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Representations of Simple Lie Groups with Regular Rings of Invariants

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§ 0. Introduction

In the theory of compact transformation groups, an important class of actions have been those which locally look like the "simplest" representations of the classical groups. These simplest representations all share the property that their rings of invariants are *regular*, i.e. are isomorphic to a polynomial ring in some number of indeterminants. (We will call such representations *coregular*.) One is naturally led to the question: Which representations of the compact Lie groups are coregular? More generally, which representations of the reductive complex algebraic groups are coregular? Wu-Yi Hsiang has answered the first question. In this paper we determine the coregular representations of the connected simple complex algebraic groups. This classification has also been carried out by O.M. Adamovič and E.O. Golovina, and their work is to appear in a collection of the University of Yaroslav.

In §1 we discuss necessary conditions for coregularity. One such condition is that every irreducible factor of a coregular representation must be coregular. Thus it is important to know the list of irreducible coregular representations of the connected simple complex algebraic groups, and this list was determined by Kac, Popov, and Vinberg [12]. Not surprisingly then, the difficult part in our classification is the determination of the rings of invariants of those representations satisfying our necessary conditions. For most representations of the groups $\mathsf{A}_n = SL_{n+1} = SL(n+1, \mathbf{C})$, classical invariant theory is sufficient for our calculations. These cases are discussed in §2. For spin representations and representations of the exceptional groups, there is no analogue of classical invariant theory, but the techniques developed by the author in [29] suffice. These matters are discussed in §3. In §4 we consider several miscellaneous topics related to coregularity. In particular, we discuss the question of when the ring of all polynomial functions on a representation space is a free module over the ring of invariants. (We call such a representation *cofree*. In a forthcoming paper we classify the cofree

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representations of the connected simple complex algebraic groups.) In an appendix we comment upon the techniques and results of [12].

§ 1. Necessary Conditions and Main Results

G will always denote a reductive complex linear algebraic group. Recall that a *representation* of G is a complex vector space V (the *representation space*) together with a morphism of algebraic groups $\varphi: G \to GL(V)$. We shall denote a representation as above by φ or by the pair (V, G). V will always denote a representation space of G. The algebra of invariants $\mathbf{C}[V]^G$ is always noetherian [21]. We say that a coregular representation (V, G) is *maximally coregular* provided that $V^G = \{0\}$ and that $(V \oplus V_1, G)$ is not coregular for any non-trivial representation space V_1 of G. It is "obvious" that every coregular representation (V, G) with $V^G = \{0\}$ is a subrepresentation of a maximally coregular representation. For G simple and connected this fact follows from:

Classification I. *Let G be connected, simple, and simply connected. Suppose that $V^G = \{0\}$ and that (V, G) is coregular. Then, up to an outer automorphism of G, (V, G) is a subrepresentation of one of the representations listed in Tables 1a, 2a, 3a, 4a, and 5a, and the representations in these tables are maximally coregular.*

We now recall some standard facts about representations of G; proofs can be found in [18] or [29]. Let $x \in V$. The conjugacy class (G_x) of the isotropy group G_x is called an *isotropy class* of $\varphi = (V, G)$. We say that the isotropy class (or isotropy group) is *closed* if the orbit Gx is closed. In this case G_x is reductive and we call the representation φ_x of G_x on $T_x V / T_x(Gx)$ the *slice representation* at x (or the slice representation of G_x). The slice representation φ_x and isotropy class (G_x) are said to be *proper* if $G_x \neq G$. G_x determines φ_x since

$$\varphi_x \oplus (\mathrm{Ad}\, G)|_{G_x} = \varphi|_{G_x} \oplus \mathrm{Ad}\, G_x.$$

The (finitely many) closed isotropy classes of φ are partially ordered, where $(L) \leqq (M)$ if L is conjugate to a subgroup of M. If $(L) \leqq (M)$, then an element of (L) generates a closed isotropy class of the slice representation ψ of M, and every closed isotropy class of ψ arises from some $(L) \leqq (M)$.

There is a unique minimal closed isotropy class (H) called the *principal isotropy class*. H is called a *principal isotropy group* and closed orbits Gx with $G_x \in (H)$ are said to be *principal orbits*. The slice representation of H is trivial if and only if φ has *generically closed orbits*, i.e. if and only if the set of closed G-orbits contains a non-empty Zariski open subset of V. In this case, $\dim \mathbf{C}[V]^G = \dim V - \dim G + \dim H$. If H is finite or if G is semisimple and the points with reductive isotropy group contain a non-empty Zariski open subset of V, then φ has generically closed orbits [20, 25].

We denote by V/G the complex variety associated to $\mathbf{C}[V]^G$. The canonical map $V \to V/G$ sets up a bijection between V/G and the closed orbits in V. Thus $\mathbf{C}[V]^G$ separates closed orbits, and $\mathbf{C}[V]^G = \mathbf{C}$ if and only if $\{0\}$ is the only closed orbit in V.

Suppose now that G is finite. A theorem of Chevalley [4] says that φ is coregular (and cofree) if (and only if ([2], Ch. V, § 5, Thm. 4)) the image of G in $GL(V)$ is generated by generalized reflections (elements fixing a hyperplane of V).

Lemma 1.1 ([12]). *Suppose that V is the direct sum of G-representation spaces V_1 and V_2.*

(1) *If (V, G) is coregular, then (V_1, G) and (V_2, G) are coregular.*

(2) *If (V, G) is coregular, then every slice representation of (V, G) is coregular.*

(3) *Suppose that (V_1, G) has finite principal isotropy groups (H) and that the image of H in $GL(V_2)$ is a non-trivial subgroup of $SL(V_2)$. Then (V, G) is not coregular.*

Proof. Clearly $\mathbf{C}[V]^G$ has minimal generating sets consisting of elements homogeneous in both V_1 and V_2. Each such set contains generators of $\mathbf{C}[V_1]^G$ and $\mathbf{C}[V_2]^G$, hence (1) holds. Part (2) follows from Luna's slice theorem [18], and (3) follows from (2) and Chevalley's theorem. □

A *minimally non-coregular* representation is a non-coregular representation all of whose proper slice representations and proper subrepresentations are coregular. There are minimally non-coregular representations of the connected simple complex algebraic groups, and by [12] such representations are necessarily reducible.

Classification II. *Let G be connected, simple, and simply connected. Suppose that (V, G) is a minimally non-coregular representation of G. Then G is of type* A *and (V, G) or its dual is listed in Table 1a'.*

After some preliminaries on notation, we recall the results of [12]: $\mathbf{Z}_n$ will denote $\mathbf{Z}/n\mathbf{Z}$, $n\in\mathbf{Z}^+$. The one-dimensional representation of $\mathbf{C}^*$ of weight n is denoted v_n.

Let φ and ψ be representations of G. Then $\varphi+\psi$ denotes the direct sum of φ and ψ, $k\varphi$ denotes the direct sum of k copies of φ, and φ^* denotes the representation dual to φ. (V^*, G) denotes the dual of (V, G). If G is connected and φ and ψ are irreducible, then $\varphi\psi$ will denote the irreducible component of heighest weight in $\varphi\otimes\psi$. (All tensor products are over $\mathbf{C}$.) If more than one group is being considered, we will use the notation $\varphi(G)$ or (φ, G) to emphasize that φ is a representation of G. The tensor product of representations (φ, G) and (φ', G') is denoted $(\varphi\otimes\varphi', G\times G')$, and $\varphi+\varphi'$ is shorthand for $\varphi\otimes\theta_1(G')+\theta_1(G)\otimes\varphi'$. We will confuse φ with its corresponding representation space, so $\mathbf{C}[\varphi]^G$ denotes the ring of invariants of φ. Any trivial representation of G is denoted θ; θ_n denotes the trivial representation of dimension n. If H is a subgroup of G, then $(\varphi(G), H)$ denotes the restriction of φ to H.

Let G be simple, connected, and simply connected; rank $G=r$. Let $\varphi_1, \ldots, \varphi_r$ denote the basic representations of G (we use the ordering of [7]). Then $\varphi_1^2\varphi_2$, $\varphi_1\varphi_2\varphi_3$, etc. will denote the irreducible component of highest weight in $S^2(\varphi_1)\otimes\varphi_2$, $\varphi_1\otimes\varphi_2\otimes\varphi_3$, etc. Both φ_0 and $\varphi_1^0\varphi_2^0\ldots\varphi_r^0$ will be interpreted as θ_1.

Lemma 1.2 ([12]). *The following and their duals are all the irreducible non-trivial coregular representations of the simple, connected, simply connected complex algebraic groups:*

(1) *Infinite principal isotropy groups: adjoint representations;* $\varphi_1(\mathsf{A}_n)$, $n \geqq 1$; $\varphi_1(\mathsf{B}_n)$, $n \geqq 2$; $\varphi_1(\mathsf{C}_n)$, $n \geqq 2$; $\varphi_1(\mathsf{D}_n)$, $n \geqq 3$; $\varphi_1(\mathsf{G}_2)$; $\varphi_1(\mathsf{F}_4)$; $\varphi_1(\mathsf{E}_6)$; $\varphi_1(\mathsf{E}_7)$; $\varphi_1^2(\mathsf{A}_n)$, $n \geqq 2$; $\varphi_2(\mathsf{A}_n)$, $n \geqq 4$; $\varphi_2(\mathsf{C}_n)$, $n \geqq 3$; $\varphi_3(\mathsf{C}_3)$; $\varphi_3(\mathsf{A}_n)$, $n=5,6,7$; $\varphi_n(\mathsf{B}_n)$, $n=3,4,5,6$; $\varphi_{n-1}(\mathsf{D}_n)$ *and* $\varphi_n(\mathsf{D}_n)$, $n=4,5,6,7$.

(2) *Finite principal isotropy groups:* $\varphi_1^2(\mathsf{B}_n)$, $n \geqq 2$; $\varphi_1^2(\mathsf{D}_n)$, $n \geqq 3$; $\varphi_1^3(\mathsf{A}_1)$; $\varphi_1^4(\mathsf{A}_1)$; $\varphi_1^3(\mathsf{A}_2)$; $\varphi_4(\mathsf{C}_4)$; $\varphi_4(\mathsf{A}_7)$; $\varphi_3(\mathsf{A}_8)$; $\varphi_7(\mathsf{D}_8)$ *and* $\varphi_8(\mathsf{D}_8)$. □

The principal isotropy groups of the representations in 1.2.2 are finite and non-central [23], so by 1.1.3 these representations are maximally coregular. The remaining candidates for coregular representations were found by applying Lemma 1.1 to sums of representations listed in 1.2.1. The main difficulty here was finding slice representations. Establishing the non-coregularity of a slice representation (if it held) was always an easy application of Lemma 1.1 or classical invariant theory.

The easiest way to obtain slice representations (equivalently, closed isotropy classes) is to take the principal isotropy class (H) of a subrepresentation. For adjoint representations H is a maximal torus. For the non-adjoint representations in 1.2.1 which have non-constant rings of invariants, H is a connected group whose Lie algebra is given in [7]. (Regarding the connectivity of H, see Remark 4.10.) Principal isotropy classes are also listed in [10]. We note here that ([24] Table 2) purports to list the sums of representations in 1.2.1 which have finite non-central principal isotropy groups, but the table is missing (perhaps only) the representation $(\varphi_3+\varphi_1^2, \mathsf{A}_5)$.

The following result is useful for finding closed isotropy classes:

Lemma 1.3 ([19]). *Let L be a reductive algebraic subgroup of G. Suppose that $v \in V^L$. Then Gv is closed if and only if $N_G(L)v$ is closed. In particular,* $\mathbf{C}[V^L]^{N_G(L)} \neq \mathbf{C}$ *if and only if* $\mathrm{res}_L\, \mathbf{C}[V]^G \neq \mathbf{C}$. □

Here res_L denotes the restriction map from $\mathbf{C}[V]^G$ to $\mathbf{C}[V^L]^{N_G(L)}$.

Example 1.4. Let $(V, G)=(3\varphi_2, \mathsf{A}_4)$, and let $L=\mathsf{A}_1$ where $(\varphi_1(\mathsf{A}_4), \mathsf{A}_1)=\varphi_1+\theta_3$. Then $(V^L, N_G(L)/L) \simeq (3\varphi_1 \otimes \nu_2 + 3\nu_{-3}, (\mathsf{A}_2 \times \mathbf{C}^*)/\mathbf{Z}_3)$. This last representation has trivial principal isotropy groups, so there is a $v \in V^L$ such that Gv is closed, $G_v \cap N_G(L)=L$. Since $N_G(L)$ contains all maximal tori of G which contain the maximal torus of L, $G_v \cap N_G(L) \neq L$ if rank $G_v>1$. Thus rank $G_v=1$, and it follows easily that $G_v=L=\mathsf{A}_1$. The slice representation of A_1 is $3\varphi_1+\theta_3$. If one considers the representations $(3\varphi_2, \mathsf{A}_{2k})$, $k \geqq 2$, then one similarly shows that (A_1) is a closed isotropy class with associated slice representation $(2k-1)\varphi_1+\theta$. But $(l\varphi_1, \mathsf{A}_1)$ is only coregular for $l \leqq 3$, so $(3\varphi_2, \mathsf{A}_{2k})$ is not coregular for $k \geqq 3$.

To arrive at the list of representations in Tables 1 through 5 we had to employ some slice representations which are (perhaps) not completely obvious. We list them in (1.5) through (1.9). In each case the arrow indicates that the right hand side is a slice representation of the left hand side. One can verify the slice representations more or less as in Example 1.4.

(1.5) $(\varphi_2+\varphi_1, \mathsf{A}_{2k}) \to (\theta_1, \mathsf{C}_k)$; $k \geqq 1$, $(\varphi_1(\mathsf{A}_{2k}), \mathsf{C}_k)=\varphi_1+\theta_1$.

(1.6) $(\varphi_2+2\varphi_1^*, \mathsf{A}_n) \to (\varphi_2+\theta_1, \mathsf{A}_{n-2})$; $n \geqq 4$, $(\varphi_1(\mathsf{A}_n), \mathsf{A}_{n-2})=\varphi_1+\theta_2$.

(1.7) $(\varphi_2+\varphi_2^*+a\,\varphi_1+b\,\varphi_1^*,\ \mathsf{A}_n)\to(\varphi_2+\varphi_2^*+a\,\varphi_1+b\,\varphi_1^*+\theta,\ \mathsf{A}_{n-2})$; $n\geqq 4$, $a+b\geqq 2$, $(\varphi_1(\mathsf{A}_n),\ \mathsf{A}_{n-2})=\varphi_1+\theta_2$.

