Chapter 1

Introduction

1.1 Overview

We study a family of algebro-geometric objects called character varieties. These are spaces of homomorphisms $\pi_1(\Sigma) \to G$, where $\pi_1(\Sigma)$ is the fundamental group of an orientable surface and G is a connected reductive group. Character varieties are related to various topics in mathematics and physics, including the Yang-Mills equations, Hitchin's equations, the $P = W$ conjecture and mirror symmetry [Sim91, Sim92, Hau13, BPGPNT14]. In general, character varieties are not well-understood. However, there is significant progress when $G = \mathrm{GL}_n$, due to the seminal work of Hausel–Letellier–Rodriguez-Villegas [HRV08, HLRV11] and subsequent work [Let15, Mel18, HMMS22, Bal23, LRV23]. In light of the Langlands program, which aims to connect number theory, representation theory and algebraic geometry, the goal of this thesis is to understand character varieties for general reductive G.

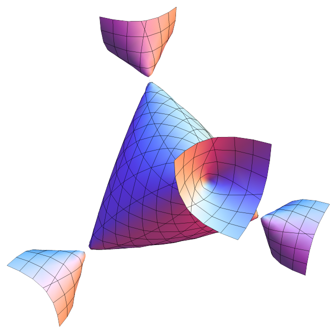


Figure 1.1: The Cayley cubic is closely related to an SL_2-character variety [CFLO16].

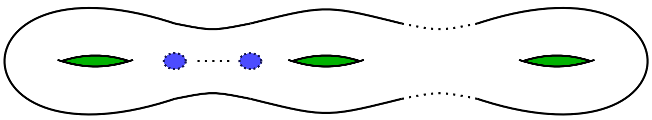
The novelty of this thesis is our type-independent approach; i.e., we do not make assumptions about the type of G and our proofs are not case-by-case in the type of G. To our knowledge, only three papers address character varieties associated to reductive groups beyond type A [Cam17, BK22, KNP23]. In [Cam17], the author considers the case when $G = \mathrm{Sp}_{2n}$ and Σ has one puncture. Since the surface is punctured, it is advantageous to select an auxiliary piece of data; to each puncture, we associate a conjugacy class of G. At the puncture, the conjugacy class is semisimple and regular. In [BK22], the authors consider a general reductive group and an orientable surface with no punctures. Rather than study the character variety directly, the authors study a closely related space called the character stack. In [KNP23], the authors add punctures to the setting of [BK22]; they choose both semisimple regular and unipotent regular conjugacy classes, and they study the character variety directly. In this thesis, we study the same character varieties but only semisimple regular conjugacy classes are chosen.

To study character varieties, we investigate their cohomology, obtaining useful topological invariants such as dimension, Euler characteristic, and the number of irreducible components. We access these invariants through techniques of arithmetic geometry. Specifically, we count points of character varieties over finite fields. Our work relies on the Weil conjectures, a jewel of 20th century mathematics. Their statements are complicated, but they teach us an important philosophy: cohomological information can be obtained by counting points over finite fields. A formula first revealed by Frobenius links the number of points of the character variety over finite fields to the representation theory of finite reductive groups (see §1.3.2). This provides a clear strategy to analyse character varieties: use the representation theory of finite reductive groups to evaluate Frobenius' formula and extract cohomological information from our expression. This is the strategy employed in this thesis.

The rest of this introduction is as follows. In §1.2, we define the character varieties that we study. In §1.3, we explain why we count points over finite fields and what cohomological information this yields. In §1.4, we state the main theorems of this thesis, and we discuss further directions in §1.5.

1.2 Character varieties

Let Σ be an orientable surface with genus $g \geq 0$ and $n \geq 0$ punctures, depicted as follows.



This surface has the fundamental group

$$\pi_1(\Sigma) \simeq \left\langle a_1, b_1, \dots, a_g, b_g, y_1, \dots, y_n \;\middle|\; [a_1, b_1] \cdots [a_g, b_g] y_1 \cdots y_n = 1 \right\rangle$$

and therefore a group homomorphism $\pi_1(\Sigma) \to G$ is determined by the images of the generators, subject to the relation of the fundamental group. Thus, we have a bijection

$$\mathrm{Hom}(\pi_1(\Sigma), G) \simeq \left\{ (A_1, B_1, \dots, A_g, B_g, Y_1, \dots, Y_n) \in G^{2g+n} \;\middle|\; [A_1, B_1] \cdots [A_g, B_g] Y_1 \cdots Y_n = 1 \right\}.$$

Reductive groups carry the structure of a variety, so this space of homomorphisms does too. Moreover, supposing that $n \geq 1$, we choose conjugacy classes $\mathcal{C} = (C_1, \dots, C_n)$ in G and define

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G) &:= \{ f \in \mathrm{Hom}(\pi_1(\Sigma), G) \mid f(y_i) \in C_i \} \\ &\simeq \left\{ (A_1, B_1, \dots, A_g, B_g, Y_1, \dots, Y_n) \in \mathrm{Hom}(\pi_1(\Sigma), G) \;\middle|\; Y_i \in C_i \right\}. \end{aligned}$$

We call $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)$ the representation variety. Recall that two representations $\pi_1(\Sigma) \to G$ are equivalent if they are conjugate by an element of G. Under the identification $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G) \subseteq G \times \cdots \times G$, the representation variety admits an action of G by simultaneous conjugation in each entry. Thus, it is natural to consider the collection of orbits $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)/G$. The structure of this orbit space is subtle; it does not necessarily inherit the algebro-geometric structure of G. However, there are two common ways to equip an algebro-geometric structure to $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)/G$:

(i) We consider the geometric-invariant-theory (GIT) quotient, denoted $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G) /\!\!/ G$. Historically, this was the first solution to the orbit-space problem, due to Mumford [Mum65]. Over algebraically closed fields, the points of the GIT quotient are in bijection with the closed orbits.

(ii) We consider the stack quotient, denoted $[\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)/G]$. Stacks are a higher algebraic object defined in the wake of Grothendieck. These solve the orbit-space problem by keeping track of more data than the GIT quotient does. In a sense, the stack quotient is the 'correct' quotient, but its construction requires some work.

The GIT quotient $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)/\!\!/G$ is called the character variety and has a close relationship to the problems from other areas stated earlier. In general, its point-count is difficult since the action of G is not necessarily free. On the other hand, the stack quotient $[\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)/G]$ is called the character stack and its point-count is essentially equal to that of $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)$. This point-count is still difficult but it is generally easier than the point-count of the character variety.

1.3 Counting points

1.3.1 Polynomial count and rational count

Our strategy to analyse character varieties is to prove that they are polynomial count. We say that a variety X defined over $\mathbb{F}_q$ is polynomial count with counting polynomial $\|X\| \in \mathbb{Q}[t]$ if

$$|X(\mathbb{F}_{q^n})| = \|X\|(q^n) \text{ for all } n \geq 1.$$

More generally, we say that X is potentially polynomial count if it becomes polynomial count after passing to a finite extension of $\mathbb{F}_q$. Fine cohomological information is encoded in the coefficients of counting polynomials (see [LRV23, §2.2] for details). From these coefficients, we can extract geometric information. For example,

(i) The dimension of X is the degree of $\|X\|$,

(ii) The Euler characteristic of X is given by $\|X\|(1)$, and

(iii) The number of irreducible components of X is the leading coefficient of $\|X\|$.[1]

[1] The leading coefficient is actually the number of irreducible components of maximum dimension. This is because the counting polynomial captures information about the compactly supported cohomology of X.

It is also worthwhile asking if a counting polynomial is palindromic. A polynomial is palindromic if its coefficients are the same when read backwards and forwards (after placing the terms in ascending/descending order of degree). If X is smooth, projective and polynomial count then Poincaré duality implies that $\|X\|$ is palindromic. The character varieties in this project are affine but we find that they have palindromic counting polynomials too. This suggests that character varieties obey a 'curious' form of Poincaré duality [Hau13, §5.1].

The story above can be extended to a larger class of algebraic objects. We will need this extended story in order to analyse the character stack. We say that an algebraic stack of finite type $\mathfrak{X}$ defined over $\mathbb{F}_q$ is rational count with counting function $\|\mathfrak{X}\| \in \mathbb{Q}(t)$ if

$$|\mathfrak{X}(\mathbb{F}_{q^n})| = \|\mathfrak{X}\|(q^n) \text{ for all } n \geq 1,$$

and potentially rational count stacks are defined analogously. For varieties, rational count and polynomial count are equivalent [LRV23, Lemma 2.8].

1.3.2 Counting points on character varieties

In general, counting points over finite fields is not an easy problem. However, in our setting, there is a formula due to Frobenius telling us how to point-count on the representation variety [HLRV11, Proposition 3.1.4]. Assume that the centraliser of each C_i in G is connected (we will soon specialise G and C_i so that this is always the case). This is done so that $C_i(\mathbb{F}_q) \subseteq G(\mathbb{F}_q)$ is a single $G(\mathbb{F}_q)$-conjugacy class [GM20, §2.7.1].[2] Then Frobenius' formula tells us

$$\frac{|\operatorname{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} = \sum_{\chi \in \operatorname{Irr}(G(\mathbb{F}_q))} \left(\frac{|G(\mathbb{F}_q)|}{\chi(1)} \right)^{2g-2} \prod_{i=1}^{n} \frac{\chi(C_i(\mathbb{F}_q))}{\chi(1)} |C_i(\mathbb{F}_q)|,$$

where $\operatorname{Irr}(G(\mathbb{F}_q))$ is the set of irreducible complex characters of the finite group $G(\mathbb{F}_q)$.[3]

We see that evaluating Frobenius' formula is a problem in the world of the representation theory of finite reductive groups. Note that we do not need to impose reductivity on the algebraic group G

[2] In other words, the geometric conjugacy class of an element of C_i is the same as its rational conjugacy class.

[3] In the numerator, $\operatorname{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)(\mathbb{F}_q)$ is the collection of elements $(A_1, B_1, \ldots, A_g, B_g, Y_1, \ldots, Y_n) \in G(\mathbb{F}_q)^{2g+n}$ such that $Y_i \in C_i(\mathbb{F}_q)$ and $[A_1, B_1] \cdots [A_g, B_g] Y_1 \cdots Y_n = 1$.

to make use of Frobenius' formula. However, if we were not in the reductive setting, then the task of evaluating Frobenius' formula leaves our reach and we would lose connections with other areas of mathematics and physics. Moreover, since G is reductive, we may apply powerful techniques of Deligne–Lusztig theory to evaluate this sum.

1.4 Results

Let G be a connected split reductive group over $\mathbb{F}_q$ with connected centre $Z = Z(G)$ and split maximal torus $T \subseteq G$. Let Σ be an orientable surface with genus $g \geq 0$ and $n \geq 1$ punctures and fix conjugacy classes $\mathcal{C} = (C_1, \dots, C_n)$ in G.

We denote the representation variety by $\mathbf{R} := \mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma), G)$. Recall that G acts on $\mathbf{R}$ by simultaneous conjugation, with Z acting trivially. Thus, we form the character variety

$$\mathbf{X} := \mathbf{R} /\!\!/ (G/Z) = \mathbf{R} /\!\!/ G$$

and the character stack

$$\mathfrak{X} := [\mathbf{R}/(G/Z)].$$

For the rest of this thesis, we make the following assumptions:

Assumption 1. *We assume that*

(i) Each C_i is the conjugacy class of a strongly regular element $S_i \in T$, and

(ii) The product $S_1 \cdots S_n$ lies in $[G, G]$.

Strongly regular is meant in the sense of [Ste65]; i.e., $C_G(S_i) = T$. The first assumption ensures that $C_i(\mathbb{F}_q)$ is a single $G(\mathbb{F}_q)$-conjugacy class, as promised, and the second assumption is necessary for $\mathbf{R}$ to be non-empty. Our main theorem is the following:

Theorem 2. *Away from finitely many primes, the character stack $\mathfrak{X}$ is potentially rational count with an expression given in Theorem 41, and if $g \geq 1$ then $\mathfrak{X}$ is potentially polynomial count.*

We exclude finitely many primes so that the representation theory of $G(\mathbb{F}_q)$ is well-behaved and the representation variety is smooth. The primes that we exclude depend only on the root datum of G. Specifically, we assume the following:

Assumption 3. *We assume that* $\operatorname{char}(\mathbb{F}_q)$ *is a very good prime for G not dividing $2\check{h}$, where $\check{h}$ is the dual Coxeter number defined in* [Kac90, Chapter 6].[4] *Overall, this means that*

(i) $p \neq 2$,

(ii) $p \nmid r+1$ *if the root system of G has a component of type A_r or C_r,*

(iii) $p \nmid 2r-1$ *if the root system of G has a component of type B_r,*

(iv) $p \nmid r-1$ *if the root system of G has a component of type D_r,*

(v) $p \neq 3$ *if the root system of G has a component of exceptional type, and*

(vi) $p \neq 5$ *if the root system of G has a component of type E_8.*

We sketch the proof of Theorem 2. It is well-known that $|\mathfrak{X}(\mathbb{F}_q)|$ equals $|\mathbf{R}(\mathbb{F}_q)|/|(G/Z)(\mathbb{F}_q)|$ [Beh93, Theorem 2.5.1]. Therefore, we need to point-count on the representation variety using Frobenius' mass formula. We rewrite this formula in terms of data ('types') arising naturally in light of Lusztig's Jordan decomposition of $\operatorname{Irr}(G(\mathbb{F}_q))$. Using Deligne-Lusztig theory, we reduce the determination of $|\mathbf{R}(\mathbb{F}_q)|$ to the determination of certain character sums previously studied in [KNP23, §5]. The novelty here is that we obtain a new simplified expression for these character sums.

