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On the Chow ring of a geometric quotient

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Dedicated to Professor Ernst S. Selmer
on the occasion of his 70th birthday, February 11, 1990.

0. Introduction

This paper grew out of an attempt to calculate the Chow ring of the component of the Hilbert scheme containing the twisted cubic curves. We soon discovered that a fruitful approach was to formulate the problem in terms of actions of reductive groups. (Some of these ideas can be found in the paper [E-P-S].)

Our setting is a reductive group G acting linearly on a vector space V, which we identify with the affine variety $\mathrm{Spec}(\mathrm{Sym}(V^\vee))$. The image of G in $\mathrm{Gl}(V)$ is supposed to contain the homotheties, so in reality we are looking at an action on the projectivized space $\mathbf{P}(V^\vee)$. The notion of stable point is lifted from projective space: a vector $v \in V$ is called stable if its image in $\mathbf{P}(V^\vee)$ is stable with respect to $G \cap \mathrm{Sl}(V)$.

The main theorem (4.4) gives a complete description of the ring $A^*(V^s(G)/G)_{\mathbf{Q}}$, where $V^s(G) \subseteq V$ is the set of stable points, provided that all semistable points are stable and the action is free on $V^s(G)$. The description is in terms of ring generators and relations. In addition, we give a scheme for computing in this Chow ring that may simplify calculations considerably (4.7).

Our methods work over any algebraically closed field; over $\mathbf{C}$ they work as well when the rational Chow ring is replaced by singular cohomology with rational coefficients; hence the cycle map is an isomorphism $A^*(V^s(G)/G)_{\mathbf{Q}} \simeq H^*(V^s(G)/G, \mathbf{Q})$.

There are versions of these statements over the integers under suitable strengthened hypotheses, the most significant being the condition that the group G be *special* in the sense of [Se1] (see 1.3).

The structure of the Chow ring depends on two combinatorial aspects of the representation: First, the induced action of a maximal torus T in G brings in the rich structure of torus imbeddings, in particular the Jurkiewicz-Danilov theorem on the Chow ring of a quasi-smooth toric variety [Da, Thm. 10.8 and Rem. 10.9]. Second, the Weyl group W acts on everything, and its anti-invariants turn out to play a crucial part. In fact, the Chow ring of $V^s(G)/G$ depends only on the following data:

(i) The action of T on V.
(ii) The action of W on T.

The paper is organized as follows. In Section 1 we recall some concepts and properties associated with reductive groups. In Section 2 we develop some lemmas on the behaviour of Chow groups under certain fibrations. In Section 3 we do the toric part. It can also be seen as the special case $G = T$ of the main theorem. Then everything is brought together in Section 4 to prove the main theorem.

The last three sections are devoted to two examples. As an easy illustration, we compute in Section 5 the degree of a Grassmann variety in its Plücker imbedding.

In Sections 6 and 7 we consider the case when V is of the form $V = \mathrm{Hom}(F \otimes U, E)$ acted on by $\mathrm{Gl}(E) \times \mathrm{Gl}(F)$. Here U, E, and F are vector spaces of dimension q, n, and m, respectively. In this case we give explicit generators and relations for the Chow ring. In Section 7 we apply methods from the representation theory of finite groups to obtain a formula for the Betti numbers of the quotient in the case $n = 3$, $m = 2$.

F. Kirwan [Ki] and J.-M. Drezet [Dr2] have proved recurrence formulas for the Betti numbers of geometric quotients. Their methods are similar, although Kirwan addresses the problem in full generality, whereas Drezet concentrates on the matrix varieties which are the subject of our example. It seems that those methods, based on a stratification of the unstable locus, are more efficient than ours for computing Betti numbers, whereas our approach yields more insight into the multiplicative structure of the Chow ring.

Notation. We follow the notation of [Fu] for intersection theory. In particular, if X is an n-dimensional complete variety and $x \in A^n(X)$, the degree of x is denoted $\int_X x$.

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considerable progress due to a resolution Macaulay computed for us. It can be very useful if you're looking at a ring and wondering whether it's Gorenstein!

1. Preliminaries on reductive groups

(1.1) Let G be a reductive group and fix a maximal torus T in G. Denote by W the corresponding Weyl group. Let B be a Borel group containing T.

Denote by $M(T)$ and $N(T)$ the groups of characters and of 1-parameter subgroups respectively, i.e.,

$$M(T) = \operatorname{Hom}(T, \mathbf{G}_m) \quad \text{and} \quad N(T) = \operatorname{Hom}(\mathbf{G}_m, T).$$

There is a perfect pairing of $M(T)$ and $N(T)$ given by

$$(\chi, \lambda) = \chi \circ \lambda \in \operatorname{Hom}(\mathbf{G}_m, \mathbf{G}_m) = \mathbf{Z}.$$

Let $M_{\mathbf{Q}}(T) = M(T) \otimes \mathbf{Q}$ and $N_{\mathbf{Q}}(T) = N(T) \otimes \mathbf{Q}$.

The Weyl group W acts on T by conjugation and hence on the abelian groups $M(T)$ and $N(T)$. Put

$$R = \operatorname{Sym}_{\mathbf{Z}}(M(T)) = \mathbf{Z}[\chi_1, \dots, \chi_t]$$

where $\chi_1, \dots, \chi_t$ is a basis for $M(T)$. The action of W on $M(T)$ induces an action on R. Let $R_{\mathbf{Q}} = R \otimes \mathbf{Q}$.

Recall that an element $w \in W$ is called a *reflection* if there is a linear form $L_w \in M_{\mathbf{Z}}(T)$ such that $w(L_w) = -L_w$ where L_w has a linear complement in $M_{\mathbf{Q}}(T)$ on which w acts trivially. It is a classical fact that W is generated by reflections (see for example [Hu, 27.1]). One consequence of this is that the ring of invariants $R_{\mathbf{Q}}^W$ is a polynomial ring of the same dimension as R, say $R_{\mathbf{Q}}^W = \mathbf{Q}[\zeta_1, \dots, \zeta_t]$.

For any $w \in W$ let $\operatorname{sign}(w) = \det(w)$ where w is regarded as an automorphism of $M_{\mathbf{Q}}(T)$. An element $r \in R_{\mathbf{Q}}$ is called *anti-invariant* if $w(r) = \operatorname{sign}(w)r$ for all $w \in W$. There is one fundamental anti-invariant, the *discriminant*. It is defined, uniquely up to sign, by $\Delta = \prod_{w \in W_1} L_w$, where W_1 is the subset of W consisting of reflections, and the L_w are *primitive* linear forms as above. Let $R^a = \{ r \in R \mid r \text{ is anti-invariant} \}$. Then $R_{\mathbf{Q}}^a$ is an $R_{\mathbf{Q}}^W$-module which is a direct summand of $R_{\mathbf{Q}}$. A splitting is given by $r \mapsto |W|^{-1} s(r)$, where $s(r) = \sum_{w \in W} \operatorname{sign}(w) w(r)$.

(1.2) Lemma. *$R_{\mathbf{Q}}^a$ is a free $R_{\mathbf{Q}}^W$-module of rank 1 generated by Δ. Furthermore, Δ is homogeneous of degree $\delta = \dim G/B$.*

Proof. It is sufficient to check that any element $r \in R_{\mathbf{Q}}^a$ is divisible by Δ. Let w be a reflection, and let $\overline{R} = R_{\mathbf{Q}}/(L_w)R_{\mathbf{Q}}$. If $\bar{r} \in \overline{R}$ is the image of r, we have, since w acts trivially on $\overline{R}$, that $\bar{r} = w(\bar{r}) = \operatorname{sign}(w)\bar{r} = -\bar{r}$. It follows that $\bar{r} = 0$ and we are done. The statement about the degree of the discriminant is just a rephrasing of the well-known fact that the number of reflections is half the number of roots, and the formulas (see for example [Hu, 26.2]):

$$\dim B = \frac{1}{2}|\Phi| + \dim T$$

$$\dim G = |\Phi| + \dim T$$

where Φ denotes the set of roots of G relative to T. □

(1.3) We shall need some information about the geometry of G/B. Recall from [De2, 1.5] the *characteristic homomorphism* c: $R \to A^*(G/B)$. It is defined by sending $\chi \in M(T)$ to $c_1(L(\chi))$, the first Chern class of the line bundle $L(\chi)$ on G/B associated to χ. The basic facts about c are the following:

(i) There is an exact sequence

$$0 \to R_+^W R \to R \xrightarrow{c} A^*(G/B) \xrightarrow{\rho^*} A^*(G)/A^0(G) \to 0,$$

where ρ: $G \to G/B$ is the projection and R_+^W are the invariants of positive degree in R.

(ii) $c \otimes 1_{\mathbf{Q}}$ is surjective; in other words, $A^*(G/B)_{\mathbf{Q}} \simeq R_{\mathbf{Q}}/R_+^W R_{\mathbf{Q}}$.

For some groups, the so-called *torsion free* groups, the homomorphism c is surjective. Recall [Se1, 2.2] that a morphism f: $X \to Y$ is called a *locally isotrivial fibre space with group G and fiber F* if G acts on X and F, f is G-invariant, and there exists an open covering $\{U_i\}$ of Y and finite, étale morphisms $U_i' \to U_i$ such that $X \times_{U_i} U_i' \simeq F \times U_i'$ over U_i'. Grothendieck has proved that G is torsion-free if and only if G is *special*; i.e., all locally isotrivial fibre spaces with group G are (Zariski) locally trivial [Gr2, Thm. 3].

(1.4) The morphism ρ: $G \to G/B$ factors through a morphism f: $G/T \to G/B$ which induces an isomorphism f^*: $A^*(G/B) \simeq A^*(G/T)$. The Weyl group W acts on G/T, hence on $A^*(G/T)$, and by transporting this action by $(f^*)^{-1}$ we obtain an action on $A^*(G/B)$. With respect to this action, the sequence (i) is equivariant.

(1.5) We will also use the fact (see [De2, Prop. 3.1]) that the pairing in $A^*(G/B)$ is unimodular; specifically: There exists a homogeneous basis $\{\eta_j\}_{j \in J}$ such that to each j there is associated a j' with $\int_{G/B} \eta_{j'} \eta_k = \delta_{k,j}$ (Kronecker's delta) for all $k \in J$.

2. On the Chow groups in fibrations

(2.1) Let X be any quasiprojective variety (reduced but not necessarily irreducible) on which G acts. Throughout this section we shall assume that for some ample line bundle $\mathscr{L}$, *all* the points of X are *stable* with respect to $\mathscr{L}$ and G. Furthermore, we assume that the action is *scheme-theoretically free*; i.e., the morphism $G \times X \to X \times X$ given by $(g, x) \mapsto (gx, x)$ is a closed immersion.

