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Convex Bodies and Algebraic Geometry

An Introduction to the Theory of Toric Varieties

With 42 Figures

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Introduction

The theory of toric varieties, which are also called torus embeddings, relates algebraic geometry to the geometry of convex figures in real affine spaces. Ever since the foundation of the theory was laid down at the beginning of the 1970's, tremendous progress has been made and various interesting applications have been found. This book tries to explain in unified form as many results as possible obtained so far on toric varieties.

This book is also meant to serve as an accessible introduction to algebraic geometry. Thanks to the many good textbooks now available, algebraic geometry seems to have become much more accessible than it was twenty years ago. Beginners, however, have to learn so many basic concepts. It may not be easy for them to learn many general theorems, get accustomed to them and then apply them effectively.

For this reason, we chose to construct toric varieties as complex analytic spaces, so that they can be understood more easily without much prior knowledge of algebraic geometry. Not only can some of the important complex analytic properties of these spaces be translated into easily visualized elementary geometry of convex figures, but many interesting examples of complex analytic spaces can be easily constructed by means of this theory.

Nevertheless, toric varieties are in essense algebraic varieties, and for the proof of the basic properties we cannot avoid using algebraic geometry through the GAGA theorems. We tried to include in this book the necessary general results in algebraic geometry, in as easily understandable a way as possible, although we mostly omit the proofs. Hopefully, the essential difference between complex analytic geometry and algebraic geometry becomes more transparent as a by-product.

Chapter 1 is devoted to the more basic part of the theory of toric varieties. We first define "fans" in real affine spaces and then maps among them. The theory will then enable us to associate toric varieties to them and equivariant holomorphic maps among them. Usually, it is easier to construct examples of fans and maps among them. Accordingly, we can construct many interesting examples of complex analytic spaces and holomorphic maps. We can also reduce interesting basic questions on the birational geometry of toric varieties into those on subdivisions of fans.

In Chapter 2, we deal with the cohomology of compact toric varieties as well as toric varieties embeddable into projective spaces. Some questions on the lattice points in convex bodies and the isoperimetric problem can then be translated into algebro-geometric questions on toric projective varieties. Mori's powerful theory on general projective varieties can be applied to toric projective varieties and interpreted in terms of the elementary geometry of fans.

In Chapter 3, we deal with holomorphic differential forms on toric varieties. Not only are they related to deformations and degenerations of complex analytic spaces, but they might also be of some interest in commutative algebra. We also deal with the Cremona groups, since they are related to the automorphism groups of toric varieties, whose structure, in turn, can be determined by means of holomorphic vector fields.

There already exist many important applications of the theory of toric varieties. Most of them are exposed already in the literature, which the reader might find easier to understand after going through this book. Due to lack of space here, we explain them only briefly at the beginning of Chapter 4. We devote the chapter rather to more elementary but illustrative applications: the relationship between periodic continued fractions and two-dimensional cusp singularities as well as higher-dimensional analogues. Together with the construction of the class VII compact complex surfaces which we also briefly mention, they amount to the construction of certain complex manifolds as quotients of open subsets of toric varieties with respect to the actions of discrete groups without fixed points.

In the Appendix, we collect together, without proof, basic results on convex sets which we need in this book.

Many results in this book are due to other people and are scattered throughout the literature. We try to give credit and precise reference(s) each time we mention these results. The author would like to thank them for letting him use their results. As is evident throughout this book, he owes much to Katsuya Miyake of Nagoya University, his first collaborator on this topic, as well as to his second collaborator Masa-Nori Ishida of Tohoku University. Ishida himself also contributed a great deal to the development of the theory. He kindly went through the manuscript for the Japanese edition as well as this English edition, and provided many valuable suggestions. The author takes this opportunity to thank both of them for their contribution.

Thanks are due to Professors Masayoshi Nagata and Takeyuki Hida, who suggested to the author to write the Japanese edition of this book.

In this English edition, we tried to incorporate, or at least mention, more recent results which came to our attention since the publication of the Japanese edition in January, 1985 from Kinokuniya, Tokyo. Errors in the Japanese edition were also corrected.

Thanks are due to the editors of the Ergebnisse series as well as to the staff of Springer-Verlag, Heidelberg, who made possible the publication of this English edition. Deep appreciation goes to Professor Izumi Kubo of Hiroshima University, whose software "Word Processor for Mathematical Papers" was so helpful in writing up this English edition. It is also a pleasure to thank Park Hye Sook for her help in proofreading.

The author dedicates both Japanese and English editions of this book to the memory of his good friend Professor Takehiko Miyata, who gave him so much mathematical stimulus over the years.

Sendai, August, 1987 Tadao Oda

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Chapter 1. Fans and Toric Varieties

The most basic in the theory are elementary objects in convex geometry called fans. The theory enables us to associate to each fan a complex analytic space called a toric variety (also called a torus embedding) in a very simple manner. Mathematically interesting fans abound. Accordingly, we obtain interesting complex analytic spaces in a very natural manner. As this book tries to show, the theory has rather extensive applications.

The gist of the theory lies in the fact that complex analytic properties of toric varieties can very often be described in terms of elementary geometry of fans. We explain the more basic part of the description in this chapter and leave the rest to later chapters.

For simplicity, we construct toric varieties as complex analytic spaces. In the same manner, however, we can construct toric varieties as algebraic varieties or schemes over arbitrary fields or rings and obtain easily understandable examples regardless of the characteristic.

The foundation of the theory was laid down at the beginning of the 1970's independently by Demazure [D5], Mumford et al. [TE], Satake [S5] and Miyake-Oda [MO′]. Algebro-geometric survey of the theory can be found in [TE], Danilov [D1], Miyake-Oda [MO], Brylinski [B5] and Teissier [T2].

1.1 Strongly Convex Rational Polyhedral Cones and Fans

We fix a free module $N \cong \mathbb{Z}^r$ of rank r over the ring $\mathbb{Z}$ of rational integers. For the dual $\mathbb{Z}$-module $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, we have a canonical $\mathbb{Z}$-bilinear pairing

$$\langle\ ,\ \rangle : M \times N \to \mathbb{Z}\ .$$

By scalar extension to the field $\mathbb{R}$ of real numbers, we have r-dimensional $\mathbb{R}$-vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ with a canonical $\mathbb{R}$-bilinear pairing $\langle\ ,\ \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$.

Definition. A subset σ of $N_{\mathbb{R}}$ is called a *strongly convex rational polyhedral cone* (with apex at the origin O), if there exist a finite number of elements $n_1, n_2, \ldots, n_s$ in N such that

$$\begin{aligned}\sigma &= \mathbb{R}_{\geq 0} n_1 + \ldots + \mathbb{R}_{\geq 0} n_s \\ &:= \{a_1 n_1 + \ldots + a_s n_s\,;\ a_i \in \mathbb{R},\ a_i \geq 0 \text{ for all } i\}\end{aligned}$$

and that $\sigma \cap (-\sigma) = \{O\}$, where we denote by $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers.

First of all, σ above is a convex polyhedral cone in the $\mathbb{R}$-vector space $N_{\mathbb{R}}$ in the sense of Section A.1 of the appendix. The strong convexity $\sigma \cap (-\sigma) = \{O\}$ means that it contains no nonzero $\mathbb{R}$-subspace of $N_{\mathbb{R}}$. We also require σ to be rational, i.e., spanned by vectors integral (or, equivalently, rational) with respect to the given lattice N.

Various notions in Section A.1 are applicable to a strongly convex rational polyhedral cone σ: Its *dimension* $\dim \sigma$ is that of the smallest $\mathbb{R}$-subspace $\sigma + (-\sigma) = \mathbb{R}\sigma$ of $N_{\mathbb{R}}$ containing σ. The cone in $M_{\mathbb{R}}$ *dual to* σ is defined to be

$$\sigma^{\vee} := \{x \in M_{\mathbb{R}}; \langle x, y \rangle \geq 0 \text{ for all } y \in \sigma\} ,$$

which is not only a convex polyhedral cone in $M_{\mathbb{R}}$ but also is rational (cf. Proposition 1.3). Since σ is strongly convex, we have $\sigma^{\vee} + (-\sigma^{\vee}) = M_{\mathbb{R}}$, hence $\dim \sigma^{\vee} = r$. A subset τ of σ is called a *face* and is denoted $\tau < \sigma$, if

$$\tau = \sigma \cap \{m_0\}^{\perp} := \{y \in \sigma; \langle m_0, y \rangle = 0\}$$

for an $m_0 \in \sigma^{\vee}$. As we see in Proposition 1.3 below, m_0 above can be chosen in $M \cap \sigma^{\vee}$, so that a face τ is also a strongly convex rational polyhedral cone. σ itself is a face of σ. Moreover, $\{O\}$ is a face of σ by the strong convexity (cf. Proposition A.5).

We are now ready to introduce the most basic notion of the theory.

Definition. A *fan* in N is a nonempty collection Δ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the following conditions:

(i) Every face of any $\sigma \in \Delta$ is contained in Δ.

(ii) For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

The union $|\Delta| := \cup_{\sigma \in \Delta} \sigma$ is called the *support* of Δ.

We sometimes call the pair (N, Δ) a fan. A fan is also called a *rational partial polyhedral decomposition* (an r.p.p. decomposition, for short). A fan gives a decomposition of a part $|\Delta|$ of $N_{\mathbb{R}}$ by strongly convex rational polyhedral cones (not necessarily finite in number) which do not intersect each other in the relative interior. For theoretical convenience, we count all the faces as members of Δ. We give examples for $r = 2$ in Fig. 1.1, which illustrate the reason why we call them (possibly torn) fans. The shaded areas are supposed to be two-dimensional cones.

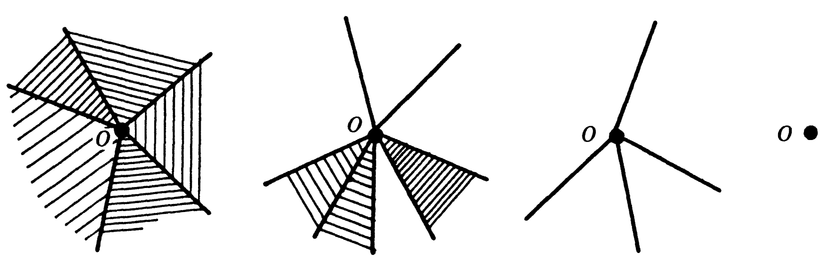


Fig. 1.1. Fans for $r = 2$

Proposition 1.1. *Let σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. Then the following hold:*

(i) $\mathscr{S}_\sigma := M \cap \sigma^\vee = \{m \in M; \langle m, y\rangle \geq 0$ for all $y \in \sigma\}$ is an additive subsemigroup, containing O, of M. Namely, $O \in \mathscr{S}_\sigma$ and $m+m' \in \mathscr{S}_\sigma$ for all $m, m' \in \mathscr{S}$.

(ii) $\mathscr{S}_\sigma$ is finitely generated as an additive semigroup, i.e., there exist $m_1, \ldots, m_p \in \mathscr{S}_\sigma$ such that

$$\mathscr{S}_\sigma = \mathbb{Z}_{\geq 0} m_1 + \ldots + \mathbb{Z}_{\geq 0} m_p := \{a_1 m_1 + \ldots + a_p m_p;\, a_i \in \mathbb{Z}, a_i \geq 0 \text{ for all } i\}\ .$$

Here, $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers.

(iii) $\mathscr{S}_\sigma$ generates M as a group, i.e., $\mathscr{S}_\sigma + (-\mathscr{S}_\sigma) = M$.

(iv) $\mathscr{S}_\sigma$ is saturated, i.e., $cm \in \mathscr{S}_\sigma$ for $m \in M$ and a positive integer c implies $m \in \mathscr{S}_\sigma$.

Conversely, for any additive subsemigroup $\mathscr{S}$ of M satisfying the properties (i) *through* (iv), *there exists a unique strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$ such that $\mathscr{S} = \mathscr{S}_\sigma$.*

Proof. (i) and (iv) are clear by definition.

As for (iii), we have $\dim \sigma^\vee = r$, as we remarked above, since σ is strongly convex, hence $\sigma^\vee + (-\sigma^\vee) = M_{\mathbb{R}}$ by Theorem A.1. Thus there exist $\mathbb{R}$-linearly independent $m_1, \ldots, m_r$ in $M \cap \sigma^\vee = \mathscr{S}_\sigma$. The simplicial rational cone $\varrho := \mathbb{R}_{\geq 0} m_1 + \ldots + \mathbb{R}_{\geq 0} m_r$ is contained in $\sigma^\vee$. For any $m \in M$, we can obviously find m' in $\mathbb{Z}_{\geq 0} m_1 + \ldots + \mathbb{Z}_{\geq 0} m_r \subset M \cap \varrho$ such that $m+m' \in \varrho$. Thus $m+m'$ is contained in $M \cap \varrho \subset M \cap \sigma^\vee = \mathscr{S}_\sigma$.

We next show (ii), which is known as Gordan's lemma. $\sigma^\vee$ is r-dimensional, as we saw above. Hence it is a finite union of simplicial rational cones by Carathéodory's theorem (cf. Theorem A.3). Thus, without loss of generality, we may assume $\sigma^\vee$ to be simplicial, i.e., $\sigma^\vee = \mathbb{R}_{\geq 0} m_1 + \ldots + \mathbb{R}_{\geq 0} m_r$ for $\mathbb{R}$-linearly independent $m_1, \ldots, m_r \in \mathscr{S}_\sigma$. Then $M' := \mathbb{Z} m_1 + \ldots + \mathbb{Z} m_r$ is a subgroup of M of finite index such that $M' \cap \sigma^\vee = \mathbb{Z}_{\geq 0} m_1 + \ldots + \mathbb{Z}_{\geq 0} m_r$. For any $m \in \mathscr{S}_\sigma$, we can obviously find $m' \in M' \cap \sigma^\vee$ in such a way that the coefficients $a_1, \ldots, a_r$ of $m - m'$ with respect to $m_1, \ldots, m_r$ are all nonnegative and less than one. This means that $m - m'$ is a point of M contained in the fundamental parallelepiped in $M'_{\mathbb{R}} = M_{\mathbb{R}}$ at O spanned by $m_1, \ldots, m_r$. The intersection of M with the parallelepiped is obviously finite and together with $m_1, \ldots, m_r$ generates $\mathscr{S}_\sigma$ as a semigroup.

It remains to show the converse. Suppose $\mathscr{S}$ is generated as a semigroup by $m_1, \ldots, m_p$. Then $\varrho := \mathbb{R}_{\geq 0} m_1 + \ldots + \mathbb{R}_{\geq 0} m_p$ is a convex polyhedral cone in $M_{\mathbb{R}}$ and $\sigma := \varrho^\vee$ is a rational convex polyhedral cone in $N_{\mathbb{R}}$ (cf. the proof of Proposition 1.3). σ is strongly convex, since $\mathscr{S} + (-\mathscr{S}) = M$ and $\varrho + (-\varrho) = M_{\mathbb{R}}$. Obviously, we have $\mathscr{S} \subset M \cap \sigma^\vee = M \cap \varrho$. To show the opposite inclusion, we may again assume, without loss of generality, that $p = r$, i.e., ϱ is simplicial, by Carathéodory's theorem. By the same argument as above, any m in $M \cap \varrho$ can be written as a nonnegative *rational* linear combination of $m_1, \ldots, m_r$. Thus a suitable positive integral multiple of m is a nonnegative integral linear combination of $m_1, \ldots, m_r$ and belongs to $\mathscr{S}$. Since $\mathscr{S}$ is saturated by assumption, we are done. q.e.d.

1.2 Toric Varieties

We denote by $\mathbb{C}^\times$ the multiplicative group of nonzero complex numbers. For $N \cong \mathbb{Z}^r$ we define an r-dimensional *algebraic torus* $T_N \cong \mathbb{C}^\times \times \ldots \times \mathbb{C}^\times$ (r times) by

$$T_N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) = N \otimes_{\mathbb{Z}} \mathbb{C}^\times \ .$$

Each $m \in M$ gives rise to a *character* $\mathbf{e}(m)$, which is a homomorphism $\mathbf{e}(m): T_N \to \mathbb{C}^\times$ defined by

$$\mathbf{e}(m)(t) := t(m) \quad \text{for} \quad t \in T_N \ .$$

We have the exponential law $\mathbf{e}(m+m') = \mathbf{e}(m) \cdot \mathbf{e}(m')$ for $m, m' \in M$. In particular, we have $\mathbf{e}(O) = 1$ and can identify M with the character group of T_N.

On the other hand, each $n \in N$ gives rise to a *one-parameter subgroup* $\gamma_n: \mathbb{C}^\times \to T_N$, which is a homomorphism defined by

$$\gamma_n(\lambda)(m) := \lambda^{\langle m, n\rangle} \quad \text{for} \quad \lambda \in \mathbb{C}^\times \quad \text{and} \quad m \in M \ .$$

The right hand side above denotes λ to the power $\langle m, n\rangle$. We have $\gamma_{n+n'} = \gamma_n \cdot \gamma_{n'}$ for $n, n' \in N$, so that we can and do identify N with the group of one-parameter subgroups of T_N.

In down-to-earth terms, what we defined above amount to the following: Choose a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N and let $\{m_1, \ldots, m_r\}$ be the dual basis of M. If we denote $u_j := \mathbf{e}(m_j)$, then we have an isomorphism

$$T_N \xrightarrow{\sim} (\mathbb{C}^\times)^r \quad \text{sending} \quad t \quad \text{to} \quad (u_1(t), \ldots, u_r(t)) \ .$$

Thus $(u_1, \ldots, u_r)$ is a coordinate system for T_N. For $m = \sum_{1 \le j \le r} a_j m_j$ we have $\mathbf{e}(m) = u_1^{a_1} u_2^{a_2} \ldots u_r^{a_r}$, which is a Laurent monomial on T_N. On the other hand, $n = \sum_{1 \le j \le r} b_j n_j$ gives rise to a homomorphism $\gamma_n: \mathbb{C}^\times \to T_N$ which sends $\lambda \in \mathbb{C}^\times$ to $(\lambda^{b_1}, \lambda^{b_2}, \ldots, \lambda^{b_r}) \in (\mathbb{C}^\times)^r$.

An algebraic torus is a commutative algebraic group. Besides, it has the following key property, the proof of which is not so hard and can be found in any standard textbook on algebraic groups:

The Complete Reducibility Theorem. *Suppose an algebraic torus T_N acts* algebraically *on a $\mathbb{C}$-vector space W, i.e., there is given a group homomorphism $\varrho: T_N \to GL(W)$ with the matrix entries expressed as Laurent polynominals in $u_1, \ldots, u_r$. If we denote by $W_m := \{w \in W; \varrho(t)w = t(m)w$ for all $t \in T_N\}$ the eigenspace with respect to each character $m \in M$, then W decomposes as the direct sum $W = \bigoplus_{m \in M} W_m$.*

We associated to M an algebraic torus $T_N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times)$. In our construction of the toric variety corresponding to a fan, we generalize this procedure to additive subsemigroups $\mathscr{S}_\sigma$ in Proposition 1.1 as follows:

Proposition 1.2. *Let $\mathscr{S}_\sigma = M \cap \sigma^\vee = \mathbb{Z}_{\ge 0} m_1 + \ldots \mathbb{Z}_{\ge 0} m_p$ be the finitely generated subsemigroup of M determined by a strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$ as in Proposition 1.1. Let*

$$U_\sigma := \{u: \mathscr{S}_\sigma \to \mathbb{C};\ u(O) = 1,\ u(m+m') = u(m)u(m'),\ \forall\, m, m' \in \mathscr{S}_\sigma\}$$

and let $\mathbf{e}(m)(u) := u(m)$ *for* $m \in \mathcal{S}_\sigma$ *and* $u \in U_\sigma$. *Then the map*

$$(\mathbf{e}(m_1), \ldots, \mathbf{e}(m_p)) : U_\sigma \to \mathbb{C}^p = \mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C}$$

is one-to-one. Identified with its image under this map, U_σ *is an algebraic subset of* $\mathbb{C}^p$ *defined as the set of solutions of a system of equations of the form (monomial) = (monomial). The structure of an r-dimensional irreducible normal complex analytic space on* U_σ *induced by the usual complex analytic structure of* $\mathbb{C}^p$ *is independent of the system* $\{m_1, \ldots, m_p\}$ *of semigroup generators chosen. Each* $m \in \mathcal{S}_\sigma$ *gives rise to a polynominal function* $\mathbf{e}(m)$ *on* U_σ *which is a holomorphic function with respect to the above structure.*

Remark. We here adopt the simple description of U_σ due to S. Ramanan. U_σ in fact has a natural structure of an r-dimensional irreducible affine algebraic variety over $\mathbb{C}$. The complex analytic space structure associated to it is what we described above. We utilize this fact to reduce to algebraic geometry the proof for various complex analytic properties of toric varieties.

In terms of algebraic geometry, the structure of an algebraic variety on U_σ can be described very concisely as follows: Let $\mathbb{C}[M] := \bigoplus_{m \in M} \mathbb{C}\mathbf{e}(m)$ be the group algebra of M over $\mathbb{C}$, where $\mathbf{e}(m)$'s are just symbols now and the ring multiplication is defined by $\mathbf{e}(m) \cdot \mathbf{e}(m') := \mathbf{e}(m+m')$ for $m, m' \in M$. Consider the affine scheme $\operatorname{Spec}(\mathbb{C}[M])$. The set of its $\mathbb{C}$-valued points (i.e., $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[M] \to \mathbb{C}$) obviously coincides with our algebraic torus $T_N = \operatorname{Hom}_{\mathbf{Z}}(M, \mathbb{C}^\times)$. Since $\mathcal{S}_\sigma$ is an additive subsemigroup of M, its semigroup algebra $\mathbb{C}[\mathcal{S}_\sigma]$ over $\mathbb{C}$ is the $\mathbb{C}$-subalgebra of $\mathbb{C}[M]$ with $\{\mathbf{e}(m); m \in \mathcal{S}_\sigma\}$ as a $\mathbb{C}$-basis. Then U_σ obviously coincides with the set of $\mathbb{C}$-valued points of the affine scheme $\operatorname{Spec}(\mathbb{C}[\mathcal{S}_\sigma])$, i.e., the set of $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[\mathcal{S}_\sigma] \to \mathbb{C}$.

Proof of Proposition 1.2. Since $m_1, \ldots, m_p$ generate $\mathcal{S}_\sigma$ as an additive semigroup, $u \in U_\sigma$ is uniquely determined by $u(m_j) = \mathbf{e}(m_j)(u) \in \mathbb{C}$ for $1 \le j \le p$. Suppose $a = (a_1, \ldots, a_p) \in \mathbb{C}^p$ is given. If we use the above remark, we can describe a necessary and sufficient condition for the existence of $u \in U_\sigma$ with $u(m_j) = a_j$ for $1 \le j \le p$ as follows: Consider the $\mathbb{C}$-algebra homomorphism from the polynominal ring $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_p]$ onto $\mathbb{C}[\mathcal{S}_\sigma] = \mathbb{C}[\mathbf{e}(m_1) \ldots, \mathbf{e}(m_p)]$ which sends x_j to $\mathbf{e}(m_j)$ for each j. Choose generators $f_1(x), \ldots, f_q(x)$ of the kernel as an ideal of $\mathbb{C}[x]$. Then there exists $u \in U_\sigma$ with $u(m_j) = a_j$ for all j if and only if $f_1(a) = \ldots = f_q(a) = 0$. Each $f(x) \in I$ is of the form $f(x) = \sum b(\nu_1, \ldots, \nu_p) x_1^{\nu_1} x_2^{\nu_2} \ldots x_p^{\nu_p}$ for $b(\nu_1, \ldots, \nu_p) \in \mathbb{C}$. Hence $0 = f(\mathbf{e}(m_1), \ldots, \mathbf{e}(m_p)) = \sum b(\nu_1, \ldots, \nu_p) \mathbf{e}(\nu_1 m_1 + \ldots + \nu_p m_p) = \sum'_m (\sum'' b(\nu_1, \ldots, \nu_p)) \mathbf{e}(m)$, where $\sum'_m$ denotes the summation over $m \in \mathcal{S}_\sigma$ and $\sum''$ denotes that over the nonnegative integers $\nu_1, \ldots, \nu_p$ satisfying $\nu_1 m_1 + \ldots + \nu_p m_p = m$. As simple calculation shows, the ideal I consisting of such f is generated by a finite number of polynomials of the form

$$x_1^{\nu_1} x_2^{\nu_2} \ldots x_p^{\nu_p} - x_1^{\mu_1} x_2^{\mu_2} \ldots x_p^{\mu_p}$$

for nonnegative integers $\nu_1, \ldots, \nu_p$, $\mu_1, \ldots, \mu_p$ satisfying $\nu_1 m_1 + \ldots + \nu_p m_p = \mu_1 m_1 + \ldots + \mu_p m_p$.

Thus it suffices to show that the coordinate ring $\mathbb{C}[\mathcal{S}_\sigma]$ of U_σ as an affine algebraic variety is an r-dimensional normal integral domain. $\mathbb{C}[M]$ coincides with the Laurent polynomial ring $\mathbb{C}[u_1, u_1^{-1}, \ldots, u_r, u_r^{-1}]$ as we saw above. Thus it is an integral domain, hence so is its subring $\mathbb{C}[\mathcal{S}_\sigma]$. The field of fractions of $\mathbb{C}[\mathcal{S}_\sigma]$ coincides with that of $\mathbb{C}[M]$, since each $m \in M = \mathcal{S}_\sigma + (-\mathcal{S}_\sigma)$ is of the form $m = m' - m''$ for $m', m'' \in \mathcal{S}_\sigma$, hence $\mathbf{e}(m) = \mathbf{e}(m')/\mathbf{e}(m'')$. In particular, $\mathbb{C}[\mathcal{S}_\sigma]$ is an r-dimensional integral domain. It remains to show that $\mathbb{C}[\mathcal{S}_\sigma]$ is integrally closed in its field of fractions. Since $\mathbb{C}[M]$ is normal, the integral closure R of $\mathbb{C}[\mathcal{S}_\sigma]$ in its field of fractions is contained in $\mathbb{C}[M]$. T_N acts algebraically on $\mathbb{C}[M]$ by $t \cdot \mathbf{e}(m) := t(m)\mathbf{e}(m)$ for $t \in T_N$ and $m \in M$. Obviously, $\mathbb{C}[\mathcal{S}_\sigma]$ and R are T_N-stable $\mathbb{C}$-subspaces. By the complete reducibility theorem above, they are direct sums of T_N-eigenspaces, which are either zero or are one-dimensional generated by an element of the form $\mathbf{e}(m)$ for $m \in M$. To prove $R = \mathbb{C}[\mathcal{S}_\sigma]$, it thus suffices to show that $m \in M$ belongs to $\mathcal{S}_\sigma$ if $\mathbf{e}(m)$ is integral over $\mathbb{C}[\mathcal{S}_\sigma]$. Hence suppose $\mathbf{e}(m)^\nu + a_1\mathbf{e}(m)^{\nu-1} + \ldots + a_\nu = 0$ for $a_1, \ldots, a_\nu \in \mathbb{C}[\mathcal{S}_\sigma]$. If we look at the component of each $a_j\mathbf{e}(m)^{\nu-j}$ with respect to the T_N-eigenspace for the character νm, we easily see that $\nu m = m' + (\nu - j)m$ for some $1 \le j \le \nu$ and some $m' \in \mathcal{S}_\sigma$, hence $jm = m' \in \mathcal{S}_\sigma$. Since $\mathcal{S}_\sigma$ is saturated by assumption, we have $m \in \mathcal{S}_\sigma$. q.e.d.

Examples. Here are simple examples for $r = 2$ (cf. Fig. 1.2): Fix a $\mathbb{Z}$-basis $\{n_1, n_2\}$ of N and let $\{m_1, m_2\}$ be the dual $\mathbb{Z}$-basis of M.

(i) If $\sigma = \mathbb{R}_{\ge 0}n_1 + \mathbb{R}_{\ge 0}n_2$, then $\sigma^\vee = \mathbb{R}_{\ge 0}m_1 + \mathbb{R}_{\ge 0}m_2$. Hence $\mathcal{S}_\sigma = \mathbb{Z}_{\ge 0}m_1 + \mathbb{Z}_{\ge 0}m_2$ and we have an isomorphism $(\mathbf{e}(m_1), \mathbf{e}(m_2)) : U_\sigma \xrightarrow{\sim} \mathbb{C}^2$.

(ii) If $\sigma = \mathbb{R}_{\ge 0}n_1$, then $\sigma^\vee = \mathbb{R}_{\ge 0}m_1 + \mathbb{R}m_2$. Thus $\mathcal{S}_\sigma = \mathbb{Z}_{\ge 0}m_1 + \mathbb{Z}_{\ge 0}m_2 + \mathbb{Z}_{\ge 0}(-m_2)$ and the embedding $(\mathbf{e}(m_1), \mathbf{e}(m_2), \mathbf{e}(-m_2)) : U_\sigma \to \mathbb{C}^3$ gives rise to an isomorphism $U_\sigma = \{(u_1, u_2, u_3) \in \mathbb{C}^3 ; u_2u_3 = 1\} \xrightarrow{\sim} \{(u_1, u_2) \in \mathbb{C}^2 ; u_2 \ne 0\} = \mathbb{C} \times \mathbb{C}^\times$.

(iii) If $\sigma = \{O\}$, then $\mathcal{S}_\sigma = M = \mathbb{Z}_{\ge 0}m_1 + \mathbb{Z}_{\ge 0}(-m_1) + \mathbb{Z}_{\ge 0}m_2 + \mathbb{Z}_{\ge 0}(-m_2)$. As above, we thus have $U_\sigma \cong \{(u_1, u_2) \in \mathbb{C}^2 ; u_1 \ne 0, u_2 \ne 0\} = \mathbb{C}^\times \times \mathbb{C}^\times$.

(iv) If $\sigma = \mathbb{R}_{\ge 0}n_1 + \mathbb{R}_{\ge 0}(n_1 + 2n_2)$, then simple computation shows $\sigma^\vee = \mathbb{R}_{\ge 0} \cdot (2m_1 - m_2) + \mathbb{R}_{\ge 0}m_2$ and $\mathcal{S}_\sigma = \mathbb{Z}_{\ge 0}m_1 + \mathbb{Z}_{\ge 0}m_2 + \mathbb{Z}_{\ge 0}(2m_1 - m_2)$. By the embedding $(\mathbf{e}(m_1), \mathbf{e}(m_2), \mathbf{e}(2m_1 - m_2)) : U_\sigma \to \mathbb{C}^3$, we have $U_\sigma = \{(u_1, u_2, u_3) \in \mathbb{C}^3 ; u_1^2 = u_2u_3\}$, which has an isolated singularity at the origin $O = (0, 0, 0)$. See Section 1.6 for systematic treatment of similar examples.

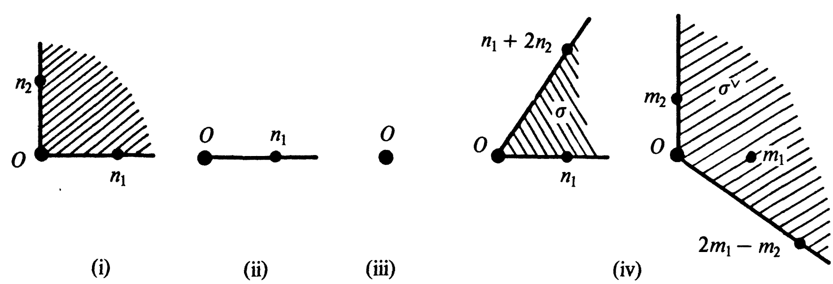


Fig. 1.2

The following plays a key rôle in our construction of toric varieties, where we glue together complex analytic spaces of the form U_σ:

Proposition 1.3. *For a strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$, its dual cone $\sigma^\vee$ is a rational polyhedral cone in $M_{\mathbb{R}}$. If τ is a face of σ, then there exists $m_0 \in M \cap \sigma^\vee$ such that*

$$\tau = \sigma \cap \{m_0\}^\perp = \{y \in \sigma; \langle m_0, y\rangle = 0\} ,$$

hence τ is also a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. In this case, we have $\mathscr{S}_\tau = \mathscr{S}_\sigma + \mathbb{Z}_{\geq 0}(-m_0)$, hence

$$U_\tau = \{u \in U_\sigma ; u(m_0) \neq 0\} ,$$

which is an open subset of U_σ.

Proof. Since σ is the set of nonnegative linear combinations of a finite number of elements in N, we see that $\sigma^\vee$ consists of the solutions of a finite system of homogeneous linear inequalities with integral coefficients. $\sigma^\vee$ is rational, since we can obviously choose elements of M as its generators in Theorem A.2. If τ is a face of σ, we have $\tau = \sigma \cap \{m_0'\}^\perp$ for some $m_0' \in \sigma^\vee$ by definition. Consider the face of $\sigma^\vee$ which contains m_0' in its relative interior. This face is again rational, since it is the set of solutions of a system of homogeneous linear inequalities. Thus the face contains an element of $M \otimes_{\mathbb{Z}} \mathbb{Q}$ in its relative interior. A suitable positive integral multiple m_0 of this element is in M as well as in the relative interior of the face in question. As in the proof of Proposition A.6, we have $\tau = \sigma \cap \{m_0\}^\perp$.

By Corollary A.7, we have $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-m_0)$, hence $\mathscr{S}_\tau = M \cap \tau^\vee \supset M \cap \sigma^\vee + \mathbb{Z}_{\geq 0}(-m_0) = \mathscr{S}_\sigma + \mathbb{Z}_{\geq 0}(-m_0)$. For any $m \in \mathscr{S}_\tau$, however, we can find a large enough positive integer a so that $m + am_0$ is in $\sigma^\vee$, hence in $M \cap \sigma^\vee = \mathscr{S}_\sigma$. Thus we have $\mathscr{S}_\tau = \mathscr{S}_\sigma + \mathbb{Z}_{\geq 0}(-m_0)$. Since m_0 is in $\mathscr{S}_\sigma$, we obviously have $U_\tau = \{u \in U_\sigma ; u(m_0) \neq 0\}$. q.e.d.

We are now in a position to construct toric varieties.

Theorem 1.4. *For a fan Δ in N, we can naturally glue $\{U_\sigma ; \sigma \in \Delta\}$ together to obtain a Hausdorff complex analytic space*

$$T_N \operatorname{emb}(\Delta) := \bigcup_{\sigma \in \Delta} U_\sigma ,$$

which is irreducible and normal with dimension equal to $r = \operatorname{rank} N$. *We call it a* toric variety *or a* torus embedding *associated to the fan* (N, Δ).

Proof. By Proposition 1.2, U_σ for each $\sigma \in \Delta$ is an r-dimensional irreducible normal algebraic subset, hence an analytic subset, in a complex affine space. On the other hand, $\sigma \cap \tau$ is a face of both σ and τ for each $\sigma, \tau \in \Delta$ by the definition of a fan. Thus by Proposition 1.3, $U_{\sigma\cap\tau}$ is naturally an open subset of both U_σ and U_τ. We can then naturally glue $\{U_\sigma ; \sigma \in \Delta\}$ together to obtain an r-dimensional irreducible normal complex analytic space $X = T_N \operatorname{emb}(\Delta)$. It thus remains to show that X is a Hausdorff space, i.e., the diagonal of the product space $X \times X$ is a closed subset. Obviously, it suffices to show that for any $\sigma, \tau \in \Delta$, the diagonal map $U_{\sigma\cap\tau} \to U_\sigma \times U_\tau$ sending u to the pair (u', u'') is a closed map, where u' (resp. u'') is the restriction of u to $\mathscr{S}_\sigma$ (resp. $\mathscr{S}_\tau$) contained in $\mathscr{S}_{\sigma\cap\tau}$.

For that purpose, we first show $\mathscr{S}_{\sigma\cap\tau} = \mathscr{S}_\sigma + \mathscr{S}_\tau$. By Theorem A.1, (2), we have $(\sigma \cap \tau)^\vee = \sigma^\vee + \tau^\vee$, hence the right hand side is contained in the left hand side. On the other hand, since σ and τ intersect each other along the common face $\sigma \cap \tau$, we can use Theorem A.1, (3) as in the proof of Proposition 1.3 to show the existence of $m_0 \in M$ such that σ and τ lie on mutually opposite sides with respect to the hyperplane $\{m_0\}^\perp$. Hence $m_0 \in \mathscr{S}_\sigma$, $-m_0 \in \mathscr{S}_\tau$ and $\sigma \cap \tau = \sigma \cap \{m_0\}^\perp = \tau \cap \{m_0\}^\perp$. By Proposition 1.3, we thus have $\mathscr{S}_{\sigma\cap\tau} = \mathscr{S}_\sigma + \mathbb{Z}_{\geq 0}(-m_0) \subset \mathscr{S}_\sigma + \mathscr{S}_\tau$.

Let $\{m_1, \ldots, m_p\}$ and $\{m'_1, \ldots, m'_q\}$ be systems of generators of $\mathscr{S}_\sigma$ and $\mathscr{S}_\tau$ as additive semigroups, respectively. By what we have just seen, $m_1, \ldots, m_p, m'_1, \ldots, m'_q$ then generate $\mathscr{S}_{\sigma\cap\tau}$ as an additive semigroup. We may identify $U_{\sigma\cap\tau}$ (resp. U_σ, resp. U_τ) with its image in $\mathbb{C}^{p+q}$ (resp. $\mathbb{C}^p$, resp. $\mathbb{C}^q$) under the closed embedding $(\mathbf{e}(m_1), \ldots, \mathbf{e}(m_p), \mathbf{e}(m'_1), \ldots, \mathbf{e}(m'_q))$ (resp. $(\mathbf{e}(m_1), \ldots, \mathbf{e}(m_p))$, resp. $(\mathbf{e}(m'_1), \ldots, \mathbf{e}(m'_q))$). Thus $U_{\sigma\cap\tau}$ and $U_\sigma \times U_\tau$ can be regarded as closed subsets of $\mathbb{C}^{p+q} = \mathbb{C}^p \times \mathbb{C}^q$. Moreover, $U_{\sigma\cap\tau}$ is contained in $U_\sigma \times U_\tau$. q.e.d.

The above construction of a toric variety from a fan might look redundant, since we twice perform the operation of taking a dual: we first take the dual cone $\sigma^\vee$ of each $\sigma \in \Delta$ and obtain the additive subsemigroup $\mathscr{S}_\sigma = M \cap \sigma^\vee$. We then take $U_\sigma = \{u : \mathscr{S}_\sigma \to \mathbb{C}; u(O) = 1, u(m+m') = u(m)u(m'), \forall m, m' \in \mathscr{S}_\sigma\}$, which is the dual of a sort for $\mathscr{S}_\sigma$. We then glue U_σ's together to get the toric variety $T_N \operatorname{emb}(\Delta)$. Thanks to the double dual operation, however, we get $U_\sigma \cap U_\tau = U_{\sigma\cap\tau}$, which enables us to relate complex analytic properties of $T_N \operatorname{emb}(\Delta)$ rather directly with elementary convex geometry of a fan Δ. The above equality is equivalent to $(\sigma \cap \tau)^\vee = \sigma^\vee + \tau^\vee$, hence to $\mathscr{S}_{\sigma\cap\tau} = \mathscr{S}_\sigma + \mathscr{S}_\tau$. It is not so easy, however, to visualize and manipulate $\{\sigma^\vee; \sigma \in \Delta\}$ and $\{\mathscr{S}_\sigma; \sigma \in \Delta\}$ satisfying these equalities.

Remark. As we observed in the remark immediately after Proposition 1.2, each U_σ is an affine algebraic variety over $\mathbb{C}$. The natural gluing maps among U_σ's are obviously algebraic by Proposition 1.3. Thus $T_N \operatorname{emb}(\Delta)$ has the natural structure of a scheme locally of finite type over $\mathbb{C}$ obtained as the union of possibly infinitely many affine open sets. The complex analytic structure naturally associated to this algebraic structure is the one we considered in Theorem 1.4. We often use this fact below in the proofs of our results. The construction works also over arbitrary fields or commutative rings instead of $\mathbb{C}$ and gives rise to toric varieties as schemes over them. For details, the reader is referred to the literature mentioned at the beginning of this chapter.

Here are easy examples of toric varieties of dimension $r \leq 2$.

Example. For $N = \mathbb{Z}$ let $\sigma := \mathbb{R}_{\geq 0} \subset N_\mathbb{R}$. Then $\Delta := \{\sigma, -\sigma, \{O\}\}$ is a fan. We obtain $T_N \operatorname{emb}(\Delta)$ by gluing $U_\sigma \cong \mathbb{C}$ and $U_{-\sigma} \cong \mathbb{C}$ along their common open subset $U_{\{O\}} = T_N = \mathbb{C}^\times$. It is easily seen to coincide with the complex projective line $\mathbb{P}_1(\mathbb{C})$ (cf. Fig. 1.3).

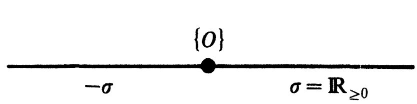


Fig. 1.3

Example. Let $\{n_1, n_2\}$ be a $\mathbb{Z}$-basis of $N \cong \mathbb{Z}^2$ and let $\{m_1, m_2\}$ be the dual basis of M (cf. Fig. 1.4).

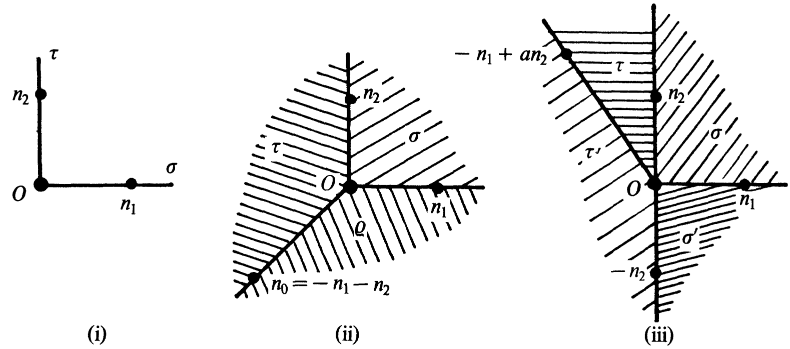


Fig. 1.4

(i) $\Delta := \{\sigma, \tau, \{O\}\}$ with $\sigma := \mathbb{R}_{\geq 0} n_1$ and $\tau := \mathbb{R}_{\geq 0} n_2$ is a fan. We get $T_N \operatorname{emb}(\Delta) = \mathbb{C}^2 \setminus \{(0,0)\}$ by gluing $U_\sigma = \mathbb{C} \times \mathbb{C}^\times$ and $U_\tau = \mathbb{C}^\times \times \mathbb{C}$ along their common open subset $U_{\{O\}} = T_N = \mathbb{C}^\times \times \mathbb{C}^\times$.

(ii) Let $n_0 := -n_1 - n_2$ and $\sigma := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2$, $\tau := \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_2$, $\varrho := \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_1$. We obtain a fan Δ consisting of σ, τ, ϱ as well as their faces $\mathbb{R}_{\geq 0} n_0$, $\mathbb{R}_{\geq 0} n_1$, $\mathbb{R}_{\geq 0} n_2$ and $\{O\}$. We have $T_N \operatorname{emb}(\Delta) = \mathbb{P}_2(\mathbb{C})$, the complex projective plane. In terms of its homogeneous coordinate $[z_0 : z_1 : z_2]$, we have $z_1/z_0 = \mathbf{e}(m_1)$, $z_2/z_0 = \mathbf{e}(m_2)$. Moreover, U_σ, U_τ and U_ϱ are the affine planes $\{z_0 \neq 0\}$, $\{z_1 \neq 0\}$ and $\{z_2 \neq 0\}$, respectively.

(iii) For an integer a let $\sigma := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2$, $\sigma' := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0}(-n_2)$, $\tau = \mathbb{R}_{\geq 0}(-n_1 + a n_2) + \mathbb{R}_{\geq 0} n_2$ and $\tau' := \mathbb{R}_{\geq 0}(-n_1 + a n_2) + \mathbb{R}_{\geq 0}(-n_2)$. Then the collection Δ of σ, σ', τ, τ' as well as their faces is a fan. In this case, $T_N \operatorname{emb}(\Delta)$ is the *Hirzebruch surface*, usually denoted by F_a or Σ_a, which is a $\mathbb{P}_1(\mathbb{C})$-bundle over $\mathbb{P}_1(\mathbb{C})$. Since F_a and F_{-a} are easily seen to be isomorphic, we usually assume $a \geq 0$.

$T_N \operatorname{emb}(\Delta)$ is called a toric variety or a torus embedding for the following reasons: A fan Δ always contains $\{O\}$. Obviously, $\mathscr{S}_{\{O\}} = M$ and $U_{\{O\}} = T_N$. Moreover, $\{O\}$ is a face of every $\sigma \in \Delta$, hence T_N is an open subset of U_σ by Proposition 1.3. Consequently, $T_N \operatorname{emb}(\Delta)$ canonically contains the algebraic torus T_N as an open subset.

On the other hand, T_N acts on $T_N \operatorname{emb}(\Delta)$ *algebraically* with respect to its structure of an algebraic variety mentioned in the remark immediately after Proposition 1.2. Indeed, let $t \in T_N$ and $u \in U_\sigma$. Then $t : M \to \mathbb{C}^\times$ is a homomorphism and $u : \mathscr{S}_\sigma \to \mathbb{C}$ satisfies $u(O) = 1$ and $u(m + m') = u(m)u(m')$ for $m, m' \in \mathscr{S}_\sigma$. We then define $tu : \mathscr{S}_\sigma \to \mathbb{C}$ by

$$(tu)(m) := t(m)u(m) \quad \text{for} \quad m \in \mathscr{S}_\sigma \; .$$

Obviously, tu is an element of U_σ. This gives rise to an action of T_N on U_σ, hence on $T_N \operatorname{emb}(\Delta)$ by natural gluing. The above action for $\sigma = \{O\}$ coincides with the group multiplication on $U_{\{O\}} = T_N$.

In fact, we have the following converse to what we have just seen. For the proof, we refer the reader to [MO, Theorem 4.1], for instance.

Theorem 1.5. *Suppose the algebraic torus T_N acts algebraically on an irreducible normal algebraic variety X locally of finite type over $\mathbb{C}$. If X contains an open orbit isomorphic to T_N, then there exists a unique fan Δ in N such that X is equivariantly isomorphic to* $T_N \operatorname{emb}(\Delta)$.

For the proof, we take advantage of the complete reducibility theorem and the following basic result.

Sumihiro's Theorem (cf. [S15, I, Lemma 8, Corollary 2]). *Suppose a connected linear algebraic group G acts algebraically on an irreducible normal algebraic variety X. Then X is a union of G-stable* quasi-projective *open subsets. If G is an algebraic torus, then X is a union of G-stable* affine *open subsets.*

1.3 Orbit Decomposition, Manifolds with Corners and the Fundamental Group

In the previous section, we constructed the toric variety $T_N \operatorname{emb}(\Delta)$ associated to a fan Δ in $N \cong \mathbb{Z}^r$. We also observed that the algebraic torus T_N acts on $T_N \operatorname{emb}(\Delta)$ as follows: U_σ for each $\sigma \in \Delta$ is a T_N-stable open subset. For $t \in T_N = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times)$ and $u \in U_\sigma = \{u : \mathscr{S}_\sigma \to \mathbb{C}; u(O)=1, u(m+m')=u(m)u(m'),\ \forall m, m' \in \mathscr{S}_\sigma\}$, we have $tu \in U_\sigma$, where

$$(tu)(m) := t(m)u(m) \quad \text{for} \quad m \in \mathscr{S}_\sigma \ .$$

We can describe the T_N-orbits in terms of Δ as follows:

Proposition 1.6. *For each $\sigma \in \Delta$ we can regard the quotient algebraic torus*

$$\operatorname{orb}(\sigma) := \{u : M \cap \sigma^\perp \to \mathbb{C}^\times;\ \textit{group homomorphisms}\}$$

of T_N as a T_N-orbit in T_N $\operatorname{emb}(\Delta)$. *Every T_N-orbit is of this form and in this way, Δ is in one-to-one correspondence with the set of T_N-orbits in T_N* $\operatorname{emb}(\Delta)$. *Moreover, the following hold:*

(i) $\operatorname{orb}(\{O\}) = U_{\{O\}} = T_N$.

(ii) *For $\sigma \in \Delta$, the complex analytic dimension of* $\operatorname{orb}(\sigma)$ *coincides with the codimension $r - \dim \sigma$ of σ in $N_{\mathbb{R}}$.*

(iii) *For $\sigma, \tau \in \Delta$, τ is a face of σ if and only if* $\operatorname{orb}(\sigma)$ *is contained in the closure of* $\operatorname{orb}(\tau)$.

(iv) *For $\sigma \in \Delta$,* $\operatorname{orb}(\sigma)$ *is the unique closed T_N-orbit in U_σ and we have* $U_\sigma = \coprod_{\tau < \sigma} \operatorname{orb}(\tau)$.

(v) *Let $n \in N$ and $\sigma \in \Delta$. Then we have $n \in \sigma$ if and only if the one-parameter subgroup γ_n corresponding to n has the property that $\lim_{\lambda \to 0} \gamma_n(\lambda)$ exists in U_σ. In this case, the limit coincides with the indentity element of* $\operatorname{orb}(\tau)$ *regarded as an algebraic torus, where τ is the face of σ which contains n in its relative interior.*

Remark. One-parameter subgroups as above play key rôles in geometric invariant theory of Hilbert and Mumford. [TE] develops the theory of toric varieties using (v) above as a key.

Proof. (i) and (ii) are obvious.

As for (v), the one-parameter subgroup $\gamma_n : \mathbb{C}^\times \to T_N$ corresponding to $n \in N$ is given by

$$\gamma_n(\lambda)(m) = \lambda^{\langle m,n\rangle} \quad \text{for} \quad \lambda \in \mathbb{C}^\times \quad \text{and} \quad m \in M \ .$$

Thus the limit as $\lambda \to 0$ exists in U_σ if and only if $\lambda^{\langle m,n\rangle}$ converges as $\lambda \to 0$ for every $m \in \mathscr{S}_\sigma$, i.e., $\langle m,n\rangle \geq 0$ for every $m \in \mathscr{S}_\sigma$. Obviously, this is equivalent to $n \in \sigma$.

We next show that orb (σ) is a T_N-orbit. Indeed, for $\sigma \in \Delta$, $\sigma^\perp$ is the largest $\mathbb{R}$-subspace contained in $\sigma^\vee$ by Proposition A.6, hence $M \cap \sigma^\perp$ is the largest subgroup contained in $\mathscr{S}_\sigma = M \cap \sigma^\vee$. If we extend a group homomorphism $u : M \cap \sigma^\perp \to \mathbb{C}^\times$ to a map $u : \mathscr{S}_\sigma \to \mathbb{C}$ by letting $u(m) := 0$ for any $m \in \mathscr{S}_\sigma$ not in $M \cap \sigma^\perp$, then it is seen to be an element of U_σ by Proposition A.9. In this way, we can regard orb (σ) naturally as a subset of U_σ. Clearly, it is a T_N-orbit in U_σ.

We now generalize the above consideration to show (iv). Let $u : \mathscr{S}_\sigma \to \mathbb{C}$ be an element of U_σ. Thus $u(O) = 1$ and $u(m + m') = u(m)u(m')$ for m, $m' \in \mathscr{S}_\sigma$. By Proposition A.9, $\{m \in \mathscr{S}_\sigma ; u(m) \neq 0\}$ is the intersection of M with a face of $\sigma^\vee$. Since every face of $\sigma^\vee$ is of the form $\sigma^\vee \cap \tau^\perp$ for a unique face τ of σ by Proposition A.6, we see that $u \in U_\sigma$ determines a unique face τ of σ such that

$$\{m \in \mathscr{S}_\sigma ; u(m) \neq 0\} = M \cap \sigma^\vee \cap \tau^\perp = \mathscr{S}_\sigma \cap \tau^\perp \ .$$

Clearly, $\mathscr{S}_\sigma \cap \tau^\perp$ generates $M \cap \tau^\perp$ as a group. Hence we can identify $\{u \in U_\sigma ; u(\mathscr{S}_\sigma \cap \tau^\perp) \subset \mathbb{C}^\times, u(\mathscr{S}_\sigma \setminus \mathscr{S}_\sigma \cap \tau^\perp) = \{0\}\}$ with the set orb (τ) of homomorphisms from $M \cap \tau^\perp$ to $\mathbb{C}^\times$. Thus we get the latter half of (iv).

Let us show (iii). If orb (σ) is contained in the closure of orb (τ), then the open neighborhood U_σ intersects, hence contains, orb (τ). Thus τ is a face of σ by the latter half of (iv) we have just proved. Conversely, suppose τ is a face of σ. Then $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$. Let u be the element of orb (τ) defined by $u(m) := 1$ for all $m \in M \cap \tau^\perp$. Choose $n \in N$ which is in the relative interior of σ. Then by Lemma A.4, $\langle m', n\rangle$ is positive for any $m' \in M \cap \sigma^\vee \cap \tau^\perp$ not in $\sigma^\perp$. On the other hand, we have $\gamma_n(\lambda) \cdot u \in$ orb (τ) and

$$(\gamma_n(\lambda) \cdot u)(m) = \lambda^{\langle m,n\rangle} \quad \text{for} \quad m \in M \cap \sigma^\vee \cap \tau^\perp \ .$$

Thus the limit $u_0 := \lim_{\lambda \to 0} \gamma_n(\lambda) \cdot u$ satisfies $u_0(m) = 1$ if m is in $M \cap \sigma^\perp$, while $u_0(m) = 0$ if m is in $M \cap \sigma^\vee \cap \tau^\perp$ but not in $\sigma^\perp$. By our identification above, u_0 is thus an element of orb (σ). Therefore, the closure of orb (τ) intersects, hence contains, orb (σ).

The former half of (iv) follows immediately from (iii) and the latter half of (iv).

q.e.d.

Remark. If we regard the toric variety T_N emb (Δ) as an algebraic variety over $\mathbb{C}$, we can show that $\{U_\sigma ; \sigma \in \Delta\}$ coincides with the set of T_N-stable affine open subsets of T_N emb (Δ). In particular, T_N emb (Δ), as an algebraic variety over $\mathbb{C}$, has a closed embedding into a complex affine space if and only if there exists $\pi \in \Delta$ such that Δ coincides with the set of faces of π. We call such T_N emb (Δ) a *toric affine variety*. For the proof, we refer the reader to [MO, Theorem 4.2, (ii)], for instance.

Corollary 1.7. *For $\tau \in \Delta$ denote by $\mathbb{Z}(\tau \cap N)$ the subgroup of N generated by $\tau \cap N$ and let $\bar{N}(\tau) := N/\mathbb{Z}(\tau \cap N)$. For $\sigma \in \Delta$ with $\tau < \sigma$, let $\bar{\sigma} := (\sigma + \mathbb{R}\tau)/\mathbb{R}\tau$ be the image of σ in* *the quotient vector space* $\bar{N}(\tau)_{\mathbb{R}} = N_{\mathbb{R}}/\mathbb{R}\tau$. *Then*

$$\bar{\Delta}(\tau) := \{\bar{\sigma}\,;\sigma \in \Delta, \tau < \sigma\}$$

is a fan in $\bar{N}(\tau)$ *and* $T_{\bar{N}(\tau)}\operatorname{emb}(\bar{\Delta}(\tau))$ *coincides with the closure* $V(\tau)$ *of* $\operatorname{orb}(\tau)$ *in* $T_N \operatorname{emb}(\Delta)$. *In particular,* $V(\tau)$ *is always normal. Moreover,* $V(\tau)$ *is nonsingular, if so is* $T_N \operatorname{emb}(\Delta)$.

Proof. For simplicity, let us denote $\bar{N}(\tau)$ and $\bar{\Delta}(\tau)$ by $\bar{N}$ and $\bar{\Delta}$, respectively. By Proposition A.8, we easily see that $\bar{\sigma}$ is a strongly convex rational polyhedral cone in $\bar{N}_{\mathbb{R}}$ if $\tau < \sigma$ and that $\bar{\Delta}$ is a fan in $\bar{N}$. By Proposition 1.6, (iii), $V(\tau)$ is the union of $\{\operatorname{orb}(\sigma)\,;\sigma \in \Delta, \tau < \sigma\}$ and is covered by $\{U_\sigma\,;\sigma \in \Delta, \tau < \sigma\}$. $M \cap \tau^\perp$ is the $\mathbb{Z}$-module dual to $\bar{N}$. Moreover, if $\tau < \sigma$, then $(\bar{\sigma})^\vee = \sigma^\vee \cap \tau^\perp$, hence $\mathscr{S}_{\bar{\sigma}} := (M \cap \tau^\perp) \cap (\bar{\sigma})^\vee = \mathscr{S}_\sigma \cap \tau^\perp$.

It thus suffices to show that

$$\bar{U}_{\bar{\sigma}} := \{\bar{u} : \mathscr{S}_{\bar{\sigma}} \to \mathbb{C}\,;\bar{u}(O) = 1, \bar{u}(m+m') = \bar{u}(m)\bar{u}(m'), \forall\, m, m' \in \mathscr{S}_{\bar{\sigma}}\}$$

coincides with $V(\tau) \cap U_\sigma$. The restriction of $u \in U_\sigma$ to $\mathscr{S}_\sigma \cap \tau^\perp = \mathscr{S}_{\bar{\sigma}}$ obviously gives rise to an element of $\bar{U}_{\bar{\sigma}}$. On the other hand, each $\bar{u} \in \bar{U}_{\bar{\sigma}}$ can be extended to a map $u : \mathscr{S}_\sigma \to \mathbb{C}$ by $u(m) := \bar{u}(m)$ if $m \in \mathscr{S}_\sigma \cap \tau^\perp$ while $u(m) := 0$ otherwise. u is easily seen to be in U_σ by Proposition A.9.

The rest of the assertion is clear, if we take Theorem 1.10 on nonsingularity in the following section into account. q.e.d.

We now state two results on toric varieties as complex analytic spaces.

The first is the (real) manifold with corners associated to a toric variety. As a topological group we have $\mathbb{C}^\times = U(1) \times \mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ is the multiplicative group of positive real numbers and $U(1) := \{z \in \mathbb{C}\,; |z| = 1\}$ is the one-dimensional unitary group. Hence the algebraic torus $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^\times = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times)$ determined by $N \cong \mathbb{Z}^r$ also has the decomposition $T_N = CT_N \times (N \otimes_{\mathbb{Z}} \mathbb{R}_{>0})$, where

$$CT_N := N \otimes_{\mathbb{Z}} U(1) = \operatorname{Hom}_{\mathbb{Z}}(M, U(1))$$

is the *compact torus* associated to N, which is a real r-dimensional torus. Consider the surjective homomorphism $-\log|\ | : \mathbb{C}^\times \to \mathbb{R}$ obtained as the composite of the surjective homomorphism $|\ | : \mathbb{C}^\times \to \mathbb{R}_{>0}$ with the isomorphism $-\log : \mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{R}$. This induces a surjective homomorphism

$$-\log|\ | : T_N \to N_{\mathbb{R}}$$

with kernel CT_N, whose restriction to the subgroup $N \otimes_{\mathbb{Z}} \mathbb{R}_{>0}$ induces an isomorphism $N \otimes_{\mathbb{Z}} \mathbb{R}_{>0} \xrightarrow{\sim} N_{\mathbb{R}}$.

For each σ in a given fan Δ, consider

$$(U_\sigma)_{\geq 0} := \{w : \mathscr{S}_\sigma \to \mathbb{R}_{\geq 0}\,; w(O) = 1, w(m+m') = w(m)w(m'),\ \forall\, m, m' \in \mathscr{S}_\sigma\}$$

analogous to U_σ in Proposition 1.2. $(U_\sigma)_{\geq 0}$ can be regarded as a topological subspace of U_σ, if we consider $\mathbb{R}_{\geq 0}$ to be a subset of $\mathbb{C}$. When $\mathscr{S}_\sigma = \mathbb{Z}_{\geq 0}\mathrm{m}_1 + \ldots + \mathbb{Z}_{\geq 0} m_p$ as in Proposition 1.2, we have a closed immersion $(U_\sigma)_{\geq 0} \to (\mathbb{R}_{\geq 0})^p$ by sending w to $(w(m_1), \ldots, w(m_p))$. Gluing $(U_\sigma)_{\geq 0}$'s together, we have a topological subspace

$$T_N \operatorname{emb}(\Delta)_{\geq 0} := \bigcup_{\sigma \in \Delta} (U_\sigma)_{\geq 0}$$

of $T_N \operatorname{emb}(\Delta)$. When $N \cong \mathbb{Z}^r$ and Δ is nonsingular in the sense of Section 1.4, we easily see that $T_N \operatorname{emb}(\Delta)_{\geq 0}$ is a real r-dimensional manifold with corners in the usual sense (cf. Fig. 1.5). Moreover, the topological group $N \otimes_{\mathbb{Z}} \mathbb{R}_{>0} = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{R}_{>0})$ acts on $T_N \operatorname{emb}(\Delta)_{\geq 0}$ with the orbits

$$\operatorname{orb}(\tau)_{\geq 0} := \operatorname{Hom}_{\mathbb{Z}}(M \cap \tau^\perp, \mathbb{R}_{>0}) \quad \text{for} \quad \tau \in \Delta \ .$$

When $\tau < \sigma$, we can regard $\operatorname{orb}(\tau)_{\geq 0}$ as a subset of $(U_\sigma)_{\geq 0}$ as in Proposition 1.6. Namely, we extend each homomorphism $w: M \cap \tau^\perp \to \mathbb{R}_{>0}$ to a map $w: \mathscr{S}_\sigma \to \mathbb{R}_{\geq 0}$ by $w(m) := 0$ if m is an element of $\mathscr{S}_\sigma$ not in $\mathscr{S}_\sigma \cap \tau^\perp$.

This topological space $T_N \operatorname{emb}(\Delta)_{\geq 0}$ was considered by Ehlers [E2, Chap. IV] and Jurkiewicz [J1], [J2]. Theoretically more convenient, however, is the following topological space $\operatorname{Mc}(N, \Delta)$, which is homeomorphic to $T_N \operatorname{emb}(\Delta)_{\geq 0}$ and which was considered by [SC] and Ehlers [E2, Chap. IV] (see also [MO, Proposition 10.1]):

Proposition 1.8. *Let Δ be a fan in N. We denote the projection map to the quotient space of $T_N \operatorname{emb}(\Delta)$ with respect to the compact torus CT_N by*

$$\operatorname{ord}: T_N \operatorname{emb}(\Delta) \to \operatorname{Mc}(n, \Delta) := T_N \operatorname{emb}(\Delta)/CT_N$$

and call $\operatorname{Mc}(N, \Delta)$ *the* manifold with corners *associated to* (N, Δ). *$N_{\mathbb{R}}$ acts continuously on* $\operatorname{Mc}(N, \Delta)$ *so that* ord *is equivariant with respect to the group homomorphism* $-\log|\ |: T_N \to N_{\mathbb{R}}$. *The $N_{\mathbb{R}}$-orbits in* $\operatorname{Mc}(N, \Delta)$ *are*

$$\operatorname{ord}(\operatorname{orb}(\tau)) = \operatorname{orb}(\tau)/CT_N \cong N_{\mathbb{R}}/\mathbb{R}\tau \quad \textit{for} \quad \tau \in \Delta \ .$$

In particular, $\operatorname{ord}(\operatorname{orb}(\{O\})) = T_N/CT_N \cong N_{\mathbb{R}}$ *is the unique open $N_{\mathbb{R}}$-orbit in* $\operatorname{Mc}(N, \Delta)$. *If we restrict* ord *to the subspace* $T_N \operatorname{emb}(\Delta)_{\geq 0}$, *we have a homeomorphism*

$$\operatorname{ord}: T_N \operatorname{emb}(\Delta)_{\geq 0} \xrightarrow{\sim} \operatorname{Mc}(N, \Delta)$$

which is equivariant with respect to the group isomorphism $-\log: N \otimes_{\mathbb{Z}} \mathbb{R}_{>0} \xrightarrow{\sim} N_{\mathbb{R}}$. *For each $\tau \in \Delta$, we have* $\operatorname{ord}(\operatorname{orb}(\tau)_{\geq 0}) = \operatorname{orb}(\tau)/CT_N \cong N_{\mathbb{R}}/\mathbb{R}\tau$.

The proof should be obvious. Set-theoretically, $\operatorname{Mc}(N, \Delta)$ is the union of $N_{\mathbb{R}} = \operatorname{ord}(T_N)$ and the $\mathbb{R}$-vector spaces $N_{\mathbb{R}}/\mathbb{R}\tau$ "at infinity and perpendicular to τ" for all $\tau \neq \{O\}$ in Δ. In this way, $\operatorname{Mc}(N, \Delta)$ rather faithfully describes the interrelationship among the $N_{\mathbb{R}}$-orbits, hence the T_N-orbits in the complex analytic space $T_N \operatorname{emb}(\Delta)$ (cf. Fig. 1.5). We utilize this convenient device immediately in Section 1.4. As we also see in the next chapter, a toric projective variety naturally gives rise to a convex polytope (i.e., a compact convex polyhedron) in $M_{\mathbb{R}}$ which is homeomorphic to $\operatorname{Mc}(N, \Delta)$ and whose faces are in one-to-one correspondence with the $N_{\mathbb{R}}$-orbits in $\operatorname{Mc}(N, \Delta)$.

The other result on toric varieties as complex analytic spaces is that on the fundamental group. N is canonically isomorphic to the fundamental group $\pi_1(T_N)$

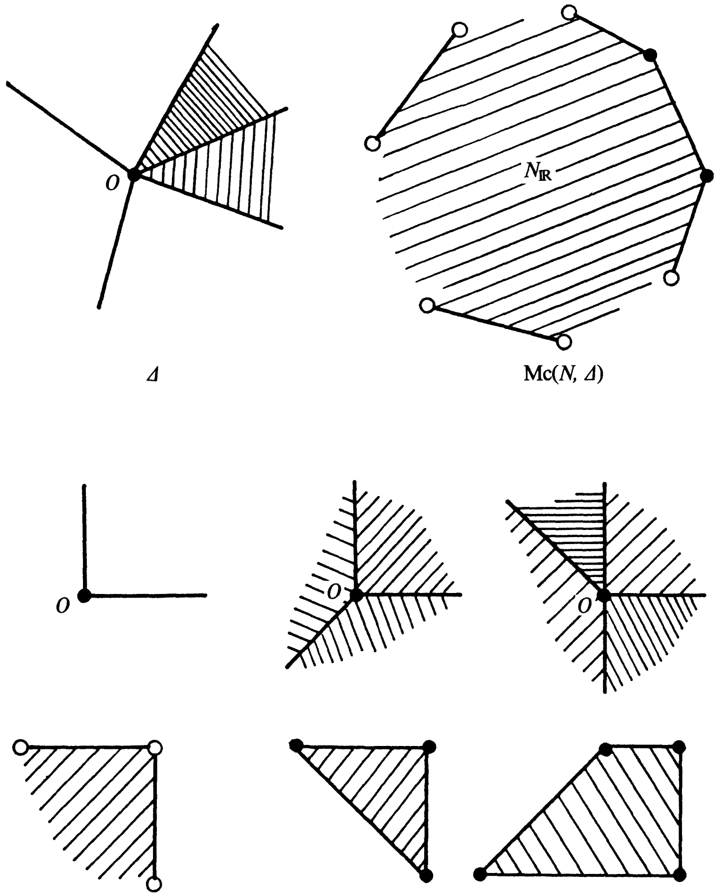


Fig. 1.5

of T_N. Namely, $n \in N$ gives rise to a one-parameter subgroup $\gamma_n : \mathbb{C}^\times \to T_N$. Its restriction to the unit circle $U(1) = \{z \in \mathbb{C}^\times : |z| = 1\}$ determines a closed path in T_N, hence an element of the fundamental group.

Proposition 1.9. *For a fan Δ in N, the fundamental group of the toric variety T_N emb (Δ) is canonically isomorphic to N/N' where N' is the subgroup of N generated by $\cup_{\sigma \in \Delta}(\sigma \cap N)$.*

The proof can be found in [SC] and [MO, Proposition 10.2]. We need to show first that the homomorphism of the fundamental groups

$$N = \pi_1(T_N) \to \pi_1(T_N \text{ emb } (\Delta))$$

arising from the open embedding $T_N \subset T_N$ emb (Δ) is surjective. We then show that for $n \in \sigma \cap N$ the corresponding closed path $U(1) \to T_N$ is homotopic to zero in T_N emb (Δ) due to the existence of orb (σ).

1.4 Nonsingularity and Compactness

Theorem 1.10. *The toric variety* $T_N \operatorname{emb}(\Delta)$ *associated to a fan* Δ *in* N *is nonsingular, i.e., is a complex manifold if and only if each* $\sigma \in \Delta$ *is* nonsingular *in the following sense: There exist a* $\mathbb{Z}$*-basis* $\{n_1, \ldots, n_r\}$ *of* N *and* $s \le r$ *such that* $\sigma = \mathbb{R}_{\ge 0} n_1 + \ldots + \mathbb{R}_{\ge 0} n_s$. *We call* Δ *itself* nonsingular *in this case.*

Proof. The sufficiency is easier. For each $\sigma \in \Delta$ choose a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N as above and the dual basis $\{m_1, \ldots, m_r\}$ of M. Then obviously, we have $\sigma^\vee = \sum_{1 \le j \le s} \mathbb{R}_{\ge 0} m_j + \sum_{s+1 \le j \le r} \mathbb{R} m_j$ and $\mathscr{S}_\sigma = \sum_{1 \le j \le s} \mathbb{Z}_{\ge 0} n_j + \sum_{s+1 \le j \le r} \mathbb{Z} m_j$. Since $\mathbb{Z} m_j = \mathbb{Z}_{\ge 0} m_j + \mathbb{Z}_{\ge 0}(-m_j)$, we have an isomorphism $U_\sigma \xrightarrow{\sim} \{(z_1, \ldots, z_r) \in \mathbb{C}^r;\ z_j \ne 0,\ s+1 \le \forall j \le r\} = \mathbb{C}^s \times (\mathbb{C}^\times)^{r-s}$ which sends $u \in U_\sigma$ to $(u(m_1), \ldots, u(m_r)) \in \mathbb{C}^r$. Thus U_σ is nonsingular.

We now show that σ is spanned by a part of a $\mathbb{Z}$-basis of N if U_σ is nonsingular. Without loss of generality, we may assume that $\dim \sigma = r$, i.e., $\sigma + (-\sigma) = N_{\mathbb{R}}$ hence $\sigma^\perp = \sigma^\vee \cap (-\sigma^\vee) = \{O\}$. Indeed in the general case, let N' be the subgroup of N generated by $N \cap \sigma$ and let M' be its dual $\mathbb{Z}$-module. Thus $M'_{\mathbb{R}} = M_{\mathbb{R}}/\sigma^\perp$ and the image σ^* of $\sigma^\vee$ in $M'_{\mathbb{R}}$ is the dual cone of σ regarded as a strongly convex rational polyhedral cone in $N'_{\mathbb{R}}$. Obviously, we have a splitting $M = (M \cap \sigma^\perp) \oplus M''$ with M'' isomorphic to M'. Choose $m_1, \ldots, m_q \in M''$ in such a way that their images in M' generate $M' \cap \sigma^*$ over $\mathbb{Z}_{\ge 0}$, and let $\mathscr{S}' := \sum_{1 \le j \le q} \mathbb{Z}_{\ge 0} m_j$. Then we can identify $M \cap \sigma^\vee = \mathscr{S}_\sigma$ with the product $\mathscr{S}' \times (M \cap \sigma^\perp)$, hence $U_\sigma \cong U' \times \mathbb{C}^\times \times \ldots \times \mathbb{C}^\times$, where $U' := \{u : \mathscr{S}' \to \mathbb{C};\ u(O) = 1,\ u(m+m') = u(m)u(m'),\ \forall m, m' \in \mathscr{S}'\}$. This U' is the toric affine variety for σ regarded as a strongly convex rational polyhedral cone in $N'_{\mathbb{R}}$ and is nonsingular if so is U_σ.

If $\dim \sigma = r$, we have $\sigma^\vee \cap (-\sigma^\vee) = \{O\}$, hence $\mathscr{S}_\sigma \cap (-\mathscr{S}_\sigma) = \{O\}$. Take a minimal system of generators $\{m_1, \ldots, m_p\}$ for $\mathscr{S}_\sigma$ over $\mathbb{Z}_{\ge 0}$ and consider the closed embedding $U_\sigma \to \mathbb{C}^p$ which sends u to $(\mathbf{e}(m_1)(u), \ldots, \mathbf{e}(m_p)(u))$. It suffices to show that $p = r$ and $\{m_1, \ldots, m_r\}$ is a $\mathbb{Z}$-basis for M if U_σ is nonsingular.

As in the proof of Proposition 1.2, consider the surjective $\mathbb{C}$-algebra homomorphism from the polynominal ring $\mathbb{C}[x_1, \ldots, x_p]$ to the ring $R := \mathbb{C}[\mathscr{S}_\sigma]$ of $\mathbb{C}$-valued polynominal functions on U_σ which sends x_j to $u_j := \mathbf{e}(m_j)$ for each $1 \le j \le p$. Denote by $I \subset \mathbb{C}[x_1, \ldots, x_p]$ the kernel of the surjective $\mathbb{C}$-algebra homomorphism above. Since $\mathscr{S}_\sigma \cap (-\mathscr{S}_\sigma) = \{O\}$, the origin $O = (0, \ldots, 0)$ of $\mathbb{C}^p$ is contained in the closed subset $U_\sigma \subset \mathbb{C}^p$. Hence the maximal ideal $J := (x_1, \ldots, x_p)$ of $\mathbb{C}[x_1, \ldots, x_p]$ consisting of the functions on $\mathbb{C}^p$ which vanish at O contains I, and J/I is a maximal ideal of R. If U_σ does not have singularity at O, then $J/(I+J^2) = (J/I)/(J/I)^2$ is an r-dimensional $\mathbb{C}$-vector space.

As we saw in the proof of Proposition 1.2, T_N acts on R and we have a decomposition $R = \oplus_m \mathbb{C}\mathbf{e}(m)$, where $\mathbf{e}(m) \in R$ for $m \in \mathscr{S}_\sigma$ are the eigenfunctions with respect to the action. Obviously, $\{\mathbf{e}(m); m \in \mathscr{S}_\sigma, m \ne 0\}$ and $\{\mathbf{e}(m'+m'');\ m', m'' \in \mathscr{S}_\sigma, m' \ne 0, m'' \ne 0\}$ form the $\mathbb{C}$-bases of J/I and $(I+J^2)/I$, respectively. Thus there exist exactly r nonzero elements m in $\mathscr{S}_\sigma$ such that $m \ne m' + m''$ for any pair of nonzero elements m$'$, $m'' \in \mathscr{S}_\sigma$. Obviously, they are $m_1, \ldots, m_r$ with $p = r$ and necessarily form a $\mathbb{Z}$-basis of M, since they generate $\mathscr{S}_\sigma$ over $\mathbb{Z}_{\ge 0}$, hence M over $\mathbb{Z}$.

q.e.d.

Example. Look at $\sigma = \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0}(n_1 + 2n_2)$ in the example (iv) immediately before Proposition 1.3. In this case, $U_\sigma = \{(u_1, u_2, u_3) \in \mathbb{C}^3; u_1^2 = u_2 u_3\}$ has singularity at $O = (0, 0, 0)$. The minimal system of generators $\{n_1, n_1 + 2n_2\}$ of σ over $\mathbb{R}_{\geq 0}$ generates over $\mathbb{Z}$ a subgroup of N of index two.

Theorem 1.11. *The toric variety* T_N emb (Δ) *associated to a fan* Δ *in* N *is compact if and only if* Δ *is a* finite *and* complete *fan, i.e.,* Δ *is a finite set with the* support $|\Delta| := \cup_{\sigma \in \Delta} \sigma$ *coinciding with the entire* $N_{\mathbb{R}}$.

Proof. The necessity is easier. $\{U_\sigma; \sigma \in \Delta\}$ is an open covering of T_N emb (Δ). Let Δ' be the subset of Δ consisting of the maximal elements with respect to the face relation $<$. As we have seen in Proposition 1.6, (iv), $\{U_\sigma; \sigma \in \Delta'\}$ covers T_N emb (Δ) but $\{U_\sigma; \sigma \in \Delta''\}$ does not for any proper subset Δ'' of Δ'. Thus if T_N emb (Δ) is compact, Δ' is necessarily finite. Since Δ consists of the faces of the members of Δ', we conclude that Δ is also finite by Proposition A.5. For each $n \in N$, the one-parameter subgroup $\gamma_n : \mathbb{C}^\times \to T_N$ is algebraic, while T_N emb (Δ) is compact. Hence $\lim_{\lambda \to 0} \gamma_n(\lambda)$ exists in T_N emb (Δ), and is contained in U_σ for some $\sigma \in \Delta$. Thus $n \in \sigma$ by Proposition 1.6, (v). Consequently, we have $|\Delta| = N_{\mathbb{R}}$. (More precisely, we need to resort to the compactness criterion mentioned in the remark immediately below.)

As for the sufficiency, it suffices to show that the manifold with corners $\mathrm{Mc}(N, \Delta) = T_N$ emb $(\Delta)/CT_N$ is compact for a finite and complete fan Δ, since CT_N is a compact group.

If (N, Δ) is a finite and complete fan, then so is $(\bar{N}(\tau), \bar{\Delta}(\tau))$ in Corollary 1.7 for each $\tau \in \Delta$. Thus $N_{\mathbb{R}}/\mathbb{R}\tau = \bar{N}(\tau)_{\mathbb{R}} = \cup_{\sigma \in \Delta, \tau < \sigma}(\sigma + \mathbb{R}\tau)/\mathbb{R}\tau$. Hence

$$\mathrm{Mc}(N, \Delta) = \bigcup_{\tau \in \Delta} N_{\mathbb{R}}/\mathbb{R}\tau = \bigcup_{\tau \in \Delta} \left(\bigcup_{\sigma \in \Delta, \tau < \sigma} (\sigma + \mathbb{R}\tau)/\mathbb{R}\tau \right) = \bigcup_{\sigma \in \Delta} W'(\sigma) ,$$

where $W'(\sigma) := \cup_{\tau < \sigma}(\sigma + \mathbb{R}\tau)/\mathbb{R}\tau$. Since $\mathrm{Mc}(N, \Delta)$ is a finite union of $W'(\sigma)$'s, we need only to show that each $W'(\sigma)$ is compact.

Let $\mathscr{S}_\sigma = \mathbb{Z}_{\geq 0} m_1 + \ldots + \mathbb{Z}_{\geq 0} m_p$ for $\sigma \in \Delta$. $W'(\sigma)$ is a subspace of $U_\sigma/CT_N = \cup_{\tau < \sigma} N_{\mathbb{R}}/\mathbb{R}\tau$ identified by Proposition 1.8. The inverse image of $W'(\sigma)$ by the homeomorphism ord : $(U_\sigma)_{\geq 0} \xrightarrow{\sim} U_\sigma/CT_N$ is

$$W(\sigma) := \{w : \mathscr{S}_\sigma \to \mathbb{R}_{\geq 0};\ w(O) = 1, w(m + m') = w(m)w(m'), \forall m, m' \in \mathscr{S}_\sigma \text{ and } w(m_j) \leq 1 \text{ for } 1 \leq \forall j \leq p\}.$$

Under the closed embedding $(u_\sigma)_{\geq 0} \to (\mathbb{R}_{\geq 0})^p$ which sends w to $(w(m_1), \ldots, w(m_p))$, $W(\sigma)$ coincides with the inverse image of the product $[0, 1]^p$ of the interval $[0, 1]$, hence is compact. Thus its homeomorphic image $W'(\sigma)$ under ord is also compact.

q.e.d.

Example. Among the examples for $r = 2$ immediately after Theorem 1.4, $\mathbb{C}^2 \setminus \{(0, 0)\}$ in (i) is not compact, while $\mathbb{P}_2(\mathbb{C})$ and the Hirzebruch surface F_a in (ii) and (iii) are compact.

Remark. The proof above for the sufficiency is due to Ehlers [E2, Theorem 1]. Here are two more proofs, which are algebro-geometric:

The first is due to Danilov [D1], which mimics Serre's proof of the result quoted immediately below. A finite and complete fan Δ has a *projective* subdivision Δ' by toric Chow's lemma stated as Proposition 2.17 in Section 2.3. By Theorem 1.13 in the following section, we have a surjective holomorphic map $T_N \operatorname{emb}(\Delta') \to T_N \operatorname{emb}(\Delta)$. We are done, since $T_N \operatorname{emb}(\Delta')$ is projective, hence compact.

For the second algebro-geometric proof, we use the following general fact:

The Compactness Criterion. *Let X be an algebraic variety of finite type over $\mathbb{C}$.*

(1) (Serre [S10, § 7, Proposition 6]) *The complex analytic space X' associated to X is compact if and only if X is* complete *(also called* proper *over $\mathbb{C}$).*

(2) (The valuative criterion. See, eg., Grothendieck-Dieudonné [EGA, Chap. II, Corollary 7.3.10]) *X is complete if and only if any discrete valuation ring $R \supset \mathbb{C}$ of the rational function field $\mathbb{C}(X)$* dominates *the local ring $\mathcal{O}_{X,Y}$ of the generic point of an irreducible algebraic subvariety Y of X. Namely, if we denote by $\mathfrak{m}$ and $\mathfrak{m}_{X,Y}$ the maximal ideals of R and $\mathcal{O}_{X,Y}$, respectively, then R contains $\mathcal{O}_{X,Y}$ and satisfies $\mathfrak{m} \cap \mathcal{O}_{X,Y} = \mathfrak{m}_{X,Y}$.*

To prove the sufficiency of Theorem 1.11 again, let us now apply this criterion to the algebraic variety X which gives rise to the toric variety $T_N \operatorname{emb}(\Delta)$ as in the remark immediately after Theorem 1.4. Since we assume Δ to be finite, X is an algebraic variety of finite type over $\mathbb{C}$. Since the algebraic torus T_N is a Zariski open set of X, the rational function field $\mathbb{C}(X)$ coincides with that of T_N, which is the field of fractions of the group algebra $\mathbb{C}[M]$ of M.

It suffices to show that any discrete valuation ring $R \supset \mathbb{C}$ of $\mathbb{C}(X)$ contains the semigroup algebra $\mathbb{C}[\mathscr{S}_\sigma]$ for some $\sigma \in \Delta$. Then the localization $\mathcal{O}$ of $\mathbb{C}[\mathscr{S}_\sigma]$ with respect to the prime ideal $\mathfrak{p} := \mathfrak{m} \cap \mathbb{C}[\mathscr{S}_\sigma]$ would be dominated by R and be the local ring of the generic point of an algebraic subvariety of X.

Denote by $v : \mathbb{C}(X) \to \mathbb{Z} \cup \{\infty\}$ the surjective discrete valuation corresponding to R. Namely, we have $v(f) = \infty$ if and only if $f = 0$; $v(c) = 0$ for $c \in \mathbb{C}^\times$; $v(fg) = v(f) + v(g)$ and $v(f+g) \geq \min\{v(f), v(g)\}$ for $f, g \in \mathbb{C}(X)$; moreover,

$$R = \{f \in \mathbb{C}(X); v(f) \geq 0\} .$$

The composite $v \circ \mathbf{e} : M \to \mathbb{Z}$ of $\mathbf{e} : M \to \mathbb{C}[M] \setminus \{0\} \subset \mathbb{C}(X) \setminus \{0\}$ with $v : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}$ is obviously an additive homomorphism. Thus we can find an $n \in N$ so that $v(\mathbf{e}(m)) = \langle m, n \rangle$ for all $m \in M$. Since $|\Delta| = N_{\mathbb{R}}$ by assumption, n is contained in some $\sigma \in \Delta$. Hence we have

$$\mathscr{S}_\sigma \subset \{m \in M; \langle m, n \rangle \geq 0\} = \{m \in M; v(\mathbf{e}(m)) \geq 0\} ,$$

which implies $\mathbb{C}[\mathscr{S}_\sigma] \subset R$.

We refer the reader to De Concini-Procesi [DP], Luna-Vust [LV] and Uzawa [U5] for related results.

We have the following which generalizes Nagata's theorem in [N1] to the equivariant case:

The Equivariant Compactification Theorem (Sumihiro [S15, I, Theorem 3]). *Suppose a connected linear algebraic group G acts algebraically on an irreducible normal* *algebraic variety X of finite type over $\mathbb{C}$. Then X can be embedded as a G-stable open subset of a complete irreducible normal algebraic variety $\tilde{X}$ on which G acts algebraically.*

In view of Theorem 1.11, the above theorem applied to toric varieties asserts the following: If Δ is a finite fan in N, then there exists a finite and complete fan $\tilde{\Delta}$ in N containing Δ, so that $T_N \operatorname{emb}(\tilde{\Delta})$ is a T_N-equivariant compactification of $T_N \operatorname{emb}(\Delta)$. No systematic way of constructing such $\tilde{\Delta}$ for a given Δ seems to be known in general, except in the following special case where Δ consists of the faces of a cone, i.e., $T_N \operatorname{emb}(\Delta)$ is a *toric affine variety* in the sense of the remark immediately before Corollary 1.7.

Proposition 1.12. *For $N \cong \mathbb{Z}^r$, let π be an r-dimensional strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. Choose a $\mathbb{Q}$-valued nondegenerate symmetric bilinear form $q: N \times N \to \mathbb{Q}$ to identify $N_{\mathbb{R}}$ with its dual $M_{\mathbb{R}}$. For each face σ of π,*

$$\tilde{\sigma} := \sigma + (-\pi^{\vee} \cap \sigma^{\perp})$$

is an r-dimensional strongly convex rational polyhedral cone in $N_{\mathbb{R}} = M_{\mathbb{R}}$. If we denote by Δ the fan consisting of the faces of π, then the set $\tilde{\Delta}$ of the faces of $\tilde{\sigma}$ for all $\sigma \in \Delta$ is a finite and complete fan in N containing Δ. Consequently, T_N emb $(\tilde{\Delta})$ is a T_N-equivariant compactification of $U_\sigma = T_N$ emb (Δ).

Proof. We identify $n \in N_{\mathbb{R}}$ with the element of $M_{\mathbb{R}}$ which sends $n' \in N_{\mathbb{R}}$ to $q(n, n') \in \mathbb{R}$. Since $\pi^{\vee} \subset M_{\mathbb{R}}$ is a strongly convex rational polyhedral cone (cf. Proposition 1.3) and since q is $\mathbb{Q}$-valued by assumption, we can regard $\pi^{\vee}$ as a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ by the above identification. If σ is a face of π, then by Proposition A.6 $\pi^{\vee} \cap \sigma^{\perp}$ is a face of $\pi^{\vee}$. Moreover, $\tilde{\sigma} = \sigma + (-\pi^{\vee} \cap \sigma^{\perp})$ is then obviously an r-dimensional strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. In particular, we have $\tilde{\pi} = \pi$.

It certainly suffices to show that $N_{\mathbb{R}} = \cup_{\sigma < \pi} \tilde{\sigma}$ and that

$$\tilde{\sigma} \cap \tilde{\tau} = (\sigma \cap \tau) + (-\pi^{\vee} \cap \sigma^{\perp} \cap \tau^{\perp}) \quad \text{for all faces} \quad \sigma, \tau \quad \text{of} \quad \Delta \ .$$

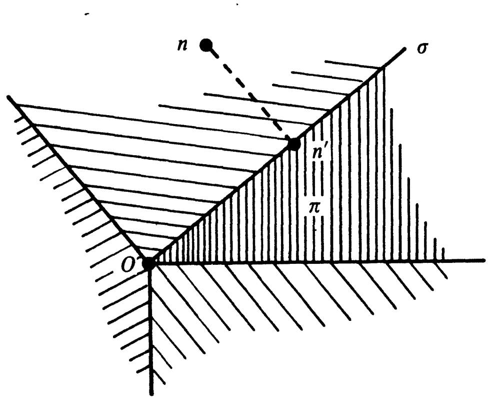


Fig.1.6

For that purpose, consider the norm $\|n\| := q(n,n)^{1/2}$ on $N_{\mathbb{R}}$. For each $n \in N_{\mathbb{R}}$ there exists a unique $n' \in \pi$ such that $\|n - n'\|$ is the smallest, since π is a closed convex cone. Let σ be the unique face of π which contains n' in its relative interior. Then $n - n'$ is in $-\pi^\vee \cap \sigma^\perp$ and we have $n = n' + (n - n') \in \tilde{\sigma}$. The latter assertion is then obvious. (See Fig. 1.6.) q.e.d.

1.5 Equivariant Holomorphic Maps

We consider fans (N, Δ) and (N', Δ') for $N \cong \mathbb{Z}^r$ and $N' \cong \mathbb{Z}^{r'}$.

Definition. A *map of fans* $\varphi : (N', \Delta') \to (N, \Delta)$ is a $\mathbb{Z}$-linear homomorphism $\varphi : N' \to N$ whose scalar extension $\varphi : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ satisfies the following property: For each $\sigma' \in \Delta'$ there exists $\sigma \in \Delta$ such that $\varphi(\sigma') \subset \sigma$.

Theorem 1.13. *A map of fans $\varphi : (N', \Delta') \to (N, \Delta)$ gives rise to a holomorphic map*

$$\varphi_* : T_{N'}\,\mathrm{emb}\,(\Delta') \to T_N\,\mathrm{emb}\,(\Delta) \ ,$$

whose restriction to the open subset $T_{N'}$ coincides with the homomorphism of algebraic tori

$$\varphi \otimes 1 : T_{N'} = N' \otimes_{\mathbb{Z}} \mathbb{C}^\times \to T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^\times$$

arising from φ. Through this homomorphism, φ_* is equivariant with respect to the actions of $T_{N'}$ and T_N on the toric varieties.

Conversely, suppose $f' : T_{N'} \to T_N$ is a homomorphism of algebraic tori and $f : T_{N'}\,\mathrm{emb}\,(\Delta') \to T_N\,\mathrm{emb}\,(\Delta)$ is a holomorphic map equivariant with respect to f'. Then there exists a unique $\mathbb{Z}$-linear homomorphism $\varphi : N' \to N$ which gives rise to a map of fans $(N', \Delta') \to (N, \Delta)$ such that $f = \varphi_*$.

Remark. This is a generalization of [MO, Theorem 4.1]. As Namikawa noted already in [N7, Definition 1.10 and Proposition 1.11], there is no need whatsoever to assume that $f : T_{N'} \to T_N$ is surjective nor $\varphi : N' \to N$ is of finite cokernel. See also Ishida [I5, § 3].

Proof. A $\mathbb{Z}$-linear map $\varphi : N' \to N$ gives rise to its dual $\mathbb{Z}$-linear map $\varphi^* : M \to M'$, its scalar extension $\varphi^* : M_{\mathbb{R}} \to M'_{\mathbb{R}}$ and the homomorphism of algebraic tori $\varphi \otimes 1 : T_{N'} \to T_N$. Suppose $\varphi(\sigma') \subset \sigma$ for $\sigma' \in \Delta'$ and $\sigma \in \Delta$. Then obviously we have $\varphi^*(\sigma^\vee) \subset (\sigma')^\vee$ and $\varphi^*(\mathscr{S}_\sigma) \subset \mathscr{S}'_{\sigma'} := M' \cap (\sigma')^\vee$. As in Sect. 1.2, let $U'_{\sigma'} := \{u' : \mathscr{S}'_{\sigma'} \to \mathbb{C};\ u'(O) = 1,\ u'(m' + m'') = u'(m')u'(m''),\ \forall m', m'' \in \mathscr{S}'_{\sigma'}\}$. Then by composition with $\varphi^* : \mathscr{S}_\sigma \to \mathscr{S}'_{\sigma'}$, we get a holomorphic map $\varphi_* : U'_{\sigma'} \to U_\sigma$, that is, $(\varphi_* u')(m) := u'(\varphi^*(m))$ for $m \in \mathscr{S}_\sigma$. This φ_* is equivariant, since $(\varphi_*(t'u'))(m) = (t'u')(\varphi^*(m)) = t'(\varphi^*(m))u'(\varphi^*(m)) = (\varphi_* t')(m)\,(\varphi_* u')(m)$ for $t' \in T_{N'}$ and $m \in \mathscr{S}_\sigma$. By gluing affine pieces together, we thus obtain an equivariant holomorphic map $\varphi_* : T_{N'}\,\mathrm{emb}\,(\Delta') \to T_N\,\mathrm{emb}\,(\Delta)$.

As for the converse, a homomorphism of algebraic tori $f': T_{N'} \to T_N$ gives rise to a homomorphism of the character groups $M \to M'$ and its dual $\mathbb{Z}$-homomorphism $\varphi: N' \to N$. Since f is assumed to be equivariant through f', the image under f of each $T_{N'}$-orbit in $T_{N'}$ emb (Δ') is contained in a T_N-orbit in T_N emb (Δ).

If τ' is a face of $\sigma' \in \Delta'$, then orb (σ') is contained in the closure of orb (τ') by Proposition 1.6, (iii). By what we have just seen, there exist $\tau, \sigma \in \Delta$ such that $f(\text{orb}(\sigma')) \subset \text{orb}(\sigma)$ and $f(\text{orb}(\tau')) \subset \text{orb}(\tau)$. Since f is continuous, orb (σ) is then contained in the closure of orb (τ), hence τ is a face of σ by Proposition 1.6, (iii) again. Consequently, we have $f(U'_{\sigma'}) \subset U_\sigma$ by Proposition 1.6, (iv).

By Proposition 1.6, (v), $\lim_{\lambda \to 0} \gamma_{n'}(\lambda)$ exists in $U'_{\sigma'}$ for each $n' \in \sigma'$. Since $f \circ \gamma_{n'} = \gamma_{\varphi(n')}$ and since f is continuous, we see that $\lim_{\lambda \to 0} \gamma_{\varphi(n')}(\lambda) = f(\lim_{\lambda \to 0} \gamma_{n'}(\lambda))$ exists in U_σ. By Proposition 1.6, (v) again, we then get $\varphi(n') \in \sigma$. Consequently, we obtain a map of fans $\varphi: (N', \Delta') \to (N, \Delta)$, which obviously satisfies $f = \varphi_*$. q.e.d.

The above theorem turns out to be quite convenient. In terms of elementary geometry of fans, it describes certain holomorphic maps among toric varieties which were defined in terms of elementary geometry to start with. In down-to-earth terms, we are dealing with holomorphic maps with monomials as coordinates among toric varieties which we obtain by gluing together the solution spaces of systems of equations of the type (monomial) = (monomial).

We would like to point out that a map of fans gives rise to an equivariant morphism of algebraic varieties mentioned in the remark immediately after Theorem 1.4. The holomorphic map associated to this morphism is what we described in Theorem 1.13.

The following is easy to show:

Proposition 1.14. *For a map of fans $\varphi: (N', \Delta') \to (N, \Delta)$, let N'' be the smallest $\mathbb{Z}$-submodule of N such that N'' contains $\varphi(N')$ and that N'' is a direct summand of N. Then $\Delta'' := \{\sigma \cap N''_{\mathbb{R}}; \sigma \in \Delta\}$ is a fan in N'' and φ factors as*

$$(N', \Delta') \xrightarrow{\psi} (N'', \Delta'') \xrightarrow{j} (N, \Delta) ,$$

where j is the canonical injection. Consequently, the equivariant holomorphic map $\varphi_*: T_{N'}$ emb $(\Delta') \to T_N$ emb (Δ) factors as

$$T_{N'} \operatorname{emb}(\Delta') \xrightarrow{\psi_*} T_{N''} \operatorname{emb}(\Delta'') \xrightarrow{j_*} T_N \operatorname{emb}(\Delta) .$$

The restriction of ψ_* to $T_{N'}$ is a surjective homomorphism of algebraic tori $T_{N'} \to T_{N''}$, while j_* coincides with the normalization of the closure of the image of φ_*.

Remark. If we consider the fan $\{\sigma \cap \varphi(N')_{\mathbb{R}}; \sigma \in \Delta\}$ in $\varphi(N')$ instead, we get the *Stein factorization* for φ_* in the sense of [EGA].

As a generalization of Theorem 1.11, we have the following:

Theorem 1.15. *Let $\varphi: (N', \Delta') \to (N, \Delta)$ be a map of fans. The equivariant holomorphic map $f = \varphi_*: T_{N'}$ emb $(\Delta') \to T_N$ emb (Δ) is* proper *(i.e., the inverse image of each compact subset is compact) if and only if for each $\sigma \in \Delta$, the set*

$\Delta'_\sigma := \{\sigma' \in \Delta'; \varphi(\sigma') \subset \sigma\}$ *is finite and*

$$\varphi^{-1}(\sigma) = |\Delta'_\sigma| := \bigcup_{\sigma' \in \Delta'_\sigma} \sigma' .$$

Proof. The necessity is easier. For each $\sigma \in \Delta$ the inverse image $f^{-1}(U_\sigma)$ is covered by the open sets $U'_{\sigma'}$ with σ' running through the maximal elements in Δ'_σ with respect to the face relation $<$ by the definition of f. By Proposition 1.6, (iv), we cannot omit any of the maximal elements above. Since the inverse image under f of a point of $\operatorname{orb}(\sigma)$ is compact, we see that the maximal elements in Δ'_σ are finite in number. Hence Δ'_σ is necessarily finite, since it consists of the faces of the maximal elements. Let $\sigma \in \Delta$. For any n' in $N' \cap \varphi^{-1}(\sigma)$, we have $\varphi(n') \in \sigma$, hence $\lim_{\lambda \to 0} f(\gamma_{n'}(\lambda)) = \lim_{\lambda \to 0} \gamma_{\varphi(n')}(\lambda)$ exists in U_σ by Proposition 1.6, (v). Since f is proper and $\gamma_{n'}$ is algebraic, $\lim_{\lambda \to 0} \gamma_{n'}(\lambda)$ exists in $T_{N'} \operatorname{emb}(\Delta')$, hence in $U'_{\sigma'}$ for some $\sigma' \in \Delta'$. By Proposition 1.6, (v) again, we thus have $n' \in \sigma'$. Consequently, we get $\varphi(\sigma') \subset \sigma$ and $\varphi^{-1}(\sigma) = |\Delta'_\sigma|$. (More precisely, we need to resort to the properness criterion below.)

For the sufficiency, we use the following generalization of the compactness criterion immediately after Theorem 1.11:

The Properness Criterion. *Let $g: X \to Y$ be a morphism of finite type between separated algebraic varieties over $\mathbb{C}$. Suppose that the image $g(X)$ is dense in Y so that the rational function field $\mathbb{C}(Y)$ of Y can be regarded as a subfield of the rational function field $\mathbb{C}(X)$ of X.*

(1) (The GAGA theorem, cf. Grothendieck [G4] and [G5]) *The holomorphic map $g': X' \to Y'$, associated to g, between the Hausdorff complex analytic spaces is a proper map if and only if g is* proper *as a morphism of algebraic varieties.*

(2) (The valuative criterion, cf. Grothendieck-Dieudonné [EGA, Chap. II, Corollary 7.3.10]) *g is proper as a morphism of algebraic varieties if and only if any discrete valuation ring $R \supset \mathbb{C}$ of $\mathbb{C}(X)$ which dominates the local ring $\mathcal{O}_{Y,W}$ of the generic point of an irreducible algebraic subvariety W of Y necessarily dominates the local ring $\mathcal{O}_{X,Z}$ of the generic point of an irreducible algebraic subvariety Z of X.*

Proof of Theorem 1.15 (continued). $f: T_{N'} \operatorname{emb}(\Delta') \to T_N \operatorname{emb}(\Delta)$ arises from a morphism of finite type between algebraic varieties, since Δ'_σ is assumed to be finite for each $\sigma \in \Delta$. Since the normalization morphism is a proper morphism (cf. [EGA, Chap. II, § 6]), we may assume without loss of generality by Proposition 1.14 that $\varphi(N')$ is of finite index in N. Thus M can be regarded as a $\mathbb{Z}$-submodule of the $\mathbb{Z}$-module M' dual to N'. Consequently, the fields of fractions of the group algebra $\mathbb{C}[M']$ and its subalgebra $\mathbb{C}[M]$ are the rational function fields of the algebraic varieties which give rise to $T_{N'} \operatorname{emb}(\Delta')$ and $T_N \operatorname{emb}(\Delta)$, respectively.

