

STABILITY OF AFFINE G -VARIETIES AND IRREDUCIBILITY IN REDUCTIVE GROUPS

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Let G be a reductive affine algebraic group and let X be an affine algebraic G -variety. We establish a (poly)stability criterion for points $x \in X$ in terms of intrinsically defined closed subgroups H_x of G and relate it with the numerical criterion of Mumford and with Richardson and Bate–Martin–Röhrle criteria, in the case $X = G^N$. Our criterion builds on a close analogue of a theorem of Mundet and Schmitt on polystability and allows the generalization to the algebraic group setting of results of Johnson–Millson and Sikora about complex representation varieties of finitely presented groups. By well established results, it also provides a restatement of the non-abelian Hodge theorem in terms of stability notions.

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1. Introduction and Main Results

Let G be a connected complex affine reductive group and Γ a finitely presented group. The G -representation variety of Γ , defined as $X := \text{Hom}(\Gamma, G)$, is a complex affine G -variety under the canonical conjugation action of G on X . In the context of geometric invariant theory, and building on earlier work by Johnson–Millson, Richardson and others (see [12, 23]), a recent article of Sikora [27] establishes a close relationship between stability properties of a point $x \in X$ and the property that

the image of the representation, $x(\Gamma) \subset G$, is irreducible or completely reducible as a subgroup of G .

In this paper, we generalize these relationships to a bigger class of affine G -varieties, where G is an affine reductive group, not necessarily irreducible, defined over an algebraically closed field k , of characteristic zero.

Besides its intrinsic relevance as stability criteria, the constructions examined here perfectly agree with other known results for certain specific classes of G -varieties. For example, the non-abelian Hodge theorem (see [9] or [32]), in the case of complex reductive groups, can be restated as a correspondence between stable points in distinct (however homeomorphic) varieties. Also, from our set up, one can recover the Mundet–Schmitt criterion for polystability [18].

To describe our main results, let $Y(G)$ denote the set of one parameter subgroups (1PS for short) of G , that is, homomorphisms λ from the multiplicative group k^\times to G . Given a 1PS $\lambda \in Y(G)$ and $g \in G$, the morphism $t \mapsto \lambda(t)g\lambda(t)^{-1}$ may or may not extend to k (as a morphism of affine varieties). In case there is such an extension, we say that $\lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1}$ exists. It is known that

$$P(\lambda) = \left\{ g \in G : \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists} \right\}$$

is always a parabolic subgroup of G .

Let X be an affine variety with an action of G , that is, an *affine G -variety*. Recall that a point $x \in X$ is called *polystable* if its G -orbit is closed. Let G_x denote the stabilizer of $x \in X$ and consider the normal subgroup

$$G_X := \bigcap_{x \in X} G_x \subset G.$$

A point $x \in X$ will be called *stable* if it is polystable and the quotient G_x/G_X is finite. Note that there are many slightly different notions of stability in the literature (see [17, 19, 23]), but all of them coincide when G_X is finite. We also define x to be *equivocal* if x is polystable and $G_x = G_X$.

We denote by Λ_x the subset of $Y(G)$ consisting of 1PS λ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists (where $g \cdot x$ denotes the action of $g \in G$ on $x \in X$, see Sec. 3 for details). We say that a subset $\Lambda \subset Y(G)$ is *symmetric* if given any $\lambda \in \Lambda$, there is another 1PS $\lambda' \in \Lambda$ such that $P(\lambda) \cap P(\lambda')$ is a Levi subgroup of both $P(\lambda)$ and $P(\lambda')$. Following the constructions in Mundet–Schmitt [18], one can show (see Theorem 4.9).

Theorem 1.1. *Let G be a reductive algebraic group and X be an affine G -variety, both defined over k . Then, a point $x \in X$ is polystable if and only if Λ_x is symmetric.*

Consider now another natural construction. Given $x \in X$, define the closed subgroup of G ,

$$H_x := \bigcap_{\lambda \in \Lambda_x} P(\lambda).$$

Recall that a closed subgroup H of G is called *irreducible* if it is not contained in a proper parabolic subgroup of G and it is called *completely reducible* if, for

any inclusion of H in a parabolic P of G , there is a Levi subgroup L of P such that $H \subset L$. These natural notions, generalizing the well-known definitions for the general linear group (see [26]), can be extended to algebraic groups which are not necessarily connected (see Sec. 2, below).

Our main result applies to certain affine G -varieties where the existence of limits under one parameter subgroups in X is related to the existence of limits under conjugation in G . More precisely, consider the following condition for every pair $x \in X$ and $\lambda \in Y(G)$:

$$\text{if } H_x \subset P(\lambda), \quad \text{then } \lambda \in \Lambda_x. \quad (1.1)$$

The main result can be presented as follows (see Theorems 4.16 and 4.19). Let $Z(G)$ denote the center of G .

Theorem 1.2. *Let G be reductive and X be an affine G -variety, as above. Suppose condition (1.1) is satisfied. Then:*

- (1) *The subgroup H_x is completely reducible if and only if x is polystable;*
- (2) *If $G_X = Z(G)$, then $H_x = G$ if and only if x is stable.*

The proof of (1) above follows similar arguments in Mundet–Schmitt (see [18]), which in turn rely on several techniques available, such as the strengthening of Hilbert–Mumford criterion by Birkes/Richardson and Kempf [13, 4]. The proof of (2) uses the notion of 1PS stability: A point x is called 1PS stable if $\lambda \in \Lambda_x$ implies $\lambda \in Y(G_X) \subset Y(G)$. Although this looks more general than the notion of stability above, we show that the two definitions agree (see Theorem 3.10).

Suppose now X is a closed G -invariant subvariety of G^N , where G acts on G^N diagonally by conjugation. For a given $x \in X$, let ϕ_x be the subgroup of G generated by the elements $f_1(x), \dots, f_N(x) \in G$, where $f = (f_1, \dots, f_N): X \hookrightarrow G^N$ is the natural inclusion. For this class of G -varieties X , condition (1.1) is automatically satisfied (although $G_X \neq Z(G)$ in general).

Our setup provides another approach to results of Richardson (see [23, Theorems 3.6 and 4.1]), Martin and Bate–Martin–Röhrle (see [16, 1, Proposition 2.13 and Corollary 3.7]) relating stability (respectively, polystability) of $x \in X$ with irreducibility (respectively, complete reducibility) of ϕ_x . Together with a corresponding relation for equicentral points of X , they can be stated as follows (see Theorem 5.5). We say that a subgroup $H \subset G$ is *isotropic* if it is completely reducible and its centralizer equals $Z(G)$. An isotropic subgroup is always irreducible, but not conversely (see Sec. A.2 in Appendix).

Theorem 1.3. *Let X be a closed G -invariant subvariety of G^N , as above. Then,*

- (1) *A point x is stable (respectively, polystable) if and only if ϕ_x is an irreducible (respectively, completely reducible) subgroup of G .*
- (2) *If $G_X = Z(G)$, then x is equicentral if and only if ϕ_x is isotropic.*

This result also generalizes to the algebraic group setting the recent result of Sikora ([27, Theorem 29, Corollary 31]), as representation varieties over \mathbb{C} of finitely generated groups are a particular class of closed G -invariant subvarieties of G^N . As a corollary of Theorems 1.2 and 1.3, we can write the relation between H_x and ϕ_x as follows.

Theorem 1.4. *Let X be a closed G -invariant subvariety of G^N as above. Then H_x is completely reducible if and only if ϕ_x is completely reducible. Moreover, assuming $G_X = Z(G)$, then $H_x = G$ if and only if ϕ_x is irreducible.*

Another consequence of Theorem 1.3 is that we can restate one form of the non-abelian Hodge theorem (see, for example [9]) as a relation between stability of G -Higgs bundles over a Riemann surface and stability of representations in the character variety of its fundamental group, in a framework also valid for reductive groups which are not necessarily connected.

The paper is organized as follows. In Sec. 2, after introducing the basic definitions, we describe some properties of irreducible and completely reducible subgroups of a given affine reductive group and in Sec. 3, we provide the stability definitions we will need, present relations between them and prove the relevant versions of the Hilbert–Mumford numerical criterion. In Sec. 4, we prove our main results, and re-derive the Mundet–Schmitt criterion for polystability. We also analyze condition (1.1) showing that it holds for the adjoint representation of G (in the complex case). In Sec. 5, we apply our results to the setting of closed subvarieties of G^N and of G -representation varieties of finitely presented groups, adding to the work of Richardson and Sikora alluded above. We also briefly recall G -Higgs bundles theory (in the complex case) in order to present a restatement of the non-abelian Hodge theorem.

In the appendix, we collect some results that are useful in comparing the distinct notions of irreducibility and related properties that appear in the literature.

2. Irreducible and Completely Reducible Subgroups

We start by recalling some important definitions and properties of the main objects considered and by fixing terminology.

Let G be an affine algebraic group (not necessarily irreducible) over an algebraically closed field k of characteristic zero and let G^0 be the connected component of the unit element of G . The unipotent radical of G , denoted by $R_u(G)$, is the maximal connected unipotent normal subgroup of G .

An affine algebraic group G is called *reductive* if the unipotent radical of G^0 is the trivial group.^a In the representation theory of reductive groups, parabolic and Levi subgroups play an important role. Recall that a *parabolic subgroup* P of G is

^aNote that, contrary to many references, we do not assume that a reductive group is itself connected.

a closed subgroup such that the coset space G/P is a complete variety and a *Levi subgroup* of G is a connected subgroup isomorphic to $G/R_u(G)$.

Note that, by definition, any parabolic subgroup P of G is identified with the semidirect product $P = L \ltimes R_u(P)$, where L is a Levi subgroup of P .

We will use the notation $Z(H)$ for the center of a group H and when H is a subgroup of G , $Z_G(H)$ will denote the centralizer of H in G .

2.1. Parabolic and one parameter subgroups

In practice, we need another characterization of parabolic subgroups for which we recall the definition of one-parameter subgroup. Let k^\times be the multiplicative group of invertible elements of the field k .

A morphism of algebraic groups $\lambda: k^\times \rightarrow G$ is called a one parameter subgroup (1PS), or cocharacter of G . The trivial 1PS is the morphism given by $\lambda(t) = e, \forall t \in k^\times$, where $e \in G$ is the identity element. The set of cocharacters of G will be denoted by $Y(G)$.

