AN INTRODUCTION TO AFFINE GEOMETRIC INVARIANT THEORY

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1. Introduction

Geometric invariant theory (GIT), developed by Mumford in [2], provides powerful machinery for constructing group quotients in algebraic geometry.

Let us consider the topological quotient: given a group G acting on a topological space X, we can define the quotient space $X/G = \{G \cdot x : x \in X\}$. We endow X/G with the quotient topology so that the map $\pi : X \to X/G$ is continuous. Concretely, we say a set $U \subset X/G$ is open if $\pi^{-1}(U)$ is open in X. If we impose further geometric structure on X, for instance if X is a smooth manifold, then a natural question to ask is whether the quotient retains this geometric structure. In the case that G is a Lie group acting on a smooth manifold X, the quotient space will, in general, not be a smooth manifold. As an example, take $\mathsf{GL}_n(\mathbb{R})$ acting on $\mathbb{R}^n$ by left multiplication. The orbits of this action are $\{0\}$ and $\mathbb{R}^n \setminus \{0\}$. With the quotient topology, this space is not even Hausdorff, let alone a smooth manifold. However, if we assume G acts properly and freely on X, then X/G has the structure of a smooth manifold and the map $\pi : X \to X/G$ is smooth [6, Proposition 4.3].

Our focus lies in the action of an algebraic group G on an affine variety X. Roughly, an affine variety is a geometric object that is 'cut out' by the common zeros of a set of polynomials. The problem that GIT addresses is the following:

Does the orbit space X/G carry the structure of an affine variety?

This problem turns out to be closely related to Hilbert's fourteenth problem, which asks if the ring of G-invariant polynomials in the coordinate ring of X, denoted $k[X]^G$, is finitely generated. In 1959, Nagata [7] presented counterexamples in which $k[X]^G$ was not finitely generated. Despite this, Hilbert's fourteenth problem was answered positively by Nagata, in the case where G is linearly reductive.

In the affine setting, GIT gives the construction of a more general quotient, known as a categorical quotient, and denoted by $X/\!\!/G$. Intuitively, $X/\!\!/G$ is the set of all closed orbits. In this article, we introduce affine varieties and linearly reductive groups. Furthermore, we describe the construction of this new quotient, considering concrete examples.

We make no comment on the more general theory of projective varieties or their corresponding GIT quotients. However, this provides us with an excellent motivation since the general theory is constructed by 'glueing' affine GIT quotients [1].

2. Affine varieties

In this section, we introduce the class of geometric objects which we aim to consider quotients of. We spend time developing a dictionary, which we use to translate problems in the language of affine varieties to problems involving rings, ideals, and algebras.

Throughout, we only consider algebraically closed fields k of characteristic zero, although it is worth mentioning that some results will hold more generally.

Definition 2.1. We define *affine n-space over k*, denoted $\mathbb{A}^n_k$, to be the set of all n-tuples of elements in k. In other words, $\mathbb{A}^n_k := \{(x_1, \ldots, x_n) : x_i \in k \text{ for all } i \leq n\}$.

Let $k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. We begin by describing the maximal ideals in $k[x_1, \ldots, x_n]$.

Theorem 2.2. *For points $c_1, \ldots, c_n \in k$, the ideal $(x_1 - c_1, \ldots, x_n - c_n)$ in $k[x_1, \ldots, x_n]$ is maximal. Furthermore, every maximal ideal is of this form.*

Proof. Let $I = (x_1 - c_1, \ldots, x_n - c_n)$ be an ideal of $k[x_1, \ldots, x_n]$. Consider the map $\phi : k[x_1, \ldots, x_n] \to k$ defined by $f(x_1, \ldots, x_n) \mapsto f(c_1, \ldots, c_n)$. This is clearly surjective and has kernel I. The first isomorphism theorem implies that $k[x_1, \ldots, x_n]/I \cong k$. Since the right hand side is a field, I is maximal.

Consider a maximal ideal $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$. Then, $k[x_1, \ldots, x_n]/\mathfrak{m} \cong k$, and so for every x_i, the coset $x_i\mathfrak{m} = c_i$ for some $c_i \in k$. Consequently, $x_i - c_i \in \mathfrak{m}$ for all $1 \leq i \leq n$, which implies $(x_1 - c_1, \ldots, x_n - c_n) \subset \mathfrak{m}$. This ideal is maximal by the argument above, hence showing equality: $(x_1 - c_1, \ldots, x_n - c_n) = \mathfrak{m}$.