(1.8) $(\varphi_1^2+\varphi_2^*+a\,\varphi_1+b\,\varphi_1^*,\ \mathsf{A}_n)\to(\varphi_1^2+\varphi_2^*+a\,\varphi_1+b\,\varphi_1^*+\theta,\ \mathsf{A}_{n-2})$; $n\geqq 4$, $a+b\geqq 1$, $(\varphi_1(\mathsf{A}_n),\ \mathsf{A}_{n-2})=\varphi_1+\theta_2$.

(1.9) $(\varphi_3+\varphi_1,\ \mathsf{A}_6)\to(\varphi_1\otimes\varphi_1'+\theta_1,\ \mathsf{A}_2\times\mathsf{A}_2')$; $(\varphi_1(\mathsf{A}_6),\ \mathsf{A}_2\times\mathsf{A}_2')=\varphi_1+\varphi_1'+\theta_1$.

§ 2. Calculating Invariants Using Classical Invariant Theory

In this section we indicate how one can compute invariants using classical invariant theory (abbreviated CIT henceforth). This method allowed us to establish the coregularity of "most" of the representations of SL_n satisfying our necessary conditions. We also derive information about the decomposition of symmetric powers of certain representations of SL_n.

CIT for SL_n (see [32]) says that the invariants of copies of $\mathbf{C}^n$ (typical points denoted $x_1, x_2, \ldots$) and copies of $(\mathbf{C}^n)^*$ (typical points denoted $\xi_1, \xi_2, \ldots$) are generated by determinants $[x_{i_1}, \ldots, x_{i_n}]$ and $[\xi_{i_1}, \ldots, \xi_{i_n}]$ and by contractions $\langle x_i, \xi_j\rangle$. The relations of these invariants have generators of the following form:

$$\begin{aligned}(2.1)\quad &[x_1,\ldots,x_n]\,[x_0,x_{n+1},\ldots,x_{2n-1}]\\ &=[x_0,x_2,\ldots,x_n]\,[x_1,x_{n+1},\ldots,x_{2n-1}]+\cdots\\ &\quad+[x_1,x_2,\ldots,x_{n-1},x_0]\,[x_n,x_{n+1},\ldots,x_{2n-1}].\end{aligned}$$

$$\begin{aligned}(2.2)\quad &[x_1,\ldots,x_n]\,\langle x_0,\xi_0\rangle\\ &=[x_0,x_2,\ldots,x_n]\,\langle x_1,\xi_0\rangle+\cdots+[x_1,\ldots,x_{n-1},x_0]\,\langle x_n,\xi_0\rangle.\end{aligned}$$

$$(2.3)\quad [x_1,\ldots,x_n]\,[\xi_1,\ldots,\xi_n]=\det(\langle x_i,\xi_j\rangle).$$

One also has relations (2.1') and (2.2') where the roles of the x_i and ξ_j are interchanged.

As an example of a typical computation we prove:

Proposition 2.4. *For the groups A_n, $n\geqq 2$, one has:*

(1) $\varphi_1^2+\varphi_2^*+\varphi_1^*$ *is coregular.*

(2) $S^j(\varphi_1^2)=\bigoplus\{\varphi_1^{2\,a_1}\ldots\varphi_n^{2\,a_n}\colon a_1+2a_2+\cdots+n\,a_n+m(n+1)=j$ *for some* $m\in\mathbf{Z}^+\}$.

(3) $S^j\varphi_2=\bigoplus\{\varphi_2^{a_1}\ldots\varphi_{2k}^{a_k}\colon a_1+2a_2+\cdots+k\,a_k=j\}$; $n=2k$.

(4) $S^j\varphi_2=\bigoplus\{\varphi_2^{a_1}\ldots\varphi_{2k}^{a_k}\colon a_1+2a_2+\cdots+k\,a_k+m(k+1)=j$ *for some* $m\in\mathbf{Z}^+\}$; $n=2k+1$.

Proof. Let f be a generator of the invariants of $(V,G)=(\varphi_1^2+\varphi_2^*+\varphi_1^*,\ \mathsf{A}_n)$. We may assume that f is homogeneous in each of the three irreducible factors of

(V, G), and polarizing f we obtain a multilinear invariant $\tilde{f}(s_1, s_2, \ldots, \omega_1, \omega_2, \ldots, \eta_1, \eta_2, \ldots)$ where the s_i, ω_j, and η_k are in copies of φ_1^2, φ_2^*, and φ_1^*, respectively. Now $\tilde{f}$ is determined by its action on decomposable tensors, so we replace each s_i by $x_{i1} \otimes x_{i2} + x_{i2} \otimes x_{i1}$ and each ω_j by $\xi_{j1} \otimes \xi_{j2} - \xi_{j2} \otimes \xi_{j1}$ where the x_{il} and ξ_{jm} lie in new copies of φ_1 and φ_1^*. We thus obtain a multilinear invariant $\bar{f}(x_{il}, \xi_{jm}, \eta_k)$ which is symmetric in i, j, and k; symmetric in l for fixed i; and skew in m for fixed j. By CIT, we may express $\bar{f}$ as a sum of products of determinants and contractions of the x_{il}, ξ_{jm}, and η_k. Let P denote one of these products.

Case 1. P is of the form $[x_{01}, \ldots, x_{n1}] P_0$: Suppose that P_0 has the form $[x_{02}, \ldots, x_{r2}, \alpha_{r+1}, \ldots, \alpha_n][x_{(r+1)2}, \ldots] P_1$ or $[x_{02}, \ldots, x_{r2}, \alpha_{r+1}, \ldots, \alpha_n] \langle x_{(r+1)2}, \cdot \rangle P_2$ where no α_s is an x_{i2} for $i \leqq n$, $r+1 \leqq s \leqq n$. In the former case the symmetries of the x_{il} force $\bar{f}$ to contain the term

$$P - P_1[x_{01}, \ldots, x_{n1}]([x_{(r+1)2}, x_{12}, \ldots, x_{r2}, \alpha_{r+1}, \ldots, \alpha_n][x_{02}, \ldots] + \cdots$$
$$+ [x_{02}, \ldots, x_{(r-1)2}, x_{(r+1)2}, \alpha_{r+1}, \ldots, \alpha_n][x_{r2}, \ldots]),$$

and using (2.1) we see that this term is a sum of terms of the form

$$[x_{01}, \ldots, x_{n1}][x_{02}, \ldots, x_{(r+1)2}, \beta_{r+2}, \ldots, \beta_n] P_3.$$

The case $P_0 = [\ldots] \langle x_{(r+1)2}, \cdot \rangle P_2$ leads to terms of the same form; one uses (2.2) rather than (2.1). Continuing inductively we may reduce to the case where $P = [x_{01}, \ldots, x_{n1}][x_{02}, \ldots, x_{n2}] P_4$. But $[x_{01}, \ldots, x_{n1}][x_{02}, \ldots, x_{n2}]$ plus terms guaranteed by symmetry is the polarization of the degree $n+1$ generator of $\mathbf{C}[\varphi_1^2]^{A_n}$, so P plus all terms demanded by symmetry corresponds to an invariant divisible by this generator.

There remains the case $P = [x_{01}, \ldots, x_{n1}] \langle x_{02}, \alpha_0 \rangle \ldots \langle x_{n2}, \alpha_n \rangle P_1$ where the α_s are ξ's or η's, $0 \leqq s \leqq n$. But the x_{i1} are skew in i, hence so are the x_{i2}, and from (2.3) we see that P plus other terms of $\bar{f}$ equals

$$[x_{01}, \ldots, x_{n1}][x_{02}, \ldots, x_{n2}][\alpha_0, \ldots, \alpha_n] P_1.$$

Thus we find no new generators.

Case 2. P contains a determinant of the ξ_{jm} and η_k: Using (2.1′) and (2.2′) as in ([29] § 15) we may reduce to the case that P is of the form $[\xi_{11}, \xi_{12}, \ldots, \xi_{k1}, \xi_{k2}] P_0$ $(n = 2k-1)$, or of the form $[\xi_{11}, \xi_{12}, \ldots, \xi_{k1}, \xi_{k2}, \eta_1] P_0$ $(n = 2k)$. Thus we find one new generator.

Case 3. P is a product of contractions: If no ξ_{jm} appears in P, then the only type of invariant one sees is of the form $\langle x_{11}, \eta_1 \rangle \langle x_{12}, \eta_2 \rangle + \langle x_{12}, \eta_1 \rangle \langle x_{11}, \eta_2 \rangle$. If some ξ_{jm} appears in P, then the symmetries of the x_{il}, ξ_{jm}, and η_k show that P must contain a factor like

$$(2.5) \quad \langle x_{11}, \xi_{11} \rangle \langle x_{21}, \xi_{12} \rangle \langle x_{31}, \xi_{21} \rangle \langle x_{41}, \xi_{22} \rangle \cdots \langle x_{(2r)1}, \xi_{r2} \rangle$$
$$\cdot \langle x_{22}, \xi_{(r+1)1} \rangle \langle x_{32}, \xi_{(r+1)2} \rangle \langle x_{42}, \xi_{(r+2)1} \rangle \cdots$$
$$\langle x_{(2r)2}, \xi_{(2r)1} \rangle \langle x_{12}, \xi_{(2r)2} \rangle$$

or

$$(2.6)\quad \langle x_{11}, \xi_{11}\rangle \langle x_{21}, \xi_{12}\rangle \langle x_{31}, \xi_{21}\rangle \dots \langle x_{(2r)1}, \xi_{r2}\rangle \langle x_{(2r+1)1}, \eta_1\rangle$$
$$\cdot \langle x_{22}, \xi_{(r+1)1}\rangle \langle x_{32}, \xi_{(r+1)2}\rangle \langle x_{42}, \xi_{(r+2)1}\rangle \dots$$
$$\langle x_{(2r+1)2}, \xi_{(2r)2}\rangle \langle x_{12}, \eta_2\rangle.$$

If $2r=n+1$ in (2.5), then (2.3) shows that we may re-express (2.5) plus other terms of $\bar{f}$ as a product of determinants, and if $2r>n+1$ then the result is zero. Thus (2.5) can only correspond to generators of $\mathbf{C}[V]^G$ for $2r<n+1$. Similarly we may restrict to $2r+1<n+1$ in (2.6).

We have shown that $\mathbf{C}[V]^G$ has at most $3+[n/2]+[(n-1)/2]=n+2$ generators, where $\dim \mathbf{C}[V]^G=n+2$. Thus (V, G) is coregular, as claimed in (1).

We now prove (2). Let $(V, G)=(\varphi_1^2+s\,\varphi_1, \mathsf{A}_n)$, s arbitrarily large. Then $\mathbf{C}[V]^G$ is generated over $\mathbf{C}[s\,\varphi_1]^G$ by contractions $\theta_1 \subset \varphi^* \otimes \varphi$ where $\varphi^* \subset \mathbf{C}[\varphi_1^2]$ and $\varphi \subset \mathbf{C}[s\,\varphi_1]$ are irreducible representations of G. We use CIT to determine $\mathbf{C}[V]^G$, and then we are able to determine the G-module structure of $\mathbf{C}[\varphi_1^2]$ and its dual $S^*(\varphi_1^2)$.

As before, to compute generators of $\mathbf{C}[V]^G$ we reduce to considering multilinear invariants $\bar{f}(x_{ik}, y_l)$ where $\bar{f}$ is symmetric in i and symmetric in k for fixed i, $k=1,2$. Here the x_{ik} and y_l lie in copies of φ_1. Now $\bar{f}$ is a sum $P_1+\dots+P_t$ where each P_i is a product of determinants. Using (2.1) (see [8] p. 328) one can reduce to the case that each P_i is a product of factors like

$$(2.7)_r\quad [x_{11}, \dots, x_{r1}, y_1, \dots, y_{n-r+1}]\,[x_{12}, \dots, x_{r2}, y_{n-r+2}, \dots, y_{2n-2r+2}]$$

where $0 \leqq r \leqq n+1$. Let P denote one of the P_i. We may assume that $\bar{f}$ equals P plus all terms guaranteed by symmetry, and we assume that P contains no factor which is a determinant of the y_l. Let a_r denote the number of factors of type $(2.7)_r$ occurring in P, $1 \leqq r \leqq n+1$. Then clearly the representation φ corresponding to $\bar{f}$ is determined by the sequence $(a_1, \dots, a_{n+1})$, and $\varphi \subset S^j(\varphi_1^2)$ where $j=a_1+2a_2+\dots+(n+1)a_{n+1}$. Letting the x_{ik} be weight vectors for the action of a maximal torus of G one can see that $\varphi=\varphi_1^{a_1}\dots\varphi_n^{a_n}$. We have proved (2), and similar arguments prove (3). $\square$

§ 3. Other Methods

We present three other techniques for calculating rings of invariants.

In case the representation (V, G) has non-trivial principal isotropy class, one can apply the following result of Luna and Richardson:

Proposition 3.1. *Let H be a principal isotropy group of (V, G), and let N denote $N_G(H)/H$.*

(1) $\mathrm{res}_H\colon \mathbf{C}[V]^G \to \mathbf{C}[V^H]^N$ *is an isomorphism* [19].

(2) *Let (L) be a closed isotropy class of (V, G) where we may assume that $H \subset L$.*

Then $(M=N_L(H)/H)$ *is a closed isotropy class of* (V^H, N), *and the correspondence* $(L)\mapsto(M)$ *is a bijection* ([29] § 11). □

Remark 3.2. Let H be as above, and let W be a representation space of G. It is not in general true that the inclusions $V^H \to V$ and $W^H \to W$ induce an isomorphism of $(\mathbf{C}[V]\otimes W)^G$ with $(\mathbf{C}[V^H]\otimes W^H)^N$. Sufficient conditions are given in ([29] § 11).

Example 3.3. Let $(V, G)=(3\varphi_2, \mathsf{A}_5)$. The principal isotropy class is $(H=\mathbf{C}^*)$ where $(\varphi_1(\mathsf{A}_5), \mathbf{C}^*)=3\nu_1+3\nu_{-1}$. The universal cover N of $N_G(H)/H$ is the semi-direct product $\mathsf{A}_2\times\mathsf{A}_2'\times_\pi\mathbf{Z}_2$, where $\mathbf{Z}_2$ interchanges the two factors of type A. By 3.1, $\mathbf{C}[V]^G\simeq\mathbf{C}[V^H]^N$ where $(V^H, N)=(3\varphi_1\otimes\varphi_1', \mathsf{A}_2\times\mathsf{A}_2'\times_\pi\mathbf{Z}_2)$. If $(V, G)=(3\varphi_1, \mathsf{E}_6)$, then A_2 is a principal isotropy group where $(\varphi_1(\mathsf{E}_6), \mathsf{A}_2)=3\varphi_1+3\varphi_2+\theta_9$. One obtains the same representation (V^H, N) as above.