As we saw immediately after Theorem 1.11, the discrete valuation v associated to a discrete valuation ring $R \supset \mathbb{C}$ of the field of fractions of $\mathbb{C}[M']$ gives rise to an additive homomorphism $v \circ \mathbf{e}: M' \to \mathbb{Z}$, hence there exists $n' \in N'$ such that $v(\mathbf{e}(m')) = \langle m', n' \rangle$ for any $m' \in M'$. If R dominates the local ring of the generic point of an irreducible algebraic subvariety of the algebraic variety for $T_N \operatorname{emb}(\Delta)$, then the local ring coincides with a localization of $\mathbb{C}[\mathcal{S}_\sigma]$ for some $\sigma \in \Delta$. Thus $v(\mathbf{e}(\varphi^*(m))) = \langle \varphi^*(m), n' \rangle = \langle m, \varphi(n') \rangle$ is nonnegative for each $m \in \mathcal{S}_\sigma$, hence n' is contained in

$\varphi^{-1}(\sigma)$. n' is then contained in some $\sigma' \in \Delta'_\sigma$, since $\varphi^{-1}(\sigma) = |\Delta'_\sigma|$ by assumption. Thus $v(\mathbf{e}(m')) = \langle m', n' \rangle$ is nonnegative for all $m' \in \mathscr{S}'_\sigma$, that is, $R \supset \mathbb{C}[\mathscr{S}'_{\sigma'}]$. Consequently, R dominates a localization of $\mathbb{C}[\mathscr{S}'_{\sigma'}]$, which is the local ring of the generic point of an irreducible algebraic subvariety of the algebraic variety giving rise to $T_{N'}\operatorname{emb}(\Delta')$. By the properness criterion above, f is thus a proper holomorphic map. q.e.d.

Let us now consider a special but important map of fans and the associated equivariant holomorphic map: If N' is a $\mathbb{Z}$-submodule of *finite index* in N, we have $N'_\mathbb{R} = N_\mathbb{R}$ and can regard a fan Δ for N also as one for N'. Thus we get a map of fans $\varphi : (N', \Delta) \to (N, \Delta)$ and the corresponding equivariant holomorphic map

$$f = \varphi_* : X' := T_{N'}\operatorname{emb}(\Delta) \to X := T_N \operatorname{emb}(\Delta) \ .$$

We show that this coincides with the projection for the quotient of X' with respect to a natural action on X' of the finite group $\ker[T_{N'} \to T_N] = \operatorname{Hom}_\mathbb{Z}(M'/M, \mathbb{C}^\times) \cong N/N'$, where we regard the $\mathbb{Z}$-module M dual to N canonically as a $\mathbb{Z}$-submodule of the $\mathbb{Z}$-module M' dual to N'.

Indeed, we have a unique $\mathbb{Z}$-bilinear map $\langle \, , \, \rangle : M' \times N \to \mathbb{Q}$ which is a common extension of $\langle \, , \, \rangle : M \times N \to \mathbb{Z}$ and $\langle \, , \, \rangle : M' \times N' \to \mathbb{Z}$. It gives rise to a nondegenerate $\mathbb{Z}$-bilinear map $(M'/M) \times (N/N') \to \mathbb{Q}/\mathbb{Z}$, hence a natural isomorphism

$$N/N' = \operatorname{Hom}_\mathbb{Z}(M'/M, \mathbb{Q}/\mathbb{Z}) \ .$$

By composition with the injection $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}^\times$ which sends $\theta + \mathbb{Z}$ to $\exp(2\pi i\theta)$, we thus get an isomorphism

$$N/N' \xrightarrow{\sim} \operatorname{Hom}_\mathbb{Z}(M'/M, \mathbb{C}^\times) \ .$$

The homomorphism of algebraic tori $T_{N'} = \operatorname{Hom}_\mathbb{Z}(M', \mathbb{C}^\times) \to T_N = \operatorname{Hom}_\mathbb{Z}(M, \mathbb{C}^\times)$ is surjective with the kernel $K := \ker[T_{N'} \to T_N]$ equal to $\operatorname{Hom}_\mathbb{Z}(M'/M, \mathbb{C}^\times)$. Thus K is isomorphic to N/N'.

If we identify $k \in K$ with the homomorphism $k : M' \to \mathbb{C}^\times$ satisfying $k(M) = 1$ as above, the action of k on the holomorphic function $\mathbf{e}(m')$ on $T_{N'}$ for $m' \in M'$ is given by $\mathbf{e}(m') \mapsto k(m')\mathbf{e}(m')$. Thus $\{\mathbf{e}(m); m \in M\}$ coincides with the subset of invariant elements in $\{\mathbf{e}(m'); m' \in M'\}$ with respect to the action of K. Each $\sigma \in \Delta$ determines open sets U_σ and U'_σ in X and X', respectively. As we saw in Proposition 1.2 and the remark immediately after that, the polynomial functions on U'_σ coincide with the $\mathbb{C}$-linear combinations of $\{\mathbf{e}(m'); m' \in M' \cap \sigma^\vee\}$. Thus the subring of K-invariant polynomial functions on U'_σ consists of the $\mathbb{C}$-linear combinations of $\{\mathbf{e}(m); m \in M \cap \sigma^\vee\}$, hence coincides with the ring of polynomial functions on U_σ. As is well-known, U_σ is then the quotient space of U'_σ with respect to the action of K.

Combining Theorem 1.15 and what we have just observed, we have the following:

Corollary 1.16. *Let $\varphi : (N', \Delta') \to (N, \Delta)$ be a map of fans. The corresponding equivariant holomorphic map*

$$\varphi_* : X' := T_{N'}\operatorname{emb}(\Delta') \to X := T_N \operatorname{emb}(\Delta)$$

is proper with $\mathbb{C}(X')$ *a finite extension of* $\mathbb{C}(X)$ *if and only if* $\varphi: N' \to N$ *is injective with finite cokernel and* Δ' *is a* locally finite subdivision *of* Δ *under the identification* $N'_{\mathbb{R}} = N_{\mathbb{R}}$, *i.e.,* $\{\sigma' \in \Delta'; \sigma' \subset \sigma\}$ *for each* $\sigma \in \Delta$ *is finite and* σ *coincides with the union of such* σ'*'s.*

In particular, if N' *is a* $\mathbb{Z}$*-submodule of* N *of finite index and* $\Delta' = \Delta$, *then* $\varphi_*: X' \to X$ *coincides with the projection for the quotient of* X' *with respect to the natural action of the finite group*

$$\ker[T_{N'} \to T_N] = \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^\times) \cong N/N' \ ,$$

where M *and* M' *are the dual* $\mathbb{Z}$*-modules of* N *and* N', *respectively.*

Corollary 1.17. *The equivariant holomorphic map* $\varphi_*: T_{N'}\,\mathrm{emb}\,(\Delta') \to T_N\,\mathrm{emb}\,(\Delta)$ *is proper and birational if and only if* $\varphi: N' \to N$ *is an isomorphism and* Δ' *is a locally finite subdivision of* Δ *under the identification* $N'_{\mathbb{R}} = N_{\mathbb{R}}$.

Combining Corollary 1.17 and Theorem 1.10, we have:

Corollary 1.18. *Let* Δ' *be a locally finite nonsingular subdivision of a fan* Δ *in* N. *Then the equivariant holomorphic map* $\mathrm{id}_*: T_N\,\mathrm{emb}\,(\Delta') \to T_N\,\mathrm{emb}\,(\Delta)$ *corresponding to the map of fans* $\mathrm{id}: (N, \Delta') \to (N, \Delta)$ *is proper birational and is an equivariant resolution of singularities for* $T_N\,\mathrm{emb}\,(\Delta)$.

Resolution of Singularities. *Any toric variety* $T_N\,\mathrm{emb}\,(\Delta)$ *admits an equivariant resolution of singularities.*

We need to find a locally finite nonsingular subdivision Δ' of Δ. We refer the reader to [TE, Chap. I, § 2, Theorem 11] and Brylinski [B5, p. 273, Theorem 11] for the construction of such a Δ'. When a finite group acts on $T_N\,\mathrm{emb}\,(\Delta)$, Brylinski [B6] constructs a nonsingular subdivision Δ' which is invariant with respect to the action of the finite group. [TE, Chap. I, Theorem 11] and Namikawa [N8, Theorem 7.20] deal with the case where an infinite discontinuous group acts, as in the case of tube domains.

Hironaka showed the existence of a *canonical* resolution of singularities for an arbitrary complex analytic space. Hence it is automatically equivariant when a group acts. The case of toric varieties above is a very special case of this general result of Hironaka's. Needless to say, the proof for toric varieties is much more elementary.

As we explain in Sect. 3.2, Kempf [TE, p. 52, Theorem 14] showed that toric singularities (i.e., singularities appearing in toric varieties) are always rational, hence are Cohen-Macaulay singularities. As we also see in Proposition 1.26, an equivariant blowing-up of a nonsingular toric variety along an invariant irreducible closed subspace corresponds to a star subdivision of the corresponding fan.

1.6 Low Dimensional Toric Singularities and Finite Continued Fractions

We first apply general results in the previous section to two-dimensional toric singularities. Not only can we describe their minimal resolutions in terms of elementary convex geometry, but we can also relate them to finite continued fractions and to cyclic quotient singularities. As we see in Chap. 4, the same technique applies to the relationship between periodic continued fractions and Hilbert modular cusp singularities and can be generalized to the higher dimensional analogues due to Tsuchihashi.

The situation is more complicated in dimension three or higher, as we also see briefly in this section. One of the reasons is the clear distiction in difficulty between the plane convex geometry and that in higher dimension.

For $N \cong \mathbb{Z}^2$, let σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ with $\dim \sigma = 2$ and consider $U_\sigma = T_N \operatorname{emb}(\{\text{the faces of } \sigma\})$. In this case, $\operatorname{orb}(\sigma)$ is a single point fixed by T_N. Since U_σ is normal, $\operatorname{orb}(\sigma)$ is the only possible singular point of U_σ. We can choose *primitive* elements $n, n' \in N$ (i.e., they are not positive integral multiples of any elements of N except themselves) such that n, n' are $\mathbb{R}$-linearly independent and that $\sigma = \mathbb{R}_{\geq 0} n + \mathbb{R}_{\geq 0} n'$. If $\{n, n'\}$ happens to be a $\mathbb{Z}$-basis of N, then U_σ is nonsingular at the point $\operatorname{orb}(\sigma)$ by Theorem 1.10.

Proposition 1.19. *For a two-dimensional cone σ as above, let Θ be the convex hull in $N_{\mathbb{R}}$ of the set $(\sigma \cap N) \setminus \{O\}$. As in Fig. 1.7, let $l_0 = n, l_1, \dots, l_s, l_{s+1} = n'$ in this order be the points of N lying on the compact edges of the boundary polygon $\partial\Theta$ of Θ.*

(i) *Let Δ' be the subdivision of σ obtained as the set of faces of $\{\mathbb{R}_{\geq 0} l_{j-1} + \mathbb{R}_{\geq 0} l_j;\ 1 \leq j \leq s+1\}$. Then $T_N \operatorname{emb}(\Delta') \to U_\sigma$ is the minimal equivariant resolution of singularities, i.e., Δ' is the coarsest nonsingular subdivision of σ.*

(ii) *There exist integers $a_j \leq -2$ for $1 \leq \forall j \leq s$ such that*

$$l_{j-1} + l_{j+1} + a_j l_j = O \ .$$

The closure $C_j := V(\mathbb{R}_{\geq 0} l_j)$ of $\operatorname{orb}(\mathbb{R}_{\geq 0} l_j)$ in $T_N \operatorname{emb}(\Delta')$ is isomorphic to the complex projective line $\mathbb{P}_1(\mathbb{C})$ and a_j coincides with its self-intersection number (C_j^2).

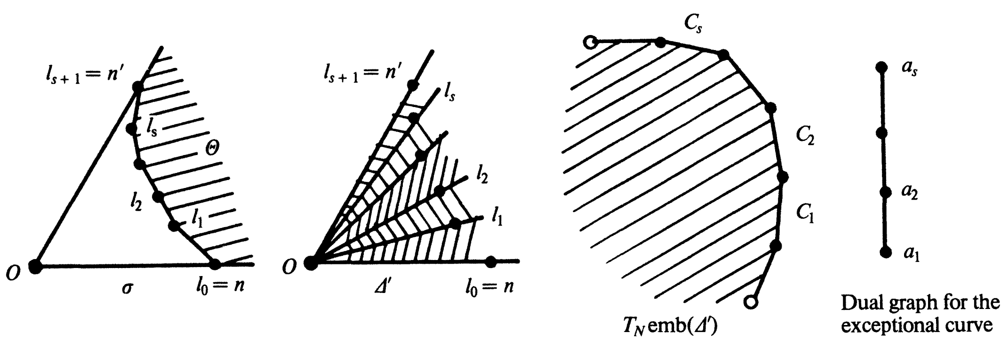


Fig. 1.7

The exceptional curve for the minimal resolution is a chain $C_1 + C_2 + \dots + C_s$ of nonsingular rational curves having a linear graph of length s as its dual graph.

Proof. The boundary polygon $\partial\Theta$ is the union of two half lines starting from n and n' and the line segments joining l_{j-1} and l_j for $1 \le \forall j \le s+1$. By our choice of $l_0, \dots, l_{s+1}$, the triangle with vertices at O, l_{j-1}, l_j contains no points in N other than the vertices. Thus $\{l_{j-1}, l_j\}$ is necessarily a $\mathbb{Z}$-basis of N thanks to the two-dimensionality of our situation. Hence Δ' is a nonsingular fan and easily seen to be the coarsest nonsingular subdivision of σ. Since l_{j-1} and l_{j+1} lie on mutually opposite sides with respect to l_j and since $\{l_{j-1}, l_j\}$ and $\{l_j, l_{j+1}\}$ are both $\mathbb{Z}$-bases of N, there exists $a_j \in \mathbb{Z}$ such that $l_{j-1} + l_{j+1} + a_j l_j = O$. By the convexity of Θ, we easily see that $a_j \le -2$. We show $(C_j^2) = a_j$ in Sect. 2.2.

The above construction for Δ' is conceptual and theoretically simple. However, it gives, for any given σ, no recipe for explicit computation of the convex hull Θ, the lattice points $l_1, \dots, l_s$ on $\partial\Theta$ and the integers $a_1, \dots, a_s$. As Cohn [C2] showed, finite continued fractions are exactly what we need:

Since $n, n' \in N$ are $\mathbb{R}$-linearly independent primitive elements with $\sigma = \mathbb{R}_{\ge 0} n + \mathbb{R}_{\ge 0} n'$, we can find a primitive element $n_1 \in N$ and relatively prime integers p, q with $0 \le p < q$ such that $\{n, n_1\}$ is a $\mathbb{Z}$-basis of N and that $n' = pn + qn_1$. Note that $q = 1$ implies $p = 0$ and σ nonsingular. We have a continued fraction expansion in terms of integers $b_j \ge 2$:

$$\frac{q}{q-p} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{}{\ddots - \cfrac{1}{b_{s'}}}}}$$

which, for typographical reasons, is usually denoted by

$$[[b_1, \dots, b_{s'}]] \quad \text{or} \quad b_1 - 1 \rceil b_2 - 1 \rceil b_3 \dots - 1 \rceil b_{s'} \; .$$

Note that this is different from the so-called regular continued fractions for which all the minus signs above are replaced by the plus signs. Those latter arise, for instance, in connection with Hirzebruch's resolution of the cusp singularity $\{(x, y) \in \mathbb{C}^2;\ x^a = y^b\}$ at the origin. We refer the reader to Jurkiewicz [J2, § 1.4] for a toric interpretation of this fact.

Lemma 1.20. *In the above notation, we have $s' = s$ and $b_j = -a_j$ for $1 \le \forall j \le s$. Moreover, if we define the subsets $\{l_0, l_1, \dots, l_{s+1}\}$ and $\{n_2, \dots, n_{s+1}\}$ of N inductively by*

$$l_0 := n,\ l_j := l_{j-1} + n_j \quad \textit{and} \quad n_{j+1} := (b_j - 2) l_{j-1} + (b_j - 1) n_j \ ,$$

then $l_0 = n$, $l_1, \dots, l_s$, $l_{s+1} = n'$ in this order are the points of N lying on the compact edges of the boundary polygon $\partial\Theta$ and we have $l_{j-1} + l_{j+1} = b_j l_j$.

Proof. Let $q_1 := q$ and $q_2 := q - p$. The definition of the continued fraction expansion determines $\{q_3, \dots, q_{s'+2}\}$ inductively by $q_{j-1} = b_{j-1}q_j - q_{j+1}$, that is, $q_{j-1}/q_j = b_{j-1} - (q_j/q_{j+1})^{-1}$. Obviously, we have $q_{s'+1} = 1$, $q_{s'+2} = 0$ and $0 < q_{j+1} < q_j$ for $j \leq s'$. Moreover, we can inductively show that $\{l_j, n_{j+1}\}$ is a $\mathbb{Z}$-basis of N and that $n' = (q_j - q_{j+1})l_{j-1} + q_j n_j$.

$\sigma_0 := \sigma = \mathbb{R}_{\geq 0} l_0 + \mathbb{R}_{\geq 0} n'$ is contained in the nonsingular $\mathbb{R}_{\geq 0} l_0 + \mathbb{R}_{\geq 0} n_1$. Besides, l_0 and $l_1 = l_0 + n_1$ are the first two lattice points (i.e., elements belonging to N) on the compact edges of the boundary polygon $\partial\Theta$, since $n' = (q_1 - q_2)l_0 + q_1 n_1$ and $0 < q_2 < q_1$. If we divide σ_0 into two cones by the half line $\mathbb{R}_{\geq 0} l_1$, then one half $\mathbb{R}_{\geq 0} l_0 + \mathbb{R}_{\geq 0} l_1$ is nonsingular, while the other half $\sigma_1 := \mathbb{R}_{\geq 0} l_1 + \mathbb{R}_{\geq 0} n'$ is contained in the nonsingular $\mathbb{R}_{\geq 0} l_1 + \mathbb{R}_{\geq 0} n_2$. Since $n' = (q_2 - q_3) l_1 + q_2 n_2$, we can apply the induction hypothesis to σ_1. (See Fig. 1.8.) q.e.d.

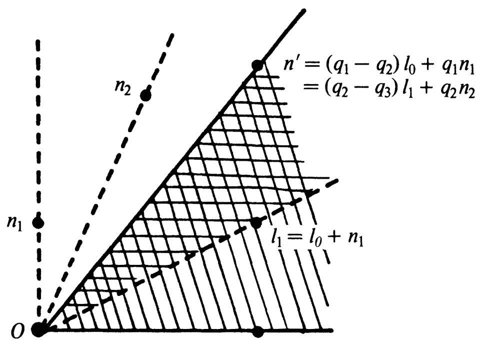


Fig. 1.8

We have $U_\sigma = \{u : \mathscr{S}_\sigma \to \mathbb{C}; u(O) = 1, u(m+m') = u(m)u(m'), \forall m, m' \in \mathscr{S}_\sigma\}$ for a two-dimensional cone σ as above. By Proposition 1.2, we get a minimal equivariant embedding

$$(\mathbf{e}(k_0), \dots, \mathbf{e}(k_{t+1})) : U_\sigma \to \mathbb{C}^{t+2} ,$$

if we choose a minimal system of generators $\{k_0, k_1, \dots, k_{t+1}\}$ for the semigroup $\mathscr{S}_\sigma = M \cap \sigma^\vee$ as follows by applying our consideration in Proposition 1.19 to $\sigma^\vee$.

Proposition 1.21. *Let $\Theta^\vee$ be the convex hull in $M_{\mathbb{R}}$ of $(\sigma^\vee \cap M) \setminus \{O\}$ for the two-dimensional strongly convex rational polyhedral cone $\sigma^\vee$ in $M_{\mathbb{R}}$. If $k_0, k_1, \dots, k_{t+1}$ in this order are the points of M lying on the compact edges of the boundary polygon $\partial\Theta^\vee$ of $\Theta^\vee$, then they form a minimal system of generators of the semigroup $\mathscr{S}_\sigma = \sigma^\vee \cap M$ and there exist integers $c_i \geq 2$ such that*

$$k_{i-1} + k_{i+1} = c_i k_i \quad \textit{for} \quad 1 \leq \forall i \leq t .$$

(See Fig. 1.9.)

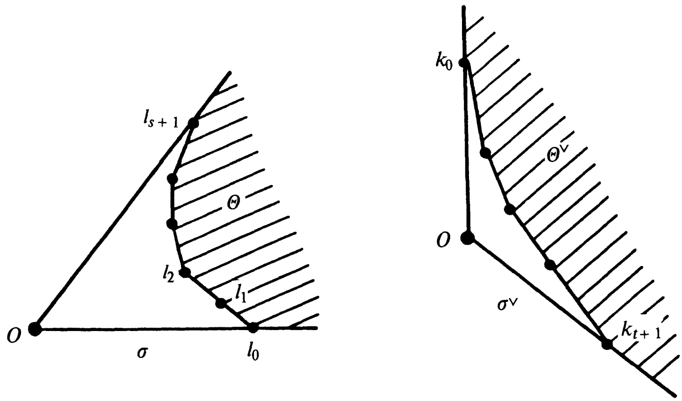


Fig. 1.9

Proof. By our argument in Proposition 1.19 applied to $\sigma^\vee$, we see that $\{k_{i-1}, k_i\}$ is a $\mathbb{Z}$-basis of M and is a minimal system of generators for the semigroup $M \cap (\mathbb{R}_{\geq 0} k_{i-1} + \mathbb{R}_{\geq 0} k_i)$. Since $\sigma^\vee \cap M$ is the union of these semigroups, we see that $k_0, \ldots, k_{t+1}$ generate $\sigma^\vee \cap M$. The rest of the assertion is clear by our argument in Proposition 1.19 applied to $\sigma^\vee$. q.e.d.

Θ and $\Theta^\vee$ are dual to each other in the following sense: By Section A.3, we define the support function $h^\vee : \sigma^\vee \to \mathbb{R}_{\geq 0}$ for Θ by

$$h^\vee(m) := \min\{\langle m, n \rangle ; n \in \Theta\} \quad \text{for} \quad m \in \sigma^\vee .$$

Since Θ is the convex hull of $(\sigma \cap N) \setminus \{O\}$, it suffices to take the minimum on the right hand side when n runs through $(\sigma \cap N) \setminus \{O\}$, or simply through $\{l_0, \ldots, l_{s+1}\}$ in our notation above. Let $\{l_{j(\alpha)} ; 0 \leq \alpha \leq v\}$ be the set of vertices of Θ with $0 = j(0) < j(1) < \ldots < j(v) = s+1$. Then we further have

$$h^\vee(m) = \min\{\langle m, l_{j(\alpha)} \rangle ; 0 \leq \alpha \leq v\} .$$

We define the *polar polyhedron* $\Theta^\circ \subset \sigma^\vee$ for Θ by

$$\Theta^\circ := \{m \in \sigma^\vee ; h^\vee(m) \geq 1\} ,$$

which is a convex polyhedron contained entirely in the interior of $\sigma^\vee$, since $h^\vee$ vanishes on the boundary of $\sigma^\vee$ and since $h^\vee$ is *positively homogeneous* and *upper convex*, i.e., $h^\vee(cm) = ch^\vee(m)$ and $h^\vee(m+m') \geq h^\vee(m) + h^\vee(m')$ for $c \in \mathbb{R}_{\geq 0}$ and $m, m' \in \sigma^\vee$.

Clearly, $m_{j(0)} := k_0$ (resp. $m_{j(v+1)} := k_{t+1}$) is the unique primitive element of M lying on the face $\sigma^\vee \cap \{l_0\}^\perp$ (resp. $\sigma^\vee \cap \{l_{s+1}\}^\perp$) of $\sigma^\vee$. We can define primitive elements $m_{j(\alpha)}$ of M for $1 \leq \forall \alpha \leq v$ by

$$\langle m_{j(\alpha)}, l_j \rangle = 1 \quad \text{for} \quad j(\alpha - 1) \leq \forall j \leq j(\alpha) ,$$

since $\{l_j; j(\alpha-1)\leq j\leq j(\alpha)\}$ are collinear elements of N with each adjacent pair forming a $\mathbb{Z}$-basis of N. Thus we get

$$h^\vee(m_{j(\alpha)})=1 \quad \text{for} \quad 1\leq\alpha\leq v \ ; \quad h^\vee(m_{j(0)})=h^\vee(m_{j(v+1)})=0 \ .$$

As Fig. 1.10 shows, we have

$$h^\vee(m)=\langle m, l_{j(\alpha)}\rangle \quad \text{if} \quad m\in\mathbb{R}_{\geq 0}m_{j(\alpha)}+\mathbb{R}_{\geq 0}m_{j(\alpha+1)}$$

for $0\leq\alpha\leq v$.

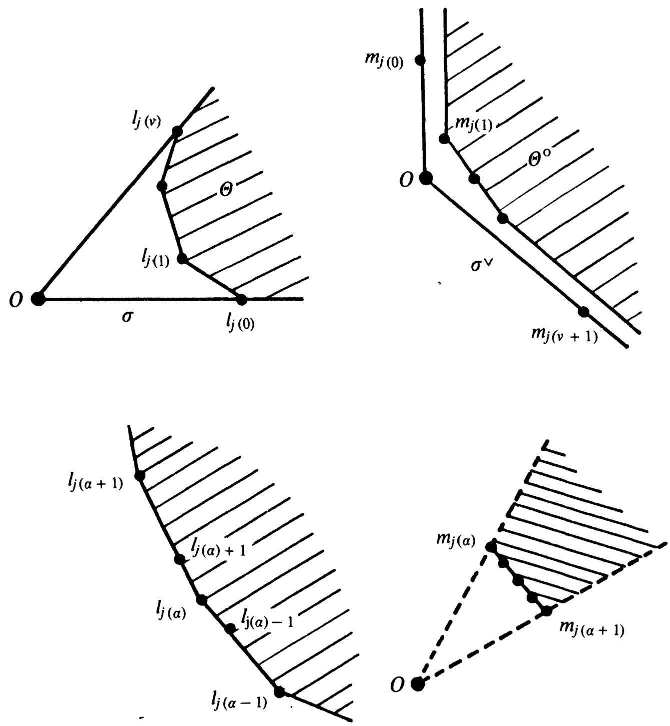


Fig. 1.10

For $1\leq\alpha\leq v-1$, the line segment joining $m_{j(\alpha)}$ and $m_{j(\alpha+1)}$ contains exactly $b_{j(\alpha)}-1$ elements of M, which together with O are all the elements of M lying on the triangle with vertices O, $m_{j(\alpha)}$ and $m_{j(\alpha+1)}$. Indeed, $\{l_{j(\alpha)-1}, l_{j(\alpha)}\}$ is a $\mathbb{Z}$-basis of N and $l_{j(\alpha)-1}+l_{j(\alpha)+1}=b_{j(\alpha)}l_{j(\alpha)}$. Moreover, we have $\langle m_{j(\alpha)}, l_{j(\alpha)-1}\rangle=\langle m_{j(\alpha)}, l_{j(\alpha)}\rangle=1$ and $\langle m_{j(\alpha+1)}, l_{j(\alpha)}\rangle=\langle m_{j(\alpha+1)}, l_{j(\alpha)+1}\rangle=1$. If we denote by $\{m, m'\}$ the $\mathbb{Z}$-basis dual to $\{l_{j(\alpha)-1}, l_{j(\alpha)}\}$, we then get $m_{j(\alpha)}=m+m'$ and $m_{j(\alpha+1)}=(b_{j(\alpha)}-1)m+m'$.

$\Theta^\vee$ is obviously the convex hull of $\Theta^\circ\cup\mathbb{R}_{\geq 0}k_0\cup\mathbb{R}_{\geq 0}k_{t+1}$, hence that of $\{m_{j(\alpha)}; 0\leq\alpha\leq v+1\}$. The only elements of M lying on the triangles with vertices O, $m_{j(0)}$, $m_{j(1)}$ and O, $m_{j(v)}$, $m_{j(v+1)}$ are their respective vertices. Thus the cardinality $t+2$ of the elements of M lying on the compact edges of $\partial\Theta^\vee$ coincides with

$3+\sum_{1\leq\alpha\leq v-1}(b_{j(\alpha)}-2)$, which is also equal to $3+\sum_{1\leq j\leq s}(b_j-2)$, since $b_j=2$ if and only if l_{j-1}, l_j, l_{j+1} are collinear (i.e., $j\neq j(\alpha)$ for $1\leq\forall\alpha\leq v-1$).

By what we have just seen and the dual version, we have:

Lemma 1.22. *Let $l_0, l_1, \ldots, l_{s+1}$ in this order be the elements of N lying on the compact edges of the boundary $\partial\Theta$ of the convex hull Θ of $(\sigma\cap N)\setminus\{O\}$. There exist integers $b_j\geq 2$ such that $l_{j-1}+l_{j+1}=b_j l_j$ for $1\leq j\leq s$. Similarly, let $k_0, k_1, \ldots, k_{t+1}$ in this order be the elements of M lying on the compact edges of the boundary $\partial\Theta^\vee$ of the convex hull $\Theta^\vee$ of $(\sigma^\vee\cap M)\setminus\{O\}$. There exist integers $c_j\geq 2$ such that $k_{i-1}+k_{i+1}=c_i k_i$. We then have the following relation:*

$$\sum_{1\leq j\leq s}(b_j-2)=t-1 \quad \textit{and} \quad \sum_{1\leq i\leq t}(c_i-2)=s-1\ .$$

Remark. This lemma has the following geometric significance: Let $C_1+\ldots+C_s$ be the exceptional curve for the minimal resolution for the singularity of U_σ at orb (σ) as in Proposition 1.19. Since $(C_j^2)=a_j=-b_j$, $(C_j\,.\,C_{j+1})=1$ and $(C_j\,.\,C_i)=0$ for nonadjacent C_i, C_j, we get

$$-(C_1+\ldots+C_s)^2=2+\sum_{1\leq j\leq s}(b_j-2)=t+1\ ,$$

which is also known to coincide with the multiplicity of U_σ at the point orb (σ). If vol_2 is the Lebesgue measure of $M_{\mathbb{R}}$ so normalized that $1/2$ is the area of the triangle with vertices at O and a $\mathbb{Z}$-basis of M, then

$$(t+1)/2=\mathrm{vol}_2(\sigma^\vee\setminus\Theta^\vee)\ ,$$

since the right hand sinde is the sum of the areas of the triangles with vertices at O, k_{i-1}, k_i for $1\leq i\leq t+1$.

Corollary 1.23 (Riemenschneider [R5, § 3, Lemma 4]). *Let p and q be relatively prime integers satisfying $0<p<q$. For the continued fraction expansions*

$$q/(q-p)=b_1-1\rceil b_2-\ldots-1\rceil b_s \quad \textit{with} \quad b_j\geq 2 \quad \textit{for} \quad 1\leq\forall j\leq s$$

$$q/p=c_1-1\rceil c_2-\ldots-1\rceil c_t \quad \textit{with} \quad c_i\geq 2 \quad \textit{for} \quad 1\leq\forall i\leq t\ ,$$

we have

$$(b_1+\ldots+b_s)-s=(c_1+\ldots+c_t)-t=s+t-1\ .$$

Proof. The first and the third terms are equal by Lemmas 1.20 and 1.22. Let us apply the same argument to $\sigma^\vee$ instead of σ. Since $\sigma=\mathbb{R}_{\geq 0}n_1+\mathbb{R}_{\geq 0}(pn_1+qn_2)$ for a $\mathbb{Z}$-basis $\{n_1,n_2\}$ of N, we have $\sigma^\vee=\mathbb{R}_{\geq 0}m_2+\mathbb{R}_{\geq 0}(qm_1-pm_2)=\mathbb{R}_{\geq 0}m_2+\mathbb{R}_{\geq 0}((q-p)m_2+q(m_1-m_2))$ for the dual basis $\{m_1,m_2\}$. The $\mathbb{Z}$-basis $\{m_2, m_1-m_2\}$ of M and the continued fraction expansion $q/(q-(q-p))=q/p=c_1-1\rceil c_2-\ldots-1\rceil c_t$ are of the form in Lemma 1.20. Thus by Lemma 1.22, the second and third terms are equal. q.e.d.

The singularity of U_σ at orb (σ) is a *cyclic quotient singularity*. Using a map of fans, we can show this fact as a special case of Corollary 1.16 as follows:

Proposition 1.24. *Let p and q be relatively prime integers satisfying $0 \leq p < q$. Suppose the generator $1+q\mathbb{Z}$ of the cyclic group $\mathbb{Z}/q\mathbb{Z}$ of order q acts on the complex plane $\mathbb{C}^2$ with coordinate (z_1, z_2) by*

$$(z_1, z_2) \mapsto (\varepsilon^{-p} z_1, \varepsilon z_2) \quad \textit{with} \quad \varepsilon := \exp(2\pi i/q) .$$

For a $\mathbb{Z}$-module N with a $\mathbb{Z}$-basis $\{n_1, n_2\}$, define a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ by

$$\sigma := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0}(pn_1 + qn_2) .$$

Then the toric affine variety U_σ is the quotient of $\mathbb{C}^2$ with respect to the action of $\mathbb{Z}/q\mathbb{Z}$.

Proof. n_1 and pn_1+qn_2 generate a $\mathbb{Z}$-submodule N' of N of index q. Let Δ be the fan in $N_{\mathbb{R}} = N'_{\mathbb{R}}$ consisting of the faces of σ. Then the natural inclusion $(N', \Delta) \to (N, \Delta)$ induces an equivariant holomorphic map $U'_\sigma = T_{N'} \operatorname{emb}(\Delta) \to U_\sigma = T_N \operatorname{emb}(\Delta)$. As in Corollary 1.16, this is the quotient of $U'_\sigma = \mathbb{C}^2$ with respect to the action of the cyclic group $\ker[T_{N'} \to T_N]$ of order q.

To be more explicit, let $\{m_1, m_2\}$ be the $\mathbb{Z}$-basis for M dual to $\{n_1, n_2\}$ and let $\{m'_1, m'_2\}$ be the $\mathbb{Z}$-basis for M' dual to the $\mathbb{Z}$-basis $\{n_1, pn_1+qn_2\}$ of N'. Regarding M canonically as a submodule of M', we have $m_1 = m'_1 + pm'_2$, $m_2 = qm'_2$ and the extension $\langle\ ,\ \rangle : M' \times N \to (1/q)\mathbb{Z}$ of the duality pairing. Since $\sigma^\vee \cap M' = \mathbb{Z}_{\geq 0} m'_1 + \mathbb{Z}_{\geq 0} m'_2$, we have an isomorphism $(z_1, z_2) : U'_\sigma \xrightarrow{\sim} \mathbb{C}^2$ with $z_1 := \mathbf{e}(m'_1)$ and $z_2 := \mathbf{e}(m'_2)$. Under the isomorphisms $\ker[T_{N'} \to T_N] = \operatorname{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^\times) \cong N/N' \cong \mathbb{Z}/q\mathbb{Z}$, we send $a + q\mathbb{Z} \in \mathbb{Z}/q\mathbb{Z}$ to $an_2 + N' \in N/N'$ and $n + N' \in N/N'$ to the homomorphism $w : M'/M \to \mathbb{C}^\times$ with $w(m' + M) := \exp(2\pi i \langle m', n \rangle)$. We are done, since $\exp(2\pi i \langle m'_1, n_2 \rangle) = \exp(2\pi i(-p/q)) = \varepsilon^{-p}$ and $\exp(2\pi i \langle m'_2, n_2 \rangle) = \exp(2\pi i/q) = \varepsilon$. q.e.d.

Remark. Brieskorn [B3] showed that every two-dimensional cyclic quotient singularity is of the form given in Proposition 1.24. We can use Proposition 1.21 to compute all the monomials in z_1 and z_2 which are invariant with respect to the action of $\mathbb{Z}/q\mathbb{Z}$ on $\mathbb{C}^2$ in Proposition 1.24 (cf. Riemenschneider [R5]).

Example. The special case of $p=1$ in Proposition 1.24 is as in Fig. 1.11. We have $U_\sigma = \{(x, y, z) \in \mathbb{C}^3 ; xy = z^q\}$. The origin $(0,0,0)$ is a singular point with multiplicity two and is known as a *rational double point of type* $\mathbf{A}_{q-1}$.

As Abhyankar [A1] showed, simultaneous resolution of singularities is impossible in general already in dimension two. We give a simple counterexample due to T. Katsura using what we have just seen.

In the above example for $p=1$, a generator of $G := \mathbb{Z}/q\mathbb{Z}$ acts on $Z := \mathbb{C}^2$ by $(z_1, z_2) \mapsto (\varepsilon^{-1} z_1, \varepsilon z_2)$ with the quotient $U := U_\sigma = Z/G$. On the other hand, the *Kummer holomorphic map* $k : Z \to W := \mathbb{C}^2$ which sends (z_1, z_2) to $k(z_1, z_2) := (z_1^q, z_2^q)$ is the quotient with respect to $G \times G$. We have a factorization

$$\mathbb{C}^2 = Z \xrightarrow{\varphi} U \xrightarrow{\psi} W = \mathbb{C}^2$$

of k into quotient maps φ and ψ with respect to actions of G.

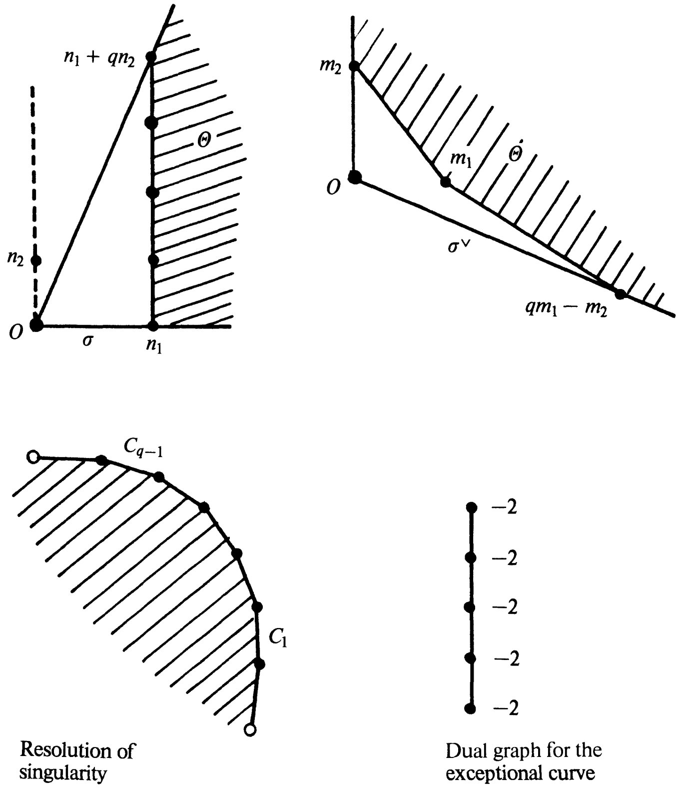


Fig. 1.11

In general, suppose a finite morphism $\psi: U \to W$ from a normal variety U to a nonsingular variety W is given. A *simultaneous resolution of singularities* is a finite succession $W' \to W$ of blowing-ups along nonsingular subvarieties such that the normalization of the pull-back $U \times_W W'$ is nonsingular.

On the other hand, suppose a quotient $\varphi: Z \to U = Z/G$ of a nonsingular variety Z with respect to an action of a finite group G is given. We can ask if there exists a succession $Z' \to Z$ of G-equivariant blowing-ups along G-stable nonsingular subvarieties such that the quotient Z'/G is nonsingular.

In our example, this latter question on $\varphi: Z \to U$ is equivalent to the existence of a simultaneous resolution of singularities for $\psi: U \to W$, since $k = \psi \circ \varphi$ is a Kummer holomorphic map.

Here is what Katsura actually showed: Suppose the generator $g := 1 + q\mathbb{Z}$ of the cyclic group $G := \mathbb{Z}/q\mathbb{Z}$ of order $q \geq 2$ acts on $Z := \mathbb{C}^2$ with coordinate (z_1, z_2) by

$$g: (z_1, z_2) \mapsto (\varepsilon^a z_1, \varepsilon z_2) \quad \text{with} \quad \varepsilon := \exp(2\pi i/q)$$

for an integer $a \not\equiv 0, 1 \pmod q$. If $q \geq 4$, then there cannot exist a succession $Z' \to Z$ of G-equivariant blowing-ups along G-stable nonsingular subvarieties such that Z'/G is nonsingular.

To prove this assertion, consider more generally the action of type $[a, b]_q$, where the generator $g := 1 + q\mathbb{Z}$ of $G := \mathbb{Z}/q\mathbb{Z}$ acts on $Z := \mathbb{C}^2$ by

$$g : (z_1, z_2) \mapsto (\varepsilon^a z_1, \varepsilon^b z_2)$$

for integers a and b modulo q. We then have the following:

(i) The blowing-up $Z_1 \to Z$ of the origin $(0, 0)$ is the only possible nontrivial blowing-up we are concerned with. Obviously, we have $Z_1 = V \cup V'$ with $V \cong \mathbb{C}^2$ and $V' \cong \mathbb{C}^2$, which are of types $[a, b-a]_q$ and $[a-b, b]_q$, respectively. Indeed, V (resp. V') has coordinate (v_1, v_2) (resp. (v_1', v_2')) such that $v_1 = z_1$, $v_2 = z_2/z_1$ when $z_1 \neq 0$ (resp. $v_1' = z_1/z_2$, $v_2' = z_2$ when $z_2 \neq 0$). Let us denote this fact symbolically by

$$[a, b-a]_q \to [a, b]_q \leftarrow [a-b, b]_q \ .$$

(ii) The quotient Z/G is clearly nonsingular for the types $[a, 0]_q$ and $[0, b]_q$.

(iii) The quotient Z/G is singular for the types $[a, 1]_q$ and $[a, -1]_q$ unless $a \equiv 0 \pmod{q}$. Indeed, if a and q are relatively prime, then we are done by Proposition 1.24. Otherwise, let $d := (a, q)$ be the greatest common divisor of $0 < a < q$ and q and denote $q = q'd$ and $a = a'd$. Then the holomorphic map $Z \to \mathbb{C}^2$ which sends (z_1, z_2) to (z_1, z_2^d) is the quotient with respect to the subgroup of G generated by $g^{q'}$. On the quotient we obviously have actions of types $[a', 1]_{q'}$ and $[a', -1]_{q'}$, respectively, with a' and q' relatively prime.

(iv) For $4 \leq q$ and $2 \leq a < q$, consider the action of type $[a, 1]_q$. Any finite succession of nontrivial G-equivariant blowing-ups gives rise to types which we know to have singular quotients by (iii) above. Indeed, if $a \neq q-1$, then we have the following loop:

$$\begin{aligned}
&[a, 1]_q \leftarrow [a-1, 1]_q \leftarrow [a-2, 1]_q \leftarrow \dots \leftarrow [2, 1]_q \\
&\quad \leftarrow [2, -1]_q \leftarrow [3, -1]_q \leftarrow \dots \leftarrow [q-2, -1]_q \leftarrow [q-2, 1-q]_q \\
&\quad = [q-2, 1]_q \leftarrow [q-3, 1]_q \leftarrow \dots \leftarrow [a, 1]_q \ .
\end{aligned}$$

If $a = q-1$, then the blowing-up of the origin gives rise to $[q-1, 1]_q \leftarrow [q-2, 1]_q$, and we are reduced to the case $a = q-2$ above.

(v) On the other hand for $q = 2$ and $q = 3$, all the actions we obtain after a succession of at most two G-equivariant blowing-ups clearly have nonsingular quotients.

Although things are not so simple in higher dimension, at least the following generalization of Proposition 1.24 is possible. It is again a special case of Corollary 1.16 (cf. [TE, p. 40] and [MO, p. 42]):

Proposition 1.25. *Let $N \cong \mathbb{Z}^r$ and let $\sigma := \mathbb{R}_{\geq 0} n_1' + \dots + \mathbb{R}_{\geq 0} n_r'$ be a simplicial rational polyhedral cone in $N_{\mathbb{R}}$ spanned by $\mathbb{R}$-linearly independent $n_1', \dots, n_r' \in N$. Denote by N' the $\mathbb{Z}$-submodule of N of finite index with $\{n_1', \dots, n_r'\}$ as a $\mathbb{Z}$-basis and let M' be the $\mathbb{Z}$-module dual to N'. Under the natural identification $N_{\mathbb{R}}' = N_{\mathbb{R}}$, $U_\sigma' := \{u' : M' \cap \sigma^\vee \to \mathbb{C};\, u'(O) = 1,\ u'(m' + m'') = u'(m')u'(m''),\ \forall m', m'' \in M' \cap \sigma^\vee\}$ is isomorphic to $\mathbb{C}^r$ and the equivariant holomorphic map $U_\sigma' \to U_\sigma$ coincides with the projection for the quotient with respect to the finite commutative group* $\ker[T_{N'} \to T_N] \cong N/N'$.

Proof. The $\mathbb{Z}$-dual M of N is canonically a $\mathbb{Z}$-submodule of M' and we have a $\mathbb{Z}$-biliniear map $\langle\ ,\ \rangle: M' \times N \to \mathbb{Q}$ as an extension of the duality pairing. For the $\mathbb{Z}$-basis $\{m'_1, \ldots, m'_r\}$ of M' dual to $\{n'_1, \ldots, n'_r\}$, we have $M' \cap \sigma^\vee = \mathbb{Z}_{\geq 0} m'_1 + \ldots + \mathbb{Z}_{\geq 0} m'_r$ hence an isomorphism $(\mathbf{e}(m'_1), \ldots, \mathbf{e}(m'_r)): U'_\sigma \xrightarrow{\sim} \mathbb{C}^r$. The surjective homomorphism $T_{N'} \to T_N$ has kernel $\mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^\times)$, which is isomorphic to N/N' by the map sending $n + N' \in N/N'$ to the homomorphism $m' + M \mapsto \exp(2\pi i \langle m', n\rangle)$ of M'/M to $\mathbb{C}^\times$. Since the action of $n + N' \in N/N'$ on the holomorphic functions $\mathbf{e}(m')$ on U'_σ is then given by $\mathbf{e}(m') \mapsto \exp(2\pi i\langle m', n\rangle)\mathbf{e}(m')$, we see that $\{\mathbf{e}(m); m \in M\}$ is exactly the set of N/N'-invariants among them. Thus $U'_\sigma \to U_\sigma$ is the quotient with respect to N/N'. q.e.d.

Remark. In the situation of Proposition 1.25 above, the multiplicity of U_σ at the point orb (σ) also turns out to be $(r!)\,\mathrm{vol}_r(\sigma^\vee \setminus \Theta^\vee)$, where $\Theta^\vee$ is the convex hull of $\sigma^\vee \cap M \setminus \{O\}$ and vol_r is the Lebesgue measure of $M_{\mathbb{R}}$ so normalized that the fundamental r-simplex with vertices at O and a $\mathbb{Z}$-basis of M has measure $1/r!$.

The following example already shows one of the complications we are to anticipate in higher dimension:

Example. Let $\{n_1, n_2, n_3\}$ be a $\mathbb{Z}$-basis of $N \cong \mathbb{Z}^3$ and let $n_0 := -n_1 + n_2 + n_3$. Consider the convex quadrangular cone $\sigma := \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_3$. Notice that n_0, n_1, n_2, n_3 are coplanar and that O is the only point belonging to $\sigma \cap N$ and contained in the open half space which the plane determines and which contains O. These facts turn out to have geometric significance as we see below. The above σ has two different nonsingular subdivisions Δ' and Δ'' as follows:

$$\Delta' := \{\mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_3,\ \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_3 \text{ and their faces}\}$$

$$\Delta'' := \{\mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_1,\ \mathbb{R}_{\geq 0} n_3 + \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_1 \text{ and their faces}\}\ .$$

Furthermore, we have a common nonsingular subdivision $\tilde{\Delta}$ of Δ' and Δ'' defined for $n_4 := n_0 + n_1 = n_2 + n_3$ by

$$\tilde{\Delta} := \{\mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_4,\ \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_4,\ \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_3 + \mathbb{R}_{\geq 0} n_4,\ \mathbb{R}_{\geq 0} n_3 + \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_4 \text{ and their faces}\}\ .$$

Accordingly, we have three different resolutions of singularities of U_σ by the corresponding toric varieties related among themselves as in Fig. 1.12. This example is already in [TE, pp. 38–39]. In terms of the dual $\mathbb{Z}$-basis $\{m_1, m_2, m_3\}$ for M, we have

$$M \cap \sigma^\vee = \mathbb{Z}_{\geq 0}(m_1 + m_2) + \mathbb{Z}_{\geq 0} m_3 + \mathbb{Z}_{\geq 0}(m_1 + m_3) + \mathbb{Z}_{\geq 0} m_2\ .$$

Thus by $x := \mathbf{e}(m_1 + m_2)$, $y := \mathbf{e}(m_3)$, $z := \mathbf{e}(m_1 + m_3)$, $w := \mathbf{e}(m_2)$ we have

$$U_\sigma = \{(x, y, z, w) \in \mathbb{C}^4;\ xy = zw\}\ ,$$

which is a three-dimensional nondegenerate quadric cone.

$$\begin{tikzcd} & T_N\mathrm{emb}(\Delta') \arrow[dr] & \\ T_N\mathrm{emb}(\tilde{\Delta}) \arrow[ur] \arrow[dr] & & U_\sigma \\ & T_N\mathrm{emb}(\Delta'') \arrow[ur] & \end{tikzcd}$$

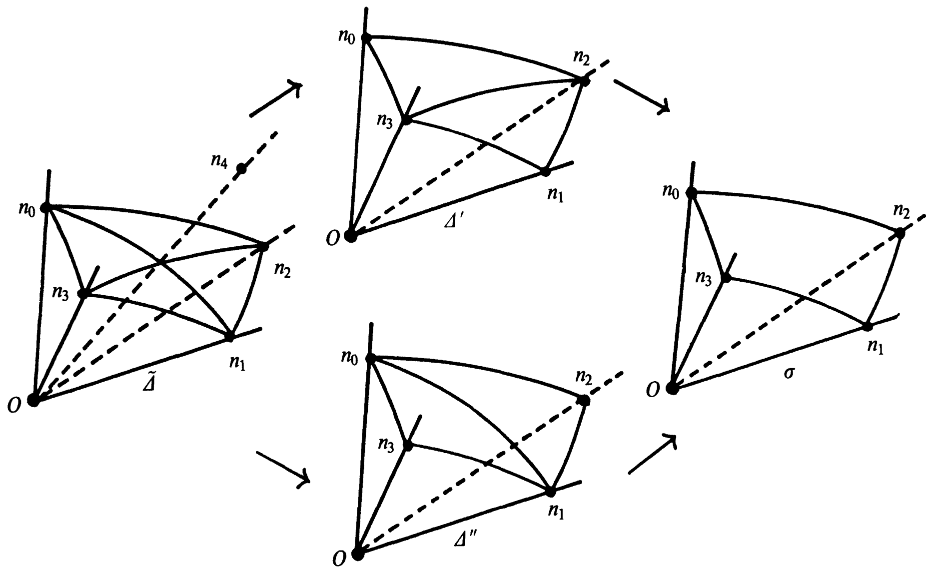


Fig. 1.12

Here are further complications in convex geometry already in dimension $r=3$: Even if σ is a three-dimensional simplicial rational cone in $N_{\mathbb{R}}$, the compact faces of the convex hull Θ of $\sigma \cap N \setminus \{O\}$ need not be triangles. Even if some of them are, the triangular cone we obtain by joining O with the three vertices may be singular. As a generalization of what we did in dimension two, we might also try to choose three points n_1', n_2', n_3' belonging to $N \cap \partial\Theta$ in such a way that the tetrahedron they span with O contains no points belonging to N other than the four vertices. Even then, there is no guarantee any longer that $\{n_1', n_2', n_3'\}$ form a $\mathbb{Z}$-basis of N, as the following lemma shows:

The Terminal Lemma. *Let n_1', n_2', and n_3' be $\mathbb{R}$-linearly independent primitive elements in $N \cong \mathbb{Z}^3$. If the tetrahedron T in $N_{\mathbb{R}}$ with vertices O, n_1', n_2', n_3' contains no points belonging to the lattice N other than the four vertices, then there exists a*

$\mathbb{Z}$-basis $\{n_1, n_2, n_3\}$ and relatively prime integers p, q with $0 \leq p < q$ such that $\{n'_1, n'_2, n'_3\}$ coincides, up to order, with

$$\{n_1, n_2, n_1 + pn_2 + qn_3\}$$

(cf. Fig. 1.13a).

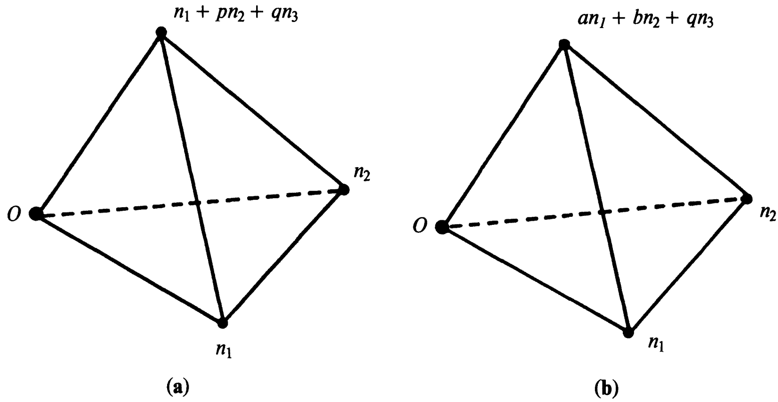


Fig. 1.13

The sufficiency is obvious. The necessity is shown as follows: Since the triangle S with vertices O, n'_1, n'_2, contains no points of N other than the vertices by assumption, there exists a $\mathbb{Z}$-basis $\{n_1, n_2, n_3\}$ of N such that $n'_1 = n_1$ and $n'_2 = n_2$. We can certainly so modify n_3 that the vertex of T not on S is $an_1 + bn_2 + qn_3$ for integers a, b, q with $0 \leq a < q$ and $0 \leq b < q$ (cf. Fig. 1.13b). In this case, a and q (resp. b and q, resp. $a + b - 1$ and q) are relatively prime, since the triangle with vertices $\{O, n_2, an_1 + bn_2 + qn_3\}$ (resp. $\{O, n_1, an_1 + bn_2 + qn_3\}$, resp. $\{n_1, n_2, an_1 + bn_2 + qn_3\}$) contains no points of N other than the vertices by assumption.

Furthermore, there exist no points on N contained in the interior of the tetrahedron T with vertices $O, n_1, n_2, an_1 + bn_2 + qn_3$ by assumption. Thus for any triple $\lambda_1, \lambda_2, \lambda_3$ of real numbers satisfying $0 < \lambda_1 < 1$, $0 < \lambda_2 < 1$, $0 < \lambda_3 < 1$ and $0 < \lambda_1 + \lambda_2 + \lambda_3 < 1$, the interior point

$$\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3(an_1 + bn_2 + qn_3) = (\lambda_1 + a\lambda_3)n_1 + (\lambda_2 + b\lambda_3)n_2 + q\lambda_3 n_3$$

of T does not belong to N. Consequently, if $\lambda_3 = k/q$ for an integer k satisfying $0 < k < q$, then at least one of $\lambda_1 + ak/q$ and $\lambda_2 + bk/q$ is not an integer. If we denote the fractional part of a real number ξ by $0 \leq \langle \xi \rangle < 1$, then what we just observed amounts to

$$\langle -ak/q \rangle + \langle -bk/q \rangle + \langle k/q \rangle > 1 \quad \text{for} \quad k = 1, 2, \ldots, q-1 \ ,$$

since a (resp. b, resp. $a + b - 1$) and q are relatively prime. However, we have the following general result:

White's Theorem (cf. [W3, Theorem 1]). *Suppose an integer α (resp. β, resp. γ) and q are relatively prime. If*

$$\langle \alpha k/q\rangle + \langle \beta k/q\rangle + \langle \gamma k/q\rangle > 1 \quad \text{for} \quad k = 1, 2, \ldots, q-1 ,$$

then at least one of $\alpha+\beta$, $\beta+\gamma$ and $\gamma+\alpha$ is divisible by q.

We can apply this theorem to our situation above with $\alpha = -a$, $\beta = -b$ and $\gamma = 1$ and conclude that at least one of $a-1$, $b-1$ and $a+b$ is divisible by q. We necessarily have $a=1$, $b=1$ or $a+b=q$, since $0<a<q$ and $0<b<q$. Obviously we are done if $a=1$ or $b=1$.

It remains to consider the case $a+b=q$. Since a and q are relatively prime, we have $ca=dq+1$, hence $cb=(c-d)q-1$, for integers c and d with $0<c<q$ and with c and q relatively prime. Thus $n_1 = n_2 + c(an_1+bn_2+qn_3) + q(-dn_1-(c-d)n_2-cn_3)$. Clearly,

$$\{n_2, an_1+bn_2+qn_3, \quad -dn_1-(c-d)n_2-cn_3\}$$

is a $\mathbb{Z}$-basis of N and we conclude the proof of the terminal lemma.

Remark. Morrison-Stevens [MS] gave a proof of White's theorem using Bernoulli functions. Ishida could reduce its proof to the classification of algebraic surfaces. We refer the reader to the next section for a generalization, due to Frumkin, of White's theorem to arbitrary convex polyhedra which need not be tetrahedra.

If we apply Proposition 1.25 to $\sigma = \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}(an_1+bn_2+qn_3)$, then U_σ is the quotient of $\mathbb{C}^3$ with respect to the following action of the cyclic group $\mathbb{Z}/q\mathbb{Z}$: The generator $1+q\mathbb{Z}$ of $\mathbb{Z}/q\mathbb{Z}$ acts on $\mathbb{C}^3$ by

$$(z_1, z_2, z_3) \mapsto (\varepsilon^{-a}z_1, \varepsilon^{-b}z_2, \varepsilon z_3) \quad \text{with} \quad \varepsilon := \exp(2\pi i/q) .$$

The origin of the word "terminal" is as follows:

Definition (Reid [R4, (1.11)]). Let $N \cong \mathbb{Z}^r$ with the dual $\mathbb{Z}$-module M. For an r-dimensional strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$, denote by $\{n'_1, \ldots, n'_s\}$ the set of primitive elements of N lying on the one-dimensional faces of σ. Le j be a positive integer.

(i) σ is *canonical* of index j, if there exists a primitive element m_0 of M such that $\langle m_0, n'_1\rangle = \ldots = \langle m_0, n'_s\rangle = j$ and that $\langle m_0, n\rangle \geq j$ for all $n \in \sigma \cap N \setminus \{O\}$.

(ii) σ is *terminal* of index j, if there exists a primitive element m_0 of M such that $\langle m_0, n'_1\rangle = \ldots = \langle m_0, n'_s\rangle = j$ and that $\langle m_0, n\rangle > j$ for all $n \in \sigma \cap N \setminus \{O, n'_1, \ldots, n'_s\}$.

Certainly, "terminal" implies "canonical". According to Reid and Danilov [R2, p. 294], σ is canonical (resp. terminal) if and only if U_σ has *canonical* (resp. *terminal*) *singularity* at the point orb (σ). As we see in Sect. 1.7 briefly in connection with the birational geometry of toric varieties, the notions of canonical and terminal singularities were introduced by Reid and Danilov [D2] and are expected to play important rôles in higher dimensional algebraic geometry.

The terminal lemma above classifies three-dimensional terminal cones σ which are simplicial. The quadrangular cone in our example immediately after Proposition 1.25 turns out to be the unique three-dimensional strongly convex rational polyhedral cone, up to isomorphism, which is terminal but not simplicial. We refer the reader to Ishida-Iwashita [II], Morrison-Stevens [MS] and Morrison [M11] more generally for the classification of three-dimensional canonical cones.

For $N \cong \mathbb{Z}^r$, let σ be an r-dimensional strongly convex rational polyhedral cone. In principle we can describe, in terms of the convex geometry of σ, the blowing-up of U_σ at the point orb (σ) and the Nash blowing-up of U_σ. The case $r=2$ is treated in González-Sprinberg [G2], [G3]. See also Ewald [E5] for related results. As we see in the next section, an equivariant blowing-up of a nonsingular toric variety along a nonsingular center is nonsingular. Ishida showed that the blowing-up of U_σ at orb (σ) is normal, hence is a toric variety in our sense, if $r \leq 3$, although it need not be normal if $r \geq 4$.

1.7 Birational Geometry of Toric Varieties

All toric varieties of the same dimension r contain an r-dimensional algebraic torus $T_N = (\mathbb{C}^\times)^r$ as an open set. Thus they are birationally equivalent as algebraic varieties, hence bimeromorphically equivalent as complex analytic spaces.

For simplicity, let us restrict ourselves to nonsingular complex algebraic varieties and call them complex manifolds here. We then use the words "birational" and "bimeromorphic" interchangeably. Of basic importance in birational geometry are the following interrelated problems on complex manifolds (or complex analytic spaces with mild singularities, as we see below):

(i) Can a proper birational morphism $f\colon Y \to X$ of complex manifolds be always decomposed as a finite succession of blowing-ups of complex manifolds along closed irreducible submanifolds?

(ii) For any given pair X, X' of birationally equivalent compact complex manifolds, can we always find a compact complex manifold X'' together with finite successions $X'' \to X$ and $X'' \to X'$ of blowing-ups of compact complex manifolds along closed irreducible submanifolds?

(ii′) Instead of finding X'' as in (ii) above, can we always find compact complex manifolds $X_0 = X, X_1, \ldots, X_{2l} = X'$ so that for each odd j with $1 \leq j \leq 2l-1$, there exist finite successions $X_j \to X_{j-1}$ and $X_j \to X_{j+1}$ of blowing-ups of compact complex manifolds along closed irreducible submanifolds?

(iii) In each birational equivalence class of compact complex manifolds, classify, up to isomorphism, those X which are *minimal with respect to blowing-ups*. Namely, if $f\colon X \to Y$ happens to be the blowing-up of a compact complex manifold Y along a closed irreducible submanifold, then f is necessarily an isomorphism.

(iii′) As a problem slightly coarser than (iii) above, classify, up to isomorphism, those X which are *minimal with respect to birational morphisms*. Namely, a birational morphism $f\colon X \to Y$ is necessarily an isomorphism.

According to **Hironaka's theorem on elimination of points of indeterminacy,** we can find, for any pair X, X' of birationally equivalent compact complex manifolds, a finite succession $X'' \to X$ of blowing-ups of compact complex manifolds along closed irreducible submanifolds in such a way that there exists a birational morphism $g: X'' \to X'$. Note, however, that (ii) asks something stronger.

The above problems are all trivial in the one-dimensional case, since proper birational morphisms for curves are necessarily isomorphisms.

In the two-dimensional case, the answer to (i), hence also to (ii) by the remark above, is affirmative. The problems (iii) and (iii′) coincide in this case and have the following answer: Each birational equivalence class of non-ruled algebraic surfaces contains a unique minimal surface up to isomorphism; minimal surfaces in each birational equivalence class of irrational ruled surfaces are $\mathbb{P}_1(\mathbb{C})$-bundles on compact Riemann surfaces of positive genera; up to isomorphism, the minimal surfaces in the birational equivalence class of rational surfaces are the projective plane $\mathbb{P}_2(\mathbb{C})$ and the Hirzebruch surfaces F_a for $0 \le a \ne 1$ (cf. Theorem 1.28 below). Together with results on nonalgebraic compact complex analytic surfaces, these results form a basic part of the **Enriques-Kodaira classification** of compact complex surfaces.

In higher dimension, however, nothing in general is known yet except the negative answer to (i). Immediately before Proposition 2.17 in Chap. 2, we give a three-dimensional counterexample to (i) using toric varieties.

As recent works due to Mori [M10], Reid [R2], [R3] and others show, it is more natural to allow complex analytic spaces with mild singularities in the above problems. Kawamata-Matsuda-Matsuki [KMM] is an excellent up-to-date survey in this connection. Closely related are the results due to Reid and Danilov given later in this section as well as Mori's theory recalled briefly in Sect. 2.5.

In this section, we consider the toric variants of the above problems involving compact nonsingular toric varieties, equivariant birational morphisms and equivariant blowing-ups. The problems can be translated into those on elementary convex geometry of fans. Hopefully, the results here give good insight into general birational geometry in higher dimension.

From now on, we fix $N \cong \mathbb{Z}^r$ and consider various toric varieties $X = T_N \operatorname{emb}(\Delta)$ associated to finite fans Δ in N. A subdivision Δ' of Δ gives rise to a map of fans $(N, \Delta') \to (N, \Delta)$ induced by the identity map of N, hence, by Corollary 1.17, an equivariant holomorphic map $f: X' = T_N \operatorname{emb}(\Delta') \to X$ which is proper and birational. We call such f an *equivariant birational morphism.*

A key rôle is played by the following special case of Corollary 1.17 on equivariant birational morphisms of toric varieties.

Proposition 1.26 ([MO, Proposition 7.4]). *For a nonsingular toric variety* $X = T_N \operatorname{emb}(\Delta)$ *and* $\tau \in \Delta$, *the* equivariant blowing-up *of* X *along the closure* $V(\tau)$ *of the orbit* $\operatorname{orb}(\tau)$ *is the equivariant birational morphism* $T_N \operatorname{emb}(\Delta^*(\tau)) \to X = T_N \operatorname{emb}(\Delta)$ *corresponding to the following nonsingular* star subdivision $\Delta^*(\tau)$ *of* Δ *with respect to* τ: *Denote* $\Delta(1) := \{\varrho \in \Delta;\ \dim \varrho = 1\}$ *and let* $\{\varrho \in \Delta(1);\ \varrho < \tau\} =: \{\varrho_1, \ldots, \varrho_k\}$. *Each* ϱ_j *contains a unique primitive element* $n_j := n(\varrho_j) \in N$. *Let* $n_0 := n_1 + \ldots + n_k$ *and*

$$\tau_j := \mathbb{R}_{\ge 0} n_1 + \ldots + \mathbb{R}_{\ge 0} n_{j-1} + \mathbb{R}_{\ge 0} n_0 + \mathbb{R}_{\ge 0} n_{j+1} + \ldots + \mathbb{R}_{\ge 0} n_k$$

for $1 \le j \le k$. *Each* $\sigma \in \Delta$ *with* $\tau < \sigma$ *can be written as*

$$\sigma = \tau + \sigma' \quad \text{with} \quad \sigma' \in \Delta \quad \text{and} \quad \sigma' \cap \tau = \{O\}\ .$$

We then let

$$\sigma_j := \tau_j + \sigma' \quad \textit{for} \quad 1 \le j \le k \quad \textit{and}$$

$$\Delta^*(\tau) := (\Delta \setminus \{\sigma \in \Delta; \tau < \sigma\}) \cup \{\textit{the faces of } \sigma_j;\ \sigma \in \Delta,\ \tau < \sigma,\ 1 \le j \le k\} \ .$$

Proof. By Corollary 1.7, $V(\tau)$ is a T_N-stable closed irreducible submanifold of X. Hence for the blowing-up $f: X^* \to X$ along $V(\tau)$, X^* is nonsingular with a natural action of T_N, with respect to which f is equivariant. By definition, the blowing-up f can be described on each open set U_σ of X as follows:

We may assume $U_\sigma \cap V(\tau)$ to be nonempty, since otherwise, the restriction of f would induce an isomorphism $f^{-1}(U_\sigma) \xrightarrow{\sim} U_\sigma$. By Proposition 1.6, (iii), (iv), we thus have $\tau < \sigma$. Hence there exists a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N and $k \le l$ such that

$$\tau = \mathbb{R}_{\ge 0} n_1 + \ldots + \mathbb{R}_{\ge 0} n_k \quad \text{and} \quad \sigma = \mathbb{R}_{\ge 0} n_1 + \ldots + \mathbb{R}_{\ge 0} n_l \ .$$

Let $\{m_1, \ldots, m_r\}$ be the $\mathbb{Z}$-basis of M dual to $\{n_1, \ldots, n_r\}$ and let $u_j := \mathbf{e}(m_j)$ for $1 \le j \le r$. Thus $(u_1, \ldots, u_r)$ is a coordinate system for $\mathbb{C}^r$ and, by Corollary 1.7,

$$U_\sigma = \{(u_1, \ldots, u_r) \in \mathbb{C}^r;\ u_{l+1} u_{l+2} \ldots u_r \ne 0\}$$

$$U_\sigma \cap V(\tau) = \{(u_1, \ldots, u_r) \in U_\sigma;\ u_1 = u_2 = \ldots = u_k = 0\} \ .$$

By definition, we have $f^{-1}(U_\sigma) = \cup_{1 \le j \le k} W_j$, where W_j is an open set with coordinate

$$\left(u_j, \frac{u_1}{u_j}, \ldots, \frac{u_{j-1}}{u_j}, \frac{u_{j+1}}{u_j}, \ldots, \frac{u_k}{u_j}, u_{k+1}, \ldots, u_l, \ldots, u_r\right)$$

satisfying $u_{l+1} u_{l+2} \ldots u_r \ne 0$. The subsemigroup of M defined by

$$\begin{aligned} \mathscr{S}_j := {} & \mathbb{Z}_{\ge 0} m_j + \mathbb{Z}_{\ge 0}(m_1 - m_j) + \ldots + \mathbb{Z}_{\ge 0}(m_{j-1} - m_j) \\ & + \mathbb{Z}_{\ge 0}(m_{j+1} - m_j) + \ldots + \mathbb{Z}_{\ge 0}(m_k - m_j) \\ & + \mathbb{Z}_{\ge 0} m_{k+1} + \ldots + \mathbb{Z}_{\ge 0} m_l + \mathbb{Z} m_{l+1} + \ldots + \mathbb{Z} m_r \end{aligned}$$

clearly satisfies $W_j = \{u : \mathscr{S}_j \to \mathbb{C};\ u(O) = 1, u(m + m') = u(m) u(m') \text{ for all } m, m' \in \mathscr{S}_j\}$ and $\mathscr{S}_j^\vee = \sigma_j$. Thus we obviously get $X^* = T_N \operatorname{emb}(\Delta^*(\tau))$. q.e.d.

For any pair Δ, Δ' of finite nonsingular fans in $N \cong \mathbb{Z}^r$ with $|\Delta| = |\Delta'|$, a finite nonsingular subdivision of the fan $\{\sigma \cap \sigma';\ \sigma \in \Delta, \sigma' \in \Delta'\}$, which exists as we saw in Sect. 1.5, is a common nonsingular subdivision of Δ and Δ'. On the other hand, the toric version of Hironaka's theorem on elimination of points of indeterminacy guarantees the existence of a finite nonsingular subdivision Δ'' of Δ' obtained from Δ by a finite succession of star subdivisions as defined in Proposition 1.26. In fact, we have the following result stronger than this latter result:

De Concini-Procesi's Theorem on Elimination of Points of Indeterminacy ([DP, II]). *Let Δ and Δ' be finite nonsingular fans in $N \cong \mathbb{Z}^r$ with $|\Delta| = |\Delta'|$ and denote by $X := T_N \operatorname{emb}(\Delta)$ and $X' := T_N \operatorname{emb}(\Delta')$ the birationally equivalent toric varieties corresponding to them. Then there exists an equivariant birational morphism $X'' \to X'$*

from a nonsingular toric variety $X'' = T_N \operatorname{emb}(\Delta'')$ *obtained from* X *by a finite succession of equivariant blowing-ups along* T_N*-stable closed irreducible submanifolds of* codimension two.

In other words, we obtain a subdivision Δ'' of Δ' by a finite succession of star subdivisions of Δ with respect to *two-dimensional* cones. It suffices to show the following. For details, we refer the reader to [DP, II]: For any nonzero $m \in M$, a suitable finite succession of star subdivisions of Δ with respect to two-dimensional cones gives rise to a fan Δ''' each cone of which is entirely contained in one of the two closed half spaces determined by the hyperplane $\{m\}^{\perp}$ in $N_{\mathbb{R}}$.

For later reference, let us state explicitly the special cases $r = 2, 3$ for Proposition 1.26:

Corollary 1.27 ([MO, Corollaries 7.5 and 7.6]). *Let* $X = T_N \operatorname{emb}(\Delta)$ *be an r-dimensional compact nonsingular toric variety.*

(a) *If* $r = 2$ *and if* Δ *contains* $\tau := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2$ *for a* $\mathbb{Z}$*-basis* $\{n_1, n_2\}$ *of* N, *then the star subdivision* $\Delta^*(\tau)$ *of* Δ *corresponding to the equivariant blowing-up* $f\colon T_N \operatorname{emb}(\Delta^*(\tau)) \to X$ *with the* T_N*-fixed point* $V(\tau)$ *as center is of the form*

$$\Delta^*(\tau) = (\Delta \setminus \{\tau\}) \cup \{\mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_2, \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_0, \mathbb{R}_{\geq 0} n_0\}\ ,$$

where $n_0 := n_1 + n_2$ (cf. Fig. 1.14a).

(b) *If* $r = 3$ *and if* Δ *contains* $\tau := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_3$ *for a* $\mathbb{Z}$*-basis* $\{n_1, n_2, n_3\}$ *of* N, *then the star subdivision* $\Delta^*(\tau)$ *of* Δ *corresponding to the equivariant blowing-up* $f\colon T_N \operatorname{emb}(\Delta^*(\tau)) \to X$ *with the* T_N*-fixed point* $V(\tau)$ *as center is of the form*

$$\Delta^*(\tau) = (\Delta \setminus \{\tau\}) \cup \{\textit{the faces of } \tau_1, \tau_2, \tau_3\}\ ,$$

where $n_0 := n_1 + n_2 + n_3$ *and*

$$\tau_1 := \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_3$$

$$\tau_2 := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_3$$

$$\tau_3 := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_0$$

(cf. Fig. 1.14b).

(c) *If* $r = 3$ *and if* Δ *contains* $\tau := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2$ *for a part* $\{n_1, n_2\}$ *of a* $\mathbb{Z}$*-basis of* N, *then the star subdivision* $\Delta^*(\tau)$ *of* Δ *corresponding to the equivariant blowing-up* $f\colon T_N \operatorname{emb}(\Delta^*(\tau)) \to X$ *along the* T_N*-stable one-dimensional closed irreducible submanifold* $V(\tau) \cong \mathbb{P}_1(\mathbb{C})$ *is of the following form: There exist exactly two cones* σ, σ' *of dimension three having* τ *as a face. We have*

$$\sigma = \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_3 \quad \textit{and} \quad \sigma' = \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_3'$$

for $\mathbb{Z}$*-bases* $\{n_1, n_2, n_3\}$ *and* $\{n_1, n_2, n_3'\}$. *Then*

$$\Delta^*(\tau) = (\Delta \setminus \{\tau, \sigma, \sigma'\}) \cup \{\textit{the faces of } \sigma_1, \sigma_2, \sigma_1', \sigma_2'\}\ ,$$

where $n_0 := n_1 + n_2$ *and*

$$\sigma_1 := \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_3\ , \qquad \sigma_2 := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_3$$

$$\sigma_1' := \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_3'\ , \qquad \sigma_2' := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_3'$$

(cf. Fig. 1.14c).

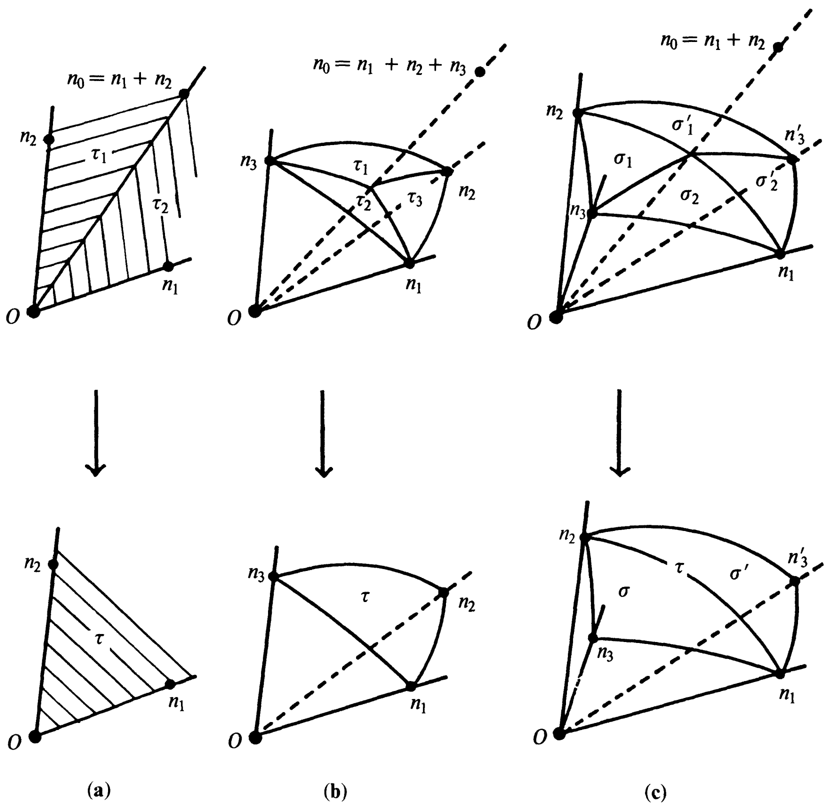


Fig. 1.14

For $r=1$, the complex projective line $\mathbb{P}_1(\mathbb{C})$ is obviously the unique one-dimensional compact toric variety up to isomorphism.

As for $r=2$, a finite complete fan Δ in N determines and is uniquely determined by a sequence $n_1, \ldots, n_s, n_{s+1} = n_1$ of elements in N which counter-clockwise go around the origin of the plane $N_{\mathbb{R}} \cong \mathbb{R}^2$ exactly once in this order in such a way that n_{j+1} comes strictly before $-n_j$ for each $1 \leq j \leq s$. Then for each j the two-dimensional strongly convex polyhedral cone $\sigma_j := \mathbb{R}_{\geq 0} n_j + \mathbb{R}_{\geq 0} n_{j+1}$ satisfies $n_k \notin \sigma_j$ for $k \neq j, j+1$ and $N_{\mathbb{R}} = \sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_s$. Hence $\Delta = \{\sigma_1, \sigma_2, \ldots, \sigma_s, \mathbb{R}_{\geq 0} n_1, \ldots, \mathbb{R}_{\geq 0} n_s, \{O\}\}$ is the corresponding finite complete fan.

Any further classification of two-dimensional compact toric varieties in general would be meaningless, since there exist too many sequences $n_1, \ldots, n_s$, $n_{s+1} = n_1$ satisfying the above properties. According to Theorem 1.10, however, $X = T_N \operatorname{emb}(\Delta)$ is nonsingular in this case if and only if $\{n_1, n_2\}$, $\{n_2, n_3\}, \ldots$, $\{n_{s-1}, n_s\}$ and $\{n_s, n_{s+1}\}$ are all $\mathbb{Z}$-bases of N. The classification becomes meaningful under this strong restriction, and we have the following complete answer to our problems on the birational geometry of two-dimensional compact nonsingular toric varieties:

Theorem 1.28 ([MO, Theorem 8.2]). (1) *If an equivariant holomorphic map between two-dimensional nonsingular toric varieties is proper and birational, then it can be decomposed as a finite succession of equivariant blowing-ups with T_N-fixed points as centers.*

(2) *For any given pair X, X' of two-dimensional compact nonsingular toric varieties, we can always find a two-dimensional compact nonsingular toric variety X'' together with finite successions $X'' \to X$ and $X'' \to X'$ of equivariant blowing-ups with T_N-fixed points as centers.*

(3) *Any two-dimensional compact nonsingular toric variety is isomorphic to the one obtained from the following* (a), (b) *by a finite succession of equivariant blowing-ups with T_N-fixed points as centers. Moreover, those in* (a) *and* (b) *are minimal and are not isomorphic to one another:*

(a) *the complex projective plane $\mathbb{P}_2(\mathbb{C})$;*

(b) *the Hirzebruch surfaces F_a for $0 \le a \ne 1$; namely, the $\mathbb{P}_1(\mathbb{C})$-bundle of degree a over $\mathbb{P}_1(\mathbb{C})$ usually written as*

$$F_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1(\mathbb{C})} \oplus \mathcal{O}_{\mathbb{P}_1(\mathbb{C})}(a))$$

(cf. Proposition 1.33 and the comment immediately after that).

If a $\mathbb{Z}$-basis $\{n, n'\}$ of N is chosen, then the fans Δ corresponding to (a) *and* (b) *are as in Fig. 1.15 up to isomorphism.*

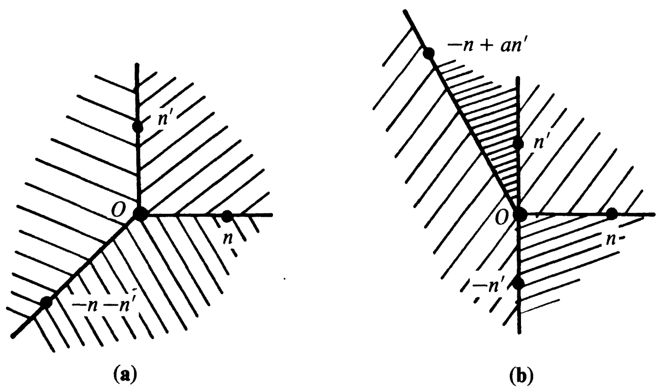


Fig. 1.15

Proof. We first show (2). Let Δ, Δ' be finite nonsingular complete fans in $N \cong \mathbb{Z}^2$ corresponding to X, X'. Then $\tilde{\Delta} := \{\sigma \cap \sigma';\ \sigma \in \Delta, \sigma' \in \Delta'\}$ is a finite complete fan and is a common subdivision of Δ, Δ'. As we saw in Sects. 1.5 and 1.6, there exists a nonsingular finite subdivision Δ'' of $\tilde{\Delta}$. It thus suffices to apply the proof below for (1) to the pairs Δ'', Δ as well as Δ'', Δ'.

To show (1), let $f: X' := T_N \operatorname{emb}(\Delta') \to X := T_N \operatorname{emb}(\Delta)$ be the equivariant holomorphic map corresponding by Corollary 1.17 to a nonsingular fan Δ in $N \cong \mathbb{Z}^2$ and a locally finite subdivision Δ' of Δ. It suffices to show that Δ' is obtained from Δ by a finite succession of star subdivisions of the form given in Corollary 1.27,(a).

If Δ happens to have no two-dimensional cone, then we obviously have $\Delta' = \Delta$ and f is an isomorphism.

On the other hand, if $\sigma \in \Delta$ is a two-dimensional cone, then we have $\sigma = \mathbb{R}_{\geq 0} n + \mathbb{R}_{\geq 0} n'$ for a $\mathbb{Z}$-basis $\{n, n'\}$ of N, since σ is assumed to be nonsingular. By assumption, $\{\sigma' \in \Delta'; \sigma' \subset \sigma\}$ is a finite nonsingular subdivision of σ. Thus we have a sequence $n_0 = n, n_1, \ldots, n_l = n'$ of elements of N such that $\{n_{j-1}, n_j\}$ for each $1 \leq j \leq l$ is a $\mathbb{Z}$-basis of N and that the two-dimensional cones $\sigma'_j := \mathbb{R}_{\geq 0} n_{j-1} + \mathbb{R}_{\geq 0} n_j$ for $1 \leq j \leq l$ do not intersect among themselves in the interior. As in Danilov [D2], let us look at the convex hull P of $\{O, n_0, n_1, \ldots, n_l\}$. If n_k for some $1 \leq k \leq l-1$ is a vertex of P, then the quadrilateral K with vertices O, n_{k-1}, n_k, n_{k+1} is necessarily convex. Moreover, we have $K \cap N = \{O, n_{k-1}, n_k, n_{k+1}\}$, since $\{n_{k-1}, n_k\}$ and $\{n_k, n_{k+1}\}$ are $\mathbb{Z}$-bases of N. Then obviously $n_k = n_{k-1} + n_{k+1}$ holds and $\{n_{k-1}, n_{k+1}\}$ is a $\mathbb{Z}$-basis of N. Hence Δ' is a star subdivision of the nonsingular fan

$$(\Delta' \setminus \{\sigma'_k, \sigma'_{k+1}, \mathbb{R}_{\geq 0} n_k\}) \cup \{\mathbb{R}_{\geq 0} n_{k-1} + \mathbb{R}_{\geq 0} n_{k+1}\} .$$

By induction, we are reduced to the case where P is a triangle with vertices $O, n_0 = n, n_l = n'$. Since $\{n, n'\}$ is a $\mathbb{Z}$-basis of N, we necessarily have $l = 1$. The same argument applied to the other two-dimensional cones in Δ completes the proof for (1).

We now sketch the proof for (3) leaving the details to [MO, Theorem 8.2].

As we remarked earlier, a two-dimensional compact nonsingular toric variety $X = T_N \operatorname{emb}(\Delta)$ is determined by a sequence $n_1, n_2, \ldots, n_s, n_{s+1} = n_1$ of primitive elements in N going counterclockwise around the origin exactly once in this order. Furthermore, $\{n_j, n_{j+1}\}$ for each $1 \leq j \leq s$ is a $\mathbb{Z}$-basis of N and n_{j+1} comes strictly before $-n_j$ if we start counterclockwise from n_j.

Obviously, we have $s \geq 3$. If $s = 3$, then $n_3 = -n_1 - n_2$ and $X = \mathbb{P}_2(\mathbb{C})$.

If $s \geq 4$, then after a suitable renumbering, if necessary, there exists $3 \leq j \leq s-1$ such that $n_j = -n_1$ (cf. [MO, Lemma 8.3]).

If $s = 4$, we then necessarily have $n_3 = -n_1$ and $n_4 = -n_2 - a n_1$ for an integer a. Replacing n_1 by $-n_1$, if necessary, we may assume $a \geq 0$ without any loss of generality. Moreover, if $a = 1$, then $n_2 + n_4 = n_3$ and Δ is a star subdivision of the complete nonsingular fan determined by the sequence $n_1, n_2, n_4 = -n_2 - n_1, n_1$. Thus the minimal ones with $s = 4$ are F_a for $0 \leq a \neq 1$.

When $s \geq 5$, we may replace n_1 by $-n_1$, if necessary, and assume $4 \leq j$ without loss of generality. Then $\{n_2, n_j\}$ is a $\mathbb{Z}$-basis of N, since so is $\{n_1, n_2\}$ by assumption. Thus $n_3, n_4, \ldots, n_{j-1}$ appear between n_2 and $n_j = -n_1$. Hence by the proof for (1), $n_3, \ldots, n_{j-1}$ are obtained from $\mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_j$ as a result of a finite succession of star subdivisions. Consequently, X is obtained from one of the Hirzebruch surfaces F_a by a finite succession of equivariant blowing-ups with T_N-fixed points as centers. q.e.d.

Much more convenient than sequences of primitive vectors are weighted dual graphs, which combinatorially describe two-dimensional compact nonsingular toric varieties up to isomorphism: As we have just seen, a two-dimensional compact nonsingular toric variety $X = T_N \operatorname{emb}(\Delta)$ is described up to isomorphism by a cycle $n_1, n_2, \ldots, n_s, n_{s+1} = n_1$ of primitive elements in $N \cong \mathbb{Z}^2$ which go counterclockwise around the origin exactly once in this order such that $\{n_j, n_{j+1}\}$ for each $1 \leq j \leq s$ is a $\mathbb{Z}$-basis for N. Obviously, there exists an integer a_j for each $1 \leq j \leq s$ such that

$$(*) \qquad n_{j-1} + n_{j+1} + a_j n_j = O \quad \text{for} \quad 1 \leq j \leq s ,$$

where $n_0 = n_s$ and $n_{s+1} = n_1$. As we see in Sect. 2.2, these integers a_j have the following geometric significance: The irreducible submanifold $D_j := V(\mathbb{R}_{\geq 0} n_j)$ is isomorphic to the complex projective line $\mathbb{P}_1(\mathbb{C})$ with the self-intersection number in X equal to

$$(D_j^2) = a_j \quad \text{for} \quad 1 \leq j \leq s \ .$$

Each adjacent pair of curves among $D_1, D_2, \ldots, D_s, D_{s+1} = D_1$ intersect transversally at one point, while the other pairs do not intersect. In view of our numbering, we thus have

$$(D_j . D_{j+1}) = 1 \quad \text{and} \quad (D_j . D_k) = 0 \quad \text{if} \quad k \neq j-1, j, j+1$$

for each $1 \leq j \leq s$. Consequently, the Cartier divisor $D := D_1 + \ldots + D_s$ on X is a *cycle of rational curves* which can be described completely in terms of the *weighted dual graph* of D defined as follows: To each irreducible component D_j corresponds a vertex with weight (D_j^2), while two vertices corresponding to distinct D_j, D_k are joined by an edge if and only if $(D_j . D_k) \neq 0$. In the present case, the weighted dual graph for the cycle D of rational curves is a circular graph with s vertices with weights $a_1, a_2, \ldots, a_s$ in this order as in Fig. 1.16.

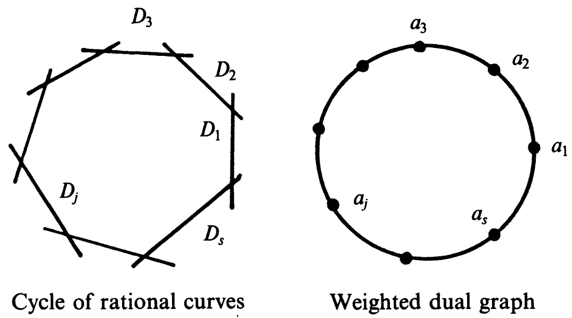


Fig. 1.16

$a_k = -1$ is equivalent to $n_k = n_{k-1} + n_{k+1}$. In this case, $n_1, \ldots, n_{k-1}, n_{k+1}, \ldots, n_s, n_{s+1} = n_1$ obviously determine another two-dimensional compact nonsingular toric variety X'. By Corollary 1.27, (a), $X \to X'$ is the equivariant blowing-up with the T_N-fixed point $V(\mathbb{R}_{\geq 0} n_{k-1} + \mathbb{R}_{\geq 0} n_{k+1})$ as the center. Since $n_{k-2} + n_{k+1} + (a_{k-1}+1) n_{k-1} = O$ and $n_{k-1} + n_{k+2} + (a_{k+1}+1) n_{k+1} = O$ with the indices counted modulo s, the weighted dual graph for X' is a circular graph with $s-1$ vertices and weights $a_1, \ldots, a_{k-1}+1, a_{k+1}+1, \ldots, a_s$ in this order.

Conversely, the weighted dual graph determines X uniquely up to isomorphism, since for an arbitrarily chosen $\mathbb{Z}$-basis $\{n_1, n_2\}$ of N, the equalities $(*)$ $n_{j-1} + n_{j+1} + a_j n_j = O$ for $2 \leq j \leq s-1$ successively determine $n_3, \ldots, n_s$ in N. Not all weighted circular graphs, however, give rise to two-dimensional compact nonsingular toric varieties in this way: Unless a_j's satisfy certain conditions, $n_s + n_2 + a_1 n_1 = O$ may not be satisfied. Neither is there any guarantee that n_{s+1}, determined by $n_{s-1} + n_{s+1} + a_s n_s = O$, coincides with n_1. Even if this is the case, $n_1, n_2, \ldots, n_s, n_{s+1} = n_1$ may go around the origin more than once. Theorem 1.28, (3) gives the following characterization for weighted circular graphs arising from two-dimensional compact nonsingular toric varieties:

Corollary 1.29. *The set of isomorphism classes of two-dimensional compact nonsingular toric varieties is in one-to-one correspondence with the set of weighted circular graphs of the following form* (cf. Fig. 1.17, (i), (ii), (iii)):

(i) *the circular graph having three vertices with* $1, 1, 1$ *as weights*;

(ii) *the circular graph having four vertices with weights* $a, 0, -a, 0$ *in this order, where* a *is a nonnegative integer*;

(iii) *the weighted circular graphs with* $s \geq 5$ *vertices which we obtain from those with* $s-1$ *vertices by adding a vertex of weight* -1 *and subtracting one from the weight of each of the two adjacent vertices.*

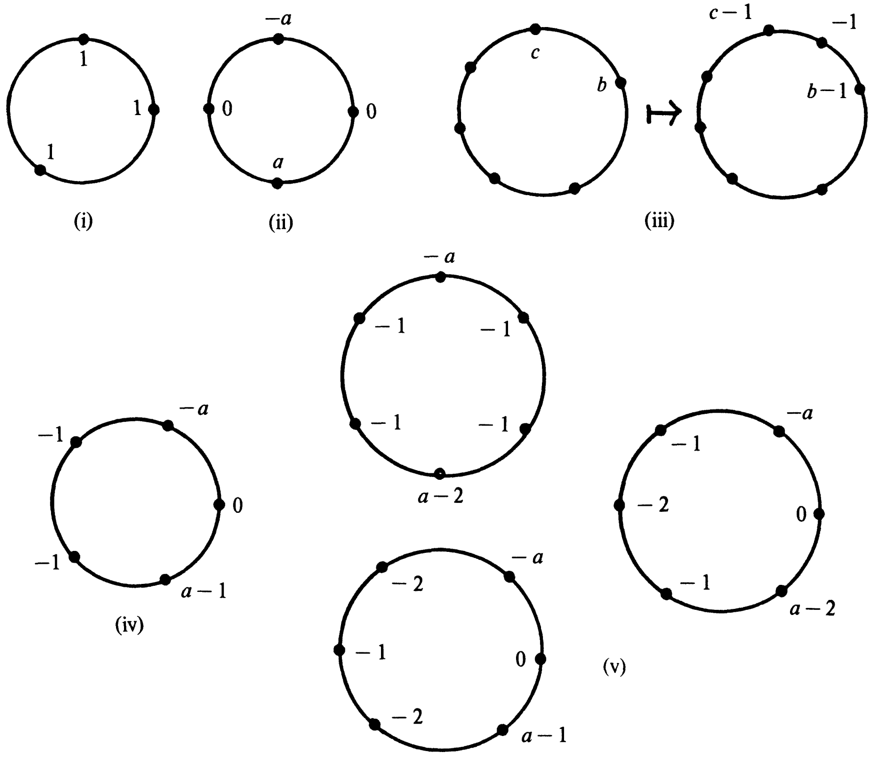


Fig. 1.17

Remark. A sequence $a_1, a_2, \ldots, a_s$ of integers in this order appear in a weighted circular graph arising from a two-dimensional compact nonsingular toric variety if and only if the following conditions are satisfied:

(1) $$\sum_{j=1}^{s} a_j = 12 - 3s$$

and

$$(2) \qquad \begin{pmatrix} 0 & -1 \\ 1 & -a_s \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & -a_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -a_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

The necessity follows from Corollary 1.29 and the equality (*) rewritten as

$$(n_{j+1}, n_j) \begin{pmatrix} 0 & -1 \\ 1 & -a_j \end{pmatrix} = (n_j, n_{j-1}) .$$

The sufficiency can be shown exactly as in Corollary 1.32 for the case of dimension three.

Illustrated in Fig. 1.17, (iv) and (v) are the weighted circular graphs with five or six vertices arising from two-dimensional compact nonsingular toric varieties, where a is an integer and where we ignore the orientation of the cycles.

Much more involved is the birational geometry of toric varieties in dimension $r=3$. Concerning the toric variants of our problems at the beginning of this section, no general result seems to be known except the following affirmative answer to the toric variant of (ii′). Later in this section, we mention partial answers to other problems.

Danilov's Factorization Theorem (cf. [D2]). (i) *Suppose a finite nonsingular fan Δ in $N \cong \mathbb{Z}^3$ is a subdivision of another nonsingular fan Δ_0. Then there exist finite nonsingular fans $\Delta_1, \Delta_2, \ldots, \Delta_{2k} = \Delta$ in N such that Δ_j for each odd j is obtained from Δ_{j-1} and from Δ_{j+1}, respectively, by a finite succession of star subdivisions of the form given in Corollary 1.27,* (b) *or* (c).

(ii) *For any given pair X, X' of three-dimensional compact nonsingular toric varieties, there exist compact nonsingular toric varieties $X_0 = X, X_1, \ldots, X_{2l} = X'$ together with finite successions $X_j \to X_{j-1}$ and $X_j \to X_{j+1}$, for each odd $1 \leq j \leq 2l-1$, of equivariant blowing-ups along T_N-invariant closed irreducible submanifolds.*

To show (ii), we let $X = T_N \operatorname{emb}(\Delta)$ and $X' = T_N \operatorname{emb}(\Delta')$. Then as we stated earlier, there exists a nonsingular subdivision Δ'' common to Δ and Δ'. It then suffices to apply (i) to the pairs Δ, Δ' as well as Δ, Δ''.

We here give a gist of the proof for (i) and leave the details to [D2]. Let us denote by $\operatorname{Reg}(\Delta_0)$ the set of all finite nonsingular subdivisions of a given finite nonsingular fan Δ_0. For Δ and Δ' in $\operatorname{Reg}(\Delta_0)$, we denote $\Delta' \to \Delta$ if Δ' is obtained from Δ by a finite succession of star subdivisions of the form given in Corollary 1.27, (b) or (c). In this way, $\operatorname{Reg}(\Delta_0)$ becomes an oriented graph. (i) certainly follows from the connectedness of this graph, when the orientation is disregarded.

(A) To show the connectedness of $\operatorname{Reg}(\Delta_0)$, we need to embed its vertex set into a larger set $\operatorname{Term}(\Delta_0)$ which provides more room for maneuver. A finite fan Δ in N is said to be *terminal* if each $\sigma \in \Delta$ is a terminal cone in the sense of Sect. 1.6. The set $\operatorname{Term}(\Delta_0)$ of finite terminal subdivisions of Δ_0 obviously contains $\operatorname{Reg}(\Delta_0)$.

(B) Here is what Danilov first shows. For any Δ' in $\operatorname{Term}(\Delta_0)$, there exist $\Delta'_0 = \Delta_0, \Delta'_1, \ldots, \Delta'_l = \Delta'$ in $\operatorname{Term}(\Delta_0)$ such that Δ'_j for each $1 \leq j \leq l$ is obtained from Δ'_{j-1} by one of the following transfromations or their inverses:

Type I *transformation.* As in Fig. 1.18, (I), suppose Δ in Term (Δ_0) contains three cones

$$\sigma_{jk} = \mathbb{R}_{\geq 0}n + \mathbb{R}_{\geq 0}n_j + \mathbb{R}_{\geq 0}n_k \quad \text{for} \quad 1 \leq j < k \leq 3$$

of dimension three, where n, n_1, n_2, n_3 are primitive elements in N and where O and n are separated by the plane in $N_{\mathbb{R}}$ which pass through n_1, n_2, n_3. In this case, $\sigma := \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}n_3$ is a three-dimensional terminal cone. We then transform Δ to

$$\Delta' := (\Delta \setminus \{\text{the faces of } \sigma_{12}, \sigma_{23}, \sigma_{13}\}) \cup \{\text{the faces of } \sigma\} \ .$$

(This may no longer be a subdivision of Δ_0, hence it may not belong to Term (Δ_0).)

Type II *transformation.* As in Fig. 1.18, (II), suppose Δ in Term (Δ_0) contains four cones

$$\sigma'_j = \mathbb{R}_{\geq 0}n + \mathbb{R}_{\geq 0}n' + \mathbb{R}_{\geq 0}n_j \quad \text{and} \quad \sigma''_j = \mathbb{R}_{\geq 0}n + \mathbb{R}_{\geq 0}n'' + \mathbb{R}_{\geq 0}n_j$$

for $j = 1, 2$ of dimension three, where n, n', n'', n_1, n_2 are primitive elements in N satisfying $n = n_1 + n_2$. In this case,

$$\sigma' := \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}n' \quad \text{and} \quad \sigma'' := \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}n''$$

turn out to be nonsingular. We then transform Δ to

$$\Delta' := (\Delta \setminus \{\text{the faces of } \sigma'_1, \sigma'_2, \sigma''_1, \sigma''_2\}) \cup \{\text{the faces of } \sigma', \sigma''\} \ .$$

Thus Δ is the star subdivision of Δ' with respect to $\mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2$ in the sense of Corollary 1.27, (c).