It stands to show that the c_i defined above are unique. Suppose that $\mathfrak{m} = (c_1, \ldots, c_n)$ and $\mathfrak{m} = (c'_1, \ldots, c'_n)$. Then, for each $1 \leq i \leq n$, $c_i - c'_i \in (x_1 - c_1, \ldots, x_n - c_n)$. Since this is a proper ideal, it cannot contain units of k. Hence, $c_i - c'_i = 0$. □

This shows that the maximal ideals in $k[x_1, \ldots, x_n]$ are the polynomials vanishing at a point, thus establishing the correspondence

$$\left\{ \begin{array}{c} \text{Maximal ideals in} \\ k[x_1, \ldots, x_n] \end{array} \right\} \longleftrightarrow \left\{ \text{ Points in } \mathbb{A}^n_k \right\}.$$

We now want to construct a link between the geometry of subsets in $\mathbb{A}^n_k$ and the algebraic data of polynomials in $k[x_1, \ldots, x_n]$. Each polynomial f in $k[x_1, \ldots, x_n]$ defines a map $f : \mathbb{A}^n \to k$ by evaluation. Given $f \in k[x_1, \ldots, x_n]$, denote by $Z(f)$ the set of zeros of f; more concretely, $Z(f) := \{p \in \mathbb{A}^n : f(p) = 0\}$. We can generalise this notion and consider subsets $T \subset k[x_1, \ldots, x_n]$. We define the *zero set* of T to be

$$Z(T) := \{p \in \mathbb{A}^n_k : f(p) = 0 \text{ for all } f \in T\}.$$

A subset $Y \subset \mathbb{A}^n$ is said to be *algebraic* if there exists some $T \subset k[x_1, \ldots, x_n]$ such that $Z(T) = Y$.

Proposition 2.3. *A subset $X \subset \mathbb{A}^n_k$ is algebraic if and only if $X = Z(I)$ for some ideal I of $k[x_1, \ldots, x_n]$.*

Proof. Suppose $X = Z(T)$ for some $T \subset k[x_1, \ldots, x_n]$ and consider the ideal generated by T, $\langle T \rangle$. Let $f \in \langle T \rangle$, then $f = \sum_i g_i h_i$ for $g_i \in T$ and $h_i \in k[x_1, \ldots, x_n]$. For each x, we thus have $f(x) = \sum_i g_i(x) h_i(x) = 0$ and so $X \subset Z(\langle T \rangle)$. Conversely, for $x \in Z(\langle T \rangle)$, $f(x) = 0$ for all $f \in \langle T \rangle$. Hence, $f(x) = 0$ for all $f \in T \subset \langle T \rangle$. This shows $Z(\langle T \rangle) \subset X$. □

Definition 2.4. The *Zariski topology* on $\mathbb{A}^n$ is the topology defined by declaring a set closed if and only if it is algebraic.

Proposition 2.5. *The Zariski topology is a topology on $\mathbb{A}^n_k$.*

Proof. First notice that $Z(\{1\}) = \emptyset$ and $Z(\{0\}) = \mathbb{A}^n_k$. Let I, J be ideals in $k[x_1, \ldots, x_n]$ and let $\{I_\alpha\}$ be a family of ideals in $k[x_1, \ldots, x_n]$, indexed by α. We need to show three properties:

(i) $I \subset J \implies Z(I) \supset Z(J)$,
(ii) $Z(I) \cup Z(J) = Z(I \cap J)$, and
(iii) $\bigcap_\alpha Z(I_\alpha) = Z(\sum_\alpha I_\alpha)$.

Firstly, suppose that $I \subset J$ and let $x \in Z(J)$. Then $f(x) = 0$ for all $f \in I$, hence $x \in Z(I)$.

We know that $I, J \supset I \cap J$ so part (i) implies that $Z(I) \cup Z(J) \subset Z(I \cap J)$. Conversely, for each $x \in Z(I \cap J)$, $f(x) = 0$ for all $f \in I \cap J$. If $x \in V(I)$, then we are done. Assume $x \notin V(I)$.

Then, there exists $f_0 \in I$ with $f_0(x) \neq 0$. In particular, for all $g \in J$, we have $f_0 g \in I \cap J$, and so $f_0 g(x) = f_0(x)g(x) = 0$. This implies that $g(x) = 0$ for all $g \in J$. Hence, $x \in Z(J)$.

Finally, let $x \in \bigcap_\alpha Z(I_\alpha)$. Then, for all $f \in I_\alpha$, $f(x) = 0$. This holds for any α, so $x \in Z(\sum_\alpha I_\alpha)$. On the other hand, part (i) implies that $Z(\sum_\alpha I_\alpha) \subset Z(I_\beta)$ for all indices β. This gives the reverse inclusion: $Z(\sum_\alpha I_\alpha) \subset \bigcap_\alpha Z(I_\alpha)$. □

Example 2.6. *Consider the Zariski topology on the affine line $\mathbb{A}^1$. Recall that the polynomial ring $A = k[x]$ is a principal ideal domain. In particular, this means that every algebraic set is the set of zeros of a single polynomial. The fundamental theorem of algebra implies that for every $f \in A$, we may write*

$$f(x) = c(x - \lambda_1) \cdots (x - \lambda_n),$$

where $c, \lambda_1, \ldots, \lambda_n \in k$ are not necessarily distinct. We find then that $Z(\{f\}) = \{\lambda_1, \ldots, \lambda_n\}$. The open sets of $\mathbb{A}^1$ in the Zariski topology are thus the complements of finite tuples of elements in k.