Remarks 3.4. There is an even closer relationship between the representations $(3\varphi_1, \mathsf{E}_6)$ and $(3\varphi_2, \mathsf{A}_5)$. From the tables of [5] one sees that there is an inclusion $(\mathsf{A}_5\times\mathsf{A}_1')/\mathbf{Z}_2\subset\mathsf{E}_6$ such that $(\varphi_1(\mathsf{E}_6), \mathsf{A}_5\times\mathsf{A}_1')=\varphi_4+\varphi_1\otimes\varphi_1'$. Then using 3.1 one sees that the inclusion $\varphi_1(\mathsf{E}_6)^{\mathsf{A}_1'}\subset\varphi_1(\mathsf{E}_6)$ induces isomorphisms of the invariants of $(a\varphi_4+b\varphi_2, \mathsf{A}_5)$ and $(a\varphi_1+b\varphi_5, \mathsf{E}_6)$, $a+b=3$.

Similarly there is an inclusion $(\mathsf{C}_3\times\mathsf{A}_1')/\mathbf{Z}_2\subset\mathsf{F}_4$ such that $(\varphi_1(\mathsf{F}_4), \mathsf{C}_3\times\mathsf{A}_1')=\varphi_2+\varphi_1\otimes\varphi_1'$, and this leads to an isomorphism of $\mathbf{C}[2\varphi_1]^{\mathsf{F}_4}$ with $\mathbf{C}[2\varphi_2]^{\mathsf{C}_3}$. Finally, we mention the inclusions of $\mathsf{C}_3\times\mathsf{G}_2'$, $(\mathsf{A}_5\times\mathsf{A}_2')/\mathbf{Z}_3$, and $(\mathsf{D}_6\times\mathsf{A}_1')/\mathbf{Z}_2$ in E_7 where $\varphi_1(\mathsf{E}_7)$ restricts to $\varphi_3+\varphi_1\otimes\varphi_1'$, $\varphi_3+\varphi_1\otimes\varphi_1'+\varphi_5\otimes\varphi_2'$, and $\varphi_6+\varphi_1\otimes\varphi_1'$, respectively. One obtains isomorphisms of the invariants of $(2\varphi_3, \mathsf{C}_3)$, $(2\varphi_3, \mathsf{A}_5)$, $(2\varphi_6, \mathsf{D}_6)$, and $(2\varphi_1, \mathsf{E}_7)$. Except for $(2\varphi_6, \mathsf{D}_6)$, all these representations are maximally coregular.

In [29] Ch. IV one can find many examples of our next method as well as proofs of our claims below: Let (L) be a closed isotropy class of (V, G). We say that (L) is *subprincipal* if it is minimal among non-principal closed isotropy classes. We say that (L) is 1-*subprincipal* if $\operatorname{Im}(V^L\to V/G)$ has codimension one. Equivalently, $\dim V^L/N_G(L)=\dim V/G-1$ or the slice representation of L is of the form $(W+\theta, L)$ where $W^L=\{0\}$ and $\dim W/L=1$. If G is connected and semisimple and (L) is 1-subprincipal, then the ideal $I(V, L)$ in $\mathbf{C}[V]^G$ vanishing on V^L is principal and prime.

We leave the proof of the following result to the reader:

Lemma 3.5. *Let* (L) *be a* 1-*subprincipal isotropy class of* (V, G). *Let* $d=\dim V/G$, *and let* $p_1, \ldots, p_r$ *be forms in* $\mathbf{C}[V]^G$ *such that the* $p_i'=\operatorname{res}_L p_i$ *are a minimal generating set for* $\operatorname{res}_L\mathbf{C}[V]^G$. *Suppose that* $I(V, L)$ *has a homogeneous generator* f_L.

(1) *If* $\operatorname{res}_L\mathbf{C}[V]^G$ *is a regular ring (i.e.* $r=d-1$*), then* $\mathbf{C}[V]^G$ *is the regular ring* $\mathbf{C}[p_1, \ldots, p_{d-1}, f_L]$.

Suppose that $r=d$ *and that the relations of the* p_i' *have generator* $f(p_1', \ldots, p_d')$. *Then either*

(2) f_L *is a multiple of* $f(p_1, \ldots, p_d)$ *and* $\mathbf{C}[V]^G\simeq\mathbf{C}[p_1, \ldots, p_d]$ *is a regular ring, or*

(3) $\mathbf{C}[V]^G$ *is generated by the* p_i *and* f_L *with a relation of the form* $f_L h(p_1, \ldots, p_d, f_L) = f(p_1, \ldots, p_d)$. *(h may be zero.)* □

Examples 3.6. The representations $(\varphi_1^2+\varphi_1, \mathsf{A}_1)$ and $(2\varphi_1^2, \mathsf{A}_1)$ have 1-subprincipal isotropy class ($L=\mathbf{C}^*$), and they illustrate, respectively, possibilities 3.5.1 and 3.5.2. The representation $(\varphi_1^2+2\varphi_1, \mathsf{A}_2)$ (again $L=\mathbf{C}^*$) illustrates 3.5.3. Here one gets a relation $\alpha f_L = f$ where $\deg f_L = 5$ and α is the degree 3 generator of $\mathbf{C}[\varphi_1^2]^{\mathsf{A}_2}$. One gets a similar type of relation for the more exotic case of $(\varphi_3+3\varphi_1, \mathsf{A}_6)$.

We have the following guarantee that 3.5.2 occurs:

Lemma 3.7. *Let L, f, f_L, etc. be as in 3.5. Suppose further that (L) is the unique subprincipal isotropy class of (V, G) and that $\deg f(p_1, \ldots, p_d) \leqq \deg f_L$. Then 3.5.2 occurs, i.e. $\mathbf{C}[V]^G \simeq \mathbf{C}[p_1, \ldots, p_d]$ is a regular ring.* □

In order to apply Lemma 3.7 we need

(a) ways to find all closed isotropy classes (or just all subprincipal isotropy classes) of (V, G).

(b) a priori estimates of $\deg f_L$.

(c) knowledge of $\operatorname{res}_L \mathbf{C}[V]^G \subset \mathbf{C}[V^L]^{N_G(L)}$.

Most of the time (a) is not difficult; the majority of the cases follow from

Lemma 3.8. *Let $V = V_1 \oplus V_2$ be a direct sum of representation spaces of G. Let (H) be a proper closed isotropy class of (V, G) and suppose that for all reductive algebraic subgroups M of G, $V_1^M \neq \{0\}$ implies that $\operatorname{res}_M \mathbf{C}[V_1]^G \neq \mathbf{C}$. Then*

(1) *$(H) \leqq (L)$ where (L) is a proper closed isotropy class of (V_1, G) or (V_2, G).*

(2) *Suppose that (H) is subprincipal and that the principal isotropy class (H_2) of (V_2, G) is not the principal isotropy class of (V, G). Then $(H) = (H_2)$ or $(H) \leqq (L)$ where (L) is a proper closed isotropy class of (V_1, G).*

Proof. If $V_1^H \neq \{0\}$, then $\operatorname{res}_H \mathbf{C}[V_1]^G \neq \mathbf{C}$ and 1.3 shows that there are closed orbits Gx, $x \in V_1 - \{0\}$, with $H \subset L = G_x$. If $V_1^H = \{0\}$, then $\operatorname{res}_H \mathbf{C}[V_2]^G \neq \mathbf{C}$ and $H \subset L$ where (L) is a proper closed isotropy class of (V_2, G). We have proved (1), and (2) is proved similarly. □

Note that 3.8 applies if (V_1, G) is orthogonal. To find the closed isotropy classes of (V_1, G) and (V_2, G) one can often apply 3.1. In some cases, however, 3.1 and 3.8 do not suffice, and one must apply techniques similar to the following:

Example 3.9. Let $(V, G) = (\varphi_3 + 3\varphi_6, \mathsf{A}_6)$. Using 1.3 one can show that there are closed isotropy classes (G_2), (C_3), and (A_3) where $\varphi_1(\mathsf{A}_6)$ restricts to $(\varphi_1, \mathsf{G}_2)$, $(\varphi_1+\theta_1, \mathsf{C}_3)$, and $(\varphi_1+\theta_3, \mathsf{A}_3)$, respectively. We indicate how to show that these are the maximal proper closed isotropy classes of (V, G), and from their slice representations one easily calculates all closed isotropy classes of (V, G). In particular, one sees that there is a unique subprincipal (and 1-subprincipal) isotropy class ($L = \mathsf{A}_1$), where $(\varphi_1(\mathsf{A}_6), \mathsf{A}_1) = 2\varphi_1 + \theta_3$.

Let (H) be a proper closed isotropy class of (V, G). If $\varphi_6(\mathsf{A}_6)^H = \{0\}$, then $\operatorname{res}_H \mathbf{C}[\varphi_3]^{\mathsf{A}_6} \neq \mathbf{C}$ and $(H) = (\mathsf{G}_2)$—the only proper closed isotropy class of $\varphi_3(\mathsf{A}_6)$. Suppose that $(\varphi_6(\mathsf{A}_6), H) = (\psi + \theta_1, H)$ where $\psi^H = \{0\}$. Then a finite cover of $N_G(H)$ contains at least $H \times \mathbf{C}^*$, where $(\varphi_6(\mathsf{A}_6), H \times \mathbf{C}^*) = \psi \otimes \nu_{-1} + \nu_6$. The representation of $\mathbf{C}^*$ on V^H is $(\Lambda^3 \psi)^H \otimes \nu_3 + (\Lambda^2 \psi)^H \otimes \nu_{-4} + 3\nu_6$. Since $\operatorname{res}_H \mathbf{C}[V]^G \neq \mathbf{C}$, we must have $(\Lambda^2 \psi)^H \neq \{0\}$. Since H is reductive, if H fixes a 2-form of rank 2 in $\Lambda^2\psi$ then it fixes a complementary rank 1 form. Hence H fixes a form of rank 3 (and then $(H)\leqq(\mathsf{C}_3)$), or H fixes only rank 1 forms and $(\psi+\theta_1, H)=(\psi_1+\psi_2+\theta_1, H)$ where $\dim\psi_1=2$, $\dim\psi_2=4$, and $\operatorname{Im}(H\to GL(\psi_i))\subset SL(\psi_i)$, $i=1,2$. If ψ_1 is the trivial representation, then $(H)\leqq(\mathsf{A}_3)$, and if $(\Lambda^2\psi_2)^H\neq\{0\}$ or $(\psi_1\otimes\psi_2)^H\neq\{0\}$, then $(H)\leqq(\mathsf{C}_3)$. If all three of the above fail, then one can see that $(V^H, N_G(H))$ has only constant invariants. Thus we may assume that $\dim\varphi_6(\mathsf{A}_6)^H\geqq 2$, and proceeding as above one finds that $(\varphi_3+3\varphi_6, \mathsf{A}_6)$ has the claimed maximal proper closed isotropy classes.

We now turn to problem (b)—estimating $\deg f_L$. Let $V=\bigoplus_{i=1}^{n} V_i$ be a direct sum of G-representation spaces. We say that $f\in\mathbf{C}[V]$ has degree $(a_1,\dots,a_n)$ if f is homogeneous of degree a_i in V_i, $i=1,\dots,n$. It should be clear from the context whether $\deg f$ refers to the n-tuple $(a_1,\dots,a_n)$ or to the total degree $a_1+\dots+a_n$. We write $(a_1,\dots,a_n)>(b_1,\dots,b_n)$ if $a_1+\dots+a_n>b_1+\dots+b_n$ and $a_i\geqq b_i$, $i=1,\dots,n$. If L is a subgroup of G, then clearly $I(V,L)$ has minimal generating sets consisting of elements simultaneously homogeneous in each V_i.

Lemma 3.10. *Let $V=V_1\oplus V_2$ be a direct sum of G-representation spaces, and let H_1 be a principal isotropy group of (V_1, G). Assume that (V,G) has a unique 1-subprincipal isotropy class (L) and that $I(V,L)$ is generated by a form f_L. Further suppose that*

(1) (V_1, G) has generically closed orbits.

(2) (V_2, H_1) has 1-subprincipal isotropy classes, and the ideal in $\mathbf{C}[V_2]^{H_1}$ vanishing on the corresponding orbits has a homogeneous generator f_2.

Then $\deg f_L\geqq(0,\deg f_2)$, and if V_2 contains no principal orbits of (V,G), then $\deg f_L>(0,\deg f_2)$. If $h\in I(V,L)$ and $\deg h=(a,\deg f_2)$, then $h=ff_L$ where $f\in\mathbf{C}[V_1]^G$. □

An example where $\deg f_L=(0,\deg f_2)$ is $(\varphi_2+4\varphi_1, \mathsf{A}_3)$, $L=\mathsf{A}_1$; and $\deg f_L=(1,\deg f_2)$ for $(\varphi_1^2+\varphi_1, \mathsf{A}_1)$, $L=\mathbf{C}^*$.

Remarks 3.11. Suppose that $V=\bigoplus_{i=1}^{n} V_i$, (V,G) has a 1-subprincipal isotropy class (L), $I(V,L)$ has a homogeneous generator f_L, and $\deg f_L=(a_1,\dots,a_n)$. If $(V_1,G)\simeq(V_2,G)$, then $a_1=a_2$. If (V,G) is orthogonal, then $\deg f_L$ is even.

We now come to problem (c): Determine $\operatorname{res}_L\mathbf{C}[V]^G\subset\mathbf{C}[V^L]^{N_G(L)}$. One can often use the following to prove that res_L is surjective:

Proposition 3.12. *Suppose that (V,G) has a unique 1-subprincipal isotropy class (L), $I(V,L)$ is a principal ideal, and $\dim V/G\geqq 3$. Further assume that if (W,M) is a proper slice representation of (V,G) with $L\subset M$, then (W,M) has a unique 1-subprincipal isotropy class generated by L and*

$$\operatorname{res}_L\mathbf{C}[W]^M=\mathbf{C}[W^L]^{N_M(L)}.$$

Then

$$\operatorname{res}_L\mathbf{C}[V]^G=\mathbf{C}[V^L]^{N_G(L)}.\quad\square$$

In the cases where res_L is not surjective the following always occurs: There is a closed isotropy class (M), $M \supset L$, where $\mathrm{Im}(V^M \to V/G)$ has codimension two, and $\mathrm{res}_L\, \mathbf{C}[V]^G$ is the subalgebra of $\mathbf{C}[V^L]^{N_G(L)}$ consisting of those elements whose restriction to V^M is $N_G(M)$-invariant. An example is $(4\varphi_1+\varphi_3, \mathsf{D}_4)$ (see [29] § 17).