(2.2) By Mumford's theorem [GIT, Thm. 1.10] there is a morphism ϕ: $X \to Y = X/G$ which is a uniform geometric quotient. With our hypothesis that the action of G is scheme-theoretically free, it follows [GIT, Prop. 0.9] that ϕ is a principal fiber bundle with group G; i.e., ϕ is flat and the obvious morphism $G \times X \to X \times_Y X$ is an isomorphism. It also follows that ϕ is a universal geometric quotient. In characteristic 0 this follows from [GIT, Amp. 1.3]; in any characteristic it may be seen as follows: Let $Z \to Y$ be a morphism. We have the diagram

$$\begin{tikzcd}
 & Z \times_Y X \arrow[r] \arrow[dl, "\phi'"'] \arrow[d, "\phi_z"] & X \arrow[d, "\phi"] \\
(Z \times_Y X)/G \arrow[r] & Z \arrow[r] & Y.
\end{tikzcd}$$

Since the property being a principal fiber bundle is preserved under base extensions, ϕ_Z is a principal fiber bundle, and again by [GIT, Prop. 0.9], ϕ' is one. We conclude with:

(2.3) Lemma. *Let $A \to B$ be an injective homomorphism of k-algebras. Let C be a faithfully flat B-algebra which is also faithfully flat as an A-algebra. If the obvious homomorphism $C \otimes_A C \to C \otimes_B C$ is an isomorphism, then $A = B$.*

Proof. It follows that B is A-flat, so that $B \otimes_A B \to C \otimes_A C$ is injective, and if $A \neq B$, the homomorphism $B \to B \otimes_A B$ given by $b \mapsto b \otimes 1 - 1 \otimes b$ is non-zero. This gives a nonzero element in the kernel of $C \otimes_A C \to C \otimes_B C$. □

(2.4) Returning to the quotient morphism ϕ: $X \to Y$, we note that if Y' is an irreducible component of Y, there is an open subset $U' \subseteq Y'$ such that $\phi \times \mathrm{id}_{U'}$: $X_{U'} := X \times_Y U' \to U'$ is locally isotrivial. Indeed, let Y'' be the normalization of Y' in $X' := \phi^{-1}(Y')$. Now ϕ is separable since $X \times_Y X \simeq X \times G$; hence the morphism $Y'' \to Y'$ is étale and finite over some open dense $U' \subseteq Y'$. Let $U'' \subseteq Y''$ be the inverse image of U'. Then

$$X_{U'} \times_{U'} U'' \simeq X \times_Y U'' \simeq X \times_Y X \times_X U'' \simeq G \times X \times_X U'' \simeq G \times U''.$$

(2.5) We may factor ϕ as $\phi = \pi \circ f \circ h$:

$$\phi\colon X \xrightarrow{h} X/T \xrightarrow{f} X/B \xrightarrow{\pi} X/G = Y.$$

Indeed, the only non-trivial fact here is the existence of X/B. It follows from

[GIT, Prop. 2.18] that all points of $X \times G/B$ are stable with respect to the ample line bundle $p_1^*\mathscr{L}^N \otimes p_2^*\mathscr{M}$ when $N \gg 0$ and $\mathscr{M}$ is any ample line bundle on G/B. Hence the quotient $X \times_G G/B := (X \times G/B)/G$ exists, and it is easily verified that this is a geometric quotient of X by B.

All the maps in this factorization are flat. In fact, we have the following commutative diagram with cartesian squares:

$$\begin{array}{ccccc}
X \times G & \longrightarrow & X \times_G G & = & X \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle h} \\
X \times G/T & \longrightarrow & X \times_G G/T & = & X/T \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\
X \times G/B & \longrightarrow & X \times_G G/B & = & X/B \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\
X \times G/G & \longrightarrow & X \times_G G/G & = & X/G = Y,
\end{array}$$

where the horizontal maps are principal fiber bundles with group G, and the left vertical maps are the natural ones. It follows that X, X/T, and X/B are fiber spaces over Y, with fibers G, G/T, and G/B respectively. It also follows that f is a fibration with fiber B/T, which is isomorphic to an affine space of dimension δ. In fact, f: $X/T \to X/B$ is an *affine bundle* in the strong sense: It is locally (on X/B) isomorphic to the projection $X/B \times \mathbf{A}^\delta \to X/B$, and with *linear affine* transition functions. (Note that this is stronger than the concept used in [Fu, 1.9].) This is because $G/T \to G/B$ is such a bundle, and the property descends to the quotients. It follows from [Gr1, p. 4–35] that:

(2.6) LEMMA. *The flat pullback f^*: $A_*(X/B) \to A_*(X/T)$ is an isomorphism.*

(2.7) We just noted that all fibers of π are isomorphic to G/B. Let i: $G/B \to X/B$ be an isomorphism onto one such fiber followed by the inclusion. Let $\{\eta_j\}_{j \in J}$ be a homogeneous basis for $A^*(G/B)$ such that for any $j \in J$ there exists a $j' \in J$ such that $\int_{G/B} \eta_{j'} \eta_k = \delta_{k,j}$ (Kronecker's delta) for all $k \in J$. The following lemma is a Leray-Hirsch type result for the morphism π.

(2.8) LEMMA. *Assume that i^*: $A^*(X/B)_{\mathbf{Q}} \to A^*(G/B)_{\mathbf{Q}}$ is surjective, and choose $e_j \in A^*(X/B)_{\mathbf{Q}}$ such that $i^*(e_j) = \eta_j$.*

(i) *If i_1: $G/B \to X/B$ is the inclusion of any fiber, then i_1^*: $A^*(X/B)_{\mathbf{Q}} \to A^*(G/B)_{\mathbf{Q}}$ is also surjective.*

(ii) *The group homomorphism*

$$\Psi\colon A^*(G/B)_{\mathbf{Q}} \otimes A_*(Y)_{\mathbf{Q}} \to A_*(X/B)_{\mathbf{Q}}$$
$$\sum_{j\in J} \eta_j \otimes \xi_j \mapsto \sum_{j\in J} e_j \cap \pi^*(\xi_j)$$

is an isomorphism.

(iii) *If G is a special group and $i^*\colon A^*(X/B) \to A^*(G/B)$ is surjective, then the map*

$$\Psi\colon A^*(G/B) \otimes A_*(Y) \to A_*(X/B)$$

defined by the same formula as in (ii) *is an isomorphism.*

Proof. To prove (i), simply note that since the fibers of π are numerically equivalent, we get

$$\int_{G/B} i_1^*(e_j e_k) = \int_{i_1(G/B)} e_j e_k = \int_{i(G/B)} e_j e_k = \int_{G/B} i^*(e_j e_j) = \int_{G/B} \eta_j \eta_k = \delta_{k,j'},$$

from which the assertion easily follows.

To see that Ψ is injective, assume that $\sum e_j \cap \pi^*(\xi_j) = 0$. By induction on $m = \max\{\deg e_j \mid \xi_j \neq 0\}$ it suffices to show that $\xi_{j_0} = 0$ whenever $\deg e_{j_0} = m$. Let $e_{j_0'}$ be such that $\int_{G/B} \eta_{j_0'} \eta_k = \delta_{k, j_0}$. Then

$$0 = \pi_*\Big(e_{j_0'} \sum_{k\in J} e_k \cap \pi^*(\xi_k)\Big) = \sum_{k\in J} \pi_*(e_{j_0'} e_k) \cap \xi_k = \xi_{j_0},$$

since for an element $\alpha \in A^{\dim G/B}(X/B)$ we have that $\pi_*(\alpha) = (\int_{\pi^{-1}y} \alpha)[Y]$, where $y \in Y$ is a general point. This proves that Ψ is injective in statements (ii) and (iii).

We prove that Ψ is surjective by induction on the dimension of X. We may assume that Y is irreducible since there is a diagram (see [Fu, Ex. 1.3.1(c) and Prop. 1.7]):

$$\begin{tikzcd} \bigoplus A_*(Y_i) \otimes A^*(G/B) \arrow[r] \arrow[d, "\oplus \Psi_i"] & A_*(Y) \otimes A^*(G/B) \arrow[r] \arrow[d, "\Psi"] & 0 \\ \bigoplus A_*(X_i/B) \arrow[r] & A_*(X/B) \arrow[r] & 0, \end{tikzcd}$$

where $Y_1, \ldots, Y_s$ are the irreducible components of Y and $X_i = \phi^{-1} Y_i$. Hence X_i/B are the irreducible components of X/B, so that if all the Ψ_i are surjective, then so is Ψ.

Let $U \subseteq Y$ be an open set over which X is isotrivial. If $Z = Y - U$, we know that Z is the geometric quotient of $X_Z = X \times_Y Z$ by G and that this quotient is a principal fiber bundle with group G; so we may apply our induction hypothesis to Z. There is a commutative diagram with exact rows (see [Fu, Prop. 1.7 and 1.8]):

$$\begin{tikzcd} A^*(G/B)_{\mathbf{Q}} \otimes A_*(Z)_{\mathbf{Q}} \arrow[r] \arrow[d, "\Psi_Z"] & A^*(G/B)_{\mathbf{Q}} \otimes A_*(X)_{\mathbf{Q}} \arrow[r] \arrow[d, "\Psi"] & A^*(G/B)_{\mathbf{Q}} \otimes A_*(U)_{\mathbf{Q}} \arrow[r] \arrow[d, "\Psi_U"] & 0 \\ A_*(X_Z/B)_{\mathbf{Q}} \arrow[r] & A_*(X/B)_{\mathbf{Q}} \arrow[r] & A_*(X_U/B)_{\mathbf{Q}} \arrow[r] & 0 \end{tikzcd}$$

where Ψ_Z is surjective by induction. Thus Ψ is surjective whenever Ψ_U is. Now there is a finite and flat morphism α: $U' \to U$ such that $X/B \times_U U' \simeq G/B \times U'$, and hence a cartesian square:

$$\begin{tikzcd} G/B \times U' \arrow[r, "\alpha'"] \arrow[d] & X_U \arrow[d] \\ U' \arrow[r, "\alpha"] & U \end{tikzcd}$$

which gives rise to the diagram

$$\begin{tikzcd} A_*(G/B \times U')_{\mathbf{Q}} \arrow[r, "\alpha'_*"] & A_*(X_U)_{\mathbf{Q}} \\ A^*(G/B)_{\mathbf{Q}} \otimes A_*(U')_{\mathbf{Q}} \arrow[r, "\alpha_*"] \arrow[u, "\Psi_{U'}"] & A^*(G/B)_{\mathbf{Q}} \otimes A_*(U)_{\mathbf{Q}}. \arrow[u, "\Psi_U"] \end{tikzcd}$$

Since α is finite and flat, so is α'. Hence α'^* is defined and satisfies $\alpha'_*\alpha'^*\xi = [U'\colon U]\xi$ for all $\xi \in A_*(X_U)_{\mathbf{Q}}$; in particular, α'_* is surjective. By [Fu, Ex. 1.10.2] it is easily seen that $\Psi_{U'}$ is an isomorphism, and so Ψ_U is surjective.