Definition 2.1. Given a 1PS $\lambda \in Y(G)$ and an element $g \in G$, consider the morphism $\lambda_g: k^\times \rightarrow G$, defined by

$$\lambda_g(t) := \lambda(t)g\lambda(t)^{-1}, \quad t \in k^\times.$$

We say that $\lim_{t \rightarrow 0} \lambda_g(t)$ exists when λ_g can be extended to a morphism $\tilde{\lambda}_g: k \rightarrow G$ (that is, we have $\lambda_g = \tilde{\lambda}_g \circ i$, for the obvious inclusion $i: k^\times \rightarrow k$). In this case we write

$$\lambda^+g = \lim_{t \rightarrow 0} \lambda_g(t)$$

for the element $\tilde{\lambda}_g(0) \in G$, which is uniquely defined.

We follow the convention that, whenever a formula with limits is written, we are assuming that they exist.

Definition 2.2. Given a 1PS $\lambda \in Y(G)$, define the following subsets of G :

$$P(\lambda) := \{g \in G: \lambda^+g \text{ exists}\},$$

$$U(\lambda) := \{g \in G: \lambda^+g = e\},$$

$$L(\lambda) := \{g \in G: \lambda^+g = g\}.$$

We call $P(\lambda)$ a *R-parabolic* subgroup of G , $U(\lambda)$ a *R-unipotent* subgroup of $P(\lambda)$ and $L(\lambda)$ a *R-Levi* subgroup of $P(\lambda)$.

The terminology R-parabolic and R-Levi was introduced in [1] referring to its relevance in the work of Richardson. For a 1PS $\lambda \in Y(G)$, $\lambda(k^\times)$ denotes the subgroup $\lambda(k^\times) := \{\lambda(t): t \in k^\times\} \subset G$. When G is reductive, the following is well-known.

Proposition 2.3. *Let G be an affine reductive group and let $\lambda \in Y(G)$. Then,*

- (i) $P(\lambda)$ is a parabolic subgroup of G , $L(\lambda)$ is a Levi subgroup of $P(\lambda)$, and $U(\lambda)$ is the unipotent radical of $P(\lambda)$; in particular $P(\lambda) = L(\lambda) \ltimes U(\lambda)$.
- (ii) $L(\lambda)$ coincides with the centralizer of the subgroup $\lambda(k^\times)$.
- (iii) In the case that G is connected, then all parabolic subgroups of G are R -parabolic subgroups.

Proof. See [17] or [31, Proposition 8.4.5 and Theorem 13.4.2]. □

Note that $P(\lambda) = G$ if and only if $\lambda(k^\times)$ is contained in the center of G , $Z(G)$. To rule out this trivial case, the parabolics $P(\lambda) \neq G$ will be called proper R -parabolics.

2.2. Irreducible and completely reducible subgroups

Assume, from now on that G is an affine reductive group.

Definition 2.4. Let H be a subgroup of G .

- (1) We say that $H \subset G$ is *irreducible* (in G) if H is not contained in any proper R -parabolic subgroup of G .
- (2) We say that $H \subset G$ is *completely reducible* (in G) if for every R -parabolic subgroup P which contains H there is a R -Levi subgroup L of P , such that $H \subset L$.

Note that in particular, an irreducible subgroup is completely reducible. These definitions are given in [1, Sec. 6] and coincide with the original definition of Serre ([26]), when G is connected; they were introduced to generalize standard methods in the representation theory of $GL(V)$ to arbitrary reductive algebraic groups.

Remark 2.5. Observe that a subgroup $H \subset G$ is irreducible (respectively, completely reducible) if and only if \overline{H} , its Zariski closure in G , is irreducible (respectively, completely reducible). Indeed, this easily follows from the fact that any R -parabolic subgroup of G is Zariski closed in G .

Here and except when explicitly mentioned otherwise, all topological notions on algebraic varieties will refer to the Zariski topology.

3. Stability Notions for Affine G -Varieties

In this section we study the natural notions of (poly)stability that turn out to be most useful for the statement of Theorem 1.2. Here, X denotes a G -variety and G an affine reductive group (neither of which is assumed to be irreducible).

3.1. Stability, proper stability and polystability

The structure of G -variety on X assumes the existence of a morphism satisfying the usual axioms for an action

$$\psi: G \times X \rightarrow X$$

and will be denoted by $(g, x) \mapsto g \cdot x$, ($g \in G$, $x \in X$) where no confusion arises. The orbit space X/G is generally not an algebraic set. However, since G is reductive, there exists a categorical quotient $X//G$ which is an affine algebraic variety. In fact, this quotient is also the so-called *affine quotient* defined as the spectrum of the ring $k[X]^G$ of regular G -invariants inside the coordinate ring $k[X]$. It can also be shown that the affine quotient parameterizes the set of closed G -orbits in X .

For a given point $x \in X$, denote by O_x or by $G \cdot x$ its G -orbit and by G_x its stabilizer in G .

Definition 3.1. The closed subgroup $G_X := \bigcap_{x \in X} G_x \subset G$ will be called the *center of the action* of G on X , or simply the *center of X* when the action is understood. When G_X is not trivial, it makes sense to consider the *normalized stabilizer*, defined as $\text{NG}_x := G_x/G_X$.

Remark 3.2. Note that for any G -variety, G_X is a normal subgroup of G . Indeed, for any $x \in X$ the stabilizers verify $gG_xg^{-1} = G_{g \cdot x}$ for all $g \in G$.

Let $\hat{G} = G/G_X$ denote the space of (left) cosets of G_X , which is again an affine reductive group (see Borel [5, Theorem 6.8]). Then, the G -variety X has also the structure of a \hat{G} -variety, since the action ψ factors through a map (henceforth called the *reduced action*)

$$\hat{\psi}: \hat{G} \times X \rightarrow X$$

defined by $(gG_X, x) \mapsto g \cdot x$ (well defined since G_X is normal).

It is clear that, as affine algebraic varieties, both quotients $X//G$ and $X//\hat{G}$ are isomorphic. This idea has been explored in affine GIT, and because of this, one usually considers the reduced action $\hat{\psi}$.

In contrast, in this paper, we want to consider the original action ψ because it turns out to be more adapted to relate with the concept of irreducibility in reductive groups.

More abstractly, this construction can be seen as a correspondence from the category of G -spaces to the category of G/G_X -spaces and we want to understand how stability behaves under this correspondence. For this, we will use the following definitions of stability closely related to the original definitions given in [17]. Recall that we are using the Zariski topology.

Definition 3.3. Let X be a G -variety and x be a point in X .

- (1) We say that x is *polystable* if $O_x = G \cdot x$ is closed;
- (2) We say that x is *properly stable* if it is polystable and G_x is finite;

- (3) We say that x is *stable* if it is polystable and NG_x is finite;
 (4) We say that x is *equicentral* if it is polystable and NG_x is trivial, that is $G_x = G_X$.

Remark 3.4. As far as we known, the definition of stability in (3) was introduced by Richardson ([23]) and coincides with proper stability (as defined by Mumford), when G_X is finite. Note also that, if either $x \in X$ is properly stable, or x is equicentral, then x is stable.

Let X be an affine G -variety, and consider, for all $x \in X$, the orbit morphism, defined as

$$\psi_x : G \rightarrow X, \quad g \mapsto g \cdot x.$$

It is clear that ψ_x is an affine morphism and that its image $\psi_x(G)$ coincides with the orbit of x , $G \cdot x \subset X$. In the same way as proper stability is equivalent to properness of the orbit morphism (see Newstead [19, Lemma 3.17]), stability is equivalent to properness of the reduced orbit morphism.

Proposition 3.5. *Let X be an affine G -variety. Then, $x \in X$ is stable if and only if the reduced orbit morphism $\hat{\psi}_x : G/G_X \rightarrow X$, $\hat{\psi}_x(gG_X) := g \cdot x$ is proper.*

Proof. It is clear that $\hat{\psi}_x$ is well defined. If $\hat{\psi}_x$ is proper, then the image of $\hat{\psi}_x$ is closed. First we show that $\hat{\psi}_x(G/G_X) = G \cdot x$, the inclusion $\hat{\psi}_x(G/G_X) \subset G \cdot x$ being clear. Note that $\hat{\psi}_x$ factors through the canonical projection $\pi_x : G/G_X \rightarrow G/G_x$ so that we also have $G \cdot x = (G/G_x) \cdot x = \hat{\psi}_x(G/G_x) \subset \hat{\psi}_x(G/G_X)$. So the image of $\hat{\psi}_x$ coincides with the orbit through x and we have concluded that the orbit is closed. Moreover, since every proper map is a finite morphism onto its image, $\hat{\psi}_x : G/G_X \rightarrow G \cdot x$ is a finite morphism. Its fibers are the stabilizer of the reduced action, that is, $(G/G_X)_x$. This means that $(G/G_X)_x = G_x/G_X$ is finite, and we have concluded that x is stable.

Conversely, if $G \cdot x$ is closed and G_x/G_X is finite, then the induced morphism $\hat{\psi}_x : G/G_X \rightarrow G \cdot x$ is finite, because it has finite fibers (see [19, Lemma 3.17]), and therefore it is proper. \square

3.2. The numerical criterion

The following notions extend, to any G -variety X , the definitions given before for the action of G on itself under conjugation (Definition 2.1).

Definition 3.6. Given a 1PS $\lambda \in Y(G)$ and a point $x \in X$, consider the morphism $\lambda_x : k^\times \rightarrow X$, defined by

$$\lambda_x(t) := \lambda(t) \cdot x, \quad t \in k^\times.$$

It will be called the λ -ray through x . We say that $\lim_{t \rightarrow 0} \lambda_x(t)$ exists when λ_x can be extended to a morphism $\bar{\lambda}_x : k \rightarrow X$, and write

$$\lambda^+ x = \lim_{t \rightarrow 0} \lambda_x(t)$$

for the unique element $\bar{\lambda}_x(0) \in X$. As before, whenever a formula with limits is written, we are assuming that they exist.

Consider the natural map $\pi_*: Y(G) \rightarrow Y(G/G_X)$ defined by composition with the canonical projection $\pi: G \rightarrow G/G_X$. The following are easily deduced properties of λ -rays.