Type II′ *transformation.* As in Fig. 1.18, (II′), suppose Δ in Term (Δ_0) contains two cones

$$\sigma'_j = \mathbb{R}_{\geq 0}n + \mathbb{R}_{\geq 0}n' + \mathbb{R}_{\geq 0}n_j \quad \text{for} \quad j = 1, 2$$

of dimension three, where n, n', n_1, n_2 are primitive elements in N satisfying $n = n_1 + n_2$ and, moreover, $\mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2$ is on the boundary of the support $|\Delta|$. In this case, $\sigma' := \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}n'$ is necessarily nonsingular. We then transform Δ to

$$\Delta' := (\Delta \setminus \{\text{the faces of } \sigma'_1, \sigma'_2\}) \cup \{\text{the faces of } \sigma'\} \ .$$

Type III *transformation.* As in Fig. 1.18, (III), suppose Δ in Term (Δ_0) contains two cones

$$\sigma' = \mathbb{R}_{\geq 0}n' + \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 \quad \text{and} \quad \sigma'' = \mathbb{R}_{\geq 0}n'' + \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2$$

of dimension three, where n', n'', n_1, n_2 are primitive elements in N such that the two tetrahedra with respective vertex sets $\{O, n', n_1, n_2\}$ and $\{O, n'', n_1, n_2\}$ have convex union. In this case, $\sigma_1 := \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n' + \mathbb{R}_{\geq 0}n''$ and $\sigma_2 := \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}n' + \mathbb{R}_{\geq 0}n''$ are both terminal. We then transform Δ to

$$\Delta' := (\Delta \setminus \{\text{the faces of } \sigma', \sigma''\}) \cup \{\text{the faces of } \sigma_1, \sigma_2\} \ .$$

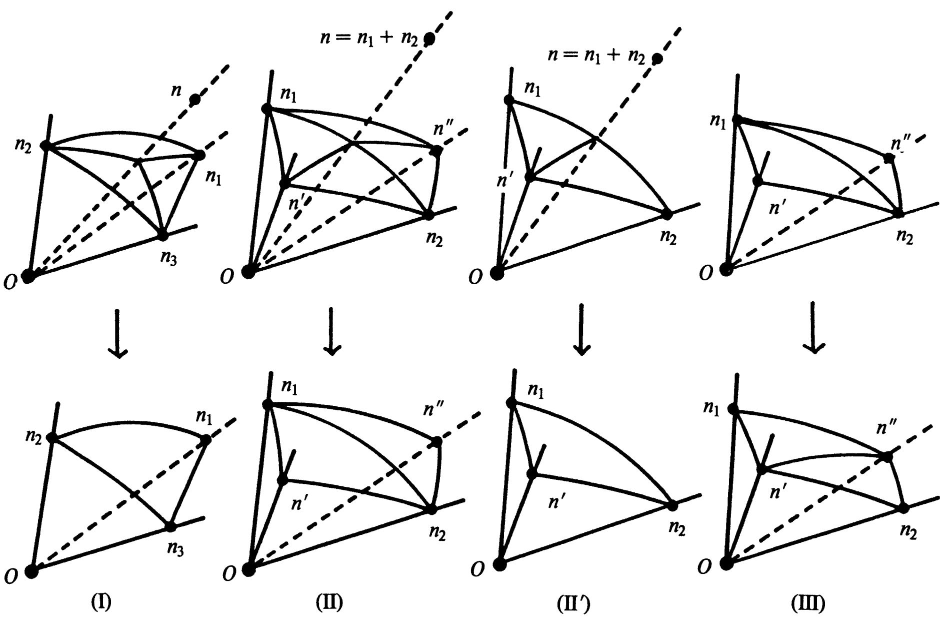


Fig. 1.18

(C) For each $\Delta' \in \operatorname{Term}(\Delta_0)$, Danilov gives a recipe for constructing a nonsingular subdivision $\tilde{\Delta}' \in \operatorname{Reg}(\Delta_0)$ of Δ'. By the very recipe, such $\tilde{\Delta}'$ is determined by Δ' only up to elementary transformations defined below in Proposition 1.30.

(D) Suppose $\Delta'' \in \operatorname{Term}(\Delta_0)$ is obtained from $\Delta' \in \operatorname{Term}(\Delta_0)$ by a finite succession of transformations of types (I), (II), (II'), (III) in (B) or their inverses. Let $\tilde{\Delta}'$ and $\tilde{\Delta}''$ in $\operatorname{Reg}(\Delta_0)$ be those obtained respectively from Δ' and Δ'' according to the recipe in (C). Danilov shows that $\tilde{\Delta}'$ and $\tilde{\Delta}''$ as vertices in the graph $\operatorname{Reg}(\Delta_0)$, can be connected by a path. Namely, one of them is obtained from the other by a finite succession of star subdivisions or inverse operations. In his proof, he uses the following generalization of the terminal lemma in Sect. 1.6:

The Terminal Lemma of White-Frumkin (cf. [D2]). *Suppose P is a convex polytope (i.e., compact convex polyhedron as in Theorem A.12) in $N_{\mathbb{R}}$ such that $N \cap P$ coincides with the vertex set of P. Then there exist a $\mathbb{Z}$-module homomorphism $m: N \to \mathbb{Z}$ and $a \in \mathbb{Z}$ such that the scalar extension $m: N_{\mathbb{R}} \to \mathbb{R}$ satisfies*

$$a \leq \langle m, y \rangle \leq a+1 \quad \textit{for all } y \in P\ .$$

(E) After these preparatory steps (A) through (D), Danilov shows that any given $\Delta \in \operatorname{Reg}(\Delta_0)$ can be joined with Δ_0 by a path in the graph $\operatorname{Reg}(\Delta_0)$ as follows: If we regard Δ, Δ_0 as elements of $\operatorname{Term}(\Delta_0)$, then by (A) there exist $\Delta'_0 = \Delta_0, \Delta'_1, \ldots, \Delta'_l = \Delta$ in $\operatorname{Term}(\Delta_0)$ such that Δ'_j for each $1 \leq j \leq l$ is obtained from Δ'_{j-1} by one of the transformations of types (I), (II), (II'), (III) or their inverses.

Danilov calls $\Delta_1', \ldots, \Delta_{l-1}'$ "milestones" on a path connecting Δ_0 and Δ. (In view of recent developments, it seems to be more natural in higher dimensional birational geometry to take into account varieties with mild singularities such as terminal and canonical singularities.) Danilov then considers $\tilde{\Delta}_j'$ in Reg (Δ_0) constructed for each Δ_j' according to the recipe in (C). Then by (D), we can connect $\tilde{\Delta}_{j-1}'$ and $\tilde{\Delta}_j'$ by a path in Reg (Δ_0), thereby completing the proof for (i).

The elementary transformations below not only play important rôles in the proof for (C) and (D) above, but are of independent interest in the birational geometry of three-dimensional nonsingular varieties (cf. Sect. 2.3, Proposition 1.31, (6) and the remark (iv) immediately after Corollary 1.32).

As in Sect. 2.1, we denote by $\Delta(j) := \{\sigma \in \Delta;\ \dim \sigma = j\}$ the set of j-dimensional cones in a fan Δ for each integer j. In particular for each $\varrho \in \Delta(1)$, there clearly exists a unique primitive element $n(\varrho) \in N$ such that $\varrho = \mathbb{R}_{\geq 0} n(\varrho)$.

Proposition 1.30 (Danilov [D2, Propositions 1 and 2]). *Let* $N \cong \mathbb{Z}^3$.

(i) *Suppose a nonsingular fan* Δ *in* N *contains two cones* $\sigma' = \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n'$ *and* $\sigma'' = \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n''$ *of dimension three adjacent along* $\tau = \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2$, *where* $\{n_1, n_2, n'\}$ *and* $\{n_1, n_2, n''\}$ *are* $\mathbb{Z}$*-bases of* N. *If* $n_1 + n_2 = n' + n''$ *holds (hence* n_1, n_2, n', n'' *are coplanar), then* $\sigma_1 := \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n' + \mathbb{R}_{\geq 0} n''$ *and* $\sigma_2 := \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n' + \mathbb{R}_{\geq 0} n''$ *are three-dimensional cones adjacent along the two-dimensional cone* $\tau' := \mathbb{R}_{\geq 0} n' + \mathbb{R}_{\geq 0} n''$ *so that*

$$\Delta' := (\Delta \setminus \{\sigma', \sigma'', \tau\}) \cup \{\sigma_1, \sigma_2, \tau'\}$$

is a nonsingular fan. Moreover, we have $\Delta^*(\tau) = \Delta'^*(\tau')$, *i.e., the star subdivision of* Δ *with respect to* τ *coincides with that of* Δ' *with respect to* τ'. *In this case,* Δ' *is said to be obtained from* Δ *by an* elementary transformation (cf. Fig. 1.19).

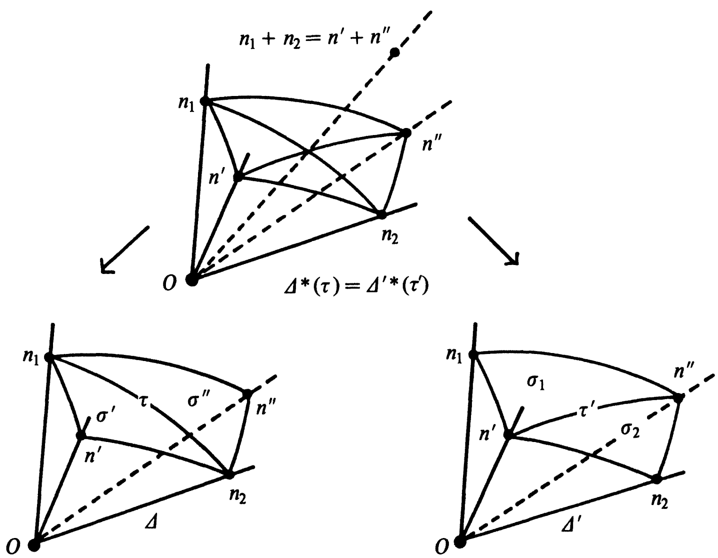


Fig. 1.19

(ii) *Let Δ' and Δ'' be nonsingular fans in N satisfying $|\Delta'|=|\Delta''|$. Suppose there exists m_0 in M such that*

$$\langle m_0, n(\varrho')\rangle = 1 \quad \textit{and} \quad \langle m_0, n(\varrho'')\rangle = 1$$

for all $\varrho' \in \Delta'(1)$ and all $\varrho'' \in \Delta''(1)$. Then Δ'' is obtained from Δ' by a finite succession of elementary transformations as defined in (i) *above.*

(iii) *Let Δ be a finite nonsingular fan in N. Each three-dimensional cone σ in Δ can be written as $\sigma = \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}n_3$ for a $\mathbb{Z}$-basis $\{n_1, n_2, n_3\}$ of N. The* shed *$S(\sigma)$ of σ is then defined to be the tetrahedron with vertices O, n_1, n_2, n_3. The* shed *of Δ is defined to be the union*

$$S(\Delta) := \bigcup_{\sigma \in \Delta(3)} S(\sigma) \ .$$

If finite nonsingular fans Δ' and Δ'' are subdivisions of a common nonsingular fan Δ_0 and if $S(\Delta') = S(\Delta'')$ holds, then Δ'' is obtained from Δ' by a finite succession of elementary transformations as defined in (i) *above.*

Proof. (i) is obvious. To derive (iii) from (ii), we proceed as follows: For each $\sigma_0 \in \Delta_0$ and each two-dimensional face F of the polyhedron $\sigma_0 \cap S(\Delta') = \sigma_0 \cap S(\Delta'')$ such that $O \notin F$, we apply (ii) to the fans $\{\sigma' \in \Delta';\, S(\sigma')$ intersects $F\}$ and $\{\sigma'' \in \Delta'';\, S(\sigma'')$ intersects $F\}$, which satisfy the requirement in (ii).

To show (ii), consider the affine plane $H := \{y \in N_{\mathbb{R}};\, \langle m_0, y\rangle = 1\}$ and the polygon $P := |\Delta'| \cap H = |\Delta''| \cap H$ on it. By assumption, $\bar{\Delta}' := \{\sigma' \cap H;\, \sigma' \in \Delta'\}$ and $\bar{\Delta}'' := \{\sigma'' \cap H;\, \sigma'' \in \Delta''\}$ are triangulations of P by means of triangles with none of their points other than the vertices belonging to the lattice $N \cap H$. For a fixed $n_0 \in N \cap H$, we have $H = n_0 + \{m_0\}^{\perp}$ and $N \cap H = n_0 + (N \cap \{m_0\}^{\perp})$. Thus we can introduce a measure on H by parallel translation of the Lebesgue measure on $\{m_0\}^{\perp}$ so normalized that the parallelogram spanned by a $\mathbb{Z}$-basis of $N \cap \{m_0\}^{\perp}$ has measure one. Then for n_1, n_2, n_3 in $N \cap H$, the following are obviously equivalent: (a) $\{n_1, n_2, n_3\}$ is a $\mathbb{Z}$-basis of N; (b) the triangle (n_1, n_2, n_3) with vertices n_1, n_2, n_3 has measure $1/2$; (c) the three vertices n_1, n_2, n_3 are the only points in the triangle (n_1, n_2, n_3) belonging to the lattice $N \cap H$.

On the other hand, the elementary transformation in (i) has the following form on H: Let $Q = (n_1, n_2, n', n'')$ be a convex quadrilateral on H with vertices $n_1, n_2, n', n'' \in N \cap H$. Suppose that the vertices are the only points of Q belonging to the lattice $N \cap H$ and that the middle point $(n_1 + n_2)/2$ of the diagonal (n_1, n_2) joining n_1 and n_2 coincides with the middle point $(n' + n'')/2$ of the diagonal (n', n''). Then the elementary transformation in question replaces the triangulation of Q via the diagonal (n_1, n_2) by that via the diagonal (n', n'').

By induction on the measure of P, we show that the triangulation $\bar{\Delta}''$ of P is obtained from $\bar{\Delta}'$ by a finite succession of elementary transformations of this form.

If P has measure zero, then we certainly have $\bar{\Delta}' = \bar{\Delta}''$, hence $\Delta' = \Delta''$. Otherwise, choose a pair $n_1, n_2 \in N \cap H$ of adjacent lattice points on the boundary of P. Clearly, the line segment (n_1, n_2) belongs to both $\bar{\Delta}'$ and $\bar{\Delta}''$. Moreover, $n', n'' \in N \cap H$ are uniquely determined in such a way that the triangle (n_1, n_2, n') (resp. (n_1, n_2, n'')) belongs to $\bar{\Delta}'$ (resp. $\bar{\Delta}''$).

If $n' = n''$, then we are done by the induction hypothesis applied to $\bar{\Delta}' \setminus \{(n_1, n_2, n')\}$ and $\bar{\Delta}'' \setminus \{(n_1, n_2, n'')\}$. Otherwise, there exists an integer a such that $n''-n'=a(n_2-n_1)$, since both triangles (n_1,n_2,n') and (n_1,n_2,n'') have measure 1/2. Without loss of generality, we may assume $a\geq 1$.

If $a=1$ and if the line segment (n',n'') is an edge of both $\bar{\Delta}'$ and $\bar{\Delta}''$, then the middle points of (n_1,n'') and of (n_2,n') coincide. Thus inside the convex quadrilateral $Q:=(n',n_1,n_2,n'')$, an elementary transformation is applicable to the triangulations $\bar{\Delta}'$ and $\bar{\Delta}''$ as in Fig. 1.20a. We are done by the induction hypothesis applied to $\bar{\Delta}'\setminus\{(n_1,n_2,n'),(n'',n',n_2)\}$ and $\bar{\Delta}''\setminus\{(n_1,n_2,n''),(n_1,n',n'')\}$.

As Ewald [E7] pointed out, we need to apply elementary transformations before dealing with the general case when $a\geq 2$ or when the line segment (n',n'') is not an edge of $\bar{\Delta}'$ or $\bar{\Delta}''$ (cf. Fig. 1.20b).

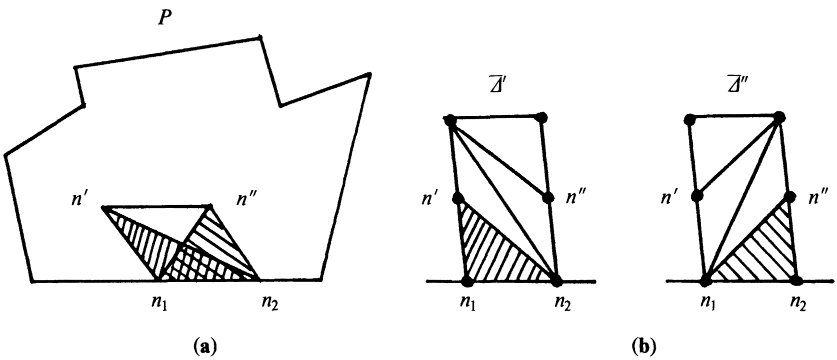


Fig. 1.20

What we have just seen is closely related to the proof by Reid [R4] of the *flip conjecture* in the case of toric morphisms. We refer the reader to Kawamata-Matsuda-Matsuki [KMM] for the current state of the minimal model problem in general. We briefly touch on it in Sect. 2.5. There is an attempt on our problem (ii) by Ewald [E7], who reduces (ii) to the projective case.

Generalizing the notion of weighted dual graphs in the two-dimensional case, we now introduce a combinatorial device to describe three-dimensional compact nonsingular toric varieties up to isomorphism. We will then be able to classify three-dimensional compact nonsingular toric varieties with the Picard number ≤ 5, getting a partial answer to the toric variant of our problem (iii) as well as a counterexample to our problem (i) (cf. the remark immediately before Proposition 2.17 in Chap. 2). We do not reproduce here the long and tedious classification process, carried out jointly with K. Miyake and improved by O. Nagaya. The details can be found in [MO, § 9]. Instead, we describe the result itself in as easily understandable a manner as possible.

Batyrev [B2] and Watanabe-Watanabe [WW] classified toric Fano threefolds using our classification (cf. Sect. 2.3).

As a result of our classification, we obtain several examples of projective as well as nonprojective threefolds, which indicate some of the possible complications in three-dimensional birational geometry (cf. Sect. 2.3 and [MO, p. 80]).

Kleinschmidt [K7] deals with higher dimensional analogues of some of our results.

As we see in Chap. 4, Tsuchihashi [T4], [T5], [T6] extended our method here to define higher dimensional analogues of cusp singularities and periodic continued fractions.

We now introduce the notion of doubly $\mathbb{Z}$-weighted triangulations of a 2-sphere. They combinatorially describe compact nonsingular toric varieties in dimension three up to isomorphism, exactly as weighted circular graphs did in the two-dimensional case (cf. Corollary 1.29).

For this purpose, let us first consider the case $N \cong \mathbb{Z}^r$ in general. We can obviously identify the set of half lines issuing from the origin O of $N_{\mathbb{R}} \cong \mathbb{R}^r$ with

$$S_N := (N_{\mathbb{R}} \setminus \{O\})/\mathbb{R}_{>0} ,$$

which is naturally homeomorphic to an $(r-1)$-sphere. Let

$$\pi : N_{\mathbb{R}} \setminus \{O\} \twoheadrightarrow S_N$$

be the projection. Clearly, π induces a bijection from the set of primitive elements in N to a dense subset in S_N. For brevity, let us call $\pi(n)$ a *rational point* of S_N corresponding to a primitive element $n \in N$, while n is called the *N-weight* of the rational point $\pi(n)$. If $\sigma := \mathbb{R}_{\geq 0}n_1 + \ldots + \mathbb{R}_{\geq 0}n_s$ is a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ for the irredundant set $\{n_1, \ldots, n_s\}$ of primitive elements in N, then $\pi(\sigma \setminus \{O\})$ is a convex spherical cell in S_N with rational points $\pi(n_1), \ldots, \pi(n_s)$ as vertices. Thus for a fan Δ in N, we get a convex spherical cell decomposition $\{\pi(\sigma \setminus \{O\}); \sigma \in \Delta\}$ of $\pi(|\Delta| \setminus \{O\}) \subset S_N$. In particular, a finite complete fan in N gives rise to a finite convex spherical cell decomposition of the sphere S_N.

If a finite complete fan Δ in N is nonsingular, the corresponding cell decomposition of S_N is simplicial, i.e., is a triangulation of S_N. Moreover, for each r-dimensional $\sigma \in \Delta$, the corresponding spherical $(r-1)$-simplex $\pi(\sigma \setminus \{O\})$ has vertices whose N-weights $n_1, \ldots, n_r$ form a $\mathbb{Z}$-basis of N. For each $\tau \in \Delta$ with $\dim \tau = r-1$, there exist exactly two r-dimensional $\sigma', \sigma'' \in \Delta$ such that $\tau = \sigma' \cap \sigma''$. In this case, the sets $\{n', n_2, \ldots, n_r\}$ and $\{n'', n_2, \ldots, n_r\}$ of N-weights for the vertices of $\pi(\sigma' \setminus \{O\})$ and $\pi(\sigma'' \setminus \{O\})$, respectively, are $\mathbb{Z}$-bases of N. Furthermore, we have

$$n' + n'' + \sum_{j=2}^{r} a_j n_j = O$$

for $a_2, \ldots, a_r \in \mathbb{Z}$ uniquely determined by τ. These integers again have the following geometric significance, as we see at the end of Sect. 2.2: For $\varrho' := \mathbb{R}_{\geq 0}n'$, $\varrho'' := \mathbb{R}_{\geq 0}n''$, $\varrho_j := \mathbb{R}_{\geq 0}n_j$ in $\Delta(1)$, let $V(\varrho')$, $V(\varrho'')$, $V(\varrho_j)$ be the corresponding T_N-invariant irreducible Cartier divisors. Then in terms of r-ple intersection numbers, we have

$$a_j = (V(\varrho_j) \cdot V(\varrho_2) \cdot \ldots \cdot V(\varrho_r)) \quad \text{for} \quad 2 \leq j \leq r .$$

We are now ready to come back to the case $r=3$ and define doubly $\mathbb{Z}$-weighted triangulations of a 2-sphere.

A finite nonsingular complete fan Δ in $N \cong \mathbb{Z}^3$ gives rise to a finite triangulation $\{\pi(\sigma \setminus \{O\}); \sigma \in \Delta\}$ of the 2-sphere S_N by spherical triangles. The vertices of each spherical triangle are rational points, and the corresponding N-weights form a

$\mathbb{Z}$-basis of N. Each edge (i.e., spherical 1-simplex) $\pi(n_2)\pi(n_3)$ is the intersection of exactly two spherical triangles $\pi(n')\pi(n_2)\pi(n_3)$ and $\pi(n'')\pi(n_2)\pi(n_3)$. There exist unique $a_2, a_3 \in \mathbb{Z}$ such that

$$n' + n'' + a_2 n_2 + a_3 n_3 = O \ .$$

We then endow the edge $\pi(n_2)\pi(n_3)$ with the *double $\mathbb{Z}$-weight* a_2, a_3, where we place a_2 (resp. a_3) on the side of the vertex $\pi(n_2)$ (resp. $\pi(n_3)$) as in Fig. 1.21. If we let $\varrho' := \mathbb{R}_{\geq 0} n'$, $\varrho'' := \mathbb{R}_{\geq 0} n''$, $\varrho_2 := \mathbb{R}_{\geq 0} n_2$ and $\varrho_3 := \mathbb{R}_{\geq 0} n_3$, then we have

$$a_2 = (V(\varrho_2)^2 \,.\, V(\varrho_3)) \quad \text{and} \quad a_3 = (V(\varrho_2) \,.\, V(\varrho_3)^2) \ .$$

Consequently, the self-intersection numbers of the rational curve $V(\varrho_2 + \varrho_3) \cong \mathbb{P}_1(\mathbb{C})$ in the two-dimensional nonsingular complete toric varieties $V(\varrho_2)$ and $V(\varrho_3)$ are given respectively by

$$(V(\varrho_2 + \varrho_3)^2)_{V(\varrho_2)} = a_3 \quad \text{and} \quad (V(\varrho_2 + \varrho_3)^2)_{V(\varrho_3)} = a_2 \ .$$

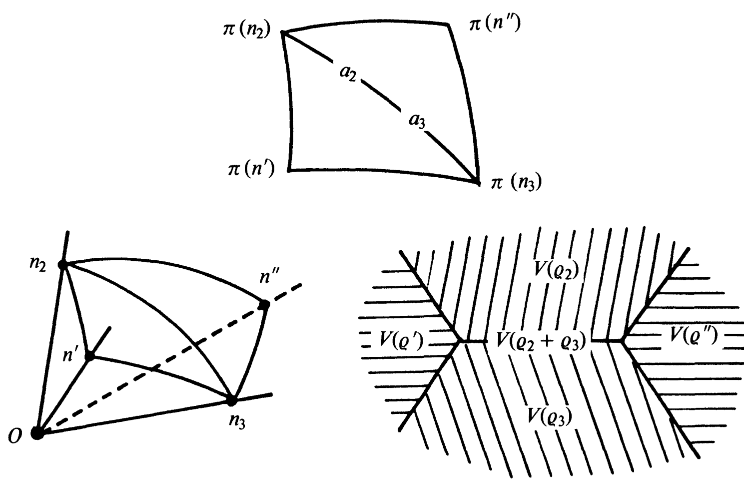


Fig. 1.21

As in Fig. 1.22, consider the *star* of a vertex $\pi(n)$ of the triangulation. Its boundary is called the *link* of $\pi(n)$. Let $\pi(n_1), \pi(n_2), \ldots, \pi(n_v)$ in this order be the vertices on the link of $\pi(n)$, where we choose $n, n_1, \ldots, n_v$ to be primitive elements of N. In particular, the vertex $\pi(n)$ has the *valency* v in the sense of Sect. A.5. For each $1 \leq j \leq v$, let us denote the double $\mathbb{Z}$-weight for the edge $\pi(n)\pi(n_j)$ by a_j on the side of the vertex $\pi(n_j)$ and b_j on the side of the vertex $\pi(n)$. Thus we have

$$(**) \qquad n_{j+1} + n_{j-1} + a_j n_j + b_j n = O \quad \text{for} \quad 1 \leq j \leq v \ ,$$

where we let $n_0 := n_v$ and $n_{v+1} := n_1$. Corollary 1.32 below characterizes the integers $a_1, \ldots, a_v, b_1, \ldots, b_v$ which appear in this way as weights in the star of a vertex.

In the above situation, consider the two-dimensional compact nonsingular toric subvariety $V(\mathbb{R}_{\geq 0} n)$ in $X := T_N\,\mathrm{emb}(\Delta)$. The rational curves $C_j := V(\mathbb{R}_{\geq 0} n + \mathbb{R}_{\geq 0} n_j)$

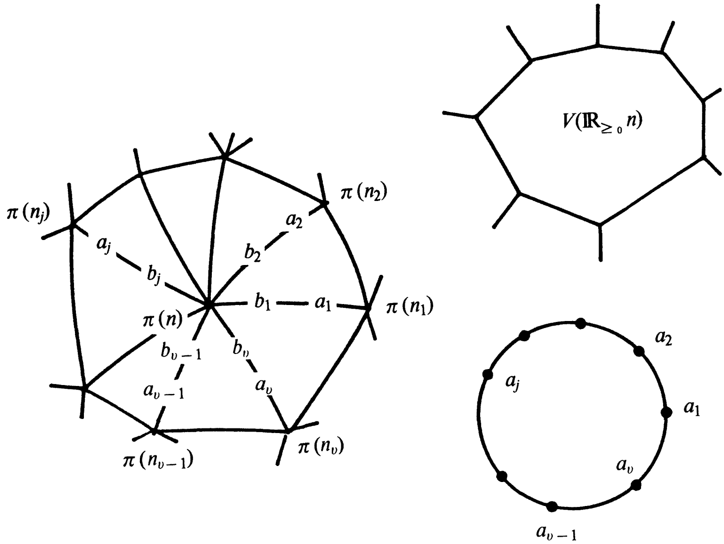


Fig. 1.22

$\cong \mathbb{P}_1(\mathbb{C})$ for $1 \leq j \leq v$ form a cycle on the surface $V(\mathbb{R}_{\geq 0}n)$. Since $n_{j+1}+n_{j-1}+a_j n_j \equiv O(\text{mod } \mathbb{Z}n)$ for $1 \leq j \leq v$, we see that C_j has the self-intersection number $(C_j^2)=a_j$ on $V(\mathbb{R}_{\geq 0}n)$. Thus $a_1, \ldots, a_v$, $a_{v+1}=a_1$ necessarily satisfy the conditions in Corollary 1.29 for weights in a weighted circular graph, hence (1), (2) in the remark immediately after that. For brevity, we define the $\mathbb{Z}$-*weighted link* of the vertex $\pi(n)$ to be the link with the weights $a_1, \ldots, a_v$ attached to the vertices $\pi(n_1), \ldots, \pi(n_v)$. It is thus a weighted dual graph we considered earlier in this section.

The following is immediate from what we have just seen:

Proposition 1.31 ([MO, Corollary 9.2]). *Consider a triangulation of the 2-sphere S_N arising from a finite nonsingular complete fan Δ in $N \cong \mathbb{Z}^3$. For a vertex of valency v and with N-weight n, let $n_1, \ldots, n_v$ in this order be the N-weights of the vertices on the link of $\pi(n)$. For each $1 \leq j \leq v$, let the double $\mathbb{Z}$-weight of the edge $\pi(n)\pi(n_j)$ be a_j on the side of $\pi(n_j)$ and b_j on the side of $\pi(n)$.*

(1) *We necessarily have $v \geq 3$ and*

$$(**) \qquad n_{j+1}+n_{j-1}+a_j n_j+b_j n=O \quad \text{for} \quad 1 \leq j \leq v ,$$

where $n_0 := n_v$ and $n_{v+1} := n_1$. The $\mathbb{Z}$-weighted link of the vertex $\pi(n)$ with weights $a_1, \ldots, a_v$ satisfies the conditions in Corollary 1.29 for weighted circular graphs, hence (1), (2) *in the remark immediately after that.*

(2) *If $v \geq 4$, then there exist integers k, l satisfying $k \geq 1$ and $v \geq l \geq k+2$ such that n_k, n, n_l are $\mathbb{R}$-linearly dependent, hence $\pi(n_k)$, $\pi(n)$, $\pi(n_l)$ are on a great circle for the sphere S_N.*

(3) *For $v=3$, we necessarily have $a_1=a_2=a_3=1$ and $b_1=b_2=b_3$, hence $n_1+n_2+n_3+b_1 n=O$. In this case, the vertex $\pi(n)$ arises from a star subdivision in Corollary 1.27,* (b) *corresponding to an equivariant blowing-up if and only if $b_1=b_2=b_3=-1$.*

(4) *For* $v=4$, *we can renumber the vertices to obtain*

$$a_1=a_3=0\ ,\qquad a_4=-a_2\ ,\qquad b_1=b_3\ ,\qquad b_4=b_2-a_2b_1\ ,$$

hence $n_2+n_4+b_1n=O$. *In this case, the vertex* $\pi(n)$ *arises from a star subdivision in Corollary 1.27,* (c) *corresponding to an equivariant blowing-up if and only if* $b_1=b_3=-1$.

(5) *For* $v=5$, *we can renumber the vertices to obtain*

$$a_1=0\ ,\qquad a_2+a_5=-1\ ,\qquad a_3=a_4=-1\ ,$$

$$b_1=b_3+b_4\ ,\qquad a_2b_3+a_3b_2=a_4b_5+a_5b_4\ ,$$

hence $n_2+n_5+b_1n=O$.

(6) *The elementary transformation in Proposition 1.30,* (i) *is applicable to the edge* $\pi(n)\pi(n_2)$ *if and only if* $n_1+n_3=n_2+n$, *or equivalently,* $a_2=b_2=-1$. *In this case,* $\pi(n)\pi(n_2)$ *is replaced by the other diagonal* $\pi(n_1)\pi(n_3)$ *of the spherical quadrilateral* $\pi(n)\pi(n_1)\pi(n_2)\pi(n_3)$.

We are now led to the following two notions:

Definition. Let $\mathscr{T}$ be a finite combinatorial triangulation of a 2-sphere S^2.

(i) Let $N\cong\mathbb{Z}^3$. An *N-weighting* for $\mathscr{T}$ is an assignment of a primitive element of N to each vertex of $\mathscr{T}$ so that for each triangle of $\mathscr{T}$, the three primitive elements for the vertices form a $\mathbb{Z}$-basis of N. It is said to be *admissible*, if there exists an N-weight preserving combinatorial isomorphism from $\mathscr{T}$ to the N-weighted triangulation of S_N arising from a finite nonsingular complete fan Δ in the way we described earlier in this section.

(ii) A *double* $\mathbb{Z}$*-weighting* for $\mathscr{T}$ assigns to each of its edges a pair of integers, one on each side of the two vertices. It is said to be *admissible*, if there exists a double $\mathbb{Z}$-weight preserving combinatorial isomorphism from $\mathscr{T}$ to the doubly $\mathbb{Z}$-weighted triangulation of S_N arising from a finite nonsingular complete fan Δ in the way described above.

An admissible N-weighting for a finite combinatorial triangulation $\mathscr{T}$ of S^2 clearly determines an admissible double $\mathbb{Z}$-weighting for $\mathscr{T}$ by the equalities (**). Moreover, the double $\mathbb{Z}$-weighting remains unchanged even if we transform all the N-weights simultaneously by a $\mathbb{Z}$-automorphism of N. Conversely, an admissible double $\mathbb{Z}$-weighting for $\mathscr{T}$ determines an admissible N-weighting for $\mathscr{T}$ up to $\mathrm{Aut}_{\mathbb{Z}}(N)$. Indeed, fix a triangle in $\mathscr{T}$ and assign to its vertices, as N-weights, the members of a $\mathbb{Z}$-basis of N. Then the N-weights for the other vertices of $\mathscr{T}$ are successively determined by the given double $\mathbb{Z}$-weights on the edges of $\mathscr{T}$ via the equalities (**). A different $\mathbb{Z}$-basis at the start obviously gives rise to N-weights which are the simultaneous transforms by a $\mathbb{Z}$-automorphism of N.

Consequently, the classification up to isomorphism of three-dimensional compact nonsingular toric varieties amounts to the following:

(I) Combinatorial classification of finite triangulations of a 2-sphere S^2.

(II) The classification up to $\mathrm{Aut}_{\mathbb{Z}}(N)$ of admissible N-weightings for each of the triangulations in (I), or

(II′) the classification of admissible double $\mathbb{Z}$-weightings for each of the triangulations in (I).

As we see in Sect. A.5, (I) is already known for triangulations with eight or less vertices. In carrying out the classification of admissible weightings for each of them, it is convenient to use both (II) and (II′) simultaneously. (II) is more suitable toward the end of the case-by-case classification, while (II′) is convenient at the beginning of the classification as well as in stating the result itself as in Theorem 1.34 below.

The following is handy in deciding when a double $\mathbb{Z}$-weighting is admissible:

Corollary 1.32 (Tsuchihashi [T6]). *A double $\mathbb{Z}$-weighting for a finite combinatorial triangulation $\mathcal{T}$ of S^2 is admissible if and only if each of its vertices satisfies the* monodromy condition: *Let $P_1, \dots, P_v$ in this order be the vertices on the link of a v-valent vertex P of $\mathcal{T}$. For each $1 \leq j \leq v$, denote the double $\mathbb{Z}$-weight on the edge PP_j of $\mathcal{T}$ by a_j on the side of P_j and b_j on the side of P. Then the monodromy condition at P is*

(1)
$$\sum_{j=1}^{v} a_j = 12 - 3v$$

and

(2)
$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -a_v & 0 \\ 0 & -b_v & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 & 0 \\ 1 & -a_2 & 0 \\ 0 & -b_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & -a_1 & 0 \\ 0 & -b_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We may replace the condition (1) *by*
(1′) *$a_1, \dots, a_v$ in this order appear as weights in one of the weighted circular graphs in Corollary 1.29.*

Remark. In view of Proposition 1.31, the monodromy condition at a vertex P of valency $v \leq 5$ can be stated more explicitly as follows (cf. Fig. 1.23):

(i) For $v = 3$, the link of P has weights $1, 1, 1$, while the same integer b appears as the weights on the side of P for the three edges meeting at P. The vertex P arises from a star subdivision in Corollary 1.27, (b) corresponding to an equivariant blowing-up if and only if $b = -1$. In this case, the exceptional divisor is the projective plane $\mathbb{P}_2(\mathbb{C})$.

(ii) For $v = 4$, the link of P has weights $a, 0, -a, 0$ in this order for an integer a. The weights on the side of P for the corresponding edges are $c, b, c-ab, b$, respectively, for integers b, c. The vertex P arises from a star subdivision in Corollary 1.27, (c) corresponding to an equivariant blowing-up if and only if $b = -1$. In this case, the exceptional divisor is the Hirzebruch surface F_a.

(iii) For $v = 5$, the link of P has weights $a-1, 0, -a, -1, -1$ in this order for an integer a. The weights on the side of P for the corresponding edges are $b_1, b_4+b_5, b_1-ab_4-(a-1)b_5, b_4, b_5$, respectively, for integers b_1, b_4, b_5.

(iv) The elementary transformation in Proposition 1.30, (i) is applicable to an edge if and only if its double $\mathbb{Z}$-weight is $-1, -1$. In this case, the elementary transformation replaces the edge by the other diagonal of the quadrilateral determined by the edge. In this case, the projective line $\mathbb{P}_1(\mathbb{C})$ corresponding to the edge is first blown up to $\mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$, which is then blown down in the direction of the other ruling.

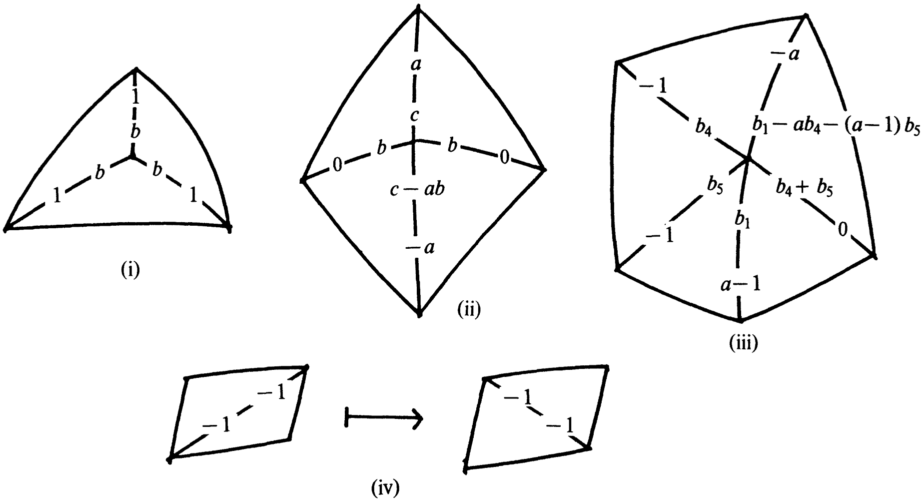


Fig. 1.23

Proof of Corollary 1.32. Let us first interpret, in terms of N-weights, the monodromy condition at a v-valent vertex P as in the statement. Choose and fix a $\mathbb{Z}$-basis $\{n, n_1, n_2\}$ of N and assign n, n_1, n_2 as N-weights to P, P_1, P_2, respectively. Then the double $\mathbb{Z}$-weights a_j, b_j successively determine the N-weights $n_3, n_4, \ldots, n_v$ for $P_3, P_4, \ldots, P_v$ by the equalities

$$(**)\qquad n_{j+1}+n_{j-1}+a_j n_j+b_j n=O$$

for $j=2, \ldots, v-1$. The same equalities $(**)$ for $j=v, v+1$ define n_{v+1} and n_{v+2} in N. Obviously, $\{n, n_2, n_3\}, \ldots, \{n, n_{v-1}, n_v\}$, $\{n,. n_v, n_{v+1}\}$ are $\mathbb{Z}$-bases of N.

The above equalities $(**)$ can be rewritten as

$$(n_{j+1}, n_j, n)\begin{pmatrix}0 & -1 & 0\\ 1 & -a_j & 0\\ 0 & -b_j & 1\end{pmatrix}=(n_j, n_{j-1}, n)$$

for $1 \leq j \leq v+1$. Hence both $n_{v+1}=n_1$ and $n_{v+2}=n_2$ hold if and only if the condition (2) in the statement is satisfied.

Suppose this is already the case. Then the rays $\mathbb{R}_{\geq 0}n_1$, $\mathbb{R}_{\geq 0}n_2, \ldots, \mathbb{R}_{\geq 0}n_v$, $\mathbb{R}_{\geq 0}n_{v+1}=\mathbb{R}_{\geq 0}n_1$ make a complete turn around the ray $\mathbb{R}_{\geq 0}n$. It is not hard to show that they actually go around exactly t times if and only if $\sum_{1\leq j\leq v} a_j=12t-3v$, in view of Corollary 1.29. (See also the remark after that.) Thus the condition (1) in the statement is satisfied if and only if the rays go around $\mathbb{R}_{\geq 0}n$ exactly once. It is also obvious that (1) and (1′) are equivalent under (2).

If the given double $\mathbb{Z}$-weighting is admissible, then the monodromy condition is satisfied at each vertex in view of what we have just seen.

Conversely, suppose the monodromy condition is satisfied at each vertex. Choose a triangle in $\mathscr{T}$ and assign the members of a $\mathbb{Z}$-basis of N as N-weights of its vertices. We can then determine the N-weights of all the other vertices in $\mathscr{T}$ by the equalities (**) involving double $\mathbb{Z}$-weights applied to sequences of successively pairwise adjacent triangles. Thanks to the monodromy condition at each vertex, the N-weights thus determined are independent of the particular sequences of triangles used. We can then determine a continuous map $\psi: S^2 \to S_N$ as follows: For each triangle $PP'P''$ in $\mathscr{T}$, let n, n', n'' be the respective N-weights for P, P', P''. We then define a homeomorphism ψ from the triangle $PP'P''$ on S^2 onto the spherical triangle $\pi(n)\pi(n')\pi(n'')$ on S_N by $\psi(P) := \pi(n)$, $\psi(P') := \pi(n')$, $\psi(P'') := \pi(n'')$ and by suitable interpolation. We can obviously choose the interpolation for all the triangles in $\mathscr{T}$ in such a way that the restriction to each edge coincides.

Let $P_1, \ldots, P_v$ in this order with N-weights $n_1, \ldots, n_v$ be the vertices on the link of a vertex P. Then by the monodromy condition at P, the continuous map ψ defined above obviously induces a homeomorphism from the star of P to the closed spherical polyhedron $\pi(n_1)\pi(n_2)\cdots\pi(n_v)$, which is a closed neighborhood of the point $\pi(n)$. Since S_N is simply connected, ψ is necessarily a homeomorphism from S^2 onto S_N.

Each triangle $PP'P''$ of $\mathscr{T}$ with respective N-weights n, n', n'' at the vertices determines a nonsingular cone $\mathbb{R}_{\geq 0}n + \mathbb{R}_{\geq 0}n' + \mathbb{R}_{\geq 0}n''$ in $N_{\mathbb{R}}$. In this way, $\mathscr{T}$ gives rise to a finite nonsingular complete fan Δ in N. Δ in turn determines a finite spherical triangulation of S_N, which is combinatorially isomorphic to $\mathscr{T}$ via ψ. Moreover, the N-weighting and the double $\mathbb{Z}$-weighting are obviously preserved under the combinatorial isomorphism. q.e.d.

To state the result of our classification, we need to recall fiber bundles related to toric varieties. The proof of the following is immediate:

Proposition 1.33 ([MO, Proposition 7.3]). *Consider a map of fans $h: (N, \Delta) \to (N', \Delta')$ and the corresponding equivariant holomorphic map $f: X := T_N \operatorname{emb}(\Delta) \to X' := T_{N'} \operatorname{emb}(\Delta')$ of toric varieties. Denote by N'' the kernel of the $\mathbb{Z}$-homomorphism $h: N \to N'$ and let Δ'' be a fan in N''. Then $f: X \to X'$ is an equivariant fiber bundle with $X'' := T_{N''} \operatorname{emb}(\Delta'')$ as typical fiber if and only if the following is satisfied: $h: N \to N'$ is surjective and there exists a subfan $\tilde{\Delta}' \subset \Delta$ such that h induces a homeomorphism $|\tilde{\Delta}'| \xrightarrow{\sim} |\Delta'|$ and $\Delta = \{\tilde{\sigma}' + \sigma''; \tilde{\sigma}' \in \tilde{\Delta}', \sigma'' \in \Delta''\}$. In this case, the open set $T_N \operatorname{emb}(\tilde{\Delta}') \subset X$ is a principal $T_{N''}$-bundle over X'.*

In general, we can associate a $\mathbb{P}_{l-1}(\mathbb{C})$-bundle $g: \mathbb{P}(E) \to Y$ to a vector bundle E of rank l on a complex analytic space Y. For each $y \in Y$, the fiber $g^{-1}(y)$ is the set of one-dimensional $\mathbb{C}$-subspaces in the fiber $E(y)$ of E over y. We have a tautological line bundle on $\mathbb{P}(E)$, which associates to each point of $\mathbb{P}(E)$ the corresponding one-dimensional $\mathbb{C}$-vector space. In [EGA], Grothendieck denotes E (resp. $\mathbb{P}(E)$) by $\mathbb{V}(\mathscr{E})$ (resp. $\mathbb{P}(\mathscr{E})$), where $\mathscr{E}$ is the $\mathscr{O}_Y$-dual of the sheaf of germs of sections of E and is a locally free $\mathscr{O}_Y$-module of rank l. Thus $E(y)$ is the $\mathbb{C}$-vector space dual to the fiber $\mathscr{E}_y \otimes \mathbb{C}(y)$ of $\mathscr{E}$, where $\mathbb{C}(y)$ is the residue field of the local ring $\mathscr{O}_{Y,y}$ of Y at y and the tensor product $\otimes$ is taken over $\mathscr{O}_{Y,y}$. The sheaf of germs of the tautological line bundle on $\mathbb{P}(E)$ is the tautological invertible sheaf denoted by $\mathscr{O}_{\mathbb{P}(E)}(1)$ or $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$.

In particular, let us consider on $X' = T_{N'} \operatorname{emb}(\Delta')$ equivariant line bundles $L_1, \ldots, L_l$ introduced later in Sect. 2.1. Then the $\mathbb{P}_{l-1}(\mathbb{C})$-bundle $\mathbb{P}(L_1 \oplus L_2 \oplus \ldots \oplus L_l)$ itself is a toric variety. The fan corresponding to it as in Proposition 1.33 can be described as follows: In the notation of Proposition 2.1 below, suppose Δ'-linear support functions $h_1, \ldots, h_l \in \mathrm{SF}(N', \Delta')$ give rise to the equivariant line bundles $L_1, \ldots, L_l$ on X', respectively. Introduce a $\mathbb{Z}$-module N'' with a $\mathbb{Z}$-basis $\{n_2'', n_3'', \ldots, n_l''\}$ and let $N := N' \oplus N''$ and $n_1'' := -(n_2'' + \ldots + n_l'')$. Denote by $\tilde{\sigma}'$ the image of each $\sigma' \in \Delta'$ under the $\mathbb{R}$-linear map $N'_{\mathbb{R}} \to N_{\mathbb{R}}$ which sends $y' \in N'_{\mathbb{R}}$ to $(y', -\sum_{1 \le j \le l} h_j(y') n_j'')$. We then let $\tilde{\Delta}' := \{\tilde{\sigma}' ; \sigma' \in \Delta'\}$.

On the other hand, let $\sigma_i'' := \mathbb{R}_{\ge 0} n_1'' + \ldots + \overset{i}{\vee} + \ldots + \mathbb{R}_{\ge 0} n_l''$ for each $1 \le i \le l$ and let Δ'' be the fan in N'' consisting of the faces of $\sigma_1'', \ldots, \sigma_l''$. Thus $T_{N''}\operatorname{emb}(\Delta'') = \mathbb{P}_{l-1}(\mathbb{C})$. It is obvious that for $\Delta := \{\tilde{\sigma}' + \sigma'' ; \tilde{\sigma}' \in \tilde{\Delta}', \sigma'' \in \Delta''\}$ we have

$$\mathbb{P}(L_1 \oplus \ldots \oplus L_l) = T_N \operatorname{emb}(\Delta) .$$

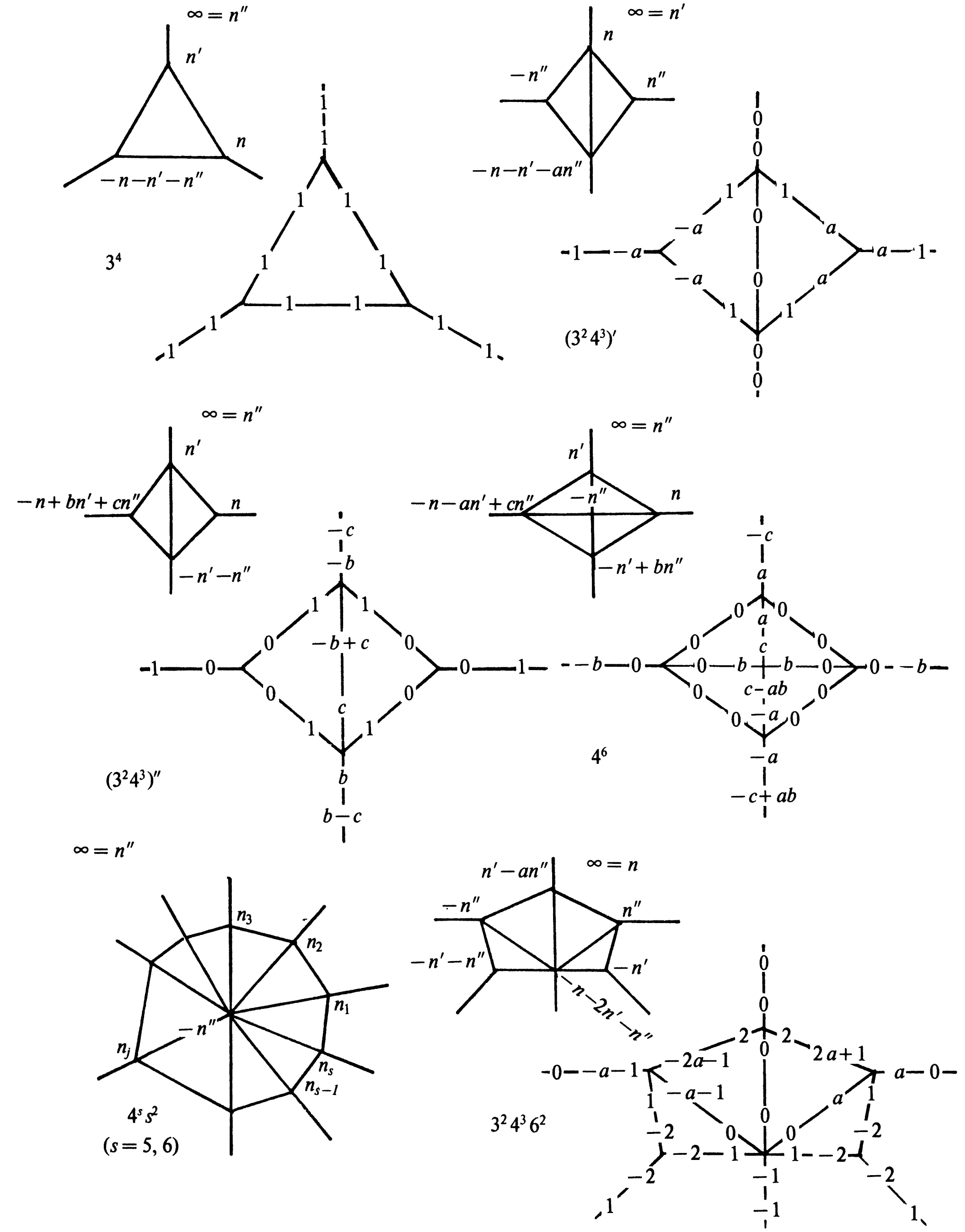
We are finally in a position to state, without proof, the classification of three-dimensional compact nonsingular toric varieties with the Picard number five or less which are minimal in the sense of equivariant blowing-ups. In view of Corollary 2.5 on the Picard number stated in Sect. 2.2, we are reduced to looking at each of the combinatorial triangulations of S^2 with eight or less vertices, which are classified in Sect. A.5. For each of them we classify the double $\mathbb{Z}$-weightings which satisfy the monodromy condition at each vertex as stated in Corollary 1.32. The remarks (i), (ii) immediately after Corollary 1.32 enable us to decide the minimality with respect to equivariant blowing-ups.

Theorem 1.34 ([MO, § 9]). *A three-dimensional compact nonsingular toric variety with the Picard number five or less which is minimal in the sense of equivariant blowing-ups is isomorphic to one of those listed below. In the list, we describe the admissible double $\mathbb{Z}$-weightings as well as the corresponding admissible N-weightings for $N \cong \mathbb{Z}^3$. Each triangulation of S^2 is represented as a planar graph with one vertex at infinity obtained from it by the stereographic projection from one of its vertices with the highest valency. We use the same labeling as that in Sect. A.5, except that we put the label in ( )′ and ( )″, when there exist more than one non-isomorphic admissible weightings for a triangulation. In the list, a, b, c, d are integers, while $\{n, n', n''\}$ is a $\mathbb{Z}$-basis of N.*

3^4: *the complex projective 3-space $\mathbb{P}_3(\mathbb{C})$;*
$(3^2 4^3)'$: *the $\mathbb{P}_1(\mathbb{C})$-bundle $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$ over $Y = \mathbb{P}_2(\mathbb{C})$ for $1 \ne a \ge 0$;*
$(3^2 4^3)''$: *the $\mathbb{P}_2(\mathbb{C})$-bundle $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(b) \oplus \mathcal{O}_Y(c))$ over $Y = \mathbb{P}_1(\mathbb{C})$;*
4^6: *the $\mathbb{P}_1(\mathbb{C})$-bundle $\mathbb{P}(\mathcal{O}_Y \oplus L)$ over a Hirzebruch surface $Y = F_a$ with $1 \ne a \ge 0$, where $L = \mathcal{O}_Y(bs + cf)$ for a fiber f of the projection $Y \to \mathbb{P}_1(\mathbb{C})$ and the minimal section s with the self-intersection number $s^2 = -a$;*
$4^s s^2$ *for* $s = 5, 6$: *a $\mathbb{P}_1(\mathbb{C})$-bundle $\mathbb{P}(\mathcal{O}_Y \oplus L)$ over a two-dimensional compact nonsingular toric variety Y with the Picard number 3 or 4;*

as well as $3^2 4^3 6^2$, $3^1 4^3 5^3$, $3^2 4^4 7^2$, $3^3 4^1 5^1 6^3$, $3^2 4^2 5^2 6^2$ (ii), $3^1 4^4 5^1 6^2$, $3^2 4^1 5^4 6^1$, $(3^1 4^3 5^3 6^1)'$, $(3^1 4^3 5^3 6^1)''$, $(3^2 5^6)'$, $(3^2 5^6)''$, $(4^4 5^4)'$, $(4^4 5^4)''$ *described in Fig. 1.24.*

Remark. For the triangulations of S^2 labeled $3^2 4^2 5^2$, $3^3 5^3 6^1$, $3^2 4^2 5^2 6^1$, $3^3 4^1 5^2 6^1 7^1$, $3^2 4^3 5^1 6^1 7^1$, $3^2 4^2 5^3 7^1$, $3^4 6^4$ and $3^2 4^2 5^2 6^2$ (i) in Sect. A.5, there exist no admissible weightings corresponding to toric varieties which are minimal in the sense of



∞ = n''
n'
n
−n−n'−n''
3⁴
∞ = n'
−n''
n''
−n−n'−an''
(3² 4³)'
∞ = n''
−n+bn'+cn''
−n'−n''
(3²4³)''
−n−an'+cn''
−n'+bn''
4⁶
∞ = n''
4ˢs²
(s = 5, 6)
n'−an''
∞ = n
−n'−n''
−n'
−n−2n'−n''
3² 4³ 6²

Fig. 1.24a

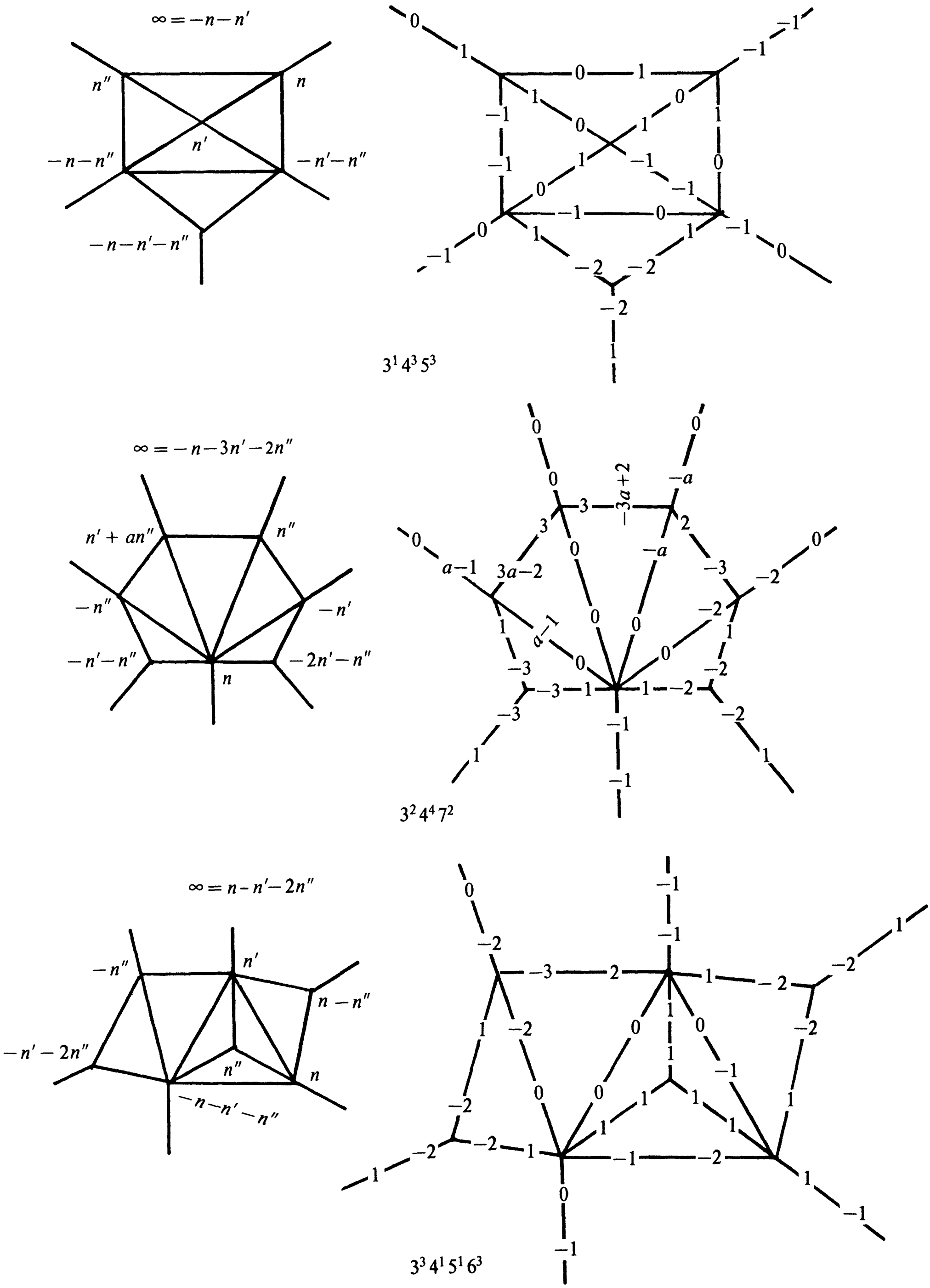

∞ = −n − n′
n″
n
n′
−n − n″
−n′ − n″
−n − n′ − n″
3¹ 4³ 5³
∞ = −n − 3n′ − 2n″
n′ + an″
n″
−n″
−n′
−n′ − n″
−2n′ − n″
n
3² 4⁴ 7²
∞ = n − n′ − 2n″
−n″
n′
n − n″
−n′ − 2n″
n″
n
−n − n′ − n″
3³ 4¹ 5¹ 6³

Fig. 1.24b

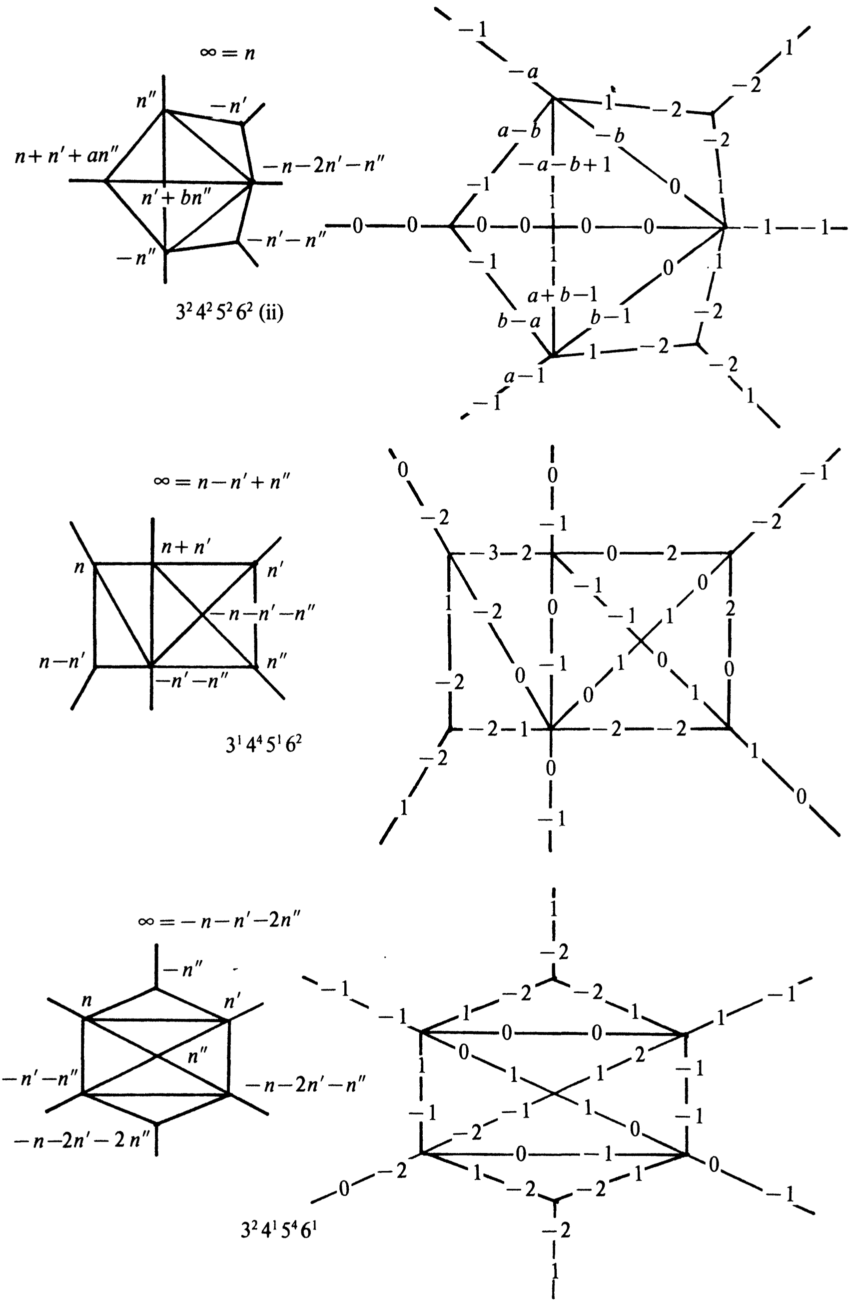

∞ = n
n + n' + an''
n' + bn''
−n − 2n' − n''
−n' − n''
3² 4² 5² 6² (ii)
∞ = n − n' + n''
n + n'
−n − n' − n''
−n' − n''
n − n'
3¹ 4⁴ 5¹ 6²
∞ = − n − n' − 2n''
−n − 2n' − n''
−n' − n''
−n − 2n' − 2 n''
3² 4¹ 5⁴ 6¹

Fig. 1.24c

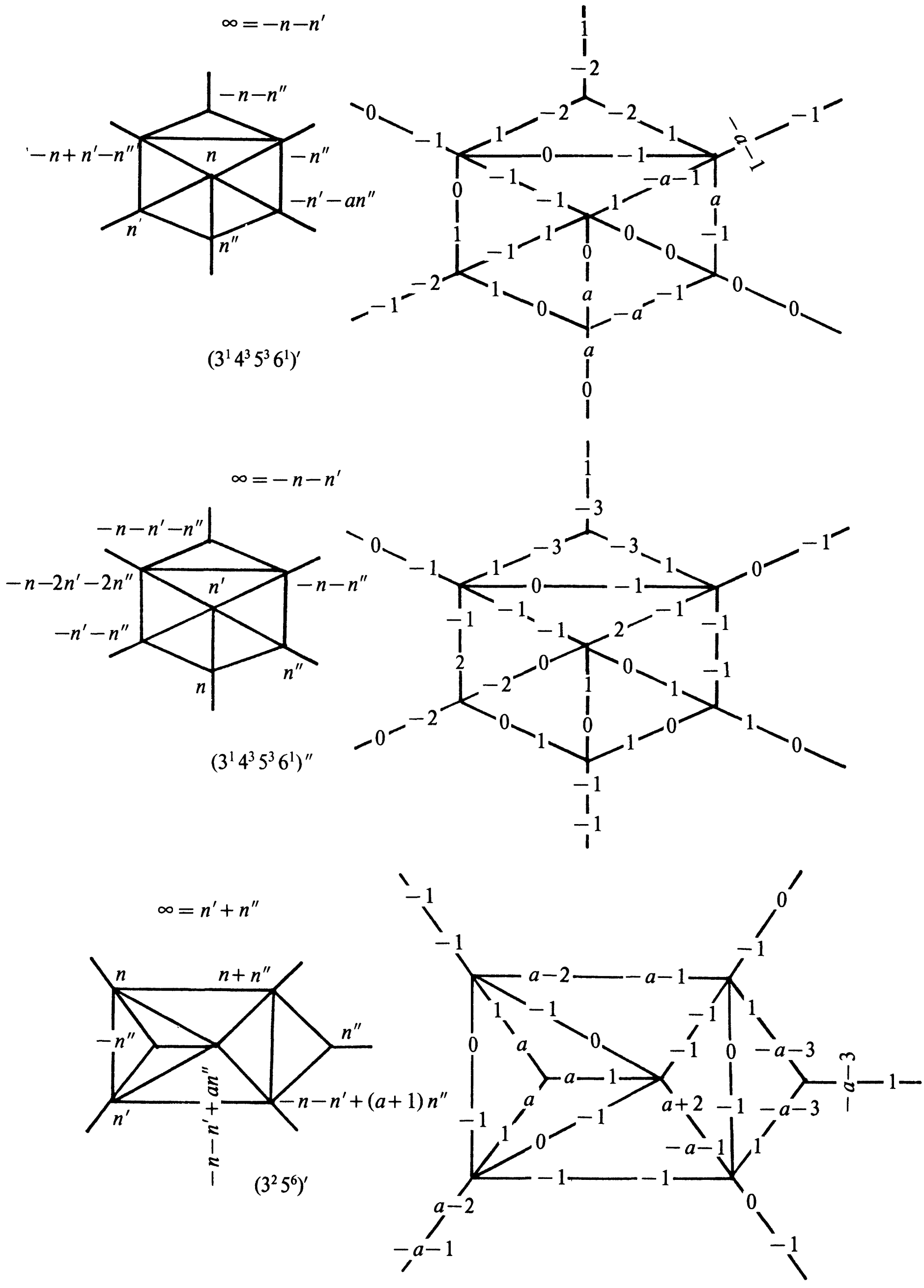
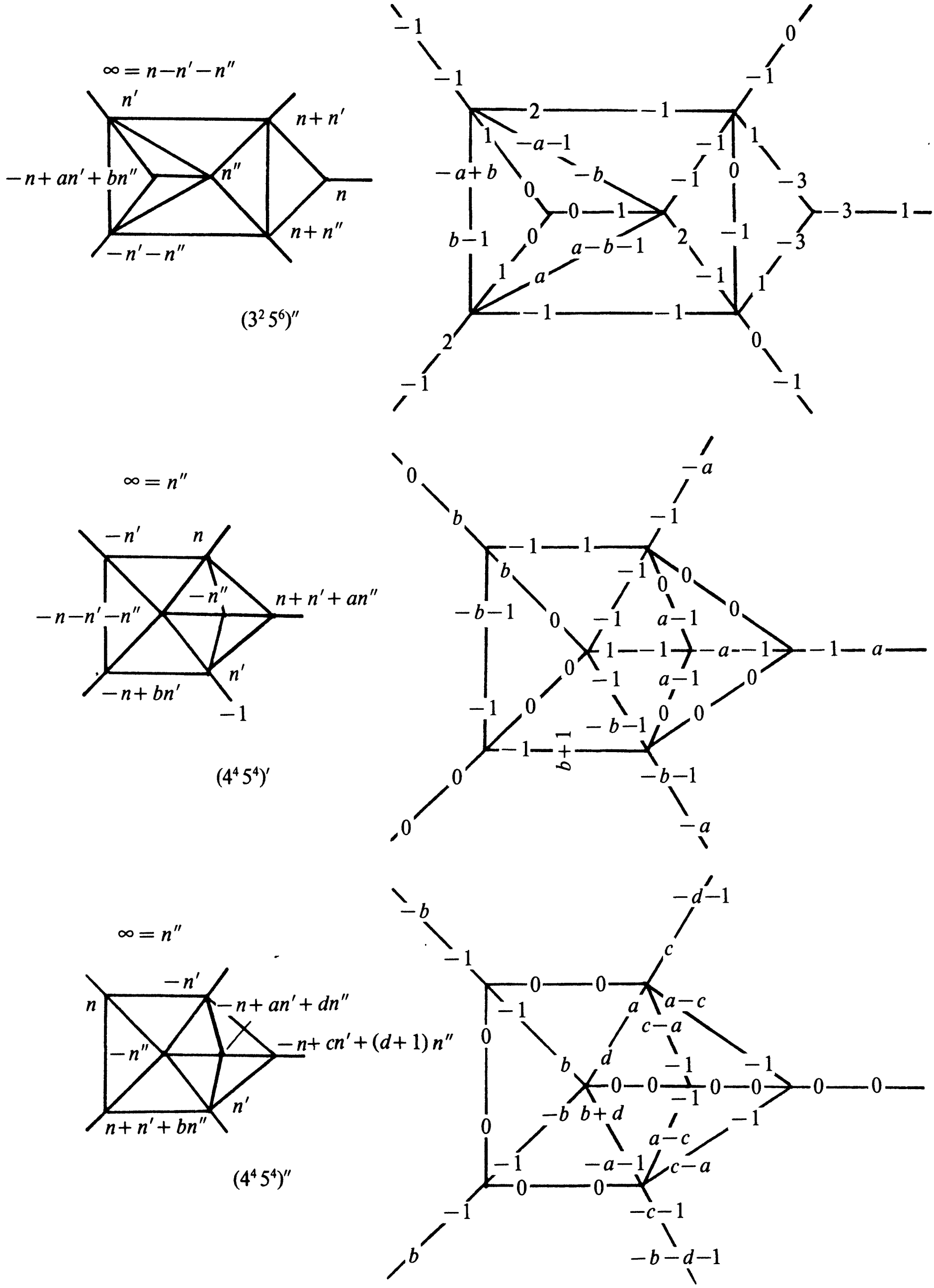


Fig. 1.24d



$(3^2 5^6)''$

$(4^4 5^4)'$

$(4^4 5^4)''$

Fig. 1.24e

equivariant blowing-ups. When the Picard number is six, N. Takeyama classified all the admissible weightings for $3^2 4^1 5^5 7^1$ among fifty different triangulations with nine vertices. On the other hand, the icosahedral triangulation 5^{12} has thirty-two different admissible weightings. (See [MO, § 9], where (1) on p. 105 should have been $n_7 = n - n''$.) The corresponding toric varieties with the Picard number nine are necessarily minimal, since all the vertices are 5-valent. As Ishida pointed out, they are all nonprojective, in view of the toric variant, Theorem 2.27 due to Reid, of Mori's theorem.

It should be noted that for certain special values for a, b, c, d, the toric varieties corresponding to those in Theorem 1.34 may not be minimal in the sense of equivariant blowing-ups.

Among those listed in Theorem 1.34, there exist admissible weightings corresponding to projective as well as nonprojective toric varieties (cf. Sect. 2.3).

Chapter 2. Integral Convex Polytopes and Toric Projective Varieties

We begin this chapter by introducing support functions linear with respect to fans. They enable us to describe equivariant line bundles and invariant Cartier divisors on toric varieties.

For compact toric varieties, we can describe the cohomology groups with coefficients in equivariant line bundles very easily in terms of the support functions. Equivariant holomorphic maps from toric varieties to projective spaces can be dealt with by means of upper convex support functions.

Toric projective varieties are compact toric varieties which can be embedded into projective spaces. We can sharpen the relationship given in Sect. A.3 between convex polytopes and their support functions. Integral convex polytopes, i.e., those with lattice points as vertices, then turn out to have close connection with toric projective varieties via their support functions. As a result, we get an interesting correspondence between elementary geometric properties of integral convex polytopes and algebro-geometric properties of toric projective varieties.

The moment maps introduced by Jurkiewicz, Atiyah, Gullemin, Sternberg and others provide another connection between toric projective varieties and convex polytopes.

As Reid showed, we can describe Mori's theory and the minimal model problem on general projective varieties in terms of a rather interesting geometry of fans in the case of toric projective varieties.

2.1 Equivariant Line Bundles, Invariant Cartier Divisors and Support Functions

Let $N \cong \mathbb{Z}^r$ and let M be its dual $\mathbb{Z}$-module. For a fixed fan Δ in N, consider the corresponding toric variety

$$X := T_N \operatorname{emb}(\Delta) \ .$$

We are going to use the following data for Δ to define equivariant line bundles and T_N-invariant Cartier divisors on X:

Definition. A real valued function $h : |\Delta| \to \mathbb{R}$ on the support $|\Delta| := \cup_{\sigma \in \Delta} \sigma$ is said to be a *Δ-linear support function* if it is $\mathbb{Z}$-valued on $N \cap |\Delta|$ and is linear on each $\sigma \in \Delta$.

Namely, there exists $l_\sigma \in M$ for each $\sigma \in \Delta$ such that $h(n) = \langle l_\sigma, n \rangle$ for $n \in \sigma$ and that $\langle l_\sigma, n \rangle = \langle l_\tau, n \rangle$ holds whenever $n \in \tau < \sigma$. We denote by SF (N, Δ) the additive group consisting of Δ-linear support functions.

Since $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \subset \mathrm{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R}) = M_{\mathbb{R}}$, we may regard $m \in M$ as a Δ-linear support function. Hence we get a natural homomorphism $M \to \mathrm{SF}(N, \Delta)$. It is injective if $|\Delta|$ generates $N_{\mathbb{R}}$ over $\mathbb{R}$. In this case, we usually identify M with its image in SF (N, Δ).

When $h \in \mathrm{SF}(N, \Delta)$ is given, the above $\{l_\sigma; \sigma \in \Delta\} \subset M$ may not be uniquely determined, since $\{l'_\sigma; \sigma \in \Delta\}$ satisfying $l'_\sigma - l_\sigma \in M \cap \sigma^\perp$ for each $\sigma \in \Delta$ gives rise to the same h. As in Sect. 1.7 immediately before Proposition 1.30, let us denote

$$\Delta(1) := \{\varrho \in \Delta;\ \dim \varrho = 1\} .$$

For each $\varrho \in \Delta(1)$, there exists a unique primitive element $n(\varrho)$ in $N \cap \varrho$ such that $\varrho = \mathbb{R}_{\geq 0} n(\varrho)$, while $\sigma \in \Delta$ in general can be written as

$$\sigma = \sum_{\varrho \in \Delta(1),\ \varrho < \sigma} \mathbb{R}_{\geq 0} n(\varrho) .$$

Thus $h \in \mathrm{SF}(N, \Delta)$ is determined by the integers $h(n(\varrho))$ for all $\varrho \in \Delta(1)$, and we obtain an injective homomorphism

$$\mathrm{SF}(N, \Delta) \hookrightarrow \mathbb{Z}^{\Delta(1)}$$

which sends h to $(h(n(\varrho)); \varrho \in \Delta(1))$. Note that $l_\sigma \in M$ is a solution in M of the system of equations $\{\langle l_\sigma, n(\varrho) \rangle = h(n(\varrho));\ \varrho \in \Delta(1),\ \varrho < \sigma\}$. Since such a solution may not exist in general, the above homomorphism need not be surjective. If σ is nonsingular, however, $\{n(\varrho); \varrho \in \Delta(1), \varrho < \sigma\}$ is a part of a $\mathbb{Z}$-basis of N by Theorem 1.10, hence such a solution $l_\sigma \in M$ exists. In particular, if X is nonsingular, we have an isomorphism

$$\mathrm{SF}(N, \Delta) \xrightarrow{\sim} \mathbb{Z}^{\Delta(1)} .$$

An *equivariant line bundle* on X is a fiber bundle $\pi: L \to X$ with fiber $\mathbb{C}$ and with an algebraic action of T_N on L such that π is equivariant with respect to T_N (i.e., $\pi(tz) = t\pi(z)$ for all $t \in T_N$ and $z \in L$) and that the action of each $t \in T_N$ on L induces a linear map from $\pi^{-1}(x)$ to $\pi^{-1}(tx)$ for each $x \in X$. The set ELB (X) of isomorphism classes of equivariant line bundles on X is a commutative group with respect to the tensor product. The identity element $\mathbf{1}_0$ is the trivial line bundle $X \times \mathbb{C}$ endowed with the action $t(x, c) := (tx, c)$ for each $t \in T_N$. More generally, each $m \in M$ gives rise to an equivariant line bundle $\mathbf{1}_m$, which is the trivial line bundle $X \times \mathbb{C}$ endowed with the action $t(x, c) := (tx, \mathbf{e}(m)(t)c)$ for each $t \in T_N$, where $\mathbf{e}(m)$ is the character of T_N corresponding to m as in Sect. 1.2. The map which sends m to $\mathbf{1}_m$ is a homomorphism

$$M \to \mathrm{ELB}(X) .$$

The *Picard group* Pic (X) of X consists of the isomorphism classes of line bundles on X. By disregarding the action of T_N on each equivariant line bundle, we obtain a homomorphism

$$\mathrm{ELB}(X) \to \mathrm{Pic}(X) ,$$

whose kernel obviously contains $\{\mathbf{1}_m; m \in M\}$.

For simplicity, let us restrict ourselves to the case where Δ is finite, hence X is of finite type. Then since X is an irreducible normal complex analytic space, we can also consider the commutative group $\mathrm{Div}(X)$ consisting of the *(Weil) divisors* on X, i.e., formal finite $\mathbb{Z}$-linear combinations of closed irreducible subspaces of X of codimension one. We denote by $T_N \mathrm{Div}(X)$ its subgroup consisting of T_N-invariant divisors on X. By Proposition 1.6 in Sect. 1.3, each $\varrho \in \Delta(1)$ determines a codimension one T_N-orbit $\mathrm{orb}(\varrho)$, whose closure in X is denoted by $V(\varrho)$ as in Corollary 1.7 in Sect. 1.3. Then $\{V(\varrho); \varrho \in \Delta(1)\}$ is a $\mathbb{Z}$-basis of $T_N \mathrm{Div}(X)$ so that we get

$$T_N \mathrm{Div}(X) = \bigoplus_{\varrho \in \Delta(1)} \mathbb{Z} V(\varrho) \ .$$

$D = \sum_{\varrho \in \Delta(1)} a_\varrho V(\varrho)$ is said to be *effective* and denoted $D \geq 0$ if a_ϱ is nonnegative for all ϱ.

Let us denote by $\mathrm{PDiv}(X)$ the subgroup of $\mathrm{Div}(X)$ consisting of *principal divisors* on X, i.e., those of the form

$$\mathrm{div}(f) := \sum_V v_V(f) V$$

for a nonzero rational function f on X, where $v_V(f)$ is the order of zero of f along each closed irreducible subspace V of X of codimension one. If f has a pole along V, then we define $v_V(f)$ to be the negative of the order of pole. Each $m \in M$ gives rise to a holomorphic function $\mathbf{e}(m)$ on T_N, hence a rational function on X. Obviously, $\mathrm{div}(\mathbf{e}(m))$ belongs to $T_N \mathrm{Div}(X) \cap \mathrm{PDiv}(X)$.

Of particular importance is the subgroup $\mathrm{CDiv}(X)$ of $\mathrm{Div}(X)$ consisting of *Cartier divisors*, i.e., locally principal (Weil) divisors. We denote by $T_N \mathrm{CDiv}(X) := T_N \mathrm{Div}(X) \cap \mathrm{CDiv}(X)$ the subgroup consisting of T_N-invariant Cartier divisors, which obviously contains $T_N \mathrm{Div}(X) \cap \mathrm{PDiv}(X)$. When X is nonsingular, all divisors are well known to be Cartier divisors, hence $\mathrm{CDiv}(X) = \mathrm{Div}(X)$ holds.

To understand our sign convention below, let us recall how we associate a line bundle to a Cartier divisor D. By definition, there exists an open covering $X = \cup_j U_j$ and nonzero rational functions f_j such that D coincides with the principal divisor $\mathrm{div}(f_j^{-1})$ on U_j. Thus both f_j/f_k and f_k/f_j are holomorphic functions on $U_j \cap U_k$ for each pair j, k. We obtain a line bundle $L := \cup_j (U_j \times \mathbb{C})$ by gluing $U_j \times \mathbb{C}$ and $U_k \times \mathbb{C}$ along $(U_j \cap U_k) \times \mathbb{C}$ via the map

$$U_j \times \mathbb{C} \supset (U_j \cap U_k) \times \mathbb{C} \xrightarrow{\sim} (U_j \cap U_k) \times \mathbb{C} \subset U_k \times \mathbb{C}$$

which sends (x, c) to $(x, (f_j/f_k)(x)c)$. The projections to the first factors are glued together to $\pi: L \to X$.

We are now ready to relate the notions of Δ-linear support functions, equivariant line bundles and T_N-invariant Cartier divisors.

Proposition 2.1. *Let $X := T_N \mathrm{emb}(\Delta)$ be the toric variety associated to a finite fan Δ in N.*

(i) *We have a natural homomorphism $\mathrm{SF}(N, \Delta) \to \mathrm{ELB}(X)$ which associates an equivariant line bundle L_h to each Δ-linear support function h. For $m \in M$, we have $L_m = \mathbf{1}_{-m}$.*

(ii) *Suppose* $h \in \mathrm{SF}(N, \Delta)$. *If* $m \in M$ *satisfies*

$$\langle m, n\rangle \geq h(n) \quad \text{for all} \quad n \in |\Delta| \ ,$$

then we have a section $\varphi : X \to L_h$ *for* L_h *which is* T_N*-semi-invariant, i.e.,* $\varphi(tx) = \mathbf{e}(m)(t)(t\varphi(x))$, *where the right hand side is the scalar multiple by* $\mathbf{e}(m)(t) \in \mathbb{C}$ *of the element* $t\varphi(x)$ *in the fiber over* tx.

(iii) *We have an injective homomorphism* $\mathrm{SF}(N, \Delta) \hookrightarrow T_N \mathrm{CDiv}(X)$ *by sending each* Δ*-linear support function* h *to the* T_N*-invariant Cartier divisor*

$$D_h := -\sum_{\varrho \in \Delta(1)} h(n(\varrho))\, V(\varrho) \ .$$

In particular, $D_m = \mathrm{div}\,(\mathbf{e}(-m))$ *for* $m \in M$. *We have* $D_h \geq 0$ *if and and only if* h *has nonpositive values.*

(iv) *For* $m \in M$ *and* $h \in \mathrm{SF}(N, \Delta)$ *the following are equivalent:* $\langle m, n\rangle \geq h(n)$ *for all* $n \in |\Delta| \Leftrightarrow -m+h$ *has nonpositive values* $\Leftrightarrow D_{-m+h} \geq 0 \Leftrightarrow \mathrm{div}\,(\mathbf{e}(m)) + D_h \geq 0$.

(v) *If* X *is nonsingular,* (iii) *gives rise to an isomorphism*

$$\mathrm{SF}(N, \Delta) \xrightarrow{\sim} T_N \mathrm{CDiv}(X) = T_N \mathrm{Div}(X) = \bigoplus_{\varrho \in \Delta(1)} \mathbb{Z}V(\varrho) \ .$$

More generally, if each $\sigma \in \Delta$ *is simplicial, we have*

$$\mathrm{SF}(N, \Delta) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} T_N \mathrm{CDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = T_N \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{\varrho \in \Delta(1)} \mathbb{Q}V(\varrho) \ .$$

(vi) *For* $h \in \mathrm{SF}(N, \Delta)$ *the sheaf of germs of sections of* L_h *coincides with the invertible sheaf* $\mathcal{O}_X(D_h)$ *associated to the* T_N*-invariant Cartier divisor* D_h. *This sheaf has an action of* T_N *and can be regarded naturally as a* T_N*-stable* $\mathcal{O}_X$*-submodule of the direct image* $j_* \mathcal{O}_{T_N}$ *with respect to the embedding* $j : T_N \to X$. *If* $m \in M$ *satisfies* $\langle m, n\rangle \geq h(n)$ *for all* $n \in |\Delta|$, *then* $\mathbf{e}(m)$ *is a* T_N*-semi-invariant section of* $\mathcal{O}_X(D_h)$.

Proof. (i) Let $\{l_\sigma ; \sigma \in \Delta\} \subset M$ be as in the definition of $h \in \mathrm{SF}(N, \Delta)$. We have $\sigma \cap \tau \in \Delta$ for σ, $\tau \in \Delta$ by the definition of a fan, while $h(n) = \langle l_\sigma, n\rangle = \langle l_{\sigma\cap\tau}, n\rangle = \langle l_\tau, n\rangle$ holds for $n \in \sigma \cap \tau$ by the definition of h. Thus $l_\sigma - l_\tau$ is contained in $M \cap (\sigma \cap \tau)^\perp \subset \mathcal{S}_{\sigma\cap\tau}$, hence both $\mathbf{e}(l_\sigma - l_\tau)$ and $\mathbf{e}(l_\tau - l_\sigma)$ are holomorphic functions on the open set $U_{\sigma\cap\tau}$ of X (cf. Sect. 1.2). By Theorem 1.4 in Sect. 1.2, $\{U_\sigma ; \sigma \in \Delta\}$ is an open covering of X and $U_\sigma \cap U_\tau = U_{\sigma\cap\tau}$ holds for each pair σ, $\tau \in \Delta$. We thus obtain a line bundle $L_h := \cup_{\sigma \in \Delta} (U_\sigma \times \mathbb{C})$ over X by gluing $U_\sigma \times \mathbb{C}$ and $U_\tau \times \mathbb{C}$ along $U_{\sigma\cap\tau} \times \mathbb{C}$ by the isomorphism $g_{\tau\sigma} : U_\sigma \times \mathbb{C} \supset U_{\sigma\cap\tau} \times \mathbb{C} \xrightarrow{\sim} U_{\sigma\cap\tau} \times \mathbb{C} \subset U_\tau \times \mathbb{C}$ defined by

$$g_{\tau\sigma}(x, c) := (x, \mathbf{e}(l_\sigma - l_\tau)(x)c) \quad \text{for} \quad (x, c) \in U_{\sigma\cap\tau} \times \mathbb{C} \ .$$

The projections to the first factors are glued together to $\pi : L_h \to X$. We can define an action of T_N on L_h by

$$t(x, c) := (tx, \mathbf{e}(-l_\sigma)(t)c) \quad \text{for} \quad t \in T_N \quad \text{and} \quad (x, c) \in U_\sigma \times \mathbb{C}$$

for all $\sigma \in \Delta$, which are obviously compatible with the gluing maps above.

If $\{l_\sigma ; \sigma \in \Delta\}$ and $\{l'_\sigma ; \sigma \in \Delta\}$ happen to define the same h, we have $h(n) = \langle l_\sigma, n\rangle = \langle l'_\sigma, n\rangle$ for all $n \in \sigma$, hence $l'_\sigma - l_\sigma$ belongs to $M \cap \sigma^\perp$. The resulting isomorphisms

$g_\sigma : U_\sigma \times \mathbb{C} \xrightarrow{\sim} U_\sigma \times \mathbb{C}$ for all $\sigma \in \Delta$ defined by

$$g_\sigma(x,c) := (x, \mathbf{e}(l'_\sigma - l_\sigma)(x)c) \quad \text{for} \quad (x,c) \in U_\sigma \times \mathbb{C}$$

are glued together to a bundle isomorphism $g : L'_h \xrightarrow{\sim} L_h$ with L'_h defined for $\{l'_\sigma; \sigma \in \Delta\}$. The T_N-action on L'_h determined by $\{l'_\sigma; \sigma \in \Delta\}$ is obviously sent to that on L_h determined by $\{l_\sigma; \sigma \in \Delta\}$. Thus L_h is determined by h up to isomorphism of equivariant line bundles.

For $m \in M$, we obviously have $L_m = \mathbf{1}_{-m}$, which is the trivial line bundle $X \times \mathbb{C}$ with the T_N-action $t(x,c) := (tx, \mathbf{e}(-m)(t)c)$.

(ii) Suppose $m \in M$ satisfies $\langle m, n\rangle \geq h(n)$ for all $n \in |\Delta|$. If $\{l_\sigma; \sigma \in \Delta\}$ is as above for h, then $\langle m, n\rangle \geq \langle l_\sigma, n\rangle$ holds for all $n \in \sigma$. Hence $m - l_\sigma \in M \cap \sigma^\vee = \mathscr{S}_\sigma$ and $\mathbf{e}(m - l_\sigma)$ is a holomorphic function on U_σ. The holomorphic maps $\varphi_\sigma : U_\sigma \to U_\sigma \times \mathbb{C}$ for all $\sigma \in \Delta$ defined by $\varphi_\sigma(x) := (x, \mathbf{e}(m - l_\sigma)(x))$ are naturally glued together to a section $\varphi : X \to L_h$. It is also obvious that $\varphi(tx) = \mathbf{e}(m)(t)(t\varphi(x))$ for all $t \in T_N$ and $x \in X$.

(iii), (iv) and (v) are obvious, since the map $h \mapsto D_h$ is just the negative of the injection $\mathrm{SF}(N, \Delta) \hookrightarrow \mathbb{Z}^{\Delta(1)}$ at the beginning of this section. Note that D_h is a Cartier divisor, since for each $\sigma \in \Delta$, it coincides on U_σ with $\mathrm{div}(\mathbf{e}(-l_\sigma))$ by the very definition.

(vi) For each $\sigma \in \Delta$, the restriction to the open set U_σ of the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(D_h)$ is $\mathcal{O}_X(D_h)_{|U_\sigma} := \mathcal{O}_{U_\sigma} \cdot \mathbf{e}(l_\sigma)$. Thus it is easily seen to be the sheaf of germs of sections of L_h. It is obviously a T_N-stable $\mathcal{O}_X$-submodule of the direct image $j_* \mathcal{O}_{T_N}$. Suppose $m \in M$ satisfies $\langle m, n\rangle \geq h(n)$ for all $n \in |\Delta|$. Then $m - l_\sigma \in M \cap \sigma^\vee$, hence $\mathbf{e}(m) = \mathbf{e}(m - l_\sigma) \cdot \mathbf{e}(l_\sigma) \in \mathcal{O}_{U_\sigma} \cdot \mathbf{e}(l_\sigma)$ for each $\sigma \in \Delta$. q.e.d.

We adopt the sign convention above because of its connection with the existing literature as well as the geometry of convex bodies, although the equality $\mathrm{div}(\mathbf{e}(m)) = D_{-m}$ is rather unnatural.

Remark. As in [TE, Chapt. I, §2, Theorem 9], we can generalize the notion of Δ-linear support functions in the following manner to deal not only with T_N-invariant (Weil) divisors but also T_N-stable and *complete* fractional $\mathcal{O}_X$-ideals $\mathscr{F}$ contained in $j_* \mathcal{O}_{T_N}$. Namely, we consider functions $h = \mathrm{ord}_{\mathscr{F}} : |\Delta| \to \mathbb{R}$ such that

(1) it is positively homogeneous, i.e., $h(cn) = ch(n)$ holds for any $c \in \mathbb{R}_{\geq 0}$ and $n \in |\Delta|$;

(2) h is $\mathbb{Z}$-valued on $N \cap |\Delta|$;

(3) h is upper convex and piecewise linear on each $\sigma \in \Delta$ (cf. Corollary A.19).

Corollary 2.2. *For a fan (N, Δ) we have a commutative diagram of commutative groups*

$$\begin{tikzcd} & \mathrm{Pic}(X) \\ \operatorname{coker}[M \to \mathrm{SF}(N, \Delta)] \arrow[ur] \arrow[dr] & \\ & T_N \mathrm{CDiv}(X)/(T_N \mathrm{Div}(X) \cap \mathrm{PDiv}(X)) \arrow[uu] \end{tikzcd}$$

Example. If X is nonsingular, there exists by Proposition 2.1, (v) a unique $k \in \mathrm{SF}(N, \Delta)$ such that $k(n(\varrho)) := 1$ for any $\varrho \in \Delta(1)$. The corresponding T_N-invariant Cartier divisor is $D_k = -\sum_{\varrho \in \Delta(1)} V(\varrho)$, which is a *canonical divisor* for X so that $\mathcal{O}_X(D_k)$

coincides with the sheaf Ω_X^r of germs of holomorphic r-forms on X with $r = \dim X$. In Sect. 3.2, D_k will turn out to be a canonical divisor in a generalized sense, even if X is singular.

Indeed, take a $\mathbb{Z}$-basis $\{m_1, \ldots, m_r\}$ of M. Then

$$\omega := \frac{d\mathbf{e}(m_1)}{\mathbf{e}(m_1)} \wedge \frac{d\mathbf{e}(m_2)}{\mathbf{e}(m_2)} \wedge \ldots \wedge \frac{d\mathbf{e}(m_r)}{\mathbf{e}(m_r)}$$

is an invariant holomorphic r-form on T_N, hence a rational r-form on X. The divisor div (ω) of ω is independent of any particular choice of the $\mathbb{Z}$-basis, since, due to one of the basic properties of the logarithmic derivative, the r-form ω will at most have opposite sign, when $\{m_1, \ldots, m_r\}$ is replaced by another $\mathbb{Z}$-basis. For each $\sigma \in \Delta$ there exist, by Theorem 1.10, a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ and $s \leq r$ such that $\sigma = \mathbb{R}_{\geq 0} n_1 + \ldots + \mathbb{R}_{\geq 0} n_s$. For the $\mathbb{Z}$-basis $\{m_1, \ldots, m_r\}$ of M dual to $\{n_1, \ldots, n_r\}$, we have

$$\operatorname{div}(\omega) = - \sum_{\varrho \in \Delta(1),\, \varrho < \sigma} V(\varrho) \quad \text{on} \quad U_\sigma .$$

In Corollary 3.3, we are going to have another look at the above argument.

Remark. We refer the reader to Kaneyama [K1], [K1′] for an attempt at the construction of equivariant vector bundles on $T_N \operatorname{emb}(\Delta)$ analogous to that of equivariant line bundles described in Proposition 2.1, (i).

2.2 Cohomology of Compact Toric Varieties

On compact toric varieties $X = T_N \operatorname{emb}(\Delta)$, all equivariant line bundles and T_N-invariant Cartier divisors arise from Δ-linear support functions. The cohomology groups with coefficients in an equivariant line bundle can be described completely in terms of the fan Δ and the corresponding Δ-linear support function.

One reason for these facts is the complete reducibility of algebraic representations of the algebraic torus T_N as explained in Sect. 1.2. Another reason is the following GAGA theorem, which enables us to describe complex analytic information on a compact algebraic variety in terms of algebro-geometric information. It was Serre [S10] who first proved it for projective varieties. The general case as well as the relative version given below are due to Grothendieck [G4], [G5].

The GAGA Theorem. *Let Z' be a complete algebraic variety over $\mathbb{C}$ with the structure sheaf $\mathcal{O}'$. Denote by Z with the structure sheaf $\mathcal{O}$ the compact complex analytic space associated to Z'.*

(i) *The category of coherent $\mathcal{O}'$-modules is equivalent to that of coherent $\mathcal{O}$-modules via the functor which sends each coherent $\mathcal{O}'$-module $\mathcal{F}'$ to the coherent $\mathcal{O}$-module $\mathcal{F} := \mathcal{F}' \otimes_{\mathcal{O}'} \mathcal{O}$.*

(ii) *For coherent sheaves, algebraic cohomology groups coincide with complex analytic cohomology groups. Namely, with $\mathcal{F} := \mathcal{F}' \otimes_{\mathcal{O}'} \mathcal{O}$ for any coherent $\mathcal{O}'$-module $\mathcal{F}'$, we have a canonical isomorphism*

$$H^q(Z', \mathcal{F}') \xrightarrow{\sim} H^q(Z, \mathcal{F}) \quad \textit{for} \quad q = 0, 1, 2, \ldots ,$$

where the left hand side is the cohomology group with respect to the Zariski topology, while the right hand side is that with respect to the classical topology.

(iii) *The set of morphisms from Z' to a complete algebraic variety W' over $\mathbb{C}$ is in natural one-to-one correspondence with the set of holomorphic maps from Z to the compact complex analytic space W associated to W'.*

The Relative GAGA Theorem. *For a proper morphism $f': Z' \to W'$ of algebraic varieties over $\mathbb{C}$, let $f: Z \to W$ be the associated proper holomorphic map of complex analytic spaces. For any coherent $\mathcal{O}_{Z'}$-module $\mathcal{F}'$ on Z' and any nonnegative integer q, the higher direct image $R^q f'_*(\mathcal{F}')$ is a coherent $\mathcal{O}_{W'}$-module. The corresponding coherent $\mathcal{O}_W$-module coincides with the complex analytic higher direct image $R^q f_*(\mathcal{F})$ of $\mathcal{F} := \mathcal{F}' \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_Z$, that is, we have a canonical isomorphism*

$$R^q f'_*(\mathcal{F}') \otimes_{\mathcal{O}_{W'}} \mathcal{O}_W \cong R^q f_*(\mathcal{F}' \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_Z) \ .$$

Let $X := T_N \operatorname{emb}(\Delta)$ be a compact toric variety. Hence Δ is a finite complete fan by Theorem 1.11 in Sect. 1.4. As we pointed out immediately after Theorem 1.4, the fan (N, Δ) determines a complete algebraic variety X' over $\mathbb{C}$ and our toric variety X is the complex analytic space associated to this X'. Hence (i) in the GAGA theorem is applicable. The equivariant line bundle L_h, the T_N-invariant Cartier divisor D_h and the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(D_h)$, which are determined by a Δ-linear support function h, are easily seen to be those associated to the algebraic counterparts L'_h, D'_h and $\mathcal{O}_{X'}(D'_h)$. Thus (ii) in the GAGA theorem is also applicable. To simplify notation, we denote the algebraic counterparts also by X, L_h, D_h and $\mathcal{O}_X(D_h)$.

The following lemma is a special case of Theorem 2.6 below on the cohomology groups. We state it here, since it is of independent importance and since we use it in the proof of Proposition 2.4.

