Toric Varieties

Bridges in Algebra and Geometry

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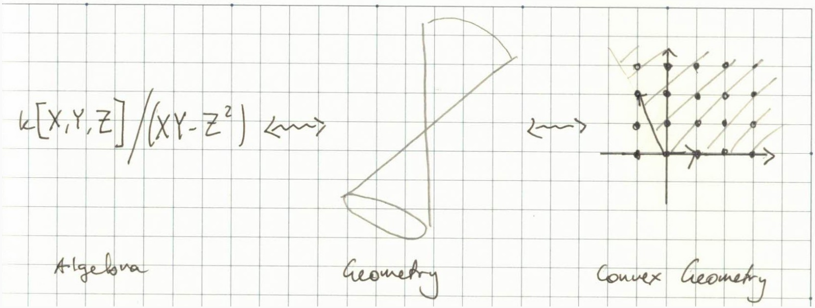
Two bridges

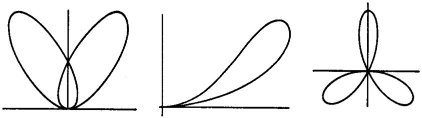


Overview

Affine varieties

Let k be a field. The set of n-tuples of elements of k is called affine space, $\mathbb{A}^n$. Given a polynomial f in $k[X_1, \ldots, X_n]$, we can consider its zero set

$$\mathbf{V}(f) := \{(a_1, \ldots, a_n) \in \mathbb{A}^n : f(a_1, \ldots, a_n) = 0\}.$$



We can also consider spaces cut out by sets of polynomials $I \subseteq k[X_1, \ldots, X_n]$:

$$\mathbf{V}(I) := \{(a_1, \ldots, a_n) \in \mathbb{A}^n : f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}.$$

These spaces are called affine varieties.

Varieties throughout mathematics

Choose $k = \mathbb{R}$

and consider $\mathbb{V}(X^2+Y^2+Z^2-1)$:

Choose $k = \mathbb{Q}$

and consider $\mathbb{V}(X^2+Y^2-1)$:

Integer solutions

to $\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1$.

The Zariski topology

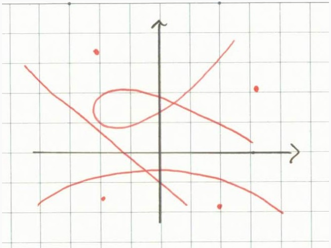
Affine space is endowed with the Zariski topology.

This is the topology defined by declaring $S \subseteq \mathbb{A}^n$ closed if

$$S = \mathbf{V}(I)$$

for some $I \subseteq k[X_1, \ldots, X_n]$.

The Zariski topology on a variety V is the induced topology from $\mathbb{A}^n$.



Polynomial functions

We say a function $V \to k$ is polynomial if it is the restriction of a polynomial function $\mathbb{A}^n \to k$.

Problem: different polynomials give the same function. For example, on the circle $V = \mathbf{V}(X^2 + Y^2 - 1)$, the polynomials

$$2XY^2, \qquad 2XY^2 - 2X(X^2 + Y^2 - 1)$$

define the same function.

Solution: consider functions in the quotient

$$k[V] := k[X_1, \ldots, X_n]/\mathbf{I}(V),$$

where

$$\mathbf{I}(V) := \{\text{polynomials vanishing on } V\}.$$

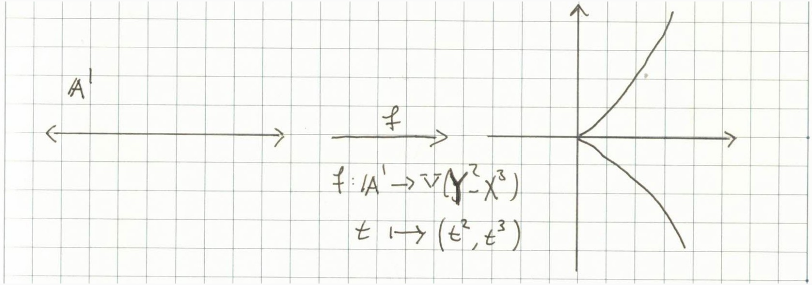
We call $k[V]$ the coordinate ring of V, and $\mathbf{I}(V)$ the ideal of V.

Polynomial maps

A map $\varphi : V \subseteq \mathbb{A}^n \to W \subseteq \mathbb{A}^m$ is called polynomial if its components are polynomial functions on V, i.e., if

$$\varphi = (\varphi_1, \dots, \varphi_m)$$

for some $\varphi_i \in k[V]$.



We say φ is an isomorphism if it is bijective with polynomial inverse.