In order to analyse counting functions, we make another assumption:

Assumption 4. *We assume that* $2g-2+n \geq 1$, *in addition to our requirement that* $g \geq 0$ *and* $n \geq 1$.

This new assumption excludes the cases $(g,n) = (0,1)$ and $(0,2)$. These cases are straightforward since $\pi_1(\Sigma) \simeq 1$ or $\pi_1(\Sigma) \simeq \mathbb{Z}$ so $\mathbf{R}$ is only a point, if non-empty. We exclude these cases because they do not make sense in the following theorems and are easily studied by hand.

[4] The prime being very good for G is a crucial assumption used in many places throughout this thesis, but the condition involving the dual Coxeter number is only needed to ensure smoothness of $\mathbf{R}$, c.f. [KNP23, §2.2].

From the counting function, we extract the following information:

Theorem 5. *The character stack has dimension equal to*

$$\dim(\mathfrak{X}) = (2g-2+n)\dim G + 2\dim Z - n \cdot \operatorname{rank} G$$

and number of components equal to

$$|\pi_0(\mathfrak{X})| = |\pi_0(Z(\check{G}))|,$$

where $Z(\check{G})$ is the centre of the Langlands dual group $\check{G}$.

So far, we have only analysed the character stack and it is not clear how to relate $|\mathfrak{X}(\mathbb{F}_q)|$ and $|\mathbf{X}(\mathbb{F}_q)|$. To analyse the character variety, we will choose conjugacy classes generically:

Definition 6. *We say that the tuple $\mathcal{C} = (C_1, \ldots, C_n)$ of conjugacy classes of G is generic if*

$$\prod_{i=1}^{n} X_i \notin [L, L]$$

for all proper Levi subgroups $L \subset G$ containing T and for all $X_i \in C_i \cap L$.

This notion of choosing conjugacy classes generically is a generalisation of the one seen in [HLRV11]. In this paper, the authors consider $G = \mathrm{GL}_n$, where a generic choice of conjugacy classes implies that G/Z acts freely on $\mathbf{R}$. Thus, in their setting, it is straightforward to show that $\mathfrak{X}$ and $\mathbf{X}$ have the same number of points over finite fields. In our setting, we have the following theorem:

Theorem 7. *If $\mathcal{C}$ is generic then*

(i) The action of G/Z on $\mathbf{R}$ has finite stabilisers,

(ii) $\mathfrak{X}$ is a smooth Deligne-Mumford stack, and

(iii) $\mathfrak{X}$ and $\mathbf{X}$ have the same number of points over finite fields.

Combining Theorems 2, 5 and 7 yields the following:

Corollary 8. *If $\mathcal{C}$ is generic then $\mathbf{X}$ is potentially polynomial count, its counting polynomial is equal to that of $\mathfrak{X}$, and the dimension and number of components of $\mathbf{X}$ and $\mathfrak{X}$ are the same.*

Unlike Theorem 5, we allow for $g = 0$ above. We can also compute the Euler characteristic of $\mathbf{X}$:

Theorem 9. *Suppose that $\mathcal{C}$ is generic.*

(i) If $g > 1$ or $g = 1$ and $\dim Z > 0$ then $\chi(\mathbf{X}) = 0$,

(ii) If $g = 1$ and $\dim Z = 0$ then $\chi(\mathbf{X})$ is non-zero with an expression given in Theorem 43, and

(iii) If $g = 0$ and $n \geq 3$ then $\chi(\mathbf{X})$ is non-zero with an expression given in Theorem 43.

Furthermore, we prove that the character variety exhibits curious Poincaré duality:

Theorem 10. *If $\mathcal{C}$ is generic then $\|\mathbf{X}\|$ is a palindromic polynomial; i.e.,*

$$\|\mathbf{X}\|(q) = q^{\dim \mathbf{X}} \|\mathbf{X}\|(1/q).$$

1.5 Further directions

We conclude by discussing research directions that warrant further attention:

(i) Associated to $\mathbf{X}$ is the mixed Poincaré polynomial $H_c(\mathbf{X}; q, t)$. Computing such a polynomial provides important information about the Frobenius' action on the (compactly supported) cohomology of $\mathbf{X}$ [LRV23, §2.2]. When $\mathbf{X}$ is polynomial count, the counting polynomial $\|\mathbf{X}\|(q)$ is equal to the specialisation $H_c(\mathbf{X}; q, -1)$ [LRV23, Theorem 2.9]. Therefore, proving that $\mathbf{X}$ is polynomial count and obtaining an explicit expression for $\|\mathbf{X}\|(q)$ is a step towards a formula for $H_c(\mathbf{X}; q, t)$. For instance, when $G = \mathrm{GL}_n$, there is a conjectural formula for $H_c(\mathbf{X}; q, t)$ [HLRV11, Conjecture 1.2.1] and a known formula when $G = \mathrm{GL}_2$ [HRV08, Theorem 1.1.3].[5]

[5] In [HRV08, HLRV11], the authors consider the mixed Hodge polynomial rather than the mixed Poincaré polynomial.

(ii) There is an additive analogue to our situation. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and recall that G acts on $\mathfrak{g}$ by the adjoint action $g \cdot x := \mathrm{Ad}(g)(x)$. Fix adjoint orbits $O_1, \ldots, O_n$ of semisimple regular $x_1, \ldots, x_n \in \mathfrak{g}$ and define the additive representation variety

$$\mathbf{A} := \left\{ (a_1, b_1, \ldots, a_g, b_g, y_1, \ldots, y_n) \in \mathfrak{g}^{2g+n} \,\middle|\, [a_1, b_1] + \cdots + [a_g, b_g] + y_1 + \cdots + y_n = 0,\ y_i \in O_i \right\}.$$

Note that $\mathbf{A}$ inherits the adjoint action of G, so we form the additive character variety $\mathbf{Y} := \mathbf{A} /\!\!/ G$. It is known that $\mathbf{Y}$ is polynomial count when G is of arbitrary type [Gia24]. Moreover, this variety is conjectured to have deep links to the (multiplicative) character variety $\mathbf{X}$. Specifically, it is conjectured that the cohomology of $\mathbf{Y}$ is isomorphic to the 'pure part' of the cohomology of $\mathbf{X}$ [HLRV11, Remark 1.3.2]. At the level of polynomials, this means that $\|\mathbf{Y}\|(q)$ is equal to the 'pure part' of $H_c(\mathbf{X}; q, t)$ [HLRV11, §1.2.1] and provides constraints for what $H_c(\mathbf{X}; q, t)$ can be.

(iii) The expression for our counting function is in terms of well-known representation-theoretic data. There are several computer algebra systems which calculate this data, allowing one to quickly compute our counting function. These systems include the Chevie project [Mic15] and Magma [BCP97]. It would be interesting to implement the counting polynomials $\|\mathbf{X}\|(q)$ and $\|\mathbf{Y}\|(q)$ in these systems, especially in light of the purity conjecture above.

(iv) We wish to remove the split assumption on our conjugacy classes. Under this assumption, we require an understanding of principal series representations, which are controlled by a well-known Hecke algebra [CR81, Theorem 11.25]. To remove this assumption, we must instead understand the irreducible constituents of Deligne–Lusztig characters, which are conjecturally controlled by cyclotomic Hecke algebras [GM20, §A.6].

(v) We expect that our results hold when $Z(G)$ is disconnected. One reason for expecting generalisation is that our results are in line with those of [Cam17]. In this paper, the author considered an Sp_{2n}-character variety with one regular semisimple and generic conjugacy class. Our work does not address this character variety since the centre of Sp_{2n} is disconnected.

Chapter 2

Representation theory of reductive groups

Frobenius' formula involves the irreducible characters of a finite reductive group. Therefore, we dedicate this section to recalling the relevant representation theory. The primary references are [Car93, GM20, DM20].

Throughout this chapter, suppose that G is a connected split reductive group over $\mathbb{F}_q$ with connected centre $Z(G) = Z$. Fix a split maximal torus $T \subseteq G$ so that (G,T) has root datum $(X,\Phi,\check{X},\check{\Phi})$ and Weyl group W. Then G has Langlands dual $\check{G}$, which is the connected split reductive group over k with split maximal torus $\check{T} := \mathrm{Spec}(k[X])$, where $(\check{G},\check{T})$ has root datum $(\check{X},\check{\Phi},X,\Phi)$.

2.1 Order polynomials and degree polynomials

It is well-known that the orders of connected split reductive groups over $\mathbb{F}_q$ and the degrees of their irreducible characters are polynomials in q [GM20, §1.6, §2.3]. That is, $|G(\mathbb{F}_q)| = \|G\|(q)$ for some polynomial $\|G\|(q)$ and, given $\chi \in \mathrm{Irr}(G(\mathbb{F}_q))$, we have $\chi(1) = \|\chi\|(q)$ for some polynomial $\|\chi\|(q)$.

To point-count later, we need an explicit description of $\|G\|(q)$ coming from a well-known formula for $|G(\mathbb{F}_q)|$ [GM20, Theorem 1.6.7]. To this end, let B be a Borel subgroup of G containing T, let U be the unipotent radical of B and write $P_W(q) := \sum_{w\in W} q^{\ell(w)}$ for the Poincaré polynomial of W. Then

$$|G(\mathbb{F}_q)| = |U(\mathbb{F}_q)| \cdot |T(\mathbb{F}_q)| \cdot |(G/B)(\mathbb{F}_q)| = q^{|\Phi^+|}(q-1)^{\dim(T)} P_W(q) =: \|G\|(q).$$

We can also describe the order polynomials of subgroups of G. If H is a connected split reductive subgroup of G containing T then (H,T) has root datum $(X,\Phi,\check{X},\check{\Phi})$ for some closed root subsystem $\Psi\subseteq\Phi$. Recall that a root subsystem $\Psi\subseteq\Phi$ is closed if $\alpha+\beta\in\Phi$ implies $\alpha+\beta\in\Psi$ for all $\alpha,\beta\in\Psi$. Writing $W(H)=W(\Psi)$ for the Weyl group of H, we have

$$|H(\mathbb{F}_q)| = q^{|\Psi^+|}(q-1)^{\dim(T)}P_{W(\Psi)}(q) =: \|H\|(q).$$

2.2 Lusztig's Jordan decomposition

Lusztig's Jordan decomposition allows us to parameterise $\mathrm{Irr}(G(\mathbb{F}_q))$ in terms of simpler data. Specifically, to each irreducible $G(\mathbb{F}_q)$-character, we associate the conjugacy class of a semisimple element $x\in\check{G}$ and a so-called unipotent character of the centraliser subgroup $\check{G}_x$.

We say that a $G(\mathbb{F}_q)$-character is unipotent if it appears as a summand in the Deligne–Lusztig character $R_T^G 1$ for some maximal torus $T\subseteq G$, not necessarily the split maximal T we fixed earlier (but this will end up being the only case we are interested in). The set of unipotent characters is denoted $\mathrm{Uch}(G(\mathbb{F}_q))$. If a character appears as a summand in $R_T^G 1$, where T is the split maximal torus fixed earlier, then the unipotent character is called principal.

Remarkably, in contrast to all irreducible characters, the subcollection of unipotent characters is understood and well-behaved. Specifically, after excluding finitely many primes, unipotent characters are independent of the ground field, they admit straightforward parameterisations and their degrees are known [GM20, Theorem 4.5.8]. This is due to Lusztig's theory of symbols in classical type [Lus77] and case-by-case analysis in exceptional type [Lus84].

In general, the semisimple centraliser $\check{G}_x$ is not a connected reductive subgroup of $\check{G}$ containing $\check{T}$. However, this is the case if the derived subgroup of $\check{G}$ is simply connected [Car93, Theorem 3.5.4, Theorem 3.5.6]. This is implied if G has connected centre and $\mathrm{char}(\mathbb{F}_q)$ is large enough [DL76, Proposition 5.23]. Specifically, we need $\mathrm{char}(\mathbb{F}_q)$ large enough to ensure that Z is smooth. This happens if $X/\langle\Phi\rangle$ is torsion-free [DL76, p. 131], which is ensured since p is very good for G.

