Toric Varieties

Declan Fletcher

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What are toric varieties, and why study them?

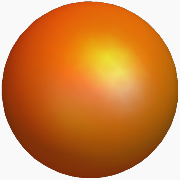
Algebraic varieties are geometric spaces defined by solutions to polynomial equations.

In general, these are complicated and difficult to study.

Toric varieties are a special class of varieties determined by a convex cone.

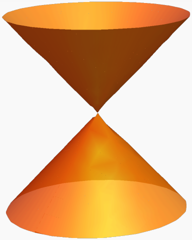
Algebraic varieties

$x^2 + y^2 = 1$



$x^2 + y^2 + z^2 = 1$

$y^2 = x^3 + 2x^2$



$xy = z^2$

The definition of a variety

Varieties are sets of solutions $(a_1, \ldots, a_n) \in \mathbb{C}^n$ to poly. equations

$$f_1(a_1, \ldots, a_n) = 0, \quad \ldots, \quad f_s(a_1, \ldots, a_n) = 0.$$

Choose the zero polynomial:

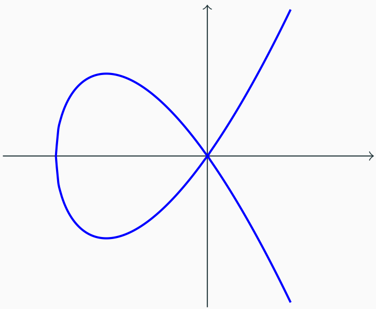
$\rightsquigarrow \mathbb{C}^n$.

Choose $y^2 = x^3 + 2x^2$:

$\rightsquigarrow$ singular curve.

Choose $xy = 1$:

$\rightsquigarrow \{(t, t^{-1}) : t \in \mathbb{C}^\times\} \cong \mathbb{C}^\times$.



$\mathbb{C}^\times$ is called an algebraic torus. The d-dimensional torus is $(\mathbb{C}^\times)^d$.

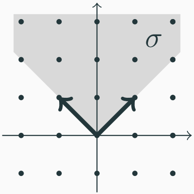
Convex cones

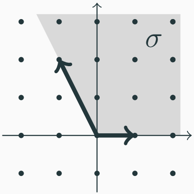
To understand toric varieties, we need to understand convex cones.

We consider polyhedral cones. These are sets in $\mathbb{R}^n$ of the form

$$\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{v_1, \ldots, v_r\},$$

for some $v_1, \ldots, v_r \in \mathbb{R}^n$. σ is rational if we can take each $v_i \in \mathbb{Z}^n$.

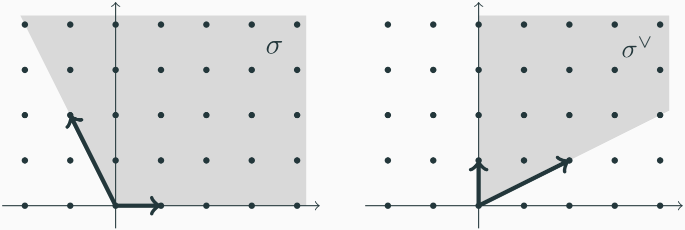




Dual cones

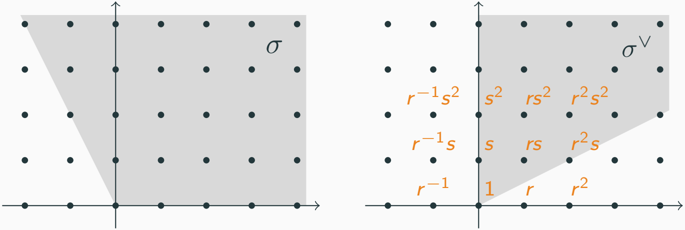
Fix: cone σ. The dual cone is the set of linear functionals which are non-negative on σ:

$$\sigma^\vee := \{u \in (\mathbb{R}^n)^* : u(v) \geq 0 \text{ for all } v \in \sigma\}.$$



An example of a toric variety

Fix: cone σ, dual $\sigma^\vee$. Monomials live on integer points in $(\mathbb{R}^n)^*$:



We create an algebra using the monomials in $\sigma^\vee$:

$$\begin{aligned}\mathbb{C}[1, s, r^2s, rs, s^2, rs^2, \ldots] &= \mathbb{C}[s, r^2s, rs] \\ &\cong \mathbb{C}[x, y, z]/(xy - z^2).\end{aligned}$$

The toric variety U_σ is the set of solutions to $xy - z^2 = 0$ in $\mathbb{C}^3$:

$$xy = z^2.$$

The definition of a toric variety

Fix: cone σ, dual $\sigma^\vee$. The previous construction generalises.

We associate monomials $x_1^{i_1} \cdots x_n^{i_n}$ to integer points in $(\mathbb{R}^n)^*$.

We create an algebra using the monomials lying in $\sigma^\vee$, called $\mathbb{C}[S_\sigma]$.

$\mathbb{C}[S_\sigma]$ is finitely generated, so it's given by generators and relations:

$$\mathbb{C}[S_\sigma] = \mathbb{C}[y_1, \ldots, y_m]/(f_1, \ldots, f_s).$$

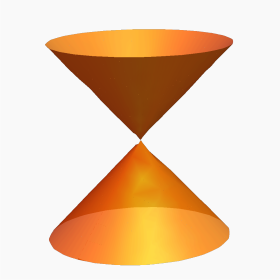
The toric variety U_σ is the subset of $\mathbb{C}^m$ defined by the equations

$$f_1(a_1, \ldots, a_m) = 0, \quad \ldots, \quad f_s(a_1, \ldots, a_m) = 0.$$

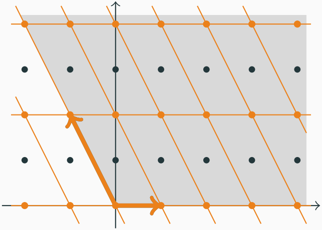
Cones detect singularities

Theorem

The toric variety U_σ is non-singular if and only if σ is generated by a subset of a basis for $\mathbb{Z}^n$.



$xy = z^2$



Torus quotients

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