COUNTING POINTS ON GENERIC CHARACTER VARIETIES OF REDUCTIVE GROUPS

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Abstract. We count points on the character varieties associated with punctured surfaces and regular semisimple generic conjugacy classes in reductive groups. We find that the number of points are palindromic polynomials. This suggests a curious Poincare duality, a curious hard Lefschetz, and a $P = W$ conjecture for these varieties. We also count points on the corresponding additive character varieties and find that the number of points are also polynomials, which we conjecture have non-negative coefficients. These polynomials can be considered as reductive analogues of Kac polynomials associated to comet-shaped quivers.

Contents

1. Introduction

1.1. **Overview.** The relationship between additive and multiplicative character variety becomes tighter when $g = 0$. In this case, the Riemann-Hilbert correspondence implies that multiplicative character variety is the space of isomorphism classes of regular singular connections on G-bundles on punctured $\mathbf{P}^1$, where as additive character variety is the space of isomorphism classes of regular singular connections on the trivial G-bundle on punctured $\mathbf{P}^1$. In particular, when $g = 0$, $\mathbf{Y}$ is a dense open subset of the Hitchin? component of $\mathbf{X}$.

1.2. **Generic character varieties.** Let g and n be integers satisfying $g \geq 0$ and $n > 0$. Let $\Gamma = \Gamma_{g,n}$ be the fundamental group of an orientable surface with genus g and n punctures

$$\Gamma := \frac{\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \rangle}{[a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_n}.$$

Let G be a connected split reductive group over a finite field $k = \mathbb{F}_q$ with centre $Z = Z(G)$ and a maximal split torus T. We assume throughout that Z is connected and that the

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characteristic of k is very good and q is large enough. Let $C = (C_1, \dots, C_n)$ be a tuple of conjugacy classes of G. The representation variety associated to (Γ, G, C) is

$$\mathbf{R} := \left\{ (a_1, a_1, \dots, a_g, b_g, c_1, \dots, c_n) \in G^{2g} \times \prod_{i=1}^{n} C_i \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] \prod_{i=1}^{n} c_i = 1 \right\}. \tag{1}$$

This is an affine scheme of finite type over k with an action of G by conjugation. The associated character variety is

$$\mathbf{X} := \mathbf{R} /\!\!/ G. \tag{2}$$

Assumption 1. Throughout the paper, we will be working under the following assumptions:

(A) $\prod_{i=1}^{n} C_i \subseteq [G, G]$ (otherwise, $\mathbf{R}$ would be empty).

(B) Each conjugacy class C_i is split, semisimple, and strongly regular. In other words, C_i is the conjugacy class of an element $S_i \in T$ satisfying $C_G(S_i) = T$.

(C) The tuple $(C_1, ..., C_n)$ is *generic*; i.e., whenever there exists a proper Levi subgroup $L \subset G$ containing $X_i \in C_i$, $i \in \{1, 2, ..., n\}$, then

$$\prod_{i=1}^{n} X_i \notin [L, L].$$

It is easy to show that such tuples exists, cf. [Boa14, §9] or [Whi24, §5.2]. The generic assumption ensures that the action of G/Z on $\mathbf{R}$ has finite stabiliser. Then the same argument as [Boa14, §9] or [KNP23, §2] implies that $\mathbf{R}$ is smooth and equidimensional. Thus, $\mathbf{X}$ is an orbifold of pure dimension

$$\dim(\mathbf{X}) = (2g - 2 + n) \dim(G) + 2 \dim(Z) - n. \dim(T). \tag{3}$$

The following is our first main result:

Theorem 2. *The number $|\mathbf{X}(\mathbb{F}_q)|$ is a polynomial in q given explicitly in §3.7. This polynomial depends only on (G, g, n); i.e., it is independent of the conjugacy classes (provided they satisfy Assumptions 1). Moreover, this polynomial is palindromic.*

The starting point of the proof is the observation that since G/Z acts on $\mathbf{R}$ with finite stabilisers, the number of points over finite fields of the generic character variety and the character stack coincide. Next, the Frobenius Mass Formula allows us to write the number of points in terms of irreducible complex representations of $G(\mathbb{F}_q)$. We then use results of Deligne and Lusztig to analyse this formula and prove the theorem; see §3 for details.

Remark 3. (i) The leading coefficient of the polynomial $|\mathbf{X}(\mathbb{F}_q)|$ equals the number of connected components of $\mathbf{X}$. It is easy to analyse this polynomial to conclude that if $2g + n \geq 3$, then $|\pi_0(\mathbf{X})| = |\pi_0(Z(\check{G}))|$; see [Whi24, §7.5] for details.