The Nullstellensatz

We have maps

$$\{\text{varieties in } \mathbb{A}^n\} \underset{\mathbf{V}}{\overset{\mathbf{I}}{\rightleftarrows}} \{\text{ideals in } k[X_1, \ldots, X_n]\}.$$

The Nullstellensatz (Moral)

Let k be algebraic closed. There is a bijection

$$\{\text{varieties in } \mathbb{A}^n\} \leftrightarrow \{\text{radical ideals in } k[X_1, \ldots, X_n]\}.$$

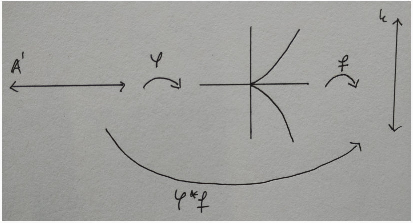
An ideal I is called radical if whenever $f^n \in I$ for some $n \in \mathbb{Z}_{>0}$, we have $f \in I$. These are ideals containing all their square roots, cube roots, etc.

Henceforth, we assume k is algebraically closed.

Maps and homomorphisms

A polynomial map $\varphi : V \to W$ gives us a rule for turning polynomial functions on W into polynomial functions on V. Specifically, it gives a ring homomorphism

$$\varphi^* : k[W] \to k[V], \qquad f \mapsto \varphi^* f := f \circ \varphi.$$



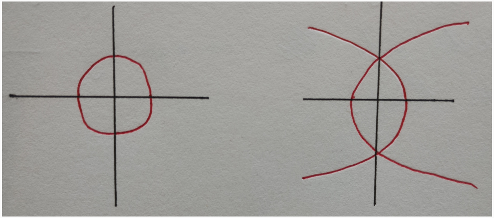
The map $\varphi \mapsto \varphi^*$ is in fact a bijection

$$\left\{ \text{ poly. maps } V \to W \right\} \longleftrightarrow \left\{ \text{ ring homs } k[W] \to k[V] \right\}.$$

Moreover, φ is an isomorphism if and only if φ^* is. Then V and W are isomorphic if and only if $k[V]$ and $k[W]$ are.

Irreducibility and prime ideals

We say a variety is irreducible if it is not the union of two proper closed subsets.



The ideal $\mathbf{I}(V)$ is prime if and only if $k[V] = k[X_1, \ldots, X_n]/\mathbf{I}(V)$ has no zero divisors.

Proposition

The variety V is irreducible if and only if the ideal $\mathbf{I}(V)$ is prime.

Tangent spaces and dimension

Tangent spaces are an important tool for studying smooth surfaces in differential geometry. There is an analog for varieties.

We define tangent spaces for the hypersurface $V = \mathbf{V}(f)$. Let $P = (a_1, \ldots, a_n)$ be a point in V. The first-order part of f at P is

$$f_P^{(1)} := \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(P)(X_i - a_i).$$

The tangent space of V at P is

$$T_P V := \mathbf{V}(f_P^{(1)}).$$

The dimension of V is

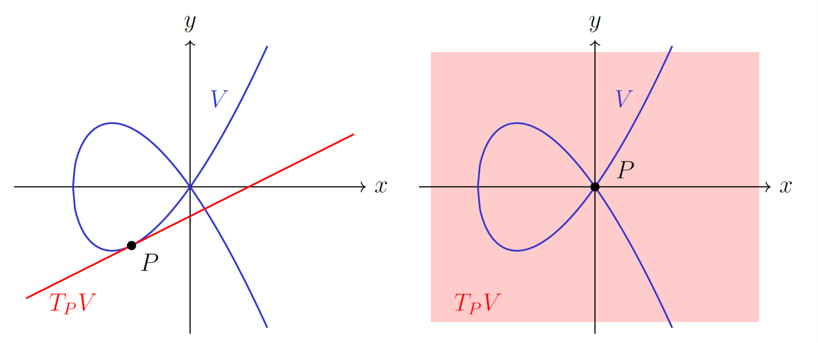
$$\dim(V) := \min_{P \in V} \dim(T_P V).$$

Singular and non-singular points

We say a point P is singular if

$$\dim(T_P V) > \dim(V),$$

and non-singular otherwise.



Convex cones

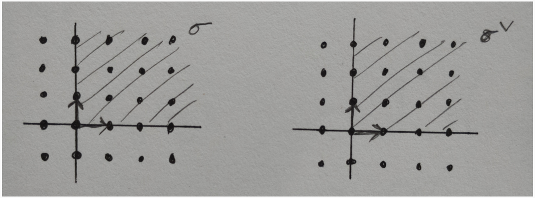
A polyhedral cone in the vector space $\mathbb{R}^n$ is a set of the form

$$\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{v_1, \ldots, v_r\},$$

for some vectors $v_1, \ldots, v_r$ in $\mathbb{Z}^n$.

The dual cone to a polyhedral cone σ is

$$\sigma^\vee := \{u \in (\mathbb{R}^n)^* : u(v) \geq 0 \text{ for all } v \in \sigma\}.$$



Cones and their duals

A trivial (but important) example: if $\sigma = \{0\}$, then $\sigma^\vee = (\mathbb{R}^n)^*$.