Lemma 2.3. *Let $X = T_N \operatorname{emb}(\Delta)$ be the compact toric variety determined by a finite complete fan (N, Δ). For $h \in \mathrm{SF}(N, \Delta)$,*

$$\square_h := \{m \in M_{\mathbb{R}} \,;\, \langle m, n\rangle \geq h(n),\ \forall n \in N_{\mathbb{R}}\}$$

is a (possibly empty) convex polytope in $M_{\mathbb{R}}$. The set $H^0(X, \mathcal{O}_X(D_h))$ of global sections of the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(D_h)$ is a finite dimensional $\mathbb{C}$-vector space with $\{\mathbf{e}(m); m \in M \cap \square_h\}$ as a basis.

Proof. Since $|\Delta| = N_{\mathbb{R}}$ and Δ is finite, $\square_h$ is a convex polytope by Theorem A.18 and Corollary A.19.

By (ii) in the GAGA theorem, we may regard X as an algebraic veriety. By definition, there exists $\{l_\sigma ; \sigma \in \Delta\} \subset M$ such that $h(n) = \langle l_\sigma, n\rangle$ for all $n \in \sigma$. The restriction of $\mathcal{O}_X(D_h)$ to U_σ coincides with that of $\mathcal{O}_X \cdot \mathbf{e}(l_\sigma)$ by Proposition 2.1, (vi). Thus the $\mathbb{C}$-vector subspace $H^0(U_\sigma, \mathcal{O}_X(D_h))$ of $H^0(U_\sigma, j_*\mathcal{O}_{T_N}) = H^0(T_N, \mathcal{O}_{T_N}) = \mathbb{C}[M]$ has, as a $\mathbb{C}$-basis, the set $\{\mathbf{e}(m); m \in l_\sigma + M \cap \sigma^\vee\}$ of T_N-semi-invariants. We also have $H^0(X, \mathcal{O}_X(D_h)) = \bigcap_{\sigma \in \Delta} H^0(U_\sigma, \mathcal{O}_X(D_h))$. Thus we are done, since $l_\sigma + M \cap \sigma^\vee = \{m \in M; \langle m, n\rangle \geq h(n), \forall n \in \sigma\}$ by the choice of l_σ. q.e.d.

Remark. For $m \in M$, Proposition 2.1, (iv) can be rewritten as

$$m \in \square_h \Leftrightarrow D_{-m+h} \geq 0 \Leftrightarrow \operatorname{div}(\mathbf{e}(m)) + D_h \geq 0$$

in the notation above. Thus $\{D_{-m+h}; m \in M \cap \square_h\}$ is the set of T_N-invariant Cartier divisors in the complete linear system $|D_h|$.

Proposition 2.4. *Let $X = T_N \operatorname{emb}(\Delta)$ be a compact toric variety.*

(i) *Every T_N-invariant Cartier divisor D on X coincides with a principal divisor on the open set U_σ for each $\sigma \in \Delta$ so that $D = D_h$ for some $h \in \mathrm{SF}(N, \Delta)$. Thus the homomorphism in Proposition 2.1,* (iii) *is an isomorphism* $\mathrm{SF}(N, \Delta) \xrightarrow{\sim} T_N \operatorname{CDiv}(X)$.

(ii) *For any Cartier divisor D on X, we have an $\mathcal{O}_X$-module isomorphism $\mathcal{O}_X(D) \cong \mathcal{O}_X(D_h)$ for some $h \in \mathrm{SF}(N, \Delta)$. Hence the composite of natural homomorphisms* $\mathrm{SF}(N, \Delta) \to \operatorname{CDiv}(X) \to \operatorname{Pic}(X)$ *is surjective.*

(iii) *The following are equivalent for $h \in \mathrm{SF}(N, \Delta)$:*

(a) *$h \in M$.*

(b) *$D_h \in \operatorname{PDiv}(X)$.*

(c) *L_h is trivial as a line bundle.*

(d) *$\mathcal{O}_X(D_h) \cong \mathcal{O}_X$ as $\mathcal{O}_X$-modules.*

Proof. We may assume X to be an algebraic variety by (i) of the GAGA theorem.

(i) Let $D = \sum_{\varrho \in \Delta(1)} b_\varrho V(\varrho)$ be a T_N-invariant Cartier divisor on X. For $\sigma \in \Delta$, both $F_\sigma := H^0(U_\sigma, \mathcal{O}_X(D))$ and $A_\sigma := H^0(U_\sigma, \mathcal{O}_X)$ are $\mathbb{C}$-vector subspaces of $\mathbb{C}[M]$ stable under the natural algebraic action of T_N. Moreover, A_σ is a subring and F_σ is an A_σ-submodule. Since D is a Cartier divisor, F_σ is a projective A_σ-module of rank one. Consequently, the T_N-stable A_σ-submodule $(A_\sigma : F_\sigma) := \{a \in \mathbb{C}[M]; aF_\sigma \subset A_\sigma\}$ of $\mathbb{C}[M]$ is the A_σ-module dual to F_σ and satisfies $(A_\sigma : F_\sigma) \cdot F_\sigma = A_\sigma$. If we set $\Phi_\sigma := \{m \in M; \langle m, n(\varrho)\rangle \geq -b_\varrho, \sigma > \forall \varrho \in \Delta(1)\}$, then $\{\mathbf{e}(m); m \in M \cap \sigma^\vee\}$ and $\{\mathbf{e}(m); m \in \Phi_\sigma\}$ are $\mathbb{C}$-bases of A_σ and F_σ, respectively, so that $\{\mathbf{e}(m'); m' \in M, m' + \Phi_\sigma \subset M \cap \sigma^\vee\}$ is a $\mathbb{C}$-basis of $(A_\sigma : F_\sigma)$. Thus the above equality is reduced to

$$\{m' \in M; m' + \Phi_\sigma \subset M \cap \sigma^\vee\} + \Phi_\sigma = M \cap \sigma^\vee .$$

Since O is in $M \cap \sigma^\vee$, there exists $l_\sigma \in \Phi_\sigma$ such that $(-l_\sigma) + \Phi_\sigma \subset M \cap \sigma^\vee$, hence $\Phi_\sigma = l_\sigma + M \cap \sigma^\vee$. Consequently, we have $F_\sigma = A_\sigma \cdot \mathbf{e}(l_\sigma)$ and $D \cap U_\sigma = \operatorname{div}(\mathbf{e}(-l_\sigma)) \cap U_\sigma$. Obviously, $\{l_\sigma; \sigma \in \Delta\}$ thus determined gives rise to $h \in \mathrm{SF}(N, \Delta)$ which satisfies $D = D_h$.

(ii) The Cartier divisor $T_N \cap D$ on T_N is a principal divisor, since the coordinate ring $\mathbb{C}[M]$ of T_N as an affine algebraic variety is a unique factorization domain. Hence there exists a nonzero rational function f on X such that $D' := D - \operatorname{div}(f)$ satisfies $D' \cap T_N = 0$. Consequently, D' is a Cartier divisor with support in $X \setminus T_N = \cup_{\varrho \in \Delta(1)} V(\varrho)$ and is necessarily T_N-invariant. We have $D' = D_h$ for some $h \in \mathrm{SF}(N; \Delta)$ by (i) above, hence $D = D_h + \operatorname{div}(f)$ and $\mathcal{O}_X(D) \cong \mathcal{O}_X(D_h)$.

(iii) The equivalence of (b), (c), (d) is well-known, while (a) obviously implies (b). It thus remains to show that (d) implies (a). By assumption, we have $H^0(X, \mathcal{O}_X(D_h)) \neq 0$ and $H^0(X, \mathcal{O}_X(D_{-h})) \neq 0$. Hence both $M \cap \square_h$ and $M \cap \square_{-h}$ are nonempty by Lemma 2.3. For $m' \in M \cap \square_h$ and $m'' \in M \cap \square_{-h}$, we have

$$\langle m', n \rangle \geq h(n) \geq \langle -m'', n \rangle \quad \text{for all} \quad n \in N_{\mathbb{R}} .$$

Thus we necessarily have $m' = -m''$ and $h(n) = \langle m', n \rangle$ for all n in $N_{\mathbb{R}}$. q.e.d.

M is obviously a direct factor in SF (N, Δ) so that SF $(N, \Delta)/M$ is a free $\mathbb{Z}$-module of finite rank. Thus Propositions 2.1 and 2.4 and Corollary 2.2 imply the following result found already in Demazure [D5], [TE, Chap. I, § 2], Danilov [D1] and [MO]:

Corollary 2.5. *For any compact toric variety $X = T_N \operatorname{emb}(\Delta)$, we have canonical isomorphisms*

$$\begin{aligned} \mathrm{SF}(N, \Delta)/M &\xrightarrow{\sim} T_N \operatorname{CDiv}(X)/(T_N \operatorname{CDiv}(X) \cap \operatorname{PDiv}(X)) \\ &\xrightarrow{\sim} \operatorname{ELB}(X)/\{\mathbf{1}_m ; m \in M\} \xrightarrow{\sim} \operatorname{Pic}(X) . \end{aligned}$$

In particular, for nonsingular X, we have an exact sequence

$$0 \to M \to T_N \operatorname{Div}(X) = \bigoplus_{\varrho \in \Delta(1)} \mathbb{Z} V(\varrho) \to \operatorname{Pic}(X) \to 0 ,$$

hence $\operatorname{Pic}(X)$ *is a free $\mathbb{Z}$-module of rank* $^{\#}\Delta(1) - \dim X$, *where* $^{\#}\Delta(1)$ *is the cardinality of* $\Delta(1)$.

To state Theorem 2.6 below, let us recall the following: Let $h \in \mathrm{SF}(N, \Delta)$ for a finite complete fan (N, Δ). For each $m \in M$, we define a closed subset of $N_{\mathbb{R}}$ by

$$Z(h, m) := \{n \in N_{\mathbb{R}} ; \langle m, n \rangle \geq h(n)\}$$

and denote the cohomology groups of $N_{\mathbb{R}}$ with support $Z(h, m)$ and with coefficients in $\mathbb{C}$ by

$$H^q_{Z(h,m)}(N_{\mathbb{R}}, \mathbb{C}) \quad \text{for} \quad q = 0, 1, 2, \ldots$$

(see, for instance, Godement [G1, Chap. II, § 4]). They are the values for $\mathbb{C}$ of the right derived functors of $H^0_{Z(h,m)}(N_{\mathbb{R}},\ )$, which takes the commutative group of global sections with support in $Z(h, m)$ of an abelian sheaf. In the present case, $H^q_{Z(h,m)}(N_{\mathbb{R}}, \mathbb{C})$ can also be defined as the relative cohomology group $H^q(N_{\mathbb{R}}, N_{\mathbb{R}} \setminus Z(h, m), \mathbb{C})$. Since

$$H^q(N_{\mathbb{R}}, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{for} \quad q = 0 \\ 0 & \text{for} \quad q \geq 1 \end{cases} ,$$

the long exact sequence in [G1, Chap. II, § 4, 4.10] gives rise to an isomorphism

$$H^{q-1}(N_{\mathbb{R}} \setminus Z(h, m), \mathbb{C}) \xrightarrow{\sim} H^q_{Z(h,m)}(N_{\mathbb{R}}, \mathbb{C}) \quad \text{for} \quad q \geq 2$$

as well as an exact sequence

$$0 \to H^0_{Z(h,m)}(N_{\mathbb{R}}, \mathbb{C}) \to \mathbb{C} \to H^0(N_{\mathbb{R}} \setminus Z(h, m), \mathbb{C}) \to H^1_{Z(h,m)}(N_{\mathbb{R}}, \mathbb{C}) \to 0$$

which implies

$$H^0_{Z(h,m)}(N_{\mathbb{R}}, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if} \quad Z(h, m) = N_{\mathbb{R}} \\ 0 & \text{if} \quad Z(h, m) \neq N_{\mathbb{R}} \end{cases} .$$

Since

$$Z(h,m)=N_{\mathbb{R}} \Leftrightarrow \langle m,n\rangle \geq h(n) \quad \text{for all} \quad n\in N_{\mathbb{R}} \Leftrightarrow m\in \square_h$$

by definition, Lemma 2.3 is the special case $q=0$ of Theorem 2.6 below.

Demazure [D5] proved Theorem 2.6 in the nonsingular case. Kempf [TE, p. 42, Theorem 12] and Danilov [D1] generalized it into the present form. As Ishida pointed out, however, the proof in [TE] seems to be incorrect. [TE] obtains a more general result valid for T_N-stable and complete $\mathcal{O}_X$-submodules of $j_*\mathcal{O}_{T_N}$ (cf. the remark immediately after Proposition 2.1 in Sect. 2.1).

Theorem 2.6. *Let $X=T_N \operatorname{emb}(\Delta)$ be a compact toric variety. For each $h\in \mathrm{SF}(N,\Delta)$ and each nonnegative integer q, the algebraic torus T_N acts algebraically on the cohomology group $H^q(X, \mathcal{O}_X(D_h))$. For each $m\in M$, the eigenspace $H^q(X,\mathcal{O}_X(D_h))_m$ of the T_N-action with respect to the character $\mathbf{e}(m)$ can be identified canonically as*

$$H^q(X,\mathcal{O}_X(D_h))_m=H^q_{Z(h,m)}(N_{\mathbb{R}},\mathbb{C})\mathbf{e}(m)$$

so that we have a direct sum decomposition

$$H^q(X,\mathcal{O}_X(D_h))=\bigoplus_{m\in M} H^q_{Z(h,m)}(N_{\mathbb{R}},\mathbb{C})\mathbf{e}(m) \ .$$

Proof. We may assume X to be an algebraic variety by (ii) of the GAGA theorem. Let us denote $\mathcal{L}:=\mathcal{O}_X(D_h)$ and compute $H^q(X,\mathcal{L})$ as the Čech cohomology group with respect to the open covering $\mathfrak{U}=\{U_\sigma;\sigma\in\Delta\}$. $\mathcal{L}(U_\sigma):=H^0(U_\sigma,\mathcal{L})$ for each $\sigma\in\Delta$ is a T_N-stable $\mathbb{C}$-subspace of $\mathbb{C}[M]$ with $\{\mathbf{e}(m);m\in M, \langle m,n\rangle\geq h(n), \forall n\in\sigma\}$ $=\{\mathbf{e}(m);\ m\in M,\ \sigma\subset Z(h,m)\}$ as a $\mathbb{C}$-basis. Thus we can identify each element $f=\sum_{m\in M} f_m\mathbf{e}(m)$ of $\mathcal{L}(U_\sigma)$ with the sequence of complex numbers $\{f_m;m\in M,$ $f_m=0$ if $\sigma\not\subset Z(h,m)\}$.

The group $\check{C}^q(\mathfrak{U},\mathcal{L})$ of q-th Čech cochains is the direct sum of $\mathcal{L}(U_{\sigma_0}\cap\ldots\cap U_{\sigma_q})=\mathcal{L}(U_{\sigma_0\cap\ldots\cap\sigma_q})$ with $(\sigma_0,\ldots,\sigma_q)$ running through Δ^{q+1}. Hence if we let

$$C^q_{Z(h,m)}(N_{\mathbb{R}},\mathbb{C}):=\{c_m:\Delta^{q+1}\to\mathbb{C};\ c_m(\sigma_0,\ldots,\sigma_q)=0 \quad \text{if} \quad \sigma_0\cap\ldots\cap\sigma_q\not\subset Z(h,m)\} \ ,$$

then we get

$$\check{C}^q(\mathfrak{U},\mathcal{L})=\bigoplus_{m\in M} C^q_{Z(h,m)}(N_{\mathbb{R}},\mathbb{C})\mathbf{e}(m) \ .$$

Obviously, the usual coboundary maps on both sides coincide under this identification.

Denote by $\mathcal{H}^q(\mathcal{L})$ the presheaf on X whose value at each open set $U\subset X$ is $H^q(U,\mathcal{L})$. Since $H^q(U,\mathcal{L})=0$ for $q>0$ and an affine open set U, the Leray spectral sequence (cf. [G1, Chap. II, Corollary to Theorem 5.4.1])

$$E_2^{p,q}=\check{H}^p(\mathfrak{U},\mathcal{H}^q(\mathcal{L}))\Rightarrow H^{p+q}(X,\mathcal{L})$$

degenerates so that $H^p(X,\mathcal{L})$ in question coincides with the Čech cohomology group $E_2^{p,0}=\check{H}^p(\mathfrak{U},\mathcal{L})$ of the complex $\{\check{C}^q(\mathfrak{U},\mathcal{L});q\geq 0\}$.

On the other hand, the fan Δ defines a finite closed covering $N_{\mathbb{R}} = \cup_{\sigma\in\Delta}\sigma$ satisfying $H^q_{Z(h,m)}(\sigma, \mathbb{C}) = 0$ for each σ and each $q > 0$. Indeed, in the long exact sequence

$$\ldots \to H^{q-1}(\sigma, \mathbb{C}) \to H^{q-1}(\sigma \setminus Z(h,m), \mathbb{C}) \to H^q_{Z(h,m)}(\sigma, \mathbb{C}) \to H^q(\sigma, \mathbb{C}) \to \ldots ,$$

the fourth term vanishes by the convexity of σ, while the map from the first term to the second is surjective, since $\sigma \setminus Z(h,m)$ is either empty or convex. Denote by $\mathscr{H}^q_{Z(h,m)}(\mathbb{C})$ the coefficient system on $N_{\mathbb{R}}$ which associates $H^q_{Z(h,m)}(\sigma, \mathbb{C})$ to each $\sigma \in \Delta$. Then an analogous spectral sequence (cf. [G1, Chap. II, Theorem 5.2.4])

$$\tilde{E}_2^{p,q} = \check{H}^p(\Delta, \mathscr{H}^q_{Z(h,m)}(\mathbb{C})) \Rightarrow H^{p+q}_{Z(h,m)}(N_{\mathbb{R}}, \mathbb{C})$$

degenerates so that $\tilde{E}_2^{p,0} \xrightarrow{\sim} H^p_{Z(h,m)}(N_{\mathbb{R}}, \mathbb{C})$. Since

$$H^0_{Z(h,m)}(\sigma, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if} \quad \sigma \subset Z(h,m) \\ 0 & \text{if} \quad \sigma \not\subset Z(h,m) \end{cases}$$

as we saw earlier, we conclude that $\tilde{E}_2^{p,0}$ coincides with the p-th cohomology group of the complex $\{C^q_{Z(h,m)}(N_{\mathbb{R}}, \mathbb{C}); q \geq 0\}$. q.e.d.

Let $h \in \mathrm{SF}(N, \Delta)$ for a finite complete fan Δ in $N \cong \mathbb{Z}^r$. By definition, there exists $\{l_\sigma; \sigma \in \Delta\} \subset M$ such that $h(n) = \langle l_\sigma, n\rangle$ for all $n \in \sigma$. If we denote

$$\Delta(r) := \{\sigma \in \Delta;\ \dim \sigma = r\}$$

as in Sect. 1.7 immediately before Proposition 1.30, then l_σ for $\sigma \in \Delta(r)$ is uniquely determined by h. Indeed, if $\langle l'_\sigma, n\rangle = \langle l_\sigma, n\rangle$ holds for all $n \in \sigma$, then $l'_\sigma - l_\sigma \in M \cap \sigma^\perp = \{O\}$. Alternatively, let us consider the *Fréchet derivative*

$$\delta h(n; n') := \lim_{\lambda \downarrow 0} \{h(n + \lambda n') - h(n)\}/\lambda$$

for the $\mathbb{R}$-valued function h on $N_{\mathbb{R}}$ in the direction n'. It is a positively homogeneous function in n'. Since h is linear on each $\sigma \in \Delta(r)$, we see that $\delta h(n; n') = \langle l_\sigma, n'\rangle$ for all n in the interior of σ and for all $n' \in N_{\mathbb{R}}$. Thus h is differentiable at such n with the derivative l_σ.

Theorem 2.7. *Let $X = T_N\,\mathrm{emb}(\Delta)$ be a compact toric variety with $N \cong \mathbb{Z}^r$. The following are equivalent for $h \in \mathrm{SF}(N, \Delta)$:*

(a) *The $\mathcal{O}_X$-module $\mathcal{O}_X(D_h)$ is generated by global sections.*

(b) *The linear system $|D_h|$ has no base points.*

(c) *h is upper convex, i.e.,*

$$h(n) + h(n') \leq h(n + n') \quad \textit{for all} \quad n, n' \in N_{\mathbb{R}} .$$

(d) *The convex polytope $\square_h$ coincides with the convex hull in $M_{\mathbb{R}}$ of the finite set $\{l_\sigma; \sigma \in \Delta(r)\}$.*

When these equivalent conditions are satisfied, we have

$$h(n) = \inf\{\langle m, n\rangle; m \in M \cap \square_h\} = \inf\{\langle l_\sigma, n\rangle; \sigma \in \Delta(r)\}$$

and h is the support function for $\square_h$ in the sense of Sect. A.3. Moreover, the following vanishing theorem *holds:*

$$H^q(X, \mathcal{O}_X(D_h)) = 0 \quad \textit{for} \quad q > 0 \ .$$

Proof. By the GAGA theorem, we may again assume X to be an algebraic variety. The equivalence of (a) and (b) is well-known.

As in the proof of Proposition 2.4, the $\mathbb{C}$-subspace $H^q(U_\sigma, \mathcal{O}_X(D_h))$ of $\mathbb{C}[M]$ has $\{\mathbf{e}(m); m \in l_\sigma + M \cap \sigma^\vee\}$ as a $\mathbb{C}$-basis. Thus by Lemma 2.3, we see that (a) is equivalent to

$$(*) \qquad l_\sigma + M \cap \sigma^\vee = M \cap \square_h + M \cap \sigma^\vee \quad \text{for all} \quad \sigma \in \Delta \ .$$

Furthermore, it suffices that the above equality holds only for all $\sigma \in \Delta(r)$. In that case, l_σ is uniquely determined by h as we remarked above, hence for all $n \in \sigma \in \Delta(r)$ we have

$$h(n) = \langle l_\sigma, n\rangle = \inf\{\langle m, n\rangle; m \in M \cap \square_h + M \cap \sigma^\vee\} \geq \inf\{\langle m, n\rangle; m \in \square_h\} \geq h(n) \ .$$

Consequently, the equalities hold above and h coincides with the support function of the convex polytope $\square_h$. In particular, h is upper convex, hence (a) implies (c).

The equivalence of (c) and (d) follows from Corollary A.19. Moreover, l_σ belongs to $\square_h$ for each $\sigma \in \Delta(r)$ in this case, hence $(*)$ holds.

The vanishing theorem remains to be shown. If h is upper convex, then for each $m \in M$, we see that

$$N_{\mathbb{R}} \setminus Z(h, m) = \{n \in N_{\mathbb{R}}; \langle m, n\rangle < h(n)\}$$

is either empty or a convex set. In either case, $H^q(N_{\mathbb{R}} \setminus Z(h, m), \mathbb{C}) = 0$ for $q > 0$ and $H^0(N_{\mathbb{R}}, \mathbb{C}) \to H^0(N_{\mathbb{R}} \setminus Z(h, m), \mathbb{C})$ is surjective. Thus by the long exact sequence mentioned before, we conclude

$$H^q_{Z(h,m)}(N_{\mathbb{R}}, \mathbb{C}) = 0 \quad \text{for all} \quad q > 0 \quad \text{and all} \quad m \in M \ ,$$

hence $H^q(X, \mathcal{O}_X(D_h)) = 0$ for $q > 0$ by Theorem 2.6. q.e.d.

Since $h := 0$ is upper convex, Theorem 2.7 implies:

Corollary 2.8. *For a compact toric variety X, we have*

$$H^q(X, \mathcal{O}_X) = 0 \quad \textit{for all} \quad q > 0 \ .$$

Corollary 2.9. *For a compact toric variety $X = T_N \operatorname{emb}(\Delta)$ and an upper convex $h \in \mathrm{SF}(N, \Delta)$, we have*

$$\dim_{\mathbb{C}} H^q(X, \mathcal{O}_X(D_h)) = \begin{cases} {}^\#(M \cap \square_h) & \textit{for} \quad q = 0 \\ 0 & \textit{for} \quad q \geq 1 \end{cases} ,$$

where ${}^\#(M \cap \square_h)$ is the cardinality of the set $M \cap \square_h$ of lattice points in the convex polytope $\square_h$.

This result enables us to relate algebraic geometry to the geometry of convex bodies as in Sect. 2.4. For that purpose, we here derive from Corollary 2.9 a basic relationship between the intersection numbers for Cartier divisors and the mixed volumes of compact convex bodies.

The *Euler-Poincaré characteristic* of a coherent $\mathcal{O}_Y$-module on an r-dimensional compact algebraic variety Y over $\mathbb{C}$ is defined to be

$$\chi(Y, \mathscr{F}) := \sum_{j=0}^{r} (-1)^j \dim_{\mathbb{C}} H^j(Y, \mathscr{F}) \ .$$

By a basic result due to Snapper and Kleiman (see, e.g., [K5]), Cartier divisors $D_1, \dots, D_s$ on Y determine a polynomial $P(z_1, \dots, z_s)$ with coefficients in $\mathbb{Q}$ such that

$$\chi(Y, \mathscr{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(v_1 D_1 + \dots + v_s D_s)) = P(v_1, \dots, v_s)$$

for all integers $v_1, \dots, v_s$. In particular, $P(z_1, \dots, z_s)$ is $\mathbb{Z}$-valued on $\mathbb{Z}^s$ and has total degree not greater than the dimension of the support $\operatorname{supp}(\mathscr{F})$ of $\mathscr{F}$.

When $s \geq \dim \operatorname{supp}(\mathscr{F})$, we define the intersection number

$$(D_1 \,.\, D_2 \,.\, \dots \,.\, D_s ; \mathscr{F})$$

to be the coefficient of the monomial $z_1 z_2 \dots z_s$ in $P(z_1, \dots, z_s)$. It is actually an integer and depends only on the invertible $\mathcal{O}_Y$-modules $\mathscr{L}_1 = \mathcal{O}_Y(D_1), \dots, \mathscr{L}_s = \mathcal{O}_Y(D_s)$ by definition. We thus denote it also by $(\mathscr{L}_1 \,.\, \mathscr{L}_2 \,.\, \dots \,.\, \mathscr{L}_s ; \mathscr{F})$. The maps

$$\operatorname{CDiv}(Y) \times \dots \times \operatorname{CDiv}(Y) \to \mathbb{Z} \quad \text{and} \quad \operatorname{Pic}(Y) \times \dots \times \operatorname{Pic}(Y) \to \mathbb{Z}$$

which send $(D_1, \dots, D_s)$ to $(D_1 \,.\, D_2 \,.\, \dots \,.\, D_s ; \mathscr{F})$ and $(\mathscr{L}_1, \dots, \mathscr{L}_s)$ to $(\mathscr{L}_1 \,.\, \mathscr{L}_2 \,.\, \dots \,.\, \mathscr{L}_s ; \mathscr{F})$, respectively, are seen to be s-ply linear.

When $\mathscr{F} = \mathcal{O}_V$ for an s-dimensional closed subvariety $V \subset Y$, we denote the intersection numbers above simply by

$$(D_1 \,.\, D_2 \,.\, \dots \,.\, D_s ; V) \quad \text{and} \quad (\mathscr{L}_1 \,.\, \mathscr{L}_2 \,.\, \dots \,.\, \mathscr{L}_s ; V) \ .$$

When Y is nonsingular, the former coincides with the usual intersection number $(D_1 \,.\, D_2 \,.\, \dots \,.\, D_s \,.\, V)$ as cycles. In particular when $V = Y$ and $s = r$, we write them simply as

$$(D_1 \,.\, D_2 \,.\, \dots \,.\, D_r) \quad \text{and} \quad (\mathscr{L}_1 \,.\, \mathscr{L}_2 \,.\, \dots \,.\, \mathscr{L}_r) \ .$$

Furthermore, we denote them by (D^r) and $(\mathscr{L}^r)$, when $D_1 = \dots = D_r = D$ and $\mathscr{L}_1 = \dots = \mathscr{L}_r = \mathscr{L}$. In this case, we have a prototype for the Riemann-Roch theorem, that is,

$$\chi(Y, \mathcal{O}_Y(vD)) = \frac{(D^r)}{r!} v^r + (\text{lower order terms in } v)$$

for integers v.

On the other hand, let us normalize the *Lebesgue measure* vol_r on $M_{\mathbb{R}} \cong \mathbb{R}^r$ so that the r-dimensional parallelepiped $\{\sum_{1 \leq j \leq r} \lambda_j m_j ; 0 \leq \lambda_j \leq 1 \text{ for all } j\}$ for a $\mathbb{Z}$-basis $\{m_1, \dots, m_r\}$ of M has volume one. We have $\operatorname{vol}_r(vK) = v^r \operatorname{vol}_r(K)$ for each compact convex body $K \subset M_{\mathbb{R}}$ and each positive real number v. As in Sect. A.4, we define the Minkowski sum for compact convex bodies $K_1, \ldots, K_r \subset M_{\mathbb{R}}$ and positive real numbers $v_1, \ldots, v_r$ to be $v_1 K_1 + \ldots + v_r K_r := \{v_1 x_1 + \ldots + v_r x_r \,;\, x_1 \in K_1, \ldots, x_r \in K_r\}$. Then

$$\frac{1}{r!} \operatorname{vol}_r(v_1 K_1 + \ldots + v_r K_r)$$

turns out to be a homogeneous polynomial of degree r in $v_1, \ldots, v_r$. We denote its coefficient for $v_1 v_2 \ldots v_r$ by $V_r(K_1, K_2, \ldots, K_r)$ and call it the *mixed volume* of $K_1, \ldots, K_r$. When $K_1 = \ldots = K_r = K$, we have $V_r(K, K, \ldots, K) = \operatorname{vol}_r(K)$.

A convex polytope $\square$ in $M_{\mathbb{R}}$ is said to be *integral* if all of its vertices belong to M. The Minkowski sum $v_1 \square_1 + \ldots + v_s \square_s$ for integral convex polytopes $\square_1, \ldots, \square_s$ and positive integers $v_1, \ldots, v_s$ is an integral convex polytope. The cardinality

$$^{\#}(M \cap (v_1 \square_1 + \ldots + v_s \square_s))$$

of the set of its elements belonging to M is known to be a polynomial in $v_1, \ldots, v_s$ of total degree not greater than r with coefficients in $\mathbb{Q}$. In particular, $^{\#}(M \cap v\square)$ for an integral convex polytope $\square$ and a positive integer v is a polynomial in v of degree not greater than r and is called the *Hilbert polynomial* for $\square$. This fact is originally due to Macdonald [M4]. Corollary 2.9 gives another proof, as we see immediately below.

The coefficient for v^r of the Hilbert polynomial for an integral convex polytope $\square$ coincides with $\operatorname{vol}_r(\square)$, since

$$\lim_{v \to \infty} \frac{^{\#}(M \cap v\square)}{v^r} = \lim_{v \to \infty} {}^{\#}(\square \cap (1/v)M) \cdot (1/v)^r = \operatorname{vol}_r(\square) \ .$$

By Corollary 2.9, we have

$$\chi(X, \mathcal{O}_X(vD_h)) = {}^{\#}(M \cap v\square_h) \quad \text{for positive integers} \quad v \ .$$

The left hand side, hence the right hand side, are polynomials in v of degree not greater than r. Thus we can relate the intersection numbers and the mixed volumes as follows:

Proposition 2.10. *Let $X = T_N \operatorname{emb}(\Delta)$ be an r-dimensional compact toric variety. If $h \in \mathrm{SF}(N, \Delta)$ is upper convex, then $\mathcal{O}_X(D_h)$ is generated by global sections, while $\square_h$ is an integral convex polytope. We then have*

$$\frac{1}{r!} (D_h^r) = \operatorname{vol}_r(\square_h) \ .$$

More generally, for upper convex $h_1, \ldots, h_r \in \mathrm{SF}(N, \Delta)$, we get

$$\frac{1}{r!} (D_{h_1} \cdot D_{h_2} \cdot \ldots \cdot D_{h_r}) = V_r(\square_{h_1}, \ldots, \square_{h_r}) \ .$$

Note that (D_h^r) and $(D_{h_1} \cdot \ldots \cdot D_{h_r})$ on the left hand side make sense for $h, h_1, \ldots, h_r \in \mathrm{SF}(N, \Delta)$ which need not be upper convex. As we see in Sect. 2.4, Teissier [T3] used this fact to generalize a question on the isoperimetric inequality for compact convex bodies to one in algebraic geometry.

Let us now compute intersection numbers in some cases which we already encountered in Chap. 1.

The proof for the following result is obvious. We need it to apply what we have just seen to T_N-stable closed irreducible subvarieties.

Lemma 2.11. *Let $X = T_N\operatorname{emb}(\Delta)$ be a compact toric variety. For $\tau \in \Delta$, consider $\bar{N}(\tau) := N/\mathbb{Z}(\tau \cap N)$, the fan $\bar{\Delta}(\tau)$ in it and the T_N-stable closed irreducible subvariety $V(\tau) = T_{\bar{N}(\tau)}\operatorname{emb}(\bar{\Delta}(\tau))$ of X as in Corollary 1.7. If the restriction of $h \in \mathrm{SF}(N, \Delta)$ to τ coincides with that of $l_\tau \in M$, then $\bar{h} := h - l_\tau$ is a well-defined element in $\mathrm{SF}(\bar{N}(\tau), \bar{\Delta}(\tau))$ independent of a particular choice of l_τ and we have a canonical isomorphism*

$$\mathcal{O}_X(D_h) \otimes_{\mathcal{O}_X} \mathcal{O}_{V(\tau)} = \mathcal{O}_{V(\tau)}(D_{\bar{h}}) \ .$$

Example. Let $X = T_N\operatorname{emb}(\Delta)$ be an r-dimensional compact toric variety. We have $V(\tau) \cong \mathbb{P}_1(\mathbb{C})$ for any $\tau \in \Delta(r-1)$. τ is the common face of exactly two σ', $\sigma'' \in \Delta(r)$ and we have

$$n' + n'' \in \mathbb{R}\tau \ , \qquad \sigma' + \mathbb{R}\tau = \mathbb{R}_{\geq 0}n' + \mathbb{R}\tau \ , \qquad \sigma'' + \mathbb{R}\tau = \mathbb{R}_{\geq 0}n'' + \mathbb{R}\tau$$

for primitive elements n' and n'' in N by Corollary 1.7. If the restriction of $h \in \mathrm{SF}(N, \Delta)$ to τ coincides with that of $l_\tau \in M$, then the restriction of $\mathcal{O}_X(D_h)$ to $V(\tau) \cong \mathbb{P}_1(\mathbb{C})$ coincides with $\mathcal{O}_{V(\tau)}(D_{\bar{h}})$ for $\bar{h} := h - l_\tau$ by Lemma 2.11. Its degree as an invertible sheaf on $\mathbb{P}_1(\mathbb{C})$ is

$$(D_h ; V(\tau)) = -(\bar{h}(n') + \bar{h}(n'')) \ .$$

Example. Let us look at the above example when X is two-dimensional and nonsingular. In this case, we have $\tau = \mathbb{R}_{\geq 0}n(\tau)$ for a primitive element $n(\tau) \in N$. For a suitable choice of n', n'' in N, $\{n', n(\tau)\}$ and $\{n'', n(\tau)\}$ are $\mathbb{Z}$-bases of N. As we saw in Proposition 1.19, there exists an integer a such that

$$n' + n'' + an(\tau) = 0 \ , \qquad \sigma' = \mathbb{R}_{\geq 0}n' + \tau \ , \qquad \sigma'' = \mathbb{R}_{\geq 0}n'' + \tau \ .$$

Since X is nonsingular, $V(\tau)$ itself is a T_N-invariant Cartier divisor on X. The corresponding $h \in \mathrm{SF}(N, \Delta)$ vanishes on $N_{\mathbb{R}} \setminus (\sigma' \cup \sigma'')$, while $h(\xi n' + \eta n(\tau)) = -\eta$ and $h(\xi n'' + \eta n(\tau)) = -\eta$ for nonnegative real numbers ξ and η. Let $\{m', m\}$ be the $\mathbb{Z}$-basis of M dual to $\{n', n(\tau)\}$. We may take $l_\tau = -m$, since $h(n(\tau)) = -1 = \langle -m, n(\tau) \rangle$ and $h(n') = 0 = \langle -m, n' \rangle$. Hence we may let $\bar{h} := h + m$. Since $h(n'') = 0$ and $n'' = -n' - an(\tau)$, we get

$$(V(\tau)^2) = (D_h ; V(\tau)) = -(\bar{h}(n') + \bar{h}(n'')) = a$$

(cf. Fig. 2.1).

Example. More generally, let $X = T_N\operatorname{emb}(\Delta)$ be an r-dimensional compact nonsingular toric variety. $V(\varrho_1), \ldots, V(\varrho_r)$ for $\varrho_1, \ldots, \varrho_r \in \Delta(1)$ are T_N-invariant Cartier divisors on X. If $\varrho_1, \ldots, \varrho_r$ are all distinct, then we obviously have the

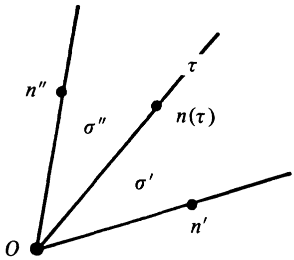


Fig. 2.1

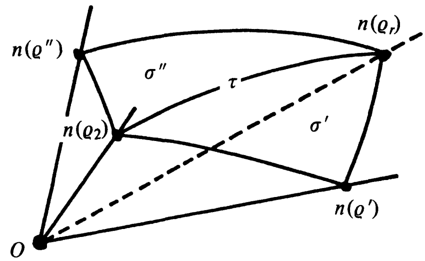


Fig. 2.2

following intersection number

$$(V(\varrho_1) \,.\, V(\varrho_2) \,.\, \ldots \,.\, V(\varrho_r)) = \begin{cases} 1 & \text{if} \quad \varrho_1 + \varrho_2 + \ldots + \varrho_r \in \Delta \\ 0 & \text{otherwise} \end{cases} .$$

Let us consider the case where $\varrho_1 = \varrho_2$ and where $\varrho_2, \ldots, \varrho_r$ are distinct. If $\tau := \varrho_2 + \ldots + \varrho_r$ does not belong to Δ, then the intersection number vanishes. On the other hand, if $\tau \in \Delta$, then as in the previous example, there exist $\varrho', \varrho'' \in \Delta(1)$ and integers $a_2, \ldots, a_r$ such that

$$n(\varrho') + n(\varrho'') + \sum_{j=2}^{r} a_j n(\varrho_j) = O \ , \qquad \varrho' + \tau \in \Delta(r) \ , \qquad \varrho'' + \tau \in \Delta(r) \ .$$

Since X is nonsingular, $\{n(\varrho'), n(\varrho_2), \ldots, n(\varrho_r)\}$ is a $\mathbb{Z}$-basis of N. Let $\{m', m_2, \ldots, m_r\}$ be the dual basis for M. $h \in \mathrm{SF}(N, \Delta)$ corresponding to the T_N-invariant Cartier divisor $V(\varrho_2)$ satisfies $h(n(\varrho_2)) = -1$ and $h(n(\varrho')) = h(n(\varrho''))$ $= h(n(\varrho_3)) = \ldots = h(n(\varrho_r)) = 0$. In particular, the restriction of h to τ coincides with that of $-m_2$ and we may let $\bar{h} := h + m_2$. Hence

$$(V(\varrho_2) \,.\, V(\varrho_2) \,.\, V(\varrho_3) \,.\, \ldots \,.\, V(\varrho_r)) = (V(\varrho_2); V(\tau)) = -(\bar{h}(n(\varrho')) + \bar{h}(n(\varrho''))) = a_2$$

(cf. Fig. 2.2).

2.3 Equivariant Holomorphic Maps to Projective Spaces

In this section, we study equivariant holomorphic maps from compact toric varieties to projective spaces. As in the previous section, we fix a compact toric variety $X = T_N \operatorname{emb}(\Delta)$ corresponding to a finite complete fan (N, Δ) and a T_N-invariant Cartier divisor D_h on X determined by $h \in \mathrm{SF}(N, \Delta)$.

When $\mathcal{O}_X(D_h)$ is generated by global sections, i.e., h is upper convex, we have a holomorphic map from X to a projective space. Namely, we denote $M \cap \square_h = \{m_0, \ldots, m_s\}$ and define a holomorphic map $\psi_h : X \to \mathbb{P}_s(\mathbb{C})$ by

$$\psi_h(x) := [\mathbf{e}(m_0)(x) : \mathbf{e}(m_1)(x) : \ldots : \mathbf{e}(m_s)(x)] \quad \text{for} \quad x \in X$$

in homogeneous coordinates. ψ_h is clearly equivariant with respect to the action of T_N on X and that of $(\mathbb{C}^\times)^{s+1}/\mathbb{C}^\times \cong (\mathbb{C}^\times)^s$ on $\mathbb{P}_s(\mathbb{C})$.

We now determine when ψ_h is a closed embedding. We remark in passing that [TE, Chap. I, § 3, Theorem 9] is a generalization valid even if Δ need not be complete. First of all, the following should be obvious by Corollary A.19:

Lemma 2.12. *Let Δ be a finite complete fan in $N \cong \mathbb{Z}^r$. The following conditions are equivalent for $h \in \mathrm{SF}(N, \Delta)$ and the subset $\{l_\sigma; \sigma \in \Delta(r)\} \subset M$ determined uniquely by h. In this case, h is said to be* strictly upper convex *with respect to Δ.*

(a) *h is upper convex, and Δ is the coarsest among the fans Δ' in N for which h is Δ'-linear.*

(b) *For any $\sigma \in \Delta(r)$ and any $n \in N_{\mathbb{R}}$, we have $\langle l_\sigma, n\rangle \geq h(n)$ with the equality holding if and only if $n \in \sigma$.*

(c) *The integral convex polytope $\square_h := \{m \in M_{\mathbb{R}}; \langle m, n\rangle \geq h(n), \forall n \in N_{\mathbb{R}}\}$ is r-dimensional and has exactly $\{l_\sigma; \sigma \in \Delta(r)\}$ as the set of its vertices. Moreover, $l_\sigma \neq l_\tau$ holds for each pair $\sigma \neq \tau$ in $\Delta(r)$.*

When these equivalent conditions are satisfied, the following correspondence between Δ and the set of nonempty faces of $\square_h$ are inverse to each other:

$$\square_h > F \mapsto F^\dagger := \{n \in N_{\mathbb{R}}; \langle m, n\rangle = h(n), \forall m \in F\} \in \Delta$$

$$\Delta \ni \sigma \mapsto \sigma^\dagger := \{m \in \square_h; \langle m, n\rangle = h(n), \forall n \in \sigma\} < \square_h .$$

We have $\dim F + \dim F^\dagger = r$ *and* $\dim \sigma + \dim \sigma^\dagger = r$.

Theorem 2.13. *Let $X = T_N\,\mathrm{emb}(\Delta)$ be an r-dimensional compact toric variety. Then the following are equivalent for $h \in \mathrm{SF}(N, \Delta)$:*

(a) *$\mathcal{O}_X(D_h)$ is* very ample, *i.e., it is generated by global sections and $\psi_h: X \to \mathbb{P}_s(\mathbb{C})$ is a closed embedding.*

(b) *h is strictly upper convex with respect to Δ. Moreover, for each $\sigma \in \Delta(r)$, the subset $M \cap \square_h - l_\sigma$ generates the semigroup $M \cap \sigma^\vee$.*

(c) *The integral convex polytope $\square_h$ is r-dimensional and has exactly $\{l_\sigma; \sigma \in \Delta(r)\}$ as the set of its vertices. Moreover, $M \cap \square_h - l_\sigma$ generates the semigroup $M \cap \sigma^\vee$ for each $\sigma \in \Delta(r)$.*

Proof. The equivalence of (b) and (c) is clear by Lemma 2.12.

By Theorem 2.7, $\mathcal{O}_X(D_h)$ is generated by global sections if and only if h is upper convex. In that case, $\square_h$ is the convex hull of $\{l_\sigma; \sigma \in \Delta(r)\}$ and the equalities $h(n) = \inf\{\langle m, n\rangle; m \in M \cap \square_h\} = \inf\{\langle l_\sigma, n\rangle; \sigma \in \Delta(r)\}$ hold for all $n \in N_{\mathbb{R}}$. The holomorphic map $\psi_h: X \to \mathbb{P}_s(\mathbb{C})$ is then defined for $x \in X$ by $\psi_h(x) := [\mathbf{e}(m_0)(x): \ldots : \mathbf{e}(m_s)(x)]$ with $M \cap \square_h = \{m_0, \ldots, m_s\}$.

For $\sigma \in \Delta(r)$, we have $l_\sigma \in M \cap \square_h$ and $M \cap \square_h - l_\sigma \subset M \cap \sigma^\vee$. By renumeration, if necessary, we may assume that $l_\sigma = m_0$, hence $\psi_h(x) = [\mathbf{e}(m_0 - l_\sigma)(x): \ldots : \mathbf{e}(m_s - l_\sigma((x)] = [1 : \mathbf{e}(m_1 - m_0)(x): \ldots : \mathbf{e}(m_s - m_0)(x)]$ for $x \in U_\sigma$. Thus the restriction $U_\sigma \to \{[z_0 : \ldots : z_s] \in \mathbb{P}_s(\mathbb{C}); z_0 \neq 0\}$ of ψ_h to U_σ is a closed embedding if and only if $\{m_1 - m_0, \ldots, m_s - m_0\} = M \cap \square_h - l_\sigma$ generates the semigroup $M \cap \sigma^\vee$.

Suppose (a) is satisfied. By what we have just seen, $M \cap \square_h - l_\sigma$ generates the semigroup $M \cap \sigma^\vee$ for each $\sigma \in \Delta(r)$. Taking the dual cone, we get

$$\sigma = (M \cap \sigma^\vee)^\vee = (M \cap \square_h - l_\sigma)^\vee = \{n \in N_{\mathbb{R}}; \langle m - l_\sigma, n\rangle \geq 0, \forall m \in M \cap \square_h\}$$
$$= \{n \in N_{\mathbb{R}}; h(n) \geq \langle l_\sigma, n\rangle\} .$$

Since the opposite inequality $\langle l_\sigma, n\rangle \geq h(n)$ holds for all $n \in N_{\mathbb{R}}$ by assumption and by Theorem 2.7, we have (b) by Lemma 2.12.

To show (c)$\Rightarrow$(a), let $V_j := \{[z_0 : \ldots : z_s] \in \mathbb{P}_s(\mathbb{C}); z_j \neq 0\}$ for $0 \leq j \leq s$. If $m_k = l_\sigma$ for some $\sigma \in \Delta(r)$, then the restriction $U_\sigma \to V_k$ of ψ_h is a closed embedding by (c) and by what we have seen above.

As for general j, $\psi_h^{-1}(V_j) = \cup_{\sigma \in \Delta(r)} \psi_h^{-1}(V_j) \cap U_\sigma$ coincides with U_τ for some $\tau \in \Delta$. Indeed, $\psi_h^{-1}(V_j) \cap U_\sigma = \{x \in U_\sigma; \mathbf{e}(m_j - l_\sigma)(x) \neq 0\}$ is the open subset of U_σ corresponding to the face $\sigma \cap (m_j - l_\sigma)^\perp = \{n \in \sigma; \langle m_j, n\rangle = h(n)\}$ of σ. But by (c) and Lemma 2.12, $\tau := \{n \in N_{\mathbb{R}}; \langle m_j, n\rangle = h(n)\}$ necessarily belongs to Δ.

Since τ is a face of some $\sigma \in \Delta(r)$ and since the restriction $U_\sigma \to V_k$ of ψ_h is a closed embedding if $l_\sigma = m_k$, we see that $\psi_h^{-1}(V_j) = U_\tau \to V_j \cap V_k \subset V_j$ is a closed embedding as well. Thus so is $\psi_h : X \to \mathbb{P}_s(\mathbb{C})$. q.e.d.

Corollary 2.14. *Let $X = T_N \operatorname{emb}(\Delta)$ be an r-dimensional compact toric variety. Then the following are equivalent for $h \in \mathrm{SF}(N, \Delta)$:*

(a) *$\mathcal{O}_X(D_h)$ is* ample, *i.e., $\mathcal{O}_X(\nu D_h)$ is very ample for large enough positive integers ν.*

(b) *h is strictly upper convex with respect to Δ.*

(c) *The integral convex polytope $\square_h$ is r-dimensional and has exactly $\{l_\sigma; \sigma \in \Delta(r)\}$ as the set of its vertices. Moreover, $l_\sigma \neq l_\tau$ holds for each pair $\sigma \neq \tau$ in $\Delta(r)$.*

Proof. By Theorem 2.13, it suffices to show that the semigroup $M \cap \sigma^\vee$ for each $\sigma \in \Delta(r)$ is generated by $M \cap \square_{\nu h} - \nu l_\sigma$ for ν large enough, if h is strictly upper convex with respect to Δ.

Let $m \in M \cap \sigma^\vee$. If n belongs to σ, then we have $\langle m, n\rangle \geq 0$ hence $\langle m + \nu l_\sigma, n\rangle \geq \nu h(n)$ for ν nonnegative. On the other hand, if n is not in σ, then $\langle l_\sigma, n\rangle > h(n)$ by assumption, hence $\langle m + \nu l_\sigma, n\rangle = \langle m, n\rangle + \nu(\langle l_\sigma, n\rangle - h(n)) + \nu h(n) \geq \nu h(n)$ for ν large enough. Consequently, we get $m + \nu l_\sigma \in M \cap \square_{\nu h}$.

The assertion follows by the above argument applied to each member m of a finite system of generators of the semigroup $M \cap \sigma^\vee$. q.e.d.

In general, we cannot let $\nu = 1$ in Corollary 2.14, (a). For instance, take as h the function for which $\square_h$ is the tetrahedron appearing in the terminal lemma in Sect. 1.6. When X is nonsingular, however, we can always let $\nu = 1$ as follows:

Corollary 2.15 (Demazure [D5]). *Let $X = T_N \operatorname{emb}(\Delta)$ be an r-dimensional compact toric variety which is nonsingular. The following are equivalent for $h \in \mathrm{SF}(N, \Delta)$:*

(a) *$\mathcal{O}_X(D_h)$ is very ample.*

(b) *$\mathcal{O}_X(D_h)$ is ample.*

(c) *h is strictly upper convex with respect to Δ.*

(d) *The integral convex polytope $\square_h$ is r-dimensional and has exactly $\{l_\sigma; \sigma \in \Delta(r)\}$ as the set of its vertices. Moreover, $l_\sigma \neq l_\tau$ holds for each pair $\sigma \neq \tau$ in $\Delta(r)$.*

If these equivalent conditions are satisfied, then the r-dimensional integral convex polytope $\square_h$ is absolutely simple, *in the sense that each vertex l_σ is incident to exactly r edges and that $\{m^{(1)} - l_\sigma, \ldots, m^{(r)} - l_\sigma\}$ is a $\mathbb{Z}$-basis of M, where $m^{(1)}, \ldots, m^{(r)} \in M$ on the r edges are right next to the vertex l_σ.*

Proof. Suppose h is strictly upper convex with respect to Δ. By Theorem 2.13 and Corollary 2.14, it suffices to show that the semigroup $M \cap \sigma^\vee$ for each $\sigma \in \Delta(r)$ is generated by $M \cap \square_h - l_\sigma$.

Since σ is nonsingular by definition, there exists, by Theorem 1.10 in Sect. 1.4, a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N such that $\sigma = \mathbb{R}_{\geq 0} n_1 + \ldots + \mathbb{R}_{\geq 0} n_r$. For $j = 1, \ldots, r$, let τ_j be the $(r-1)$-dimensional face of σ which we obtain by omitting $\mathbb{R}_{\geq 0} n_j$ in the above summation. By Lemma 2.12, the corresponding $\tau_1^\dagger, \ldots, \tau_r^\dagger$ are the edges of $\square_h$ incident to the vertex $l_\sigma = \sigma^\dagger$. Suppose $m^{(j)} \in M$ on the edge $\tau_j^\dagger$ is right next to the vertex l_σ. Then for $k = 1, \ldots, r$, we have

$$\langle m^{(j)}, n_k \rangle = h(n_k) = \langle l_\sigma, n_k \rangle \quad \text{if} \quad k \neq j$$

$$\langle m^{(j)}, n_j \rangle > h(n_j) = \langle l_\sigma, n_j \rangle \ .$$

Hence $\langle m^{(j)} - l_\sigma, n_k \rangle = 0$ for $k \neq j$, while $\langle m^{(j)} - l_\sigma, n_j \rangle > 0$. By the very choice of $m^{(j)}$, we have $\langle m^{(j)} - l_\sigma, n_j \rangle = 1$. Hence $\{m^{(1)} - l_\sigma, \ldots, m^{(r)} - l_\sigma\}$ is the $\mathbb{Z}$-basis of M dual to $\{n_1, \ldots, n_r\}$, and obviously generates the semigroup $M \cap \sigma^\vee$. q.e.d.

By Corollaries 2.5 and 2.14 we have the following:

Corollary 2.16. *A compact toric variety $X = T_N \operatorname{emb}(\Delta)$ is a projective variety, i.e., can be embedded holomorphically into a projective space as a closed subvariety, if and only if there exists h in* SF(N, Δ) *which is strictly upper convex with respect to Δ.*

On the one hand, compact toric varieties of dimensions one and two are necessarily projective varieties (cf. [MO, §8, Proposition 8.1]). On the other hand, there exist many compact nonsingular toric varieties of dimension three which are nonprojective. Demazure [D5] constructed a rather complicated example. Thanks to our classification in Sect. 1.7, the following example turned out to be the simplest among nonprojective but compact nonsingular toric varieties of dimension three. Much earlier, Nagata and Hironaka constructed examples of nonprojective but complete nonsingular algebraic varieties in dimension three.

Example (cf. [MO, Proposition 9.4]). In a $\mathbb{Z}$-module N with a $\mathbb{Z}$-basis $\{n_1, n_2, n_3\}$, let $n_0 := -n_1 - n_2 - n_3$, $n_1' := n_0 + n_1 = -n_2 - n_3$, $n_2' := n_0 + n_2 = -n_1 - n_3$, $n_3' := n_0 + n_3 = -n_1 - n_2$. Then the fan Δ in N consisting of the faces of the following ten three-dimensional cones is a nonsingular and finite complete fan (cf. Fig. 2.3):

$$\begin{array}{ll}
\mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_3 \ , & \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_1' \ , \\
\mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_3 + \mathbb{R}_{\geq 0} n_2' \ , & \mathbb{R}_{\geq 0} n_3 + \mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_3' \ , \\
\mathbb{R}_{\geq 0} n_1 + \mathbb{R}_{\geq 0} n_1' + \mathbb{R}_{\geq 0} n_3' \ , & \mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_2' + \mathbb{R}_{\geq 0} n_1' \ , \\
\mathbb{R}_{\geq 0} n_3 + \mathbb{R}_{\geq 0} n_3' + \mathbb{R}_{\geq 0} n_2' \ , & \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_1' + \mathbb{R}_{\geq 0} n_2' \ , \\
\mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_2' + \mathbb{R}_{\geq 0} n_3' \ , & \mathbb{R}_{\geq 0} n_0 + \mathbb{R}_{\geq 0} n_3' + \mathbb{R}_{\geq 0} n_1' \ .
\end{array}$$

In this case, $T_N \operatorname{emb}(\Delta)$ cannot be a projective variety. Indeed, suppose $h \in \mathrm{SF}(N, \Delta)$ were strictly upper convex with respect to Δ. Since n_1', n_2, $n_1' + n_2$ belong to $\mathbb{R}_{\geq 0} n_2 + \mathbb{R}_{\geq 0} n_2' + \mathbb{R}_{\geq 0} n_1' \in \Delta$, we would have $h(n_1' + n_2) = h(n_1') + h(n_2)$. Since no cone in Δ contains n_1, n_2', $n_1 + n_2' = n_1' + n_2$ simultaneously, however, we

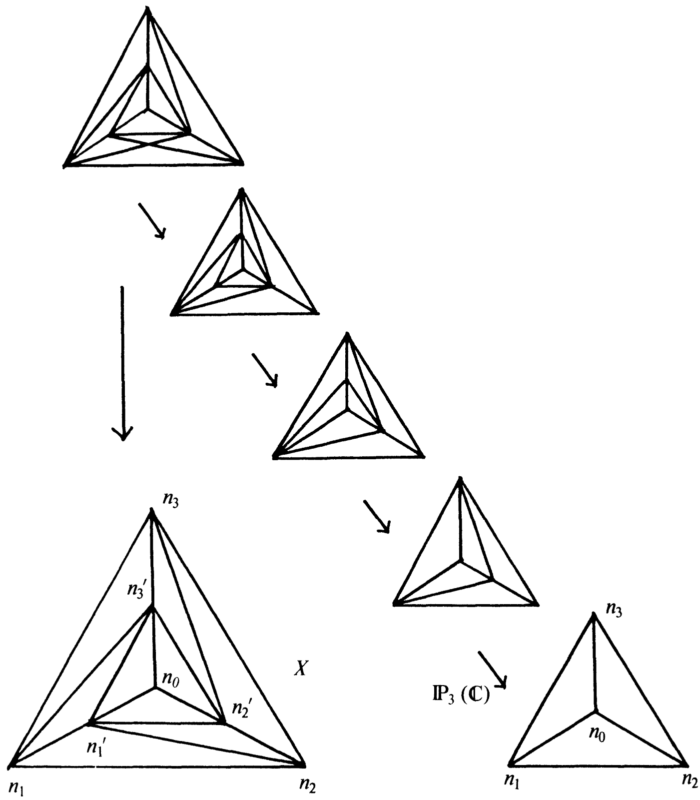


Fig. 2.3

would have $h(n_1+n_2')>h(n_1)+h(n_2')$, hence $h(n_1)+h(n_2')<h(n_1')+h(n_2)$. Similarly, we would have $h(n_2)+h(n_3')<h(n_2')+h(n_3)$ and $h(n_3)+h(n_1')<h(n_3')+h(n_1)$, an obvious contradiction. This $T_N \operatorname{emb}(\Delta)$ might also be of interest from the viewpoint of birational geometry in Sect. 1.7: There exists an equivariant birational holomorphic map from $T_N \operatorname{emb}(\Delta)$ to $\mathbb{P}_3(\mathbb{C})$, which cannot be factored into a composite of equivariant blowing-ups along closed nonsingular subvarieties. From the equivariant blowing-up of $T_N \operatorname{emb}(\Delta)$ along a one-dimensional closed T_N-stable nonsingular subvariety, however, there exists an equivariant holomorphic map to $\mathbb{P}_3(\mathbb{C})$ which can be factored as a composite of equivariant blowing-ups along one-dimensional closed T_N-stable nonsingular subvarieties (see Fig. 2.3 and [MO, Proposition 9.4]).

A convenient sufficient condition can be found in [MO, p. 68, Proposition 9.3] for a three-dimensional compact nonsingular toric variety to be nonprojective. Using it, we can easily determine if each of the toric varieties in Theorem 1.34 is projective or not (cf. [MO, p. 80]).

Proposition 2.17. (Toric Chow's Lemma, cf. Danilov [D1]). *For any compact toric variety* $T_N \operatorname{emb}(\Delta)$, *there exists an equivariant birational holomorphic map* $T_N \operatorname{emb}(\Delta') \to T_N \operatorname{emb}(\Delta)$ *from a toric projective variety.*

Proof. By Corollaries 1.17 and 2.16, it suffices to construct a finite subdivision Δ' of Δ such that some $h \in \mathrm{SF}(N, \Delta')$ is strictly upper convex with respect to Δ'. For that purpose, take all the hyperplanes in $N_{\mathbb{R}}$ each of which contains one of the $(r-1)$-dimensional cones $\tau \in \Delta(r-1)$. The resulting arrangement of hyperplanes certainly determines a finite complete fan Δ' which is also a subdivision of Δ.

Each $\tau \in \Delta(r-1)$ uniquely determines a set $\{m(\tau), -m(\tau)\}$ of primitive elements in M such that $\{m(\tau)\}^{\perp} = \{m(\tau), -m(\tau)\}^{\vee}$ is the hyperplane in $N_{\mathbb{R}}$ containing τ. The function $h: N_{\mathbb{R}} \to \mathbb{R}$ determined by

$$h(n) := -\sum_{\tau \in \Delta(r-1)} |\langle m(\tau), n\rangle| \quad \text{for} \quad n \in N_{\mathbb{R}}$$

certainly belongs to $\mathrm{SF}(N, \Delta')$ and is strictly upper convex with respect to Δ'. q.e.d.

Remark. We refer the reader to Sumihiro [S15, I, Theorem 2] for G-equivariant Chow's lemma, when, more generally, a connected linear algebraic group G acts on an algebraic variety.

The *Nakai criterion* gives, in terms of intersection numbers, a necessary and sufficient condition for a Cartier divisor D on an r-dimensional complete algebraic variety X to be ample. Namely,

$$(D^s; V) := (D.D.\ldots.D; V) > 0$$

for any s with $0 \leq s \leq r$ and for any s-dimensional closed subscheme $V \subset X$, where the intersection number is that considered in Sect. 2.2. When X is a compact nonsingular toric variety, we may assume D to be a T_N-invariant Cartier divisor by Proposition 2.4, (ii). In that case, the above criterion has the following form:

Theorem 2.18. (The Toric Nakai Criterion). *Let $X = T_N \operatorname{emb}(\Delta)$ be an r-dimensional compact nonsingular toric variety. A T_N-invariant Cartier divisor D on X is ample if and only if*

$$(D.V(\tau)) > 0 \quad \textit{for any} \quad \tau \in \Delta(r-1),$$

where $V(\tau) \cong \mathbb{P}_1(\mathbb{C})$ is the T_N-stable closed subvariety corresponding to $\tau \in \Delta(r-1)$ as in Corollary 1.7.

Proof. By Proposition 2.4, (i), we have $D = D_h$ for some $h \in \mathrm{SF}(N, \Delta)$. By Corollary 2.14, it suffices to interpret, in terms of intersection numbers, the condition for h to be strictly upper convex with respect to Δ.

It is not so hard to show that h is strictly upper convex with respect to Δ if and only if h satisfies $h(n'') < \langle l', n''\rangle$ for each $\tau \in \Delta(r-1)$, where n'' and l' are as follows: There exist exactly two $\sigma', \sigma'' \in \Delta(r)$ such that $\tau = \sigma' \cap \sigma''$. We have $\sigma' = \mathbb{R}_{\geq 0} n' + \tau$ and $\sigma'' = \mathbb{R}_{\geq 0} n'' + \tau$ for primitive elements n' and n'' in N. Moreover, by the Δ-linearity of h, there exists $l' \in M$ such that $h(n) = \langle l', n\rangle$ for all $n \in \sigma'$.

Since X is assumed to be nonsingular, there exist $n_2, \ldots, n_r \in N$ such that $\{n', n_2, \ldots, n_r\}$ and $\{n'', n_2, \ldots, n_r\}$ are $\mathbb{Z}$-bases of N and that $\tau = \sum_{2 \leq j \leq r} \mathbb{R}_{\geq 0} n_j$.

Moreover, there exist $a_2, \dots, a_r \in \mathbb{Z}$ such that

$$n' + n'' + \sum_{j=2}^{r} a_j n_j = O$$

as we saw at the end of the previous section. If we denote $\varrho' := \mathbb{R}_{\geq 0} n'$, $\varrho'' := \mathbb{R}_{\geq 0} n''$ and $\varrho_j := \mathbb{R}_{\geq 0} n_j$ for $2 \leq j \leq r$, which all belong to $\Delta(1)$, then for any $\varrho \in \Delta(1)$ we have

$$(V(\varrho) \,.\, V(\tau)) = \begin{cases} 1 & \text{if} \quad \varrho = \varrho' \quad \text{or} \quad \varrho = \varrho'' \\ a_j & \text{if} \quad \varrho = \varrho_j \quad \text{with} \quad 2 \leq j \leq r \\ 0 & \text{otherwise} \end{cases}$$

as in the example at the end of the previous section. Since $D_h = -\sum_{\varrho \in \Delta(1)} h(n(\varrho))\, V(\varrho)$, we have

$$\begin{aligned} (D_h \,.\, V(\tau)) &= -h(n') - h(n'') - \sum_{j=2}^{r} a_j h(n_j) \\ &= -h(n'') - \left\langle l', n' + \sum_{j=2}^{r} a_j n_j \right\rangle \\ &= -h(n'') + \langle l', n'' \rangle \ . \end{aligned}$$

q.e.d.

Remark. By a slight modification, we have a similar result when $X = T_N \operatorname{emb}(\Delta)$ may be singular but Δ is simplicial.

By Proposition A.20 in Sect. A.3, we have the following:

Proposition 2.19. *Suppose that a convex polytope $Q \subset N_{\mathbb{R}}$ contains the origin O in its interior and that all of its vertices belong to $N \otimes_{\mathbb{Z}} \mathbb{Q}$. Define $h: N_{\mathbb{R}} \to \mathbb{R}$ and Δ by*

$$h(n) := \begin{cases} -\inf\{\lambda \in \mathbb{R}_{\geq 0} \,;\, n \in \lambda Q\} & \text{if} \quad n \neq O \\ 0 & \text{if} \quad n = O \end{cases}$$

$$\Delta := \{\mathbb{R}_{\geq 0} Q' \,;\, Q' < Q,\ Q' \neq Q\} \ .$$

Then Δ is a finite complete fan in N and we have $vh \in \mathrm{SF}(N, \Delta)$ for a suitable positive integer v. Moreover, vh is strictly upper convex with respect to Δ, and $Q = \{n \in N_{\mathbb{R}} \,;\, h(n) \geq -1\}$.

As an application, let us now study toric Fano varieties.

In general, an r-dimensional compact complex manifold X is called a *Fano variety*, if it has an ample anticanonical divisor $-K_X$, i.e., the highest exterior power $\bigwedge^r \Theta_X$ of its tangent sheaf Θ_X is an ample invertible $\mathcal{O}_X$-module. A Fano variety of dimension $r = 2$, usually called a *del Pezzo surface*, is known to be either $\mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$ or to be obtained from $\mathbb{P}_2(\mathbb{C})$ by the blowing-up of at most eight points in general position. Iskovskih and Mori-Mukai [MM1], [MM2] completed the classification of *Fano threefolds*, i.e., Fano varieties of dimension three. The reader is refered also to Miyanishi [M8]. Maeda [M5] deals with "logarithmic" Fano varieties.

As a good application of what we have seen in this section, we now explain the classification, due to Batyrev [B2] and Watanabe-Watanabe [WW], of toric Fano varieties in dimensions two and three. A more recent paper Voskresenskii-Klyachko [VK] shows that in each dimension there exist only a finite number of toric Fano varieties up to isomorphism. It also deals with "centrally symmetric" toric Fano varieties in arbitrary dimension and relates them to root systems. As Sakane asked, it might be interesting to find out which toric Fano varieties admit Einstein-Kähler metrics. There has been an attempt in this direction by Mabuchi and Bando (see [BM], [M2], [M3]). The example in Sakane [S2] is a toric Fano threefold. Closely related to this problem is the special case for toric Fano varieties of the moment maps introduced by Jurkiewicz [J1], [J2], Atiyah [A4], [A5], Guillemin-Sternberg [GS] and Ness [N9] (see also Bruguières [B4]). We explain them briefly in Sect. 2.4.

Let us call an r-dimensional compact nonsingular toric variety $X=T_N\operatorname{emb}(\Delta)$ a *toric Fano variety* if it is a Fano variety as well. We call it a *toric del Pezzo surface* if $r=2$, and a *toric Fano threefold* if $r=3$. As we saw in the last example in Sect. 2.1, $-K_X=\sum_{\varrho\in\Delta(1)}V(\varrho)$ is an anticanonical divisor of a toric variety and corresponds to the Δ-linear support function $-k\in\mathrm{SF}(N,\Delta)$ defined by

$$-k(n(\varrho))=-1 \quad \text{for all} \quad \varrho\in\Delta(1)\ .$$

Thus by Corollary 2.15, Theorem 2.18 and Proposition 2.19 we have:

Lemma 2.20. *The following statements are equivalent for an r-dimensional compact nonsingular toric variety* $X=T_N\operatorname{emb}(\Delta)$.

(a) *X is a Fano variety, i.e., $-K_X$ is ample.*

(b) *$-K_X$ is very ample.*

(c) *The Δ-linear support function $-k$ is strictly upper convex with respect to Δ.*

(d) *The integral convex polytope $\square_{-k}:=\{m\in M_{\mathbb{R}};\langle m,n\rangle\geq -k(n),\forall n\in N_{\mathbb{R}}\}$ in $M_{\mathbb{R}}$ is r-dimensional and has exactly $\{l_\sigma;\sigma\in\Delta(r)\}$ as the set of its vertices with $l_\sigma\neq l_\tau$ for each pair $\sigma\neq\tau$ in $\Delta(r)$, where l_σ for $\sigma\in\Delta(r)$ is the unique element of M satisfying*

$$\langle l_\sigma,n\rangle=-k(n) \quad \textit{for all} \quad n\in\sigma\ .$$

(e) *$Q:=\{n\in N_{\mathbb{R}};-k(n)\geq -1\}$ is a simplicial convex polytope with $\{n(\varrho);\varrho\in\Delta(1)\}$ as the set of its vertices.*

(f) *$(-K_X\,.\,V(\tau))>0$ holds for any $\tau\in\Delta(r-1)$.*

Remark. $\square_{-k}$ and Q above are mutually polar convex polytopes in the sense of Sect. A.2.

Proposition 2.21 (Batyrev [B2, Proposition 6] and Watanabe-Watanabe [WW, Proposition (2.7)]). *There exist five distinct toric del Pezzo surfaces up to isomorphism. They correspond to the weighted circular graphs with all weights ≥ -1 in Corollary 1.29. They are* (i) $\mathbb{P}_2(\mathbb{C})$, (ii) $\mathbb{P}_1(\mathbb{C})\times\mathbb{P}_1(\mathbb{C})$, (iii) *the Hirzebruch*

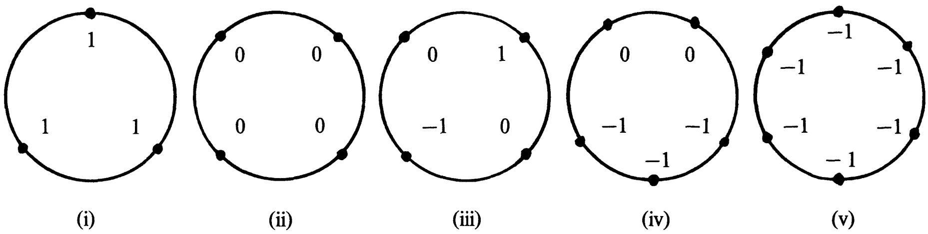


Fig. 2.4

surface F_1, (iv) *the equivariant blowing-up of* $\mathbb{P}_2(\mathbb{C})$ *at two of the* T_N*-fixed points, and* (v) *the equivariant blowing-up of* $\mathbb{P}_2(\mathbb{C})$ *at the three* T_N*-fixed points. The corresponding weighted circular graphs are as in Fig. 2.4.*

Proof. As in the proof of Theorem 1.28, (3), suppose primitive elements $n_1, \ldots, n_s \in N$ go around the origin O of $N_{\mathbb{R}}$ exactly once in this order. If we identify the indices modulo s, then $\{n_{j-1}, n_j\}$ is a $\mathbb{Z}$-basis of N and $n_{j+1}+n_{j-1}+a_j n_j=O$ holds for each j. Let $\varrho_j := \mathbb{R}_{\geq 0} n_j$ for $1 \leq j \leq s$. Then we have $(V(\varrho_j) \,.\, V(\varrho_j))=a_j$ as in the previous section. Moreover, $(V(\varrho_l) \,.\, V(\varrho_j))=1$ for $l=j-1$, $j+1$, while $(V(\varrho_l) \,.\, V(\varrho_j))=0$ for $l \neq j-1, j, j+1$. Since $-K_X=\sum_{1 \leq l \leq s} V(\varrho_l)$, we thus get

$$(-K_X \,.\, V(\varrho_j))=a_j+2 \;.$$

Hence by Theorem 2.18, $-K_X$ is ample if and only if $a_j \geq -1$ for each $1 \leq j \leq s$. By Corollary 1.29, the five cases listed are exactly those satisfying this condition. q.e.d.

Batyrev [B2] and Watanabe-Watanabe [WW] carry out the classification in dimension three as follows:

By Corollary 1.32, an admissible doubly $\mathbb{Z}$-weighted triangulations of S^2 describe each isomorphism class of a three-dimensional compact nonsingular toric variety $X=T_N \operatorname{emb}(\Delta)$. For $\tau \in \Delta(2)$, let a, b be the double $\mathbb{Z}$-weight for the corresponding edge in the triangulation. As in the two-dimensional case above, we have

$$(-K_X \,.\, V(\tau))=a+b+2$$

by what we saw in the last example of Sect. 2.2. By the toric Nakai criterion in Theorem 2.18, X is a Fano variety, i.e., $-K_X$ is ample, if and only if the sum of the double $\mathbb{Z}$-weight satisfies $a+b \geq -1$ for each edge of the corresponding triangulation.

On the other hand, the Picard number of any toric Fano threefold is not greater than five, i.e., ${}^{\#}\Delta(1) \leq 8$ by Corollary 2.5. Batyrev [B2] shows this by clever use of Lemma 2.20, (d), (e), while intersection numbers are used in Watanabe-Watanabe [WW].

Among the admissible doubly $\mathbb{Z}$-weighted triangulations listed in Theorem 1.34, exactly twelve are easily seen to have the sum of the double $\mathbb{Z}$-weight not less than -1 for each edge.

However, those in Theorem 1.34 do not exhaust all Δ with ${}^{\#}\Delta(1) \leq 8$. We also have to take into account those obtained from them by finite successions of star subdivisions described in Corollary 1.27, (b), (c) and Proposition 1.31, (3), (4). In this way, exactly six more turn out to have the property that the sum of the double $\mathbb{Z}$-weight is not less than -1 for each edge. In the examination process, we need the information in Fig. 2.5 on the change of double $\mathbb{Z}$-weights under star subdivisions (cf. (i) and (ii) in the remark immediately after Corollary 1.32).

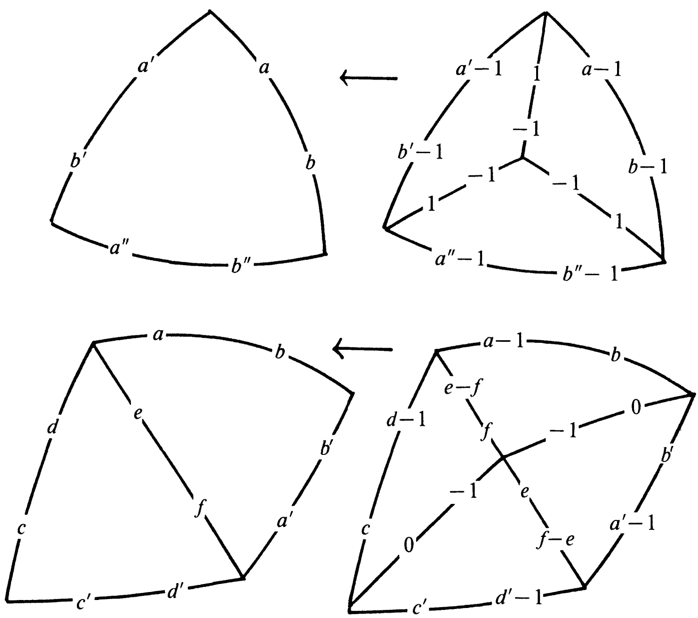


Fig. 2.5

Here is the final result due to Batyrev [B2] and Watanabe-Watanabe [WW], although we use different notation:

Classification of Toric Fano Threefolds. *Up to isomorphism, there exist eighteen distinct toric Fano threefolds. By Corollary 1.32, they correspond to the admissible doubly $\mathbb{Z}$-weighted triangulations of S^2 with the sum of the double $\mathbb{Z}$-weight not less than -1 for each edge. Among them, each of* (11), (12), (14), (15), (16) *and* (18) *below is obtained from one of the others by a finite succession of equivariant blowing-ups.*

(1) $\mathbb{P}_3(\mathbb{C})$.

(2) $\mathbb{P}_2(\mathbb{C})\times\mathbb{P}_1(\mathbb{C})$.

(3) *The* $\mathbb{P}_1(\mathbb{C})$*-bundle* $\mathbb{P}(\mathcal{O}_Y\oplus\mathcal{O}_Y(1))$ *over* $Y=\mathbb{P}_2(\mathbb{C})$.

(4) *The* $\mathbb{P}_1(\mathbb{C})$*-bundle* $\mathbb{P}(\mathcal{O}_Y\oplus\mathcal{O}_Y(2))$ *over* $Y=\mathbb{P}_2(\mathbb{C})$.

(5) *The* $\mathbb{P}_2(\mathbb{C})$*-bundle* $\mathbb{P}(\mathcal{O}_Y\oplus\mathcal{O}_Y\oplus\mathcal{O}_Y(1))$ *over* $Y=\mathbb{P}_1(\mathbb{C})$.

(6) $\mathbb{P}_1(\mathbb{C})\times\mathbb{P}_1(\mathbb{C})\times\mathbb{P}_1(\mathbb{C})$.

(7) *The* $\mathbb{P}_1(\mathbb{C})$*-bundle* $\mathbb{P}(\mathcal{O}_Y\oplus\mathcal{O}_Y(f_1+f_2))$ *over* $Y=\mathbb{P}_1(\mathbb{C})\times\mathbb{P}_1(\mathbb{C})$, *where* f_1 *and* f_2 *are fibers of the two projections from* Y *to* $\mathbb{P}_1(\mathbb{C})$.

(8) $\mathbb{P}(\mathcal{O}_Y\oplus\mathcal{O}_Y(f_1-f_2))$ *in the notation of* (7).

(9) $\mathbb{P}_1(\mathbb{C})\times F_1$ *for the Hirzebruch surface* F_1.

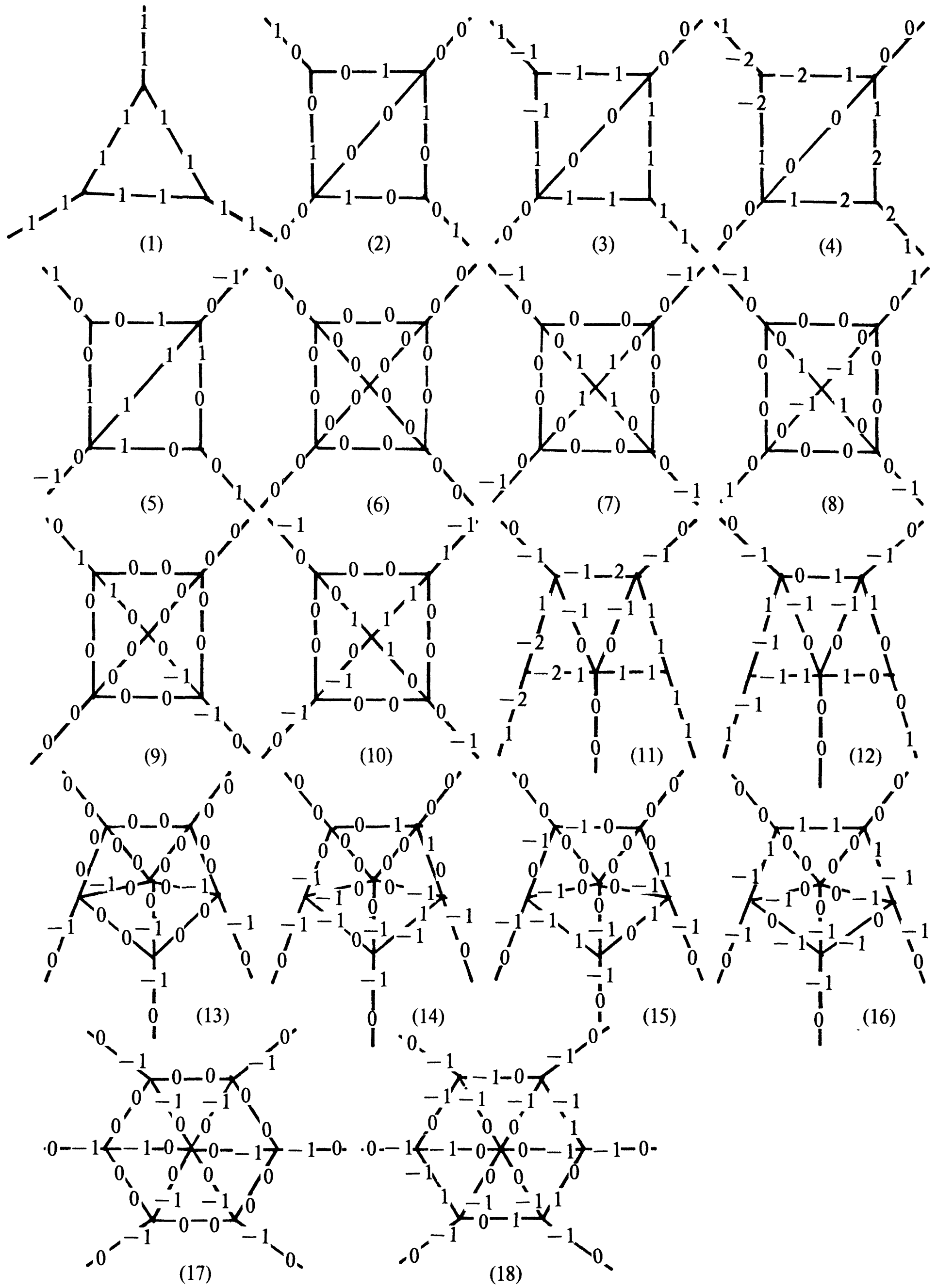


Fig. 2.6

(10) *The* $\mathbb{P}_1(\mathbb{C})$*-bundle* $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(s+f))$ *over* $Y = F_1$, *where* f *is a fiber of the projection from* F_1 *to* $\mathbb{P}_1(\mathbb{C})$, *while* s *is the minimal cross section for the projection with* -1 *as the self-intersection number.*

(13) $\mathbb{P}_1(\mathbb{C}) \times Y_2$, *where* Y_2 *is the toric del Pezzo surface obtained from* $\mathbb{P}_2(\mathbb{C})$ *by the equivariant blowing-up at two of the* T_N*-fixed points.*

(17) $\mathbb{P}_1(\mathbb{C}) \times Y_3$, *where* Y_3 *is the toric del Pezzo surface obtained from* $\mathbb{P}_2(\mathbb{C})$ *by the equivariant blowing-up at the three* T_N*-fixed points.*

Listed in Fig. 2.6 are the admissible doubly $\mathbb{Z}$*-weighted triangulations of* S^2 *corresponding to the isomorphism classes of toric Fano threefolds (see Theorem 1.34 for our convention).*

As in Watanabe-Watanabe [WW], *we show in Fig. 2.7 the equivariant blowing-ups which exist among* (1) *through* (18), *where a solid arrow is an equivariant blowing-up along a* T_N*-stable closed irreducible subvariety of dimension one, while a dotted arrow is an equivariant blowing-up at a* T_N*-fixed point.*

Remark. Let $X' \to X$ be a blowing-up of a compact complex manifold X along a closed submanifold. Obviously, X' need not be a Fano variety even if X is one. Conversely, even if X' is a Fano variety, X may not be one. Indeed, some among (1) through (18) are obtained by the blowing-up of a three-dimensional compact nonsingular toric variety X, which is not a Fano variety, along a T_N-stable closed subvariety of dimension one. As in the remark (iv) immediately after Corollary 1.32, the doubly $\mathbb{Z}$-weighted triangulation of S^2 corresponding to X turns out to have an edge with double $\mathbb{Z}$-weight -1, -1. This seems to be a general phenomenon, as Mori pointed out.

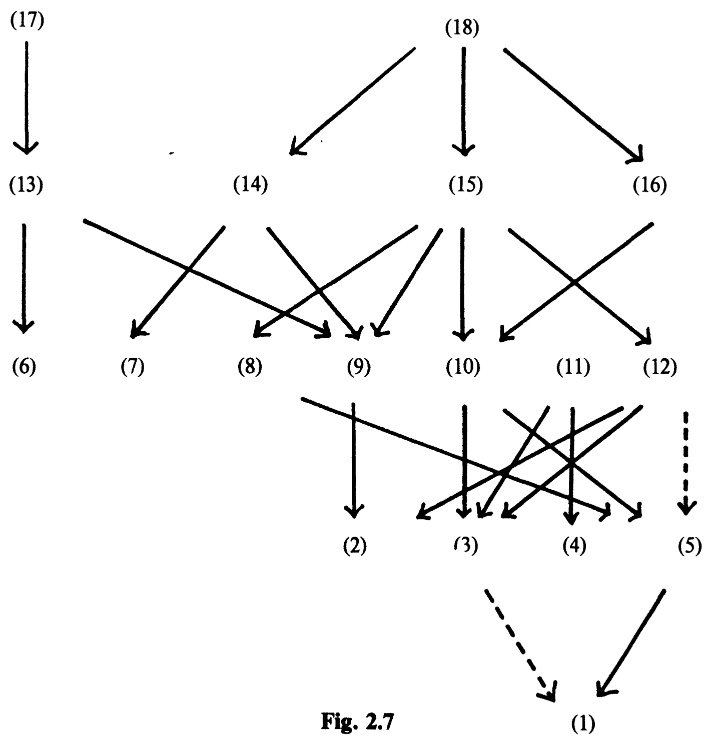


Fig. 2.7

2.4 Toric Projective Varieties

In the previous section, we started with an r-dimensional compact toric variety $X = T_N \operatorname{emb}(\Delta)$ and determined when it is a *toric projective variety*, i.e., can be embedded into a projective space as a closed analytic subspace. In this section, we look at the problem from another direction. Namely, we start with an r-dimensional integral convex polytope in $M_{\mathbb{R}}$ and construct an r-dimensional toric projective variety. We will then be able to interpret convex geometric properties of the former in terms of algebro-geometric properties of the latter. Among relevant examples which we look at is the problem posed by Teissier.

As we saw in Corollary 2.16, an r-dimensional compact toric variety $X = T_N \operatorname{emb}(\Delta)$ is a projective variety if and only if there exists a Δ-linear support function $h \in \mathrm{SF}(N, \Delta)$ which is strictly upper convex with respect to Δ. If that is the case, then

$$\square_h := \{m \in M_{\mathbb{R}}; \langle m, n\rangle \geq h(n),\ \forall n \in N_{\mathbb{R}}\}$$

is an r-dimensional integral convex polytope in $M_{\mathbb{R}}$ by Lemma 2.12. Moreover, we get mutually inverse one-to-one correspondences

$$\square_h > F \mapsto F^{\dagger} := \{n \in N_{\mathbb{R}}; \langle m, n\rangle = h(n),\ \forall m \in F\} \in \Delta$$

$$\Delta \ni \sigma \mapsto \sigma^{\dagger} := \{m \in \square_h; \langle m, n\rangle = h(n),\ \forall n \in \sigma\} < \square_h$$

between Δ and the set of nonempty faces of $\square_h$ such that

$$\dim F + \dim F^{\dagger} = r \quad \text{and} \quad \dim \sigma + \dim \sigma^{\dagger} = r\ .$$

By Theorem 2.13 and Corollary 2.14, $\mathcal{O}_X(\nu D_h)$ for ν large enough is a very ample invertible $\mathcal{O}_X$-module and gives rise to an equivariant closed embedding $\psi_{\nu h}: X \to \mathbb{P}_s(\mathbb{C})$, where $M \cap \nu\square_h = \{m_0, \ldots, m_s\}$ and

$$\psi_{\nu h}(x) := [\mathbf{e}(m_0)(x) : \mathbf{e}(m_1)(x) : \ldots : \mathbf{e}(m_s)(x)] \quad \text{for} \quad x \in X\ .$$

If X is nonsingular, we may choose $\nu = 1$ above by Corollary 2.15, and $\square_h$ is absolutely simple (cf. Corollary 2.15) as an r-dimensional integral convex polytope in $M_{\mathbb{R}}$.

Alternatively, we could also start with an r-dimensional convex polytope Q in $N_{\mathbb{R}}$ which contains the origin O in the interior and whose vertices all belong to $N \otimes_{\mathbb{Z}} \mathbb{Q}$. By Proposition 2.19, there exist a finite complete fan Δ and a Δ-linear support function $\nu h \in \mathrm{SF}(N, \Delta)$ strictly upper convex with respect to Δ. Hence $X := T_N \operatorname{emb}(\Delta)$ is a toric projective variety. According to Proposition A.20, $\square_{\nu h} \subset M_{\mathbb{R}}$ is then the polar polytope of $(1/\nu)Q$ and contains the origin O in the interior.

We now start with an r-dimensional integral convex polytope $\square$ in $M_{\mathbb{R}} \cong \mathbb{R}^r$ and construct a toric projective variety $X_{\square}$.

Theorem 2.22. *For $M \cong \mathbb{Z}^r$, let $\square$ be an r-dimensional integral convex polytope in $M_{\mathbb{R}}$.*

(i) *There exists a unique finite complete fan Δ in $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ such that the support function $h: N_{\mathbb{R}} \to \mathbb{R}$ for $\square$ defined by*

$$h(n) := \inf\{\langle m, n\rangle; m \in \square\} \quad \textit{for} \quad n \in N_{\mathbb{R}}$$

is a Δ-linear support function strictly upper convex with respect to Δ. We denote the corresponding r-dimensional toric projective variety and an ample T_N-invariant Cartier divisor on it by

$$X_\square := T_N \operatorname{emb}(\Delta) \quad \textit{and} \quad D_\square := D_h \ .$$

By parallel translation with respect to $m \in M$, we have $X_{m+\square} = X_\square$, while $D_{m+\square} = D_\square + \operatorname{div}(\mathbf{e}(-m))$. The linear system $|D_\square|$ has no base points.

(ii) *For each face F of $\square$, let $F^\dagger := \{n \in N_{\mathbb{R}}; \langle m, n\rangle = h(n), \forall m \in F\} \in \Delta$ and let $V(F^\dagger)$ be the closure of the T_N-orbit* $\operatorname{orb}(F^\dagger)$. *In this way, we get a bijective correspondence from the set $\mathscr{F}(\square)\setminus\{\emptyset\}$ of nonempty faces of $\square$ and that of T_N-stable irreducible closed subspaces of $X_\square$. We have* $\dim F = \dim_{\mathbb{C}} V(F^\dagger)$ *and*

$$F_1 < F_2 \Leftrightarrow V(F_1^\dagger) \subset V(F_2^\dagger)$$

for F_1, $F_2 \in \mathscr{F}(\square)\setminus\{\emptyset\}$. Furthermore, $\square$ is homeomorphic to the manifold with corners $\operatorname{Mc}(N, \Delta) = X_\square / CT_N$ *in Proposition 1.8.*

(iii) *If $\square$ is simple as a convex polytope, then $X_\square$ has at worst quotient singularities.*

(iv) *$X_\square$ is nonsingular if and only if $\square$ is absolutely simple as an integral convex polytope. In this case, $D_\square$ is very ample.*

Proof. By Corollary A.19, the support function h of $\square$ is positively homogeneous and upper convex, and satisfies $\square = \{m \in M_{\mathbb{R}}; \langle m, n\rangle \geq h(n), \forall n \in N_{\mathbb{R}}\}$. Since $\square$ is an integral convex polytope, we clearly have $h(N) \subset \mathbb{Z}$. Moreover, the coarsest decomposition Δ of $N_{\mathbb{R}}$ into a finite union of convex polyhedral cones as in Corollary A.19 is obviously a finite and complete fan in N such that h is a Δ-linear support function and is strictly upper convex with respect to Δ.

(ii) follows from Proposition 1.6, Lemma 2.12 and Proposition 1.8. If $\square$ is simple in the sense of Sect. A.2, then Δ is obviously a simplicial fan, hence we get (iii) by Proposition 1.25. (iv) follows immediately from Corollary 2.15. q.e.d.

Remark. As in Jurkiewicz [J1], [J2], Atiyah [A4], [A5], Guillemin-Sternberg [GS] and Ness [N9] (see also Bruguières [B4]), we can define a *moment map* $\mu: X_\square \to M_{\mathbb{R}}$ by

$$\mu(x) := \sum_{j=1}^{v} \frac{|\mathbf{e}(m_j)(x)|}{\sum_{1 \leq l \leq v} |\mathbf{e}(m_l)(x)|} m_j \quad \text{for} \quad x \in X_\square \ ,$$

where $\{m_1, \ldots, m_v\}$ is the set of vertices of $\square$ and $\mathbf{e}(m_j)$ is the character of T_N corresponding to m_j (cf. Sect. 1.2). Then as a key result, μ induces a homeomorphism from $X_\square / CT_N = \operatorname{Mc}(N, \Delta)$ onto the convex polytope $\square$. As Ishida pointed out, the following generalization might be useful: Consider a weight map $w: M \cap \square \to \mathbb{R}_{\geq 0}$ such that $w(m) > 0$ whenever m is a vertex of $\square$. Then we define a generalized moment map $\mu^w: X_\square \to M_{\mathbb{R}}$ of weight w by

$$\mu^w(x) := \sum_{m \in M \cap \square} \frac{w(m)|\mathbf{e}(m)(x)|}{\sum_{m' \in M \cap \square} w(m')|\mathbf{e}(m')(x)|} m \quad \text{for} \quad x \in X_\square \ .$$

According to Bando and Mabuchi, the moment maps as well as their restrictions to the real and the nonnegative real loci of $X_\square$ are closely related to Einstein-Kähler metrics on toric Fano varieties considered in Sect. 2.3. For related results, we refer the reader to Bando [B1], Bando-Mabuchi [BM], Mabuchi [M1], [M2], [M3] and Sakane [S2].

We can look at Theorem 2.22 more naïvely as follows: First of all, $T_N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times)$ and $\mathbf{e}(m)(t) := t(m)$ for $m \in M$ and $t \in T_N$. When a finite subset $\{m_0, m_1, \ldots, m_s\} \subset M$ is given, we have a homomorphism of algebraic tori

$$T_N \ni t \mapsto (\mathbf{e}(m_0)(t), \ldots, \mathbf{e}(m_s)(t)) \in (\mathbb{C}^\times)^{s+1} .$$

Its composite with the projection $\mathbb{C}^{s+1} \setminus \{O\} \ni (z_0, \ldots, z_s) \mapsto [z_0 : z_1 : \ldots : z_s] \in \mathbb{P}_s(\mathbb{C})$ gives rise to a holomorphic map

$$T_N \ni t \mapsto \Psi(t) := [\mathbf{e}(m_0)(t) : \ldots : \mathbf{e}(m_s)(t)] \in \mathbb{P}_s(\mathbb{C}) .$$

The closure Y in $\mathbb{P}_s(\mathbb{C})$ of the image $\Psi(T_N)$ is a reduced compact complex analytic space, which contains the algebraic torus $\Psi(T_N)$ as an open set and on which T_N acts algebraically through Ψ. We have $\Psi(T_N) = \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{C}^\times)$, where M' is the $\mathbb{Z}$-submodule of M generated by $\{m_j - m_k ; 0 \le j \le s, 0 \le k \le s\}$. Hence Ψ is injective if and only if $M' = M$. Even if this is the case, however, Y may not be a toric variety for $T_N \xrightarrow{\sim} \Psi(T_N)$ in the sense of Chap. 1, since Y may not be normal.

Examples. Let $r = 1$. For $\{0, 2, 3\} \subset M = \mathbb{Z}$, we have

$$\mathbb{C}^\times \ni t \mapsto \Psi(t) := [1 : t^2 : t^3] \in \mathbb{P}_2(\mathbb{C}) .$$

The closure of its image is $Y = \{[z_0 : z_1 : z_2] \in \mathbb{P}_2(\mathbb{C}); z_0 z_2^2 = z_1^3\}$, which is a plane cubic curve with a cusp at $[1:0:0]$. On the other hand, for $\{0, 1, 2, 3\} \subset M = \mathbb{Z}$, the closure Y of the image of

$$\mathbb{C}^\times \ni t \mapsto \Psi(t) := [1 : t : t^2 : t^3] \in \mathbb{P}_3(\mathbb{C})$$

is a twisted cubic curve in $\mathbb{P}_3(\mathbb{C})$ and is isomorphic to $\mathbb{P}_1(\mathbb{C})$.

To go back to the general case, let $U_j := \{[z_0 : \ldots : z_s] \in \mathbb{P}_s(\mathbb{C}); z_j \neq 0\}$ for $0 \le j \le s$. It is isomorphic to $\mathbb{C}^s$ by the map $[z_0 : \ldots : z_s] \mapsto (z_0/z_j, \ldots, z_s/z_j)$ and we get an open covering $\mathbb{P}_s(\mathbb{C}) = U_0 \cup U_1 \cup \ldots \cup U_s$.

For simplicity, let us restrict ourselves to the case $j = 0$ and consider $\mathscr{S}_0 := \mathbb{Z}_{\ge 0}(m_1 - m_0) + \ldots + \mathbb{Z}_{\ge 0}(m_s - m_0)$. Then obviously, we have $M' = \mathscr{S}_0 + (-\mathscr{S}_0)$ and can identify $Y \cap U_0$ with

$$\{u : \mathscr{S}_0 \to \mathbb{C} ; u(O) = 1,\ u(m + m') = u(m)u(m'),\ \forall m, m' \in \mathscr{S}_0\}$$

via the map $u \mapsto [1 : u(m_1 - m_0) : \ldots : u(m_s - m_0)]$ as in the proof of Proposition 1.2. Thus $Y \cap U_0$ is normal if and only if $\mathscr{S}_0$ coincides with $M' \cap (\mathbb{R}_{\ge 0}(m_1 - m_0) + \ldots + \mathbb{R}_{\ge 0}(m_s - m_0))$.

The convex hull $\square$ of $\{m_0, \ldots, m_s\}$ is an integral convex polytope. For each $m' \in M' \cap \square$, we necessarily have $m' - m_0 \in \mathbb{R}_{\ge 0}(m_1 - m_0) + \ldots + \mathbb{R}_{\ge 0}(m_s - m_0)$. Hence the above condition in particular requires $m' - m_0 \in \mathscr{S}_0$ for each $m' \in M' \cap \square$.

For an interpretation of Theorem 2.22, we may thus restrict ourselves to the case

$$M \cap \square = \{m_0, m_1, \ldots, m_s\}$$

for an r-dimensional integral convex polytope $\square$ in $M_{\mathbb{R}}$. In particular, the $\mathbb{Z}$-submodule M' of $M \cong \mathbb{Z}^r$ generated by $\{m' - m''; m', m'' \in M \cap \square\}$ has finite index in M. Note, however, that the closure Y of the image of Ψ may not be normal. For each j, consider the dual cone σ_j in $N_{\mathbb{R}}$ of the convex polyhedral cone $\sum_{k \neq j} \mathbb{R}_{\geq 0}(m_k - m_j)$ in $M_{\mathbb{R}} = M'_{\mathbb{R}}$. Obviously, σ_j is a strongly convex rational polyhedral cone, and

$$\Delta := \{\text{the faces of } \sigma_0, \ldots, \sigma_s\}$$

is a finite and complete fan in N. The toric projective variety considered in Theorem 2.22 is nothing but $X_\square := T_N \operatorname{emb}(\Delta)$ corresponding to this fan Δ. We thus have a T_N-equivariant finite holomorphic surjection

$$\psi : X_\square \to Y \subset \mathbb{P}_s(\mathbb{C}) \quad \text{with} \quad s+1 = {}^{\#}(M \cap \square) .$$

The restriction of this ψ to T_N is nothing but $\Psi : T_N \to \Psi(T_N)$ which we started with.

