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LIE GROUP REPRESENTATIONS ON POLYNOMIAL RINGS.* [1]

By BERTRAM KOSTANT.[2]

0. Introduction. 1. Let G be a group of linear transformations on a finite dimensional real or complex vector space X. Assume X is completely reducible as a G-module. Let S be the ring of all complex-valued polynomials on X, regarded as a G-module in the obvious way, and let $J \subseteq S$ be the subring of all G-invariant polynomials on X.

Now let J^+ be the set of all $f \in J$ having zero constant term and let $H \subseteq S$ be any graded subspace such that $S = J^+S + H$ is a G-module direct sum. It is then easy to see that

$$S = JH. \tag{0.1.1}$$

(Under mild assumptions H may be taken to be the set of all G-harmonic polynomials on X. That is, the set of all $f \in S$ such that $\partial f = 0$ for every homogeneous differential operator ∂ with constant coefficients, of positive degree, that commutes with G.)

One of our main concerns here is the structure of S as a G-module. Regard S as a J-module with respect to multiplication. Matters would be considerably simplified if S were free as a J-module. One shows easily that S is J-free if and only if $S = J \otimes H$. This, however, is not always the case. For example S is not J-free if G is the two element group $\{I, -I\}$ and $\dim X \geqq 2$. On the other hand one has

Example 1. It is due to Chevalley (see [2]) that if G is a finite group generated by reflections then indeed $S = J \otimes H$. Furthermore the action of G on H is equivalent to the regular representation of G.

Example 2. S is J-free in case G is the full rotation group (with respect to some Euclidean metric on X. For convenience assume in this example that $\dim X \geqq 3$). Note that the decomposition of a polynomial according to the relation $S = J \otimes H$ is just the so-called "separation of variables" theorem for polynomials. This is so because J is the ring of radical polynomials and H is the space of all harmonic polynomials (in the usual sense).

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Now, for any $x \in X$, let $O_x \subseteq X$ denote the G-orbit of x and let $S(O_x)$ be the ring of all functions on O_x defined by restricting S to O_x. Since J reduces to constants on any orbit it follows that (0.1.1) induces a G-module epimorphism

$$H \to S(O_x). \tag{0.1.2}$$

Since our major concern is the case where X is a reductive Lie algebra and G is the adjoint group and since the methods used there belong to algebraic geometry we will assume now that X is complex and the G is algebraic and reductive. All varieties considered are over $\boldsymbol{C}$. If Y has an algebraic structure $R(Y)$ will denote the ring of everywhere defined rational functions in Y. Obviously one always has

$$S(O_x) \subseteq R(O_x). \tag{0.1.3}$$

On the other hand if $G^x \subseteq G$ is the isotropy group defined by $x \in X$ then one has a G-module isomorphism

$$R(G/G^x) \to R(O_x). \tag{0.1.4}$$

The significance of (0.1.4) is that one knows the G-module structure of $R(G/G^x)$ completely by a very simple algebraic Frobenius reciprocity theorem (even though G^x may not be reductive). In fact if V^λ is any irreducible G-module with respect to the representation ν^λ and V_λ is the dual module then one has

$$\text{mult. of } \nu^\lambda \text{ in } R(G/G^x) = \dim V_\lambda{}^{G^x} \tag{0.1.5}$$

where $V_\lambda{}^{G^x}$ is the space of vectors in V_λ fixed under G^x.

Now in Examples 1 and 2 (assume complexified) the following three optimum situations occur:

(a) S is J-free so that $S = J \otimes H$

(b) the map $H \to S(O_x)$ is an isomorphism for certain $x \in X$ and for those x

(c) $R(O_x) = S(O_x)$.

But one observes that if in any general case, (b) and (c) hold then, clearly, upon combining (0.1.4) and (0.1.5) one gets the G-module structure of H. If one gets in addition the "graded" G-module structure of H and knows the structure of J then one gets the full graded G-module structure of S in case (a) also holds.

In Example 2 the conditions (b) and (c) hold for any $x \neq 0$ (even if $(x,x) = 0$). In fact, classically, one has exploited (b) and (c) for $(x,x) > 0$

to solve the Dirichlet problem with the sphere as boundary. That is, if f is any continuous function on the sphere one first expands f as a Fourier development of spherical harmonics f_m. The sphere is $O_x \cap \boldsymbol{R}^n$ and the f_m are in $R(O_x)$. The equality $R(O_x) = S(O_x)$ and the isomorphism $H \to S(O_x)$ then yields the extension of f_m uniquely as harmonic polynomials h_m on X. But this yields the desired extension of f.

In Example 1 the conditions (b) and (c) are satisfied for any "regular" element $x \in X$.

Our first concern in this paper is to give criteria for (a), (b) and (c) to hold in general. Since our interest is in the continuous case we will assume G is connected (and hence a variety). Thus Example 2 rather than Example 1 serves as a model.

Now let $P \subseteq X$ be the cone of common zeros defined by the ideal J^+S in S. Let X^* be the dual space to X and let $P^* \subseteq X^*$ be defined in a similar way with the roles of X and X^* interchanged. As a criterion to establish (a) and more we prove

PROPOSITION 0.1. *Assume* (1) *that J^+S is a prime ideal in S and* (2) *there exists an orbit $O_e \subseteq P$ which is dense in P. Then $S = J \otimes H$. Furthermore if G is a subgroup of the complex rotation group then H may be taken as the space of all G-harmonic polynomials. Moreover H then coincides with the space spanned by all powers f^k where $f \in P^*$.*

It may be observed that the criterion is satisfied in Example 2.

An element $x \in X$ is called quasi-regular if $P \subseteq \overline{\boldsymbol{C}^* \cdot O_x}$. A criterion to establish (b) is given by

PROPOSITION 0.2. *Assume conditions* (1) *and* (2) *of Proposition* 0.1 *are satisfied. Then the G-module epimorphism $H \to S(O_x)$ is an isomorphism for any quasi-regular element $x \in X$.*

It may be observed that in Example 2 every nonzero $x \in X$ is quasi-regular.

From known facts in algebraic geometry one has the following criterion to insure (c).

PROPOSITION 0.3. *Let $x \in X$ and assume* (1) *the closure $\bar{O}_x$ is a normal variety and* (2) *$\bar{O}_x - O_x$ has a codimension of at least 2 in $\bar{O}_x$. Then $R(O_x) = S(O_x)$.*

It may be observed that the conditions of Proposition 0.3 are satisfied for every $x \in X$ in Example 2.

Now assume that $X=\mathfrak{g}$ is a complex reductive Lie algebra and G is the adjoint group. Here the structure of J is given by a theorem of Chevalley. This aserts that J is a polynomial ring in l (the rank of $\mathfrak{g}$) homogeneous generators u_i, $i=1,2,\cdots,l$ with $\deg u_i=m_i+1$ where the m_i are the exponents of $\mathfrak{g}$.

Now one knows that here P is the set of all nilpotent elements of $\mathfrak{g}$ ([13], Theorem 9.1). But then by [13], Corollary 5.5, P does contain a dense orbit O_e, namely, the set of all principal nilpotent elements in $\mathfrak{g}$. Thus to apply Propositions 0.1 and 0.2 one must prove that J^+S is a prime ideal.

If $n=\dim\mathfrak{g}$ (all dimensions are over $\boldsymbol{C}$) then one sees easily that $n-l$ is the maximal dimension of any orbit. Let $\mathfrak{r}=\{x\in\mathfrak{g}\mid \dim O_x=n-l\}$. Any regular element $x\in\mathfrak{g}$ belongs to $\mathfrak{r}$. But also $e\in\mathfrak{r}$ for any principal nilpotent element. These in fact are extreme cases.

PROPOSITION 0.4. *Let $x\in\mathfrak{g}$ be arbitrary. Write (uniquely) $x=y+z$ where y is semi-simple, z is nilpotent and $[y,z]=0$. Let $\mathfrak{g}^y$ be the centralizer of y in $\mathfrak{g}$ so that $\mathfrak{g}^y$ is a reductive Lie algebra and $z\in\mathfrak{g}^y$. Then $x\in\mathfrak{r}$ if and only if z is principal nilpotent in $\mathfrak{g}^y$.*

Let $x\in\mathfrak{g}$. Consider the values $(du_i)_x$ of the l-differential forms du_i, $i=1,2,\cdots,l$, at x. It is known that these covectors are linearly independent whenever x is regular. (One recalls that the product of the positive roots is the determinant of an $l\times l$ minor of a certain $n\times l$ matrix determined by the du_i.) But to prove the primeness of the ideal J^+S one needs to know that these covectors are linearly independent if x is a principal nilpotent element. This fact is contained in

THEOREM 0.1. *Let $x\in\mathfrak{g}$. Then the $(du_i)_x$ is linearly independent if and only if $x\in\mathfrak{r}$.*

Proposition 0.1 may now be applied.

THEOREM 0.2. *One has $S=J\otimes H$ where H is the space of all G-harmonic polynomials on $\mathfrak{g}$. Furthermore H coincides with the space of all polynomials spanned by all powers of "nilpotent" linear functionals.*

Since Theorem 0.1 shows also that P is a complete intersection the decomposition $S=J\otimes H$ when combined with Proposition 5, § 78, in FAC [15], gives, in the notation of FAC, all the sheaf cohomology groups $H^j(\boldsymbol{P},\mathcal{O}(m))$ where $\boldsymbol{P}$ is the projective variety defined by P.

Another application of the primeness of J^+S in algebraic geometry is

THEOREM 0.3. *The intersection multiplicity of P, at the origin, with any Cartan subalgebra is w, where w is the order of the Weyl group.*

Next, Proposition 0.2 is put into effect for all orbits of maximal dimension by

THEOREM 0.4. *The set* $\mathfrak{r}$ *coincides with the set of all quasi-regular elements in* $\mathfrak{g}$. *(Thus* H *and* $S(O_x)$ *are isomorphic as* G*-modules for any* $x \in \mathfrak{r}$.*)*

As a consequence of Theorems 0.2 and 0.4 one shows that not only is the ideal J^+S prime in S but J_1S is prime for any prime ideal $J_1 \subseteq J$. Furthermore one gets the following characterization of all the invariant prime ideals in S which are generated by elements of J.

THEOREM 0.5. *Let* $I \subseteq S$ *be any* G*-invariant prime ideal. Let* $\mathfrak{u} \subseteq \mathfrak{g}$ *be the affine variety of zeros of* I. *Then* I *is of the form* $I = J_1S$, *for* J_1 *a prime ideal in* J, *if and only if* $\mathfrak{u} \cap \mathfrak{r}$ *is not empty.*

Since $R(O_x) = S(O_x)$ in case O_x is closed and since O_x is closed if x is regular one gets the G-module structure of H by applying Theorem 0.4 and (0.1.5) for x regular. Thus if D denotes the set of dominant integral forms corresponding to a Cartan subgroup A, so that D indexes all the irreducible representations of G as highest weights, then one has

$$\text{mult } \nu^\lambda \text{ in } H = l_\lambda \tag{0.1.6}$$

where $l_\lambda = \dim V_\lambda{}^A$ is the multiplicity of the zero weight of ν_λ.

In order to determine the G-module structure of S^k, the space of homogeneous polynomials on $\mathfrak{g}$ of degree k, one must know more than (0.1.6). In fact using the relation $S = J \otimes H$ what one wants is the multiplicity of ν^λ in $H^j = S^j \cap H$ for any λ and j. As it turns out, for this, one needs $R(O_e) = S(O_e)$ where e is a principal nilpotent element. To show the latter using Proposition 0.3 it is enough to show that P is a normal variety and $P - O_e$ has a codimension of at least 2 in P.

Let $\mathcal{O}_\mathfrak{r}$ be the set of all orbits of maximal dimension $(n - l)$. The set $\mathcal{O}_\mathfrak{r}$ may be parametrized by $\boldsymbol{C}^l$ in the following way. Let

$$u\colon \mathfrak{g} \to \boldsymbol{C}^l$$

be the morphism given by putting $u(x) = (u_1(x), \cdots, u_l(x))$ for any $x \in \mathfrak{g}$. Since u reduces to a constant on any orbit it induces a map

$$\eta_\mathfrak{r}\colon \mathcal{O}_\mathfrak{r} \to \boldsymbol{C}^l.$$

It is known that u induces a bijection from the set of all orbits consisting of semi-simple elements onto $\boldsymbol{C}^l$ (for completeness a proof of this fact will be given here). Combining this with Proposition 0.4 one obtains

THEOREM 0.6. *$\eta_{\mathfrak{r}}$ is a bijection.*

Thus to each $\xi \in \boldsymbol{C}^l$ there exists a unique orbit, $O^{\mathfrak{r}}(\xi)$, of dimension $n-l$ which correspond to ξ under $\eta_{\mathfrak{r}}$. Now let $P(\xi) = u^{-1}(\xi)$ for any $\xi \in \boldsymbol{C}^l$ so that

$$\mathfrak{g} = \bigcup_{\xi \in \boldsymbol{C}^l} P(\xi)$$

is a disjoint union. Note that $P(\xi) = P$ and $O^{\mathfrak{r}}(\xi) = O_e$ if ξ is the origin of $\boldsymbol{C}^l$. One proves

THEOREM 0.7. *For any $\xi \in \boldsymbol{C}^l$ one has*

$$P(\xi) = \overline{O^{\mathfrak{r}}(\xi)}$$

so that $P(\xi)$ is a variety of dimension $n-l$. Moreover $P(\xi)$ is a complete intersection and $O^{\mathfrak{r}}(\xi)$ coincides with the set of non-singular points on $P(\xi)$. Finally $P(\xi)$ is a finite union of orbits so that $\bar{O}_x$ is a finite union of orbits for any $x \in \mathfrak{g}$.

Since $P(\xi)$ is a complete intersection and since its singular locus is the complement (a finite union of orbits) of $O^{\mathfrak{r}}(\xi)$ in $P(\xi)$ one would get the normality of $P(\xi)$ by a theorem of Seidenberg if one knew the dimension of the other orbits in $P(\xi)$ were at most $n-l-2$.

Now it is well known that dim O_x is even (and hence $\dim_{\boldsymbol{R}} O_x$ is a multiple of 4) for any semi-simple element $x \in \mathfrak{g}$. Less known is the following proposition observed independently by the author, Borel, and (most simply proved by) Kirillov.

PROPOSITION 0.5. *The dimension of O_x is even for any $x \in \mathfrak{g}$.*

Combining Theorem 0.7 and Proposition 0.5 one obtains

THEOREM 0.8. *Let $\xi \in \boldsymbol{C}^l$ be arbitrary. Then $P(\xi)$ is a normal variety and the codimension of $P(\xi) - O^{\mathfrak{r}}(\xi)$ in $P(\xi)$ is at least 2.*

Applying Proposition 0.3 one then has

THEOREM 0.9. *Let $x \in \mathfrak{r}$. Then $R(O_x) = S(O_x)$. (This implies that all $R(O_x)$ for $x \in \mathfrak{r}$ are isomorphic as G-modules; even though they are not in general as rings.) Let $\xi = u(x)$. Then $R(O_x)$ $(= R(G/G^x))$ is an affine algebra (even though O_x is not necessarily an affine variety) and $P(\xi)$ is the variety of all maximal ideals of $R(O_x)$. Thus the embedding of G/G^x in $\mathfrak{g}$ as O_x is special in that any morphism of G/G^x (or O_x) into any affine variety extends uniquely to a morphism of $P(\xi) = \bar{O}_x$ into the variety. (In particular* *this holds for* O_e *and* $\bar{O}_x = P$.) *Finally* (*using* (0.1.5) *and the equality* $R(O_x) = S(O_x)$) *one has, for any* $\lambda \in D$

$$\dim V_\lambda{}^{G^x} = l_\lambda \tag{0.1.7}$$

so that the left side of (0.1.7) *is independent of* $x \in \mathfrak{r}$.

Now let $\{e_-, x_0, e\}$ be a principal S-triple (that is, a "canonical" basis of a principal three dimensional simple Lie subalgebra). In particular then e is a principal nilpotent element. Used heavily in the theorems above is the result of [13] which asserts that $\mathfrak{g}^e$ is l-dimensional and has a basis z_i, $i = 1, 2, \cdots, l$, such that

$$[x_0, z_i] = m_i z_i \tag{0.1.8}$$

where, we recall, the m_i are the exponents of $\mathfrak{g}$. The main application of this is the following result: Let $\mathfrak{a}$ be any subspace of $\mathfrak{g}$ such that (1) $\mathfrak{g} = \mathfrak{a} + [e_-, \mathfrak{g}]$ is a direct sum and (2) $\mathfrak{a}$ is stable under $\operatorname{ad} x_0$ (e.g. take $\mathfrak{a} = \mathfrak{g}^e$). Then if $\mathfrak{v}$ is the l-plane defined by the translation $\mathfrak{v} = e_- + \mathfrak{a}$ one has

THEOREM 0.10. *The variety* $\mathfrak{v}$ *is contained in* $\mathfrak{r}$. *Moreover each orbit in* $\mathcal{O}_{\mathfrak{r}}$ *intersects* $\mathfrak{v}$ *in one and only one point. Finally the mapping* $f \to f \mid \mathfrak{v}$ *induces an isomorphism of* J *onto* $R(\mathfrak{v})$.

Remark. If $\mathfrak{g}$ is the set of all $l \times l$ complex matrices then one shows easily that $\mathfrak{r}$ is the set of all matrices whose characteristic polynomial is equal to their minimal polynomial. An example of the subvariety $\mathfrak{v}$ is the set of all "companion" matrices. Here the validity of Theorem 0.10 is a well-known fact in matrix theory.

Now since $\mathfrak{g}^e = \mathfrak{g}^{G^e}$ (because $\mathfrak{g}^e$ is commutative) and since (0.1.7) holds for $x = e$ this suggests a generalization of the notion of exponent. Let V be any finite dimensional G-module with respect to a representation ν. If l^ν is the multiplicity of the zero weight of ν then by (0.1.7) one has $\dim V^{G^e} = l^\nu$. It follows therefore that there exists a unique non-decreasing sequence of non-negative integers $m_i(\nu)$, $i = 1, 2, \cdots, l^\nu$, such that one has

$$\nu(x_0) z_i = m_i(\nu) z_i$$

for a basis z_i of V^{G^e}. If ν is the adjoint representation the $m_i(\nu)$ are the usual exponents. If $\nu = \nu^\lambda$ we will write $m_i(\lambda)$ for $m_i(\nu^\lambda)$ and note (because the highest weight has multiplicity one) that

$$m_j(\lambda) = o(\lambda) \text{ for } j = l_\lambda$$

where $o(\lambda)$ is the sum of the coefficients of λ relative to the simple roots and that this highest value occurs with multiplicity one among the generalized exponents $m_i(\lambda)$. (This specializes to the familiar relation $m_l = o(\psi)$ when $\mathfrak{g}$ is simple and ψ is the highest root.)

The following theorem now gives the G-module structure of H^j and hence S^k for any j and k.

THEOREM 0.11. *Let $\lambda \in D$ be arbitrary and let $H(\lambda)$ be the set of G-harmonic polynomials which transform under G according to ν^λ. Let (by (0.1.6)) $H(\lambda) = \sum_{j=1}^{l_\lambda} H_j(\lambda)$ be a decomposition into irreducible components so that $H_j(\lambda) \subseteq H^{n_j}$ where n_j, $j = 1, 2, \cdots, l_\lambda$, is a non-decreasing sequence of integers. Then $n_j = m_j(\lambda)$ for all j. In particular then $k = o(\lambda)$ is the highest degree k such that ν^λ occurs in H^k. Moreover it occurs with multiplicity one for this value of k.*

Assume for convenience that $\mathfrak{g}$ is simple and let $\psi \in D$ be the highest root. Let x_i, $i = 1, 2, \cdots, n$ be a basis of $\mathfrak{g}$. If the $u_j \in J$ are chosen properly one sees that $\dfrac{\partial u_j}{\partial x_j}$, $i = 1, 2, \cdots, n$, is a basis of $H_j(\psi)$. One notes then that Theorem 0.11 is a generalization of the result in [13] given by (0.1.8).

H. S. M. Coxeter observed and A. J. Coleman proved in [4] that if W is the Weyl group and $\sigma \in W$ is the Coxeter-Killing transformation then the eigenvalues of σ operating on the Cartan subalgebra are $e^{2\pi i m_j/s}$, $j = 1, 2, \cdots, l$, where s is order of σ. Now more generally W operates on the zero weight space of V^λ for any $\lambda \in D$ according (say) to some representation π^λ of W. As a generalization of the Coxeter-Coleman theorem one now has

THEOREM 0.12. *For any $\lambda \in D$ the eigenvalues of $\pi^\lambda(\sigma)$ are $e^{2\pi i m_j(\lambda)/s}$, $j = 1, 2, \cdots, l_\lambda$.*

0.2. By applying the Birkhoff-Witt theorem the results above carry over from S to U, the universal enveloping of $\mathfrak{g}$ (U is obviously a G-module in a natural way).

THEOREM 0.13. *Let U be the universal enveloping algebra over $\mathfrak{g}$ and let $Z \subseteq U$ be the center of U. Then U is free as a Z-module (under multiplication). In fact*

$$U = Z \otimes E \tag{0.2.1}$$

where E is the subspace (and G-submodule) of U spanned by all powers x^k for all nilpotent elements $x \in \mathfrak{g}$. Moreover E is equivalent to H as a G-module so that every irreducible representation of G occurs with finite multiplicity in E (in fact ν^λ occurs l_λ times in E for any $\lambda \in D$).

Let V be a finite dimensional irreducible U-module so that one has a G-module algebra epimorphism

$$\rho\colon U \to \operatorname{End} V$$

Since $\rho(Z)$ reduce to the scalars it follows from (0.2.1) that $\rho(E) = \operatorname{End} V$. Now let Y be any subspace of U. If Y is one-dimensional then it is due to Harish-Chandra that there exists an irreducible U-module V such that ρ is faithful on Y. This is not true in general if $\dim Y \geqq 2$. However it is true if $Y \subseteq E$.

THEOREM 0.14. *Let $Y \subseteq E$ be any finite dimensional subspace. Then there exists an irreducible U-module V such that ρ is faithful on Y.*

I would like to express my thanks to C. Chevalley, M. Rosenlicht and A. Seidenberg for helpful conversations about questions in algebraic geometry. In particular to Siedenberg for making me aware of his criterion for normality and to Chevalley for simplifying my proof of the primeness of J^+S.

1. Consequences of the primeness of J^+S and a dense orbit in P. 1. Let X be a n-dimensional vector space over the complex numbers $\boldsymbol{C}$. Let $S_* = S_*(X)$ symmetric algebra over X. One knows that S_* may be regarded as the algebra of all differential operators ∂ on X which may be put in the form

$$\partial = \sum a_{i_1\cdots i_n}\left(\frac{\partial}{\partial z_i}\right)^{i_1}\cdot\cdot\left(\frac{\partial}{\partial z_n}\right)^{i_n}$$

where the $a_{i_1\cdots i_n}$ are complex constants and $z_1, \cdots, z_n$ are the affine coordinates of X.

Let $S^* = S^*(X)$ (or just S) be the symmetric algebra over the dual space to X. Then S is just the ring of all polynomials on X. In fact we take the point of view that X is an affine variety (over $\boldsymbol{C}$) and S is its ring of everywhere defined rational functions. The algebra S (resp. S_*) is graded in the obvious way and a subspace $L \subseteq S$ (resp. $L \subseteq S_*$) will be called graded if it is spanned by its homogeneous components $L^j = L \cap S^j$ (resp. $L \cap S_j$).

Now one knows that a non-singular pairing of S_* and S into $\boldsymbol{C}$ is established by putting

$$\langle \partial, f\rangle = \partial f(0) \tag{1.1.1}$$

where $\partial \in S_*$, $f \in S$ and $\partial f(0)$ denotes the value of the function ∂f at the origin. In this way S_k is orthogonal to S^j if $j \neq k$ and becomes isomorphic to its dual if $k = j$. It is obvious from (1.1.1) that

$$\langle \partial_1\partial_2, f\rangle = \langle \partial_2, \partial_1 f\rangle \tag{1.1.2}$$

for any $\partial_1, \partial_2 \in S_*$ and $f \in S$ and hence in particular if $f \in S^m$ and $x \in X$ then by the Taylor expansion

$$\langle \frac{(\partial_x)^m}{m!}, f \rangle = f(x) \tag{1.1.3}$$

where ∂_x is the element of $S_1(X) \cong X$ corresponding to x.

Now assume that $G \subseteq \operatorname{Aut} X$ is a connected linear reductive algebraic group, i. e., G is the complexification of a connected compact subgroup of $\operatorname{Aut} X$. We regard G as not only operating on X, but by unique extension, as a group of algebra automorphisms of $S_*(X)$ and also as a group of algebra automorphisms of $S(X)$. The action on the latter is also uniquely defined by requiring that

$$\langle a \cdot \partial, a \cdot f \rangle = \langle \partial, f \rangle \tag{1.1.4}$$

for all $a \in G$, $\partial \in S_*$ and $f \in S$. Note by (1.1.3) that

$$(a \cdot f)(x) = f(a^{-1}x) \tag{1.1.5}$$

for any $x \in X$, $f \in S$ and $a \in G$.

Now let $J \subseteq S$ be the graded subring of G-invariant polynomials in S. That is

$$J = \{f \in S \mid a \cdot f = f \text{ for all } a \in G\},$$

and let

$$J^+ = \{f \in J \mid f(0) = 0\}.$$

We will often be concerned with the homogeneous ideal J^+S in S generated by J^+.

Proposition 1. *Let L be any graded subspace of S such that $S = J^+S + L$ is a direct sum. Then*

$$S = JL.$$

Proof. We must show $S^k \subseteq JL$ for all k. This is obvious if $k = 0$ since $S^0 \subseteq L$. Assume it is true for S^j where $j \leqq k$. But since one clearly has $S^{k+1} \subseteq J^+S^{(k)} + L$ where $S^{(k)} = \sum_{i=0}^{k} S^i$ it is then obviously true for S^{k+1}. Q. E. D.

We are interested first of all in the question as to when S is free over J, or more specifically, as to when S is a tensor product of J and L. (Choosing L to be G-stable, such a decomposition of S reduces the study of its G-module structure to that of L.) We first observe that the two conditions are equivalent.

The expression linear independence (resp. basis) without any reference to a ring always means linear independence (resp. basis) with respect to $\boldsymbol{C}$. Furthermore, tensor product without reference to a ring means tensor product over $\boldsymbol{C}$.

LEMMA 1. *The following conditions are equivalent:*

1. *Let L be as in Proposition* 1. *Then the map*

$$J \otimes L \rightarrow S$$

given by $f \otimes g \rightarrow fg$ is an isomorphism.

2. *S is free over J.*

3. *Let $M \subseteq S$ be any subspace such that $M \cap J^+S = (0)$. Then for elements in M linear independence is equivalent to linear independence over J.*

Proof. Obviously $(1) \Rightarrow (2)$ since a $\boldsymbol{C}$ basis of L defines a J basis of S. Assume (2) and let e_i, $i = 1, 2, \cdots$, define a J basis of S. Let $f_1, \cdots, f_k$ be linearly independent elements of M. For $j = 1, \cdots, k$ write $f_j = \sum_i a_{ij} e_i$ where we may assume $i = 1, 2, \cdots, p$ and $a_{ij} \in J$. To show that the f_j are linearly independent over J it clearly suffices to see that the $k \times p$ matrix (a_{ij}) with entries in J is of rank k. If f' denotes the image of $f \in S$ in S/J^+S it is clear that $f'_j = \sum a_{ij}(0) e'_i$. Since the f'_j are obviously linearly independent it is clear that the $\boldsymbol{C}$ matrix $a_{ij}(0)$ is of rank k. Hence not all $k \times k$ minors of the J matrix (a_{ij}) can be zero. This proves that the f_j are linearly independent over J.

It is trivial that linear independence over J implies linear independence. To obtain (1) from (3) we simply put $M = L$ and apply Proposition 1.

Q. E. D.

1.2. Now for any $x \in X$ let

$$O_x = \{y \in X \mid y = ax, \text{ for some } a \in G\}.$$

A subset $O \subseteq X$ is said to be an orbit if $O = O_x$ for some $x \in X$.

For any $x \in X$ let

$$G^x = \{a \in G \mid ax = x\}$$

It is clear that G^x is an algebraic (hence a closed, complex Lie subgroup) subgroup of G. Furthermore if

$$\beta'_x\colon G \rightarrow X \tag{1.2.1}$$

is the map given by $\beta'_x(a) = ax$ then β'_x induces a bijection

$$\beta_x\colon G/G^x \rightarrow O_x \tag{1.2.2}$$

Now let U be universal enveloping algebra of the Lie algebra of G. Then since the representation of G on S induces a representation of its Lie algebra on S it is clear that S becomes a U-module by further extension to U. Obviously S^k is a U-submodule for any k. We denote by $p \cdot f \in S$ the effect of applying p to f where $p \in U$ and $f \in S$.