For (iii), note that when G is special, we could take $U' = U$ (after shrinking U if necessary), and then the same proof works with integral coefficients. □

(2.9) *Remarks.* (a) If the Weyl group W acts on $A_*(X/B)$ in such a way that i^* is equivariant, it follows that π^* induces an isomorphism $A_*(Y)_{\mathbf{Q}} \simeq A_*(X/B)_{\mathbf{Q}}^W$. Under the additional hypotheses of (iii) we even have an isomorphism $A_*(Y) \simeq A_*(X/B)^W$. In particular, $A_*(Y)$ is torsion-free if $A_*(X)$ is. This will be used later.

(b) If the ground field $k = \mathbf{C}$ then the analogue of (2.4) remains true with Chow groups replaced with ordinary singular homology/cohomology groups. Indeed, this is the classical Leray-Hirsch theorem, see for example [Hus, 16.1.1].

(c) There is a paper by A. Vistoli [Vi2] where a more general result is proved. See also [Vi1].

3. The case of a linear action of a torus

(3.1) Let the t-dimensional torus T act on a vector space V of dimension N. We assume that the action is faithful and that the image of T in $\mathrm{Gl}(V)$, which we identify with T, contains the homotheties.

Call a point $v \in V$ *stable* (resp. *semistable*) if the corresponding point $\bar{v} \in \mathbf{P}(V^\vee)$ is stable (resp. semistable) with respect to the action of $T_0 := T \cap \mathrm{Sl}(V)$. The open subsets of stable and semistable points are denoted by $V^s(T)$ and $V^{ss}(T)$ respectively.

Choose a basis $v_1, \dots, v_N$ of V diagonalizing the action of T and let D be the corresponding maximal torus of $\mathrm{Gl}(V)$; then D contains T. Let $\psi_1, \dots, \psi_N$ be the basis of the character group $M(D)$ such that $g \cdot v_i = \psi_i(g)v_i$ for any $g \in D$ and $i = 1, \dots, N$. Let $x_1, \dots, x_N$ be the basis of $V^\vee$ dual to $v_1, \dots, v_N$ so that $V = \mathrm{Spec}\, k[x_1, \dots, x_N]$, and let $\lambda_1, \dots, \lambda_N$ be the basis of $N(D)$ dual to $\psi_1, \dots, \psi_N$. For any vector $v = \sum_{i=1}^N a_i v_i \in V$ where $a_1, \dots, a_N$ are in k, let $\mathrm{Supp}(v) = \{ i \mid a_i \neq 0 \}$.

By Mumford's criterion, a vector $v \in V$ is stable (resp. semistable) if and only if for any $0 \neq \lambda \in N_{\mathbf{Q}}(T_0)$, there is an $i \in \mathrm{Supp}(v)$ such that $(\psi_i, \lambda) < 0$ (resp. $(\psi_i, \lambda) \leq 0$). We shall call a subset $I \subseteq [1, N] = \{1, \dots, N\}$ *(semi)stable* if $I = \mathrm{Supp}(v)$ for some (semi)stable vector v. If $I \subseteq I'$ and I is (semi)stable, then clearly I' is (semi)stable as well. This means that the sets

$$\Phi_s = \{ J \subseteq [1, N] \mid J^c \text{ is stable} \} \quad \text{and} \quad \Phi_{ss} = \{ J \subseteq [1, N] \mid J^c \text{ is semistable} \}$$

where J^c is the complement of J in $[1, N]$, form simplicial complexes with vertices $1, \dots, N$.

We proceed to give a geometrical realization of these complexes. The inclusion of T in D gives rise to an exact sequence

$$0 \to N(T) \to N(D) \xrightarrow{\phi} N(D/T) \to 0.$$

For each subset $I \subseteq [1, N]$, let S_I be the convex hull of $\{ \phi(\lambda_i) \mid i \in I \}$ in $N_{\mathbf{R}}(D/T) = N(D/T) \otimes \mathbf{R}$. We are going to show that the boundary of $S_{[1,N]}$ is the union of the S_I for $I \in \Phi_{ss}$. Say that S_I is a *geometric face* of $\partial S_{[1,N]}$ if there exists an affine hyperplane in $N_{\mathbf{R}}(D/T)$ which intersects $S_{[1,N]}$ exactly along S_I and does not cut "through" $S_{[1,N]}$, or equivalently, there exist a linear form $\psi \in M(D/T) \subseteq M(D)$ and a number b with $(\psi, \lambda_i) < b$ for $i \notin I$ and $(\psi, \lambda_i) = b$ for $i \in I$.

(3.2) Lemma. *Given $I \subseteq [1, N]$, the following are equivalent*:

(i) $I \in \Phi_{ss}$, *i.e., I^c is semistable.*

(ii) *The convex hull of $\{ \bar{\psi}_i \mid i \in I^c \}$ in $M_{\mathbf{R}}(T_0)$ contains the origin, where $\bar{\psi}_i \in M(T_0)$ denotes the restriction of the character ψ_i to T_0.*

(iii) *There exist rational numbers* $a_1, \dots, a_N$ *and* b *such that* $0 \le a_i \le 1$ *for all* $i \in [1, N]$, $a_i = 0$ *for* $i \in I$, $\Sigma a_i = 1$, *and* $\Sigma_{i=1}^N (b - a_i)\psi_i \in M(D/T)$.

(iv) S_I *is contained in a geometric face of* $\partial S_{[1,N]}$.

Proof. The equivalence of (i) and (ii) is Mumford's criterion mentioned above. (ii) and (iii) are seen to be equivalent if we write the relation in the form $b\Sigma\psi_i \equiv \Sigma_{i\in I} a_i\psi_i (\mathrm{mod}\ M(T))$, and note that $\Sigma\psi_i$ is the determinant character, hence a generator of the kernel of $M(T) \to M(T_0)$. Assume that (iii) holds. Put $\psi = \Sigma(b - a_i)\psi_i$; then $(\psi, \lambda_i) = b - a_i$. Let $I' = \{ i \in [1, N] \mid a_i = 0\}$; then $I \subseteq I'$ and S_I is contained in $S_{I'}$, which is a geometric face. Conversely, assume that S_I is contained in a geometric face $S_{I'}$, and let $\psi \in M(D/T)$ define a corresponding touching affine hyperplane, i.e., $(\psi, \lambda_i) = b - a_i$, where $a_i = 0$ for $i \in I'$, and $a_i > 0$ for $i \notin I'$. Dividing ψ by Σa_i, we may assume that $\Sigma a_i = 1$. With these numbers, (iii) holds. □

(3.3) Lemma. *If* I^c *is stable, then* S_I *is a simplex.*

Proof. The assertion is that $\{\phi(\lambda_i) \mid i \in I\}$ is linearly independent. Assume the contrary, and let $\Sigma_{i\in I} a_i\phi(\lambda_i) = 0$ be a nontrivial relation. Then $\Sigma_{i\in I} a_i\lambda_i \in N(T)$. Put $\alpha = \Sigma a_i$, and let

$$\mu = \frac{\alpha}{N}\sum_{i=1}^{N}\lambda_i - \sum_{i\in I} a_i\lambda_i.$$

Then $(\Sigma\psi_i, \mu) = 0$, so that $0 \neq \mu \in N_{\mathbf{Q}}(T_0)$. We get $(\psi_i, \mu) = \alpha/N$ for all $i \notin I$. By Mumford's criterion, this contradicts the stability of I^c. □

(3.4) Proposition. *Assume that* $V^{ss}(T) = V^s(T)$. *Then the geometric faces of* $\partial S_{[1,N]}$, *which is homeomorphic to the* $(N - t - 1)$*-dimensional sphere, are exactly* S_I *for* $I \in \Phi_s$. *In particular, all these faces are simplices.*

Proof. If S_I is a geometric face, then I^c is semistable by (3.2), hence stable. Conversely, any stable I is contained in some J with S_J a geometric face. By (3.2-3), S_J is a simplex. From this it follows easily that I also is a geometric face. □

(3.5) We shall connect these observations with the theory of torus imbeddings. All the varieties $V^{ss}(T)$, $V^s(T)$, $V^{ss}(T)/T$, and $V^s(T)/T$ are toric varieties; $V^s(T)$ and $V^{ss}(T)$ are invariant under any maximal torus D of $\mathrm{GL}(V)$ containing T, and of course D has a dense orbit in $V^s(T)$. It follows that D/T acts on $V^s(T)/T$ and $V^{ss}(T)/T$ with a dense orbit. We shall describe all these torus embeddings by means of rational polyhedral fans (see [Da] for the basics on this theory.)

Let Σ'_s (resp. Σ'_{ss}) be the fan in $N_{\mathbf{Q}}(D)$ corresponding to $V^s(T)$ (resp. $V^{ss}(T)$). Likewise, denote by Σ_s (resp. Σ_{ss}) the fan in $N_{\mathbf{Q}}(D/T)$ corresponding to $V^s(T)/T$ (resp. $V^{ss}(T)/T$). To each $I \subseteq [1, N]$ we associate the cone

$$\tau'_I := \left\{ \sum_{i \in I} a_i \lambda_i \mid a_i \geq 0 \right\}$$

in $N_{\mathbf{Q}}(D)$ and let $\tau_I = \phi(\tau'_I)$ be the image cone in $N_{\mathbf{Q}}(D/T)$.

(3.6) Proposition. (i) *Σ'_s (resp. Σ'_{ss}) consists of those τ'_I for which $I \in \Phi_s$ (resp. $I \in \Phi_{ss}$).*

(ii) *Σ_s (resp. Σ_{ss}) is the image of Σ'_s (resp. Σ'_{ss}) under the map ϕ, hence consists of those τ_I for which $I \in \Phi_s$ (resp. $I \in \Phi_{ss}$).*

Proof. (i) Let U be the toric variety defined by these cones. Since all the cones τ'_I are contained in the first orthant $\{\sum_{i=1}^N a_i \lambda_i \mid a_i \geq 0\}$, U is an open subset of V; indeed, U is the union of the open, affine sets

$$U_I = \operatorname{Spec} k\left[M(D) \cap \tau_I'^{\vee}\right] = \operatorname{Spec} k\left[x_1, \ldots, x_N, x_{i_1}^{-1}, \ldots, x_{i_q}^{-1}\right]$$

where $I^c = \{i_1, \ldots, i_q\}$, as I runs through Φ_s (resp. Φ_{ss}) (see [Da, 2.2 and 5.2]). To see that $U = V^s(T)$ (resp. $U = V^{ss}(T)$), it suffices to remark that a vector $v = \sum a_i v_i$ is in U_I if and only if $\operatorname{Supp}(v) \supseteq I$; so if I is (semi)stable, then all the points in U_I are (semi)stable.

