*\right\|_2^2 - \left\|x^{(k)} - x^*\right\|_2^2\right) \\
&\leq \frac{1}{2t}(||x^{(0)} - x^*||_2^2).
\end{aligned}$$

Since the sequence of objection function evaluations is nonincreasing,

$$\begin{aligned} f(x^{(k)}) - f(x^*) &\leq \frac{1}{k}\sum_{i=1}^{k} f(x^{(i)}) - f(x^*) \\ &\leq \frac{||x^{(0)} - x^*||_2^2}{2kt} \end{aligned}$$

as desired.

Question 4.(f)

The method of selecting the step size according to backtracking line search consists of fixing some $0 < \beta < 1$ and starting with $t = 1$. Then, at each iteration, while

$$f(x - t\nabla f(x)) > f(x) - \frac{t}{2}||\nabla f(x)||_2^2,$$

shrink the step size $t = \beta t$.

We will show that the convergence rate is anagolous to the one above when the step sizes are chosen according to backtracking line search. The equations are the same as before, but we now replace t by $t_{min} = \min_{i=1,...,k} t_i$. It follows that $t_{min} = \min\{1, \frac{\beta}{L}\}$, where ∇g is Lipschitz, with constant $L > 0$. We now define

$$s_{min} = G_{t_{min}}(x^{(i-1)}) - \nabla g(x^{(i-1)}),$$

where

$$G_{t_{min}}(x^{(i-1)}) = \frac{1}{t_{min}}(x^{(i-1)} - x^{(i)}).$$

It follows that $s_{min} \in \partial h(x^{(i)})$. Furthermore, the following inequality still holds,

$$f\left(x^{(i)}\right) \leq f(z) + G_{t_{min}}\left(x^{(i-1)}\right)^T\left(x^{(i-1)} - z\right) - \frac{t_{min}}{2}\left\|G_{t_{min}}\left(x^{(i-1)}\right)\right\|_2^2.$$

and therefore the sequence of objective functions defined above is nonincreasing, for $i = 0, ..., k$. As per part d),

$$f(x^{(i)}) - f(x^*) \leq \frac{1}{2t_{min}}\left(\left\|x^{(i-1)} - x^*\right\|_2^2 - \left\|x^{(i)} - x^*\right\|_2^2\right).$$

Finally, it follows from the proof in part e), that

$$f(x^{(k)}) - f(x^*) \leq \frac{1}{2kt_{min}}\left\|x^{(0)} - x^*\right\|_2^2,$$

as desired. We conclude that the convergence rate result is anagolous when the step sizes are chosen according to backtracking line search.

Section 5: Proximal Gradient Descent for Group Lasso

Question 5.(a)

Consider design matrix $X \in \mathbb{R}^{n\times(p+1)}$ split in J *groups* such that we may express as

$$X = \left[\mathbf{1}\ X_{(1)}\ X_{(2)}\ \cdots\ X_{(J)}\right],$$

where $\mathbf{1} = [1, ..., 1] \in \mathbb{R}^n$ and $X_{(j)} \in \mathbb{R}^{n \times p_j}$ for $\sum_j^J p_j = p$. The *group lasso* problem seeks to estimate grouped coefficients $\beta = \left[\beta_{(0)}, \beta_{(1)}, ..., \beta_{(J)}\right]$ through the minimization problem

$$\widehat{\beta} = \arg\min_{\beta \in \mathbb{R}^{p+1}} \{g(\beta) + h(\beta)\},$$

such that g is a convex and differentiable loss function, and the group-lasso-specific h is defined as

$$h(\beta) = \lambda \sum_{j=1}^{J} w_j \left\|\beta_{(j)}\right\|_2,$$

for tuning parameter $\lambda > 0$ and weights $w_j > 0$.

