MATH 680: Assignment 3

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Section 1: Subgradients and Proximal Operators

Question 1.1

1.1.(i)

Recall that a *subgradient* of f at point $x \in \mathbb{R}^n$ is defined as a vector $g \in \mathbb{R}^n$ satisfying the inequality

$$f(y) \geq f(x) + g^T(y - x), \quad \forall\, y.$$

The *subdifferential* of f at x is the set of all subgradients at x

$$\partial f(x) = \{g \in \mathbb{R}^n \ : \ g \text{ is a subgradient of } f \text{ at } x\}.$$

Let $g_1, g_2 \in \partial f(x)$ be two subgradients of f at x so that

$$\begin{aligned} f(y) &\geq f(x) + g_1^T(y - x) \\ f(y) &\geq f(x) + g_2^T(y - x). \end{aligned}$$

Let $\lambda \in [0, 1]$ and consider the linear combination of the above two inequalities, yielding

$$\begin{aligned} \lambda f(y) + (1 - \lambda) f(y) &\geq \lambda \left[f(x) + g_1^T(y - x) \right] + (1 - \lambda) \left[f(x) + g_2^T(y - x) \right] \\ \iff f(y) &\geq f(x) + \left[\lambda g_1^T + (1 - \lambda) g_2^T \right] (y - x) \\ &= f(x) + \left[\lambda g_1 + (1 - \lambda) g_2 \right]^T (y - x). \end{aligned}$$

That is, vector $\lambda g_1 + (1 - \lambda) g_2$ is a valid subgradient of f at x since it satisfies the subgradient inequality. Therefore,

$$g_1, g_2 \in \partial f(x) \implies \lambda g_1 + (1 - \lambda) g_2 \in \partial f(x), \quad \lambda \in [0, 1]$$

which informs us that $\partial f(x)$ is indeed a convex set for all $x \in \mathrm{dom}(f)$. To show that $\partial f(x)$ is a closed set we first note that for fixed $y \in \mathrm{dom}(f)$ the set

$$H_y = \left\{ g \mid f(y) \geq f(x) + g^T(y - x) \right\} = \left\{ g \mid f(y) - f(x) \geq g^T(y - x) \right\}$$

defines a halfspace $\left\{ z \mid b \geq a^T z \right\}$. It's easy to see that the complement $H_y^c = \left\{ g \mid f(y) - f(x) < g^T(y - x) \right\}$ is an open set since, for $a_x < b_x$, $a_x, b_x \in \mathbb{R}$,

$$\forall\, x \in H_y^c, \ \exists\, (a_x, b_x) \subset H_y^c.$$

Therefore, each H_y must be a closed set. Next, note that we may express $\partial f(x)$ as the intersection of all halfspaces H_y over all $y \in \mathrm{dom}(f)$, i.e.,

$$
\begin{aligned}
\partial f(x) &= \left\{g \mid f(y) \geq f(x) - g^T(y-x),\ \forall\, y \in \operatorname{dom}(f)\right\} \\
&= \bigcap_{y \in \operatorname{dom}(f)} \left\{g \mid f(y) \geq f(x) - g^T(y-x)\right\}.
\end{aligned}
$$

Recall that a (potentially uncountable) intersection of closed sets is closed. Therefore, $\partial f(x)$ is indeed a closed set, as desired.

1.1.(ii)

Note that f is differentiable for all $x \neq 0$. Therefore, the subgradient of f at x is simply the gradient given by

$$
\nabla f = \frac{x}{\|x\|_2}.
$$

However, at $x = 0$, we apply the definition of the subgradient

$$
\begin{aligned}
\partial f(x)\Big]_{x=0} &= \left\{z \mid f(y) \geq f(0) + z^T(y-0),\ \forall\, y \in \operatorname{dom}(f)\right\} \\
&= \left\{z \mid \|y\|_2 \geq z^T y,\ \forall\, y \in \operatorname{dom}(f)\right\} \\
&= \left\{z \mid 1 \geq \|z\|_2\right\}.
\end{aligned}
$$

Thus,

$$
\partial f(x) = \begin{cases} \frac{x}{\|x\|_2} & \text{if } x \neq 0 \\ \{z \mid \|z\|_2 \leq 1\} & \text{if } x = 0, \end{cases}
$$

as desired.

1.1.(iii)

Let $p, q > 0$ be conjugates so that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we can express the p-norm through the q-norm via the relationship

$$
\|x\|_p = \max_{\|z\|_q \leq 1} z^T x.
$$

To prove Holder's inequality we define vectors z and w such that

$$
z = \frac{x}{\|x\|_p} \quad \text{and} \quad w = \frac{y}{\|y\|_q}.
$$

Hence, by Young's inequality,

$$
\sum_k |z_k w_k| \leq \sum_k \left(\frac{|z_k|^p}{p} + \frac{|w_k|^q}{q}\right).
$$

However, by construction we find that both z and w have unit length

$$
\|z\|_p^p = 1 \quad \text{and} \quad \|w\|_q^q = 1.
$$

Thus,

$$\sum_k |z_k w_k| = \sum_k \left(\frac{|z_k|^p}{p} + \frac{|w_k|^q}{q} \right) = \frac{1}{p} + \frac{1}{q} = 1$$

so

$$\sum_k |z_k w_k| \leq 1.$$

That is,

$$\begin{aligned} &\sum_k \left| \frac{x_k}{\|x\|_p} \cdot \frac{y_k}{\|y\|_q} \right| \leq 1 \\ \iff &\frac{1}{\|x\|_p \|y\|_q} \sum_k |x_k y_k| \leq 1 \\ \iff &x^T y \leq \|x^T y\|_1 \leq \|x\|_p \|y\|_q, \end{aligned}$$

as desired.