Proposition 3.7. *Let X, Y be affine G -varieties.*

- (i) *Let $f: X \rightarrow Y$ be a G -morphism (i.e. a morphism that is G -equivariant), $x \in X$, and $\lambda \in Y(G)$. Then, the λ -rays in X and Y (through x and $f(x)$, respectively) are related by $\lambda_{f(x)} = f \circ \lambda_x$.*
- (ii) *If $f: X \rightarrow Y$ is an inclusion of affine G -varieties, λ^+x exists in X if and only if $\lambda^+(f(x))$ exists in Y .*
- (iii) *We can define a λ -ray through x , for each $\lambda \in Y(G/G_X)$, since $G_X \subset G_x$ for all $x \in X$. Then, μ^+x exists, $\mu \in Y(G)$, if and only if $(\pi_*\mu)^+x$ exists.*

Using these notions, the numerical criterion of Mumford ([17]) can be rephrased in the following way.

Theorem 3.8. *Let X be a G -variety and $x \in X$. Then, x is properly stable for the G/G_X action if and only if for every non-trivial 1PS $\lambda \in Y(G/G_X)$, the limit λ^+x does not exist.*

Proof. This is the direct application of Mumford's numerical criterion to the G/G_X action, the reduced action on X (see [17]). \square

Obviously, given a 1PS $\lambda \in Y(G)$ contained in G_X (that is, such that $\lambda(k^\times) \subseteq G_X$), the limit λ^+x exists for all $x \in X$. To avoid these redundant one-parameter subgroups, we introduce the following notion.

Definition 3.9. Let $x \in X$. We say that x is 1PS stable if it is polystable and for every 1PS $\lambda \in Y(G)$ not contained in G_X , the limit λ^+x does not exist.

From Mumford's criterion above, one easily sees that if $x \in X$ is stable, then it is also 1PS stable. However, it turns out that the two concepts are indeed equivalent, as follows.

Theorem 3.10. *Let X be a G -variety and $x \in X$. Then, the following are equivalent:*

- (1) *The point x is stable;*
- (2) *The point x is properly stable for the G/G_X action;*
- (3) *The point x is 1PS stable for the G/G_X action;*
- (4) *The point x is 1PS stable.*

Proof. Let x be properly stable for the G/G_X action. Then, by Theorem 3.8, this is equivalent to the non-existence of the limit λ^+x , for all $\lambda \in Y(G/G_X)$ non-trivial. Since

$$(G/G_X)_X = \bigcap_{x \in X} (G/G_X)_x = \left(\bigcap_{x \in X} G_x \right) / G_X = \{e\},$$

this means that x is 1PS stable for the G/G_X action. So (2) and (3) are equivalent. Now, suppose $x \in X$ is properly stable for the action of G/G_X . Then, $\text{NG}_x = (G/G_X)_x = G_x/G_X$ is finite and so, x is stable for the G -action, so (2) implies (1). Conversely if $x \in X$ is stable, then the reduced orbit morphism $\hat{\psi}_x: G/G_X \rightarrow X$ is proper, by Proposition 3.5. This means that x is properly stable for the reduced G/G_X action. The equivalence of (1) and (4) is the statement of Theorem 3.12 below. \square

To proceed, we need the following lemma.

Lemma 3.11. *Let $f: G \rightarrow G'$ be a surjective morphism of algebraic groups, and let $f_*: Y(G) \rightarrow Y(G')$ be the canonical push-forward map. For every $\mu \in Y(G')$, there exists $\lambda \in Y(G)$ and $p \in \mathbb{N}$ such that $f_*(\lambda) = \mu^p$.*

Proof. Assume, first, that G and G' are tori. So there are $n, m \in \mathbb{N}$ with $m \leq n$ such that $G \cong (k^\times)^n$ and $G' \cong (k^\times)^m$. Let $\mu \in Y(G') = \mathbb{Z}^m$, $\mu(t) = (t^{l_1}, \dots, t^{l_m})$ for every $t \in k^\times$ and some $(l_1, \dots, l_m) \in \mathbb{Z}^m$. The map $f: G \cong (k^\times)^n \rightarrow G' \cong (k^\times)^m$ applied to $(t_1, \dots, t_n) \in (k^\times)^n$ is $f(t_1, \dots, t_n) = (t_1^{p_1}, \dots, t_m^{p_m})$ for $(p_1, \dots, p_m) \in \mathbb{Z}^m$. Let p be the least common multiple of $|p_1|, \dots, |p_m|$. In particular, there exist $\alpha_1, \dots, \alpha_m \in \mathbb{Z}$, so that $p = \alpha_1 p_1 = \dots = \alpha_m p_m$. We define $\lambda \in Y(G)$ by $\lambda(t) = (t^{\alpha_1 l_1}, \dots, t^{\alpha_m l_m}, t^{x_{m+1}}, \dots, t^{x_n})$, for each $t \in k^\times$ and some $x_{m+1}, \dots, x_n \in \mathbb{Z}$. It is easy to check that $f_*(\lambda) = \mu^p$, where $\mu^p(t) = (t^{pl_1}, \dots, t^{pl_m})$.

In the general case, let $\mu \in Y(G')$. Then, there is a maximal torus T of G' so that $\mu(k^\times) \subset T$. By [5, 22.6], there exists a maximal torus S of G such that $f(S) = T$. Now, we apply the previous case to prove the statement. \square

This allows to complete the proof of Theorem 3.10.^b

Theorem 3.12. *Let $x \in X$. Then, x is stable if and only if x is 1PS stable.*

Proof. If x is not 1PS stable, then there exists a 1PS $\lambda \in Y(G)$ such that $\lambda(k^\times) \not\subseteq G_X$ and λ^+x exists. So, $\tilde{\lambda}^+x$ exists as well, where $\tilde{\lambda}$ is the (non-trivial) push forward 1PS of G/G_X . Thus, x is not stable by Theorem 3.10.

Now assume that x is not stable. By Theorem 3.10, there exists a non-trivial $\mu \in Y(G/G_X)$ such that μ^+x exists. By Lemma 3.11, applied to the canonical

^bWe thank A. Schmitt for suggesting the use of tori in showing the equivalence of stability and 1PS stability.

surjection $\pi: G \rightarrow G/G_X$, we have $\mu^p = \pi_*\lambda$ for some $\lambda \in Y(G)$ and $p \in \mathbb{N}$. Moreover μ^p is non-trivial and $(\mu^p)^+x$ exists (because $p \in \mathbb{N}$). So $\lambda(k^\times) \not\subseteq G_X$ since μ is non-trivial. If the limit λ^+x did not exist, then $(\mu^p)^+x$ would not exist as well, by Proposition 3.7(ii), so we conclude that λ^+x exists, and so, x is not 1PS stable. \square

Corollary 3.13. *Let X be a G -variety with G_X finite. Then all three stability notions are equivalent (stable, properly stable and 1PS stable).*

Proof. By definition, x is stable if x is polystable and G_x/G_X is finite, but since G_X is finite, this means that G_x is finite, so x is stable if and only if x is properly stable. The equivalence of stability and 1PS stability was established in the previous theorem. \square

To summarize, we showed the following relations between stability notions of a point x in a general affine G -variety X : x properly stable $\Rightarrow x$ stable $\Leftrightarrow x$ 1PS stable $\Rightarrow x$ 1PS polystable $\Leftrightarrow x$ polystable.

4. Stability and Polystability Criteria

In this section, we prove our main results (in particular Theorem 1.2) on the relation between polystability of a point x in an affine G -variety and the complete reducibility of a subgroup of G , naturally associated to x . Along the way, we prove an analogue of a theorem of Mundet-Schmitt in the context of affine G -varieties and provide some examples of the constructions.

4.1. 1PS polystability

Let G be an affine reductive group and X be an affine G -variety. Recall that, for any $x \in X$, the closure of its orbit $\overline{G \cdot x}$ contains a unique closed orbit. The following well-known statement characterizes non-closed orbits.

Theorem 4.1. *If $x \in X$ is such that $\overline{G \cdot x} \setminus G \cdot x$ is non-empty then there is a $y \in X$ in the unique closed orbit of $\overline{G \cdot x}$ and a one parameter subgroup $\lambda \in Y(G)$ such that $\lambda^+x = \lim_{t \rightarrow 0} \lambda(t) \cdot x = y$.*

Proof. See [4] or [13]. \square

This result motivates the following definition introduced by Levy (see [15] and also [3, Definition 3.8]).

Definition 4.2. Let $x \in X$. We say that x is *1PS polystable* if $\lambda^+x \in G \cdot x$ for all $\lambda \in Y(G)$ such that λ^+x exists.

With this terminology, Theorem 4.1 can be restated as an equivalence between the notions of polystability and 1PS polystability.

Proposition 4.3. *Let $x \in X$. Then, x is polystable if and only if x is 1PS polystable.*

Proof. If x is polystable, then $G \cdot x$ is closed and the orbit $G \cdot x$ is a subvariety of X . Thus, if λ^+x exists in X , for some $\lambda \in Y(G)$, then λ^+x is in fact in $G \cdot x$ (see Proposition 3.7). So x is 1PS polystable. Conversely, if x is not polystable, so that $\overline{G \cdot x} \setminus G \cdot x$ is non-empty, then by Theorem 4.1, there exists $\lambda \in Y(G)$ and $y \notin G \cdot x$ such that $y = \lambda^+x$, which, by definition, implies that x is not 1PS polystable. \square

Remark 4.4. In [3, Corollary 4.11] it is shown that this proposition also holds for affine G -varieties defined over non-algebraically closed perfect fields.

4.2. Opposite 1PS and the theorem of Mundet–Schmitt

Let λ be a 1PS of G . Recall the notions of R-parabolic and R-Levi subgroups, given in Definition 2.2.

Recall that, given $\lambda \in Y(G)$, one says that another 1PS $\tilde{\lambda}$ is opposite to λ , if $P(\lambda) \cap P(\tilde{\lambda})$ is a R-Levi subgroup of both $P(\lambda)$ and $P(\tilde{\lambda})$. For later convenience, we record the following fact.

Lemma 4.5. *Let $\lambda, \tilde{\lambda} \in Y(G)$ be opposite 1PS lying in the same maximal torus of G . Then $P(\lambda^{-1}) = P(\tilde{\lambda})$.*

Proof. We will use two uniqueness results, valid also for non-connected reductive G : (i) Given a R-parabolic subgroup P and a maximal torus T of G contained in P , there is a unique R-Levi subgroup of P containing T ; (ii) Given a R-parabolic subgroup P and one of its R-Levi subgroups L , there is a unique opposite R-parabolic P' such that $P \cap P' = L$ (see [1, Corollary 6.5, Lemma 6.11, respectively]).