Our final method of calculating invariants is the most cumbersome of all, but it avoids the problem of calculating the functions f_L and even works when (V, G) has no 1-subprincipal isotropy classes: Let Y be a complex affine variety, $f \in \mathbf{C}[Y]$. Then Y_f denotes the points of Y where f does not vanish. $\mathbf{C}[Y]_f$ denotes $\mathbf{C}[Y]$ localized at f, so we have $\mathbf{C}[Y_f] \simeq \mathbf{C}[Y]_f$.

Lemma 3.13 ([29] §15). *Suppose that (V, G) has generically closed orbits and that there is a form f which generates the ideal in $\mathbf{C}[V]^G$ vanishing on the non-principal orbits. Let H be a principal isotropy group of (V, G), and let W be a representation space of G. Then the inclusion $V^H \subset V$ induces an isomorphism*

$$\mathbf{C}[V+W]^G_f \simeq \mathbf{C}[V^H+W]^{N_G(H)}_f. \quad \square$$

Lemma 3.14 ([29] § 15). *Let S be a noetherian domain over $\mathbf{C}$, and let R be the subdomain generated by non-zero elements $f_1, \ldots, f_d \in S$, where $\dim S = d$. Suppose that $R_{f_1} = S_{f_1}$ and $R_{f_2} = S_{f_2}$. Then $R = S$.* $\square$

Let V, W, f, etc. be as in 3.13. Let $S = \mathbf{C}[V \oplus W]^G$, let $f = f_1, f_2, \ldots, f_d$ be non-zero elements of S, $\dim S = d$, and let R denote $\mathbf{C}[f_1, \ldots f_d]$. If $R = S$, then this fact can be established as follows: Calculate the ring of invariants $\bar{S}$ of $(V^H \oplus W, N_G(H))$ and show that restriction to $V^H \oplus W$ induces an isomorphism of R_f with $\bar{S}_f$. (One has to be very careful here to see how the f_i restrict to $V^H \oplus W$.) Then 3.13 shows that $R_f \simeq S_f$. Establish that $R_h \simeq S_h$ where h is one of $f_2, \ldots f_d$. Then 3.14 shows that $R = S$. The point of the method is that $\bar{S}$ is easier to compute than S. See ([29] § 15) for a (very simple) example of this technique.

In the tables below, φ indicates a representation of the given group G and d denotes $\dim \mathbf{C}[\varphi]^G$. The notation $l\langle m \rangle$ in an "a" table refers to l generators of the type described in entry m of the corresponding "b" table. An entry $(a_1, \ldots, a_n)$ denotes a generator of the indicated degree. In each case the precise description of the generator (in terms of tensor decompositions) is obvious or is obvious from the tensor decompositions we list. We consider each entry and tensor decomposition in Tables 1b, 2b, and 5b as representing itself and the corresponding dual entry. For example, entry 3 of Table 1b describes invariants of representations containing $(\varphi_2 + l\varphi_1, \mathsf{A}_{n-1})$ or $(\varphi_2^* + l\varphi_1^*, \mathsf{A}_{n-1})$ as a subrepresentation, $l \in \mathbf{Z}^+$. In Table 3b we subject entries and tensor decompositions for D_n to the automorphism interchanging φ_{n-1} and φ_n.

To calculate tensor decompositions we mainly used CIT and the methods of Krämer ([15, 17]). (Part of Krämer's results are already in Ch. III § 2 of [6].) Also, the existence of invariants (say using 3.1 or 3.12) or the existence of covariants (see 3.2) often gave useful information.

We used CIT to compute the generators of the invariants for representations listed in Tables 1a and 1a′. (Entry 1a.20 is an exception; see Remark 4.8.) In the other tables, the notations CIT, VIN, LR, (L), and SLICE under "Method" indicate, respectively, that we used CIT, the results of Vinberg [30], Lemma 3.1,

Table 1a ($G=SL_n=\mathsf{A}_{n-1}$)

	φ	d	Generators
1	$(n+1)\varphi_1$	$n+1$	$(n+1)\langle 1\rangle$
2	$n\varphi_1+(n-1)\varphi_1^*$	n^2-n+1	$\langle 1\rangle, n(n-1)\langle 2\rangle$
3	$\varphi_2+3\varphi_1+(n-2)\varphi_1^*$; $n\geqq 4$	$\frac{1}{2}n(n+1)+1$	$(3n-6)\langle 2\rangle, 4\langle 3\rangle, \frac{1}{2}(n-2)(n-3)\langle 4\rangle$
4	$\varphi_2+2\varphi_1+(n-1)\varphi_1^*$; $n\geqq 4$	$\frac{1}{2}n(n+1)+1$	$(2n-2)\langle 2\rangle, 2\langle 3\rangle, \frac{1}{2}(n-1)(n-2)\langle 4\rangle$
5	$\varphi_2+n\varphi_1^*$; $n\geqq 5$, n odd	$\frac{1}{2}n(n-1)+1$	$\langle 1\rangle, \frac{1}{2}n(n-1)\langle 4\rangle$
6	$2\varphi_2+3\varphi_1$; $n\geqq 4$	$2n+1$	$8\langle 3\rangle, (2n-7)\langle 5\rangle$
7	$2\varphi_2+2\varphi_1+\varphi_1^*$; $n\geqq 4$	$2n+1$	$2\langle 2\rangle, 4\langle 3\rangle, (2n-5)\langle 5\rangle$
8	$2\varphi_2+\varphi_1+2\varphi_1^*$; $n\geqq 4$	$2n+1$	$2\langle 2\rangle, 2\langle 3\rangle, 2\langle 4\rangle, (2n-5)\langle 5\rangle$
9	$2\varphi_2+3\varphi_1^*$; $n\geqq 4$, n even	$2n+1$	$2\langle 3\rangle, 6\langle 4\rangle, (2n-7)\langle 5\rangle$
9′	$2\varphi_2+3\varphi_1^*$; $n\geqq 7$, n odd	$2n+1$	$6\langle 4\rangle, (2n-5)\langle 5\rangle$
10	$\varphi_2+\varphi_2^*+3\varphi_1$; $n\geqq 4$, n even	$2n+1$	$5\langle 3\rangle, 3\langle 4\rangle, (2n-7)\langle 6\rangle$
11	$\varphi_2+\varphi_2^*+2\varphi_1+\varphi_1^*$; $n\geqq 4$	$2n+1$	$2\langle 2\rangle, 3\langle 3\rangle, \langle 4\rangle, (2n-5)\langle 6\rangle$
12	$\varphi_1^2+\varphi_1+(n-2)\varphi_1^*$	$\frac{1}{2}n(n-1)+1$	$(n-2)\langle 2\rangle, \langle 7\rangle, \langle 8\rangle, \frac{1}{2}(n-1)(n-2)\langle 9\rangle$
13	$\varphi_1^2+(n-1)\varphi_1^*$	$\frac{1}{2}n(n-1)+1$	$\langle 7\rangle, \frac{1}{2}n(n-1)\langle 9\rangle$
14	$\varphi_1^2+\varphi_2+\varphi_1$; $n\geqq 4$	$n+1$	$\langle 3\rangle, \langle 7\rangle, \langle 8\rangle, (n-2)\langle 10\rangle$
15	$\varphi_1^2+\varphi_2+\varphi_1^*$; $n\geqq 4$, n even	$n+1$	$\langle 3\rangle, \langle 7\rangle, \langle 9\rangle, (n-2)\langle 10\rangle$
15′	$\varphi_1^2+\varphi_2+\varphi_1^*$; $n\geqq 5$, n odd	$n+1$	$\langle 7\rangle, \langle 9\rangle, (n-2)\langle 10\rangle, \langle 11\rangle$
16	$\varphi_1^2+\varphi_2^*+\varphi_1$; $n\geqq 4$, n even	$n+1$	$\langle 3\rangle, \langle 7\rangle, \langle 8\rangle, (n-2)\langle 12\rangle$
17	$\varphi_1^2+\varphi_2^*+\varphi_1^*$; $n\geqq 4$	$n+1$	$\langle 3\rangle, \langle 7\rangle, \langle 9\rangle, (n-2)\langle 12\rangle$
18	$2\varphi_1^2$	$n+1$	$2\langle 7\rangle, (n-1)\langle 13\rangle$
19	$\varphi_1^2+\varphi_{n-1}^2$	$n+1$	$2\langle 7\rangle, (n-1)\langle 14\rangle$
20	$\varphi_1\varphi_{n-1}+\varphi_1$	n	$(2,0), (3,0), \ldots, (n,0), \langle 16\rangle$

Table 1a′ ($G=SL_n=\mathsf{A}_{n-1}$)

	φ	d	Generators
1	$(n+2)\varphi_1$	$2n+1$	$\frac{1}{2}(n+1)(n+2)\langle 1\rangle$
2	$(n+1)\varphi_1+\varphi_1^*$	$2n+1$	$(n+1)\langle 1\rangle, (n+1)\langle 2\rangle$
3	$\varphi_2+4\varphi_1$; $n=4$	7	$\langle 1\rangle, 7\langle 3\rangle$
3′	$\varphi_2+4\varphi_1$; $n\geqq 5$	7	$8\langle 3\rangle$
4	$\varphi_1^2+2\varphi_1$; $n\geqq 3$	4	$\langle 7\rangle, 3\langle 8\rangle, \langle 15\rangle$
5	$2\varphi_2+3\varphi_1^*$; $n=5$	11	$6\langle 4\rangle, 6\langle 5\rangle$

Table 1b ($G=SL_n=\mathsf{A}_{n-1}$)

	Invariants
1	$\theta_1=\Lambda^n\varphi_1\subset S^n(n\varphi_1)$
2	$\theta_1\subset\varphi_1\otimes\varphi_1^*$
3	$\theta_1\subset\varphi_{2i}\otimes\varphi_{n-2i}\subset S^i\varphi_2\otimes S^{n-2i}((n-2i)\varphi_1)$; $2\leqq 2i\leqq n$
4	$\theta_1\subset\varphi_2\otimes\varphi_2^*\subset\varphi_2\otimes S^2(2\varphi_1^*)$
5	$\theta_1\subset\varphi_{2i+j}\otimes\varphi_{2k-l}\subset\varphi_{2i}\otimes\varphi_j\otimes\varphi_{2k}\otimes\varphi_l^*\subset S^i\varphi_2\otimes S^j(j\varphi_1)\otimes S^k\varphi_2\otimes S^l(l\varphi_1^*)$; $2i+j+2k-l=n$; $i,k\geqq 1$; $i,k\geqq l$
6	$\theta_1\subset\varphi_{2i+j}\otimes\varphi_{2k+l}^*\subset S^i\varphi_2\otimes S^j(j\varphi_1)\otimes S^k(\varphi_2^*)\otimes S^l(l\varphi_1^*)$; $2i+j=2k+l<n$; $i,k\geqq 1$
7	$\theta_1\subset S^n(\varphi_1^2)$

Table 1b (continued)

	Invariants
8	$\theta_1 \subset \varphi_{n-1}^2 \otimes \varphi_1^2 \subset S^{n-1}(\varphi_1^2) \otimes [S^2\varphi_1 \text{ or } \varphi_1 \otimes \varphi_1]$
9	$\theta_1 \subset \varphi_1^2 \otimes (\varphi_1^*)^2 \subset \varphi_1^2 \otimes [S^2\varphi_1^* \text{ or } \varphi_1^* \otimes \varphi_1^*]$
10	$\theta_1 \subset \varphi_{i+j}^2 \otimes \varphi_{2k-l}^2 \subset S^i(\varphi_1^2) \otimes S^{2j}(j\varphi_1) \otimes S^{2k}\varphi_2 \otimes S^{2l}(l\varphi_1^*)$; $i+j+2k-l=n$; $k \geqq 1 \geqq l$; $i>2l$
11	$\theta_1 \subset \varphi_1^2 \otimes \varphi_1^* \otimes \varphi_1^* \subset \varphi_1^2 \otimes S^k\varphi_2 \otimes \varphi_1^*$; $n=2k+1$
12	$\theta_1 \subset \varphi_{i+j}^2 \otimes (\varphi_{2k+l}^2)^* \subset S^i(\varphi_1^2) \otimes S^{2j}(j\varphi_1) \otimes S^{2k}\varphi_2^* \otimes S^{2l}(l\varphi_1^*)$; $i+j=2k+l<n$; $i, k \geqq 1$
13	$\theta_1 \subset \varphi_i^2 \otimes \varphi_{n-i}^2 \subset S^i(\varphi_1^2) \otimes S^{n-i}(\varphi_1^2)$; $1 \leqq i < n$
14	$\theta_1 \subset \varphi_i^2 \otimes \varphi_{n-i}^2 \subset S^i(\varphi_1^2) \otimes S^i(\varphi_{n-1}^2)$; $1 \leqq i < n$
15	$\theta_1 \subset \varphi_{n-2}^2 \otimes \varphi_2^2 \subset S^{n-2}(\varphi_1^2) \otimes S^4(2\varphi_1)$; $n \geqq 3$
16	$\theta_1 \subset \varphi_{n-1}^n \otimes \varphi_1^n \subset S^{\frac{1}{2}n(n-1)}(\varphi_1\varphi_{n-1}) \otimes S^n\varphi_1$