(ii) The value at 1 of the E-polynomial gives the Euler characteristic of X. This can be used to show that if $g > 1$, or $g = 1$ and $\dim(Z) > 0$, then the Euler characteristic of $\mathbf{X}$ is 0. Expressions for the Euler characteristic when $g = 0$ or $g = 1$ and $\dim(Z) = 0$ are given in §6.3; see [Whi24, §7.6] for details.

In lieu of the above theorem, it is reasonable to expect:

Conjecture 4. *The variety $\mathbf{X}$ is Hodge–Tate, and satisfies the curious Poincare duality and curious hard Lefschetz. Moreover, there is a version of the $P = W$ conjecture for $\mathbf{X}$.*

Remark 5. (i) We expect that the above theorems and conjectures hold without the regularity assumption. However, the generic assumption and semisimplicity are crucial.
(ii) When $G = \mathrm{GL}_n$, Mellit constructed a cell decomposition for $\mathbf{X}$, thus providing a conceptual understanding of the fact that $\mathbf{X}$ is polynomial count and proving that it is Hodge–Tate and satisfies curious Hard Lefschetz [Mel20]. We expect that $\mathbf{X}$ has an analogous cell decomposition for arbitrary reductive G.
(iii) When $G = \mathrm{GL}_n$, conjectural formulas for the mixed Hodge polynomial of $\mathbf{X}$ was given in [HRV08, HLRV11]. Finding the reductive analogues of these formulas is an interesting open problem.

1.3. **Generic additive character varieties.** Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\mathcal{O} = (\mathcal{O}_1, \cdots, \mathcal{O}_n)$ be a tuple of G-orbits on $\mathfrak{g}$. Define additive representation variety by

$$\mathbf{A} := \left\{ (a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n) \in \mathfrak{g}^{2g} \times \prod_{i=1}^{n} \mathcal{O}_i \;\middle|\; \sum_{i=1}^{g} [a_i, b_i] + \sum_{i=1}^{n} c_i = 0 \right\}.$$

We have the adjoint action of G on $\mathbf{A}$. We call the GIT quotient

$$\mathbf{Y} := \mathbf{A} /\!\!/ G.$$

the additive character variety associated $(\Gamma, G, \mathcal{O})$. We will be working under the additive analogues of Assumptions §1. Then $\mathbf{A}$ is smooth, the action of G/Z on $\mathbf{A}$ has finite stabilisers, and $\mathbf{Y}$ is an orbifold of the same dimension as $\mathbf{X}$.

Theorem 6. *The number $|\mathbf{Y}(\mathbb{F}_q)|$ is a polynomial in q given explicitly in §4. This polynomial depends only on (Γ, g, n); in particular, it is independent of the chosen orbits (provided they satisfy the additive analogue of Assumptions 1.)*

The starting point of the proof is an additive analogue of the Frobenius formula which gives an expression for $|\mathbf{Y}(\mathbb{F}_q)|$ in terms of the Fourier transforms of G-invariant orbits on $\mathfrak{g}$. We then use results of Kazhdan and Letellier on these Fourier transforms [Let05] to evaluate the sum and prove that it is a polynomial.

Remark 7. Using the explicit expression of $|\mathbf{Y}(\mathbb{F}_q)|$, one can show that if $2g + n \geq 3$, then $\mathbf{Y}$ is non-empty and connected.

Next, we have:

Theorem 8. *The cohomology of $\mathbf{Y}$ has a pure Hodge structure. Thus, $|\mathbf{Y}(\mathbb{F}_q)|$ has non-negative coefficients.*

Remark 9. (i) In [HLRV11], it is proved that when $G = \mathrm{GL}_n$, the counting polynomial of $\mathbf{Y}$ is the Kac polynomial associated to a comet shaped quiver. Thus, one can think of the counting polynomial of Theorem 6 as reductive analogues of the Kac polynomials of comet shape quivers.
(ii) The coefficients of the Kac polynomials are expected to be the graded dimensions of the associated graded Borcherd algebra [Sch18, §3]. It would be interesting to have an analogous understanding of the counting polynomial of $\mathbf{Y}$ associated to an arbitrary reductive group G.
(iii) For $G = \mathrm{GL}_n$, it is conjectured in [HRV08, HLRV11] that $H^*(\mathbf{Y})$ is isomorphic to the pure part of $H^*(\mathbf{X})$. It would be interesting to understand the relationship between $H^*(\mathbf{X})$ and $H^*(\mathbf{Y})$ for arbitrary reductive groups.

1.4. **Structure of the text.**

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2. Preliminaries

Give an over view of this section.

2.1. **Root systems.** Let Φ be a (reduced crystallographic) root system in a finite dimensional Euclidean vector space $(V, (.,.))$. We assume that Φ spans V. We fix a base Δ of Φ. Then Δ is a basis of V.