To start, let T be a maximal torus containing both λ and $\tilde{\lambda}$. Then, $T \subset L(\lambda)$ and $T \subset L(\tilde{\lambda})$ so both $P(\lambda)$ and $P(\tilde{\lambda})$ contain T . Since λ and $\tilde{\lambda}$ are opposite, $P(\lambda) \cap P(\tilde{\lambda})$ is a R-Levi subgroup L of both, and clearly L contains T . By the uniqueness result (i) above, and since $L(\lambda)$ is a R-Levi of $P(\lambda)$ containing T , we conclude that $L = L(\lambda)$. Finally, since both $P(\lambda^{-1})$ and $P(\tilde{\lambda})$ are opposite to $P(\lambda)$, and $L(\lambda)$ is a R-Levi of $P(\lambda)$, the uniqueness result (ii) implies that $P(\lambda^{-1}) = P(\tilde{\lambda})$, as wanted. \square

Definition 4.6. A subset $\Lambda \subset Y(G)$ is called *symmetric* if for any 1PS $\lambda \in \Lambda$ there exists another 1PS $\tilde{\lambda} \in \Lambda$, which is opposite to λ .

Given $x \in X$, we use the following notation:

$$\Lambda_x := \{\lambda \in Y(G) : \lambda^+x \text{ exists}\}.$$

Note that if λ^+x exists and $g \in P(\lambda)$, then a computation yields

$$\lambda^+(g \cdot x) = (\lambda^+g) \cdot (\lambda^+x).$$

In particular, if $u \in U(\lambda)$, then $\lambda^+(u \cdot x) = \lambda^+x$. So, it is clear that $\Lambda_x = \Lambda_{u \cdot x}$ for any $u \in U(\lambda)$ such that $\lambda \in \Lambda_x$. The following properties of Λ_x are straightforward.

Proposition 4.7. *Consider the natural inclusion $Y(G_x) \subset Y(G)$. For all $x \in X$, we have*

- (i) *If $G \cdot x = x$ then $\Lambda_x = Y(G)$.*
- (ii) *$Y(G_X) \subset Y(G_x) \subset \Lambda_x \subset Y(G)$.*
- (iii) *If x is stable, then $Y(G_x) = \Lambda_x = Y(G_X)$.*

Let X and Y be affine G -varieties, and $f: X \rightarrow Y$ be a G -equivariant morphism. The following properties are also easily established.

Proposition 4.8. (i) *For all $x \in X$, $\Lambda_x \subset \Lambda_{f(x)}$.*

(ii) *If $f: X \rightarrow Y$ is an inclusion, then $\Lambda_x = \Lambda_{f(x)}$ for all $x \in X$.*

(iii) *If $X \times Y$ is the product variety endowed with the natural product G -action, then $\Lambda_{(x,y)} = \Lambda_x \cap \Lambda_y$ for every $(x,y) \in X \times Y$.*

Given a 1PS $\lambda \in Y(G)$ and $x \in X$, we have defined the limit λ^+x . Similarly, we define λ^-x as the limit (when it exists)

$$\lambda^-x := \lim_{t \rightarrow \infty} \lambda(t) \cdot x = \lim_{t \rightarrow 0} \lambda^{-1}(t) \cdot x.$$

We now prove Theorem 1.1.

Theorem 4.9. *$x \in X$ is polystable if and only if Λ_x is symmetric.*

Proof. Let $x \in X$ be polystable and suppose that $\lambda \in \Lambda_x$. Then λ^+x exists and because $G \cdot x$ is closed, $\lambda^+x \in G \cdot x$. By the general theorems, [3] and references in [18], $\lambda^+x = u \cdot x$ for some $u \in U(\lambda)$. Now, it is clear that the limit $u \cdot x$ centralizes λ , so that $\lambda^{-1} \cdot (u \cdot x) = u \cdot x$. This means that $\lambda^{-1} \in \Lambda_{u \cdot x} = \Lambda_x$. Since λ^{-1} and λ are obviously opposite, we conclude that Λ_x is symmetric.

Conversely, if $G \cdot x$ is not closed, then by Theorem 4.1, there exists $\lambda \in Y(G)$ such that $\lambda^+x \in \overline{G \cdot x} \setminus G \cdot x$. This implies that $\{\lambda(t) \cdot x : t \in k^\times\} \subset X$ is not a single point. Suppose to get a contradiction, that Λ_x was symmetric: there is an opposite $\tilde{\lambda} \in Y(G)$ such that $\tilde{\lambda}^+x$ exists. By the same argument in [18, Appendix] we can find a $u \in U(\lambda)$ such that $u\lambda(t)u^{-1}$ and $\tilde{\lambda}(t)$ lie on the same maximal torus (and are still opposite). So, by Lemma 4.5 and because $u \in U(\lambda) \subset P(\lambda)$

$$P(\tilde{\lambda}^{-1}) = P(u\lambda u^{-1}) = P(\lambda).$$

Now, this means that $\tilde{\lambda}^{-1} \in \Lambda_x$, so that

$$\tilde{\lambda}^+x = (\tilde{\lambda}^{-1})^-x = \lambda^-x.$$

Then, $\lambda_x : k \rightarrow X$ extends to a morphism $\lambda : \mathbb{P}_k^1 \rightarrow X$ which has to be constant, contradicting the previous claim; we conclude that Λ_x is not symmetric. \square

We now summarize the setting in Schmitt's appendix to Mundet's article [18]. Suppose our G -variety is a k -vector space V . We say that v is *semistable* if $0 \notin \overline{G \cdot v}$. Given a 1PS $\lambda \in Y(G)$ and a weight $n \in \mathbb{Z}$, define

$$V(n) := \{v \in V \mid \lambda(t) \cdot v = t^n v \ \forall t \in k^\times\}.$$

Then, we have a finite vector space decomposition in non-empty subspaces $V = V(n_1) \oplus \cdots \oplus V(n_m)$, where $n_1 < \cdots < n_m$. With respect to this decomposition, write $v = v_1 + \cdots + v_m$ and define

$$\mu(v, \lambda) := \min\{n_j \mid v_j \neq 0\}$$

Then, we have

Lemma 4.10. *Let v be semistable. Then $\mu(v, \lambda) = 0$ if and only if $\lambda^+ v$ exists.*

Proof. By construction

$$\lambda(t) \cdot v = \lambda(t) \cdot (v_1 + \cdots + v_m) = t^{n_1} v_1 + \cdots + t^{n_m} v_m,$$

with $n_1 < \cdots < n_m$, so the limit $\lambda^+ v = \lim_{t \rightarrow 0} \lambda(t) \cdot v$ exists if and only if $\mu(v, \lambda) \geq 0$. On the other hand, if $\mu(v, \lambda) > 0$, we would have $\lambda^+ v = 0$, contradicting the semistability of v . \square

Corollary 4.11. (Mundet–Schmitt [18, Theorem A.5]) *A semistable $v \in V$ is polystable if and only if for every $\lambda \in Y(G)$ with $\mu(v, \lambda) = 0$ there exists an opposite λ' such that $\mu(v, \lambda') = 0$.*

Proof. In view of the previous lemma, this is a simple restatement of Theorem 4.9. \square

4.3. Polystability and completely reducible subgroups

In this subsection we provide a correspondence between the notion of polystability (respectively, stability) for points of the G -variety X and the notion of complete reducibility (respectively, irreducibility) for subgroups of G . In order to obtain this, we introduce the following construction:

Given $\Lambda \subset Y(G)$ we define $H_\Lambda := \bigcap_{\lambda \in \Lambda} P(\lambda)$; For any such Λ , H_Λ is a closed subgroup of G .

Let $S(G)$ be the set of closed subgroups of G . We consider the map $\eta : X \rightarrow S(G)$ given by

$$x \mapsto H_x := H_{\Lambda_x} \subset G.$$

The following properties are clear, in view of Propositions 4.7 and 4.8. Again, $f: X \rightarrow Y$ denotes a G -equivariant morphism of affine G -varieties and $X \times Y$ the product G -variety.

Proposition 4.12. *Let $Z = Z(G)$ and $x \in X$.*

- (i) *If $\Lambda_x \subset Y(Z)$, then $H_x = G$.*
- (ii) *If $G_x = G$ then $H_x = Z$.*
- (iii) *$Z \subset H_x \subset \bigcap_{\lambda \in Y(G_x)} P(\lambda)$.*
- (iv) *H_x is irreducible if and only if $H_x = G$.*
- (v) *$H_{f(x)} \subset H_x$ for all $x \in X$.*
- (vi) *If $f: X \rightarrow Y$ is an inclusion, then $H_x = H_{f(x)}$ for all $x \in X$.*
- (vii) *For every $(x, y) \in X \times Y$, $H_x \cup H_y \subset H_{(x, y)}$.*

Now, we give a name to the condition (1.1) alluded in Sec. 1.

Definition 4.13. We say that $\eta: X \rightarrow S(G)$ is *proper* if for every $x \in X$ and $\lambda \in Y(G)$, $H_x \subset P(\lambda)$ implies that $\lambda \in \Lambda_x$.

Note that, for any action, with η proper or not, the converse implication is valid i.e. if $\lambda \in \Lambda_x$ then $H_x \subset P(\lambda)$. Indeed, assume that $\lambda \in \Lambda_x$, and that $g \in H_x = \bigcap_{\mu \in \Lambda_x} P(\mu)$; then, $g \in P(\mu)$, for every $\mu \in \Lambda_x$, in particular $g \in P(\lambda)$.

Lemma 4.14. *Let $\eta: X \rightarrow S(G)$ be proper, $x \in X$ and $\lambda \in Y(G)$. Then $H_x \subset P(\lambda)$ if and only if $\lambda \in \Lambda_x$.*

Proposition 4.15. *Let η be proper and $x \in X$. If Λ_x is symmetric then $\eta(x) = H_x$ is completely reducible in G .*

Proof. Suppose that $\eta(x) = H_x = \bigcap_{\lambda \in \Lambda_x} P(\lambda) \subset P(\mu)$ for some $\mu \in Y(G)$. So, for every $g \in H_x$, there exists μ^+g . As η is proper, μ^+x exists, then $\mu \in \Lambda_x$. We have that Λ_x is symmetric, this implies that there is $\tilde{\mu} \in \Lambda_x$ opposite to μ . Thus $\eta(x) \subset P(\mu) \cap P(\tilde{\mu}) = L$ with L a R-Levi subgroup of $P(\mu)$. We can conclude that $\eta(x) = H_x$ is completely reducible in G . \square

Now, we prove the first part of our main result.

