Toric Varieties

Declan Fletcher

October 2024

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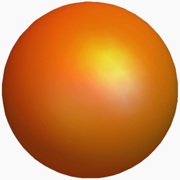

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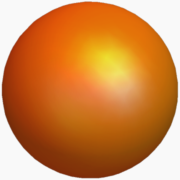

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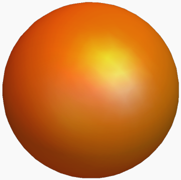

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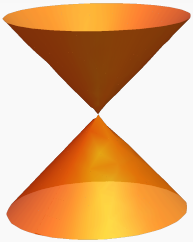

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Goal of the talk: introduce toric varieties and explain some of their properties.

The definition of a variety

Varieties are sets of solutions $(a_1, \ldots, a_n) \in \mathbb{C}^n$ to poly. equations

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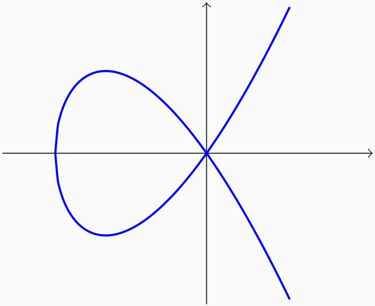

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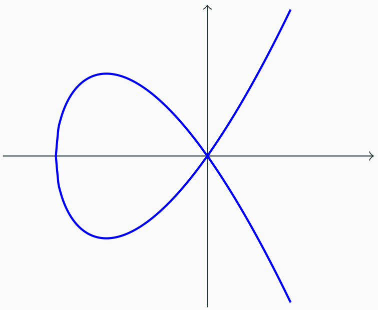

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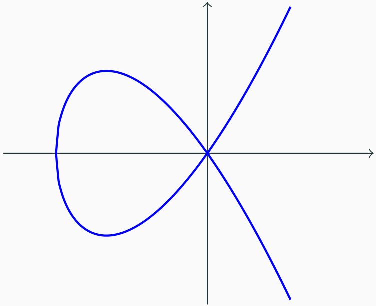

$\mathbb{C}^\times$ is called the algebraic torus. The d-dimensional torus is $(\mathbb{C}^\times)^d$.

Convex cones

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We consider polyhedral cones. These are sets in $\mathbb{R}^n$ of the form

$$\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{v_1, \ldots, v_r\},$$

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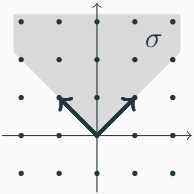

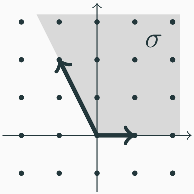

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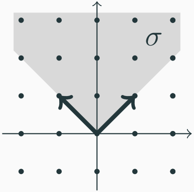
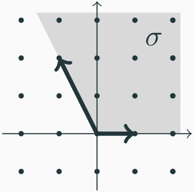
We call σ rational if we can take each $v_i \in \mathbb{Z}^n$.





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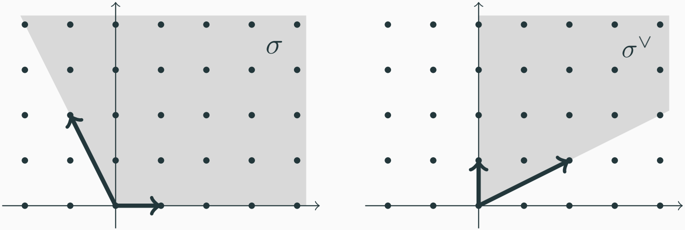

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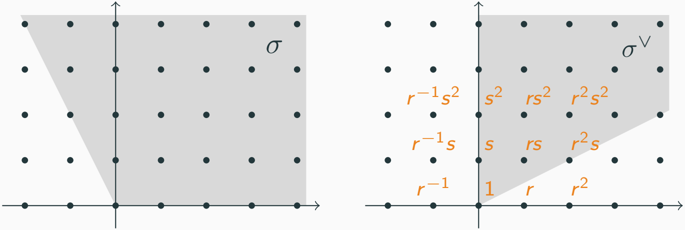

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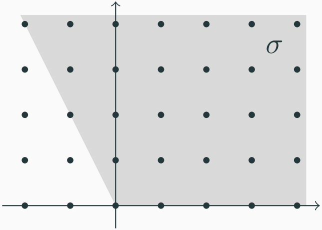