Considering $\mathbb{A}^1$ as a subset of itself gives us a first example of an irreducible subset. We say a closed subset Y of a topological space X is *irreducible* if it cannot be expressed as the union of two proper, closed subsets of X. That is, we cannot find $Y_1, Y_2 \subset X$ such that $Y = Y_1 \cup Y_2$.

Definition 2.7. An *affine variety* is an irreducible, closed subset of $\mathbb{A}^n$.

Remark 2.8. *We give an affine variety X the subspace topology from $\mathbb{A}^n_k$. In this sense, the basic open sets are $X_f := \{x \in X : f(x) \neq 0\}$. To show that this forms a basis for a topology, one requires Hilbert's basis theorem [5].*

Example 2.9. *The following are basic examples of affine varieties:*

(i) The equation $x - c = 0$ defines an affine variety in $\mathbb{A}^1_k$ for all $c \in k$. The solution set is a single point $\{c\}$.
(ii) In $\mathbb{A}^2_k$, the solution set of the equation $xy - 1 = 0$ forms an affine variety. One may recognise this curve as a hyperbola.
(iii) Similarly, a parabola, given by the solutions to $y - x^2 = 0$, is an affine variety in $\mathbb{A}^2_k$.
(iv) The twisted cubic is an affine variety in $\mathbb{A}^3_k$ defined by $Z(\{x_1^2 - x_2, x_1^3 - x_3\})$. This is clearly an algebraic set. To see that it is irreducible, consider the surjective morphism $\phi : k[x_1, x_2, x_3] \to k$ defined by $x_i \mapsto x_i^i$. This has kernel $(x_1^2 - x_2, x_1^3 - x_3)$, and so the first isomorphism theorem gives us $k[x_1, x_2, x_3]/(x_1^2 - x_2, x_1^3 - x_3) \cong k$. The right hand side being a field implies that $(x_1^2 - x_2, x_1^3 - x_3)$ is irreducible.

Definition 2.10. Let $X \subset \mathbb{A}^n_k$ be an affine variety. The *ideal associated with* X is

$$\mathcal{I}_X := \{f \in k[x_1, \ldots, x_n] : f(x) = 0 \text{ for all } x \in X\}.$$

We now have the beginning of our dictionary. Let $X \subset \mathbb{A}^n$ be an affine variety, and I be an ideal of $k[x_1, \ldots, x_n]$. Then, X corresponds to an ideal $\mathcal{I}_X$, and I corresponds to an algebraic set $Z(I)$. Applying these associations twice gives us

$$I \subset \mathcal{I}_{Z(I)}, \text{ and } X \subset Z(\mathcal{I}_X).$$

A natural question that arises is whether the above inclusions are equalities. For the first inclusion, this is not true in general. For example, consider the ideal $I = (x^2)$. The zero set $Z(I) = \{0\}$ but $\mathcal{I}_{Z(I)} = (x)$, which contains I. For the second inclusion, however, this is always true.

Theorem 2.11. *Let $X \subset \mathbb{A}^n_k$ be an affine variety. Then $X = Z(\mathcal{I}_X)$.*

Proof. Since X is an affine variety, there exists an ideal $I \subset k[x_1, \dots, x_n]$ such that $X = Z(I)$. The first inclusion above gives $I \subset \mathcal{I}_X$. So, for each point $x \in Z(\mathcal{I}_X)$, $f(x) = 0$ for all $f \in \mathcal{I}_X$. Thus, $f(x) = 0$ for all $f \in I$, and $x \in Z(I)$. $\square$

Using this fact, we can see that if $\mathcal{I}_X$ is a maximal ideal, then X must be a singleton. Indeed, $\mathcal{I}_X = (x_1 - c_1, \dots, x_n - c_n)$ for some $c_1, \dots, c_n \in k$ so $X = Z(\mathcal{I}_X) = \{c\}$ where $c = (c_1, \dots, c_n) \in \mathbb{A}^n_k$. In fact, the converse holds too: $X = \{c\} \implies \mathcal{I}_X = (x_1 - c_1, \dots, x_n - c_n)$ which, by Theorem 2.2, is maximal.

Theorem 2.12 (Weak Nullstellansatz)**.** *Let I be a proper ideal of $k[x_1, \dots, x_n]$. Then $Z(I) \neq \emptyset$.*

Proof. Since $k[x_1, \dots, x_n]$ is a Noetherian ring (i.e. it satisfies the ascending chain condition), there exists a maximal ideal $\mathfrak{m}$ containing I. Then, $Z(\mathfrak{m}) \subset Z(I)$. It suffices to show that $Z(\mathfrak{m})$ is not empty, but this is clear by Theorem 2.2. $\square$

The weak Nullstellansatz equivalently says that a set of polynomials $f_1, \dots, f_r$ in $k[x_1, \dots, x_n]$ has a common solution if and only if $(f_1 \cdots f_r)$ is a proper ideal. The ideal associated to an affine variety is distinguished: if $f \in k[x_1, \dots, x_n]$ and $f^m \in \mathcal{I}_X$ for some m, then $f \in \mathcal{I}_X$. To see this, notice that $f(x)^m = 0 \implies f(x) = 0$.