Theorem 4.16. *Let $\eta: X \rightarrow S(G)$ be proper. Then, $x \in X$ is polystable if and only if $H_x = \eta(x)$ is completely reducible in G .*

Proof. From Proposition 4.15 and Theorem 4.9, H_x is completely reducible in G , for any $x \in X$ which is polystable. Conversely, suppose that $H_x \subset G$ is completely reducible and let $\lambda \in \Lambda_x$. Then $H_x = \bigcap_{\mu \in \Lambda_x} P(\mu) \subset P(\lambda)$, by definition. Since H_x is completely reducible, there is $\tilde{\lambda}$, opposite to λ such that $H_x \subset P(\tilde{\lambda})$. Since η is proper, this implies that $\tilde{\lambda} \in \Lambda_x$. We have concluded that Λ_x is symmetric, so that by Theorem 4.9, x is polystable. \square

Corollary 4.17. *Let $\eta: X \rightarrow S(G)$ be proper and $x \in X$. Then Λ_x is symmetric if and only if $H_x = \eta(x)$ is completely reducible in G .*

Proof. This is an immediate consequence of Theorems 4.9 and 4.16. \square

Definition 4.18. We say that the action of G on X is *central* (or X is a *central G -variety*) if $G_X = Z(G)$.

Theorem 4.19. *Let X be a G -variety and assume that $Z(G) \subset G_X$. Let η be proper and $x \in X$. Then*

- (1) *If x is stable, then $H_x = G$.*
- (2) *Assume that X is central. Then x is stable if and only if $H_x = G$.*

Proof. By Theorem 3.12, x is stable in X if and only if it is 1PS stable, so that for every 1PS, λ such that $\lambda(k^*) \not\subseteq G_X$ it does not exist λ^+x . As η is proper, for every such 1PS, λ , of G , λ^+g does not exist for some $g \in \eta(x) = H_x$. But this is equivalent to saying that $H_x \not\subseteq P(\lambda)$ for every 1PS, λ , of G which is not in G_X . Since $Z(G) \subset G_X$ this means that H_x is not in $P(\lambda)$ for every non-central λ . Thus H_x is not contained in any proper R-parabolic, so that H_x is irreducible in G . Then, by Proposition 4.12(iv) $H_x = G$, and we have shown (1). To prove (2) suppose $G_X = Z(G)$. If x is stable, by (1) $H_x = G$. Conversely, if x is not stable, then it is not 1PS stable which means that λ^+x exists for some $\lambda \in Y(G)$, with $\lambda(k^\times) \not\subseteq G_X = Z(G)$. So $\lambda \in \Lambda_x$, which by properness of η means $H_x \subset P(\lambda)$, a proper R-parabolic. So $H_x \neq G$. \square

Theorems 4.16 and 4.19 finish the proof of Theorem 1.2.

Example 4.20. Let $G = SL_2(k)$ and $X = M_2(k)$, the vector space of 2×2 matrices, where G acts by conjugation on X . By general facts, any non-trivial 1PS in G is conjugated to

$$\lambda_n(t) := \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix},$$

for some $n = 1, 2, \dots$. Let $\lambda = h\lambda_n h^{-1}$, for $h \in G$ and $x \in X$. Then, by definition, λ^+x exists, if and only if

$$\lim_{t \rightarrow 0} \lambda_n(t) h^{-1} x h \lambda_n(t)^{-1} \quad (4.1)$$

exists, which implies, by a simple computation, that $h^{-1}xh \in U$, where U is the set of upper triangular matrices in $M_2(k)$. First, let us compute Λ_x and $H_x = \eta(x)$ for x in the form:

$$x = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in X.$$

Writing

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(k), \quad (4.2)$$

another computation shows that, in the case that x is generic, that is $\alpha \neq \beta$, $h^{-1}xh \in U$ is equivalent to $ac = 0$. So, we conclude that

$$\Lambda_x = \{h\lambda_n h^{-1} : h \in H_1\} \cup \{h\lambda_n h^{-1} : h \in H_2\},$$

where H_1 is the set of elements $h \in SL_2(k)$ in the form (4.2) such that $a = 0$ and $H_2 = U$ the set of those h with $c = 0$. We note that the decomposition of Λ_x above, corresponds to opposite 1PS, in the sense that, for $h \in H_1$ and $\tilde{h} \in H_2$ the 1PS $h\lambda_n h^{-1}$ and $\tilde{h}\lambda_n \tilde{h}^{-1}$ are opposite.

Now, let $h \in H_1$ be fixed. Computing as in (4.1), with $g \in G$ instead of $x \in X$,

$$P(h\lambda_n h^{-1}) = \{g \in G : h^{-1}gh \in U \cap SL_2(k)\} = U \cap SL_2(k)$$

and, in the case $h \in H_2$, we have

$$P(h\lambda_n h^{-1}) = \{g \in G : h^{-1}gh \in U \cap SL_2(k)\} = L \cap SL_2(k),$$

where L is the set of lower triangular matrices in $M_2(k)$. Finally, then

$$H_x = \bigcap_{\lambda \in \Lambda_x} P(\lambda) = \bigcap_{h \in H_1} P(h\lambda_n h^{-1}) \bigcap_{h \in H_2} P(h\lambda_n h^{-1}) = U \cap L \cap SL_2(k)$$

is the subgroup of diagonal matrices in $SL_2(k)$, a completely reducible subgroup in $SL_2(k)$.

When x is scalar, it is easy to see that $\Lambda_x = Y(G)$ and $H_x = \{e\}$. Finally, when x is of the form

$$x = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in X,$$

for $\beta \neq 0$, by similar computations we obtain $H_x = U \cap SL_2(k)$, which is not completely reducible in G .

To conclude, because in this case the map η is clearly proper (as the action of G on itself is the restriction of the action of G on X), this example serves as a simple illustration of our main results, namely that the orbit $G \cdot x$ is closed if and only if H_x is completely reducible.

The above constructions can be interpreted in the language of buildings. In fact, there is a natural way to pass from $Y(G)$ to the building of G , under which Λ_x corresponds to a convex subset in this building (see [17, Sec. 2]). This observation, for which we thank Röhrlé, will be explored in a later work.

4.4. A sufficient criterion for properness of η

Above, we have discussed actions with η proper. However, it is easy to find examples where η is not proper. For instance, consider $G = \mathbb{C}^*$ and $X = \mathbb{C}^n$, with the action of G on X given by $t \cdot (x_1, \dots, x_n) = (t^{-1}x_1, \dots, t^{-1}x_n)$ (where $t \in G$ and

$(x_1, \dots, x_n) \in X$). Trivially, $H_x = G$ for all $x \in X$ because for every $\lambda \in Y(G)$, $P(\lambda) = G$. But $\lambda \notin \Lambda_x$ for the identity character ($\lambda(t) := t$ for $t \in \mathbb{C}^*$) and every $x \in X \setminus \{0\}$.

Some easy properties of the properness notion are the following. Let again X , Y be affine G -varieties and $X \times Y$ the product G -variety, with respective maps η_X , η_Y and $\eta_{X \times Y}$. Let $f: X \rightarrow Y$ be a G -morphism.

Proposition 4.21. (i) *If $f: X \rightarrow Y$ is an inclusion and η_Y is proper, then η_X is also proper.*

(ii) *If η_X and η_Y are proper then $\eta_{X \times Y}$ is also proper.*

Proof. (i) Let $x \in X$, $\lambda \in Y(G)$, and suppose that $H_x \subset P(\lambda)$. By Propositions 4.8(i) and 4.12(vi), we have $\Lambda_x = \Lambda_{f(x)}$ and $H_{f(x)} = H_x$, so $H_{f(x)} \subset P(\lambda)$. Since η_Y is proper, by hypothesis, $\lambda^+ f(x)$ exists in Y and so $\lambda \in \Lambda_{f(x)} = \Lambda_x$. Thus η_X is proper.

(ii) Let $(x, y) \in X \times Y$ and $\lambda \in Y(G)$. Suppose that $H_{(x,y)} \subset P(\lambda)$, by 4.12(vii) $H_x \cup H_y \subset H_{(x,y)}$ so H_x and H_y are contained on $P(\lambda)$. By hypothesis η_X and η_Y are proper then $\lambda \in \Lambda_x \cap \Lambda_y$. Thus $\lambda \in \Lambda_{(x,y)}$ by Proposition 4.8. Therefore $\eta_{X \times Y}: X \times Y \rightarrow S(G)$ is proper. \square

The following is a simple sufficient condition to obtain properness.

Lemma 4.22. *Let $\psi: X \rightarrow G$ be a map that verifies the following condition: For every pair $x \in X$, $\lambda \in Y(G)$, $\lambda^+(\psi(x))$ exists if and only if λ^+x exists. Then, η is proper.*

Proof. The condition implies that if $\lambda \in \Lambda_x$, then $\psi(x) \in P(\lambda)$, so $\psi(x) \in H_x = \bigcap_{\lambda \in \Lambda_x} P(\lambda)$. Now let $H_x \subset P(\mu)$, for some $\mu \in Y(G)$. Then, $\psi(x) \in H_x \subset P(\mu)$, so that $\mu^+(\psi(x))$ exists. Then, the condition implies that $\mu \in \Lambda_x$, so η is proper. \square

Such a ψ does not exist in general (see the example in the beginning of this subsection), but when it is a G -equivariant morphism, the existence of λ^+x implies that of $\lambda^+(\psi(x))$, by a simple calculation.

We now apply this Lemma to the adjoint representation, restricting to the case of complex affine reductive groups (i.e. to the field $k = \mathbb{C}$). Let $\text{Lie}(G)$ denote the Lie algebra of G , which is naturally a G -variety with the conjugation action of G and let $\exp: \text{Lie}(G) \rightarrow G$ be the exponential map. Note however, that \exp is not an algebraic morphism.