1.1.(iv)

We wish to show that $g \in \partial f(x) \iff g = \arg\max_{\|z\|_q \leq 1} z^T x$. First, let $g \in \partial f(x)$, then

$$f(y) \geq f(x) + g^T(y - x) \iff \|y\|_p \geq \|x\|_p + g^T(y - x)$$

Taking $y = 0$

$$0 \geq \|x\|_p - g^T x \iff g^T x \geq \|x\|_p.$$

Taking $y = 2x$

$$\|2x\|_p = 2\|x\|_p \geq \|x\|_p + g^T x \iff g^T x \leq \|x\|_p.$$

Applying both inequalities we find

$$g^T x = \|x\|_p \iff g^T x = \max_{\|z\|_q \leq 1} z^T x \iff g = \arg\max_{\|z\|_q \leq 1} z^T x.$$

Next, suppose $g = \arg\max_{\|z\|_q \leq 1} z^T x$. Then, $\|g\|_q \leq 1$ and

$$g^T x = \|x\|_p.$$

However, recall that $\partial f(x)$ is defined as the set of vectors z satisfying $\|z\|_q \leq 1$ and $z^T x = \|x\|_p$. Therefore,

$$g \in \partial f(x) = \left\{ z \mid \|z\|_q \leq 1 \text{ and } z^T x = \|x\|_p \right\},$$

as desired.

Question 1.2

1.2.(i)

If $h(z) = \frac{1}{2}z^T Az + b^T z + c$, $A \in \mathbb{S}^n_+$ then our proximal operator is the minimizier

$$\text{prox}_{h,t}(x) = \arg\min_z \left\{ \frac{1}{2}\|z - x\|_2^2 + t\left(\frac{1}{2}z^T Az + b^T z + c\right)\right\}.$$

Since the proximal objective is continuous with respect to z, we may simply take the gradient of our objective to obtain

$$\begin{aligned}\frac{\partial}{\partial z}\left[\frac{1}{2}(z-x)^T(z-x) + t\left(z^T Az + b^T z + c\right)\right] &= \frac{\partial}{\partial z}\left[\frac{1}{2}z^T z - z^T x + \frac{1}{2}x^T x + t\left(z^T Az + b^T z + c\right)\right] \\ &= z - x + tz^T A + tb\end{aligned}$$

Setting this quantity to zero

$$0 = z - x + tAz + tb \implies z = (\mathbb{I} + tA)^{-1}(x - tb).$$

Therefore,

$$\text{prox}_{h,t}(x) = (\mathbb{I} + tA)^{-1}(x - tb),$$

as desired.

1.2.(ii)

Taking $h(z) = -\sum_{i=1}^n \log z_i$, $z \in \mathbb{R}^n_{++}$, we seek to solve the proximal operator

$$\text{prox}_{h,t}(x) = \arg\min_z \left\{\frac{1}{2}\|z - x\|_2^2 - t\sum_{i=1}^n \log z_i\right\}.$$

Noting that the objective is once again continuous (on $\mathbb{R}_{++}$), we take the gradient with respect to each z_i

$$\frac{\partial}{\partial z_i}\left[\frac{1}{2}\|z - x\|_2^2 - t\sum_{i=1}^n \log z_i\right] = z_i - x_i - \frac{t}{z_i}.$$

Setting this equal to zero yields

$$0 = z_i - x_i - \frac{t}{z_i} \iff z_i = \frac{1}{2}\left(x_i - \sqrt{x_i^2 - 4t}\right).$$

Thus, for $i = 1, ..., n$, we find the i^{th} component of the proximal operator to be

$$\left[\text{prox}_{h,t}(x)\right]_i = \frac{1}{2}\left(x_i - \sqrt{x_i^2 - 4t}\right),$$

as desired.

1.2.(iii)

Consider the proximal operator

$$\mathrm{prox}_{h,t}(x) = \arg\min_z \left\{ \frac{1}{2}\|z-x\|_2^2 + t\|z\|_2 \right\}.$$

Recall that we had found the subgradient of $\|z\|_2$ to be

$$\partial h(z) = \begin{cases} \frac{z}{\|z\|_2} & \text{if } z \neq 0 \\ \{g \mid 1 \geq \|g\|_2\} & \text{if } z = 0. \end{cases}$$

Omitting the point $z = 0$ we take the gradient of our proximal objective function and set it to zero,

$$\frac{z-x}{t} + \frac{z}{\|z\|_2} = 0.$$

To solve this we consider the map to polar coordinates $x \mapsto (r_x, \theta_x)$ where

$$\begin{aligned} r_x &= \|x\|_2 \\ \theta_x &= \tan^{-1}\left(\frac{x_1}{x_2}\right). \end{aligned}$$

Note that both terms of the above gradient $\frac{z}{\|z\|_2}$ and $x - z$ must have the same angle such that the angle of $\frac{z}{\|z\|_2}$ and z must be equation to either the positive or negative angle of x. This informs us that $z = ax$, for any $a \in \mathbb{R}$. Substituting this expression for z into our gradient yields

$$\frac{a-1}{t} r_x + \mathrm{sign}(a) = 0$$

and so

$$a = \begin{cases} \frac{r_x - t}{r_x} & \text{if } r_x > t \\ 0 & \text{else.} \end{cases} \tag{1}$$

Now, if $z = 0$, we see that $r_x \leq t$ and $\frac{1}{t}x \in \{\|x\|_2 \leq 1\}$. Therefore, we conclude that

$$\mathrm{prox}_{h,t}(x) = \begin{cases} x\frac{\|x\|_2 - t}{\|x\|_2} & \text{if } ||x||_2 > t \\ 0 & \text{else,} \end{cases} \tag{2}$$

as desired.

