Parabolic & stability.

Let G be a conn. reductive group $/\mathbb{C}$.
Let T be a max split torus.

Let us call a parabolic subgroup $P \leq G$ admissible if P contains T. Similarly we have admissible Borels.

Example: $G = GL_3$. Then admissible Borels are

$$\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$$ and its W-conjugates:

$$\begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & 0 & \\ * & * & * \\ * & 0 & * \end{pmatrix}$$

$$\begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}$$

In addition to these, we have the following maximal parabolics:

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}, \quad \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$\begin{pmatrix} * & * & * \\ & * & \\ * & * & * \end{pmatrix} \quad \begin{pmatrix} * & & * \\ * & * & * \\ * & & * \end{pmatrix}$$

<u>Upshot</u>: We have 6 minimal admissible parabolics & 6 maximal (& proper) admissible parabolics.

Def: An element $x \in \mathfrak{g}$ is called T-stable if it is not in any admissible parabolic $\mathfrak{p} \subseteq \mathfrak{g}$.

Note: This is not the original definition of stability but convenient for our purposes.

Remark: This is equivalent to saying that $x \notin \mathfrak{p}$ for any maximal (proper) admissible parabolic.

Now consider the adjoint action of T on $\mathfrak{g}$.

Theorem: If x is T-stable, then $T_x := \mathrm{Stab}_T(x)$ is finite.

This follows from the Hilbert–Mumford criteria for stability.

It would be nice to have a direct proof.

<u>Question</u>: Let B be a Borel & x a T-stable element. Is the stabiliser

$$B_x = \mathrm{Stab}_B(x)$$

finite?

I checked the case of GL_2 & GL_3 and it seems $B_x = T_x$ for T-stable elements. Is that always the case?