Lemma 4.23. *Let V be a finite dimensional complex vector space, and $G = GL(V)$ be the complex general linear group. Then, for all $v \in \text{Lie}(G) \cong \text{End}(V)$ and $\lambda \in Y(G)$, $\lambda^+\exp(v)$ exists if and only if λ^+v exists.*

Proof. By choosing basis for V , we can assume that $\lambda(t)$ is represented by a diagonal matrix $\lambda(t) = \text{diag}(t^{k_1}, \dots, t^{k_n})$, for some weights $k_1, \dots, k_n \in \mathbb{Z}$ with

$k_1 \geq k_2 \geq \cdots \geq k_n$. Since $\lambda^+ \exp(v)$ equals $\lim_{t \rightarrow 0} \lambda(t) \exp(v) \lambda(t^{-1})$, it exists if and only if the matrix representing $\exp(v)$ has zero entries below the main diagonal. By an elementary calculation, this is equivalent to the matrix $v \in \text{Lie}(G)$ having zero entries below the main diagonal, which in turn means that $\lambda^+ v$ exists. \square

Corollary 4.24. *Let G be a complex affine reductive group, and let $\text{Lie}(G)$ be its Lie algebra. Then, $\exp : \text{Lie}(G) \rightarrow G$ verifies the condition in Lemma 4.22. Thus, η is proper for the adjoint representation.*

Proof. Assume that G is equivariantly embedded in $GL(V)$ for some vector space V . Then, we have the commutative diagram,

$$\begin{array}{ccc} \text{End } V & \xrightarrow{\exp} & GL(V) \\ \varphi \uparrow & & \uparrow \varphi \\ \text{Lie}(G) & \xrightarrow{\exp} & G. \end{array}$$

Suppose that $\lambda^+(\exp(v))$ exists in G , for $v \in \text{Lie}(G)$ so that $\lambda^+(\varphi(\exp(v))) = \lambda^+(\exp(\varphi(v)))$ exists in $GL(V)$. By the previous lemma, this means that $\lambda^+ \varphi(v)$ exists in $\text{End } V$, which implies that $\lambda^+ v$ exists in $\text{Lie}(G)$ (note that φ are inclusions of closed varieties). Similarly, one shows that the existence of $\lambda^+ v$ implies that of $\lambda^+(\exp(v))$. \square

Remark 4.25. There are many other examples of affine G -varieties with η proper. For instance, one obtains a new example by removing some closed G -invariant algebraic subset from a G -variety with η proper. Naturally, it is more interesting to work in the opposite direction: Start from the proper case and extend the action of G to a bigger affine variety, preserving properness. We plan to exploit these natural questions in a future work.

5. Application to G -Invariant Subvarieties of G^N

Let G be an affine reductive group defined over k . In this section, we apply the techniques of the last sections to a particularly interesting class of G -varieties, namely to closed G -invariant subvarieties of the variety of N -tuples in G . This class includes the representation varieties of finitely generated groups, which are varieties of the form $\text{Hom}(\Gamma, G)$, consisting of homomorphisms from a fixed finitely generated group Γ into G . Because of the natural conjugation action, the variety $X = \text{Hom}(\Gamma, G)$ becomes a G -variety. An application of these results is a restatement of the well-known correspondence between G -Higgs bundles over an algebraic curve and character varieties of surface groups.

5.1. Polystability for subvarieties of G^N

Let X be an affine closed G -subvariety of G^N , where G^N is the G -variety on which G acts component-wise by simultaneous conjugation. Let $f : X \hookrightarrow G^N$ be

the G -equivariant inclusion. Fix $x \in X$ and let $f(x) = (f_1(x), \dots, f_N(x))$. Let ϕ_x be the subgroup of G generated by the elements $f_1(x), \dots, f_N(x) \in G$.

Recall the construction of the map $\eta: X \rightarrow S(G)$, with $\eta(x) = H_x = \bigcap_{\lambda \in \Lambda_x} P(\lambda)$. We have:

Proposition 5.1. *Let X be an affine closed G -subvariety of G^N .*

- (i) *Let $x \in X$ and $\mu \in Y(G)$, then $\mu \in \Lambda_x$ if and only if $\phi_x \subseteq P(\mu)$.*
- (ii) *Let $x \in X$ then $\phi_x \subset H_x$.*
- (iii) *η is proper.*

Proof. To show (i), let $x \in X$ and $\mu \in Y(G)$. Consider $\mu_x: k^\times \rightarrow X$ and $\mu_{f(x)}: k^\times \rightarrow G^N$ the μ -rays through x and $f(x)$, respectively. We have that $\mu_{f(x)} = f \circ \mu_x$ by Proposition 3.7(i).

Suppose that $\mu \in \Lambda_x$, meaning that μ^+x exists. So if μ^+x exists then $\mu^+f(x)$ exists. This is equivalent to the existence of $\mu^+f_i(x)$, for every $i = 1, \dots, N$. But this one implies the existence of μ^+g , for every $g \in \phi_x$. Thus $\phi_x \subset P(\mu)$.

Conversely, suppose now that $\phi_x \subseteq P(\mu)$, this implies, in particular, that $\mu^+f_i(x)$ exists for every $i = 1, \dots, N$. Thus, $\mu^+f(x)$ exists. Since X is a closed subset of G^N , we can guarantee the existence of μ^+x (see Proposition 3.7(ii)). Thus if $\phi_x \subseteq P(\mu)$ then $\mu \in \Lambda_x$.

To show (ii), using (i) we have $\phi_x \subseteq P(\mu)$ for every $\mu \in \Lambda_x$ so $\phi_x \subset \bigcap_{\lambda \in \Lambda_x} P(\lambda) = H_x$.

Suppose that $H_x \subset P(\mu)$. Then, by (ii) $\phi_x \subset H_x \subset P(\mu)$, which means that $\mu \in \Lambda_x$, according to (i). So, η is proper. \square

For this class of varieties we have the next properties.

Proposition 5.2. *Let $x \in X$. Then H_x is completely reducible in G if and only if ϕ_x is completely reducible in G .*

Proof. By (ii) of Proposition 5.1, we have that $\phi_x \subset H_x$.

Suppose that $H_x = \bigcap_{\lambda \in \Lambda_x} P(\lambda)$ is completely reducible. If $\phi_x \subset P(\mu)$ for some $\mu \in Y(G)$, then by (i) of Proposition 5.1 $\mu \in \Lambda_x$ and so $\phi_x \subset P(\mu)$. As H_x is completely reducible, $H_x \subset L$, with L a R-Levi subgroup of $P(\mu)$. Thus $\phi_x \subset L$ and then ϕ_x is completely reducible in G .

Conversely, suppose that ϕ_x is completely reducible in G and $H_x = \bigcap_{\lambda \in \Lambda_x} P(\lambda) \subset P(\mu)$ for some $\mu \in Y(G)$. By (ii) of Proposition 5.1, $\phi_x \subset H_x$ then $\rho(\Gamma) \subset P(\mu)$, so by (i) of Proposition 5.1, $\mu \in \Lambda_x$. On the other hand, as ϕ_x is completely reducible in G , there exists a R-Levi subgroup $L = L(u\mu u^{-1})$ of $P(\mu)$, with $u \in U(\mu)$, such that $\rho(\Gamma) \subset L$. Now, we can find a R-parabolic subgroup of G , P , opposite to $P(\mu)$ so that $P(\mu) \cap P = L$ and $P = P(u\mu^{-1}u^{-1})$. Thus $\phi_x \subset P$, so again by (i) of Proposition 5.1 $u\mu^{-1}u^{-1} \in \Lambda_x$. Then $H_x \subset P$ and as also $H_x \subset P(\mu)$ we get that $H_x \subset L$. Thus H_x is completely reducible in G . \square

Proposition 5.3. *Let $x \in X$. If $H_x = G$, then ϕ_x is irreducible in G .*

Proof. If ϕ_x is not irreducible in G , by definition, there exists a non-central $\mu \in Y(G)$ such that $\phi_x \subset P(\mu)$. So by (i) of Proposition 5.1, $\mu \in \Lambda_x$, thus $H_x \subset P(\mu)$ and we conclude that H_x is not irreducible. \square

Proposition 5.4. *For any $x \in X$, we have $G_x = Z_G(\phi_x)$, where $Z_G(\phi_x)$ is the centralizer of ϕ_x and $Z(G) \subset G_X$.*

Proof. The first statement is clear. To show the second one, first note that the stabilizer of any $x \in X$ contains the center of the group. Thus, $Z(G) \subset \bigcap_{x \in X} G_x = G_X$. \square

The proof of next result (Theorem 1.3) follows easily from the work of several authors on N -tuples in algebraic groups (see [23, Theorems 3.6 and 4.1], [1, Corollary 3.7] and [16]). We include a self-contained proof for the convenience of the reader. Recall that a G -variety is called central when $G_X = Z(G)$.

Theorem 5.5. *Let $x \in X \subset G^N$. Then*

- (1) *x is polystable if and only if ϕ_x is completely reducible.*
- (2) *x is stable if and only if ϕ_x is irreducible.*
- (3) *If X is central then x is equicentral if and only if ϕ_x is isotropic.*

Proof. (1) By Proposition 5.2 and Theorem 4.16 we just need to see that η is proper. But this is true by (iii) of Proposition 5.1.

(2) If x is stable in X , as $Z(G) \subset G_X$ (Proposition 5.4) and η is proper ((iii) of Proposition 5.1), we can apply Theorem 4.19(2) and therefore H_x is irreducible in G . Thus by Proposition 5.3, ϕ_x is irreducible.

Conversely, if ϕ_x is irreducible in G , since ϕ_x is completely reducible in G , by (1) we get that x is polystable, that is, $G \cdot x$ is closed in G . Then by [23, 1.3.3] G_x is reductive. If we suppose that G_x/G_X is infinite, this implies that $G_x/Z(G)$ is infinite, as $Z(G) \subset G_X$. So there exists $\lambda \in Y(G_x)$ such that $\lambda(k^\times) \not\subset Z(G)$. Then $P(\lambda)$ is a proper parabolic subgroup of G containing ϕ_x . This contradicts the assumption, concluding that G_x/G_X is finite and thus x is stable.