5.(a).1

Recall that for convex, differentiable g and convex h, we define the proximal operator of the minimization problem

$$\min_\beta f(\beta) = \min_x \{g(\beta) + h(\beta)\}$$

to be the mapping

$$\operatorname{prox}_{h,t}(\beta) = \arg\min_\beta \left\{\frac{1}{2}\|\beta - z\|_2^2 + t \cdot h(z)\right\}.$$

Therefore, to find the proximal operator for the group lasso problem we seek to solve

$$\operatorname{prox}_{h,t}(\beta) = \arg\min_\beta \left\{\frac{1}{2}\|\beta - z\|_2^2 + \lambda t \sum_{j=1}^{J} w_j \left\|z_{(j)}\right\|_2\right\}.$$

Proceeding in the typical manner, we find the subgradient of the corresponding objective function to our proximal operator (with respect to group component (j))

$$\begin{aligned}\partial_{(j)}\left\{\frac{1}{2}\|\beta - z\|_2^2 + \lambda t \sum_{j=1}^{J} w_j \left\|z_{(j)}\right\|_2\right\} &= \beta_{(j)} - z_{(j)} + \lambda t \cdot \partial_{(j)}\left\{\sum_{j=1}^{J} w_j \left\|z_{(j)}\right\|_2\right\} \\ &= \beta_{(j)} - z_{(j)} + \lambda t w_j \cdot \partial_{(j)} \left\|z_{(j)}\right\|_2.\end{aligned}$$

From question 1.1.(ii) we find the final subgradient to be

$$\partial_{(j)} \left\|z_{(j)}\right\|_2 = \begin{cases} \frac{z_{(j)}}{\left\|z_{(j)}\right\|_2} & \text{if } z_{(j)} \neq \mathbf{0} \\ \{v \,:\, \|v\|_2 \leq 1\} & \text{if } z_{(j)} = \mathbf{0}. \end{cases}$$

Therefore, if $z_{(j)} \neq \mathbf{0}$ we find the subgradient to be

$$\partial_{(j)}\left\{\frac{1}{2}\|\beta - z\|_2^2 + \lambda t \sum_{j=1}^{J} w_j \left\|z_{(j)}\right\|_2\right\} = \beta_{(j)} - z_{(j)} + \lambda t w_j \frac{z_{(j)}}{\left\|z_{(j)}\right\|_2}.$$

We obtain the proximal operator by setting this quantity to zero, yielding optimum

$$
\begin{aligned}
0 &= \beta_{(j)} - z_{(j)} + \lambda t w_j \frac{z_{(j)}}{\left\| z_{(j)} \right\|_2} \\
\iff z_{(j)} &= \left[\widetilde{S}_{\lambda t} \left(\beta \right) \right]_{(j)},
\end{aligned}
$$

where $\widetilde{S}$ is the group soft thresholding operator

$$
\left[\widetilde{S}_{\lambda t} \left(\beta \right) \right]_{(j)} = \begin{cases} \beta_{(j)} - \lambda t w_j \frac{\beta_{(j)}}{\left\| \beta_{(j)} \right\|_2} & \text{if } \left\| \beta_{(j)} \right\|_2 > \lambda t \\ \mathbf{0} & \text{otherwise.} \end{cases}
$$

Note that in the case where $J = p$ we find $\beta_{(j)} = \beta_j \in \mathbb{R}$, so

$$
\frac{\beta_{(j)}}{\left\| \beta_{(j)} \right\|_2} = \frac{\beta_j}{\left\| \beta_j \right\|_2} = \frac{\beta_j}{|\beta_j|} = \text{sign} \left(\beta_j \right) =: s_j
$$

Therefore,

$$
\beta_j - \lambda t w_j \frac{\beta_j}{\left\| \beta_j \right\|_2} = \beta_j - \lambda t w_j s_j.
$$

So, if we set $w_j \equiv 1$ for all j, we obtain

$$
\left[\widetilde{S}_{\lambda t} \left(\beta \right) \right]_{j} = \begin{cases} \beta_j - \lambda t s_j & \text{if } \beta_j > \lambda t \\ 0 & \text{otherwise,} \end{cases}
$$

which is precisely the proximal operator for the (ungrouped) lasso problem.