Table 2a

	G	φ	d	Generators	Method	deg f_L
1	A_1	φ_1^3	1	(4)	LR	
2		φ_1^4	2	(2), (3)	from 1a.18	
3	A_2	φ_1^3	2	(4), (6)	VIN	
4	A_4	$3\varphi_2$	6	$3\langle 2\rangle$, $3\langle 3\rangle$	CIT	
5		$2\varphi_2+\varphi_3$	6	$2\langle 1\rangle$, $3\langle 4\rangle$, $\langle 5\rangle$	(L)	(3, 3, 1)
6	A_5	$3\varphi_2$	11	$10\langle 6\rangle$, $\langle 7\rangle$	CIT	
7		$2\varphi_2+\varphi_4$	11	$2\langle 1\rangle$, $5\langle 6\rangle$, $\langle 7\rangle$, $3\langle 8\rangle$	CIT	
8		$\varphi_3+4\varphi_1$	9	(4, 0, 0, 0, 0), $8\langle 9\rangle$	CIT	
9		$\varphi_3+3\varphi_1+2\varphi_5$	15	(4, 0, 0, 0, 0, 0), $6\langle 1\rangle$, $2\langle 9\rangle$, $6\langle 10\rangle$	CIT	
10		$\varphi_3+\varphi_2+\varphi_1$	6	(4, 0, 0), (0, 3, 0), (4, 3, 0), (1, 1, 1), (3, 1, 1), (2, 2, 2) $=\langle 11\rangle$	from 2a.14	(6, 5, 2)
11		$\varphi_3+\varphi_2+\varphi_5$	6	(4, 0, 0), (0, 3, 0), (4, 3, 0), (2, 1, 2) $=\langle 12\rangle$, (1, 2, 1) and (3, 2, 1) $=2\langle 13\rangle$	from 2a.15	(6, 4, 2)
12		$2\varphi_3$	7	(4, 0), (3, 1), (1, 3), (0, 4), (1, 1), (2, 2), (3, 3)	$\simeq$ 5a.8	(12, 12)
13		$\varphi_3+\varphi_1^2$	6	(4, 0), (0, 6), (4, 6) $=\langle 14\rangle$, $3\langle 15\rangle$	SLICE	
14	A_6	$\varphi_3+2\varphi_1+\varphi_6$	8	(7, 0, 0, 0), $2\langle 1\rangle$, $3\langle 16\rangle$, $\langle 17\rangle$, $\langle 18\rangle$	(L)	(11, 2, 2, 2)
15		$\varphi_3+\varphi_1+2\varphi_6$	8	(7, 0, 0, 0), $2\langle 1\rangle$, $\langle 16\rangle$, $3\langle 17\rangle$, $\langle 19\rangle$	(L)	(10, 2, 2, 2)
16		$\varphi_3+3\varphi_6$	8	(7, 0, 0, 0), $6\langle 17\rangle$, (1, 1, 1, 1)	(L)	(9, 2, 2, 2)
17	A_7	$\varphi_3+\varphi_1$	3	(16, 0), (10, 2), (7, 3)	LR	
18		$\varphi_3+\varphi_7$	3	(16, 0), (6, 2), (9, 3)	LR	
19		φ_4	7	(2), (6), (8), (10), (12), (14), (18)	VIN	
20	A_8	φ_3	4	(12), (18), (24), (30)	VIN	

Table 2b

	G	Tensor decompositions and invariants
1	A_n	$\theta_1 \subset \varphi_i \otimes \varphi_i^*$; $1 \leqq i \leqq n$
	A_4	$\Lambda^2 \varphi_2 = \varphi_1 \varphi_3$; $\varphi_2 \subset \varphi_2 \varphi_4 \otimes \varphi_2 \varphi_4$; also see Prop. 2.4
2		$\theta_1 \subset \varphi_3 \otimes \varphi_2 \subset \varphi_4 \otimes \varphi_4 \otimes \varphi_2 \subset S^2 \varphi_2 \otimes S^2 \varphi_2 \otimes \varphi_2$
3		$\theta_1 \subset \varphi_2 \varphi_4 \otimes \varphi_1 \varphi_3 \subset S^3 \varphi_2 \otimes \varphi_2 \otimes \varphi_2$
4		$\theta_1 \subset \varphi_4 \otimes \varphi_1 \subset [S^2 \varphi_2$ or $\varphi_2 \otimes \varphi_2] \otimes S^2 \varphi_3$
5		$\theta_1 \subset \varphi_2 \otimes \varphi_3 \subset \varphi_2 \varphi_4 \otimes \varphi_2 \varphi_4 \otimes \varphi_3 \subset S^3 \varphi_2 \otimes S^3 \varphi_2 \otimes \varphi_3$
	A_5	$S^2 \varphi_3 = \varphi_3^2 + \varphi_1 \varphi_5$; $S^3 \varphi_3 = \varphi_3^3 + \varphi_1 \varphi_3 \varphi_5 + \varphi_3$
		$S^4 \varphi_3 = \varphi_3^4 + \varphi_1 \varphi_3^2 \varphi_5 + \varphi_1^2 \varphi_5^2 + \varphi_3^2 + \varphi_2 \varphi_4 + \theta_1$; also see Prop. 2.4
6		$\theta_1 \subset S^3 \varphi_2$, $S^2 \varphi_2 \otimes \varphi_2$, or $\varphi_2 \otimes \varphi_2 \otimes \varphi_2$
7		$\theta_1 \subset \varphi_4 \otimes \varphi_4 \otimes \varphi_4 \subset S^2 \varphi_2 \otimes S^2 \varphi_2 \otimes [S^2 \varphi_2$ or $\varphi_4]$
8		$\theta_1 \subset \varphi_4 \otimes \varphi_2 \subset [S^2 \varphi_2$ or $\varphi_2 \otimes \varphi_2] \otimes S^2 \varphi_4$
9		$\theta_1 \subset \varphi_3 \otimes \varphi_3 \subset [S^3 \varphi_3$ or $\varphi_3] \otimes S^3 (3 \varphi_1)$
10		$\theta_1 \subset \varphi_1 \varphi_5 \otimes \varphi_1 \varphi_5 \subset S^2 \varphi_3 \otimes \varphi_1 \otimes \varphi_5$
11		$\theta_1 \subset \varphi_1 \varphi_5 \otimes \varphi_1 \varphi_5 \subset \varphi_1 \varphi_5 \otimes \varphi_4 \otimes \varphi_1^2 \subset S^2 \varphi_3 \otimes S^2 \varphi_2 \otimes S^2 \varphi_1$
12		$\theta_1 \subset \varphi_1 \varphi_5 \otimes \varphi_1 \varphi_5 \subset S^2 \varphi_3 \otimes \varphi_2 \otimes S^2 \varphi_5$
13		$\theta_1 \subset \varphi_3 \otimes \varphi_3 \subset \varphi_3 \otimes \varphi_4 \otimes \varphi_5 \subset [S^3 \varphi_3$ or $\varphi_3] \otimes S^2 \varphi_2 \otimes \varphi_5$
14		$\theta_1 \subset \varphi_1^2 \varphi_5^2 \otimes \varphi_1^2 \varphi_5^2 \subset S^4 \varphi_3 \otimes S^6 (\varphi_1^2)$
15		$\theta_1 \subset \varphi_3^2 \otimes \varphi_3^2 \subset S^3 (\varphi_1^2) \otimes [$the generator in $S^j \varphi_3$ of the occurrences of φ_3^2 in the $S^* (\varphi_3)^{\mathsf{A}_5}$-module $S^* (\varphi_3)$; $j = 2, 4, 6]$
	A_6	$S^2 \varphi_3 = \varphi_3^2 + \varphi_1 \varphi_5$; $S^3 \varphi_3 = \varphi_3^3 + \varphi_3 \varphi_6 + \varphi_1 \varphi_3 \varphi_5 + \varphi_1^2$
		$S^4 \varphi_3 = \varphi_3^4 + \varphi_1 \varphi_3^2 \varphi_5 + \varphi_1^2 \varphi_5^2 + \varphi_3^2 \varphi_6 + \varphi_2 \varphi_4 \varphi_6 + \varphi_6^2 + \varphi_1^2 \varphi_3 + \varphi_1 \varphi_4$
		$S^5 \varphi_3 \supset \varphi_2 \varphi_6$; $S^7 \varphi_3 \supset \theta_1$
16		$\theta_1 \subset \varphi_6^2 \otimes \varphi_1^2 \subset S^4 \varphi_3 \otimes [S^2 \varphi_1$ or $\varphi_1 \otimes \varphi_1]$
17		$\theta_1 \subset \varphi_1^2 \otimes \varphi_6^2 \subset S^3 \varphi_3 \otimes [S^2 \varphi_6$ or $\varphi_6 \otimes \varphi_6]$
18		$\theta_1 \subset \varphi_1 \varphi_5 \otimes \varphi_2 \otimes \varphi_6 \subset S^2 \varphi_3 \otimes S^2 (2 \varphi_1) \otimes \varphi_6$
19		$\theta_1 \subset \varphi_2 \varphi_6 \otimes \varphi_1 \varphi_5 \subset S^5 \varphi_3 \otimes \varphi_1 \otimes S^2 (2 \varphi_6)$

Lemma 3.5, or Lemmas 3.13 and 3.14 to establish the coregularity of the given representation φ. (For the adjoint representations of the exceptional Lie groups we used LR and ([2] Ch. V § 6 Prop. 3 and Planche V–X).) The term "from Ya.Z" (resp. "$\simeq$Ya.Z") means that (modulo trivial factors) φ is a slice representation of (resp. has a ring of invariants isomorphic to those of) entry Z of Table Ya. Under "deg f_L" we list the degree of f_L, provided that (modulo trivial factors) φ is a slice representation of (or is) a representation to which we applied method (L). In all instances but one, Lemma 3.10 and Remark 3.11 allowed us to correctly estimate deg f_L. For $\varphi = (\varphi_3 + 3\varphi_6, \mathsf{A}_6) = 2$a.16 these methods only obtain

Table 3a

	G	φ	d	Generators	Method	$\deg f_L$
1	D_n,	$(2n-1)\varphi_1$	$n(2n-1)$	$n(2n-1)\langle 1\rangle$	from 1a.13	
2	$n\geqq 3$	$\mathrm{Ad}+\varphi_1$	$2n$	$(2,0),(4,0),\ldots,(2n-2,0),$ $(n,0),(0,2),(2,2),\ldots,(2n-2,2)$	from 1a.14	
3		φ_1^2	$2n-1$	$(2),(3),\ldots,(2n)$	from 1a.18	
4	B_n,	$2n\varphi_1$	$n(2n+1)$	$n(2n+1)\langle 1\rangle$	from 1a.13	
5	$n\geqq 2$	$\mathrm{Ad}+\varphi_1$	$2n+1$	$(2,0),(4,0),\ldots,(2n,0),$ $(0,2),(2,2),\ldots,(2n-2,2),(n,1)$	from 1a.14	
6		φ_1^2	$2n$	$(2),(3),\ldots,(2n+1)$	from 1a.18	
7[a]	D_3	$3\varphi_1+\varphi_2$	7	$6\langle 1\rangle,\langle 2\rangle$	(L)	(1, 1, 1, 2)
8		$2\varphi_1+\varphi_2^2$	7	$3\langle 1\rangle,(0,0,3),3\langle 7\rangle$	CIT	
9	B_3	$3\varphi_1+\varphi_3$	8	$6\langle 1\rangle,\langle 2\rangle,\langle 5\rangle$	from 3a.13	(2, 2, 2, 4)
10		$2\varphi_1+2\varphi_3$	9	$3\langle 1\rangle,\langle 3\rangle,2\langle 4\rangle,3\langle 5\rangle$	from 3a.14	(2, 2, 2, 2)
11		$\varphi_1+3\varphi_3$	10	$\langle 1\rangle,3\langle 4\rangle,6\langle 5\rangle$	from 3a.14	(2, 2, 2, 2)
12		$4\varphi_3$	11	$10\langle 5\rangle,\langle 8\rangle$	from 3a.13	(2, 2, 2, 2)
13	D_4	$4\varphi_1+\varphi_3$	12	$10\langle 1\rangle,\langle 2\rangle,\langle 6\rangle$	(L)	(2, 2, 2, 2, 4)
14		$3\varphi_1+2\varphi_3$	12	$6\langle 1\rangle,3\langle 4\rangle,3\langle 6\rangle$	(L)	(2, 2, 2, 2, 2)
15		$3\varphi_1+\varphi_3+\varphi_4$	12	$6\langle 1\rangle,\langle 3\rangle,3\langle 5\rangle,2\langle 6\rangle$	from 3a.17	(2, 2, 2, 2, 2)
16		$2\varphi_1+2\varphi_3+\varphi_4$	12	$3\langle 1\rangle,\langle 4\rangle,4\langle 5\rangle,4\langle 6\rangle$	from 3a.18	(2, 2, 2, 2, 2)
17	B_4	$4\varphi_1+\varphi_4$	16	$10\langle 1\rangle,\langle 2\rangle,4\langle 5\rangle,\langle 6\rangle$	from 3a.20	(2, 2, 2, 2, 4)
18		$2\varphi_1+2\varphi_4$	14	$3\langle 1\rangle,\langle 4\rangle,6\langle 5\rangle,3\langle 6\rangle,\langle 9\rangle$	from 3a.22	(2, 2, 4, 4)
19		$3\varphi_4$	12	$6\langle 6\rangle,6\langle 9\rangle$	from 3a.23	
20	D_5	$5\varphi_1+\varphi_4$	21	$15\langle 1\rangle,\langle 2\rangle,5\langle 6\rangle$	(L)	(2, 2, 2, 2, 2, 4)
21		$3\varphi_1+2\varphi_4$	17	$6\langle 1\rangle,\langle 4\rangle,9\langle 6\rangle,\langle 10\rangle$	SLICE	
22		$3\varphi_1+\varphi_4+\varphi_5$	17	$6\langle 1\rangle,3\langle 5\rangle,6\langle 6\rangle,\langle 10\rangle,\langle 11\rangle$	from 3a.25	(2, 2, 2, 4, 4)
23		$\varphi_1+3\varphi_4$	13	$\langle 1\rangle,6\langle 6\rangle,6\langle 10\rangle$	SLICE	
24		$\varphi_1+2\varphi_4+\varphi_5$	13	$\langle 1\rangle,4\langle 6\rangle,4\langle 10\rangle,2\langle 11\rangle,2\langle 12\rangle$	SLICE	
25	B_5	$4\varphi_1+\varphi_5$	21	$10\langle 1\rangle,6\langle 5\rangle,4\langle 6\rangle,(0,0,0,0,4)$	from 3a.27	(2, 2, 2, 2, 8)
26		$2\varphi_5$	9	$(4,0),(3,1),(1,3),(0,4),$ $(1,1),2(2,2),(3,3),(4,4)$	from 3a.28	(12, 12)
27	D_6	$5\varphi_1+\varphi_5$	26	$15\langle 1\rangle,10\langle 6\rangle,(0,0,0,0,0,4)$	(L)	(2, 2, 2, 2, 2, 8)
28		$\varphi_1+2\varphi_5$	10	$(0,4,0),(0,3,1),(0,1,3),(0,0,4),$ $(0,1,1),(0,2,2),(0,3,3),(2,0,0),$ $(2,2,2),(2,4,4)$	(L)	(6, 12, 12)
29		$\varphi_1+\varphi_5+\varphi_6$	10	$(0,4,0),(0,0,4),(0,2,2),(0,4,4),$ $(2,0,0),(1,1,1),(1,3,1),(1,1,3),$ $(1,3,3),(2,2,2)$	from 3a.30	(4, 12, 12)
30	B_6	$2\varphi_1+\varphi_6$	12	$3\langle 1\rangle,(0,0,4),(0,0,8),$ $(1,0,4),(0,1,4),(2,0,4),(1,1,4),$ $(0,2,4),(1,1,2),(1,1,6)$	from 3a.31	(4, 4, 24)
31	D_7	$3\varphi_1+\varphi_6$	15	$6\langle 1\rangle,(0,0,0,8),(1,1,1,2),$ $(1,1,1,6),6\langle 13\rangle$	(L)	(4, 4, 4, 24)
32	D_8	φ_7	8	$(2),(8),(12),(14),(18),(20),$ $(24),(30)$	VIN	