2.1.1. *Subsystems.* A root subsystem of Φ is a subset $\Psi \subseteq \Phi$ which is itself a root system. This is equivalent to the requirement that for all $\alpha, \beta \in \Psi$, $s_\alpha(\beta) \in \Psi$. A subsystem $\Psi \subseteq \Phi$ is *closed* if

$$\alpha, \beta \in \Psi \quad \text{and} \quad \alpha + \beta \in \Phi \quad \implies \quad \alpha + \beta \in \Psi.$$

Given a subset $S \subseteq \Phi$, let $\langle S \rangle_{\mathbb{Z}}$ denote the subgroup of V generated by S. The subsystem of Φ generated by S is defined by

$$\langle S \rangle := \langle S \rangle_{\mathbb{Z}} \cap \Phi.$$

Note that $\langle S \rangle$ is a closed subsystem of Φ.

2.1.2. *Dual root system.* For each root $\alpha \in \Phi$, define the coroot $\check{\alpha} := \frac{2}{(\alpha,\alpha)}\alpha \in V$. The set of coroots forms the dual root system $\check{\Phi} \subset V$. If Ψ is a subsystem of Φ, then $\check{\Psi}$ is a subsystem of $\check{\Phi}$. However, if Ψ is closed in Φ, then $\check{\Psi}$ need not be closed in $\check{\Phi}$. For instance, $C_n \times C_n$ is a closed subsystem of C_{2n} but if $n > 1$, then $B_n \times B_n$ is not closed in B_{2n}. Indeed, it is easy to see that $B_n \times B_n$ does not arise in the Borel–de Siebenthal algorithm applied to B_{2n}.

2.1.3. *Levi subsystems.* A *Levi subsystem* of Φ is a subsystem of the form $\Phi \cap E$ where $E \subseteq V$ is a subspace. A Levi subsystem of Φ is closed in Φ. Moreover, every Levi subsystem of Φ is of the form $w.\langle S \rangle$, where S is a subset of Δ. Note that Ψ is a Levi subsystem of Φ if and only $\check{\Psi}$ is a Levi subsystem of $\check{\Phi}$.

2.1.4. *Isolated subsystems.* A subset $S \subseteq \Phi$ is called *isolated in* Φ if S is not contained in a proper Levi subsystem of Φ. One can check that the following are equivalent (recall, we assume Φ spans V):

(1) $S \subseteq \Phi$ is isolated.
(2) $\check{S}$ is isolated in $\check{\Phi}$.
(3) S does not lie in a proper subspace of V.
(4) $\langle S \rangle_{\mathbb{Z}} \otimes \mathbb{R} = V$
(5) The root systems $\langle S \rangle$ and Φ have the same rank.
(6) $\bigcap_{\alpha \in S} \ker(\alpha) = 0$.

2.1.5. *Pseudo-Levi subsystems.* Assume Φ is irreducible. Let α_0 be the highest root of Φ, and $\widetilde{\Delta} := \Delta \sqcup \{-\alpha_0\}$. A pseudo-Levi subsystem is a subsystem of Φ of the form $w.\langle S\rangle$, where $w \in W$ and S is a subset of $\widetilde{\Delta}$. More generally, if $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_r$, then a subsystem $\Psi \subseteq \Phi$ is said to be pseudo-Levi if $\Psi_i := \Psi \cap \Phi_i$ is a pseudo-Levi inside Φ_i. It is clear that every Levi is a pseudo-Levi. The converse is true in type A but is false outside of type A. For instance, B_2 has a pseudo-Levi $A_1 \times A_1$ which is not a Levi.

2.1.6. *Isolated pseudo-Levis.* Let Ψ be a pseudo-Levi subsystem of an irreducible root system Φ. One can check that the following are equivalent:

(1) Ψ is isolated in Φ.
(2) $\Psi = w.\langle S\rangle$ where $S \subset \tilde{\Delta}$ has the same size as Δ.

In other words, up to W-conjugation, isolated pseudo-Levis are those which are obtained by removing just a single element of $\widetilde{\Delta}$ (equivalently, a single node from the extended Dynkin diagram). We leave it to the reader to generalise these statements to non-irreducible root systems. The following table lists isolated pseudo-Levi subsystems of irreducible root systems.

Type of Φ	Isolated pseudo-Levi subsystems of Φ
$A_n,\ n \geq 1$	A_n only
$B_n,\ n \geq 2$	$B_n,\ A_1 \times A_1 \times B_{n-2},\ A_3 \times B_{n-3},$ $A_1 \times D_{n-1},\ D_n,\ B_{n-r} \times D_r\ (r > 3)$
$C_n,\ n \geq 2$	$C_n,\ A_1 \times C_{n-1},\ C_r \times C_{n-r}\ (r > 1)$
$D_n,\ n \geq 3$	$D_n,\ A_1 \times A_1 \times D_{n-2},\ D_r \times D_{n-r}\ (r > 3)$
G_2	$G_2,\ A_2,\ A_1 \times A_1$
F_4	$F_4,\ A_1 \times C_3,\ A_2 \times A_2,\ A_1 \times A_3,\ B_4$
E_6	$E_6,\ A_1 \times A_5,\ A_2 \times A_2 \times A_2$
E_7	$E_7,\ A_7,\ A_1 \times D_6,\ A_2 \times A_5,\ A_1 \times A_3 \times A_3$
E_8	$E_8,\ A_1 \times E_7, A_2 \times E_6,\ A_3 \times D_5,\ A_4 \times A_4,$ $A_1 \times A_2 \times A_5,\ A_1 \times A_7,\ A_8,\ D_8$

FIGURE 1. The isolated pseudo-Levi subsystems of irreducible root systems.