(3) It is enough to show that $G_x = G_X$ if and only if $Z_G(\phi_x) = Z(G)$. This is immediate by the assumption $G_X = Z(G)$ and the fact proven in Proposition 5.4 that $G_x = Z_G(\phi_x)$. \square

From Proposition 5.2 and Theorem 5.5(2), Theorem 1.4 follows. Consider now, for $N \in \mathbb{N}$ and a complex reductive G , with Lie algebra $\text{Lie}(G)$, the vector space $X = (\text{Lie}(G))^N$, with the diagonal adjoint action of G . The next result follows easily from Richardson ([23, Theorems 3.6 and 4.1]), being also a natural corollary of our approach.

Theorem 5.6. *Let $k = \mathbb{C}$ and X be a closed G -invariant subvariety of $(\mathrm{Lie}(G))^N$. An element $x \in X$ is polystable (respectively, stable) if and only if H_x is completely reducible in G (respectively, $H_x = G$).*

Proof. Consider first the case $X = (\mathrm{Lie}(G))^N$, in which $Z(G) = G_X$. Then, the statement is an easy consequence of Corollary 4.24 and Proposition 4.21(ii), in view of Theorem 1.2. The case when $X \subset (\mathrm{Lie}(G))^N$ then follows from Proposition 4.21(i). \square

5.2. Stability in representation varieties

Let Γ be a finitely generated group. In this subsection we let X be the G -representation variety of Γ , that is $X = \mathrm{Hom}(\Gamma, G)$, where the G -action is by conjugation. It is an affine closed G -invariant subvariety of G^N , so the results obtained in the last subsection can be immediately applied to this case. We will phrase them using the following standard terminology.

Definition 5.7. Let $\rho \in \mathrm{Hom}(\Gamma, G)$ and $\rho(\Gamma)$ be its image in G . The representation ρ is called:

- (1) *Irreducible*, if $\rho(\Gamma)$ is an irreducible subgroup in G .
- (2) *Reductive*, if $\rho(\Gamma)$ is completely reducible in G .
- (3) *Good*, if it is reductive and $G_\rho = Z(G)$.

Remark 5.8. (1) Because of these definitions, all properties satisfied by irreducible (respectively, completely reducible) subgroups are satisfied by irreducible (respectively, reductive) representations. In particular, we have the analogues of the Propositions in Subsec. 2.2(2) Another definition of irreducible representation, in the case k algebraically closed and G connected, appears in the work of Ramanathan ([21]) which is equivalent to our definition if we use Proposition A.4 and [21, Proposition 2.1.2].

Suppose now $\Gamma = \langle \gamma_1, \dots, \gamma_N \mid r_1, \dots, r_N \rangle$ is a presentation of the group Γ , where $\gamma_1, \dots, \gamma_N$ are generators for some $N \in \mathbb{N}$. From this, we form the canonical projection map

$$\pi: F_N \twoheadrightarrow \Gamma$$

where F_N is a free group of rank N . This allows us to view any representation variety $X = \mathrm{Hom}(\Gamma, G)$, by composing any representation with π , as a closed G -invariant affine subvariety of $G^N = \mathrm{Hom}(F_N, G)$. This allows the direct application of the previous results to this class of varieties.

Remark 5.9. Note that, however, not all G -invariant closed affine subvarieties of G^N are character varieties for some Γ , as simple examples show.

Recalling the notation of the previous subsection applied to this particular case, we have $f: \mathrm{Hom}(\Gamma, G) \hookrightarrow \mathrm{Hom}(F_N, G)$ the inclusion such that for

$\rho \in \text{Hom}(\Gamma, G)$, $f(\rho) = (\rho(\gamma_1), \dots, \rho(\gamma_N)) = \phi \in \text{Hom}(F_N, G) = G^N$ and $\phi_\rho = \rho(\Gamma) = \rho(\pi(F_N)) = \phi(F_N)$. By Proposition 5.4, $Z(G) \subseteq G_X$ and for each $\rho \in \text{Hom}(\Gamma, G)$, $G_\rho = Z_G(\rho(\Gamma))$. So it is easy to notice that a representation ρ is good if and only if $\rho(\Gamma)$ is isotropic.

Because we are in the conditions of Proposition 5.1 with $X = \text{Hom}(\Gamma, G)$, we conclude the following.

Proposition 5.10. *Let $\rho \in \text{Hom}(\Gamma, G)$.*

- (i) *Let $\mu \in Y(G)$; Then $\mu \in \Lambda_\rho$ if and only if $\rho(\Gamma) \subseteq P(\mu)$.*
- (ii) *$\rho(\Gamma) \subset \eta(\rho) = H_\rho$.*
- (iii) *The map η is proper.*
- (iv) *H_ρ is completely reducible in G if and only if $\rho(\Gamma)$ is completely reducible in G .*

Proof. As $\phi_\rho = \rho(\Gamma)$, all claims follow from Propositions 5.1 and 5.2. \square

As a consequence of Theorem 5.5, we obtain an equivalence between stability and irreducibility notions.

Proposition 5.11. *Let $X = \text{Hom}(\Gamma, G)$ and $\rho \in X$. We have:*

- (i) *ρ is polystable if and only if ρ is reductive*
- (ii) *If $\eta(\rho)$ is irreducible in G then ρ is irreducible.*
- (iii) *ρ is stable if and only if ρ is irreducible.*
- (iv) *If X is central, then ρ is equicentral if and only if ρ is good.*

Proof. To show (i), (iii) and (iv), we use again the equality $\rho(\Gamma) = \phi_\rho$ and Theorem 5.5. To prove (ii) note that showing irreducibility of ρ is the same proving that $\rho(\Gamma)$ is irreducible in G . We know that $\rho(\Gamma) = \phi_\rho$, so if H_ρ is irreducible, by Proposition 5.3 we get that $\rho(\Gamma) = \phi_\rho$ is irreducible. The last statement follows from (i) and Theorem 5.5(3). \square

Remark 5.12. It is very common for representation varieties to be central. A sufficient condition is that for every $g \in G$, there exists a $\rho \in \text{Hom}(\Gamma, G)$ and an element γ of Γ such that $\rho(\gamma) = g$, as one easily checks.

We therefore obtained the following table that summarizes the situation for a character variety $X = \text{Hom}(\Gamma, G)$. In each line we have equivalences, and at the bottom line, the * indicates that we require that X is central: $G_X = Z(G)$.

Representation ρ	Image subgroup $H = \rho(\Gamma)$	Affine GIT $\rho \in X$
Reductive	Completely reducible (CR)	Polystable
Reductive, $\text{Lie}(G_\rho) = \text{Lie}(Z)$	Irreducible	Stable
Good (reductive, $G_\rho = Z$)*	Isotropic (CR, $Z_G(H) = Z$)*	Equicentral*

5.3. Stability in character varieties and Higgs bundles

In this last subsection, we present a restatement of the well-known correspondence between G -Higgs bundles and representations in terms of the stability notions considered above, restricting to the case $k = \mathbb{C}$. We start with a brief review of the theory of G -Higgs bundles, following mainly the article [9] and references therein.

As mentioned in Sec. 3.1, the affine GIT quotient $X_\Gamma(G) := \text{Hom}(\Gamma, G) // G$ is an affine algebraic set defined over \mathbb{C} called the G -character variety of Γ . It is an elementary fact that the properties of irreducibility and complete reducibility are invariant by conjugation, so they can be defined unambiguously for points of the character variety.

Let Σ be a compact Riemann surface of genus $g \geq 2$ and let $\pi_1(\Sigma)$ be the fundamental group of Σ . The space $\text{Hom}(\pi_1(\Sigma), G)$ is naturally identified with an algebraic subvariety of G^{2g} .

Let us consider the universal central extension Γ of $\pi_1(\Sigma)$ given by the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \Gamma \rightarrow \pi_1(\Sigma) \rightarrow 1$ and Ω_Σ^1 denote the canonical line bundle of Σ .

Recall that a G -Higgs bundle over Σ is a pair (E, φ) , where E is a holomorphic G -bundle over Σ and φ is a holomorphic section of $E(\mathfrak{g}) \otimes \Omega_\Sigma^1$ with $E(\mathfrak{g})$ being the adjoint bundle of E . In particular, a $GL_n(\mathbb{C})$ -Higgs bundle is what is called a Higgs bundle, that it is a pair (V, ϕ) with V a holomorphic rank n vector bundle and $\phi \in H^0(X, \text{End}(V) \otimes \Omega_\Sigma^1)$.

There is a notion of (poly)stability for G -Higgs bundles. When $G = GL_n(\mathbb{C})$ stability can be defined using what is called the slope of a Higgs bundle. The notion was later generalized to the case of G -Higgs bundles for G complex (for the details, see [28] or [9]).

Consider now the moduli space of G -Higgs bundles over X , $\mathcal{M}_X(G)$, that is the set of isomorphism classes of polystable G -Higgs bundles. This moduli space has the structure of a complex algebraic variety, for the details of its construction see Kobayashi ([14]), Ramanathan ([22]), Simpson ([29, 30]) and Schmitt ([24, 25]).

Denote by $X_\Gamma(G)$ the character variety of representations $\rho: \Gamma \rightarrow G$ such that $\rho(J) \in Z(G)^0$, where J is a central element of Γ and by $X_\Gamma^r(G)$ the moduli space of reductive representations in $X_\Gamma(G)$. In the Appendix (Proposition A.3), we recall that a representation ρ is reductive (that is, $\rho(\Gamma)$ is completely reducible in G), if and only if the Zariski closure of $\rho(\Gamma)$ in G is a reductive group.

Now, we are ready to state the non-abelian Hodge theorem. For details see [11, 28, 9] and [6, 7].

Theorem 5.13. *Let G be a connected reductive complex Lie group. There is a homeomorphism $X_\Gamma^r(G) \simeq \mathcal{M}_X(G)$ under which a G -Higgs bundle is stable if and only if it comes from an irreducible representation. Moreover, the G -Higgs bundle is stable and simple if and only if the representation is reductive with $Z_G(\rho) = Z(G)$.*

Note that this result generalizes both the theorem of Narasimhan–Seshadri which states that a vector bundle on X of rank n is stable if and only if it is associated to a irreducible representation of Γ in $U(n)$ and the Theorem of Ramanathan which states that a holomorphic principal G -bundle is stable if and only if it is associated to an irreducible unitary representation of Γ in G , satisfying $\rho(\Gamma) \subset K$, where K is the maximal compact subgroup of G . We observe that in [9] this result is extended also to the case of real reductive groups, which we do not consider here.