We now state Lusztig's Jordan decomposition:

Theorem 11 (Theorem 4.23 of [Lus84]). *If G has connected centre then there is a bijection*

$$\operatorname{Irr}(G(\mathbb{F}_q)) \longleftrightarrow \bigsqcup_{\substack{[x]\subseteq \check{G}(\mathbb{F}_q) \\ x\, semisimple}} \operatorname{Uch}(\check{G}_x(\mathbb{F}_q))$$

such that if $\chi \in \operatorname{Irr}(G(\mathbb{F}_q))$ is paired with $\rho \in \operatorname{Uch}(\check{G}_x(\mathbb{F}_q))$ then $\chi(1)$ and $\rho(1)$ are related by

$$\frac{|G(\mathbb{F}_q)|}{\chi(1)} = q^{r(x)}\frac{|\check{G}_x(\mathbb{F}_q)|}{\rho(1)}$$

where $r(x) := |\Phi(\check{G})^+| - |\Phi(\check{G}_x)^+|$ is the difference between the number of positive roots.

Our goal is to express the quantity $|G(\mathbb{F}_q)|/\chi(1)$ on the left in terms of data independent of q. The quantity $q^{r(x)}|\check{G}_x(\mathbb{F}_q)|/\rho(1)$ on the right is close to achieving this goal; clearly $r(x)$ is independent of q and we noted earlier that $\rho(1)$ is too. Next, we deal with the semisimple centraliser $\check{G}_x$.

2.3 Pseudo-Levi subgroups and endoscopy groups

In general, if $\check{\Phi}$ is the root system of $\check{G}$ and $x \in \check{G}$ is a semisimple element, then $\check{G}_x$ has root data $\Psi \subseteq \check{\Phi}$ and Ψ is a closed root subsystem of $\check{\Phi}$. However, due to the Deriziotis Criterion [Hum95, §2.15], more can be said about the root data of semisimple centralisers. To state the criterion, we need to distinguish a subcollection of closed root subsystems.

Definition 12. *Suppose that Φ is irreducible with simple roots Δ and highest root α_0. We say that a closed root subsystem $\Psi \subseteq \Phi$ is a pseudo-Levi subsystem if $\Psi = w.\langle S\rangle$, where $w \in W$ and S is a proper subset of $\tilde{\Delta} := \Delta \sqcup \{-\alpha_0\}$. Here, $\langle S\rangle$ is the closed subsystem* $\operatorname{span}_{\mathbb{Z}}(S) \cap \Phi$.

Theorem 13 (Deriziotis Criterion). *Let $H \subseteq G$ be a connected reductive subgroup containing T with root system $\Psi \subseteq \Phi$. Then H is the centraliser of a semisimple element in G if and only if Ψ is a pseudo-Levi subsystem of Φ. I omitted the assumption that G is simple and simply connected.*

This leads to the notion of pseudo-Levi subgroups:

Definition 14. *We say that a connected split reductive subgroup $L \subseteq G$ containing T is a pseudo-Levi subgroup of G if its root system is a pseudo-Levi subsystem.*

In Definition 6, we refered to the so-called Levi subgroups, which are defined as follows:

Definition 15. *A pseudo-Levi subsystem Ψ is a Levi subsystem if $\Psi = w.\langle S\rangle$ for some $S \subseteq \Delta$. Then a pseudo-Levi subgroup is a Levi subgroup if its root system is a Levi subsystem.*

We distinguish another important family of pseudo-Levi subgroups, which will be key to our point-count in Chapter 3.

Definition 16. *A pseudo-Levi subsystem is isolated if it is not contained in a proper Levi subsystem. Then an isolated pseudo-Levi subgroup is a pseudo-Levi subgroup whose pseudo-Levi subsystem is isolated (equivalently, it is not contained in a proper Levi subgroup).*

Often, we will work with the so-called endoscopy subsystems of Φ, rather than the pseudo-Levi subsystems of $\check{\Phi}$, so that we can work in G and Φ rather than $\check{G}$ and $\check{\Phi}$.

Definition 17. *A closed root subsystem $\Psi \subseteq \Phi$ is an endoscopy system if $\check{\Psi}$ is a pseudo-Levi subsystem of $\check{\Phi}$. Isolated endoscopy systems and isolated endoscopy groups are defined analogously.*

This aligns with the notion of endoscopy groups of G:

Definition 18. *A connected split reductive group K contained in G (not necessarily a subgroup of G) is an endoscopy group of G if the dual group $\check{K}$ is the centraliser of a semisimple element in $\check{G}$.*

Note that an endoscopy group of G must contain T since its dual group contains $\check{T}$.

2.4 Types

We introduce the notion of the type of a $G(\mathbb{F}_q)$-character. This is data independent of the ground field which remembers enough information about χ to evaluate expressions appearing in Frobenius' formula. The idea of a type follows naturally from Theorem 11. Our definition generalises and unifies those used in [HLRV11, Cam17] to count points on GL_n- and Sp_{2n}-character varieties.[1]

We define types as follows. Consider the collection of pairs (L,ρ) where L is an endoscopy group of G and ρ is a unipotent character of $L(\mathbb{F}_q)$. Since W acts on the root system of G, it also acts on the collection of pairs (L,ρ).

Definition 19. *A G-type is an orbit of this action, denoted* $\tau = [(L,\rho)] = [L,\rho]$.

The set of G-types is denoted $\mathcal{T}(G)$ and is advantageous for two reasons:

(i) Theorem 11 implies that there is a type map

$$\mathrm{Irr}(G(\mathbb{F}_q)) \to \mathcal{T}(G), \quad \chi \mapsto \tau = [L,\rho]$$

where L and ρ are determined by Lusztig's Jordan decomposition.

(ii) The set of types only depends on the root datum of G, in particular, it is independent of q.

The fibre a type τ under the type map is denoted $\mathrm{Irr}(G(\mathbb{F}_q))_\tau$ and we will rewrite Frobenius' mass formula in terms of these fibres. In light of Theorem 11, if $\chi \in \mathrm{Irr}(G(\mathbb{F}_q))$ has type $\tau = [L,\rho]$ then

$$\frac{|G(\mathbb{F}_q)|}{\chi(1)} = q^{|\Phi(G)^+|-|\Phi(L)^+|}\frac{|L(\mathbb{F}_q)|}{\rho(1)} = q^{|\Phi(G)^+|-|\Phi(L)^+|}\frac{\|L\|(q)}{\|\rho\|(q)}.$$

The right-hand side of this equality only depends on τ. Moreover, it is actually a polynomial in q since $\|\rho\|(q)$ always divides $\|L\|(q)$ [GM20, Remark 2.3.27] and $\Phi(G) \supseteq \Phi(L)$. Thus, defining

$$\|\tau\|(q) := q^{r(G)-r(L)}\frac{\|L\|(q)}{\|\rho\|(q)},$$

Frobenius' mass formula becomes

$$\frac{|\mathbf{R}(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} = \frac{1}{|T(\mathbb{F}_q)|^n}\sum_{\tau\in\mathcal{T}(G)} \|\tau\|(q)^{2g-2+n} S_\tau(q),$$

where

$$S_\tau(q) := \sum_{\chi\in\mathrm{Irr}(G(\mathbb{F}_q))_\tau} \prod_{i=1}^n \chi(C_i(\mathbb{F}_q)).$$

[1] The unification is explained in Appendix C.

The equality $|C_i(\mathbb{F}_q)|/|G(\mathbb{F}_q)| = 1/|T(\mathbb{F}_q)|$ follows from the orbit-stabiliser theorem applied to the conjugation action $G(\mathbb{F}_q) \curvearrowright G(\mathbb{F}_q)$.

We have reduced the determination of $|\mathbf{R}(\mathbb{F}_q)|$ to the determination of $S_\tau(q)$. The polynomiality of $S_\tau(q)$ and explicitly evaluating it is addressed in Chapter 3. Any reader may skip to Chapter 3 now and refer to §2.5 and §2.6 when necessary.

2.5 Principal series representations

To evaluate S_τ, we must evaluate $G(\mathbb{F}_q)$-characters at strongly regular elements of T. In §3.4, this evaluation is perform and involves the principal series characters of $G(\mathbb{F}_q)$. To this end, we recall the relevant properties of these characters now.

A principal series character is an irreducible summand of the Deligne-Lusztig character

$$R_T^G\,\theta = \operatorname{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}\tilde{\theta}$$

for some $\theta \in T(\mathbb{F}_q)$. Above, $\tilde{\theta}$ is the usual extension of $\theta \in T(\mathbb{F}_q)^\vee$ from $T(\mathbb{F}_q)$ to $B(\mathbb{F}_q)$. The fact that Deligne-Lusztig induction reduces to plain induction when T is split is proven in [Car93, Proposition 7.2.4]. A key observation is principal series characters behave in the following manner:

Proposition 20 (Corollary 6.3 of [DL76]). *Either $R_T^G\,\theta$ and $R_T^G\,\theta'$ share no irreducible constituents (up to isomorphism), or θ and θ' are related by the natural action $W \curvearrowright T(\mathbb{F}_q)^\vee$ and $R_T^G\,\theta \simeq R_T^G\,\theta'$ as $G(\mathbb{F}_q)$-representations.*

A deeper understanding of principal series character is afforded by the Hecke algebra, denoted $\mathcal{H}(G,\theta)$. This is the unital associative $\mathbb{C}$-algebra of functions $f\colon G(\mathbb{F}_q) \to \mathbb{C}$ satisfying $f(bgb') = \tilde{\theta}(b)f(g)\tilde{\theta}(b')$ for all $g \in G(\mathbb{F}_q)$ and $b, b' \in B(\mathbb{F}_q)$, with convolution product

$$(ff')(g) := \sum_{xy=g} f(x)f'(y).$$

The utility of the Hecke algebra is as follows. Irreducible $\mathcal{H}(G,\theta)$-characters are in bijection with irreducible constituents of $R_T^G\,\theta$. Moreover, the multiplicity of an irreducible consistituent of $R_T^G\,\theta$

is recorded by the dimension of the associated $\mathcal{H}(G,\theta)$-character. Furthermore, it is well-known (e.g., via Tits' deformation theorem) that $\mathcal{H}(G,\theta)$ is isomorphic to the group algebra $\mathbb{C}[W_\theta]$, where W_θ is the stabiliser subgroup of θ under the action $W \curvearrowright T(\mathbb{F}_q)^\vee$, and this isomorphism preserves isomorphism classes of irreducible representations.

We summarise these two parameterisations in the following proposition:

Proposition 21 (Theorem 11.25 of [CR81] and Theorem 7.4.6 of [GP00])**.** *There are bijections*

$$\left\{\begin{array}{c}\textit{Irreducible}\\ \textit{constituents of } R_T^G\,\theta\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Irreducible}\\ \textit{representations of } \mathcal{H}(G,\theta)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Irreducible}\\ \textit{representations of } W_\theta\end{array}\right\}$$

such that if $\chi \in R_T^G\,\theta$ has image $\zeta \in \mathrm{Irr}(\mathcal{H}(G,\theta))$ and $\phi \in \mathrm{Irr}(W_\theta)$, then

$$\langle \chi, R_T^G\,\theta\rangle = \dim(\zeta) = \dim(\phi).$$

We will see in §3.4 that an important collection of G-types are those arising from principal series representations. We take a moment to determine such types. A principal series representation $\chi \in R_T^G\,\theta$ can be viewed as an element of $\mathrm{Uch}(\check{G}_\theta(\mathbb{F}_q))$ using the identification $T(\mathbb{F}_q)^\vee \simeq \check{T}(\mathbb{F}_q)$ [DL76, §5] and the bijections

$$R_T^G\,\theta \longleftrightarrow \mathrm{Irr}(W_\theta) \longleftrightarrow R_{\check{T}}^{\check{G}_\theta}\,1 \subseteq \mathrm{Uch}(\check{G}_\theta(\mathbb{F}_q)).$$

Then the semisimple centraliser subgroup $\check{G}_\theta$ is a connected split reductive group with pseudo-Levi subsystem $\Psi \subseteq \check{\Phi}$. Then picking $\chi \in R_T^G\,\theta$ is the same as picking $\rho \in R_{\check{T}}^{\check{G}_\theta}\,1 \subseteq \mathrm{Uch}(\check{G}_\theta(\mathbb{F}_q))$. This motivates the definition of a principal type:

Definition 22. *A G-type $[L,\rho]$ is principal if ρ is principal.*

The collection of principal G-types is denoted $\mathcal{T}_0(G)$. The above discussion shows a character is a principal series character if and only if its type is principal.

We have developed the general theory in order to elucidate the general situation and connections to [HLRV11, Cam17]. However, we conclude by remarking that one can define a restricted type map

$$\{\text{Principal series representations of } G(\mathbb{F}_q)\} \to \mathcal{T}_0(G)$$

without appealing to Lusztig's Jordan decomposition of $\mathrm{Irr}(G(\mathbb{F}_q))$. This is because principal unipotent characters of $\check{G}_\theta$ are in canonical bijection with irreducible characters of W_θ, so long as the centre of G is connected. This is the approach taken in [KNWG24].

