GIT and Affine Toric Varieties

Mid-year Review

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Affine GIT

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Suppose a group G acts on a complex affine variety $X = \mathrm{Spec}(R)$. The G-invariant functions on X are

$$R^G := \{f \in R : f(g \cdot P) = f(P) \text{ for all } g \in G, P \in X\}.$$

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Example: $\mathbb{C}^\times$ acting on $\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[X, Y])$ by

$$g \cdot (x, y) = (gx, g^{-1}y), \qquad g \in \mathbb{C}^\times, (x, y) \in \mathbb{C}^2.$$

Then,

$$\mathbb{C}[X, Y]^{\mathbb{C}^\times} = \mathbb{C}[XY], \quad \text{and} \quad \mathbb{C}^2 /\!\!/ \mathbb{C}^\times \cong \mathbb{C}.$$

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Example: $G = \mathrm{GL}_2(\mathbb{C})$, T is the subgroup of diagonal matrices, and $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C})$. Then T acts on $\mathfrak{g}$ by

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & ab^{-1}y \\ a^{-1}bz & w \end{pmatrix},$$

and therefore

$$\mathbb{C}[\mathfrak{g}]^T = \mathbb{C}[X, W, YZ], \quad \text{and} \quad \mathfrak{g}/\!\!/T \cong \mathbb{C}^3.$$

Affine toric varieties

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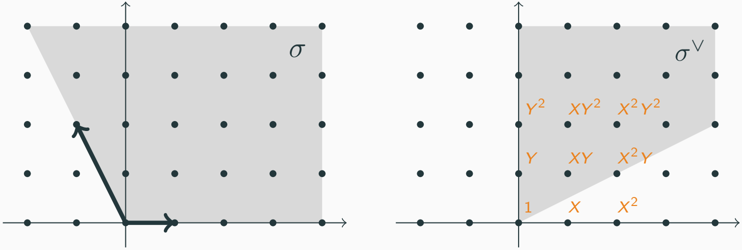

$$U_\sigma = \operatorname{Spec}(\mathbb{C}[Y, XY, X^2Y]) \cong \operatorname{Spec}(\mathbb{C}[U, V, W]/(UW - V^2)).$$

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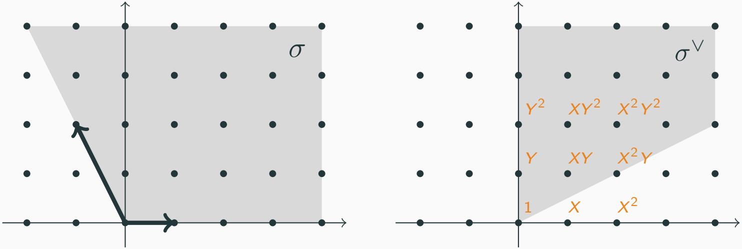

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Properties of the variety are informed by properties of the cone.

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2. Computed a basis for the invariant ring $\mathbb{C}[\mathfrak{g}]^T$ for general G:

$$\mathbb{C}[X^\eta Y^\mu : \eta \in \mathcal{A}, \mu \in (\mathbb{Z}_{\geq 0})^r],$$

where

$$\mathcal{A} := \left\{ \eta \in (\mathbb{Z}_{\geq 0})^\Phi : \sum_{\alpha \in \Phi} \eta_\alpha \alpha = 0 \right\}.$$

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3. Proved that $\mathfrak{g} /\!\!/ T$ has the structure of a toric variety.

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1. Compute the cone corresponding to the affine toric variety $\mathfrak{g}/\!\!/T$, and use this to study features of the variety.
2. Investigate the concepts of stability and semi-stability in GIT, and compute the stable and semi-stable points for the action of T on $\mathfrak{g}$.
3. Learn about the projective GIT quotient and projective toric varieties; the projective GIT quotient $\mathfrak{g}/\!\!/_{\chi}T$, where χ is a character of T, is a projective toric variety.