1.2.(iv)

Finally, consider $h(z) = t\|z\|_0$ in the proximal operator

$$\mathrm{prox}_{h,t}(x) = \arg\min_z \left\{ \frac{1}{2}\|z-x\|_2^2 + t\|z\|_0 \right\},$$

where $\|z\|_0$ denotes the sum of indicators

$$h(z) = \|z\|_0 = \sum_i \mathbb{I}_{\{z_i \neq 0\}}.$$

Note that,

$$t \cdot \mathbb{I}_{\{z_i \neq 0\}} = \begin{cases} t, & z_i \neq 0 \\ 0, & z_i = 0. \end{cases}$$

We can express this indicator as the sum $t \cdot \mathbb{I}(z_i) = t \cdot \mathbb{J}(z_i) + t$ for $\mathbb{J}$ given by

$$t \cdot \mathbb{J}(z_i) = \begin{cases} 0, & z_i \neq 0 \\ -t, & z_i = 0. \end{cases}$$

Section 2: Properties of Proximal Mappings and Subgradients

2.(b)

We wish to show that, for $\forall x, y \in \mathbb{R}$, $u \in \partial f(x)$, and $v \in \partial f(y)$,

$$(x-y)^T(u-v) \geq 0.$$

To see this, first note that if $u \in \partial f(x)$ then by definition

$$f(y) \geq f(x) + u^T(y-x).$$

It follows that (result from Stanford's notes)

$$f(y) \leq f(x) \implies u^T(y-x) \leq 0$$

Similarly, if $v \in \partial f(y)$ then

$$f(x) \geq f(y) + v^T(x-y),$$

so

$$f(x) \leq f(y) \implies v^T(x-y) \leq 0.$$

Therefore, putting these two inequalities together,

$$\begin{aligned} & u^T(y-x) + v^T(x-y) \leq 0 \\ \implies & (x-y)^T(v-u) \leq 0 \\ \implies & (x-y)^T(u-v) \geq 0, \end{aligned}$$

as desired.

2.(d)

We wish to show

$$\text{prox}_t(x) = u \iff h(y) \geq h(u) + \frac{1}{t}(x-u)^T(y-u), \quad \forall y.$$

First, recall that

$$\text{prox}_t(x) = \arg\min_u \left\{ \frac{1}{2t}||x-u||_2^2 + h(u) \right\}.$$

If h is closed and convex, then the proximal mapping exists and is unique for all x. That is, it is closed, bounded, and strongly convex. It follows, from these optimality conditions that

$$\begin{aligned} u = \text{prox}_t(x) &\iff x - u \in \partial h(u) \\ &\iff h(y) \geq h(u) + \frac{1}{t}(x-u)^T(y-u) \end{aligned}$$

as desired.

2.(e)

Section 3: Properties of Lasso

Question 3.1

First, note that the Lagrangian of the Lasso problem is

$$\widehat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2n}\|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda\|\beta\|_1 \right\}$$

for centered response vector $\mathbf{y} \in \mathbb{R}^n$ and centered design matrix $\mathbf{X} \in \mathbb{R}^{n\times p}$. The solution $\widehat{\beta}_j$, $j = 1, ..., p$, to the above minimization problem must satisfy the subgradient condition

$$0 = -\frac{1}{n}\langle X_j, \mathbf{y} - X_j\widehat{\beta}_j\rangle + \lambda s_j,$$

where X_j denotes the j^{th} column/predictor of $\mathbf{X}$ and s_j is

$$s_j = \text{sign}\left(\widehat{\beta}_j\right).$$

Therefore, for $\widehat{\beta}_j = 0$, $j = 1, ..., p$, we find that λ must satisfy

$$0 = -\frac{1}{n}\langle X_j, \mathbf{y}\rangle + \lambda s_j.$$

and so, for $\widehat{\beta}_j \equiv 0$, we find

$$\lambda = \left|\frac{1}{n}\langle X_j, \mathbf{y}\rangle\right|.$$

Hence, for all $\widehat{\beta}_j \equiv 0$ we must set

$$\lambda_{\max} = \max_j \left| \frac{1}{n} \langle X_j, \mathbf{y} \rangle \right|,$$

as desired.

Question 3.2

3.2.(a)

Suppose solutions $\widehat{\beta}$, $\widehat{\gamma}$ have common optimum c^* such that

$$\mathbf{X}\widehat{\beta} \neq \mathbf{X}\widehat{\gamma}.$$

Recall that the squared-loss function $f(a) = \|y - a\|_2^2$ is strictly convex, and that the ℓ_1 norm is convex, implying that the lasso minimization problem must also be strictly convex. Therefore, the solution set $\mathcal{B}$ to the lasso problem must also be convex. Thus, by convexity of $\mathcal{B}$,

$$\alpha\widehat{\beta} + (1-\alpha)\widehat{\gamma} \in \mathcal{B}$$

for $0 < \alpha < 1$. It follows that

$$\begin{aligned}
\frac{1}{2}\|\mathbf{y} - \mathbf{X}\left[\alpha\widehat{\beta} + (1-\alpha)\widehat{\gamma}\right]\|_2^2 + \lambda\|\alpha\widehat{\beta} + (1-\alpha)\widehat{\gamma}\|_1 &< \alpha\left(\frac{1}{2}\|\mathbf{y} - \mathbf{X}\widehat{\beta}\|_2^2 + \lambda\|\widehat{\beta}\|_1\right) + (1-\alpha)\left(\frac{1}{2}\|\mathbf{y} - \mathbf{X}\widehat{\gamma}\|_2^2 + \lambda\|\widehat{\gamma}\|_1\right) \\
&= \alpha c^* + (1-\alpha)c^* \\
&= c^*.
\end{aligned}$$

This implies that the solution of $\alpha\widehat{\beta} + (1-\alpha)\widehat{\gamma}$ attains a new optima $c^{\text{new}} < c^*$, which is a contradiction. Therefore, we must conclude

$$\mathbf{X}\widehat{\beta} = \mathbf{X}\widehat{\gamma},$$

as desired.