2.6 Alvis–Curtis duality

It has been noted as early as [HRV08, Hau13] that Alvis–Curtis duality is responsible for the palindromicity of the counting polynomials in this thesis. Alvis–Curtis duality was originally defined in [Alv79, Cur80] and one may view it as a generalisation of the relationship between the trivial and Steinberg representations of $G(\mathbb{F}_q)$. For us, its main utility is that it yields a useful expression for $\|\chi\|(1/q)$ [Alv82, Corollary 3.6]. We recall the necessary properties of Alvis–Curtis duality now:

Proposition 23 (§3.4 of [GM20])**.** *There is an involution D_G on the space of complex-valued $G(\mathbb{F}_q)$-class functions with the following properties:*

(i) If $\chi \in \mathrm{Irr}(G(\mathbb{F}_q))$ then $D_G(\chi) \in \mathrm{Irr}(G(\mathbb{F}_q))$ (up to a sign),

(ii) If $x \in \check{G}(\mathbb{F}_q)$ is semisimple and $\rho \in \mathrm{Uch}(\check{G}_x(\mathbb{F}_q))$ then $D_{\check{G}_x}(\rho) \in \mathrm{Uch}(\check{G}_x(\mathbb{F}_q))$, and

(iii) We have $\|D_G(\chi)\|(q) = q^{|\Phi(G)^+|}\|\chi\|(1/q)$.

For later use in §3.9, we record a corollary relating the types of χ and $D_G(\chi)$ when χ is a principal series representation of $G(\mathbb{F}_q)$:

Corollary 24. *Suppose that $\chi \in R_T^G\,\theta$ has type $\tau = [L,\rho]$ and ρ is matched with $\phi \in \mathrm{Irr}(W(L))$ according to the bijection in Proposition 21. Then $D_G(\chi)$ has type $[L, D_L(\rho)]$ where $D_L(\rho)$ is matched with $\phi \otimes \mathrm{sgn} \in \mathrm{Irr}(W(L))$. Hence, since $\mathrm{sgn} \otimes \mathrm{sgn} = \mathrm{triv}$, there is an involution (and therefore bijection)*

$$D_G \colon \mathcal{T}_0(G) \to \mathcal{T}_0(G), \quad [L,\rho] \mapsto [L, D_L(\rho)].$$

In particular, for principal series representations, Alvis–Curtis duality commutes with the type map.

Proof. This is a combination of Proposition 23 and [DM20, Proposition 7.2.13]. □

Chapter 3

Counting points

Recall from §2.4 the definition

$$S_\tau(q) := \sum_{\chi \in \mathrm{Irr}(G(\mathbb{F}_q))_\tau} \prod_{i=1}^{n} \chi(C_i(\mathbb{F}_q)).$$

We dedicate this chapter to the determination of $S_\tau(q)$ under Assumptions 1 and 3. In particular, each C_i is the conjugacy class of a strongly regular $S_i \in T$, and $\mathrm{char}(\mathbb{F}_q)$ is not too small. To aid in our determination of $S_\tau(q)$, we develop a reductive notion of a generic choice of semisimple conjugacy classes, generalising the notion seen in [HLRV11].

We saw in Chapter 1 what it means for conjugacy classes to be chosen generically:

Definition 25. *We say the tuple $(C_1, \ldots, C_n)$ of conjugacy classes of G is generic if*

$$\prod_{i=1}^{n} X_i \notin [L, L]$$

for all proper Levi subgroups $L \subset G$ containing T and for all $X_i \in C_i \cap L$.

The presence of a generic collection of conjugacy classes has two advantages:

(i) We relate the point-counts of the character stack and the character variety, and

(ii) We obtain an expression for $S_\tau(q)$ which is straightforward to compute.[1]

[1] We still obtain a computable expression for $S_\tau(q)$ when conjugacy classes are not chosen generically, but its computation is less clear.

We address the first advantage in §3.3 and the second advantage in §3.4. Before doing so, we prove conjugacy classes can be chosen generically in the first place.

3.1 Conjugacy classes can be chosen generically

We are working under Assumption 1 so we only need to show a generic collection of strongly regular conjugacy classes exists.

Theorem 26. *There exists a generic collection of conjugacy classes satisfying Assumption 1.*

This theorem directly follows from the following lemma. Let $\mathcal{L}$ be the set of proper Levi subgroups of G containing T. It is important to note $\mathcal{L}$ is a finite set because all Levis are W-conjugate to a standard Levi and standard Levis are in bijection with subsets of simple reflections in W.

Lemma 27. *There exist strongly regular $S_1,\dots,S_n \in T$ such that*

$$(w_1.S_1)\cdots(w_n.S_n) \in [G,G] \setminus \bigcup_{L\in\mathcal{L}} [L,L]$$

for all $w_i \in W$. Moreover, if $S_1,\dots,S_n \in T$ are as above, then $(C_1,\dots,C_n)$ is generic.

Note such S_i obviously satisfy $S_1\cdots S_n \in [G,G]$ (c.f. Assumption 1).

Proof. Such S_i existing is the same as finding strongly regular elements $(S_1,\dots,S_n)$ in the set

$$\bigcap_{\underline{w}\in W^n}\left(\left\{(S_1,\dots,S_n)\in T^n \;\middle|\; \underline{w}.\underline{S}\in[G,G]\right\}\setminus\bigcup_{L\in\mathcal{L}}\left\{(S_1,\dots,S_n)\in T^n \;\middle|\; \underline{w}.\underline{S}\in[L,L]\right\}\right).$$

Basic set identities imply this set is equal to

$$A\setminus B := \left(\bigcap_{\underline{w}\in W^n}\left\{(S_1,\dots,S_n)\in T^n \;\middle|\; \underline{w}.\underline{S}\in[G,G]\right\}\right)\setminus\left(\bigcup_{L\in\mathcal{L}}\bigcup_{\underline{w}\in W^n}\left\{(S_1,\dots,S_n)\in T^n \;\middle|\; \underline{w}.\underline{S}\in[L,L]\right\}\right)$$

which is a non-empty open subvariety of T^n since $\dim(A) > \dim(B)$, verified later in Lemma 28. Strongly regular elements form an open dense subset of T [Ste65, §2] so the desired S_i exist.

We verify that $(C_1,\dots,C_n)$ is generic. For want of a contradiction, assume the collection is not; i.e., assume $X_1\cdots X_n \in [L,L]$ for some proper Levi $L \subset G$ and some $X_i \in C_i \cap L$. Write $X_i = g_i S_i g_i^{-1}$

for some $g_i \in G$. Then T and $g_iTg_i^{-1}$ are maximal tori inside of L, so they are conjugate by some $l_i \in L$. That is, $l_iTl_i^{-1} = g_iTg_i^{-1}$, meaning that $g_i^{-1}l_i$ normalises T. Thus, we can write $g_i = l_i\dot{w}_i$ for some $l_i \in L$ and $w_i = \dot{w}_iT \in W$; i.e., we can write $X_i = l_i(w_i.S_i)l_i^{-1}$. Then

$$\begin{aligned} X_1\cdots X_n &= l_1(w_1.S_1)l_1^{-1}\cdots l_n(w_n.S_n)l_n^{-1} \\ &= [l_1, w_1.S_1](w_1.S_1)[l_2, w_2.S_2](w_2.S_2)\cdots[l_n, w_n.S_n](w_n.S_n), \end{aligned}$$

where the second equality is obtained by inserting $(w_i.S_i)^{-1}(w_i.S_i)$ after $l_i(w_i.S_i)l_i^{-1}$. Notice $w_i.S_i$ lies in L since $X_i = l_i(w_i.S_i)l_i^{-1}$ does. Therefore, the above expression is a product of elements in L and $[L,L]$. By assumption, this product lies in $[L,L]$, so the product $(w_1.S_1)\cdots(w_n.S_n)$ does too. This is a contradiction. □

Lemma 28. *Let*

$$A := \bigcap_{\underline{w}\in W^n} \left\{ (S_1,\dots,S_n) \in T^n \,\middle|\, \underline{w}.\underline{S} \in [G,G] \right\}$$

and

$$B := \bigcup_{\underline{w}\in W^n} \bigcup_{L\in\mathcal{L}} \left\{ (S_1,\dots,S_n) \in T^n \,\middle|\, \underline{w}.\underline{S} \in [L,L] \right\}.$$

Then $\dim(A) > \dim(B)$.

Proof. We claim

$$\dim(A) = \dim(T) + \operatorname{rank}[G,G]$$

and

$$\dim(B) = \dim(T) + \max_{L\in\mathcal{L}} \operatorname{rank}[L,L].$$

The desired inequality follows since $\mathcal{L}$ is finite and $\operatorname{rank}[G,G] > \operatorname{rank}[L,L]$ for all $L \in \mathcal{L}$. To compute $\dim(A)$ and $\dim(B)$, note that

$$\underline{w}.\underline{S} = (\dot{w}_1S_1\dot{w}_1^{-1})\cdots(\dot{w}_nS_n\dot{w}_n^{-1}) = [\dot{w}_1, S_1]S_1[\dot{w}_2, S_2]S_2\cdots[\dot{w}_n, S_n]S_n.$$

Therefore $\underline{w}.\underline{S} \in [G,G]$ if and only if $S_1 \cdots S_n \in [G,G]$ and similarly $\underline{w}.\underline{S} \in [L,L]$ if and only if $S_1 \cdots S_n \in [L,L]$. Then

$$\left\{ (S_1, \ldots, S_n) \in T^n \;\middle|\; \underline{w}.\underline{S} \in [G,G] \right\} = \left\{ (S_1, \ldots, S_n) \in T^n \;\middle|\; S_1 \cdots S_n \in [G,G] \right\}$$

and

$$\left\{ (S_1, \ldots, S_n) \in T^n \;\middle|\; \underline{w}.\underline{S} \in [L,L] \right\} = \left\{ (S_1, \ldots, S_n) \in T^n \;\middle|\; S_1 \cdots S_n \in [L,L] \right\}.$$

These have dimensions

$$\dim(T) + \operatorname{rank}[G,G]$$

and

$$\dim(T) + \operatorname{rank}[L,L],$$

respectively. □

3.2 Stabilisers are finite

We prove every point on the representation variety has finite stabilisers when conjugacy classes are chosen generically. Throughout this section, assume $p = (A_1, B_1, \ldots, A_g, B_g, X_1, \ldots, X_n)$ is a point in $\mathbf{R}$. Moreover, define the following sets:

(i) Let $\mathcal{I}$ be the set of isolated pseudo-Levi subgroups of G containing T, and

(ii) Let $\mathcal{Z}$ be the subgroup of G generated by the centres of the isolated pseudo-Levi subgroups in $\mathcal{I}$.

A key observation is the following:

Lemma 29. *The group $\mathcal{Z}$ is an abelian group containing the centre Z of G, and $\mathcal{Z}/Z$ is finite.*

Proof. Each isolated pseudo-Levi $L \in \mathcal{I}$ has a centre contained in T:

$$Z(L) = C_L(L) \subseteq C_G(L) \subseteq C_G(T) = T.$$

Then $\mathcal{Z} \subseteq T$ so $\mathcal{Z}$ is abelian. Next, G is isolated so $\mathcal{Z}$ contains Z. Lastly, $\mathcal{Z}/Z$ is generated by finitely many elements in the abelian group T/Z. These elements have finite order, so $\mathcal{Z}/Z$ is finite. □

Lemma 30. *The stabiliser* $\mathrm{Stab}_{G/Z}(p)$ *lies in* $\mathcal{Z}/Z$.

Proof. Note that $\mathrm{Stab}_{G/Z}(p) = \mathrm{Stab}_G(p)/Z$, both considered as subsets of G/Z. Fix $h \in \mathrm{Stab}_G(p)$. This means $A_1, B_1, \ldots, A_g, B_g, X_1, \ldots, X_n$ all lie in $C_G(h)$, which implies the inclusion

$$X_1 \cdots X_n = ([A_1,B_1]\cdots[A_g,B_g])^{-1} \in [C_G(h), C_G(h)].$$

Now $h \in C_G(X_i) = g_i T g_i^{-1}$ for some $g_i \in G$, so $C_G(h)$ is a pseudo-Levi subgroup. In particular, $C_G(h)$ must be isolated. If it were not, then $C_G(h) \subset L$ for some proper Levi subgroup L, meaning $X_1 \cdots X_n \in [L,L]$, which contradicts genericity of $\mathcal{C}$. Clearly $h \in Z(C_G(h))$, so $h \in \mathcal{Z}$. □

Corollary 31. *The stabiliser* $\mathrm{Stab}_{G/Z}(p)$ *is finite.*

3.3 $\mathfrak{X}$ and X have the same point-count

Corollary 31 allows us to relate the point-counts of the character variety and the character stack. Recall that the character variety is the GIT quotient

$$\mathbf{X} := \mathbf{R}/\!\!/(G/Z) = \mathbf{R}/\!\!/G$$

and the character stack is the quotient stack

$$\mathfrak{X} := [\mathbf{R}/(G/Z)].$$

We now prove Theorem 7, restated here:

Theorem 32. *If* $\mathcal{C}$ *is generic then*

(i) The action of G/Z *on* $\mathbf{R}$ *has finite stabilisers,*

(ii) $\mathfrak{X}$ *is a smooth Deligne-Mumford stack, and*

(iii) $\mathfrak{X}$ *and* $\mathbf{X}$ *have the same number of points over finite fields. In particular,*

$$|\mathbf{X}(\mathbb{F}_q)| = |\mathfrak{X}(\mathbb{F}_q)| = \frac{|\mathbf{R}(\mathbb{F}_q)|}{|(G/Z)(\mathbb{F}_q)|} = \frac{|Z(\mathbb{F}_q)|}{|T(\mathbb{F}_q)|^n} \sum_{\tau \in \mathfrak{T}(G)} \|\tau\|(q)^{2g-2+n} S_\tau(q).$$

Proof. *(i)* This is Corollary 31.