Definition 2.13. An ideal I in a ring R is *radical* if for every $a \in R$, $a^m \in I \implies a \in I$ for $m \geq 1$.

The set of $a \in R$ for which the above property holds is called the *radical of I* and often denoted $\sqrt{I}$. Thus, we equivalently say an ideal is radical if $I = \sqrt{I}$.

Theorem 2.14 (Hilbert's Nullstellensatz)**.** *Let k be algebraically closed and let I be an ideal of $k[x_1, \dots, x_n]$. Then $\mathcal{I}_{Z(I)} = \sqrt{I}$.*

Proof. See [4, Theorem 1.3A] $\square$

The Nullstellensatz gives us the one-to-one correspondence

$$\left\{ \begin{array}{c} \text{Radical ideals of} \\ k[x_1, \dots, x_n] \end{array} \right\} \longleftrightarrow \left\{ \text{ Closed sets of } \mathbb{A}^n_k \right\}.$$

As a corollary, an algebraic set $X \subset \mathbb{A}^n$ is irreducible if and only if $\mathcal{I}_X$ is prime. In this context, we have the one-to-one correspondence

$$\left\{ \text{ Affine Varieties } \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Zero sets of prime} \\ \text{ideals in } k[x_1, \dots, x_n] \end{array} \right\}.$$

2.1. **Morphisms of affine varieties.** The following definitions are more general than we require. They appear more natural when considering the class of projective varieties (which contains the class of affine varieties). We will see shortly that, in the setting of affine varieties, there is a more concrete definition of a morphism.

Definition 2.15. Let $X \subset \mathbb{A}^n_k$ be an affine variety. We say that a map $f : X \to k$ is *regular* at a point $p \in X$ if there exists an open neighbourhood $U \subset X$ of p such that $f|_U = g/h$ for $g, h \in k[x_1, \dots, x_n]$ where $h \neq 0$ on U. If a map is regular for all points in a subset $S \subset X$, we say f is regular on S.

If we identify k with $\mathbb{A}^1_k$, then regular functions are continuous with respect to the Zariski topology. Denote by $\mathcal{O}(X)$ the ring of regular functions on X.

Definition 2.16. Let X and Y be affine varieties. We say that $\phi : X \to Y$ is a *morphism of varieties* if

(i) ϕ is continuous (i.e. for every open $V \subset Y$, $\phi^{-1}(V) \subset X$ is open), and

(ii) for each open set $V \subset Y$ and all $f \in \mathcal{O}(V)$, the composition $f \circ \phi : \phi^{-1}(V) \to k$ is regular.

From the above definition, it is easy to see that the composition of two morphisms remains a morphism of varieties. In particular, we say that a morphism $\varphi : X \to Y$ of two varieties is an *isomorphism* if it admits an inverse morphism $\psi : Y \to X$ such that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$.

Recall that every affine variety is the zero set of some prime ideal in $k[x_1, \ldots, x_n]$, called the associated ideal. Let X be an affine variety with associated ideal $\mathcal{I}_X$. The quotient ring $k[X] := k[x_1, \ldots, x_n]/\mathcal{I}_X$ is known as the *coordinate ring of* X. It can be shown that $k[X] \cong \mathcal{O}(X)$.

We have the following result, which gives an equivalent (and potentially more natural) definition of a morphism between affine varieties.

Theorem 2.17. *Let $X \subset \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$ be affine varieties. A map $\phi : X \to Y$ is a morphism of affine varieties if and only if $\phi(x_1, \ldots, x_n) = (y_1, \ldots, y_m)$, where each $y_i := \phi_i(x_1, \ldots, x_n)$ for some regular map $\phi_i \in \mathcal{O}(X)$.*

Proof. Suppose $\phi : X \to Y$ is a morphism of affine varieties and let $\phi(x_1, \ldots, x_n) = (y_1, \ldots, y_m)$, where $y_i = \phi_i(x_1, \ldots, x_n)$ for maps $\phi_i : X \to k$. The projection mapping $(y_1, \ldots, y_m) \mapsto y_i$ is regular on Y. As ϕ is regular, $p_i \circ \phi = \phi_i$ is regular on $\phi^{-1}(Y) = X$. Conversely, let $\phi(x_1, \ldots, x_n) = (y_1, \ldots, y_m)$ with $y_i = \phi_i(x_1, \ldots, x_n)$ for regular functions ϕ_i. Let $U \subset Y$ be open. Consider a decomposition of U into basic open sets $Y_{f_1}, \ldots, Y_{f_r}$. Then,

$$\phi^{-1}(U) = \phi^{-1}(Y_{f_1} \cup \ldots \cup Y_{f_r}) = \phi^{-1}(Y_{f_1}) \cup \ldots \cup \phi^{-1}(Y_{f_r}).$$