For (ii), let Σ''_{ss} be the image fan of Σ'_{ss} under ϕ, and let Y be the toric variety defined by Σ''_{ss}. Then by the functoriality of toric varieties [Da, 5.5], there is a map η: $V^{ss}(T) \to Y$ which clearly is T-invariant. Hence there is an induced morphism $\bar{\eta}$: $V^{ss}(T)/T \to Y$. This morphism is proper, birational and D/T-equivariant, hence induced from a subdivision of fans $\Sigma_{ss} \to \Sigma''_{ss}$ ([Da, 5.5.1]). On the other hand, the map of fans ϕ: $\Sigma'_{ss} \to \Sigma''_{ss}$ factors through Σ_{ss}. This forces equality between the two fans in $N_{\mathbf{Q}}(D/T)$, proving the proposition. □

(3.7) The inclusion $T \subseteq D$ induces an exact sequence

$$0 \to M(D/T) \to M(D) \to M(T) \to 0$$

and hence a surjective graded homomorphism

$$\mathbf{Z}[\psi_1, \ldots, \psi_N] \to \mathbf{Z}[\chi_1, \ldots, \chi_t],$$

which is nothing but dividing out by the ideal generated by the linear forms in $M(D/T)$. In the polynomial ring $\mathbf{Z}[\psi_1, \ldots, \psi_N]$, let $\mathbf{a}'$ be the ideal generated by the monomials $\prod_{i \in I} \psi_i$ for $I \notin \Phi_{ss}$. Let $\mathbf{a}$ be the image of $\mathbf{a}'$ in $\mathbf{Z}[\chi_1, \ldots, \chi_t]$.

(3.8) THEOREM. *In the notation above, assume that* $V^s(T) = V^{ss}(T)$. *Then there is an isomorphism of graded rings*

$$\mathbf{Q}[\chi_1, \dots, \chi_t]/\mathbf{a}_{\mathbf{Q}} \simeq A^*(V^s(T)/T)_{\mathbf{Q}}.$$

Assume furthermore that $V^{ss}(T)/T$ *is smooth* (*e.g., that the action is free*). *Then there is an isomorphism of graded rings*

$$\mathbf{Z}[\chi_1, \dots, \chi_t]/\mathbf{a} \simeq A^*(V^s(T)/T),$$

and $A^*(V^s(T)/T)$ *is a free* **Z**-*module.*

Proof. In view of (3.6), this is just a translation of [Da, Thm. 10.8-9]. □

(3.9) *Remark.* If the ground field is **C**, the same statements hold with the Chow rings replaced by singular cohomology.

4. The case of a linear action of a reductive group

(4.1) Assume that the reductive group G acts on a vector space V such that the image of G in $\mathrm{GL}(V)$ contains the homotheties. A point $v \in V$ is as usual called stable (resp. semistable) if the image of v in $\mathbf{P}(V^\vee)$ is stable (resp. semistable) under the group $G \cap \mathrm{SL}(V)$. The set of stable (resp. semistable) points will be denoted by $V^s(G)$ (resp. $V^{ss}(G)$).

(4.2) Assume that for a maximal torus T in G we have $V^s(T) = V^{ss}(T)$. This is then true for all maximal tori. Indeed, if T' is another one, then T' is conjugate to T, say $T' = gTg^{-1}$. From the formula $\mu(v, g^{-1}\lambda g) = \mu(gv, \lambda)$ in [GIT, Def. 2.2], where $v \in V$ and λ is any one-parameter subgroup in T, it follows that $V^s(T') = V^{ss}(T')$. It also follows that $V^s(G) = V^{ss}(G)$, and by Mumford's theorem we know that $V^s(T)/T$ and $V^s(G)/G$ are projective varieties.

(4.3) The normalizer of T acts on $V^s(T)$; in fact, if $n \in G$ normalizes T, then $\mu(nv, \lambda) = \mu(v, n^{-1}\lambda n)$, so that $v \in V^s(T)$ if and only if $nv \in V^s(T)$. Consequently, the Weyl group W acts on $V^s(T)/T$.

Recall from Theorem (3.8) that $A^*(V^s(T)/T)_{\mathbf{Q}} = \mathbf{Q}[\chi_1, \dots, \chi_t]/\mathbf{a}_{\mathbf{Q}}$, where $\chi_1, \dots, \chi_t$ is a basis for $M(T)$ and where the ideal $\mathbf{a}$ is as defined in (3.7). The action of W on $A^*(V^s(T)/T)_{\mathbf{Q}}$ induced from the action of W on $M(T)$ is easily checked to be equal to the one induced by the action of W on $V^s(T)/T$.

It follows from (1.1) and (1.2) that there is a map

$$p\colon \mathbf{Z}[\chi_1, \dots, \chi_t] \to \mathbf{Z}[\chi_1, \dots, \chi_t]^W$$

given by $p(r) = \Delta^{-1}\Sigma_{w \in W} \operatorname{sign}(w)w(r)$. This map induces an isomorphism

$$\mathbf{Q}[\chi_1, \ldots, \chi_t]^a \to \mathbf{Q}[\chi_1, \ldots, \chi_t]^W.$$

Indeed, if r is invariant, then $p(\Delta r) = |W|r$, and if r is anti-invariant, then $\Delta p(r) = |W|r$.

Our main theorem is the following:

(4.4) THEOREM. *Assume that G acts freely on $V^s(G)$ and that for some maximal torus T in G we have $V^s(T) = V^{ss}(T)$. Then:*

(i) *There is an isomorphism of graded rings*

$$\mathbf{Q}[\chi_1, \ldots, \chi_t]^W/p(\mathfrak{a}_{\mathbf{Q}}) \simeq A^*(V^s(G)/G)_{\mathbf{Q}}.$$

If furthermore $k = \mathbf{C}$, the natural map

$$A^*(V^s(G)/G)_{\mathbf{Q}} \simeq H^*(V^s(G)/G, \mathbf{Q})$$

is an isomorphism.

(ii) *If in addition the group G is special, and if T acts freely on $V^s(T)$, there is an isomorphism of graded rings*

$$\mathbf{Z}[\chi_1, \ldots, \chi_t]^W/p(\mathfrak{a})_{\mathbf{Q}} \cap \mathbf{Z}[\chi_1, \ldots, \chi_t]^W \simeq A^*(V^s(G)/G),$$

and $A^*(V^s(G)/G)$ is a free $\mathbf{Z}$-module. If furthermore $k = \mathbf{C}$, the natural map

$$A^*(V^s(G)/G) \simeq H^*(V^s(G)/G, \mathbf{Z})$$

is an isomorphism.

Proof. In the proof we shall use the following morphisms:

$$\begin{array}{ccccc} V^s(T)/T & & & & \\ \uparrow{\scriptstyle i} & & & & \\ V^s(G)/T & \xrightarrow{f} & V^s(G)/B & \xrightarrow{\pi} & V^s(G)/G \end{array}$$

where i is the open immersion induced from the open immersion $V^s(G) \subseteq V^s(T)$. The morphisms f and π are as in (2.5) with $X = V^s(G)$. Hence by (2.6), we know that f^*: $A^*(V^s(G)/B) \to A^*(V^s(G)/T)$ is an isomorphism (both being smooth, their Chow rings equal their Chow groups).

From remark (2.9) we know that $A^*(V^s(G)/G)_{\mathbf{Q}} \simeq A^*(V^s(G)/T)^W_{\mathbf{Q}}$. Since the ring $A^*(V^s(G)/T)_{\mathbf{Q}}$ is generated by $\mathbf{Q}[\chi_1, \ldots, \chi_t]$ and taking invariants is an exact functor on $\mathbf{Q}W$-modules, it follows that $A^*(V^s(G)/G)_{\mathbf{Q}}$ is a quotient of $\mathbf{Q}[\chi_1, \ldots, \chi_t]^W$. Hence there is a surjective homomorphism β as in the diagram below

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{a}_{\mathbf{Q}} & \longrightarrow & \mathbf{Q}[\chi_1,\dots,\chi_t] & \xrightarrow{\alpha} & A^*(V^s(T)/T)_{\mathbf{Q}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_1} & & \\ 0 & \longrightarrow & p(\mathbf{a}_{\mathbf{Q}}) & \longrightarrow & \mathbf{Q}[\chi_1,\dots,\chi_t]^W & \xrightarrow{\beta} & A^*(V^s(G)/G)_{\mathbf{Q}} & \longrightarrow & 0. \end{array}$$

Here $p_1 = \pi_* \circ (f^*)^{-1} \circ i^*$. The upper sequence is exact by (3.4), and the exactness of the lower one is what we are aiming at.

To see that the right-hand square is commutative, choose a homogeneous basis $\{e_j\}_{j \in J}$ for $\mathbf{Q}[\chi_1,\dots,\chi_t]$ over $\mathbf{Q}[\chi_1,\dots,\chi_t]^W$. Then $\deg e_j < \delta$ for all but one j, say $j = \omega$, and $\deg e_\omega = \delta$. (We may take $e_\omega = \Delta$.) Since δ is the fiber dimension of π, it follows that both p and p_1 lower degrees by δ. Thus $\beta \circ p$ and $p_1 \circ \alpha$ both annihilate all e_j, $j \neq \omega$. It follows that the right-hand square commutes up to a constant factor, since all the maps are $\mathbf{Q}[\chi_1,\dots,\chi_t]^W$-linear. Now $p(\Delta) = |W|$, and by [Del, Eq. 11], $\alpha(\Delta)$ induces $\pm|W|$ times the class of a point on all the fibers of π. It follows that $p_1(\alpha(\Delta)) = \pm|W|$. By normalizing Δ suitably (note that p is defined in terms of Δ!), we can make the right-hand square commute.

It follows that β induces a surjection $\bar{\beta}$: $R^W_{\mathbf{Q}}/p(\mathbf{a}_{\mathbf{Q}}) \to A^*(V^s(G)/G)_{\mathbf{Q}}$, where $R_{\mathbf{Q}} = \mathbf{Q}[\chi_1,\dots,\chi_t]$. The following lemmas show that $\bar{\beta}$ is injective on the scale of $R^W_{\mathbf{Q}}/p(\mathbf{a}_{\mathbf{Q}})$, and thus injective.

(4.5) Lemma. (i) *There is an isomorphism* $A^*(V^s(T)/T)^a_{\mathbf{Q}} \simeq R^W_{\mathbf{Q}}/p(\mathbf{a}_{\mathbf{Q}})$ *of graded* $R^W_{\mathbf{Q}}$*-modules decreasing degrees by* δ.