3.2.(b)

The statement $\|\widehat{\beta}\|_1 = \|\widehat{\gamma}\|_1$, for $\lambda > 0$, is directly implied by the above proof. Specifically, since $\mathbf{X}\widehat{\beta} = \mathbf{X}\widehat{\gamma}$, we must have that both solutions must have the same squared residuals

$$\|\mathbf{y} - \mathbf{X}\widehat{\beta}\|_2^2 = \|\mathbf{y} - \mathbf{X}\widehat{\gamma}\|_2^2,$$

and since both Lagrangian loss functions attain the same optimum c^* we find that the penalty terms must also be equal

$$\lambda\|\widehat{\beta}\|_1 = \lambda\|\widehat{\gamma}\|_1,$$

as desired.

Section 4: Convergence Rates for Proximal Gradient Descent

Question 4.(a)

We wish to show that

$$s = G_t(x^{(i-1)}) - \nabla g(x^{(i-1)})$$

is a subgradient of h evaluated at $x^{(i)}$. Note that h is convex, but not necessarily differentiable, and recall that from Question 2.(d) we had shown that

$$\text{prox}_t(x) = u \iff h(y) \geq h(u) + \frac{1}{t}(x-u)^T(y-u), \quad \forall y.$$

In this case,

$$\begin{aligned}
x^{(i)} &= \text{prox}_{t,h}(x^{(i-1)} - t\nabla g(x^{(i-1)})) \\
\iff h(y) &\geq h(x^{(i)}) + \frac{1}{t}(x^{(i-1)} - x^{(i)} - t\nabla g(x^{(i-1)}))^T(y - x^{(i)}) \\
&= h(x^{(i)}) + (G_t(x^{(i-1)}) - \nabla g(x^{(i-1)}))^T(y - x^{(i)}) \\
&= h(x^{(i)}) + s^T(y - x^{(i)}).
\end{aligned}$$

That is, s precisely satisfies the definition of a subgradient, $s \in \partial h(x^{(i)})$, as desired.

Question 4.(b)

We wish to derive the following inequality

$$f\left(x^{(i)}\right) \leq f(z) + G_t\left(x^{(i-1)}\right)^T\left(x^{(i-1)} - z\right) - \frac{t}{2}\left\|G_t\left(x^{(i-1)}\right)\right\|_2^2.$$

Recall that our objective function f can be decomposed as

$$f(x) = g(x) + h(x),$$

for g convex and differentiable, with ∇g being Lipschitz, and h convex. Therefore, f must also be convex. Now, by (A4),

$$g\left(x^{(i)}\right) \leq g\left(x^{(i-1)}\right) - t\nabla g\left(x^{(i-1)}\right)^T G_t\left(x^{(i-1)}\right) + \frac{t}{2}\left\|G_t\left(x^{(i-1)}\right)\right\|_2^2.$$

Furthermore, since ∇g is Lipschitz,

$$\begin{aligned}
\left\|\nabla g(x^{(i-1)}) - g(x^{(i)})\right\|_2^2 &\leq L\left\|x^{(i-1)} - x^{(i)}\right\|_2^2 \\
&= \frac{1}{t}\left\|x^{(i-1)} - x^{(i)}\right\|_2^2 \\
&= \left\|G_t(x^{(i-1)})\right\|_2^2.
\end{aligned}$$

On the other hand, s is a subgradient of h at $x^{(i)}$, $s \in \partial h\left(x^{(i)}\right)$, so

$$
\begin{aligned}
&h\left(x^{(i-1)}\right) \geq h\left(x^{(i)}\right) + s^T\left(x^{(i-1)} - x^{(i)}\right) \\
&\iff h\left(x^{(i)}\right) < h\left(x^{(i-1)}\right) + G_t\left(x^{(i-1)}\right) - \nabla g\left(x^{(i-1)}\right)^T\left(x^{(i)} - x^{(i-1)}\right).
\end{aligned}
$$

Rearranging the above expressions, it follows that

$$
\begin{aligned}
f\left(x^{(i)}\right) &= g\left(x^{(i)}\right) + h\left(x^{(i)}\right) \\
&\leq h\left(x^{(i-1)}\right) + g\left(x^{(i-1)}\right) + G_t\left(x^{(i-1)}\right) - \nabla g\left(x^{(i-1)}\right)^T\left(x^{(i)} - x^{(i-1)}\right) - \\
&\qquad\qquad t\nabla g\left(x^{(i-1)}\right) G_t\left(x^{(i-1)}\right) + \frac{t}{2}\left\|G_t\left(x^{(i-1)}\right)\right\|_2^2 \\
&\leq f(z) + G_t\left(x^{(i-1)}\right)^T\left(x^{(i-1)} - z\right) - \frac{t}{2}\left\|G_t\left(x^{(i-1)}\right)\right\|_2^2,
\end{aligned}
$$

for $z \in \mathbb{R}^n$, as desired.