Theorem 2.22 assigns, to each nonempty face F of $\square$, an irreducible closed T_N-stable subspace $V(F^\dagger)$ of $X_\square$. We have

$$(*) \qquad \psi(V(F^\dagger)) = Y \cap \{[z_0 : \ldots : z_s] \in \mathbb{P}_s(\mathbb{C}); z_j = 0 \text{ if } m_j \notin F\} .$$

Indeed, $F^\dagger = \{n \in N_{\mathbb{R}}; \langle m, n\rangle = h(n), \forall m \in F\} \in \Delta$ and $h(n) = \inf\{\langle m, n\rangle; m \in \square\}$. If $n \in N \cap F^\dagger$, then $\lim_{\lambda \to 0} \gamma_n(\lambda)$ exists in $\operatorname{orb}(F^\dagger) \subset V(F^\dagger) \subset X_\square$ by Proposition 1.6, (v), hence $\psi(\lim_{\lambda \to 0} \gamma_n(\lambda)) = \lim_{\lambda \to 0} \psi(\gamma_n(\lambda))$ exists in $\psi(\operatorname{orb}(F^\dagger)) \subset Y$. Since

$$\begin{aligned} \psi(\gamma_n(\lambda)) &= [\lambda^{\langle m_0, n\rangle} : \ldots : \lambda^{\langle m_s, n\rangle}] \\ &= [\lambda^{\langle m_0, n\rangle - h(n)} : \ldots : \lambda^{\langle m_s, n\rangle - h(n)}] , \end{aligned}$$

the limit $[\varepsilon_0 : \varepsilon_1 : \ldots : \varepsilon_s]$ as λ tends to 0 satisfies

$$\begin{aligned} m_j \notin F &\Leftrightarrow \langle m_j, n\rangle > h(n) \Rightarrow \varepsilon_j = 0 \\ m_j \in F &\Leftrightarrow \langle m_j, n\rangle = h(n) \Rightarrow \varepsilon_j = 1 . \end{aligned}$$

We thus have the assertion $(*)$ by the equivariance of ψ.

Remark. Satake [S 5] deals with an infinite dimensional analogue of what we have just seen: He considers certain infinite subsets $\{m_0, m_1, \ldots\}$ of M and the closure of the image of a similar holomorphic map from T_N to an infinite dimensional projective space. We touch on related results in Chap. 4.

Examples. (i) Let $\{m_1, \ldots, m_r\}$ be a $\mathbb{Z}$-basis of M. The r-simplex $\square$ with vertices $m_0 = O$, $m_1, \ldots, m_r$ is an absolutely simple integral convex polytope. We have $X_\square = \mathbb{P}_r(\mathbb{C})$ with $D_\square$ equal to the hyperplane at infinity $\{[z_0 : z_1 : \ldots : z_r] \in \mathbb{P}_r(\mathbb{C}); z_0 = 0\}$. Indeed, let $\{n_1, \ldots, n_r\}$ be the dual $\mathbb{Z}$-basis for N and let $n_0 := -(n_1 + \ldots + n_r)$. The support function

$$h(n) := \inf\{\langle m, n\rangle; m \in \square\} \quad \text{for} \quad n \in N_{\mathbb{R}}$$

of $\square$ then has the form

$$h(n)=\langle m_j, n\rangle \quad \text{when} \quad n\in\sigma_j:=\sum_{k\neq j}\mathbb{R}_{\geq 0}n_k$$

for $0\leq j\leq r$. In particular, we have $h(n_0)=-1$, $h(n_1)=\ldots=h(n_r)=0$ and Δ is the fan in N consisting of the faces of the r-dimensional simplicial cones $\sigma_0, \sigma_1, \ldots, \sigma_r$.

(ii) For a positive integer v, consider the multiple $v\square$ of the r-simplex $\square$ in (i) above. $v\square$ is an r-simplex with vertices $vm_0=O$, $vm_1, \ldots, vm_r$ and is absolutely simple. In this case, we have $s+1:={}^{\#}(M\cap v\square)=(v+1)(v+2)\ldots(v+r)/r!$ and $X_{v\square}\subset\mathbb{P}_s(\mathbb{C})$ coincides with the image of the v-th *Veronese embedding* of $\mathbb{P}_r(\mathbb{C})=X_\square$, which sends $[z_0:\ldots:z_r]\in\mathbb{P}_r(\mathbb{C})$ to the point of $\mathbb{P}_s(\mathbb{C})$ with the homogeneous coordinates consisting of all the monomials of degree v in $z_0, \ldots, z_r$ arranged in a prescribed manner.

(iii) Let $\{m_1, \ldots, m_r\}$ be a $\mathbb{Z}$-basis of M and let $0<q<r$. Then the product $\square:=\square'\times\square''$ of the q-simplex $\square'$ with vertices O, $m_1, \ldots, m_q$ and the $(r-q)$-simplex $\square''$ with vertices O, $m_{q+1}, \ldots, m_r$ is again an r-dimensional absolutely simple integral convex polytope. In this case, we have $s+1:={}^{\#}(M\cap\square)=(q+1)(r-q+1)$, and $X_\square\subset\mathbb{P}_s(\mathbb{C})$ coincides with the image of the *Segre embedding* of $\mathbb{P}_q(\mathbb{C})\times\mathbb{P}_{r-q}(\mathbb{C})$, which sends $([z_0:\ldots:z_q], [w_0:\ldots:w_{r-q}])\in\mathbb{P}_q(\mathbb{C})\times\mathbb{P}_{r-q}(\mathbb{C})$ to $[z_0w_0:\ldots:z_jw_k:\ldots:z_qw_{r-q}]\in\mathbb{P}_s(\mathbb{C})$.

(iv) Let us consider the 3-simplex appearing in the terminal lemma in Sect. 1.6. Namely, for a $\mathbb{Z}$-basis $\{m_1, m_2, m_3\}$ of $M\cong\mathbb{Z}^3$ and relatively prime integers p, q satisfying $0\leq p<q$, let $\square$ be the 3-simplex with vertices O, m_1, m_2, $m_1+pm_2+qm_3$ which is simple but not absolutely simple for $q\geq 2$. We have $M\cap\square=\{O, m_1, m_2, m_1+pm_2+qm_3\}$, hence $\psi: X_\square\to Y=\mathbb{P}_3(\mathbb{C})$ in our notation. $X_\square$ has four cyclic quotient singularities.

We very often construct compact complex manifolds as closed submanifolds of a projective space, which is of the form $X_\square$ as in (i) above. The r-dimensional toric projective varieties $X_\square$ for more general r-dimensional integral convex polytopes $\square$ are expected to be convenient ambient spaces for the construction of interesting compact complex manifolds. Here are two such examples:

Example (Khovanski [K3], [K4] and Ishida [I3]). Let $M\cong\mathbb{Z}^r$. $\mathbb{R}$-linearly independent elements $m_1', \ldots, m_r'\in M$ generate a $\mathbb{Z}$-submodule M' of finite index. Hence there exists a positive integer d such that $dM\subset M'$. The r-simplex $\square$ in $M_{\mathbb{R}}$ with vertices $m_0':=O, m_1', \ldots, m_r'$ is an integral convex polytope with respect to any of the three lattices $M'\subset M\subset(1/d)M'$ in $M_{\mathbb{R}}$, and is absolutely simple with respect to M' and $(1/d)M'$. Let N (resp. N') be the $\mathbb{Z}$-module dual to M (resp. M'). Then $N'\supset N\supset dN'$ is obviously the sequence of lattices dual to that above. The finite complete fan Δ for N determined by $\square$ as in Theorem 2.22 is also a fan with respect to N' as well as dN'. Thus by Theorem 1.13 we have equivariant finite holomorphic maps

$$X'':=T_{dN'}\operatorname{emb}(\Delta)\xrightarrow{f}X:=T_N\operatorname{emb}(\Delta)\xrightarrow{g}X':=T_{N'}\operatorname{emb}(\Delta)\ .$$

As in Example (i) above, we have $X'=\mathbb{P}_r(\mathbb{C})$ and $X''=\mathbb{P}_r(\mathbb{C})$. Moreover, $g\circ f$ clearly coincides with the ramified map which sends $[z_0:\ldots:z_r]\in X''$ to $[z_0^d:\ldots:z_r^d]\in X'$

and is the quotient with respect to the action of $N'/dN' \cong (\mathbb{Z}/d\mathbb{Z})^r$ as in Corollary 1.16.

The inverse image with respect to $g \circ f$ of the hyperplane

$$H := \{[w_0 : w_1 : \ldots : w_r] \in X' ; w_0 + w_1 + \ldots + w_r = 0\} \subset X'$$

is

$$F := \{[z_0 : z_1 : \ldots : z_r] \in X'' ; z_0^d + z_1^d + \ldots + z_r^d = 0\} ,$$

which is called the *Fermat variety* of degree d and dimension $r-1$. We refer the reader to Shioda-Katsura [SK], Shioda [S11], Ran [R1], Aoki [A2], Aoki-Shioda [AS] etc. for Hodge cycles on Fermat varieties.

Note that $G := g^{-1}(H) \subset X$ is sandwiched between F and H, and that it is the closure in X of

$$G \cap T_N = \{t \in T_N ; \mathbf{e}(m_0')(t) + \mathbf{e}(m_1')(t) + \ldots + \mathbf{e}(m_r')(t) = 0\} ,$$

which is the hypersurface in $T_N \cong (\mathbb{C}^\times)^r$ *defined by the sum of Laurent monomials* $\mathbf{e}(m_0') = 1, \mathbf{e}(m_1'), \ldots, \mathbf{e}(m_r')$.

Conversely, when we seek to compactify a hypersurface G' in T_N defined by a sum of Laurent monomials, it is sometimes convenient to take the closure G in a toric projective variety $X_\square$ for an integral convex polytope $\square$ closely related to the Laurent monomials in question. We could sandwich G between certain F and H as above so that $F \to H$ and $F \to G$ are quotients with respect to a finite group and a subgroup. We can use these facts to determine algebro-geometric invariants for G or its resolution $\tilde{G}$ in terms of those for $\square$. Khovanski [K3], [K4] and Nakamura [N6] compute algebro-geometric invariants of the surface

$$\{(x, y, z) \in \mathbb{C}^3 ; x^p + y^q + z^r - xyz = 0\} ,$$

which plays an important rôle in the study of cusp singularities. In this case, we may first take its intersection G' with $(\mathbb{C}^\times)^3$, and then construct G as above.

It may be an interesting problem to look at weighted homogeneous hypersurfaces and complete intersections in weighted projective spaces from this point of view (cf. Mori [M9] and Saito [S1]).

Example (Hirzebruch [H4] and Ishida [I2], [I3]). An *arrangement of lines* in $\mathbb{P}_2(\mathbb{C})$ is closely related to the example above. Let $l_0, l_1, \ldots, l_r$ with $r+1>4$ be lines in the projective plane $\mathbb{P}_2(\mathbb{C})$ which do not pass through a point in common, i.e., $l_0 \cap l_1 \cap \ldots \cap l_r = \emptyset$. Let z_0, z_1, z_2 be homogeneous coordinates of $\mathbb{P}_2(\mathbb{C})$ and let $l_j(z)$ be a linear form defining the line l_j. Let us denote the rational function field of $\mathbb{P}_2(\mathbb{C})$ by $K_0 := \mathbb{C}(\mathbb{P}_2(\mathbb{C})) = \mathbb{C}(z_1/z_0, z_2/z_0)$.

For a positive integer d, let Y be the normalization of $\mathbb{P}_2(\mathbb{C})$ in the Galois extension

$$K := K_0((l_1(z)/l_0(z))^{1/d}, (l_2(z)/l_0(z))^{1/d}, \ldots, (l_r(z)/l_0(z))^{1/d})$$

of K_0 with the Galois group $(\mathbb{Z}/d\mathbb{Z})^r$. When the lines $l_0, \ldots, l_r$ are in a special position and d is suitably chosen, Hirzebruch showed that the minimal resolution $\tilde{Y}$ of singularities of Y is a compact algebraic surface of general type satisfying the equality in *Miyaoka's inequality* $c_1^2 \leq 3c_2$ for the Chern numbers.

To compute such algebro-geometric invariants as the Chern numbers c_1^2, c_2 and the irregularity for $\tilde{Y}$ in general, we need to construct $\tilde{Y}$ in a manner closely related to $l_0, \ldots, l_r$. One such way is to take the normalization $\tilde{Y}$ in K of the blowing-up $Z \to \mathbb{P}_2(\mathbb{C})$ along all the points through which more than two lines pass.

As another way, Ishida made clever use of toric varieties as follows:

$$\mathbb{P}_2(\mathbb{C}) \ni [z_0 : z_1 : z_2] \mapsto [l_0(z) : \ldots : l_r(z)] \in \mathbb{P}_r(\mathbb{C})$$

is a closed embedding and allows us to identify $\mathbb{P}_2(\mathbb{C})$ with a two-dimensional linear subspace, i.e., a complete intersection of $r-2$ hyperplanes in $X := \mathbb{P}_r(\mathbb{C})$. As in the previous example, consider

$$X' := \mathbb{P}_r(\mathbb{C}) \ni [w_0 : \ldots : w_r] \overset{f}{\mapsto} [w_0^d : \ldots : w_r^d] \in X \ .$$

Then we have $Y = f^{-1}(\mathbb{P}_2(\mathbb{C}))$, which is a complete intersection in X' of $r-2$ Fermat hypersurfaces of degree d. We now choose a $\mathbb{Z}$-module N with a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ and let $n_0 := -(n_1 + \ldots + n_r)$. Then

$$\Delta := \{\mathbb{R}_{\geq 0} n_{j(1)} + \ldots + \mathbb{R}_{\geq 0} n_{j(p)} ; 0 \leq p \leq r,\ 0 \leq j(1) < j(2) < \ldots < j(p) \leq r\}$$

is a fan for N as well as dN. The equivariant holomorphic map induced by the inclusion $dN \subset N$ is nothing but

$$X' = T_{dN}\,\mathrm{emb}\,(\Delta) \xrightarrow{f} \mathrm{X} = T_N\,\mathrm{emb}\,(\Delta) \ .$$

It is convenient to choose the following noncompact but nonsingular toric variety $T_N\,\mathrm{emb}\,(\Phi)$ as the ambient space for the blowing-up Z of $\mathbb{P}_2(\mathbb{C})$ above: $\Phi = \Phi(0) \cup \Phi(1) \cup \Phi(2)$ is a fan in N consisting of cones of dimension ≤ 2 defined by

$$\begin{aligned}
&\Phi(0) = \Delta(0) = \{\{O\}\} \\
&\Phi(1) = \Delta(1) \cup \{\mathbb{R}_{\geq 0} n_Q ; Q \in \mathbb{P}_2(\mathbb{C}) \text{ is on more than two lines}\} \\
&\Phi(2) = \{\mathbb{R}_{\geq 0} n_j + \mathbb{R}_{\geq 0} n_k ;\ 0 \leq j < k \leq r,\ l_j \cap l_k \text{ is on no line other than } l_j \text{ and } l_k\} \\
&\qquad \cup \{\mathbb{R}_{\geq 0} n_Q + \mathbb{R}_{\geq 0} n_j ;\ Q \in l_j \text{ is on more than two lines}\} \ ,
\end{aligned}$$

where we let $n_Q := \sum_{1 \leq i \leq s} n_{j(i)}$ when $l_{j(1)}, l_{j(2)}, \ldots, l_{j(s)}$ with $s > 2$ are the lines passing through $Q \in \mathbb{P}_2(\mathbb{C})$.

We have a natural equivariant birational morphism $g : T_N\,\mathrm{emb}\,(\Phi) \to T_N\,\mathrm{emb}\,(\Delta) = X$ such that $Z = g^{-1}(\mathbb{P}_2(\mathbb{C}))$. We also have $\tilde{Y} = h^{-1}(Z)$ for the equivariant holomorphic map $h : T_{dN}\,\mathrm{emb}\,(\Phi) \to T_N\,\mathrm{emb}\,(\Phi)$ determined by the inclusion $dN \subset N$. By Corollary 1.16, h and its restriction $\tilde{Y} \to Z$ to $\tilde{Y}$ turn out to be the quotients with respect to a natural action of $N/dN \cong (\mathbb{Z}/d\mathbb{Z})^r$. We can rather easily characterize $\mathbb{Z}$-submodules $N' \supset dN$ of N such that N'/dN acts freely on $T_{dN}\,\mathrm{emb}\,(\Phi)$. The quotient of $\tilde{Y}$ with respect to such an N'/dN turns out to be an interesting algebraic surface.

Although this $T_{dN}\,\mathrm{emb}\,(\Phi)$ is not compact, it is thus a nice nonsingular ambient space for $\tilde{Y}$ closely related to the very construction of $\tilde{Y}$.

Let us now return to general $X_\square$ and $D_\square$ in Theorem 2.22. By Corollary 2.9 and Proposition 2.10, we first have the following:

Corollary 2.23. *For an r-dimensional integral convex polytope $\Box$ in $M_{\mathbb{R}} \cong \mathbb{R}^r$, we have*

$$\dim_{\mathbb{C}} H^j(X_\Box, \mathcal{O}_{X_\Box}(D_\Box)) = \begin{cases} {}^\#(M \cap \Box) & \text{for } j=0 \\ 0 & \text{for } j \neq 0 \end{cases} .$$

The Hilbert polynomial for $\Box$ in the sense of Sect. 2.2 coincides with the usual Hilbert polynomial for $(X_\Box, D_\Box)$, i.e., the polynomial

$$\chi(X_\Box, \mathcal{O}_{X_\Box}(\nu D_\Box)) := \sum_{j \geq 0} (-1)^j \dim_{\mathbb{C}} H^j(X_\Box, \mathcal{O}_{X_\Box}(\nu D_\Box))$$

in $\nu \in \mathbb{Z}$ of degree r with the leading coefficient equal to

$$\mathrm{vol}_r(\Box) = \frac{1}{r!}(D_\Box \,.\, D_\Box \,.\, \ldots \,.\, D_\Box) .$$

Moreover for $\nu \geq 0$, we have

$${}^\#(M \cap \nu\Box) = \dim_{\mathbb{C}} H^0(X_\Box, \mathcal{O}_{X_\Box}(\nu D_\Box)) .$$

The following reciprocity law for the Hilbert polynomial gives geometric significance to its value at negative integers ν. It was conjectured by Ehrhart and proved by Macdonald [M4]. We here give another proof due to Danilov [D1, 11.12.4].

Proposition 2.24. *The Hilbert polynomial $P(\nu)$ for an r-dimensional integral convex polytope $\Box$ in $M_{\mathbb{R}} \cong \mathbb{R}^r$ satisfies*

$${}^\#(M \cap \mathrm{int}(\nu\Box)) = (-1)^r P(-\nu) \quad \text{for} \quad \nu > 0 ,$$

where $\mathrm{int}(\nu\Box)$ *denotes the interior of the convex polytope $\nu\Box$.*

Proof. For simplicity, let us denote by $X := X_\Box$ and $D := D_\Box$ the toric projective variety and the invariant ample Cartier divisor on it determined by $\Box$ as in Theorem 2.22. We have $X = T_N \mathrm{emb}(\Delta)$ for the finite complete fan Δ determined by $\Box$.

By Lemma 2.3, $\{\mathbf{e}(m); m \in M \cap \nu\Box\}$ is a basis for the $\mathbb{C}$-vector space $H^0(X, \mathcal{O}_X(\nu D))$. To each one-dimensional cone $\varrho \in \Delta(1)$ correspond a codimension one face $\varrho^\dagger$ of $\nu\Box$ and a T_N-stable closed irreducible subvariety $V(\varrho) \subset X$ of codimension one. The image of $\mathbf{e}(m)$ for $m \in M \cap \nu\Box$ under the restriction map

$$H^0(X, \mathcal{O}_X(\nu D)) \to H^0(X, \mathcal{O}_{V(\varrho)} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\nu D))$$

does not vanish if and only if m is on the face $\varrho^\dagger$. Since the complement of $\mathrm{int}(\nu\Box)$ in $\nu\Box$ is $\cup_{\varrho \in \Delta(1)} \varrho^\dagger$, we see that ${}^\#(M \cap \mathrm{int}(\nu\Box))$ coincides with the dimension of the kernel of the direct sum

$$H^0(X, \mathcal{O}_X(\nu D)) \to H^0(X, \bigoplus_{\varrho \in \Delta(1)} \mathcal{O}_{V(\varrho)} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\nu D))$$

of the restriction maps.

By Theorem 3.6, (2) concerning Ishida's complex $\mathcal{K}^\cdot(X; r)$ of degree r (cf. Sect. 3.2), we have an exact sequence

$$0 \to \tilde{\Omega}_X^r \to \mathcal{K}^0(X; r) \xrightarrow{\delta} \mathcal{K}^1(X; r) ,$$

where $\mathscr{K}^0(X;r)=\mathcal{O}_X$, $\mathscr{K}^1(X;r)=\bigoplus_{\varrho\in\Delta(1)}\mathcal{O}_{V(\varrho)}$ and δ is the direct sum of the restriction maps $\mathcal{O}_X\to\mathcal{O}_{V(\varrho)}$. By the exact sequence which we obtain by tensoring the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(vD)$ with the exact sequence above, we conclude that

$$\begin{aligned}{}^{\#}(M\cap\operatorname{int}(vD)) &= \dim_{\mathbb{C}} H^0(X,\tilde{\Omega}_X^r\otimes_{\mathcal{O}_X}\mathcal{O}_X(vD))\\ &=\chi(X,\tilde{\Omega}_X^r\otimes_{\mathcal{O}_X}\mathcal{O}_X(vD)) \ .\end{aligned}$$

The second equality follows from Bott's vanishing theorem in Sect. 3.3, which in the present case is nothing but the toric Kodaira vanishing theorem.

As we see in Corollary 3.9, X is a Cohen-Macaulay variety with $\tilde{\Omega}_X^r$ as the dualizing $\mathcal{O}_X$-module. Hence by the Serre-Grothendieck duality theorem applied to the third term in the above equality we get

$${}^{\#}(M\cap\operatorname{int}(vD))=(-1)^r\chi(X,\mathcal{O}_X(-vD)) \ .$$

We are done, since the Hilbert polynomial for $\square$ is given by

$$P(v)=\chi(X,\mathcal{O}_X(vD)) \quad \text{for} \quad v\in\mathbb{Z}$$

by Corollary 2.23. q.e.d.

Remark. It may be easier to prove the above proposition as follows: Let Δ' be a finite nonsingular subdivision of Δ as in Sect. 1.5, and let D' be the pull-back of $D_\square$ by the equivariant resolution of singularities

$$X'=T_N\operatorname{emb}(\Delta')\to X_\square=T_N\operatorname{emb}(\Delta) \ .$$

Then $\{\mathbf{e}(m); m\in M\cap\operatorname{int}(v\square)\}$ is seen to be a $\mathbb{C}$-basis for

$$H^0(X',\Omega_{X'}^r\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{X'}(vD')) \ .$$

We can now apply the usual Serre duality to the nonsingular X'.

Denote by $\partial\square := \square\setminus\operatorname{int}(\square)$ the *boundary* of an r-dimensional integral convex polytope $\square$. The Hilbert polynomial for $\square$ is of the form $P(v)=\sum_{0\le j\le r}a_jv^j$ with $a_r=\operatorname{vol}_r(\square)$ and $a_0=P(0)=1$.

If $r=1$, i.e., $\square$ is a finite interval with integral endpoints, then ${}^{\#}(M\cap v\square)=P(v)$ $=\operatorname{vol}_1(\square)v+1$.

Suppose $r=2$, i.e., $\square$ is a plane convex polygon with vertices in the lattice M. Then ${}^{\#}(M\cap v\square)=P(v)=\operatorname{vol}_2(\square)v^2+a_1v+1$, and ${}^{\#}(M\cap\operatorname{int}(v\square))=P(-v)$ $=\operatorname{vol}_2(\square)v^2-a_1v+1$ by Proposition 2.24. Thus ${}^{\#}(M\cap\partial(v\square))=2a_1v$, the left hand side of which equals ${}^{\#}(M\cap\partial\square)v$, since $\partial\square$ is one-dimensional. Hence

$$P(v)=\operatorname{vol}_2(\square)v^2+\frac{{}^{\#}(M\cap\partial\square)}{2}v+1 \ .$$

Letting $v=1$, we get *G. Pick's formula*

$$\operatorname{vol}_2(\square)={}^{\#}(M\cap\square)-\frac{{}^{\#}(M\cap\partial\square)}{2}-1 \ ,$$

which in fact holds also for nonconvex polygons $\Box$, as we can show in various elementary ways.

When $r \geq 3$, Macdonald expressed $\mathrm{vol}_r(\Box)$ in terms of the numbers of lattice points $^{\#}(M \cap \nu\Box)$ for finitely many positive integers ν. We refer the reader to Hammer [H1] for various results and problems concerning lattice points in convex polytopes.

When $\Box$ is a simple integral convex polytope, the toric projective variety $X_\Box$ has at worst quotient singularities. As we see in Sect. 3.3, Jurkiewicz-Danilov's theorem describes the structure of the Chow ring $A^{\cdot}(X_\Box) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the cohomology ring $H^{\cdot}(X_\Box, \mathbb{Q})$.

As we see in Sect. A.5, the Poincaré duality theorem for $H^{\cdot}(X_\Box, \mathbb{Q})$ turns out to be equivalent to the Dehn-Sommerville equalities for the simplicial convex polytope $\Box^\circ$, which is the polar of $\Box$. By a result of Stanley, the strong Lefschetz theorem for the cohomology class ω of $D_\Box$ in $H^2(X_\Box, \mathbb{Q})$ will be reflected in an interesting property enjoyed by the numbers of faces in various dimensions of $\Box^\circ$ (see also the end of Sect. 3.3).

In Sect. A.4, we discuss results concerning mixed volumes, inradii, circumradii and the isoperimetric inequality for compact convex sets as well as a problem for them posed by Teissier [T3]. In the special case of integral convex polytopes $\Box$, they have algebro-geometric interpretation in terms of the corresponding toric projective varieties $X_\Box$ and ample Cartier divisors $D_\Box$. Teissier's problem can then be generalized to one on Cartier divisors on compact algebraic varieties. Here are what Teissier [T3] showed in this connection:

Let D', D be Cartier divisors on an r-dimensional compact algebraic variety Y. For $0 \leq j \leq r$, let us denote the intersection number (cf. Sect. 2.2) of j copies of D' and $r-j$ copies of D simply by

$$s_j := (D'^{[j]} . D^{[r-j]}) \quad \text{for} \quad 0 \leq j \leq r \ .$$

Thus for ν', $\nu \in \mathbb{Z}$, the self-intersection number of $\nu' D' + \nu D$ becomes

$$((\nu' D' + \nu D)^{[r]}) = \sum_{j=0}^{r} \binom{r}{j} s_j \nu'^j \nu^{r-j} \ .$$

Teissier [T1] then showed the inequalities

$$s_j^2 \geq s_{j-1} s_{j+1} \quad \text{for} \quad 1 \leq j \leq r-1$$

by reducing them to the following famous theorem:

The Hodge Index Theorem. *For a two-dimensional nonsingular projective variety Z, let* $\mathrm{NS}(Z)$ *be its Néron-Severi group, i.e., the commutative group of a finite rank p consisting of the numerical equivalence classes of Cartier divisors on Z. The quadratic form on the scalar extension* $\mathrm{NS}(Z) \otimes_{\mathbb{Z}} \mathbb{R}$ *determined by the self-intersection number* $(D^{[2]})$ *of Cartier divisors has signature equal to* $(1, p-1)$.

The above inequalities for s_j have exactly the same form as the Alexandrov-Fenchel inequalities for mixed volumes in Sect. A.4. Thus we similarly get

$$s_j^r \geq s_0^{r-j} s_r^j \quad \text{for} \quad 0 \leq j \leq r$$

as well as an analogue of the Brünn-Minkowski inequality

$$((D'+D)^{[r]})^{1/r} \geq (D'^{[r]})^{1/r} + (D^{[r]})^{1/r} \ .$$

We might as well extend the analogy further to define the *inradius* and the *circumradius* respectively by

$$\varrho(D:D') := \sup\{a/b;\ \text{positive integers } a, b \text{ with } H^0(Y, \mathcal{O}_Y(-aD'+bD)) \neq 0\}$$

$$R(D:D') := \inf\{a/b;\ \text{positive integers } a, b \text{ with } H^0(Y, \mathcal{O}_Y(aD'-bD) \neq 0\} \ .$$

Teissier's Problem (Algebro-Geometric Version). Suppose that Cartier divisors D', D on an r-dimensional compact algebraic variety Y satisfy $(D'^{[r]}) > 0$, $(D^{[r]}) > 0$ and that the invertible $\mathcal{O}_Y$-modules $\mathcal{O}_Y(D')$ and $\mathcal{O}_Y(D)$ are generated by global sections. Give a good estimate for the inradius $\varrho(D:D')$ and the circumradius $R(D:D')$ in terms of

$$s_j := (D'^{[j]} . D^{[r-j]}) \quad \text{for} \quad 0 \leq j \leq r \ .$$

Can we use the roots of the algebraic equation

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} s_j t^j = 0$$

for the estimate?

When $r=2$ and $D'+D$ is ample, Teissier [T3] obtained the same inequalities as those due to Flanders and Bonnesen for compact convex sets (cf. Sect. A.4):

$$(s_1 + \sqrt{s_1^2 - s_0 s_2})/s_2 \geq R(D:D') \geq s_1/s_2 \geq s_0/s_1 \geq \varrho(D:D') \geq (s_1 - \sqrt{s_1^2 - s_0 s_2})/s_2$$

hence

$$s_1^2 - s_0 s_2 \geq (R(D:D') - \varrho(D:D'))^2 s_2^2/4 \ .$$

Furthermore, when Y is an Abelian variety of an arbitrary dimension and $D'+D$ is ample, he showed that the roots of

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} s_j t^j = 0$$

are all real and that $\varrho(D:D')$ is not less than the smallest among them.

In the rest of this section, we show that Teissier's problem (cf. Sect. A.4) for integral convex polytopes in $M_{\mathbb{R}}$ is a special case of the algebro-geometric version described above. In particular when $r=2$, the algebro-geometric proof for algebraic surfaces described above gives alternative proof for the results in Sect. A.4 for two-dimensional compact convex sets, since they can always be approximated by integral convex polytopes with respect to fine enough lattices.

For r-dimensional integral convex polytopes $\square$, $\square'$ in $M_{\mathbb{R}}$, let h, $h': N_{\mathbb{R}} \to \mathbb{R}$ be their respective support functions. Thus they are positively homogeneous and upper convex, and satisfies $\square = \{m \in M_{\mathbb{R}};\ \langle m, n\rangle \geq h(n), \forall n \in N_{\mathbb{R}}\}$, $\square' = \{m \in M_{\mathbb{R}};\ \langle m, n\rangle \geq h'(n), \forall n \in N_{\mathbb{R}}\}$, $h(N) \subset \mathbb{Z}$ and $h'(N) \subset \mathbb{Z}$. Since h and h' are piecewise linear by Corollary A.19, there exists a finite complete fan Δ in N such that both h and h'

are Δ-linear support functions. With respect to the coarsest such Δ, we see that $h+h'$ is strictly upper convex. Let

$$X := T_N \operatorname{emb}(\Delta) , \quad D := D_h \quad \text{and} \quad D' := D_{h'} .$$

Then by Theorem 2.7 and Corollary 2.14, $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ are generated by global sections and $\mathcal{O}_X(D+D')$ is ample. Furthermore,

$$(D^{[r]})/r! = \operatorname{vol}_r(\square) > 0 , \quad (D'^{[r]})/r! = \operatorname{vol}_r(\square') > 0$$

$$(D'^{[j]} . D^{[r-j]})/r! = V_r(\square'^{[j]}, \square^{[r-j]}) \quad \text{for} \quad 0 \leq j \leq r$$

by Proposition 2.10. Thus the following lemma finishes the interpretation of relevant invariants for $\square'$, $\square$ completely in terms of D', D on X:

Lemma 2.25 (Teissier [T3, 2.1]). *In the above situation, we have*

$$\varrho(D:D') = \varrho(\square:\square') \quad \textit{and} \quad R(D:D') = R(\square:\square') .$$

Proof. We only prove the first equality, since the second can be proved similarly.

For nonnegative integers a, b, the T_N-invariant Cartier divisor $-aD'+bD$ corresponds to $-ah'+bh \in \mathrm{SF}(N, \Delta)$ by Proposition 2.1. Thus Lemma 2.3 allows us to choose $\{\mathbf{e}(m); m \in M, \langle m, n\rangle \geq -ah'(n)+bh(n), \forall n \in N_{\mathbb{R}}\}$ as a $\mathbb{C}$-basis for $H^0(X, \mathcal{O}_X(-aD'+bD))$. Since h' and h are the support functions for $\square'$ and $\square$, respectively, we see that the above $\mathbb{C}$-basis coincides with $\{\mathbf{e}(m); m \in M, m+a\square' \subset b\square\}$. Consequently,

$$H^0(X, \mathcal{O}_X(-aD'+bD)) \neq 0 \Leftrightarrow \exists m \in M , \quad m+a\square' \subset b\square .$$

$\varrho(D:D')$ is the supremum of a/b for a and b satisfying the left hand side, while $\varrho(\square:\square') := \sup\{\lambda \in \mathbb{R}_{\geq 0}; \exists x \in M_{\mathbb{R}}, x+\lambda\square' \subset \square\}$ is obviously equal to

$$\sup\{a/b; a, b \text{ positive integers}, \exists m \in M, m+a\square' \subset b\square\} ,$$

since $\cup_{b>0} b^{-1}M = M_{\mathbb{Q}}$ is dense in $M_{\mathbb{R}}$. q.e.d.

2.5 Mori's Theory and Toric Projective Varieties

Mori's theory in [M10] deals with rational curves on projective varieties. It has already found important applications and is expected to keep playing important rôles in the birational geometry of algebraic varieties. Its "logarithmic" version was formulated by Tsunoda and applied to degenerations of algebraic surfaces. In this section, we first recall Mori's theory briefly, following Mori [M10] and Miyanishi [M8]. (The reader is referred to Kawamata [K2] and Kawamata-Matsuda-Matsuki [KMM] for more recent developments.) We then explain the toric version of Mori's theory due to Reid [R4].

For simplicity, we restrict ourselves to Mori's theory for *nonsingular* projective varieties X of dimension $r \geq 2$ over the field $\mathbb{C}$ of complex numbers.

As in Sect. 2.1, we denote by $\mathrm{Div}(X)$ the additive group of divisors on X. Thus an element D in it is a formal finite $\mathbb{Z}$-linear combination $D=\sum a_j D_j$ of closed irreducible subvarieties D_j of codimension one. D is said to be *effective* (denoted $D\geq 0$) if $a_j\geq 0$ for all j. We denote by $\mathrm{Div}^+(X)$ the additive subsemigroup of effective divisors on X.

On the other hand, an *algebraic* 1-*cycle* on X is a formal finite $\mathbb{Z}$-linear combination $z=\sum b_j C_j$ of one-dimensional closed irreducible subvarieties C_j. We denote by $Z_1(X)$ the additive group of algebraic 1-cycles on X. z is said to be *effective* if $b_j\geq 0$ for all j. The additive subsemigroup of effective algebraic 1-cycles on X is denoted by $Z_1^+(X)$.

As in Sect. 2.2, the intersection numbers $(D\,.\,z)$ give rise to a $\mathbb{Z}$-bilinear map

$$\mathrm{Div}(X)\times Z_1(X)\to\mathbb{Z}\ .$$

$D\in\mathrm{Div}(X)$ is said to be *numerically equivalent to zero* (denoted $D\equiv 0$) if $(D\,.\,z)=0$ holds for every $z\in Z_1(X)$. The *Néron-Severi group* $\mathrm{NS}(X)$ of X is the quotient of $\mathrm{Div}(X)$ modulo its subgroup $\{D\in\mathrm{Div}(X); D\equiv 0\}$. By Matsusaka's theorem, $\mathrm{NS}(X)$ is a free $\mathbb{Z}$-module of finite rank $\varrho(X)$, which is called the *Picard number* of X. Let us denote by $[D]$ the image in $\mathrm{NS}(X)$ of $D\in\mathrm{Div}(X)$.

Similarly, $z\in Z_1(X)$ is said to be *numerically equivalent to zero* (also denoted $z\equiv 0$) if $(D\,.\,z)=0$ for every $D\in\mathrm{Div}(X)$. We denote by $Z_1(X)/(\equiv)$ the quotient of $Z_1(X)$ modulo its subgroup $\{z\in Z_1(X); z\equiv 0\}$ and let

$$\mathbb{N}(X):=(Z_1(X)/(\equiv))\otimes_{\mathbb{Z}}\mathbb{R}\ ,$$

which is a $\varrho(X)$-dimensional $\mathbb{R}$-vector space and plays an important rôle in Mori's theory. The dual $\mathbb{R}$-vector space $\mathbb{N}(X)^*$ can be identified with $\mathrm{NS}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ via the bilinear intersection pairing mentioned above. Let us denote by $[z]$ the image in $\mathbb{N}(X)$ of $z\in Z_1(X)$. For $D\in\mathrm{Div}(X)$, we also denote $([D]\,.\,[z]):=(D\,.\,z)$.

$\mathbb{NE}(X):=\mathbb{R}_{\geq 0}[Z_1^+(X)]$ is the smallest convex cone in $\mathbb{N}(X)$ containing the image $[Z_1^+(X)]$ of the additive subsemigroup $Z_1^+(X)$ of effective algebraic 1-cycles. $\mathbb{NE}(X)$ is a $\varrho(X)$-dimensional convex cone and consists of formal finite $\mathbb{R}_{\geq 0}$-linear combinations $\sum b_j[C_j]$ of the images $[C_j]$ of one-dimensional closed irreducible subvarieties C_j. We denote by $\overline{\mathbb{NE}}(X)$ the closure of $\mathbb{NE}(X)$ in $\mathbb{N}(X)$ with respect to the usual topology as a $\varrho(X)$-dimensional $\mathbb{R}$-vector space.

The closed cone in $\mathbb{N}(X)^*=\mathrm{NS}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ dual (cf. Sect. A.1) to $\overline{\mathbb{NE}}(X)$ coincides with the *pseudo-ample cone* $\mathbb{PA}(X)$ considered by Kleiman [K5]. Namely, $\delta\in\mathbb{N}(X)^*$ belongs to $\mathbb{PA}(X)$ if and only if $(\delta\,.\,[C])\geq 0$ holds for every closed one-dimensional irreducible subvariety C.

Since we assume X to be a projective variety, it follows from Kleiman [K5] that the interior $\mathrm{int}(\mathbb{PA}(X))$ of $\mathbb{PA}(X)$ is nonempty and consists of the finite $\mathbb{R}_{\geq 0}$-linear combinations $\sum a_j[D_j]$ for ample divisors D_j. In particular, $\overline{\mathbb{NE}}(X)$ is strongly convex, i.e.,

$$\overline{\mathbb{NE}}(X)\cap(-\overline{\mathbb{NE}}(X))=\{0\}\ .$$

Dually, we consider the smallest convex cone $\mathbb{R}_{\geq 0}[\mathrm{Div}^+(X)]$ in $\mathbb{N}(X)^*$ containing $\mathrm{Div}^+(X)$ and its dual cone

$$\{\zeta\in\mathbb{N}(X); ([D]\,.\,\zeta)\geq 0,\ \forall D\in\mathrm{Div}^+(X)\}$$

in $\mathbb{N}(X)$. An element ζ of it is said to be *numerically effective*. $z \in Z_1(X)$ is said to be numerically effective if so is $[z]$. Note, however, that $\mathbb{NE}$ does not stand for numerical effectiveness. The fact that $\zeta \in \mathbb{N}(X)$ is numerically effective is not equivalent to $\zeta \in \mathbb{NE}(X)$. Since $\mathbb{PA}(X)$ is contained in the closure of $\mathbb{R}_{\geq 0}[\mathrm{Div}^+(X)]$ by a result of Kleiman [K5], we see by taking the dual cone that $\{\zeta \in \mathbb{N}(X); \zeta$ is numerically effective$\}$ is contained in $\overline{\mathbb{NE}}(X)$.

In the notation introduced so far, we are ready to state:

Mori's Theorem. *Let X be an r-dimensional nonsingular projective variety and let K_X be a canonical divisor.*

(1) *Fix a very ample divisor H on X and for a real $\varepsilon > 0$ sufficiently small, let*

$$\overline{\mathbb{NE}}_\varepsilon(X) := \{\zeta \in \overline{\mathbb{NE}}(X);\ ([-K_X].\zeta) \leq \varepsilon([H].\zeta)\} \ .$$

Then there exist a finite number of rational curves $l_1, \ldots, l_s$ (that is, one-dimensional closed subvarieties whose normalizations are isomorphic to $\mathbb{P}_1(\mathbb{C})$) satisfying $0 < (-K_X . l_j) \leq r+1$ for all j such that

$$\overline{\mathbb{NE}}(X) = \mathbb{R}_{\geq 0}[l_1] + \ldots + \mathbb{R}_{\geq 0}[l_s] + \overline{\mathbb{NE}}_\varepsilon(X) \ .$$

Namely, $\overline{\mathbb{NE}}(X)$ is a convex polyhedral cone slightly inside the half space $\{\zeta \in \mathbb{N}(X); ([-K_X].\zeta) \geq 0\}$ in $\mathbb{N}(X)$.

(2) *A half line $R = \mathbb{R}_{\geq 0}\zeta$ in $\overline{\mathbb{NE}}(X)$ is called an* extremal ray, *if $([-K_X].\zeta) > 0$ and if $\zeta_1 + \zeta_2 \in R$ for $\zeta_1, \zeta_2 \in \overline{\mathbb{NE}}(X)$ implies $\zeta_1, \zeta_2 \in R$. A rational curve l on X is called an* extremal rational curve, *if $\mathbb{R}_{\geq 0}[l]$ is an extremal ray and if $0 < (-K_X . l) \leq r+1$. Then any extremal ray R is of the form $R = \mathbb{R}_{\geq 0}[l]$ for an extremal rational curve l.*

(3) *X contains at least one extremal rational curve if and only if $K_X \notin \mathbb{PA}(X)$, i.e., K_X is not numerically effective.*

(4) *Suppose X is a Fano variety, i.e., $-K_X$ is ample. Then $\overline{\mathbb{NE}}(X) = \mathbb{NE}(X)$ holds and it is a convex polyhedral cone. Moreover, we have*

$$\mathbb{NE}(X) = \mathbb{R}_{\geq 0}[l_1] + \ldots + \mathbb{R}_{\geq 0}[l_s]$$

for a finite number of extremal rational curves $l_1, \ldots, l_s$.

(5) *Suppose* $\dim X \leq 3$. *For each extremal ray R of $\overline{\mathbb{NE}}(X)$, there exists a morphism* $\mathrm{cont}_R : X \to X_R$ *to a normal projective variety, which is uniquely determined by R up to isomorphism and is called the contraction for R, such that*

(i) $(\mathrm{cont}_R)_* \mathcal{O}_X = \mathcal{O}_Y$ *and*

(ii) *a one-dimensional closed irreducible subvariety $C \subset X$ satisfies $[C] \in R$ if and only if* $\dim(\mathrm{cont}_R)(C) = 0$.

We have $\mathrm{Pic}(X_R) = \{L \in \mathrm{Pic}(X); ([L].R) = 0\}$ *and $-K_X$ turns out to be relatively ample with respect to* cont_R, *hence in particular, we get*

$$R^j(\mathrm{cont}_R)_* \mathcal{O}_X = 0 \quad \textit{for all} \quad j > 0 \ .$$

(6) *If R in (5) is not numerically effective, then* cont_R *is a birational morphism with a closed irreducible subvariety $D \subset X$ of codimension one as the exceptional set. That is, the codimension of* $\mathrm{cont}_R(D)$ *is greater than one and the restriction of* cont_R *to $X \setminus D$ is an isomorphism onto its image.*

(6′) *Conversely, suppose* $\dim X \le 3$ *and* $f: X \to Z$ *is a birational morphism, which is not an isomorphism, onto a nonsingular projective variety Z. Then we have* $\dim f(l) = 0$ *for an extremal rational curve* $l \subset X$. *In this case, the ray* $R := \mathbb{R}_{\ge 0}[l]$ *is not numerically effective and f factors into the composite of* $\mathrm{cont}_R: X \to X_R$ *and a birational morphism* $X_R \to Z$.

(7) *If R in* (5) *is numerically effective, then* X_R *is nonsingular and* $\dim X_R < \dim X$.

(7′) *Conversely, suppose* $\dim X \le 3$ *and* $f: X \to Z$ *is a morphism to a normal projective variety Z such that* $f_* \mathcal{O}_X = \mathcal{O}_Z$. *If* $\dim f(C) = 0$ *and* $(-K_X . C) > 0$ *for a one-dimensional closed irreducible subvariety* $C \subset X$, *then f factors into the composite of the contraction* $\mathrm{cont}_R: X \to X_R$ *for an extremal ray R in* $\overline{\mathrm{NE}}(X)$ *and a morphism* $X_R \to Z$. *In this case, the Picard numbers satisfy* $\varrho(X) \ge \varrho(Z) + 1$. *If the equality holds, then we necessarily have* $f = \mathrm{cont}_R$ *and* $R = \mathbb{R}_{\ge 0}[C]$.

The dimension restriction above were later removed by Kawamata [K2]. We have the following "logarithmic" version of the above theorem (cf. Miyanishi [M8′] and Tsunoda [T9]):

Tsunoda's Theorem. *Let X be an r-dimensional nonsingular projective variety over* $\mathbb{C}$ *and let* $D = D_1 + \ldots + D_k$ *be a reduced effective divisor on X with only simple normal crossings. Fix an ample divisor H on X and let*

$$\overline{\mathrm{NE}}_\varepsilon(X, D) := \{\zeta \in \overline{\mathrm{NE}}(X); ([-K_X - D] . \zeta) \le \varepsilon([H] . \zeta)\}$$

for a real $\varepsilon > 0$ *sufficiently small. Then there exist a finite number of rational curves* $l_1, \ldots, l_t$ *satisfying* $0 < ((-K_X - D) . l_j) \le r + 1$ *such that*

$$\overline{\mathrm{NE}}(X) = \mathbb{R}_{\ge 0}[l_1] + \ldots + \mathbb{R}_{\ge 0}[l_t] + \overline{\mathrm{NE}}_\varepsilon(X, D) .$$

K_X tells us that a certain part of $\overline{\mathrm{NE}}(X)$ is a polyhedral cone by Mori's theorem, while $K_X + D$ tells us that another part of $\overline{\mathrm{NE}}(X)$ is a polyhedral cone by Tsunoda's theorem.

We now explain the toric version of Mori's theorem due to Reid [R4]. He studies $\overline{\mathrm{NE}}(X)$ for toric projective varieties $X = T_N \operatorname{emb}(\Delta)$ even when the fans Δ are only assumed to be simplicial, namely, X might have quotient singularities (cf. Proposition 1.25). As we saw in Sect. 1.7, it may be more natural to deal with this general case for birational geometry. For simplicity, however, we here restrict ourselves to r-dimensional *nonsingular* projective toric varieties $X = T_N \operatorname{emb}(\Delta)$.

Proposition 2.26 (Reid [R4, Proposition 1.6]). *For an r-dimensional nonsingular toric projective variety* $X = T_N \operatorname{emb}(\Delta)$, *there exist* $\tau_1, \ldots, \tau_s \in \Delta(r-1)$ *such that*

$$\mathrm{NE}(X) = \overline{\mathrm{NE}}(X) = \mathbb{R}_{\ge 0}[V(\tau_1)] + \ldots + \mathbb{R}_{\ge 0}[V(\tau_s)] .$$

Thus $\mathrm{NE}(X)$ *is a convex polyhedral cone, and is in fact strongly convex, i.e.,* $\mathrm{NE}(X) \cap (-\mathrm{NE}(X)) = \{0\}$ *holds. Each one-dimensional face R of* $\mathrm{NE}(X)$, *which we call an* extremal ray *(in a generalized sense), is of the form* $R = \mathbb{R}_{\ge 0}[V(\tau)]$ *for some* $\tau \in \Delta(r-1)$. *Such* $V(\tau)$ *is called a* T_N-stable extremal rational curve *contained in R.*

Proof. As we recalled above, the strong convexity of $\mathbb{NE}(X)$ is a consequence of the projectivity of X by Kleiman [K5]. By Jurkiewicz-Danilov's theorem explained in Sect. 3.3, we know the structure of the Chow ring $A^{\cdot}(X)$ and the cohomology ring $H^{\cdot}(X, \mathbb{Z})$ of X completely. According to this result, every one-dimensional closed irreducible subvariety C of X is rationally equivalent, hence numerically equivalent, to a finite nonnegative integral linear combination of $\{V(\tau); \tau \in \Delta(r-1)\}$. Hence $\mathbb{NE}(X)$ is a convex polyhedral cone. q.e.d.

Thus for a nonsingular toric projective variety X, we get the entire picture of $\mathbb{NE}(X)$. Since $V(\tau) \cong \mathbb{P}_1(\mathbb{C})$ for each $\tau \in \Delta(r-1)$, an extremal ray R in our generalized sense also contains a rational curve. The main theorem of Reid [R4], which we now explain, asserts further that for each such R, the contraction cont_R exists in the context of toric projective varieties exactly as in Mori's theorem.

Remark. The above proposition is compatible with Tsunoda's result as follows: As in the last example in Sect. 2.1, we have $-K_X = \sum_{\varrho \in \Delta(1)} V(\varrho)$, which has simple normal crossings. Thus so does any effective divisor D satisfying $D \leq -K_X$. For any $\tau \in \Delta(r-1)$, we have $(V(\varrho_0) . V(\tau)) > 0$ for some $\varrho_0 \in \Delta(1)$, since there certainly exists ϱ_0 such that $\tau + \varrho_0$ belongs to $\Delta(r)$ and has τ as a face. Choose D so that $-K_X - D = V(\varrho_0)$ and apply Tsunoda's theorem to this D. Then we see that $\overline{\mathbb{NE}}(X)$ is a convex polyhedral cone near $[V(\tau)]$.

Example. For $r=2$, we generally have $\mathbb{N}(X)^* = \mathbb{N}(X)$ and $\mathbb{PA}(X) \subset \overline{\mathbb{NE}}(X)$. Let us examine $\mathbb{NE}(X)$ for each of the two-dimensional nonsingular toric projective varieties X classified in Theorem 1.28. Note that they are projective as we remarked immediately after Corollary 2.16.

(i) For $X = \mathbb{P}_2(\mathbb{C})$, we have $\mathbb{N}(X) \cong \mathbb{R}$ and $\mathbb{NE}(X) \cong \mathbb{R}_{\geq 0}$.

(ii) Let X be the Hirzebruch surface F_a for $a \geq 0$. Since $\Delta(1) = \{\mathbb{R}_{\geq 0} n, \mathbb{R}_{\geq 0} n', \mathbb{R}_{\geq 0}(-n'), \mathbb{R}_{\geq 0}(-n+an')\}$, we have four T_N-stable closed irreducible subvarieties of dimension one.

$$f := [V(\mathbb{R}_{\geq 0} n)] = [V(\mathbb{R}_{\geq 0}(-n+an'))]$$

is the numerical equivalence class of the fibers for the $\mathbb{P}_1(\mathbb{C})$-bundle structure $X \to \mathbb{P}_1(\mathbb{C})$, while

$$s := [V(\mathbb{R}_{\geq 0} n')]$$

is the numerical equivalence class of the section with the smallest self-intersection number $-a$. By Sect. 2.2, we see that

$$(f.f) = 0 \ , \quad (f.s) = 1 \ , \quad (s.s) = -a \ ,$$

and that $[V(\mathbb{R}_{\geq 0}(-n'))] = s + af$ has the self-intersection number a. It is not hard to show that

$$\mathbb{NE}(X) = \mathbb{R}_{\geq 0} s + \mathbb{R}_{\geq 0} f \ , \quad \mathbb{PA}(X) = \mathbb{R}_{\geq 0}(s+af) + \mathbb{R}_{\geq 0} f$$

and $[-K_X] = 2s + (a+2)f$.

For the extremal ray $R := \mathbb{R}_{\geq 0} f$ of $\mathbb{NE}(X)$, the contraction $\mathrm{cont}_R : X \to X_R = \mathbb{P}_1(\mathbb{C})$ coincides with the $\mathbb{P}_1(\mathbb{C})$-bundle structure. Let us consider the contraction $\mathrm{cont}_{R'} : X \to X_{R'}$ for the other extremal ray $R' := \mathbb{R}_{\geq 0} s$. If $a=0$, then it is the other projection of $X = \mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$ to $\mathbb{P}_1(\mathbb{C})$. If $a>0$, then it is the birational morphism which collapses $V(\mathbb{R}_{\geq 0} n')$ to a point. For $a=1$, we see that $\mathrm{cont}_{R'}$ is an equivariant blowing-up of $X_{R'} = \mathbb{P}_2(\mathbb{C})$, while for $a \geq 2$, $\mathrm{cont}_{R'}(V(\mathbb{R}_{\geq 0} n'))$ is a singular point of $X_{R'}$. This latter case is not dealt with in Mori's theorem, since $([-K_X].s) = 2-a \leq 0$. (cf. Fig. 2.8.)

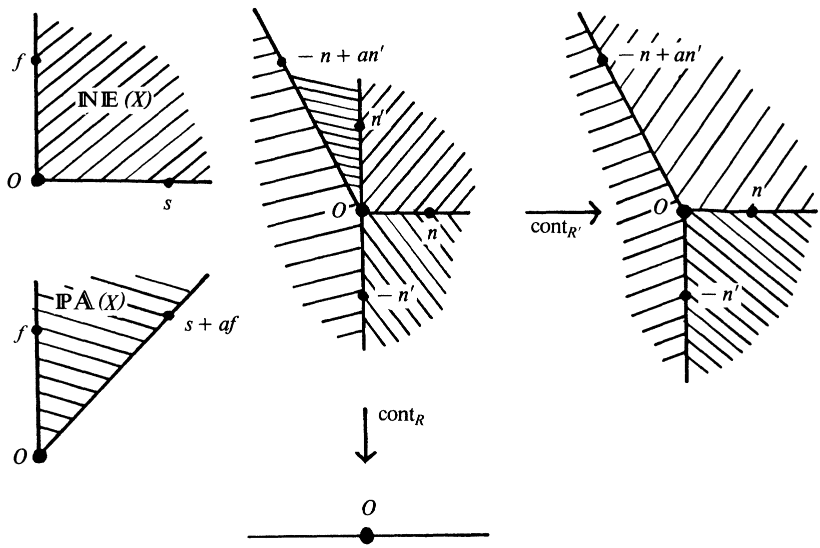


Fig. 2.8

(iii) By Theorem 1.28, $X = T_N \operatorname{emb}(\Delta)$ with $^{\#}\Delta(1) \geq 5$ is obtained from some F_a by a succession of equivariant blowing-ups at T_N-fixed points. Corresponding to each equivariant blowing-up, we get one additional exceptional curve $V(\tau)$, and $\mathbb{R}_{\geq 0}[V(\tau)]$ becomes a new extremal ray. Thus $\mathbb{NE}(X)$ is a simplicial cone of dimension $^{\#}\Delta(1) - 2$.

Consider, for instance, the equivariant blowing-up $X \to F_a$ with $a \geq 2$ at the point $V(\mathbb{R}_{\geq 0}(-n') + \mathbb{R}_{\geq 0}(-n+an'))$. Then $\mathbb{NE}(X)$ has three extremal rays R, R', R'' with respective generators $[V(\mathbb{R}_{\geq 0}(-n+(a-1)n')]$, $[V(\mathbb{R}_{\geq 0}(-n+an'))]$, $[V(\mathbb{R}_{\geq 0} n')]$. The maps of fans corresponding to the contractions cont_R, $\mathrm{cont}_{R'}$ and $\mathrm{cont}_{R''}$ are as in Fig. 2.9.

Let us go back to the general situation of an r-dimensional nonsingular toric projective variety $X = T_N \operatorname{emb}(\Delta)$ and construct the contraction $\mathrm{cont}_R : X \to X_R = T_{N(R)} \operatorname{emb}(\Delta_R)$ for each one-dimensional face R of $\mathbb{NE}(X)$ following Reid [R4]. For that purpose, let

$$A(R) := \{\rho \in \Delta(1); ([V(\rho)].\zeta) < 0,\ R \ni \forall \zeta \neq 0\}$$

$$B(R) := \{\rho \in \Delta(1); ([V(\rho)].\zeta) > 0,\ R \ni \forall \zeta \neq 0\}$$

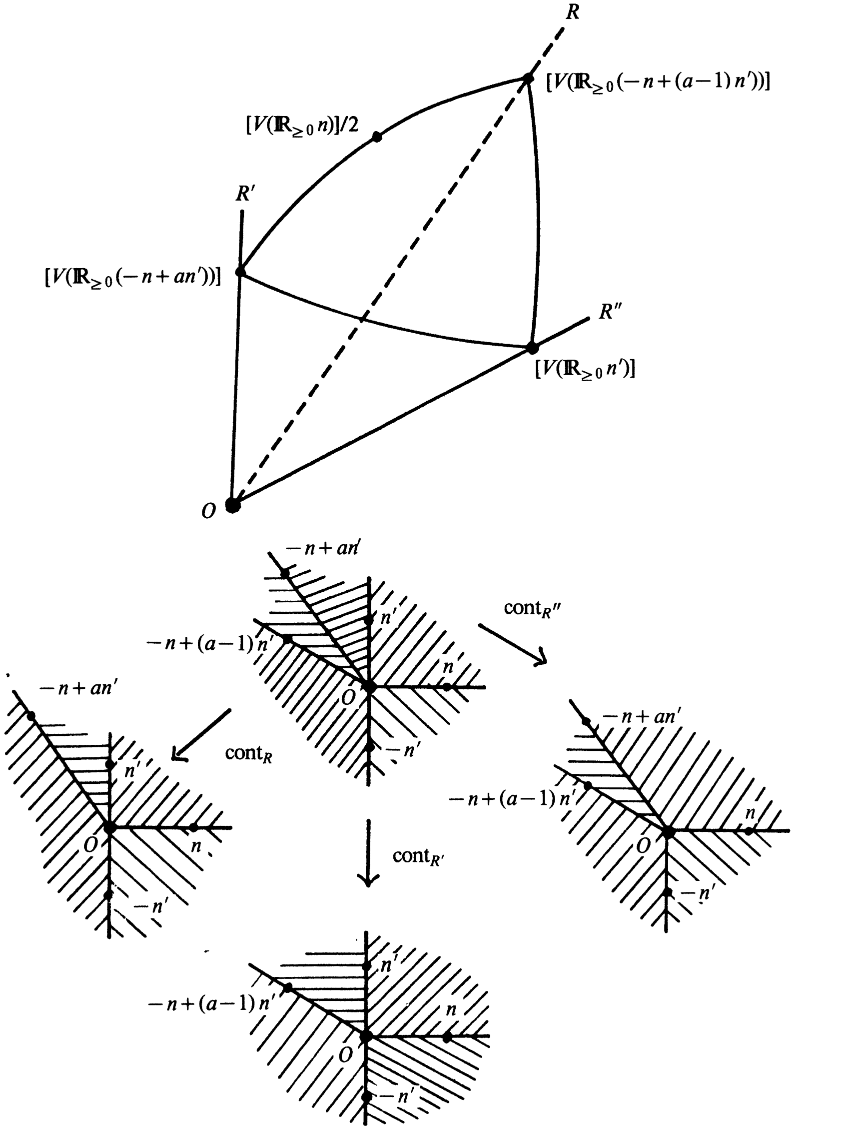


Fig. 2.9

and define convex polyhedral cones in $N_{\mathbb{R}}$ by

$$\pi_-(R) := \sum_{\varrho \in A(R)} \varrho\ , \quad \pi_+(R) := \sum_{\varrho \in B(R)} \varrho \quad \text{and} \quad \pi(R) := \pi_-(R) + \pi_+(R)\ .$$

Thus R is not numerically effective if and only if $A(R)$ is nonempty, i.e., $\pi_-(R) \neq \{O\}$. Each $\tau \in \Delta(r-1)$ is a face of exactly two $\sigma_1, \sigma_2 \in \Delta(r)$. We then define an r-dimensional convex polyhedral cone $P(\tau)$ by

$$P(\tau) := \sigma_1 + \sigma_2\ .$$

Theorem 2.27 (Reid [R4, Theorem 2.4, Corollaries 2.5 and 2.7]). *Let $X = T_N \operatorname{emb}(\Delta)$ be an r-dimensional nonsingular toric projective variety and let R be a one-dimensional face of $\mathbb{NE}(X)$, i.e., an extremal ray in a generalized sense. With $A(R)$, $B(R)$, $\pi_-(R)$, $\pi_+(R)$ and $\pi(R)$ defined as above, we have the following:*

(1) *If R is not numerically effective, i.e., $A(R)$ is nonempty, then there exists a unique finite complete fan Δ_R in $N(R) := N$ such that Δ is a subdivision of Δ_R and that*

$$\Delta_R(r-1) = \{\tau \in \Delta(r-1);\ [V(\tau)] \notin R\} .$$

Furthermore, Δ_R satisfies:

(i) $\pi_-(R) \in \Delta$, $\pi(R) \in \Delta_R$ *and*

$$\{\sigma \in \Delta ; \sigma \not> \pi_-(R)\} = \{\sigma' \in \Delta_R ; \sigma' \not> \pi(R)\} .$$

(ii) $\{\sigma' \in \Delta_R(r); \sigma' > \pi(R)\} = \{P(\tau); \tau \in \Delta(r-1),\ [V(\tau)] \in R\}$.

(2) *If R is numerically effective, i.e., $A(R)$ is empty, then $\pi(R) = \pi_+(R)$ is an $\mathbb{R}$-subspace of $N_{\mathbb{R}}$. Denote by ψ the canonical projection $N \to N(R) := N/N \cap \pi(R)$ as well as its scalar extension $N_{\mathbb{R}} \to N(R)_{\mathbb{R}} = N_{\mathbb{R}}/\pi(R)$. Then there exists a unique finite complete nonsingular fan Δ_R in $N(R)$ such that*

(i) Δ *is a subdivision of* $\Delta_R^* := \{\psi^{-1}(\sigma'); \sigma' \in \Delta_R\}$

and that

(ii) $\{\psi^{-1}(\sigma') \in \Delta_R^* ; \dim \psi^{-1}(\sigma') = r\} = \{\tau + \pi(R); \tau \in \Delta(r-1),\ [V(\tau)] \in R\}$.

Corollary 2.28. *In the above notation, the following hold:*

(1) *If R is not numerically effective, then the map of fans $(N, \Delta) \to (N(R), \Delta_R)$ gives rise to an equivariant birational morphism (called the* contraction *for R)*

$$\operatorname{cont}_R : X = T_N \operatorname{emb}(\Delta) \to X_R := T_{N(R)} \operatorname{emb}(\Delta_R) ,$$

which induces an isomorphism $X \setminus V(\pi_-(R)) \xrightarrow{\sim} X_R \setminus V(\pi(R))$ and which satisfies $(\operatorname{cont}_R)_* \mathcal{O}_X = \mathcal{O}_{X_R}$. For $\tau \in \Delta(r-1)$, $\operatorname{cont}_R(V(\tau))$ is a point if and only if $[V(\tau)] \in R$. X_R is a toric projective variety.

(2) *If R is numerically effective, then $X_R := T_{N(R)} \operatorname{emb}(\Delta_R)$ is a nonsingular toric projective variety of dimension less than r. The map of fans $\psi : (N, \Delta) \to (N(R), \Delta_R)$ induced by ψ gives rise to an equivariant morphism (called the* contraction *for R)*

$$\operatorname{cont}_R : X \to X_R$$

which again satisfies $(\operatorname{cont}_R)_* \mathcal{O}_X = \mathcal{O}_{X_R}$. For $\tau \in \Delta(r-1)$, $\operatorname{cont}_R(V(\tau))$ is a point if and only if $[V(\tau)] \in R$.

The above (1) and (2) correspond respectively to (6) and (7) in Mori's theorem. The one-dimensional face R of $\mathbb{NE}(X)$ corresponds, by Proposition A.6, to a codimension one face R^* of the dual polyhedral cone $\mathbb{PA}(X)$. In fact, R^* can be canonically identified with $\mathbb{PA}(X_R)$. The projectivity of X_R is a reflection of this fact. We omit the proofs of these facts as well as Corollary 2.28.

Here is a sketch of the proof of Theorem 2.27:

By definition, Δ_R in (1) and Δ_R^* in (2) are decompositions of $N_{\mathbb{R}}$ which we obtain by removing the codimension one walls $\{\tau \in \Delta(r-1); [V(\tau)] \in R\}$ from the decomposition $N_{\mathbb{R}} = \cup_{\sigma \in \Delta} \sigma$. We need to show that Δ_R is a fan in both cases.

For that purpose, let us examine those $\tau\in\Delta(r-1)$ which satisfy $[V(\tau)]\in R$. There exist exactly two cones in $\Delta(r)$ which have τ as a face. As in the last example of Sect. 2.2, there thus exist two $\mathbb{Z}$-bases $\{n_1,\ldots,n_{r-1},n_r\}$ and $\{n_1,\ldots,n_{r-1},n_{r+1}\}$ of N such that

$$\tau=\varrho_1+\ldots+\varrho_{r-1}\ ,\qquad \tau+\varrho_r\in\Delta(r)\ ,\quad \tau+\varrho_{r+1}\in\Delta(r)\ ,$$

where $\varrho_j:=\mathbb{R}_{\geq 0}n_j$ for all j. Moreover, there exist integers $a_1,\ldots,a_{r-1}$, $a_r=1$, $a_{r+1}=1$ such that

$$\sum_{j=1}^{r+1} a_jn_j=O \quad\text{and}\quad (V(\varrho_j).\,V(\tau))=a_j \quad\text{for}\quad 1\leq j\leq r+1\ .$$

Since $[V(\tau)]$ is assumed to be a generator of R and since $V(\varrho)$ for $\varrho\in\Delta(1)\setminus\{\varrho_1,\ldots,\varrho_{r+1}\}$ has no point in common with $V(\tau)$, we get

$$A(R)=\{\varrho_j;\ 1\leq j\leq r-1,\ a_j<0\}$$
$$B(R)=\{\varrho_j;\ 1\leq j\leq r+1,\ a_j>0\}\supset\{\varrho_r,\varrho_{r+1}\}\ .$$

By renumeration, if necessary, we may assume

$$A(R)=\{\varrho_1,\ldots,\varrho_\alpha\}\quad\text{and}\quad B(R)=\{\varrho_{\beta+1},\ldots,\varrho_{r-1},\varrho_r,\varrho_{r+1}\}$$

for $0\leq\alpha\leq\beta\leq r-1$. For $1\leq j\leq r-1$,

$$\omega_j:=\varrho_1+\ldots+\hat{\varrho}_j+\ldots+\varrho_{r-1}\qquad (\varrho_j\text{ is omitted})$$

is a face of τ of dimension $r-2$ and $\tau=\omega_j+\varrho_j$ holds.

Lemma 2.29 (Reid [R4, Lemma 3.2]). *In the above notation, we get a decomposition*

$$P(\tau)=\tau+\varrho_r+\varrho_{r+1}=(\tau+\varrho_r)\cup(\tau+\varrho_{r+1})\cup\left(\bigcup_{j=\beta+1}^{r-1}(\omega_j+\varrho_r+\varrho_{r+1})\right)\ .$$

$\pi(R)=\sum_{1\leq j\leq\alpha}\varrho_j+\sum_{\beta+1\leq j\leq r+1}\varrho_j$ is a face of $P(\tau)$ of dimension $r-\beta+\alpha$. If $\alpha=0$, i.e., R is numerically effective, then $\pi(R)=\sum_{\beta+1\leq j\leq r+1}\varrho_j$ is an $\mathbb{R}$-subspace of $N_{\mathbb{R}}$ of dimesnion $r-\beta$ such that $P(\tau)=\tau+\pi(R)$.

Proof. An element n of $P(\tau)=\tau+\varrho_r+\varrho_{r+1}$ is of the form $n=\sum_{1\leq j\leq r+1}y_jn_j$ for $y_1,\ldots,y_{r+1}\in\mathbb{R}_{\geq 0}$. Since $a_1,\ldots,a_\alpha$ are negative, while $a_{\beta+1},\ldots,a_{r+1}$ are positive and since $\sum_{1\leq j\leq\alpha}a_jn_j+\sum_{\beta+1\leq j\leq r+1}a_jn_j=O$, we see that $n=n-(y_k/a_k)\sum_{1\leq j\leq r+1}a_jn_j$ belongs to $\varrho_1+\ldots+\hat{\varrho}_k+\ldots+\varrho_{r+1}$ where $y_k/a_k:=\min\{y_j/a_j;\ \beta+1\leq j\leq r+1\}$. We have the decomposition in question, since this sum equals $\omega_k+\varrho_r+\varrho_{r+1}$ if $k\leq r-1$, while it equals $\tau+\varrho_{r+1}$ and $\tau+\varrho_r$, if $k=r$ and $k=r+1$, respectively.

$P(\tau)$ is contained in

$$\Pi:=\sum_{1\leq j\leq\beta}\mathbb{R}_{\geq 0}n_j+\sum_{\beta+1\leq j\leq r}\mathbb{R}n_j\ ,$$

since $n_{r+1}=\sum_{1\le j\le\alpha}(-a_j)n_j-\sum_{\beta+1\le j\le r-1}a_jn_j-n_r$. Moreover, $\pi(R)$ is exactly the intersection of $P(\tau)$ with the face $\sum_{1\le j\le\alpha}\mathbb{R}_{\ge 0}n_j+\sum_{\beta+1\le j\le r}\mathbb{R}n_j$ of Π. Thus $\pi(R)$ is a face of $P(\tau)$. We have $\dim \pi(R)=r-\beta+\alpha$, since the $\mathbb{R}$-subspace spanned by $\pi(R)$ has $\{n_1,\dots,n_\alpha,n_{\beta+1},\dots,n_r\}$ as a basis.

If $\alpha=0$, then the coefficients of $n_{r+1}=\sum_{\beta+1\le j\le r}(-a_j)n_j$ are all negative. Thus we obviously get $\pi(R)=\sum_{\beta+1\le j\le r}\varrho_j+\varrho_{r+1}=\sum_{\beta+1\le j\le r}\mathbb{R}n_j$ and $P(\tau)=\tau+\pi(R)$.

q.e.d.

Remark. Reid [R4, Theorem 3.4] obtains an analogue of the elementary transformation in Sect. 1.7 by looking at another decomposition $P(\tau) = \bigcup_{1\le j\le\alpha}(\omega_j+\varrho_r+\varrho_{r+1})$ obtained similarly.

Suppose $\tau\in\Delta(r-1)$ satisfies $[V(\tau)]\in R$. Using the fact that R is a one-dimensional face of $\mathbb{NE}(X)$, we can find $\tau'\in\Delta(r-1)$ with $[V(\tau')]\in R$ in the "vicinity" of τ as follows:

Proposition 2.30 (Reid [R4, Corollary 2.10]). *Let the notation be as above.*

(a) *If $\varrho_j\in B(R)$, i.e., $\beta+1\le j\le r-1$, then $\omega_j\in\Delta(r-2)$ is a face of exactly three cones $\tau=\omega_j+\varrho_j$, $\omega_j+\varrho_r$ and $\omega_j+\varrho_{r+1}$ in $\Delta(r-1)$. In this case, we have $[V(\omega_j+\varrho_r)]\in R$ and $[V(\omega_j+\varrho_{r+1})]\in R$. Moreover, ω_j is a face of exactly three cones $\tau+\varrho_r$, $\tau+\varrho_{r+1}$ and $\omega_j+\varrho_r+\varrho_{r+1}$ in $\Delta(r)$ and we have $P(\tau)=P(\omega_j+\varrho_r)=P(\omega_j+\varrho_{r+1})$.* (cf. Fig. 2.10a.)

(b) *If $\alpha+1\le j\le\beta$, i.e., $([V(\varrho_j)].[V(\tau)])=0$, then there exists a unique $\varrho'_j\in\Delta(1)$ such that $\omega_j\in\Delta(r-2)$ is a face of exactly four cones $\tau+\varrho_r$, $\tau+\varrho_{r+1}$, $\omega_j+\varrho'_j+\varrho_r$ and $\omega_j+\varrho'_j+\varrho_{r+1}$ in $\Delta(r)$. In this case, we have $[V(\omega_j+\varrho'_j)]\in R$ and see that $P(\tau)$ and $P(\omega_j+\varrho'_j)$ lie on mutually opposite sides with respect to the common $(r-1)$-dimensional face $\omega_j+\varrho_r+\varrho_{r+1}$* (see Fig. 2.10b).

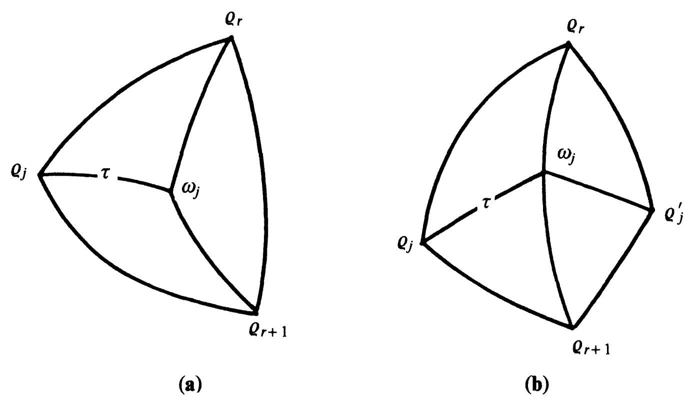


Fig. 2.10

Proof. Let $\tau_1:=\tau=\omega_j+\varrho_j$, $\tau_2:=\omega_j+\varrho_r$, $\tau_3:=\omega_j+\varrho_{r+1}$, $\tau_4,\dots,\tau_v$ be the cones in $\Delta(r-1)$ which have ω_j as a face.

(a) Let $\{m_1,\dots,m_r\}$ be the $\mathbb{Z}$-basis for M dual to $\{n_1,\dots,n_r\}$. Thus $\langle m_j,n_r\rangle=0$ and $\langle m_j,n_{r+1}\rangle=-a_j<0$. Since m_j is in $\omega_j^\perp$, we may regard the restriction of $\mathbf{e}(m_j)$ to the two-dimensional subvariety $V(\omega_j)$ as a rational function on $V(\omega_j)$ in view of Lemma 2.11. Then its divisor on $V(\omega_j)$ is

$$V(\tau_1)+(-a_j)V(\tau_3)+\sum_{k=4}^{v} b_k V(\tau_k) \; ,$$

where $\tau_k=\omega_j+\mathbb{R}_{\geq 0}n'_k$ for a primitive element $n'_k\in N$ and $b_k:=\langle m_j, n'_k\rangle \leq 0$. Consequently, $V(\tau_1)$ is rationally equivalent to $a_j V(\tau_3)+\sum_{4\leq k\leq v}(-b_k)V(\tau_k)$ on $V(\omega_j)$, hence on X. Since $\tau_1=\tau$ and $\tau_3=\omega_j+\varrho_{r+1}$ by definition, we have

$$[V(\tau)]=a_j[V(\omega_j+\varrho_{r+1})]+\sum_{k=4}^{v}(-b_k)[V(\tau_k)] \; .$$

We thus conclude $[V(\omega_j+\varrho_{r+1})]\in R$, since $a_j>0$, $-b_k\geq 0$ and since the one-dimensional face R of $\mathbb{NE}(X)$ is generated by $[V(\tau)]$.

Using the $\mathbb{Z}$-basis for M dual to $\{n_1,\dots,n_{r-1},n_{r+1}\}$ instead, we similarly conclude $[V(\omega_j+\varrho_r)]\in R$.

We have $(V(\varrho_{r+1}) \,.\, V(\omega_j+\varrho_r))\neq 0$, while ϱ_{r+1} is obviously not a face of $\omega_j+\varrho_r$. Thus applying to $\omega_j+\varrho_r\in\Delta(r-1)$ what we saw immediately before Lemma 2.29, we see that $(\omega_j+\varrho_r)+\varrho_{r+1}\in\Delta(r)$. Hence $v=3$.

(b) The $\mathbb{Z}$-basis $\{m_1,\dots,m_r\}$ of M dual to $\{n_1,\dots,n_r\}$ satisfies $\langle m_j,n_r\rangle=\langle m_j,n_{r+1}\rangle=0$. As in (a) above, the divisor of the rational function obtained as the restriction of $\mathbf{e}(m_j)$ to $V(\omega_j)$ is $V(\tau)+\sum_{4\leq k\leq v}b_k V(\tau_k)$ for some $b_k\leq 0$. Hence $[V(\tau)]=\sum_{4\leq k\leq v}(-b_k)[V(\tau_k)]$. Consequently, we have $[V(\tau_k)]\in R$ for every k satisfying $b_k\neq 0$ and $4\leq k\leq v$. At least one such k exists and we get $\tau_k=\varrho'_j+\omega_j$ for some $\varrho'_j\in\Delta(1)$. Obviously, ϱ_r and ϱ_{r+1} are not faces of $\omega_j+\varrho'_j$, while we have $(V(\varrho_r)\,.\,V(\tau_k))\neq 0$ and $(V(\varrho_{r+1})\,.\,V(\tau_k))\neq 0$. Hence we can again apply to $\omega_j+\varrho'_j$ our consideration immediately before Lemma 2.29 and conclude $k=v=4$. q.e.d.

By Lemma 2.29 and Proposition 2.30, distinct r-dimensional polyhedral cones among $\{P(\tau);\tau\in\Delta(r-1),\,[V(\tau)]\in R\}$ do not intersect with each other in the interior. These cones thus give rise to a polyhedral decomposition of the union P. We see easily that P is a convex neighborhood in $N_{\mathbb{R}}$ of $\pi(R)$ and that the boundary of P is a union of convex polyhedral cones belonging to Δ. Hence Δ_R is a fan in $N(R)$.

For lack of space, we do not mention many other results in Reid [R4] which are of interest from the point of view of birational geometry.

Chapter 3. Toric Varieties and Holomorphic Differential Forms

In this chapter, we deal with topics related to holomorphic differential forms on toric varieties.

More natural for toric varieties, however, are differential forms with logarithmic poles, which we first describe in Sect. 3.1. Using the result, we define Ishida's complexes of coherent sheaves on toric varieties in Sect. 3.2. For toric varieties which are nonsingular or which have at worst quotient singularities, these complexes turn out to be resolutions of the sheaves of germs of holomorphic differential forms. As a result, we will be able to compute, directly from fans, the Hodge cohomology, the cohomology ring, the Euler number, the index and so forth of a toric variety.

Even for general toric varieties with singularities, however, one of Ishida's complexes remains a dualizing complex. Using this fact, we can show that toric varieties in general have at worst Cohen-Macaulay singularities.

These results are originally due to Hochster [H5], [TE], Demazure [D5], Danilov [D1], Ehlers [E2], etc. We formulate them in a more unified and clear-cut manner using Ishida's complexes in Ishida [I1] as well as results in Ishida-Oda [IO]. Hopefully, the methods leads us to a theory of "degenerate varieties" which generalizes that of toroidal embeddings. There has been an attempt in this direction by Ishida [I5] as we see at the end of Sect. 3.2. See also the introduction to Chap. 4.

We also explain the description due to Demazure [D5] of the automorphism groups of nonsingular compact toric varieties as well as related results on the Cremona groups due to Umemura [U1], [U2], [U3], [U4].

3.1 Differential Forms with Logarithmic Poles

For $N \cong \mathbb{Z}^r$ and its dual $\mathbb{Z}$-module M, we already defined in Sect. 1.2 an algebraic torus by

$$T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^\times = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^r \ .$$

The Lie algebra of T_N is canonically identified as

$$\mathrm{Lie}\,(T_N) = \mathbb{C} \otimes_{\mathbb{Z}} N \ ,$$

hence the sheaf of germs of holomorphic vector fields on T_N and that of holomorphic 1-forms on T_N are identified as

$$\Theta_{T_N} = \mathcal{O}_{T_N} \otimes_{\mathbb{Z}} N \quad \text{and} \quad \Omega^1_{T_N} = \mathcal{O}_{T_N} \otimes_{\mathbb{Z}} M \ .$$

Indeed, for each $n \in N$ we define a $\mathbb{C}$-linear endomorphism δ_n of $\mathbb{C}[M] = \oplus_{m \in M} \mathbb{C}\mathbf{e}(m)$ by

$$\delta_n(\mathbf{e}(m)) := \langle m, n \rangle \mathbf{e}(m) \quad \text{for every} \quad m \in M \ .$$

Clearly, δ_n is then a derivation of the $\mathbb{C}$-algebra $\mathbb{C}[M]$. Choose a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N and let $\{m_1, \ldots, m_r\}$ be the dual $\mathbb{Z}$-basis for M. With $u_j := \mathbf{e}(m_j)$ for $1 \le j \le r$, we have $\mathbb{C}[M] = \mathbb{C}[u_1, \ldots, u_r, u_1^{-1}, \ldots, u_r^{-1}]$ and $\delta_{n_j} = u_j \partial/\partial u_j$. Obviously, $\{u_1 \partial/\partial u_1, \ldots, u_r \partial/\partial u_r\}$ is a $\mathbb{C}$-basis for Lie (T_N). Moreover, $d\mathbf{e}(m)/\mathbf{e}(m)$ for each $m \in M$ is a T_N-invariant holomorphic 1-form on T_N. As is well-known, $\{du_1/u_1, \ldots, du_r/u_r\}$ is a $\mathbb{C}$-basis for the space of T_N-invariant holomorphic 1-forms on T_N.

We now extend the above consideration to toric varieties in general.

Let D be a closed analytic subspace of a complex analytic space V. Then $\Theta_V(-\log D)$ is the *sheaf of germs of holomorphic vector fields with logarithmic zeros along D*, while $\Omega_V^p(\log D)$ for $p = 0, \ldots, r$ denote the *sheaves of germs of p-forms with logarithmic poles along D*. They are defined as follows: Let I be the sheaf of ideals of $\mathcal{O}_V$ defining D. Then $\Theta_V(-\log D)$ is the sheaf of germs of holomorphic derivations in Θ_V which leave the ideal I stable. Thus

$$\Theta_V(-\log D) := \{\delta \in \Theta_V \, ; \, \delta(I) \subset I\} \subset \Theta_V \ .$$

We denote its dual $\mathcal{O}_V$-module by

$$\Omega_V^1(\log D) := \mathcal{H}om_{\mathcal{O}_V}(\Theta_V(-\log D), \mathcal{O}_V)$$

and denote the exterior powers of this latter by

$$\Omega_V^p(\log D) := \textstyle\bigwedge^p \Omega_V^1(\log D) \quad \text{for} \quad p = 0, 1, \ldots, r \ .$$

Thus we have a natural $\mathcal{O}_V$-linear map $\Omega_V^p \to \Omega_V^p(\log D)$ from the sheaf Ω_V^p of germs of Kähler p-forms on V (cf. [EGA]). When V is nonsingular, this map is injective and Ω_V^p coincides with the sheaf of germs of holomorphic p-forms.

Proposition 3.1 (Ishida-Oda [IO, (1.12), Proposition]). *On an r-dimensional toric variety $X = T_N \operatorname{emb}(\Delta)$, consider the T_N-invariant effective Weil divisor $D := \sum_{\varrho \in \Delta(1)} V(\varrho)$, where $\Delta(1) := \{\varrho \in \Delta \, ; \dim \varrho = 1\}$ and where $V(\varrho)$ is the closure of the codimension one T_N-orbit* orb(ϱ) *corresponding to $\varrho \in \Delta(1)$ as in Proposition 1.6 and Corollary 1.7. Then we have the following canonical isomorphisms of $\mathcal{O}_X$-modules:*

$$\mathcal{O}_X \otimes_{\mathbb{Z}} N = \Theta_X(-\log D)$$

$$\mathcal{O}_X \otimes_{\mathbb{Z}} \textstyle\bigwedge^p M = \Omega_X^p(\log D) \quad \textit{for} \quad p = 0, 1, \ldots, r \ .$$

Proof. It suffices to prove the former, since the latter is the p-th exterior power of the $\mathcal{O}_X$-dual of the former. Moreover, we may regard X as an algebraic variety, since the $\mathcal{O}_X$-modules in question are analytic coherent sheaves associated to algebraic coherent sheaves (cf. Remarks immediately after Proposition 1.2 and Theorem 1.4).

We have a canonical $\mathcal{O}_X$-homomorphism $\mathcal{O}_X \otimes_{\mathbb{Z}} N \to \Theta_X$, since δ_n for each $n \in N$ clearly belongs to $\operatorname{Lie}(T_N) \subset H^0(X, \Theta_X)$. It is enough to show that this $\mathcal{O}_X$-homomorphism is an isomorphism onto $\Theta_X(-\log D) \subset \Theta_X$.

Let us consider the restriction to the affine open set U_σ corresponding to each $\sigma \in \Delta$. As we saw in Sect. 1.2, $\mathbb{C}[M \cap \sigma^\vee]$ is the ring of polynomial functions on U_σ with $\{\mathbf{e}(m); m \in M \cap \sigma^\vee\}$ as a $\mathbb{C}$-basis. Furthermore, $\{\mathbf{e}(m); m \in M \cap \operatorname{int}(\sigma^\vee)\}$ is a $\mathbb{C}$-basis for the ideal I_σ of $\mathbb{C}[M \cap \sigma^\vee]$ defining the closed set $U_\sigma \cap D$, where $\operatorname{int}(\sigma^\vee)$ is the interior of the polyhedral cone $\sigma^\vee$. For each $n \in N$, the $\mathbb{C}$-linear endomorphism δ_n defined by $\delta_n(\mathbf{e}(m)) := \langle m, n \rangle \mathbf{e}(m)$ is clearly a derivation of the $\mathbb{C}$-algebra $\mathbb{C}[M \cap \sigma^\vee]$ such that $\delta_n(I_\sigma) \subset I_\sigma$.

Conversely, let δ be a derivation of $\mathbb{C}[M \cap \sigma^\vee]$ such that $\delta(\mathbb{C}) = 0$ and $\delta(I_\sigma) \subset I_\sigma$. We show that $\delta = \sum_{1 \leq j \leq s} \varphi_j \delta_{n_j}$ for a suitable element $\sum_{1 \leq j \leq s} \varphi_j \otimes n_j$ of $\mathbb{C}[M \cap \sigma^\vee] \otimes_{\mathbb{Z}} N$. Indeed, each one-dimensional face ϱ of σ gives rise to a closed irreducible subvariety $U_\sigma \cap V(\varrho)$ of U_σ, hence a prime ideal $\mathfrak{p}(\varrho)$ of height one in $\mathbb{C}[M \cap \sigma^\vee]$. As we saw in the proof of Proposition 1.6, $\{\mathbf{e}(m); m \in M \cap \sigma^\vee, m \notin \sigma^\vee \cap \varrho^\perp\}$ is a $\mathbb{C}$-basis for $\mathfrak{p}(\varrho)$. Moreover, we know that the irredundant prime ideal decomposition

$$I_\sigma = \bigcap_{\varrho \in \Delta(1), \varrho < \sigma} \mathfrak{p}(\varrho)$$

is unique. Since $\delta(I_\sigma) \subset I_\sigma$, we necessarily have $\delta(\mathfrak{p}(\varrho)) \subset \mathfrak{p}(\varrho)$ for each ϱ.

On the other hand for each $m \in M \cap \sigma^\vee$, the principal ideal of $\mathbb{C}[M \cap \sigma^\vee]$ generated by $\mathbf{e}(m)$ has the primary ideal decomposition

$$\mathbf{e}(m) \mathbb{C}[M \cap \sigma^\vee] = \bigcap_{\varrho \in \Delta(1), \varrho < \sigma} \mathfrak{p}(\varrho)^{(\langle m, n(\varrho) \rangle)} ,$$

where for a positive integer l, the symbolic l-th power $\mathfrak{p}(\varrho)^{(l)}$ of the prime ideal $\mathfrak{p}(\varrho)$ is the primary ideal defined as the pull-back to $\mathbb{C}[M \cap \sigma^\vee]$ of the l-th power of the maximal ideal of the localization of $\mathbb{C}[M \cap \sigma^\vee]$ with respect to $\mathfrak{p}(\varrho)$. This decomposition is nothing but the ring-theoretic re-interpretation of the equality

$$\operatorname{div}(\mathbf{e}(m)) = \sum_{\varrho \in \Delta(1)} \langle m, n(\varrho) \rangle V(\varrho)$$

which we had in Proposition 2.1. Since δ preserves $\mathfrak{p}(\varrho)$, it also preserves the symbolic powers of $\mathfrak{p}(\varrho)$. Thus δ preserves the right hand side, hence the left hand side, of the primary ideal decomposition above. Namely, there exists $\xi(m) \in \mathbb{C}[M \cap \sigma^\vee]$ for each $m \in M \cap \sigma^\vee$ such that

$$\delta(\mathbf{e}(m)) = \mathbf{e}(m) \xi(m) \quad \text{for all} \quad m \in M \cap \sigma^\vee .$$

The map $\xi: M \cap \sigma^\vee \ni m \mapsto \xi(m) \in \mathbb{C}[M \cap \sigma^\vee]$ thus obtained is additive, since δ is a derivation by assumption, hence $\mathbf{e}(m+m')\xi(m+m') = \delta(\mathbf{e}(m+m'))$ $= \delta(\mathbf{e}(m) \cdot \mathbf{e}(m')) = \mathbf{e}(m)\delta(\mathbf{e}(m')) + \mathbf{e}(m')\delta(\mathbf{e}(m)) = \mathbf{e}(m+m')\{\xi(m) + \xi(m')\}$. We know that $M \cap \sigma^\vee$ generates M as a group. Hence ξ can be extended uniquely to an additive homomorphism $\xi: M \to \mathbb{C}[M \cap \sigma^\vee]$. Consequently, ξ can be identified with an element $\sum_{1 \leq j \leq s} \varphi \otimes n_j$ of $\mathbb{C}[M \cap \sigma^\vee] \otimes_{\mathbb{Z}} N$. Thus $\xi(m) = \sum_{1 \leq j \leq s} \langle m, n_j \rangle \varphi_j$ and $\delta(\mathbf{e}(m)) = \mathbf{e}(m) \sum_{1 \leq j \leq s} \langle m, n_j \rangle \varphi_j = \sum_{1 \leq j \leq s} \varphi_j \delta_{n_j}(\mathbf{e}(m))$. q.e.d.

Corollary 3.2. *Let $X = T_N \operatorname{emb}(\Delta)$. For each $\sigma \in \Delta$, define a T_N-invariant effective Weil divisor on the closed irreducible subspace $V(\sigma)$ of X by*

$$D(\sigma) := \sum_{\tau} V(\tau) \;,$$

where the summation is taken over all $\tau \in \Delta$ satisfying $\sigma < \tau$ and $\dim \sigma + 1 = \dim \tau$ (cf. Corollary 1.7). *Then we have canonical isomorphisms of $\mathcal{O}_X$-modules*

$$\Theta_{V(\sigma)}(-\log D(\sigma)) = \mathcal{O}_{V(\sigma)} \otimes_{\mathbb{Z}} N/\mathbb{Z}(\sigma \cap N)$$

$$\Omega^p_{V(\sigma)}(\log D(\sigma)) = \mathcal{O}_{V(\sigma)} \otimes_{\mathbb{Z}} \textstyle\bigwedge^p (M \cap \sigma^\perp) \quad \textit{for} \quad 0 \le p \le \dim V(\sigma) \;.$$

Proof. By Corollary 1.7, we have $V(\sigma) = T_{\bar{N}(\sigma)} \operatorname{emb}(\bar{\Delta}(\sigma))$, while $M \cap \sigma^\perp$ is the dual $\mathbb{Z}$-module of $\bar{N}(\sigma) = N/\mathbb{Z}(\sigma \cap N)$. Thus the results follow from Proposition 3.1 with the fan (N, Δ) replaced by $(\bar{N}(\sigma), \bar{\Delta}(\sigma))$. q.e.d.

Corollary 3.3. *Let $X = T_N \operatorname{emb}(\Delta)$ be an r-dimensional nonsingular toric variety. Then $-D := -\sum_{\varrho \in \Delta(1)} V(\varrho)$ is a canonical divisor for X. Namely, we have an isomorphism of $\mathcal{O}_X$-modules $\Omega^r_X \cong \mathcal{O}_X(-D)$.*

Proof. We have $\Omega^r_X(\log D) = \mathcal{O}_X \otimes_{\mathbb{Z}} \bigwedge^r M \cong \mathcal{O}_X$ by Proposition 3.1, while $\Omega^r_X(\log D) = \Omega^r_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ by the nonsingularity of X. q.e.d.

Remark. Corollary 3.3 is a re-interpretation of what we saw in the last example of Sect. 2.1. Even when X is singular, X turns out to have at worst Cohen-Macaulay singularities, as we see in the next section. The $\mathcal{O}_X$-fractional ideal $\mathcal{O}_X(-D)$ will then be a dualizing $\mathcal{O}_X$-module and coincide with the double $\mathcal{O}_X$-dual of Ω^r_X.

3.2 Ishida's Complexes

Ishida's complexes are complexes of coherent sheaves on toric varieties. As a preparation for constructing them, we first introduce Ishida's p-th complex of $\mathbb{Z}$-modules

$$C^\cdot(\Delta; p) = (0 \to C^0(\Delta; p) \xrightarrow{\delta} C^1(\Delta; p) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^p(\Delta; p) \to 0)$$

for a fan Δ in $N \cong \mathbb{Z}^r$ and $p = 0, 1, \ldots, r$.

For that purpose, we let $\Delta(j) := \{\sigma \in \Delta; \dim \sigma = j\}$ as before and define $\mathbb{Z}$-modules by

$$C^j(\Delta; p) := \bigoplus_{\sigma \in \Delta(j)} \textstyle\bigwedge^{p-j}(M \cap \sigma^\perp) \quad \text{for} \quad 0 \le j \le p \;.$$

The coboundary homomorphism $\delta : C^j(\Delta; p) \to C^{j+1}(\Delta; p)$ is defined to be the direct sum $\delta = \oplus_{\tau,\sigma} \delta_{\tau,\sigma}$ of the following components $\delta_{\tau,\sigma} : \bigwedge^{p-j}(M \cap \sigma^\perp) \to \bigwedge^{p-j-1}(M \cap \tau^\perp)$ with respect to $\sigma \in \Delta(j)$ and $\tau \in \Delta(j+1)$. We let $\delta_{\tau,\sigma} = 0$ if σ is not a face of τ, while for $\sigma < \tau$, a primitive element $n \in N$ is uniquely determined modulo $\mathbb{Z}(\sigma \cap N)$ so that $\tau + (-\sigma) = \mathbb{R}_{\ge 0} n + \mathbb{R}\sigma$. Moreover, $M \cap \tau^\perp$ is a $\mathbb{Z}$-submodule of rank $r-j-1$ in the $\mathbb{Z}$-module $M\cap\sigma^\perp$ of rank $r-j$. Each element of $\bigwedge^{p-j}(M\cap\sigma^\perp)$ is a finite linear combination of elements of the form

$$m_1\wedge m_2\wedge\ldots\wedge m_{p-j}\quad\text{with}\quad m_1\in M\cap\sigma^\perp\quad\text{and}\quad m_2,\ldots,m_{p-j}\in M\cap\tau^\perp\ .$$

We then define $\delta_{\tau,\sigma}\colon \bigwedge^{p-j}(M\cap\sigma^\perp)\to\bigwedge^{p-j-1}(M\cap\tau^\perp)$ by

$$\delta_{\tau,\sigma}(m_1\wedge m_2\wedge\ldots\wedge m_{p-j}):=\langle m_1,n\rangle m_2\wedge\ldots\wedge m_{p-j}\ .$$

This is well-defined and is independent of any particular choice of n.

Lemma 3.4 (Ishida [I1, Proposition 1.6]). *$C^\cdot(\Delta;p)$ is a complex of $\mathbb{Z}$-modules. We call it* Ishida's p-th complex of $\mathbb{Z}$-modules *for* (N,Δ).

Proof. We show the composite $\delta^2: C^j(\Delta;p)\to C^{j+1}(\Delta;p)\to C^{j+2}(\Delta;p)$ to vanish. It suffices to show that for all $\sigma\in\Delta(j)$ and $\pi\in\Delta(j+2)$ with $\sigma<\pi$, the (π,σ)-component of δ^2 vanishes.

Since $(\pi+(-\sigma))/\mathbb{R}\sigma$ is a two-dimensional convex polyhedral cone, it has two faces of dimension one. Thus there exist exactly two $\tau,\tau'\in\Delta(j+1)$ such that $\sigma<\tau<\pi$ and $\sigma<\tau'<\pi$. Furthermore by Proposition A.8, primitive elements $n,n'\in N$ are uniquely determined modulo $\mathbb{Z}(\sigma\cap N)$ so that $\pi+(-\sigma)=\mathbb{R}_{\geq0}n+\mathbb{R}_{\geq0}n'+\mathbb{R}\sigma$, $\tau+(-\sigma)=\mathbb{R}_{\geq0}n+\mathbb{R}\sigma$ and $\tau'+(-\sigma)=\mathbb{R}_{\geq0}n'+\mathbb{R}\sigma$. The (π,σ)-component of δ^2 is easily seen to be $\delta_{\pi,\tau}\circ\delta_{\tau,\sigma}+\delta_{\pi,\tau'}\circ\delta_{\tau',\sigma}$. Each element of $\bigwedge^{p-j}(M\cap\sigma^\perp)$ is a finite linear combination of elements of the form

$$m_1\wedge m_2\wedge m_3\wedge\ldots\wedge m_{p-j}$$

with $m_1\in M\cap(\tau')^\perp$, $m_2\in M\cap\tau^\perp$ and $m_3,\ldots,m_{p-j}\in M\cap\pi^\perp$. By definition, we have

$$\begin{aligned}(\delta_{\pi,\tau}\circ\delta_{\tau,\sigma})(m_1\wedge m_2\wedge m_3\wedge\ldots\wedge m_{p-j})&=\langle m_1,n\rangle\delta_{\pi,\tau}(m_2\wedge m_3\wedge\ldots\wedge m_{p-j})\\&=\langle m_1,n\rangle\langle m_2,n'\rangle m_3\wedge\ldots\wedge m_{p-j}\ ,\end{aligned}$$

while

$$\begin{aligned}(\delta_{\pi,\tau'}\circ\delta_{\tau',\sigma})(m_1\wedge m_2\wedge m_3\wedge\ldots\wedge m_{p-j})&=-\langle m_2,n'\rangle\delta_{\pi,\tau'}(m_1\wedge m_3\wedge\ldots\wedge m_{p-j})\\&=-\langle m_2,n'\rangle\langle m_1,n\rangle m_3\wedge\ldots\wedge m_{p-j}\ .\end{aligned}$$

q.e.d.

Remark. More generally, Ishida [I1] defines complexes of $\mathbb{Z}$-modules $C^\cdot(\Phi,p)$ for a "*locally star closed*" subset Φ of a fan Δ. Namely, Φ is required to contain τ whenever $\sigma<\tau<\pi$ for $\sigma,\pi\in\Phi$. Then

$$C^j(\Phi;p):=\bigoplus_{\sigma\in\Phi\cap\Delta(j)}\bigwedge^{p-j}(M\cap\sigma^\perp)$$

and $\delta: C^j(\Phi;p)\to C^{j+1}(\Phi;p)$ is defined to be the direct sum of $\delta_{\tau,\sigma}$ for $\sigma,\tau\in\Phi$. The same proof shows that $C^\cdot(\Phi;p)$ is a complex. Particularly important are the cases where Φ is a *subcomplex* of Δ (i.e., $\tau\in\Phi$ and $\sigma<\tau$ imply $\sigma\in\Phi$) and where Φ is *star closed* in Δ (i.e., $\sigma\in\Phi$ and $\sigma<\tau$ imply $\tau\in\Phi$).

Remark. When $p=r$, we have $\bigwedge^{r-j}(M\cap\sigma^{\perp})\cong\mathbb{Z}$ for all σ in $\Delta(j)$. In this case, we alternatively obtain $C^{\cdot}(\Delta;r)$ as the $\mathbb{Z}$-cochain complex for the abstract complex Δ with an "orientation" chosen and fixed for each $\sigma\in\Delta$. A fan Δ gives rise to a cell decomposition of the subset $(|\Delta|\setminus\{O\})/\mathbb{R}_{>0}$ of the $(r-1)$-sphere $(N_{\mathbb{R}}\setminus\{O\})/R_{>0}$. For $l\geq 0$, the l-th reduced cohomology group $\tilde{H}^l((|\Delta|\setminus\{O\})/\mathbb{R}_{>0};\mathbb{Z})$ with coefficients in $\mathbb{Z}$ clearly coincides with the $(l+1)$-st cohomology group of the complex $C^{\cdot}(\Delta;r)$. Similar remark applies more generally to locally star closed subsets Φ of Δ considered in the previous remark. Important, however, is the fact that the complexes $C^{\cdot}(\Delta;p)$ and $C^{\cdot}(\Phi;p)$ are defined independently of any orientation.