[a] Also see representations of $\mathsf{A}_3\simeq\mathsf{D}_3$ in Table 1a

Table 3b

	G	Tensor decompositions and invariants
	B_n,	$S^2\varphi_n=\varphi_n^2+\varphi_{n-3}+\varphi_{n-4}$ $(n\geqq 4)$; $\varphi_n^2=\Lambda^n\varphi_1$
	$3\leqq n\leqq 6$	$\Lambda^2\varphi_n=\varphi_{n-1}+\varphi_{n-2}+\varphi_{n-5}$ $(n\geqq 5)+\varphi_{n-6}$ $(n\geqq 6)$
	D_n,	$S^2\varphi_n=\varphi_n^2+\varphi_{n-4}$ $(n\geqq 4)$; $\varphi_{n-1}^2+\varphi_n^2=\Lambda^n\varphi_1$
	$3\leqq n\leqq 7$	$\Lambda^2\varphi_{n-1}=\Lambda^2\varphi_n=\varphi_{n-2}+\varphi_{n-6}$ $(n\geqq 6)$
		$\varphi_{n-1}\otimes\varphi_n=\Lambda^{n-1}\varphi_1+\varphi_{n-3}+\varphi_{n-5}$ $(n\geqq 5)+\varphi_{n-7}$ $(n\geqq 7)$
1	B_n or D_n	$\theta_1\subset S^2\varphi_1$ or $\varphi_1\otimes\varphi_1$
	B_n,	In 2, 4, and 6 below, $\sigma=\varphi_{n-1}(\mathsf{D}_n)$, $\varphi_n(\mathsf{D}_n)$, or $\varphi_n(\mathsf{B}_n)$
2	$3\leqq n\leqq 6$	$\theta_1\subset\Lambda^n\varphi_1\otimes\sigma^2\subset S^n(n\varphi_1)\otimes[S^2\sigma$ or $\sigma\otimes\sigma]$
3	and	$\theta_1\subset\Lambda^{n-1}\varphi_1\otimes\Lambda^{n-1}\varphi_1\subset S^{n-1}((n-1)\varphi_1)\otimes[S^2(2\varphi_n(\mathsf{B}_n))$ or
	D_n,	$(\varphi_{n-1}\otimes\varphi_n)(\mathsf{D}_n)]$
4	$3\leqq n\leqq 7$	$\theta_1\subset\varphi_{n-2}\otimes\varphi_{n-2}\subset S^{n-2}((n-2)\varphi_1)\otimes S^2(2\sigma)$
5		$\theta_1\subset\varphi_{n-3}\otimes\varphi_{n-3}\subset S^{n-3}((n-3)\varphi_1)\otimes[S^2\varphi_n(\mathsf{B}_n)$, $(\varphi_n\otimes\varphi_n)(\mathsf{B}_n)$, or
		$(\varphi_n\otimes\varphi_{n-1})(\mathsf{D}_n)]$
6		$\theta_1\subset\varphi_{n-4}\otimes\varphi_{n-4}\subset S^{n-4}((n-4)\varphi_1)\otimes[S^2\sigma$ or $\sigma\otimes\sigma]$; $n\geqq 4$
7	D_3	$\theta_1\subset\varphi_1^2\otimes\varphi_1^2\subset[S^2\varphi_1$ or $\varphi_1\otimes\varphi_1]\otimes S^2(\varphi_2^2)$
8	B_3	$\theta_1\subset\Lambda^4\varphi_3\subset S^4(4\varphi_3)$
9	B_4	$\theta_1\subset\varphi_1\otimes\varphi_1\subset S^2\varphi_4\otimes[S^2\varphi_4$ or $\varphi_4\otimes\varphi_4]$
10	D_5	$\theta_1\subset\varphi_1\otimes\varphi_1$ where each φ_1 is in $S^2\sigma$ or $\sigma\otimes\sigma$; $\sigma=\varphi_4$ or φ_5
11		$\theta_1\subset\varphi_4\otimes\varphi_5$
12		$\theta_1\subset\varphi_2\otimes\varphi_2\subset\varphi_1\otimes\varphi_1\otimes\varphi_2\subset\varphi_1\otimes S^2\varphi_4\otimes\varphi_4\otimes\varphi_5$
	B_5	$S^2\varphi_5=\varphi_5^2+\varphi_2+\varphi_1$; $S^3\varphi_5=\varphi_5^3+\varphi_2\varphi_5+\varphi_1\varphi_5+\varphi_5$
		$S^4\varphi_5=\varphi_5^4+\varphi_2\varphi_5^2+\varphi_1\varphi_5^2+\varphi_5^2+\varphi_2^2+\varphi_1^2+\varphi_4+\varphi_3+\theta_1$
	D_6	$S^2\varphi_5=\varphi_5^2+\varphi_2$; $S^3\varphi_5=\varphi_5^3+\varphi_2\varphi_5+\varphi_5$
		$S^4\varphi_5=\varphi_5^4+\varphi_2\varphi_5^2+\varphi_5^2+\varphi_2^2+\varphi_4+\theta_1$
		$S^2\varphi_2\supset\varphi_1^2$; $S^2\varphi_4\supset\varphi_1^2$; $\varphi_5\otimes\varphi_6\supset\varphi_1$
	B_6	$S^2\varphi_6\supset\varphi_2$; $S^4\varphi_6\supset\varphi_1^2+\varphi_1+\theta_1$; $S^6\varphi_6\supset 2\varphi_2$; $S^8\varphi_6\supset\theta_2$
	D_7	$S^2\varphi_6\supset\varphi_3$; $S^4\varphi_6\supset\varphi_1^2$; $S^6\varphi_6\supset\varphi_3$; $S^8\varphi_6\supset\theta_1$
13		$\theta_1\subset\varphi_1^2\otimes\varphi_1^2\subset[S^2\varphi_1$ or $\varphi_1\otimes\varphi_1]\otimes S^4\varphi_6$

the estimate $\deg f_L\geqq(2,2,2,2)$, while we can construct an element of degree $(9,2,2,2)$ in $I(V,L)$. But the only invariant in degree $(2,2,2,2)$ is the square of the generator α in degree $(1,1,1,1)$, and $\operatorname{res}_L\alpha\neq 0$. Using 3.10 and the fact that $\mathbf{C}[\varphi_3]^{\mathsf{A}_6}$ is generated by a form of degree 7 we then obtain $\deg f_L=(9,2,2,2)$.

Table 4a

	G	φ	d	Generators	Method	deg f_L
1	C_n,	$(2n+1)\varphi_1$	$n(2n+1)$	$n(2n+1)\langle 1\rangle$	from 1a.3	
2	$n\geqq 2$	$\varphi_2+3\varphi_1$	$4n-1$	$3\langle 1\rangle, (3n-3)\langle 2\rangle, (2,0,0,0), (3,0,0,0), \ldots, (n,0,0,0)$	from 1a.6	
3		$\varphi_1^2+\varphi_1$	$2n$	$(2,0), (4,0), \ldots, (2n,0), (1,2), (3,2), \ldots, (2n-1,2)$	from 1a.14	
4[a]	C_2	$2\varphi_2+\varphi_1$	4	$(1,1,2), (2,0,0), (1,1,0), (0,2,0)$	from 3a.7	$(1,1,2)$
5	C_3	$2\varphi_2$	8	$(3,0), (2,1), (1,2), (0,3), (2,0), (1,1), (0,2), (2,2)$	from 2a.6	
6		$\varphi_3+2\varphi_1$	5	$(4,0,0), (0,1,1), (2,2,0), (2,1,1), (2,0,2)$	from 2a.10	$(4,2,2)$
7		$2\varphi_3$	7	$(4,0), (3,1), (1,3), (0,4), (1,1), (2,2), (3,3)$	$\simeq$ 5a.8	$(12,12)$
8	C_4	φ_4	6	$(2), (5), (6), (8), (9), (12)$	VIN	

[a] Also see representations of $\mathsf{B}_2\simeq\mathsf{C}_2$ in Table 3a

Table 4b

	G	Tensor decompositions and invariants
1	C_n	$\theta_1\subset \Lambda^2\varphi_1\subset S^2(2\varphi_1)$
2	C_n, $n\geqq 2$	$\theta_1\subset\varphi_2\otimes\varphi_2\subset S^2(2\varphi_1)\otimes$[the generator in $S^j\varphi_2$ of the occurrences of φ_2 in the $S^*(\varphi_2)^{\mathsf{C}_n}$-module $S^*(\varphi_2)$; $j=1,2,\ldots,n-1$]
	C_2	$S^2\varphi_2=\varphi_2^2+\theta_1$; $\quad \Lambda^2\varphi_2=\varphi_1^2$
	C_3	$S^2\varphi_2=\varphi_2^2+\varphi_2+\theta_1$; $\quad S^3\varphi_2=\varphi_2^3+\varphi_2^2+\varphi_1\varphi_3+\varphi_2+\theta_1$ $S^2\varphi_3=\varphi_3^2+\varphi_1^2$; $\quad S^3\varphi_3=\varphi_3^3+\varphi_1^2\varphi_3+\varphi_3$ $S^4\varphi_3=\varphi_3^4+\varphi_1^2\varphi_3^2+\varphi_1^4+\varphi_3^2+\varphi_2^2+\theta_1$

Table 5a

	G	φ	d	Generators	Method	deg f_L
1	G_2	$3\varphi_1$	7	$6\langle 1\rangle, \langle 2\rangle$	from 3a.12	$(2,2,2)$
2		$\varphi_2=\mathrm{Ad}$	2	$(2), (6)$	LR	
3	F_4	$2\varphi_1$	8	$(3,0), (2,1), (1,2), (0,3), (2,0), (1,1), (0,2), (2,2)$	from 5a.5	
4		$\varphi_4=\mathrm{Ad}$	4	$(2), (6), (8), (12)$	LR	
5	E_6	$3\varphi_1$	11	$10\langle 3\rangle, \langle 4\rangle$	$\simeq$ 2a.6	
6		$2\varphi_1+\varphi_5$	11	$5\langle 3\rangle, \langle 4\rangle, 2\langle 5\rangle, 3\langle 6\rangle$	$\simeq$ 2a.7	
7		$\varphi_6=\mathrm{Ad}$	6	$(2), (5), (6), (8), (9), (12)$	LR	
8	E_7	$2\varphi_1$	7	$(4,0), (3,1), (1,3), (0,4), (1,1), (2,2), (3,3)$	$\simeq(2\varphi_5, \mathsf{D}_6)$ (see 3a.28)	$(12,12)$
9		$\varphi_6=\mathrm{Ad}$	7	$(2), (6), (8), (10), (12), (14), (18)$	LR	
10	E_8	$\varphi_1=\mathrm{Ad}$	8	$(2), (8), (12), (14), (18), (20), (24), (30)$	LR	

Table 5b

	G	Tensor decompositions and invariants
1	G_2	$\theta_1 \subset S^2\varphi_1$ or $\varphi_1 \otimes \varphi_1$
2		$\theta_1 \subset \Lambda^3\varphi_1$
	F_4	$S^2\varphi_1 = \varphi_1^2 + \varphi_1 + \theta_1$
		$S^3\varphi_1 = \varphi_1^3 + \varphi_1^2 + \varphi_2 + \varphi_1 + \theta_1$
	E_6	$S^i\varphi_1 = \bigoplus\{\varphi_1^j\varphi_5^k\colon j+2k+3m=i$ for some $m\in\mathbf{Z}^+\}$
3		$\theta_1 \subset S^3\varphi_1$, $S^2\varphi_1\otimes\varphi_1$, or $\varphi_1\otimes\varphi_1\otimes\varphi_1$
4		$\theta_1 \subset \varphi_5\otimes\varphi_5\otimes\varphi_5 \subset S^2\varphi_1\otimes S^2\varphi_1\otimes[S^2\varphi_1$ or $\varphi_5]$
5		$\theta_1 \subset \varphi_1\otimes\varphi_5$
6		$\theta_1 \subset \varphi_5\otimes\varphi_1 \subset [S^2\varphi_1$ or $\varphi_1\otimes\varphi_1]\otimes S^2\varphi_5$
	E_7	$S^2\varphi_1 = \varphi_1^2 + \varphi_6$; $S^3\varphi_1 = \varphi_1^3 + \varphi_1\varphi_6 + \varphi_1$
		$S^4\varphi_1 = \varphi_1^4 + \varphi_1^2\varphi_6 + \varphi_6^2 + \varphi_1^2 + \varphi_2 + \theta_1$

§ 4. Remarks

Recall that if G is finite, then (V, G) is coregular if and only if it is cofree. This phenomenon does not hold for general reductive G. In [13, 14, and 28] one can find sufficient conditions for cofreeness. Below we present conditions which are more general and often easier to check. We also show how to calculate the G-module structure of $\mathbf{C}[V]$, provided that (V, G) is cofree.

Remark 4.1. If $\dim V/G = 1$, then (V, G) is cofree [27]. Suppose that G is connected and semisimple and that $\dim V/G = 2$. V.L. Popov has conjectured that (V, G) is then coregular. The conjecture holds if (V, G) is orthogonal ([29] § 8). In any case, if (V, G) is as above, then (V, G) coregular implies that (V, G) is cofree [27].

Remark 4.2. One can easily show that Lemma 1.1 remains true if one replaces "coregular" by "cofree" throughout.

The following result has surely been noticed many times before. The proof uses only standard facts from commutative algebra.

Proposition 4.3 ([27, 29]). *The following are equivalent:*
(1) *(V, G) is cofree.*
(2) *(V, G) is coregular and* $\operatorname{codim} Z_G(V) = \dim V/G$. □

Here $Z_G(V)$ denotes the zero set of the elements of positive degree in $\mathbf{C}[V]^G$.

Let T be a maximal torus of G, and let μ_0 denote the number of zero weights of V relative to T. Using the Hilbert-Mumford criterion ([21] Ch. 2) one proves:

Proposition 4.4 ([29] § 10). *Let T and μ_0 be as above. Then*

$$\operatorname{codim} Z_G(V) \geqq \dim V - \dim Z_T(V) - \tfrac{1}{2}(\dim G - \operatorname{rank} G).$$

If (V, G) is self dual, then

$$\operatorname{codim} Z_G(V) \geqq \tfrac{1}{2}(\dim V - \dim G + \operatorname{rank} G + \mu_0). \quad \square$$

Remarks 4.5

(1) Using 4.1, 4.3, and 4.4 one can show that most of the representations listed in Lemma 1.2 are cofree. These methods fail only for $\varphi_3(\mathsf{A}_8)$ and $\varphi_2(\mathsf{C}_n)$, $n \geqq 4$. But $(\varphi_2+2\varphi_1, \mathsf{C}_n)$ is cofree by 4.3 and 4.4, and then 4.2 implies that φ_2 is also cofree.