Notice that $\phi^{-1}(Y_{f_i}) := \{p \in X : \phi(p) \in Y_{f_i}\} = \{p \in X : f_i(\phi(p)) = 0\} = X_{f_i \circ \phi}$, which is a basic open set of X, and so ϕ is continuous. Let f be a regular function on U. Thus, there exist $g, h \in k[y_1, \ldots, y_m]$ with $h|_U \neq 0$ such that $f = g/h$. It is clear that for any polynomial g, $g \circ \phi$ is polynomial too. Consequently, $(g/h) \circ \phi = (g \circ \phi)/(h \circ \phi)$, where for any $x \in \phi^{-1}(U)$, $(h \circ \phi)(x) = h \circ \phi(x) \neq 0$. Thus, $f \circ \phi$ is regular and ϕ is a morphism of varieties. □

Notice that the coordinate ring of an affine variety is in fact a k-algebra as well. To each affine variety, we have the corresponding k-algebra $k[X] \cong \mathcal{O}(X)$. We will come to see that studying the morphisms of affine varieties $\phi : X \to Y$ is equivalent to studying the morphisms between their coordinate rings: $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Here the map ϕ^* is defined by $\phi^*(f)(x) = f(\phi(x))$, and is known as the *pull-back* of ϕ.

Proposition 2.18. *Let $\phi : X \to Y$ be a morphism of affine varieties. Then the pull back of ϕ is a morphism of k-algebras $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$.*

Proof. For each $f \in \mathcal{O}(Y)$, $\phi^*(f) = f \circ \phi$. Since ϕ is a morphism, $f \circ \phi$ is a regular function on X, and so $\phi * (f) \in \mathcal{O}(X)$. For all $\lambda \in k$, $x \in X$, and $f, g \in \mathcal{O}(Y)$, we have:

(i) $(f + g)(\phi(x)) = f(\phi(x)) + g(\phi(x))$,
(ii) $(fg)(\phi(x)) = (f(\phi(x)))(g(\phi(x)))$,
(iii) $(\lambda f)(\phi(x)) = \lambda(f(\phi(x)))$, and
(iv) $\phi^*(1)(x) = 1 \circ \phi(x) = 1 \implies \phi^*(1) = 1$.

Hence, ϕ^* is a morphism of k-algebras. □

Proposition 2.19. *Let $X \subset \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$ be affine varieties, and let $\varphi : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a morphism of k-algebras. There exists a unique morphism $\phi : X \to Y$ such that $\phi^* = \varphi$.*

Proof. See [4, Proposition 3.5]. □

We have thus shown the following one-to-one correspondence

$$\left\{\begin{array}{c}\text{Morphisms of affine varieties}\\ \phi : X \to Y\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Morphisms of } k\text{-algebras}\\ \varphi : \mathcal{O}(Y) \to \mathcal{O}(X)\end{array}\right\}.$$

We know that an affine variety X corresponds to its ring of regular functions $\mathcal{O}(X)$. Since $\mathcal{O}(X) \cong k[X] = k[x_1, \ldots, x_n]/\mathcal{I}_X$ where $\mathcal{I}_X$ is a prime ideal, $\mathcal{O}(X)$ is a finitely generated k-algebra. In fact, it is an integral domain as well. We claim that any finitely generated k-algebra that is also an integral domain has a corresponding affine variety. Indeed, let $A = \langle a_1, \ldots, a_n \rangle$ be an integral domain and a k-algebra. Consider the k-algebra homomorphism

$$\varphi : k[x_1, \ldots, x_n] \to A, \quad x_i \mapsto a_i.$$

The kernel $\ker \varphi$ must be prime since A is an integral domain. As a result of the first isomorphism theorem, $A \cong k[x_1, \ldots, x_n]/\ker \varphi$, which is an affine variety. This shows the one-to-one correspondence

$$\left\{\text{ Affine varieties }\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Finitely generated } k\text{-algebras}\\ \text{that are integral domains}\end{array}\right\}.$$

Remark 2.20. *The two above correspondences show an anti-equivalence of categories.*

3. Algebraic groups and their actions on affine varieties

3.1. **Algebraic Groups.** We very briefly introduce the notion of an algebraic group. These are varieties with a group structure.

Definition 3.1. Let G be a variety equipped with multiplication and inversion maps, $m : G \times G \to G$ and $\iota : G \to G$, such that G has the structure of a group. If m and ι are morphisms of varieties, we say that G is an *algebraic group.*

We remark that the product $G \times G$ is furnished with the Zariski topology. This is in contrast to the topological group setting, where $G \times G$ is given the product topology.

Definition 3.2. Let G be an algebraic group. We say that a subgroup $H \leq G$ is an *algebraic subgroup of* G if H is closed (in the Zariski topology). We say that an algebraic subgroup is *normal* if it is a normal subgroup of G.