(ii) *If* $i < \delta$ *or* $i > e - \delta$, *where* $e = \dim V^s(T)/T$, *then*

$$A^i(V^s(T)/T)^a_{\mathbf{Q}} = 0.$$

Proof. (i) The isomorphism is the restriction to $A^*(V^s(T)/T)^a_{\mathbf{Q}}$ of the map

$$\rho\colon A^*(V^s(T)/T)_{\mathbf{Q}} \to R^W_{\mathbf{Q}}/p(\mathbf{a}_{\mathbf{Q}})$$

induced by p. Since taking anti-invariants is exact,

$$A^*(V^s(T)/T)^a_{\mathbf{Q}} = R^a_{\mathbf{Q}}/\mathbf{a}^a_{\mathbf{Q}} \simeq \Delta/R^W_{\mathbf{Q}} \cdot \Delta \cap \mathbf{a}_{\mathbf{Q}} \simeq \left(R^W_{\mathbf{Q}}/p(\mathbf{a}_{\mathbf{Q}})\right)\Delta \simeq R^W_{\mathbf{Q}}/p(\mathbf{a}_{\mathbf{Q}}).$$

(ii) For a moment let $A^* = A^*(V^s(T)/T)_{\mathbf{Q}}$. First, note that there are no anti-invariants in $R_{\mathbf{Q}}$ of degree less than δ, hence neither in $A^*(V^s(T)/T)_{\mathbf{Q}}$.

The multiplication in A^* is a perfect pairing which is W-equivariant. Since the action of W on A^* is induced by an action of W on $V^s(T)/T$, the class of a point is invariant; i.e., W acts trivially on A^e. Hence the pairing induces isomorphisms of $\mathbf{Q}W$-modules $A^i \simeq (A^{e-i})^\vee$. For any $\mathbf{Q}W$-module M, there is a canonical isomorphism $(M^a)^\vee \simeq (M^\vee)^a$. Therefore $(A^i)^a \simeq ((A^{e-i})^a)^\vee$, and (ii) follows. □

(4.6) LEMMA. (i) $(R_{\mathbf{Q}}^W/p))_i = 0$ *for* $i > d = \dim V^s(G)/G$.

(ii) $(R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}}))_d = \mathbf{Q}$, *and the multiplication in* $R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}})$ *induces perfect pairings*

$$\left(R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}})\right)_i \otimes \left(R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}})\right)_{d-i} \to \left(R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}})\right)_d = \mathbf{Q}.$$

Proof. (i) is just a rephrasing of (i) and (ii) of Lemma (4.5). To prove (ii), note first that although the map ρ from (4.5) is not multiplicative, it has the property that

$$\rho(x)\rho(x') = |W|\rho(\Delta^{-1}xx')$$

for $x, x' \in (A^*)^a$. Assume that $y \in (R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}}))_i$; then $y = \rho(x)$ for some $x \in (A^{i+\delta})^a$. By the end of the proof of (4.5), there exists an element $x' \in (A^{e-i-\delta})^a$ with $xx' \neq 0$. Write $x = \Delta z$ and $x' = \Delta z'$; then

$$y\rho(x') = \rho(x)\rho(x') = \rho(\Delta z)\rho(\Delta z') = |W|\rho(\Delta zz') \neq 0$$

since $\Delta zz'$ is a non-zero anti-invariant. □

To finish the proof of the first part of (i) of the theorem, note that $\bar{\beta}$ maps the highest degree part $(R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}}))_d$ isomorphically onto $A^d(V^s(G)/G)_{\mathbf{Q}}$, since by (4.6), $(R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}}))_d \simeq \mathbf{Q}$. Let $0 \neq x \in (R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}}))_i$. The pairing is perfect, so there exists a $y \in (R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}}))_{d-i}$ with $xy \neq 0$. Hence $0 \neq \bar{\beta}(xy) = \bar{\beta}(x)\bar{\beta}(y)$, and $\bar{\beta}(x) \neq 0$.

The second statement of (i) follows because all the arguments above remain valid if we replace the Chow ring by singular cohomology with rational coefficients. Hence both of the rings $A^*(V^s(G)/G)_{\mathbf{Q}}$ and $H^*(V^s(G)/G, \mathbf{Q})$ are isomorphic to $R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}})$.

To prove (ii), note that by (2.9a) we have an isomorphism

$$\pi^*\colon A^*(V^s(G)/G) \simeq A^*(V^s(G)/B)^W.$$

It follows that the diagram in the proof of Theorem 4.4 can be defined over the integers in this case:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{a} & \longrightarrow & \mathbf{Z}[\chi_1,\dots,\chi_t] & \xrightarrow{\alpha} & A^*(V^s(T)/T) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p_1} & & \\
0 & \longrightarrow & p(\mathbf{a}) & \longrightarrow & \mathbf{Z}[\chi_1,\dots,\chi_t]^W & \xrightarrow{\beta} & A^*(V^s(G)/G) & \longrightarrow & 0.
\end{array}$$

This still commutes, but the map β is no longer a priori surjective. The top row is exact by (3.4). Since G is torsion-free, there exists an element

$u \in \mathbf{Z}[\chi_1, \dots, \chi_t]$ which induces the class of a point in G/B under the characteristic homomorphism c of (1.3). Then $(f^*)^{-1} i^* \alpha(u) \in A^*(V^s(G)/B)$ induces the class of a point on each fiber $\pi^{-1}(y) \simeq G/B$ of π. Therefore $p_1(\alpha(u)) = 1$, and it follows that β is surjective. Now by (i) above, $\operatorname{Ker}\beta/p(\mathbf{a})$ is torsion. Since $A^*(V^s(G)/G)$ is torsion-free by (2.9a), $\operatorname{Ker}\beta$ must be the saturation $p(\mathbf{a})_{\mathbf{Q}} \cap \mathbf{Z}[\chi_1, \dots, \chi_t]^W$ of $p(\mathbf{a})$. □

Note that the inverse of the isomorphism in Lemma (4.5) may be thought of in the following way: send an element $\bar{r} \in R_{\mathbf{Q}}^W/p(\mathbf{a}_{\mathbf{Q}})$ to the element $\widetilde{\Delta r}$ in $A^*(V^s(T)/T)^a_{\mathbf{Q}}$, where $\bar{r}$ denotes the class of $r \in R_{\mathbf{Q}}^W \bmod p(\mathbf{a}_{\mathbf{Q}})$ and $\tilde{r}$ the class of $r \bmod \mathbf{a}_{\mathbf{Q}}$. The following corollary is useful for calculating in the Chow ring:

(4.7) Corollary. *Let $r_1, r_2 \in \mathbf{Q}[\chi_1, \dots, \chi_t]^W$. Then*

$$\bar{r}_1 = \bar{r}_2 \quad (\text{in } A^*(V^s(G)/G)_{\mathbf{Q}}) \quad \Leftrightarrow \quad \widetilde{\Delta r_1} = \widetilde{\Delta r_2} \quad (\textit{in } \mathbf{Q}[\chi_1, \dots, \chi_t]/\mathbf{a}_{\mathbf{Q}}).$$

We end this section by noting that (4.4) also gives the structure of the Chow ring of $V^s(G)/B$:

(4.8) Corollary. *In the situation of* (4.4), *there is an isomorphism of graded rings*:

$$\mathbf{Q}[\chi_1, \dots, \chi_t]/p(\mathbf{a}_{\mathbf{Q}})\mathbf{Q}[\chi_1, \dots, \chi_t] \simeq A^*(V^s(G)/B)_{\mathbf{Q}}.$$

Proof. There is a surjection $(f^*)^{-1} \circ i^*\colon \mathbf{Q}[\chi_1, \dots, \chi_t] \to A^*(V^s(G)/B)_{\mathbf{Q}}$. The latter ring is an $A^*(V^s(G)/G)_{\mathbf{Q}}$-algebra via π^*. It follows that $p(\mathbf{a}_{\mathbf{Q}})\mathbf{Q}[\chi_1, \dots, \chi_t]$ is contained in the kernel of $(f^*)^{-1} \circ i^*$. The resulting map

$$\mathbf{Q}[\chi_1, \dots, \chi_t]/p(\mathbf{a}_{\mathbf{Q}})\mathbf{Q}[\chi_1, \dots, \chi_t] \to A^*(V^s(G)/B)_{\mathbf{Q}}$$

is a surjective map of free $A^*(V^s(G)/G)_{\mathbf{Q}}$-modules of the same rank, in view of (2.8), (ii) with $X = V^s(G)$. □

5. Illustration: The degree of a Grassmann variety

(5.1) This example is the Grassmann variety $G(d, n)$ of d-planes in $\mathbf{P}^n$, or equivalently, of rank-$(d+1)$ quotients of k^{n+1}. Its dimension is $N = (n-d)(d+1)$.

Put $V = \operatorname{Hom}(k^{n+1}, k^{d+1})$ and let $G = \operatorname{Gl}(d+1)$ act on V via the canonical representation in k^{d+1}. Then it turns out that $V^s(G) = V^{ss}(G) = \{\text{surjective maps}\}$, the action is free on $V^s(G)$, and the quotient is exactly $G(d, n)$.

Put $R_{\mathbf{Q}} = \mathbf{Q}[\chi_0, \dots, \chi_d]$. The ideal $\mathbf{a}$ becomes $\mathbf{a} = (\chi_0^{n+1}, \dots, \chi_d^{n+1})$. It turns out that under the isomorphism of Theorem (4.4), the Chern classes c_i of the universal quotient bundle on $G(d, n)$ correspond to the elementary symmetric polynomials in the χ_i. In particular, $c_1 = \Sigma\bar{\chi}_i$ and $c_{d+1} = \Pi\bar{\chi}_i$. The class of a point in $G(d,n)$ is $c_{d+1}^{n-d} = (\bar{\chi}_0 \cdots \bar{\chi}_d)^{n-d}$. The degree d_0 of the Grassmannian in its Plücker imbedding is the degree of c_1^N.

Let us use (4.7) to compute d_0. (The following calculation was worked out in collaboration with Kristian Ranestad.) We have the equation $(\Sigma\bar{\chi}_i)^N = d_0\Pi\bar{\chi}_i^{n-d}$ in $A^N(G(d,n))$. By (4.7) we get the following congruence in R_Q:

$$\Delta\left(\sum \chi_i\right)^N \equiv d_0\,\Delta\prod\chi_i \quad \left(\operatorname{mod}\left(\chi_0^{n+1},\dots,\chi_d^{n+1}\right)\right).$$

Now in this case, $W = S_{d+1}$ is the symmetric group, and the discriminant is $\Delta = \Sigma_{\tau\in S_{d+1}} \operatorname{sign}(\tau)\Pi_{i=0}^d \chi_i^{\tau(i)}$. Compare coefficients for the monomial $\Pi_{i=0}^d \chi_i^{n-d+i}$ on the two sides of the congruence above. On the right-hand side this coefficient is d_0, and on the left-hand side we find:

$$\begin{aligned}
&\sum_{\tau\in S_{d+1}} \operatorname{sign}(\tau)\frac{N!}{\prod_{i=0}^d(n-d+i-\tau(i))!}\\
&\quad = N!\det\left(\frac{1}{(n-d+i-j)!}\right)\\
&\quad = N!\det\left(\binom{n-j}{d-i}\frac{(d-i)!}{(n-j)!}\right)\\
&\quad = \frac{N!\,1!\,2!\cdots d!}{(n-d)!(n-d+1)!\cdots n!}\det\left(\binom{n-j}{d-i}\right)\\
&\quad = \frac{N!\,1!\,2!\cdots d!}{(n-d)!(n-d+1)!\cdots n!},
\end{aligned}$$

which is thus the degree of $G(d,n)$. (See [Fu, Ex. 14.7.11] for a different derivation of this number.)