For any subset $P \subseteq X$ and $f \in S$ let $f \mid P$ be the restriction of f to P.

LEMMA 2. *Let $f_j \in S$, $j = 1, 2, \cdots, k$, and let p_i, $i = 1, 2, \cdots$, be a basis of U. Consider the k-column matrix $D = (d_{ij})$ where $d_{ij} = p_i \cdot f_j$. (The matrix thus has entries in S and hence, for any $x \in X$, $D(x) = (d_{ij}(x))$ is a $\boldsymbol{C}$ matrix.)*

Then if $x \in X$ the functions $f_i \mid O_x$ on O_x, $j = 1, 2, \cdots, k$, are linearly independent if and only if $D(x)$ has rank k.

Proof. Let C be the algebra of all holomorphic functions on the complex homogeneous space G/G^x. It is clear that C is a module for the Lie algebra of G since the latter defines holomorphic vector fields on G/G^x. Hence C also becomes a U-module and in such a way that if

$$\alpha : S \to C$$

is the homomorphism given by $\alpha f = f \circ \beta_x$ one has

$$\alpha(p \cdot f) = p \cdot \alpha f \tag{1.2.3}$$

for any $f \in S$. Furthermore if $s \in G/G^x$ denotes the point corresponding to G^x and $g \in C$ then since the Lie algebra of G spans the holomorphic tangent space at s it follows from the Taylor expansion that g vanishes identically on G/G^x if and only if $(p \cdot g)(s) = 0$ for all $p \in U$.

Now assume that rank $D(x) < k$. Then there exists a non-zero vector $(c_1, \cdots, c_k) \in \boldsymbol{C}^k$ such that $\sum_j d_{ij}(x) c_j = 0$ for all i. Thus if $f = \sum_j c_j f_j \in S$ one has $(p \cdot f)(x) = 0$ for all $p \in U$. Thus by (1.2.3) $(p \cdot \alpha f)(s) = 0$ for all $p \in U$ and hence αf vanishes identically on G/G^x; or equivalently $f \mid O_x$ is zero. Hence the $f_j \mid O_x$ are linearly dependent. Conversely assume that the $f_j \mid O_x$ are linearly dependent so that $(\sum c_j f_j) \mid O_x$ is zero for a non-zero vector $(c_1, \cdots, c_k) \in \boldsymbol{C}^k$. But then $\alpha(\sum_j c_j f_j) = 0$ and hence for $i = 1, 2, \cdots$,

$$\begin{aligned} \sum_j c_j p_i \cdot f_j(x) &= (p_i \cdot \alpha(\sum_j c_j f_j))(s) \\ &= 0 \end{aligned}$$

Q. E. D.

Thus rank $D(x) < k$.

As a corollary we obtain the following criterion for linear independence over J.

LEMMA 3. *Let $f_j \in S$, $i = 1, 2, \cdots, k$. Assume the functions $f_j \mid O$ are linearly independent for some orbit $O \subseteq X$. Then the f_j are linearly independent over J.*

Proof. Assume $\sum_j g_j f_j = 0$ where $g_j \in J$, $j = 1, 2, \cdots, k$. Let D be the matrix given in Lemma 2 and let $x \in O$. Then by Lemma 2 there exists a $k \times k$ minor of D whose determinant $\alpha \in S$ is such that $\alpha(x) \neq 0$. But then there exists a neighborhood W of x such that $\alpha(y) \neq 0$ for all $y \in W$. Thus $D(y)$ has rank k and hence the $f_j \mid O_y$ are linearly independent for all $y \in W$. But since the g_j reduce to constants on any orbit O_y it follows from the relation $\sum_j g_j f_j = 0$ that the g_j vanish identically on W. This implies that the g_j vanish on X since the g_j are polynomials. Q. E. D.

1.3. For any subset $Y \subseteq X$ we will let

$$I(Y) = \{f \in S \mid f \mid Y = 0\}$$

be the ideal in S defined by Y.

Now let $P \subseteq X$ be the cone (since J^+ is homogeneous) given by

$$P = \{x \in X \mid f(x) = 0 \text{ for all } f \in J^+\}$$

Since P is defined by the ideal J^+S one knows that $I(P)$ is the radical of J^+S and that the cone P is irreducible (in the sense of algebraic geometry) and $J^+S = I(P)$ if and only if J^+S is prime.

We now give a criterion for the conditions of Lemma 1 to be satisfied.

PROPOSITION 2. *Assume* (1) *J^+S is prime and* (2) *there exists an orbit O such that $\bar{O} = P$. Then the conditions of Lemma* 1, § 1.1, *are satisfied. In particular if $S = J^+S + L$ is a direct sum where L is a graded subspace of S then the map*

$$J \otimes L \to S \tag{1.3.1}$$

given by $f \otimes g \to fg$, is an isomorphism.

Proof. Let $M \subseteq S$ be any subspace such that $M \cap J^+S = (0)$. Since $J^+S = I(P)$ it is clear that if $f_j \in M$, $j = 1, 2, \cdots, k$, are linearly independent then the $f_j \mid P$ are linearly independent. But this obviously implies that the $f_j \mid O$ are linearly independent since $\bar{O} = P$. But then the f_j are linearly independent over J by Lemma 3 and thus the result follows by (1) and (3) of Lemma 1. Q. E. D.

Remark 1. In the proof we have only used the fact that $J^+S = I(P)$ and not that J^+S is prime. However assumption (2) already implies that the cone P is irreducible (recall that G is connected) so that there is no loss in assuming that J^+S is prime.

1.4. Now assume that B is a symmetric non-singular G-invariant bilinear form on X. Then, as one knows, B induces a unique G-module ring isomorphism (also written B)

$$B\colon S_* \to S \tag{1.4.1}$$

of degree zero where $\langle \partial_x, B\partial_y \rangle = B(x,y)$ for any $x, y \in X$. Obviously for any $\partial_1, \partial_2 \in S_*$ one has

$$\langle \partial_1, B\partial_2 \rangle = \langle \partial_2, B\partial_1 \rangle. \tag{1.4.2}$$

Now let

$$J_* = \{\partial \in S_* \mid a \cdot \partial = \partial \text{ for all } a \in G\}$$

and let $J_*{}^+$ be the space of elements in J_* having zero constant term. It is obvious that

$$B(J_*) = J \text{ and } B(J_*{}^+) = J^+.$$

An element $f \in S$ is called G-harmonic in case

$$\partial f = 0$$

for all $\partial \in J_*{}^+$. Let H be the (obviously graded) space of all G-harmonic polynomials in S. By (1.1.2) and (1.1.4) it is clear that H is a G-submodule of S.

One obtains a class of G-harmonic polynomials in the following way: Let P' be the cone in S^1 defined by putting

$$P' = B\{\partial_x \in S_1 \mid x \in P\}$$

and let $H_P \subseteq S$ be the (graded) space spanned by all powers z^m, $m = 0, 1, \cdots$, for all $z \in P'$. The following proposition was proved independently by Helgason (see [10]).

Proposition 3. *One has $H_P \subseteq H$. Furthermore $S = J^+S + H$ is a G-module direct sum so that by Proposition* 1, §1.1,

$$S = JH.$$

Proof. Let $z \in P'$ so that $z = B(\partial_x)$ for $x \in P$. Let $\partial \in J_k$ where $k > 0$. We wish to show that $\partial z^m = 0$. We may assume that $m \geqq k$ and hence by

(1.1.2) it suffices to show that $\langle \partial\partial_1, z^m \rangle = 0$ for all $\partial_1 \in S_{m-k}$. But by (1.4.2)

$$\begin{aligned}\langle \partial\partial_1, z^m \rangle &= \langle \partial\partial_1, B(\partial_x{}^m) \rangle \\ &= \langle \partial_x{}^m, B(\partial\partial_1) \rangle \\ &= \langle \partial_x{}^m, ff_1 \rangle\end{aligned}$$

where $f = B\partial$ and $f_1 = B\partial_1$. But $f \in J^k$ and hence $ff_1 = g \in J^+S$. Thus $\langle \partial\partial_1, z^m \rangle = \langle \partial_x{}^m, g \rangle = m!g(x) = 0$ since $x \in P$ and $g \in I(P)$. Hence $H_P \subseteq H$.

Now let $H' \subseteq S$ be the orthocomplement of $J_*{}^+S_*$ in S. Thus if $f \in S$ then $f \in H'$ if and only if

$$\begin{aligned}\langle \partial\partial_1, f \rangle &= \langle \partial_1, \partial f \rangle \\ &= 0\end{aligned}$$

for all $\partial_1 \in S$ and $\partial \in J_*{}^+$. It follows immediately then that $H' = H$. To prove that $S = J^+S + H$ is a direct sum therefore it suffices, by dimension, (one restricts to S^k) to show that $H \cap J^+S = 0$. (One already uses that $B(J_*{}^+S_*) = J^+S$). But to show $H \cap J^+S = 0$ it suffices to show that B induces a non-singular bilinear form on $J_*{}^+S_*$. Now let K be a maximal compact subgroup of G. Then one knows there exists a real subspace $X_{\boldsymbol{R}} \subseteq X$ such that (1) $X_{\boldsymbol{R}}$ is stable under K, (2) B is positive definite on $X_{\boldsymbol{R}}$ and (3) $X = X_{\boldsymbol{R}} + iX_{\boldsymbol{R}}$ is a real direct sum. But by (2) it is obvious that B induces a positive definite bilinear form on $S_{*,\boldsymbol{R}}(X_{\boldsymbol{R}}) = S_{\boldsymbol{R}}$. But by (1) and (3) $J_*{}^+$ is the complexification of its intersection with $S_{\boldsymbol{R}}$ since G is the complexification of K. Hence $J_*{}^+S_*$ is also the complexification of its intersection with $S_{\boldsymbol{R}}$ and consequently B induces a non-singular bilinear form on $J_*{}^+S_*$. Q. E. D.

Now combining Propositions 2, § 1.3, and 3, § 1.4, we obtain the following "separation of variables" theorem.

Proposition 4. *Assume that G leaves invariant a symmetric non-singular bilinear form on X.*

Let S, H and J be, respectively, the ring of all polynomials on X, the space of G-harmonic polynomials on X and the ring of invariant polynomials on X.

Let $P \subseteq X$ be the homogeneous affine variety defined by the ideal J^+S in S.

Now assume (1) *that there exists an orbit O such that $\bar{O} = P$ and* (2) *the ideal J^+S is prime. Then the mapping*

$$J \otimes H \rightarrow S \qquad (1.4.3)$$

given by $f \otimes g \to fg$ *is a* G*-module isomorphism. Furthermore if* $P' \subseteq S^1$ *is the cone corresponding (by means of the bilinear form) to* P *and* H_P *is the subspace of* S *generated by all powers* z^m, $z \in P'$, $m = 0, 1, \cdots$, *then*

$$H = H_P. \tag{1.4.4}$$

Proof. Since G operates as algebra automorphisms of S it is obvious that (1.4.3) is a G-module map. But it is also an isomorphism by Propositions 2 and 3. Now let $H_* \subseteq S_*$ be the space generated by all powers $\partial_x{}^m$ where $x \in P$. Let B be as in (1.4.1). Then clearly $B(H_*) = H_P$. Now if $H_P{}^m \neq H^m$ then by dimension and Proposition 3 there exist a non-zero $f \in H^m$ such that $\langle \partial, f \rangle = 0$ for all $\partial \in H_m$. But then putting $\partial = \partial_x{}^m/m!$ for $x \in P$ it follows from (1.1.3) that $f(x) = 0$ for all $x \in P$. But then $f \in J^+S$ since J^+S is prime. This is a contradiction since $J^+S \cap H = (0)$. Q. E. D.

Remark 2. A familiar instance as to when the conclusion of Proposition 4 holds is the case where $X = \boldsymbol{C}^n$, $n \geqq 2$, and G is the full complex rotation group. Here J is the algebra generated by 1 and $z_1{}^2 + \cdots + z_n{}^2$, H is space of all polynomials which satisfy Laplace's equation.

$$\sum_{i=1}^{n} \frac{\partial^2 f}{\partial z_i{}^2} = 0$$

and P is the conic given by $z_1{}^2 + \cdots + z_n{}^2 = 0$. If $n \geqq 3$ one obtains the classical separation of variables theorem as a consequence of Proposition 4 since (1) $P = \bar{O}$, where O is the set (easily seen to be an orbit) of all vectors $x \in \boldsymbol{C}^n$ such that $x \in P$ and $x \neq 0$, and (2) J^+S is a prime ideal, because it is generated by $z_1{}^2 + \cdots + z_n{}^2$ and this polynomial is irreducible if $n \geqq 3$.

1.5. For any element $x \in X$ besides O_x, we may consider the orbit O_{cx} where $c \in \boldsymbol{C}^*$ is any non-zero scalar. It is obvious of course that $O_{cx} = cO_x$. Now where $P \subseteq X$ is the cone defined in § 1.3 and $x \in X$ is arbitrary put

$$P_x = \overline{\bigcup_{c \in \boldsymbol{C}^*} O_{cx}} \cap P$$

It is clear of course that $P_x \subseteq P$ is stable under the action of G. An element $x \in X$ will be called quasi-regular if

$$P_x = P$$

Remark 3. Note that if $P = \bar{O}$ for an orbit O then any element of O is quasi-regular.

Now for any subset $W \subseteq X$ let $S(W)$ be the ring of all functions on W of the form $g \mid W$ where $g \in S$. Note that if W is stable under G then $S(W)$ is a G-module with respect to the action of G given by (1.1.5) where $f \in S(W)$.

Let $x \in X$. We are particularly interested in the ring $S(O_x)$ of functions on the orbit O_x.

Obviously the map

$$(1.5.1) \qquad S \to S(O_x)$$

defined by the correspondence $f \to f \mid O_x$ is a G-module epimorphism.

PROPOSITION 5. *As in Proposition* 1, §1.1, *let* $L \subseteq S$ *be any graded subspace such that* $S = J^+S + L$ *is a direct sum. Also for any* $x \in X$ *let*

$$\gamma_x \colon L \to S(O_x)$$

be the linear map obtained by restricting (1.5.1) *to* L. *Then* γ_x *is an epimorphism.*

Assume that conditions (1) *and* (2) *of Proposition* 2, §1.3, *are satisfied. Then* γ_x *is an isomorphism for any quasi-regular element* $x \in X$.

In particular if G *leaves invariant a non-singular symmetric bilinear form on* X *and, as in Proposition* 4, §1.4, $L = H$ *is the space of* G-*harmonic polynomials on* X *then*

$$(1.5.2) \qquad \gamma_x \colon H \to S(O_x)$$

is a G-*module isomorphism for any quasi-regular element* $x \in X$.

Proof. Since J reduces to scalars on any orbit O_x it follows from Proposition 1 that γ_x maps L surjectively onto $S(O_x)$. Assume now that conditions (1) and (2) of Proposition 2 are satisfied. Let x be quasi-regular. We must show that γ_x is injective. Let $f \in L$. Since L is graded we may write $f = \sum_{i=1}^{k} c_i f_i$ where $f_i \in L$ is homogeneous of degree n_i, and the f_i, $i = 1, 2, \cdots, k$, are linearly independent.

But now since J^+S is the prime ideal corresponding to P it follows that the functions $f_i \mid P$ are linearly independent. But since $P = \bar{O}$ for an orbit O one has also that the functions $f_i \mid O$ are linearly independent. The argument of Lemma 2, § 1.2, shows that for any $y \in O$ there exists a neighborhood W of y in X such that for any $z \in W$ the functions $f_i \mid O_z$ are linearly independent. But now since $y \in P = P_x$ there exists a non-zero scalar c such that $O_{cx} = O_z$ for some $z \in W$. Hence there exists a non-zero scalar c such that the functions $f_i \mid O_{cx}$ are linearly independent. Now let

$$\mu \colon O_{cx} \to O_x$$

be the bijection defined by $y \to 1/c \cdot y$. If μ^* is then the corresponding contravariant isomorphism on functions one has

$$\mu^*(f_i \mid O_x) = 1/c^{n_i}(f_i \mid O_{cx})$$

But then since the $f_i \mid O_{cx}$ or $1/c^{n_i}(f_i \mid O_{cx})$ are linearly independent it follows that the $f_i \mid O_x$ are linearly independent. But then if $f \mid O_x$ is zero it follows that the c_i are all zero and hence f is identically zero. Thus γ_x is injective.

The isomorphism (1.5.2) is a G-module map since (1.5.1) is a G-module map. Q. E. D.

Remark 4. In the example of Remark 2, § 1.4, not that $x \in \boldsymbol{C}^n$ is quasi-regular if and only if $x \neq 0$.

Thus in that example one has that

$$\gamma_x \colon H \to S(O_x) \tag{1.5.3}$$

is an isomorphism for any $x \neq 0$. Hence all $S(O_x)$ where $x \neq 0$ are equivalent as G-modules.

1.6. In order to apply Propositions 3, 4 in § 1.4 or Proposition 5, § 1.5, one needs to know that J^+S is a prime ideal. In general this appears to be difficult to ascertain even if one knows J completely. (Except of course if J has only one ring generator, as in the example of Remark 2.) However we will now observe (Proposition 6, § 1.6) that in the familiar case when J is a polynomial ring the question of the primeness of J^+S reduces to a more manageable one.

Throughout much of the remainder of the paper we will need to draw upon techniques and results in algebraic geometry. Our reference for all definitions will be [3] where for us the fixed algebraically closed field is of course $\boldsymbol{C}$. We recall in particular that by definition, among other things, a variety is irreducible in its Zariski topology.

To avoid confusion of terminology we remark here that the words open, closed, closure and denseness, etc. will have their usual Hausdorff topological meaning unless stated otherwise (i.e., unless preceded by "Zariski").

If $f_i \in S$, $i = 1, 2, \cdots, l$, are arbitrary let $(f_1, \cdots, f_l)$ denote the ideal in S that they generate. If $Y \subseteq X$ is a Zariski closed subvariety of X of dimension $n - l$ then we recall that Y is called a complete intersection in case

$$I(Y) = (f_1, \cdots, f_l)$$

for some $f_i \in I(Y)$, $i = 1, \cdots, l$.

Now for any $f \in S$ and $x \in X$ let $(df)_x$ be the value of the differential df

at x. If $f_i \in S$, $i = 1, \cdots, l$, then one knows that the $(df_i)_x$ are linearly independent if and only if the $n \times l$ matrix $(\partial_{x_j} f_i)(x)$, $j = 1, \cdots, n$, has rank l where the x_j is any basis of X.

The following lemma in one form or another is well known in algebraic geometry.

LEMMA 4. *Let $f_i \in S$, $i = 1, 2, \cdots, l$, and let Y be the Zariski closed set given by*

$$Y = \{x \in X \mid f_i(x) = 0, i = 1, \cdots, l\}.$$

Assume (1) *Y is a subvariety of X (that is, assume Y is irreducible) and* (2) *there exists $y \in Y$ such that $(df_i)_y$, $i = 1, 2, \cdots, l$, are linearly independent. Then Y is a subvariety of* $\dim n - l$. *Furthermore*

$$I(Y) = (f_1, \cdots, f_l) \tag{1.6.1}$$

so that (a) *$(f_1, \cdots, f_l)$ is a prime ideal and* (b) *Y is a complete intersection.*

Proof. Let S_y be the local ring of X at y. Let $I = (f_1, \cdots, f_l)$. Since the $(df_i)_y$ are linearly independent the f_i may be included in a complete system of uniformizing variables at y. Thus by [3], Proposition 3, p. 219, IS_y is a prime ideal of S_y. Furthermore since $I(Y)$ is the radical of I in S it is clear that IS_y is the ideal of Y at y (that is, $I(Y)S_y = IS_y$) so that, by the same reference, $\dim Y = n - l$ and $IS_y \cap S = I(Y)$. To prove (1.6.1) it suffices to show that I is primary for $I(Y)$ since in that case $IS_y \cap S = I$ (a primary ideal is equal to the contraction of its extension; see [13], Theorem 19, p. 228). But I is primary by MacCaulay's theorem (see [19], p. 203) which asserts that there are no embedded primes for I so that $I(Y)$ is the only associated prime ideal. Q.E.D.

Now we recall that G is a connected algebraic reductive group. Hence G has the structure of an affine variety. (It is Zariski closed in $\operatorname{Aut} X$ but not necessarily Zariski closed in $\operatorname{End} X$.) Since (1.2.1) is obviously a morphism it follows that any orbit $O \subseteq X$ is an irreducible constructible set. In fact since O is epais ([3], Proposition 4, p. 95) and G operates transitively on it, it follows that O is a subvariety ([3], Theorem 5, p. 68) of X. It follows therefore that its (usual) closure $\bar{O}$ is a Zariski closed subvariety of the same dimension as O.

As an application of Lemma 4, § 1.6, we have

PROPOSITION 6. *Assume J, as a ring, is generated by l homogeneous algebraically independent polynomials u_i, $i = 1, 2, \cdots, l$.*

Now let $\xi \in \mathbf{C}^l$, $\xi = (\xi_1, \cdots, \xi_l)$, *be an arbitrary complex* l*-tuple and let*

$$P(\xi) = \{x \in X \mid u_i(x) = \xi_i, i = 1, 2, \cdots, l\}$$

Assume $P(\xi)$ *is not empty and there exists an orbit* $O(\xi)$ *such that*

$$P(\xi) = \overline{O(\xi)}. \tag{1.6.2}$$

Then $P(\xi)$ *is a Zariski closed subvariety (of* X*) of dimension* $n - l$. *Furthermore the ideal* $(u_1 - \xi_1, \cdots, u_l - \xi_l)$ *in* S *is prime if and only if there exists* $y \in P(\xi)$ *such that the* $(du_i)_y$ *are linearly independent. In such a case* $P(\xi)$ *is a complete intersection and the set* $P(\xi)_s$ *of simple points on* $P(\xi)$ *is given by*

$$P(\xi)_s = \{x \in P(\xi) \mid (du_i)_x, i = 1, 2, \cdots, l, \text{ are linearly independent}\}. \tag{1.6.3}$$

Proof. Since $O(\xi)$ is irreducible it follows from (1.6.2) that $P(\xi)$ is a subvariety of the same dimension as $O(\xi)$. Now by [3], Corollary, p. 102, it is clear that

$$\dim P(\xi) \geqq n - l.$$

To prove that $\dim P(\xi) = n - l$ it suffices to show that

$$\dim O \leqq n - l \tag{1.6.4}$$

for any orbit O.

Let m be the maximum of the dimensions of all orbits. Let $\mathfrak{u} \subseteq \operatorname{End} X$ be the Lie algebra of G and for each $x \in X$ let ϕ_x be the homomorphism of $\mathfrak{u}$ into X given by $\phi_x(z) = z(x)$. It is then obvious that $\operatorname{rank} \phi_x = \dim O_x$. By consideration of minors it is then clear that

$$U = \{x \in X \mid \dim O_x = m\}$$

is a non-empty Zariski open subset of X. Now let V be the Zariski open subset of X consisting of all $x \in X$ such that the $(du_i)_x$ are linearly independent. To see that $V \cap U$ is not empty it clearly suffices to see that V is not empty. But this is a known consequence of algebraic independence. Indeed if $u_j \in S$, $j = l+1, \cdots, n$, are chosen so that u_i, $i = 1, 2, \cdots, n$, is a transcendental basis of S then each element of a coordinate basis z_j, $j = 1, 2, \cdots, n$ of X is algebraically dependent upon the u_i. Hence on a non-empty Zariski open set each dz_j is in the span of the du_i. This proves that V and hence $V \cap U$ is not empty. Now let $x \in V \cap U$ and let W be the $n - l$ dimensional variety (see [3], Proposition 3, p. 219) containing x whose prime ideal at x is IS_x where $I = (u_1 - u_1(x), \cdots, u_l - u_l(x))$ and S_x is the local ring at x.

Since, obviously, $O_x \subseteq W$ and dim $O_x = m$ it follows that $m \leqq n - l$ and this proves (1.6.4) and hence $\dim P(\xi) = n - l$.

Now if the ideal $(u_1 - \xi_1, \cdots, u_l - \xi_l)$ is prime it must equal $I(P(\xi))$ and hence $(df)_x$ for any $x \in P(\xi)$ and $f \in I(P(\xi))$ lies in the span of the $(d(u_i - \xi_i))_x = (du_i)_x$. But since $\dim P(\xi) = n - l$ one immediately obtains (1.6.3) by Zariski's criterion and since $(P(\xi))_s$ is not empty there exists $y \in P(\xi)$ such that the $(du_i)_y$ are linearly independent. Conversely if the latter holds $(u_1 - \xi_1, \cdots, u_l - \xi_l)$ is prime by Lemma 4, § 1.6, and hence $P(\xi)$ is a complete intersection. Q. E. D.

For us Proposition 4, § 1.4, will be put into effect by

PROPOSITION 7. *Assume J, as a ring, is generated by l algebraically independent homogeneous polynomials u_i, $i = 1, 2, \cdots, l$.*

Assume also that there exists an orbit O such that $P = \bar{O}$. Then P is a subvariety of dimension $n - l$. Moreover J^+S is prime if and only if there exists $y \in P$ such that $(du_i)_y$, $i = 1, 2, \cdots, l$, are linearly independent.

Proof. This is just the special case $\xi = 0$ of Proposition 6. Q. E. D.

2. Normality and the closure of an orbit. 1. If Y is any variety we let $R(Y)$ denote the ring of everywhere defined rational functions on Y.

Now let C be the ring of all holomorphic functions on G. If $f \in C$, $a \in G$, then the left (resp. right) translate $a \cdot f$ (resp. $f \cdot a$) of f by a is the function defined by putting $(a \cdot f)(b) = f(a^{-1}b)$ (resp. $(f \cdot a)(b) = f(ba^{-1})$). It is obvious that $a \cdot f \in C$ (resp. $f \cdot a \in C$) for all $f \in C$, $a \in G$.

One knows that $R(G)$ is a subring of C which in fact may be given by (see [9], Theorem 5.2)

$R(G) = \{f \in C \mid$ space spanned by all $a \cdot f$, $a \in G$, is finite dimensional$\}$

It is obvious that $R(G)$ is stable under left and right translations.

Now let D denote the set of equivalence classes of all irreducible rational (equivalently, holomorphic) finite dimensional representations of G. For each $\lambda \in D$ choose a fixed irreducible representation.

$$\nu^\lambda\colon G \to \operatorname{Aut} V^\lambda$$

belonging to λ. The dual space to V^λ will be denoted by V_λ and the irreducible representation of G on V_λ contragradient to ν^λ will be denoted by ν_λ.

If M is any G-module we will let M^λ denote the set of all vectors in M which transform according to the irreducible representation ν^λ. Since G is assumed to be reductive one knows that if each vector in M generates a finite dimensional cyclic G-module then M is in fact a direct sum of the M^λ. In particular regarding $R(G)$ as a G-module under left translation one has that

$$R(G) = \sum_{\lambda \in D} R^\lambda(G)$$

is a direct sum. Since $\nu^\lambda(G)$ generates $\operatorname{End} V^\lambda$ one can be very explicit about the structure of $R^\lambda(G)$. In fact let d_λ be the dimension of V_λ and let v_i and v'_j, $i, j = 1, 2, \cdots, d_\lambda$, be, respectively, a basis of V^λ and a basis of its dual space V_λ. Now let $g_{ij}{}^\lambda$ be the function on G defined by

$$g_{ij}{}^\lambda(b) = \langle v_i, \nu_\lambda(b) v'_j \rangle. \tag{2.1.1}$$

Then one knows that the $d_\lambda{}^2$ functions defined in this way form a basis of $R^\lambda(G)$. In particular $R^\lambda(G)$ is finite dimensional and in fact

$$\dim R^\lambda(G) = d_\lambda{}^2.$$

Now assume that $F \subseteq G$ is an algebraic (and hence closed, Lie) subgroup. Then one knows that G/F (space of left coset $aF, a \in G$) has the structure of an irreducible algebraic variety where

$$\dim G/F = \dim G - \dim F$$

and the ring $R(G/F)$ of everywhere defined rational functions on G/F may be identified, in the obvious way, with the set of elements in $R(G)$ that are right invariant under F.

Now for any $\lambda \in D$ let $V_\lambda{}^F$ be the space of all vectors in the dual space V_λ to V^λ that are fixed under all transformations on V_λ of the form $\nu_\lambda(a)$ where $a \in F$. Put

$$d_\lambda{}^F = \dim V_\lambda{}^F.$$

Now it is obvious that $R(G/F)$ is a G-submodule (by left translations) of $R(G)$. It is furthermore obvious that

$$R^\lambda(G/F) \subseteq R^\lambda(G).$$

The following is a special case of an algebraic Frobenius reciprocity theorem. We prove for it for completeness.

