Toric Varieties

Three perspectives

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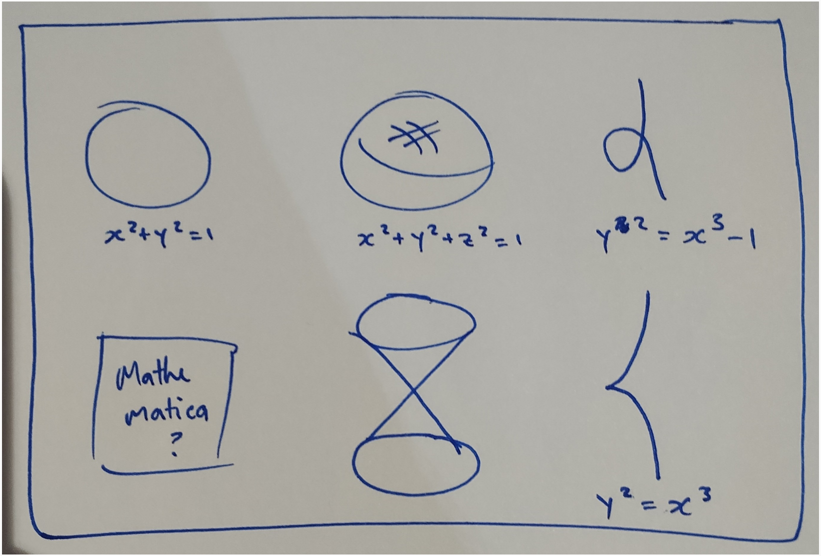
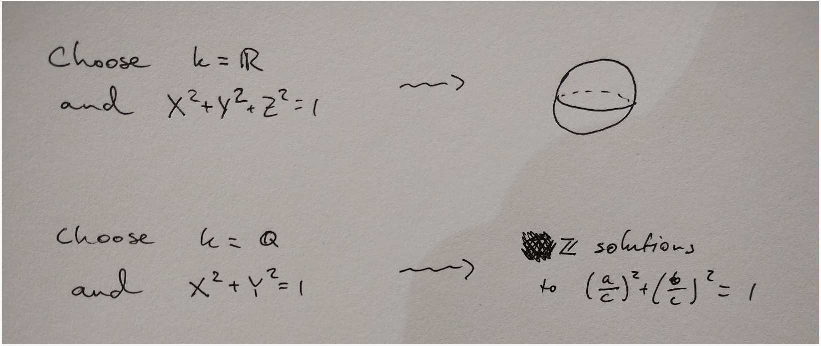
Algebraic varieties





We choose $k = \mathbb{C}$ for simplicity.

The definition of a variety

Varieties are solution sets to polynomials in several variables:

$$\mathbf{V}(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in \mathbb{C}^n : f_i(a_1, \dots, a_n) = 0 \, \forall i\}.$$

Choose the zero polynomial:

$$\rightsquigarrow \mathbb{C}^n.$$

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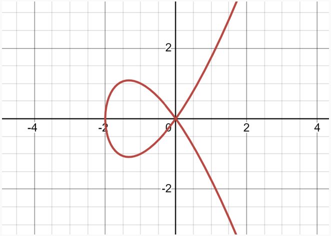
Choose $Y^2 = X^3 + 2X^2$:



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Choose $XY = 1$:

$$\rightsquigarrow \{(t, t^{-1}) : t \in \mathbb{C}^\times\} \cong \mathbb{C}^\times.$$

This variety is called an algebraic torus. The n-dimensional torus is $(\mathbb{C}^\times)^n$ and is defined by $X_1 \cdots X_n Y = 1$.

What are toric varieties, and why study them?

In general, algebraic varieties are complicated.

Toric varieties are a rich but tractable class of varieties.

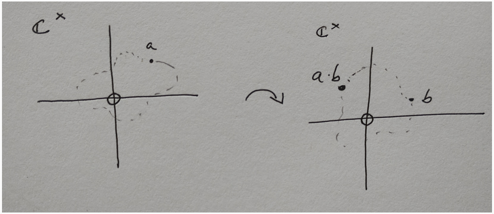
- The first way to understand them is as generalisations of tori.
- The second way to understand them is using convex cones.
- A third way to understand them is as quotient varieties.

Symmetry in tori

$(\mathbb{C}^\times)^n$ is a group:

$$(s_1, \ldots, s_n) \cdot (t_1, \ldots, t_n) = (s_1 t_1, \ldots, s_n t_n).$$

Groups encode symmetry:



This is formalised by group actions. Above, $\mathbb{C}^\times$ acts on itself, but tori act on other varieties.

The first perspective

A toric variety has two properties:

(1) It has a dense torus. Think:

$$(\mathbb{C}^\times)^n \hookrightarrow \mathbb{C}^n.$$

(2) The torus acts on the variety. Think:

$$\underbrace{(t_1, \ldots, t_n)}_{\in(\mathbb{C}^\times)^n} \cdot \underbrace{(a_1, \ldots, a_n)}_{\in\mathbb{C}^n} = \underbrace{(t_1 a_1, \ldots, t_n a_n)}_{\in\mathbb{C}^n}.$$

Convex cones

To understand the second perspective, we need convex cones.