Using Theorem 5.11, we can restate the non-abelian Hodge theorem as follows. Note that, by Theorem 5.11 and Proposition A.3 we have the identity $X_{\Gamma}^r(G) = X_{\Gamma}^{ps}(G)$, where $X_{\Gamma}^{ps}(G)$ denotes the moduli space of polystable representations inside $X_{\Gamma}(G)$.

Corollary 5.14. *Let G be a complex connected reductive Lie group. There is a homeomorphism $X_{\Gamma}^{ps}(G) \simeq \mathcal{M}_X(G)$ under which a G -Higgs bundle is stable if and only if it comes from a stable representation. Moreover, the G -Higgs bundle is stable and simple if and only if the representation is good.*

As a concluding remark, we note that the setup in this paper allows the consideration of the moduli space $X_{\Gamma}^r(G)$ of polystable representations of Γ in a complex reductive group G which is not necessarily connected. As far as we know, Theorem 5.13 has not been proved in general for the case of G -Higgs bundles with G not connected (see however [8, 20] for definitions and some non-connected real reductive groups). On the other hand, our results seem to indicate that this form of the non-abelian Hodge theorem should still be valid in this case, for an adequate notion of stability of G -Higgs bundles, defined in terms of reduction of structure group to R -parabolics.

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Appendix A. Completely Reducible, Irreducible and Isotropic Subgroups

Let G be a reductive affine algebraic group (not necessarily irreducible) defined over an algebraically closed field k of characteristic 0. For the convenience of the reader,

in this appendix we state some known properties of completely reducible, irreducible, isotropic and Ad -irreducible subgroups of G and compare to other relevant results in the literature (see [1, 2, 16, 27]).

A.1. *Alternative characterizations of complete reducibility and irreducibility*

We begin by recalling the following characterization of the centralizer of an irreducible subgroup.

Proposition A.1. *The centralizer of an irreducible subgroup of G is a finite extension of $Z(G)$.*

Proof. See [16, Lemma 6.2]. We give an alternative proof analogous to the proof of Theorem 5.5(2). Let H be an irreducible subgroup of G then $Z_G(H)$ is reductive (see [1, Sec. 6.3]). Suppose that $Z_G(H)$ is an infinite extension of $Z(G)$. As $Z_G(H)$ is reductive, this implies that there exists $\lambda \in Y(Z_G(H))$ such that $\lambda(k^\times) \not\subseteq Z(G)$. So $P(\lambda)$ is a proper R -parabolic of G and $H \subset P(\lambda)$. Then H is not irreducible. \square

To see if a completely reducible subgroup is irreducible we have the next criterion. We denote by $\text{Lie}(H)$ the Lie algebra of an algebraic group H .

Proposition A.2. *For a completely reducible subgroup H of G , the following claims are equivalent.*

- (1) H is irreducible in G ;
- (2) $\dim Z_G(H) = \dim Z(G)$;
- (3) $\text{Lie}(Z_G(H)) = \text{Lie } Z(G)$.

Proof. (1) \Rightarrow (2) See A.1.

(2) \Rightarrow (1) [27, Corollary 16].

(2) \Leftrightarrow (3) By Corollary 3.6 of [5] we have $\dim \text{Lie}(Z_G(H)) = \dim Z_G(H)$ and $\dim \text{Lie}(Z(G)) = \dim Z(G)$. So it is easy to see that (3) \Rightarrow (2). For the other implication, if $\dim Z_G(H) = \dim Z(G)$ then $\dim \text{Lie}(Z_G(H)) = \dim \text{Lie}(Z(G))$, as $\text{Lie}(Z(G))$ is a k -vector subspace of $\text{Lie}(Z_H(G))$, they must be equal. \square

The next results provide other ways to check complete reducibility and irreducibility.

Proposition A.3. *Consider a subgroup $H \subset G$. Then H is completely reducible in G if and only if \overline{H} , the Zariski closure of H in G , is a reductive group.*

Proof. See [2, Sec. 2.2]. \square

Proposition A.4. Assume that G is connected. A subgroup $H \subset G$ is irreducible if and only if H does not leave any parabolic subalgebra of $\mathrm{Lie}(G)$ invariant.

Proof. If H is irreducible, suppose that there is a parabolic subalgebra, \mathfrak{p} , of $\mathrm{Lie}(G)$, such that, H leaves it invariant, i.e. $\mathrm{Ad}(H)(\mathfrak{p}) \subset \mathfrak{p}$ (Ad is the adjoint representation of G in $\mathrm{Lie}(G)$). Let P be the parabolic group whose Lie algebra is \mathfrak{p} . So $H \subset N_G(P) = P$ (where $N_G(P)$ is the normalizer of P in G and it is equal to P because this one is parabolic). This contradicts the assumption of H being irreducible.

If H is not irreducible then there is a parabolic subgroup P such that $H \subset P$. Let \mathfrak{p} be the Lie algebra of P which is a parabolic subalgebra of $\mathrm{Lie}(G)$. Via the adjoint representation, P acts on \mathfrak{p} . Now we just have to prove that $\mathrm{Ad}(H)(\mathfrak{p}) \subset \mathfrak{p}$. We can suppose that $G \subset \mathrm{GL}(n)$ for some $n \in \mathbb{N}$, in this case $\mathrm{Ad}(g)(Y) = gYg^{-1}$ for every $g \in G$ and $Y \in \mathrm{Lie}(G)$. So if we restrict to $g \in H$ and $Y \in \mathfrak{p}$, as $H \subset P$, $gYg^{-1} \in \mathfrak{p}$. Thus H leaves invariant a parabolic subalgebra of $\mathrm{Lie}(G)$. \square

A.2. Irreducible, Ad-irreducible and isotropic subgroups

Definition A.5. A completely reducible subgroup $H \subset G$ is called *isotropic* in G if $Z_G(H) = Z(G)$.

Remark A.6. For $\rho \in \mathrm{Hom}(\Gamma, G)$ is trivial to see that ρ is good in $\mathrm{Hom}(\Gamma, G)$ if and only if $\rho(\Gamma)$ is isotropic in G .

By Proposition A.2, we can see that if H is isotropic, then it is irreducible. But the converse is not true, as the following example shows.

Example A.7. Let $G = \mathrm{PGL}_2(\mathbb{C})$ and H be the subgroup of $\mathrm{PGL}_2(\mathbb{C})$ generated by

$$\gamma_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \gamma_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The subgroup H is reductive because it is finite and by Proposition A.3 it is completely reducible. We can prove that $H = Z_{\mathrm{PGL}_2(\mathbb{C})}(H) \neq Z(\mathrm{PGL}_2(\mathbb{C})) = 1$, to see this it is enough to check that all $h \in H$ centralizes γ_1 and γ_2 . Thus, H is not isotropic, but it is irreducible because $\dim Z_{\mathrm{PGL}_2(\mathbb{C})}(H) = \dim Z(\mathrm{PGL}_2(\mathbb{C})) = 0$ by Proposition A.2.

Remark A.8. In [10] for the case $G = \mathrm{PSL}_2(\mathbb{C})$ and in [27] for G an algebraic group over \mathbb{C} , it is also introduced the definition of Ad-irreducible subgroup of G which is a subgroup whose image by the adjoint representation is irreducible in $GL(\mathrm{Lie}(G))$. By [27, Proposition 13] every Ad-irreducible subgroup of G is isotropic.

In the remainder of the appendix, we will see that the notion of irreducible subgroup is unaffected when we restrict the group to its semisimple quotient. The

same will hold for the notion of completely reducible subgroup. On the contrary, this will not hold for the notion of an isotropic subgroup.

Proposition A.9. *Let $\phi: G_1 \rightarrow G_2$ be a homomorphism of reductive algebraic groups.*

- (i) *If H is a completely reducible subgroup of G_1 then $\phi(H)$ is a completely reducible subgroup of G_2 .*
- (ii) *Suppose that ϕ is an epimorphism, then if H is an irreducible subgroup of G_1 then $\phi(H)$ is an irreducible subgroup of G_2 .*

Proof. See [1, 6.2]. □

As G is reductive, we know that $G/Z(G)$ is a semisimple group. Let $\phi: G \rightarrow G/Z(G)$ be the canonical projection. We have the following results.

Proposition A.10. *Let $H \subset G$ be a subgroup of G . Then*

- (i) *H is completely reducible if and only if $\phi(H)$ is completely reducible.*
- (ii) *H is irreducible if and only if $\phi(H)$ is irreducible.*

Proof. For the proof of (i) and the implication (ii) \Rightarrow , see [1, 6.2].

To show (ii) \Leftarrow , suppose $\phi(H)$ is irreducible. If $H \subset P$ for P a proper R-parabolic subgroup of G , by [1, Lemma 6.15] $\phi(P)$ is a R-parabolic subgroup of $G/Z(G)$.

If $\phi(P) = G/Z(G) = \phi(G)$, as $Z(G) \subset P$ we have that $P/Z(G) = G/Z(G)$ and then $P = G$. So $\phi(P)$ is proper. And $\phi(H) \subset \phi(P)$ which is impossible because $\phi(H)$ is irreducible. □

By Proposition A.1, if a subgroup H of G is irreducible then the centralizer of H in G , $Z_G(H)$, is a finite extension of the center of G , $Z(G)$. Using Proposition A.10(ii), we see that $Z_{G/Z(G)}(\phi(H))$ is also a finite extension of the center of $G/Z(G)$. The same does not happen for the property $Z_G(H) = Z(G)$, as we see next.

Example A.11. Let $G = GL_2(\mathbb{C})$ and H be the subgroup of $GL_2(\mathbb{C})$ generated by the matrices γ_1 and γ_2 of Example A.7. The subgroup H is reductive because it is finite and by Proposition A.3 it is completely reducible. We can prove that $Z_{GL_2(\mathbb{C})}(H) = Z(GL_2(\mathbb{C})) = \mathbb{C}^*$, so that H is isotropic. However, consider now the image $\phi(H)$ of H under the quotient $\phi: G \rightarrow PGL_2(\mathbb{C}) = G/Z(G)$. Using Example A.7 we know that $\phi(H)$ is not isotropic because $Z_{PGL_2(\mathbb{C})}(\phi(H)) \neq Z(PGL_2(\mathbb{C})) = 1$.

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