PROPOSITION 8. *For $i = 1, \cdots, d_\lambda$ and $j = 1, \cdots, d_\lambda{}^F$ let v_i be a basis of V^λ and let w'_j be a basis of $V_\lambda{}^F$. Also let h_{ij} be the function on G given by*

$$h_{ij}(b) = \langle v_i, \nu_\lambda(b) w'_j \rangle.$$

Then $h_{ij} \in R^\lambda(G/F)$ and in fact the $d_\lambda d_\lambda{}^F$ functions defined in this way are a basis of $R^\lambda(G/F)$.

Thus

$$\dim R^\lambda(G/F) = d_\lambda d_\lambda^F$$

so that ν^λ *occurs with multiplicity* d_λ^F *in* $R(G/F)$. *Furthermore*

$$R(G/F) = \sum_{\lambda \in D} R^\lambda(G/F) \tag{2.1.2}$$

is a direct sum.

Proof. The decomposition (2.1.2) is obvious since each element of $R(G)$ is an element of $R(G)$ and hence generates a finite dimensional subspace under the action (left translation) of G.

Furthermore it is also obvious that the $d_\lambda d_\lambda^F$ functions h_{ij} defined in the proposition are in $R^\lambda(G/F)$ and (see (2.1.1)) are linearly independent. To prove the proposition therefore one simply has to show that every element of $R^\lambda(G)$ invariant under right translation by elements of F is in the span of the h_{ij}. Assume that $g \in R^\lambda(G)$ and $g \cdot a = g$ for all $a \in F$. Let g_{ij}^λ be as in (2.1.1) (a basis of $R^\lambda(G)$). Write $g = \sum g_{ij}^\lambda c_{ji}$ where $c_{ij} \in \boldsymbol{C}$ defines a matrix and hence, relative to the basis v'_j, a linear transformation α of V_λ. It suffices only to show that $\operatorname{Im} \alpha \subseteq V_\lambda^F$. But the condition on g implies that $(\nu_\lambda(a) - 1)\alpha = 0$ for all $a \in F$. This proves $\operatorname{Im} \alpha \subseteq V_\lambda^F$. Q. E. D.

Remark 5. A case of importance for us is the case where $F = A$ is a Cartan subgroup of G. Here V_λ^A is just the zero weight subspace, corresponding to A, of V_λ. To make it independent of A we will put $l_\lambda = d_\lambda^A$ so that $l_\lambda =$ multiplicity of the zero weight of ν_λ (2.1.3).

Remark 6. Since one knows that the multiplicity of any weight μ for ν_λ is equal to the multiplicity of $-\mu$ for ν^λ it follows that l_λ is also the multiplicity of the zero weight of ν^λ.

2.2. Now we wish to apply the considerations of § 2.1 to the case where $F = G^x$ for any $x \in X$. See § 1.2. By Proposition 8 any question as to the complete reduction of $R(G/G^x)$ as a G-module becomes a question in the finite dimensional representation theory of G and how such representations restrict to G^x.

Now, as we observed in § 1.6, the orbit O_x is a subvariety of X. Furthermore the bijection $\beta_x\colon G/G^x \to O_x$ induced by β'_x is an algebraic isomorphism (this follows easily from the transitivity of G together with [3], Corollary, p. 53 and Corollary 2, p. 90. (See also [1], § 2.2.) Thus if $R(O_x)$ is regarded as a G-module, using the action of G in O_x, it follows that β_x induces a G-module and ring isomorphism

$$R(G/G^x) \to R(O_x). \tag{2.2.1}$$

Now we recall that $S(O_x)$ is the ring of functions on O_x obtained by restricting S (the ring of polynomials on X) to O_x. Since β_x is a morphism one obviously has

$$S(O_x) \subseteq R(O_x)$$

for any $x \in X$ and in fact it is clear that $S(O_x)$ is a G-submodule of $R(O_x)$. Unlike $R(O_x)$ whose G-module structure is completely determined by Proposition 8 because (2.2.1) is a G-module isomorphism, in the general case it seems (to us) to be quite difficult to describe how $S(O_x)$ decomposes as a G-module.

In many instances, however, $S(O_x) = R(O_x)$ (and hence, in such cases, one knows the G-module structure of $S(O_x)$). Indeed, in the general case since $\bar{O}_x$ is Zariski closed in X one has

$$S(\bar{O}_x) = R(\bar{O}_x). \tag{2.2.2}$$

Thus

$$S(O_x) = R(O_x) \text{ if } O_x = \bar{O}_x. \tag{2.2.3}$$

Remark 7. In the example of Remark 2, § 1.4, one depends upon the equality $S(O_x) = R(O_x)$ for a particular x in order to solve the Dirichlet problem in $\boldsymbol{R}^n$. Indeed let $x \in \boldsymbol{R}^n$ where $(x, x) = \alpha > 0$ and let f be a continuous function on the sphere $S^{n-1} = O_x \cap \boldsymbol{R}^n$ of radius $\sqrt{\alpha}$. The problem is to extend f as a harmonic function f' defined in the interior of S^{n-1}. To do this one expands f

$$f = \sum_{\lambda \in D} c_\lambda f_\lambda,$$

using some limiting process (e.g., L_2), as an infinite sum of spherical harmonics f_λ. That is, here $c_\lambda \in \boldsymbol{C}$ and $f_\lambda = g_\lambda \mid S^{n-1}$ where

$$g_\lambda \in R^\lambda(O_x)$$

However since $R(O_x) = S(O_x)$ it follows from (1.5.3) that there exists a unique harmonic polynomial $h_\lambda \in H$ on $\boldsymbol{C}^n$ such that $h_\lambda \mid O_x = g_\lambda$. One then puts

$$f' = \sum_{\lambda \in D} c_\lambda h'_\lambda$$

where h'_λ is the restriction of h_λ to the interior of S^{n-1}.

Now it is not necessarily true, in general, that $S(O_x) = R(O_x)$. For example let X be the m^2 dimensional space of all complex $m \times m$ matrices and G is the general linear group $Gl(m, \boldsymbol{C})$ regarded as operating on X by left matrix multiplication. Then if x is the identity matrix O_x is isomorphic to

$Gl(m,\boldsymbol{C})$. But $S(O_x) \neq R(O_x)$ since in particular if $f(a) = (\det a)^{-1}$ for $a \in G$ then $f \in R(O_x)$ but $f \notin S(O_x)$.

The equality $S(O_x) = R(O_x)$ in the example of Remark 7 when $(x,x) > 0$ may be established either using the fact that O_x is closed (see (2.2.3)) or by applying the Stone-Weierstrass theorem to both $S(O_x)$ and $R(O_x)$ restricted to $O_x \cap \boldsymbol{R}^n$. These methods also work more generally in case $(x,x) \neq 0$. However, they do not apply to O_x where $x \neq 0$ and $(x,x) = 0$. Nevertheless it is still true in this case that $R(O_x) = S(O_x)$. The more powerful tool (and the one that will be required in § 5.1) needed to establish the equality for this case is given in the next proposition.

For any $x \in X$ let C_x be the Zariski closed subset of X defined by taking the complement of O_x in $\bar{O}_x$. If we put

$$\operatorname{codim} C_x = \dim \bar{O}_x - \dim C_x$$

then of course one has

$$\operatorname{codim} C_x \geqq 1.$$

An affine variety Y is called normal in case the ring $R(Y)$ is integrally closed in its quotient field.

PROPOSITION 9. *Let* $x \in X$. *Assume* (1) *that* $\bar{O}_x$ *is a normal variety and* (2) *that* $\operatorname{codim} C_x \geqq 2$. *Then*

$$S(O_x) = R(O_x).$$

Proof. If Y is any variety let $Q(Y)$ denote the field of all rational functions on Y. In any $f \in Q(O_x)$ let $\bar{f}$ denote its image in $Q(\bar{O}_x)$ under the canonical isomorphism $Q(O_x) \to Q(\bar{O}_x)$ defined by extension.

Now let $f \in R(O_x)$. Then obviously $\bar{f} \in Q(\bar{O}_x)$ is defined at every point of O_x. Thus if T is the set of points of $\bar{O}_x$ where f is not defined then $T \subseteq C_x$. Since $\operatorname{codim} C_x \geqq 2$ one also must have $\operatorname{codim} T \geqq 2$. But now for a normal affine variety Y one knows (see [3], Proposition 2, p. 166 and 10, p. 134. Also Corollary, p. 135), that if $g \in Q(Y)$ then either $g \in R(Y)$ or the set of points where g is not defined has codimension 1. Since $\bar{O}_x$ is assumed to be normal it follows that the first alternative must hold for $\bar{f}$. That is, $\bar{f}$ is everywhere defined on $\bar{O}_x$. But then $\bar{f}$, as a function on $\bar{O}_x$, is the restriction of a polynomial on X to $\bar{O}_x$. (See (2.2.2).) But then this is certainly true of f so that $f \in S(O_x)$. Q. E. D.

Remark 8. Proposition 9 is stronger than the criterion $O_x = \bar{O}_x$ for insuring $S(O_x) = R(O_x)$. In fact if $O_x = \bar{O}_x$ (in which case we may take (2) to be trivially satisfied) then C_x is empty and $\bar{O}_x$ is non-singular. But since non-singularity implies normality the conditions of Proposition 9 are satisfied in case O_x is closed.

The proof that $\bar{O}_x$ is normal for the example of Remark 2 where $(x, x) = 0$, $x \neq 0$, and $d \geqq 3$ follows from a result of Seidenberg (see § 5.1).

3. The orbit structure for the adjoint representation. 1. Let $\mathfrak{g}$ be a complex reductive Lie algebra of dimension n. Then $\mathfrak{g}$ is a Lie algebra direct sum

$$\mathfrak{g} = \mathfrak{z} + [\mathfrak{g}, \mathfrak{g}] \tag{3.1.1}$$

where $\mathfrak{z}$ is the center of $\mathfrak{g}$. The commutator $[\mathfrak{g}, \mathfrak{g}]$ is, as one knows, the maximal semi-simple ideal in $\mathfrak{g}$.

A subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ is said to be reductive in $\mathfrak{g}$ if the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$ is completely reducible. Such a subalgebra is necessarily reductive (in itself).

Let $\mathfrak{g}^x$, for any $x \in \mathfrak{g}$, denote the centralizer of x. An element $x \in \mathfrak{g}$ is called semi-simple if $\operatorname{ad} x$ is diagonalizable. One knows that $\mathfrak{g}^x$ is reductive in $\mathfrak{g}$ for any semi-simple element $x \in \mathfrak{g}$ (see e.g. Theorem 7 in [11]).

An element $x \in \mathfrak{g}$ is called nilpotent in case (1) $x \in [\mathfrak{g}, \mathfrak{g}]$ and (2) $\operatorname{ad} x$ is a nilpotent endomorphism.

Remark 9. If $x \in \mathfrak{a} \subseteq \mathfrak{g}$ where $\mathfrak{a}$ is reductive in $\mathfrak{g}$ then x is semi-simple (resp. nilpotent) with respect to $\mathfrak{a}$ if and only if it is semi-simple (resp. nilpotent) with respect to $\mathfrak{g}$. The proof of these statements are immediate consequences of the representation theory of reductive Lie algebras.

Now one knows that the most general element $x \in \mathfrak{g}$ may be uniquely written

$$x = y + z \tag{3.1.2}$$

where y is semi-simple, z is nilpotent and $[y, z] = 0$. We will speak of y and z, respectively, as the semi-simple and nilpotent components of x. See [11], Theorem 6.

Remark 10. If $x \in \mathfrak{a} \subseteq \mathfrak{g}$ where $\mathfrak{a}$ is a subalgebra reductive in $\mathfrak{g}$ then by Remark 9 the decomposition (3.1.2) formed in $\mathfrak{g}$ is the same as the decomposition (3.1.2) formed in $\mathfrak{a}$.

In particular given the decomposition (3.1.2) one should observe that z is not only nilpotent in $\mathfrak{g}$ but also in the "reductive in $\mathfrak{g}$" subalgebra $\mathfrak{g}^y$. In particular then

$$z \in [\mathfrak{g}^y, \mathfrak{g}^y]. \tag{3.1.3}$$

Conversely if $y \in \mathfrak{g}$ is semi-simple and z is nilpotent in $\mathfrak{g}^y$ and one puts $x = y + z$ then y and z are respectively the semi-simple and nilpotent components of x.

3.2. We wish to apply the considerations of §§1 and 2 to the case where $X = \mathfrak{g}$ and $G \subseteq \operatorname{Aut} \mathfrak{g}$ is the adjoint group of $\mathfrak{g}$. Thus not only is G a connected algebraic reductive linear group but in fact G is then a semi-simple Lie group whose Lie algebra is isomorphic to $[\mathfrak{g}, \mathfrak{g}]$.

In this case we observe that the orbit O_x defined by any $x \in \mathfrak{g}$ is just the set of elements of $\mathfrak{g}$ that are conjugate to x.

If $\mathfrak{a} \subseteq \mathfrak{g}$ is any subalgebra then under the adjoint representation $\mathfrak{a}$ corresponds to a Lie subgroup $A \subseteq G$. Indeed A is the group generated by all $\exp \operatorname{ad} x$ where x ranges over $\mathfrak{a}$. In this way $\mathfrak{g}^x$ clearly corresponds to the identity component of the algebraic subgroup G^x.

We recall that an element $x \in \mathfrak{g}$ is semi-simple if and only if x may be embedded in a Cartan subalgebra (C. S.) of $\mathfrak{g}$. Equivalently $x \in \mathfrak{g}$ is semi-simple if and only if $\mathfrak{g}^x$ contains a C. S. of $\mathfrak{g}$.

The following lemma is known. We will prove it for completeness and also because, as noted in Remark 11 below, the proof may be used to give a more general result.

Lemma 5. *Assume $x \in \mathfrak{g}$ is semi-simple. Then* (1) *G^x is connected and* (2) *O_x is closed in $\mathfrak{g}$.*

Proof. We first show G^x is connected. Let $b \in G^x$. Then by Theorem 2, p. 108, in [6], one knows that b may be uniquely written

$$b = a \exp \operatorname{ad} y \tag{3.2.1}$$

where $a \in G$ is diagonalizable and $y \in \mathfrak{g}$ is nilpotent and $a(y) = y$. Put $c_t = \exp t \operatorname{ad} x$. Then $b = c_t b c_t^{-1} = (c_t a v_t^{-1}) \exp \operatorname{ad} c_t(y)$. By the uniqueness of the decomposition (3.2.1) it follows that $a = c_t a c_t^{-1}$ and $c_t(y) = y$. Hence $a \in G^x$ and $y \in \mathfrak{g}^x$. But then b is "connected" to a in G^x by means of the curve $a \exp s \operatorname{ad} y$, $s \in \boldsymbol{R}$. Thus we may assume that b is diagonalizable. But now by Theorem 10, p. 117 in [6], if $\mathfrak{g}^b$ is the Lie subalgebra of all y such that $b(y) = y$ then $\mathfrak{g}^b$ contains a C. S. $\mathfrak{h}$ of $\mathfrak{g}$ and if $\mathfrak{h}$ is any C. S. in $\mathfrak{g}^b$ then $b = \exp \operatorname{ad} z$ for some $z \in \mathfrak{h}$.

But now $x \in \mathfrak{g}^b$ and since $\operatorname{ad} x \mid \mathfrak{g}^b$ is semi-simple there exists a C. S. $\mathfrak{h}$ such that $x \in \mathfrak{h} \subseteq \mathfrak{g}^b$. But $b = \exp \operatorname{ad} z$ for some $z \in \mathfrak{h}$. However since $\mathfrak{h} \subseteq \mathfrak{g}^x$ it follows that b may be joined to the identity in G^x by a curve; indeed one uses the curve $\exp t \operatorname{ad} z$. Hence G^x is connected.

To show that O_x is closed let $\mathfrak{h}$ be a Cartan subalgebra such that $x \in \mathfrak{h}$. By the Iwasawa decomposition we may write $G = KMH_0$ where K and M are connected Lie groups which are, respectively, compact and unipotent (an endomorphism u is called unipotent if $u - 1$ is nilpotent; a group is called unipotent if all its elements are unipotent) and H_0 is an abelian Lie group corresponding to a subalgebra of $\mathfrak{h}$. Since $x \in \mathfrak{h}$ it follows then that x is fixed under H_0. Thus $O_x = KMx$. We have proved (unpublished) that any orbit of a connected unipotent Lie group is closed. Rosenlicht [14] has generalized this to the case of a field of arbitrary characteristic. Thus we may use the reference [14] to establish that Mx is a closed subset of $\mathfrak{g}$. But since O_x is obtained by applying a compact group to a closed set it follows easily that O_x is closed.

Remark 11. Another proof that O_x is closed if x is semi-simple follows from Theorem 4, § 3.8. In fact one sees there that O_x is closed if and only if x is semi-simple. This observation was also made in [1]. Note however the proof given above, that O_x is closed when x is semi-simple generalizes and shows that the orbit of any zero weight vector for any representation of G is closed.

As a consequence of the connectivity of G^x for x semi-simple one has

LEMMA 6. *Assume $x \in \mathfrak{g}$ is semi-simple. Then $\mathfrak{g}^x$ is stable under G^x and the restriction of G^x to $\mathfrak{g}^x$ is the adjoint group of $\mathfrak{g}^x$.*

Proof. It is trivial that $\mathfrak{g}^x$ is stable under G^x. Furthermore as we have observed in the beginning of this section the identity component of G^x corresponds to $\mathfrak{g}^x$ under the adjoint representation of $\mathfrak{g}$ and hence its restriction to $\mathfrak{g}^x$ is the adjoint group of $\mathfrak{g}^x$. But G^x is connected by Lemma 5. Q. E. D.

3.3. Now for the case at hand S is just the symmetric algebra $S^*(\mathfrak{g})$ over the dual space to $\mathfrak{g}$. The well known description of the ring of invariants J given below is due to Chevalley.

If l is the rank of $\mathfrak{g}$ then J is generated by l algebraically independent homogeneous polynomials. That is, there exist homogeneous elements $u_i \in J$, $i = 1, \cdots, l$, such that if $\boldsymbol{C}[Y_1, \cdots, Y_l]$ denotes the polynomial ring, over $\boldsymbol{C}$, in l indeterminates and

$$\boldsymbol{C}[Y_1, \cdots, Y_l] \rightarrow J \tag{3.3.1}$$

is the homomorphism given by $p(Y_1, \cdots, Y_l) \rightarrow p(u_1, \cdots, u_l)$ then (3.3.1) is an isomorphism. Moreover, if we write $\deg u_i = m_i + 1$ then the integers

m_i, called the exponents of $\mathfrak{g}$, are those special integers such that $\prod_{j=1}^{l}(1+t^{2m_j+1})$ is the Poincaré polynomial of $\mathfrak{g}$.

Throughout we will assume that the u_i are ordered so that

$$m_1 \leqq m_2 \leqq \cdots \leqq m_l$$

We will refer to the u_i, $i=1,2,\cdots,l$, as the primitive invariants.

Remark 12. One knows that the primitive invariants and even the l-dimensional space they span is not unique. However, in § 5.4 in connection with G-harmonic polynomials one normalizes the space they span in a natural way. See Remark 26, § 5.4.

We now define a mapping

$$u\colon \mathfrak{g} \rightarrow \boldsymbol{C}^l \tag{3.3.2}$$

by putting

$$u(x) = (u_1(x), \cdots, u_l(x)).$$

It is obvious that u is a morphism.

Now let $\mathcal{O}$ be the set of all obits $O \subseteq \mathfrak{g}$. Since u obviously maps any orbit into a point it is clear that u induces a map

$$\eta\colon \mathcal{O} \rightarrow \boldsymbol{C}^l$$

Now if $\mathfrak{u} \subseteq \mathfrak{g}$ is any subset stable under the action of G it is obvious that $\mathfrak{u}$ is a union of orbits. Let

$$\mathcal{O}_{\mathfrak{u}} = \{O \in \mathcal{O} \mid O \subseteq \mathfrak{u}\}$$

and we will let $\eta_{\mathfrak{u}}$ be the restriction of η to $\mathcal{O}_{\mathfrak{u}}$.

Let $\mathfrak{s}$ be the set of all semi-simple elements in $\mathfrak{g}$. Obviously $\mathfrak{s}$ is stable under G so that we may consider the case where $\mathfrak{u}=\mathfrak{s}$.

Now it is easy to see that η is not one-one, that is it does not separate all orbits. One observes, however, that not only does η separate the orbits in $\mathfrak{s}$ but also that $\eta_{\mathfrak{s}}$ is a surjection. The following proposition is no doubt known. We prove it for completeness.

PROPOSITION 10. *Let $\mathfrak{s}$ be the set of all semi-simple elements in $\mathfrak{g}$. Then the map*

$$\eta_{\mathfrak{s}}\colon \mathcal{O}_{\mathfrak{s}} \rightarrow \boldsymbol{C}^l$$

induced by u (see (3.3.2)) is a bijection.

Proposition 10 permits us to parameterize $\mathcal{O}_{\mathfrak{s}}$ by all complex l-tuples. In order to prove Proposition 10 we need some further notation and Lemma 7 below.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ regarded as fixed once and for all. Let W be the Weyl group of $\mathfrak{g}$ regarded as operating in $\mathfrak{h}$. Let $\Delta \subseteq S^1(\mathfrak{h})$ be the set of roots and let $\Delta_+ \subseteq \Delta$ be a system of positive roots fixed once and for all.

An element $x \in \mathfrak{g}$ is called regular if $\mathfrak{g}^x$ is a Cartan subalgebra. If $x \in \mathfrak{h}$ one knows that x is regular if and only if $\langle x, \phi \rangle \neq 0$ for all $\phi \in \Delta$. Now let

$$u_{\mathfrak{h}} : \mathfrak{h} \to \boldsymbol{C}^l$$

be the restriction of u to $\mathfrak{h}$.

LEMMA 7. *The map $u_{\mathfrak{h}}$ is proper. (That is, the inverse image of any compact set is compact).*

Proof. Let $\pi : \mathfrak{g} \to \operatorname{End} V$ be a faithful completely reducible representation of $\mathfrak{g}$ and let $m = \dim V$. For any positive number k let r_k be a positive number such that for any monic polynomial $Y^m + \sum_{i=0}^{m-1} c_i Y^i = p(Y)$ in the indeterminate Y, where $c_i \in \boldsymbol{C}$, one has $|c_i| \leqq k$, $i = 0, 1, \cdots, m-1$, implies $|\lambda| \leqq r_k$ for any root λ of $p(Y)$. In fact, it suffices to take $r_k = mk + 1$.

Now let $f_i \in J$ be the invariant polynomials defined so that

$$\det(Y - \pi(x)) = Y^m + \sum_{i=0}^{m-1} f_i(x) Y^i \tag{3.3.3}$$

for any $x \in \mathfrak{g}$. Now there exist unique polynomials

$$p_i(Y_1, \cdots, Y_l) \in C[Y_1, \cdots, Y_l], \ i = 0, 1, \cdots, m-1,$$

so that $f_i = p_i(u_1, \cdots, u_l)$. Thus regarding $C[Y_1, \cdots, Y_l]$ as the polynomial ring on $\boldsymbol{C}^l$ it follows that

$$f_i(x) = p_i(u_{\mathfrak{h}}(x)) \tag{3.3.4}$$

for any $x \in \mathfrak{h}$.

Now let $E \subseteq \boldsymbol{C}^l$ be any compact set. We wish to show that $u_{\mathfrak{h}}^{-1}(E)$ is compact. Let

$$k = \sup_{\substack{\xi \in E \\ i = 0, 1, \cdots, m-1}} |p_i(\xi)|.$$

It follows therefore from (3.3.4) that $|f_i(x)| \leqq k$ for all $x \in u_{\mathfrak{h}}^{-1}(E)$. Hence if λ is a root of (3.3.3) it follows that $|\lambda| \leqq r_k$ for any $x \in u_{\mathfrak{h}}^{-1}(E)$.

Now if $\Delta(\pi) \subseteq S^1(\mathfrak{h})$ is the set of weights of π and we put

$$|x| = \max_{\psi \in \Delta(\pi)} |\psi(x)|$$

then since π is faithful it is clear that $|x|$ defines a norm on the space $\mathfrak{h}$. But for any $\psi \in \Delta(\pi)$, $\lambda = \psi(x)$ is a root of (3.3.3). Hence $|x| \leqq r_k$ for any $x \in u_{\mathfrak{h}}^{-1}(E)$. That is, $u_{\mathfrak{h}}^{-1}(E)$ lies in the ball $|x| \leqq r_k$ and consequently is compact. Q. E. D.

Proof of Proposition 10. Let x_i, $i = 1, 2, \cdots, l$, be a basis of $\mathfrak{h}$ and let $r = \operatorname{Card} \Delta_+$. It is then a well-known result that (the generalization from the semi-simple to the reductive case is trivial)

$$\det \partial_{x_i} u_j \mid \mathfrak{h} = c \prod_{\phi \in \Delta_+} \phi \tag{3.3.5}$$

where c is a non-zero scalar. See e.g. [17]. But

$$c \prod_{\phi \in \Delta_+} \phi(x) \neq 0 \text{ if and only if } x \text{ is regular.} \tag{3.3.6}$$

Hence the Jacobian (3.3.5) of $u_{\mathfrak{h}}$ does not vanish identically on $\mathfrak{h}$ and consequently the Zariski closure of $u_{\mathfrak{h}}(\mathfrak{h})$ equals $\boldsymbol{C}^l$. But since $u_{\mathfrak{h}}$ is a morphism $u_{\mathfrak{h}}(\mathfrak{h})$ contains a Zariski open subset of its closure; that is, a Zariski open subset of $\boldsymbol{C}^n$. But any Zariski open set is dense in the usual topology. Thus in the usual sense

$$\overline{u_{\mathfrak{h}}(\mathfrak{h})} = \boldsymbol{C}^l$$

But now let $\xi \in \boldsymbol{C}^l$ and let $y_j \in \mathfrak{h}$, $j = 1, 2, \cdots$, be such that $u_{\mathfrak{h}}(y_j)$ converges to ξ. Since $u_{\mathfrak{h}}$ is proper the set y_j has a cluster point y. Obviously $u_{\mathfrak{h}}(y) = \xi$ and hence $u_{\mathfrak{h}}$ is surjective. It follows therefore that $\eta_{\mathfrak{s}}$ is surjective since every element of $\mathfrak{h}$ is semi-simple.

To show that $\eta_{\mathfrak{s}}$ is injective we must show that if $x, y \in \mathfrak{s}$ and $u(x) = u(y)$ then x and y are conjugate. Since every element in $\mathfrak{s}$ is conjugate to an element in $\mathfrak{h}$ we may assume $x, y \in \mathfrak{h}$. But now by Lemma 9.2 in [13] (the extension to the reductive case in trivial) if $u_j(x) = u_j(y)$ for $j = 1, 2, \cdots, l$, then x and y are conjugate under the Weyl group and consequently are conjugate with respect to G. One explicitly uses here the well known theorem that, under the mapping $S^*(\mathfrak{g}) \to S^*(\mathfrak{h})$ induced by injection $\mathfrak{h} \to \mathfrak{g}$, J maps onto the algebra of Weyl group invariants. Q. E. D.

Since $\eta_{\mathfrak{s}}$ is a bijection we may invert it. For any $\xi \in \boldsymbol{C}^l$ we will let $O^{\mathfrak{s}}(\xi)$ be the unique semi-simple orbit O such that $\eta_{\mathfrak{s}}(O) = \xi$.

3.4. We now wish to look at the orbits of maximal dimension. By adding and subtracting the dimension of the center of $\mathfrak{g}$ it is obvious from that for any $x \in \mathfrak{g}$

$$(3.4.1) \qquad \begin{aligned} \dim O_x &= \dim \mathfrak{g} - \dim \mathfrak{g}^x \\ &= n - \dim \mathfrak{g}^x. \end{aligned}$$

PROPOSITION 11. *Let $x \in \mathfrak{g}$ be arbitrary. Then $\mathfrak{g}^x$ contains an l dimensional commutative subalgebra.*

Proof. This is just Theorem 5.7 in [13]. (Using the grassmannian of all l-planes in $\mathfrak{g}$ the proof is an easy consequence of the fact that the set of regular elements is dense in $\mathfrak{g}$.) Q. E. D.

