Toric Varieties as Quotients of Affine Space

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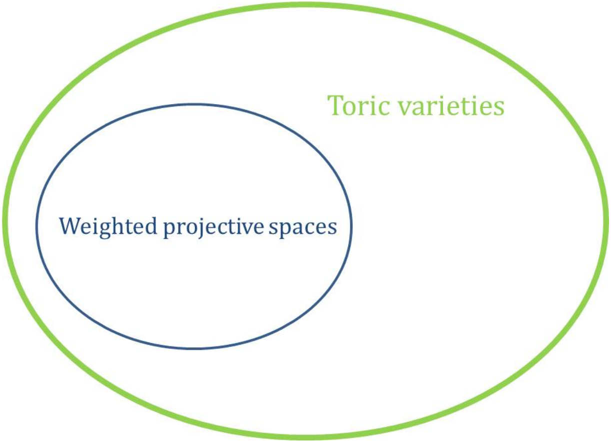
Motivation



Toric Varieties

A toric variety is an irreducible variety X containing an algebraic torus $T \simeq (\mathbb{C}^*)^n$ such that

- T is a Zariski open subset of X
- Action of T on $T \longrightarrow$ action of T on X

Toric Varieties: Examples

Example. $(\mathbb{C}^*)^n$

Example. $\mathbb{C}^n$

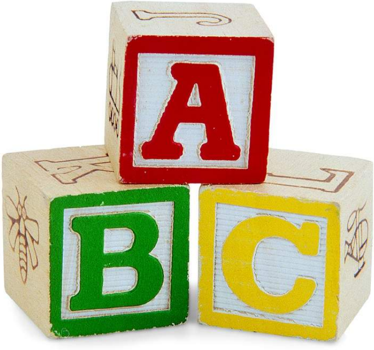
Example. $\mathbb{P}^n$

Example. Weighted projective space

Example. $V = \mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$

Toric Varieties

But first, some building blocks of toric varieties...



Lattice Notation

$N \simeq \mathbb{Z}^r$: a lattice (free abelian group of finite rank)

$M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$: the dual lattice of N

$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$

$\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$ dual pairing

Polyhedral Cones

A rational polyhedral cone in $N_{\mathbb{R}}$ is a collection

$$\text{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0 \right\}$$

where S is a finite subset of N.

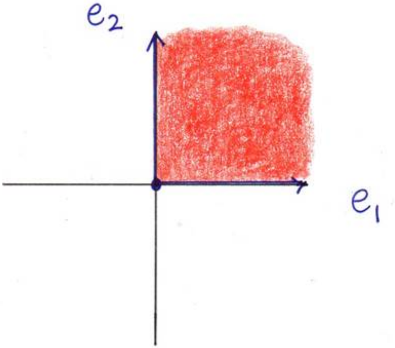
Set $\text{Cone}(\varnothing) = \{0\}$

Notation: $\sigma = \text{Cone}(S)$ "S generates σ"

The dimension of a polyhedral cone σ is the dimension of $\text{Span}(S)$ of $N_{\mathbb{R}}$.