Example 3.3. *The following are examples of algebraic groups:*

(i) Consider the group $\mathsf{GL}_n(k)$*, the* $n \times n$ *invertible matrices over* k*. This has the structure of an affine variety. To see this, notice that* $\mathsf{GL}_n(k)$ *can be written as* $Z(\{t \det A - 1 : t \in k, A \in \mathsf{M}_n(k)\})$*. This can be given the structure of an affine variety by embedding it into the affine space* $\mathbb{A}_k^{n^2+1}$*. Hence,* $\mathsf{GL}_n(k)$ *is an algebraic group.*

(ii) Any closed subgroup of $\mathsf{GL}_n(k)$ *is an algebraic group.*

(iii) The multiplicative and additive groups $\mathbb{C}^*$ *and* $\mathbb{C}$*, respectively, are algebraic groups. Indeed,* $\mathbb{C}^* \cong \mathsf{GL}_1(\mathbb{C})$*, while* $\mathbb{C}$ *is isomorphic to the closed subgroup*

$$\left\{\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix} : z \in \mathbb{C}\right\} \leq \mathsf{GL}_2(\mathbb{C}).$$

(iv) All finite groups are algebraic. To see this, we view S_n *as a closed subgroup of* $\mathsf{GL}_n(k)$*. All finite groups are subgroups of some* S_n*.*

Definition 3.4. A *homomorphism* of algebraic groups $\varphi : G \to \tilde{G}$ is a morphism of varieties that is also a group homomorphism. We say a homomorphism of algebraic groups is an *isomorphism* if it is an isomorphism of abstract groups as well as varieties.

3.2. **The action of algebraic groups on affine varieties.** The following definitions are natural extensions of the usual notions when considering group actions.

Definition 3.5. Let G be an algebraic group and X an affine variety. We that that G *acts morphically* (or just acts) on X if there exists a morphism $\varphi : G \times X \to X$ denoted by $(g, x) \mapsto g \cdot x$ such that for $x \in X$ and $g, h \in G$,

(i) $g \cdot (h \cdot x) = (gh) \cdot x$, and
(ii) $e \cdot x = x$.

Let G be an algebraic group acting on an affine variety X. We have the following standard definitions:

(i) for each $x \in X$, the set $G \cdot x := \{g \cdot x : g \in G\}$ is the *orbit of* x;
(ii) for each $x \in X$, the set $G_x := \{g \in G : g \cdot x = x\}$ is the *isotropy group of* x;
(iii) we say that G acts *transitively* if for each $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$;

Example 3.6. *Let G be an algebraic group. We present two important actions of G on itself.*

(i) Consider the action of G on itself by conjugation: for each $g, h \in G$, $g \cdot h := ghg^{-1}$. The orbit of $h \in G$ is $G \cdot h = \{ghg^{-1} : g \in G\}$, which one may recognise as the conjugacy class of h. The isotropy group $G_h := \{g \in G : ghg^{-1} = h\}$ is the centraliser of h.
(ii) Consider the action of G on itself by left (resp. right) translation: for each $g, h \in G$, $g \cdot h := gh$ (resp. $g \cdot h = hg^{-1}$). These actions are both transitive and hence there is a single orbit, G. The isotropy groups are all trivial.

4. Quotients and Their Challenges

Consider the action of an algebraic group G on an affine variety X. We consider the following problem:

Does the orbit space X/G have the structure of an affine variety?

In general, the answer is negative, as the following example makes clear.

Example 4.1. *Let the multiplicative group $\mathbb{C}^*$ act on $\mathbb{C}^2$ by $\lambda \cdot (x, y) := (\lambda x, \lambda^{-1} y)$ for each $(x, y) \in \mathbb{C}^2$ and $\lambda \in \mathbb{C}^*$. The orbits of this action are:*

- *the punctured axes $\{(x, 0) : x \neq 0\}$ and $\{(0, y) : y \neq 0\}$;*
- *the hyperbola $\{(x, y) : xy = \lambda\}$;*
- *the origin $\{(0, 0)\}$.*

The usual quotient is the collection of these orbits. However, we cannot give this set the structure of a variety. If it were a variety, then for each pair of points, there would exist a $\mathbb{C}^*$-invariant polynomial that separates them (i.e. attains different values at each point). Consider the punctured x-axis and the origin; each $\mathbb{C}^*$-invariant polynomial is continuous in the Zariski topology and so cannot separate the two orbits.

Notice that the action of G on X induces an action of G on the coordinate algebra $k[X]$ defined by $g \cdot f(x) := f(g^{-1} \cdot x)$ for all $g \in G$, $x \in X$ and $f \in k[X]$. A possible quotient variety should then have coordinate ring $k[X]^G$, the G-invariant elements of $k[X]$. Recall that affine varieties are in one-to-one correspondence with finitely generated k-algebras that are integral domains. The question of whether a quotient variety exists then reduces to asking when $k[X]^G$ is finitely generated.

This is exactly Hilbert's fourteenth problem. In 1959, Nagata [7] described a counterexample to $k[X]^G$ being finitely generated in general. However, Nagata proved that $k[X]^G$ is finitely generated when G is linearly reductive. We now define linearly reductive groups, for which we must recall some representation theory.

Let G be an algebraic group. A *linear representation of G* (or *G-module*) is a pair (V, ρ) consisting of a vector space V and a group homomorphism $\rho : G \to \mathsf{GL}(V)$. A *$G$-submodule* is a subspace $W \subset V$ that is invariant under ρ. Every G-module V has at least two submodules: V itself, and the trivial submodule $\{0\}$. We say that a G-module is *irreducible* if it has no proper, non-trivial G-submodules. A G-module is *completely reducible* if it can be decomposed as the direct sum of irreducible G-submodules.