As a corollary one has

PROPOSITION 12. *Let $x \in \mathfrak{g}$. Then*

$$(3.4.2) \qquad \dim O_x \leqq n - l$$

and the set of x for which equality holds in (3.4.2) *is not empty.*

Proof. The equality in (3.4.2) clearly holds for all regular elements in $\mathfrak{g}$. (It also holds for a larger collection of elements. See (3.4.3) and Theorem 2, § 3.5). The inequality (3.4.2) for all elements follows from (3.4.1) and Proposition 11. Q. E. D.

By Proposition 12 $n - l$ is the maximal dimension of any orbit. We now wish to consider the set of elements in $\mathfrak{g}$ which define orbits of this dimension. Put

$$(3.4.3) \qquad \mathfrak{r} = \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = l\}.$$

It is obvious that $\mathfrak{r}$ is stable under G so that we may consider the subset $\boldsymbol{\mathcal{O}}_{\mathfrak{r}} \subseteq \boldsymbol{\mathcal{O}}$. In fact $\boldsymbol{\mathcal{O}}_{\mathfrak{r}}$ is the set of orbits given by

$$(3.4.4) \qquad \boldsymbol{\mathcal{O}}_{\mathfrak{r}} = \{O \in \boldsymbol{\mathcal{O}} \mid \dim 0 = n - l\}.$$

The structure of $\mathfrak{r}$ and $\boldsymbol{\mathcal{O}}_{\mathfrak{r}}$ will be known as soon as we establish certain facts about principal nilpotent elements (Theorem 1). We will use a different definition of principal nilpotent than that given in [13].

Let $\mathfrak{p}$ be the set of all nilpotent elements in $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is called principal nilpotent if and only if

$$x \in \mathfrak{p} \cap \mathfrak{r}$$

that is, if and only if (1) x is nilpotent and (2) $\dim \mathfrak{g}^x = l$.

Obviously $\mathfrak{p}$ is stable under G so that we may consider the set of orbits $\mathcal{O}_{\mathfrak{p}}$. The following was established in [13].

THEOREM 1. *There are only a finite number of elements in $\mathcal{O}_{\mathfrak{p}}$. Furthermore the set $\mathfrak{p} \cap \mathfrak{r}$ of all principal nilpotent is not empty and is in fact a single orbit $O \in \mathcal{O}_{\mathfrak{p}}$. That is, for any $e \in \mathfrak{p} \cap \mathfrak{r}$ one has* (1)

$$\dim O_e = n - l$$

and (2)

$$\mathfrak{p} \cap \mathfrak{r} = O_e. \tag{3.4.5}$$

Moreover, this orbit is dense in $\mathfrak{p}$. That is,

$$\mathfrak{p} = \bar{O}_e \tag{3.4.6}$$

for any principal nilpotent element e.

If $O \in \mathcal{O}_{\mathfrak{p}}$ is any orbit other than the orbit of principal nilpotent elements one has

$$\dim O < n - l. \tag{3.4.7}$$

Proof. Theorem 1 above is a restatement of Corollaries 5.3 and 5.5 in [13]. Q.E.D.

Remark 13. It follows from (3.4.6) that the set of nilpotent elements $\mathfrak{p} \subseteq \mathfrak{g}$ is an affine variety of dimension $n - l$. With regard to all the orbits in $\mathcal{O}_{\mathfrak{p}}$ it should perhaps be recalled that in [13] it was shown that excluding the orbit consisting of zero alone they are in a natural one-one correspondence with the conjugacy classes of all 3-dimensional simple subalgebras of $\mathfrak{g}$.

If $x \in \mathfrak{g}$ is arbitrary and $x = y + z$ is the decomposition (3.1.2) then by the uniqueness of the decomposition, clearly,

$$G^x = G^y \cap G^z \tag{3.4.8}$$

and hence

$$\mathfrak{g}^x = \mathfrak{g}^y \cap \mathfrak{g}^z. \tag{3.4.9}$$

The subset $\mathfrak{r}$ may be characterized as follows:

PROPOSITION 13. *Let $x \in \mathfrak{g}$ be arbitrary. Write $x = y + z$ where y and z are, respectively, the semi-simple and nilpotent components of x.*

Then $x \in \mathfrak{r}$ if and only if z is a principal nilpotent of the reductive Lie algebra $\mathfrak{g}^y$.

Proof. Now $\mathfrak{g}^y$ is a reductive Lie algebra of rank l, and z is a nilpotent element of $\mathfrak{g}^y$. (Remark 10, § 3.1.), Furthermore $\mathfrak{g}^y \cap \mathfrak{g}^z$ is exactly the centralizer of z in $\mathfrak{g}^y$. Thus by definition z is a principal nilpotent element of $\mathfrak{g}^y$ if and only if $\dim \mathfrak{g}^y \cap \mathfrak{g}^z = l$. But then by (3.4.9) one has $x \in \mathfrak{r}$ if and only if z is a principal nilpotent element of $\mathfrak{g}^y$. Q. E. D.

3.5. Now let $x \in \mathfrak{g}$ and let $x = y + z$ be the decomposition (3.1.2) for x. Then as was observed in [13]

$$f(x) = f(y) \tag{3.5.1}$$

for any invariant polynomial $f \in J$. That is, $f(x)$ depends only upon the semi-simple component of x. Indeed (3.5.1) is an immediate consequence of the fact (see [13], p. 1031) that

$$y + z \text{ and } y + cz \text{ are conjugate} \tag{3.5.2}$$

for any non-zero complex number c.

We can now completely describe $\mathcal{O}_{\mathfrak{r}}$. Proposition 10, § 3.3, shows that the orbits of semi-simple elements are in a natural one-one correspondence with $\boldsymbol{C}^l$. We now observe (and this is more important for us) that the set $\mathcal{O}_{\mathfrak{r}}$ of all orbits of maximal dimension $(n - l)$ is also in a natural one-one correspondence with $\boldsymbol{C}^l$.

THEOREM 2. *The map*

$$\eta_{\mathfrak{r}}\colon \mathcal{O}_{\mathfrak{r}} \to \boldsymbol{C}^l$$

(given by the primitive invariant polynomials $u_1, \cdots, u_l$; see § 3.3) *is a bijection.*

Proof. Let $\xi \in \boldsymbol{C}^l$. Then by Proposition 10, § 3.3, there exists a semi-simple element $y \in \mathfrak{g}$ such that $u(y) = \xi$. Now let z be a principal nilpotent element in $\mathfrak{g}^y$. Then by Proposition 13, and (3.5.1) if $x = y + z$ then $x \in \mathfrak{r}$ and $f(x) = f(y)$ for all $f \in J$. Hence $u(x) = \xi$. Thus $\eta_{\mathfrak{r}}(O_x) = \xi$ so that $\eta_{\mathfrak{r}}$ is surjective.

To show that $\eta_{\mathfrak{r}}$ is injective we must show that if $x_1, x_2 \in \mathfrak{r}$ and $u(x_1) = u(x_2)$ then x_1 and x_2 are conjugate. Let $x_i = y_i + z_i$, $i = 1, 2$, be the decomposition (3.1.2) for x_i. By (3.5.1) one has $u(y_1) = u(y_2)$. But then, by Proposition 10, § 3.3, y_1 and y_2 are conjugate. Hence we may assume that $y_1 = y_2 = y$. But since $x_1, x_2 \in \mathfrak{r}$ it follows from Proposition 13 that z_1 and z_2 are principal nilpotent elements of $\mathfrak{g}^y$. But then by Theorem 1 applied to $\mathfrak{g}^y$ it follows from Lemma 6, § 3.2, that there exists $a \in G^y$ such that $az_1 = z_2$. Thus $ax_1 = x_2$. Q. E. D.

Since $\eta_{\mathfrak{r}}$ is a bijection we may invert it. For any $\xi \in \boldsymbol{C}^l$ let $O^{\mathfrak{r}}(\xi) \in \mathcal{O}_{\mathfrak{r}}$ be the unique orbit O of dimension $n - l$ such that $\eta_{\mathfrak{r}}(O) = \xi$.

3.6. Let $\{e_\phi\}$, $\phi \in \Delta$, be a set of root vectors belonging to Δ. Let

$$\mathfrak{m} = \sum_{\phi \in \Delta_+} (e_\phi) \tag{3.6.1}$$

be the maximal Lie algebra of nilpotent elements defined by Δ_+. Let $\mathfrak{m}^*$ be the corresponding nilpotent Lie algebra defined by the negative roots $\Delta_- = -\Delta_+$ so that

$$\mathfrak{g} = \mathfrak{m}^* + \mathfrak{h} + \mathfrak{m} \tag{3.6.2}$$

is a linear direct sum.

Let A be the Cartan subgroup of G coressponding to $\mathfrak{h}$ and let

$$\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_k\} \subseteq \Delta_+$$

be the set of simple positive roots. (Here $k = l - \dim \mathfrak{z}$). For any $\phi \in \Delta$ let $a^\phi \in \boldsymbol{C}^*$ be the non-zero scalar defined by

$$a(e_\phi) = a^\phi e_\phi. \tag{3.6.3}$$

Since G is the adjoint group of $\mathfrak{g}$ one knows that the mapping

$$A \to (\boldsymbol{C}^*)^k \tag{3.6.4}$$

given by $a \to (a^{\alpha_1}, \cdots, a^{\alpha_k})$ is an isomorphism.

LEMMA 8. *Let $y \in \mathfrak{g}$ be semi-simple. Then the center of G^y is connected.*

Proof. If x is conjugate to y then G^x is conjugate to G^y and hence it is enough to show the center of G^x is connected for some $x \in O_y$. Since all Cartan subalgebras are conjugate we may choose $x \in \mathfrak{h}$. Moreover, by applying an element of the Weyl group W to x, if necessary, we may assume that $\mathfrak{g}^x$ is of the form

$$\mathfrak{g}^x = \mathfrak{h} + \sum_{\phi \in \Delta_1} (e_\phi) \tag{3.6.5}$$

where Π_1 is some subset of the set of simple roots Π and $\Delta_1 \subseteq \Delta$ is the set all roots ϕ of the form

$$\phi = \sum_{\alpha \in \Pi_1} n_\alpha \alpha,$$

for integers n_α.

Let Z be the center of G^x and let $a \in Z$. It is then obvious that $av = v$ for any $v \in \mathfrak{g}^x$. Hence $a \in A$ and $a^\alpha = 1$ for all $\alpha \in \Pi_1$. Since G^x is connected (see Lemma 5, § 3.2) the converse is clearly true. Hence

$$Z = \{a \in A \mid a^\alpha = 1 \text{ for all } \alpha \in \Pi_1\}.$$

Using the isomorphism (3.6.4) it is then obvious that Z is connected.

Q. E. D.

Remark 14. The structure of G^x is not as simple as Lemma 8, § 3.6 seems to indicate. In particular even though the center of G^x is connected the subgroup of G^x corresponding to the maximal semi-simple ideal $[\mathfrak{g}^x, \mathfrak{g}^x]$ of $\mathfrak{g}^x$ may have a non-trivial discrete center. The structure of G^x is analogous to that of the general linear group $Gl(d, \boldsymbol{C})$.

Let M and B be respectively the unipotent and Borel subgroups of G corresponding to $\mathfrak{m}$ and $\mathfrak{b} = \mathfrak{h} + \mathfrak{m}$.

The orbits of maximal dimension $(n - l)$ are uniform in the following respect.

PROPOSITION 14. *For any $x \in \mathfrak{r}$ the group G^x is an abelian, connected, algebraic subgroup of G of dimension $l - \dim \mathfrak{z}$. (Recall that $\mathfrak{z}$ is the center of $\mathfrak{g}$).*

Proof. By Proposition 11, § 3.4, it is immediate that $\mathfrak{g}^x$ is a commutative Lie algebra of dimension l. Therefore to prove the proposition it is enough to show that G^x is connected.

Let $e \in \mathfrak{g}$ be given by

$$e = \sum_{\alpha \in \Pi} e_\alpha$$

Then by [13], Theorem 5.3, e is a principal nilpotent element of $\mathfrak{g}$. We first show that G^e is connected. Let $h \in G^e$. But [13], Corollary 5.6, $\mathfrak{m}$ must be stable under h (this corollary asserts that a principal nilpotent element lies in one and only one nilpotent Lie algebra of the form (3.6.1)). But now since $\mathfrak{m}$ is stable under h it follows that $h \in B$. This may be proved in the following way. According to the Bruhat decomposition of G (see [7]) we may write

$$h = bs(\sigma)g$$

where $b \in B$, $g \in M$ and $s(\sigma)$ is in the normalizer of A inducing the element $\sigma \in W$ on $\mathfrak{h}$. To prove $h \in B$ it suffices to show that σ is the identity of W. But this is obvious since h, b, g and hence $s(\sigma)$ leaves $\mathfrak{m}$ stable (only the identity element of W leaves Δ_+ stable). Thus $a \in B$. Write $h = da$ where $a \in A$ and $d \in M$. Now since $d \in M$

$$\begin{aligned} h(e) &= da(e) \\ &= a(e) + w \end{aligned}$$

where $w \in [\mathfrak{m}, \mathfrak{m}]$. But since $h(e) = e$ and

$$a(e) = \sum_{\alpha \in \Pi} a^\alpha e_\alpha$$

(see (3.6.3)) it follows that $w = 0$ and $a(e) = e$. The latter implies that $a^\alpha = 1$ for all $\alpha \in \Pi$ and hence by (3.6.4) a is the identity of G. Thus $h = d \in M$. But then h may be uniquely written $h = \exp \operatorname{ad} v$ where $v \in \mathfrak{m}$. But then by well known properties of nilpotent Lie algebras one has $v \in \mathfrak{g}^e$ and hence h lies in the identity component of G^e or since h was arbitrary G^e is connected.

Since all principal nilpotents of $\mathfrak{g}$ are conjugate it follows that G^z is connected for any principal nilpotent element $z \in \mathfrak{g}$.

Now let x be an arbitrary element of $\mathfrak{r}$ and let $x = y + z$ be the decomposition (3.1.2) for x. We recall that, by Proposition 13, § 3.4, z is a principal nilpotent element of the reductive Lie algebra $\mathfrak{g}^y$. Let F be the adjoint group of $\mathfrak{g}^y$ so that applying what just proved (in the case of $\mathfrak{g}$) to $\mathfrak{g}^y$ it follows that F^z is connected. Now by Lemma 6, § 3.2, the restriction of G^y to $\mathfrak{g}^y$ induces an epimorphism

$$G^y \to F \tag{3.6.6}$$

with kernel Z equal to center of G^y. But now the full inverse image of F^z under the map (3.6.6) is just $G^y \cap G^z = G^x$ (see (3.4.8)). But since F^z is connected and Z is connected by Lemma 8, § 3.6, it follows that G^x is connected. Q. E. D.

Remark 15. Note that if $x \in \mathfrak{r}$ the abelian group G^x ranges all the way from a reductive group, (in the case where x is a regular element) to a unipotent group (in the case where x is a principal nilpotent element). In the general case G^x, for any $x \in \mathfrak{r}$, is the direct product of a abelian reductive group and an abelian unipotent group.

It seems suggestive from Lemma 5, § 3.2, and Proposition 14, that possibly G^x is always connected for any $x \in \mathfrak{g}$. This, however, is false. If $\mathfrak{g}$ is the Lie algebra of the exceptional simple Lie group G_2 and $\psi = 3\alpha + 2\beta$ is the highest root where $\Pi = \{\alpha, \beta\}$, then one can show that G^x is not connected in case $x = e_\alpha + e_\psi$. In fact one sees easily that G^x contains the non-trivial diagonalizable element $a \in A$ where $a^\alpha = 1$ and $a^\beta = -1$ whereas on the other hand the identity component of G^x is unipotent. One proves the latter statement by using the first line in Table 21, p. 186 in [5] (no X_0 term) together with the argument in the proof of Theorem 3, § 3.7.

3.7. It was pointed out to us by Dixmier that the following proposition is a special case of a more general result of Kirillov (all orbits of *any* Lie group for the representation contragredient to the adjoint representation are even dimensional). The simple proof given here is due to Kirillov and is easily modified to give the more general result.

PROPOSITION 15. *For any* $x \in \mathfrak{g}$ *one has* $\dim O_x$ *is even.*

Proof. Let B be a G-invariant non-singular symmetric bilinear form on $\mathfrak{g}$. Now for any $x \in \mathfrak{g}$ let B_x be the alternating bilinear form on $\mathfrak{g}$ given by

$$B_x(y, z) = B(x, [y, z]).$$

From the invariance of B it is clear that $B_x(y, z) = 0$ for all $z \in \mathfrak{g}$ if and only if $y \in \mathfrak{g}^x$. It follows therefore that B_x defines a non-singular alternating bilinear form on $\mathfrak{g}/\mathfrak{g}^x$. But since such a bilinear form can only be carried by an even dimensional space it follows that $\dim \mathfrak{g} - \dim \mathfrak{g}^x$ is even. The proposition then follows from (3.4.1). Q. E. D.

3.8. We now consider the cone $P \subseteq \mathfrak{g}$, defined as in § 1.3 by J^+. That is, P is the set of common zeros for all the polynomials in J^+.

The following proposition was essentially proved in [13] (it was proved for the case when $\mathfrak{g}$ is semi-simple).

PROPOSITION 16. *The cone* P *is identical with the set* $\mathfrak{p} \subseteq \mathfrak{g}$ *of all nilpotent elements in* $\mathfrak{g}$.

Proof. If $x \in P$ then clearly $\operatorname{ad} x$ is nilpotent. Indeed if $f_j \in S$ is defined by $f_j(y) = \operatorname{tr}(\operatorname{ad} y)^j$ for any $y \in \mathfrak{g}$ and positive integer j one has $f_j \in J^+$ and hence $f_j(x) = 0$ for all such j implies $\operatorname{ad} x$ is nilpotent. On the other hand if $x \in P$ then also $x \in [\mathfrak{g}, \mathfrak{g}]$. In fact let $x = x_1 + x_2$ be the decomposition of x according to (3.1.1) where $x_1 \in \mathfrak{z}$ and $x_2 \in [\mathfrak{g}, \mathfrak{g}]$. To see that x_1 is zero observe that every linear functional (element of S^1) on $\mathfrak{g}$ which vanishes on $[\mathfrak{g}, \mathfrak{g}]$ lies in J^1. But then one must have $f(x) = 0$ for all such linear functionals. Hence $x \in [\mathfrak{g}, \mathfrak{g}]$ and thus $P \subseteq \mathfrak{p}$ (see the definition of nilpotent elements). But $\mathfrak{p} \subseteq P$ by (3.5.2) since $y = 0$ when x is nilpotent and $f(0) = 0$ for any $f \in J^+$. Q. E. D.

Now Theorem 1 together with Propostiion 16 above implies that (1)

P has a dense orbit (the set of principal nilpotent elements) (2) P is of dimension $n-l$ and (3) P is a finite union of orbits. On the other hand from the particular structure of J the cone P can be given by considering only the primitive invariants $u_i \in J$. That is,

$$P=\{x \in \mathfrak{g} \mid u_i(x)=0, i=1,2,\cdots,l\}.$$

We now generalize Theorem 1, and thereby encompass every point of $\mathfrak{g}$, by proving a similar theorem after substituting any point in $\boldsymbol{C}^l$ for the l-tuple of zeros above.

For any $\xi \in \boldsymbol{C}^l$, $\xi=(\xi_1,\cdots,\xi_l)$, let $P(\xi)$, as in Proposition 6, § 1.6, be the affine variety in $\mathfrak{g}$ given by

$$P(\xi)=\{x \in g \mid u_i(x)=\xi_i,\ i=1,\cdots,l\}.$$

That is, with respect to the map u (see (3.3.2) one has

$$P(\xi)=u^{-1}(\xi). \tag{3.8.1}$$

Thus

$$\mathfrak{g}=\bigcup_{\xi \in \boldsymbol{C}^l} P(\xi) \tag{3.8.2}$$

is a disjoint union.

Now recall (see end of § 3.3 and § 3.5) that $O^{\mathfrak{s}}(\xi)$ and $O^{\mathfrak{r}}(\xi)$ are, respectively, the orbit of semi-simple elements and orbit of maximal dimension corresponding to any $\xi \in \boldsymbol{C}^l$ under $\eta_{\mathfrak{s}}$ and $\eta_{\mathfrak{r}}$.

Now obviously $P(\xi)$ is a union of orbits. In particular $O^{\mathfrak{s}}(\xi)$ and $O^{\mathfrak{r}}(\xi)$ are contained in $P(\xi)$. Furthermore every orbit O lies in some $P(\xi)$. Theorem 1 generalizes in the following way.

THEOREM 3. *Let $\xi \in \boldsymbol{C}^l$ be arbitrary. Then $P(\xi)$ is the set of all $x \in \mathfrak{g}$ whose semi-simple component lies in $O^{\mathfrak{s}}(\xi)$.*

$$P(\xi)=O^{\mathfrak{r}}(\xi) \cup \cdots \cup O^{\mathfrak{s}}(\xi) \tag{3.8.3}$$

is a union of a finite number of orbits. Moreover

$$O^{\mathfrak{r}}(\xi)=P(\xi) \cap \mathfrak{r} \tag{3.8.4}$$

and $O^{\mathfrak{r}}(\xi)$ is the unique orbit of maximal dimension $(n-l)$ in $P(\xi)$ and in fact

$$\dim O \leqq (n-l)-2 \tag{3.8.5}$$

for any other orbit in $P(\xi)$. Next (2)

$$O^{\mathfrak{s}}(\xi)=P(\xi) \cap \mathfrak{s} \tag{3.8.6}$$

and $O^{\mathfrak{s}}(\xi)$ *is the orbit of minimal dimension in* $P(\xi)$. *Finally*

$$P(\xi)=\overline{O^{\mathfrak{r}}(\xi)} \tag{3.8.7}$$

so that the Zariski closed set $P(\xi)$ *is irreducible and*

$$\dim P(\xi)=n-l. \tag{3.8.8}$$

Proof. The statement that $P(\xi)$ is the set of all $x\in\mathfrak{g}$ whose semi-simple component lies in $O^{\mathfrak{s}}(\xi)$ is an immediate consequence of (3.5.1) and Proposition 10, § 3.3. This of course implies (3.8.6). Furthermore using (3.5.2) it follows from (3.8.5) that for any orbit $O\subseteq P(\xi)$ one has

$$O^{\mathfrak{s}}(\xi)\subseteq\bar{O} \tag{3.8.9}$$

so that $O^{\mathfrak{s}}(\xi)$ is the orbit of minimal dimension in $P(\xi)$.

Now let $y\in O^{\mathfrak{s}}(\xi)$ so that $\mathfrak{g}^y$ is a reductive Lie algebra. Let $\mathfrak{p}^y$ denote the set of nilpotent elements of $\mathfrak{g}^y$. Applying Theorem 1 to $\mathfrak{g}^y$ there exist k elements $z_i\in\mathfrak{p}^y$, $i=1,\cdots,k$, such that under the adjoint group F of $\mathfrak{g}^y$ every nilpotent element in $\mathfrak{g}^y$ is conjugate to one and only one of the z_i. Furthermore Theorem 1 asserts that if N is the set of all principal nilpotent elements of $\mathfrak{g}^y$ then N is an orbit under F and $\bar{N}=\mathfrak{p}^y$.

Now put $x_i=y+z_i$, $i=1,\cdots,k$, so that by Remark 10, § 3.1, y and z_i are, respectively, the semi-simple and nilpotent components of x_i.

We now assert that every element in $P(\xi)$ is conjugate to one and only one of the x_i. We first show that the x_i lie in different conjugate classes. Assume $ax_i=x_j$ for $a\in G$. Then by the uniqueness of the decomposition (3.1.2) one has $ay=y$ so that $a\in G^y$ and $az_i=z_j$. But, by Lemma 6, § 3.2, z_i is then conjugate to z_j under F and hence $i=j$. Now let $v\in P(\xi)$ be arbitrary and let w and e be, respectively, its semi-simple and nilpotent compotents. We show that v is conjugate to one of the x_i. By the first statement of the theorem (already proved above) there exists $a\in G$ such that $aw=y$. Hence we may assume $w=y$. But then $e\in\mathfrak{p}^y$ and hence by Lemma 6 there exists $a\in G^y$ such that $ae=z_i$ for some i so that $av=x_i$. Thus there are only a finite (k) number of orbits in $P(\xi)$.

The statement (3.8.4) and the fact that $O^{\mathfrak{r}}(\xi)$ is the unique orbit of maximal dimension in $P(\xi)$ is just a restatement of Theorem 2. The inequality (3.8.5) then follows by Proposition 15.

Finally since N (see above) is dense in $\mathfrak{p}^y$, by Theorem 1, each x_i is in the closure of $O^{\mathfrak{r}}(\xi)$ so that, clearly, one has (3.8.7) and hence also (3.8.8).

Q.E.D.

Remark 16. We can be very explicit about the number of orbits in $P(\xi)$ for any $\xi \in \boldsymbol{C}^l$. Let $y \in O^s(\xi)$. By the argument above and Remark 13, § 3.4, if k is the number of orbits in $P(\xi)$ then $k-1$ is the number of conjugacy classes of three dimensional simple Lie algebras in $\mathfrak{g}^y$. But such classes have been listed by Dynkin (see [5]) for every simple Lie algebra and hence one knows k as soon as one knows the maximal semi-simple ideal $[\mathfrak{g}^y, \mathfrak{g}^y]$ in $\mathfrak{g}^y$.

We note also that (3.8.9) together with Lemma 5, § 3.2, implies that the semi-simple orbits are the only closed orbits

COROLLARY 1. *Let $x \in \mathfrak{g}$ be arbitrary. Then $\bar{O}_x$ is a union of only a finite number of orbits. Moreover if C_x is the (Zariski closed) complement of O_x in $\bar{O}_x$ then, where* codim C_x *is, as in* § 2.2, *defined with respect to O_x, one has*

$$\operatorname{codim} C_x \geqq 2.$$

Proof. If $u(x) = \xi$ then obviously $\bar{O}_x \subseteq P(\xi)$ and since $P(\xi)$ is composed of only a finite number of orbits the same is true for $\bar{O}_x$. But if $O \subseteq C_x$ then certainly

$$\dim O_x - \dim O \geqq 2$$

by Proposition 15 and the fact that $\dim O \leqq \dim C_x < \dim \bar{O}_x = \dim O_x$. But then Corollary 1 follows immediately since C_x must be a finite union of varieties of the form $\bar{O}$ where $O \subseteq C_x$. Q. E. D.

As an immediate application one has

COROLLARY 2. *Let O be any orbit and let T be the set (Zariski closed) of all non-simple points of the affine variety $\bar{O}$. Then if* codim T *is defined with respect to $\bar{O}$ one has*

$$\operatorname{codim} T \geqq 2.$$

Proof. One knows that the set of all simple points of $\bar{O}$ is Zariski open in $\bar{O}$ and non-empty. It therefore meets O. Since G is transitive on O it follows that all the points of O are simple. Thus $T \subseteq C_x$ where $O = O_x$ and hence the result follows from Corollary 1.

Remark 17. It is suggestive from Corollary 2 that possibly $\bar{O}$ is a normal variety for any orbit O. This will be proved later (see § 5.1) for all orbits of maximal dimension. For such orbits it will also be seen that T is exactly the complement of O in $\bar{O}$.

4. The transversal $\boldsymbol{l}$-plane $\mathfrak{v}$. 1. Now we recall that S, the ring of polynomials on $\mathfrak{g}$ and S_*, the symmetric algebra over $\mathfrak{g}$ (the ring of differential operators with constant coefficients on $\mathfrak{g}$) are G-modules and are paired by (1.1.1). By taking the differential (via the adjoint representation) they become $\mathfrak{g}$-modules and by (1.1.4) one has

$$\langle x\cdot\partial, f\rangle + \langle\partial, x\cdot f\rangle = 0 \tag{4.1.1}$$

for all $x\in\mathfrak{g}$, $\partial\in S_*$ and $f\in S$. Furthermore any $x\in\mathfrak{g}$ operates as a derivation of degree 0 of S and S_* and hence, by (4.1.1), its action is completely determined by its restriction to S_1. But the latter is given by

$$x\cdot\partial_y = \partial_{[x,y]} \tag{4.1.2}$$

for any $y\in\mathfrak{g}$.

Note that if $\partial\in S_*$ is of the form $\partial = x\cdot\partial_1$ where $x\in\mathfrak{g}$ and $\partial_1\in S_*$ then by (4.1.1)

$$\langle\partial, f\rangle = 0 \text{ for all } f\in J. \tag{4.1.3}$$

This criterion for an element $\partial\in S_*$ to be orthogonal to J is especially convenient to use when x equals a certain element $x_0\in\mathfrak{h}$, now to be defined.