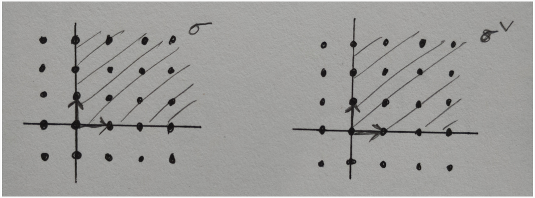
Polyhedral cones in the vector space $\mathbb{R}^n$ are sets

$$\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{v_1, \ldots, v_r\},$$

where $v_1, \ldots, v_r \in \mathbb{Z}^n$.

The dual cone to σ is

$$\sigma^\vee := \{u \in (\mathbb{R}^n)^* : u(v) \geq 0 \text{ for all } v \in \sigma\}.$$



Cones and their duals

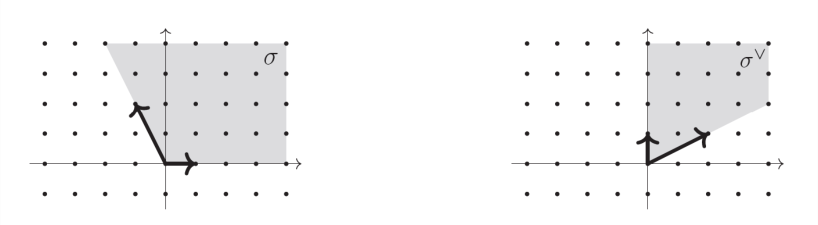
Trivial (but important) example: if $\sigma = \{0\}$, then $\sigma^\vee = (\mathbb{R}^n)^*$.

Non-trivial example:

$$\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{e_1, -e_1 + 2e_2\}.$$

Then,

$$\sigma^\vee = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{2e_1 + e_2, e_2\}.$$



Polynomial functions

We study a variety V using polynomial functions.

Problem: Redundancy. On the circle $X^2 + Y^2 = 1$,

$$2XY^2, \qquad 2XY^2 - 2X(X^2 + Y^2 - 1)$$

agree.

Solution: Remove redundancy.

$$\mathbb{C}[X_1, \ldots, X_n]/\{\text{polys vanishing on } V\}.$$

For rings given by generators and relations

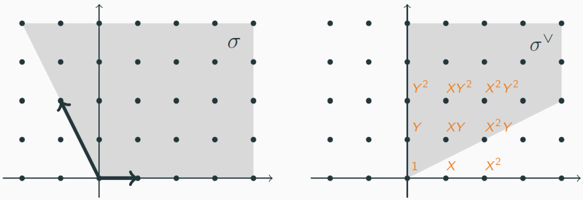
$$A = \mathbb{C}[Y_1, \ldots, Y_m]/\{\text{ring relations}\},$$

there is a variety with A as its ring of functions:

$$\mathbf{V}(\{\text{ring relations}\}) \subseteq \mathbb{C}^m.$$

An example of a toric variety

Given σ and $\sigma^\vee$, monomials $X^i Y^j$ live on integer points in $(\mathbb{R}^n)^*$:



We create a ring using the monomials in $\sigma^\vee$:

$$\begin{aligned} k[1, Y, XY, X^2Y, Y^2, XY^2, \ldots] &= k[Y, XY, X^2Y] \\ &= k[R, S, T]/(RT - S^2). \end{aligned}$$

The toric variety U_σ has this ring of functions:

$$U_\sigma := \mathbf{V}(RT - S^2).$$

The second perspective

The integer points in $\sigma^\vee$ form a semigroup,

$$S_\sigma := \sigma^\vee \cap (\mathbb{Z}^n)^*.$$

We form the semigroup algebra $\mathbb{C}[S_\sigma]$. This has the basis

$$\{\chi^u : u \in S_\sigma\}$$

with multiplication

$$\chi^u \chi^{u'} = \chi^{u+u'}.$$

Write $\mathbb{C}[S_\sigma] = \mathbb{C}[Y_1, \ldots, Y_m]/\{\text{relations}\}$. The toric variety U_σ is

$$\mathbf{V}(\{\text{relations}\}).$$

The torus in toric varieties

When $\sigma = \{0\}$, we know $\sigma^\vee = (\mathbb{R}^n)^*$. Then S_σ is

$$S_\sigma = (\mathbb{R}^n)^* \cap (\mathbb{Z}^n)^* = (\mathbb{Z}^n)^*.$$

We see

$$\begin{aligned}\mathbb{C}[S_\sigma] &= \mathbb{C}[\chi^{e_1^*}, \chi^{-e_1^*}, \dots, \chi^{e_n^*}, \chi^{-e_n^*}] \\ &= \mathbb{C}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}].\end{aligned}$$

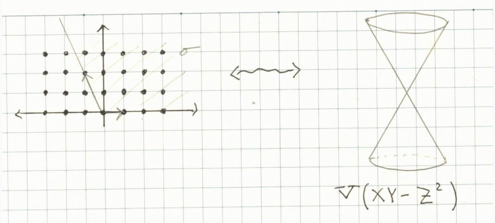
These are the polynomial functions on $(\mathbb{C}^\times)^n$.

$$\rightsquigarrow U_\sigma = (\mathbb{C}^\times)^n.$$

Singularities of toric varieties

Cones detect singularities.

A toric variety U_σ is non-singular if and only if σ is generated by a subset of a basis for $\mathbb{Z}^n$.



The third perspective

References

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