2.1.7. *Endoscopy subsystems.* Let $\check{\Psi}$ be a pseudo-Levi subsystem of $\check{\Phi}$. Then the dual root system Ψ is called an *endoscopy subsystem* of Φ. The example in §2.1.2 shows that an endoscopy subsystem of Φ is not necessarily closed in Φ.

2.2. **Reductive groups.** Let k be a field, G a connected split reductive group over k, $Z = Z(G)$ the centre of G, T a maximal split torus of G, B a Borel subgroup containing T, U the unipotent radical of B, W the Weyl group, $(X, \Phi, \check{X}, \check{\Phi})$ the root datum, $\langle\Phi\rangle_{\mathbb{Z}}$ root lattice, and $\langle\check{\Phi}\rangle_{\mathbb{Z}}$ the coroot lattice. Let $\Delta \subseteq \Phi$ (resp. $\check{\Delta} \subseteq \check{\Phi}$) denote the base defined by B and $V := X \otimes \mathbb{R}$. We assume throughout that the ground characteristic is very good for G and that Z is connected (equivalently, $X/\langle\Phi\rangle$ is torsion free).

2.2.1. *Dual group.* Let $\check{T} := \operatorname{Spec} k[X]$ be the dual torus and $\check{G}$ the Langlands dual group of G over k. In other words, $\check{G}$ is a connected split reductive group over k with maximal split torus $\check{T}$ and root datum $(\check{X}, X, \check{\Phi}, \Phi)$. Since $Z(G)$ is assumed to be connected, the derived subgroup $[\check{G}, \check{G}]$ is simply connected.

2.2.2. *Pseudo-Levis and endoscopy groups.* Let $\Psi \subseteq \Phi$ be a subsystem and $H := G(\Psi)$ the connected split reductive group over k with root datum $(X, \check{X}, \Psi, \check{\Psi})$. Note that H is not necessarily a subgroup of G; however, they share the maximal torus T. A Levi subroup of G containing T is by definition a group of the form $G(\Psi)$ where Ψ is a Levi subsystem of Φ. Similarly, one defines pseudo-Levi subgroup and endoscopy groups of G containing T. Below, we explain that one can give a characterisation of Levi and pseudo-Levis subgroups as centralisers.

2.2.3. *Centralisers of elements of* $\mathfrak{t}$. Let $y \in \mathfrak{t}(k)$ and $\Phi_y := \{\alpha \in \Phi \,|\, \alpha(y) = 0\}$. Then, the centraliser G_y is a Levi subgroup of G with root system Φ_y. Note that G_y is connected, cf. [Ste75, Theorem 0.2]. If k is algebraically closed, then every Levi subgroup arises as a centraliser.

2.2.4. *Centralisers of elements of* $\check{T}$. Let $x \in \check{T}(k)$ and $\check{\Phi}_x := \{\check{\alpha} \in \check{\Phi} \,|\, \check{\alpha}(x) = 1\}$. Then, the centraliser $\check{G}_x$ is a pseudo-Levi subgroup of $\check{G}$ with root system $\check{\Phi}_x$. Note that $\check{G}_x$ is connected [Ste75, Theorem 2.15]. Let G_x be the Langlands dual of $\check{G}_x$. Then G_x an endoscopy group for G. If k is algebraically closed of good characteristic, then every pseudo-Levi subgroup of $\check{G}$ arises as a centraliser [MS03, §9].

2.2.5. *Isolated subgroups.* Let Ψ be a subsystem in Φ. We say that the group $H = G(\Psi)$ is isolated with respect to G if Ψ is isolated in Φ. It follows from the discussions of §2.1.4 that if an endoscopy group H is isolated with respect to G, then

$$T \cap [H, H] = T \cap [G, G]. \tag{4}$$

2.2.6. *Pontryagin duality.* Suppose $k = \mathbb{F}_q$ is a finite field of characteristic p. Then, we can relate Langlands duality for tori with Pontriyagin duality. Namely, for a finite abelian group A, let $A^\vee := \operatorname{Hom}(A, \mathbb{C}^\times)$ denote its Pontryagin dual. Let $\mu_{\infty,p'}(\mathbb{C})$ denote the set of roots of unity in $\mathbb{C}^\times$ whose order is prime to p. We choose, once and for all, isomorphisms

$$\overline{\mathbb{F}_p}^\times \simeq (\mathbb{Q}/\mathbb{Z})_{p'} \qquad \text{and} \qquad (\mathbb{Q}/\mathbb{Z})_{p'} \simeq \mu_{\infty,p'}(\mathbb{C}). \tag{5}$$