(ii) The character variety arises as the coarse moduli space associated to the character stack [Beh14, Example 1.165]. In light of [Ols16, Remark 8.3.4], it is sufficient to show that stabilisers are finite, which we did in Corollary 31. Moreover, when the classes are chosen generically, $\mathbf{R}$ is smooth, with the same proof as [KNP23, §2]. Since G/Z is also smooth, the smoothness of $\mathfrak{X}$ follows, c.f. [Ols16, §8.2].

(iii) We show that all orbits of the action $G/Z \curvearrowright \mathbf{R}$ are closed. Observe that the action map

$$G/Z \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}, \quad (gZ, p) \mapsto (g \cdot p, p)$$

is proper by [MFK94, Proposition 0.8]. Proper maps are closed, so the image of the closed set $G/Z \times \{x\}$ is closed. The image is $\mathrm{Orb}_{G/Z}(x) \times \{x\}$, so the orbit $\mathrm{Orb}_{G/Z}(x)$ is closed. □

3.4 A formula for $S_\tau(q)$: once-punctured case

In this section, we explain the evaluation of $S_\tau(q)$ in the once-punctured case (i.e., $n = 1$). We split into cases to elucidate the important ideas, and we will find that the multi-punctured case immediately follows. In the once-punctured case, there is one conjugacy class containing a strongly regular S, and $S_\tau(q)$ is given by

$$S_\tau(q) = \sum_{\chi \in \mathrm{Irr}(G(\mathbb{F}_q))_\tau} \chi(S).$$

Our evaluation is centered around a theorem of Deligne–Lusztig which tells us how to evaluate characters at strongly regular elements:

Proposition 33 (Corollary 7.6 of [DL76]). *If $S \in T(\mathbb{F}_q)$ is strongly regular then*

$$\chi(S) = \sum_{\theta \in T(\mathbb{F}_q)^\vee} \langle \chi, R_T^G \theta \rangle \, \theta(S).$$

In particular, $\chi(S) = 0$ unless χ is a principal series representation.

In light of §2.5, this means $S_\tau(q) = 0$ unless τ is principal, which we assume from now on. Furthermore, recall from §2.5 that constituents of $R_T^G\,\theta$ are in bijection with characters of W_θ. Given a character $\phi \in \mathrm{Irr}(W_\theta)$, denote the corresponding constituent of $R_T^G\,\theta$ by $\chi_{\theta,\phi}$. In view of Propositions 20, 21 and 33, we compute

$$\chi_{\theta,\phi}(S) = \dim(\phi) \sum_{w \in W/W_\theta} (w.\theta)(S) = \frac{\dim(\phi)}{|W_\theta|} \sum_{w \in W} \theta(w.S).$$

If $\chi_{\theta,\phi}$ has (principal) type $\tau = [L,\rho]$ then the dual of $\check{G}_\theta$, which we denote by G_θ, is an endoscopy group of G which lies in the W-orbit of L, and $\phi \in \mathrm{Irr}(W_\theta)$ is paired with $\rho \in \mathrm{Uch}(L)$ according to the bijections

$$\mathrm{Irr}(W_\theta) \longleftrightarrow R_{\check{T}}^{\check{G}_\theta}\,1 \longleftrightarrow R_T^L\,1 \subseteq \mathrm{Uch}(L).$$

Denote by $\tilde{\rho}$ the character in $\mathrm{Irr}(W(L))$ corresponding to $\rho \in \mathrm{Uch}(L)$. Then

$$S_\tau(q) = \sum_{\chi \in \mathrm{Irr}(G(\mathbb{F}_q))_\tau} \chi(S) = \sum_{\substack{[\theta] \in T(\mathbb{F}_q)^\vee/W \\ G_\theta \in [L]}} \chi_{\theta,\tilde{\rho}}(S).$$

Note that $G_\theta \in [L]$ if and only if $W_\theta \in [W(L)]$ [DL76, Theorem 5.13]. This means that

$$S_\tau(q) = \sum_{\substack{[\theta] \in T(\mathbb{F}_q)^\vee/W \\ W_\theta \in [W(L)]}} \chi_{\theta,\tilde{\rho}}(S).$$

We write this as a sum over all $\theta \in T(\mathbb{F}_q)^\vee$, rather than W-orbits $[\theta] \in T(\mathbb{F}_q)^\vee/W$. To do so, notice that $\chi_{\theta,\tilde{\rho}} = \chi_{w.\theta,\tilde{\rho}}$ since $R_T^G\,\theta \simeq R_T^G\,w.\theta$ by Proposition 20. Therefore

$$S_\tau(q) = \frac{|W(L)|}{|W|} \sum_{\substack{\theta \in T(\mathbb{F}_q)^\vee \\ W_\theta \in [W(L)]}} \chi_{\theta,\tilde{\rho}}(S).$$

Lastly, we can replace the condition $W_\theta \in [W(L)]$ with the condition $W_\theta = W(L)$. We do so by averaging over the orbit size $|[W(L)]| = |W|/|N_W(W(L))|$. This gives

$$S_\tau(q) = \frac{|W(L)|}{|N_W(W(L))|} \sum_{\substack{\theta \in T(\mathbb{F}_q)^\vee \\ W_\theta = W(L)}} \chi_{\theta,\tilde{\rho}}(S).$$

We now substitute in our formula for $\chi_{\theta,\tilde{\rho}}(S)$, giving

$$S_\tau(q) = \frac{\dim(\tilde{\rho})}{|N_W(W(L))|} \sum_{w\in W} \sum_{\substack{\theta\in T(\mathbb{F}_q)^\vee \\ W_\theta=W(L)}} \theta(w.S).$$

Lastly, define the character sum

$$\alpha_{L,S}(q) := \sum_{\substack{\theta\in T(\mathbb{F}_q)^\vee \\ W_\theta=W(L)}} \theta(S).$$

Then our formula for $S_\tau(q)$ is

$$S_\tau(q) = \frac{\dim(\tilde{\rho})}{|N_W(W(L))|} \sum_{w\in W} \alpha_{L,w.S}(q).$$

We have reduced the computation of $S_\tau(q)$ to the computation of $\alpha_{L,S}(q)$, which we address now.

3.4.1 A formula for $\alpha_{L,S}(q)$

Our evaluation of $\alpha_{L,S}(q)$ follows [KNP23, §5]. We omit any proof already appearing there in order to increase clarity and to emphasise a new idea appearing in this thesis: genericity of S yields a simple formula for $\alpha_{L,S}(q)$, written in terms of isolated endoscopy groups.

The evaluation of $\alpha_{L,S}(q)$ centers around an application of Möbius inversion. Consider the partially ordered set P of endoscopy groups of G, ordered by inclusion, which comes with Möbius function $\mu\colon P\times P\to\mathbb{Z}$. Define the sum

$$\Delta_{L,S}(q) := \sum_{\substack{\theta\in T(\mathbb{F}_q)^\vee \\ W_\theta\supseteq W(L)}} \theta(S),$$

so that the Möbius inversion formula [Sta12, §3.7] yields

$$\alpha_{L,S}(q) = \sum_{L'\supseteq L} \mu(L,L')\,\Delta_{L',S}(q).$$

We turn our attention to evaluating $\Delta_{L',S}(q)$, starting with an application of Pontryagin duality:

Proposition 34 (Lemma 26 of [KNP23]). *Let $f\colon A \to B$ be a surjective homomorphism of finite abelian groups and $f^\vee\colon B^\vee \to A^\vee$ be the pullback map $f^\vee(\varphi) = \varphi \circ f$. Then for each $a \in A$ we have*

$$\sum_{\theta \in f^\vee(B^\vee)} \theta(a) = \begin{cases} |B|, & \text{if } f(a) = 1, \\ 0, & \text{else.} \end{cases}$$

To alleviate notation, let $k = \mathbb{F}_q$. We apply the above result to the (dual of the) natural map

$$f_L\colon T(k) \to \frac{T(k)}{T(k) \cap [L(k), L(k)]}.$$

Corollary 35 (Corollary 27 and Proposition 28 of [KNP23]). *We have*

$$\Delta_{L,S}(q) = \begin{cases} |\check{T}(k)^{W(L)}|, & \text{if } f_L(S) = 1, \\ 0, & \text{else.} \end{cases}$$

In particular, $\Delta_{L,S}(q)$ is zero unless $S \in [L(k), L(k)]$.

In light of the above, we must understand the fixed points $\check{T}(k)^{W(L)}$. To this end, let

$$\pi_0^L := |\pi_0(\check{T}^{W(L)})|$$

be the size of the group of components of $\check{T}^{W(L)}$. We recall precisely what this means now (c.f. [KNP23, §4]). Recall that the cocharacters $\check{X}$ admits an action of $W(L)$, and define the $W(L)$-coinvariants of $\check{X}$ by

$$\check{X}_{W(L)} := \check{X}/\langle x - w.x \mid x \in \check{X},\ w \in W(L)\rangle,$$

so that $\check{T}^{W(L)} = \operatorname{Spec} k[\check{X}_{W(L)}]$. Then $\check{X}_{W(L)}$ is an abelian group with torsion part $\mathscr{T} := \operatorname{Tor}(\check{X}_{W(L)})$, and the group of components is the k-group scheme $\pi_0(\check{T}^{W(L)}) := \operatorname{Spec} k[\mathscr{T}]$.

The following proposition explains why we introduced the group of components:

Proposition 36 (Proposition 23 of [KNP23]). *(i) We have $|\check{T}(k)^{W(L)}| = |\pi_0(\check{T}^{W(L)})| \times |(\check{T}^{W(L)})^\circ(k)|$.*

(ii) Assume that $q \equiv 1 \bmod d(\check{G})$, where $d(\check{G})$ is the modulus of $\check{G}$, defined to be the least common multiple of $|\operatorname{Tor}(\check{X}/\langle\Psi\rangle)|$ as Ψ ranges over all closed subsystems of $\check{\Phi}$. Then $\check{T}(k)^{W(L)}$ and $(\check{T}^{W(L)})^\circ$ are polynomial count.

We take stock of our progress. For any strongly regular $S \in T(\mathbb{F}_q)$, we have shown that

$$\alpha_{L,S}(q) = \sum_{\substack{L' \supseteq L \\ S \in [L'(k), L'(k)]}} \mu(L, L') \, |\check{T}^{W(L)}(\mathbb{F}_q)|.$$

We have not yet used our generic assumption. We do so now, providing a major simplification:

Proposition 37. *(i) An element of G lies in $[L, L]$ if and only if it lies in $[L(k'), L(k')]$ for some finite extension k' of k,*

(ii) The generic element S lies in $[L, L]$ if and only if L is isolated, and

(iii) If L is isolated then $|(\check{T}^{W(L)})^\circ(\mathbb{F}_q)| = |Z(\mathbb{F}_q)|$.

Proof. *(i)* If $x \in [L(k'), L(k')]$ for some finite extension k'/k then $x \in [L, L](k') \cap G$ so $x \in [L, L]$. On the other hand, if $x \in [L, L]$ then $x \in [L, L](\bar{k})$ so $x \in [L(\bar{k}), L(\bar{k})]$. In other words, there exists $a_1, b_1, \ldots, a_r, b_r \in L(\bar{k})$ such that $x = [a_1, b_1] \ldots [a_r, b_r]$. However, $L(\bar{k})$ equals the union of all $L(k')$ as k' ranges over all finite extensions of k, proving the result.

(ii) If L is not isolated then there exists a proper Levi $L' \subset G$. Since S is generic, we must have $S \notin [L', L']$. On the other hand, if L is isolated then $[L, L] = [G, G]$ which contains S.