For a non-trivial example, consider

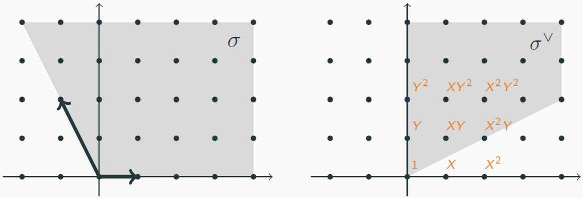
$$\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{e_1, -e_1 + 2e_2\}.$$

Then,

$$\sigma^\vee = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{2e_1 + e_2, e_2\}.$$

Toric varieties, the intuition

Given a cone and its dual, we place a monomal $X^i Y^j$ on the integer points (i, j) in the dual space:



We form a ring using the monomials lying in the dual cone:

$$k[1, Y, XY, X^2Y, Y^2, XY^2, \ldots] = k[Y, XY, X^2Y].$$

The toric variety U_σ is defined by taking this as its coordinate ring:

$$U_\sigma := \mathrm{Spec}(k[Y, XY, X^2Y]) = \mathrm{Spec}(k[R, S, T]/(RT - S^2)).$$

Toric varieties, the definition

The integer points in $\sigma^\vee$ form a semigroup,

$$S_\sigma := \sigma^\vee \cap (\mathbb{Z}^n)^*.$$

We form the semigroup algebra $k[S_\sigma]$. This has the basis of formula symbols

$$\{\chi^u : u \in S_\sigma\}$$

with multiplication

$$\chi^u \chi^{u'} = \chi^{u+u'}.$$

The toric variety associated with σ is defined as

$$U_\sigma := \mathrm{Spec}(k[S_\sigma]).$$

The torus in toric varieties

When $\sigma = \{0\}$, we know $\sigma^\vee = (\mathbb{R}^n)^*$. The semigroup S_σ is

$$S_\sigma = (\mathbb{R}^n)^* \cap (\mathbb{Z}^n)^* = (\mathbb{Z}^n)^*.$$

We see

$$\begin{aligned} k[S_\sigma] &= k[\chi^{e_1^*}, \chi^{-e_1^*}, \dots, \chi^{e_n^*}, \chi^{-e_n^*}] \\ &= k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] \end{aligned}$$

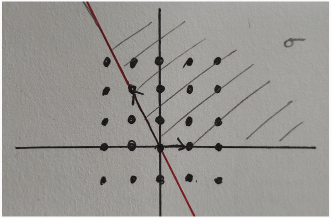
Then

$$U_\sigma = \mathrm{Spec}(k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]) = (k^\times)^n$$

is the algebraic torus.

Faces

A face of a cone σ is the intersection of σ with a hyperplane which contains σ in a half-space.



If τ is a face of σ, then $\tau \subseteq \sigma$ but $\tau^\vee \supseteq \sigma^\vee$ and $S_\tau \supseteq S_\sigma$. We then have an inclusion

$$k[S_\sigma] \hookrightarrow k[S_\tau].$$

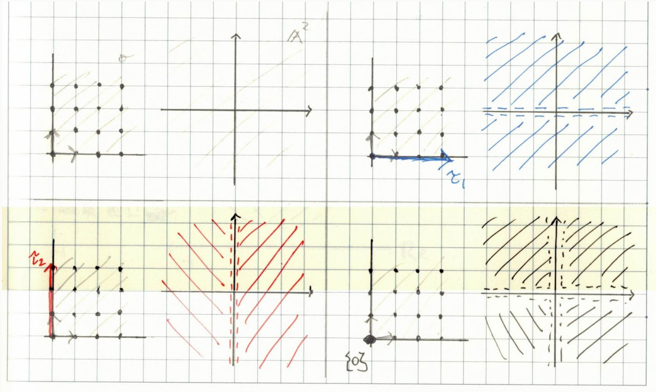
Corresponding to this, there is a map

$$U_\tau \to U_\sigma.$$

Theorem

Let τ be a face of σ. Then the map $U_\tau \to U_\sigma$ embeds U_τ as a principal open subset of U_σ.

Smaller faces correspond to smaller open subsets.



Singularities of toric varieties

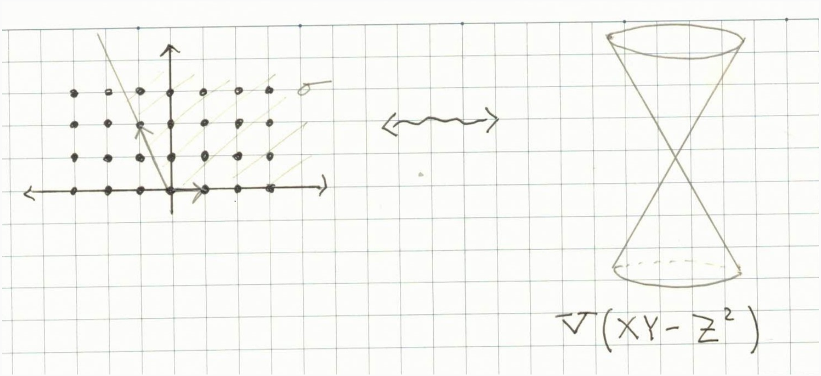
Theorem

An toric variety U_σ is non-singular if and only if σ is generated by a subset of a basis for $\mathbb{Z}^n$. In this case,

$$U_\sigma \cong k^r \times (k^\times)^{n-r},$$

where $r = \dim(\sigma)$.

Cones detect singularities



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