As noted in [DL76, §5], this induces an isomorphism

$$T(k)^\vee \simeq \check{T}(k). \tag{6}$$

2.3. **Generic elements and characters.** An element $S \in T$ is called *generic* if S is not in $[L, L]$ for every proper Levi subgroup $T \subseteq L \subset G$. More generally, let $S_1, \ldots, S_n$ be elements of T. We say the tuple $(S_1, \ldots, S_n)$ is *generic* if for all proper Levi subgroup $T \subseteq L \subset G$ and all $(w_1, \ldots, w_n)$ of elements of W, we have $\prod_{i=1}^n w_i.S_i \notin [L, L]$.

2.3.1. Let $C_i \subset G$ be the conjugacy class of S_i, $i = 1, \ldots, n$. Then one can show that $(S_1, \ldots, S_n)$ is generic if and only if $(C_1, \ldots, C_n)$ is generic in the sense of Assumption 1.

2.3.2. Let $S_1, \dots, S_n$ be a generic tuple of elements of T and $w_1, \dots, w_n \in T$. Let E be an endoscopy group of G containing T. Then $\prod_{i=1}^n w_i.S_i \in T \cap [E,E]$ if and only if E is isolated with respect to G. Indeed, if E is isolated, then $T \cap [E,E] = T \cap [G,G]$. Conversely, if E is not isolated, then it lies inside a proper Levi subgroup $L \subset G$. By genericness, $\prod_{i=1}^n w_i.S_i \notin [L,L]$. Thus, $\prod_{i=1}^n w_i.S_i \notin [E,E]$.

2.3.3. In the same manner, one defines the notion of generic tuples of elements of $\mathfrak{t}$. Namely, $H \in \mathfrak{t}$ is generic if $H \notin [\mathfrak{l}, \mathfrak{l}]$ for every proper Levi subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$.

2.3.4. Next, we turn our attention to the dual notion of generic characters. A (rational) character $\chi : T \to \mathbb{G}_m$ is called *generic* if $\chi(Z(L)) \neq \{1\}$ for every proper Levi subgroups $T \subseteq L \subset G$. More generally, let $\chi_1, \dots, \chi_n$ be characters of T. We say the tuple $(\chi_1, \dots, \chi_n)$ is *generic* if for all proper Levi subgroup $T \subseteq L \subset G$ and all tuples $(w_1, \dots, w_n)$ of elements of W, the character $\prod_{i=1}^n w_i.\chi_i$ is nontrivial on $Z(L)$.[1]

Lemma 10. *Let b be a non-zero integer. Then χ is generic if and only if χ^b is generic.*

Proof. Note that $\overline{\chi(Z(L))}$ is a Zariski closed connected subgroup of $\mathbb{G}_m$; thus, it is either $\{1\}$ or $\mathbb{G}_m$. Now we have

$$\chi(Z(L)) \neq \{1\} \quad \Longleftrightarrow \quad \overline{\chi(Z(L))} = \mathbb{G}_m \quad \Longleftrightarrow \quad \overline{\chi^b(Z(L))} = \mathbb{G}_m. \qquad \square$$

Lemma 11. *Let S be subset of Φ and $\chi : T \to \mathbb{G}_m$ a character of the form*

$$\chi = \sum_{\alpha \in S} a_\alpha \alpha, \qquad a_\alpha \in \mathbb{Q}^\times.$$

If χ is generic, then S is isolated in Φ.

Proof. Suppose S is not isolated in Φ. Then S is in some proper Levi subsystem of Φ. Let L denote the corresponding proper Levi subgroup of G. Note that

$$Z(L) = \bigcap_{\alpha \in \Phi(L)} \ker(\alpha) \implies \alpha(Z(L)) = \{1\}, \ \forall \alpha \in S.$$

Now, let b be a positive integer such that $ba_\alpha \in \mathbb{Z}$ for all $\alpha \in S$. Thus,

$$\alpha^{ba_\alpha}(Z(L)) = \{1\}, \ \forall \alpha \in S \implies \chi^b(Z(L)) = \{1\}.$$

Thus, χ^b is not generic. By the previous lemma, χ is not generic. $\square$

2.4. **Action of T on $\mathfrak{g}$.** In this section, we collect some facts about the action of T on $\mathfrak{g}$ by conjugation. Let us choose a generator g_α for each root subspace $\mathfrak{g}_\alpha$. Then we can write each $x \in \mathfrak{g}$ as

$$x = x_0 + \sum_{\alpha \in \Phi} x_\alpha g_\alpha, \qquad x_0 \in \mathfrak{t}, \quad x_\alpha \in k.$$

Let

$$\boxed{S_x := \{\alpha \in \Phi \,|\, x_\alpha \neq 0\} \subseteq \Phi.}$$

[1]Since we have assumed G has connected centre, every Levi subgroup of G also has connected centre, cf. [MS03, Lemma 33].