Definition 4.2. A group G is called *linearly reductive* if every G-module is completely reducible.

There are two related notions: geometrically reductive, and reductive. Work by Haboush, Mumford, Nagata, and Weyl [2,3,7,8] shows that in characteristic zero, the three coincide. The following gives us equivalent conditions for a group to be linearly reductive.

Proposition 4.3. *Let G be a group. Then the following are equivalent:*

(i) G is reductive;
(ii) for any G-module V, if V is a G-submodule then it has a module complement;
(iii) any surjective morphism of G-modules $\phi : V \to W$ induces a surjective map

$$\phi^G : V^G \to W^G.$$

We can now state Nagata's result, and an important consequence.

Theorem 4.4. *Let k be an algebraically closed field of characteristic 0. If G is a linearly reductive group then $k[x_1, \ldots, x_n]^G$ is finitely generated.*

Proof. See [3]. □

Corollary 4.5. *Let X be an affine variety, k an algebraically closed field of characteristic 0, and G a linearly reductive group. Then $k[X]^G$ is finitely generated.*

Proof. As $k[X]$ is finitely generated, we have the homomorphism of k-algebras $\phi : k[x_1, \ldots, x_n] \to k[X]$ defined by $x_i \mapsto X_i$ for each $1 \leq i \leq n$, where each X_i is a generator of $k[X]$. The induced G-module morphism $k[x_1, \ldots, x_n]^G \to k[X]^G$ is surjective, and so $k[X]^G$ is finitely generated. □

We now present the usual notion of quotient in a slightly more abstract way.

Definition 4.6. Let G be an algebraic group acting on an affine variety X. A *geometric quotient* of X by G is an affine variety Y and a morphism $\pi : X \to Y$ such that the following hold:

(i) π is surjective, and its fibres are the orbits in X;
(ii) a subset $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is open;
(iii) for any open $U \subset Y$, the associated morphism $k[U] \to k[\pi^{-1}(U)]^G$ is an isomorphism.

With this definition, Y is identified with the orbit space X/G equipped with the quotient topology. Example 4.1 admits no geometric quotient since the orbits are not separated by G-invariant functions. This suggests that we need a different definition of quotient.

Definition 4.7. Let G be an algebraic group acting on an affine variety X. A *categorical quotient* of X by G is an affine variety Y and a G-invariant morphism $\pi : X \to Y$ such that for every affine variety Z and G-invariant morphism $\phi : X \to Z$, there exists a unique morphism $\overline{\phi} : Y \to Z$ where the following commutes:

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\overline{\phi}"] \\ & Z & \end{tikzcd}.$$

Let A be a k-algebra. Denote by $\mathrm{Spm}(A)$ the set of maximal ideals in A. We claim that every affine variety X can be regarded as $\mathrm{Spm}(k[X])$. Indeed, the maximal ideals of $k[X] = k[x_1, \dots, x_n]/\mathcal{I}_X$ are the ideals of $k[x_1, \dots, x_n]$ containing $\mathcal{I}_X$. Since maximal ideals of $k[x_1, \dots, x_n]$ are in correspondence with points in $\mathbb{A}^n_k$, we can view the maximal ideals of $k[X]$ as points in $\mathbb{A}^n_k$ where polynomials in $\mathcal{I}_X$ vanish. In other words, $\mathrm{Spm}(k[X]) = Z(\mathcal{I}_X)$, but by Theorem 2.11, this is just X.

Similarly, consider a reductive group G acting on an affine variety $X \subset \mathbb{A}^n_k$. We know $k[X]^G$ is finitely generated; let $\{f_1, \dots, f_r\}$ be the set of its generators. This induces a G-invariant map

$$\phi : X \to \mathbb{A}^r_k, \quad x \mapsto (f_1(x), \dots, f_r(x)).$$

We claim that $k[\phi(X)] \cong k[X]^G$. Indeed, $k[\phi(X)] := k[x_1, \dots, x_r]/\mathcal{I}_{\phi(X)}$ and $k[X]^G = k[f_1, \dots, f_r]$. Define a surjective k-algebra homomorphism

$$\psi : k[x_1, \dots, x_r] \to k[f_1, \dots, f_r], \quad g(x_1, \dots, x_r) \mapsto g(f_1(x_1), \dots, f_r(x_r)).$$

This has kernel

$$\ker \psi = \{g \in k[x_1, \dots, x_r] : g(f_1(x_1), \dots, f_r(x_r)) = 0\} = \mathcal{I}_{\phi(X)}.$$

Hence, by the first isomorphism theorem, $k[\phi(X)] \cong k[X]^G$.

Remark 4.8. *Since ϕ is G-invariant, it is constant on the orbits of X under G.*

With the above discussion in mind, if a categorical quotient Z exists, then $k[Z]$ should be isomorphic to $k[X]^G$. Thus, if the maximal spectrum $\mathrm{Spm}(k[X]^G)$ is an affine variety, it is a good candidate for the quotient variety Z. The following result makes this concrete.