Question 5.(i)

5.(i).(a)

For $g(\beta) = \|y - X\beta\|_2^2$ we find the gradient

$$
\begin{aligned}
\nabla g(\beta) &= \nabla \left(y - X\beta \right)^T \left(y - X\beta \right) \\
&= \nabla \left[y^T y - 2\beta^T X^T y + \beta^T X^T X \beta \right] \\
&= -X^T y + X^T X \beta,
\end{aligned}
$$

as desired.

5.(i).(b)

We load our data

```
X <- as.matrix(read.csv("../data/birthwt/X.csv"))
y <- as.matrix(read.csv("../data/birthwt/y.csv"))

yc <- scale(y, scale = F)
Xc <- scale(X, scale = F)
ybar <- attributes(yc)$`scaled:center`
Xbar <- attributes(Xc)$`scaled:center`
```

and define some useful functions

```
norm_p <- function(v, p) {
  sum(abs(v)^p)^(1/p)
}
grad_g <- function(X, y, b) {
  -crossprod(X, y - X %*% b)
}
Stilde_groupj <- function(beta_groupj, lambda, t_step, w_groupj) {
  beta_groupj_norm2 <- norm_p(beta_groupj, 2)

  beta_groupj/beta_groupj_norm2 *
    max(beta_groupj_norm2 - lambda * t_step * w_groupj, 0)
}
```

Next, we set some parameters and define the group structure, as well as initialize our solution $\beta^{(0)} = \mathbf{0}$

```
fstar <- 84.5952
lambda <- 4
t_step <- 0.002
max_steps <- 1e3
group_idx <- list()
group_idx[[1]] <- 1:3 # age1, age2, age3
group_idx[[2]] <- 4:6 # lwt1, lwt2, lwt3
group_idx[[3]] <- 7:8 # white, black
group_idx[[4]] <- 9 # smoke
group_idx[[5]] <- 10:11 # ptl1, ptl2m
group_idx[[6]] <- 12 # ht
group_idx[[7]] <- 13 # ui
group_idx[[8]] <- 14:16 # ftv1, ftv2, ftv3m
n_groups <- length(group_idx)

w <- sapply(group_idx, function(groupj) sqrt(length(groupj)))
```

First, we compute the traditional proximal gradient descent algorithm

```
beta_init <- rep(0, ncol(Xc))
beta <- matrix(nrow = max_steps, ncol = length(beta_init))
beta[1, ] <- beta_init - t_step * grad_g(Xc, yc, beta_init)

for (k in 2:max_steps) {
  # update step
  beta[k,] <- beta[k - 1,] - t_step * grad_g(Xc, yc, beta[k - 1,])

  # proximal step
  for (j in 1:n_groups) {
    beta[k, group_idx[[j]]] <-
      Stilde_groupj(beta[k,  group_idx[[j]]], lambda, t_step, w[j])
  }
}

beta_prox_sol <- beta[max_steps,] # extract solution

f <- apply(beta, 1, function(b) {
  h <- lambda * sum(w * sapply(group_idx, function(groupj) norm_p(b[groupj], 2)))
  crossprod(yc - Xc %*% b) + h
```