(iii) By definition, we have $(\check{T}^{W(L)})^\circ = Z(\check{L})^\circ$. But $|Z(\check{L})^\circ(\mathbb{F}_q)| = |Z(L)^\circ(\mathbb{F}_q)|$ [Car93, Proposition 4.4.5]). Since L is isolated, we have $Z(L)^\circ = Z^\circ = Z$. □

Corollary 38. *If $S \in T(\mathbb{F}_q)$ is strongly regular and generic then*

$$\alpha_{L,S}(q) = |Z(\mathbb{F}_q)| \sum_{\substack{L' \supseteq L \\ L' \text{ isolated}}} \mu(L, L') \, \pi_0^{L'}.$$

In particular, $\alpha_{L,S}(q)$ is independent of S.

Corollary 39. *If $S \in T(\mathbb{F}_q)$ is strongly regular and generic then*

$$S_\tau(q) = |Z(\mathbb{F}_q)| \dim(\tilde{\rho}) \, |[L]| \sum_{\substack{L' \supseteq L \\ L' \text{ isolated}}} \mu(L, L') \, \pi_0^{L'}.$$

3.5 A formula for $S_\tau(q)$: multi-punctured case

In this section, we explain the evaluation of $S_\tau(q)$ in the multi-punctured case (i.e., $n \geq 1$). In this case, we have a generic collection $\mathcal{C} = (C_1, \ldots, C_n)$ containing strongly regular $S_i \in T(\mathbb{F}_q)$, and $S_\tau(q)$ is given by

$$S_\tau(q) = \sum_{\chi \in \mathrm{Irr}(G(\mathbb{F}_q))_\tau} \chi(S_1) \cdots \chi(S_n).$$

The evaluation of $S_\tau(q)$ begins with an observation: if τ is not principal then $S_\tau(q) = 0$. Therefore, assume that $\tau = [L, \rho]$ is principal, and recall from the once-punctured case that $\chi \in \mathrm{Irr}(G(\mathbb{F}_q))_\tau$ if and only if $\chi = \chi_{\theta,\tilde{\rho}}$ for some $\theta \in T(\mathbb{F}_q)^\vee$ with $G_\theta \in [L]$. Therefore

$$S_\tau(q) = \sum_{\substack{[\theta] \in \check{T}(\mathbb{F}_q)/W \\ G_\theta \in [L]}} \chi_{\theta,\tilde{\rho}}(S_1) \cdots \chi_{\theta,\tilde{\rho}}(S_n) = \frac{|W(L)|}{|N_W(W(L))|} \sum_{\substack{\theta \in \check{T}(\mathbb{F}_q) \\ W_\theta = W(L)}} \chi_{\theta,\tilde{\rho}}(S_1) \cdots \chi_{\theta,\tilde{\rho}}(S_n).$$

Let $\underline{S}$ denote the tuple $(S_1, \ldots, S_n)$ and, for each $\underline{w} := (w_1, \ldots, w_n) \in W^n$, let $\underline{w}.\underline{S}$ denote the product $w_1.S_1 \cdots w_n.S_n$. From the once-punctured case, we know that

$$\chi_{\theta,\tilde{\rho}}(S_1) \cdots \chi_{\theta,\tilde{\rho}}(S_n) = \frac{\dim(\tilde{\rho})^n}{|W_\theta|^n} \sum_{w_1, \ldots, w_n \in W} \theta(\underline{w}.\underline{S}).$$

Plugging this into the previous expression for $S_\tau(q)$ and rearranging, we get

$$S_\tau(q) = \frac{\dim(\tilde{\rho})^n}{|N_W(W(L))|\,|W(L)|^{n-1}} \sum_{\underline{w} \in W^n} \alpha_{L,\underline{w}.\underline{S}}.$$

We obtained in Corollary 38 an expression for $\alpha_{L,S}(q)$, yielding the following:

Corollary 40. *If $\mathcal{C} = (C_1, \ldots, C_n)$ is a generic collection of strongly regular conjugacy classes then*

$$S_\tau(q) = |Z(\mathbb{F}_q)| \dim(\tilde{\rho})^n \, |[L]| \, \frac{|W|^{n-1}}{|W(L)|^{n-1}} \sum_{\substack{L' \supseteq L \\ L' \text{ isolated}}} \mu(L, L') \, \pi_0^{L'}.$$

3.6 Counting functions for $\mathfrak{X}$ and X

We are now ready to prove Theorem 2, restated here in detail.

Theorem 41. *The character stack $\mathfrak{X}$ is potentially rational count with counting function*

$$\|\mathfrak{X}\|(q) = \frac{\|Z\|(q)}{\|T\|(q)^n} \sum_{\tau \in \mathcal{T}_0(G)} \|\tau\|(q)^{2g-2+n} S_\tau(q),$$

where:

(i) $\|Z\|(q) = |Z(\mathbb{F}_q)| = (q-1)^{\dim Z}$ *is the counting polynomial of the centre Z,*

(ii) $\|T\|(q) = |T(\mathbb{F}_q)| = (q-1)^{\dim T}$ *is the counting polynomial of the split maximal torus T,*

(iii) $\mathcal{T}_0(G)$ *is the set of principal types of G, defined to be the W-orbit, denoted $[L,\rho]$, of a pair (L,ρ) where L is an endoscopy group of G and ρ is a principal unipotent character of $L(\mathbb{F}_q)$,*

(iv) For a principal type $\tau = [L,\rho]$, we have

$$\|\tau\|(q) = q^{|\Phi(G)^+|-|\Phi(L)^+|} \frac{\|L\|(q)}{\|\rho\|(q)}$$

and

$$S_\tau(q) = \|Z\|(q)\, \dim(\tilde{\rho})^n\, |[L]|\, \frac{|W|^{n-1}}{|W(L)|^{n-1}} \sum_{\substack{L' \supseteq L \\ L' \text{ isolated}}} \mu(L,L')\, \pi_0^{L'},$$

where:

(a) $\Phi(G)^+$ *and* $\Phi(L)^+$ *are the positive roots of G and L, respectively,*

(b) $W(L)$ *is the Weyl group of L,*

(c) $\tilde{\rho}$ *is the character of $W(L)$ corresponding to the principal unipotent character ρ of $L(\mathbb{F}_q)$,*

(d) μ *is the Möbius function on the poset of endoscopy groups of G, ordered by inclusion, and*

(e) $\pi_0^{L'} = |\pi_0(\check{T}^{W(L')})|$ *is the number of components of $\check{T}^{W(L')}$.*

Moreover, if $g \geq 1$ then $\mathfrak{X}$ is potentially polynomial count.

Proof. In view of Frobenius' mass formula (see §2.4) and Corollary 39, we have

$$|\mathfrak{X}(\mathbb{F}_q)| = \frac{\|Z\|(q)}{\|T\|(q)^n} \sum_{\tau \in \mathcal{T}_0(G)} \|\tau\|(q)^{2g-2+n} S_\tau(q).$$

If q is replaced by q^m, we have

$$|\mathfrak{X}(\mathbb{F}_{q^m})| = \frac{\|Z\|(q^m)}{\|T\|(q^m)^n} \sum_{\tau \in \mathfrak{T}_0(G)} \|\tau\|(q^m)^{2g-2+n} S_\tau(q^m).$$

The functions $\|Z\|(q)$, $\|T\|(q)$, $\|\tau\|(q)$ and $S_\tau(q)$ are polynomials, so $\mathfrak{X}$ is rational count. Moreover, if $g \geq 1$ then $\|T\|(q)$ divides $\|\tau\|(q)$ [GM20, Remark 2.3.27], so $\mathfrak{X}$ is polynomial count. □

3.7 Dimension and components

In this section, we prove Theorem 5, restated here:

Theorem 42. *The character stack has dimension equal to*

$$\dim(\mathfrak{X}) = (2g-2+n)\dim G + 2\dim Z - n \cdot \operatorname{rank} G$$

and number of components equal to

$$|\pi_0(\mathfrak{X})| = |\pi_0(Z(\check{G}))|,$$

where $Z(\check{G})$ is the centre of the Langlands dual group $\check{G}$.

Proof. We compute the degree and leading coefficient of $\|\mathfrak{X}\|(q)$. First, we determine which principal type τ maximises the degree of $\|\tau\|(q)^{2g-2+n} S_\tau(q)$. Clearly, for any principal type τ, the degree of $S_\tau(q)$ is equal to $\dim Z$. Therefore, we just need to maximise the degree of $\|\tau\|(q)$. Using formulas from §2.1, the degree of $\|\tau\|(q)$ equals

$$|\Phi(G)^+| - |\Phi(L)^+| + \dim L - \deg\|\rho\| = |\Phi(G)^+| + \dim T + \deg P_{W(L)} - \deg\|\rho\|.$$

This is maximised by the type $\tau = [G, \mathrm{triv}]$. In this case, the degree of $\|\tau\|(q)$ equals $\dim G$, and the degree of $\|\mathfrak{X}\|(q)$ follows. The leading coefficient of $\|\mathfrak{X}\|(q)$ is the leading coefficient of

$$\frac{\|Z\|(q)}{\|T\|(q)^n} \|\tau\|(q)^{2g-2+n} S_\tau(q)$$

when $\tau = [G, \mathrm{triv}]$. In this case, the above expression equals

$$\|Z\|(q)^2 \|G\|(q)^{2g-2} \pi_0^G$$

which has leading coefficient $\pi_0^G = |\pi_0(Z(\check{G}))|$. □

3.8 Euler characteristic

In this section, we prove Theorem 9, restated here:

Theorem 43. *Suppose that $\mathcal{C}$ is generic.*

(i) If $g > 1$ or $\dim Z > 0$ then $\chi(\mathbf{X}) = 0$.

(ii) If $g = 1$ and $\dim Z = 0$ then

$$\chi(\mathbf{X}) = |W|^{n-1} \sum_{L} |\operatorname{Irr}(W(L))|\, |W(L)|\, \nu(L),$$

where the sum is over all endoscopy groups L of G, and

$$\nu(L) := \sum_{\substack{L' \supseteq L \\ L' \text{ isolated}}} \mu(L, L')\, \pi_0^{L'}.$$

In the definition of $\nu(L)$, the sum is over all isolated endoscopy groups of G containing L.

(iii) If $g = 0$ and $n \geq 3$ then

$$\chi(\mathbf{X}) = \frac{1}{\ell!} \frac{d^\ell}{dq^\ell}\bigg|_{q=1} \xi(q),$$

where $\ell := 2(\dim T - \dim Z)$ is twice the semisimple rank of G, and

$$\xi(q) := \frac{|W|^{n-1}}{q^{|\Phi(G)^+|}} \sum_{L} \frac{P_{W(L)}(q)^{n-2}}{q^{2|\Phi(L)^+|} |W(L)|^{n-1}} \nu(L) \sum_{\rho} \left(\frac{\dim(\tilde{\rho})}{\|\rho\|(q)} \right)^n \|\rho\|(q)^2.$$

In the definition of $\xi(q)$, the first sum is over all endoscopy groups L of G, and the second sum is over all principal unipotent characters of $L(\mathbb{F}_q)$.

It is not obvious the ℓ^{th} derivative of $\xi(q)$ can be evaluated at $q=1$. By linearity of the derivative and in view of the quotient rule, we just need to check that $q^{|\Phi(G)^+|}$, $q^{2|\Phi(L)^+|}$ and $\|\rho\|(q)$ (the functions of q appearing in denominators) are non-zero when $q=1$, which they are.

Proof. *(i)* We rewrite

$$\|\mathbf{X}\|(q) = \|Z\|(q) \sum_{\tau \in \mathcal{T}_0(G)} \|\tau\|(q)^{2g-2} \left(\frac{\|\tau\|(q)}{\|T\|(q)} \right)^n S_\tau(q).$$

We saw in the proof of Theorem 41 that $\|T\|(q)$ divides $\|\tau\|(q)$, so $\|Z\|(q)$, $\|\tau\|(q)^{2g-2}$, $\frac{\|\tau\|(q)}{\|T\|(q)}$ and $S_\tau(q)$ are all polynomial, as long as $g \geq 1$. If $g > 1$ then $2g-2 > 0$ and one checks $\|\tau\|(1)^{2g-2} = 0$, so $\chi(\mathbf{X}) = \|\mathbf{X}\|(1) = 0$. Similarly, if $g=1$ and $\dim Z > 0$ then $\|Z\|(1) = 0$, so $\chi(\mathbf{X}) = \|\mathbf{X}\|(1) = 0$ too.