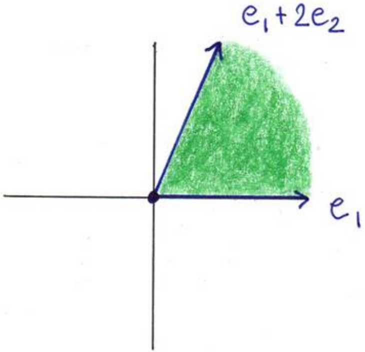
Polyhedral Cones

Example. First quadrant in $\mathbb{R}^2$



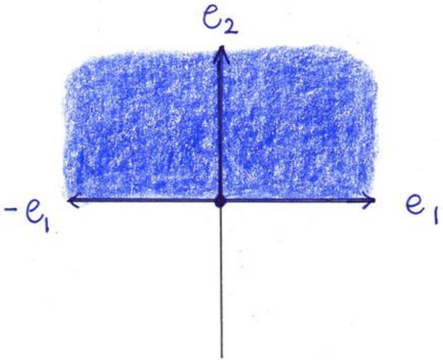
Polyhedral Cones

Example. $\sigma = \mathrm{Cone}(e_1, e_1 + 2e_2) \subset \mathbb{R}^2$



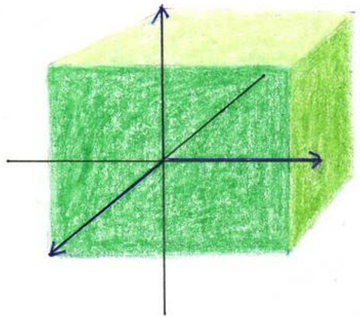
Polyhedral Cones

Example. Upper-half plane in $\mathbb{R}^2$



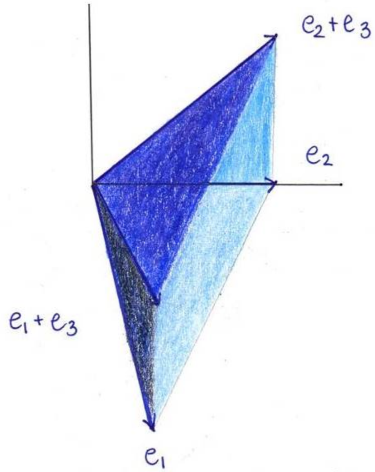
Polyhedral Cones

Example. First octant in $\mathbb{R}^3$



Polyhedral Cones

Example. $\sigma = \text{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subset \mathbb{R}^3$



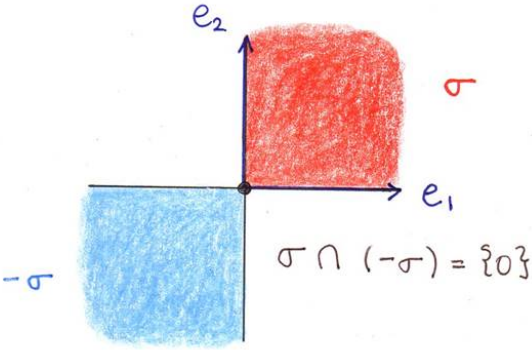
Strongly Convex Cones

A cone is strongly convex if it does not contain a straight line through the origin.

$$\sigma \bigcap (-\sigma) = \{0\}$$

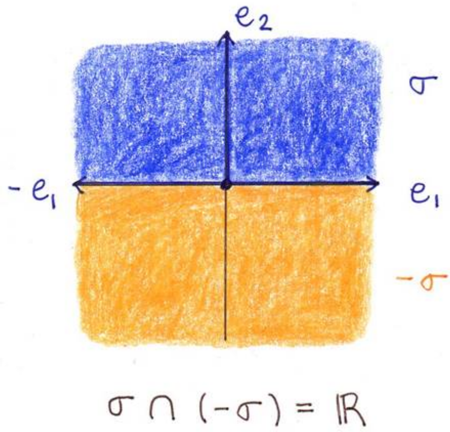
Strongly Convex Cones

Example. The cone σ is strongly convex.



Strongly Convex Cones

Example. The cone σ is not strongly convex.



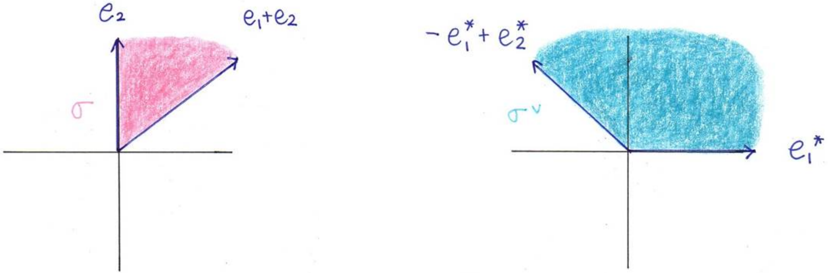
Dual Cones

Let σ be a polyhedral cone. The dual cone of σ is defined by

$$\sigma^\vee = \{m \in M_\mathbb{R} \mid \langle m, u \rangle \geq 0 \; \forall \; u \in \sigma\}$$