2.4.1. *Stabilisers.* Note that $t \in T$ acts on x by

$$t.x = x_0 + \sum_{\alpha \in \Phi} \alpha(t) x_\alpha g_\alpha$$

Thus, we see t is in the stabiliser T_x if and only if $\alpha(t) = 1$ for all $\alpha \in S_x$; in other words,

$$T_x = \bigcap_{\alpha \in S_x} \operatorname{Ker} \alpha.$$

2.4.2. *Limits.* Observe that for $c \in \mathbb{G}_m$ and a cocharacter $\lambda : \mathbb{G}_m \to T$, we have

$$\lambda(c).x_\alpha = c^{\langle \lambda, \alpha \rangle} x_\alpha.$$

Thus, $\lim_{c \to 0} \lambda(c).x$ exists if and only if either $x_\alpha = 0$ or $\langle \lambda, \alpha \rangle \geq 0$. In other words, $\lim_{c \to 0} \lambda(c).x$ exists if and only if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in S_x$.

2.4.3. *Stability vs. semistability.* Let $\chi : T \to \mathbb{G}_m$ a rational character. Consider the action of T on $\mathfrak{g}$. The notion of χ-stable and χ-semistable points of $\mathfrak{g}$ is defined in [Kin94]. As usual in GIT, one expects these two notions to coincide for almost all χ's. The following proposition makes this more precise:

Proposition 12. *Suppose χ is generic. Then every χ-semistable element $x \in \mathfrak{g}$ is χ-stable.*

To prove this proposition, we need to recall some facts about cones.

2.4.4. A *cone* in a Euclidean vector space $(V, (.,.))$ is a subset which is closed under positive scalar multiplication. A *convex* cone is a cone which is closed under addition. Let S be a subset of V and $C(S) := \mathrm{Span}_{\mathbb{R}_{\geq 0}} S$ the convex cone generated by S. Define

$$S^* := \{ f \in V \mid (f, s) \geq 0, \ \forall s \in S \}.$$

Then S^* is a convex cone. In fact, it is clear that $S^* = C(S)^*$. By a standard result in convex geometry (cf. [`?Fulton`, §1.2]), if S is finite, we have

$$(S^*)^* = (C(S)^*)^* = C(S). \tag{7}$$

2.4.5. For a subset $\Lambda \subseteq V$, define

$$S^\Lambda := \{ \lambda \in \Lambda \mid \lambda(s) \geq 0, \ \forall s \in S \} = C(S)^* \cap \Lambda \subseteq S^* = C(S)^*.$$

Lemma 13. *Suppose Λ is a lattice of full rank in V. Then $(S^\Lambda)^* = (S^*)^*$.*

Proof. By definition, $(S^\Lambda)^* \supseteq (S^*)^*$. For the reverse inclusion, let $v \in (S^\Lambda)^*$. Then for all $\lambda \in S^\Lambda$, we have $(\lambda, v) \geq 0$. Thus, we also have $(\frac{1}{n}\lambda, v) \geq 0$ for all positive integers n. Setting

$$\tilde{\Lambda} := \bigcup_{n \geq 1} \frac{1}{n} \Lambda,$$

we conclude that $(f, v) \geq 0$ for all $f \in S^{\tilde{\Lambda}}$. Since $\tilde{\Lambda}$ is dense in V, we also have that $S^{\tilde{\Lambda}}$ is dense in S^*. By continuity, $(f, v) \geq 0$ for all $f \in S^*$; i.e. $v \in (S^*)^*$. ∎

2.4.6. *Proof of Proposition 12.* By the Hilbert–Mumford criterion for semistability [Kin94, Proposition 2.5], being χ-semistable means that whenever $\lim_{c\to 0}\lambda(c).x$ exists for some cocharacter $\lambda : \mathbb{G}_m \to T$, we have that $\langle \chi, \lambda\rangle \geq 0$. In view of the above discussions, this is equivalent to requiring that whenever $\lambda \in \check{X}$ is such that $\langle \alpha, \lambda\rangle \geq 0$ for all $\alpha \in S_x$, then we have that $\langle \chi, \lambda\rangle \geq 0$. In view of Lemma 13, this means that $\chi \in (S_x^{\check{X}})^* = (S_x^*)^*$. By (7), χ can be written as

$$\chi = \sum_{\alpha\in S_x} a_\alpha \alpha, \qquad a_\alpha \in \mathbb{R}_{\geq 0}.$$

Let us choose one such a representation for χ and let

$$S := \{\alpha \in S_x \,|\, a_\alpha \neq 0\} \subseteq S_x \subseteq \Phi.$$

Since χ is generic, S is an isolated subset of Φ.