6. Example: Linear matrices

(6.1) The next example is the quotient space $N(q;m,n)$ of $n\times m$-matrices with entries in a q-dimensional vector space, modulo row and column operations, where q, n, and m are natural numbers. (In [Dr1], such matrices are called "Kronecker modules".) Specifically, let U, E, and F be vector spaces of dimensions q, n, and m, respectively. Put $G' = \mathrm{Gl}(E)\times\mathrm{Gl}(F)$, and let it act on $V = \mathrm{Hom}(F\otimes U, E)$ via

$$(g,h)\cdot\alpha = g\circ\alpha\circ\left(h^{-1}\otimes \mathrm{id}_U\right)$$

if $(g,h)\in G'$ and $\alpha\in V$. The closed, normal subgroup $\Gamma = \{(t,t)\,|\,t\in k^*\}\subseteq G'$ acts trivially; hence there is induced an action of $G = G'/\Gamma$ on V. This action induces an imbedding of G in $\mathrm{Gl}(V)$, and the composed morphism

$$S = \mathrm{Sl}(E) \times \mathrm{Sl}(F) \to G' \to G$$

is a finite surjection onto $G_0 = G \cap \mathrm{Sl}(V)$, with kernel isomorphic to the group of d-th roots of unity, where $d = \mathrm{lcd}(n, m)$. For the analysis of stability we may therefore look at the action of S.

If we choose bases for E and F, an element $\alpha \in V$ can be identified with an $n \times m$-matrix with coefficients in $U^\vee$. There is an obvious G-equivariant isomorphism $V \simeq \mathrm{Hom}(E^\vee \otimes U, F^\vee)$ which corresponds to taking the transpose of this matrix. With no loss of generality we shall therefore assume that $m \leq n$.

Let $T_E \subseteq \mathrm{Gl}(E)$ and $T_F \subseteq \mathrm{Gl}(F)$ be maximal tori; then $T = T_E \times T_F/\Gamma$ is a maximal torus in G.

(6.2) LEMMA. *Let $\alpha \in V$. (i) α is stable (resp. semistable) with respect to T if and only if for any non-zero T-invariant subspace $F' \subseteq F$ and any proper T-invariant subspace $E' \subset E$ such that $\alpha(F' \otimes U) \subseteq E'$,*

$$\frac{\dim E'}{\dim F'} > \frac{n}{m} \qquad \left(\textit{resp.}\ \frac{\dim E'}{\dim F'} \geq \frac{n}{m}\right).$$

(ii) α is stable (resp. semistable) with respect to G if and only if for any non-zero subspace $F' \subseteq F$ and proper subspace $E' \subset E$ such that $\alpha(F' \otimes U) \subseteq E'$,

$$\frac{\dim E'}{\dim F'} > \frac{n}{m} \qquad \left(\textit{resp.}\ \frac{\dim E'}{\dim F'} \geq \frac{n}{m}\right).$$

Proof. Part (ii) of this lemma is [Dr1, Prop. 15], and the proof given there actually proves part (i) as well. □

(6.3) If n and m are relatively prime, all semistable points are stable, both with respect to G and T. Henceforth we shall assume that this is the case.

(6.4) PROPOSITION. *The action of G on $V^s(G)$ is free.*

Proof. The following lemma implies immediately that the action is set-theoretically free. It is not hard, using the same lemma, to show that it is scheme-theoretically free as well.

(6.5) LEMMA. *Let $\alpha \in V^s(G)$, $g \in \mathrm{End}(E)$, and $h \in \mathrm{End}(F)$. If $\alpha \circ (h \otimes \mathrm{id}_U) = g \circ \alpha$, then there exists a $t \in k$ such that $g = t\,\mathrm{id}_E$ and $h = t\,\mathrm{id}_F$.*

Proof. Let t be an eigenvalue of h. Then clearly $\alpha \circ ((h - t\,\mathrm{id}_F) \otimes \mathrm{id}_U) = (g - t\,\mathrm{id}_E) \circ \alpha$. Hence we may assume that $\mathrm{Ker}\, h \neq 0$ and then prove that $g = h = 0$.

From the assumption of the lemma it follows that $\alpha(\operatorname{Ker} h \otimes U) \subseteq \operatorname{Ker} g$ and $\alpha(\operatorname{Im} h \otimes U) \subseteq \operatorname{Im} g$. Put $k = \dim \operatorname{Ker} h$, and assume that $0 < k < m$. Then (6.2(ii)) gives $\dim \operatorname{Ker} g > kn/m$ and $\dim \operatorname{Im} g > (m-k)n/m$. But this contradicts the fact that $\dim \operatorname{Ker} g + \dim \operatorname{Im} g = n$. It follows that $h = 0$. But α itself is surjective by stability, which implies that $g = 0$. □

(6.6) *Remark*. There is an analogous lemma valid for $\alpha \in V^s(T)$, $g \in \operatorname{Lie}(T_E)$, and $h \in \operatorname{Lie}(T_F)$, implying that T acts freely on $V^s(T)$. The proof is the same if one notes that the kernels and images are T-invariant in this case.

(6.7) We are now in position to apply Theorem (4.4) to the present situation. Given the product structure of V and G, it is convenient to use multi-indices in the constructions of Sections 3 and 4.

Fix bases for U, E, and F. Let $D \subseteq \mathrm{Gl}(V)$, $T_E \subseteq \mathrm{Gl}(E)$, and $T_F \subseteq \mathrm{Gl}(F)$ be the corresponding maximal tori; then

$$T = (T_E \times T_F)/\Gamma = G \cap D$$

is a maximal torus in G. Let $\{\psi_{ij\mu}\}$, $\{\varepsilon_i\}$, and $\{\phi_j\}$ be corresponding bases for the character groups $M(D)$, $M(T_E)$, and $M(T_F)$ respectively, $i \in [1, n]$, $j \in [1, m]$, $\mu \in [1, q]$. The morphism $G' \to \mathrm{Gl}(V)$ from the action induces a diagram

$$\begin{tikzcd} 0 \arrow[r] & M(D/T) \arrow[r] & M(D) \arrow[r, "a"] & M(T) \arrow[r] \arrow[d, "i"] & 0 \\ & & & M(T_E) \oplus M(T_F) & \end{tikzcd}$$

where the composition $i \circ a$ is given by $i \circ a(\psi_{ij\mu}) = \varepsilon_i - \phi_j$, all i, j, and μ. The inclusion i identifies $M(T)$ with the subgroup

$$\left\{ \sum k_i \varepsilon_i + \sum l_j \phi_j \mid k_i, l_j \in \mathbf{Z}, \sum k_i + \sum l_j = 0 \right\} \subseteq M(T_E) \oplus M(T_F).$$

To find a generator set for this subgroup, let a and b be integers such that $an + bm = 1$, let $\omega = a\sum_{i=1}^n \varepsilon_i + b\sum_{j=1}^m \phi_j$, and put $\beta_i = \varepsilon_i - \omega$ and $\gamma_j = \phi_j - \omega$. Then the polynomial ring $R = \operatorname{Sym}_{\mathbf{Z}}(M(T))$ can be written

$$R = \mathbf{Z}[\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m],$$

where the variables are free except for the relation $a\sum_{i=1}^n \beta_i + b\sum_{j=1}^m \gamma_j = 0$.

(6.8) Fix an element $\alpha \in V$. With respect to the bases fixed in (6.7), there corresponds an $n \times m$ matrix A with entries in $U^\vee$. From (6.2(i)) it follows that

α is not stable with respect to T if and only if A contains an $(n-l) \times k$ submatrix consisting entirely of zeros from some $k \in [1, m]$, $l \in [1, n]$ with $l/k < n/m$, or equivalently, $l \leq \lceil kn/m \rceil - 1$, where $\lceil x \rceil = \min\{p \in \mathbf{Z} | x \leq p\}$ for any real number x. In short, α is T-unstable if and only if A has an $(n + 1 - \lceil kn/m \rceil) \times k$ submatrix consisting of zeros, for some $k \in [1, m]$.

Referring to the construction in Section 3, we get from the above that the ideal $\mathbf{a}' \subseteq \mathbf{Z}[\psi_{ij\mu}]$ is generated by the monomials

$$M'_{\pi, k} = \prod_{j=1}^{k} \prod_{i=\lceil kn/m \rceil}^{n} \prod_{\mu=1}^{q} \psi_{\sigma(i)\tau(j)\mu}, \qquad \pi = (\sigma, \tau) \in S_n \times S_m,\ k \in [1, m].$$

The image $\mathbf{a} \subseteq R$ of $\mathbf{a}'$ is generated by the polynomials

$$M_{\pi, k} = \prod_{j=1}^{k} \prod_{i=\lceil kn/m \rceil}^{n} \left(\beta_{\sigma(i)} - \gamma_{\tau(j)}\right)^q, \qquad \pi = (\sigma, \tau) \in S_n \times S_m,\ k \in [1, m].$$

The Weyl group $W = S_n \times S_m$ acts by $(\sigma, \tau) \cdot \varepsilon_i = \varepsilon_{\sigma(i)}$ and $(\sigma, \tau) \cdot \phi_j = \phi_{\tau(j)}$ for $(\sigma, \tau) \in W$. The action on R is similar: $(\sigma, \tau) \cdot \beta_i = \beta_{\sigma(i)}$ and $(\sigma, \tau) \cdot \gamma_j = \gamma_{\tau(j)}$.