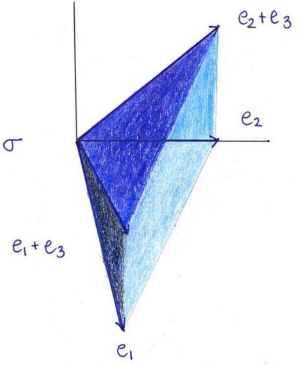
Dual Cones

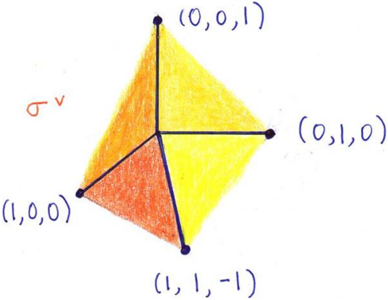
Example. The cones below are dual.



Dual Cones: Examples

Example. The cones below are dual.





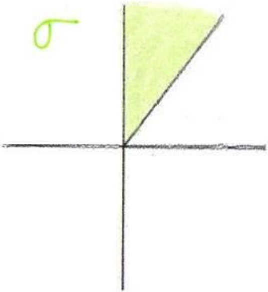
Faces

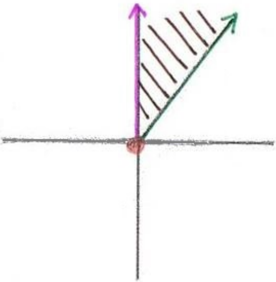
A face τ of a cone σ is the intersection of σ with the orthogonal complement to some $m \in \sigma^\vee \cap M$,

$$\tau = \sigma \cap m^\perp = \{v \in \sigma \mid \langle m, v \rangle = 0\}.$$

Faces

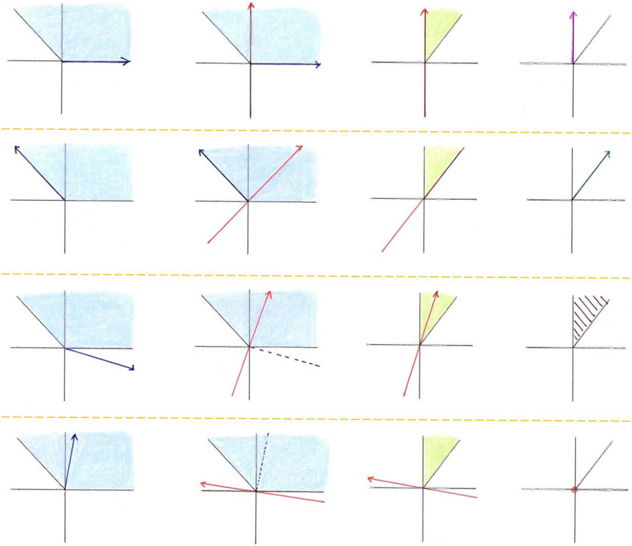
Example. Let $\sigma = \text{Cone}(e_1 + e_2, e_2) \subset \mathbb{R}^2$





Faces

Example (cont.ed).



Semigroup

Recall: A semigroup is a set S together with a binary operation $+$ such that
(i) S contains an identity element 0
(ii) operation $+$ is associative

An affine semigroup S has the additional properties
(iii) operation $+$ is commutative
(iv) S is finitely generated $A\mathbb{N} = S$, where A finite set
(v) S may be embedded in a lattice

Semigroup

Given $\sigma \subseteq N_{\mathbb{R}}$: a rational polyhedral cone.

Then the lattice points

$$R_\sigma = \sigma^\vee \cap M \subseteq M$$

form a semigroup.

Gordan's Lemma. The semigroup R_σ is a finitely generated affine semigroup.