(ii) Suppose that $g=1$ and $\dim Z = 0$. Then expanding the sum over τ and rearranging yields

$$\|\mathbf{X}\|(q) = \sum_L q^{|\Phi(G)^+|-|\Phi(L)^+|} \left(\frac{\|L\|(q)}{\|T\|(q)} \right)^n |[L]| \frac{|W|^{n-1}}{|W(L)|^{n-1}} \nu(L) \sum_\rho \left(\frac{\dim(\tilde{\rho})}{\|\rho\|(q)} \right)^n,$$

where the first sum is over all endoscopy groups L of G and the second sum is over all principal unipotent characters of $L(\mathbb{F}_q)$. The endoscopy L contains T, so we have $\|L\|(q)/\|T\|(q) = q^{|\Phi(L)^+|} P_{W(L)}(q)$ (c.f. §2.1). Moreover, for each principal unipotent character ρ of $L(\mathbb{F}_q)$, we have $\|\rho\|(1) = \dim(\tilde{\rho})$. That is, the degree of a principal unipotent character is a q-deformation of the degree of the corresponding Weyl group character, which is recovered by sending $q \mapsto 1$. Then $\sum_\rho (\dim(\tilde{\rho})/\|\rho\|(q))^n$ is a rational function in q which equals the number of principal unipotent characters of $L(\mathbb{F}_q)$ when it is evaluated at $q=1$. The number of principal unipotent characters of $L(\mathbb{F}_q)$ is equal to the number of irreducible characters of the Weyl group $W(L)$. Then evaluating the above expression at $q=1$ yields the Euler characteristic.

(iii) Suppose that $g=0$ and $n \geq 3$. Then expanding the sum over τ and rearranging yields

$$\|\mathbf{X}\|(q) = \frac{\|Z\|(q)^2}{\|T\|(q)^2} \frac{|W|^{n-1}}{q^{|\Phi(G)^+|}} \sum_L \frac{P_{W(L)}(q)^{n-2}}{q^{2|\Phi(L)^+|}|W(L)|^{n-1}} \nu(L) \sum_\rho \left(\frac{\dim(\tilde{\rho})}{\|\rho\|(q)} \right)^n \|\rho\|(q)^2.$$

In other words, we have $\|\mathbf{X}\|(q)(q-1)^{\ell} = \xi(q)$. Applying $\frac{d^{\ell}}{dq^{\ell}}$ to both sides and evaluating at $q = 1$ yields the Euler characteristic. □

It would be interesting to properly understand the function $\xi(q)$ as it governs the Euler characteristic of $\mathbf{X}$. We do not know of any interpretation of this function. Add examples of $\xi(q)$ in appendix.

3.9 Palindromicity

In this section, we prove Theorem 10, restated here:

Theorem 44. *If $\mathcal{C}$ is generic then $\|\mathbf{X}\|$ is a palindromic polynomial; i.e.,*

$$\|\mathbf{X}\|(q) = q^{\dim \mathbf{X}} \|\mathbf{X}\|(1/q).$$

Proof. We have $\dim \mathbf{X} = (2g-2+n)\dim G - 2\dim Z + n \cdot \operatorname{rank} G$ (c.f. Corollary 8), so we just need to understand $\|\mathbf{X}\|(1/q)$. Using formulas from §2.1, it is straightforward to show the identities:

$$\|Z\|(1/q) = (-1)^{\dim Z} q^{-\dim Z} \|Z\|(q),$$

$$\|T\|(1/q) = (-1)^{\dim T} q^{-\dim T} \|T\|(q),$$

$$\|L\|(1/q) = (-1)^{\dim T} q^{-3|\Phi(L)^{+}|-\dim T} \|L\|(q).$$

Next, by Corollary 24, if $\tau = [L, \rho]$ is a principal G-type then $D_G(\tau) = [L, D_L(\rho)]$ and

$$\|\tau\|(1/q) = q^{-2|\Phi(G)^{+}|-\dim T} (-1)^{\dim T} \|D_G(\tau)\|(q).$$

Lastly, recall from Corollary 40 the formula

$$S_{\tau}(q) = \|Z\|(q) \dim(\tilde{\rho})^n \, |[L]| \, \frac{|W|^{n-1}}{|W(L)|^{n-1}} \sum_{\substack{L' \supseteq L \\ L' \text{ isolated}}} \mu(L, L') \, \pi_0^{L'}.$$

Then $S_{D_G(\tau)}(q) = S_{\tau}(q)$ since $\dim(\tilde{\rho} \otimes \operatorname{sgn}) = \dim(\tilde{\rho})$. Therefore

$$S_{\tau}(1/q) = (-1)^{\dim Z} q^{-\dim Z} S_{D_G(\tau)}(q),$$

allowing us to compute

$$\|\mathbf{X}\|(1/q) = q^{-\dim \mathbf{X}} \frac{\|Z\|(q)}{\|T\|(q)^n} \sum_{\tau \in \mathcal{T}_0(G)} \|D_G(\tau)\|(q)^{2g-2+n} S_{D_G(\tau)}(q).$$

The result follows after recalling from §2.6 that $D_G \colon \mathcal{T}_0(G) \to \mathcal{T}_0(G)$ is a bijection. $\square$

Appendix A

Examples

In this appendix, we provide examples of counting polynomials. To compute examples, we provide tables whose rows are labelled by principal G-types and whose columns contain the minimum data required to compute the counting polynomial according to our formula. Recall that our formula is

$$\|\mathbf{X}\|(q) = \frac{\|Z\|(q)}{\|T\|(q)^n} \sum_{\tau \in \mathcal{T}_0(G)} \|\tau\|(q)^{2g-2+n} S_\tau(q),$$

where:

(i) $\|Z\|(q) = |Z(\mathbb{F}_q)| = (q-1)^{\dim Z}$ is the counting polynomial of the centre $Z(G) = Z$ of G,

(ii) $\|T\|(q) = |T(\mathbb{F}_q)| = (q-1)^{\dim T}$ is the counting polynomial of the split maximal torus $T \subseteq G$,

(iii) $\mathcal{T}_0(G)$ is the set of principal types of G, defined to be the W-orbit, denoted $[L,\rho]$, of a pair (L,ρ) where L is an endoscopy group of G and ρ is a principal unipotent character of $L(\mathbb{F}_q)$,

(iv) For a principal type $\tau = [L,\rho]$, we have

$$\|\tau\|(q) = q^{|\Phi(G)^+|-|\Phi(L)^+|} \frac{\|L\|(q)}{\|\rho\|(q)}$$

and

$$S_\tau(q) = \|Z\|(q) \dim(\tilde{\rho})^n \, |[L]| \, \frac{|W|^{n-1}}{|W(L)|^{n-1}} \nu(L),$$

where:

(a) $\Phi(G)^+$ and $\Phi(L)^+$ are the positive roots of G and L, respectively,

(b) $\|L\|(q) = |L(\mathbb{F}_q)| = q^{|\Phi(L)^+|}(q-1)^{\dim T} P_{W(L)}(q)$ is the counting polynomial of L,

(c) $P_{W(L)}(q) = \sum_{w \in W(L)} q^{\mathrm{length}(w)}$ is the Poincaré polynomial of $W(L)$,

(d) $W(L)$ is the Weyl group of L,

(e) $\tilde{\rho}$ is the character of $W(L)$ corresponding to the principal unipotent character ρ of $L(\mathbb{F}_q)$,

(f) $\nu(L) = \sum_{\substack{L' \supseteq L \\ L' \text{ isolated}}} \mu(L, L')\, \pi_0^{L'}$,

(g) μ is the Möbius function on the poset of endoscopy groups of G, ordered by inclusion, and

(h) $\pi_0^{L'} = |\pi_0(\check{T}^{W(L')})|$ is the number of components of $\check{T}^{W(L')}$.

A.1 A GL_2-character variety

$[L,\rho]$	$\lvert\Phi(L)^+\rvert$	$\rho(1)$	$\lvert L(\mathbb{F}_q)\rvert$	$\tilde{\rho}(1)$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	L isolated?	π_0^L	$\nu(L)$
$[\mathrm{GL}_2,(2)]$	1	1	$q\Phi_1^2\Phi_2$	1	2	1	Yes	1	1
$[\mathrm{GL}_2,(1\,1)]$	1	q	$q\Phi_1^2\Phi_2$	1	2	1	Yes	1	1
$[T,\mathrm{triv}]$	0	1	Φ_1^2	1	1	1	No		-1

Table A.1: The three GL_2-types. The unipotent characters of $\mathrm{GL}_2(\mathbb{F}_q)$ are denoted by partitions. The i^{th} cyclotomic polynomial is denoted Φ_i.

A.2 A GL_3-character variety

$[L,\rho]$	$\lvert\Phi(L)^+\rvert$	$\rho(1)$	$\lvert L(\mathbb{F}_q)\rvert$	$\tilde{\rho}(1)$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	L isolated?	π_0^L	$\nu(L)$
$[\mathrm{GL}_3,(3)]$	3	1	$q^3\Phi_1^3\Phi_2\Phi_3$	1	6	1	Yes	1	1
$[\mathrm{GL}_3,(2\,1)]$	3	$q\Phi_2$	$q^3\Phi_1^3\Phi_2\Phi_3$	2	6	1	Yes	1	1
$[\mathrm{GL}_3,(1\,1\,1)]$	3	q^3	$q^3\Phi_1^3\Phi_2\Phi_3$	1	6	1	Yes	1	1
$[G_{A_1},(2)]$	1	1	$q\Phi_1^3\Phi_2$	1	2	3	No		-1
$[G_{A_1},(1\,1)]$	1	q	$q\Phi_1^3\Phi_2$	1	2	3	No		-1
$[T,\mathrm{triv}]$	0	1	Φ_1^3	1	1	1	No		2

Table A.2: The six GL_3-types. The unipotent characters of $\mathrm{GL}_3(\mathbb{F}_q)$ and $\mathrm{GL}_2(\mathbb{F}_q)$ are denoted by partitions. The i^{th} cyclotomic polynomial is denoted Φ_i.

A.3 An SO_5-character variety

$[L,\rho]$	$\lvert\Phi(L)^+\rvert$	$\rho(1)$	$\lvert L(\mathbb{F}_q)\rvert$	$\tilde{\rho}(1)$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	L isolated?	π_0^L	$\nu(L)$
$[\mathrm{SO}_5, \binom{2}{\ }]$	4	1	$q^4\Phi_1^2\Phi_2^2\Phi_4$	1	8	1	Yes	2	2
$[\mathrm{SO}_5, \binom{0\,1}{2}]$	4	$\frac{1}{2}q\Phi_4$	$q^4\Phi_1^2\Phi_2^2\Phi_4$	1	8	1	Yes	2	2
$[\mathrm{SO}_5, \binom{1\,2}{0}]$	4	$\frac{1}{2}q\Phi_4$	$q^4\Phi_1^2\Phi_2^2\Phi_4$	1	8	1	Yes	2	2
$[\mathrm{SO}_5, \binom{0\,2}{1}]$	4	$\frac{1}{2}q\Phi_2^2$	$q^4\Phi_1^2\Phi_2^2\Phi_4$	2	8	1	Yes	2	2
$[\mathrm{SO}_5, \binom{0\,1\,2}{1\,2}]$	4	q^4	$q^4\Phi_1^2\Phi_2^2\Phi_4$	1	8	1	Yes	2	2
$[G_{A_1\times A_1}, (2)\otimes(2)]$	2	1	$q^2\Phi_1^2\Phi_2^2$	1	4	1	Yes	4	2
$[G_{A_1\times A_1}, (2)\otimes(1\,1)]$	2	q	$q^2\Phi_1^2\Phi_2^2$	1	4	1	Yes	4	2
$[G_{A_1\times A_1}, (1\,1)\otimes(2)]$	2	q	$q^2\Phi_1^2\Phi_2^2$	1	4	1	Yes	4	2
$[G_{A_1\times A_1}, (1\,1)\otimes(1\,1)]$	2	q^2	$q^2\Phi_1^2\Phi_2^2$	1	4	1	Yes	4	2
$[G_{A_1}, (2)]$	1	1	$q\Phi_1^2\Phi_2$	1	2	2	No		-4
$[G_{A_1}, (1\,1)]$	1	q	$q\Phi_1^2\Phi_2$	1	2	2	No		-4
$[G_{A_1'}, (2)]$	1	1	$q\Phi_1^2\Phi_2$	1	2	2	No		-2
$[G_{A_1'}, (1\,1)]$	1	q	$q\Phi_1^2\Phi_2$	1	2	2	No		-2
$[T, \mathrm{triv}]$	0	1	Φ_1^2	1	1	1	No		8

Table A.3: The fourteen principal SO_5-types. The principal unipotent characters of $\mathrm{SO}_5(\mathbb{F}_q)$ are denoted using Lusztig's symbols [GM20, §4.4]. We distinguish the copies of A_1 with longer and shorter roots by A_1 and A_1', respectively. The i^{th} cyclotomic polynomial is denoted Φ_i.