Question 4.(c)

We now wish to show that the sequence $\left\{f(x^{(i)})\right\}$ is nonincreasing for $i = 0, ..., k$. That is, we wish to show, for $i = 1, ..., k$,

$$f(x^{(i)}) \leq f(x^{(i-1)}).$$

We recall the inequality from the previous question,

$$f(x^{(i)}) \leq f(z) + G_t(x^{(i-1)})^T(x^{(i-1)} - z) - \frac{t}{2}||G_t(x^{(i-1)})||_2^2, \quad z \in \mathbb{R}^n.$$

If we let $z = x^{(i-1)}$, we see that

$$
\begin{aligned}
f(x^{(i)}) &\leq f(x^{(i-1)}) + G_t(x^{(i-1)})^T(x^{(i-1)} - x^{(i-1)}) - \frac{t}{2}\left\|G_t(x^{(i-1)})\right\|_2^2 \\
&= f(x^{(i-1)}) - \frac{t}{2}\left\|G_t(x^{(i-1)})\right\|_2^2.
\end{aligned}
$$

Note that $\frac{t}{2}\left\|G_t(x^{(i-1)})\right\|_2^2$ will always be positive unless $G_t(x^{(i-1)}) = 0$. This implies that

$$f(x^{(i)}) \leq f(x^{(i-1)}), \quad i = 1, ..., k,$$

as desired.

Question 4.(d)

We will now derive the following inequality

$$f(x^{(i)}) - f(x^*) \leq \frac{1}{2t}\left(\left\|x^{(i-1)} - x^*\right\|_2^2 - \left\|x^{(i)} - x^*\right\|_2^2\right),$$

for x^* the minimizer of f (where $f(x^*)$ is assumed to be finite). Using the previous inequality, and the fact that $f(x^{(i)}) \leq f(x^*)$,

$$
\begin{aligned}
& f(x^{(i)}) \leq f(x^*) + G_t(x^{(i-1)})^T(x^{(i-1)} - x^*) - \frac{t}{2}\left\|G_t(x^{(i-1)})\right\|_2^2 \\
\iff\ & f(x^{(i)}) - f(x^*) \leq G_t(x^{(i-1)})^T(x^{(i-1)} - x^*) - \frac{t}{2}\left\|G_t(x^{(i-1)})\right\|_2^2 \\
\iff\ & f(x^{(i)}) - f(x^*) \leq \frac{1}{2t}\left(2t \cdot G_t(x^{(i-1)})^T(x^{(i-1)} - x^*) - t^2\left\|G_t(x^{(i-1)})\right\|_2^2\right) \\
\iff\ & f(x^{(i)}) - f(x^*) \leq \frac{1}{2t}\left(2t \cdot G_t(x^{(i-1)})^T(x^{(i-1)} - x^*) - t^2\left\|G_t(x^{(i-1)})\right\|_2^2 - \left\|x^{(i-1)} - x^*\right\|_2^2 + \left\|x^{(i-1)} - x^*\right\|_2^2\right).
\end{aligned}
$$

Note that

$$
\left\|x^{(i-1)} - x^* - tG_t(x^{(i-1)})\right\|_2^2 = \left\|x^{(i-1)} - x^*\right\|_2^2 - 2t \cdot G_t(x^{(i-1)})^T(x^{(i-1)} - x^*) + t^2\left\|G_t(x^{(i-1)})\right\|_2^2.
$$

Therefore

$$
f(x^{(i)}) - f(x^*) \leq \frac{1}{2t}\left(\left\|x^{(i-1)} - x^*\right\|_2^2 - \left\|x^{(i-1)} - tG_t(x^{(i-1)}) - x^*\right\|_2^2\right).
$$

Furthermore, we have that $G_t(x^{(i-1)}) = \frac{1}{t}(x^{(i-1)} - x^{(i)})$. Hence,

$$
\begin{aligned}
f(x^{(i)}) - f(x^*) &\leq \frac{1}{2t}\left(\left\|x^{(i-1)} - x^*\right\|_2^2 - \left\|x^{(i-1)} - t\left(\frac{1}{t}(x^{(i-1)} - x^{(i)})\right) - x^*\right\|_2^2\right) \\
&= \frac{1}{2t}\left(\left\|x^{(i-1)} - x^*\right\|_2^2 - \left\|x^{(i)} - x^*\right\|_2^2\right),
\end{aligned}
$$

as desired.

Question 4.(e)

We will now show

$$
f(x^{(k)}) - f(x^*) \leq \frac{1}{2kt}\left\|x^{(0)} - x^*\right\|_2^2.
$$

We begin with the result above, summing over all k iterations,

$$
\begin{aligned}
\sum_{i=1}^{k} f(x^{(i)}) - f(x^*) &\leq \sum_{i=1}^{k} \frac{1}{2t}\left(\left\|x^{(i-1)} - x^*\right\|_2^2 - \left\|x^{(i)} - x^*\right\|_2^2\right) \\
&= \frac{1}{2t}\left(\left\|x^{(0)} - x^*\right\|_2^2 - \left\|x^{(k)} - x^*\right\|_2^2\right) \\
&\leq \frac{1}{2t}(||x^{(0)} - x^*||_2^2).
\end{aligned}
$$

Since the sequence of objection function evaluations is nonincreasing,

$$
\begin{aligned}
f(x^{(k)}) - f(x^*) &\le \frac{1}{k}\sum_{i=1}^{k} f(x^{(i)}) - f(x^*) \\
&\le \frac{||x^{(0)} - x^*||_2^2}{2kt}
\end{aligned}
$$

as desired.