(6.9) Theorem. *Let V and G be as in* (6.1), *and assume that $an + bm = 1$ for some integers a and b.*

(i) *There is an isomorphism of rings*

$$\mathbf{Z}[\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_m]/\left(L, \{M_{\pi, k} | \pi \in W,\ k \in [1, m]\}\right) \simeq A^*(V^s(T)/T),$$

where

$$M_{\pi, k} = \prod_{j=1}^{m} \prod_{i=\lceil kn/m \rceil}^{n} \left(\beta_{\sigma(i)} - \gamma_{\tau(j)}\right)^q, \quad \textit{and} \quad L = a \sum_{i=1}^{n} \beta_i + b \sum_{j=1}^{m} \gamma_j.$$

(ii) *Let b_i (resp. c_j) be the elementary symmetric polynomials in the β_i (resp. γ_j). Then there is an isomorphism of rings*

$$\mathbf{Z}[b_1, \dots, b_n, c_1, \dots, c_m]/p(\mathbf{a}_{\mathbf{Q}}) \cap \mathbf{Z}[b_1, \dots, b_n, c_1, \dots, c_m] \simeq A^*(V^s(G)/G),$$

where $\mathbf{a}$ *is the ideal in* (i) *and*

$$p \colon \mathbf{Q}[\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_m] \to \mathbf{Q}[b_1, \dots, b_n, c_1, \dots, c_m]$$

is the morphism defined by $p(r) = \Delta^{-1} \Sigma_{w \in W} \operatorname{sign}(w) w(r)$, *and*

$$\Delta = \prod_{1 \leq k < i \leq n} (\beta_i - \beta_k) \prod_{1 \leq l < j \leq m} (\gamma_j - \gamma_l).$$

(iii) *If the base field is* $\mathbf{C}$, *the cycle map is an isomorphism*

$$A^*(V^s(G)/G) \simeq H^*(V^s(G)/G, \mathbf{Z}).$$

Proof. We just apply (3.8) and (4.4) in this situation, noting that G is a special group and using (6.3) and (6.6). □

(6.10) *Remark.* The theorem above gives the ring structure of the Chow ring of the quotient $N(q; m, n) = A^*(V^s(G)/G)$ in terms of algebra generators and relations. When it comes to actually computing intersection products, however, the easier approach is often to do the calculations "upstairs" in $A^*(V^s(T)/T)$, which has significantly simpler structure. Corollary (4.7) is one way of formulating this principle. The example of the degree of the Grassmannian goes to show the same thing. In the next section we also exploit the fact that $A^*(V^s(G)/G)$ is isomorphic to the anti-invariant part of $A^*(V^s(T)/T)$ in computing the Betti numbers.

(6.11) In the case of the Grassmannian $G(d, n)$ (which, by the way, is the special case $N(n+1; 1, d+1)$ of the present example), there exists a universal bundle whose Chern classes generate the Chow ring. What about the $n \times m$ matrix variety $N(q; n, m)$? One could try to push down the trivial vector bundles $E_{V^s(G)}$ and $F_{V^s(G)}$ to vector bundles on $N(q; m, n) = V^s(G)/G$. This does not work directly, as the group G does not act on E and F, the action of Γ being non-trivial. However, Γ does act trivially on the G'-modules $E' = E \otimes \det(F)^{-b} \otimes \det(E)^{-a}$ and $F' = F \otimes \det(F)^{-b} \otimes \det(E)^{-a}$; hence these are G-modules. Since the action of G is free on $V^s(G)$, these G-modules descend to vector bundles $\mathscr{E}$ and $\mathscr{F}$ on $N(q; m, n)$. (There is even a "universal" map $\tilde{\alpha}$: $\mathscr{E} \otimes U \to \mathscr{F}$ of bundles on $N(q; m, n)$.) One can check that

(i) $V^s(G)/B$ is the fiber product over $N(q; m, n)$ of the flag bundles associated to the vector bundles $\mathscr{E}$ and $\mathscr{F}$,

(ii) The β_i (resp. γ_j) correspond to the Chern roots of $\mathscr{E}$ (resp. $\mathscr{F}$).

(iii) The b_i (resp. c_j) correspond to the Chern classes of $\mathscr{E}$ (resp. $\mathscr{F}$).

It seems reasonable to believe that in the situation of Theorem (4.4), there will always exist naturally given vector bundles on $V^s(G)/G$, the Chern classes of which will generate the Chow ring, at least over the rational numbers.

7. Betti numbers for 3 × 2 matrices

(7.1) We shall proceed to compute the Betti numbers of the matrix variety $N(q; m, n) = V^s(G)/G$. The strategy is to determine the class of the ring $R/\mathbf{a} = A^*(V^s(T)/T)$ in the representation ring $R(W)$ of the Weyl group. Once we have this class, the anti-invariant part can be read off by taking the inner product of $R/\mathbf{a}$ with the one-dimensional representation Λ of W given by the character "sign". To obtain a closed formula for the Betti numbers in general seems to require more technical skill than we have at our disposal; so we shall be content with the case of 3×2 matrices. However, we shall present much of the argument for general n, m.

One reason that the 3×2 case is of particular interest is the connection with the Hilbert scheme of three points in the plane and of twisted cubics, which we sketch below. Details can be found in [E-P-S].

There is a birational map h: $N(q; 3, 2) \cdots \to H$, where H is the irreducible component of the Hilbert scheme of $\mathbf{P}^{q-1}$ containing the codimension 2, degree 3, arithmetically Cohen Macaulay subschemes. For $q = 3$, H is the whole of $\mathrm{Hilb}^3(\mathbf{P}^2)$. For $q = 4$, H is the closure of the set parametrizing twisted cubics. By [P-S], this is smooth.

The map h is given by sending a 3×2 matrix α to the subscheme of $\mathbf{P}^{q-1}$ defined by the vanishing of the 2×2 minors of α, this map being defined off the closed subvariety $I \subseteq N(q; 3, 2)$ of those α for which the minors have a common linear factor. The inverse of h is given generically by sending a subscheme Z to the matrix of a minimal resolution of $\mathcal{O}_Z$. This extends to a morphism $H \to N(q; 3, 2)$. It can be shown that at least for $q \leq 4$ this morphism is the blowing up of $N(q; 3, 2)$ along I. The structure of I and its normal bundle are easy to determine; in fact, $I \simeq F(q, q-1, q-3)$ is the flag variety parametrizing a hyperplane of $\mathbf{P}^{q-1}$ and a codimension 3 linear subspace of $\mathbf{P}^{q-1}$ contained in the hyperplane. Hence the Chow ring of H can be computed from that of $N(q; 3, 2)$.

For generalities and terminology on representation theory for finite groups, we refer to Serre's treatment [Se2]. We are concerned with the group $W = S_n \times S_m$. Abusing notation, if $\sigma \in S_n$, we shall also denote its image $(\sigma, 1) \in W$ by σ, and likewise with the second factor. In the same vein, if M is a representation of S_m, we consider it a representation of W with S_n acting trivially, and so on. Almost all the representations we shall use are of the form $M \otimes N$, where M is a $\mathbf{Q}S_m$-module and N is a $\mathbf{Q}S_n$-module.

Some important representations are the permutation representations:

$$M_m = \mathbf{Q}^m \text{ with } S_m \text{ permuting the coordinates, and}$$

$$N_n = \mathbf{Q}^n \text{ with } S_n \text{ permuting the coordinates.}$$

These decompose as a direct sum of a trivial one-dimensional space and an invariant irreducible hyperplane:

$$M_{m-1} = \left\{(x_1, \ldots, x_m) \in M_m \mid \sum x_j = 0\right\},$$

$$N_{n-1} = \left\{(x_1, \ldots, x_n) \in N_n \mid \sum x_i = 0\right\}.$$

The anti-invariant representation Λ above is nothing but $\Lambda = (\wedge^m M_m) \otimes (\wedge^n N_n)$.

(7.2) Let $P = \mathbf{Q}[X]$ be a polynomial ring in the variables $\{X_{ij} | (i, j) \in [1, n] \times [1, m]\}$. Write a typical monomial in P as $X^d = \prod X_{ij}^{d_{ij}}$, where $d = (d_{ij})$ is an $n \times m$ matrix with nonnegative integers as entries.

Let $\Psi \subseteq \{0, 1\}^{nm}$ consist of those $n \times m$-matrices with entries in $\{0, 1\}$ that do not contain any $(n + 1 - \lceil kn/m \rceil \times k$ submatrix consisting entirely of 1's, for any $k \in [1, m]$. The set Ψ is invariant under the operation of changing entries from 1 to 0. This induces a simplicial structure on Ψ, a matrix d' being a face of d if all entries in $d - d'$ are in $\{0, 1\}$.

Next let $I \subseteq P$ be the ideal generated by the monomials X^d for $d \notin \Psi$. There is a natural action of W on P, with S_n permuting the first set of indices, and S_m permuting the second set. The ideal I is invariant under this action. As a graded $\mathbf{Q}W$-module, I has a graded invariant complement in P: Let $K \subseteq P$ be the vector space generated by the monomials X^d for which $d_{\mathrm{red}} \in \Psi$, where $(d_{\mathrm{red}})_{i,j} = \min\{1, d_{ij}\}$. Then K is a graded $\mathbf{Q}W$-module which maps isomorphically onto P/I. We can decompose K into pieces indexed by Ψ as follows:

$$K \simeq \bigoplus_{d \in \Psi} X^d \operatorname{Sym}_{\mathbf{Q}}\langle d \rangle,$$

where $\langle d \rangle$ denotes the linear span of the set of variables $\{X_{ij} | d_{ij} \neq 0\} \subseteq P$.

For reasons that will become clear later, we shall equip everything here with the grading such that all the variables X_{ij} have degree q. This permits us to extract some qth roots:

Let $S = \mathbf{Q}[Y]$ be the polynomial ring in the variables Y_{ij} indexed by the same set as the X_{ij}. Now S is graded by assigning each Y_{ij} degree 1. Consider P as a subring of S under the identification $X_{ij} = Y_{ij}^q$. If we put $H = \mathbf{Q}[Y]/(Y_{ij}^q)$, there is an isomorphism of graded $\mathbf{Q}W$-modules $P \otimes_{\mathbf{Q}} H \simeq S$ induced by $X_{ij} \otimes \overline{Y}_{kl} \mapsto Y_{ij}^q Y_{kl}$.

(7.3) Proposition. *The ring homomorphism* κ: $S \to R_{\mathbf{Q}}$ *given by* $\kappa(Y_{ij}) = \beta_i - \gamma_j$ *induces a W-equivariant isomorphism of graded rings*:

$$S/IS \otimes_S R_{\mathbf{Q}} \simeq R_{\mathbf{Q}}/IR_{\mathbf{Q}} \simeq R_{\mathbf{Q}}/\mathbf{a}_{\mathbf{Q}}.$$

Furthermore, $\operatorname{Tor}_k^S(R_{\mathbf{Q}}, S/IS) = 0$ *for* $k > 0$.

Proof. Construct a homomorphism $\mathbf{Q}[\psi_{ij\mu}] \to S$ by sending $\psi_{ij\mu} \mapsto Y_{ij}$. Clearly, this is a W-equivariant surjection, and the homomorphism $\mathbf{Q}[\psi_{ij\mu}] \to R_{\mathbf{Q}}$ which comes from the map a of (6.7) factors through κ. The image of $\mathbf{a}'$ in S is nothing but IS, and the first statement follows. The second statement is a consequence of the following facts:

(i) S/IS is Cohen-Macaulay of Krull dimension $(m-1)(n-1)$. (By [Ho, Cor. 6.8], the ring P/I (and hence also S/IS) is Gorenstein of dimension $d+1$ if Ψ is a triangulation of a d-dimensional sphere. But Ψ is exactly equal to what Φ_s of (3.1) would have been in the special case $q = 1$. By (3.4), Ψ is a triangulation of the $(mn - m - n)$-sphere.)

(ii) $\operatorname{codim}(R_{\mathbf{Q}}, S) = (m-1)(n-1)$. This is clear.