Recall that $\Pi\subseteq\Delta_+$ is the set of simple positive roots. We now put x_0 equal to the unique element in $\mathfrak{h}\cap[\mathfrak{g},\mathfrak{g}]$ such that (see [13], § 5.2)

$$\langle x_0,\alpha\rangle = 1 \text{ for all } \alpha\in\Pi.$$

If $\phi\in\Delta$ is arbitrary and the order $o(\phi)$ of ϕ is the integer defined by

$$o(\phi) = \sum_{\alpha\in\Pi} n_\alpha(\phi) \tag{4.1.4}$$

where

$$\phi = \sum_{\alpha\in\Pi} n_\alpha(\phi)\,\alpha \tag{4.1.5}$$

then clearly

$$\langle x_0,\phi\rangle = o(\phi) \tag{4.1.6}$$

and hence

$$[x_0, e_\phi] = o(\phi)\,e_\phi. \tag{4.1.7}$$

As usual let $\mathbf{Z}$ denote the set of all integers. For every integer $j\in\mathbf{Z}$ let

$$S_*^{(j)} = \langle\partial\in S_* \mid x_0\cdot\partial = j\partial\rangle.$$

It is obvious that $S_*^{(j)}$ is a graded subspace of S_* and since x_0 operates as a derivation of S it follows immediately from (4.1.7) that

$$S_* = \sum_{j\in\mathbf{Z}} S_*^{(j)}$$

is a direct sum and

$$S_*^{(i)} S_*^{(j)} \subseteq S_*^{(i+j)}. \tag{4.1.8}$$

Similarly let $\mathfrak{g}^{(j)}$ be the eigenspace of $\operatorname{ad} x_0$ for the eigenvalue j so that $\mathfrak{g}$ is a direct sum of the $\mathfrak{g}^{(j)}$. Since $\operatorname{ad} x_0$ is a derivation of $\mathfrak{g}$ clearly

$$[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subseteq \mathfrak{g}^{(i+j)}. \tag{4.1.9}$$

The decomposition (3.6.2) is related to x_0 in the following way.

LEMMA 9. *The nilpotent Lie algebras $\mathfrak{m}$ and $\mathfrak{m}^*$ may be expressed in terms of the eigenspaces $\mathfrak{g}^{(j)}$ of $\operatorname{ad} x_0$ as follows:*

$$\mathfrak{m} = \sum_{j>0} \mathfrak{g}^{(j)}, \qquad \mathfrak{m}^* = \sum_{j<0} \mathfrak{g}^{(j)}.$$

Moreover $h = \mathfrak{g}^{(0)} = \mathfrak{g}^{x_0}$ (i.e. *x_0 is regular*).

Proof. Obvious from (4.1.7) and the fact $o(\phi)$ is positive for positive roots ϕ and negative for negative roots ϕ. Q. E. D.

Since $S_*^{(j)}$ is in the range of the action of x_0 whenever $j \neq 0$ one has, by (4.1.3),

$$\langle \partial, f \rangle = 0 \text{ if } f \in J,\ \partial \in S_*^{(j)} \text{ where } j \neq 0. \tag{4.1.10}$$

In the obvious way the symmetric algebra $S_*(\mathfrak{u})$ over any subspace $\mathfrak{u} \subseteq \mathfrak{g}$ may be regarded as a subalgebra of S_*.

Let $\mathfrak{b}$ be the maximal solvable Lie subalgebra of $\mathfrak{g}$ given by the direct sum

$$\mathfrak{b} = \mathfrak{h} + \mathfrak{m} \tag{4.1.11}$$

(resp. put $\mathfrak{b}^* = \mathfrak{m}^* + \mathfrak{h}$).

One knows that if $\mathfrak{g} = \mathfrak{gl}(d, \boldsymbol{C})$ then $f(x)$ depends only on the diagonal entries of x in case x is a triangular matrix and $f \in J$. More generally one has

PROPOSITION 17. *Let $x \in \mathfrak{b}^*$ so that $x = y + v$ where $y \in \mathfrak{h}$ and $v \in \mathfrak{m}^*$. Then for any $f \in J$ one has $f(x) = f(y)$. In particular*

$$u(x) = u(y)$$

where u is the map (3.3.2).

Proof. Since J is graded we may assume $f \in J^k$. Then, by (1.1.3), $k! f(x) = \langle (\partial_x)^k, f \rangle = \langle (\partial_y + \partial_v)^k, f \rangle = \langle (\partial_y)^k + \partial, f \rangle = k! f(y) + \langle \partial, f \rangle$ where, by binomial expansion, $\partial \in \mathfrak{m}^* \cdot S_*(\mathfrak{b}^*)$. But now by Lemma 9, § 4.1 and (4.1.8) it follows that

$$\mathfrak{m}^* \cdot S_*(\mathfrak{b}^*) \subseteq \sum_{j<0} S_*^{(j)}$$

and hence $\langle \partial, f \rangle = 0$ by (4.1.10). Thus $f(x) = f(y)$. Q. E. D.

4.2. The following simple characterization of the principal nilpotent elements in $\mathfrak{m}$ was given in [13].

THEOREM 4. *Let* $e \in \mathfrak{m}$. *Write*

$$e = \sum_{\phi \in \Delta_+} c_\phi e_\phi$$

then e is principal nilpotent if and only if $c_\alpha \neq 0$ *for every simple root* $\alpha \in \Pi$.

Proof. This is just Theorem 5.3 in [13]. Q. E. D.

Now for every simple $\alpha \in \Pi$ let c'_α be an arbitrary *non-zero* complex number (normalized in § 4.4). We isolate a particular principal nilpotent element (by Theorem 5, after interchanging the roles of Δ_+ and Δ_-) e_- by putting

$$e_- = \sum_{\alpha \in \Pi} c'_\alpha e_{-\alpha}. \tag{4.2.1}$$

The following lemma gives a very simple method for constructing elements in $\mathfrak{r}$ (in fact by Lemma 11 and Proposition 10, § 3.3, at least a representative for every orbit of maximal dimension is constructed in this way).

Recall that $\mathfrak{b}$ is the maximal solvable Lie subalgebra $\mathfrak{h} + \mathfrak{m}$.

LEMMA 10. *One has the relation*

$$e_- + \mathfrak{b} \subseteq \mathfrak{r}$$

where, we recall, $\mathfrak{r}$ *is the set of all* $x \in \mathfrak{g}$ *such that* $\dim \mathfrak{g}^x = l$.

Proof. For any $j \in \mathbf{Z}$ put

$$\mathfrak{a}^{(j)} = \mathfrak{g}^{e_-} \cap \mathfrak{g}^{(j)}.$$

Then since $e_- \in \mathfrak{g}^{(-1)}$ it is clear from (4.1.9) that

$$\operatorname{ad} e_-(\mathfrak{g}^{(j)}) \subseteq \mathfrak{g}^{(j-1)} \tag{4.2.2}$$

and hence

$$\mathfrak{g}^{e_-} = \sum_{j \in \mathbf{Z}} \mathfrak{a}^{(j)} \tag{4.2.3}$$

is a direct sum.

Since $e_- \in \mathfrak{r}$ one therefore has

$$\sum_{j \in Z} \dim \mathfrak{a}^{(j)} = l. \tag{4.2.4}$$

Now filter $\mathfrak{g}$ by putting

$$\mathfrak{g}_j = \sum_{i \geqq j} \mathfrak{g}^{(i)}. \tag{4.2.5}$$

Thus (1) $\mathfrak{g}_j \supseteq \mathfrak{g}_{j+1}$ for all j (2) $\mathfrak{g} = \mathfrak{g}_j$ for j sufficiently small and (3) $\mathfrak{g}_j = 0$ for j sufficiently big.

Now let $v \in \mathfrak{b}$ and $x = e_- + v$. But now by (4.1.9) one has

$$\operatorname{ad} v(\mathfrak{g}_j) \subseteq \mathfrak{g}_j \tag{4.2.6}$$

for all j. On the other hand since the $\mathfrak{g}_j$ induce a filtration

$$\mathfrak{g}_j{}^x = \mathfrak{g}^x \cap g_j$$

on $\mathfrak{g}^x$ one has

$$\sum_{j \in Z} \dim \mathfrak{g}_j{}^x / \mathfrak{g}_{j+1}{}^x = \dim \mathfrak{g}^x. \tag{4.2.7}$$

But now if

$$\mathfrak{g}_j{}^x / \mathfrak{g}_{j+1}{}^x \to \mathfrak{g}^{(j)} \tag{4.2.8}$$

is the obvious injective map induced by (4.2.5) it follows immediately from (4.2.2) and (4.2.6) that the image of (4.2.8) lies in $\mathfrak{a}^{(j)}$. Thus

$$\dim \mathfrak{g}_j{}^x / \mathfrak{g}_{j+1}{}^x \leqq \dim \mathfrak{a}^{(j)} \tag{4.2.9}$$

for any j. Comparing (4.2.4) and (4.2.7) it follows that $\dim \mathfrak{g}^x \leqq l$. But, by (3.4.2), $\dim \mathfrak{g}^x \geqq l$. Hence $\dim \mathfrak{g}^x = l$ (and hence also the equality holds in (4.2.9) for any j). Q.E.D.

LEMMA 11. *Let $y \in \mathfrak{h}$ be arbitrary. Put $x = e_- + y$. Then $x \in \mathfrak{r}$ and also $u(x) = u(y)$.*

Proof. Since $y \in \mathfrak{b}$ one has $x \in \mathfrak{r}$ by Lemma 10. On the other hand since $e_- \in \mathfrak{m}^*$ it follows that $u(x) = u(y)$ by Proposition 17, § 4.1. Q.E.D.

4.3. In § 1.5 we defined the notion of a quasi-regular element $x \in X$ for the general case of a linear group G operating on a vector space X. The notion is important for us because of Proposition 5, § 1.5. The question as to which elements are quasi-regular, for the case at hand, is settled by

PROPOSITION 18. *The set $\mathfrak{r}$ (all elements $x \in \mathfrak{g}$ such that $\dim \mathfrak{g}^x = l$) is identical with the set of all quasi-regular elements in $\mathfrak{g}$.* (See § 1.5.)

Proof. In the case at hand P is the set $\mathfrak{p}$ of all nilpotent elements. If $x \in \mathfrak{g}$ is quasi-regular then by definition $P_x = \mathfrak{p}$. Hence in particular if $e \in \mathfrak{p}$ is a principal nilpotent element there exists a sequence x_j, $j = 1, 2, \cdots$, such that $x_j \in O_{c_j x}$ where $c_j \in \boldsymbol{C}^*$ and such that x_j converges to e.

Now let $k = \dim \mathfrak{g}^x$. We wish to show that $k = l$. By Proposition 12, § 3.4, it suffices to show that $k \leqq l$. Obviously $\dim \mathfrak{g}^{c_j x} = k$ so that $\dim \mathfrak{g}^{x_j} = k$ for all j. Now consider the Grassmanian (which one recalls is compact) of of all subspaces of dimension k. Let $\mathfrak{u}$ be a cluster point of the $\mathfrak{g}^{x_j}$. It then follows easily (see argument in [13], p. 1003) that $\mathfrak{u} \subseteq \mathfrak{g}^e$. But then $k \leqq l$ since $\dim \mathfrak{g}^e = l$ and $\dim \mathfrak{u} = k$. Thus $k = l$ and consequently $x \in \mathfrak{r}$.

Now conversely assume that $x \in \mathfrak{r}$. We wish to show that x is quasi-regular. Since P_x is closed and stable under G to prove $P_x = \mathfrak{p}$ it suffices by (3.4.6) to show that P_x contains a principal nilpotent element.

Now let $\xi = u(x)$. Then by Proposition 10, § 3.3, there exists $y \in \mathfrak{h}$ such that $u(y) = \xi$. Put $x_1 = e_- + y$. Then by Lemma 11, § 4.2, and Theorem 2 it follows that $x_1 \in O_x$. The same argument shows that for any $c \in \boldsymbol{C}^*$ one has that $x_c \in O_{cx}$ where $x_c = e_- + cy$. (One uses the fact that $u(x) = u(y)$ implies $u(cx) = u(cy)$; an immediate consequence of the homogeneity of the u_i.) But now, obviously, $x_c \to e_-$ as $c \to 0$. Hence $e_- \in P_x$. But since e_- is principal nilpotent this proves x is quasi-regular. Q. E. D.

4.4. Now for every simple root $\alpha \in \Pi$ let $c_\alpha \in \boldsymbol{C}^*$ be any arbitrary non-zero complex number. Let e_+ be the principal nilpotent element (see Theorem 5, § 4.2) given by

$$e_+ = \sum_{\alpha \in \Pi} c_\alpha e_\alpha. \tag{4.4.1}$$

Since e_+ is principal nilpotent one has $\dim \mathfrak{g}^{e_+} = l$.

The following description of $\mathfrak{g}^{e_+}$ proved in [13] will play a fundamental role in this paper.

THEOREM 5. *There exists a basis z_i, $i = 1, 2, \cdots, l$, of $\mathfrak{g}^{e_+}$ such that (see* § 4.1*)*

$$z_i \in \mathfrak{g}^{(m_i)} \tag{4.4.2}$$

where, we recall, m_i is that integer given by

$$\deg u_i = m_i + 1$$

and $u_i \in J$ is the i-th primitive invariant polynomial (see § 3.3*).*

In particular then

$$\mathfrak{g}^{e_+} \subseteq \mathfrak{b}. \tag{4.4.3}$$

Proof. When $\mathfrak{g}$ is simple the first statement here is just the 2nd and 3rd from the last statements of Theorem 6.7 in [13] (our z_i here is the u_i of that theorem) together with Corollary 8.7 (which shows that $k_i = m_i$) of [13].

If $\mathfrak{g}$ is semi-simple the first statement is still true since e_+ may be written

$$e_+ = \sum_i e_{+,i}$$

where the $e_{+,i}$ are principal nilpotent elements of the various simple components of $\mathfrak{g}$ and each is of the form (4.4.1) for the corresponding simple component. One then uses the fact that the exponents (the m_i) of $\mathfrak{g}$ are composed of the exponents of the various simple components of $\mathfrak{g}$.

In the general case the first statement of the theorem also follows since if $\mathfrak{z}$ is the center of $\mathfrak{g}$ then $m_i = 0$ if and only if $i \leqq \dim \mathfrak{z}$. But clearly $\mathfrak{z} = \mathfrak{g}^{e_+} \cap \mathfrak{g}^{(0)}$.

Since the m_i are non-negative integers the relation (4.4.3) follows from Lemma 9, § 4.1. Q.E.D.

Remark 18. Subject only to the conditions of Theorem 5, § 4.4, it is obvious that the basis of primitive polynomial invariants u_i does not uniquely determine the basis z_i of $\mathfrak{g}^{e_+}$. However with the further relations uncovered in § 4.6 we wish to note that the u_i do uniquely determine a basis z_i of $\mathfrak{g}^{e_+}$.

Now one knows the elements $x_\alpha \in \mathfrak{h}$, $\alpha \in \Pi$, given by $x_\alpha = [e_\alpha, e_{-\alpha}]$, form a basis of $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$. Hence we may write

$$x_0 = \sum_{\alpha \in \Pi} r_\alpha x_\alpha$$

where x_0 is defined as in § 4.1.

Now define

$$e'_- = \sum_{\alpha \in \Pi} r_\alpha / c_\alpha \, e_{-\alpha}$$

where the c_α define the principal nilpotent element e_+ (see (4.4.1)). Then as observed in [13], § 5.2, the elements e'_-, x_0 and e_+ form a basis of a principal three dimensional simple subalgebra $\mathfrak{a}_0$ of $\mathfrak{g}$. Furthermore this basis satisfies the commutation relations of an S-triple (see [13], p. 996). It is obvious then that e'_- is conjugate to e_+ and hence e'_- is a principal nilpotent element of $\mathfrak{g}$. By Theorem 4, §4.2, one must therefore have that $r_\alpha \neq 0$ for any $\alpha \in \Pi$ and hence we may normalize the c'_α of (4.2.1) by putting

$$c'_\alpha = r_\alpha / c_\alpha$$

so that $e_- = e'_-$.

Now if V is any finite dimensional irreducible module for the three dimensional simple Lie algebra $\mathfrak{a}_0$ with respect to a representation π then one knows that V is a direct sum of $\operatorname{Ker}\pi(e_+)$ and $\operatorname{Im}\pi(e_-)$. Since any finite dimensional $\mathfrak{a}_0$-module is completely reducible the same must be true without the assumption of irreducibility. It follows therefore from (4.2.2) and (4.4.3) that by restriction to $\mathfrak{b}$ we have proved

LEMMA 12. *Let* $\mathfrak{b}$ *be the maximal solvable subalgebra of* $\mathfrak{g}$ *given by* (4.1.11). *Then*

$$\mathfrak{b} = \mathfrak{g}^{e_+} + \operatorname{ad} e_-(\mathfrak{m}) \tag{4.4.4}$$

is a direct sum.

4.5. A subset $\mathfrak{u} \subseteq \mathfrak{g}$ will be called a plane if it is of the form $\mathfrak{u} = w + \mathfrak{a}$ where $w \in \mathfrak{g}$ is an element and $\mathfrak{a}$ is a subspace. (It is called a k-plane if $\dim \mathfrak{a} = k$.)

It is obvious that a k-plane $\mathfrak{u}$ is a k-dimensional affine subvariety of $\mathfrak{g}$. Furthermore $S(\mathfrak{u})$, the restriction of S to $\mathfrak{u}$, is the affine algebra of $\mathfrak{u}$ and in fact if, as above, $\mathfrak{u} = w + \mathfrak{a}$ then writing an arbitrary $x \in \mathfrak{u}$ in the form $x = w + \sum_{i=1}^{k} r_i(x) y_i$ where y_i is a basis of $\mathfrak{a}$ it is clear that the r_i are in $S(\mathfrak{u})$ and define a coordinate system on $\mathfrak{u}$. Moreover one obviously has

$$S(\mathfrak{u}) = \boldsymbol{C}[r_1, \cdots, r_k]. \tag{4.5.1}$$

Note also that if $f \in S$ and $g = f \mid \mathfrak{u}$ then

$$\partial g / \partial r_i = \partial_{y_i} f \mid \mathfrak{u}. \tag{4.5.2}$$

Now let

$$u_{\mathfrak{u}} \colon \mathfrak{u} \to \boldsymbol{C}^l$$

be the restriction of u to $\mathfrak{u}$. The plane $\mathfrak{u}$ will be called transversal if

$$\dim u_{\mathfrak{u}}(\mathfrak{u}) = l.$$

Since $u_{\mathfrak{u}}$ is a morphism it is clear that $\mathfrak{u}$ is transversal if and only if the functions $v_i \in S(\mathfrak{u})$, where $v_i = u_i \mid \mathfrak{u}$, are algebraically independent. But this is the case if and only if the Zariski open subset $\mathfrak{u}_0$ of $\mathfrak{u}$ given by

$$\mathfrak{u}_0 = \{x \in \mathfrak{u} \mid (dv_i)_x,\ i = 1, 2, \cdots, l, \text{ are linearly independent}\}$$

is non-empty (see proof of Proposition 6, §1.6). But by (4.5.2) $\mathfrak{u}_0$ is the set of all points x in $\mathfrak{u}$ where the $k \times l$ matrix $(\partial_{y_j} u_i)(x)$, $i = 1, \cdots, l$, $j = 1, \cdots, k$, is of rank l. Thus (to make it independent of the basis y_j of $\mathfrak{a}$)

if $d(\mathfrak{u}) \subseteq S(\mathfrak{u})$ is the space of functions on $\mathfrak{u}$ spanned by the determinants of all $l \times l$ minors of $\partial_{y_j} u_i \mid \mathfrak{u}$ (put equal to zero if $k < l$) then

(4.5.3) $\qquad \mathfrak{u}$ is transversal if and only if $\dim d(\mathfrak{u}) \geqq 1$

and

(4.5.4) $$\mathfrak{u}_0 = \{x \in \mathfrak{u} \mid g(x) \neq 0 \text{ for some } g \in d(\mathfrak{u})\}.$$

Now if $\mathfrak{u}$ is an l-plane then obviously $\dim \mathfrak{u}$ is either 1 or 0 (according as to whether $\mathfrak{u}$ is transversal or not). In case $\mathfrak{u}$ is a Cartan subalgebra, say $\mathfrak{h}$, then we have already observed that $\mathfrak{u}$ is transversal. In fact

(4.5.5) $$d(\mathfrak{h}) = (\prod_{\phi \in \Delta_+} \phi)$$

and $\mathfrak{h}_0 = \mathfrak{u}_0$ is the set of all $x \in \mathfrak{h}$ which are regular in $\mathfrak{g}$. (See (3.3.5).)

Although $\mathfrak{h}$ is transversal it is not suitable for our needs, mainly because $\mathfrak{h}_0 \neq \mathfrak{h}$.

On the other hand if we put $\mathfrak{v}$ equal to the l-plane given by

(4.5.6) $$\mathfrak{v} = e_- + \mathfrak{g}^{e_+}$$

then not only is $\mathfrak{v}$ transversal but it will also be shown (1) that $d(\mathfrak{v}) = C$ so that $\mathfrak{v}_0 = \mathfrak{v}$. Moreover it will be seen that every element of $\mathfrak{v}$ lies on an orbit of maximal dimension and every such orbit meets $\mathfrak{v}$ in one and only one point.

Remark 19. In a sense $\mathfrak{v}$ is to $\mathfrak{r}$ as $\mathfrak{h}$ is to $\mathfrak{s}$, the set of all semi-simple elements in $\mathfrak{g}$. However $\mathfrak{v}$ has the advantage in that there are no "Weyl group ambiguities" with regard to conjugation. Furthermore the restriction of J to $\mathfrak{h}$ induces only a monomorphism of J into $S(\mathfrak{h})$ whereas (by Theorem 8, § 4.7) the restriction of J to $\mathfrak{v}$ induces an isomorphism of J onto $S(\mathfrak{v})$.

Remark 19′ (added in proof). If $\mathfrak{a}$ is *any* linear complement of $\operatorname{ad} e_-(\mathfrak{m})$ in $\mathfrak{b}$ which is stable under $\operatorname{ad} x_0$ it is clear that $\mathfrak{a}$ may be substituted for $\mathfrak{g}^{e_+}$ in Theorem 5. We now wish to observe that if $\mathfrak{a}$ is substituted for $\mathfrak{g}^{e_+}$ in the definition of $\mathfrak{v}$ above (see (4.5.6)) then all the results to be proved henceforth about $\mathfrak{v}$ will still hold true. That is, the only properties of $\mathfrak{g}^{e_+}$ needed are Theorem 5 and (4.4.4). In this generality the results contain Theorem 0.10 of the Introduction. In particular they apply to the special case of the plane of companion matrices. (See Remark following Theorem 0.10.)

In order to show first that $\mathfrak{v}$ is transversal the following obvious fact will be useful. Assume that $\mathfrak{u}$ and $\mathfrak{w}$ are planes and that $\psi: \mathfrak{u} \to \mathfrak{w}$ is a morphism defined so that

(4.5.7) $$u_{\mathfrak{w}} \circ \psi = u_{\mathfrak{u}}.$$

Then clearly (since $u_{\mathfrak{u}}(\mathfrak{u}) \subseteq u_{\mathfrak{w}}(\mathfrak{w})$)

(4.5.8) $\qquad \mathfrak{u}$ is transversal implies $\mathfrak{w}$ is transversal.

The following proposition asserts among other things that every element of the l-plane $e_- + \mathfrak{h}$ is conjugate to an element in the l-plane $\mathfrak{v}$.

PROPOSITION 19. *For each element $x \in e_- + \mathfrak{h}$ there exists a unique element $a_x \in M$ such that $a_x(x) \in \mathfrak{v}$. Furthermore the map*

(4.5.9)
$$e_- + \mathfrak{h} \to M$$

given by $x \to a_x$ is a morphism.

Proof. Let e_i, $i = 1, 2, \cdots, m$, be a basis of $\mathfrak{m}$ so that $e_i \in \mathfrak{g}^{(j)}$ for some $j > 0$. (See Lemma 9, § 4.1.) In fact let $r(i)$ be that positive number such that $e_i \in \mathfrak{g}^{(r(i))}$. We may then order the basis e_i so that $r(i) \leqq r(i+1)$ for all i.

Now let $w_i = [e_i, e_-]$. Then since $\mathfrak{g}^{e_-} \subseteq \mathfrak{b}^*$ (by (4.4.3) after interchanging Δ_+ and Δ_-), so that $g^{e_-} \cap \mathfrak{m} = (0)$, it follows that

$$w_i \in \mathfrak{g}^{(r(i)-1)}$$

and the w_i form a basis of ad $e_-(\mathfrak{m})$.

Obviously one has

(4.5.10)
$$M \times (e_- + \mathfrak{h}) \to e_- + \mathfrak{b}$$

for the map $(a, x) \to ax$.

Now for any $v \in e_- + \mathfrak{b}$ let $c_i(v)$, $i = 1, 2, \cdots, m$, be the scalar defined so that if v is uniquely written $v = e_- + v_1 + v_2$ where $v_1 \in \mathfrak{g}^{e_+}$ and $v_2 \in \operatorname{ad} e_-(\mathfrak{m})$, according to the decomposition (4.4.4), then

$$v_2 = \sum_{i=1}^{m} c_i(v) w_i.$$

We now make the following inductive assumption about a positive integer k. There exists k functions $g_i \in S(e_- + \mathfrak{h})$, $i = 1, 2, \cdots, k$, such that if $z \in \mathfrak{m}$ where

$$z = \sum_{i=1}^{m} b_i e_i.$$

Then one has for any $x \in e_- + \mathfrak{h}$,

$$c_j(\exp \operatorname{ad} z(x)) = 0 \text{ for all } j \leqq k$$

if and only if

$$b_i = g_i(x) \tag{4.5.11}$$

for all $i \leqq k$.

We now show that the assumption holds for $k+1$. Indeed if we compute $c_{k+1} = c_{k+1}(\exp \operatorname{ad} z(x))$ where z satisfies (4.5.11) it is straightforward, using (4.1.9), to see that

$$c_{k+1} = b_{k+1} + f_0(g_1(x), \cdots, g_k(x)) + \sum_{i=1}^{l} f_i(g_1(x), \cdots, g_k(x)) r_i(x) \tag{4.5.12}$$

where f_i, $i = 0, 1, \cdots, l$, are polynomials in k variables and $r_i \in S(e_- + \mathfrak{h})$ is the same as in (4.5.1) with $\mathfrak{u} = e_- + \mathfrak{h}$.

Now consider the equation $c_{k+1} = 0$. Since (4.5.12) is linear in b_{k+1} we can obviously uniquely solve for b_{k+1} obtaining $b_{k+1} = g_{k+1}(x)$ where $g_{k+1} \in S(e_- + \mathfrak{h})$. Thus the induction assumptions hold for $k+1$. On the other hand (4.5.12) is also valid for $k = 0$ provided $f_0 = 0$ and f_i are constants for $i = 1, 2, \cdots, l$. Thus, similarly, the induction assumption holds for $k = 1$.

We have thus proved inductively that given $x \in e_- + \mathfrak{h}$ there exists a unique element $z = \sum_{i=1}^{m} g_i(x) e_i$ in $\mathfrak{m}$ such that $c_i(\exp \operatorname{ad} z(x)) = 0$ for $i = 1, 2, \cdots, m$. That is, such that $\exp \operatorname{ad} z(x) \in \mathfrak{v} = e_- + \mathfrak{g}^{e_+}$ and that furthermore $g_i \in S(e_- + \mathfrak{h})$. But if $a_x = \exp \operatorname{ad} z$ this proves the lemma since one knows the map $\mathfrak{m} \to M$ given by $z \to \exp \operatorname{ad} z$ is an algebraic isomorphism.

Q. E. D.

PROPOSITION 20. *For every $x \in e_- + \mathfrak{h}$ let $a_x \in M$ be defined as in Proposition 19. If now*

$$\rho \colon e_- + \mathfrak{h} \to \mathfrak{v}$$

is the map given by $x \to a_x(x)$ then ρ is a morphism.

Proof. Obviously the map (4.5.10) is rational and everywhere defined. But by Proposition 19 so is the map

$$e_- + \mathfrak{h} \to M \times (e_- + \mathfrak{h}) \tag{4.5.13}$$

given by $x \to (a_x, x)$. One obtains the proposition by composing (4.5.10) with (4.5.13).