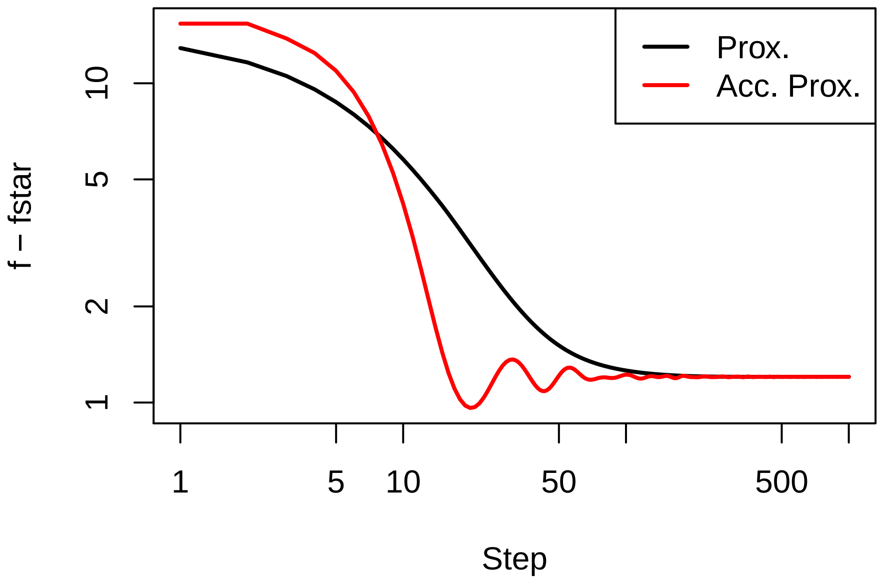
```
})
```

Next, we implement the accelerated proximal algorithm

```
beta_init_m1 <- rep(0, ncol(Xc))
beta_init_00 <- rep(0, ncol(Xc))

beta <- matrix(nrow = max_steps + 2, ncol = ncol(Xc))
beta[1, ] <- beta_init_m1
beta[2, ] <- beta_init_00
#beta[1, ] <- beta_init - t_step * grad_g(Xc, yc, beta_init)

for (k in 3:nrow(beta)) {
  # momentum step
  v <- beta[k - 1,] + (k - 4)/(k - 1) * (beta[k - 1,] - beta[k - 2,])
  # update step
  beta[k,] <- v - t_step * grad_g(Xc, yc, beta[k - 1,])

  # proximal step
  for (j in 1:n_groups) {
    beta[k, group_idx[[j]]] <-
      Stilde_groupj(beta[k,  group_idx[[j]]], lambda, t_step, w[j])
  }
}

f_acc <- apply(beta, 1, function(b) {
  h <- lambda * sum(w * sapply(group_idx, function(groupj) norm_p(b[groupj], 2)))
  crossprod(yc - Xc %*% b) + h
})

acc_min_idx <- which(f_acc == min(f_acc))
beta_acc_prox_sol <- beta[acc_min_idx,] # extract solution
```

Finally, we visualize the results

```
plot(f - fstar, ylim = range(c(f, f_acc) - fstar),
     xlab = "Step", ylab = "f - fstar",
     log = 'xy', type = 'l', lwd = 2)
lines(f_acc - fstar, col = 'red', lwd = 2)
legend("topright", legend = c("Prox.", "Acc. Prox."),
       lwd = 2, seg.len = 1.5, col = c("black", "red"))
```



5.(i).(c)

We now display the estimated coefficients of both the proximal and accelerated proximal algorithms

```
round(beta_prox_sol, 4)
```

```
##  [1]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.2449 -0.0616
##  [9] -0.2443 -0.1166  0.0110 -0.0962 -0.3786  0.0000  0.0000  0.0000
```

```
round(beta_acc_prox_sol, 4)
```

```
##  [1]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.2680 -0.0626
##  [9] -0.2720 -0.1378  0.0130 -0.0648 -0.4036  0.0000  0.0000  0.0000
```

In both algorithms we see that predictors ($7 =$ `white`, $8 =$ `black`), ($9 =$ `smoke`), ($10 =$ `ptl1`, $11 =$ `ptl2`), ($12 =$ `ht`), and ($13 =$ `ui`) are selected, corresponding to groups 3, 4, 5, 6, 7.