(iii) $R_{\mathbf{Q}}/IR_{\mathbf{Q}}$ is of Krull dimension 0. Indeed, it is a Chow ring. □

(7.4) Let us determine all these objects in the representation ring $R(W)$ of W. To keep track of the grading, we shall use a free variable T, so that we are working in the ring $R(W)[T, T^{-1}]$ (which can be identified with $R(W \times \mathbf{G}_m)$, if we think of the grading as induced by a $\mathbf{G}_m$-action). Thus for a graded module M we shall write $M = \sum_k M_k T^k$. With this convention

$$P = \sum_k \operatorname{Sym}_k(M_m \otimes N_n) T^{qk},$$

$$S = \sum_k \operatorname{Sym}_k(M_m \otimes N_n) T^k,$$

$$H = S \otimes \sum_k (-1)^k \wedge^k (M_m \otimes N_n) T^{qk},$$

$$R_{\mathbf{Q}} = S \otimes \sum_k (-1)^k \wedge^k (M_{m-1} \otimes N_{n-1}) T^k.$$

The last two arise from the Koszul resolutions of H and $R_{\mathbf{Q}}$ as S-modules: The relations Y_{ij}^m clearly span a copy of $(M_m \otimes N_n)T^q$, whereas the kernel of κ is generated by the linear forms $Y_{11} + Y_{ij} - Y_{1j} - Y_{i1}$, and these forms span a $(M_{m-1} \otimes N_{n-1})T$.

Putting all this together, we get the following equalities in the representation ring:

$$\begin{aligned} R_{\mathbf{Q}}/IR_{\mathbf{Q}} &= P/I \otimes_P S \otimes_{\mathbf{Q}} B \\ &= P/I \otimes_P P \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B \\ &= K \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B, \end{aligned}$$

where $B = \sum_k (-1)^k \wedge^k (M_{m-1} \otimes N_{n-1}) T^k$.

The Betti numbers rank $A^k(N(q; m, n))$ can now be computed as

$$\operatorname{rank} A^k(N(q; m, n)) = \left\langle \Lambda, (R_{\mathbf{Q}}/IR_{\mathbf{Q}})_{\delta+k} \right\rangle_W$$

where $\langle\ ,\ \rangle_W$ is the scalar product of representations. Note the shift in degrees; $\delta = \binom{m}{2} + \binom{n}{2}$ is the degree of the discriminant. We get the following formula for the Betti series of $N(q; m, n)$:

$$\sum_k \operatorname{rank} A^k(N(q; m, n))T^k = T^{-\delta}\langle \Lambda, K \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B\rangle_W.$$

(7.5) To compute this scalar product, we pass to the characters of the representations. The character of a $\mathbf{Q}W$-module M is by definition the function χ_M: $W \to \mathbf{C}$ given by $\chi_M(s) = \operatorname{Trace}(s\colon M \to M)$. For a graded $\mathbf{Q}W$-module M, we put $\chi_M(s, T) = \sum_k \chi_{M_k}(s)T^k$. The scalar product of two representations can be computed by the formula

$$\langle M, N\rangle_W = \frac{1}{|W|} \sum_{s \in W} \chi_M(s^{-1})\chi_N(s).$$

In the present case, s and s^{-1} are always conjugate in W. This implies that the formula above can be simplified to

$$\langle M, N\rangle_W = \frac{1}{|W|} \sum_{s \in W} \chi_M(s)\chi_N(s).$$

The character of the symmetric algebra on a representation and the character of a Koszul complex (essentially an exterior algebra) are given in [Se2, Ex. 9.3]. Applied to our modules, they give the following formulas:

$$\begin{aligned}
\chi_S(s, T) &= 1/\det(1 - Tr(s)),\\
\chi_P(s, T) &= 1/\det(1 - T^q r(s)),\\
\chi_H(s, T) &= \det(1 - T^q r(s))/\det(1 - Tr(s)),\\
\chi_B(s, T) &= \det(1 - Tr'(s)),
\end{aligned}$$

where r: $W \to \mathrm{Gl}(M_m \otimes N_n)$ and r': $W \to \mathrm{Gl}(M_{m-1} \otimes N_{n-1})$ are the homomorphisms giving the representations.

(7.6) From now on, we shall assume that $m = 2$ and $n = 3$. Then there are six conjugacy classes in W. We pick representatives of each class as follows: Let

σ = (3-cycle, identity),

τ = (transposition, identity),

ρ = (identity, transposition).

Then every element of W is conjugate to exactly one element in $C = \{1, \sigma, \tau, \rho, \rho\sigma, \rho\tau\}$. Let $n(s)$ be the number of elements in the conjugacy class of s. The following table shows for each representative s the values of $n(s)$, sign(s), and the rational functions (in fact, polynomials) $\chi_H(s, T)$ and $\chi_B(s, T)$.

s	$n(s)$	sign(s)	$\chi_H(s, T)$	$\chi_B(s, T)$
1	1	1	$\dfrac{(1-T^q)^6}{(1-T)^6}$	$(1-T)^2$
σ	2	1	$\dfrac{(1-T^{3q})^2}{(1-T^3)^2}$	$(1+T+T^2)$
τ	3	-1	$\dfrac{(1-T^q)^4(1+T^q)^2}{(1-T)^4(1+T)^2}$	$(1-T)(1+T)$
ρ	1	-1	$\dfrac{(1-T^q)^3(1+T^q)^3}{(1-T)^3(1+T)^3}$	$(1+T)^2$
$\rho\sigma$	2	-1	$\dfrac{(1-T^{3q})(1+T^{3q})}{(1-T^3)(1+T^3)}$	$(1-T+T^2)$
$\rho\tau$	3	1	$\dfrac{(1-T^q)^3(1+T^q)^3}{(1-T)^3(1+T)^3}$	$(1-T)(1+T)$

We also need to know a little more about K. Recall from (7.2) that it has a decomposition into pieces of the form $X^d\,\mathrm{Sym}\langle d\rangle$. Let $\Psi_k = \{d \in \Psi \mid \Sigma d_{ij} = k\}$. In the 3×2 case it is easy to verify that the complex Ψ consists of the three pieces Ψ_0, which consists of the zero matrix alone, Ψ_1, which consists of the W-orbit of the matrix

$$d^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and Ψ_2, which is the orbit of

$$d^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There is a corresponding decomposition of $K = K^{(0)} \oplus K^{(1)} \oplus K^{(2)}$. Here $K^{(0)}$ is just the trivial one-dimensional representation, but the other two are more interesting. If we realize $\tau \in W$ as the transposition of the last two rows and ρ as the transposition of the two columns, then the stabilizer of $d^{(1)}$ is $W_1 = \{1, \tau\}$ and the stabilizer of $d^{(2)}$ is $W_2 = \{1, \rho\tau\}$. Furthermore, W_1 acts trivially on

$\langle d^{(1)}\rangle$, whereas $\langle d^{(2)}\rangle$ is the permutation representation of W_2. In each case, $K^{(k)} = \mathrm{Ind}_{W_k}^{W}(\mathrm{Sym}\langle d^{(k)}\rangle)$ is the induced representation. This gives us the opportunity to use Frobenius reciprocity [Se2, Thm. 13] to calculate the scalar product.

(7.7) The last formula of (7.4) can be rewritten as follows:

$$\begin{aligned} T^4\sum_k \operatorname{rank} A^k\big(N(q;2,3)T^k &= \langle K, \Lambda \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B\rangle_W \\ &= \langle K^{(0)}, \Lambda \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B\rangle_W \\ &\quad + \langle K^{(1)}, \Lambda \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B\rangle_W \\ &\quad + \langle K^{(2)}, \Lambda \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B\rangle_W. \end{aligned}$$

The first of the three terms is

$$\left\langle K^{(0)}, \Lambda \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B\right\rangle_W = \frac{1}{12}\sum_{s\in C} n(s)\operatorname{sign}(s)\chi_H(s,T)\chi_B(s,T).$$

For the second one, use the reciprocity formula to get

$$\begin{aligned} \left\langle K^{(1)}, \Lambda \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B\right\rangle_W &= \left\langle \mathrm{Sym}\left\langle d^{(1)}\right\rangle, \mathrm{Res}_{W_1}^{W}(\Lambda \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B)\right\rangle_{W_1} \\ &= \frac{1}{2}\sum_{s\in\{1,\tau\}} \chi_{\mathrm{Sym}\langle d^{(1)}\rangle}(s,T)\operatorname{sign}(s)\chi_H(s,T)\chi_B(s,T). \end{aligned}$$

Now

$$\chi_{\mathrm{Sym}\langle d^{(1)}\rangle}(s,T) = T^q/(1-T^q)$$

for both values $s = 1$ and $s = \tau$.

For the third, use the reciprocity formula again:

$$\begin{aligned} \left\langle K^{(2)}, \Lambda \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B\right\rangle_W &= \mathrm{Sym}\left\langle d^{(2)}\right\rangle, \mathrm{Res}_{W_2}^{W}(\Lambda \otimes_{\mathbf{Q}} H \otimes_{\mathbf{Q}} B)\right\rangle_{W_2} \\ &= \frac{1}{2}\sum_{s\in\{1,\rho\tau\}} \chi_{\mathrm{Sym}\langle d^{(2)}\rangle}(s,T)\operatorname{sign}(s)\chi_H(s,T)\chi_B(s,T). \end{aligned}$$

Here we have the formulas

$$\begin{aligned} \chi_{\mathrm{Sym}\langle d^{(2)}\rangle}(1,T) &= T^{2q}/(1-T^q)^2, \\ \chi_{\mathrm{Sym}\langle d^{(2)}\rangle}(\rho\tau,T) &= T^{2q}/(1-T^q)(1+T^q). \end{aligned}$$

(7.8) THEOREM. *The Betti series of* $N(q;2,3)$ *is given by*

$$\sum_k \operatorname{rank}\bigl(A^k(N(q;2,3))\bigr)T^k$$
$$= \frac{(1-T^q)(1-T^{q-1})\bigl(T^{q-1}(1-T^{q-1})^2(1+T)+(1-T^{2q})(1-T^{2q-3})\bigr)}{(1-T)^4(1+T)^2(1+T+T^2)}.$$

The topological Euler characteristic is

$$\sum_k \operatorname{rank}\bigl(A^k(N(q;2,3))\bigr) = \frac{1}{6}q(q-1)(3q^2-5q+1).$$

Proof. The Betti series results from what we just did above. If one cancels the factor $(1-T)^4$ and substitutes $T=1$, one gets the Euler characteristic. □

(7.9) *Remarks.* (a) The Euler characteristic can also be calculated as the number of fixed points in $N(q;2,3)$ for the natural action of a maximal torus in $\mathrm{Gl}(U)$. This gives the number with far less effort.

(b) F. Kirwan has proved a recurrence formula for the Betti numbers of quotients of a projective variety by a reductive group; see [Ki]. Similarly, J.-M. Drezet [Dr2] has a recurrence formula for the Betti numbers of the varieties $N(q;m,n)$ (with n,m relatively prime).

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