We can now prove

LEMMA 13. *The l-plane $\mathfrak{v} = e_- + \mathfrak{g}^{e_+}$ is transversal.*

Proof. Let ρ be as in Proposition 20. Since $\rho(x) \in O_x$ for any $x \in e_- + \mathfrak{h}$ it is obvious that

$$u_{\mathfrak{v}} \circ \rho = u_{\mathfrak{u}}$$

where $\mathfrak{u} = e_- + \mathfrak{h}$. Thus, by (4.5.8), to prove $\mathfrak{v}$ is transversal it suffices to show that $\mathfrak{u}$ is transversal. But if

$$\tau : \mathfrak{h} \to \mathfrak{u}$$

is the map by $\tau(y) = e_- + y$ then of course τ is a morphism. On the other hand by Lemma 11, § 4.2,

$$u_{\mathfrak{u}} \circ \tau = u_{\mathfrak{h}}.$$

Thus by (4.5.8) to prove $\mathfrak{u}$ is transversal it suffices to show that $\mathfrak{h}$ is transversal. But $\mathfrak{h}$ is transversal by (4.5.5). Q. E. D.

4.6. Now let z_i, $i = 1, 2, \cdots, l$, be the basis of $\mathfrak{g}^{e_+}$ given by Theorem 5, § 4.4. We recall that $z_i \in \mathfrak{g}^{(m_i)}$. On the other hand one has $\deg u_i = m_i + 1$. Hence if we put

$$g_i = \frac{(\partial_{e_-})^{m_i}}{m_i!} u_i \tag{4.6.1}$$

then $g_i \in S^1$. That is, g_i is just a linear functional on $\mathfrak{g}$.

LEMMA 14. *Let $1 \leqq i, j \leqq l$. Then*

$$g_i(z_j) = 0$$

whenever $m_i \neq m_j$.

Proof. Since g_i is a linear function on $\mathfrak{g}$ one has, by (1.1.2) and (1.1.3),

$$m_i! \, g_i(z) = \langle (\partial_{e_-})^{m_i} \partial_z, u_i \rangle \tag{4.6.2}$$

for any $z \in \mathfrak{g}$. But since $e_- \in \mathfrak{g}^{(-1)}$ one has $(\partial_{e_-})^{m_i} \in S_*^{(-m_i)}$ by (4.1.8) and, if $z \in \mathfrak{g}^{(k)}$, then also $(\partial_{e_-})^{m_i}\partial_z \in S_*^{(k-m_i)}$. But if $k \neq m_i$, that is, if $k - m_i \neq 0$ then $g_i(z) = 0$ by (4.1.10) and (4.6.2). In particular $g_i(z_j) = 0$ if $m_j \neq m_i$. Q. E. D.

The following lemma is crucial. Recall that $\mathfrak{b}$ is the maximal solvable algebra given by (4.1.11).

LEMMA 15. *Let $1 \leqq i, j \leqq l$. Then if $m_i \leqq m_j$ the function $\partial_{z_j} u_i$ reduces to a constant on $e_- + \mathfrak{b}$. In fact*

$$\partial_{z_j} u_i \mid e_- + \mathfrak{b} = g_i(z_j). \tag{4.6.3}$$

Furthermore, if $m_i < m_j$ then the constant is zero. That is, in this case

$$\partial_{z_j} u_i \mid e_- + \mathfrak{b} = 0. \tag{4.6.4}$$

Proof. We first observe that if $x \in e_- + \mathfrak{b}$ then for any non-negative integer k one has

$$(\partial_x)^k \in \sum_{p \geqq -k} S_*^{(p)}. \tag{4.6.5}$$

Indeed this is clear from (4.1.8) and Lemma 9, § 4.1, upon writing $x = e_- + y$ where $y \in \mathfrak{b}$ and using binomial expansion. In fact from the binomial expansion it is obvious that $(\partial_{e_-})^k$ is the component of $(\partial_x)^k$ in $S_*^{(-k)}$. That is

$$(\partial_x)^k - (\partial_{e_-})^k \in \sum_{p > -k} S_*^{(p)}. \tag{4.6.6}$$

Now if $f \in J$ it follows, since $z_j \in \mathfrak{g}^{(m_j)}$, that by (1.1.2), (4.1.8) and (4.1.10)

$$\langle \partial, \partial_{z_j} f \rangle = 0 \tag{4.6.7}$$

for all $\partial \in S_*^{(p)}$ where $p \neq -m_j$; in particular for all $p > -m_j$.

But now if $k = m_i$ in (4.6.6) then the sum there is over all p where $p > -m_i$. Hence if $m_i \leqq m_j$, so that $-m_i \geqq -m_j$, the sum in (4.6.6) is over all p, where $p > -m_j$. Thus, by (4.6.7),

$$\langle (\partial_x)^{m_i}, \partial_{z_j} f \rangle = \langle (\partial_{e_-})^{m_i}, \partial_{z_j} f \rangle \tag{4.6.8}$$

for all $x \in e_- + \mathfrak{b}$ whenever $m_i \leqq m_j$.

We now assert that this implies

$$(\partial_{z_j} u_i)(x) = g_i(z_j) \tag{4.6.9}$$

for any $x \in e_- + \mathfrak{b}$ whenever $m_i \leqq m_j$. Indeed replace f by u_i and divide by $m_i!$ in (4.6.8). Recalling that $\deg \partial_{z_j} u_i = m_i$ the left side of (4.6.8) becomes the left side of (4.6.9) by (1.1.3). On the other hand by (1.1.2) the right side of (4.6.8) becomes the right side of (4.6.9) by (4.6.2). (Recall that S_* is commutative.) This proves (4.6.3).

But now if $m_i < m_j$ then the right side of (4.6.9) vanishes by Lemma 14. Hence one obtains (4.6.4). Q. E. D.

We can now show that the Jacobian matrix of functions $\partial_{z_j} u_i \mid \mathfrak{v}$ of the map $u_{\mathfrak{v}}$ takes triangular form and reduces to non-zero constants along the diagonal.

THEOREM 6. *There exists a unique basis z_j, $j=1,2,\cdots,l$, of $\mathfrak{g}^{e_+}$ such that for $i=1,2,\cdots,l$,*

$$g_i(z_j)=\delta_{ij}. \tag{4.6.10}$$

Furthermore the basis satisfies the condition of Theorem 5. That is $z_j\in\mathfrak{g}^{(m_j)}$ for all j. Furthermore

$$\partial_{z_j}u_i \mid \mathfrak{v}=\begin{cases}0 \text{ for } i<j\\ 1 \text{ for } i=j\end{cases} \tag{4.6.11}$$

so that not only is $\mathfrak{v}$ transversal but in fact

$$\det \partial_{z_j}u_i \mid \mathfrak{v}=1 \tag{4.6.12}$$

and hence (see § 4.5)

$$d(\mathfrak{v})=\boldsymbol{C}. \tag{4.6.13}$$

Proof. An integer k will be called an exponent if $k=m_i$ for some i. Let E be the set of exponents and for any $k\in E$ let $P_k\subseteq\{1,2,\cdots,l\}$ be the set of all i such that $m_i=k$. Now, for any $k\in E$ put

$$b_k=\det_{i,j\in P_k} g_i(z_j).$$

It then follows from Lemma 15 that $\det\partial_{z_j}u_i$ is a constant on $e_-+\mathfrak{b}$ and in fact

$$\det\partial_{z_j}u_i \mid e_-+\mathfrak{b}=\prod_{k\in E} b_k.$$

But since $\mathfrak{v}\subseteq e_-+\mathfrak{b}$ and since $\mathfrak{v}$ is transversal (Lemma 13) this constant can not be zero. Thus $b_k\neq 0$ for any $k\in E$. That is, the matrix $g_i(z_j)$, $i,j\in P_k$, is non-singular and this holds for any $k\in E$. It follows immediately then from Lemma 14 that a unique basis z_j of $\mathfrak{g}^{e_+}$ exists so that (4.6.10) is satisfied. It is also clear from Lemma 14 that the z_j necessarily satisfy the condition of Theorem 5. Since $\mathfrak{v}\subseteq e_-+\mathfrak{b}$ the remaining statements follow from Lemma 15. Q.E.D.

4.7. We will assume from here on that the basis z_j of $\mathfrak{g}^{e_+}$ is given by Theorem 6. Now let $s_j\in S(\mathfrak{v})$ be the coordinate functions on $\mathfrak{v}$ corresponding to the z_j. That is, s_j is such that $x=e_-+\sum s_j(x)z_j$. We have already noted that $S(\mathfrak{v})=\boldsymbol{C}[s_1,\cdots,s_l]$ (see (4.5.1)).

In notational simplicity let

$$t\colon\mathfrak{v}\to\boldsymbol{C}^l$$

(instead of $u_{\mathfrak{v}}$) denote the restriction of u to $\mathfrak{v}$. Thus for any $x\in\mathfrak{v}$

$$t(x)=(t_1(x),\cdots,t_l(x))$$

where $t_i = u_i \mid \mathfrak{v}$. It follows therefore from (4.5.2) that

$$\frac{\partial t_i}{\partial s_j} = \partial_{z_j} u_i \mid \mathfrak{v}. \tag{4.7.1}$$

Now if $\mathfrak{u}$ is an arbitrary k-plane in $\mathfrak{g}$ let

$$J \to S(\mathfrak{u}) \tag{4.7.2}$$

be the ring homomorphism obtained by restricting an invariant polynomial to $\mathfrak{u}$. Now in general one could hardly expect (4.7.2) to be an isomorphism. Indeed if (4.7.2) is an epimorphism one must have $k \geqq l$ and if (4.7.2) is a monomorphism one must have $k \leqq l$ (since the u_i are algebraically independent). Hence the possibility could only exist if $k = l$. If $\mathfrak{u}$ is a Cartan subalgebra the one knows that (4.7.2) is a monomorphism and the image is the space of Weyl group invariants. Hence in such a case (4.7.2) is an isomorphism only when $\mathfrak{g}$ is abelian. On the other hand when $\mathfrak{u} = \mathfrak{v}$ we have, in general, the following corollary of Theorem 6

THEOREM 7. *If $\mathfrak{u} = \mathfrak{v}$ then (4.7.2) is an isomorphism. Moreover the map*

$$t \colon \mathfrak{v} \to \boldsymbol{C}^l \tag{4.7.3}$$

obtained by restricting u to $\mathfrak{v}$ is an algebraic isomorphism so that $t_1, \cdots, t_l$ define a global coordinate system on $\mathfrak{v}$.

Furthermore the relationship between the t_i and the linear coordinates s_j on $\mathfrak{v}$ is as follows: For $i = 1, 2, \cdots, l$, there exists polynomials p_i and q_i in $i-1$ variables without constant term such that

$$t_i = s_i + p_i(s_1, \cdots, s_{i-1}) \tag{4.7.4}$$

and

$$s_i = t_i + q_i(t_1, \cdots, t_{i-1}) \tag{4.7.5}$$

Proof. To prove the theorem observe that it suffices only to prove (4.7.4). Indeed using (4.7.4) we can solve for s_i obtaining (4.7.5). It is then immediate that t is one-one, onto and is in fact a biregular birational map. Since the t_i generate the image of J in $S(\mathfrak{v})$ it is then also obvious that (4.7.2) is an isomorphism.

But now (4.7.4) is immediate from (4.6.11) and (4.7.1).

Finally by definition of the coordinate system s_i one has $s_i(e_-) = 0$ for all i. On the other hand $t_i(e_-) = 0$ for all i since $t(e_-) = u(e_-) = 0$ (recall that e_- is nilpotent). Thus the p_i and q_i have no constant term. Q.E.D.

Any orbit O of semi-simple elements (i.e., $O \in \mathcal{O}_{\mathfrak{s}}$) intersects $\mathfrak{h}$ in a finite number but in general more than one point. We now find that any orbit O of maximal dimension (i.e. $O \in \mathcal{O}_{\mathfrak{r}}$) intersects $\mathfrak{v}$ in one and only one point.

THEOREM 8. *One has* $\mathfrak{v} \subseteq \mathfrak{r}$. *Furthermore if*

$$\mathfrak{v} \to \mathcal{O}_{\mathfrak{r}} \tag{4.7.6}$$

is the map given by $x \to O_x$ *then* (4.7.6) *is a bijection. That is, no two distinct elements of* $\mathfrak{v}$ *are conjugate and every element in* $\mathfrak{r}$ *is conjugate to one and only one element in* $\mathfrak{v}$.

Proof. Since $\mathfrak{v} \subseteq e_- + \mathfrak{b}$ one has $\mathfrak{v} \subseteq \mathfrak{r}$ by Lemma 10, § 4.2. But now if we compose (4.7.6) with the bijection $\eta_{\mathfrak{r}}$ (Theorem 2, § 3.5) we obtain the bijection t (see Theorem 7). Hence (4.7.6) must be a bijection. Q. E. D.

We can now obtain the following characterization of the set $\mathfrak{r}$.

THEOREM 9. *Let* $x \in \mathfrak{g}$. *Then* $x \in \mathfrak{r}$ *if and only if* $(du_i)_x$, $i = 1, 2, \cdots, l$, *are linearly independent.*

Proof. By (4.6.12) the matrix $(\partial_{z_j} u_i)(x)$ is of rank l for any $x \in \mathfrak{v}$. Thus $(du_i)_x$, $i = 1, \cdots, l$, are linearly independent for any $x \in \mathfrak{v}$. But then by Theorem 9 and conjugation the same is true for any $x \in \mathfrak{r}$.

Now let $x \in \mathfrak{g}$ but where $x \notin \mathfrak{r}$. We must prove that the $(du_i)_x$, $i = 1, 2, \cdots, l$, are linearly dependent. Assume first that $x \in \mathfrak{s}$ (that is, x is semi-simple). Then x is not regular so that $\mathfrak{g}^x$ contains a Cartan subalgebra as a proper subalgebra. It follows therefore that if $\mathfrak{u}$ is the center of $\mathfrak{g}^x$ and $l_x = \dim \mathfrak{u}$ one has $l_x < l$.

Furthermore it is also clear that $\mathfrak{u}$ is the set of fixed vectors for the action of G^x on $\mathfrak{g}$ (recall that G^x is connected. See Lemma 5, § 3.2). Thus there exists a non-abelian simple component $\mathfrak{g}_1$ of $\mathfrak{g}$ of rank, say l_1, such that in the notation of § 2.1

$$d_\psi{}^{G^x} < l_1$$

where $\psi \in D$ and the irreducible representation ν^ψ is equivalent to the adjoint action of G on $\mathfrak{g}_1$. (One uses here that ν^ψ is self-contragredient.) But then by Proposition 8 the multiplicity of ν^ψ in the G-module $R(G/G^x)$ or $R(O_x)$ is less than l_1. But $S(O_x) \subseteq R(O_x)$ (in fact here $S(O_x) = R(O_x)$ since O_x is closed. See (2.2.3)) so that the same is true for the G-module $S(O_x)$.

Now let $\mathfrak{g}_2$ be an ideal in $\mathfrak{g}$ complementary to $\mathfrak{g}_1$ so that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a direct sum. It is obvious that we may choose the primitive invariants so that for $1 \leqq i_1 < i_2 < \cdots < i_{l_1} \leqq l$ one has

$$u_{i_k}(x_1 + x_2) = u_{i_k}(x_1)$$

where $x_j \in \mathfrak{g}_j$, $j = 1, 2$, and $k = 1, 2, \cdots, l_1$. (The dependence or independence of the $(du_i)_x$ obviously does not depend upon the choice of the primitive generators u_i). That is, for any such k,

$$\partial_y(u_{i_k}) = 0 \text{ for any } y \in \mathfrak{g}_2 \tag{4.7.7}$$

Now for any $u \in J$, $y \in \mathfrak{g}$ and $a \in G$ one clearly has

$$a \cdot \partial_y u = \partial_{ay} u. \tag{4.7.8}$$

Now let y be the root vector e_ϕ where ϕ is the highest root of $\mathfrak{g}_1$. It follows then from (4.7.8) that $\partial_y u \in S^\psi$, and hence $\partial_y u \mid O_x \in S^\psi(O_x)$, and in fact, if not zero, these functions are highest weight vectors. But then since the multiplicity of ν^ψ in $S(O_x)$ is less than l_1 it follows that $\partial_y u_{i_k} \mid O_x$, $k = 1, 2, \cdots, l_1$, must be linearly dependent. Thus there exists scalars c_{i_k}, not all zero, such that if $u = \sum_k c_{i_k} u_{i_k}$ then $\partial_y u \mid O_x = 0$. But then by (4.7.7) and (4.7.8) one has $\partial_z u \mid O_x = 0$ for all $z \in \mathfrak{g}$. In particular then

$$(\partial_z u)(x) = 0 \text{ for all } z \in \mathfrak{g}.$$

Thus $(du)_x = 0$ and hence the $(du_i)_x$, $i = 1, 2, \cdots, l$, are linearly dependent.

Now assume that $x \in \mathfrak{g}$ is arbitrary where $x \notin \mathfrak{r}$. Since the set of all y such that the $(du_i)_y$ are linearly independent is an open set (see § 1.6) to prove the theorem it suffices from above to show that there exists a sequence x_k such that $x_k \to x$ where the x_k are semi-simple but not regular. In fact it suffices to show this for the case where x is nilpotent. Indeed if $x = y + z$ is the decomposition (3.1.2) for x then by Proposition 13 z is not a principal nilpotent element if $\mathfrak{g}^y$. Hence if such a sequence has been shown to exist in the nilpotent case there exists a sequence y_k of non-regular semi-simple elements in $\mathfrak{g}^y$ such that $y_k \to z$. Hence $x_k = y + y_k$ converges to x. But clearly x_k is semi-simple and non-regular in $\mathfrak{g}$.

Hence we may assume that x is a non-principal nilpotent element of $\mathfrak{g}$. By conjugation we may also assume $x \in \mathfrak{m}$ so that, by Theorem 4, § 4.2, $x = \sum_{\phi \in \Delta_+} c_\phi e_\phi$ where there exists a simple root $\alpha \in \Pi$ such that $c_\alpha = 0$. Let

$$\mathfrak{n} = \sum_{\substack{\phi \in \Delta_+ \\ \phi \neq \alpha}} (e_\phi)$$

so that $x \in \mathfrak{n}$. Also let $y \in \mathfrak{h}$ be such that $\langle \beta, y \rangle$ is positive for $\beta \in \Pi$ where $\beta \neq \alpha$ and $\langle \alpha, y \rangle = 0$. It follows therefore that y is semi-simple and non-regular. Furthermore it is clear that $\mathfrak{n}$ is in the range of ad y. In fact $[y, \mathfrak{n}] = \mathfrak{n}$ so that by the argument of Theorem 3.6 in [13] (See Note following this theorem) every element in $y + \mathfrak{n}$ is conjugate to y. In particular

$y + x$ is conjugate to y and hence is semi-simple and non-regular. But we may substitute y/k for y. Hence if $x_k = y/k + x$ then $x_k \to x$ where x_k is semi-simple and non-regular. Q. E. D.

4.8. Generalizing the definition of J^+, let J^ξ, for any $\xi \in \boldsymbol{C}^l$, be the (maximal) ideal in J generated by $u_i - \xi_i$, $i = 1, 2, \cdots, l$. Obviously then

$$J^\xi S = (u_1 - \xi_1, \cdots, u_l - \xi_l)$$

Recall that $P(\xi) \subseteq \mathfrak{g}$ is the set of zeros of $J^\xi S$. We can now prove

THEOREM 10. *Let $\xi \in \boldsymbol{C}^l$ be arbitrary. Then $P(\xi)$ is a Zariski closed subvariety (of X) of dimension $n - l$ and its ideal $I(P(\xi))$ is given by*

$$I(P(\xi)) = J^\xi S \tag{4.8.1}$$

so that (a) *$P(\xi)$ is a complete intersection and* (b) *$J^\xi S$ is a prime ideal. Furthermore if $P(\xi)_s$ is the set of simple points of $P(\xi)$ then*

$$P(\xi)_s = O^r(\xi) \tag{4.8.2}$$

where $O^{\mathfrak{r}}(\xi)$ is the unique orbit of dimension $n - r$ in $P(\xi)$. (See Theorem 4, § 3.8). Moreover the set of non-simple points in $P(\xi)$ is a finite union of orbits and has a codimension of at least 2 in $P(\xi)$.

Proof. By Theorem 3, § 3.8, $P(\xi)$ is a Zariski closed subvariety of dimension $n - l$ and $P(\xi) \cap \mathfrak{r}$ is the orbit $O^{\mathfrak{r}}(\xi)$ of dimension $n - l$ defined in § 3.5.

But if $x \in P(\xi)$ then $(du_i)_x$, $i = 1, 2, \cdots, l$, are linearly independent if and only if $x \in O^{\mathfrak{r}}(\xi)$ by Theorem 9, § 4.7. Hence by Proposition 6, § 1.6, one obtains (4.8.1) and (4.8.2). Furthermore the set of non-simple points of $P(\xi)$ is a finite union of orbits and have a codimension of at least two in $P(\xi)$ by Theorem 3, § 3.8. Q. E. D.

Remark 20. By (4.8.2) note that $P(\xi)$ is a non-singular variety if and only if $O^{\mathfrak{r}}(\xi) = P(\xi)$; that is, if and only if $O^{\mathfrak{r}}(\xi)$ is an orbit of regular elements.

Now let B be a non-singular symmetric G-invariant bilinear form on $\mathfrak{g}$. (One extends the Cartan-Killing form of $[\mathfrak{g}, \mathfrak{g}]$ in an obvious way.)

Let $H \subseteq S$ be the graded space of G-harmonic polynomials defined as in § 1.4 so that $S = J^+S + H$ is a G-module direct sum.

Remark 21. By Proposition 16, § 3.8, note that the subspace $H_P \subseteq S$ (see § 1.4) is the space of polynomials spanned by all powers of $g^k \in S^k$, $k = 0, 1, \cdots$, and all linear functionals $g \in S^1$ corresponding to ∂_x, under the map (1.4.1), where x is an arbitrary nilpotent element of $\mathfrak{g}$.

Finally for any $\lambda \in D$ recall that l_λ is the multiplicity the zero weight for the representation ν^λ. That is, $l_\lambda = d_\lambda{}^A$ for any Cartan subgroup $A \subseteq G$. (See § 2.1.)

We can now prove

THEOREM 11. *Let* $\mathfrak{g}$ *be a complex reductive Lie algebra and let* S *be the ring of all polynomials on* $\mathfrak{g}$. *Let* G *be the adjoint group of* $\mathfrak{g}$ *and let* $J \subseteq S$ *be the subring of* G*-invariant polynomials. Then* S *is free as a* J*-module (under multiplication). Furthermore*

$$H = H_P \tag{4.8.3}$$

and if

$$J \otimes H \to S \tag{4.8.4}$$

is the map given by $u \otimes h \to uh$ *then* (4.8.4) *is a* G*-module isomorphism. Also for any* $\xi \in \boldsymbol{C}^l$ *the ideal* $J^\xi S$ *is prime in* S *and*

$$S = J^\xi S + H \tag{4.8.5}$$

is a direct sum.

Moreover H *is completely reducible as a* G*-module and for any* $\lambda \in D$ *the irreducible representation* ν^λ *of* G *occurs with multiplicity* l_λ *in* H *(so that* $H = \sum_{\lambda \in D} H^\lambda$ *is a direct sum and* $\dim H^\lambda = l_\lambda d_\lambda$ *where* d_λ *is the dimension of* ν^λ*).*

Let $\mathfrak{r}$ *be the set of all* $x \in \mathfrak{g}$ *whose corresponding orbit* O_x *has maximal dimension* $(n - l)$. *Let* $x \in \mathfrak{r}$ *and let* $S(O_x)$ *be the ring of functions on* $S(O_x)$ *obtained by restricting* S *to* O_x *and let*

$$H \to S(O_x) \tag{4.8.6}$$

be the map obtained by restricting G*-harmonic functions to* O_x. *Then* (4.8.6) *is a* G*-module isomorphism so that all* $S(O_x)$, *for* $x \in \mathfrak{r}$, *are isomorphic as* G*-modules.*

Proof. By Theorem 1, § 3.4, and Proposition 15, § 3.8, the cone P has a dense orbit and by Theorem 10 J^+S is a prime ideal (case where $\xi = 0$). One obtains (4.8.3) and (4.8.4) as a consequence of Proposition 4, § 1.4. Moreover the map (4.8.6) is an isomorphism by (1.5.2) and Proposition 18, § 4.3. Since $J^\xi S$ is prime by Theorem 10 one therefore obtains the direct sum decomposition (4.8.5) (using 3.8.7)).

Obviously H is a completely reducible G-module. To find the multiplicity of ν^λ in H one uses the isomorphism (4.8.6) and chooses x to be regular.

In such a case $R(O_x) = S(O_x)$ by (2.2.3) since $O_x = \bar{O}_x$. But the multiplicity of ν^λ in $R(O_x)$ is l_λ by Proposition 8 since G^x is a Cartan subgroup of G. Q.E.D.

Remark 22. Except for (4.8.3) note that by Proposition 2, § 1.3, one may replace H in Theorem 11 by any G-stable complement of J^+S in S.

Also we wish to note that every irreducible representation of G appears with positive multiplicity in H. That is $l_\lambda \geqq 1$ for any $\lambda \in D$. Indeed let $Z \subseteq \mathfrak{h}'$ be the discrete subgroup generated by all roots $\phi \in \Delta$. Since G is the adjoint group one knows that every weight of ν^λ can be regarded as an element of Z. On the other hand if D is identified with the subset of all $\mu \in Z$ such that $\mu(x_\phi) \geqq 0$, for all $\phi \in \Delta_+$, where $x_\phi \in \mathfrak{h}$ is the root normal corresponding to ϕ, in such a way that λ is the highest weight of ν^λ then it is known that any $\mu \in D$ is a weight of ν^λ if and only if $\lambda - \mu$ is a non-negative integral combination of positive roots. Since $\mu = 0$ always satisfies this condition it follows that $l_\lambda \geqq 1$.

Remark 22′. It has been pointed out to us by Serre, as we ourselves have also noticed, one of the conclusions of Theorem 11, namely $S = J \otimes H$, can be obtained directly from the theorem of Chevalley mentioned in Example 1 of the Introduction.

(*Added in proof.*) As used above, the primeness of J^+S implies that J^+S is the ideal of the variety P. Another application of this fact in algebraic geometry is the following theorem.

Clearly P meets the Cartan subalgebra $\mathfrak{h}$ (or any Cartan subalgebra) only at the origin and $\dim P + \dim \mathfrak{h} = \dim \mathfrak{g}$.

THEOREM 12. *Let w be the intersection multiplicity of P and $\mathfrak{h}$ at the origin. Then w is the order of the Weyl group.*

Proof. Let L be the local ring at the origin (as a point of $\mathfrak{g}$) and let I and K, respectively, be the prime ideals of L corresponding to $\mathfrak{h}$ and P. Now one knows that w is the alternating sum of the integers $\dim \mathrm{Tor}_i^L(L/I, L/K)$. But clearly

$$\dim \mathrm{Tor}_i^L(L/I, L/K) = \dim \mathrm{Tor}_i^S(S(\mathfrak{h}), S(P))$$

We now observe, however, that by Theorem 11 one has $\mathrm{Tor}_i^S(S(\mathfrak{h}), S(P)) = 0$ for $i > 0$. Indeed since S is J-free and $S(P) = S/J^+S$ there is a spectral sequence converging to $\mathrm{Tor}_i^S(S(\mathfrak{h}), S(P))$ where

$$E_{p,q}^2 = \mathrm{Tor}_p^{S(P)}(\mathrm{Tor}_q^J(S(\mathfrak{h}), \boldsymbol{C}), S(P))$$

(See Cartan and Eilenberg, Homological Algebra, Chapter XVI, § 6, Theorem

6.1, p. 349). On the other hand since $S(\mathfrak{h})$ is J-free (See [2]) one has $E_{p,q}{}^2=0$ unless $p=q=0$ and $E_{0,0}{}^2=S(\mathfrak{h})/J^+S(\mathfrak{h})$. This verifies the observation and also proves that

$$w=\dim S(\mathfrak{h})/J^+S(\mathfrak{h})\,. \tag{4.8.7}$$

But by [2] (and also from the cohomology theory of the generalized flag manifold) one knows that the right side of (4.8.7) is the order of the Weyl group. Q.E.D.

4.9. Now let $p(t)$ be the formal power series

$$p(t)=\sum_{k=0}^{\infty}\dim H^k t^k.$$

Since S is isomorphic to the tensor product $J\otimes H$ by Theorem 11 it follows therefore that

$$p(t)=\frac{\prod_{i=1}^{l}1-t^{m_i}}{(1-t)^n}\,. \tag{4.9.1}$$

Now let $\bar{\mathfrak{g}}$ denote the projective space of all one dimensional subspaces of $\mathfrak{g}$ and let

$$\mathfrak{g}-(0)\to\bar{\mathfrak{g}} \tag{4.9.2}$$

be the canonical projection map. If $\mathfrak{u}$ is a homogeneous Zariski closed subvariety of $\mathfrak{g}$ let $\bar{\mathfrak{u}}\subseteq\mathfrak{g}$ be the image of $\mathfrak{u}-(0)$ under (4.9.2) so that $\bar{\mathfrak{u}}$ is a projective variety.