Question 4.(f)

The method of selecting the step size according to backtracking line search consists of fixing some $0 < \beta < 1$ and starting with $t = 1$. Then, at each iteration, while

$$f(x - t\nabla f(x)) > f(x) - \frac{t}{2}||\nabla f(x)||_2^2,$$

shrink the step size $t = \beta t$. Now, in the context of this problem,

$$f(x - t\nabla f(x)) > f(x) - \frac{t}{2}||\nabla f(x)||_2^2$$

Section 5: Proximal Gradient Descent for Group Lasso

Question 5.(a)

Consider design matrix $X \in \mathbb{R}^{n\times(p+1)}$ split in J *groups* such that we may express as

$$X = \begin{bmatrix}\mathbf{1} \ X_{(1)} \ X_{(2)} \ \cdots \ X_{(J)}\end{bmatrix},$$

where $\mathbf{1} = [1, ..., 1] \in \mathbb{R}^n$ and $X_{(j)} \in \mathbb{R}^{n\times p_j}$ for $\sum_j^J p_j = p$. The *group lasso* problem seeks to estimate grouped coefficients $\beta = \left[\beta_{(0)}, \beta_{(1)}, ..., \beta_{(J)}\right]$ through the minimization problem

$$\widehat{\beta} = \arg\min_{\beta\in\mathbb{R}^{p+1}} \left\{g(\beta) + h(\beta)\right\},$$

such that g is a convex and differentiable loss function, and the group-lasso-specific h is defined as

$$h(\beta) = \lambda \sum_{j=1}^{J} w_j \left\|\beta_{(j)}\right\|_2,$$

for tuning parameter $\lambda > 0$ and weights $w_j > 0$.

5.(a).1

Recall that for convex, differentiable g and convex h, we define the proximal operator of the minimization problem

$$\min_\beta f(\beta) = \min_x \left\{g(\beta) + h(\beta)\right\}$$

to be the mapping

$$\operatorname{prox}_{h,t}(\beta) = \arg\min_\beta \left\{\frac{1}{2}\left\|\beta - z\right\|_2^2 + t\cdot h(z)\right\}.$$

Therefore, to find the proximal operator for the group lasso problem we seek to solve

$$\text{prox}_{h,t}(\beta) = \arg\min_{\beta} \left\{ \frac{1}{2} \|\beta - z\|_2^2 + \lambda t \sum_{j=1}^{J} w_j \left\|z_{(j)}\right\|_2 \right\}.$$

Proceeding in the typical manner, we find the subgradient of the corresponding objective function to our proximal operator (with respect to group component (j))

$$\begin{aligned} \partial_{(j)} \left\{ \frac{1}{2} \|\beta - z\|_2^2 + \lambda t \sum_{j=1}^{J} w_j \left\|z_{(j)}\right\|_2 \right\} &= \beta_{(j)} - z_{(j)} + \lambda t \cdot \partial_{(j)} \left\{ \sum_{j=1}^{J} w_j \left\|z_{(j)}\right\|_2 \right\} \\ &= \beta_{(j)} - z_{(j)} + \lambda t w_j \cdot \partial_{(j)} \left\|z_{(j)}\right\|_2 . \end{aligned}$$

From question 1.1.(ii) we find the final subgradient to be

$$\partial_{(j)} \left\|z_{(j)}\right\|_2 = \begin{cases} \frac{z_{(j)}}{\left\|z_{(j)}\right\|_2} & \text{if } z_{(j)} \neq \mathbf{0} \\ \{v \, : \, \|v\|_2 \leq 1\} & \text{if } z_{(j)} = \mathbf{0}. \end{cases}$$

Therefore, if $z_{(j)} \neq \mathbf{0}$ we find the subgradient to be

$$\partial_{(j)} \left\{ \frac{1}{2} \|\beta - z\|_2^2 + \lambda t \sum_{j=1}^{J} w_j \left\|z_{(j)}\right\|_2 \right\} = \beta_{(j)} - z_{(j)} + \lambda t w_j \frac{z_{(j)}}{\left\|z_{(j)}\right\|_2}.$$

We obtain the proximal operator by setting this quantity to zero, yielding optimum

$$\begin{aligned} 0 &= \beta_{(j)} - z_{(j)} + \lambda t w_j \frac{z_{(j)}}{\left\|z_{(j)}\right\|_2} \\ \iff z_{(j)} &= \left[\widetilde{S}_{\lambda t}\left(\beta\right)\right]_{(j)}, \end{aligned}$$

where $\widetilde{S}$ is the group soft thresholding operator

$$\left[\widetilde{S}_{\lambda t}\left(\beta\right)\right]_{(j)} = \begin{cases} \beta_{(j)} - \lambda t w_j \frac{\beta_{(j)}}{\left\|\beta_{(j)}\right\|_2} & \text{if } \left\|\beta_{(j)}\right\|_2 > \lambda t \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Note that in the case where $J = p$ we find $\beta_{(j)} = \beta_j \in \mathbb{R}$, so

$$\frac{\beta_{(j)}}{\left\|\beta_{(j)}\right\|_2} = \frac{\beta_j}{\|\beta_j\|_2} = \frac{\beta_j}{|\beta_j|} = \text{sign}\left(\beta_j\right) =: s_j$$

Therefore,

$$\beta_j - \lambda t w_j \frac{\beta_j}{\|\beta_j\|_2} = \beta_j - \lambda t w_j s_j.$$

So, if we set $w_j \equiv 1$ for all j, we obtain

$$\left[\widetilde{S}_{\lambda t}\left(\beta\right)\right]_{j} = \begin{cases} \beta_j - \lambda t s_j & \text{if } \beta_j > \lambda t \\ 0 & \text{otherwise,} \end{cases}$$

which is precisely the proximal operator for the (ungrouped) lasso problem.

Question 5.(i)

5.(i).(a)

For $g(\beta) = \|y - X\beta\|_2^2$ we find the gradient

$$\begin{aligned}\nabla g(\beta) &= \nabla (y - X\beta)^T (y - X\beta) \\ &= \nabla \left[y^T y - 2\beta^T X^T y + \beta^T X^T X\beta\right] \\ &= -X^T y + X^T X\beta,\end{aligned}$$

as desired.