Semigroup algebras

The semigroup algebra of a cone σ is the $\mathbb{C}$-algebra $\mathbb{C}[R_\sigma]$ with $\mathbb{C}$-vector space basis $\{v^m \,|\, m \in R_\sigma\}$ and product

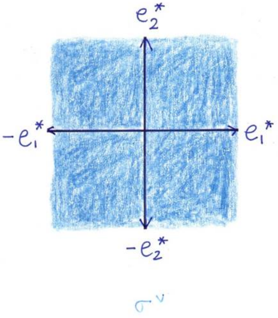
$$v^{m_1} \cdot v^{m_2} = v^{m_1+m_2}$$

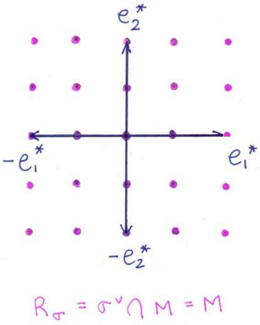
Remark. There is an inclusion $\mathbb{C}[R_\sigma] \subset \mathbb{C}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ by identifying v^m with Laurent monomial t^m.

Semigroup algebras

Example.

Cone $\sigma = \{0\}$
Dual cone $\sigma^\vee = \mathrm{Cone}(\pm e_1^*, \pm e_2^*)$



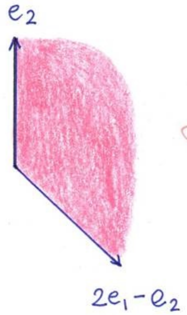


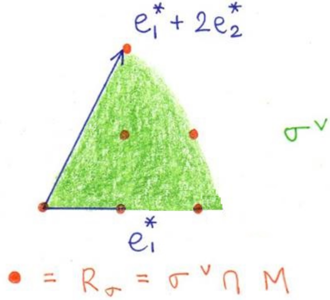
Semigroup algebra for σ is $\mathbb{C}[R_\sigma] = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$.

Semigroup algebras

Example.

Cone $\sigma = \text{Cone}(e_2, 2e_1 - e_2)$
Dual cone $\sigma^\vee = \text{Cone}(e_1^*, e_1^* + 2e_2^*)$



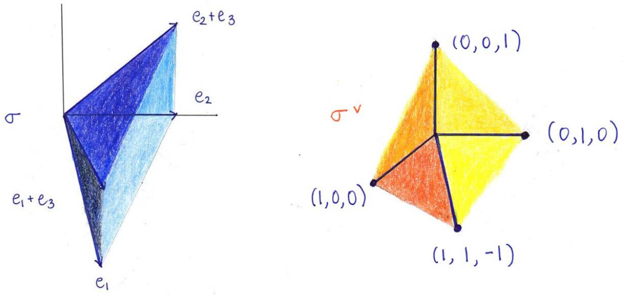


Semigroup algebra for σ is
$\mathbb{C}[R_\sigma] = \mathbb{C}[t_1, t_1t_2, t_1t_2^2] \simeq \mathbb{C}[x, y, z]/(xz - y^2)$.

Semigroup algebras

Example.

Cone $\sigma = \text{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3)$
Dual cone $\sigma^\vee = \text{Cone}(e_1^*, e_2^*, e_3^*, e_1^* + e_2^* - e_3^*)$



Semigroup algebra for σ is
$R_\sigma = \mathbb{C}[t_1, t_2, t_3, t_1 t_2 t_3^{-1}] = \mathbb{C}[x, y, z, w]/(xy - zw)$.