In the remaining portion of this section we use the notation of FAC, [15].

Consider the projective variety $\bar{\mathfrak{p}}\subseteq\bar{\mathfrak{g}}$ defined by the cone $\mathfrak{p}$ of all nilpotent elements in $\mathfrak{g}$. The dimensional determination of the sheaf cohomology groups $H^j(\bar{\mathfrak{p}},\mathcal{O}(k))$ for all $j,k\in\boldsymbol{Z}$ is given by

THEOREM 13. *Let $k\in\boldsymbol{Z}$ be arbitrary. Then*

$$H^j(\bar{\mathfrak{p}},\mathcal{O}(k))=0 \tag{4.9.3}$$

where j is any integer other than 0 or $n-l-1$. On the other hand

$$H^0(\bar{\mathfrak{p}},\mathcal{O}(k))=H^{n-l-1}(\bar{\mathfrak{p}},\mathcal{O}(\frac{l-(2k+n)}{2}))\,. \tag{4.9.4}$$

(Recall that $n-l$ is even.)

Furthermore if $q(t)$ is the formal power series defined by

$$q(i)=\sum_{k=-\infty}^{\infty}\dim H^0(\bar{\mathfrak{p}},\mathcal{O}(k))\,t^k$$

then $q(t) = p(t)$. *That is* $q(t)$ *may be given by*

$$(4.9.5) \qquad q(t) = \frac{\prod_{i=1}^{l} 1 - t^{m_i}}{(1-t)^n}.$$

Proof. By Proposition 15, § 3.8, one has $\mathfrak{p} = P$. Hence by Theorem 10, for $\xi = 0$, one has that $\bar{\mathfrak{p}}$ is a complete intersection so that Proposition 5, § 78, in [15] is applicable. This yields (4.9.3) and (4.9.4) since clearly $N = l - n$. On the other hand in the notation of [15] one has $\dim H^0(\mathfrak{p}, \mathcal{O}(k)) - \dim S^k(\bar{\mathfrak{p}})$. But since J^+S is the prime ideal of $\bar{\mathfrak{p}}$ by (4.8.1) and since $S = J^+S + H$ is a direct sum it follows that $\dim H^k = \dim S^k(\bar{\mathfrak{p}})$. Thus $p(t) = q(t)$ and hence one obtains (4.9.5) from (4.9.1). Q.E.D.

4.10. One obviously has an isomorphism $f \to f^*$ of J onto the ring $S(\boldsymbol{C}^l)$ of all polynomials on $\boldsymbol{C}^l$ by defining, for any $\xi \in \boldsymbol{C}^l$,

$$f^*(\xi) = p(\xi_1, \cdots, \xi_l)$$

where p is that polynomial in l variables such that $f = p(u_1, \cdots, u_l)$. We recall that the u_i are the primitive invariant polynomials.

Now let $\mathfrak{U}$ be the set of all Zariski closed subvarieties of $\boldsymbol{C}^l$. We may use $\mathfrak{U}$ to index all the prime ideals in J by defining $J^U \subseteq J$ for $U \in \mathfrak{U}$ to be the prime ideal consisting of all $f \in J$ such that f^* vanishes on U.

If $I \subseteq S$ is any prime ideal in S let $\mathfrak{u}(I) \subseteq \mathfrak{g}$ be the corresponding Zariski closed subvariety of $\mathfrak{g}$ of all points in $\mathfrak{g}$ at which I vanishes. It is of course obvious that I is stable under G if and only if $\mathfrak{u}(I)$ is stable under the action of G on $\mathfrak{g}$; that is, if and only if $\mathfrak{u}$ is a union of orbits. It is clear of course that if I is generated by invariant polynomials then I is G-stable. However this is not a necessary condition. The question arises: how does one characterize all those G-stable Zariski closed subvarieties $\mathfrak{u}$ of $\mathfrak{g}$ whose prime ideal $I(\mathfrak{u})$ is generated by invariant polynomials? The following theorem asserts that a necessary and sufficient condition is $\mathfrak{u} \cap \mathfrak{r}$ should not be empty. Note that since $\mathfrak{r}$ is a Zariski open subset of $\mathfrak{g}$ (obvious from its definition. Also see the proof of Proposition 6, § 1.6) then $\mathfrak{u} \cap \mathfrak{r}$ is Zariski dense in $\mathfrak{u}$ in case $\mathfrak{u} \cap \mathfrak{r}$ is not empty. Theorem 14 also generalizes most of Theorem 10 (case where U has only one point).

THEOREM 14. *Let* $J^U \subseteq J$, $U \in \mathfrak{U}$, *be any prime ideal in* J. *Then* $J^U S$ *is a prime ideal in* S *and*

$$(4.10.1) \qquad \mathfrak{u}(J^U S) = \bigcup_{\xi \in U} P(\xi).$$

That is, $\mathfrak{u}(J^U S) = u^{-1}(U)$ *where* u *is the map* (3.3.2). *Moreover*

$$U \to \mathfrak{u}(J^U S) \tag{4.10.2}$$

defines a one-one correspondence between the set of all Zariski closed subvarieties of $\boldsymbol{C}^l$ *and the set of all* G*-stable Zariski closed subvarieties* $\mathfrak{u} \subseteq \mathfrak{g}$ *such that* $\mathfrak{u} \cap \mathfrak{r}$ *is not empty. In particular* $U = u(\mathfrak{u})$ *is in* $\mathfrak{U}$ *for such a subvariety* $\mathfrak{u} \subseteq \mathfrak{g}$, $\mathfrak{u} = \mathfrak{u}(J^U S)$ *and*

$$\mathfrak{u} \cap \mathfrak{r} = \bigcup_{\xi \in U} O^{\mathfrak{r}}(\xi). \tag{4.10.3}$$

Let $U \in \mathfrak{U}$ *and put* $\mathfrak{u} = \mathfrak{u}(J^U S)$. *Then*

$$\operatorname{codim} U \text{ in } \boldsymbol{C}^l = \operatorname{codim} \mathfrak{u} \text{ in } \mathfrak{g}. \tag{4.10.4}$$

Furthermore if $x \in \mathfrak{u} \cap \mathfrak{r}$ *then* x *is a simple point of* $\mathfrak{u}$ *if and only if* $u(x)$ *is a simple point of* U. *Finally if* $\mathfrak{r}^c$ *is the (Zariski closed in* $\mathfrak{g}$*) complement of* $\mathfrak{r}$ *in* $\mathfrak{g}$ *then*

$$\operatorname{codim} \mathfrak{u} \cap \mathfrak{r}^c \text{ (in } \mathfrak{u}) \geqq 2. \tag{4.10.5}$$

Proof. Let J' be any ideal in J. It is immediate that $J'S$ is the image of $J' \otimes H$ under the isomorphism (4.8.4) and hence one has

$$J'S \cap J = J'. \tag{4.10.6}$$

Now assume that J' is a radical ideal (an ideal equal to its own radical) in J. We will show that $J'S$ is a radical ideal in S.

Let J'_* be the radical ideal in $S(\boldsymbol{C}^l)$ corresponding to J' under the isomorphism $J \to S(\boldsymbol{C}^l)$ where $f \to f^*$ and let $U \subseteq \boldsymbol{C}^l$ be the Zariski closed set of all $\xi \in \boldsymbol{C}^l$ at which J'_* vanishes. It is obvious that if $\mathfrak{u}$ is the Zariski closed set, in $\mathfrak{g}$, of all $x \in \mathfrak{g}$ at which $J'S$ vanishes then $\mathfrak{u} = u^{-1}(U)$ or

$$\mathfrak{u} = \bigcup_{\xi \in U} P(\xi). \tag{4.10.7}$$

To prove $J'S$ is a radical ideal it suffices to show that if $f \in S$ is assumed to vanish on $\mathfrak{u}$ then $f \in J'S$. By Theorem 11, § 4.8, we can write $f = \sum f_i h_i$ where $f_i \in J$, $h_i \in H$ and the h_i are linearly independent. Let $\xi \in U$. Then since the f_i reduce to constants on $P(\xi)$ it follows from the isomorphism (4.8.6) and (3.8.7) that since f vanishes on $P(\xi)$ the f_i also vanish on $P(\xi)$. Thus the f_i are in J' by the nullstellensatz and hence $f \in J'S$ so that $J'S$ is a radical ideal.

Now let $U \in \mathfrak{U}$ so that J^U is prime in J. Put $J' = J^U$ so that, from above, $J^U S$ is a radical ideal in S. To prove that $J^U S$ is prime it suffices now only to show that $\mathfrak{u}$ is irreducible.

Let $f \in S$ and let $U(f)$ be the set of all $\xi \in U$ such that $f \mid P(\xi)$ is not zero. Obviously $f \mid \mathfrak{u}$ is not zero if and only if $U(f)$ is not empty. We first show that in such a case $U(f)$ contains a non-empty Zariski open subset of U. Indeed assume $U(f)$ is not empty and $\xi \in U(f)$. Then $f \mid O^{\mathfrak{r}}(\xi)$ is not zero by (3.8.7). Hence there exists $a \in G$ such that $(a \cdot f) \mid O^{\mathfrak{r}}(\xi) \cap \mathfrak{v}$ is not zero, by Theorem 8, § 4.7, where $\mathfrak{v}$ is defined as in (4.5.6). Thus $(a \cdot f) \mid \mathfrak{v}$ does not vanish on a Zariski subset of $\mathfrak{v}$ containing $O^{r}(\xi) \cap \mathfrak{v}$. Using the isomorphism (4.7.3) it follows that $U(a \cdot f)$ contains a non-empty Zariski open subset of U. But clearly $U(a \cdot f) = U(f)$. Hence $U(f)$ contains such a subset.

Now let $f_i \in S$, $i = 1, 2$, be arbitrary except that $f_i \mid \mathfrak{u}$ is not zero. To show $\mathfrak{u}$ is irreducible we must show that $f_1 f_2 \mid \mathfrak{u}$ is not zero. From above it follows that $U(f_i)$ contain a non-empty Zariski open subset of U. But since U is irreducible it follows that $U(f_1) \cap U(f_2)$ is not empty. But then $f_1 f_2 \mid P(\xi)$ is not zero in case $\xi \in U(f_1) \cap U(f_2)$ since $P(\xi)$ is irreducible by Theorem 3, § 3.8. Thus $\mathfrak{u}$ is irreducible and hence $J^U S$ is prime.

The relation (4.10.1) is just (4.10.7). Furthermore if $\mathfrak{u} = \mathfrak{u}(J^U S)$ then (4.10.3) follows from (3.8.4).

Moreover, using (4.10.1), it is immediate that the map, given by (4.10.2), from $\mathcal{U}$ into the set of all Zariski closed G-stable subvarieties $\mathfrak{u}$ of $\mathfrak{g}$ such that $\mathfrak{u} \cap \mathfrak{r}$ is not empty is injective. Now assume that $\mathfrak{u}$ is such a subvariety. We will show that $\mathfrak{u}$ is in the image of the map defined by (4.10.2). Let the set $U \subseteq \boldsymbol{C}^l$ be defined by putting $U = u(\mathfrak{u} \cap \mathfrak{r})$. Since $\mathfrak{u}$ is Zariski irreducible and $\mathfrak{u} \cap \mathfrak{r}$ is Zariski dense in $\mathfrak{u}$ it follows that U is Zariski irreducible. On the other hand by Theorem 8, § 4.7, it is clear that U corresponds to $\mathfrak{u} \cap \mathfrak{v}$ under the isomorphism (4.7.3). But since $\mathfrak{u} \cap \mathfrak{v}$ is Zariski closed in $\mathfrak{v}$ it follows that U is Zariski closed in $\boldsymbol{C}^l$. Hence $U \in \mathcal{U}$. But U is Zariski dense in $u(\mathfrak{u})$. But this implies $u(\mathfrak{u}) = U$ since U is Zariski closed. Thus $\mathfrak{u} \subseteq u^{-1}(U)$. But $u^{-1}(U)$ is clearly in the Zariski closure of $\mathfrak{u} \cap \mathfrak{r}$ since the relation (4.10.3) obviously holds. Thus $\mathfrak{u} = u^{-1}(U)$ or $\mathfrak{u} = \mathfrak{u}(J^U S)$.

Now obviously $(du_i^*)_\xi$, $i = 1, 2, \cdots, l$, are linearly independent at any point $\xi \in \boldsymbol{C}^l$. Since $(df)_x$ is in the span of the $(du_i)_x$ for any $f \in J$ and $x \in \mathfrak{g}$ it follows from Theorem 9, § 4.7, that $(df)_x = 0$ if and only if $(df^*)_\xi = 0$ for any $f \in J$ and $x \in \mathfrak{r}$, where $\xi = u(x)$. It follows in particular that if $U \in \mathcal{U}$ and $x \in \mathfrak{u} \cap \mathfrak{r}$ where $\mathfrak{u} = \mathfrak{u}(J^U S)$ then the dimension r_x of the space spanned the $(df)_x$ for all $f \in J^U$ is the same as the dimension r_ξ of the space spanned by all $(df^*)_\xi$ where $f \in J^U$ and $\xi = u(x)$. If r is the codimension of

U in $\boldsymbol{C}^l$ and U_s is the set of simple points of U then by the Zariski criterion $r_\xi \leqq r$ for all $\xi \in U$ and $r_\xi = r$ if and only if $\xi \in U_s$. Thus $r_x \leqq r$ for all $x \in \mathfrak{u} \cap \mathfrak{r}$ and $r_x = r$ if and only if $u(x) \in U_s$. Since $\mathfrak{u} \cap \mathfrak{r}$ is Zariski open in $\mathfrak{u}$ and since J^U generates the prime ideal of $\mathfrak{u}$ then by applying the Zariski criterion to $\mathfrak{u}$ one obtains (4.10.4) and the fact that $x \in \mathfrak{u} \cap \mathfrak{r}$ is simple on $\mathfrak{u}$ if and only if $u(x)$ is simple on U.

Finally let $U \in \mathfrak{U}$ have codimension r in $\boldsymbol{C}^l$. Let $\mathfrak{a}$ be an irreducible component of $\mathfrak{u} \cap \mathfrak{r}^c$ where $\mathfrak{u} = \mathfrak{u}(J^U S)$ and let $Y \subseteq U$ be the Zariski closure of $u(\mathfrak{a})$. Let $q = \dim Y$ so that, obviously, $q \leqq l - r$. But if $p = \dim \mathfrak{a}$ then by Corollary 1, p. 109 in [3] there exists $\xi \in Y$ such that $(u \mid \mathfrak{a})^{-1}(\xi) = P(\xi) \cap \mathfrak{a}$ has dimension $p - q$. Hence $\dim P(\xi) \cap \mathfrak{a} \geqq p - l + r$. But by (3.8.5) one has $\dim P(\xi) \cap \mathfrak{r}^c \leqq n - l - 2$. Thus $n - 2 \geqq p + r$ or $n - r \geqq p + 2$. That is, by (4.10.4) $\dim \mathfrak{u} \geqq \dim \mathfrak{a} + 2$. This proves (4.10.5). Q.E.D.

Remark 23. By putting $\mathfrak{u} = \mathfrak{g}$ in (4.10.5) note that $\mathfrak{r}^c$ has a dimension of at least 2 in $\mathfrak{g}$. On the other hand if $\mathfrak{q}$ is the set of regular elements in $\mathfrak{g}$ and $v \in J$ is the invariant polynomial such that $v \mid \mathfrak{h}$ is the product of all the roots (positive as well as negative, so that it is a Weyl group invariant) then the complement $\mathfrak{q}^c$ of $\mathfrak{g}$ is the set of zeros of v and hence has codimension 1 in $\mathfrak{g}$.

5. The normality of the varieties $P(\xi)$ and the generalized exponents.

The following criterion for normality and its proof is due to Seidenberg.

THEOREM 15 (Seidenberg). *Let $\mathfrak{u} \subseteq \mathfrak{g}$ be a Zariski closed subvariety of $\mathfrak{g}$. Assume* (a) *that $\mathfrak{u}$ is a complete intersection and* (b) *the set of non-simple point of $\mathfrak{u}$ has a codimension of at least two in $\mathfrak{u}$. Then $\mathfrak{u}$ is a normal variety.*

Proof. Let $r = \dim \mathfrak{u}$. By [16], Theorem 3, one knows that $\mathfrak{u}$ is normal if (1) $\mathfrak{u}$ is free of $(r-1)$-dimensional singularities and (2), every principal ideal in the affine algebra of $\mathfrak{u}$ is unmixed. Since assumption (1) is satisfied (statement (b) in Theorem) it suffices therefore only to show that if $I(\mathfrak{u})$ is the prime ideal, in S, corresponding to $\mathfrak{u}$ and $f \in S$ then the ideal $(I(\mathfrak{u}), f)$ is unmixed. Obviously one may assume that $(I(\mathfrak{u}), f)$ has dimension $r - 1$. But then the result follows from Macaulay's theorem (see [19], p. 203) since, by (a), one has that $I(\mathfrak{u}) = (f_1, \cdots, f_{n-r})$ for some $f_i \in I(\mathfrak{u})$. Q.E.D.

Now if V is any finite dimensional G-module and $x \in \mathfrak{g}$ is arbitrary consider V^{G^x} the subspace of vectors in V that are fixed under all elements of G^x. We now find that the dimension of V^{G^x} is the multiplicity of the zero weight in V (and hence is the same) for all $x \in \mathfrak{r}$. (This, incidentally is not necessarily true for the covering group of G). Obviously it is enough to show this for irreducible G-modules.

THEOREM 16. *Let $\xi \in \mathbf{C}^l$ be arbitrary. Then $P(\xi)$ is a normal variety. That is, if $x \in \mathfrak{r}$ then $\bar{O}_x$ is a normal variety (see Theorem 4, § 3.8). Furthermore if $R(O_x)$ denotes the ring of everywhere defined functions on the orbit O_x ($R(O_x)$ is isomorphic to $R(G/G^x)$) then $R(O_x)$ is an affine algebra. (That is, it is finitely generated.) In fact $R(O_x) = S(O_x)$ where $S(O_x)$ is the restriction of S to O_x so that if $\xi = u(x)$ then $P(\xi) = \bar{O}_x$ is the affine variety of maximal ideals of $R(O_x)$.*

Moreover $R(O_x)$ is a completely reducible G-module and for any $\lambda \in D$ the multiplicity of ν^λ in $R(O_x)$ is l_λ, where l_λ is the multiplicity of the zero weight of ν_λ (or equally of ν^λ), so that $R(O_x)$, for all $x \in \mathfrak{r}$, are isomorphic as G-modules. Finally for any $x \in \mathfrak{r}$

$$\dim V_\lambda{}^{G^x} = l_\lambda \tag{5.1.1}$$

where V_λ and ν_λ is defined as in § 2.1.

Proof. By Theorem 10, § 4.8, $P(\xi)$ is a complete intersection. Also by Theorem 10 the set of non-simple points of $P(\xi)$ has a codimension of at least two in $P(\xi)$. Hence $P(\xi)$ is normal by Theorem 15. But now by Corollary 1, § 3.8, the complement of O_x in $\bar{O}_x$ has a codimension of at least two in $\bar{O}_x$ for any $x \in r$. Hence $R(O_x) = S(O_x)$ by Proposition 9, § 2.2. But in any case $S(O_x)$ is isomorphic to $S(\bar{O}_x)$. Since $S(\bar{O}_x) = R(\bar{O}_x)$ (see (2.2.2)) it follows that every element of $R(O_x)$ extends uniquely to an element of $R(\bar{O}_x)$. This induces an isomorphism

$$R(O_x) \to R(\bar{O}_x) \tag{5.1.2}$$

so that $R(O_x)$ is an affine algebra. Obviously $R(O_x)$ is a completely reducible G-module. But by Theorem 11, § 4.8, the multiplicity of ν^λ in $S(O_x)$ is l_λ. It follows therefore from Proposition 8, § 2.1, and the G-module isomorphism (2.2.1) that $\dim V_\lambda{}^{G^x} = l_\lambda$. Q. E. D.

Let $x \in \mathfrak{r}$. As a corollary of Theorem 16 we now observe that $\bar{O}_x$ is distinguished among all affine varieties into which O_x may be embedded.

COROLLARY 3. *Let $x \in \mathfrak{r}$ be arbitrary so that $\xi = u(x) \in \mathbf{C}^l$ is arbitrary. If we identify G/G^x with O_x (using the isomorphism (1.2.2)) then any morphism of G/G^x into any affine variety X extends uniquely to a morphism of the affine variety $P(\xi)$ into X.*

In particular any morphism of the orbit of principal nilpotent elements *into an affine variety X extends to a morphism of the variety of all nilpotent elements into X.*

Proof. Using the isomorphism (5.1.2) this result follows from Corollary 1, p. 58 in [3]. Q.E.D.

Remark 24. By Theorem 16 all $R(O_x)$ for $x \in \mathfrak{r}$, are isomorphic as G-modules. However it should be noted that they are not isomorphic as rings. Indeed the corresponding variety of maximal ideals of $R(O_x)$ is non-singular in case x is regular and, by Theorem 10, § 4.8, is singular otherwise (for example, in case x is principal nilpotent).

5.2. Now for any $\lambda \in D$ consider $\mathrm{Hom}_G(V^\lambda, S)$ the space of all G-module maps γ of V^λ into the ring of polynomials S. Obviously $\gamma(V^\lambda) \subseteq S^\lambda$ for any such γ.

We now observe that any $x \in \mathfrak{g}$ induces a linear map

$$\omega_x\colon \mathrm{Hom}_G(V^\lambda, S) \to V_\lambda{}^{G^x}$$

by the relation

$$\langle v, \omega_x\gamma\rangle = (\gamma(v))(x) \tag{5.2.1}$$

for all $v \in V^\lambda$ and any $\gamma \in \mathrm{Hom}_G(V^\lambda, S)$. Since γ is a G-module map it follows immediately from (1.1.5) that for any $a \in G$

$$\nu_\lambda(a)\,\omega_x(\gamma) = \omega_{ax}(\gamma) \tag{5.2.2}$$

and hence, obviously, $\omega_x(\gamma) \in V_\lambda{}^{G^x}$ for any $x \in \mathfrak{g}$.

Now by Theorem 11, § 4.8, the subspace $\mathrm{Hom}_G(V^\lambda, H)$ of $\mathrm{Hom}_G(V^\lambda, S)$ is of dimension l_λ. On the other hand, by (5.1.1), $V_\lambda{}^{G^x}$ is also l_λ-dimensional whenever $x \in \mathfrak{r}$. As a corollary of Theorem 16 we obtain

COROLLARY 4. *Let $x \in \mathfrak{r}$ and let $\lambda \in D$. Then the map*

$$\mathrm{Hom}_G(V^\lambda, H) \to V_\lambda{}^{G^x} \tag{5.2.3}$$

defined by restricting ω_x is an isomorphism.

Proof. Since both sides of (5.2.3) are vector spaces of dimension l_λ it suffices to show (5.2.3) is a monomorphism. Let γ be in the kernel of (5.2.3). Then, by (5.2.2), $\omega_{ax}(\gamma) = 0$ for all $a \in G$. Now let $v \in V^\lambda$. Then, by (5.2.1), $(\gamma(v))(ax) = 0$ for all $a \in G$. Thus $\gamma(v) \mid O_x = 0$. But since (4.8.6) is an isomorphism it follows that $\gamma(v) = 0$ for all $v \in V^\lambda$. Hence $\gamma = 0$, and consequently (5.2.3) is an isomorphism. Q.E.D.

5.3. Now since ad maps $\mathfrak{g}$ into the Lie algebra of G it is clear that any finite dimensional representation

$$\nu\colon G \to \mathrm{Aut}\, V$$

of G induces, by taking differentials, a representation of $\mathfrak{g}$, which we also denote by ν. Note that one always has $\nu(\mathfrak{z})=0$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$.

Now let Z be the subgroup of $\mathfrak{h}'$ generated by the set of roots Δ. If $o(\mu)$, the order of μ, is defined by

$$o(\mu)=\langle x_0,\mu\rangle$$

for any $\mu\in Z$ it is then clear from (4.1.6) that $o(\mu)$ is always an integer. Now since every weight of ν is clearly necessarily an element of Z it follows therefore that

$$V=\sum_{k\in Z} V^{(k)}$$

is a direct sum where $V^{(k)}$ is the eigenspace of $\nu(x_0)$ belonging to the eigenvalue k. Obviously

$$\nu(\mathfrak{g}^{(i)})\,V^{(k)}\subseteq V^{(i+k)}. \tag{5.3.1}$$

Now e_- be the principal nilpotent element defined as in § 4.2. For notational simplicity write $F=V^{G^{e_-}}$. Since G^{e_-} is connected (Proposition 14, § 3.6) one also has

$$F=\operatorname{Ker}\nu(\mathfrak{g}^{e_-}) \tag{5.3.2}$$

and by (5.1.1)

$$\dim F=l_\nu$$

where l_ν is the dimension of the zero weight of ν.

Now since x_0 lies in the normalizer of $\mathfrak{g}^{e_-}$ it follows from (5.3.2) that F is stable under $\nu(x_0)$. But then we observe that there exists a unique sequence of integers $m_i(\nu)$, $i=1,2,\cdots,l_\nu$, where

$$m_1(\nu)\leqq\cdots\leqq m_{l_\nu}(\nu)$$

such that F has a basis v_i, $i=1,2,\cdots,l_\nu$, where

$$v_i\in V^{(-m_i(\nu))}. \tag{5.3.3}$$

Remark 25. By applying the inner automorphism which carries e_+ into e_- and x_0 into $-x_0$ note that we would get the same integers $m_i(\nu)$ by using e_+ into instead of e_- and dropping the minus in (5.3.3).

Observe then that the $m_i(\nu)$ generalizes the notion of exponents. Indeed if ν is the adjoint representation then $l_\nu=l$ and, by Theorem 5, § 4.4, $m_i(\nu)=m_i$ since $F=\mathfrak{g}^{e_-}$.

Since $F\subseteq\operatorname{Ker}\nu(e_-)$ it follows from the representation theory of a three dimensional simple Lie algebra (e.g. see [13], § 2.5) that for any i

$$0\leqq m_i(\lambda) \tag{5.3.4}$$

Now, as in Remark 22, § 4.8, identify D with the (subset of Z) set of all dominant integral (with respect to G) forms on $\mathfrak{h}$ so that any $\lambda \in D$ is the highest weight of ν^{λ}. Note then that $-\lambda$ is the lowest weight of ν_{λ}.

When $V = V_{\lambda}$ and $\nu = \nu_{\lambda}$ we will write F_{λ} for F and $m_i(\lambda)$ for $m_i(\nu_{\lambda})$. The $m_i(\lambda)$, $i = 1, 2, \cdots, l_{\lambda}$, will be called the generalized exponents of $\mathfrak{g}$ (corresponding to λ). See Remark 25.

Now let $v_{\lambda} \in V_{\lambda}$ be the lowest weight $(-\lambda)$ vector. Then since V_{λ}, as one knows, is a cyclic module with respect to the universal enveloping algebra of $\mathfrak{m}$ with v_{λ} as cyclic vector it follows that

$$(V_{\lambda})^{(-k)} = \begin{cases} 0 \text{ if } k > o(\lambda) \\ (v_{\lambda}) \text{ if } k = o(\lambda). \end{cases}$$

Since $V_{\lambda}^{(-o(\lambda))}$ is obviously contained in F_{λ}. (One uses the relation $\mathfrak{g}^{e-} \subseteq \mathfrak{m}^* + \mathfrak{z}$ mentioned in the proof of Theorem 5, § 4.4). It follows then that

$$m_i(\lambda) < m_{l_{\lambda}}(\lambda) = o(\lambda) \tag{5.3.5}$$

for $1 \leqq i < l_{\lambda}$.

5.4. Let $\lambda \in D$. For convenience we will write $H(\lambda)$ and $S(\lambda)$ for the subspaces $H^{\lambda} \subseteq H$ and $S^{\lambda} \subseteq S$ respectively. See § 2.1. Clearly $H(\lambda)$ and $S(\lambda)$ are graded subspaces. In particular then

$$H(\lambda) = \sum_{j=0}^{\infty} H(\lambda)^j.$$

On the other hand by Theorem 11, § 4.8, the multiplicity of ν^{λ} in H is l_{λ}. Hence one has a direct sum

$$H(\lambda) = \sum_{i=1}^{l_{\lambda}} H_i(\lambda) \tag{5.4.1}$$

where (1) $H_i(\lambda)$ is an irreducible G-module, (2) $H_i(\lambda)$ is a space of homogeneous polynomials so that for some degree $n_i(\lambda)$ one has $H_i(\lambda) \subseteq S^{n_i(\lambda)}$ where (3) we may assume the $n_i(\lambda)$ are monotone non-decreasing with i.

