Toric Varieties

Reference: Fulton.

Fastest def: a toric variety X is a normal variety X containing an algebraic torus T as a dense open subset, such that the action of T on itself extends to an action $T \times X \to X$.

Ex: (1) $(\mathbb{C}^*)^k \times (\mathbb{C})^{n-k}$. $(\mathbb{C}^*)^n \hookrightarrow (\mathbb{C}^*)^k \times (\mathbb{C})^{n-k}$.

(2) $\mathbb{V}(\{XY - Z^2\})$. $(\mathbb{C}^*)^2 \hookrightarrow \mathbb{V}(\{XY-Z^2\})$

$$(t_1, t_2) \longmapsto (t_1^2, t_2^2, t_1 t_2)$$

This def doesn't tell the full story. Toric varieties give a link between convex geometry and algebraic geometry.

Convex geometry

Let

$$N := \mathbb{Z}^n \subseteq \mathbb{R}^n =: N_{\mathbb{R}}, \qquad M := (\mathbb{Z}^n)^* = \mathrm{Hom}(N, \mathbb{Z}) \subseteq (\mathbb{R}^n)^* =: M_{\mathbb{R}}$$

A convex polyhedral cone in $N_{\mathbb{R}}$ is a set

$$\sigma = \{ r_1 v_1 + \dots + r_s v_s \ : \ r_i \geq 0 \}$$

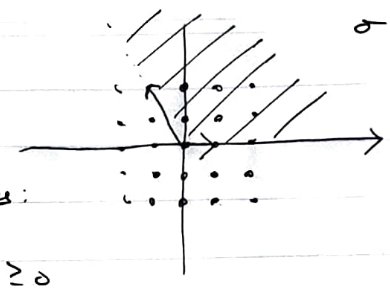
for some finite set of vectors $v_1, \dots, v_s \in N_{\mathbb{R}}$. The dual cone $\sigma^\vee$ is

$$\sigma^\vee = \{ u \in M_{\mathbb{R}} \ : \ \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma \}.$$

Ex: (1) $\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}} \{ e_1, \dots, e_k \}$ for some $1 \leq k \leq n$.

$$\longrightarrow \sigma^\vee = \mathrm{span}_{\mathbb{R}_{\geq 0}} \{ e_1^*, \dots, e_k^*, \pm e_{k+1}^*, \dots, \pm e_n^* \}.$$

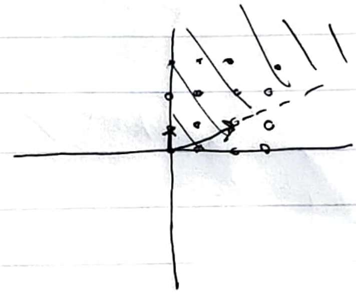
(2) $\sigma = \operatorname{span}_{\mathbb{R}_{\geq 0}} \{ e_1, \ -e_1 + 2e_2 \}$.



$(a,b) \in \sigma^\vee$ is defined by the inequalities:

$\langle (a,b), e_1 \rangle = a \geq 0, \quad \langle (a,b), -e_1 + 2e_2 \rangle = -a + 2b \geq 0$

$\rightsquigarrow \sigma^\vee = \operatorname{span}_{\mathbb{R}_{\geq 0}} \{$



A cone is called <u>rational</u> if its generators can be taken from $N = \mathbb{Z}^n$. When σ is rational, so is $\sigma^\vee$.

<u>Fact</u> (Gordan's lemma): When σ is rational, $\sigma^\vee \cap M$ is a finitely generated semigroup.

When we have a semigroup S, we can form the semigroup algebra $\mathbb{C}[S]$, which has a basis of formal symbols

$$\{ \chi^u : u \in S \},$$

with multiplication determined by addition in S:

$$\chi^u \cdot \chi^v = \chi^{u+v}.$$

For a cone σ, we look at $S_\sigma := \sigma^\vee \cap M$ and form $\mathbb{C}[S_\sigma]$.
~~The affine toric variety~~ When $\sigma = \{0\}$, $\sigma^\vee = M_{\mathbb{R}}$ so $S_{\{0\}} = \sigma^\vee \cap M = M_{\mathbb{R}} \cap M = M$. As a semigroup, $S_{\{0\}}$ is generated by $\pm e_1^*, \dots, \pm e_n^*$, so

$$\mathbb{C}[M] = \mathbb{C}[\chi^{e_1^*}, \chi^{-e_1^*}, \dots, \chi^{e_n^*}, \chi^{-e_n^*}] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

$$\hookrightarrow \chi^{e_i^*} =: x_i,$$

Upshot: $\mathbb{C}[S_\sigma]$ can be thought of as a subalg of Laurent polys.

The <u>affine toric variety</u> U_σ is

$$U_\sigma = \mathrm{Specm}(\mathbb{C}[S_\sigma]).$$

<u>Ex</u>: (0) $\sigma = \{0\} \rightsquigarrow \mathbb{C}[S_\sigma] = \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$

$\rightsquigarrow U_\sigma = (\mathbb{C}^\times)^n.$

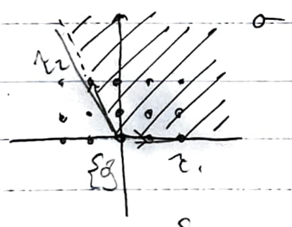
(1) $\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{e_1, \dots, e_k\}$

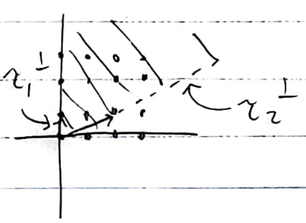
$\rightsquigarrow \sigma^\vee = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{e_1^*, \dots, e_k^*, \pm e_{k+1}^*, \dots, \pm e_n^*\}$

$\rightsquigarrow \mathbb{C}[S_\sigma] = \mathbb{C}[X_1, \dots, X_k, X_{k+1}^{\pm 1}, \dots, X_n^{\pm}]$

$\rightsquigarrow U_\sigma = \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$

(2)





$\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{e_1, -e_1 + 2e_2\}$ $\qquad \sigma^\vee = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{e_2, 2e_1 + e_2\}$

$\rightsquigarrow S_\sigma = \mathrm{span}_{\mathbb{Z}_{\geq 0}}\{e_2, 2e_1 + e_2, e_1 + e_2\}$

$\rightsquigarrow \mathbb{C}[S_\sigma] = \mathbb{C}[\chi^{e_2}, \chi^{2e_1+e_2}, \chi^{e_1+e_2}] = \mathbb{C}[X_2, X_1^2 X_2, X_1 X_2]$

$\cong \mathbb{C}[X, Y, Z]/(XY - Z^2)$

The action of the torus: When S_σ is generated by $\{a_1, \dots, a_r\} \subseteq M$, $T = (\mathbb{C}^*)^n$ acts on U_σ by

$$t \cdot (x_1, \dots, x_r) = (t^{a_1} x_1, \dots, t^{a_r} x_r).$$

In example (2), this is:

$$(t_1, t_2) \cdot (X, Y, Z) = (t_2 X, t_1^2 t_2 Y, t_1 t_2 Z).$$

~~Th~~ General fact: torus orbits are in bijection with the faces of σ:

$$\sigma \longleftrightarrow T \cdot (0,0,0)$$

$$\tau_1 \longleftrightarrow T \cdot (1,0,0)$$

$$\tau_2 \longleftrightarrow T \cdot (0,1,0)$$

$$\{0\} \longleftrightarrow T \cdot (1,1,1)$$