Affine Toric Variety

Cone σ

Affine Toric Variety

$$\begin{array}{c}\text{Cone } \sigma \\ \Big\downarrow \\ \text{Dual cone } \sigma^\vee\end{array}$$

Affine Toric Variety

$$\text{Cone } \sigma$$
$$\downarrow$$
$$\text{Dual cone } \sigma^\vee$$
$$\downarrow$$
$$\text{Semigroup } R_\sigma = \sigma^\vee \cap M$$

Affine Toric Variety

Cone σ

$\downarrow$

Dual cone $\sigma^\vee$

$\downarrow$

Semigroup $R_\sigma = \sigma^\vee \cap M$

$\downarrow$

$X_\sigma = \text{MaxSpec}(\mathbb{C}[R_\sigma])$

Affine Toric Variety

The affine toric variety X_σ corresponding to a strongly convex, rational polyhedral cone σ is defined as

$$X_\sigma = \mathrm{MaxSpec}(\mathbb{C}[R_\sigma]).$$

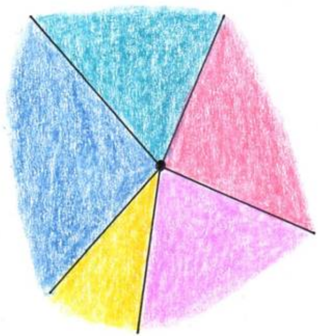
Fans

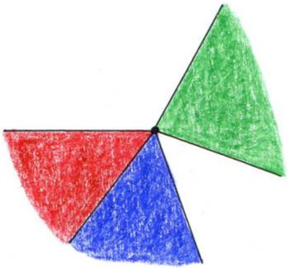
A fan in N is a nonempty finite collection Δ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying:

- For any cone $\sigma \in \Delta$, every face of σ is contained in Δ.
- For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'

Fans

Example. These are sometimes called torn fans





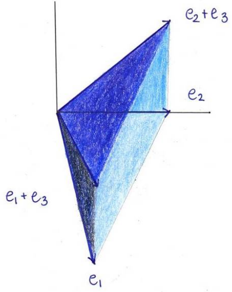
Simplicial Fans

Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.

We say that σ is simplicial if its generators are linearly independent over $N_{\mathbb{R}}$.

Simplicial Fans

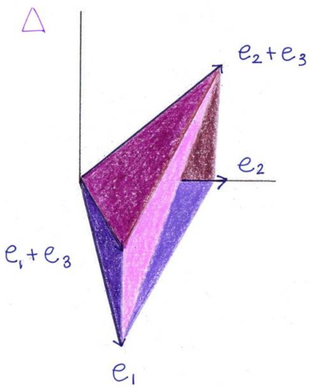
Example. Cone $\sigma = \text{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subset \mathbb{R}^3$ is not simplicial:

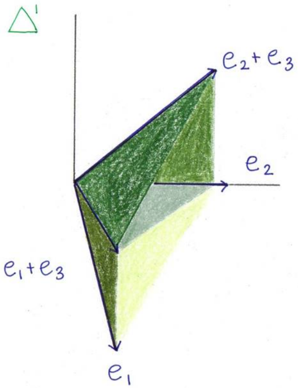


It is generated by four vectors in $\mathbb{R}^3$.

Simplicial Fans

Example (cont.ed). Subdivide $\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3)$ into simplicial fan:



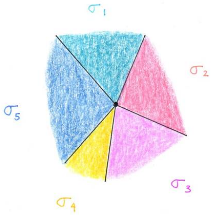


This is called the Atiyah-flop.

Building a Toric Variety

Now we have all the materials to build a toric variety!

Given a fan Δ in N with cones $\{\sigma_i\}$:



Building a Toric Variety

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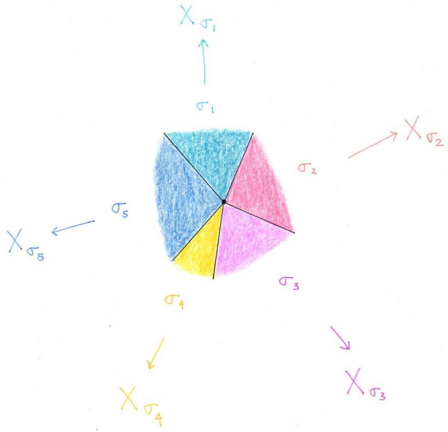


Building a Toric Variety

For all $\sigma, \sigma' \in \Delta$, glue X_σ and $X_{\sigma'}$ along their common open subset $X_{\sigma \cap \sigma'}$.