We are now ready to construct Ishida's complexes of coherent modules on toric varieties.

Let $X=T_N\operatorname{emb}(\Delta)$ be an r-dimensional toric variety. For $0\leq p\leq r$, we define $\mathcal{O}_X$-modules $\mathscr{K}^{\cdot}(X;p)$ by

$$\mathscr{K}^j(X;p)=\begin{cases}0 & \text{if } j<0 \text{ or } p<j\\ \bigoplus_{\sigma\in\Delta(j)}\Omega^{p-j}_{V(\sigma)}(\log D(\sigma)) & \text{if } 0\leq j\leq p\end{cases}.$$

By Corollary 3.2, we have $\Omega^{p-j}_{V(\sigma)}(\log D(\sigma))=\mathcal{O}_{V(\sigma)}\otimes_{\mathbb{Z}}\bigwedge^{p-j}(M\cap\sigma^{\perp})$. The coboundary map $\delta:\mathscr{K}^j(X;p)\to\mathscr{K}^{j+1}(X;p)$ is defined to be the direct sum $\delta=\bigoplus R_{\tau,\sigma}$ for $\sigma\in\Delta(j)$ and $\tau\in\Delta(j+1)$ of the following $\mathcal{O}_X$-homomorphisms $R_{\tau,\sigma}:\mathcal{O}_{V(\sigma)}\otimes_{\mathbb{Z}}\bigwedge^{p-j}(M\cap\sigma^{\perp})\to\mathcal{O}_{V(\tau)}\otimes_{\mathbb{Z}}\bigwedge^{p-j-1}(M\cap\tau^{\perp})$. We let $R_{\tau,\sigma}=0$ if σ is not a face of τ, while for $\sigma<\tau$, we define $R_{\tau,\sigma}$ to be the tensor product of the canonical restriction map $\mathcal{O}_{V(\sigma)}\to\mathcal{O}_{V(\tau)}$ and the previously defined $\delta_{\tau,\sigma}:\bigwedge^{p-j}(M\cap\sigma^{\perp})\to\bigwedge^{p-j-1}(M\cap\tau^{\perp})$. It turns out that the resulting

$$R_{\tau,\sigma}:\Omega^{p-j}_{V(\sigma)}(\log D(\sigma))\to\Omega^{p-j-1}_{V(\tau)}(\log D(\tau))$$

is nothing but the *Poincaré residue map* for the component $V(\tau)$ of the divisor $D(\sigma)$ on $V(\sigma)$.

Since $V(\{O\})=X$ and $D(\{O\})=D:=\sum_{\varrho\in\Delta(1)}V(\varrho)$, we have a canonical $\mathcal{O}_X$-homomorphism

$$\Omega^p_X\to\Omega^p_X(\log D)=\mathscr{K}^0(X;p)\ ,$$

which maps a germ $\mathbf{e}(m)d\mathbf{e}(m_1)\wedge\ldots\wedge d\mathbf{e}(m_p)$ of sections of Ω^p_X to $\mathbf{e}(m+m_1+\ldots+m_p)\otimes(m_1\wedge\ldots\wedge m_p)$. Moreover, the composite

$$\Omega^p_X\to\mathscr{K}^0(X;p)\xrightarrow{\delta}\mathscr{K}^1(X;p)$$

vanishes. Indeed, consider a section $d\mathbf{e}(m_1)\wedge\ldots\wedge d\mathbf{e}(m_p)$ of Ω^p_X over the open set U_σ of X corresponding to $\sigma\in\Delta$, where $m_1,\ldots,m_p$ are elements of $M\cap\sigma^{\vee}$. Suppose $\varrho\in\Delta(1)$ is a face of σ. Then the restriction of $\mathbf{e}(m_j)$ to $V(\varrho)$ vanishes if m_j does not belong to $M\cap\varrho^{\perp}$, while $\langle m_j,n(\varrho)\rangle=0$ if $m_j\in M\cap\varrho^{\perp}$. Thus we have

$$\delta_{\varrho,\{O\}}(\mathbf{e}(m_1+\ldots+m_p)\otimes(m_1\wedge\ldots\wedge m_p))=0\ .$$

Combining this fact with our argument in the proof of Lemma 3.4, we get:

Lemma 3.5. *Let $X = T_N\operatorname{emb}(\Delta)$ be an r-dimensional toric variety. For $0 \le p \le r$,*

$$\mathscr{K}^{\cdot}(X;p) = (0 \to \mathscr{K}^0(X;p) \xrightarrow{\delta} \mathscr{K}^1(X;p) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathscr{K}^p(X;p) \to 0)$$

is a complex of $\mathcal{O}_X$-modules. We call it Ishida's p-th complex of $\mathcal{O}_X$-modules. *Moreover, $0 \to \Omega_X^p \to \mathscr{K}^{\cdot}(X;p)$ is also a complex.*

We are now in a position to show the following theorem of fundamental importance. It is a reformulation due to Ishida of a result in Danilov [D1]. It also contains results found in Demazure [D5], Ehlers [E2] and Kempf [TE]. Geometrically important applications of this theorem will be explained in the next section.

Theorem 3.6. *Let $X = T_N\operatorname{emb}(\Delta)$ be an r-dimensional toric variety. For $0 \le p \le r$, Ishida's p-th complex $\mathscr{K}^{\cdot}(X;p)$ of $\mathcal{O}_X$-modules has the following properties:*

(1) *Denote by $i: U \hookrightarrow X$ the open embedding of the nonsingular locus U of X and define the* Zariski sheaf of germs of p-forms *by $\tilde{\Omega}_X^p := i_* \Omega_U^p$. Then it is a coherent $\mathcal{O}_X$-module and the following sequence is exact:*

$$0 \to \tilde{\Omega}_X^p \to \mathscr{K}^0(X;p) \xrightarrow{\delta} \mathscr{K}^1(X;p) \ .$$

(2) *For $p = r$, the cohomology sheaves of $\mathscr{K}^{\cdot}(X;r)$ are as follows:*

$$\mathscr{H}^j(\mathscr{K}^{\cdot}(X;r)) = \begin{cases} \tilde{\Omega}_X^r & \text{for } j=0 \\ 0 & \text{for } j>0 \end{cases} .$$

(3) *Suppose every $\sigma \in \Delta$ is simplicial. Then for all p, we have*

$$\mathscr{H}^j(\mathscr{K}^{\cdot}(X;p)) = \begin{cases} \tilde{\Omega}_X^p & \text{for } j=0 \\ 0 & \text{for } j>0 \end{cases} .$$

(4) *In particular, if X is nonsingular, then the following sequence is exact for each $p = 0, 1, \dots, r$:*

$$(0 \to \Omega_X^p \to \mathscr{K}^{\cdot}(X;p)) = (0 \to \Omega_X^p \to \mathscr{K}^0(X;p) \xrightarrow{\delta} \mathscr{K}^1(X;p) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathscr{K}^p(X;p) \to 0) \ .$$

Proof. Since all the assertions are on local properties, we may restrict ourselves to the open set $U_\pi \subset X$ corresponding to $\pi \in \Delta$. Furthermore, we may regard X as an algebraic variety (cf. Remarks immediately after Proposition 1.2 and Theorem 1.4), since the complexes of $\mathcal{O}_X$-modules in question are those associated to complexes of coherent algebraic modules.

Let us first assume X to be nonsingular and show that $H^0(U_\pi, \Omega_X^p)$ coincides with the kernel of $\delta: H^0(U_\pi, \mathscr{K}^0(X;p)) \to H^0(U_\pi, \mathscr{K}^1(X;p))$. We know that

$$\begin{aligned} H^0(U_\pi, \mathscr{K}^0(X;p)) &= \mathbb{C}[M \cap \pi^\vee] \otimes_{\mathbb{Z}} \textstyle\bigwedge^p M \quad \text{and} \\ H^0(U_\pi, \mathscr{K}^1(X;p)) &= \bigoplus_{\varrho \in \Delta(1),\, \varrho < \pi} \mathbb{C}[M \cap \pi^\vee \cap \varrho^\perp] \otimes_{\mathbb{Z}} \textstyle\bigwedge^{p-1}(M \cap \varrho^\perp) \end{aligned}$$

are modules over $H^0(U_\pi, \mathcal{O}_X) = \mathbb{C}[M \cap \pi^\vee]$. By the definition of $R_{\varrho,\{O\}}$, the kernel of $\delta = \oplus_\varrho R_{\varrho,\{O\}}$ is generated as a $\mathbb{C}$-vector space by

$$\{\mathbf{e}(m)\otimes(m'_1\wedge\ldots\wedge m'_p);\ m\in M\cap\pi^\vee,\ m'_1,\ldots,m'_p\in M \text{ such that either } m\notin\varrho^\perp \text{ or } m'_1,\ldots,m'_p\in\varrho^\perp \text{ for } \forall\varrho\in\Delta(1)\}\ .$$

Since X is assumed to be nonsingular, there exist a $\mathbb{Z}$-basis $\{m_1,\ldots,m_r\}$ for M and $s\leq r$ such that $M\cap\pi^\vee=\sum_{1\leq j\leq s}\mathbb{Z}_{\geq 0}m_j+\sum_{s+1\leq j\leq s}\mathbb{Z}m_j$ and that $\{d\mathbf{e}(m_j)=\mathbf{e}(m_j)\otimes m_j;\ 1\leq j\leq r\}$ is a basis for $H^0(U_\pi,\Omega_X^1)$ as a $\mathbb{C}[M\cap\pi^\vee]$-module. We then get $\ker(\delta)=H^0(U_\pi,\Omega_X^p)$. Indeed, note first that $\mathbf{e}(m_{s+1}),\ldots,\mathbf{e}(m_r)$ are invertible elements in $\mathbb{C}[M\cap\pi^\vee]$. Let $\{n_1,\ldots,n_r\}$ be the $\mathbb{Z}$-basis of N dual to $\{m_1,\ldots,m_r\}$ and suppose $\mathbf{e}(m)\otimes(m_{i_1}\wedge\ldots\wedge m_{i_p})$ is in $\ker(\delta)$. For $i_l\leq s$, we have $m_{i_l}\notin(\mathbb{R}_{\geq 0}n_{i_l})^\perp$, hence $m\notin(\mathbb{R}_{\geq 0}n_{i_l})^\perp$ and $\langle m,n_{i_l}\rangle>0$. Without loss of generality, we may assume that $i_1,\ldots,i_q$ are not greater than s, while $i_{q+1},\ldots,i_p$ are greater than s. Then m is contained in $(m_{i_1}+\ldots+m_{i_q})+M\cap\pi^\vee$ and $\mathbf{e}(m)\otimes(m_{i_1}\wedge\ldots\wedge m_{i_p})$ is a multiple of $d\mathbf{e}(m_{i_1})\wedge\ldots\wedge d\mathbf{e}(m_{i_p})$ by an element of $\mathbb{C}[M\cap\pi^\vee]$. We thus get (1), if we replace X by its nonsingular locus U, which is a nonsingular toric variety. We also get the case $j=0$ in (2) and (3) as well.

We now show that $\mathscr{H}^j(\mathscr{K}^\cdot(X;p))=0$ for $j>0$ under the assumptions of (2), (3), (4).

Since X is assumed to be an algebraic variety, the algebraic torus T_N acts algebraically on $H^0(U_\pi,\mathscr{K}^\cdot(X;p))$. By the complete reducibility (cf. Sect. 1.2), we have a decomposition

$$H^0(U_\pi,\mathscr{K}^\cdot(X;p))=\bigoplus_{m\in M}H^0(U_\pi,\mathscr{K}^\cdot(X;p))_m\ ,$$

where $H^0(U_\pi,\mathscr{K}^\cdot(X;p))_m$ is the eigenspace with respect to the character $\mathbf{e}(m)$ for each $m\in M$. Henceforth, we may assume m to be contained in $M\cap\pi^\vee$, since $H^0(U_\pi,\mathscr{K}^\cdot(X;p))_m=0$ otherwise. Consequently, $\pi\cap m^\perp$ is a face of π. We define a subset of Δ by

$$\Delta_m:=\{\sigma\in\Delta;\sigma<\pi\cap m^\perp\}\ .$$

As in the remark immediately after Lemma 3.4, we can define Ishida's p-th complex of $\mathbb{Z}$-modules $C^\cdot(\Delta_m;p)$. We then have canonical isomorphisms of $\mathbb{C}$-vector spaces

$$H^0(U_\pi,\mathscr{K}^\cdot(X;p))_m=\mathbb{C}\mathbf{e}(m)\otimes_{\mathbb{Z}}C^\cdot(\Delta_m;p)=\mathbb{C}\mathbf{e}(m)\otimes_{\mathbb{Q}}\{\mathbb{Q}\otimes_{\mathbb{Z}}C^\cdot(\Delta_m;p)\}\ .$$

Indeed, we have $H^0(U_\pi,\mathscr{O}_{V(\sigma)})=\mathbb{C}[M\cap\pi^\vee\cap\sigma^\perp]$ for $\sigma\in\Delta(j)$ satisfying $\sigma<\pi$. Thus $H^0(U_\pi,\Omega_{V(\sigma)}^{p-j}(\log D(\sigma)))_m\neq 0$ if and only if $m\in\pi^\vee\cap\sigma^\perp$, that is, $\sigma<\pi\cap m^\perp$. By Lemmas 3.7 and 3.8 below we have

$$H^j(\mathbb{Q}\otimes_{\mathbb{Z}}C^\cdot(\Delta_m;p))=0 \quad\text{for all}\quad j>0 \quad\text{and}\quad m\in M\cap\pi^\vee$$

under the assumptions of (2), (3), (4). q.e.d.

Lemma 3.7. *Let $N\cong\mathbb{Z}^r$ and $0\leq p\leq r$. For a strongly convex rational polyhedral cone $\pi\subset N_{\mathbb{R}}$, denote by Φ_π the finite abstract complex consisting of the faces of π. For $\tau\in\Phi_\pi$ let*

$$\Phi_\pi(\tau):=\{\sigma\in\Phi_\pi;\sigma<\tau\}\ .$$

Then Ishida's p-th complex $C^\cdot(\Phi_\pi;p)$ of $\mathbb{Z}$-modules satisfies the following properties:

(i) *If π is simplicial, then $\mathbb{Q} \otimes_{\mathbb{Z}} C^{\cdot}(\Phi_\pi; p)$ is isomorphic to a finite direct sum of $\mathbb{Q} \otimes_{\mathbb{Z}} C^{\cdot}(\Phi_\pi(\tau); r)$ with $\tau \in \Phi_\pi$.*

(ii) *If π is nonsingular, then $C^{\cdot}(\Phi_\pi; p)$ is isomorphic to a finite direct sum of $C^{\cdot}(\Phi_\pi(\tau); r)$ with $\tau \in \Phi_\pi$.*

Proof. Let $\dim \pi = s$. Since π is simplicial in either case, there exist $\mathbb{R}$-linearly independent primitive elements $n_1, \ldots, n_s$ in N such that $\pi = \sum_{1 \le k \le s} \mathbb{R}_{\ge 0} n_k$. We can choose primitive elements $n_{s+1}, \ldots, n_r \in N$ in such a way that $\{n_1, \ldots, n_r\}$ is a $\mathbb{Q}$-basis for $\mathbb{Q} \otimes_{\mathbb{Z}} N$. We have $(\mathbb{Q} \otimes_{\mathbb{Z}} M) \cap \pi^\vee = \sum_{1 \le k \le s} \mathbb{Q}_{\ge 0} m_k + \sum_{s+1 \le k \le r} \mathbb{Q} m_k$ for the dual $\mathbb{Q}$-basis $\{m_1, \ldots, m_r\}$ for $\mathbb{Q} \otimes_{\mathbb{Z}} M$. If π happens to be nonsingular, we can choose the above $\{n_1, \ldots, n_r\}$ to be a $\mathbb{Z}$-basis for N and get $M \cap \pi^\vee = \sum_{1 \le k \le s} \mathbb{Z}_{\ge 0} m_k + \sum_{s+1 \le k \le r} \mathbb{Z} m_k$.

Each $\sigma \in \Phi_\pi$ can be written as $\sigma = \sum_{k \notin \xi(\sigma)} \mathbb{R}_{\ge 0} n_k$, where

$$\xi(\sigma) := \{k; 1 \le k \le r, m_k \in (\mathbb{Q} \otimes_{\mathbb{Z}} M) \cap \pi^\vee \cap \sigma^\perp\} \ .$$

Moreover, $\mathbb{Q} \otimes_{\mathbb{Z}} (M \cap \sigma^\perp) = \sum_{k \in \xi(\sigma)} \mathbb{Q} m_k$. Thus if $\dim \sigma = j$, then $\mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge^{p-j}(M \cap \sigma^\perp)$ has a $\mathbb{Q}$-basis consisting of

$$m(\lambda, \sigma) := m_{k_1} \wedge m_{k_2} \wedge \ldots \wedge m_{k_{p-j}}$$

with $\lambda = \{k_1 < k_2 < \ldots < k_{p-j}\}$ running through all the subsets of $\xi(\sigma)$ of cardinality $p-j$. Hence we have

$$\mathbb{Q} \otimes_{\mathbb{Z}} C^{\cdot}(\Phi_\pi; p) = \bigoplus_{(\lambda, \sigma)} \mathbb{Q} m(\lambda, \sigma) \ ,$$

where σ runs through Φ_π, while λ runs through all the subsets of $\xi(\sigma)$ of cardinality $p - \dim \sigma$.

We have $\delta(m(\lambda, \sigma)) = \sum (\pm 1) m(\mu, \tau)$, where (μ, τ) runs through such pairs that $\sigma < \tau$ and $\lambda \supset \mu$ hold and that $\xi(\sigma) \setminus \xi(\tau) = \lambda \setminus \mu$ has cardinality one. The sign ± 1 is chosen according as the integer $\lambda \setminus \mu$ is in the odd or even position with respect to the order of elements in λ. If we define an order for these pairs by

$$(\lambda, \sigma) > (\mu, \tau) \Leftrightarrow \sigma < \tau \ , \quad \lambda \supset \mu \ , \quad \xi(\sigma) \setminus \xi(\tau) = \lambda \setminus \mu \ ,$$

then we clearly get a decomposition

$$\mathbb{Q} \otimes_{\mathbb{Z}} C^{\cdot}(\Phi_\pi; p) = \bigoplus_{(\mu, \tau)} \mathbb{Q} \otimes_{\mathbb{Z}} C^{\cdot}(\Phi_\pi(\tau); r) \ ,$$

where (μ, τ) runs through all the minimal pairs with respect to the above order.

If π is nonsingular, the above argument works over $\mathbb{Z}$ instead of $\mathbb{Q}$. q.e.d.

Lemma 3.8 (cf. [I1, Proposition 2.3]). *Let $N \cong \mathbb{Z}^r$. For a strongly convex rational polyhedral cone π in $N_{\mathbb{R}}$, we denote by Φ_π the finite abstract complex consisting of all the faces of π. Then Ishida's r-th complex $C^{\cdot}(\Phi_\pi; \mathbb{Z})$ of $\mathbb{Z}$-modules has the following* *cohomology groups:*

$$H^j(C^\cdot(\Phi_\pi;r))=\begin{cases}\mathbb{Z} & \text{if } j=0 \text{ and } \pi=\{O\}\\ 0 & \text{if } j\neq 0 \text{ or } \pi\neq\{O\}\end{cases} .$$

Proof. If $\pi=\{O\}$, then $C^0(\Phi_\pi:r)=\mathbb{Z}$ and $C^j(\Phi_\pi;r)=0$ for $j\neq 0$.

Suppose $\pi\neq\{O\}$. Obviously, Φ_π has support $|\Phi_\pi|=\pi$. As we noted in the second remark after Lemma 3.4, $H^j(C^\cdot(\Phi_\pi;r))$ for $j\geq 1$ coincides with the $(j-1)$-st reduced cohomology group of the space $(\pi\setminus\{O\})/\mathbb{R}_{>0}$ with coefficients in $\mathbb{Z}$. Since this space is a cell inside the $(r-1)$-sphere $(N_{\mathbb{R}}\setminus\{O\})/\mathbb{R}_{>0}$, all the reduced cohomology groups vanish. We refer the reader to Ishida [I1, Proposition 2.3] for a direct proof without resort to reduced cohomology groups. q.e.d.

Let us now recall relevant facts concerning dualizing complexes. For details, we refer the reader to Hartshorne [RD], Bănică-Stănăşilă [BS], Verdier [V3], Schenzel [S9], etc.

When Y is a complex analytic space (or an algebraic variety), we denote by $\mathbb{D}(\mathcal{O}_Y)$ the derived category for complexes of $\mathcal{O}_Y$-modules. $\mathbb{D}_c^+(\mathcal{O}_Y)$ (resp. $\mathbb{D}_c^-(\mathcal{O}_Y)$) denotes its subcategory for complexes which are bounded below (resp. above) and which have $\mathcal{O}_Y$-coherent cohomology sheaves. A complex $\omega^\cdot$ of $\mathcal{O}_Y$-modules, or its image in $\mathbb{D}_c^+(\mathcal{O}_Y)$ denoted also by $\omega^\cdot$, is said to be a *dualizing complex* for Y if it has a finite injective resolution and if the functors $D:\mathbb{D}_c^-(\mathcal{O}_Y)\to\mathbb{D}_c^+(\mathcal{O}_Y)$ and $D:\mathbb{D}_c^+(\mathcal{O}_Y)\to\mathbb{D}_c^-(\mathcal{O}_Y)$ defined by

$$D(F^\cdot):=\mathbb{R}\mathcal{H}om_{\mathcal{O}_Y}(F^\cdot,\omega^\cdot)\quad\text{for}\quad F^\cdot\quad\text{in}\quad \mathbb{D}_c^-(\mathcal{O}_Y)\quad\text{or}\quad \mathbb{D}_c^+(\mathcal{O}_Y)$$

satisfy the duality. Namely, the canonical homomorphism $F^\cdot\to DD(F^\cdot)$ is an isomorphism for every $F^\cdot$ in $\mathbb{D}_c^-(\mathcal{O}_Y)$.

A dualizing complex $\omega^\cdot$ for Y is determined only up to degree shift, tensor product of invertible $\mathcal{O}_Y$-modules and quasi-isomorphism of complexes. In fact, a *globally normalized dualizing complex* $\omega_Y^\cdot$ for Y is uniquely determined up to quasi-isomorphism of complexes and satisfies the following properties:

(i) The cohomology sheaves of $\omega_Y^\cdot$ satisfy

$$\mathcal{H}^j(\omega_Y^\cdot)=0\quad\text{if}\quad j<-\dim Y\quad\text{or}\quad 0<j .$$

(ii) If Y is nonsingular of dimension s, then $\omega_Y^\cdot=\Omega_Y^s[s]$, which is a complex with the only nonzero term in degree $-s$ being the sheaf Ω_Y^s of germs of holomorphic s-forms.

(iii) For each holomorphic map $f\colon Y\to Z$, the *twisted inverse image functor* $f^!:\mathbb{D}_c^+(\mathcal{O}_Z)\to\mathbb{D}_c^+(\mathcal{O}_Y)$ is defined and satisfies $f^!\omega_Z^\cdot=\omega_Y^\cdot$. Moreover, we have a canonical isomorphism of functors $(g\circ f)^!=f^!\circ g^!$ for another holomorphic map $g:Z\to W$.

(iv) For each proper holomorphic map $f\colon Y\to Z$, we denote the direct image functor by $\mathbb{R}f_*:\mathbb{D}_c^+(\mathcal{O}_Y)\to\mathbb{D}_c^+(\mathcal{O}_Z)$. We also have functors $\mathbb{R}\mathcal{H}om_{\mathcal{O}_Y}:\mathbb{D}_c^-(\mathcal{O}_Y)\times\mathbb{D}_c^+(\mathcal{O}_Y)\to\mathbb{D}_c^+(\mathcal{O}_Y)$ as well as $\mathbb{R}\mathcal{H}om_{\mathcal{O}_Z}:\mathbb{D}_c^-(\mathcal{O}_Z)\times\mathbb{D}_c^+(\mathcal{O}_Z)\to\mathbb{D}_c^+(\mathcal{O}_Z)$. Then the following **duality theorem** holds for all $F^\cdot\in\mathbb{D}_c^-(\mathcal{O}_Y)$ and $G^\cdot\in\mathbb{D}_c^+(\mathcal{O}_Z)$:

$$\mathbb{R}f_*\mathbb{R}\mathcal{H}om_{\mathcal{O}_Y}(F^\cdot,f^!G^\cdot)=\mathbb{R}\mathcal{H}om_{\mathcal{O}_Z}(\mathbb{R}f_*F^\cdot,G^\cdot) .$$

With $G^{\cdot} = \omega_Z^{\cdot}$ in particular, we have

$$\mathbb{R}f_*\mathbb{R}\mathscr{H}om_{\mathcal{O}_Y}(F^{\cdot}, \omega_Y^{\cdot}) = \mathbb{R}\mathscr{H}om_{\mathcal{O}_Z}(\mathbb{R}f_*F^{\cdot}, \omega_Z^{\cdot}) \quad \text{for all} \quad F^{\cdot} \in \mathbb{D}_c^-(\mathcal{O}_Y) .$$

(v) **(Serre-Grothendieck's Duality Theorem)** Suppose Y is compact and consider the case where Z in (iv) is a point. For each coherent $\mathcal{O}_Y$-module F and each integer j, the $\mathbb{C}$-vector spaces $\operatorname{Ext}_{\mathcal{O}_Y}^{-j}(F, \omega_Y^{\cdot})$ and $H^j(Y, F)$ are dual to each other.

(vi) Y is said to be a *Cohen-Macaulay space*, if

$$\mathscr{H}^j(\omega_Y^{\cdot}) = 0 \quad \text{for all} \quad j \neq -\dim Y .$$

In this case, $\omega_Y := \mathscr{H}^{-\dim Y}(\omega_Y^{\cdot})$ is called the *dualizing $\mathcal{O}_Y$-module* or the *canonical $\mathcal{O}_Y$-module*. If, in addition, this ω_Y is an invertible $\mathcal{O}_Y$-module, then Y is called a *Gorenstein space*. For instance, if Y is nonsingular of dimension s, then Y is a Gorenstein space with $\omega_Y = \Omega_Y^s$ by (ii) above.

The dualizing complexes found in the literature are complexes of *injective* $\mathcal{O}_Y$-modules and are too big. The gist of the following theorem lies in the fact that the dualizing complexes for toric varieties X can be explicitly constructed in terms of *coherent* $\mathcal{O}_X$-modules and are easy to handle. The proof, which we omit here, can be carried out by an extension of our arguments in Lemma 3.8 and Theorem 3.6, (2).

Ishida's Construction Theorem (cf. [I1, Theorem 3.5] and [I5, Theorem 3.1]). *For an r-dimensional toric variety $X = T_N \operatorname{emb}(\Delta)$, Ishida's r-th complex $\mathscr{K}^{\cdot}(X; r)$ of $\mathcal{O}_X$-modules is a dualizing complex for X. Its degree shift $\mathscr{K}^{\cdot}(X; r)[r]$ is the globally normalized dualizing complex for X, where its j-th term is $\mathscr{K}^{r+j}(X; r)$ for each j.*

More generally, let $Y \subset X$ be a T_N-stable reduced closed subspace (which may be reducible) and define a complex $\mathscr{K}^{\cdot}(Y, X; r)$ of $\mathcal{O}_Y$-modules by

$$\mathscr{K}^j(Y, X; r) := \bigoplus_{\sigma \in \Delta(j),\, V(\sigma) \subset Y} \Omega_{V(\sigma)}^{r-j}(\log D(\sigma))$$

with the obvious coboundary map induced by δ for $\mathscr{K}^{\cdot}(X; r)$. Then its degree shift $\mathscr{K}^{\cdot}(Y, X; r)[r]$ is the globally normalized dualizing complex for Y.

Theorem 3.6, (2) holds for general toric varieties thanks to the validity of Lemma 3.8 without any restriction. Together with Ishida's construction theorem, it implies the former half of the following corollary, due originally to Hochster [H5], Kempf [TE, Chap. 1, Theorems 9 and 14] and Ishida [I1, Proposition 6.2]. The latter half is due to [TE, Chap. 1, §3], although the argument found there is incorrect, as we already pointed out immediately before Theorem 2.6.

Corollary 3.9. *An r-dimensional toric variety $X = T_N \operatorname{emb}(\Delta)$ is a Cohen-Macaulay space with $\tilde{\Omega}_X^r$ as the dualizing $\mathcal{O}_X$-module. The resolution of singularities $f: X' := T_N \operatorname{emb}(\Delta') \to X$ with a locally finite nonsingular subdivision Δ' of Δ satisfies*

$$R^q f_* \mathcal{O}_{X'} = \begin{cases} \mathcal{O}_X & \text{for} \quad q = 0 \\ 0 & \text{for} \quad q > 0 \end{cases} ,$$

i.e., the singularities of X are rational singularities. *Moreover,* **Grauert-**

Riemenschneider's Vanishing Theorem *holds:*

$$R^q f_* \Omega^r_{X'} = \begin{cases} \tilde{\Omega}^r_X & \text{for} \quad q=0 \\ 0 & \text{for} \quad q>0 \end{cases} .$$

More generally, we can determine when a T_N-stable reduced closed subspace Y of a toric variety $X = T_N \operatorname{emb}(\Delta)$ is a Cohen-Macaulay space or a Gorenstein space. Indeed,

$$\Delta_Y := \{\sigma \in \Delta ; V(\sigma) \subset Y\}$$

is a star closed subset of Δ with the codimension of Y in X equal to $h := \min\{j ; \Delta_Y \cap \Delta(j) \text{ is nonempty}\}$. For $\tau \in \Delta_Y$, the subset $\Delta_Y(\tau) := \{\sigma \in \Delta_Y ; \sigma < \tau\}$ is locally star closed in Δ so that we can define Ishida's r-th complex $C^\cdot(\Delta_Y(\tau); r)$ of $\mathbb{Z}$-modules.

Ishida's Criteria. *In the above notation, we have:*

(1) ([I1, Corollary 3.5]) *Y is a Cohen-Macaulay space if and only if*

$$H^j(\mathbb{C} \otimes_{\mathbb{Z}} C^\cdot(\Delta_Y(\tau); r)) = 0 \quad \text{for all} \quad \tau \in \Delta_Y \quad \text{and} \quad j \neq h .$$

In this case, $\mathscr{H}^h(\mathscr{K}^\cdot(Y, X; r))$ is the dualizing $\mathcal{O}_Y$-module.

(2) ([I1, Theorem 5.10]) *Y is a Gorenstein space if*

$$H^j(\mathbb{C} \otimes_{\mathbb{Z}} C^\cdot(\Delta_Y(\tau); r)) = \begin{cases} \mathbb{C} & \text{for} \quad j=h \\ 0 & \text{for} \quad j \neq h \end{cases}$$

holds for every $\tau \in \Delta_Y$. In this case, the dualizing $\mathcal{O}_Y$-module is isomorphic to $\mathcal{O}_Y$ so that we have the following exact sequence with $\Delta_Y(j) := \Delta_Y \cap \Delta(j)$:

$$0 \to \mathcal{O}_Y \to \bigoplus_{\sigma \in \Delta_Y(h)} \mathcal{O}_{V(\sigma)} \to \bigoplus_{\tau \in \Delta_Y(h+1)} \mathcal{O}_{V(\tau)} \to \ldots \to \mathbb{C} \to 0 .$$

(3) *In particular, the union*

$$Y := \bigcup_{j \geq h} \bigcup_{\sigma \in \Delta(j)} V(\sigma)$$

of all the T_N-orbits in X of dimensions not greater than $r-h$ is a Cohen-Macaulay space. When $h=1$, $Y = X \setminus T_N$ is a Gorenstein space.

Remark. Most of the results given so far in this chapter are valid also for toric varieties as algebraic varieties over an arbitrary ground field k. For example, the following results hold and might be of interest in commutative algebra:

(i) Let π be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. The semigroup algebra $k[M \cap \pi^\vee]$ over a field k, defined exactly as in Chap. 1, is a Cohen-Macaulay ring, and its ideal with $\{\mathbf{e}(m); m \in M \cap \operatorname{int}(\pi^\vee)\}$ as a k-basis is the dualizing module. The semigroup algebra is a Gorenstein ring if and only if

$$M \cap \operatorname{int}(\pi^\vee) = m_0 + (M \cap \pi^\vee)$$

for some m_0. Among such, Ishida [I1, Theorem 8.1] classified those which are complete intersections in dimension three. Nakajima [N2] later carried out the classification in arbitrary dimension.

(ii) Let Ξ be a family of subsets of $\mathscr{I}:=\{1,2,\ldots,r\}$ such that $i\in\mathscr{I}$ implies $\{i\}\in\Xi$ and that $\xi\in\Xi$, $\eta\subset\xi$ imply $\eta\in\Xi$. The *Stanley-Reisner ring* R is defined to be the residue ring of the polynomial ring $k[x_1,\ldots,x_r]$ in r variables modulo its ideal generated by the set of square-free monomials

$$\{x_{i_1}x_{i_2}\ldots x_{i_s};\{i_1,\ldots,i_s\}\subset\mathscr{I},\ \{i_1,\ldots,i_s\}\notin\Xi\}\ .$$

Geometrically, we have $Y:=\mathrm{Spec}(R)=\cup_{\xi\in\Xi}V_\xi$, where V_ξ for each $\xi\in\Xi$ is the linear subspace of the r-dimensional affine space k^r defined by

$$V_\xi:=\{(a_1,\ldots,a_r)\in k^r;a_i=0\quad\text{for}\quad i\notin\xi\}\ .$$

If d is the maximum of the cardinalities of the subsets belonging to Ξ, then the above Y is a d-dimensional algebraic subvariety of the affine space k^r.

On the other hand, we can also regard Ξ as a $(d-1)$-dimensional simplicial complex with $\mathscr{I}$ as the vertex set. We can determine when R is a Cohen-Macaulay ring or a Gorenstein ring by looking at the reduced homology groups with coefficients in k of the simplicial complex Ξ or those of the link with respect to each simplex in Ξ. Reisner [R6] obtained this result by means of the Koszul complex. When Ξ is a triangulation of a $(d-1)$-sphere, for instance, we see that R is a Gorenstein ring. We come across such a ring in Sects. 3.3 and A.5 in relation to cohomology rings.

We can use Ishida's criteria for the Stanley-Reisner ring instead of the Koszul complex used by Reisner: Let N be a $\mathbb{Z}$-module with a $\mathbb{Z}$-basis $\{n_1,\ldots,n_r\}$ and consider the strongly convex rational polyhedral cone $\pi:=\mathbb{R}_{\geq 0}n_1+\ldots+\mathbb{R}_{\geq 0}n_r$ in $N_{\mathbb{R}}$. The set Δ of the faces of π consists of

$$\sigma_\zeta:=\sum_{i\notin\zeta}\mathbb{R}_{\geq 0}n_i$$

for subsets ζ of $\mathscr{I}=\{1,2,\ldots,r\}$. For a family Ξ of subsets of $\mathscr{I}$, we see that

$$\Delta_Y:=\{\sigma_\xi;\xi\in\Xi\}$$

is a star closed subset of Δ. Hence we can consider, for each τ in Δ_Y, Ishida's r-th complex $C^\cdot(\Delta_Y(\tau);r)$ of $\mathbb{Z}$-modules. The vanishing, as in Ishida's criteria (1), (2), of the cohomology groups $H^j(k\otimes_{\mathbb{Z}}C^\cdot(\Delta_Y(\tau);r))$ of its scalar extension to k enables us to decide when the Stanley-Reisner ring R is a Cohen-Macaulay ring or a Gorenstein ring.

When Ξ is a triangulation with r vertices of a $(d-1)$-sphere, for instance, we again see in this way that R is a Gorenstein ring. By Ishida's criterion (2), we further have an exact sequence of R-modules

$$0\to R\to\bigoplus_{\xi\in\Xi(d)}k[\xi]\to\bigoplus_{\eta\in\Xi(d-1)}k[\eta]\to\ldots\to\bigoplus_{i\in\mathscr{I}}k[x_i]\to k\to 0\ ,$$

where $\Xi(j)$ denotes the subset of Ξ consisting of those $\xi\in\Xi$ of cardinality j as a subset of $\mathscr{I}$ and where the polynomial ring $k[\xi]$ for $\xi\in\Xi$ has variables $\{x_i;i\in\xi\}$ and is regarded naturally as a residue ring of R. This exact sequence, which can be found already in Danilov [D1, Remark 3.8], appears in the proof of Stanley's results on the morphology of convex polytopes (cf. Sect. A.5) as well as in the proof of Jurkiewicz-Danilov's theorem in the next section.

(iii) In studying deformations and degenerations of complex manifolds, it might be useful to study the *tangent complex* of those Y which appear in Ishida's criteria and in (ii) above. We refer the reader to Ishida-Oda [IO]. It might also be useful to study the *stability* in the sense of Mumford [M12] of the singularities appearing in such Y. There was an attempt in low dimensions by Hiromichi Kagami in his master's thesis at Tohoku University (1980).

Let us go back to an r-dimensional toric variety $X = T_N \operatorname{emb}(\Delta)$ in general and study further properties of the Zariski sheaves $\tilde{\Omega}_X^p$. By Theorem 3.6, (1), $\tilde{\Omega}_X^p$ coincides with the kernel of $\delta: \mathscr{K}^0(X;p) = \Omega_X^p(\log D) \to \mathscr{K}^1(X;p)$. The exterior product $\Omega_X^p(\log D) \otimes_{\mathcal{O}_X} \Omega_X^{r-p}(\log D) \to \Omega_X^r(\log D)$ induces an $\mathcal{O}_X$-bilinear map

$$\tilde{\Omega}_X^p \otimes_{\mathcal{O}_X} \tilde{\Omega}_X^{r-p} \to \tilde{\Omega}_X^r \ ,$$

hence an $\mathcal{O}_X$-homomorphism

$$\tilde{\Omega}_X^{r-p} \to \mathscr{H}om_{\mathcal{O}_X}(\tilde{\Omega}_X^p, \tilde{\Omega}_X^r) \ .$$

Proposition 3.10 (Danilov [D1], Propositions 4.7 and 4.8, and Corollary 4.9]). *For an r-dimensional toric variety* $X = T_N \operatorname{emb}(\Delta)$ *and* $0 \le p \le r$, *the* $\mathcal{O}_X$*-homomorphism above is an isomorphism, i.e.,*

$$\tilde{\Omega}_X^{r-p} \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_X}(\tilde{\Omega}_X^p, \tilde{\Omega}_X^r) \ .$$

If each $\sigma \in \Delta$ *is simplicial, then* $\tilde{\Omega}_X^p$ *has depth r as an* $\mathcal{O}_X$*-module and satisfies*

$$\mathscr{E}xt_{\mathcal{O}_X}^k(\tilde{\Omega}_X^p, \tilde{\Omega}_X^r) = 0 \quad \text{for} \quad k \ne 0 \ .$$

Proof. Let $i: U \hookrightarrow X$ be the open embedding of the nonsingular locus U of X. As is well-known, the exterior product induces an $\mathcal{O}_U$-isomorphism

$$\Omega_U^{r-p} \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_U}(\Omega_U^p, \Omega_U^r) \ ,$$

hence an $\mathcal{O}_X$-isomorphism

$$\tilde{\Omega}_X^{r-p} = i_*\Omega_U^{r-p} \xrightarrow{\sim} i_*\mathscr{H}om_{\mathcal{O}_U}(\Omega_U^p, \Omega_U^r) \ .$$

The first assertion follows from the canonical $\mathcal{O}_X$-isomorphisms

$$\mathscr{H}om_{\mathcal{O}_X}(\tilde{\Omega}_X^p, \tilde{\Omega}_X^r) = \mathscr{H}om_{\mathcal{O}_X}(i_*\Omega_U^p, i_*\Omega_U^r) = i_*\mathscr{H}om_{\mathcal{O}_U}(i^*i_*\Omega_U^p, \Omega_U^r)$$

and $i^*i_*\Omega_U^p = \Omega_U^p$ (see, e.g., [EGA, I, (9.4.2), (i)]).

As for the proof of the second assertion, we use the exact sequence

$$0 \to \tilde{\Omega}_X^p \to \mathscr{K}^0(X;p) \to \mathscr{K}^1(X;p) \to \dots \to \mathscr{K}^p(X;p) \to 0$$

in Theorem 3.6, (3), where

$$\mathscr{K}^j(X;p) = \bigoplus_{\sigma \in \Delta(j)} \mathcal{O}_{V(\sigma)} \otimes_{\mathbf{Z}} \textstyle\bigwedge^{p-j}(M \cap \sigma^\perp) \ .$$

We have depth $(\mathcal{O}_{V(\sigma)}) = r - j$ if $j \le p$ and $\sigma \in \Delta(j)$ by Corollary 3.9 applied to $V(\sigma)$. Hence depth $(\mathscr{K}^j(X;p)) = r - j$. By descending induction on j, we conclude that the kernel of $\delta: \mathscr{K}^j(X;p) \to \mathscr{K}^{j+1}(X;p)$ has depth $r - j$. The last assertion follows from the local duality theorem (cf. Hartshorne [RD, Theorem 6.2, p. 278]) which says that the stalk at each $x \in X$ of $\mathscr{E}xt^k_{\mathcal{O}_X}(\mathscr{F}, \tilde{\Omega}^r_X[r])$ is dual to the local cohomology group $H^{-k}_{\{x\}}(\mathscr{F}_x)$ with support in $\{x\}$ of the stalk $\mathscr{F}_x$ of $\mathscr{F}$ at x. q.e.d.

Remark. Theorem 3.6, (3) asserts the existence of isomorphisms

$$\tilde{\Omega}^p_X \simeq \mathscr{K}^\cdot(X;p) \quad \text{and} \quad \tilde{\Omega}^r_X \simeq \mathscr{K}^\cdot(X;r)$$

in the derived category $\mathbb{D}^+_c(\mathcal{O}_X)$ under the assumption that each σ in Δ is simplicial. Thus by Proposition 3.10, we have an isomorphism

$$(*) \qquad \tilde{\Omega}^{r-p}_X \simeq \mathbb{R}\mathscr{H}om_{\mathcal{O}_X}(\mathscr{K}^\cdot(X;p), \mathscr{K}^\cdot(X;r))$$

in $\mathbb{D}^+_c(\mathcal{O}_X)$. Using the same method as in [I1, Theorem 3.5], however, Ishida [I5] shows that $(*)$ above holds for a general toric variety X *without* the assumption that each $\sigma \in \Delta$ is simplicial. More important still, an assertion similar to $(*)$ holds for T_N-stable reduced closed subspaces Y of X, or more generally for "degenerate varieties" which are complex analytic spaces locally isomorphic to such Y.

For the Zariski sheaves, we also have the exterior differentiation $d: \tilde{\Omega}^p_X \to \tilde{\Omega}^{p+1}_X$ for each p, hence the *de Rham complex* $\tilde{\Omega}^\cdot_X$. Indeed, $\tilde{\Omega}^p_X$ is equal to the kernel of $\delta: \mathscr{K}^0(X;p) \to \mathscr{K}^1(X;p)$ by Theorem 3.6, (1), while, as usual, d is so defined for $\Omega^p_X(\log D) = \mathscr{K}^0(X;p) = \mathcal{O}_X \otimes_{\mathbb{Z}} \bigwedge^p M$ that

$$d(\mathbf{e}(m) \otimes (m_1 \wedge \ldots \wedge m_p)) = \mathbf{e}(m) \otimes (m \wedge m_1 \wedge \ldots \wedge m_p) \ ,$$

hence $d(\tilde{\Omega}^p_X) \subset \tilde{\Omega}^{p+1}_X$ is satisfied. As in Danilov [D1, Proposition 13.4], the **Poincaré lemma** holds, hence the complex $\tilde{\Omega}^\cdot_X$ of (complex analytic) $\mathcal{O}_X$-modules is a resolution for the constant sheaf $\mathbb{C}_X$. Ishida [I5, Theorem 4.1] shows the de Rham complex here to be isomorphic to that introduced by du Bois. Consequently, the Poincaré lemma follows by definition. In fact, he shows these results to be valid more generally for T_N-stable reduced closed subspaces Y of X.

3.3 Compact Toric Varieties and Holomorphic Differential Forms

Let us derive global consequences for compact toric varieties from various local results on the Zariski sheaves in the previous section.

Serre-Grothendieck's Duality Theorem. *Let $X = T_N \operatorname{emb}(\Delta)$ be an r-dimensional compact toric variety.*

(1) *We have*

$$H^q(X, \tilde{\Omega}^r_X) = \begin{cases} 0 & \text{for} \quad q \neq r \\ \mathbb{C} & \text{for} \quad q = r \end{cases} .$$

If each $\sigma \in \Delta$ is simplicial, then for each pair p, q of integers, the exterior product induces a perfect bilinear pairing

$$H^q(X, \tilde{\Omega}^p_X) \otimes_{\mathbb{C}} H^{r-q}(X, \tilde{\Omega}^{r-p}_X) \to H^r(X, \tilde{\Omega}^r_X) = \mathbb{C}$$

so that $H^{r-q}(X, \tilde{\Omega}^{r-p}_X)$ is canonically the dual $\mathbb{C}$-vector space of $H^q(X, \tilde{\Omega}^p_X)$.

(2) $\tilde{\Omega}_X^r[r]$ *is the* globally normalized dualizing $\mathcal{O}_X$-module *in the sense of the previous section. Thus for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ and for $0 \le q \le r$, the $\mathbb{C}$-vector space* $\operatorname{Ext}^{r-q}_{\mathcal{O}_X}(\mathcal{F}, \tilde{\Omega}_X^r)$ *is canonically dual to $H^q(X, \mathcal{F})$. In particular, if $\mathcal{F}$ is $\mathcal{O}_X$-locally free with the $\mathcal{O}_X$-dual denoted by $\mathcal{F}^*$, then $H^{r-q}(X, \mathcal{F}^* \otimes_{\mathcal{O}_X} \tilde{\Omega}_X^r)$ is canonically $\mathbb{C}$-dual to $H^q(X, \mathcal{F})$.*

This is a special case of Serre-Grothendieck's duality theorem recalled in the previous section, since X is a Cohen-Macaulay space with $\tilde{\Omega}_X^r$ as the dualizing $\mathcal{O}_X$-module by Corollary 3.9 and Proposition 3.10. Danilov [D1, Proposition 7.7.1] proves (1) as well as (2) for $\mathcal{F} = \mathcal{O}_X(D_h)$ with a T_N-invariant Cartier divisor D_h as in Chap. 2 more directly using the fact that X is a toric variety.

Bott's Vanishing Theorem. *Let $X = T_N \operatorname{emb}(\Delta)$ be an r-dimensional compact toric variety and let $0 \le p \le r$. Then*

$$H^q(X, \tilde{\Omega}_X^p \otimes_{\mathcal{O}_X} \mathcal{L}) = 0 \quad \textit{for} \quad q \neq 0$$

holds for every ample invertible $\mathcal{O}_X$-module $\mathcal{L}$.

This is a toric variant of a well-known result valid for compact complex manifolds in general. We leave the proof of it as well as Danilov [D1, Theorem 7.5.2] to the reader, and only point out that $\mathcal{L} = \mathcal{O}_X(D_h)$ for a strictly upper convex support function $h \in \mathrm{SF}(N, \Delta)$ by Theorem 2.7 and Corollary 2.5.

Theorem 3.11. *Let $X = T_N \operatorname{emb}(\Delta)$ be an r-dimensional compact toric variety such that each $\sigma \in \Delta$ is simplicial.*

(1) *For each $0 \le p \le r$, we have*

$$H^q(X, \tilde{\Omega}_X^p) = 0 \quad \textit{for all} \quad q \neq p\ .$$

Moreover, we have an exact sequence

$$0 \to H^0(X, \mathcal{K}^0(X;p)) \xrightarrow{\delta} H^0(X, \mathcal{K}^1(X;p)) \xrightarrow{\delta} \dots$$
$$\dots \xrightarrow{\delta} H^0(X, \mathcal{K}^p(X;p)) \to H^p(X, \tilde{\Omega}_X^p) \to 0\ ,$$

where $H^0(X, \mathcal{K}^j(X;p)) = \bigoplus_{\sigma \in \Delta(j)} \mathbb{C} \otimes_{\mathbb{Z}} \bigwedge^{p-j}(M \cap \sigma^\perp)$.

(2) *Let $h^{p,p} := \dim_{\mathbb{C}} H^p(X, \tilde{\Omega}_X^p)$ and define the* Poincaré series *in a variable t by*

$$P(t) := \sum_{p=0}^{r} h^{p,p} t^p \in \mathbb{Z}[t]\ .$$

Then the functional equation $P(t) = t^r P(1/t)$ *holds and*

$$P(t) = \sum_{j=0}^{r} {}^{\#}\Delta(j)\, (t-1)^{r-j}\ ,$$

where ${}^{\#}\Delta(j)$ is the cardinality of $\Delta(j) := \{\sigma \in \Delta;\ \dim \sigma = j\}$.

Proof. The following sequence is exact by Theorem 3.6, (3):

$$0 \to \tilde{\Omega}_X^p \to \mathcal{K}^0(X;p) \to \mathcal{K}^1(X;p) \to \dots \to \mathcal{K}^p(X;p) \to 0\ .$$

Since $\mathscr{K}^j(X;p)=\oplus_{\sigma\in\Delta(j)}\mathcal{O}_{V(\sigma)}\otimes_{\mathbb{Z}}\bigwedge^{p-j}(M\cap\sigma^\perp)$, we can apply Corollary 2.8 to the toric varieties $V(\sigma)$ to obtain

$$H^q(X,\mathscr{K}^j(X;p))=0 \quad \text{for} \quad q\neq 0 \ .$$

Thus $H^q(X,\tilde{\Omega}_X^p)$ coincides with the q-th cohomology group of the complex $H^0(X,\mathscr{K}^{\cdot}(X;p))$ of $\mathbb{C}$-vector spaces. $H^q(X,\tilde{\Omega}_X^p)$ vanishes for $q>p$, since the complex has nonzero terms only in degrees not greater than p. By (1) of Serre-Grothendieck's duality theorem above, we get $H^q(X,\tilde{\Omega}_X^p)=0$ for $q<p$ as well.

By what we have seen so far, we clearly get the exact sequence in (1), which we can use to compute $h^{p,p}$ as

$$\begin{aligned} h^{p,p} &= \sum_{j=0}^{p} (-1)^{p-j}\dim_{\mathbb{C}} H^0(X,\mathscr{K}^j(X;p)) \\ &= \sum_{j=0}^{p} (-1)^{p-j} \sum_{\sigma\in\Delta(j)} \dim_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{Z}}\textstyle\bigwedge^{p-j}(M\cap\sigma^\perp)) \ . \end{aligned}$$

Since $M\cap\sigma^\perp$ is a free $\mathbb{Z}$-module of rank $r-j$ for $\sigma\in\Delta(j)$, we thus get

$$h^{p,p}=\sum_{j=0}^{p} (-1)^{p-j}\binom{r-j}{p-j}{}^{\#}\Delta(j) \ ,$$

hence $P(t)=\sum_{0\le j\le r}{}^{\#}\Delta(j)t^j(1-t)^{r-j}$. Moreover, (1) of Serre-Grothendieck's duality theorem gives $h^{r-p,r-p}=h^{p,p}$, which implies the functional equation $P(t)=t^rP(1/t)$. Using the expression for $P(t)$ above for the right hand side, we conclude

$$P(t)=\sum_{j=0}^{r} {}^{\#}\Delta(j)\,(t-1)^{r-j} \ . \qquad \text{q.e.d.}$$

Danilov [D1] and Ehlers [E2] gave another proof for the following:

Theorem 3.12. *Let $X=T_N\operatorname{emb}(\Delta)$ be an r-dimensional compact nonsingular toric variety and consider the T_N-invariant divisor $D:=\sum_{\sigma\in\Delta(1)}V(\varrho)$ on X.*

(1) *We have an exact sequence*

$$0\to\Theta_X(-\log D)\to\Theta_X\to\bigoplus_{\varrho\in\Delta(1)}\mathcal{O}_{V(\sigma)}\otimes_{\mathcal{O}_X}\mathcal{O}_X(V(\varrho))\to 0 \ .$$

(2) *The* total Chern class *of X is given by*

$$c(X):=c(\Theta_X)=\prod_{\varrho\in\Delta(1)}(1+c_1(\mathcal{O}_X(V(\varrho)))) \ .$$

(3) *The topological* Euler number *of X, which coincides with the r-th Chern class, is*

$$c_r(X)={}^{\#}\Delta(r)={}^{\#}\{\textit{the } T_N\textit{-fixed points in } X\} \ ,$$

while the index *(also called the* signature*) of X, which is defined to be the signature of the $\mathbb{R}$-bilinear cup product pairing*

$$H^r(X,\mathbb{R})\otimes_{\mathbb{R}}H^r(X,\mathbb{R})\to H^{2r}(X,\mathbb{R})$$

is given by

$$\tau(X)=\sum_{j=0}^{r}(-2)^j\,{}^{\#}\Delta(r-j)\ .$$

In particular, both sides vanish if r is odd.

Proof. Letting $p=1$ in Theorem 3.6, (4), we get an exact sequence $0\to\Omega_X^1\to\Omega_X^1(\log D)\to\oplus_{\varrho\in\Delta(1)}\mathcal{O}_{V(\varrho)}\to 0$. Taking the $\mathcal{O}_X$-dual, we obtain an exact sequence

$$0\to 0\to\Theta_X(-\log D)\to\Theta_X\to\bigoplus_{\varrho\in\Delta(1)}\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_{V(\varrho)},\mathcal{O}_X)\to 0\ ,$$

while the $\mathcal{O}_X$-dual of the exact sequence $0\to\mathcal{O}_X(-V(\varrho))\to\mathcal{O}_X\to\mathcal{O}_{V(\varrho)}\to 0$ gives rise to an exact sequence

$$0\to\mathcal{O}_X\to\mathcal{O}_X(V(\varrho))\to\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_{V(\varrho)},\mathcal{O}_X)=\mathcal{O}_{V(\varrho)}\otimes_{\mathcal{O}_X}\mathcal{O}_X(V(\varrho))\to 0\ .$$

Hence we get (1).

Since $V(\varrho)$ is a Cartier divisor on X, it follows from the exact sequence we just obtained, together with the additivity of the total Chern classes, that

$$c(\mathcal{O}_{V(\varrho)}\otimes_{\mathcal{O}_X}\mathcal{O}_X(V(\varrho)))=c(\mathcal{O}_X(V(\varrho)))=1+c_1(\mathcal{O}_X(V(\varrho)))\ .$$

On the other hand, (1) implies

$$c(\Theta_X)=c(\Theta_X(-\log D))\cdot\prod_{\varrho\in\Delta(1)}c(\mathcal{O}_{V(\varrho)}\otimes_{\mathcal{O}_X}\mathcal{O}_X(V(\varrho)))\ ,$$

while $c(\Theta_X(-\log D))=c(\mathcal{O}_X\otimes_{\mathbb{Z}}N)=1$ by Proposition 3.1. Hence we get (2).

It remains to show (3). By Theorem 3.11, (1), we have $h^{p,q}:=\dim_{\mathbb{C}}H^q(X,\Omega_X^p)=0$ for $p\neq q$. Hence the topological Euler number is given by

$$c_r(X)=\sum_{p=0}^{r}\sum_{q=0}^{r}(-1)^{p+q}h^{p,q}=\sum_{p=0}^{r}h^{p,p}\ .$$

By Hodge-Atiyah-Singer's index theorem (see, for instance, Hirzebruch [H2, Theorem 15.8.2 and Sect. 25]), the index of X is given by

$$\tau(X)=\sum_{p=0}^{r}\sum_{q=0}^{r}(-1)^q h^{p,q}=\sum_{p=0}^{r}(-1)^p h^{p,p}\ .$$

Hence by the definition of the Poincaré series $P(t)$, we have

$$c_r(X)=P(1)\quad\text{and}\quad\tau(X)=P(-1)\ ,$$

the right hand sides of which are computed by Theorem 3.11, (2) as

$$P(1)={}^{\#}\Delta(r)\quad\text{and}\quad P(-1)=\sum_{j=0}^{r}{}^{\#}\Delta(j)(-2)^{r-j}\ .$$

Moreover, we have $P(-1)=(-1)^r P(-1)$ by the functional equation, hence $P(-1)=0$ if r is odd. q.e.d.

For a compact complex manifold X in general, the hypercohomology of the de Rham complex $\Omega_X^{\cdot}$ gives rise to the *Hodge spectral sequence*

$$E_1^{p,q}=H^q(X,\Omega_X^p)\Rightarrow H^{p+q}(X,\mathbb{C}) ,$$

which relates the Hodge cohomology on the left hand side to the ordinary cohomology with coefficients in $\mathbb{C}$. This spectral sequence is known to degenerate at E_1 (i.e., we have $E_1^{p,q}=E_\infty^{p,q}$) when X is a Kähler manifold (by the theory of harmonic forms) or when X is associated to an algebraic variety (by Deligne [D3]).

Instead, if we consider the de Rham complex $\Omega_X^{\cdot}(\log D)$ for a reduced effective divisor D with normal crossings only, we get *Grothendieck-Deligne's spectral sequence*

$$E_1^{p,q}=H^q(X,\Omega_X^p(\log D))\Rightarrow H^{p+q}(X\setminus D,\mathbb{C}) ,$$

which plays a key rôle in the Hodge theory due to Deligne [D4].

For compact toric varieties which may be singular, Danilov obtained the following analogue of the Hodge spectral sequence:

Danilov's Spectral Sequence (cf. [D1, Theorems 12.2 and 12.5, Corollary 12.7 and Proposition 12.10]). *Let $X=T_N\operatorname{emb}(\Delta)$ be an r-dimensional compact toric variety. Then we have a spectral sequence*

$$E_1^{p,q}=H^q(X,\tilde{\Omega}_X^p)\Rightarrow H^{p+q}(X,\mathbb{C}) ,$$

which degenerates at E_1. Moreover, we have

$$H^q(X,\tilde{\Omega}_X^p)=0 \quad \text{for} \quad q<p \quad \text{and} \quad H^r(X,\tilde{\Omega}_X^p)=0 \quad \text{for} \quad r>p .$$

We refer the reader to Danilov's paper itself for an interesting proof and only point out one of the keys given in [D1, Lemma 12.3]:

$$H^{\cdot}(U_\sigma,\mathbb{C})=\mathbb{C}\otimes_{\mathbb{Z}}\textstyle\bigwedge^{\cdot}(M\cap\sigma^\perp) \quad \text{for} \quad \sigma\in\Delta .$$

Ishida [I5, Corollary 6.3] is slightly more general and obtains the degeneracy at E_1 as a consequence of [I5, Theorem 4.1] to the effect that the de Rham complex $\tilde{\Omega}_X^{\cdot}$ is isomorphic to that introduced by du Bois.

In Theorem 3.12, we did not specify where the Chern classes lie. In the finest theory, the Chern classes lie in the *Chow ring*

$$A^{\cdot}(X)=A^0(X)\oplus A^1(X)\oplus\ldots\oplus A^p(X)\oplus\ldots\oplus A^r(X) ,$$

where the toric variety X is regarded as an algebraic variety and where $A^p(X)$ is the group of codimension p algebraic cycles on X (i.e., formal finite $\mathbb{Z}$-linear combinations of closed irreducible subvarieties of X of codimension p) modulo rational equivalence. The intersection of cycles gives rise to a product $A^p(X)\times A^q(X)\to A^{p+q}(X)$ so that $A^{\cdot}(X)$ becomes a graded ring. The assignment of the fundamental cohomology class to each closed irreducible subvariety induces an additive homomorphism $A^p(X)\to H^{2p}(X,\mathbb{Z})$. The intersection of cycles corresponds to the cup product of cohomology classes so that we get a ring homomorphism

$$A^{\cdot}(X) \to H^{\cdot}(X, \mathbb{Z})$$

which doubles the degrees. In dealing with the Chern classes, we also consider the images in $H^{\cdot}(X, \mathbb{Z})$, $H^{\cdot}(X, \mathbb{Q})$, $H^{\cdot}(X, \mathbb{R})$ or $H^{\cdot}(X, \mathbb{C})$ of the finest in $A^{\cdot}(X)$.

Let us now explain the structure theorem, due to Jurkiewicz and Danilov, of the Chow ring of a compact nonsingular toric variety $X = T_N \operatorname{emb}(\Delta)$. For each $\sigma \in \Delta(j)$, let us denote by $v(\sigma) \in A^j(X)$ the rational equivalence class of the irreducible cycle $V(\sigma)$ on X of codimension j. For distinct $\varrho_1, \ldots, \varrho_s$ in $\Delta(1)$, we clearly have

$$v(\varrho_1) \cdot v(\varrho_2) \cdot \ldots \cdot v(\varrho_s) = \begin{cases} v(\varrho_1 + \ldots + \varrho_s) & \text{if} \quad \varrho_1 + \ldots + \varrho_s \in \Delta \\ 0 & \text{otherwise} \end{cases} .$$

Furthermore, each $m \in M$ gives rise to a codimension one cycle $\operatorname{div}(\mathbf{e}(m)) = \sum_{\varrho \in \Delta(1)} \langle m, n(\varrho) \rangle V(\varrho)$ which is rationally equivalent to zero. Hence we have the following equality in $A^1(X)$:

$$\sum_{\varrho \in \Delta(1)} \langle m, n(\varrho) \rangle v(\varrho) = 0 \quad \text{for every} \quad m \in M .$$

Let us introduce a variable $t(\varrho)$ for each $\varrho \in \Delta(1)$ and consider the polynomial ring

$$S := \mathbb{Z}[t(\varrho);\ \varrho \in \Delta(1)] .$$

It is the symmetric algebra associated to the $\mathbb{Z}$-module $T_N \operatorname{Div}(X)$ of T_N-invariant divisors on X. We have a homomorphism $S \to A^{\cdot}(X)$ of graded rings which sends each $t(\varrho)$ to $v(\varrho)$. Let I be the ideal in S generated by a set

$$\{t(\varrho_1)t(\varrho_2) \ldots t(\varrho_s);\ \text{distinct} \quad \varrho_1, \ldots, \varrho_s \in \Delta \quad \text{with} \quad \varrho_1 + \ldots + \varrho_s \notin \Delta\}$$

of square-free monomials. On the other hand, we define J to be the ideal of S generated by

$$\left\{ \sum_{\varrho \in \Delta(1)} \langle m, n(\varrho) \rangle t(\varrho);\ m \in M \right\} .$$

By what we have seen above, we have the following homomorphisms of graded rings:

$$S/(I+J) \to A^{\cdot}(X) \to H^{\cdot}(X, \mathbb{Z}) ,$$

where the second doubles the degrees.

Jurkiewicz-Danilov's Theorem (cf. [D1, Theorem 10.8 and Remark 10.9]). *Let $X = T_N \operatorname{emb}(\Delta)$ be an r-dimensional compact toric variety.*

(i) *If X is nonsingular, then the homomorphisms of graded rings mentioned above are isomorphisms:*

$$S/(I+J) \xrightarrow{\sim} A^{\cdot}(X) \xrightarrow{\sim} H^{\cdot}(X, \mathbb{Z}) .$$

(ii) *If each $\sigma \in \Delta$ is simplicial, we can define the Chow ring $\mathbb{Q} \otimes_{\mathbb{Z}} A^{\cdot}(X)$ with $\mathbb{Q}$-coefficients and get isomorphisms of graded $\mathbb{Q}$-algebras*

$$\mathbb{Q} \otimes_{\mathbb{Z}} (S/(I+J)) \xrightarrow{\sim} \mathbb{Q} \otimes_{\mathbb{Z}} A^{\cdot}(X) \xrightarrow{\sim} H^{\cdot}(X, \mathbb{Q}) \ .$$

For the proof, we use the Poincaré series and the behavior of $h^{p,q} = \dim_{\mathbb{C}} H^q(X, \tilde{\Omega}_X^p)$ we studied in Theorem 3.12. Note that S/I is the Stanley-Reisner ring considered in the remark (ii) immediately after Ishida's criteria in Sect. 3.2. In fact, I is the ideal corresponding to a triangulation of an $(r-1)$-sphere S^{r-1} so that S/I is a Gorenstein ring. Indeed, $(N_{\mathbb{R}} \setminus \{O\})/\mathbb{R}_{>0}$ is homeomorphic to S^{r-1} and each $\sigma \in \Delta$ gives rise to a simplex $(\sigma \setminus \{O\})/\mathbb{R}_{>0}$ on S^{r-1}. Thus the fan Δ induces a triangulation of S^{r-1}. Note further that a $\mathbb{Z}$-basis $\{m_1, \ldots, m_r\}$ for M gives rise to elements

$$l_j := \sum_{\varrho \in \Delta(1)} \langle m_j, n(\varrho) \rangle t(\varrho) \quad \text{for} \quad j = 1, \ldots, r$$

of J which turn out to form a regular sequence for the S-module S/I.

For the cohomology groups with $\mathbb{Q}$-coefficients of a projective variety, Lefschetz's hyperplane section theorem and the strong Lefschetz theorem are known to hold. Let us state the toric version of the latter, since we need it in Sect. A.5.

The Strong Lefschetz Theorem. *Let $X = T_N \operatorname{emb}(\Delta)$ be an r-dimensional toric projective variety with an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$. Denote by $\omega := c_1(\mathcal{L}) \in H^2(X, \mathbb{Q})$ the first Chern class of $\mathcal{L}$. If X is nonsingular, or if each $\sigma \in \Delta$ is simplicial, then the cup product with $\omega^q = \omega \cup \omega \cup \ldots \cup \omega \in H^{2q}(X, \mathbb{Q})$ induces an isomorphism*

$$\cup \omega^q : H^{r-q}(X, \mathbb{Q}) \xrightarrow{\sim} H^{r+q}(X, \mathbb{Q}) \quad \textit{for} \quad q = 0, \ldots, r \ .$$

In particular, the cup product induces injections

$$\cup \omega : H^{q-2}(X, \mathbb{Q}) \hookrightarrow H^q(X, \mathbb{Q}) \quad \textit{for} \quad q = 2, \ldots, r \ .$$

3.4 Automorphism Groups of Toric Varieties and the Cremona Groups

For a compact nonsingular toric variety $X = T_N \operatorname{emb}(\Delta)$, the automorphism group $\operatorname{Aut}(X)$ turns out to be a linear algebraic group with T_N as a maximal algebraic torus and with the root system computable in terms of the fan (N, Δ). Demazure [D5] obtained these results in connection with his study of linear algebraic subgroups of the Cremona groups. These results seem to have motivated him to initiate the theory of toric varieties.

In this section, we explain Demazure's results mostly without proofs and then touch on the results due to Enriques and Fano on the Cremona groups, to which Umemura [U1], [U2], [U3] gave rigorous proofs recently.

First of all, $\operatorname{Aut}(X)$ is well-known to be a complex Lie group with respect to the compact open topology. On the other hand, it coincides with the automorphism group of X as a complete algebraic variety by the GAGA theorem (iii) in Sect. 2.2. Thus Aut (X) also has a natural structure of an algebraic group as we now see, and the associated complex Lie group structure is that mentioned above.

Regard X as an algebraic variety. For any scheme S over $\mathbb{C}$ denote by $A_X(S)$ the group under composition, consisting of the S-automorphisms of $X\times S$ (i.e., automorphisms $f: X\times S \xrightarrow{\sim} X\times S$ such that $p_2\circ f=p_2$ holds for the second projection $p_2: X\times S\to S$). The pull-back by a morphism $g: S\to S'$ induces a group homomorphism $A_X(g): A_X(S')\to A_X(S)$ so that A_X becomes a contravariant functor from the category of $\mathbb{C}$-schemes to that of groups.

By **Murre-Matsumura-Oort-Artin's criterion** (cf. Artin [A3]), A_X turns out to be a representable functor, that is, there exists an algebraic group $\mathbb{A}_X$ over $\mathbb{C}$ satisfying the following properties: For any S, there exists an isomorphism $\varphi_S: \mathrm{Hom}(S, \mathbb{A}_X) \xrightarrow{\sim} A_X(S)$ from the set of morphisms from S to $\mathbb{A}_X$. Moreover, for any morphism $g: S\to S'$, the map $g^*: \mathrm{Hom}(S', \mathbb{A}_X)\to \mathrm{Hom}(S, \mathbb{A}_X)$ induced by the composition with g satisfies $\varphi_S\circ g^* = A_X(g)\circ\varphi_{S'}$.

In particular, if we consider the case $S=\mathrm{Spec}(\mathbb{C})$, then the set $\mathrm{Hom}(\mathrm{Spec}(\mathbb{C}), \mathbb{A}_X)$ of $\mathbb{C}$-valued points of $\mathbb{A}_X$ is isomorphic to $A_X(\mathrm{Spec}(\mathbb{C})) = \mathrm{Aut}(X)$. In this sense, Aut (X) has a natural structure of an algebraic group.

Let us next consider the ring of dual numbers $\mathbb{C}[\varepsilon]=\mathbb{C}\oplus\mathbb{C}\varepsilon$ with $\varepsilon^2=0$ and let $S' := \mathrm{Spec}(\mathbb{C}[\varepsilon])$, which is a single point set but has a nontrivial scheme structure. The $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\varepsilon]\to\mathbb{C}$ which sends ε to zero induces a morphism $g: S\hookrightarrow S'$. The Lie algebra Lie $(\mathbb{A}_X)$ of $\mathbb{A}_X$, which can be identified with the tangent space of $\mathbb{A}_X$ at the identity element id_X, consists of the first order infinitesimal automorphisms of X so that

$$\begin{aligned}\mathrm{Lie}(\mathbb{A}_X) &= \ker(\mathrm{Hom}(S', \mathbb{A}_X)\to \mathrm{Hom}(S, \mathbb{A}_X)) = \ker(A_X(S')\to A_X(S))\\ &= H^0(X, \Theta_X)\ ,\end{aligned}$$

where Θ_X is the sheaf of germs of holomorphic vector fields on X as before.

For simplicity, we henceforth identify $\mathbb{A}_X$ with Aut (X), and denote by $\mathrm{Aut}^o(X)$ its connected component containing id_X with respect to the Zariski topology.

Although what we stated so far holds for general complete nonsingular algebraic varieties and the associated compact complex manifolds, let us now restrict ourselves to a compact nonsingular toric variety $X=T_N\,\mathrm{emb}(\Delta)$. In this case, Aut (X) obviously contains T_N as an algebraic subgroup.

Demazure's results can be most easily understood, if we start by studying the Lie algebra of Aut (X). For the Cartier divisor $D := \sum_{\varrho\in\Delta(1)} V(\varrho)$ on X, we have by Theorem 3.12, (1) an exact sequence

$$0\to\Theta_X(-\log D)\to\Theta_X\to \bigoplus_{\varrho\in\Delta(1)} \mathcal{O}_{V(\varrho)}\otimes_{\mathcal{O}_X}\mathcal{O}_X(V(\varrho))\to 0\ .$$

Since $\Theta_X(-\log D)=\mathcal{O}_X\otimes_{\mathbb{Z}}N$ by Proposition 3.1, we thus get an exact sequence

$$\begin{aligned}0&\to\mathbb{C}\otimes_{\mathbb{Z}}N = H^0(X, \Theta_X(-\log D))\to H^0(X,\Theta_X)\\ &\to \bigoplus_{\varrho\in\Delta(1)} H^0(X, \mathcal{O}_{V(\varrho)}\otimes_{\mathcal{O}_X}\mathcal{O}_X(V(\varrho)))\to 0\end{aligned}$$

in view of Corollary 2.8. We can regard $\mathbb{C}\otimes_{\mathbb{Z}}N=\mathrm{Lie}(T_N)$ canonically as a Lie subalgebra of $\mathrm{Lie}(\mathrm{Aut}(X)) = H^0(X, \Theta_X)$ by sending $c \otimes n \in \mathbb{C} \otimes_{\mathbb{Z}} N$ to the derivation $c\delta_n \in H^0(X, \Theta_X)$ of $\mathcal{O}_X$ (cf. Sect. 3.1).

We can easily prove the following proposition by the exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(V(\varrho)) \to \mathcal{O}_{V(\varrho)} \otimes_{\mathcal{O}_X} \mathcal{O}_X(V(\varrho)) \to 0$ and Lemma 2.3 applied to the Cartier divisor $V(\varrho)$ on X, where $n(\varrho)$ for each $\varrho \in \Delta(1)$ is the unique primitive element in $N \cap \varrho$ as before.

Proposition 3.13 (Demazure [D5, Proposition 7, p. 571]). *Let $X = T_N\,\mathrm{emb}(\Delta)$ be a compact nonsingular toric variety. The subset*

$$R(N,\Delta) := \{\alpha \in M;\ \exists \varrho(\alpha) \in \Delta(1) \text{ with } \langle \alpha, n(\varrho(\alpha))\rangle = 1,$$
$$\text{but } \langle \alpha, n(\varrho')\rangle \leq 0 \text{ for } \Delta(1) \ni \forall \varrho' \neq \varrho(\alpha)\}$$

of M is called the root system *for the fan (N, Δ), and its elements are called the* roots *for (N, Δ). In this notation, $\{\mathbf{e}(-\alpha)\delta_{n(\varrho(\alpha))};\ \alpha \in R(N,\Delta)\}$ is a $\mathbb{C}$-linearly independent subset of $H^0(X, \Theta_X)$ and we get a direct sum decomposition*

$$\mathrm{Lie}(\mathrm{Aut}(X)) = H^0(X, \Theta_X) = \mathrm{Lie}(T_N) \oplus \left(\bigoplus_{\alpha \in R(N,\Delta)} \mathbb{C}\mathbf{e}(-\alpha)\delta_{n(\varrho(\alpha))} \right) .$$

Moreover, we have $\mathrm{Lie}(T_N) = \mathbb{C} \otimes_{\mathbb{Z}} N$ by identifying $c \otimes n \in \mathbb{C} \otimes_{\mathbb{Z}} N$ with $c\delta_n \in H^0(X, \Theta_X)$.

Remark. By definition, each $\alpha \in R(N, \Delta)$ uniquely determines $\varrho(\alpha) \in \Delta(1)$. However, we might have $\varrho(\alpha) = \varrho(\beta)$ for distinct α and β in $R(N, \Delta)$. Obviously, $R(N, \Delta)$ is a finite set, since $H^0(X, \Theta_X)$ is finite dimensional for compact X.

Examples. Let us compute the root systems for the two-dimensional examples (ii), (iii) immediately before Theorem 1.5 in Sect. 1.2 (cf. Fig. 1.4).

(ii) In this case, we have $X = \mathbb{P}_2(\mathbb{C})$ and

$$\Delta(1) = \{\mathbb{R}_{\geq 0} n_1, \mathbb{R}_{\geq 0} n_2, \mathbb{R}_{\geq 0}(-n_1 - n_2)\}$$

for a $\mathbb{Z}$-basis $\{n_1, n_2\}$ of N. Denote by $\{m_1, m_2\}$ the dual $\mathbb{Z}$-basis for M and identify $\alpha \in R(N, \Delta)$ with the pair $(\alpha; n(\varrho(\alpha)))$. Then we have

$$R(N,\Delta) = \{(m_1; n_1),\ (-m_1; -n_1 - n_2),\ (m_2; n_2),\ (-m_2; -n_1 - n_2),$$
$$(m_1 - m_2; n_1),\ (-m_1 + m_2; n_2)\} .$$

Obviously, this coincides with the root system for $\mathrm{Aut}(\mathbb{P}_2(\mathbb{C})) = PGL_3(\mathbb{C})$ in the usual sense.

(iii) In this case, we have $X = F_a$ and we may assume $a \geq 0$. With respect to a $\mathbb{Z}$-basis $\{n_1, n_2\}$ for N, we get

$$\Delta(1) = \{\mathbb{R}_{\geq 0} n_1, \mathbb{R}_{\geq 0} n_2, \mathbb{R}_{\geq 0}(-n_2),\ \mathbb{R}_{\geq 0}(-n_1 + a n_2)\} .$$

When $a = 0$, we easily have

$$R(N,\Delta) = \{(m_1; n_1),\ (-m_1; -n_1),\ (m_2; n_2),\ (-m_2; -n_2)\} ,$$

while for $a\geq 1$, we get

$$R(N,\Delta)=\{(m_1;n_1),\ (-m_1;\ -n_1+an_2),\ (km_1-m_2;\ -n_2)\quad \text{for}\quad -a\leq\forall k\leq 0\}\ .$$

$F_0=\mathbb{P}_1(\mathbb{C})\times\mathbb{P}_1(\mathbb{C})$ holds and the identity component of $\operatorname{Aut}(F_0)$ is $\operatorname{Aut}^o(F_0)=PGL_2(\mathbb{C})\times PGL_2(\mathbb{C})$ with the root system in the usual sense coinciding with $R(N,\Delta)$ above.

If $a\geq 1$, then F_a is a $\mathbb{P}_1(\mathbb{C})$-bundle over $\mathbb{P}_1(\mathbb{C})$. Let G_1 be the subgroup of $\operatorname{Aut}(F_a)$ consisting of those automorphisms which send each fiber into itself, while G_2 denotes the subgroup consisting of those automorphisms which send each fiber to another fiber. Then G_2 contains G_1 as a normal subgroup and G_2/G_1 is a subgroup of the automorphism group $\operatorname{Aut}(\mathbb{P}_1(\mathbb{C}))$ of the base space. In fact, G_2/G_1 turns out to coincide with $\operatorname{Aut}(\mathbb{P}_1(\mathbb{C}))$ and have the root system $\{m_1,\ -m_1\}$. The other roots in $R(N,\Delta)$ form the root system for G_1 (cf. Maruyama [M6]).

By Theorem 1.28, the blowing-up of $\mathbb{P}_2(\mathbb{C})$ at the point $V(\mathbb{R}_{\geq 0}n_1+\mathbb{R}_{\geq 0}(-n_1-n_2))$ is F_{-1}, hence is isomorphic to F_1. The root system for $\operatorname{Aut}^o(F_{-1})$ is $\{m_1,\ -m_1,m_2,\ -m_1+m_2\}$, which is a subset of the root system $\{m_1,\ -m_1,m_2,\ -m_2,\ m_1-m_2,\ -m_1+m_2\}$ for $\operatorname{Aut}(\mathbb{P}_2(\mathbb{C}))$. This is a special case of the general phenomenon we point out in Proposition 3.15 below.

For each root $\alpha\in R(N,\Delta)$, we now construct the one-parameter subgroup $x_\alpha:\mathbb{C}\to\operatorname{Aut}(X)$ corresponding to the infinitesimal automorphism $\mathbf{e}(-\alpha)\delta_{n(\varrho(\alpha))}\in\operatorname{Lie}(\operatorname{Aut}(X))$ as follows:

Proposition 3.14 (Demazure [D5, Theorem 3, p. 573]). *Let $X=T_N\operatorname{emb}(\Delta)$ be a compact nonsingular toric variety. For each root $\alpha\in R(N,\Delta)$ and for each $\lambda\in\mathbb{C}$, define a birational map $x_\alpha(\lambda)$ from T_N into itself as follows: If $u\in T_N=\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{C}^\times)$ satisfies $1+\lambda u(-\alpha)\neq 0$, then we let*

$$x_\alpha(\lambda)(u):=u\cdot\gamma_{n(\varrho(\alpha))}(1+\lambda u(-\alpha))\in T_N\ ,$$

where $\gamma_{n(\varrho(\alpha))}:\mathbb{C}^\times\to T_N$ is the one-parameter subgroup determined by $n(\varrho(\alpha))\in N$ as in Sect. 1.2. More concretely, we have

$$x_\alpha(\lambda)(u)(m)=u(m)(1+\lambda u(-\alpha))^{\langle m,n(\varrho(\alpha))\rangle}\quad \textit{for}\quad m\in M\ .$$

This $x_\alpha(\lambda)$ can be naturally extended to an automorphism of X and has the following properties:

(i) *The map $x_\alpha:\mathbb{C}\to\operatorname{Aut}(X)$ which sends λ to $x_\alpha(\lambda)$ is an injective homomorphism of algebraic groups.*

(ii) *x_α induces an injective Lie algebra homomorphism $dx_\alpha:\operatorname{Lie}(\mathbb{C})\to\operatorname{Lie}(\operatorname{Aut}(X))=H^0(X,\Theta_X)$ such that $dx_\alpha(d/d\lambda)=\mathbf{e}(-\alpha)\delta_{n(\varrho(\alpha))}$, where $d/d\lambda$ is the derivation with respect to the coordinate λ of $\mathbb{C}$ and generates* Lie $(\mathbb{C})$ *as usual.*

(iii) *For $t\in T_N$ and $\lambda\in\mathbb{C}$, the equality*

$$t\circ x_\alpha(\lambda)\circ t^{-1}=x_\alpha(t(\alpha)\lambda)$$

holds in $\operatorname{Aut}(X)$.

Proof. In the following steps (a) and (b), we first show that $x_\alpha(\lambda)$ can be extended to a holomorphic map from X to itself:

(a) If σ has $\varrho(\alpha)$ as a face, then $x_\alpha(\lambda)$ can be extended to a holomorphic map from U_σ to itself. Indeed, we have $U_\sigma=\{u: M\cap\sigma^\vee\to\mathbb{C};\ u(O)=1,\ u(m+m')=u(m)u(m')$ for $\forall m, m'\in M\cap\sigma^\vee\}$ by definition. Since $\varrho(\alpha)$ is a face of σ, we have $\langle m, n(\varrho(\alpha))\rangle\geq 0$ for $m\in M\cap\sigma^\vee$. In this case, $m-j\alpha$ belongs to $M\cap\sigma^\vee$ if $0\leq j\leq\langle m, n(\varrho(\alpha))\rangle$, since $\langle m-j\alpha, n(\varrho(\alpha))\rangle=\langle m, n(\varrho(\alpha))\rangle-j\geq 0$, while $\langle m-j\alpha, n(\varrho')\rangle=\langle m, n(\varrho')\rangle+j(-\langle\alpha, n(\varrho')\rangle)\geq 0$ if $\varrho'\in\Delta(1)$ is a face of σ different from $\varrho(\alpha)$. When $u\in T_N$ satisfies $1+\lambda u(-\alpha)\neq 0$, then for m in $M\cap\sigma^\vee$ we have

$$x_\alpha(\lambda)(u)(m)=u(m)(1+\lambda u(-\alpha))^{\langle m, n(\varrho(\alpha))\rangle}=\sum_{j=0}^{\langle m, n(\varrho(\alpha))\rangle}\binom{\langle m, n(\varrho(\alpha))\rangle}{j}\lambda^j u(m-j\alpha)$$

by definition. By what we have seen above, the third term makes sense as an element of $\mathbb{C}$ for $m\in M\cap\sigma^\vee$ and $u\in U_\sigma$ so that $x_\alpha(\lambda)(u)$ can be regarded as an element of U_σ.

(b) Suppose $\varrho(\alpha)$ is not a face of $\sigma\in\Delta$. By definition, $-\alpha$ then belongs to $M\cap\sigma^\vee$, so that $\sigma\cap\alpha^\perp$ is a face of σ. Clearly, $\tau:=\varrho(\alpha)+(\sigma\cap\alpha^\perp)$ belongs to the complete fan Δ, since $\langle\alpha, n(\varrho(\alpha))\rangle=1$ and since $\langle\alpha, n(\varrho')\rangle\leq 0$ if $\varrho'\in\Delta(1)$ is different from $\varrho(\alpha)$. Obviously, U_σ is the union of two open subsets $U':=\{u\in U_\sigma;\ 1+\lambda u(-\alpha)\neq 0\}$ and $U'':=\{u\in U_\sigma; u(-\alpha)\neq 0\}$. We now show that $x_\alpha(\lambda)$ can be extended to holomorphic maps $U'\to U_\sigma$ and $U''\to U_\tau$.

Indeed, $\sigma\cap\alpha^\perp$ is a face of τ and $U''=U_{\sigma\cap\alpha^\perp}$ by definition. Hence U'' is contained in U_τ by Proposition 1.3. Consequently, we have an extension $U''\to U_\tau$ in view of $x_\alpha(\lambda)(U'')\subset U_\tau$ obtained in (a) above. On the other hand, suppose u is in U'. Then the right hand side of

$$x_\alpha(\lambda)(u)(m)=u(m)(1+\lambda u(-\alpha))^{\langle m, n(\varrho(\alpha))\rangle}$$

makes sense as an element of $\mathbb{C}$ for $m\in M\cap\sigma^\vee$ so that $x_\alpha(\lambda)(u)$ can be regarded as an element of U_σ.

In either of the above cases, $x_\alpha(\lambda)$ has a polynomial expression in λ, hence $\mathbb{C}\times X\ni(\lambda, u)\mapsto x_\alpha(\lambda)\in X$ is a holomorphic map.

If $u\in T_N$ and $\lambda, \lambda'\in\mathbb{C}$ satisfy $1+\lambda u(-\alpha)\neq 0$, $1+\lambda' u(-\alpha)\neq 0$ and $1+(\lambda+\lambda')\times u(-\alpha)\neq 0$, then we see easily that $x_\alpha(\lambda)\circ x_\alpha(\lambda')=x_\alpha(\lambda+\lambda')$ holds. Thus $(x_\alpha(\lambda)\circ x_\alpha(\lambda'))(u)$ and $x_\alpha(\lambda+\lambda')(u)$ coincide as holomorphic maps from $(\lambda, \lambda', u)\in\mathbb{C}\times\mathbb{C}\times X$ to X. Since $x_\alpha(0)=\mathrm{id}_X$, we conclude that $x_\alpha(\lambda)$ is an automorphism of X and that $x_\alpha:\mathbb{C}\to\mathrm{Aut}(X)$ is a homomorphism. This x_α is injective, since $x_\alpha(\lambda)$ for $\lambda\neq 0$ is different from the identity map already as a birational map from T_N into itself. Since $x_\alpha(\lambda)(u)$ has a polynomial expression in (λ, u), we see that x_α is a homomorphism of algebraic groups by the very definition of the algebraic group structure on $\mathrm{Aut}(X)$ described at the beginning of this section. Thus we have (i).

We obtain (ii) simply by differentiating the defining equality for $x_\alpha(\lambda)$ with respect to λ.

If $\lambda\in\mathbb{C}$ and $t, u\in T_N$ satisfy $1+\lambda t(\alpha)u(-\alpha)\neq 0$, then for each $m\in M$, we see easily that both $(t\circ x_\alpha(\lambda)\circ t^{-1})(u)(m)$ and $x_\alpha(t(\alpha)\lambda)(u)(m)$ coincide with

$$u(m)\{1+\lambda t(\alpha)u(-\alpha)\}^{\langle m, n(\varrho(\alpha))\rangle}\ .$$

We get (iii), since the holomorphic maps $(t\circ x_\alpha(\lambda)\circ t^{-1})(u)$ and $x_\alpha(t(\alpha)\lambda)(u)$ from $(\lambda, t, u)\in\mathbb{C}\times T_N\times X$ to X coincide on an open subset. q.e.d.

We are now ready to state:

Demazure's Structure Theorem. *Let* $X = T_N\,\mathrm{emb}(\Delta)$ *be a compact nonsingular toric variety. Then its automorphism group* $\mathrm{Aut}(X)$ *is a linear algebraic group having the following properties:*

(i) T_N *is a maximal algebraic torus in the connected component* $\mathrm{Aut}^o(X)$ *containing the identity* id_X. $R(N, \Delta)$ *is the root system for* $\mathrm{Aut}^o(X)$ *with respect to the maximal algebraic torus* T_N. *In particular,* $\mathrm{Aut}^o(X)$ *is generated as a group by* T_N *and the family* $\{x_\alpha(\mathbb{C}); \alpha \in R(N, \Delta)\}$ *of unipotent one-parameter subgroups.*

(ii) *Let*

$$R_s(N, \Delta) := R(N, \Delta) \cap (-R(N, \Delta)) \quad \text{and} \quad R_u(N, \Delta) := R(N, \Delta) \setminus R_s(N, \Delta) \ .$$

Then as a variety, the unipotent radical G_u *of* $\mathrm{Aut}^o(X)$ *is isomorphic to the product of* $\{x_\alpha(\mathbb{C}); \alpha \in R_u(N, \Delta)\}$, *while there exists a reductive algebraic subgroup* G_s *with* T_N *as a maximal algebraic torus and with* $R_s(N, \Delta)$ *as the root system so that* $\mathrm{Aut}^o(X)$ *is a semidirect product of* G_s *and the normal subgroup* G_u. *In particular, we have*

$$\dim \mathrm{Aut}^o(X) = \mathrm{rank}\, N + {}^{\#}R(N, \Delta) \ ,$$

where ${}^{\#}R(N, \Delta)$ *is the cardinality of* $R(N, \Delta)$. *Moreover, each simple component of the root system* $R_s(N, \Delta)$ *is of type* **A**.

(iii) *Let* $\mathrm{Aut}(N, \Delta)$ *be the group of automorphisms of the fan* (N, Δ). *For each* $\alpha \in R_s(N, \Delta)$, *we define* $w_\alpha \in \mathrm{Aut}(N, \Delta)$ *by*

$$w_\alpha(n') := n' - \langle \alpha, n' \rangle \{n(\varrho(\alpha)) - n(\varrho(-\alpha))\} \quad \text{for} \quad n' \in N \ .$$

Then the subgroup $W(N, \Delta)$ *of* $\mathrm{Aut}(N, \Delta)$ *generated by* $\{w_\alpha; \alpha \in R_s(N, \Delta)\}$ *coincides with the Weyl group of* G_s *and we have an isomorphism*

$$\mathrm{Aut}(X)/\mathrm{Aut}^o(X) \cong \mathrm{Aut}(N, \Delta)/W(N, \Delta) \ .$$

Demazure states his results in this final form in [D5, Proposition 11, p. 581], although the proof, which is valid for (not necessarily compact) nonsingular toric varieties, is spread over the entire paper. We omit the proof here, since it requires so much space and basic knowledge on linear algebraic groups.

In trying to determine which toric Fano varieties admit Einstein-Kähler metrics (cf. Sect. 2.3), Mabuchi [M3] uses Demazure's structure theorem to rule out those with nonreductive automorphism groups in view of a result due to Matsushima. From among the remaining candidates, Mabuchi rules out some more using the moment maps we mentioned in Sect. 2.4.

Instead of going into any detail on the proof for Demazure's structure theorem, we rather state a consequence on the Cremona groups. For that purpose, let us first note the following:

Proposition 3.15. *Let* $X = T_N\,\mathrm{emb}(\Delta)$ *be a compact nonsingular toric variety. For* $\tau \in \Delta$, *let* $f: X' = T_N\,\mathrm{emb}(\Delta') \to X$ *be the equivariant blowing-up along* $V(\tau)$. *Then* $\mathrm{Aut}^o(X')$ *is naturally a subgroup of* $\mathrm{Aut}^o(X)$ *and the root system* $R(N, \Delta')$ *for* (N, Δ')

can be naturally identified with the subset

$$\{\alpha \in R(N,\Delta); \langle \alpha, n(\varrho_1)+\ldots+n(\varrho_s)\rangle \leq 0\}$$

of $R(N,\Delta)$, *where* $\varrho_1,\ldots,\varrho_s$ *are distinct elements of* $\Delta(1)$ *such that* $\tau=\varrho_1+\ldots+\varrho_s$.

Proof. The exceptional set $f^{-1}(V(\tau))$ is well-known to be the unique effective divisor linearly equivalent to it. Since $\mathrm{Aut}^o(X')$ is a connected linear algebraic group, we thus see that $f^{-1}(V(\tau))$ is stable under $\mathrm{Aut}^o(X')$. In view of the rigidity lemma and Zariski's main theorem, we have a canonical injective homomorphism $\mathrm{Aut}^o(X')\hookrightarrow \mathrm{Aut}^o(X)$, hence an injective homomorphism of Lie algebras $H^0(X',\Theta_{X'})\hookrightarrow H^0(X,\Theta_x)$.

Let $\varrho_0:=\mathbb{R}_{\geq 0}(n(\varrho_1)+\ldots+n(\varrho_s))$. Then by Proposition 1.26, we have

$$\Delta'(1)=\Delta(1)\cup\{\varrho_0\} \quad \text{and} \quad n(\varrho_0)=n(\varrho_1)+\ldots+n(\varrho_s)\ .$$

For $\alpha'\in R(N,\Delta')$, we have $\varrho(\alpha')\neq\varrho_0$, hence $\alpha'\in R(N,\Delta)$, since otherwise we have $\langle\alpha',n(\varrho')\rangle\leq 0$ for all $\varrho'\in\Delta(1)$, so that $\langle\alpha',n\rangle\leq 0$ would hold for all $n\in|\Delta|=N_{\mathbb{R}}$, a contradiction.

It is easy to see that $\alpha\in R(N,\Delta)$ belongs to $R(N,\Delta')$ if and only if $\langle\alpha,n(\varrho_0)\rangle\leq 0$. q.e.d.

For each positive integer r, consider a purely transcendental extension $\mathbb{C}(z_1,\ldots,z_r)$ of $\mathbb{C}$ in variables $z_1,\ldots,z_r$. The *Cremona group* $\mathrm{Cr}(r,\mathbb{C})$ in r variables is defined to be the group under composition, consisting of the field automorphisms of $\mathbb{C}(z_1,\ldots,z_r)$ over $\mathbb{C}$ (i.e., those which induce the identity automorphism on the subfield $\mathbb{C}$).

Its structure for $r=1$ is easy to describe: An automorphism of $\mathbb{C}(z_1)$ over $\mathbb{C}$ is uniquely determined by the image of z_1, which is given by a linear fractional transformation $z_1\mapsto(az_1+b)/(cz_1+d)$ with a, b, c, $d\in\mathbb{C}$ satisfying $ad-bc\neq 0$. Hence we have

$$\mathrm{Cr}(1,\mathbb{C})=PGL_2(\mathbb{C})=\mathrm{Aut}(\mathbb{P}_1(\mathbb{C}))\ .$$

When $r\geq 2$, however, $\mathrm{Cr}(r,\mathbb{C})$ is not an algebraic group in the ordinary sense. An *algebraic subgroup* G of $\mathrm{Cr}(r,\mathbb{C})$ is defined as follows: Let Y be an r-dimensional *rational algebraic variety* over $\mathbb{C}$, that is, the function field $\mathbb{C}(Y)$ of Y is $\mathbb{C}$-isomorphic to the purely transcendental extension $\mathbb{C}(z_1,\ldots,z_r)$. If an algebraic group G acts faithfully on Y, then we have an injective group homomorphism $G\hookrightarrow\mathrm{Cr}(r,\mathbb{C})$. The conjugacy class in $\mathrm{Cr}(r,\mathbb{C})$ of its image is independent of any particular choice of the $\mathbb{C}$-isomorphism $\mathbb{C}(Y)\cong\mathbb{C}(z_1,\ldots,z_r)$.

It is a **basic problem on the Cremona groups** to classify, up to conjugacy, all the algebraic subgroups of $\mathrm{Cr}(r,\mathbb{C})$, in particular, all the maximal connected algebraic subgroups.

As H. Umemura pointed out, we have to keep in mind the fact that the connected algebraic subgroups of $\mathrm{Cr}(r,\mathbb{C})$ do *not* form an inductively ordered set. Thus to solve the above problem, we first need to classify all the maximal connected algebraic subgroups up to conjugacy. If possible, we then try to check if a suitable conjugate of a given connected algebraic subgroup is contained in one of the maximal ones in the list.

An r-dimensional compact nonsingular toric variety $X = T_N \operatorname{emb}(\Delta)$ is a rational algebraic variety, on which the connected linear algebraic group $\operatorname{Aut}^o(X)$ faithfully acts as we saw in this section. Thus $\operatorname{Aut}^o(X)$ is a connected algebraic subgroup of $\operatorname{Cr}(r, \mathbb{C})$ and contains an r-dimensional algebraic torus T_N.

If X is obtained as an equivariant blowing-up $f'': X \to X'' := T_N \operatorname{emb}(\Delta'')$ of a compact nonsingular toric variety, then we have $\operatorname{Aut}^o(X) \subset \operatorname{Aut}^o(X'')$ by Proposition 3.15. Thus $\operatorname{Aut}^o(X)$ may be a maximal connected algebraic subgroup of $\operatorname{Cr}(r, \mathbb{C})$ only if X is *minimal* in the sense that an equivariant blowing-up $f'': X \to X''$ is necessarily an isomorphism. In this sense, our consideration in Sect. 1.7 might be meaningful.

Here are what we know about the Cremona groups:

(1) (Matsumura [M 7]) Any connected algebraic subgroup of $\operatorname{Cr}(r, \mathbb{C})$ is a linear algebraic group.

(2) (Demazure [D 5, p. 522, Corollaries 4 and 5]) Any algebraic torus contained in $\operatorname{Cr}(r, \mathbb{C})$ has dimension r or less. Any two r-dimensional algebraic tori in $\operatorname{Cr}(r, \mathbb{C})$ are conjugate to each other.

(3) (Demazure [D 5]) Any connected algebraic subgroup of $\operatorname{Cr}(r, \mathbb{C})$ of rank r (that is, containing an r-dimensional algebraic torus) is conjugate to an automorphism group $G(N, \Delta; R')$ of a (not necessarily compact) nonsingular toric variety, defined as follows: For $X = T_N \operatorname{emb}(\Delta)$ corresponding to a finite nonsingular fan Δ in $N \cong \mathbb{Z}^r$ and for a *saturated* finite subset R' of the root system (R, Δ) for (N, Δ), we define $G(N, \Delta; R')$ to be the subgroup of $\operatorname{Aut}(X)$ generated by T_N and $\{x_\alpha(\mathbb{C}); \alpha \in R'\}$. Moreover, the normalizer $\mathfrak{N}$ of $G(N, \Delta; R')$ in $\operatorname{Cr}(r, \mathbb{C})$ has the identity component equal to $G(N, \Delta; R')$ and we have an isomorphism

$$\mathfrak{N}/G(N, \Delta; R') \cong \operatorname{Aut}(N, \Delta; R')/W(N, \Delta; R') \ .$$

Here $\operatorname{Aut}(N, \Delta; R')$ is the group consisting of the automorphisms of the fan (N, Δ) which preserve R' in an obvious sense, while $W(N, \Delta; R')$ is the subgroup of $\operatorname{Aut}(N, \Delta : R')$ generated by $\{w_\alpha; \alpha \in R' \cap (-R')\}$, where for $\alpha \in R' \cap (-R')$, w_α is the $\mathbb{Z}$-automorphism of N defined by

$$w_\alpha(n') := n' - \langle \alpha, n' \rangle (n(\varrho(\alpha)) - n(\varrho(-\alpha))) \quad \text{for} \quad n' \in N \ .$$

When $X = T_N \operatorname{emb}(\Delta)$ is a nonsingular toric variety which may not be compact, we define the system $R(N, \Delta)$ of *roots* for X to consist of $\alpha \in M$ which satisfy the following (i) and (ii), slightly stronger than the condition in Proposition 3.13.

(i) $\langle \alpha, n(\varrho(\alpha)) \rangle = 1$ for some $\varrho(\alpha) \in \Delta(1)$, while $\langle \alpha, n(\varrho') \rangle \leq 0$ for all $\varrho' \in \Delta(1)$ different from $\varrho(\alpha)$.

(ii) $\sigma + \varrho(\alpha)$ belongs to Δ if $\sigma \in \Delta$ satisfies $\sigma \subset \alpha^\perp$.

Exactly as in Proposition 3.14, we have an injective homomorphism $x_\alpha : \mathbb{C} \to \operatorname{Aut}(X)$ for each $\alpha \in R(N, \Delta)$.