Now the expression for χ implies that whenever $\langle \alpha, \lambda\rangle \geq 0$ for all $\alpha \in S_x$, then we have that $\langle \chi, \lambda\rangle > 0$ or $\langle \alpha, \lambda\rangle = 0$ for all $\alpha \in S$. The latter condition means that

$$\lambda \in \bigcap_{\alpha\in S} \ker(\alpha) \implies \lambda \in (\bigcap_{\alpha\in S} \ker(\alpha))^\circ = \bigcap_{\alpha\in\Phi} \ker(\alpha) = Z(G).$$

Thus, whenever $\lim_{c\to 0}\lambda(c).x$ exists for some cocharacter $\lambda : \mathbb{C}^\times \to T$, then either $\langle \chi, \lambda\rangle > 0$ or $\lambda \in Z(G)$. Applying the Hilbert–Mumford criterion for stability, we conclude that x is χ-stable.

3. Counting points on $\mathbf{X}$

Give an overview of this section. We first consider the character variety associated to a once-punctured surface. The multi-punctured case is considered at the end of the section.

3.1. **Counting points: first steps.**

Let $\mathbf{R}$ be the representation variety associated to a strongly regular generic element $S \in T(k) \cap [G, G]$; see Assumption 1. Let $\mathbf{X} = \mathbf{R}/\!\!/G$ be the associated character variety and $[\mathbf{R}/(G/Z)]$ the associated Deligne–Mumford stack.

3.1.1. Since S is generic, the action of G/Z on $\mathbf{R}$ has finite stabilisers. Thus, the quotient stack $[\mathbf{R}/(G/Z)]$ is a Deligne-Mumford stack with the same number of k-points as $\mathbf{X}$, cf. [Beh91, §2]. Thus, we have

$$|\mathbf{X}(k)| = |[\mathbf{R}/(G/Z)](k)| = \frac{|\mathbf{R}(k)|}{|(G/Z)(k)|}.$$

3.1.2. The Frobenius Mass Formula, cf. [HLRV11, Proposition 3.1.4] then implies

$$|\mathbf{X}(k)| = \frac{|\mathbf{R}(k)|}{|(G/Z)(k)|} = \frac{|Z(k)|}{|T(k)|} \sum_{\chi\in \mathrm{Irr}(G(k))} \left(\frac{|G(k)|}{\chi(1)}\right)^{2g-1} \chi(S). \tag{8}$$

Here, $\mathrm{Irr}(G(k))$ denotes the set of irreducible complex characters of $G(k)$. Thus, counting points on $\mathbf{R}$ (and therefore $\mathbf{X}$) is a problem in complex representation theory of the finite reductive group $G(k)$. In this section, we use the Deligne–Lusztig theory to rewrite the above sum in a more convenient form.

3.1.3. Recall that a principal series representation is an irreducible constituent of

$$R_T^G\,\theta := \mathrm{Ind}_{B(k)}^{G(k)}\,\theta, \qquad \theta \in T(k)^\vee.$$

Theorem 14 (Corollary 7.6 [DL76])**.** *Let $\chi \in \mathrm{Irr}(G(k))$ be an arbitrary character and $S \in T(k)$ a strongly regular element. Then*

$$\chi(S) = \sum_{\theta \in T(k)^\vee} \langle \chi, R_T^G\,\theta \rangle\, \theta(S).$$

Corollary 15. *Under the assumptions of the previous theorem, if $\chi(S) \neq 0$ then χ is a principal series representation.*

Proof. Indeed, if $\chi(S) \neq 0$, then we must have $\langle \chi, R_T^G\,\theta \rangle \neq 0$ for some $\theta \in T(k)^\vee$. Thus, χ is a constituent of $R_T^G\,\theta$. $\square$

Thus, we see that only principal series representations contribute to the sum in (8).

3.1.4. The induced representations $R_T^G\,\theta$ and $R_T^G\,\theta'$ have a common constituent if and only if θ and θ' are W-conjugate, cf. [DL76, Corollary 6.3]. Thus, we can associate to a principal series representation $\chi \in \mathrm{Irr}(G(\mathbb{F}_q))$ a character $\theta \in T(k)^\vee$, well-defined up to W-conjugacy.