This gives the toric variety X_Δ

Building a Toric Variety

Theorem. The variety X_Δ associated to a fan Δ is a normal toric variety, with torus $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Every normal toric variety is X_Δ for some fan Δ.

The Homogeneous Coordinate Ring

X: toric variety with fan Δ in lattice N

Let

$\Delta(1) = \{\text{one-dimensional cones in } \Delta\}$
$\sigma(1) = \{\text{one-dimensional faces of } \sigma\}$

For each $\rho \in \Delta(1)$, let

$$\langle n_\rho \rangle = \rho \cap N \qquad \text{unique generator}$$

Assume: $\mathrm{Span}_{\mathbb{R}}\{n_\rho \mid \rho \in \Delta(1)\} = N_{\mathbb{R}}$

The Homogeneous Coordinate Ring

We have correspondence

$$\rho \longleftrightarrow D_\rho$$

(a 1-dimensional cone, i.e., $\rho \in \Delta(1)$) (an irreducible T_N-invariant Weil divisor)

$\mathbb{Z}^{\Delta(1)}$: free abelian group of these Weil divisors.

$\mathrm{Div}_{T_N}(X)$: the Cartier divisors

Then $\mathrm{Div}_{T_N}(X) \subset \mathbb{Z}^{\Delta(1)}$ (subgroup)

Then $\mathrm{Cl}(X) = \mathbb{Z}^{\Delta(1)} \,/\, \mathrm{Div}_0(X)$ and $\mathrm{Pic}(X) = \mathrm{Div}_{T_N}(X) \,/\, \mathrm{Div}_0(X)$.

$\mathrm{Div}_0(X) = \{\text{principal divisors}\}$

The Homogeneous Coordinate Ring

For each $m \in M$, there is a character $\chi^m : T_N \to \mathbb{C}^*$ defined by

$$\chi^m(\sum_{\rho \in \Delta(1)} n_\rho \otimes \lambda_\rho) = \prod_{\rho \in \Delta(1)} \lambda_\rho^{\langle m, n_\rho \rangle}$$

Then

- χ^m is a rational function on X
- χ^m has the principal divisor

$$D_m = \sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle D_\rho$$

The Homogeneous Coordinate Ring

Then there is an injective map $M \to \mathbb{Z}^{\Delta(1)}$. This gives the following:

Proposition. There is a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & \mathrm{Div}_{T_N}(X) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathbb{Z}^{\Delta(1)} \arrow[r] & \mathrm{Cl}(X) \arrow[r] & 0
\end{tikzcd}$$

A divisor $D \in \mathbb{Z}^{\Delta(1)} \longrightarrow \alpha = [D] \in \mathrm{Cl}(X)$.

The Homogeneous Coordinate Ring

The homogeneous coordinate ring of X is the polynomial ring S with variables that correspond to $\Delta(1)$:

$$S = \mathbb{C}[x_\rho \mid \rho \in \Delta(1)]$$

$$\prod_{\rho \in \Delta(1)} x_\rho^{a_\rho} \xrightarrow[\text{denoted by}]{} x^D$$

(a monomial in S)

(D is divisor $\sum_{\rho \in \Delta(1)} a_\rho D_\rho$)

Remark. The homogeneous coordinate ring is a graded algebra

$$S = \bigoplus_{\alpha \in \mathrm{Cl}(X)} S_\alpha,$$

where $S_\alpha = \mathrm{Span}_{\mathbb{C}}\{x^D \in S \mid [D] = \alpha\}$.

The Homogeneous Coordinate Ring

Example. Weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$

⊛ Homogeneous coordinate ring is the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$

⊛ Each variable x_i has weight a_i.

⊛ Includes the usual projective space where all weights are 1.

⊛ The grading defined above agrees with the usual grading of polynomials by degree.

The Homogeneous Coordinate Ring

How can we reconstruct a toric variety from its homogeneous coordinate ring?