A finite subset R' of $R(N, \Delta)$ is said to be *saturated*, if it satisfies the following conditions (iii) and (iv):

(iii) α, $\beta \in R'$ and $\alpha + \beta \in R(N, \Delta)$ imply $\alpha + \beta \in R'$.

(iv) α, $\beta \in R'$, $\langle \alpha, n(\varrho(\beta)) \rangle < 0$ and $\langle \beta, n(\varrho(\alpha)) \rangle < 0$ imply $\alpha + \beta = 0$.

If we let $R'_s := R' \cap (-R')$ and $R'_u := R' \setminus R'_s$ as in Demazure's structure theorem, then $G(N, \Delta; R')$ again turns out to be a semidirect product of a reductive algebraic subgroup G_s and the unipotent radical G_u. Again G_s contains T_N as a maximal algebraic torus and R_s' is its system of roots with respect to T_N. Each simple component of R_s' is of type **A**. On the other hand, G_u is isomorphic, as an algebraic variety, to the product of $\{x_\alpha(\mathbb{C}); \alpha \in R_u'\}$ so that $\dim G(N, \Delta; R') = \operatorname{rank} N + {}^\# R'$.

When $X = T_N \operatorname{emb}(\Delta)$ is compact, this new $R(N, \Delta)$ coincides with that in Proposition 3.13. In this case, $R(N, \Delta)$ itself is a saturated finite set and $G(N, \Delta; R(N, \Delta))$ coincides with $\operatorname{Aut}^o(X)$.

The result (3) above seems to have been a motivation for Demazure to introduce toric varieties. Its proof is again spread over the entire paper [D5]. He relates toric varieties to algebraic subgroups of the Cremona groups via *saturated Enriques systems* and *pseudo-projectors* from connected algebraic subgroups to their maximal algebraic tori.

When $r = 2$, we have the following three kinds of minimal compact nonsingular toric varieties up to isomorphism by Theorem 1.28 in Sect. 1.7:

$$\mathbb{P}_2(\mathbb{C}) \ , \quad F_0 = \mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C}) \ , \quad F_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1(\mathbb{C})} \oplus \mathcal{O}_{\mathbb{P}_1(\mathbb{C})}(a)) \quad (a \geq 2) \ .$$

Corresponding to each of these, we have three kinds of maximal connected algebraic subgroups in $\operatorname{Cr}(2, \mathbb{C})$ up to conjugacy:

Enriques' Theorem. *The maximal connected algebraic subgroups of* $\operatorname{Cr}(2, \mathbb{C})$ *are of the following form* (i), (ii), (iii) *up to conjugacy. Moreover, a suitable conjugate of any connected algebraic subgroup of* $\operatorname{Cr}(2, \mathbb{C})$ *is contained in one of them.*

Let us fix a $\mathbb{Z}$-basis $\{n, n'\}$ for $N \cong \mathbb{Z}^2$ with the dual $\mathbb{Z}$-basis $\{m, m'\}$ for M, and consider the fans (N, Δ) in Fig. 3.1. We describe the root systems $R(N, \Delta)$ as sets of pairs $(\alpha; n(\varrho(\alpha)))$.

(i) $\operatorname{Aut}(\mathbb{P}_2(\mathbb{C})) = PGL_3(\mathbb{C})$.

$$R(N, \Delta) = \{(m; n),\ (-m;\ -n-n'),\ (m'; n'),\ (-m';\ -n-n'),\ (m-m'; n),\ (-m+m'; n')\} \ .$$

(ii) $\operatorname{Aut}^o(\mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})) = PGL_2(\mathbb{C}) \times PGL_2(\mathbb{C})$.

$$R(N, \Delta) = \{(m; n),\ (-m;\ -n),\ (m'; n'),\ (-m';\ -n')\} \ .$$

(iii) $\operatorname{Aut}(F_a)$ *for* $a \geq 2$.

$$R(N, \Delta) = \{(m; n),\ (-m;\ -n-an'),\ (km+m'; n') \ \ \text{for} \quad -a \leq \forall k \leq 0\} \ .$$

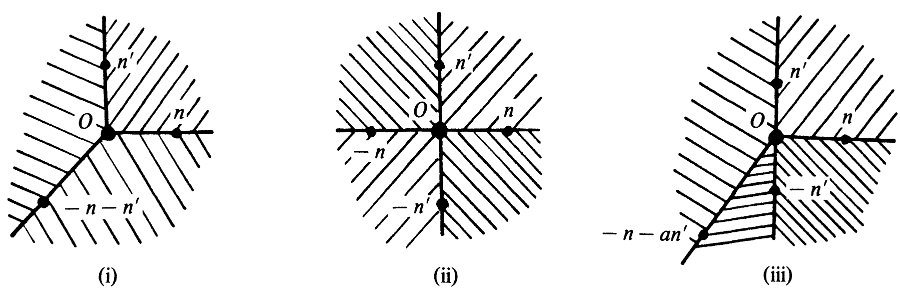


Fig. 3.1

When $r=3$, however, the maximal connected algebraic subgroups of $\mathrm{Cr}(3, \mathbb{C})$ are much more difficult to classify. Toward the end of the nineteenth century, Enriques and Fano carried out the classification, to which Umemura [U1], [U2], [U3] gave a rigorous modern proof recently. According to this result, maximal connected algebraic subgroups of rank three (i.e., those containing three-dimensional maximal algebraic tori) can be realized as the automorphism groups of three-dimensional toric varieties, but there are others of ranks two or one which can be realized only as the automorphism groups of non-toric compact rational threefolds. These threefolds are of much interest in the birational geometry of threefolds and are closely related to Fano threefolds (see Mukai-Umemura [MU] and Umemura [U4]). For instance, [MU] studies in detail the $SO_3(\mathbb{C})$-equivariant compactifications of the quotient spaces $SO_3(\mathbb{C})/\Gamma$ of $SO_3(\mathbb{C})$ with respect to finite polyhedral subgroups Γ.

Here is a summary of the classification in dimension $r=3$, where we use our notation in Sect. 1.7:

Enriques-Fano-Umemura's Theorem. *Up to conjugacy, the maximal connected algebraic subgroups of* $\mathrm{Cr}(3, \mathbb{C})$ *are divided into the following fourteen kinds, where we also add Umemura's notation in parentheses. Furthermore, a suitable conjugate of any connected algebraic subgroup of* $\mathrm{Cr}(3, \mathbb{C})$ *is contained in one of them.*

(I) **Two kinds of rank one subgroups.**

(i) $SO_3(\mathbb{C}) \cong PGL_2(\mathbb{C})$ *regarded as acting faithfully on the threefold* $SO_3(\mathbb{C})/\Gamma$, *where* Γ *is one of the following subgroups of* $SO_3(\mathbb{C})$:

(a) *the dihedral subgroups of order* ≥ 8 (J5),

(b) *the octahedral subgroup* (E1),

(c) *the icosahedral subgroup* (E2).

(ii) $PGL_2(\mathbb{C})$ *acting on a rational threefold with two-dimensional general orbits, each point of which has a maximal algebraic torus as the stabilizer.* (J12).

(II) **Three kinds of rank two subgroups.**

(iii) $PSO_5(\mathbb{C})$ *acting in the usual manner on the hyperquadric* $\{[z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{P}_4(\mathbb{C});\ z_0^2+z_1^2+z_2^2+z_3^2+z_4^2=0\}$ *in* $\mathbb{P}_4(\mathbb{C})$. (P2).

(iv) $PGL_3(\mathbb{C})$ *acting from the left on the flag manifold* $PGL_3(\mathbb{C})/B$, *where B is the Borel subgroup of* $PGL_3(\mathbb{C})$ *consisting of upper triangular matrices.* (J4).

(v) $\mathrm{Aut}^o(X)$, *where* $X \to Y$ *is an affine line bundle, not arising from a* $\mathbb{C}^\times$*-bundle, over an affine line bundle* $Y \to \mathbb{P}_1(\mathbb{C})$ *over* $\mathbb{P}_1(\mathbb{C})$. (J11).

(III) **Nine kinds of rank three subgroups.** $\mathrm{Aut}^o(X)$ *for the compact nonsingular toric varieties* $X = T_N \operatorname{emb}(\Delta)$ *corresponding to the fans* (N, Δ) *arising from the N-weighted triangulations of* S^2 *listed in Fig. 3.2. We here denote by* $\{n, n', n''\}$ *a* $\mathbb{Z}$*-basis for* $N \cong \mathbb{Z}^3$ *with the dual* $\mathbb{Z}$*-basis* $\{m, m', m''\}$ *for M, and describe the root systems* $R(N, \Delta)$ *as sets of pairs* $(\alpha; n(\varrho(\alpha)))$.

(vi) $\mathrm{Aut}(\mathbb{P}_3(\mathbb{C})) = PGL_4(\mathbb{C})$. (P1).

$$
\begin{aligned}
R(N, \Delta) = \{&(m; n),\ (-m;\ -n-n'-n''),\ (m'; n'),\ (-m';\ -n-n'-n''),\ (m''; n''),\\
&(-m'';\ -n-n'-n''),\ (m-m'; n),\ (-m+m'; n'),\ (m'-m''; n'),\\
&(-m'+m''; n''),\ (m-m''; n),\ (-m+m''; n'')\}.
\end{aligned}
$$

(vii) $\mathrm{Aut}^o(\mathbb{P}_2(\mathbb{C})\times\mathbb{P}_1(\mathbb{C}))=PGL_3(\mathbb{C})\times PGL_2(\mathbb{C})$. (J1).

$$R(N,\Delta)=\{(m;n),\,(-m;\,-n-n'),\,(m';n'),\,(-m';\,-n-n'),\,(m'';n''),\,(-m'';\,-n''),\,(m-m';n),\,(-m+m';n')\}\ .$$

(viii) $\mathrm{Aut}^o(\mathbb{P}_1(\mathbb{C})\times\mathbb{P}_1(\mathbb{C})\times\mathbb{P}_1(\mathbb{C}))=PGL_2(\mathbb{C})^3$. (J2).

$$R(N,\Delta)=\{(m;n),\,(-m;\,-n),\,(m';n'),\,(-m';\,-n'),\,(m'';n''),\,(-m'';\,-n'')\}\ .$$

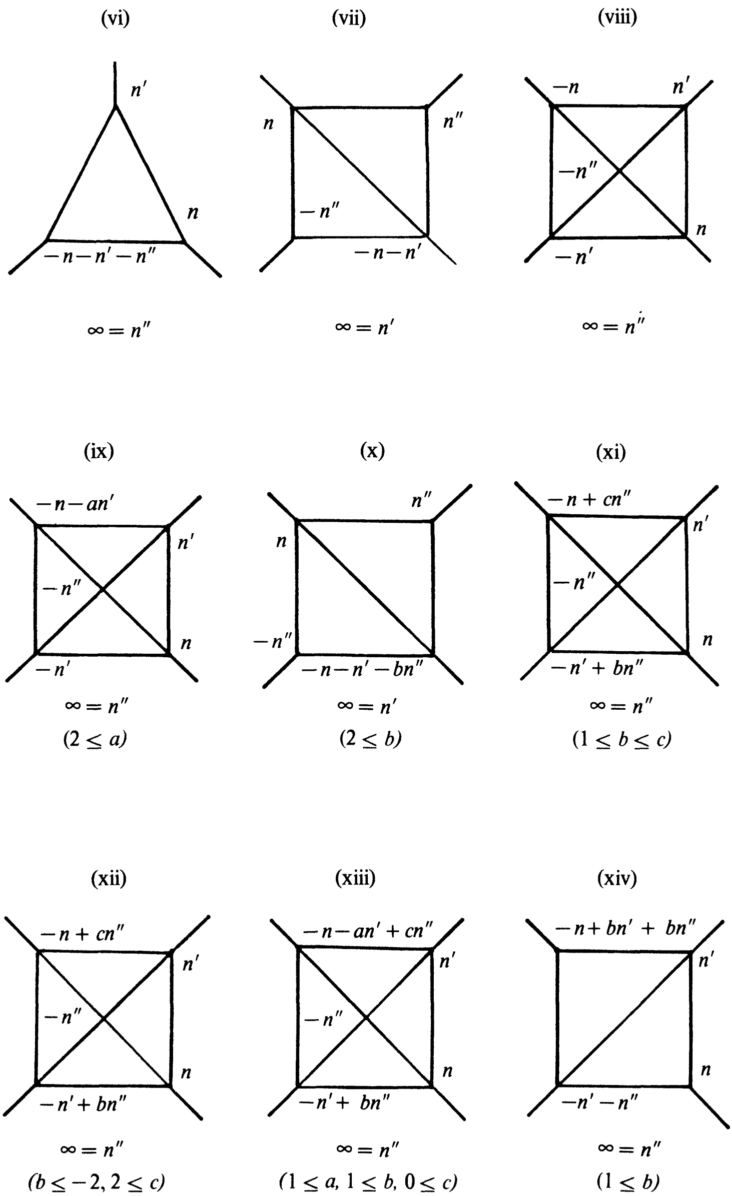


Fig. 3.2

(ix) $\mathrm{Aut}^o(F_a \times \mathbb{P}_1(\mathbb{C}))$ *with* $a \geq 2$. (J3).

$$R(N,\Delta)=\{(m;n),\ (-m;\ -n-an'),\ (m'';n''),\ (-m'';\ -n''),\ (km+m';n') \text{ for } -a \leq \forall k \leq 0\}\ .$$

(x) $\mathrm{Aut}^o(X)$ *for the* $\mathbb{P}_1(\mathbb{C})$*-bundle* $X=\mathbb{P}(\mathcal{O}_{\mathbb{P}_2(\mathbb{C})} \oplus \mathcal{O}_{\mathbb{P}_2(\mathbb{C})}(b))$ *over* $\mathbb{P}_2(\mathbb{C})$ *with* $b \geq 2$. (J7).

$$R(N,\Delta)=\{(m;n), (-m;\ -n-n'-bn''), (m';n'), (-m';\ -n-n'-bn''), (m-m';n), (-m+m';n'), (km+lm'+m'';n'') \text{ for } \forall k \leq 0,\ \forall l \leq 0 \text{ with } -b \leq k+l\}\ .$$

(xi) $\mathrm{Aut}^o(X)$ *for the* $\mathbb{P}_1(\mathbb{C})$*-bundle* $X=\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(bf_1+cf_2))$ *over* $Y=\mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$ *with* $1 \leq b \leq c$, *where* f_1 *and* f_2 *are fibers for the first and second projections from* Y *to* $\mathbb{P}_1(\mathbb{C})$, *respectively*. (J8).

$$R(N,\Delta)=\{(m;n),\ (-m;\ -n+cn''),\ (m';n'),\ (-m';\ -n'+bn''), (km+lm'-m'';\ -n'') \text{ for } -c \leq \forall k \leq 0 \text{ and } -b \leq \forall l \leq 0\}\ .$$

(xii) $\mathrm{Aut}^o(X)$ *for* X, Y, f_1, f_2 *as in* (xi) *above with* $b \leq -2$ *and* $2 \leq c$. (J6).

$$R(N,\Delta)=\{(m;n),\ (-m;\ -n+cn''),\ (m';n'),\ (-m';\ -n'+bn'')\}\ .$$

(xiii) $\mathrm{Aut}^o(X)$ *for the* $\mathbb{P}_1(\mathbb{C})$*-bundle* $X=\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(bs+cf))$ *over the Hirzebruch surface* $Y=F_a$ *with* $a \geq 1$, *where* f *is one of the fibers of* Y *over* $\mathbb{P}_1(\mathbb{C})$, *while* s *is a section satisfying* $s^2=-a$. (J9).

$$R(N,\Delta)=\{(m;n),\ (-m;\ -n-an'+cn''),\ (km+m';n') \text{ for } -a \leq \forall k \leq 0, (lm+pm'-m'';\ -n'') \text{ for } -b \leq \forall l \leq 0,\ -al-c \leq \forall p \leq 0\}\ .$$

(xiv) $\mathrm{Aut}^o(X)$ *for the* $\mathbb{P}_2(\mathbb{C})$*-bundle* $X=\mathbb{P}(\mathcal{O}_{\mathbb{P}_1(\mathbb{C})} \oplus \mathcal{O}_{\mathbb{P}_1(\mathbb{C})}(b) \oplus \mathcal{O}_{\mathbb{P}_1(\mathbb{C})}(b))$ *with* $b \geq 1$. (J10).

$$R(N,\Delta)=\{(m;n),\ (-m;\ -n+bn'+bn''),\ (m'-m'';n'),\ (-m'+m'';n''), (km-m';\ -n'-n''),\ (km-m'';\ -n'-n'') \text{ for } -b \leq \forall k \leq 0\}\ .$$

Chapter 4. Applications

By definition, a toric variety contains an algebraic torus as an open set. In particular, a compact toric variety belongs to the very special class of rational varieties.

However, if we take the quotient of an open set of a toric variety with respect to an action of a discrete group, we sometimes get an interesting complex manifold belonging to an entirely different class.

In this chapter, we first explain one of such constructions in dimension two, which is closely related to periodic continued fractions. Then we are naturally led to their higher dimensional analogues, which are closely related to compactifications of locally symmetric varieties we mention in (2) immediately below. We also briefly touch on the construction of compact complex surfaces in class VII as well as higher dimensional analogues.

The theory of toric varieties has much more important applications than those explained in this chapter. For lack of space, we only briefly mention some of them now and refer the reader to the existing literature for details.

(1) *Toroidal embeddings*, introduced by Mumford et al. [TE], are complex analytic spaces locally isomorphic to toric varieties. They share some of the properties of toric varieties, but are much more widespread. By means of toroidal embeddings, Mumford et al. [TE] could prove the very important **semistable reduction theorem** on one-parameter families of general algebraic varieties. We remark in passing that the results in Sect. 3.2 might lead us to a theory of "degenerate varieties" more general than toroidal embeddings, which might be useful for compactifications of various moduli spaces and for degenerations of algebraic varieties. There are attempts in this direction by Ishida-Oda [IO], Ishida [I1], [I5] and Danilov [D1].

(2) Closely related to (1) above is the construction of good compactifications of quotients of homogeneous bounded domains with respect to discrete groups. Quotients of toric varieties are very useful for the purpose. Mumford et al. [SC] and Satake [S5] already established the general framework, which has been playing important rôles in the theory of automorphic forms. Namikawa [N8] studied the case of Siegel upper half planes, which is related to the moduli spaces of polarized Abelian varieties. A brief introduction to it can be found in [MO, § 11], although Proposition 11.1 is incorrect. A related question on the compactifications of the generalized Jacobian varieties is dealt with in Oda-Seshadri [OS]. Chai [C1] and Faltings' recent results are also related. Hirzebruch [H3], Hirzebruch-van der Geer [HV] deal with the case of Hilbert modular varieties. Satake [S6], [S7], [SO] are also related. We touch on this latter case of $\mathbb{Q}$-rank one in Sects. 4.1 and 4.2.

(3) Convex polyhedral sets, called Newton polyhedra, play important rôles in the study of singularities of analytic functions and analytic spaces. Using toric varieties, we can describe the resolution of singularities and some of the analytic invariants in terms of the convex geometry of Newton polyhedra. We refer the reader to Teissier's results quoted in [T3], as well as to Kushnirenko [K6], Khovanski [K3], [K4], Varchenko [V1], [V2], Oka [O4], [O5] and numerous other papers.

(4) The Weyl chamber decompositions associated to semisimple algebraic groups are fans in our sense which are very rich in symmetry due to the action of the Weyl groups. Thus the compact toric varieties associated to them as in Chap. 1 have the induced action of the Weyl groups. They appear in De Concini-Procesi [DP] in connection with the intersection theory of algebraic cycles on symmetric spaces. [TE, Chap. IV, § 2] is also related. Moreover, Voskresenskii-Klyachko [VK] used them in their study of centrally symmetric toric Fano varieties, while Jun Murakami studied their group-theoretic significance. See also Wirthmüller [W4].

4.1 Periodic Continued Fractions and Two-Dimensional Toric Varieties

In Sect. 1.6, we encountered finite continued fraction expansions of rational numbers. More generally, each real number $\xi \in \mathbb{R}$ is known to have a unique continued fraction expansion of the form

$$\xi = e_0 - \cfrac{1}{e_1 - \cfrac{1}{e_2 - \ddots}} := e_0 - 1\rfloor e_1 - 1\rfloor e_2 - \ldots := [[e_0, e_1, e_2, \ldots]]$$

with integers e_ν, where $e_\nu \geq 2$ for $\nu > 0$ and where no infinite succession of 2's appear in the sequence $\{e_\nu : \nu \geq 0\}$. To be more precise, ξ coincides with the limit

$$\lim_{\nu \to \infty} [[e_0, e_1, \ldots, e_{\nu-1}, \xi_\nu]] ,$$

where $\{\xi_\nu ; \nu \geq 0\}$ is defined inductively from $\xi_0 := \xi$ as follows: e_ν for each ν is the unique integer such that $e_\nu - 1 < \xi_\nu \leq e_\nu$. If $\xi_\nu \neq e_\nu$, then we define $\xi_{\nu+1} \in \mathbb{R}$ uniquely by $\xi_\nu = e_\nu - 1/\xi_{\nu+1}$, while $\{e_\nu\}$ and $\{\xi_\nu\}$ terminate at $\nu = k$ if $\xi_k = e_k$.

$\xi = [[e_0, e_1, e_2, \ldots]]$ is well-known to be a quadratic irrational number if and only if its continued fraction expansion is *periodic*, i.e., there exist integers $k \geq 0$ and $p > 0$ such that $e_{\nu+p} = e_\nu$ holds for every $\nu \geq k$. We call the smallest such p the *minimal period* and often denote

$$\xi = [[e_0, \ldots, e_{k-1}, \overline{e_k, \ldots, e_{k+p-1}}]] .$$

Furthermore, we have $k = 0$, i.e., the continued fraction expansion is *purely periodic* if and only if ξ and its conjugate ξ' over $\mathbb{Q}$ satisfy

$$\xi > 1 > \xi' > 0 .$$

Such a ξ is called a *reduced quadratic irrational number*.

What we have seen so far has the following convex geometric interpretation in terms of two-dimensional fans, as Cohn [C2] pointed out: Let $K := \mathbb{Q}(\sqrt{D})$ be a *real quadratic field* for a square-free positive integer D. For each $\xi \in K$, we denote by ξ' its conjugate over $\mathbb{Q}$. Namely, the conjugate of $\xi = x + y\sqrt{D}$ for $x, y \in \mathbb{Q}$ is $\xi' = x - y\sqrt{D}$. The embedding $K \ni \xi \mapsto (\xi, \xi') \in \mathbb{R}^2$ induces a canonical isomorphism $K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^2$.

Let N be a $\mathbb{Z}$-*lattice* in K, i.e., a free $\mathbb{Z}$-submodule of K of rank two. Then we have $N \otimes_{\mathbb{Z}} \mathbb{Q} = K$ so that $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^2$ canonically. An invertible element in the ring $\mathfrak{o}$ of integers in K is called a *unit*. Then

$$\Gamma_N := \{\text{units } u \text{ of } K; u > 0,\ uN = N\} \quad \text{and}$$

$$\Gamma_N^+ := \{\text{units } u \text{ of } K; u > 0,\ u' > 0,\ uN = N\}$$

are known to be infinite cyclic groups under multiplication with the subgroup Γ_N^+ having index one or two in Γ_N. We can obviously regard Γ_N and Γ_N^+ as subgroups of the group $\mathrm{Aut}_{\mathbb{Z}}(N)$ of automorphisms of N.

The following result enables us to relate continued fractions to fans, and leads us to higher dimensional analogues of continued fractions in the next section:

Proposition 4.1. *Let N be a $\mathbb{Z}$-lattice in a real quadratic field K. We canonically identify $N_{\mathbb{R}} = \mathbb{R}^2$ as above and define a two-dimensional open convex cone in it by $C_N := (\mathbb{R}_{>0})^2$. Let Θ_N be the convex hull of $N \cap C_N$ with the boundary $\partial\Theta_N$. The rays from the origin O to the points in $N \cap \partial\Theta_N$ give rise to an infinite nonsingular fan Δ_N in N with $|\Delta_N| = C_N \cup \{O\}$ such that the action of Γ_N^+ on N leaves Δ_N stable. Regard $N_{\mathbb{R}}$ as an open subset of the real manifold with corners* $\mathrm{Mc}(N, \Delta_N)$ *as in Sect. 1.3, so that $\hat{C}_N := C_N \cup \{\mathrm{Mc}(N, \Delta_N) \setminus N_{\mathbb{R}}\}$ is the closure of C_N in* $\mathrm{Mc}(N, \Delta_N)$. *Denote by $\tilde{U}_N := \mathrm{ord}^{-1}(\hat{C}_N)$ the inverse image of $\hat{C}_N$ under the canonical surjection*

$$\mathrm{ord}: T_N \mathrm{emb}(\Delta_N) \twoheadrightarrow \mathrm{Mc}(N, \Delta_N)$$

and let $\tilde{X}_N := \tilde{U}_N \setminus (T_N \cap \tilde{U}_N) = T_N \mathrm{emb}(\Delta_N) \setminus T_N$. Then Γ_N^+ acts on $\tilde{U}_N$ properly discontinuously without fixed points so that

$$U_N := \tilde{U}_N / \Gamma_N^+$$

is a two-dimensional complex manifold, and $X_N := \tilde{X}_N / \Gamma_N^+$ is a finite cycle of $\mathbb{P}_1(\mathbb{C})$'s on it (cf. Fig. 4.1).

Proof. The three vertices are the only points of N belonging to the triangle determined by O and each pair of adjacent points in $N \cap \partial\Theta_N$. Thus Δ_N is a nonsingular fan as in Proposition 1.19. Choose a generator u of the infinite cyclic group Γ_N^+. Then $N \cap \partial\Theta_N$ is obviously stable under the action of u on N, hence u induces an automorphism of the fan (N, Δ_N). As in Fig. 4.1, u induces on $\mathrm{Mc}(N, \Delta_N)$ a homeomorphism with O as the unique fixed point. Hence u acts on $\hat{C}_N$ properly discontinuously without fixed points. Since $\hat{C}_N$ is the quotient of $\tilde{U}_N$ with respect to the action of the compact real torus CT_N, we conclude that the action of u on $\tilde{U}_N$ has the same property as well. The rest of the assertion is obvious. q.e.d.

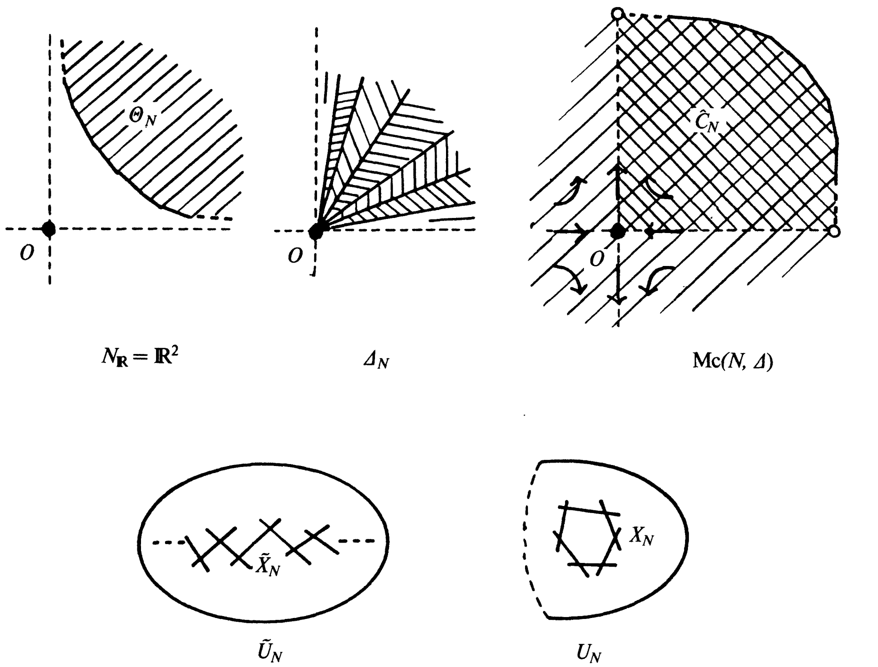


Fig. 4.1

Remark. Each subgroup $\Gamma \neq \{1\}$ of Γ_N^+ has finite index, and $\tilde{U}_N/\Gamma$ is again a two-dimensional complex manifold with a finite cycle $\tilde{X}_N/\Gamma$ of $\mathbb{P}_1(\mathbb{C})$'s.

Proposition 4.1 has the following interpretation in terms of purely periodic continued fractions:

A $\mathbb{Z}$-lattice N in K is easily seen to be expressible in the form $aN = \mathbb{Z} + \mathbb{Z}\omega$ for a reduced quadratic irrational number $\omega \in K$ and a *totally positive* $a \in K$ (i.e., $a > 0$ and $a' > 0$). Since the multiplication by a induces an isomorphism of $\mathbb{Z}$-modules from N to aN which sends Δ_N to Δ_{aN}, we may assume without loss of generality that $a = 1$, that is,

$$N = \mathbb{Z} + \mathbb{Z}\omega \quad \text{with} \quad \omega > 1 > \omega' > 0 \ .$$

Let p be the minimal period for the purely periodic continued fraction expansion

$$\omega = [[\overline{b_0, b_1, \ldots, b_{p-1}}]]$$

for ω. Hence the intermediate terms $\{\omega_\nu ; \nu \geq 0\}$ are defined inductively by $\omega_0 := \omega$ and $\omega_\nu = b_\nu - 1/\omega_{\nu+1}$, where b_ν for each ν is the unique integer such that $b_\nu - 1 < \omega_\nu < b_\nu$. In the present case, $b_{\nu+p} = b_\nu$ and $b_\nu \geq 2$ hold for all ν with not all b_ν equal to two.

Proposition 4.2. *For* $\omega = [[\overline{b_0, b_1, \ldots, b_{p-1}}]]$ *as above, consider the* $\mathbb{Z}$*-lattice* $N = \mathbb{Z} + \mathbb{Z}\omega$ *in the real quadratic field* $K = \mathbb{Q}(\omega)$. *Then* $u := 1/(\omega_1 \omega_2 \ldots \omega_p)$ *is a generator* *for* Γ_N^+. *Let*

$$n_j := 1/(\omega_1\omega_2\ldots\omega_j) \quad \textit{for} \quad 0<j\le p \;,$$
$$n_{kp+j} := u^k/(\omega_1\omega_2\ldots\omega_j) \quad \textit{for} \quad 0<j\le p \quad \textit{and} \quad k\in\mathbb{Z} \;,$$

so that $n_{\nu+p}=un_\nu$ *holds for every* $\nu\in\mathbb{Z}$. *Then* $\{n_\nu ; \nu\in\mathbb{Z}\}$ *coincides with* $N\cap\partial\Theta_N$ *arranged in the order of appearance, and we get*

$$\Delta_N=\{\{O\},\ \mathbb{R}_{\ge 0}n_\nu,\ \mathbb{R}_{\ge 0}n_\nu+\mathbb{R}_{\ge 0}n_{\nu+1}\ ;\ \nu\in\mathbb{Z}\}$$
$$n_{\nu-1}+n_{\nu+1}+(-b_\nu)n_\nu=O \quad \textit{for all} \quad \nu\in\mathbb{Z}.$$

Proof. We proceed by induction on ν. If we know that ω_ν is reduced, then $b_\nu\ge 2$ holds and $\omega_{\nu+1}$ determined by $\omega_\nu=b_\nu-1/\omega_{\nu+1}$ is again reduced. Moreover, ω_ν, 1, $1/\omega_{\nu+1}$ are three consecutive points which belong to $\mathbb{Z}+\mathbb{Z}\omega_\nu$ and which lie on the boundary of the convex hull of $C_N\cap(\mathbb{Z}+\mathbb{Z}\omega_\nu)$. We obviously have $\mathbb{Z}+\mathbb{Z}(1/\omega_\nu)$ $=(1/\omega_\nu)(\mathbb{Z}\omega_\nu+\mathbb{Z})=(1/\omega_\nu)(\mathbb{Z}+\mathbb{Z}(1/\omega_{\nu+1}))$.

Let $\zeta_{-1}:=1$ and $\zeta_\nu:=1/(\omega_0\omega_1\ldots\omega_\nu)$ for all $\nu\ge 0$. Then we have $\mathbb{Z}+\mathbb{Z}(1/\omega)=(1/\omega_0)\,(\mathbb{Z}+\mathbb{Z}(1/\omega_1))=(1/\omega_0\omega_1)\,(\mathbb{Z}+\mathbb{Z}(1/\omega_2))=\ldots=\mathbb{Z}\zeta_\nu+\mathbb{Z}\zeta_{\nu+1}$ for all $\nu\ge -1$, while $\zeta_{\nu-1}+\zeta_{\nu+1}=b_\nu\zeta_\nu$ for all $\nu\ge 0$. In particular, $\{\zeta_\nu ; \nu\ge -1\}$ consists of consecutive points which belong to $\mathbb{Z}+\mathbb{Z}(1/\omega)$ and which lie on the boundary of the convex hull of $C_N\cap(\mathbb{Z}+\mathbb{Z}(1/\omega))$.

Since $b_{\nu+p}=b_\nu$ and $\omega_{\nu+p}=\omega_\nu$ hold for all $\nu\ge 0$, we get

$$\zeta_{\nu+p}=u\zeta_\nu \quad \text{and} \quad n_\nu=\omega\zeta_\nu \quad \text{for all} \quad \nu\ge -1 \;.$$

We easily complete the proof, since $N=\mathbb{Z}+\mathbb{Z}\omega=\omega(\mathbb{Z}+\mathbb{Z}(1/\omega))$ and $\mathbb{Z}+\mathbb{Z}(1/\omega)$ $=\mathbb{Z}\zeta_{\nu+p}+\mathbb{Z}\zeta_{\nu+1+p}=u(\mathbb{Z}\zeta_\nu+\mathbb{Z}\zeta_{\nu+1})=u(\mathbb{Z}+\mathbb{Z}(1/\omega))$. q.e.d.

As in Proposition 1.19 and Sect. 2.2, we have:

Corollary 4.3. *For the* $\mathbb{Z}$*-lattice* $N=\mathbb{Z}+\mathbb{Z}\omega$ *in* $K=\mathbb{Q}(\omega)$ *with* $\omega=[[\overline{b_0,b_1,\ldots,b_{p-1}}]]$ *as above, consider the two-dimensional complex manifold* U_N *in Proposition 4.1. The finite cycle* X_N *of* $\mathbb{P}_1(\mathbb{C})$ *'s on* U_N *has a decomposition into irreducible components* $X_N=C_0+C_1+\ldots+C_{p-1}$ *such that*

$$(C_j^2)=-b_j \quad \textit{for} \quad j=0,1,\ldots,p-1$$

if $p\ge 2$, *while for* $p=1$ *we have*

$$(C_0^2)=-b_0+2 \;,$$

where C_0 *in this latter case is a rational curve with a node obtained from* $\mathbb{P}_1(\mathbb{C})$ *by the identification of* 0 *and* ∞.

In either of the two cases, the intersection matrix $((C_j\,.\,C_k))$ *is negative definite.*

Remark. We can also consider subgroups Γ of Γ_N^+ of finite index as in the remark immediately after Proposition 4.1. This amounts to considering integral multiples of the minimal period for the purely periodic continued fraction expansion for ω.

As we see more generally in the next section, we can contract the cycle X_N in U_N to an isolated singular point P_N of a two-dimensional complex analytic space V_N. In turn, U_N is the minimal resolution of singularities for V_N. Similar results hold more generally for $\tilde{U}_N/\Gamma$ corresponding to subgroups $\Gamma \subset \Gamma_N^+$ of finite index. The singular points obtained in this manner are called *two-dimensional Hilbert modular cusp singularities.*

In connection with the next section, let us reformulate what we have seen so far:

To a finite cycle $X = C_1 + C_2 + \ldots + C_s$ (with $s \geq 3$ for simplicity) of $\mathbb{P}_1(\mathbb{C})$'s on a two-dimensional complex manifold U, we associate a *weighted dual graph*, which is a triangulation of the circle S^1 with s vertices to which are attached, as weights, the self-intersection numbers $(C_1^2), (C_2^2), \ldots, (C_s^2)$ in this order. This is exactly the same as in the case of chains of $\mathbb{P}_1(\mathbb{C})$'s in Sect. 1.7.

Thus when $p \geq 3$, the weighted dual graph for the cycle X_N in U_N is the triangulation of S^1 with p vertices with weights $-b_0, -b_1, \ldots, -b_{p-1}$ in this order. The universal covering space for S^1 is $\tilde{S}^1 = \mathbb{R}$ and the fundamental group is $\pi_1(S^1) = p\mathbb{Z} \cong \Gamma_N^+$. Consider the triangulation of the real line $\mathbb{R}$ with $\mathbb{Z}$ as the set of vertices and attach the weight $-b_\nu$ to each vertex $\nu \in \mathbb{Z}$. This weighted linear graph, which can also be regarded as a weighted triangulation of $\tilde{S}^1$, is clearly stable under the action of $\pi_1(S^1) = p\mathbb{Z}$. Similarly, for subgroups $\Gamma \subset \pi_1(S^1)$ of finite index, we have a weighted triangulation of the quotient of $\tilde{S}^1$ with respect to Γ (cf. Fig. 4.2). This formulation admits higher dimensional analogues in the next section.

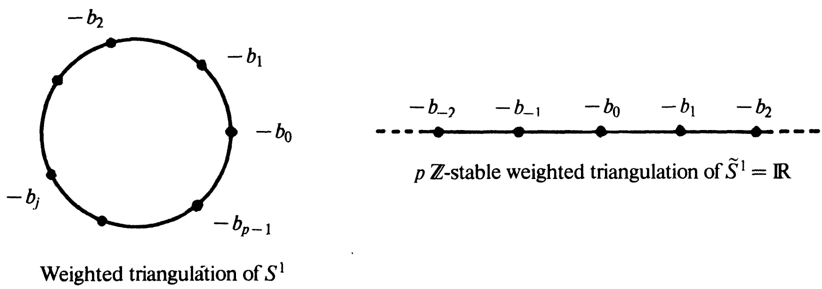


Fig. 4.2

Alternatively, the following (a) and (b) correspond to each other through the recursion formula $n_{\nu-1} + n_{\nu+1} + a_\nu n_\nu = O$:

(a) A finite or infinite sequence of integers $\{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\}$.

(b) A finite or infinite sequence of primitive elements $\{\ldots, n_{-2}, n_{-1}, n_0, n_1, n_2, \ldots\}$ in a $\mathbb{Z}$-module N such that each consecutive pair $\{n_{\nu-1}, n_\nu\}$ is a $\mathbb{Z}$-basis for N and that n_ν moves counterclockwise around the origin O as ν increases.

In an obvious manner, we obtain a two-dimensional nonsingular fan Δ in N, when the convex cones $\mathbb{R}_{\geq 0} n_{\nu-1} + \mathbb{R}_{\geq 0} n_\nu$ for increasing ν and decreasing ν never intersect in their interior.

(i) If the sequence is finite and two extreme primitive elements coincide, then Δ is a finite complete nonsingular fan appearing in Corollary 1.29.

(ii) If the sequence is finite and $|\Delta|$ is strongly convex, then Δ is a fan as in Proposition 1.19 and Lemma 1.20.

(iii) In Proposition 4.2 above, ν runs through all integers and we get $|\Delta| = \{O\} \cup C_N$. Note that C_N is a convex cone but is irrational with respect to N, since

$$C_N = \mathbb{R}_{>0}(n_{-1} - \omega' n_0) + \mathbb{R}_{>0}(-n_{-1} + \omega n_0) .$$

4.2 Cusp Singularities

In the previous section, we dealt with the two-dimensional Hilbert modular cusp singularity associated to a $\mathbb{Z}$-lattices N in a real quadratic field K. In doing so, we first constructed the minimal resolution of the singularity by means of a toric variety, and then contracted a finite cycle of $\mathbb{P}_1(\mathbb{C})$'s to the singular point. Moreover, the cycle of $\mathbb{P}_1(\mathbb{C})$'s could be described by the purely periodic continued fraction expansion of a reduced quadratic irrational number closely related to N, or alternatively, by a weighted triangulation of $\tilde{S}^1 = \mathbb{R}$ which is stable under the action of $\pi_1(S^1)$.

Tsuchihashi [T4], [T5], [T6] generalized this construction to higher dimension and constructed normal isolated singularities which are now called Tsuchihashi cusp singularities. Included among them are the cusp singularities which appear in the $\mathbb{Q}$-rank one case of the compactifications of the quotients of homogeneous bounded domains with respect to arithmetic subgroups. The so-called Hilbert modular cusp singularities are important such examples (see (2) at the beginning of this chapter).

In this section, we explain the construction of Tsuchihashi cusp singularities and describe their properties following Tsuchihashi [T4], [T5], [T6], Ogata [O1] and Ishida [I7].

In the rest of this section, we fix $N \cong \mathbb{Z}^r$ and denote by $GL(N) := \mathrm{Aut}_{\mathbb{Z}}(N)$ the group of automorphisms of N as a $\mathbb{Z}$-module. We denote by $SL(N)$ the subgroup of $GL(N)$ consisting of those with determinant one. These groups are naturally regarded as subgroups of the group $GL(N_{\mathbb{R}}) := \mathrm{Aut}_{\mathbb{R}}(N_{\mathbb{R}})$ of invertible $r \times r$ real matrices. As in Sect. 1.7, we identify the set

$$S_N := (N_{\mathbb{R}} \setminus \{O\}) / \mathbb{R}_{>0}$$

of rays from the origin O of $N_{\mathbb{R}} \cong \mathbb{R}^r$ with the $(r-1)$-sphere and denote by $\pi : N_{\mathbb{R}} \setminus \{O\} \twoheadrightarrow S_N$ the canonical projection. The following pairs (C, Γ) play key rôles in this section:

Definition. We denote by $\mathscr{P}(N)$ the family consisting of pairs (C, Γ) satisfying the following properties: C is a nonempty open convex cone in $N_{\mathbb{R}}$ with the apex at the origin O, and Γ is a subgroup of $GL(N)$ such that

(a) the closure $\bar{C}$ of C in $N_{\mathbb{R}}$ is strongly convex. i.e., $\bar{C} \cap (-\bar{C}) = \{O\}$;

(b) C is stable under the action of Γ on $N_{\mathbb{R}}$, and the induced action of Γ on $D := \pi(C) = C/\mathbb{R}_{>0}$ is properly discontinuous without fixed points and has compact quotient D/Γ.

Remark. An open convex cone C satisfying the condition (a) is said to be nondegenerate. The condition means that $\bar{C}$ does not contain any line in $N_{\mathbb{R}}$. By the condition (b), the quotient D/Γ is a compact real manifold of dimension $r-1$. It is orientable, if Γ is contained in $SL(N)$.

Here are some of the important examples of such a pair (C, Γ). For details, we refer the reader to [SC], [S6], [S7], [V4], [SO] etc.

For an open convex cone C satisfying (a), the connected open subset

$$\mathfrak{D} := N_{\mathbb{R}} + iC = \{x+iy \in N \otimes_{\mathbb{Z}} \mathbb{C};\ x \in N_{\mathbb{R}},\ y \in C\}$$

of $N \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^r$ is a tube domain called a *Siegel domain of the first kind* and is known to be biholomorphic to a bounded domain. The group $\mathrm{Aut}(\mathfrak{D})$ of analytic automorphisms of $\mathfrak{D}$ is a Lie group with respect to the compact open topology. The connected component G of the identity of $\mathrm{Aut}(\mathfrak{D})$ contains the real translation group $N_{\mathbb{R}}$ (with $n \in N_{\mathbb{R}}$ acting on $\mathfrak{D}$ by $x+iy \mapsto (x+n)+iy$) as well as the group

$$\mathrm{Aut}_{\mathbb{R}}(N_{\mathbb{R}};\ C) := \{g \in GL(N_{\mathbb{R}});\ gC = C\}$$

of linear automorphisms of $N_{\mathbb{R}}$ which preserve C (with g acting by $x+iy \mapsto g(x) + ig(y)$).

When $\mathrm{Aut}_{\mathbb{R}}(N_{\mathbb{R}};\ C)$ acts transitively on C, we call C a *homogeneous open convex cone*, and $\mathfrak{D}$ a *homogeneous Siegel domain of the first kind.* In this case, the subgroup of G generated by $N_{\mathbb{R}}$ and $\mathrm{Aut}_{\mathbb{R}}(N_{\mathbb{R}};\ C)$ (hence G itself) acts transitively on $\mathfrak{D}$ as a group of affine transformations of $N \otimes_{\mathbb{Z}} \mathbb{C}$.

Particularly important is the special case where C is self-dual with respect to a positive definite inner product of $N_{\mathbb{R}}$. In this case, $\mathfrak{D}$ is a *symmetric domain*, and G is a semisimple Lie group. An arithmetic subgroup $\mathfrak{A}$ of G, with respect to a structure on G of an algebraic group over $\mathbb{Q}$, acts properly discontinuously on $\mathfrak{D}$. When G has $\mathbb{Q}$-rank zero, the quotient $\mathfrak{D}/\mathfrak{A}$ is compact. If G has $\mathbb{Q}$-rank one and zero-dimensional rational boundary components, we can compactify $\mathfrak{D}/\mathfrak{A}$ by adding a finite number of points called cusps. If $\mathfrak{A}$ in this latter case is sufficiently small, a certain subgroup Γ of $\mathrm{Aut}_{\mathbb{R}}(N_{\mathbb{R}};\ C) \cap GL(N)$ arises for each cusp and satisfies the condition (b) above.

It might be an interesting problem to study possible differences under (a) between the homogeneity or the self-duality of C and the existence of Γ satisfying the rather strong condition (b). When $r=2$, for instance, C satisfying (b) is automatically homogeneous and self-dual, since we may regard N as a $\mathbb{Z}$-lattice in a real quadratic field K and can easily identify Γ (resp. C) with a subgroup of Γ_N^+ of finite index (resp. $C_N = (\mathbb{R}_{>0})^2 \subset \mathbb{R}^2 = N_{\mathbb{R}}$).

Examples. (i) The Hilbert modular case. Let K be a totally real algebraic number field of degree r over $\mathbb{Q}$. Using r distinct embeddings $K \hookrightarrow \mathbb{R}$, we have a canonical identification $K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^r$. For a $\mathbb{Z}$-lattice in K, i.e., a free $\mathbb{Z}$-submodule of K of rank r, we have $N \otimes_{\mathbb{Z}} \mathbb{Q} = K$, hence a canonical isomorphism

$$N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^r\ .$$

$C_N := (\mathbb{R}_{>0})^r$ is an open convex cone and satisfies the condition (a). The componentwise logarithm induces an isomorphism

$$D := \pi(C) = C/\mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{R}^r/\mathbb{R} \cong \mathbb{R}^{r-1} .$$

A *unit* of K is an invertible element in the ring $\mathfrak{o}$ of integers in K, whereas an element $\xi \in K$ is said to be *totally positive* if the images of ξ in $\mathbb{R}$ under the r distinct embeddings are all positive. The group

$$\Gamma_N^+ := \{\text{totally positive units } u \text{ in } K \text{ with } uN = N\}$$

under multiplication is a free commutative group of rank $r-1$ by Dirichlet's unit theorem, and has a canonical injective homomorphism $\Gamma_N^+ \hookrightarrow SL(N)$.

For each subgroup $\Gamma \subset \Gamma_N^+$ of finite index, the pair (C_N, Γ) satisfies the condition (b), hence belongs to our family $\mathscr{P}(N)$. In this case, D/Γ is homeomorphic to the real torus $\mathbb{R}^{r-1}/\mathbb{Z}^{r-1}$. The corresponding Siegel domain of the first kind is

$$\mathfrak{D} = \mathbb{R}^r + i(\mathbb{R}_{>0})^r = \mathfrak{H}_1 \times \ldots \times \mathfrak{H}_1 ,$$

which is the product of r copies of the usual upper half plane $\mathfrak{H}_1 = \mathbb{R} + i\mathbb{R}_{>0}$. We have dealt with the case $r=2$ in the previous section.

(ii) Let B be a central quaternion algebra over a totally real algebraic number field F of degree s over $\mathbb{Q}$ such that the scalar extension $B \otimes_F \mathbb{R}$ with respect to each of the s distinct embeddings $F \hookrightarrow \mathbb{R}$ is isomorphic to the algebra $\mathfrak{M}_2(\mathbb{R})$ of 2×2 real matrices. Real symmetric matrices in $\mathfrak{M}_2(\mathbb{R})$ form a three-dimensional subspace $\mathfrak{S}_2(\mathbb{R})$, in which lies the open convex cone $\mathfrak{P}_2(\mathbb{R})$ of positive definite symmetric matrices. Without mentioning anything as to the choice of N and Γ in this case, we only point out that the corresponding Siegel domain of the first kind is

$$\mathfrak{D} = (\mathfrak{S}_2(\mathbb{R}) \times \ldots \times \mathfrak{S}_2(\mathbb{R})) + i(\mathfrak{P}_2(\mathbb{R}) \times \ldots \times \mathfrak{P}_2(\mathbb{R})) = \mathfrak{H}_2 \times \ldots \times \mathfrak{H}_2 ,$$

which is the product of s copies of the Siegel upper half plane $\mathfrak{H}_2 = \mathfrak{S}_2(\mathbb{R}) + i\mathfrak{P}_2(\mathbb{R})$ of degree two.

Let us return to a general pair $(C, \Gamma) \in \mathscr{P}(N)$ for $N \cong \mathbb{Z}^r$ satisfying the conditions (a) and (b).

The $\mathbb{Z}$-module M dual to N admits the contragredient action of $GL(N)$ so that the $\mathbb{Z}$-bilinear duality pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$ satisfies $\langle g(m), g(n)\rangle = \langle m, n\rangle$ for all $g \in GL(N)$, $m \in M$ and $n \in N$. We define the *open dual cone* of C by

$$C' := \{m \in M_{\mathbb{R}};\ \langle m, n\rangle > 0 \quad \text{for} \quad \forall n \in \bar{C} \setminus \{O\}\}$$

as in Proposition A.10. C' coincides with the interior of the closed dual cone $\bar{C}^\vee$ of $\bar{C}$, since C is the interior of $\bar{C}$. We can regard Γ as a subgroup of $GL(M)$ through the contragredient action of Γ on M. Then C' is clearly Γ-stable and the new pair (C', Γ) satisfies the conditions (a) and (b), that is, it belongs to $\mathscr{P}(M)$ (cf. [T5, Lemma 1.6]). Since $(C')' = C$, we have a *duality* between $\mathscr{P}(N)$ and $\mathscr{P}(M)$.

The convex hull Θ of $N \cap C$ has the *support function* $h : \bar{C}' \to \mathbb{R}_{\geq 0}$ defined as in Sect. A.3 by

$$h(m) := \inf\{\langle m, n\rangle;\ n \in N \cap C\} \quad \text{for} \quad m \in \bar{C}' .$$

h is a positively homogeneous and upper convex continuous function taking nonnegative integer values on $M \cap \bar{C}'$. Furthermore, we have

$$\Theta = \{n \in C;\ \langle m, n\rangle \geq h(n) \quad \text{for} \quad \forall m \in C'\}\ .$$

As [T5, Lemmas 1.1 through 1.4] shows, the proper faces of Θ are convex polytopes with vertices in N and give rise to a *Γ-stable convex polytope decomposition* $\square$ of the boundary $\partial\Theta$ of Θ. The restriction onto $\partial\Theta$ of the projection $\pi : N_{\mathbb{R}} \setminus \{O\} \twoheadrightarrow S_N$ induces a homeomorphism

$$\pi : \partial\Theta \xrightarrow{\sim} D = \pi(C) = C/\mathbb{R}_{>0}\ ,$$

which gives rise to a Γ-stable compact spherical cell decomposition $\pi(\square)$ of D.

Define convex subsets Θ' and Θ° of C' as follows: Θ' is the convex hull of $M \cap C'$, while the *polar* Θ° of Θ is

$$\Theta^\circ := \{m \in C';\ h(m) \geq 1\} = \{m \in C';\ \langle m, n\rangle \geq 1 \quad \text{for} \quad \forall n \in N \cap C\}\ .$$

Since h has positive integer values on $M \cap C'$, we obviously have $\Theta' \subset \Theta^\circ$, but the equality may fail to hold for $r \geq 3$. The polar $(\Theta^\circ)^\circ$ of Θ° coincides with the original Θ. The boundaries $\partial\Theta'$ and $\partial\Theta^\circ$ have Γ-stable convex polytope decompositions $\square'$ and $\square^\circ$ defined in a way similar to $\square$ above. The relationship between $\square$ and $\square'$ is not clear in general. Between $\square$ and $\square^\circ$, however, we have the following Galois correspondence called the *polarity:* If a face $\alpha \in \square$ of $\partial\Theta$ has the vertex set $\{n_1, \ldots, n_s\}$, then

$$\alpha^\dagger := \{m \in \partial\Theta^\circ;\ \langle m, n_1\rangle = \ldots = \langle m, n_s\rangle = 1\}$$

belongs to $\square^\circ$ and satisfies $\dim\alpha + \dim\alpha^\dagger = r - 1$. If α is a face of $\beta \in \square$, then $\beta^\dagger$ is a face of $\alpha^\dagger$.

The Γ-stable convex polytope decomposition $\square$ of $\partial\Theta$ induces an infinite fan

$$\Sigma := \{\mathbb{R}_{\geq 0}\alpha;\ \alpha \in \square\} \cup \{\{O\}\}$$

in N which may fail to be nonsingular, where $\mathbb{R}_{\geq 0}\alpha := \{cn;\ n \in \alpha \text{ and } c \in \mathbb{R}_{\geq 0}\}$ is the convex polyhedral cone obtained as the union of rays from the origin O to the points of α. Obviously, Σ is Γ-stable and satisfies $|\Sigma| = \{O\} \cup C$.

Regard $N_{\mathbb{R}}$ as an open set of the real manifold with corners $\mathrm{Mc}(N, \Sigma)$ introduced in Sect. 1.3, and denote by

$$\hat{C} := C \cup \{\mathrm{Mc}(N, \Sigma) \setminus N_{\mathbb{R}}\}$$

the interior of the closure of C in $\mathrm{Mc}(N, \Sigma)$. The canonical projection $\mathrm{ord} : T_N\mathrm{emb}(\Sigma) \twoheadrightarrow \mathrm{Mc}(N, \Sigma)$ enables us to define

$$\tilde{U} := \mathrm{ord}^{-1}(\hat{C}) \supset \tilde{X} := \tilde{U} \setminus (T_N \cap \tilde{U}) = T_N\mathrm{emb}(\Sigma) \setminus T_N.$$

$\tilde{U}$ is a Γ-stable open set of $T_N\mathrm{emb}(\Sigma)$ containing $\tilde{X}$ as a closed analytic subspace of codimension one. The induced action of Γ on $\tilde{U}$ is properly discontinuous and without fixed points, since so is the action of Γ on $\hat{C}$ by the condition (b). Let

$$U := \tilde{U}/\Gamma \supset X := \tilde{X}/\Gamma.$$

U is a complex analytic space containing X as a *compact* analytic subspace of codimension one. X is compact, since so is D/Γ by assumption so that the set of Γ-equivalence classes in $\square$ is finite.

Although U may fail to be a complex manifold in general, Corollary 3.9 applied to $\tilde{U}$ guarantees that U has at worst *rational singularities* and the Grauert-Riemenschneider type vanishing theorem holds for U (cf. [T5, Lemma 2.5]) as we mention again later in this section. We now contract X in U to a Tsuchihashi cusp singularity.

Proposition 4.4 (Tsuchihashi [T5, Proposition 1.7]). *Let (C,Γ) be a pair in $\mathscr{P}(N)$ with $N\cong\mathbb{Z}^r$. The Γ-stable convex polytope decomposition $\square$ of the boundary $\partial\Theta$ of the convex hull Θ of $N\cap C$ induces a Γ-stable fan (N,Σ). We can contract the compact analytic subspace X of codimension one in the r-dimensional complex analytic space U above to a normal isolated singularity $P\in V$, that is, we have a surjective holomorphic map $\psi: U\twoheadrightarrow V$ such that $\psi(X)=P$ and that ψ induces a biholomorphic map $U\setminus X\xrightarrow{\sim}V\setminus\{P\}$. This (V,P) is determined by (C,Γ) uniquely up to isomorphism, and* $\mathrm{Cusp}(C,\Gamma):=(V,P)$ *is called the* Tsuchihashi cusp singularity *associated to (C,Γ).*

Proof. Since $\Gamma\cap SL(N)$ has index one or two in Γ, it suffices to consider only the case $\Gamma\subset SL(N)$ in view of [SC, Chap. II, § 1 and Chap. III, § 1].

As in Proposition A.10, define the *characteristic function* $\varphi: C\to\mathbb{R}_{>0}$ for C by

$$\varphi(n)=\int_{C'}\exp(-\langle m,n\rangle)dm \quad \text{for} \quad n\in C ,$$

where dm is a Lebesgue measure on $M_{\mathbb{R}}$. φ is Γ-invariant and strictly (lower) convex, and can be extended to a continuous function $\hat{\varphi}:\hat{C}\to\mathbb{R}_{\geq 0}$ vanishig on $\hat{C}\setminus C$. The composite function $\hat{\varphi}\circ\mathrm{ord}:\tilde{U}\to\hat{C}\to\mathbb{R}_{\geq 0}$ is a Γ-invariant C^2-function which vanishes only on $\tilde{X}$ and which is *strictly Levi-convex* on $\tilde{U}\setminus\tilde{X}$. Consequently, it induces a C^2-function on U which vanishes only on X and which is strictly Levi-convex on $U\setminus X$. Since $U\setminus X$ is a manifold, X is well-known to be uniquely contractible to a normal isolated singular point. q.e.d.

Remark. Corresponding to the pair $(C',\Gamma)\in\mathscr{P}(M)$ dual to (C,Γ), we obtain the *dual Tsuchihashi cusp singularity* $\mathrm{Cusp}(C',\Gamma)$. This duality is a generalization of that considered by Nakamura [N3], [N4] in dimension two. It comes up again in the next section in connection with hyperbolic Inoue surfaces.

Remark. We can describe the completion $\hat{\mathcal{O}}$ of the local ring $\mathcal{O}$ of a Tsuchihashi cusp singularity $P\in V$ as follows: As in Sect. 1.2, regard the family $\{\mathbf{e}(m); m\in M\}$ of holomorphic functions on T_N as that of Laurent monomials satisfying

$$\mathbf{e}(m+m')=\mathbf{e}(m)\mathbf{e}(m') \quad \text{for} \quad m,m'\in M .$$

Then the set $\mathbb{C}[[\{O\}\cup(M\cap C')]]$ of formal infinite $\mathbb{C}$-linear combinations of $\{\mathbf{e}(m); m\in\{O\}\cup(M\cap C')\}$ has a naturally well-defined multiplication and becomes a $\mathbb{C}$-algebra. Γ acts on this $\mathbb{C}$-algebra, since $M\cap C'$ is stable under the contragredient action of Γ on M. The completion $\hat{\mathcal{O}}$ can be identified with the $\mathbb{C}$-subalgebra of Γ-invariants, i.e.,

$$\hat{\mathcal{O}} = \mathbb{C}[[\{O\} \cup (M \cap C')]]^{\Gamma} .$$

As in [T5, § 2], we can use this fact to determine explicit equations defining V in a neighborhood of P, at least in principle when $\Theta' = \Theta^{\circ}$ holds.

We now mention some of the properties enjoyed by the Tsuchihashi cusp singularities.

In general, let (V, P) be an r-dimensional normal isolated singularity. Denote by ω the sheaf of germs of holomorphic r-forms on the complex manifold $V \setminus \{P\}$. For a positive integer ν, a section $\eta \in H^0(V \setminus \{P\}, \omega^{\otimes \nu})$ is said to be *locally $L^{2/\nu}$-integrable* if there exists a sufficiently small compact neighborhood Z of P such that the integral $\int (\eta \wedge \bar{\eta})^{1/\nu}$ over $Z \setminus \{P\}$ is finite. Let us denote by $L^{2/\nu}(V, P)$ the subspace consisting of such forms. Watanabe [W1] defines the *ν-genus* of (V, P) to be the codimension δ_ν of $L^{2/\nu}(V, P)$ in $H^0(V \setminus \{P\}, \omega^{\otimes \nu})$. This invariant, defined independently of any resolution of the singularity, plays an important rôle in the study of (V, P) (see Watanabe [W1], [W2]). If a resolution $\tilde{V} \to V$ of the singularity is given, then $L^{2/\nu}(V, P)$ turns out to coincide with $H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(\nu\tilde{K} + (\nu - 1)E))$ by a result of Fumio Sakai, where $\tilde{K}$ is a canonical divisor of $\tilde{V}$ and E is the exceptional divisor on $\tilde{V}$ with respect to $\tilde{V} \to V$. A normal isolated singularity (V, P) is said to be *purely elliptic* if

$$\delta_\nu = 1 \quad \text{for every positive integer } \nu .$$

Proposition 4.5 (Tsuchihashi [T5, Proposition 2.2]). *The Tsuchihashi cusp singularity* Cusp(C, Γ) *is a purely elliptic singularity if Γ is in $SL(N)$, while for $\Gamma \not\subset SL(N)$, the ν-genus for* Cusp(C, Γ) *satisfies $\delta_\nu = 1$ for even ν and $\delta_\nu = 0$ for odd ν.*

Proof. Denote $(V, P) := \mathrm{Cusp}(C, \Gamma)$. In the notation of Proposition 4.4, we have $\psi : U \setminus X \xrightarrow{\sim} V \setminus \{P\}$ and $U = \tilde{U}/\Gamma \supset X = \tilde{X}/\Gamma$. By Corollary 3.9 and Theorem 3.6, (2), the dualizing $\mathcal{O}_{\tilde{U}}$-module for $\tilde{U}$ is isomorphic to $\mathcal{O}_{\tilde{U}}(-\tilde{X})$.

For a $\mathbb{Z}$-basis $\{m_1, \ldots, m_r\}$ of M, the holomorphic r-form $\tilde{\eta} := (d\mathbf{e}(m_1)/\mathbf{e}(m_1)) \wedge \ldots \wedge (d\mathbf{e}(m_r)/\mathbf{e}(m_r))$ on T_N vanishes nowhere on $\tilde{U} \setminus \tilde{X}$. We easily see that $(\tilde{\eta})^\nu$ is Γ-invariant either when Γ is in $SL(N)$ and ν arbitrary or when ν is even and Γ arbitrary. In either case, $(\tilde{\eta})^\nu$ gives rise to a holomorphic section η^ν of $\omega^{\otimes \nu}$ on $U \setminus X \xrightarrow{\sim} V \setminus \{P\}$ which is not locally $L^{2/\nu}$-integrable, but $\varphi\eta^\nu$ is locally $L^{2/\nu}$-integrable for every holomorphic function φ on V which vanishes at P. Thus we have $\delta_\nu = 1$ in both cases. By taking a double covering of V, we easily see that $\delta_\nu = 0$ for ν odd. q.e.d.

To state other properties enjoyed by $(V, P) :=$ Cusp(C, Γ), we need the following explicit construction of a resolution of its singularity:

Recall that we constructed $\psi : U \twoheadrightarrow V$ as the map which contracts the compact analytic subspace $X = \tilde{X}/\Gamma$ of $U = \tilde{U}/\Gamma$ to a point P. Since U has at worst rational singularities, we may call $\psi : (U, X) \to (V, P)$ a "*rationalization of singularities*". $\tilde{U}$ is an open set of a toric variety $T_N\mathrm{emb}(\Sigma)$ and $\tilde{X} = \tilde{U} \setminus (T_N \cap \tilde{U})$. As we mentioned in

Sect. 1.5, [TE, Chap. I, Theorem 11], [N8, Theorem 7.20] and [S6, Appendix] guarantee the existence of a nonsingular, Γ-stable and locally finite subdivision (N, Λ) of the fan (N, Σ). The corresponding equivariant holomorphic map

$$\tilde{\Psi}: T_N\mathrm{emb}(\Lambda) \to T_N\mathrm{emb}(\Sigma)$$

is a resolution of singularities equivariant with respect to the actions of T_N and Γ. $\tilde{W} := \tilde{\Psi}^{-1}(\tilde{U})$ is a Γ-stable nonsingular open set, while $\tilde{Y} := \tilde{\Psi}^{-1}(\tilde{X}) = \tilde{W} \setminus (\tilde{W} \cap T_N)$ is a Γ-stable closed analytic subspace of codimension one which has simple normal crossings only. Since Γ obviously acts on $\tilde{W}$ properly discontinuously without fixed points, we see that $Y := \tilde{Y}/\Gamma$ is a compact reduced analytic subspace of codimension one in the r-dimensional complex manifold $W := \tilde{W}/\Gamma$. We can choose Λ so fine that Y itself has simple normal crossings only. Consequently, each irreducible component of Y is isomorphic to an irreducible component of $\tilde{Y}$ and is a compact nonsingular toric variety of dimension $r-1$. The intersections of these irreducible components are toric varieties of lower dimensions as well.

The holomorphic map $\Psi: W \to U$ induced by $\tilde{\Psi}: \tilde{W} \to \tilde{U}$ is a resolution of the rational singularities, so that the composite

$$f := \psi \circ \Psi: W \to U \to V$$

is a resolution of the singularity of V. A Grauert-Riemenschneider type vanishing theorem similar to that in Corollary 3.9 holds for this $\Psi: W \to U$. Since $\Psi^{-1}(X) = Y$ and $\psi^{-1}(P) = X$, we see that $f^{-1}(P) = Y$. Hence $f: W \to V$ is the contraction of the compact reduced analytic subspace Y in W of codimension one to a point P. When $r=2$, Δ_N in the previous section is the coarsest such Λ and $\tilde{U}_N = \tilde{W}$, $\tilde{X}_N = \tilde{Y}$.

For an r-dimensional complex analytic space V, denote by $\omega_V^{\cdot}$ the globally normalized dualizing complex for V in the sense of Sect. 3.2. Thus its cohomology sheaves satisfy

$$\mathscr{H}^j(\omega_V^{\cdot}) = 0 \quad \text{if} \quad j < -r \quad \text{or} \quad 0 < j \ .$$

The truncation of $\omega_V^{\cdot}$ in degrees $\leq -r$ is defined to be

$$\tau_{-r}(\omega_V^{\cdot}) := (\ldots \to 0 \to \mathrm{Image}\,(\omega_V^{-r} \to \omega_V^{-r+1}) \to \omega_V^{-r+1} \to \omega_V^{-r+2} \to \ldots) \ .$$

Denote by $\mathcal{O}_{V,P}$ the local ring of V at a point $P \in V$ with the maximal ideal $\mathfrak{m}_{V,P}$.

A point $P \in V$ is called a *Buchsbaum singularity* (or, $\mathcal{O}_{V,P}$ is a *Buchsbaum ring*), if the action of $\mathfrak{m}_{V,P}$ on the stalk $\tau_{-r}(\omega_{V,P}^{\cdot})$ of $\tau_{-r}(\omega_V^{\cdot})$ at P vanishes. Namely, $\tau_{-r}(\omega_{V,P}^{\cdot})$ is a complex of vector spaces over the residue field $\mathbb{C}(P) := \mathcal{O}_{V,P}/\mathfrak{m}_{V,P}$. As can be found in Schenzel [S9], for instance, Buchsbaum rings were originally defined in the context of commutative algebra as follows: Let J be the ideal in $\mathcal{O}_{V,P}$ generated by a system of parameters $\{x_1, \ldots, x_r\}$ for $\mathfrak{m}_{V,P}$. Then the difference between the multiplicity of J in $\mathcal{O}_{V,P}$ and the length of $\mathcal{O}_{V,P}/J$ is independent of the choice of any particular system of parameters.

$\mathcal{O}_{V,P}$ is a Cohen-Macaulay ring if and only if the difference is zero, or equivalently $\mathscr{H}^j(\omega_{V,P}^{\cdot}) = 0$ for $j \neq -r$. Hence a Cohen-Macaulay ring is a Buchsbaum ring.

If $\mathcal{O}_{V,P}$ is a Buchsbaum ring, then $\mathscr{H}^j(\omega_{V,P}^{\cdot})$ for $j \neq -r$ is a $\mathbb{C}(P)$-vector space. $\mathcal{O}_{V,P}$ satisfying this latter property is sometimes called a *quasi-Buchsbaum ring*.

There is another notion of *normal isolated du Bois singularities*, which Ishida [I6] showed to be Buchsbaum singularities in the same manner as in the theorem below.

The Tsuchihashi cusp singularities are not Cohen-Macaulay singularities for $r \geq 3$, but are always Buchsbaum singularities as the following theorem guarantees. We omit the proof here, which depends heavily on the duality theorem in the derived category.

The Buchsbaum Singularity Theorem (Ishida [I4, Theorems 3.5 and 2.2, Proposition 2.1], Tsuchihashi [T5, Theorem 2.3, Corollary 2.4]).

(1) *The r-dimensional Tsuchihashi cusp singularity* $(V, P) := \mathrm{Cusp}(C, \Gamma)$ *corresponding to a pair* (C, Γ) *is a Buchsbaum singularity.*

(2) *For a resolution of the singularity* $f: (W, Y) \to (V, P)$ *considered above, we have*

$$R^j f_* \mathcal{O}_W = \begin{cases} \mathcal{O}_V & \text{for } j=0 \\ H^j(D/\Gamma, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}(P) & \text{for } j>0 \end{cases}$$

$$\mathbb{R}f_* \mathcal{O}_W(-Y) = \mathfrak{m}_{V,P} \ .$$

Here, $\mathbb{C}(P)$ *is the residue field of the local ring* $\mathcal{O}_{V,P}$ *with the maximal ideal* $\mathfrak{m}_{V,P}$ *and* $H^j(D/\Gamma, \mathbb{C})$ *is the usual cohomology group with coefficients in* $\mathbb{C}$ *of the* $(r-1)$*-dimensional compact real manifold* D/Γ *obtained as the quotient of* $D = C/\mathbb{R}_{>0}$ *with respect to the induced action of* Γ*. Furthermore,* $\mathbb{R}f_* : \mathbb{D}_c^+(\mathcal{O}_W) \to \mathbb{D}_c^+(\mathcal{O}_V)$ *is the direct image functor for the derived categories.*

In Proposition 4.4, we constructed the fan (N, Σ) starting from the Γ-stable convex polytope decomposition $\square$ of $\partial\Theta$. By construction, we had $\tilde{X} = T_N\mathrm{emb}(\Sigma) \setminus T_N$. Hence by Proposition 1.6 and Corollary 1.7, we have a bijection from $\square$ to the set of T_N-stable closed irreducible analytic subspaces of $\tilde{X}$ sending $\alpha \in \square$ to $V(\mathbb{R}_{\geq 0}\alpha)$. If β is a face of α, then $V(\mathbb{R}_{\geq 0}\alpha)$ is contained in $V(\mathbb{R}_{\geq 0}\beta)$. On the other hand, the homeomorphism $\pi: \partial\Theta \xrightarrow{\sim} D = C/\mathbb{R}_{>0}$ gives rise to a Γ-stable spherical cell decomposition $\pi(\square)$ of D. Since $\mathbb{R}_{\geq 0}\alpha = \{O\} \cup \pi^{-1}(\pi(\alpha))$, we have a bijection

$$\pi(\square) \xrightarrow{\sim} \{T_N\text{-stable closed irreducible analytic subspaces of } \tilde{X}\}$$

by sending $\pi(\alpha)$ to $V(\{O\} \cup \pi^{-1}(\pi(\alpha)))$.

Similar consideration applies to a locally finite, Γ-stable and nonsingular subdivision Λ of Σ which we used for a resolution of the singularity. We again have $\tilde{Y} = T_N\mathrm{emb}(\Lambda) \setminus T_N$, hence we have a bijection from $\Lambda \setminus \{\{O\}\}$ to the set of T_N-stable closed irreducible analytic subspaces of $\tilde{Y}$ by sending $\sigma \neq \{O\}$ in Λ to $V(\sigma)$. On the other hand, $\pi(\sigma \setminus \{O\})$ for $\Lambda \ni \sigma \neq \{O\}$ is a spherical simplex on D and

$$\Delta := \{\pi(\sigma \setminus \{O\}); \sigma \in \Lambda, \sigma \neq \{O\}\}$$

is a Γ-stable triangulation for D. Consequently, we get a bijection

$$\Delta \xrightarrow{\sim} \{T_N\text{-stable closed irreducible analytic subspaces of } \tilde{Y}\}$$

sending $\pi(\sigma\setminus\{O\})$ to $V(\sigma)$. We have

$$\dim \pi(\sigma\setminus\{O\})+\dim V(\sigma)=r-1 \ .$$

Moreover, $V(\sigma)$ is contained in $V(\tau)$ if $\pi(\tau\setminus\{O\})$ is a face of $\pi(\sigma\setminus\{O\})$.

If Λ is sufficiently fine, then Δ induces a triangulation Δ/Γ of the $(r-1)$-dimensional real manifold D/Γ. Moreover, the projection $\tilde{Y}\twoheadrightarrow Y=\tilde{Y}/\Gamma$ sends each T_N-stable closed irreducible analytic subspace of $\tilde{Y}$ biholomorphically to its image. Y is a reduced effective divisor on the r-dimensional complex manifold $W=\tilde{W}/\Gamma$ and has simple normal crossings only. The triangulation Δ/Γ above is a *dual graph* for Y and combinatorially describes how the irreducible components of Y intersect one another:

Denote by $J:=\{1,2,\ldots,v\}$ the set of vertices (0-simplices) in the triangulation Δ/Γ. Let us identify each simplex ξ in Δ/Γ with the subset of J consisting of the vertices of ξ. To each $j\in J$ corresponds an irreducible component Y_j of Y so that $Y=\sum_{j\in J}Y_j$. Since Y_j is biholomorphic to an irreducible component of $\tilde{Y}$, we see that Y_j is a compact nonsingular toric variety of dimension $r-1$ by Corollary 1.7. Furthermore, for each nonempty subset ξ of J,

$$Y_\xi:=\bigcap_{j\in\xi}Y_j$$

is a compact nonsingular toric variety if it is not empty. Let

$$Y_\emptyset:=W \quad\text{and}\quad \Xi:=\{\xi\subset J;\ Y_\xi\neq\emptyset\} \ .$$

Then we can identify Δ/Γ with $\Xi\setminus\{\emptyset\}$. For each $j\in J$,

$$D_j:=\sum_{k\neq j}Y_j\cap Y_k$$

is a reduced effective divisor on Y_j with simple normal crossings. Besides, $Y_j\setminus\operatorname{supp}(D_j)$ coincides with the $(r-1)$-dimensional algebraic torus in Y_j regarded as a toric variety.

We are thus led to the following notion:

Definition. A reduced effective divisor $Y=\sum_{1\le j\le v}Y_j$ with simple normal crossings only on an r-dimesional complex manifold W is said to be a *toric divisor* (without self-intersection), if the following conditions are satisfied: Y_j for each $j\in J:=\{1,2,\ldots,v\}$ is an $(r-1)$-dimensional compact nonsingular toric variety with $Y_j\setminus\operatorname{supp}(D_j)$ coinciding with the $(r-1)$-dimensional algebraic torus in Y_j, where $D_j:=\sum_{k\neq j}Y_j\cap Y_k$. Moreover, for any subset $\xi\subset J$, the intersection $\cap_{j\in\xi}Y_j$ is either empty or irreducible.

D_j above is a reduced effective divisor on Y_j with simple normal crossing only. If we let $Y_\emptyset:=W$ and

$$\Xi:=\{\xi\subset J;\ Y_\xi:=\bigcap_{j\in\xi}Y_j\neq\emptyset\} \ ,$$

then Y_ξ for $\xi\neq\emptyset$ is a compact nonsingular toric variety of dimension equal to $r-{}^{\#}\xi$ containing a reduced effective divisor $D_\xi:=\sum_{k\notin\xi}Y_\xi\cap Y_k$ with simple normal crossings only such that $Y_\xi\setminus\operatorname{supp}(D_\xi)$ coincides with the algebraic torus in Y_ξ.

We now explain results due to Ogata in his master's thesis at Tohoku University (1984) and in [O1]. These are variants, in the case of Tsuchihashi cusp singularities, of results obtained for symmetric domains by Satake [S5], [S6], [S7]. Ehlers [E2] and Hirzebruch-van der Geer [HV] dealt with the special case of Hilbert modular cusp singularities. Ogata's proof uses results found in Chap. 3. Ishida [I7] recently introduced an abstract notion of "T-complexes" which describe toric divisors just as fans describe toric varieties. In this general set-up, he generalized Ogata's result on the zeta zero values and could prove their rationality. For Tsuchihashi cusp singularities, the T-complexes give essentially the same information as the weighted dual graphs mentioned at the end of this section.

Recall that a Tsuchihashi cusp singularity $\mathrm{Cusp}(C,\Gamma)$ is obtained by contraction to one point of a toric divisor $Y=\tilde{Y}/\Gamma$ on an r-dimensional complex manifold $W=\tilde{W}/\Gamma$ for a fine enough subdivision Λ we considered earlier.

Proposition 4.6. *For $(C,\Gamma)\in\mathscr{P}(N)$, consider a Γ-stable nonsingular subdivision Λ which is sufficiently fine.*

(1) (Satake) *Denote by $\delta_j\in H_c^2(W,\mathbb{Z})$ the cohomology class of each irreducible component Y_j of Y in the cohomology group of W with compact support. Then the rational number*

$$\mathrm{inv}(C,\Gamma):=\kappa_r\left[\prod_{j\in J}\frac{\delta_j}{1-e^{-\delta_j}}\right]$$

is an invariant for (C,Γ) and is independent of any particular choice of the Γ-stable nonsingular subdivision Λ. Here $\kappa_r: H_c^{\cdot}(W,\mathbb{Q})\to\mathbb{Q}$ sends the degree $2r$ component of each element to the image under the canonical isomorphism $H_c^{2r}(W,\mathbb{Q})\xrightarrow{\sim}\mathbb{Q}$.

(2) (Ogata) *For $\xi\neq\emptyset$ in Ξ, let $\tau(Y_\xi)$ be the index of the toric variety $Y_\xi:=\bigcap_{j\in\xi}Y_j$ so that it vanishes when $^{\#}\xi\not\equiv r$* (mod 2) (cf. Theorem 3.12, (3)). *Then*

$$2^r\cdot\mathrm{inv}(C,\Gamma)=\kappa_r\left[\prod_{j\in J}\frac{\delta_j(1+e^{-2\delta_j})}{(1-e^{-2\delta_j})}\right]+\sum_{\xi\neq\emptyset}\tau(Y_\xi)\ .$$

(3) (Satake) *Denote $\Xi(l):=\{\xi\in\Xi;\ ^{\#}\xi=l\}$ for $0\leq l\leq r$. Then the folllowing equalities hold, where the equality of the second and the third terms is nontrivial:*

$$\begin{aligned}\sum_{\xi\neq\emptyset}\tau(Y_\xi)&=(-1)^{r+1}\sum_{l=1}^{r}(-2)^{r-l}\,{}^{\#}\Xi(l)\\&=2^r\sum_{l=1}^{r}(-1)^{r-l}\,{}^{\#}\Xi(l)-\sum_{l=1}^{r}(-2)^{r-l}\,{}^{\#}\Xi(l)\ .\end{aligned}$$

If r is odd, *then the first term on the right hand side of* (2) *vanishes and*

$$\begin{aligned}\sum_{\xi\neq\emptyset}\tau(Y_\xi)&=2^{r-1}\sum_{l=1}^{r}(-1)^{r-l}\,{}^{\#}\Xi(l)\\&=2^{r-1}\ (\text{the Euler number of } D/\Gamma)\end{aligned}$$

so that $\mathrm{inv}(C,\Gamma)=(1/2)$ *(the Euler number of D/Γ).*

(4) (Ogata and Ishida) *Let* $\varphi_C: C \to \mathbb{R}_{>0}$ *be the characteristic function for* C *as in Theorem A.10. Then the* zeta function *for* (C, Γ) *is defined by*

$$\zeta(C, \Gamma; s) := \sum_{n \in (N \cap C)/\Gamma} \varphi_C(n)^s \quad \text{for} \quad s \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}(s) > 1 \ .$$

It is absolutely convergent for $\mathrm{Re}(s) > 1$ *and can be continued to a meromorphic function on the whole complex plane. It is holomorphic at* $s = 0$ *and has a rational value. When* r *is odd or* $r = 2$, *we have*

$$\zeta(C, \Gamma; 0) = -\mathrm{inv}(C, \Gamma) \ .$$

A Sketch of the Proof by Ogata. (1) We follow Ehlers [E2]. $\Theta_W(-\log Y)$ has a holomorphic integrable connection, hence its total Chern class is trivial. Indeed, $W = \tilde{W}/\Gamma$ is the quotient of $\tilde{W}$ on which Γ acts properly discontinuously without fixed points. By Proposition 3.1, $\Theta_{\tilde{W}}(-\log \tilde{Y})$ is a free module over $\mathcal{O}_{\tilde{W}}$ having an action of Γ through constant matrices. $\Theta_W(-\log Y)$ is the subsheaf of germs of Γ-invariant sections in its direct image under $\tilde{W} \to W$ so that it has a holomorphic integrable connection.

This latter fact and an exact sequence similar to that in Theorem 3.12, (1) imply, as in Theorem 3.12, (2), that the total Chern class of W in $H_c^{\cdot}(W, \mathbb{Z})$ is given by

$$c(W) = \prod_{j \in J} (1 + \delta_j) \ .$$

The r-th degree *Todd polynomial* for this $c(W)$ is nothing but the right hand side of (1). As for its independence of the choice of Λ, we may assume $(V, P) := \mathrm{Cusp}(C, \Gamma)$ to be a neighborhood of a unique singular point of a projective variety Z', in view of Artin's approximation theorem and Hironaka's theorem on the resolution of singularities. On the other hand, (V, P) is *rationally parallelizable*, i.e., $V' \setminus \{P\}$ for a compact neighborhood V' of P in V has the trivial Chern class in $H^{\cdot}(V' \setminus \{P\}, \mathbb{Q})$, since the holomorphic tangent sheaf of $\tilde{W} \setminus \tilde{Y}$ is free with an action of Γ through constant matrices so that the holomorphic tangent sheaf of $V \setminus \{P\} = W \setminus Y$ has a holomorphic integrable connection. Let $Z \to Z'$ be a resolution of the unique singularity P. By what we have seen so far, the r-th degree Todd polynomial for $c(W)$ coincides with the arithmetic genus of Z. We are done by the birational invariance of the arithmetic genus.

(2) It is well-known that $\delta/(1-e^{-\delta}) - \delta/2 = \delta(1+e^{-\delta})/2(1-e^{-\delta}) = 1 + \sum_{1 \le k \le \infty} (-1)^{k-1} B_k \delta^{2k}/(2k)!$ is a power series in δ^2 with the Bernoulli numbers B_k. Hence

$$2^r \cdot \mathrm{inv}(C, \Gamma) = \kappa_r \left[\sum_{\xi}{}' \prod_{j \in \xi} \delta_j \prod_{k \notin \xi} \frac{\delta_k(1+e^{-2\delta_k})}{(1-e^{-2\delta_k})} \right] + \kappa_r \left[\prod_{j \in J} \frac{\delta_j(1+e^{-2\delta_j})}{(1-e^{-2\delta_j})} \right] ,$$

where $\sum'_\xi$ is the summation over $\xi \in \Xi$ with $\xi \neq \emptyset$ and ${}^{\#}\xi \equiv r \pmod 2$. By the index theorem applied to Y_ξ for $\xi \neq \emptyset$, the first term on the right hand side is easily seen to coincide with $\sum'_\xi \tau(Y_\xi)$. If ${}^{\#}\xi \not\equiv r \pmod 2$, then Y_ξ is odd-dimensional so that $\tau(Y_\xi) = 0$.

(3) By Theorem 3.12, (3) applied to the toric variety Y_ξ for $\xi \neq \emptyset$, we get (cf. Corollary 1.7)

$$\tau(Y_\xi) = \sum_{l=0}^{r-{}^{\#}\xi} (-2)^l \, {}^{\#}\{\eta \in \Xi;\, \xi \subset \eta,\, {}^{\#}\eta = r-l\} \ .$$

The summation over all $\xi \neq \emptyset$ shows that the two extreme sides of (3) are equal.

Applying Theorem 3.6, (4) to $\tilde{W}$ and then taking the subsheaves of germs of Γ-invariant sections of the direct image of the exact sequences on $\tilde{W}$, we get, for $0 \leq p \leq r$, the exact sequences of $\mathcal{O}_W$-modules

$$0 \to \Omega_W^p \to \mathscr{K}^0(W;p) \xrightarrow{\delta} \mathscr{K}^1(W;p) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathscr{K}^p(W;p) \to 0 \ ,$$

where $\mathscr{K}^l(W;p) := \oplus_\xi \Omega_{Y_\xi}^{p-l}(\log D_\xi)$ with the summation taken over all $\xi \in \Xi(l)$. Denoting by $\mathscr{E}^p$ the cokernel of $\Omega_W^p \to \mathscr{K}^0(W;p) = \Omega_W^p(\log Y)$, we thus obtain an exact sequence

$$0 \to \mathscr{E}^p \to \mathscr{K}^1(W;p) \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathscr{K}^p(W;p) \to 0$$

of $\mathcal{O}_W$-modules with support in the compact analytic subspace Y. When ${}^{\#}\xi = l \neq 0$, Y_ξ is a compact nonsingular toric variety of dimension $r-l$. By Proposition 3.4, $\Omega_{Y_\xi}^{p-l}(\log D_\xi)$ is a free $\mathcal{O}_{Y_\xi}$-module so that its rank coincides with its Euler-Poincaré characteristic in view of Corollary 2.8, and is equal to

$$\binom{r-l}{p-l} \ .$$

Consequently, the Euler-Poincaré characteristic of $\mathscr{E}^p$ is

$$\chi(W, \mathscr{E}^p) = \sum_{l=1}^{p} (-1)^{l+1} \binom{r-l}{p-l} \, {}^{\#}\Xi(l) \ .$$

Summing this up over $1 \leq p \leq r$, we get

$$\sum_{1 \leq p \leq r} \chi(W, \mathscr{E}^p) = (-1)^{r+1} \sum_{1 \leq l \leq r} (-2)^{r-l} \, {}^{\#}\Xi(l) \ .$$

On the other hand, consider a *weight filtration*

$$0 \subset \omega_0 := \Omega_W^p \subset \omega_1 \subset \dots \subset \omega_p := \Omega_W^p(\log Y)$$

on $\Omega_W^p(\log Y)$ defined by $\omega_k := \Omega_W^{p-k} \wedge \Omega_W^k(\log Y)$. The Poincaré residue map as in Sect. 3.2 induces an isomorphism $\omega_k/\omega_{k-1} \xrightarrow{\sim} \oplus_\xi \Omega_{Y_\xi}^{p-k}$ with the summation taken over all $\xi \in \Xi(k)$. Since $\mathscr{E}^p = \omega_p/\omega_0$, we get

$$\chi(W, \mathscr{E}^p) = \sum_{k=1}^{p} \sum_\xi \chi(Y_\xi, \Omega_{Y_\xi}^{p-k})$$

with $\sum_\xi$ over all $\xi \in \Xi(k)$. Summing this up over $1 \leq p \leq r$, we obtain

$$\sum_{p=1}^{r} \chi(W, \mathscr{E}^p) = \sum_{k=1}^{p} \sum_{\xi \in \Xi(k)} \sum_{p=k}^{r} \chi(Y_\xi, \Omega_{Y_\xi}^{p-k}) \ ,$$

the right hand side of which is equal to $\sum_{\xi \neq \emptyset} \tau(Y_\xi)$ by the index theorem.

(4) As for the absolute convergence, we select a finite system of representatives modulo Γ of r-dimensional nonsingular convex cones in Λ in view of $C=|\Lambda|\setminus\{O\}$. It suffices to show the absolute convergence of the partial summation over each of the r-dimensional nonsingular convex cones.

As for the holomorphy at $s=0$, we look at the Dirichlet series obtained as the partial summation over each of a finite set of representatives of $\Lambda\setminus\{O\}$ modulo Γ. We then use the Euler-Maclaurin summation formula to get an asymptotic expansion for the corresponding exponential series. The resulting formula for the value at $s=0$ is quite complicated in general. Ishida makes a clever use of T-complexes to show that it is actually a rational number. When r is odd, Ogata could use (1), (2), (3) to simplify the formula. q.e.d.

Remark. The rational value $\zeta(C,\Gamma;0)$ is yet to be determined when $r\geq 4$ is even. [ADS] proves a related conjecture due to [H3] in the case of Hilbert modular cusp singularities. According to this result, it might be more natural to look at the special values of the zeta function for the pair (C',Γ) dual to (C,Γ).

When $r=2$, we have

$$\operatorname{inv}(C,\Gamma)=\sum_{j\in J}\frac{((Y_j^2)+3)}{12},$$

where (Y_j^2) is the self-intersection number of Y_j in W.

Proposition 4.6, (2), (3) and their proofs work in the compact case as well:

Let W be an r-dimensional *compact* complex manifold. Formally factor its total Chern class as $c(W):=c(\Theta_W)=\prod_{1\leq j\leq r}(1+\gamma_j)$. Then by the Riemann-Roch theorem, the *arithmetic genus* of W is given by

$$\chi(W):=\chi(W,\mathcal{O}_W)=\kappa_r\left[\prod_{i=1}^{r}\frac{\gamma_i}{(1-e^{-\gamma_i})}\right],$$

while the index theorem enables us to write the *index* of W as

$$\tau(W)=\sum_{p=0}^{r}\chi(W,\Omega_W^p)=\kappa_r\left[\prod_{i=1}^{r}\frac{\gamma_i(1+e^{-2\gamma_i})}{(1-e^{-2\gamma_i})}\right]$$

(cf. [H2]). Ogata used our results in Chap. 3 to give another proof for the following found in Sakate [S6] and Ehlers [E2]:

Proposition 4.7. *Let W be an r-dimensional* compact *complex manifold and let $Y=\sum_{j\in J}Y_j$ be a* toric divisor *on W.*

(1) (R. Tsushima) *Formally factor the* logarithmic total Chern class *of (W,Y) as*

$$\bar{c}(W,Y)=\sum_{k=0}^{r}\bar{c}_k(W,Y):=c(\Theta_W(-\log Y))=\prod_{i=1}^{r}(1+\bar{\gamma}_i)$$

and let δ_j be the first Chern class of $\mathcal{O}_W(Y_j)$ for each $j \in J$. Then we have

$$c(W) = \bar{c}(W, Y) \cdot \prod_{j \in J} (1 + \delta_j)$$

$$\bar{c}_k(W, Y)\delta_j = 0 \quad \textit{for all} \quad k > 0 \quad \textit{and} \quad j \in J \ .$$

(2) *Define the* logarithmic arithmetic genus *and the* logarithmic index *for* (W, Y) *by*

$$\bar{\chi}(W, Y) := \kappa_r \left[\prod_{i=1}^{r} \frac{\bar{\gamma}_i}{(1 - e^{-\bar{\gamma}_i})} \right]$$

and

$$\bar{\tau}(W, Y) := \kappa_r \left[\prod_{i=1}^{r} \frac{\bar{\gamma}_i(1 + e^{-2\bar{\gamma}_i})}{(1 - e^{-2\bar{\gamma}_i})} \right] ,$$

respectively. Then we have

$$\kappa_r \left[\prod_{j \in J} \frac{\delta_j}{(1 - e^{-\delta_j})} \right] = \chi(W) - \bar{\chi}(W, Y) = 2^{-r} \left\{ \sum_{p=0}^{r} \chi(W, \Omega_W^p(\log Y)) - \bar{\tau}(W, Y) \right\} .$$

(3) $$\sum_{\xi \neq \emptyset} \tau(Y_\xi) = \sum_{p=0}^{r} \chi(W, \Omega_W^p(\log Y)) - \tau(W) = (-1)^{r+1} \sum_{l=1}^{r} (-2)^{r-l}\, {}^{\#}\Xi(l)$$

$$= 2^r \sum_{l=1}^{r} (-1)^{r-l}\, {}^{\#}\Xi(l) - \sum_{l=1}^{r} (-2)^{r-l}\, {}^{\#}\Xi(l) \ .$$

(4) *When r is odd, we have*

$$\tau(W) = \bar{\tau}(W, Y) = 0 \ , \qquad \sum_{\xi \neq \emptyset} \tau(Y_\xi) = 2^{r-1} \left(\sum_{l=1}^{r} (-1)^{r-1}\, {}^{\#}\Xi(l) \right)$$

and

$$\kappa_r \left[\prod_{j \in J} \frac{\delta_j}{(1 - e^{-\delta_j})} \right] = \chi(W) - \bar{\chi}(W, Y) = 2^{-r} \sum_{p=0}^{r} \chi(W, \Omega_W^p(\log Y))$$

$$= 2^{-1} \sum_{l=1}^{r} (-1)^{r-l}\, {}^{\#}\Xi(l) \ .$$

A Sketch of the Proof. (1) As in Theorem 3.12, (1), the exact sequence

$$0 \to \Theta_W(-\log Y) \to \Theta_W \to \bigoplus_{j \in J} \mathcal{O}_{Y_j} \otimes_{\mathcal{O}_W} \mathcal{O}_W(Y_j) \to 0$$

gives $c(W) = \bar{c}(W, Y) \cdot \prod_{j \in J} (1 + \delta_j)$. By assumption, Y_j is a toric variety with $Y_j \setminus D_j$ as the algebraic torus. The Poincaré residue map on Y_j induces an exact sequence

$$0 \to \Omega_{Y_j}^1(\log D_j) \to \mathcal{O}_{Y_j} \otimes_{\mathcal{O}_W} \Omega_W^1(\log Y) \to \mathcal{O}_{Y_j} \to 0 \ ,$$

whose $\mathcal{O}_{Y_j}$-dual gives rise to an exact sequence

$$0 \to \mathcal{O}_{Y_j} \to \mathcal{O}_{Y_j} \otimes_{\mathcal{O}_W} \Theta_W(-\log Y) \to \Theta_{Y_j}(-\log D_j) \to 0 \ .$$

Hence the restriction of $\bar{c}(W, Y)$ to Y_j coincides with the logarithmic total Chern class $\bar{c}(Y_j, D_j)$. We have $\bar{c}(Y_j, D_j) = 1$, since $\Theta_{Y_j}(-\log D_j)$ is $\mathcal{O}_{Y_j}$-free by Corollary 3.2.

(2) (1) implies

$$\begin{aligned}\prod_{i=1}^{r} \frac{\gamma_i}{(1-e^{-\gamma_i})} &= \prod_{i=1}^{r} \frac{\bar{\gamma}_i}{(1-e^{-\bar{\gamma}_i})} \cdot \prod_{j\in J} \frac{\delta_j}{(1-e^{-\delta_j})} \\ &= \prod_{i=1}^{r} \frac{\bar{\gamma}_i}{(1-e^{-\bar{\gamma}_i})} - 1 + \prod_{j\in J} \frac{\delta_j}{(1-e^{-\delta_j})}\ ,\end{aligned}$$

hence $\chi(W)-\bar{\chi}(W,Y)=\kappa_r[\prod_{j\in J}\{\delta_j/(1-e^{-\delta_j})\}]$. Similarly, we get $\tau(W)-\bar{\tau}(W,Y)=\kappa_r[\prod_{j\in J}\{\delta_j(1+e^{-2\delta_j})/(1-e^{-2\delta_j})]$. By the Riemann-Roch theorem applied to each $\Omega_W^p(\log Y)$, we get

$$\sum_{p=0}^{r} \chi(W, \Omega_W^p(\log Y)) = \bar{\tau}(W, Y) + 2^r \kappa_r \left[\prod_{j\in J} \frac{\delta_j}{(1-e^{-\delta_j})} \right] .$$

The proof for (3) and (4) is similar to that of Proposition 4.6, (3) but is simpler now, since we can directly compute the Euler-Poincaré characteristic on the compact W instead of considering $\mathscr{E}^p$. q.e.d.

Remark. Let Z be a compactification we obtain by adding cusps $P_1, \ldots, P_c$ to a quotient of a symmetric tube domain $\mathfrak{D}$ with respect to a small enough arithmetic subgroup of $\mathbb{Q}$-rank one and with zero-dimensional rational boundary components. Let $f: W \to Z$ be a resolution of singularities obtained as in [SC]. Then the reduced effective divisors $f^{-1}(P_1), \ldots, f^{-1}(P_c)$ are mutually disjoint toric divisors. If we apply Proposition 4.7 to the toric divisor $Y := f^{-1}(P_1) + \ldots + f^{-1}(P_c)$, then we obtain

$$\sum_{k=1}^{c} \operatorname{inv}(P_k) = \chi(W) - \bar{\chi}(W, Y) = 2^{-r}\{\tau(W) - \bar{\tau}(W, Y) + \sum_{\xi \neq \varphi} \tau(Y_\xi)\}\ ,$$

where $\operatorname{inv}(P_k)$ is the invariant for the cusp singularity P_k in the sense of Proposition 4.6.

According to the **proportionality theorem of Hirzebruch-Mumford,** the logarithmic Chern classes for (W, Y) are proportional to the corresponding Chern classes for a compact rational manifold $\mathfrak{D}^\vee$ called the *compact dual* of $\mathfrak{D}$. In particular, we can compute $\bar{\chi}(W, Y)$ and $\bar{\tau}(W, Y)$ in terms of $\chi(\mathfrak{D}^\vee)$ and $\tau(\mathfrak{D}^\vee)$. Thus to compute $\chi(W)$, we need to know the cusp contributions $\operatorname{inv}(P_1), \ldots, \operatorname{inv}(P_c)$. More generally, we have similar results for the dimensions of various spaces of automorphic forms. See [S6], [S7] for details.

The description in Sect. 4.1 of two-dimensional cusp singularities in terms of periodic continued fractions can be generalized to the Tsuchihashi cusp singularities of an arbitrary dimension. We close this section by explaining results in Tsuchihashi [T5]. For simplicity, we restrict ourselves to the case $r=3$. As we saw at the end of Sect. 4.1, a periodic continued fraction for $r=2$ can be regarded as a $\pi_1(S^1)$-stable weighted triangulation of $\tilde{S}^1$. For $r=3$, we consider, instead, the universal covering surface of a compact real manifold of dimension two, a triangulation of it stable under the action of the fundamental group, and then a double $\mathbb{Z}$-weighting for it again stable under the action of the fundamental group.

For (C, Γ) in $\mathscr{P}(N)$, recall that the Γ-stable convex polytope decomposition $\Box$ of $\partial\Theta$ gives rise to a Γ-stable spherical cell decomposition $\pi(\Box)$ of $D=\pi(C)=C/\mathbb{R}_{>0}$. On the other hand, a Γ-stable, nonsingular and locally finite subdivision Λ of the fan Σ obtained from $\Box$ gives rise to a Γ-stable triangulation

$$\Delta := \{\pi(\sigma\setminus\{O\});\ \sigma\in\Lambda, \sigma\neq\{O\}\} ,$$

which is a subdivision of the cell decomposition $\pi(\Box)$. If Λ is fine enough, then Δ induces a triangulation Δ/Γ of the compact real manifold D/Γ of dimension $r-1$.

When $r=3$, D is a Γ-stable and simply connected open subset of the 2-sphere $S_N := (N_{\mathbb{R}}\setminus\{O\})/\mathbb{R}_{>0}$ and is contained entirely in a hemisphere. As in Corollary 1.32, the nonsingular fan (N, Λ) determines a Γ-stable triangulation Δ of D as well as a Γ-stable *double $\mathbb{Z}$-weighting* on Δ satisfying the *monodromy condition.* If Λ is fine enough, this latter determines a double $\mathbb{Z}$-weighting for the triangulation Δ/Γ of the compact real manifold D/Γ of dimension two.