5.(i).(d)

Using the same framework we now compute the lasso with $\lambda = 0.35$

```
#===== LASSO =====#
lambda <- 0.35
t_step <- 0.002
max_steps <- 1e4
group_idx <- list()
for (i in 1:ncol(Xc))
  group_idx[[i]] <- i
n_groups <- length(group_idx)

w <- sapply(group_idx, function(groupj) sqrt(length(groupj)))
beta_init <- rep(0, ncol(Xc))

beta_lasso <- matrix(nrow = max_steps, ncol = length(beta_init))
beta_lasso[1, ] <- beta_init - t_step * grad_g(Xc, yc, beta_init)
```

```r
for (k in 2:max_steps) {
  # update step
  beta_lasso[k,] <- beta_lasso[k - 1,] - t_step * grad_g(Xc, yc, beta_lasso[k - 1,])

  # proximal step
  for (j in 1:n_groups) {
    beta_lasso[k, group_idx[[j]]] <-
      Stilde_groupj(beta_lasso[k,  group_idx[[j]]], lambda, t_step, w[j])
  }
}

f <- apply(beta_lasso, 1, function(b) {
  h <- lambda * sum(w * sapply(group_idx, function(groupj) norm_p(b[groupj], 2)))
  crossprod(yc - Xc %*% b) + h
})
```

Comparing the lasso results to the proximal and accelerated proximal results

```r
round(beta_prox_sol, 4)
```

```
##  [1]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.2449 -0.0616
##  [9] -0.2443 -0.1166  0.0110 -0.0962 -0.3786  0.0000  0.0000  0.0000
```

```r
round(beta_acc_prox_sol, 4)
```

```
##  [1]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.2680 -0.0626
##  [9] -0.2720 -0.1378  0.0130 -0.0648 -0.4036  0.0000  0.0000  0.0000
```

```r
round(beta_lasso[max_steps,], 4)
```

```
##  [1]  0.0000  1.2000  0.5571  1.4376  0.0000  0.9979  0.3063 -0.1146
##  [9] -0.2846 -0.3004  0.1363 -0.5059 -0.4722  0.0827  0.0027 -0.1292
```

We find that the lasso solution does not apply groupwise sparsity, instead setting some predictors to zero in the same group as nonzero predictors.

5.3.(i).(a)

The gradient ∇g is given by the vector

$$\nabla g(\beta) = \left[\frac{\partial g}{\partial \beta_1}, ..., \frac{\partial g}{\partial \beta_p}\right],$$

whose j^{th} component is the partial derivative with respect to the j^{th} coefficient

$$\frac{\partial}{\partial \beta_j} g(\beta) = \sum_{i=1}^{n} -y_i x_{ij} + \sum_{i=1}^{n} \frac{x_{ij} e^{X_i \beta}}{1 + e^{X_i \beta}},$$

as desired.

5.3.(i).(b)

5.3.(i).(c)

Section 6: Practice with KKT Conditions and Duality

We begin with the usual least squares problem,

$$\min_{\beta \in \mathbb{R}^p} ||y - X\beta||_2^2.$$

Note that the corresponding primal problem is given by

$$\min_{v \in \mathbb{R}^n} \frac{1}{2}||v||_2^2 \quad \text{subject to } y = X\beta + v.$$

In this form we see that the Lagragian is the function

$$L(v, \beta, \lambda) = \frac{1}{2}||v||_2^2 + \lambda(y - X\beta - v).$$

It follows that the first order necessary conditions are

$$\begin{aligned}
0 &= \frac{\partial L}{\partial v} = v - \lambda \cdot \mathbf{1} \\
0 &= \frac{\partial L}{\partial \beta} = -X\lambda \\
0 &= \frac{\partial L}{\partial \lambda} = y - X\beta - v.
\end{aligned}$$

Note that from these first order conditions we find

$$v = \lambda \cdot \mathbf{1} \quad \text{and} \quad v^T v = v^T \lambda,$$

permitting us to simplify the Lagrangian as

$$\begin{aligned}
L(v, \beta, \lambda) &= \frac{1}{2}||v||_2^2 + \lambda(y - X\beta - v) \\
&= \frac{1}{2}||v||_2^2 + \lambda y - \lambda u \\
&= \frac{1}{2}||v||_2^2 - v^T y - ||v||_2^2 \\
&= ||y - v||_2^2.
\end{aligned}$$

Therefore, we conclude that the dual problem is given by

$$\min_{v \in \mathbb{R}^n} ||y - v||_2^2 \quad \text{subject to } X^T v = 0,$$

as desired.