Recall homogeneous coordinate ring definition:

$$S = \mathbb{C}[x_\rho \mid \rho \in \Delta(1)]$$

⊛ Only takes into account the one-dimensional cones of a fan.

⊛ Need more information on the structure of higher-dimensional cones in the fan

⊛ Where this information is encoded: an ideal in the homogeneous coordinate ring.

The Irrelevant Ideal

For a maximal cone $\sigma \in \Delta$, let $\hat{\sigma}$ be the divisor $\sum_{\rho \notin \sigma(1)} D_\rho$.

We define the irrelevant ideal

$$B = (x^{\hat{\sigma}} \mid \sigma \text{ maximal cone in } \Delta).$$

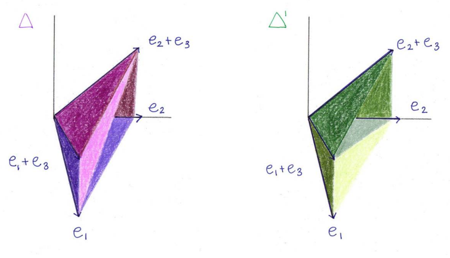
The Irrelevant Ideal

Example. For Projective space $\mathbb{P}^n$ or weighted projective space $\mathbb{P}(a_0 \ldots, a_n)$, the irrelevant ideal is:

$$B = (x_1, \ldots, x_n)$$

The Irrelevant Ideal

Example. Consider fans Δ and Δ' previous example:



- ⊛ Same one-dimensional cones: e_1, e_2, $e_1 + e_3$, $e_2 + e_3$
- ⊛ Toric varieties have same homogeneous coordinate ring

$$S = \mathbb{C}[x_{e_1}, x_{e_2}, x_{e_1+e_3}, x_{e_2+e_3}]$$

- ⊛ But their ideals are distinct:

$$B_{\Delta} = (x_{e_1+e_3}, x_{e_2}) \text{ and } B_{\Delta'} = (x_{e_2+e_3}, x_{e_1})$$

Toric Varieties as Quotients

The exceptional subset of $\mathbb{C}^{\Delta(1)}$ is the zero set of the irrelevant ideal:

$$Z = \mathbf{V}(B) = \{x \in \mathbb{C}^{\Delta(1)} \mid x^{\hat{\sigma}} = 0 \text{ for all } \sigma \in \Delta)\}$$

Toric Varieties as Quotients

We can construct toric varieties as quotients of affine space.

Projective space $\mathbb{P}^n$	Toric Variety X_Δ
$(\mathbb{C}^{n+1} \backslash \{0\}) \,/\, \mathbb{C}^*$	$(\mathbb{C}^{\Delta(1)} \backslash Z) \,/\, G$

Toric Varieties as Quotients

⊛ Let $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \mathbb{C}^*)$, a reductive group.

⊛ Action on $\mathbb{C}^{\Delta(1)}$: Let $g \in G$ and $t = (t_\rho) \in \mathbb{C}^{\Delta(1)}$

$$g \cdot t = (g([D_\rho])t_\rho)$$

⊛ There is a corresponding representation of G on S where S_α is the α-eigenspace.

⊛ Observe: applying functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the bottom sequence of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & \mathrm{Div}_{T_N}(X) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathbb{Z}^{\Delta(1)} \arrow[r] & \mathrm{Cl}(X) \arrow[r] & 0
\end{tikzcd}$$

gives short exact sequence

$$0 \to G \to (\mathbb{C}^*)^{\Delta(1)} \to T_N \to 0.$$

Toric Varieties as Quotients

Theorem. Let X be a toric variety with fan Δ and let Z be the exceptional subvariety of $\mathbb{C}^{\Delta(1)}$. Then

(i) $\mathbb{C}^{\Delta(1)} \setminus Z$ is invariant under the action of G;

(ii) X is naturally isomorphic to the categorical quotient of $\mathbb{C}^{\Delta(1)} \setminus Z$ by G;

(iii) X is the geometric quotient of $\mathbb{C}^{\Delta(1)} \setminus Z$ by G if and only if X is simplicial.