Theorem 4.9. *Let G be a reductive algebraic group acting on an affine variety X. The following hold:*

(i) The image of the morphism $\phi : X \to \mathbb{A}^r_k$, defined above, is closed in the Zariski topology and doesn't depend on the choice of $f_1, \dots, f_r$.
(ii) Denote by $\phi_X : X \to \phi(X) := X /\!\!/ G$ the surjective morphism defined above. Then, for every affine variety Y and every G-invariant morphism $\pi : X \to Y$, there exists a unique morphism $\overline{\phi} : X/\!\!/G \to Y$ such that the following commutes.

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\phi_X"'] & & Y \\ & X/\!\!/G \arrow[ur, "\overline{\phi}"'] & \end{tikzcd}$$

(iii) For any closed G-invariant subset $U \subset X$, the restriction of ϕ_X to U may be identified with ϕ_U. Moreover, given another closed G-invariant subset $V \subset X$, we have $\phi_X(U \cap V) = \phi_X(U) \cap \phi_X(V)$.
(iv) Each fibre of ϕ_X (i.e. $\phi^{-1}(p)$ for $p \in X/\!\!/G$) contains a unique closed orbit.

Proof. The proof is quite technical, and so we omit it; see [1, Theorem 1.24]. $\square$

We remark that (iv) makes precise what was mentioned in the introduction: the GIT quotient $X/\!\!/G$ may be viewed as the space of closed orbits. We now consider a subset of X that admits a geometric quotient.

Definition 4.10. Let G be a reductive group acting on an affine variety X. We say that a point $x \in X$ is *stable* if the orbit $G \cdot x$ is closed in X and G_x is finite. We denote the set of stable points in X by X^s.

Proposition 4.11. *With the preceding notation, $\phi(X^s)$ is open in $X/\!\!/G$, X^s is an open G-invariant subset of X, and $\phi^s : X^s \to \phi(X^s)$ is a geometric quotient.*

Proof. See [1, Proposition 1.26]. □

Example 4.12. *Let us consider a few examples to conclude.*

(i) *Let $\mathbb{C}^*$ act on $\mathbb{C}^n$ by scalar multiplication: for all $\lambda \in \mathbb{C}^*, (x_1, \ldots, x_n) \in \mathbb{C}^n$, $\lambda \cdot (x_1, \ldots, x_n) := (\lambda x_1, \ldots, \lambda x_n)$. The orbits are the punctured lines through the origin and the origin itself. The only closed orbit is the origin itself. The GIT quotient is $\mathbb{C}^n /\!\!/ \mathbb{C}^* = \{0\}$, and there are no stable points. This example shows that X^s can be empty.*

(ii) *Let $\mathbb{C}^*$ act on $\mathbb{C}^2$ as in Example 4.1. Recall the induced action on the coordinate ring $\mathbb{C}[x, y]$ is defined by $\lambda \cdot f(x, y) = f(\lambda^{-1}x, \lambda y)$. We can compute the $\mathbb{C}^*$ invariant polynomials; let $f \in \mathbb{C}[x, y]$ be of the form $f(x, y) = \sum_{i=1}^n \sum_{j=1}^m c_{ij} x^i y^j$ for integers n, m and constants $c_{ij} \in \mathbb{C}$. The condition $\lambda \cdot f = f$ is equivalent to having $n = m$ and $i = j$. Hence, $\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}[xy]$. Thus, the morphism $\phi_{\mathbb{C}^2} : \mathbb{C}^2 \to \mathbb{C}^2 /\!\!/ \mathbb{C}^*$ is given by $(x, y) \mapsto xy$. This map is clearly surjective, so $\mathbb{C}^2 /\!\!/ \mathbb{C}^* \cong \mathbb{C}$.*

The origin is contained in the closure of both the punctured axes, and so the categorical quotient identifies the three orbits. This shows that $\phi_{\mathbb{C}^2}$ is not a geometric quotient. The set of stable points is the complement of the union of coordinate axes since the hyperbola $\{xy = c\}$ are closed when $c \neq 0$.

(iii) *Consider the affine variety $X = \mathsf{M}_2(\mathbb{C})$ embedded in $\mathbb{A}^4_{\mathbb{C}}$ by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d).$$

Let $G = \mathsf{GL}_2(\mathbb{C})$ act on X by conjugation: $g \cdot x = gxg^{-1}$ for $g \in G$ and $x \in X$. It is easy to see that tr *and* det *are invariant polynomials. It can be shown that these are the only G invariant polynomials. Taking this for granted, we see that $\mathbb{C}[X]^G = \mathbb{C}[ad - bc, a + d]$. Hence, the quotient morphism $\phi_X : \mathsf{M}_2(\mathbb{C}) \to \mathbb{C}^2$ maps a matrix to its determinant and trace. This is surjective and so $X /\!\!/ G \cong \mathbb{C}^2$.*

There are no stable points since the subgroup $\mathbb{C}^* I_n$ acts trivially on every $x \in X$, meaning G_x is never finite. The closed orbits are the diagonalisable matrices.

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