The integers $n_i(\lambda)$ are the degrees k such that ν^{λ} occurs in H^k. The question arises: how does one determine these integers? Since $S(\lambda)$ is obviously isomorphic to the tensor product $J \otimes H(\lambda)$ by Theorem 11, § 4.8, it is clear that such information is needed if one is to determine the formal power series

$$q_{\lambda}(t) = \sum_{k=0}^{\infty} \dim S(\lambda)^k \, t^k$$

and, as a consequence, the multiplicity of ν^{λ} in S^k for any k.

The following theorem asserts that the $n_i(\lambda)$ are exactly the generalized exponents $m_i(\lambda)$.

THEOREM 17. *For any* $\lambda \in D$ *and* $i = 1, 2, \cdots, l_\lambda$, *one has* $n_i(\lambda) = m_i(\lambda)$ *so that* $H_i(\lambda) \subseteq S^{m_i(\lambda)}$. *In particular* $k = o(\lambda)$ *is the maximum degree such that* $H(\lambda)^k \neq 0$. *Furthermore* $H(\lambda)^k$ *is irreducible for this value of* k. *That is,*

$$H_{l_\lambda}(\lambda) = H(\lambda)^{o(\lambda)}.$$

Moreover the formal power series $q_\lambda(t)$ *may be given by*

$$q_\lambda(t) = d_\lambda \frac{\sum_{i=1}^{l_\lambda} t^{m_i(\lambda)}}{\prod_{i=1}^{l} (1 - t^{m_i})} \tag{5.4.2}$$

where $d_\lambda = \dim V_\lambda$.

Proof. Let v'_i, $i = 1, 2, \cdots, l_\lambda$, be a basis of F_λ such that $v'_i \in V_\lambda^{(-m_i(\lambda))}$. Now let $c \in \boldsymbol{C}^*$ be arbitrary. Let $r \in \boldsymbol{C}$ be such that $e^{-r} = c$ and let $a \in G$ be defined by putting $a = \exp r \operatorname{ad} x_0$. It is then clear that

$$\nu_\lambda(a) v'_i = c^{m_i(\lambda)} v'_i. \tag{5.4.3}$$

Also note that

$$a(e_-) = c e_-. \tag{5.4.4}$$

Now by Corollary 14, § 5.2, there exists a basis γ_i, $i = 1, 2, \cdots, l_\lambda$, of $\operatorname{Hom}_G(V^\lambda, H)$ such that

$$\omega_{e_-}(\gamma_i) = v'_i.$$

But now by (5.4.4-5) and (5.2.2) one gets the equation

$$c^{m_i(\lambda)} \omega_{e_-}(\gamma_i) = \omega_{ce_-}(\gamma_i). \tag{5.4.5}$$

Substituting in (5.2.1) this implies that for any $v \in V^\lambda$

$$(\gamma_i(v))(ce_-) = c^{m_i(\lambda)} \gamma_i(v)(e_-).$$

But then, conjugating by G (and using (1.1.5)) it also follows that

$$\gamma_i(v)(cy) = c^{m_i(\lambda)} \gamma_i(v)(y)$$

for all $y \in O_{e_-}$. But then since (4.8.6) is an isomorphism for $x = e_-$ it follows easily by choosing c, for example, to be positive that

$$\gamma_i(V^\lambda) \subseteq S^{m_i(\lambda)}.$$

On the other hand by definition of the γ_i

$$H(\lambda) = \sum_{i=1}^{l_\lambda} \gamma_i(V^\lambda)$$

is a direct sum of irreducible G-modules. By uniqueness of the $n_i(\lambda)$ it then follows that $n_i(\lambda) = m_i(\lambda)$ for $i = 1, \cdots, l_\lambda$.

The second and third statements of the theorem follow immediately from (5.3.5). The equation (5.4.2) is an immediate consequence of the obvious fact that the isomorphism (4.8.4) induces an isomorphism of $J \otimes H(\lambda)$ onto $S(\lambda)$. Q.E.D.

Remark 26. We observe here that Theorem 17 is a generalization of Theorem 5, § 4.4, asserting that $m_i(\nu) = m_i$ where ν is the adjoint representation. Indeed let U be the subspace of all $u \in J^+$ such that $\langle \partial, u \rangle = 0$ where $\partial \in (J_*{}^+)^2$. It follows immediately that $\dim U = l$ and that any homogeneous basis of U is a set of primitive invariants u_i, $i = 1, 2, \cdots, l$. But if the u_i are so chosen then by definition of U one has $\partial u_i \in J^0 = S^0$ (immediate from (1.1.2)) for any $\partial \in J_*{}^+$. It follows immediately then that, for $1 \leqq i \leqq l$,

$$(5.4.6) \qquad \partial_x u_i \in H^{m_i} \text{ for every } x \in \mathfrak{g}.$$

(Recall that S_* is commutative.) But now if g_1 is a simple component of $\mathfrak{g}$ of rank l_1 and u_{i_k}, $k = 1, 2, \cdots, l_1$, are as in the proof of Theorem 9, § 4.7, then one must have $\partial_x u_{i_k} \in H(\psi)$ for any $x \in \mathfrak{g}_1$ where $\psi \in D$ is defined as in the proof of Theorem 9. See (4.7.8). (In particular note that if $\mathfrak{g}$ is simple and x_j is a basis of $\mathfrak{g}$ then

$$(5.4.7) \qquad \partial_{x_j} u_i,\ i = 1, \cdots, l,\ j = 1, \cdots, n \text{ is a basis of } H(\psi)$$

where ψ is the highest root of $\mathfrak{g}$). It follows immediately that $n_k(\psi) = m_{i_k}$. Applying Theorem 17 one then obtains $m_k(\psi) = m_{i_k}$ which immediately yields $m_j(\nu) = m_j$.

It should also be observed then that the relation $o(\lambda) = m_{l_\lambda}(\lambda)$ generalizes the well known relation $o(\psi) = m_l$. See Corollary 8.6 and Lemma 9.1 in [13].

Remark 27. Note that the argument given in the proof of Theorem 17 may be reversed. That is, if L is an arbitrary irreducible G-submodule of $H(\lambda)$ and $\gamma \in \operatorname{Hom}_G(V^\lambda, H)$ is such that

$$(5.4.8) \qquad \gamma(V^\lambda) = L$$

then for any integer $j \geqq 0$ one has

$$(5.4.9) \qquad L \subseteq S^j \text{ if and only if } \omega_{e_-}(\gamma) \in V_\lambda{}^{(-j)}.$$

Indeed if $L \subseteq S^j$, then by (5.2.1), $\omega_{ce_-}(\gamma) = c^j \omega_{e_-}(\gamma)$ for any $c \in \boldsymbol{C}^*$. But then if $a \in G$ is defined as in the proof of Theorem 17 one obviously has

$\omega_{e_-}(\gamma) \in V_\lambda^{(-j)}$ since $\nu_\lambda(a)\omega_{e_-}(\gamma) = c^j\omega_{e_-}(\gamma)$, by (5.4.4) and (5.2.2) and $c \in \boldsymbol{C}^*$ is arbitrary. The argument for the other direction has been given in the proof of Theorem 17.

For any $\lambda \in D$ let $\gamma_i^\lambda \in \mathrm{Hom}_G(V^\lambda, H)$, $i = 1, 2, \cdots, l_\lambda$ be fixed so that $\gamma_i^\lambda(V^\lambda) = H_i(\lambda)$. Obviously the γ_i^λ are a basis of $\mathrm{Hom}_G(V^\lambda, H)$ by (5.4.1).

Thet argument in Remark 27 may be used to yield

THEOREM 18. *Let* $x \in \mathfrak{r}$ *and* $a \in G$. *Assume* $a(x) = cx$ *for some* $c \in \boldsymbol{C}^*$. *Now let* $\lambda \in D$. *Then the* l_λ*-dimensional* (by 5.1.1)) *space* $V_\lambda^{G^x}$ *is stable under* $\nu_\lambda(a)$. *Furthermore for* $i = 1, 2, \cdots, l_\lambda$,

$$\{c^{m_i(\lambda)}\} \text{ are the eigenvalues of } \nu_\lambda(a) \mid V_\lambda^{G^x}$$

and $\omega_x(\gamma_i^\lambda)$ *is a corresponding basis of eigenvectors.*

Proof. Since $H_i(\lambda) \subseteq S^{m_i(\lambda)}$ by Theorem 17 it follows from (5.2.1) that $\omega_{cx}(\gamma_i^\lambda) = c^{m_i(\lambda)}\omega_x(\gamma_i^\lambda)$. But then $\nu_\lambda(a)\omega_x(\gamma_i^\lambda) = c^{m_i(\lambda)}\omega_x(\gamma_i^\lambda)$ by (5.2.2). On the other hand by Corollary 4, § 5.2, the $\omega_x(\gamma_i^\lambda)$ are a basis of $V_\lambda^{G^x}$.

Q. E. D.

5.5. Let V be a finite dimensional G-module with respect to a representation ν. Let $A \subseteq G$ be the Cartan subgroup of G corresponding to $\mathfrak{h}$ so that V^A is the zero weight space. We recall that W is the Weyl group of G corresponding to $\mathfrak{h}$. Now since V^A is obviously stable under the normalizer of A in G it follows that ν induces a representation

$$\pi: W \to \mathrm{Aut}\, V^A$$

of the Weyl group W on V^A. One notes that this is a generalization of the usual representation of W on $\mathfrak{h}$ (case where ν is the adjoint representation).

When $V = V_\lambda$ and $\nu = \nu_\lambda$ we will write π_λ for π.

Now assume in the remainder of this section that $\mathfrak{g}$ is a non-trivial simple Lie algebra. Let $\psi \in D$ be the highest root of $\mathfrak{g}$ so that ν^ψ is the adjoint representation. Let

$$s = 1 + o(\psi)$$

or equivalently let $s = 1 + m_l$. See end of Remark 26, § 5.4.

We recall that an element $\sigma \in W$ is called a Coxeter-Killing transformation in [13], § 8.1, if it can be expressed as the product of the reflections defined by the simple roots (in any order) relative to any system of positive roots.

Let $\sigma \in W$ be a Coxeter-Killing transformation. It was observed empirically by Coxeter and then proved independently by Steinberg and in [13] that the order of σ is s. It was also observed empirically by Coxeter and proved in [4] that the eigenvalues of $\pi_\psi(\sigma)$ are $e^{2\pi i m_j/s}$, $j=1,2,\cdots,l$. We will now generalize this for all $\lambda\in D$.

By a theorem of Coleman (see [4]) there exists a regular element $z\in\mathfrak{h}$ such that

$$\sigma(z)=e^{2\pi i/s}z. \tag{5.5.1}$$

In fact, by Corollary 9.2 in [13], z is a cyclic element of $\mathfrak{g}$.

THEOREM 19. *Let $\sigma\in W$ be a Coxeter-Killing transformation and let $\lambda\in D$ be arbitrary. Then the eigenvalues of $\pi_\lambda(\sigma)$ are $e^{2\pi i m_i(\lambda)/s}$, $i=1,2,\cdots,l_\lambda$, and if $z\in\mathfrak{h}$ is the cyclic element satisfying* (5.5.1) *then $\{\omega_z(\gamma_i{}^\lambda)\}$ is a corresponding basis of eigenvectors.*

Proof. Let $a\in G$ be any element of the normalizer of A which defines $\sigma\in W$ so that $\nu_\lambda(a)\mid V_\lambda{}^A=\pi_\lambda(\sigma)$. Since $a(z)=cz$ where $c=e^{2\pi i/s}$ the result follows immediately from Theorem 18 since by Coleman's theorem z is regular and hence $z\in\mathfrak{r}$. Q.E.D.

Remark 28. If $\mathfrak{g}$ is the three dimensional simple Lie algebra we may identify D with the set of non-negative integers where $\dim V^\lambda=2\lambda+1$. Here $l_\lambda=1$ for all $\lambda\in D$ and $m_1(\lambda)=\lambda$ by (5.3.5) since $o(\lambda)=\lambda$. Also obviously $s=2$. Note then that one recovers from Theorem 19 the well known fact that $\pi_\lambda(\sigma)=(-1)^\lambda$ for $\sigma\in W$, $\sigma\neq 1$.

Remark 29. For any $k\in\mathbf{Z}$ let $[k]$ denote its canonical image in $\mathbf{Z}_s=\mathbf{Z}/s\mathbf{Z}$ and for any $m\in\mathbf{Z}$ let $r_\lambda(m)$ be the number of integers $1\leqq i\leqq l_\lambda$ such that $[m_i(\lambda)]=[m]$. Now if $\sigma\in W$ is a Coxeter-Killing transformation and $k\in\mathbf{Z}$ is prime to s then σ^k is again a Coxeter-Killing transformation by Corollary 9.2 in [13] (using (5.5.1)). It follows easily therefore from Theorem 19 that $r_\lambda(m)=r_\lambda(km)$ for any integer m. Note that this generalizes Chevalley's observation that $m_i+m_{l+1-i}=s$ since we may put $k=-1$ and one knows $m_i<s$.

6. A decomposition theorem for the universal enveloping algebra U of $\mathfrak{g}$. 1. Let T be the tensor algebra over $\mathfrak{g}$. Then one knows that in a unique way T is a G-module so that G operates as a group of algebra automorphisms extending the action of G on $\mathfrak{g}$.

Let $Q\subseteq T$ denote the G-submodule of all symmetric tensors in T.

Also for $\tau=0,1$ let I_τ be the ideal in T generated by all

$$(x\otimes y-y\otimes x)-\tau[x,y]$$

where $x,y\in\mathfrak{g}$. Then since I_τ is G-stable the algebra $T_\tau=T/I_\tau$ is a G-module and one knows that the canonical epimorphism $T \to T_\tau$ induces a G-module isomorphism

$$\delta_\tau \colon Q \to T_\tau.$$

However, by definition the G-module T_0 is the symmetric algebra S_* with the G-module structure of § 1.1 and the G-module T_1 is the universal enveloping algebra U of $\mathfrak{g}$ with G operating by the usual extension of the adjoint representation. Now if $\delta_* = \delta_1 \circ \delta_0^{-1}$ then obviously

$$\delta_* \colon S_* \to U$$

is a G-module isomorphism and that furthermore since Q is the subspace of T generated by all tensors of the form $x \otimes \cdots \otimes x$ where $x \in \mathfrak{g}$ it follows that $\delta_*((\partial_x)^k) = x^k$ for any $x \in \mathfrak{g}$. Finally then if we compose δ_* with the inverse of B (see § 1.4) one obtains a G-module isomorphism

$$\delta \colon S \to U$$

such that, for any integer $k \geqq 0$,

$$\delta(g^k) = (\delta(g))^k \tag{6.1.1}$$

for every $g \in S^1$. Furthermore, one knows, by the theorem of Birkhoff-Witt that the filtration of U defined by the G-submodules

$$U_k = \delta\Big(\sum_{i=0}^{k} S^i\Big)$$

is such that

$$\delta(fg) = \delta(f)\delta(g) \bmod U_{i+j-1} \tag{6.1.2}$$

for (not necessarily homogeneous) polynomials f and g where $\deg f \leqq i$ and $\deg g \leqq j$. In particular (6.1.2) implies

$$U_i U_j \subseteq U_{i+j}. \tag{6.1.3}$$

Now let $Z \subseteq U$ be the center of U. By a theorem of Chevalley one knows that Z, like J, is isomorphic to a polynomial ring in l generators. Furthermore since Z is clearly the subalgebra of fixed elements under the action of G on U and since δ is a G-module isomorphism it follows that

$$\delta(J) = Z. \tag{6.1.5}$$

We now introduce a G-submodule E of U by letting E be the subspace spanned by all elements of the form x^k, $k = 0, 1, \cdots$, where $x \in \mathfrak{g}$ is nilpotent.

Remark 30. Note that the universal enveloping algebra $U(\mathfrak{m})$ of $\mathfrak{m}$ is contained in E and (since every nilpotent element is conjugate to an element in $\mathfrak{m}$) that in fact E is the subspace of U spanned by all the algebras $\{U(\mathfrak{m}')\}$ where $\mathfrak{m}'$ runs through all the Lie subalgebras of $\mathfrak{g}$ conjugate to $\mathfrak{m}$.

THEOREM 20. *One has*

$$\delta(H) = E$$

where, we recall, H is the space of all G-harmonic polynomials on $\mathfrak{g}$.

Proof. Obviously $\delta(P') = P$ where P is the set of all nilpotent elements in $\mathfrak{g}$ and P' is defined as in § 1.4. But since $H = H_P$ by (4.8.3) the thecrem follows immediately from (6.1.1). Q.E.D.

The filtration on U induces a filtration on E where $E_k = E \cap U_k$.

Now regard U as a Z-module (with respect to multiplication). The following is our main result on the structure of U.

THEOREM 21. *Let U be the universal enveloping algebra of a reductive Lie algebra* $\mathfrak{g}$. *Let Z be the center of U. Then U is free as a module over Z. In fact if E is the G-submodule of U defined above (see Remark 30) then the map*

$$Z \otimes E \to U \tag{6.1.6}$$

given by $p \otimes q \to pq$ is a G-module isomorphism.

Furthermore for any $\lambda \in D$ the multiplicity of the irreducible representation ν^λ in E is l_λ (the multiplicity of the zero weight for ν^λ). Moreover the order $o(\lambda)$ of λ (see § 5.3) is the smallest integer k such that the multiplicity of ν^λ is E_k is l_λ.

Proof. Let β denote the (G-module) map (6.1.6). To show first that β is surjective assume inductively that $U_j \subseteq \operatorname{Im}\beta$ (obviously $U_0 \subseteq \operatorname{Im}\beta$ since $U_0 \subseteq E$) for some integer j. Let $r \in U_{j+1}$. Then $r = \delta(g)$ where, by Theorem 11, § 4.8,

$$g = \sum_i f_i h_i$$

with $f_i \in J$, $h_i \in H$ and $\deg f_i + \deg h_i \leq j+1$. But then

$$r = \beta(\sum_i \delta(f_i) \otimes \delta(h_i)) \bmod U_j$$

by (6.1.2). We have, of course, used (6.1.5) and Theorem 20. But since $U_j \subseteq \operatorname{Im}\beta$ it follows that $r \in \operatorname{Im}\beta$. Hence β is surjective.

Now let $p \in Z \otimes E$ where $p \neq 0$. Then p is the image of an element $e \in J \otimes H$ under the isomorphism $J \otimes H \to Z \otimes E$ induced by δ (using (6.1.5) and Theorem 20). Furthermore we may assume that $e = \sum_i f_i \otimes h_i$ where $0 \neq f_i \in J$ and the h_i are homogeneous and linearly independent in H.

Now let $k = \max_i(\deg f_i + \deg h_i)$. It follows therefore by Theorem 11, § 4.8, that if $g = \sum_i f_i h_i$ then $g \neq 0$ and $\deg g = k$. Hence

$$\delta(g) \neq 0 \bmod U_{k-1}. \tag{6.1.7}$$

But now $p=\sum_i \delta(f_i)\otimes\delta(h_i)$. Thus $\beta(p)=\sum_i \delta(f_i)\delta(h_i)$. But by (6.1.2) one has

$$\delta(g)=\beta(p) \bmod U_{k-1}. \tag{6.1.8}$$

Thus $\beta(p)\neq 0$ by (6.1.7-8) and hence β is an isomorphism. The remaining statements of the theorem follow immediately from Theorems 11, 17 and 20.

Q. E. D.

6.2. Let G_1 be any algebraic reductive group whose Lie algebra is $\mathfrak{g}$. If

$$\nu\colon G_1\to \operatorname{Aut} V \tag{6.2.1}$$

is a representation of G_1 on a finite dimensional space V then the corresponding representation of $\mathfrak{g}$ and U on V will also be denoted by ν.

An element $a\in G_1$ is called unipotent if a is of the form $a=\exp x$ where $x\in\mathfrak{g}$ is nilpotent.

LEMMA 16. *Let ν be as in* (6.2.1) *and let $W\subseteq \operatorname{End} V$ be the space spanned by all operators on V of the form $\nu(a)$ where $a\in G_1$ is unipotent. Then $W=\nu(E)$.*

Proof. By definition of E and the exponential formula it is obvious that $W\subseteq\nu(E)$. On the other hand if x is nilpotent then $\nu(\exp tx)\in W$ for all real numbers t. But, clearly, W also contains all t-derivatives of $\nu(\exp tx)$. Hence $\nu(x^k)\in W$ for $k=0,1,\cdots$. Thus $\nu(E)\subseteq W$. Q. E. D.

As a corollary of Theorem 21, § 6.1, we obtain

THEOREM 22. *Assume ν (as in* (6.2.1)) *is irreducible. Then*

$$\nu(E)=\operatorname{End} V \tag{6.2.2}$$

or equivalently (by Lemma 16), *every operator b on V may be put in the form*

$$b=\sum_{i=1}^{k} c_i\nu(a_i) \tag{6.2.3}$$

where the c_i are complex scalars and the a_i are unipotent elements of G_1.

Proof. Since ν is irreducible one has $\nu(U)=\operatorname{End} V$ and $\nu(Z)=\boldsymbol{C}$. The equation (6.2.2) then follows from the fact that (6.1.6) is an isomorphism. The second form of it follows from Lemma 16. Q. E. D.

Now let D_1, V_1^{ξ} and ν_1^{ξ}, for $\xi\in D_1$, play the same role for G_1 as the corresponding notation without the subscript 1 plays for G. (See § 2.1 and Remark 22, § 4.8).

Now $\operatorname{End} V_1^{\xi}$ is a G_1-module with respect to the tensor product of ν_1^{ξ} with the representation of G_1 contragradient to ν_1^{ξ}. But since the center of

G_1 obviously operates trivially on End $V_1{}^{\xi}$ it follows that End $V_1{}^{\xi}$ is a G-module and in fact the homomorphism of U onto End $V_1{}^{\xi}$ induced by $\nu_1{}^{\xi}$ is a G-module epimorphism. Since all modules under consideration are completely reducible it follows therefore by Theorem 21, § 6.1, that $\nu_1{}^{\xi}$ induces a G-module epimorphism

$$E^{\lambda} \to (\text{End } V_1{}^{\xi})^{\lambda} \tag{6.2.4}$$

for every $\lambda \in D$.

Since every representation of G induces a representation of G_1 it is clear that $D \subseteq D_1$. One defines a partial ordering on D_1 by putting $\xi \geqq \xi'$, for $\xi, \xi' \in D_1$, whenever $\xi - \xi' \in D_1$. It is easy to see that D_1 is a directed set with respect to this ordering. ("Sufficiently large" will mean all $\xi \in D_1$ such that $\xi \geqq \xi'$ for some $\xi' \in D_1$.)

Lemma 17. *For any $\xi \in D_1$ and $\lambda \in D$ let $n_{\lambda}(\xi)$ denote the multiplicity of ν^{λ} in* End $V_1{}^{\xi}$ *(regarded as a G-module). Then*

$$n_{\lambda}(\xi) \leqq l_{\lambda} \tag{6.2.5}$$

and (for fixed λ) the equality holds for ξ sufficiently large.

Proof. Since (6.2.4) is an epimorphism the inequality (6.2.5) is an immediate consequence of Theorem 21, § 6.1. However a much simpler and more direct proof of the inequality may be given using § 4.1 in [12].

Now identifying $V_1{}^{\xi} \otimes V_{1,\xi}$ with End $V_1{}^{\xi}$ and regarding G-modules as G_1-modules it follows immediately from Schur's lemma, upon forming the triple tensor product $V_{\lambda} \otimes V_1{}^{\xi} \otimes V_{1,\xi}$, that $n_{\lambda}(\xi)$ is also the multiplicity of $\nu_1{}^{\xi}$ in $\nu_{\lambda} \otimes \nu_1{}^{\xi}$.

We refer now to [12], § 4.4, for the definition as to when λ is totally subordinate to ξ. By Theorem 5.1, (3) in [12] λ is totally subordinate to ξ for ξ sufficiently large. But now by (6) in this theorem (where $\mu = 0$, $\lambda_2 = \lambda$, $\lambda_1 = \xi$) the multiplicity of $\nu_1{}^{\xi}$ in $\nu_{\lambda} \otimes \nu_1{}^{\xi}$ is l_{λ} whenever λ is totally subordinate to ξ. Hence $n_{\lambda}(\xi) = l_{\lambda}$ whenever λ is totally subordinate to ξ or when ξ is sufficiently large. Q. E. D.

Harish-Chandra proved in [8] that if $Y \subseteq U$ is any one-dimensional subspace there exists $\xi \in D_1$ such that $\nu_1{}^{\xi}$ is faithful on Y. This is not true in general for higher dimensional subspaces. For example if $p \in Z$ and $q \in U$ where $q \neq 0$ and $p \notin U_0$ then q and pq span a two dimensional space in $\mathbf{U}$ but its image under $\nu_1{}^{\xi}$, for any $\xi \in D_1$, is at most a one dimensional space. We now observe, however, that the generalization is true provided that $Y \subseteq E$.

Theorem 23. *Let $Y \subseteq E$ be any finite dimensional subspace. Then the irreducible representation $\nu_1{}^{\xi}$ is faithful on Y for all $\xi \in D_1$ sufficiently large.*

Proof. Since Y is finite dimensional there exists k such that $Y \subseteq E_k$.

Now let $C \subseteq D$ denote the set of all $\lambda \in D$ such that ν^{λ} occurs with positive multiplicity in E_k. Since E is finite dimensional it is obvious that C is a finite set. Now since D_1 is a directed set it follows that equality holds in (6.2.5) for all $\lambda \in C$ and all ξ sufficiently large. But then by Theorem 21, § 6.1, the map (6.2.4) is an isomorphism also for all $\lambda \in C$ and $\xi \in D_1$ sufficiently large. Thus $\nu_1{}^{\xi}$ is faithful on E_k and hence on Y for all ξ sufficiently large. Q. E. D.

REFERENCES.

[1] A. Borel and Harish-Chandra, "Arithmetic subgroups of algebraic groups," *Annals of Mathematics*, vol. 75 (1962), pp. 485-535.

[2] C. Chevalley, "Invariants of finite groups generated by reflections," *American Journal of Mathematics*, vol. 77 (1955), pp. 778-782.

[3] ———, *Fondéments de la géométrie algébrique*, Course notes at the Institut Henri Poincaré, Paris, 1958.

[4] A. J. Coleman, "The Betti numbers of the simple Lie groups," *Canadian Journal of Mathematics*, vol. 10 (1958), pp. 349-356.

[5] E. B. Dynkin, "Semi-simple subalgebras of semi-simple Lie algebras," *American Mathematical Society Translations*, ser. 2, vol. 6 (1957), pp. 111-244.

[6] F. Gantmacher, "Canonical representation of automorphisms of a complex semi-simple Lie group," *Matematičeskii Sbornik*, vol. 47 (1939), pp. 104-146.

[7] Harish-Chandra, "On a lemma of Bruhat," *Journal de Mathématiques Pures et Appliquées*, vol. 9, 315 (1956), pp. 203-210.

[8] ———, "On representations of Lie algebras," *Annals of Mathematics*, vol. 50 (1949), pp. 900-915.

[9] G. Hochschild and G. D. Mostow, "Representations and representative functions on Lie groups, III," *Annals of Mathematics* (2), vol. 70 (1959), pp. 85-100.

[10] S. Helgason, "Some results in invariant theory," *Bulletin of the American Mathematical Society*, vol. 68 (1962), pp. 367-371.

[11] N. Jacobson, "Completely reducible Lie algebras of linear transformations," *Proceedings of the American Mathematical Society*, vol. 2 (1951), pp. 105-133.

[12] B. Kostant, "A formula for the multiplicity of a weight," *Transactions of the American Mathematical Society*, vol. 9 (1959), pp. 53-73.

[13] ———, "The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group," *American Journal of Mathematics*, vol. 81 (1959), pp. 973-1032.

[14] M. Rosenlicht, "On quotient varieties and the affine embedding of certain homogeneous spaces," *Transactions of the American Mathematical Society*, vol. 101 (1961), pp. 211-223.

[15] J. P. Serre, "Faisceaux algébriques cohérent," *Annals of Mathematics*, vol. 61 1955), pp. 197-278.

[16] A. Seidenberg, "The hyperplane sections of normal varieties," *Transactions of the American Mathematical Society*, vol. 64 (1950), pp. 357-386.

[17] R. Steinberg, "Invariants of finite reflection groups," *Canadian Journal of Mathematics*, vol. 12 (1960), pp. 616-618.

[18] O. Zariski and P. Samuel, *Commutative Algebra*, vol. I, van Nostrand Company, Princeton, 1958.

[19] ———, *Commutative Algebra*, vol. II, van Nostrand Company, Princeton, 1960.