A.4 A G_2-character variety

$[L,\rho]$	$\lvert\Phi(L)^+\rvert$	$\rho(1)$	$\lvert L(\mathbb{F}_q)\rvert$	$\tilde{\rho}(1)$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	L isolated?	π_0^L	$\nu(L)$
$[G_2,\phi_{1,0}]$	6	1	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	1	12	1	Yes	1	1
$[G_2,\phi'_{1,3}]$	6	$\frac{1}{3}q\Phi_3\Phi_6$	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	1	12	1	Yes	1	1
$[G_2,\phi''_{1,3}]$	6	$\frac{1}{3}q\Phi_3\Phi_6$	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	1	12	1	Yes	1	1
$[G_2,\phi_{2,1}]$	6	$\frac{1}{6}q\Phi_2^2\Phi_3$	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	2	12	1	Yes	1	1
$[G_2,\phi_{2,2}]$	6	$\frac{1}{2}q\Phi_2^2\Phi_6$	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	2	12	1	Yes	1	1
$[G_2,\phi_{1,6}]$	6	q^6	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	1	12	1	Yes	1	1
$[G_{A_2},(2)]$	3	1	$q^3\Phi_1^2\Phi_2\Phi_3$	1	6	1	Yes	3	2
$[G_{A_2},(1\,1\,1)]$	3	q^3	$q^3\Phi_1^2\Phi_2\Phi_3$	1	6	1	Yes	3	2
$[G_{A_2},(2\,1)]$	3	$q\Phi_2$	$q^3\Phi_1^2\Phi_2\Phi_3$	2	6	1	Yes	3	2
$[G_{A_1\times A_1},(2)\otimes(2)]$	2	1	$q^2\Phi_1^2\Phi_2^2$	1	4	3	Yes	2	1
$[G_{A_1\times A_1},(2)\otimes(1\,1)]$	2	q	$q^2\Phi_1^2\Phi_2^2$	1	4	3	Yes	2	1
$[G_{A_1\times A_1},(1\,1)\otimes(2)]$	2	q	$q^2\Phi_1^2\Phi_2^2$	1	4	3	Yes	2	1
$[G_{A_1\times A_1},(1\,1)\otimes(1\,1)]$	2	q^2	$q^2\Phi_1^2\Phi_2^2$	1	4	3	Yes	2	1
$[G_{A_1},(2)]$	1	1	$q\Phi_1^2\Phi_2$	1	2	3	No		-4
$[G_{A_1},(1\,1)]$	1	q	$q\Phi_1^2\Phi_2$	1	2	3	No		-4
$[G_{A'_1},(2)]$	1	1	$q\Phi_1^2\Phi_2$	1	2	3	No		-2
$[G_{A'_1},(1\,1)]$	1	q	$q\Phi_1^2\Phi_2$	1	2	3	No		-2
$[T,\mathrm{triv}]$	0	1	Φ_1^2	1	1	1	No		12

Table A.4: The eighteen principal G_2-types. The principal unipotent characters of $G_2(\mathbb{F}_q)$ are in the notation of [Car93]. We distinguish the copies of A_1 with longer and shorter roots by A_1 and A'_1, respectively. The i^{th} cyclotomic polynomial is denoted Φ_i.

Appendix B

Counting polynomials in Julia

Appendix C

Green-types and Cambò-types

C.1 Green-types

We explain why GL_n-types are the same as the types seen in [HLRV11], which we call Green-types. To this end, we recall some definitions and fix some notation. By a non-zero partition, we mean a finite decreasing list of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

where $\ell = \ell(\lambda)$ is the length of λ. A partition is also denoted by

$$n_1^{m_1} n_2^{m_2} \ldots n_r^{m_r}$$

where the n_i are strictly decreasing positive integers and $m_i > 0$ denotes the multiplicity of n_i in λ. The sum

$$|\lambda| := \sum_{i=1}^{\ell} \lambda_i = \sum_{i=1}^{r} n_i m_i$$

is called the weight of λ. Denote the set of non-zero partitions by $\mathcal{P}^\times$.

With these notions, we define a Green-type in terms of the following total order:

Definition 45. *We define a total order $\geq$ on $\mathbb{Z}^+ \times \mathcal{P}^\times$ by the following three conditions:*

(i) If $d > d'$ then $(d, \lambda) > (d', \lambda')$,

(ii) *If* $d = d'$ *and* $|\lambda| > |\lambda'|$ *then* $(d,\lambda) > (d',\lambda')$*, and*

(iii) *If* $d = d'$*,* $|\lambda| = |\lambda'|$ *and* $\lambda > \lambda'$ *according to the lexicographic order then* $(d,\lambda) > (d',\lambda')$*.*

Definition 46. *A Green-type is a finite subset of* $\mathbb{Z}^+ \times \mathcal{P}^\times$*, denoted by* $\omega = (d_1,\lambda_1)\dots(d_s,\lambda_s)$ *where* $(d_1,\lambda_1) \geq \dots \geq (d_s,\lambda_s)$*. The weight of a type is* $|\omega| := \sum_{i=1}^s d_i|\lambda_i|$*.*

For example, the four types of weight 2 are

$$(1,1^2),\quad (1,2^1),\quad (1,1^1)(1,1^1),\quad (2,1^1)$$

and the eight types of weight 3 are

$$(1,1^3),\quad (1,2^1 1^1),\quad (1,3^1),\quad (3,1^1),$$

$$(1,1^2)(1,1^1),\quad (1,2^1)(1,1^1),\quad (1,1^1)(1,1^1)(1,1^1),\quad (2,1^1)(1,1^1).$$[1]

We recall two facts:

(i) $\mathrm{Uch}(\mathrm{GL}_n(\mathbb{F}_q))$ is in bijection with the set of partitions of n, and

(ii) Centralisers of semisimple elements of $\mathrm{GL}_n(\mathbb{F}_q)$ are of the form

$$\mathrm{GL}_{n_1}(\mathbb{F}_{q^{d_1}}) \times \dots \times \mathrm{GL}_{n_s}(\mathbb{F}_{q^{d_s}}).$$

The following table describes the translation between Green-types and GL_n-types.

Green-type	GL_n-type
$(1,1^1),\dots,(1,1^1)$	$([T],(\mathrm{triv},\dots,\mathrm{triv}))$
$(1,1^{n_1}),\dots,(1,1^{n_r})$	$([A_{n_1-1}\times\dots\times A_{n_r-1}],[1],(\mathrm{triv},\dots,\mathrm{triv}))$
$(1,\lambda_1),\dots,(1,\lambda_r)$	$([A_{\lvert\lambda_1\rvert-1}\times\dots\times A_{\lvert\lambda_r\rvert-1}],[1],(\lambda_1,\dots,\lambda_r))$
$(d_1,1^1),\dots,(d_r,1^1)$	$([\emptyset],[w_{d_1,\dots,d_r}],\mathrm{triv})$
$(d_1,1^{n_1}),\dots,(d_r,1^{n_r})$	$([A_{n_1-1}\times\dots\times A_{n_r-1}],[w_{d_1,\dots,d_r}],(\mathrm{triv},\dots,\mathrm{triv}))$
$(d_1,\lambda_1),\dots,(d_r,\lambda_r)$	$([A_{\lvert\lambda_1\rvert-1}\times\dots\times A_{\lvert\lambda_r\rvert-1}],[w_{d_1,\dots,d_r}],(\lambda_1,\dots,\lambda_r))$

Table C.1: A dictionary between Green-types and GL_n-types. The tuple of partitions $(\lambda_1,\dots,\lambda_r)$ denotes the unipotent character of $\prod_i \mathrm{GL}_{n_i}$ labeled by the λ_i.

[1]The number of types of weight n is described in the OEIS entry A003606.

C.2 Cambò-types

A notion of types was introduced in [Cam17] to perform the point count of an Sp_{2n}-character variety. Even though the center of Sp_{2n} is disconnected, we detail these types, which we call Cambò-types, and explain their relationship to G-types.

Fix $G = \mathrm{Sp}_{2n}$ with maximal split torus T. A character $\theta \in \mathrm{Irr}(T(\mathbb{F}_q))$ is identified with an element of $(\mathbb{F}_q^\times)^n \simeq C_{q-1}^n$. Under this identification, we fix a collection of W-orbit representatives of $\mathrm{Irr}(T(\mathbb{F}_q))$:

Proposition 47 (Proposition 2.4.14 of [Cam17])**.** *The set*

$$\left\{ \left(\overbrace{k_1,\dots,k_1}^{\lambda_1}, \overbrace{k_2,\dots,k_2}^{\lambda_2},\dots,\overbrace{k_\ell,\dots,k_\ell}^{\lambda_\ell}, \overbrace{0,\dots,0}^{\alpha_1}, \overbrace{\frac{q-1}{2},\dots,\frac{q-1}{2}}^{\alpha_\varepsilon} \right) : \begin{array}{c} 1 \le k_i \le \frac{q-3}{2}, \\ 0 \le |\lambda|, \alpha_1, \alpha_\varepsilon \le n, \\ k_i \textit{ distinct}, \\ k_i > k_j \textit{ if } \lambda_i = \lambda_j, \\ |\lambda| + \alpha_1 + \alpha_\varepsilon = n \end{array} \right\}$$

is a complete collection of W-orbit representatives.

Definition 48. *The Cambò-type of $\chi \in R_T^G \theta$ is $\tau = (\lambda, \alpha_1, \alpha_\varepsilon, \beta)$. Here, the triple $(\lambda, \alpha_1, \alpha_\varepsilon)$ corresponds to the W-orbit representative of θ in Proposition 47 and $\beta \in \mathrm{Irr}(W_\theta)$ corresponds to χ under the bijection of Proposition 21.*

For instance, if $n = 2$, then choosing a W-orbit amounts to choosing a pair of the form

$$\underbrace{\underbrace{\left(\frac{q-1}{2}, \frac{q-1}{2}\right)}_{\alpha_1=0,\ \alpha_\varepsilon=2}, \quad \underbrace{\left(0, \frac{q-1}{2}\right)}_{\alpha_1=\alpha_\varepsilon=1}, \quad \underbrace{(0,0)}_{\alpha_1=2,\ \alpha_\varepsilon=0}}_{|\lambda|=0}, \quad \underbrace{\underbrace{\left(k_1, \frac{q-1}{2}\right)}_{\alpha_1=0,\ \alpha_\varepsilon=1}, \quad \underbrace{(k_1,0)}_{\alpha_1=1,\ \alpha_\varepsilon=0}}_{|\lambda|=1}, \quad \underbrace{\underbrace{(k_1,k_1)}_{\lambda=(2)}, \quad \underbrace{(k_1,k_2)}_{\lambda=(1,1)}}_{|\lambda|=2}.$$

where $1 \le k_1, k_2 \le \frac{q-3}{2}$ and $k_1 > k_2$. The sixteen Cambò-types are summarised in the following table:

We explain how to encode the data of a Cambò-type as a G-type. Specifically, we state the SO_{2n+1}-type associated to the Cambò-type, since $\mathrm{Sp}_{2n}^\vee = \mathrm{SO}_{2n+1}$. Note that, by Lemma 20, the

W-orbit rep. θ	λ	α_1	α_ε	W_θ	$\beta \in \mathrm{Irr}(W_\theta)$
$\left(\frac{q-1}{2}, \frac{q-1}{2}\right)$	0^1	0	2	D_8	triv, χ_1, χ_2, χ_3, χ_{2D}
$\left(0, \frac{q-1}{2}\right)$	0^1	1	1	1	triv
$(0,0)$	0^1	2	0	D_8	triv, χ_1, χ_2, χ_3, χ_{2D}
$\left(k_1, \frac{q-1}{2}\right)$	1^1	0	1	1	triv
$(k_1,0)$	1^1	1	0	1	triv
(k_1,k_1)	2^1	0	0	$\mathbb{Z}/2\mathbb{Z}$	triv, sgn
(k_1,k_2)	1^2	0	0	1	triv

Table C.2: Cambò-types when $n = 2$. The χ_i are the 1-dimensional non-trivial characters of D_8 and χ_{2D} is the 2-dimensional irreducible character of D_8.

collection

$$\{\mathcal{B}(\theta)\}_{\theta\in\mathcal{R}}$$

is a complete collection of principal series representations. That is, every principal series representation appears in this collection and the principal series representations in the collection are mutually disjoint. This means that the choice of triple $(\lambda, \alpha_1, \alpha_\varepsilon)$ in the Cambò-type $(\lambda, \alpha_1, \alpha_\varepsilon, \beta)$ is the same as the choice of a principal series representation of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$. This is the same as choosing a closed root subsystem Ψ of C_2 (up to the action of W). This follows after recalling the bijections

$$\mathcal{B}(\theta) = R_T^G(\theta) \leftrightarrow \mathrm{Irr}(W_\theta) \leftrightarrow R_{T^\vee}^{G_\theta^\vee}(1) \subseteq \mathrm{Uch}(\check{G}_\theta(\mathbb{F}_q))$$

from §2.5. In light of these bijections, the choice of $\beta \in \mathrm{Irr}(W_\theta)$ in the Cambò-type $(\lambda, \alpha_1, \alpha_\varepsilon, \beta)$ is the same as the choice of an irreducible constituent of $\mathcal{B}(\theta)$. This is the same as choosing a principal unipotent character $\rho \in \mathrm{Uch}(\check{G}_\Psi(\mathbb{F}_q))$. Then the SO_{2n+1}-type associated to the Cambò-type $(\lambda, \alpha_1, \alpha_\varepsilon, \beta)$ is $([\Psi], [1], \rho)$.

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