(2) Suppose that $Z_G(V)$ is the closure of a finite number of orbits. Then the mapping $V \to V/G$ has equidimensional fibers [27], hence $\operatorname{codim} Z_G(V) = \dim V/G$. In [11] and [30] it is shown that $\varphi_3(\mathsf{A}_8)$ satisfies the finiteness hypothesis, so $\varphi_3(\mathsf{A}_8)$ is also cofree.

(3) Using 4.2 one can show that most of the representations of Tables 1a thorough 5a are *not* cofree. For example, entries 1a.3, 1a.4, 1a.6, ... have slice representation $(3\varphi_1+\theta, \mathsf{A}_1)$. But $(V, G)=(3\varphi_1, \mathsf{A}_1)$ is not cofree since one can easily show that $\operatorname{codim} Z_G(V)=2<3=\dim V/G$.

One may always choose a minimal graded G-invariant subspace S of $\mathbf{C}[V]$ which generates $\mathbf{C}[V]$ as a $\mathbf{C}[V]^G$-module ([13]). If (V, G) is cofree, then $\mathbf{C}[V] \simeq \mathbf{C}[V]^G \otimes S$ ([13]).

Proposition 4.6 (compare [13]). *Let H be a principal isotropy group of the cofree representation (V, G), and let $(V_1+\theta, H)$ be the slice representation of H where $V_1^H=\{0\}$. Let $\rho=(W, G)$ be an irreducible representation of G, let S be as above, and let m_ρ denote the multiplicity of ρ in S. Then $m_\rho=\dim(\mathbf{C}[V_1]\otimes W^*)^H$. In particular, if (V, G) has generically closed orbits, then $m_\rho=\dim(W^*)^H$.*

Proof. By Luna's slice theorem [18] there is a non-zero $f \in \mathbf{C}[V]^G$ and a G-equivariant isomorphism

$$(V_f \times_{V_f/G} B) \xrightarrow{\sim} (G \times_H V_1) \times B$$

where B is an affine variety. Here $G \times_H V_1$ is the bundle over the homogeneous space G/H obtained from the representation of H on V_1, and $V_f \times_{V_f/G} B$ is the fiber product of V_f and B over V_f/G via the canonical map $V_f \to V_f/G$ and an étale (hence flat) map $B \to V_f/G$. It follows that m_ρ is the multiplicity of ρ in $\mathbf{C}[G \times_H V_1]$. But by a version of Frobenius reciprocity (see the proof of Thm. 8.2 of [31]),

$$\operatorname{Hom}(W, \mathbf{C}[G \times_H V_1])^G \simeq \operatorname{Hom}(W, \mathbf{C}[V_1])^H \simeq (\mathbf{C}[V_1]\otimes W^*)^H. \quad \square$$

We say that (V, G) is *prehomogeneous* if there is an open G-orbit in V, and we say that (V, G) is *strongly prehomogeneous* if in addition the complement of the open orbit has codimension $\geqq 2$. If G is semisimple and $\mathbf{C}[V]^G=\mathbf{C}$, then (V, G) is strongly prehomogeneous ([20] Cor. 2 and Lemma 4).

Corollary 4.7. *Let V_1, H, ρ, etc. be as in 4.6. Suppose that (V_1, H) is strongly prehomogeneous with open orbit Hx. Then $m_\rho=\dim(W^*)^{H_x}$.*

Proof. The hypotheses imply that $\mathbf{C}[G \times_H V_1] \simeq \mathbf{C}[G \times_H Hx]$, and using Frobenius reciprocity we see that

$$\operatorname{Hom}(W, \mathbf{C}[G \times_H V_1])^G \simeq \operatorname{Hom}(W, \mathbf{C}[G/H_x])^G \simeq (W^*)^{H_x}. \quad \square$$

Remark 4.8. Let (V, G) be cofree, and let S be as above. One can sometimes use properties of S to prove the coregularity of a representation $(V+V_1, G)$. For example, consider $(V+V_1, G)=(\varphi_1\varphi_{n-1}+\varphi_1, \mathsf{A}_{n-1})=1\text{a}.20$. From the slice representation of a maximal torus $(\mathbf{C}^*)^{n-1}$ one sees that all bihomogeneous elements of $\mathbf{C}[V+V_1]^G$ occur in degrees $(k, l\,n)$; $k, l\in\mathbf{Z}^+$. Since $S^{ln}(\varphi_1^*)=(\varphi_1^*)^{ln}$, $\mathbf{C}[V+V_1]^G$ is generated by $\mathbf{C}[V]^G$ and the contractions of occurrences of φ_1^{ln} in S with $(\varphi_1^*)^{ln}\subset\mathbf{C}[V_1]$. But φ_1^{ln} occurs with multiplicity one in S, and the generator is in degree $l\binom{n}{2}$ ([13] Thm. 0.11). Hence $\mathbf{C}[V+V_1]^G$ is generated by $\mathbf{C}[V]^G$ and an invariant of degree $\left(\binom{n}{2}, n\right)$, and $\mathbf{C}[V+V_1]^G$ is a regular ring.

Remark 4.9. One can often generate new coregular representations from old ones by "increasing the group" and using Proposition 3.1. For example, $\rho=(2\varphi_2+2\varphi_1, \mathsf{A}_{2k-1})$ is coregular for all $k\geqq 2$, and adding the A_1 factor of $\rho'=(2\varphi_2+\varphi_1\otimes\varphi_1', \mathsf{A}_{2k-1}\times\mathsf{A}_1')$ does not change the ring of invariants. Applying 3.1 to ρ' one obtains the coregular representation $(2S^2\mathbf{C}^k+\mathbf{C}^k, SL_k^{\sim})$ where $SL_k^{\sim}$ denotes the elements of GL_k of determinant ± 1.

Finally, to help "explain" why the principal isotropy groups of all the representations in 1.2.1 are connected, we offer the following:

Proposition 4.10. *Let G be connected and semisimple, and let H be a principal isotropy group of (V, G). Assume that (V, G) has generically closed orbits,* $\dim V/G=1$, *and that $Z_G(V)$ is non-singular in codimension one. Then H is connected.*

Proof. If $\dim V<3$, one easily sees that $\operatorname{Im}(G\to GL(V))$ is trivial, so we may assume that $\dim V\geqq 3$. From the homotopy sequence of the fibration $H\to G\to G/H$ we see that $\pi_1(G/H)$ maps onto $\pi_0(H)$, so it suffices to prove that $\pi_1(G/H)$ is trivial.

Let f be a form generating $\mathbf{C}[V]^G$, $e=\deg f$. Let $\mathbf{C}^*$ act on V with weight 1 and on $\mathbf{C}$ with weight e. Luna's slice theorem shows that we have a fibration

$$G/H \longrightarrow V_f \xrightarrow{\ f\ } \mathbf{C}-\{0\},$$

and quotienting by $\mathbf{C}^*$ we see that $(G/H)/\mathbf{Z}_e\simeq P(V)_f$ where $P(V)$ is the projective space $(V-\{0\})/\mathbf{C}^*$ and $P(V)_f=V_f/\mathbf{C}^*$. It thus suffices to show that $\pi_1(P(V)_f)\simeq\mathbf{Z}_e$.

If L is a generic hyperplane in V, then $L\cap Z_G(V)$ will be non-singular in codimension one. We may choose L so that (providing $\dim V\geqq 4$) we have $\pi_1(P(V)_f)\simeq\pi_1(P(L)_f)$ ([9]). By induction $\pi_1(P(V)_f)\simeq\pi_1(P(M)_f)$ where M is a 3-dimensional subspace of V and the zero set of $f|_M$ has at most an isolated singularity at 0. Hence $f|_M=h^j$ where $j\in\mathbf{Z}^+$ and $h\in\mathbf{C}[M]$ has at most an isolated singularity at 0. Then by a theorem of Zariski (see [3]), $\pi_1(P(M)_f)\simeq\mathbf{Z}_{e/j}$. But $\pi_1(P(M)_f)$ has quotient $\mathbf{Z}_e$, hence $j=1$. $\square$

Remark 4.11. Suppose that $\dim V/G=1$. Then $Z_G(V)$ is non-singular in codimension one if and only if it is normal ([22] p. 391). If $Z_G(V)$ contains finitely many G-orbits of the same dimension whose complement has codimension $\geqq 2$ in $Z_G(V)$ (e.g. (V, G) is exceptional—see [26]), then $Z_G(V)$ is normal.

§ 5. Appendix

The completeness of our classification of coregular representations depends upon the completeness of the classification announced by Kac et al. in [12]. Because of this we have verified their results. Not all of the calculations involved are straightforward, and below we indicate how we handled most of the non-routine cases, e.g. those involving the exceptional groups. We first recall and illustrate the use of the methods of [12].

Let T denote a copy of $\mathbf{C}^*$ which is contained in a simple 3-dimensional subgroup of G. Then for any representation $\rho=(W,G)$ the non-zero weights of (W,T) occur in pairs $\pm\mu$, and we let $q(\rho)$ or $q(W)$ denote the number of such pairs. Assume that $(V^T, N_G(T)/T)$ has finite principal isotropy groups (abbreviated FPIG). Then the argument of Example 1.4 shows that there is a closed orbit Gv, $v\in V^T$, such that rank $G_v=1$. Hence the identity component $(G_v)^0$ of G_v is either T or a simple 3-dimensional subgroup of G. Let ψ denote the slice representation of G_v. Then $q(\psi)=q(V)-q(\operatorname{Ad} G)$ if $(G_v)^0=T$, else $q(\psi)=q(V)-q(\operatorname{Ad} G)+1$. A careful analysis shows that ψ (hence (V,G)) is not coregular if either $q(\psi)\geqq 4$, $q(\psi)=3$ and $(G_v)^0=T$, or $q(\psi)=2$ and G_v centralizes T ([12]). To sum up, we have the following result:

Theorem 5.1 ([12]). *Let T be as above. Suppose that*

(1) *$(V^T, N_G(T)/T)$ has FPIG.*

(2) *$q(V)-q(\operatorname{Ad} G)\geqq 3$, or $q(V)-q(\operatorname{Ad} G)=2$ and there is a closed orbit Gv such that G_v centralizes T, $(G_v)^0=T$.*

Then (V,G) is not coregular. □

Remark 5.2. Suppose that (V,G) is irreducible and not coregular, and suppose that G is connected and simple, rank $G\geqq 2$. Then there is almost always a T satisfying 5.1.1 such that $q(V)-q(\operatorname{Ad} G)\geqq 3$. The only exceptions are $(\varphi_1\varphi_2, \mathsf{A}_3)$ (here $q(V)-q(\operatorname{Ad} G)=2$ and $G_v=T$), and $(\varphi_1^3, \mathsf{A}_3)$ (in this case one finds a finite closed isotropy group with non-coregular slice representation; see [12]).

Note that one may replace the group $N_G(T)/T$ of 5.1.1 by N, where N denotes $(N_G(T)/T)^0$ or a finite cover thereof. Then to apply Theorem 5.1 one must be able to show that the representations (V,T) and (V^T, N) are "large". The following result (a special case of which is stated in [12]) allows one to estimate $(V, N_G(T)^0)$, hence also (V,T) and (V^T,N).

Theorem 5.3 ([16]). *Let G be connected, and let H be a connected reductive algebraic subgroup of G. Suppose that $\rho_1,\rho_2,\dots,\rho_m$ are irreducible representations of G such that (ρ_i,H) contains the irreducible representation τ_i of H with multiplicity n_i, $i=1,\dots,m$. Then $(\rho_1\rho_2\dots\rho_m, H)$ contains $\tau_1\tau_2\dots\tau_m$ with multiplicity at least* $\max\{n_1,\dots,n_m\}$. □

Remark 5.4. In our applications $H=N_G(T)^0$ for some $T=\mathbf{C}^*\subset \mathsf{A}_1'\subset G$, and it usually turns out that $H\subset H'=N_G(\mathsf{A}_1')^0$. It sometimes gives better results or is notationally more convenient to apply 5.3 to estimate the restriction of a representation of G to H', and then to restrict to H.

Example 5.5. Let $T \subset \mathsf{A}_1' \subset \mathsf{A}_n = G$ where $(\varphi_1(G), \mathsf{A}_1') = \varphi_1' + \theta_{n-1}$. Let $j \in \mathbf{Z}^+$, $j \geqq 1$, let $H = N_G(T)^0$, and let $H' = N_G(\mathsf{A}_1')^0$. Then $H = (T \times \mathsf{A}_{n-2} \times \mathbf{C}^*)/\mathbf{Z}_{2n-2}$ and $H' = (\mathsf{A}_1' \times \mathsf{A}_{n-2} \times \mathbf{C}^*)/\mathbf{Z}_{2n-2}$ where $(\varphi_1(G), H') = \varphi_1' \otimes \nu_{n-1} + \varphi_1 \otimes \nu_{-2}$. By 5.3, $(\varphi_1^j(G), H')$ contains each term $(\varphi_1')^k \otimes (\varphi_1)^{j-k} \otimes \nu_{kn+k-2j}$ with multiplicity at least 1, $0 \leqq k \leqq j$. Restricting the representations $(\varphi_1')^k$ to $T \subset \mathsf{A}_1'$ one obtains an estimate of $(\varphi_1^j(G), H)$. Since $\varphi_1^j = S^j \varphi_1$, one can see that the estimate of 5.3 gives the exact answer in this case.

Example 5.6. Let φ be a non-zero irreducible (perhaps trivial) representation of $G = \mathsf{A}_n$, $n \geqq 9$. We show that $\varphi\varphi_5$ is not coregular: Let T, A_1', and $H' = (\mathsf{A}_1' \times \mathsf{A}_{n-2} \times \mathbf{C}^*)/\mathbf{Z}_{2n-2}$ be as in Example 5.5. Then $N = \mathsf{A}_{n-2} \times \mathbf{C}^*$ is a finite cover of $(N_G(T)/T)^0$. If $\varphi = \varphi_2, \varphi_3, \ldots, \varphi_{n-1}, \varphi_1^k$, or φ_n^k, $k \geqq 2$, then (φ^T, N) contains a factor $\pi \otimes \nu_p + \rho \otimes \nu_{-r}$ where π and ρ are irreducible representations of A_{n-2} and $p, r \geqq 0$. In particular, $(\varphi_5^T, N) = \varphi_3 \otimes \nu_{2n-8} + \varphi_5 \otimes \nu_{-10}$. If $\varphi = \varphi_1^{a_1} \ldots \varphi_n^{a_n}$ then it follows from 5.3 that $((\varphi\varphi_5)^T, N)$ contains a factor $\sigma\varphi_3 \otimes \nu_s + \tau\varphi_5 \otimes \nu_{-t}$ where $s, t \geqq 8$ (one has estimates $s, t \geqq 10$ if a_1 and a_2 are not 1). Using the tables of [1] and [7] one sees that both $\sigma\varphi_3$ and $\tau\varphi_5$ have non-trivial rings of invariants and that $\sigma\varphi_3 + \tau\varphi_5$ has FPIG. It follows that $\sigma\varphi_3 \otimes \nu_s + \tau\varphi_5 \otimes \nu_{-t}$ and $((\varphi\varphi_5)^T, N)$ have FPIG.