Conversely, start with a compact real manifold D/Γ of dimension two with the universal covering surface D and the fundamental group Γ. Consider a Γ-stable and locally finite combinatorial triangulation Δ of D and endow it with a Γ-stable double $\mathbb{Z}$-weighting satisfying the monodromy condition. If we choose a $\mathbb{Z}$-basis for $N\cong\mathbb{Z}^3$ and regard it as the set of N-weights for the vertices of a triangle in Δ, we obtain a homomorphism $\Gamma\to GL(N)$ and an N-weight for each vertex of Δ exactly as in Corollary 1.32 thanks to the simple connectivity of D. The N-weighting thus obtained is obviously Γ-stable. Each simplex in Δ together with the N-weights for its vertices determine a nonsingular polyhedral cone in $N_{\mathbb{R}}$. Contrary to the case dealt with in Corollary 1.32, however, there is no guarantee that the collection Λ of the nonsingular polyhedral cones thus obtained is a fan, although it is locally a fan due to the monodromy condition. Even if Λ happens to be a fan, the closure of $C := |\Lambda|\setminus\{O\}$ may fail to be strongly convex. We need some "convexity" restriction on the double $\mathbb{Z}$-weighting we start with.

Even when $r=2$, we needed the following restriction on a Γ-stable weighted triangulation of $D\cong\mathbb{R}$: All the weights are not greater than -2 and at least one of them is actually less than -2. This is nothing but the condition imposed on purely periodic continued fractions and can be thought of as a convexity condition.

The following result gives a sufficient condition when $r=3$. We omit the proof, since it is rather long. [T6] generalizes this result to a necessary and sufficient condition.

Tsuchihashi's Theorem ([T5, Theorem 4.5, Proposition 4.6]). *Let D/Γ be a compact real manifold of dimension two with the universal covering surface D and the fundamental group Γ. Consider a Γ-stable and locally finite triangulation Δ of D and endow it with a Γ-stable double $\mathbb{Z}$-weighting satisfying the monodromy condition. There exists a Γ-stable and locally finite nonsingular fan Λ of $\mathbb{Z}^3$, unique up to isomorphism, such that $(|\Lambda|\setminus\{O\}, \Gamma)$ belongs to $\mathscr{P}(\mathbb{Z}^3)$ and, together with Λ, gives rise to the Γ-stable double $\mathbb{Z}$-weighted triangulation we started with, if the following two conditions are satisfied:*

(a) *The sum of the double $\mathbb{Z}$-weight for each edge is not greater than* -2.

(b) *The removal from Δ of all the edges with the sum of the double $\mathbb{Z}$-weight equal to -2 gives rise to a cell decomposition $\square$ of D.*

In this case, D/Γ has negative Euler number, hence has genus not less than two when orientable.

Remark. For a three-dimensional Hilbert modular cusp singularity, $D/\Gamma \cong \mathbb{R}^2/\mathbb{Z}^2$ is a two-dimensional real torus, hence its Euler number is zero. Consequently, the sufficient condition in the above theorem is not a necessary condition. A more recent paper [T6] gives a necessary and sufficient condition. Several examples satisfying (a) and (b) can be found in [T5, § 5].

Sankaran [S3] characterizes Hilbert modular cusp singularities among Tsuchihashi cusp singularities of an arbitrary dimension. Tsuchihashi [T7] and Ogata [O2], [O3] study infinitesimal deformations of Tsuchihashi cusp singularities and related singularities appearing in connection with compactifications of arithmetic quotients of symmetric domains of $\mathbb{Q}$-rank one.

We refer the reader also to Ishii [I8], [I9] for results somewhat related to what we have seen in this section.

4.3 Compact Quotients of Toric Varieties

When a discrete group acts on an open subset of a toric variety properly discontinuously and without fixed points, the quotient complex manifold may happen to be compact. Let us briefly mention interesting examples obtained in this manner.

(1) **Complex Tori.** For $N \cong \mathbb{Z}^r$, the algebraic torus $T_N \cong (\mathbb{C}^\times)^r$ itself is the simplest toric variety. For a free commutative subgroup $\Gamma \subset T_N$ of rank r, the quotient T_N/Γ is a complex torus. Indeed, through the surjective homomorphism

$$N \otimes_{\mathbb{Z}} \mathbb{C} \to T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^\times$$

induced by $\mathbb{C} \ni z \mapsto \exp(2\pi i z) \in \mathbb{C}^\times$, we can identify T_N with the quotient of $N \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^r$ with respect to its subgroup N, and the inverse image of $\Gamma \subset T_N$ in $N \otimes_{\mathbb{Z}} \mathbb{C}$ is a free commutative subgroup of rank $2r$. For short, we first factor $N \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^r$ out by a half of the $2r$ periods to get $T_N \cong (\mathbb{C}^\times)^r$ and then factor that out further by the remaining half of the periods to obtain a complex torus. As in [MO, § 11], it is convenient to reformulate the above consideration in the following manner:

Give Γ additively as a free $\mathbb{Z}$-module of rank r. For the $\mathbb{Z}$-module M dual to another free $\mathbb{Z}$-module N of rank r, a *period* is an injective homomorphism

$$q : \Gamma \to T_N = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \ .$$

If we denote the image of $\gamma \in \Gamma$ under q by q^γ, then the image q^Γ of q is a free commutative subgroup of T_N of rank r, and the quotient $X := T_N/q^\Gamma$ is a complex torus. Alternatively, we may define a period to be a nondegenerate $\mathbb{Z}$-bilinear map

$$Q: \Gamma \times M \to \mathbb{C}^\times ,$$

where we let $Q(\gamma, m) := q^\gamma(m)$ for $\gamma \in \Gamma$ and $m \in M$. We can then define the dual period

$$\hat{q}: M \to \hat{T} := \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^\times)$$

by $\hat{q}^m(\gamma) := q^\gamma(m) = Q(\gamma, m)$ and the *dual complex torus* $\hat{X} := \hat{T}/\hat{q}^M$.

Although [MO, Proposition 11.1] is incorrect, subsequent arguments remain unaffected and enable us to formulate theta functions in a multiplicative manner. We then have a rather transparent construction of compactifications of the moduli spaces of polarized Abelian varieties.

(2) **Compact complex surfaces of class VII.** A compact complex surface X, i.e., a compact complex manifold of dimension two, is said to be of class VII, if its first Betti number is $b_1(X) = 1$. It is said to be of class VII_0, if, in addition, X contains no exceptional curve of the first kind. These surfaces were introduced by Kodaira and studied by Kodaira himself, M. Inoue, Ma. Kato, I. Nakamura, I. Enoki and others. We refer the reader to Nakamura [N 5] and the literature quoted therein. We here mention those which are related to toric varieties.

(a) **Hopf surfaces.** An r-dimensional compact complex manifold X is called a *Hopf manifold*, if the universal covering space is biholomorphic to $\mathbb{C}^r \setminus \{O\}$. In particular, it is called a *primitive Hopf manifold*, if the fundamental group $\pi_1(X)$ is infinite cyclic.

When $r = 2$, a Hopf manifold is called a *Hopf surface*. It can be characterized as a compact complex surface X whose second Betti number is $b_2(X) = 0$ and whose fundamental group $\pi_1(X)$ contains an infinite cyclic subgroup of finite index.

As we saw in the example (i) immediately before Theorem 1.5, $\mathbb{C}^2 \setminus \{O\}$ is a two-dimensional nonsingular toric variety. When complex numbers λ, λ' satisfy $0 < |\lambda| < 1$ and $0 < |\lambda'| < 1$, the infinite cyclic group $g^{\mathbb{Z}}$ generated by $g := (\lambda, \lambda') \in (\mathbb{C}^\times)^2$ acts on $\mathbb{C}^2 \setminus \{O\}$ properly discontinuously and without fixed points. The quotient $X := (\mathbb{C}^2 \setminus \{O\})/g^{\mathbb{Z}}$ is a primitive Hopf surface. It contains mutually disjoint elliptic curves E, F such that $(E^2) = (F^2) = 0$. The real manifold with corners in Sect. 1.3 plays a convenient rôle in the proof of these facts as well as the construction of a one-parameter family of its degeneration to a rational surface. Details can be found in [MO, § 13].

(b) **Parabolic Inoue surfaces.** M. Inoue constructed compact complex surfaces X of class VII_0 with $b_2(X) = 1$ which contains, as only closed irreducible analytic subspaces of dimension one, an elliptic curve E and a rational curve C with a node such that $(E^2) = -1$, $(C^2) = 0$ and $(E \,.\, C) = 0$. As in [MO, Theorem 14.1], toric varieties are convenient in describing the construction:

In a $\mathbb{Z}$-module N with a $\mathbb{Z}$-basis $\{n, n'\}$, define a nonsingular fan Δ by

$$\Delta := \{\{O\}, \mathbb{R}_{\geq 0} n', \mathbb{R}_{\geq 0}(n + \nu n'), \mathbb{R}_{\geq 0}(n + \nu n') + \mathbb{R}_{\geq 0}(n + (\nu - 1)n');\ \nu \in \mathbb{Z}\} .$$

Then we have $|\Delta| = (\mathbb{R}_{\geq 0} n') \cup (\mathbb{R}_{>0} n + \mathbb{R} n')$. The automorphism $h \in \mathrm{Aut}_{\mathbb{Z}}(N)$ defined by

$$h(n) := n + n' \quad \text{and} \quad h(n') := n'$$

obviously induces an automorphism of the fan Δ. For a complex number λ with $0<|\lambda|<1$, define $\gamma_n(\lambda)\in T_N$ as in Sect. 1.2 by

$$\gamma_n(\lambda)(m):=\lambda^{\langle m,n\rangle} \quad \text{for} \quad m\in M \ .$$

h induces an automorphism h_* of the toric variety $T_N\mathrm{emb}(\Delta)$, while $\gamma_n(\lambda)\in T_N$ also acts on $T_N\mathrm{emb}(\Delta)$. We denote the composite action by

$$g_\lambda:=(\text{the action of } \gamma_n(\lambda))\circ h_* \ ,$$

which generates an infinite cyclic group $(g_\lambda)^{\mathbb{Z}}$ acting on $T_N\mathrm{emb}(\Delta)$. For the $\mathbb{Z}$-basis $\{m,m'\}$ of M dual to $\{n,n'\}$, define coordinates for T_N by $z:=\mathbf{e}(m)$ and $z':=\mathbf{e}(m')$. Then the action of g_λ on the point $(z,z')\in T_N$ is given by

$$g_\lambda(z,z')=(\lambda z, zz') \ .$$

Thus we have

$$(g_\lambda)^\nu(z,z')=(\lambda^\nu z, z^\nu z'\lambda^{\nu(\nu-1)/2}) \quad \text{for} \quad \nu\in\mathbb{Z} \ .$$

Looking at the action of $(g_\lambda)^{\mathbb{Z}}$ on $\mathrm{Mc}(N,\Delta)$, we see that

$$X_\lambda:=T_N\mathrm{emb}(\Delta)/(g_\lambda)^{\mathbb{Z}}$$

is a compact complex surface containing an elliptic curve

$$E_\lambda:=\mathrm{orb}(\mathbb{R}_{\geq 0}n')/(g_\lambda)^{\mathbb{Z}}$$

and a rational curve with a node

$$C_\lambda:=\left\{\bigcup_{\nu\in\mathbb{Z}} V(\mathbb{R}_{\geq 0}(n+\nu n'))\right\}\Big/(g_\lambda)^{\mathbb{Z}}$$

which is obtained from $\mathbb{P}_1(\mathbb{C})$ by the identification of 0 and ∞. This X_λ for each λ is called a *parabolic Inoue surface*. It is a compact complex surface of class VII_0 with $b_2(X_\lambda)=1$. A one-parameter family of its degeneration to a rational surface as $\lambda\to 0$ is constructed in [MO, § 14].

(c) **Hyperbolic Inoue surfaces** (also called **Inoue-Hirzebruch surfaces**). As in Proposition 4.1, choose a $\mathbb{Z}$-lattice N in a real quadratic field K. Under the canonical identification $N_{\mathbb{R}}=\mathbb{R}^2$, denote by Θ_N the convex hull of $N\cap(\mathbb{R}_{>0})^2$ and construct a fan Δ_N by subdividing the first quadrant $(\mathbb{R}_{>0})^2$ by the rays from the origin O to the points of $N\cap\partial\Theta_N$. Repeat the same procedure to the fourth quadrant $\mathbb{R}_{>0}\times\mathbb{R}_{<0}$. Namely, denote by Θ'_N the convex hull of $N\cap(\mathbb{R}_{>0}\times\mathbb{R}_{<0})$ and construct a fan Δ'_N as the subdivision of the fourth quadrant by the rays from O to the points of $N\cap\partial\Theta'_N$. Then $\Delta_N\cup\Delta'_N$ is a nonsingular fan in N with

$$|\Delta_N\cup\Delta'_N|=\{O\}\cup(\mathbb{R}_{>0}\times\mathbb{R}_{>0})\times(\mathbb{R}_{>0}\times\mathbb{R}_{<0}) \ .$$

Obviously, Γ_N^+ is a group of automorphisms of the fan $(N,\Delta_N\cup\Delta'_N)$ and acts on $T_N\mathrm{emb}(\Delta_N\cup\Delta'_N)$. Denote by $\tilde{Z}_N$ the inverse image of $(\mathbb{R}_{>0}\times\mathbb{R})\cup(\mathrm{Mc}(N,\Delta_N\cup\Delta'_N)\setminus N_{\mathbb{R}})$ under the canonical projection

$$\mathrm{ord}: T_N\mathrm{emb}(\Delta_N\cup\Delta'_N)\twoheadrightarrow\mathrm{Mc}(N,\Delta_N\cup\Delta'_N)\supset N_{\mathbb{R}} \ .$$

Then $Z_N:=\tilde{Z}_N/\Gamma_N^+$ turns out to be a compact complex surface of class VII_0. Also for a subgroup $\Gamma\subset\Gamma_N^+$ of finite index, the quotient $\tilde{Z}_N/\Gamma$ is a compact complex surface of class VII_0 and is called a *hyperbolic Inoue surface* or an *Inoue-Hirzebruch surface*.

Z_N contains U_N in Proposition 4.1 and U'_N constructed similarly for the fourth quadrant as disjoint open sets. The intersection of the closures of U_N and U'_N is an $(S^1 \times S^1)$-bundle over S^1, since its quotient under $CT_N \cong S^1 \times S^1$ is homeomorphic to S^1.

Z_N contains two disjoint cycles of $\mathbb{P}_1(\mathbb{C})$'s, one contained in U_N and the other contained in U'_N. The second Betti number $b_2(Z_N)$ turns out to be the sum of the lengths of these two cycles, hence the number of rational curves on Z_N. Similar results hold more generally for $\tilde{Z}_N/\Gamma$.

The first and the fourth quadrants are dual cones in the sense of Sect. 4.2. Hence the cycles on U_N and U'_N are contracted to mutually dual cusp singularities. Nakamura [N3], [N4], [N6] studied this duality in detail.

There seem to be generalizations to higher dimension in two different directions. *Inoue-Kato manifolds* constructed by Tsuchihashi [T8] have infinite cyclic fundamental groups and are related to the Perron-Frobenius theorem on real square matrices with nonnegative entries. Ishida [I7] deals also with T-complexes related to these manifolds. On the other hand, Sankaran [S4] considers r-dimensional compact complex manifolds with free commutative fundamental group of rank $r-1$.

(d) **Half Inoue surfaces.** If Γ_N^+ has index two in Γ_N, then elements of Γ_N not in Γ_N^+ interchange Δ_N and Δ'_N in (c) above so that the first and the fourth quadrants are "symmetric". In this case, Γ_N is a group of automorphisms of the fan $\Delta_N \cup \Delta'_N$ and $\tilde{Z}_N$ is Γ_N-stable. The quotient

$$\hat{Z}_N := \tilde{Z}_N/\Gamma_N$$

is a compact complex surface of class VII_0. For a subgroup $\Gamma \subset \Gamma_N$ of finite index not contained in Γ_N^+, the quotient $\tilde{Z}_N/\Gamma$ is again a compact complex surface of class VII_0. Any one of these is called a *half Inoue surface* and contains a unique cycle of $\mathbb{P}_1(\mathbb{C})$'s.

Appendix. Geometry of Convex Sets

The purpose of this appendix is to collect together basic results on convex cones, convex polyhedral cones, convex bodies and convex polyhedra scattered throughout the literature. Emphasis is put on combinatorial and morphological aspects, which turn out to have close connection with the algebraic geometry of toric varieties. We omit most of the proofs and refer the reader to Grunbaum [G6] and its supplement [G7], Brønsted [B7], Rockafellar [R7] and numerous other books on convex sets.

Throughout this appendix, we fix an r-dimensional vector space $V \cong \mathbb{R}^r$ over the field $\mathbb{R}$ of real numbers. We endow V with the usual topology, although usually we do not fix any Euclidean metric on V. The dual vector space V^* consists of linear functionals $u: V \to \mathbb{R}$ on V. We denote the value of u at $v \in V$ by $\langle v, u\rangle \in \mathbb{R}$ and get the $\mathbb{R}$-bilinear duality pairing

$$\langle\ ,\ \rangle : V \times V^* \to \mathbb{R}\ .$$

We denote by $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{R}_{>0}$) the set of nonnegative (resp. positive) real numbers.

A.1 Convex Polyhedral Cones

A subset $C \subset V$ is a *convex cone* (with apex at the origin O) if $av \in C$ and $v + v' \in C$ for any $v, v' \in C$ and $a \in \mathbb{R}_{>0}$. In particular, C is a *convex polyhedral cone* if

$$C = \mathbb{R}_{\geq 0}v_1 + \ldots + \mathbb{R}_{\geq 0}v_s := \{a_1v_1 + \ldots + a_sv_s\,;\ a_j \in \mathbb{R}_{\geq 0},\ 1 \leq \forall j \leq s\}$$

for a finite number of elements $v_1, \ldots, v_s \in C$. Note that these elements may not be $\mathbb{R}$-linearly independent. C is called a *simplicial cone* if $v_1, \ldots, v_s$ can be chosen to be $\mathbb{R}$-linearly independent.

For a convex cone C in V,

$$C^\vee := \{u \in V^*;\ \langle v, u\rangle \geq 0, \forall v \in C\}$$

is obviously a convex cone in V^* and is called the *dual cone* of C or the cone dual to C. Geometrically, $C^\vee$ can be thought of as the set of half spaces $H^+(u; 0) := \{v \in V;\ \langle v, u\rangle \geq 0\}$ containing C. Note that we are here calling $H^+(O; 0) = V$ also a half space for convenience when $u = O$.

We define the *dimension* $\dim C$ of a convex cone C to be the dimension of the $\mathbb{R}$-vector subspace $\mathbb{R}C = C + (-C)$ of V generated by C, where

$$-C := \{-v\,;\, v \in C\} \quad \text{and} \quad C_1 + C_2 := \{v_1 + v_2\,;\, v_1 \in C_1, v_2 \in C_2\}$$

are clearly convex cones for any given convex cones C, C_1 and C_2 in V. We obviously have $(C_1 + C_2)^\vee = C_1^\vee \cap C_2^\vee$.

The *relative interior* $\operatorname{rel\,int}(C)$ of a convex cone C is the usual interior of C regarded as a subset of the $\mathbb{R}$-vector space $\mathbb{R}C$.

The cone $C^{\vee\vee}$ in V dual to $C^\vee \subset V^*$ is a convex cone containing C and $C^{\vee\vee} = \bigcap_{u \in C^\vee} H^+(u;0)$ holds. In general, C and $C^{\vee\vee}$ may not coincide

Theorem A.1 (1) (The Duality Theorem. See, e.g., [R7, Theorem 14.1]). *If $C \subset V$ is a closed convex cone, then $C = C^{\vee\vee}$.*

(1′) *In particular, we have $C = \mathbf{C}^{\vee\vee}$ for a convex polyhedral cone C.*

(2) *$(C_1 \cap C_2)^\vee = C_1^\vee + C_2^\vee$ for closed convex cones C_1, C_2 in V.*

(3) (The Separation Theorem. See, e.g., [R7, Theorem 11.3]) *Let C_1 and C_2 be convex cones in V. Then* $\operatorname{rel\,int}(C_1)$ *and* $\operatorname{rel\,int}(C_2)$ *are disjoint if and only if there exists $u \neq O$ in V^* such that $C_1 \subset H^+(u;0)$ and $C_2 \subset H^+(-u;0)$ and that at least one of C_1 and C_2 is not contained in the hyperplane*

$$H(u;0) := \{v \in V;\, \langle v, u\rangle = 0\}$$

itself. (C_1 and C_2 then lie on mutually opposite sides with respect to $H(u;0)$.)

For a subset S of V, we define subsets of V^* by

$$S^\vee := \{u \in V^*;\, \langle v, u\rangle \geq 0, \forall v \in S\}$$

$$S^\perp := \{u \in V^*;\, \langle v, u\rangle = 0, \forall v \in S\}\ .$$

Clearly, $S^\vee$ is a convex cone in V^* and $S^\perp$ is an $\mathbb{R}$-subspace of V^*. For a subset T of V^*, we similarly define the convex cone $T^\vee$ in V and the $\mathbb{R}$-subspace $T^\perp$ of V.

Obviously, we have $S^\perp \subset S^\vee$ and $S^\vee \cap (-S^\vee) = S^\perp$. If $S_1 \supset S_2$, then we have $S_1^\vee \subset S_2^\vee$ and $S_1^\perp \subset S_2^\perp$. Moreover, $(S_1 \cup S_2)^\vee = S_1^\vee \cap S_2^\vee$ and $(S_1 \cup S_2)^\perp = S_1^\perp \cap S_2^\perp$ hold for subsets S_1 and S_2.

Theorem A.2 (Farkas' Theorem. See, e.g., [R7, §§ 19 and 22]). *The dual cone of a convex polyhedral cone is again a convex polyhedral cone.*

This theorem and the duality theorem in Theorem A.1, (1′) enable us to express a convex polyhedral cone C in two different ways:

$$C = \mathbb{R}_{\geq 0}v_1 + \ldots + \mathbb{R}_{\geq 0}v_s = \{v \in V;\, \langle v, u_j\rangle \geq 0, 1 \leq \forall j \leq t\}$$

for $v_1, \ldots, v_s \in V$ and $u_1, \ldots, u_t \in V^*$. The latter expression describes C as the intersection of the half spaces $H^+(u_1;0), \ldots, H^+(u_t;0)$. Thus C is the set of solutions for a system of t homogeneous linear inequalities and consists of the nonnegative linear combinations of "fundamental solutions" $v_1, \ldots, v_s$.

Example. Let $\{e_1, \ldots, e_r\}$ be a basis of V and let $\{e_1^*, \ldots, e_r^*\}$ be the dual basis of V^*. Then

$$C := \left\{ v \in V;\ \langle v, e_1^* \rangle \geq \langle v, e_2^* \rangle \geq \ldots \geq \langle v, e_r^* \rangle,\ \sum_{j=1}^{r} \langle v, e_j^* \rangle = 0 \right\}$$

is a convex polyhedral cone in V. We have

$$C = \mathbb{R}_{\geq 0} v_1 + \ldots + \mathbb{R}_{\geq 0} v_{r-1}$$

with

$$v_j := \frac{1}{j}(e_1 + \ldots + e_j) - \frac{1}{r-j}(e_{j+1} + \ldots + e_r) \quad \text{for} \quad 1 \leq j \leq r-1 \ .$$

Theorem A.3 (Carathéodory's Theorem. See, e.g., [G6]). *Let d be the dimension of a convex polyhedral cone $C = \mathbb{R}_{\geq 0} v_1 + \ldots + \mathbb{R}_{\geq 0} v_s$ in V. Then any $v \in C$ can be expressed as a nonnegative linear combination of d elements among $\{v_1, \ldots, v_s\}$. In particular, a d-dimensional convex polyhedral cone C is a finite union of d-dimensional simplicial cones.*

As we defined earlier, the relative interior $\operatorname{rel\,int}(C)$ of a convex cone $C \subset V$ is the interior of C regarded as a subset of $\mathbb{R}C$. We define the *relative boundary* of C to be

$$\partial C := C \setminus \operatorname{rel\,int}(C) \ .$$

A subset F of a convex polyhedral cone $C \subset V$ is called a *face* of C and denoted $F < C$ if

$$F = C \cap \{u\}^{\perp} \quad \text{for some} \quad u \in C^{\vee} \ .$$

F is thus the intersection of C with the boundary $\partial H^+(u; 0) = \{u\}^{\perp}$ of a half space $H^+(u; 0)$ containing C. Hence F is also a convex polyhedral cone in V. C itself is a face of C, since we can take $u = O$. The proper faces of C (i.e., the faces other than C itself) are contained in the relative boundary ∂C. Obviously, we have the decomposition

$$C = \coprod_{F < C} \operatorname{rel\,int}(F) \qquad \text{(disjoint union)} \ .$$

Lemma A.4. *The following are equivalent for a convex polyhedral cone $C \subset V$ and $v \in C$:*

(i) *v is contained in* $\operatorname{rel\,int}(C)$.

(ii) $\langle v, u \rangle > 0$ *for any u in $C^{\vee} \setminus C^{\perp}$.*

(iii) $C^{\vee} \cap \{v\}^{\perp} = C^{\perp}$.

(iv) $C + \mathbb{R}_{\geq 0}(-v)$ *coincides with the $\mathbb{R}$-vector subspace $\mathbb{R}C = C + (-C)$ of V generated by C.*

Proof. If (ii) does not hold, we have $\langle v, u \rangle = 0$ for some u in $C^{\vee} \setminus C^{\perp}$. Thus v is contained in the proper face $C \cap \{u\}^{\perp}$ of C and (i) does not hold. (ii) obviously implies (iii). If (i) does not hold, then v is contained in a proper face F of C. By definition, we have $F=C\cap\{u\}^{\perp}$ for some $u\in C^{\vee}$, which cannot belong to $C^{\perp}$ since $F\neq C$. Thus u is in $C^{\vee}\cap\{v\}^{\perp}$ but not in $C^{\perp}$ so that (iii) does not hold. (iii) and (iv) are equivalent by duality and Theorem A.1. q.e.d.

Proposition A.5. *The set $\mathscr{F}(C)$ of faces of a convex polyhedral cone C is a finite partially ordered set with respect to the face relation $<$. C is the largest element, while the smallest element is $C\cap(-C)$, which is the largest $\mathbb{R}$-vector subspace of V contained in C. Moreover, $\mathscr{F}(C)$ is an abstract complex in the following sense: $F\in\mathscr{F}(C)$ and $F'<F$ imply $F'\in\mathscr{F}(C)$ and the intersection $F_1\cap F_2$ of $F_1, F_2\in\mathscr{F}(C)$ is a face of F_1 as well as of F_2.*

Proof. C is obviously the largest element of $\mathscr{F}(C)$. The finiteness of $\mathscr{F}(C)$ is seen as follows: If $C=\mathbb{R}_{\geq 0}v_1+\ldots+\mathbb{R}_{\geq 0}v_s$ and $u\in C^{\vee}$, then we have $\langle v_j, u\rangle\geq 0$ for all j and the face $C\cap\{u\}^{\perp}$ of C is the set of nonnegative linear combinations of the subset $\{v_j;\langle v_j,u\rangle=0\}$ of the finite set $\{v_1,\ldots,v_s\}$.

If $C>F>F'$, then $F=C\cap\{u\}^{\perp}$ for some $u\in C^{\vee}\subset F^{\vee}$ and $F'=F\cap\{u'\}^{\perp}$ for some $u'\in F^{\vee}$. For $a\in\mathbb{R}$ large enough, $u'':=au+u'$ is contained in $C^{\vee}$ and we have $F'=C\cap\{u''\}^{\perp}$.

If $F_1<C$ and $F_2<C$, then $F_1=C\cap\{u_1\}^{\perp}$ and $F_2=C\cap\{u_2\}^{\perp}$ for $u_1,u_2\in C^{\vee}$. In this case, u_1+u_2 is contained in $C^{\vee}\subset F_1^{\vee}\cap F_2^{\vee}$ and we have $F_1\cap F_2=F_1\cap\{u_1+u_2\}^{\perp}$ $=F_2\cap\{u_1+u_2\}^{\perp}$.

$C\cap(-C)$ is the smallest element of $\mathscr{F}(C)$ as a consequence of the following proposition. q.e.d.

Proposition A.6. *If F is a face of a convex polyhedral cone C, then $F^*:=C^{\vee}\cap F^{\perp}$ is a face of the dual cone $C^{\vee}$. We have a bijection from $\mathscr{F}(C)$ to $\mathscr{F}(C^{\vee})$ by the map sending F to F^*, which is a Galois correspondence in the following sense:*

(i) *For $F_1,F_2\in\mathscr{F}(C)$, we have $F_1>F_2$ if and only if $F_1^*<F_2^*$.*

(ii) *C is the largest element of $\mathscr{F}(C)$ and we have $C^*=C^{\perp}=C^{\vee}\cap(-C^{\vee})$.*

(iii) *$C\cap(-C)$ is the smallest element of $\mathscr{F}(C)$ and we have $(C\cap(-C))^*=C^{\vee}$.*

(iv) $\dim F+\dim F^*=\dim V$ *for any* $F\in\mathscr{F}(C)$.

Proof. If v is a point in the relative interior of $F\in\mathscr{F}(C)$, then $F^{\vee}\cap\{v\}^{\perp}=F^{\perp}$ by Lemma A.4, hence $F^*=C^{\vee}\cap F^{\perp}=C^{\vee}\cap\{v\}^{\perp}$ is in $\mathscr{F}(C^{\vee})$. The map $F\mapsto F^*$ of $\mathscr{F}(C)$ to $\mathscr{F}(C^{\vee})$ is thus surjective. Moreover, $F_1>F_2$ implies $F_1^*\subset F_2^*$, hence $F_1^*<F_2^*$ by Proposition A.5.

The dual map $\mathscr{F}(C^{\vee})\to\mathscr{F}(C)$ which sends G to $G^*:=C\cap G^{\perp}$ also has the same property as above. For $F\in\mathscr{F}(C)$, we have $(F^*)^*=C\cap(F^*)^{\perp}=C\cap(C^{\vee}\cap F^{\perp})^{\perp}\supset C\cap(F^{\perp})^{\perp}=F$, hence $(F^*)^*>F$ again by Proposition A.5. Combining this result with similar one for the dual map, we get $(F^*)^*=F$ and $(G^*)^*=G$ for any $F\in\mathscr{F}(C)$ and $G\in\mathscr{F}(C^{\vee})$. Thus the mutually dual maps are inverse to each other and we get (i). (ii) is obvious, hence $C^{\vee}\cap(-C^{\vee})$ is the smallest element of $\mathscr{F}(C^{\vee})$. By duality, we thus get (iii).

The equality in (iv) clearly holds for $F=C$, hence for $F=C\cap(-C)$ as well by what we have seen so far and the dual statement. We can reduce the proof of (iv) in the general case to this latter case as follows: $C':=C+\mathbb{R}F$ is a convex polyhedral cone in V and $\mathbb{R}F:=F+(-F)$ is the smallest element of $\mathscr{F}(C')$ for $F\in\mathscr{F}(C)$.

The dual cone of C' is $(C')^\vee = C^\vee \cap (-F)^\vee = C^\vee \cap F^\perp = F^*$. Thus by the result above applied to C', we get $\dim F^* + \dim \mathbb{R}F = \dim V$. We are done, since $\dim \mathbb{R}F = \dim F$ by definition. q.e.d.

Corollary A.7. *For any face F of a convex polyhedral cone C in V, we have the equality $F^\vee = C^\vee + F^\perp$ in V^*. Furthermore, for any u in the relative interior of $C^\vee \cap F^\perp$, we have*

$$F^\vee = C^\vee + \mathbb{R}_{\geq 0}(-u) \ .$$

Proof. For any u in the relative interior of $C^\vee \cap F^\perp$, we have $F = C \cap (C^\vee \cap F^\perp)^\perp = C \cap \{u\}^\perp$ by Proposition A.6. If we take the dual of the obvious inclusions $F^\vee \supset C^\vee + F^\perp \supset C^\vee + \mathbb{R}_{\geq 0}(-u)$, we thus have $F \subset C \cap \{-u\}^\vee = C \cap \{u\}^\perp = F$. By taking the dual again, we conclude $F^\vee = C^\vee + F^\perp = C^\vee + \mathbb{R}_{\geq 0}(-u)$. q.e.d.

Proposition A.8. *For any face F of a convex polyhedral cone C in V, let $\pi : V \to V/\mathbb{R}F$ be the canonical projection to the quotient $\mathbb{R}$-vector space by the $\mathbb{R}$-subspace $\mathbb{R}F$ generated by F. Then $C + \mathbb{R}F$ is a convex polyhedral cone in V, while $\pi(C) = (C + \mathbb{R}F)/\mathbb{R}F$ is one in $V/\mathbb{R}F$. By the maps*

$$F' \mapsto F' + \mathbb{R}F \mapsto \pi(F') = (F' + \mathbb{R}F)/\mathbb{R}F \ ,$$

the three sets $\{F' \in \mathscr{F}(C); F' > F\}$, $\mathscr{F}(C + \mathbb{R}F)$ and $\mathscr{F}(\pi(C))$ are in bijective correspondence.

Proof. $C + \mathbb{R}F$ is obviously a convex polyhedral cone in V with the dual $(C + \mathbb{R}F)^\vee = C^\vee \cap F^\perp = F^*$. By Proposition A.6, we have $\mathscr{F}(F^*) = \{(F')^*; F' \in \mathscr{F}(C), F' > F\}$. Since $(F')^* = C^\vee \cap (F')^\perp = (C^\vee \cap F^\perp) \cap (F')^\perp = (C + \mathbb{R}F)^\vee \cap (F' + \mathbb{R}F)^\perp$, we obtain $\mathscr{F}(C + \mathbb{R}F) = \{F' + \mathbb{R}F; F' \in \mathscr{F}(C), F' > F\}$.

On the other hand, $F^\perp$ is the vector space dual to $V/\mathbb{R}F$ and $C^\vee \cap F^\perp = F^*$ is the dual cone of $\pi(C) = (C + \mathbb{R}F)/\mathbb{R}F$. We again have $\mathscr{F}(\pi(C)) = \{\pi(F'); F' \in \mathscr{F}(C), F' > F\}$. q.e.d.

Proposition A.9. *Let C be a convex polyhedral cone in V. A subset S of C is a face of C if and only if S satisfies the following conditions:*

(i) *S is a convex cone.*

(ii) *$v + v'$ is not contained in S for any $v \in C \setminus S$ and any $v' \in C$, that is, $C \setminus S$ is an ideal in the additive semigroup C.*

Proof. It suffices to show the "if" part. Replacing V by $\mathbb{R}C$, we may assume that $\dim C = \dim V$. We may further assume that C is the smallest face of C containing S, by replacing C by a face. We now show $S = C$ under these assumptions and (i), (ii).

S contains an interior point v_0 of C. Indeed, $C = \{u_1, \ldots, u_p\}^\vee$ for a minimal system of generators $u_1, \ldots, u_p$ for $C^\vee$ by Theorem A.2. Since we assume C to be the smallest face containing S and $\dim C = \dim V$, Lemma A.4 guarantees the existence, for each j, of $v_j \in S$ such that $\langle v_j, u_j \rangle > 0$. Then $v_0 := \sum_{1 \leq j \leq p} v_j$ satisfies $\langle v_0, u_j \rangle > 0$ for all j, hence is in the interior of C by Lemma A.4. Moreover, v_0 is in S by (i).

If there existed v in $C \setminus S$, then $av_0 = v + v'$ for some $v' \in C$ and a sufficiently large real number a, since $\langle av_0 - v, u_j \rangle \geq 0$ would then hold for all j, hence $v' := av_0 - v$ would be in C. Then av_0 would be in S by (i), while $v \in C \setminus S$ and $v' \in C$ would imply $v + v' \notin S$ by (ii), a contradiction. q.e.d.

We conclude this section by recalling the characteristic function for an open convex cone. It was introduced by Vinberg [V4] for homogenous convex cones arising in connection with homogeneous bounded domains. We refer the reader to the above paper for not so difficult proofs.

Proposition A.10. *Suppose the closure $\bar{C}$ of an open convex cone C is strongly convex, i.e., $\bar{C} \cap (-\bar{C}) = \{O\}$. Then the dual cone defined by $C' := \{u \in V^*;\ \langle v, u \rangle > 0, 0 \neq \forall v \in \bar{C}\}$ is an open convex cone with strongly convex closure and the duality $(C')' = C$ holds. For a Lebesgue measure du of V^* define the* characteristic function *$\varphi_C : C \to \mathbb{R}_{>0}$ for C by*

$$\varphi_C(v) := \int_{C'} \exp(-\langle v, u \rangle) du \quad \textit{for} \quad v \in C .$$

The integral converges and $\varphi_C(v)$ tends to $+\infty$ as v approaches the boundary of $\bar{C}$. φ_C is a (lower) convex function on C satisfying

$$\varphi_C(gv) = \varphi_C(v)/|\det(g)|$$

for g in the group $\mathrm{Aut}_{\mathbb{R}}(V; C)$ of $\mathbb{R}$-linear automorphisms of V preserving C. In particular, $\varphi_C(v)dv$ is an $\mathrm{Aut}_{\mathbb{R}}(V; C)$-invariant measure on C for a Lebesgue measure dv on V.

When C is a homogeneous cone, such a function on C is known to be essentially unique up to multiplication of positive real numbers, hence $\log \varphi_C$ has intrinsic significance for C up to addition of real constants.

Example. Let $V = \mathbb{R}^r$ with coordinate $(\xi_1, \ldots, \xi_r)$. The characteristic function for the positive orthant

$$C = (\mathbb{R}_{>0})^r = \{(\xi_1, \ldots, \xi_r) \in V;\ \xi_1 > 0, \ldots, \xi_r > 0\}$$

is easily seen to be

$$\varphi_C(\xi_1, \ldots, \xi_r) = \frac{1}{\xi_1 \xi_2 \ldots \xi_r} .$$

The characteristic function is known to have much stronger convexity property: Not only is φ_C convex, but $\log \varphi_C$ is a strictly (lower) convex function on C.

If we identify V (resp. V^*) with the tangent space (resp. the cotangent space) of C at a point v, then the exterior derivative $d\varphi_C(v) \in V^*$ of φ_C at $v \in C$ has value

$$(d\varphi_C(v))(a) = -\int_{C'} \langle a, u \rangle \exp(-\langle v, u \rangle) du \quad \text{at} \quad a \in V .$$

Moreover, the exterior derivative

$$d \log \varphi_C(v) = \frac{d\varphi_C(v)}{\varphi_C(v)} \in V^*$$

of $\log \varphi_C$ at $v \in C$, which is intrinsically defined for C, has the following interesting properties:

(i) The map $C \to C'$ sending v to $v^* := -d \log \varphi_C(v)$ is bijective.

(ii) The tangent hyperplane at $v \in C$ of the level hypersurface $\{x \in C;\ \varphi_C(x) = \varphi_C(v)\}$ for φ_C passing through v is given by

$$\{x \in V;\ \langle x, v^* \rangle = \dim V\} .$$

If we regard the symmetric quadratic differentials of φ_C and $\log \varphi_C$ at $v \in C$ as quadratic forms in $a \in V$, then we have

$$(d^2\varphi_C(v))(a) = \int_{C'} \langle a, u\rangle^2 \exp(-\langle v, u\rangle)du$$

$$d^2 \log \varphi_C(v) = \frac{d^2\varphi_C(v)}{\varphi_C(v)} - \left(\frac{d\varphi_C(v)}{\varphi_C(v)}\right)^2 .$$

In fact, the positive definiteness of this $d^2 \log \varphi_C$ implies the strict convexity of $\log \varphi_C$, hence the convexity of φ_C. Moreover, we can define an intrinsic Riemannian metric on C by means of $d^2 \log \varphi_C(v)$.

A.2 Convex Polyhedra

In connection with toric projective varieties in Chap. 2, we need to deal with general convex sets and convex polyhedra, aside from convex cones considered in the previous section.

A subset K of an r-dimensional $\mathbb{R}$-vector space V is *convex* if the line segment joining any pair v, v' of points in K is contained entirely in K, that is, $av + a'v' \in K$ for any a, a' in $\mathbb{R}_{\geq 0}$ satisfying $a + a' = 1$. A convex set K is called a *convex body* if it is compact and has an interior point.

The *convex hull* of a subset $S \subset V$ is the smallest convex set containing S. It is equal to the intersection of all the convex sets containing S.

An *affine half space* of V is a subset of the form

$$H^+(u; b) := \{v \in V;\ \langle v, u\rangle \geq b\} \quad \text{for} \quad u \in V^* \quad \text{and} \quad b \in \mathbb{R} .$$

For convenience, we sometimes call $H^+(O; b) = V$ for $b \leq 0$ and the empty set $H^+(O; b)$ for $b > 0$ also affine half spaces of V.

As we see in Proposition A.14 below, the results on convex sets, which we now recall, follow from those on convex cones in the previous section.

Theorem A.11. *The intersection of a family $\{H^+(u_i; b_i);\ i \in I\}$ of affine half spaces of V is a closed convex set. Conversely, a closed convex set in V is the intersection of affine half spaces, which are generally infinite in number.*

A *convex polyhedral set* in V is the intersection of a finite number of affine half spaces

$$\bigcap_{1\leq j\leq s} H^+(u_j\,;b_j)=\{v\in V;\,\langle v,u_j\rangle\geq b_j,1\leq\forall j\leq s\}$$

for $u_1,\ldots,u_s\in V^*$ and $b_1,\ldots,b_s\in\mathbb{R}$. It is thus the set of solutions of a finite system of linear inequalities and may be empty.

Theorem A.12. *A convex polyhedral set K in V is compact if and only if it is the convex hull of a finite subset of V. Such a K is called a* compact convex polyhedron *or a* convex polytope.

A convex polytope is necessarily the intersection of a finite number of affine half spaces, hence is a convex polyhedral set. The converse, however, does not hold in general.

For convex subsets K, K' of V, we define the *Minkowski sum* $K+K'$ and the *scalar multiple* cK for $c\in\mathbb{R}_{\geq 0}$ by

$$K+K':=\{v+v';\,v\in K, v'\in K'\}\quad\text{and}\quad cK:=\{cv;\,v\in K\}\ ,$$

which are convex sets in V. If K and K' are convex polyhedral sets, then so are $K+K'$ and cK.

Theorem A.13. *Any convex polyhedral set in V is the Minkowski sum of a convex polytope and a convex polyhedral cone with apex at the origin.*

The following easy proposition reduces the proof of all these results on convex sets to those on convex cones in the previous section:

Proposition A.14. *For a convex set $K\subset V$, let $\tilde{K}$ be the closure in the $\mathbb{R}$-vector space $\tilde{V}:=V\times\mathbb{R}$ of the subset $\mathbb{R}_{\geq 0}(K\times\{-1\}):=\{(av,\,-a);\,a\in\mathbb{R}_{\geq 0},v\in K\}$,* that is,

$$\tilde{K}=\{(av,\,-a);\,a\in\mathbb{R}_{\geq 0},v\in K\}\cup\{(v,0);\,av\in K\ \textit{for}\ a>>0\}\ .$$

Then $\tilde{K}$ is a convex cone in $\tilde{V}:=V\times\mathbb{R}$ such that

$$K\times\{-1\}=\tilde{K}\cap(V\times\{-1\})\ .$$

If K is a convex polyhedral set, then $\tilde{K}$ is a convex polyhedral cone.

We here choose rather unnatural -1, because of the way we introduce support functions in the next section. When conversely a convex cone $\tilde{K}$ in $V\times\mathbb{R}$ is given, $\tilde{K}\cap(V\times\{-1\})$ may be empty. When $\tilde{K}$ is a convex polyhedral cone, $\tilde{K}\cap(V\times\{-1\})$ is a convex polyhedral subset of $V\times\{-1\}$, but may not be a convex polytope (cf. Fig. A.1).

We may and do regard $\tilde{V}^*:=V^*\times\mathbb{R}$ as the $\mathbb{R}$-vector space dual to $\tilde{V}=V\times\mathbb{R}$ by the bilinear pairing

$$\langle(v,c),(u,b)\rangle:=\langle v,u\rangle+bc\quad\text{for}\quad(u,b)\in\tilde{V}^*\quad\text{and}\quad(v,c)\in\tilde{V}\ .$$

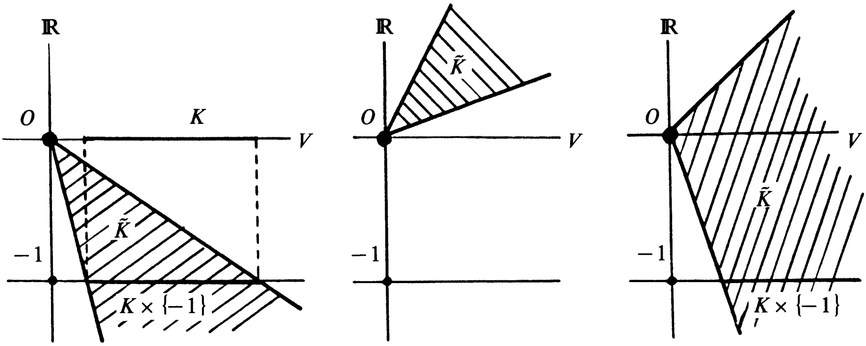


Fig. A.1

For a convex set K in V, consider the convex cone $\tilde{K}$ in $\tilde{V}$ as in Proposition A.14. The intersection of its dual cone $(\tilde{K})^\vee$ in $\tilde{V}^*$ with $V^*\times\{-1\}$ is of the form $K^\circ\times\{-1\}:=(\tilde{K})^\vee\cap(V^*\times\{-1\})$ with

$$K^\circ:=\{u\in V^*;\ \langle v,u\rangle\geq -1, \forall v\in K\} .$$

This K° is a convex subset of V^* and is called the *polar convex set* for K. If K is a convex polyhedral set, then so is K° and K° is called the *polar polyhedral set* for K.

The *dimension* $\dim K$ of a convex set $K\subset V$ is that of the smallest affine subspace of V containing K. When K is a convex cone, this definition coincides with that in the previous section.

Theorem A.15 (Carathéodory's Theorem). *Let d be the dimension of the convex polytope P obtained as the convex hull of a finite subset $\{v_1,\dots,v_s\}$ of V. Then any point of P is contained in the convex hull of $d+1$ points among $\{v_1,\dots,v_s\}$.*

Recall that a d-dimensional *simplex* is a convex polytope obtained as the convex hull of $d+1$ affinely independent points. We see easily from the above theorem that any d-dimensional convex polytope is a finite union of d-dimensional simplices.

Definition. A subset Q of a convex polyhedral set $P\subset V$ is called a *face* and denoted $Q<P$ if

$$P\subset H^+(u;b):=\{v\in V;\ \langle v,u\rangle\geq b\}\quad\text{and}$$

$$Q=P\cap\partial H^+(u;b)=\{v\in P;\ \langle v,u\rangle=b\}$$

for some $u\in V^*$ and $b\in\mathbb{R}$.

Proposition A.16. *Let $P\subset V$ be a convex polytope. The set $\mathscr{F}(P)$ of its faces is a finite partially ordered set with respect to the face relation $<$. P is its largest element, while the empty set is the smallest element. Moreover, $\mathscr{F}(P)$ is an abstract complex in the following sense: $Q\in\mathscr{F}(P)$ and $Q'<Q$ imply $Q'\in\mathscr{F}(P)$ and the intersection $Q_1\cap Q_2$ of $Q_1,Q_2\in\mathscr{F}(P)$ is a face of Q_1 as well as of Q_2.*

It suffices to apply Proposition A.5 to $\tilde{K}$ in Proposition A.14.

As the result corresponding to Proposition A.16 in a similar manner, we have the following *polarity*, which is valid only under a restriction. As we see in Corollary A.19 in the next section, we can use the notion of support functions to formulate the polarity valid without any restriction.

Proposition A.17. *Suppose a convex polytope $P \subset V$ contains the origin O in this interior. Then its polar convex set P° is a convex polytope in V^* and the polarity $(P^\circ)^\circ = P$ holds. We have a Galois correspondence $Q \mapsto Q^*$ between the sets $\mathscr{F}(P)$, $\mathscr{F}(P^\circ)$ of their faces. In particular, we have* $\dim Q + \dim Q^* = \dim V - 1$ *for any face Q of P.*

We leave to Sect. A.5 the morphology for a convex polytope P, that is, the combinatorial structure of the set $\mathscr{F}(P)$ of its faces. Here, we just point out the following:

A d-dimensional convex polytope $P \subset V$ is *simplicial*, if all the facets (i.e., codimension one faces) are $(d-1)$-dimensional simplices. On the other hand, P is *simple*, if there exist exactly d edges (i.e., one-dimensional faces) incident with each vertex (i.e., zero-dimensional face) of P, or equivalently, exactly d facets meet at each vertex. These two notions are dual to each other in the sense of the polarity above. As Grünbaum [G6, pp. 57–58] points out, d-dimensional simplices are the most generic d-dimensional convex polytopes. Slightly less generic in two different respects are d-dimensional simplicial convex polytopes and d-dimensional simple convex polytopes. A d-dimensional convex polytope P is simplicial if and only if no $d+1$ vertices lie on the same facet of P. On the other hand, P is simple if and only if no $d+1$ hyperplanes determining facets of P have any vertex in common.

A.3 Support Functions

It is an idea originally due to Minkowski to describe compact convex sets in V in terms of their support functions. Although less geometric, they are more convenient to deal with than compact convex sets themselves. Besides, they fit in nicely with the theory of toric projective varieties in Chap. 2. Convex analysis treats the theory of convex sets as a special case of the theory of convex functions by means of support functions, their variants and generalizations.

We draw the reader's attention to the fact that our convention here is different in sign from the usual one found, for instance, in Rockafellar [R7], since we would rather like to use *upper* convex (concave, in the usual terminology) functions for consistency with the existing literature on toric varieties.

For a *nonempty* compact convex set K in V, the real number

$$h_K(u) := \inf\{\langle v, u\rangle ; v \in K\} \quad \text{for} \quad u \in V^*$$

exists. We thus obtain a real-valued function $h = h_K : V^* \to \mathbb{R}$ on V^* called the *support function* for K. For each $u \in V^*$, the affine half space $H^+(u; h_K(u)) = \{v \in V; \langle v, u\rangle \geq h_K(u)\}$ contains K, and K has a nonempty intersection with its boundary $\partial H^+(u; h_K(u)) = \{v \in V; \langle v, u\rangle = h_K(u)\}$, which we call the *supporting hyperplane* for K in the direction u. By Theorem A.11, we have

$$K = \bigcap_{u \in V^*} H^+(u; h_K(u)) = \{v \in V; \langle v, u\rangle \geq h_K(u), \forall u \in V^*\} .$$

Remark. It is customary to take the supremum rather than the infimum in the definition of support functions. The result is essentially the same. When a convex set K is not compact, $\inf\{\langle v, u\rangle; v \in K\}$ may not exist for some $u \in V^*$. Even then, we can introduce the support function for K by allowing functions whose domain of definition is not the whole V^* or functions on V^* with values in $\mathbb{R} \cup \{+\infty, -\infty\}$ (cf. Rockafellar [R7]).

The following result, which can be found in [R7, Corollary 13.2.2], is a prototype for the general principle of treating the theory of convex sets as a part of the theory of convex functions.

Theorem A.18. *Let $\mathfrak{K}(V)$ be the set of nonempty compact convex sets in V. Denote by* $\mathrm{SF}(V^*)$ *the set of real-valued functions $h: V^* \to \mathbb{R}$ on V^* which are* positively homogeneous *and* upper convex, *i.e., $h(cu) = ch(u)$ and $h(u+u') \geq h(u) + h(u')$ for any $c \in \mathbb{R}_{\geq 0}$ and $u, u' \in V^*$.*

(i) *We have mutually inverse maps $\mathfrak{K}(V) \to \mathrm{SF}(V^*)$ and $\mathrm{SF}(V^*) \to \mathfrak{K}(V)$ which respectively send K to h_K and h to $\square_h$ defined as follows:*

$$h_K(u) := \inf\{\langle v, u\rangle; v \in K\} \quad \text{for} \quad u \in V^*$$

$$\square_h := \{v \in V; \langle v, u\rangle \geq h(u), \ \forall u \in V^*\} .$$

(ii) *Under the maps above, the Minkowski sum $K+K'$ and a nonnegative scalar multiple cK in $\mathfrak{K}(V)$ correspond to the sum function $h+h'$ and the nonnegative scalar multiple ch, respectively.*

Proof. (ii) is obvious by definition.

As for (i), h_K clearly belongs to $\mathrm{SF}(V^*)$. We have already seen that $K = \square_{h_K}$. On the other hand, $\square_h = \cap_{u \in V^*} H^+(u; h(u))$ for $h \in \mathrm{SF}(V^*)$ is obviously a closed convex set. It is bounded, hence belongs to $\mathfrak{K}(V)$, since $-h(-u) \geq \langle v, u\rangle \geq h(u)$ for any $u \in V^*$ and $v \in \square_h$.

Since h is positively homogeneous and upper convex, the subset

$$G_h := \{(u, \lambda) \in V^* \times \mathbb{R}; h(u) \geq \lambda\}$$

of $V^* \times \mathbb{R}$ is a closed convex cone with apex at the origin $(O, 0)$. The cone in $V \times \mathbb{R}$ dual to G_h is easily seen to be

$$\mathbb{R}_{\geq 0}(\square_h \times \{-1\}) = \{(v, \mu) \in V \times \mathbb{R}; \mu \leq 0 \text{ and } \langle v, u\rangle + \mu h(u) \geq 0, \forall u \in V^*\}$$

considered in Proposition A.14. A supporting hyperplane of G_h at any given point $(u_0, h(u_0))$ of its boundary is also easily seen to be of the form $\partial H^+((v_0, -1); 0)$ for some $v_0 \in \square_h$. In this case, we have $\langle v_0, u\rangle \geq h(u)$ for all $u \in V^*$ with the equality holding for $u = u_0$. Hence $h(u_0) = \langle v_0, u_0\rangle = \inf\{\langle v, u_0\rangle; v \in \square_h\}$ and h coincides with the support function for $\square_h$. q.e.d.

Of particular importance in Chap. 2 are the support functions for nonempty convex polytopes:

Corollary A.19. *Under the correspondence in Theorem A.*18, $h \in \mathrm{SF}(V^*)$ *is the support function for a convex polytope* P *in* V *if and only if* h *is* piecewise linear, *i.e., there exists a finite decomposition of* V^* *into a union of convex polyhedral cones which do not intersect among themselves in their relative interiors such that the restriction of* h *to each convex polyhedral cone in the decomposition is a linear function.*

For any given convex polytope P, *there exists the coarsest such decomposition* Π *of* V^*, *which satisfies the following properties:*

(i) *Define*

$$Q^\dagger := \{u \in V^*;\ \langle v, u\rangle = h(u),\ \forall v \in Q\}$$

for each nonempty face Q *of* P. *Then the map sending* Q *to* $Q^\dagger$ *gives rise to a bijection* $\mathscr{F}(P) \setminus \{\textit{the empty set}\} \xrightarrow{\sim} \Pi$.

(ii) $\dim Q + \dim Q^\dagger = \dim V$ *for each nonempty face* Q *of* P.

(iii) *If* $Q_1 > Q_2$ *for nonempty faces* Q_1 *and* Q_2 *of* P, *then* $Q_1^\dagger < Q_2^\dagger$.

(iv) *The map inverse to that in* (i) *sends* $C \in \Pi$ *to*

$$C^\dagger := \{v \in P;\ \langle v, u\rangle = h(u),\ \forall u \in C\} \in \mathscr{F}(P) \setminus \{\textit{the empty set}\}\ .$$

Proof. As we saw in Proposition A.14 and the proof of Theorem A.18, $G_h := \{(u, \lambda) \in V^* \times \mathbb{R};\ h(u) \geq \lambda\}$ is the cone in $V^* \times \mathbb{R}$ dual to the convex polyhedral cone $\tilde{P} := \mathbb{R}_{\geq 0}(P \times \{-1\})$ in $V \times \mathbb{R}$. The faces of $\tilde{P}$ other than $\{(O, 0)\}$ are $\tilde{Q} := \mathbb{R}_{\geq 0}(Q \times \{-1\})$ for nonempty faces Q of P. By the Galois correspondence in Proposition A.6, $\tilde{Q}$ corresponds to the face

$$G_h \cap (Q \times \{-1\})^\perp = \{(u, \lambda) \in G_h;\ \langle v, u\rangle = \lambda,\ \forall v \in Q\}$$

of G_h. This face coincides with

$$\{(u, h(u));\ \langle v, u\rangle = h(u),\ \forall v \in Q\}\ ,$$

since $h(u) = \inf\{\langle v, u\rangle;\ v \in P\}$ and since $h(u) \geq \lambda$ for (u, λ) in G_h. $Q^\dagger$ is nothing but the image of this face by the first projection $V^* \times \mathbb{R} \to V^*$. The remaining assertions follow easily from this fact. q.e.d.

Remark. Such h is said to be *strictly upper convex with respect to* Π in Lemma 2.12 in Sect. 2.3.

Corollary A.19 is a generalization of Proposition A.17 in the following sense: If $\dim P = \dim V$ holds in Proposition A.17, then we have $G_h \cap (-G_h) = \{(O, 0)\}$. Consequently, each cone C in the decomposition Π is strongly convex, i.e., $C \cap (-C) = \{O\}$. Furthermore, if P contains O in its interior, then its support function h satisfies $h(u) < 0$ for $u \neq O$. Hence $P^\circ \times \{-1\} = G_h \cap (V^* \times \{-1\})$ is a convex polytope, where P° is the polar convex polytope for P defined in the previous section. The decomposition Π and $\mathscr{F}(P^\circ) \setminus \{P^\circ\}$ are in one-to-one correspondence, since so are the set of faces of G_h other than $\{(O, 0)\}$ and the set of nonempty faces of P°.

Let us state an easy consequence of what we have just seen for reference in Proposition 2.19.

Proposition A.20. *For a convex polytope $Q \subset V^*$ which contains the origin O in its interior, define a function $h: V^* \to \mathbb{R}$ by*

$$h(u) := \begin{cases} -\inf\{\lambda \in \mathbb{R}_{>0};\, u \in \lambda Q\} & \text{if } u \neq O \\ 0 & \text{if } u = O \end{cases} .$$

Then we have $Q = \{u \in V^*;\, h(u) \geq -1\}$. On the other hand, for any proper face Q' of Q, the union $\mathbb{R}_{\geq 0}Q'$ of the rays from O through the points of Q' is a strongly convex polyhedral cone in V^*, and

$$\Pi := \{\mathbb{R}_{\geq 0}Q';\, \text{proper faces } Q' \text{ of } Q\}$$

is a finite decomposition of V^* by strongly convex polyhedral cones which do not intersect among themselves in their relative interiors. h is strictly upper convex with respect to Π and is the support function for the polar polytope $Q^\circ \subset V$.

h above is called the *gauge function* for the polytope $Q \subset V^*$ (cf. Rockafellar [R7, p. 28 and Theorem 14.5 on p. 125]).

Remark. We can introduce support functions also in the following situation. They can be convenient in dealing with the *Newton polyhedra* for the singularities of analytic functions and algebraic varieties (see, for instance, [K3], [K4], [K6], [O4], [O5], [V1], [V2]). They also play a key rôle in our discussion of *Tsuchihashi cusp singularities* in Chap. 4:

Suppose a closed convex cone $C \subset V$ satisfies $C \cap (-C) = \{O\}$ and $C + (-C) = V$. Its dual cone $C^\vee := \{u \in V^*;\, \langle v, u\rangle \geq 0, \forall v \in V\}$ in V^* then satisfies $(C^\vee) \cap (-C^\vee) = \{O\}$ and $C^\vee + (-C^\vee) = V^*$. If a nonempty closed convex set $\Theta \subset C$ is an *ideal* of the additive semigroup C, i.e., has the property that $\Theta \supset C + \Theta := \{v + \theta;\, v \in C, \theta \in \Theta\}$, then we obtain the *support function* $h = h_\Theta : C^\vee \to \mathbb{R}_{\geq 0}$ for Θ by

$$h(u) := \inf\{\langle \theta, u\rangle;\, \theta \in \Theta\} \quad \text{for} \quad u \in C^\vee .$$

It obviously satisfies the following properties:

(i) h is *positively homogeneous*, i.e., $h(\lambda u) = \lambda h(u)$ for any nonnegative real number λ and any $u \in C^\vee$.

(ii) h is *upper convex*, i.e., $h(u + u') \geq h(u) + h(u')$ for any $u, u' \in C^\vee$.

(iii) $\Theta = \{v \in C;\, \langle v, u\rangle \geq h(u), \forall u \in C^\vee\}$.

(iv) h is continuous with respect to the usual topology.

(v) In particular, suppose Θ is a polyhedral set as well, i.e., Θ coincides with the convex hull of $\cup_{1 \leq j \leq s}(\theta_j + C)$ for a finite subset $\{\theta_1, \ldots, \theta_s\}$ of Θ. Then there exists a *finite* decomposition of $C^\vee$ by convex polyhedral cones such that the restriction of h to each polyhedral cone is linear. In fact, the coarsest such decomposition exists.

The *polar convex set* $\Theta^\circ \subset C^\vee$ for a closed convex ideal $\Theta \subset C$ defined by

$$\Theta^\circ := \{u \in C^\vee;\, h(u) \geq 1\} = \{u \in C^\vee;\, \langle v, u\rangle \geq 1, \forall v \in \Theta\}$$

is obviously a closed convex ideal of $C^\vee$. Thus we can in turn define its support function $h^\circ := h_{\Theta^\circ} : C \to \mathbb{R}_{\geq 0}$ and its polar convex set $\Theta^{\circ\circ} = \{v \in C;\ h^\circ(v) \geq 1\}$. The *polarity* $\Theta^{\circ\circ} = \Theta$ holds for any closed convex ideal Θ of C.

A.4 The Mixed Volume of Compact Convex Sets

This section deals with results related to the basic notion of mixed volumes, due to Minkowski, for compact convex sets in $V \cong \mathbb{R}^r$. Eggleston [E1, Chaps. 4 and 5] is a basic and convenient reference for the proofs.

As before, denote by $\mathfrak{K} = \mathfrak{K}(V)$ the set of nonempty compact convex sets in V. A fixed Euclidean distance d in V makes $\mathfrak{K}$ a complete metric space with respect to the induced Hausdorff distance δ for $\mathfrak{K}$. The topology thus defined on $\mathfrak{K}$ is called the *Hausdorff topology:* As usual, define the distance of $v \in V$ from $K \in \mathfrak{K}$ to be $d(v, K) := \inf\{d(v, w);\ w \in K\}$. For $\varepsilon \in \mathbb{R}_{>0}$, define the ε-neighborhood of $K \in \mathfrak{K}$ by $\mathfrak{U}(K; \varepsilon) := \{v \in V;\ d(v, K) < \varepsilon\}$. Then the *Hausdorff distance* of $K, K' \in \mathfrak{K}$ is defined to be

$$\delta(K, K') := \inf\{\varepsilon \in \mathbb{R}_{>0};\ \mathfrak{U}(K; \varepsilon) \supset K' \quad \text{and} \quad \mathfrak{U}(K'; \varepsilon) \supset K\} ,$$

which is easily seen to satisfy the axioms for distance.

According to the **Blaschke selection theorem**, $\{K \in \mathfrak{K};\ K \subset K_0\}$ for any fixed $K_0 \in \mathfrak{K}$ is a compact subset of $\mathfrak{K}$ with respect to the Hausdorff topology. It is also known that nonempty convex polytopes in V form a dense subset of $\mathfrak{K}$, that is, any $K \in \mathfrak{K}$ can be approximated by convex polytopes.

Let us now consider the Lebesgue measure vol_r for V normalized with respect to the Euclidean distance d. It gives rise to a function

$$\mathrm{vol}_r : \mathfrak{K} \to \mathbb{R}_{\geq 0} ,$$

which turns out to be continuous with respect to the Hausdorff topology. It is monotone increasing, i.e., $K \subset K'$ implies $\mathrm{vol}_r(K) \leq \mathrm{vol}_r(K')$. Moreover, it is homogeneous of degree r, that is, $\mathrm{vol}_r(\lambda K) = \mathrm{vol}_r(K)\lambda^r$ for any $\lambda \in \mathbb{R}_{\geq 0}$ and $K \in \mathfrak{K}$.

For $\lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0}$ and $K_1, \ldots, K_r \in \mathfrak{K}$, the nonnegative linear combination $\lambda_1 K_1 + \ldots + \lambda_r K_r$ belongs to $\mathfrak{K}$. It is known that

$$\frac{1}{r!}\mathrm{vol}_r(\lambda_1 K_1 + \ldots + \lambda_r K_r)$$

is a homogeneous polynomial of degree r in $\lambda_1, \ldots, \lambda_r$. Its coefficient

$$V_r(K_1, \ldots, K_r)$$

for $\lambda_1 \lambda_2 \ldots \lambda_r$ is called the *mixed volume* of $K_1, \ldots, K_r$. Note here that r is the dimension of the ambient space V. The mixed volume satisfies the following properties:

(1) It is always nonnegative so that we have a map $V_r : \mathfrak{K} \times \mathfrak{K} \times \ldots \times \mathfrak{K} \to \mathbb{R}_{\geq 0}$.

(2) $V_r(K_1, \ldots, K_r)$ is symmetric with respect to $K_1, \ldots, K_r$.

(3) It is linear with respect to each K_i. For instance for $i = 1$, this means $V_r(\lambda K_1 + \lambda' K_1', K_2, \ldots, K_r) = \lambda V_r(K_1, K_2, \ldots, K_r) + \lambda' V_r(K_1', K_2, \ldots, K_r)$ for λ, λ' nonnegative.

(4) It is monotone increasing with respect to each K_i. For example for $i=1$, we have $V_r(K_1, K_2, \ldots, K_r) \leq V_r(K_1', K_2, \ldots, K_r)$ if $K_1 \subset K_1'$.

(5) V_r is a continuous map to $\mathbb{R}_{\geq 0}$ from $\mathfrak{K} \times \ldots \times \mathfrak{K}$ endowed with the product of the Hausdorff topology for each $\mathfrak{K}$.

(6) $V_r(K, K, \ldots, K) = \mathrm{vol}_r(K)$ for each $K \in \mathfrak{K}$.

As we see in Sects. 2.2 and 2.4, the mixed volume corresponds to the intersection number for toric varieties, which is one of the basic notions in algebraic geometry. In the geometry of convex bodies itself, the mixed volume also plays an important rôle as follows:

The r-dimensional closed *unit ball* $\mathbb{B} := \{v \in V; d(O, v) \leq 1\}$ in V is a member of $\mathfrak{K}$. For ε positive real, $K + \varepsilon\mathbb{B}$ is the closure of the ε-neighborhood $\mathfrak{U}(K; \varepsilon)$ of $K \in \mathfrak{K}$ considered before. We then define the *surface area* of $K \in \mathfrak{K}$ to be

$$A(K) := \lim_{\varepsilon \to +0} \frac{1}{\varepsilon} \{\mathrm{vol}_r(K + \varepsilon\mathbb{B}) - \mathrm{vol}_r(K)\} .$$

This limit exists and we have

$$\frac{A(K)}{r} = V_r(\mathbb{B}, K, \ldots, K) ,$$

since

$$\begin{aligned} \mathrm{vol}_r(K + \varepsilon\mathbb{B}) &= V_r(K + \varepsilon\mathbb{B}, \ldots, K + \varepsilon\mathbb{B}) \\ &= \mathrm{vol}_r(K) + \varepsilon r V_r(\mathbb{B}, K, \ldots, K) + O(\varepsilon^2) . \end{aligned}$$

Thus $A(K)$, as a function in $K \in \mathfrak{K}$, is monotone increasing and continuous with nonnegative real values.

For $K', K \in \mathfrak{K}$ and $0 \leq j \leq r$, let us simply denote the mixed volume of $K_1 = K_2 = \ldots = K_j := K'$ and $K_{j+1} = \ldots = K_r := K$ by

$$v_j := V_r(K'^{[j]}, K^{[r-j]}) .$$

Thus $v_0 = \mathrm{vol}_r(K)$ and $v_r = \mathrm{vol}_r(K')$. For $\lambda', \lambda \in \mathbb{R}_{\geq 0}$, we have

$$\mathrm{vol}_r(\lambda' K' + \lambda K) = \sum_{j=0}^{r} \binom{r}{j} v_j \lambda'^j \lambda^{r-j} .$$

The *Alexandrov-Fenchel inequalities*

$$v_j^2 \geq v_{j-1} v_{j+1} \quad \text{for} \quad 1 \leq j \leq r-1$$

is known to hold. Consequently, we have

$$v_j^r \geq v_0^{r-j} v_r^j \quad \text{for} \quad 0 \leq j \leq r$$

and the *Brünn-Minkowski inequality*

$$\mathrm{vol}_r(K' + K)^{1/r} \geq \mathrm{vol}_r(K')^{1/r} + \mathrm{vol}_r(K)^{1/r} .$$

Since $v_1 = V_r(\mathbb{B}, K^{[r-1]}) = A(K)/r$, we in particular have the so-called *isoperimetric inequality*

$$\left(\frac{A(K)}{\mathrm{r}}\right)^r \geq \mathrm{vol}_r(K)^{r-1} \mathrm{vol}_r(\mathbb{B}) .$$

When $r=2$, for instance, $\mathrm{vol}_2(K)$ is the area of K, while $A(K)$ is the circumference of K. Since $\mathrm{vol}_2(\mathbb{B})=\pi$, we have the well-known isoperimetric inequality for plane convex figures

$$A(K)^2 \geq 4\pi \,\mathrm{vol}_2(K) \ .$$

The equality here is known to hold if and only if K is a disk.

Osserman [O6] explains various results and long history concerning the isoperimetric inequality and its variants as well as the characterization for the case of equality. We here only mention the following in connection with the algebro-geometric version of Teissier's problem in Sect. 2.4:

For $K', K \in \mathfrak{K}$, we define the *inradius* $\varrho(K:K')$ and the *circumradius* $R(K:K')$ by

$$\varrho(K:K') := \sup\{\lambda \in \mathbb{R}_{\geq 0}\,;\, x+\lambda K' \subset K, \exists x \in V\}$$

$$R(K:K') := \inf\{\lambda \in \mathbb{R}_{\geq 0}\,;\, x+\lambda K' \supset K, \exists x \in V\} \ .$$

When $K'=\mathbb{B}$ in particular, $\varrho(K:\mathbb{B})$ is the radius of the largest balls contained in K, while $R(K:\mathbb{B})$ is the radius of the smallest balls containing K.

Flanders obtained the following strong result concerning the inradius and the circumradius for $r=2$: Consider the quadratic equation

$$v_0 - 2v_1 t + v_2 t^2 = 0$$

in t with $v_0 = \mathrm{vol}_2(K)$, $v_1 = V_2(K', K)$ and $v_2 = \mathrm{vol}_2(K')$. It has real roots by the Alexandrov-Fenchel inequality $v_1^2 \geq v_0 v_2$. The *Flanders inequality* asserts that these roots satisfy

$$(v_1 + \sqrt{v_1^2 - v_0 v_2})/v_2 \geq R(K:K') \geq v_1/v_2 \geq v_0/v_1 \geq \varrho(K:K') \geq (v_1 - \sqrt{v_1^2 - v_0 v_2})/v_2 \ .$$

As a consequence, we get the *Bonnesen inequality*

$$v_1^2 - v_0 v_2 \geq (R(K:K') - \varrho(K:K'))^2 v_2^2/4 \ .$$

If $v_1^2 = v_0 v_2$, then the left hand side vanishes. Hence $R(K:K') = \varrho(K:K')$, which implies $K = x + \lambda K'$ for some $x \in V$ and $\lambda \in \mathbb{R}_{\geq 0}$. The case $K' = \mathbb{B}$ thus gives the proof for the characterization above of the equality in the isoperimetric inequality for plane convex figures.

Teissier [T3] posed the following for general r.

Teissier's Problem (Compact Convex Version). Estimate the inradius $\varrho(K:K')$ and the circumradius $R(K:K')$ for compact convex sets K, K' in $V \cong \mathbb{R}^r$ in terms of the mixed volumes $v_j = V_r(K'^{[j]}, K^{[r-j]})$, $0 \leq j \leq r$. When $v_0 = v_1 = \ldots = v_r$, the estimate should guarantee the equality $\varrho(K:K') = R(K:K') = 1$ so that K is a parallel translate of K'. Can the roots of the algebraic equation

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} v_j t^j = 0$$

of degree r be used in the estimate?

As we explained earlier, the mixed volume is continuous and monotone increasing with respect to the Hausdorff topology for $\mathfrak{K}(V)$. Besides, nonempty convex polytopes in V form a dense subset of $\mathfrak{K}(V)$. Thus in answering the above problem to a considerable degree, we may restrict ourselves to the case of convex polytopes, or more specifically, those with vertices in a fixed lattice $\mathbb{Z}^r$ in V.

As we explain in Sect. 2.4, the problem above is thus reduced to a special case, for toric projective varieties, of a problem on intersection numbers of line bundles on general algebraic varieties.

A.5 Morphology for Convex Polytopes

In this section, we recall relevant results on the morphology for r-dimensional convex polytopes P in a real affine space $V \cong \mathbb{R}^r$, that is, the combinatorial study of the finite abstract complex $\mathscr{F}(P)$ consisting of the faces of P and introduced in Proposition A.16.

For $-1 \leq j \leq r$, let us denote by $f_j = f_j(P)$ the number of j-dimensional faces of P. We regard the empty set as a face of dimension -1 so that we have $f_{-1} = f_r = 1$. We have the well-known *Euler relation*

$$\sum_{j=-1}^{r} (-1)^j f_j = 0, \quad \text{that is,} \quad \sum_{j=0}^{r-1} (-1)^j f_j = 1 + (-1)^{r-1} .$$

This latter is nothing but the Euler-Poincaré characteristic for a sphere S^{r-1} homeomorphic to the boundary of P. The Euler relation is known to be the unique linear equality satisfied in general by $f_0, f_1, \ldots, f_{r-1}$ (cf. [G6, p. 98]).

For $r \geq 1$, we have a cell decomposition of a sphere S^{r-1} which gives information combinatorially equivalent to $\mathscr{F}(P)$. Indeed. P is contained in a large enough closed ball $\mathbb{B}^r$ with center at an interior point of P. For each nonempty face F of P consider the cone with apex at the center of $\mathbb{B}^r$ obtained as the union of the rays from the center to points of F. The intersection of the cone with the boundary S^{r-1} of $\mathbb{B}^r$ is a convex spherical cell.

P is a point for $r=0$ while it is a line segment for $r=1$. When $r=2$, P is a plane convex polygon with the morphology determined completely by the number $f_0 = f_1$, or equivalently, by the corresponding finite decomposition of a circle S^1.

Already for $r=3$, however, the morphology is rather complicated. For reference in Sect. 1.7, we here recall relevant results found in Grünbaum [G6, Chap. 13], [G7].

Unfortunately, the higher dimensional analogue of the following basic result for $r=3$ is known to be false.

Steinitz's Theorem (cf. [G6, § 13.1]). *Any finite cell decomposition of a 2-sphere S^2 is combinatorially equivalent to one obtained from a three-dimensional convex polytope in the manner described above. Thus the combinatorial classification of $\mathscr{F}(P)$ for three-dimensional convex polytopes P is equivalent to that of finite cell decompositions of S^2.*

By Riemann's stereographic projection from the north pole of S^2 to a plane, each finite cell decomposition of S^2 gives rise to a *finite planar graph*. A finite planar graph is known to be obtained in this way if and only if it is 3-*connected*, i.e., any pair of its vertices can be joined by at least three paths which have no common points other than the two end vertices. Consequently, the morphology for three-dimensional convex polytopes is also equivalent to the combinatorial classification of 3-connected finite planar graphs.

Let P be a three-dimensional convex polytope. f_0 is the number of its vertices (zero-dimensional faces), f_1 is that of its edges (one-dimensional faces) and f_2 is the number of two-dimensional faces (simply called faces now). The Euler relation is $f_0 - f_1 + f_2 = 2$ in this case.

Let us now restrict our attention to the morphology for *simplicial* three-dimensional convex polytopes. By the polarity in Sect. A.2, it is dual to that for *simple* three-dimensional convex polytopes. On the other hand by Steinitz's theorem above, it is equivalent to the combinatorial classification of finite *triangulations* of S^2.

We have $2f_1 = 3f_2$ for simplicial P, since each edge is common to exactly two triangles. Hence by the Euler relation, we get

$$f_1 = 3f_0 - 6 \quad \text{and} \quad f_2 = 2f_0 - 4 \ .$$

In particular, f_0 determines f_1 and f_2 in the simplicial case. For each integer $v \geq 3$, let $p(v)$ be the number of *v-valent* vertices of P (i.e., vertices incident with exactly v edges). Thus we have

$$f_0 = \sum_{v \geq 3} p(v) \quad \text{and} \quad 2f_1 = 3f_2 = \sum_{v \geq 3} vp(v) \ .$$

Consequently, f_0, f_1, f_2 are determined by a sequence of nonnegative integers $\{p(v); c \geq 3\}$ satisfying $\sum_{v \geq 3}(6-v)p(v) = 12$. We can rewrite this as an equality

$$3p(3) + 2p(4) + p(5) = 12 + \sum_{v \geq 7} (v-6)p(v)$$

with the right hand side positive. It follows that there always exist vertices of valency 3, 4 or 5. Note that the above equality does not involve the number $p(6)$ of 6-valent vertices. In fact, Eberhard showed (cf. [G6, § 13.3]) that any given sequence $\{p(v); 6 \neq v \geq 3\}$ of nonnegative integers satisfying the above equality can be realized, together with a suitable $p(6)$, as the sequence of the numbers of vertices of various valencies for a three-dimensional simplicial convex polytope. The inductive construction we now explain played a key rôle in the proof. It seems to be an interesting coincidence that two of the three inductive steps correspond to equivariant blowing-ups for three-dimensional toric varieties (cf. Sect. 1.7). Although we here describe the inductive steps for finite triangulations of S^2, which are equivalent by Steinitz's theorem to those for three-dimensional simplicial convex polytopes, it is easier and more natural to describe them dually in terms of "truncations" of edges or vertices for three-dimensional simple convex polytopes.

The Induction Theorem of Brückner-Eberhard (cf. [G6, p. 270]). *Suppose a finite triangulation T of S^2 is given. We get a triangulation T' of S^2 with one more vertex, if a vertex of T is "split into two" by one of the three steps* (i), (ii), (iii) *shown in Fig. A.2. We can obtain any given finite triangulation of S^2 from the tetrahedral triangulation by splitting vertices finitely many times.*

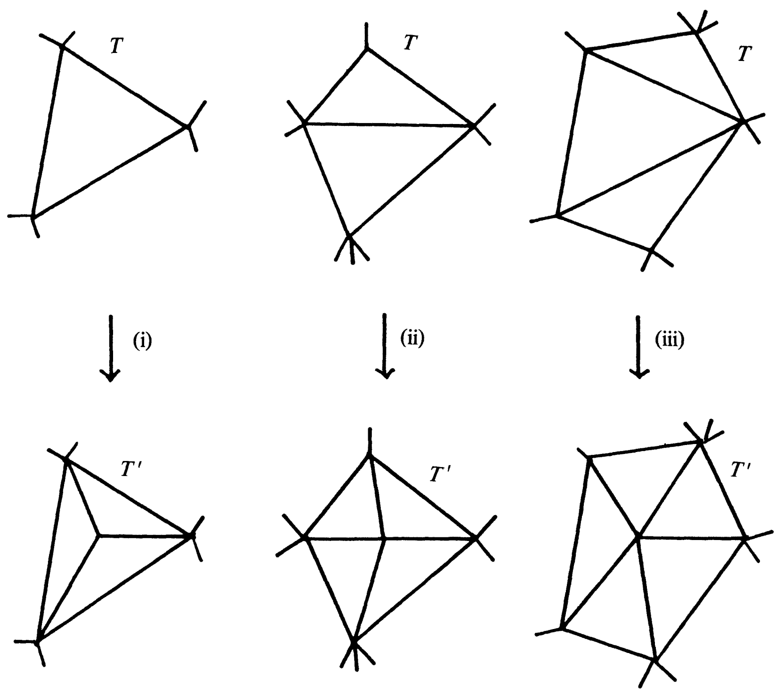


Fig. A.2

It is not so easy, however, to check possible combinatorial equivalence of two triangulations of S^2 we obtain by splitting vertices finitely often in two different ways. According to [G6, p. 424] and [G7, p 1174], there exist the following number of combinatorially different triangulations of S^2 with $f_0 \leq 12$ vertices:

$$1\ (f_0=4)\ ,\quad 1\ (f_0=5)\ ,\quad 2\ (f_0=6)\ ,\quad 5\ (f_0=7)\ ,\quad 14\ (f_0=8)\ ,$$

$$50\ (f_0=9)\ ,\quad 233\ (f_0=10)\ ,\quad 1249\ (f_0=11)\ ,\quad 7595\ (f_0=12)\ .$$

For reference in Sect. 1.7, we list up in Figs. A.3 and A.4 all the combinatorially different triangulations of S^2 with $f_0 \leq 9$. Also included in Fig. A.4 is the icosahedral triangulation, one of the 7595 different triangulations with $f_0=12$ vertices. Note that the figures shown are the stereographic projections of the triangulations from a vertex of the highest valency so that we are having one vertex at infinity. They are labeled and arranged in the following way: To a triangulation with $p(v)$ vertices of valency v for each $v \geq 3$, we give the label $\prod_{v \geq 3} v^{p(v)}$. We add (i), (ii),... to the label if there are more than one triangulations with the same label. In the label, we omit $v^{p(v)}$ if $p(v)=0$. We can read off the number of vertices as the sum $f_0 = \sum_{v \geq 3} p(v)$ of the exponents in each label. For each fixed f_0, the triangulations are arranged in the lexicographic order with respect to decreasing v. (The list for $f_0=9$ is due to Y. Kado-oka and M. Ohshima.)

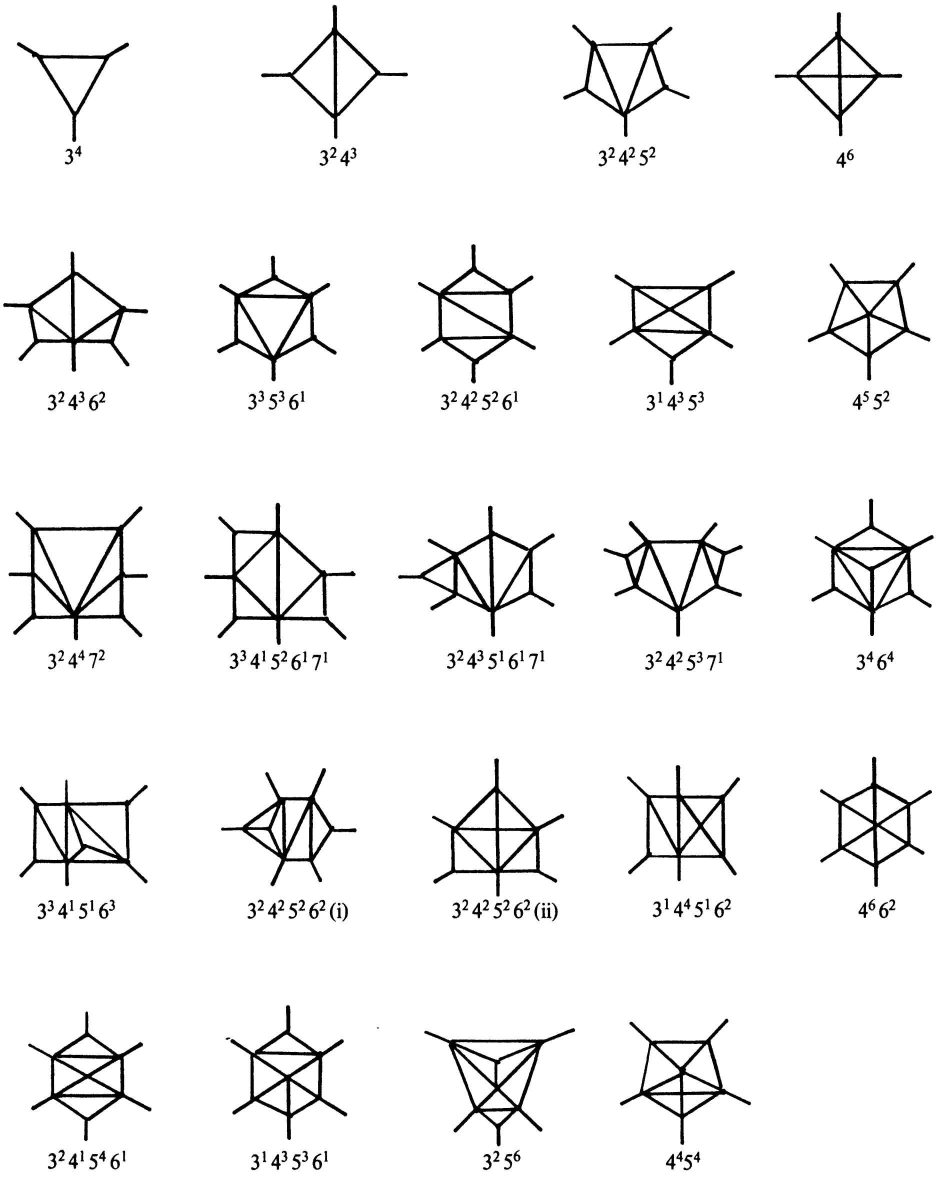


Fig. A.3. The triangulations of S^2 with $f_0 \leq 8$

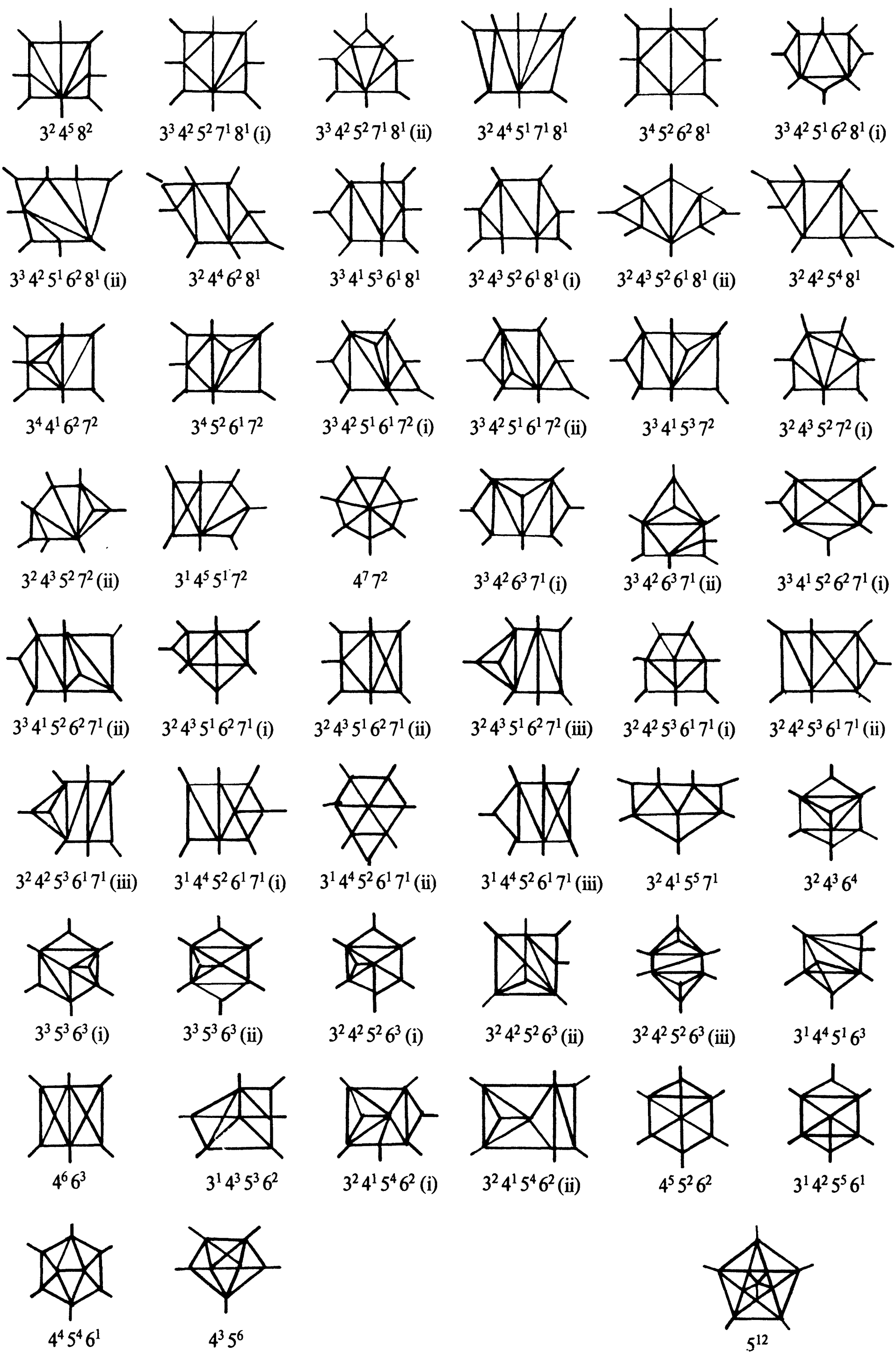


Fig. A.4. The triangulations of S^2 with $f_0 = 9$ and the icosahedral triangulation

Let us return to the morphology for convex polytopes P of an arbitrary dimension r.

(I) **Motzkin's upper bound conjecture** was proved by McMullen in 1970. Stanley [S12] later gave a new proof using commutative algebra. The number f_j of j-dimensional faces of P for $0 \le j \le r-1$ satisfies

$$f_j \le \sum_{k=1}^{r/2} \frac{f_0}{f_0-k}\binom{f_0-k}{k}\binom{k}{j+1-k} \quad \text{for } r \text{ even}$$

$$f_j \le \sum_{k=0}^{(r-1)/2} \frac{j+2}{f_0-k}\binom{f_0-k}{k+1}\binom{k+1}{j+1-k} \quad \text{for } r \text{ odd} ,$$

where $\binom{a}{b}$ for integers a, b are defined as follows: It is the usual binomial coefficient if $0 \le b \le a$, while it is zero otherwise except $\binom{-1}{0} = 1$. The complicated term on the right-hand side in each case is the number of j-dimensional faces of an r-dimensional cyclic polytope with f_0 vertices. It suffices to show the above inequalities for simplicial convex polytopes, since we can always deform a convex polytope to a simplicial one by slightly "pulling up" the vertices in such a way that f_0 is unchanged while f_j for each j does not decrease.

(II) For simplicial P, we have the **Dehn-Sommerville equalities**

$$\sum_{k=j}^{r-1} (-1)^k \binom{k+1}{j+1} f_k = (-1)^{r-1} f_j \quad \text{for} \quad -1 \le j \le r-1 .$$

It is trivial for $j = r-1$, while it coincides with the Euler relation for $j = -1$ in view of $f_{-1} = f_r = 1$.

(III) In 1971, McMullen conjectured a necessary and sufficient condition for a given sequence $\{f_j ; 0 \le j \le r-1\}$ of nonnegative integers to be realized as the numbers of faces in various dimensions of a simplicial convex polytope of dimension r. In 1980, Billera-Lee [BL1], [BL2] showed the sufficiency, while Stanley [S13] could prove the necessity. The condition requires that the Dehn-Sommerville equalities for all j hold and that a certain sequence arising from $\{f_j\}$ is an O-sequence in the sense defined below. We state the condition explicitly later in (III′).

By means of generating functions, we now reformulate these results (I), (II), (III) into ones easier to understand. In fact, the reformulation of (III) in this way is indispensable for its proof:

Inside the ring $\mathbb{Q}[[t]]$ of formal power series in t with rational coefficients, we define the *generating function* for a convex polytope P to be

$$F(t) := \sum_{j=0}^{r} f_{j-1} t^j ,$$

where $f_{-1} = 1$. Then

$$H(t) := (1-t)^r F\left(\frac{t}{1-t}\right) = \sum_{j=0}^{r} f_{j-1} t^j (1-t)^{r-j}$$

is a polynomial in t of degree r so that it can be expressed as

$$H(t)=\sum_{p=0}^{r} h_p t^p .$$

Conversely, we have $F(t)=(1+t)^r H(t/(1+t))$, hence

$$h_p=\sum_{j=0}^{p}(-1)^{p-j}\binom{r-j}{r-p} f_{j-1} \quad \text{for} \quad 0\le p\le r$$

$$f_{j-1}=\sum_{p=0}^{j}\binom{r-p}{r-j} h_p \quad \text{for} \quad 0\le j\le r .$$

Thus either one of the sequences $\{f_j\}$ and $\{h_p\}$ determines the other. It is not so hard to show that (I) above is now equivalent to

$$h_p\le\binom{f_0-r+p-1}{p} \quad \text{for} \quad 0\le p\le r .$$

On the other hand, (II) is equivalent to the functional equation $F(t-1)=(-1)^r F(-t)$, hence to $H(t)=t^r H(1/t)$. Consequently, (II) is equivalent to the duality

(II′) $$h_p=h_{r-p} \quad \text{for} \quad 0\le p\le r .$$

For any pair a, i of positive integers, a unique sequence of integers $a(i)>a(i-1)>\ldots>a(j)\ge j\ge 1$ for some j is known to exist such that

$$a=\binom{a(i)}{i}+\binom{a(i-1)}{i-1}+\ldots+\binom{a(j)}{j} .$$

We then define a new integer by

$$a^{\langle i\rangle}:=\binom{a(i)+1}{i+1}+\binom{a(i-1)+1}{i}+\ldots+\binom{a(j)+1}{j+1} .$$

A sequence $(v_0, v_1, v_2, \ldots)$ of nonnegative integers is said to be an *O-sequence* if

$$v_0=1 \quad \text{and} \quad 0\le v_{i+1}\le v_i^{\langle i\rangle} \quad \text{for each} \quad i\ge 1 .$$

This mysterious definition is better understood in the following reformulation in commutative algebra:

Macaulay's Theorem. *A sequence $(v_0, v_1, v_2, \ldots)$ of nonnegative integers is an O-sequence if and only if there exists a graded commutative algebra $R=R_0\oplus R_1\oplus R_2\oplus\ldots$ over a field k with $R_0=k$ such that R is generated by R_1 over k and that*

$$v_i=\dim_k R_i \quad \textit{for} \quad i\ge 0 ,$$

that is, $\{v_i; i\ge 0\}$ is the Hilbert-Samuel function *for the commutative k-algebra R.*

The associated *Poincaré series*

$$\sum_{i=0}^{\infty} (\dim_k R_i) t^i$$

for such R, rather than the Hilbert-Samuel function itself, turns out to be more convenient for our purpose as we now see.

The necessary and sufficient condition in (III) can be stated as:

(III′) $h_p = h_{r-p}$ holds for each $0 \leq p \leq r$ and, moreover,

$$(h_0, h_1 - h_0, \ldots, h_p - h_{p-1}, \ldots, h_{[r/2]} - h_{[r/2]-1}, 0, 0, \ldots)$$

is an O-sequence, where $[r/2]$ is the greatest integer not exceeding $r/2$.

We are now ready to explain the proofs for (I′), (II′) as well as the necessity of (III′), due to Stanley. We omit the proof for the sufficiency of (III′), since it is outside the scope of this book.

First of all, we construct a commutative algebra over the field $\mathbb{Q}$ of rational numbers with the Poincaré series equal to

$$F\left(\frac{t}{1-t}\right) = \frac{H(t)}{(1-t)^r} = \sum_{j=1}^{r} f_{j-1} \frac{t^j}{(1-t)^j}$$

as follows: Let $S := \mathbb{Q}[x_1, x_2, \ldots, x_{f_0}]$ be the polynomial ring over $\mathbb{Q}$ in the variables x_i of degree one. We index the vertices of P from 1 through f_0. To each proper face of P, we assign the subset $\xi \subset \{1, 2, \ldots, f_0\} =: \mathscr{I}$ consisting of the indices for the vertices of the face. Let Ξ be the collection of the subsets ξ of $\mathscr{I}$ obtained in this way. Then denote by I the ideal of S generated by the set of monomials

$$\{x_{i_1} x_{i_2} \ldots x_{i_s};\ \{i_1, \ldots, i_s\} \subset \mathscr{I},\ \{i_1, \ldots, i_s\} \notin \Xi\} .$$

Then $\sum_{1 \leq j \leq r} f_{j-1} t^j/(1-t)^j = H(t)/(1-t)^r$ is easily seen to be the Poincaré series for the *Stanley-Reisner ring* S/I considered in (ii) of the remark immediately after Ishida's criteria in Sect. 3.2. Since Ξ is combinatorially equivalent to a triangulation of an $(r-1)$-dimensional sphere, we see that S/I is a Gorenstein ring, hence a Cohen-Macaulay ring.

For further results, we need to fix a lattice $N \cong \mathbb{Z}^r$ so that $V \cong N_{\mathbb{R}}$. Without altering the combinatorial structure of $\mathscr{F}(P)$, we can slightly deform $P \subset V$ in such a way that each vertex of P belongs to $N_{\mathbb{Q}}$. By parallel translation, if necessary, we may further assume that the origin O is an interior point P. By Proposition 2.19, we thus have a finite complete fan Δ and a support function $h \in \mathrm{SF}(N, \Delta)$ which is strictly upper convex with respect to Δ. Corollary 2.14 then associates to them a toric projective variety $X = T_N \mathrm{emb}(\Delta)$ and an ample line bundle L on X. We have an indexing $\Delta(1) = \{\varrho_i;\ 1 \leq i \leq f_0\}$ induced by that for the vertices of P. For the $\mathbb{Z}$-basis $\{m_1, \ldots, m_r\}$ of the $\mathbb{Z}$-module M dual to N, we define elements of degree one in S by

$$y_l := \sum_{i=1}^{f_0} \langle m_l, n(\varrho_i) \rangle x_i \quad \text{for} \quad 1 \leq l \leq r .$$

Let J be the ideal of S generated by $\{y_1, \ldots, y_r\}$, which is a regular sequence for S/I as in (ii) of Jurkiewicz-Danilov's theorem in Sect. 3.3. Hence $R := S/(I+J)$

is a Cohen-Macaulay ring with the Poincaré series equal to

$$(1-t)^r \frac{H(t)}{(1-t)^r} = H(t) = \sum_{p=0}^{r} h_p t^p \; .$$

We obtain (I′), since $\dim_{\mathbb{Q}} R_1 = h_1 = f_0 - r$ and

$$h_p = \begin{cases} \dim_{\mathbb{Q}} R_p \leq \binom{f_0 - r + p - 1}{p} & \text{for} \quad 0 \leq p \leq r \\ 0 & \text{for} \quad r < p \end{cases} .$$

On the other hand, we have

$$H^{2p+1}(X, \mathbb{Q}) = 0 \quad \text{and} \quad H^{2p}(X, \mathbb{Q}) = R_p \quad \text{for} \quad 0 \leq p \leq r$$

by (ii) of Jurkiewicz-Danilov's theorem in Sect. 3.3. Hence we have $h_p = h_{r-p}$ in (II′) by the Poincaré duality theorem for X, or by Serre-Grothendieck's duality theorem in Sect. 3.3.

Furthermore, we know that X is a projective variety with an ample line bundle L. Thus as we see in Sect. 3.4, the strong Lefschetz theorem holds for X with respect to the cohomology class $\omega \in H^2(X, \mathbb{Q})$ of L. Let ω' be the element of R_1 corresponding to ω and denote by $\bar{R}$ the residue ring of R with respect to the ideal generated by ω' and $R_{[r/2]+1}$. We now get the necessity of (III′), since the Poincaré series for $\bar{R}$ is

$$h_0 + (h_1 - h_0)t + (h_2 - h_1)t^2 + \ldots + (h_{[r/2]} - h_{[r/2]-1})t^{[r/2]} \; .$$

This surprising relevance of Cohen-Macaulay rings in the morphology for convex polytopes was a starting point for the study of Cohen-Macaulay partially ordered sets in combinatorics. We refer the reader to Stanley [S14] and the results quoted therein.

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