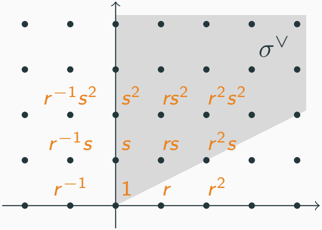

We create a ring using the monomials in $\sigma^\vee$:

$$\begin{aligned}\mathbb{C}[1, s, r^2s, rs, s^2, rs^2, \ldots] &= \mathbb{C}[s, r^2s, rs] \\ &\cong \mathbb{C}[x, y, z]/(xy - z^2).\end{aligned}$$

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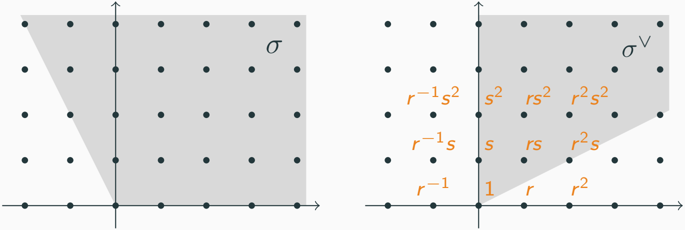

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The toric variety U_σ is the set of solutions to $xy - z^2 = 0$ in $\mathbb{C}^3$:

$$xy = z^2.$$

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The toric variety U_σ is the subset of $\mathbb{C}^m$ defined by the equations

$$f_1(a_1, \dots, a_m) = 0, \quad \dots, \quad f_s(a_1, \dots, a_m) = 0.$$

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The toric variety U_σ is non-singular if and only if σ is generated by a subset of a basis for $\mathbb{Z}^n$.

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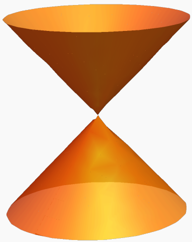

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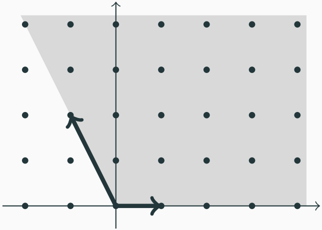

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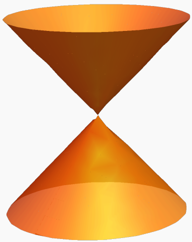

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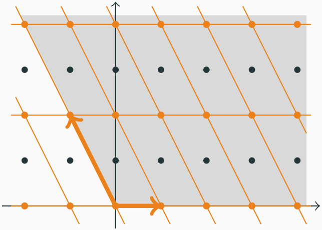

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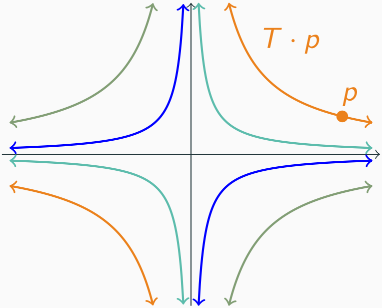

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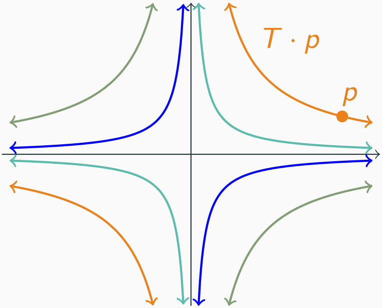

Problem: determine invariant polynomials

$$\mathbb{C}[x, y]^T = \{\text{polys } f : f(p) = f(T \cdot p)\}.$$

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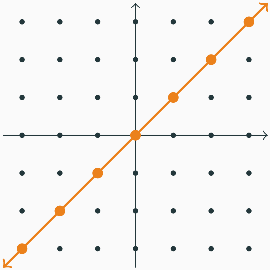

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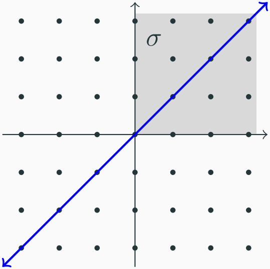
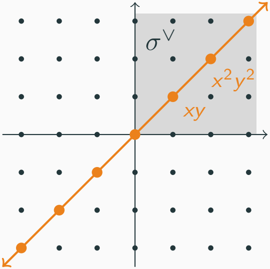
Choose σ so $\sigma^\vee$ only sees points with non-negative coordinates.





References

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