Now $(\varphi_5(\mathsf{A}_n), \mathsf{A}_1') = \binom{n-1}{4}\varphi_1' + \theta$ and $(\mathrm{Ad}\, \mathsf{A}_n, \mathsf{A}_1') = (\varphi_1')^2 + (2n-2)\varphi_1' + \theta$. Using 5.3 one sees that $q(\varphi\varphi_5) \geqq q(\varphi_5)$, so $q(\varphi\varphi_5) - q(\mathrm{Ad}\, \mathsf{A}_n) \geqq \binom{n-1}{4} - (2n-1) \geqq 53$ (since $n \geqq 9$). Thus $\varphi\varphi_5$ is not coregular. A similar argument establishes that $(\varphi\varphi_i, \mathsf{A}_n)$ and $(\varphi\varphi_{n-i}, \mathsf{A}_n)$ are not coregular, $5 \leqq i \leqq (n+1)/2$.

In a few cases one needs to use tensor product identities to improve upon the estimates of 5.3. (All the required identities can be found in [15].) For example, suppose that $(V, G) = (\varphi_3^2, \mathsf{B}_3)$ and that $T \subset \mathsf{A}_1'$ where $(\varphi_3(\mathsf{B}_3), \mathsf{A}_1') = 2\varphi_1' + \theta_4$. Then 5.1.1 holds and $q(\mathrm{Ad}\, G) = 7$. The estimate $q(V) \geqq 10$ is sufficient to establish that (V, G) is not coregular, while 5.3 only shows that $q(V) \geqq 9$. However, using the identity $S^2\varphi_3 = \varphi_3^2 + \theta_1$ one can see that $q(V) = 11$.

Sometimes one has to explicitly exhibit invariants in order to show that 5.1.1 holds:

Example 5.7. Let $(V, G) = (\varphi_1\varphi_2, \mathsf{A}_5)$, and let T and N be as in 5.6. Using 5.3 (resp. the identity $\varphi_1 \otimes \varphi_2 = \varphi_1\varphi_2 + \varphi_3$) one can show that $((\varphi_1\varphi_2)^T, N)$ contains (resp. equals) $(2\varphi_1 \otimes \nu_6 + \varphi_1\varphi_2 \otimes \nu_{-6}, \mathsf{A}_3 \times \mathbf{C}^*)$. Since $(\varphi_1\varphi_2, \mathsf{A}_3)$ has FPIG, to show that $((\varphi_1\varphi_2)^T, N)$ has FPIG it suffices to find an invariant of $(2\varphi_1 + \varphi_1\varphi_2, \mathsf{A}_3)$ of degree (a, b, c), $a + b > c$. Now Theorem 0.11 of [13] implies that $S^3(\varphi_1\varphi_2(\mathsf{A}_2)) \supset \varphi_2^3$, and then Theorem 1 of [17] implies that $S^3(\varphi_1\varphi_2(\mathsf{A}_3)) \supset \varphi_2^3\varphi_3$. Thus there are invariants of $(2\varphi_1 + \varphi_1\varphi_2, \mathsf{A}_3)$ of degrees (4, 3, 3) and (3, 4, 3), and (V^T, N) has FPIG.

We now comment upon the T's needed for the classification of [12]. We also mention any unusual aspects of the computations involved.

$G = \mathsf{A}_n$, $n \geqq 2$: The T of Examples 5.5 through 5.7 suffices for almost all irreducible non-coregular representations. Let $T' = \mathbf{C}^* \subset \mathsf{A}_n$ where $(\varphi_1(\mathsf{A}_n), \mathbf{C}^*) = k\nu_1 + k\nu_{-1} + \theta_m$, $m = k+1$, $k+2$, or $k+3$. As pointed out in [12], it is necessary or easier to use T' for the representations $\varphi_3(\mathsf{A}_n)$, $n \geqq 9$; $\varphi_4(\mathsf{A}_n)$, $n \geqq 8$, n even;

and $\varphi_1^3(\mathsf{A}_n)$, $n \geqq 6$. To this list we would add $\varphi_1\varphi_2(\mathsf{A}_n)$, $n \geqq 6$. It takes a little work to show that $(V^{T'}, N_G(T')/T')$ has FPIG in the cases of $\varphi_3(\mathsf{A}_9)$, $\varphi_3(\mathsf{A}_{10})$, $\varphi_3(\mathsf{A}_{12})$, and $\varphi_4(\mathsf{A}_8)$. Using T and T' one can handle all the irreducible non-coregular representations of G except for $\varphi_1^3(\mathsf{A}_n)$, $3 \leqq n \leqq 5$. Using $T'' = \mathbf{C}^* \subset \mathsf{A}_n$ where $(\varphi_1(\mathsf{A}_n), \mathbf{C}^*) = \nu_1 + \nu_{-1} + \nu_2 + \nu_{-2} + \theta$ one can handle $\varphi_1^3(\mathsf{A}_4)$ and $\varphi_1^3(\mathsf{A}_5)$.

$G = \mathsf{B}_n$, $n \geqq 3$ or D_n, $n \geqq 4$: Here one always can use $T \subset \mathsf{A}_1' \subset G$ where $(\varphi_1(G), \mathsf{A}_1') = 2\varphi_1' + \theta$. Note that in this case (and almost all other cases to follow) N is semisimple, so the fact that (V^T, N) has FPIG usually follows quite easily from the results of [1] and [7].

$G = \mathsf{C}_n$, $n \geqq 2$: Mainly one uses $T \subset \mathsf{C}_1' \subset \mathsf{C}_n$ where $(\varphi_1(\mathsf{C}_n), \mathsf{C}_1') = \varphi_1' + \theta$. For $\varphi_3(\mathsf{C}_4)$ one instead arranges that $(\varphi_1(\mathsf{C}_4), \mathsf{C}_1') = 3\varphi_1' + \theta_2$, and for $(\varphi_2^3, \mathsf{C}_2)$ one arranges that $(\varphi_1(\mathsf{C}_2), \mathsf{C}_1') = 2\varphi_1'$.

If G is an exceptional group we proceed as follows: Let β denote the highest root of G. We choose $T \subset \mathsf{A}_1'$ where A_1' is the 3-dimensional simple subgroup of G corresponding to β. Let N denote the connected subgroup of G whose Dynkin diagram consists of the simple roots of G orthogonal to β (see Ch. II §5 of [5]). Then $N_G(T)^0$ (resp. $N_G(\mathsf{A}_1')^0$) has finite cover $N \times T$ (resp. $N \times \mathsf{A}_1'$). We list the groups N and describe the decomposition of the adjoint representation and lowest dimensional representation of G with respect to $N \times \mathsf{A}_1'$. (One can determine these decompositions from the tables of [5]. We also show how one can read off the decompositions from the Dynkin diagram of G.) We list identities which allow one to express any basic representation in terms of the representations whose decompositions are known. One can then easily obtain enough information to apply Theorem 5.1. We were unable to find enough identities for E_8, but we get around this difficulty as indicated below.

$G = \mathsf{G}_2$: $N = \mathsf{A}_1$ where $(\varphi_1(\mathsf{G}_2), \mathsf{A}_1 \times \mathsf{A}_1') = \varphi_1 \otimes \varphi_1' + \varphi_1^2$. $\Lambda^2 \varphi_1 = (\mathrm{Ad}\ \mathsf{G}_2 = \varphi_2) + \varphi_1$.

$G = \mathsf{F}_4$: $N = \mathsf{C}_3$ where $(\varphi_1(\mathsf{F}_4), \mathsf{C}_3 \times \mathsf{A}_1') = \varphi_1 \otimes \varphi_1' + \varphi_2$; $(\mathrm{Ad}\ \mathsf{F}_4 = \varphi_4, \mathsf{C}_3 \times \mathsf{A}_1') = \mathrm{Ad}\ \mathsf{C}_3 + \mathrm{Ad}\ \mathsf{A}_1' + \varphi_3 \otimes \varphi_1'$. $\Lambda^2 \varphi_1 = \varphi_2 + \varphi_4$; $\Lambda^2 \varphi_4 = \varphi_3 + \varphi_4$.

$G = \mathsf{E}_6$: $N = \mathsf{A}_5$ where $(\varphi_1(\mathsf{E}_6), \mathsf{A}_5 \times \mathsf{A}_1') = \varphi_1 \otimes \varphi_1' + \varphi_4$; $(\mathrm{Ad}\ \mathsf{E}_6 = \varphi_6, \mathsf{A}_5 \times \mathsf{A}_1') = \mathrm{Ad}\ \mathsf{A}_5 + \mathrm{Ad}\ \mathsf{A}_1' + \varphi_3 \otimes \varphi_1'$. $\Lambda^2 \varphi_1 = \varphi_2$; $\Lambda^3 \varphi_1 = \varphi_3$; $\varphi_4 = \varphi_2^*$; $\varphi_5 = \varphi_1^*$.

$G = \mathsf{E}_7$: $N = \mathsf{D}_6$ where $(\varphi_1(\mathsf{E}_7), \mathsf{D}_6 \times \mathsf{A}_1') = \varphi_1 \otimes \varphi_1' + \varphi_6$; $(\mathrm{Ad}\ \mathsf{E}_7 = \varphi_6, \mathsf{D}_6 \times \mathsf{A}_1') = \mathrm{Ad}\ \mathsf{D}_6 + \mathrm{Ad}\ \mathsf{A}_1' + \varphi_5 \otimes \varphi_1'$. $\Lambda^2 \varphi_1 = \varphi_2 + \theta_1$; $\Lambda^3 \varphi_1 = \varphi_3 + \varphi_1$; $\Lambda^4 \varphi_1 = \varphi_4 + \varphi_2 + \theta_1$; $\Lambda^2 \varphi_6 = \varphi_5 + \varphi_6$; $\varphi_1 \otimes \varphi_6 = \varphi_1\varphi_6 + \varphi_7 + \varphi_1$. (To compute the decomposition of φ_7, we first computed the decomposition of $\varphi_1 \otimes \varphi_6$. Using 5.3 we estimated the decomposition of $\varphi_1\varphi_6$, and then using dimension arguments and the last identity above we found that

$$(\varphi_7(\mathsf{E}_7), \mathsf{D}_6 \times \mathsf{A}_1') = \varphi_1\varphi_5 + \varphi_3 \otimes \varphi_1' + \varphi_1 \otimes \varphi_1' + \varphi_6 \otimes (\varphi_1')^2.)$$

$G = \mathsf{E}_8$: $N = \mathsf{E}_7$ where $(\mathrm{Ad}\ \mathsf{E}_8 = \varphi_1, \mathsf{E}_7 \times \mathsf{A}_1') = \mathrm{Ad}\ \mathsf{E}_7 + \mathrm{Ad}\ \mathsf{A}_1' + \varphi_1 \otimes \varphi_1'$. We determine two of the components of the restrictions of $\varphi_2, \ldots, \varphi_8$ from the Dynkin diagram of E_8, and these components are (with one exception) enough to establish the needed non-coregularity results.

Let $\omega_1, \ldots, \omega_8$ denote the highest weights of $\varphi_1(\mathsf{E}_8), \ldots, \varphi_8(\mathsf{E}_8)$, and let $\alpha_1, \ldots, \alpha_8$ denote the corresponding simple roots. Then $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$. Changing the canonical inner product on the weights by a scalar, we can arrange that $(\omega_i, \omega_j) = \delta_{ij}$. Each ω_i then restricts to the highest weight of $(\varphi_{i-1} \otimes (\varphi_1')^{k_i}, \mathsf{E}_7 \times \mathsf{A}_1')$ where $k_i = (\omega_i, \beta)$. For $1 \leqq i \leqq 7$, let ω_i' denote $\omega_i - \alpha_1 - \alpha_2 - \cdots - \alpha_i$. Then each ω_i' restricts to a highest weight of $(\varphi_i(\mathsf{E}_8), \mathsf{E}_7 \times \mathsf{A}_1')$, and the corresponding representations are, respectively, $\varphi_1 \otimes \varphi_1'$, $\varphi_2 \otimes (\varphi_1')^2$, $\varphi_3 \otimes (\varphi_1')^3$, $\varphi_4 \otimes (\varphi_1')^4$, $\varphi_5 \varphi_7 \otimes (\varphi_1')^5$, $\varphi_6 \varphi_7 \otimes (\varphi_1')^3$, and $\varphi_7 \otimes \varphi_1'$. From $\omega_8' = \omega_8 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_8$ one similarly obtains that $\varphi_5 \otimes (\varphi_1')^2 \subset (\varphi_8(\mathsf{E}_8), \mathsf{E}_7 \times \mathsf{A}_1')$. Our estimates so far are sufficient to handle all irreducible non-coregular representations of E_8 except for φ_7. In this case we only have the estimate $(V^T, N) \supset \mathrm{Ad}\, N$. But using Theorem 5.3 and the identity $S^2 \varphi_1 = \varphi_1^2 + \varphi_7 + \theta_1$ one can compute that $(\varphi_7(\mathsf{E}_8), \mathsf{E}_7 \times \mathsf{A}_1') = \varphi_2 + \theta_1 + \varphi_1 \otimes \varphi_1' + \varphi_7 \otimes \varphi_1' + \varphi_6 \otimes (\varphi_1')^2$, and Theorem 5.1 applies.

Our method above estimated that $(\mathrm{Ad}\, \mathsf{E}_8, \mathsf{E}_7 \times \mathsf{A}_1')$ contains $\mathrm{Ad}\, \mathsf{A}_1' + \varphi_1 \otimes \varphi_1'$. Clearly the factor $\mathrm{Ad}\, \mathsf{E}_7$ must also be present, and then a dimension argument yields our claimed decomposition for $\mathrm{Ad}\, \mathsf{E}_8$. One can similarly prove the decompositions claimed for G_2, F_4, etc.

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