5.(i).(b)

We load our data

```
X <- as.matrix(read.csv("../data/birthwt/X.csv"))
y <- as.matrix(read.csv("../data/birthwt/y.csv"))

yc <- scale(y, scale = F)
Xc <- scale(X, scale = F)
ybar <- attributes(yc)$`scaled:center`
Xbar <- attributes(Xc)$`scaled:center`
```

and define some useful functions

```
norm_p <- function(v, p) {
  sum(abs(v)^p)^(1/p)
}
grad_g <- function(X, y, b) {
  -crossprod(X, y - X %*% b)
}
Stilde_groupj <- function(beta_groupj, lambda, t_step, w_groupj) {
  beta_groupj_norm2 <- norm_p(beta_groupj, 2)

  beta_groupj/beta_groupj_norm2 *
    max(beta_groupj_norm2 - lambda * t_step * w_groupj, 0)
}
```

Next, we set some parameters and define the group structure, as well as initialize our solution $\beta^{(0)} = \mathbf{0}$

```
fstar <- 84.5952
lambda <- 4
t_step <- 0.002
max_steps <- 1e3
group_idx <- list()
group_idx[[1]] <- 1:3 # age1, age2, age3
group_idx[[2]] <- 4:6 # lwt1, lwt2, lwt3
group_idx[[3]] <- 7:8 # white, black
group_idx[[4]] <- 9 # smoke
group_idx[[5]] <- 10:11 # ptl1, ptl2m
group_idx[[6]] <- 12 # ht
group_idx[[7]] <- 13 # ui
```

```
group_idx[[8]] <- 14:16 # ftv1, ftv2, ftv3m
n_groups <- length(group_idx)

w <- sapply(group_idx, function(groupj) sqrt(length(groupj)))
```

First, we compute the traditional proximal gradient descent algorithm

```
beta_init <- rep(0, ncol(Xc))
beta <- matrix(nrow = max_steps, ncol = length(beta_init))
beta[1, ] <- beta_init - t_step * grad_g(Xc, yc, beta_init)

for (k in 2:max_steps) {
  # update step
  beta[k,] <- beta[k - 1,] - t_step * grad_g(Xc, yc, beta[k - 1,])

  # proximal step
  for (j in 1:n_groups) {
    beta[k, group_idx[[j]]] <-
      Stilde_groupj(beta[k,  group_idx[[j]]], lambda, t_step, w[j])
  }
}

beta_prox_sol <- beta[max_steps,] # extract solution

f <- apply(beta, 1, function(b) {
  h <- lambda * sum(w * sapply(group_idx, function(groupj) norm_p(b[groupj], 2)))
  crossprod(yc - Xc %*% b) + h
})
```

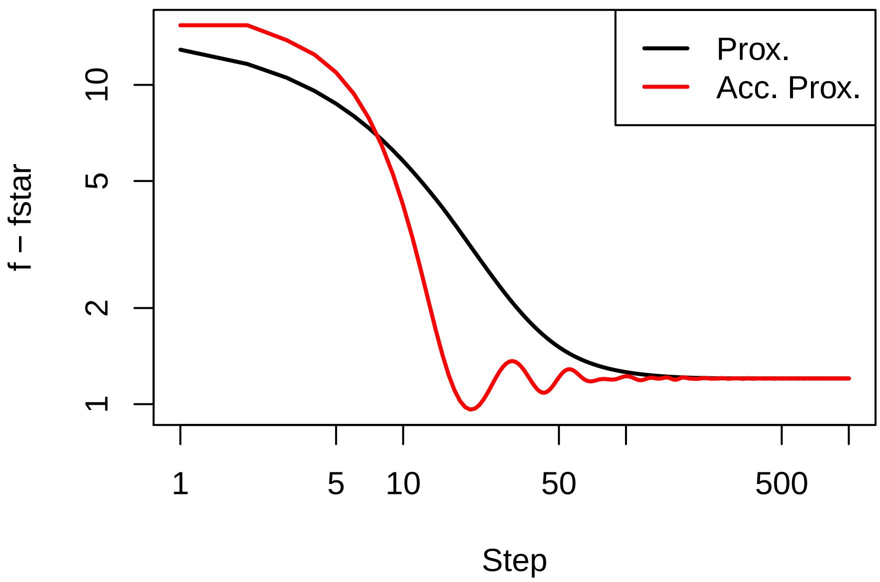
Next, we implement the accelerated proximal algorithm

```
beta_init_m1 <- rep(0, ncol(Xc))
beta_init_00 <- rep(0, ncol(Xc))

beta <- matrix(nrow = max_steps + 2, ncol = ncol(Xc))
beta[1, ] <- beta_init_m1
beta[2, ] <- beta_init_00
#beta[1, ] <- beta_init - t_step * grad_g(Xc, yc, beta_init)

for (k in 3:nrow(beta)) {
  # momentum step
  v <- beta[k - 1,] + (k - 4)/(k - 1) * (beta[k - 1,] - beta[k - 2,])
  # update step
  beta[k,] <- v - t_step * grad_g(Xc, yc, beta[k - 1,])

  # proximal step
  for (j in 1:n_groups) {
    beta[k, group_idx[[j]]] <-
      Stilde_groupj(beta[k,  group_idx[[j]]], lambda, t_step, w[j])
  }
}

f_acc <- apply(beta, 1, function(b) {
  h <- lambda * sum(w * sapply(group_idx, function(groupj) norm_p(b[groupj], 2)))
  crossprod(yc - Xc %*% b) + h
```

```
})

acc_min_idx <- which(f_acc == min(f_acc))
beta_acc_prox_sol <- beta[acc_min_idx,] # extract solution
```

Finally, we visualize the results

```
plot(f - fstar, ylim = range(c(f, f_acc) - fstar),
     xlab = "Step", ylab = "f - fstar",
     log = 'xy', type = 'l', lwd = 2)
lines(f_acc - fstar, col = 'red', lwd = 2)
legend("topright", legend = c("Prox.", "Acc. Prox."),
       lwd = 2, seg.len = 1.5, col = c("black", "red"))
```



5.(i).(c)

We now display the estimated coefficients of both the proximal and accelerated proximal algorithms

```
round(beta_prox_sol, 4)
```

```
##  [1]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.2449 -0.0616
##  [9] -0.2443 -0.1166  0.0110 -0.0962 -0.3786  0.0000  0.0000  0.0000
```

```
round(beta_acc_prox_sol, 4)
```

```
##  [1]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.2680 -0.0626
##  [9] -0.2720 -0.1378  0.0130 -0.0648 -0.4036  0.0000  0.0000  0.0000
```

In both algorithms we see that predictors ($7 =$ `white`, $8 =$ `black`), ($9 =$ `smoke`), ($10 =$ `ptl1`, $11 =$ `ptl2`), ($12 =$ `ht`), and ($13 =$ `ui`) are selected, corresponding to groups 3, 4, 5, 6, 7.

5.(i).(d)

Using the same framework we now compute the lasso with $\lambda = 0.35$

```r
#===== LASSO =====#
lambda <- 0.35
t_step <- 0.002
max_steps <- 1e4
group_idx <- list()
for (i in 1:ncol(Xc))
  group_idx[[i]] <- i
n_groups <- length(group_idx)

w <- sapply(group_idx, function(groupj) sqrt(length(groupj)))
beta_init <- rep(0, ncol(Xc))

beta_lasso <- matrix(nrow = max_steps, ncol = length(beta_init))
beta_lasso[1, ] <- beta_init - t_step * grad_g(Xc, yc, beta_init)

for (k in 2:max_steps) {
  # update step
  beta_lasso[k,] <- beta_lasso[k - 1,] - t_step * grad_g(Xc, yc, beta_lasso[k - 1,])

  # proximal step
  for (j in 1:n_groups) {
    beta_lasso[k, group_idx[[j]]] <-
      Stilde_groupj(beta_lasso[k,  group_idx[[j]]], lambda, t_step, w[j])
  }
}

f <- apply(beta_lasso, 1, function(b) {
  h <- lambda * sum(w * sapply(group_idx, function(groupj) norm_p(b[groupj], 2)))
  crossprod(yc - Xc %*% b) + h
})
```

Comparing the lasso results to the proximal and accelerated proximal results

```r
round(beta_prox_sol, 4)
```

```
##  [1]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.2449 -0.0616
##  [9] -0.2443 -0.1166  0.0110 -0.0962 -0.3786  0.0000  0.0000  0.0000
```

```r
round(beta_acc_prox_sol, 4)
```

```
##  [1]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.2680 -0.0626
##  [9] -0.2720 -0.1378  0.0130 -0.0648 -0.4036  0.0000  0.0000  0.0000
```

```r
round(beta_lasso[max_steps,], 4)
```

```
##  [1]  0.0000  1.2000  0.5571  1.4376  0.0000  0.9979  0.3063 -0.1146
##  [9] -0.2846 -0.3004  0.1363 -0.5059 -0.4722  0.0827  0.0027 -0.1292
```

We find that the lasso solution does not apply groupwise sparsity, instead setting some predictors to zero in the same group as nonzero predictors.

5.3.(i).(a)

The gradient ∇g is given by the vector

$$\nabla g(\beta) = \left[\frac{\partial g}{\partial \beta_1}, \dots, \frac{\partial g}{\partial \beta_p}\right],$$

whose j^{th} component is the partial derivative with respect to the j^{th} coefficient

$$\frac{\partial}{\partial \beta_j} g(\beta) = \sum_{i=1}^{n} -y_i x_{ij} \beta_j + \sum_{i=1}^{n} \frac{x_{ij} e^{X_i \beta}}{1 + e^{X_i \beta}},$$

as desired.

5.3.(i).(b)

5.3.(i).(c)

Section 6: Practice with KKT Conditions and Duality

We begin with the usual least squares problem,

$$\min_{\beta \in \mathbb{R}^p} ||y - X\beta||_2^2.$$

Note that the corresponding primal problem is given by

$$\min_{v \in \mathbb{R}^n} \frac{1}{2} ||v||_2^2 \quad \text{subject to } y = X\beta + v.$$

In this form we see that the Lagragian is the function

$$L(v, \beta, \lambda) = \frac{1}{2} ||v||_2^2 + \lambda(y - X\beta - v).$$

It follows that the first order necessary conditions are

$$\begin{aligned} 0 &= \frac{\partial L}{\partial v} = v - \lambda \cdot \mathbf{1} \\ 0 &= \frac{\partial L}{\partial \beta} = -X\lambda \\ 0 &= \frac{\partial L}{\partial \lambda} = y - X\beta - v. \end{aligned}$$

Note that from these first order conditions we find

$$v = \lambda \cdot \mathbf{1} \quad \text{and} \quad v^T v = v^T \lambda,$$

permitting us to simplify the Lagrangian as

$$\begin{aligned} L(v, \beta, \lambda) &= \frac{1}{2} ||v||_2^2 + \lambda(y - X\beta - v) \\ &= \frac{1}{2} ||v||_2^2 + \lambda y - \lambda u \\ &= \frac{1}{2} ||v||_2^2 - v^T y - ||v||_2^2 \\ &= ||y - v||_2^2. \end{aligned}$$

Therefore, we conclude that the dual problem is given by

$$\min_{v \in \mathbb{R}^n} ||y - v||_2^2 \quad \text{subject to } X^T v = 0,$$

as desired.