3.1.5. Identifying $T(k)^\vee$ with the dual torus $\check{T}(k)$ (as in §2.2.6), we can think of θ as living in the Langland dual group $\check{G}(k)$. Let $\check{G}_\theta$ denote the centraliser of θ in $\check{G}$. Following the discussions of §2.2, $\check{G}_\theta$ is a (connected) pseudo-Levi subgroup of $\check{G}$ (containing $\check{T}$) with Weyl group

$$W_\theta = \{w \in W \mid w.\theta = \theta\}.$$

Then the Langlands dual group $G_\theta := (\check{G}_\theta)\check{}$ is an endoscopy group for G containing T with Weyl group W_θ.

3.1.6. Since G is assumed to have connected centre, the constituents of $R_T^G\,\theta$ are in canonical bijection with $\mathrm{Irr}(W_\theta)$, cf. [Kil78, Corollary 4.20]. Let $\chi_{\theta,\rho} \in R_T^G\,\theta$ be the irreducible character corresponding to $\rho \in \mathrm{Irr}(W_\theta)$. Then, the double centraliser theorem (cf. [EGH$^+$11, Theorem 5.18.1]) gives

$$\langle \chi_{\theta,\rho}, R_T^G\,\theta \rangle = \dim(\rho).$$

3.1.7. Consider the set of pairs (L, ρ), where L is an endoscopy group of G (containing T) and ρ an irreducible character of $W(L)$. Note W acts on this set by conjugation. Let $\mathcal{T}(G)$ denote the set of W-orbits. We call the elements of $\mathcal{T}(G)$ *types* and denote them by $\tau = [L, \rho]$.[2] Note that $\mathcal{T}(G)$ depends only on the root datum $\Phi = \Phi(G)$; in particular, it is independent of the ground field. By the above discussions, we can associate a type to every principal series representation.[3] In other words, we have a map

$$\text{Principal series representations of } G(k) \to \mathcal{T}(G). \tag{9}$$

[2]Here, for ease of notation, we write $[L, \rho]$ for the orbit $[(L, \rho)]$.

[3]This is analogous to the types used in [HRV08, HLRV11, Cam17], see [Whi24, §4] for details. Using Lusztig's Jordan decomposition, one can generalise the notion of types to all irreducible $G(k)$-characters. We shall not need the more general version in this paper.

3.1.8. *Unipotent characters.* As noted above, irreducible constituents of $R_T^L(1)$ are in canonical bijection with irreducible representations of $W(L)$. Let $\tilde{\rho}$ denote the unipotent (principal series) character of $L(k)$ corresponding to $\rho \in \mathrm{Irr}(W(L))$.

3.1.9. *Mass of a type.* Let $\tau = [L, \rho] \in \mathcal{T}(G)$ be a type. Recall k is a finite field of order q.

Definition 16. We define the q-mass of τ by

$$m_\tau(q) := q^{|\Phi^+(G)|-|\Phi^+(L)|} \frac{|L(k)|}{\tilde{\rho}(1)}.$$

Here, $\Phi^+(L)$ denotes the set of positive roots of L. Note that $\tilde{\rho}(1)$ is a polynomial in q which divides $|L(k)|$, cf. [GM20, Remark 2.3.27]. Thus, $m_\tau(q)$ is a polynomial in q.

3.1.10. The relevance of $m_\tau(q)$ emerges in the following proposition which can be found in, e.g., [GM20, Corollary 2.6.6]:

Proposition 17. *Suppose χ is a principal series character of $G(\mathbb{F}_q)$ of type τ. Then*

$$\frac{|G(\mathbb{F}_q)|}{\chi(1)} = m_\tau(q).$$

3.1.11. We can now rewrite the Frobenius Mass Formula (8) in terms of types. For each type $\tau \in \mathcal{T}(G)$, let $\mathrm{Irr}(G(k))_\tau$ denote the set of irreducible principal series characters of type τ. Then we have:

$$|\mathbf{X}(k)| = \frac{|Z(k)|}{|T(k)|} \sum_{\tau \in \mathcal{T}(G)} m_\tau(q)^{2g-1} \sum_{\chi \in \mathrm{Irr}(G(k))_\tau} \chi(S).$$

3.1.12. Let

$$S_\tau(q) := \sum_{\chi \in \mathrm{Irr}(G(\mathbb{F}_q))_\tau} \chi(S).$$

Then, we have

$$\boxed{|\mathbf{X}(k)| = \frac{|Z(k)|}{|T(k)|} \sum_{\tau \in \mathcal{T}(G)} m_\tau(q)^{2g-1} S_\tau(q).}$$

Thus, computing $|\mathbf{X}(k)|$ is reduced to computing the character sums $S_\tau(q)$.[4]

3.2. **Evaluating the character sums S_τ.** In this subsection, we re-write the expression for S_τ in terms of sums of characters of $T(k)$, turning the problem of computing S_τ into a "commutative" problem. Suppose $\tau = [L, \rho] \in \mathcal{T}(G)$ where L is an endoscopy group of G (containing T) with Weyl group $W(L)$ and ρ is an irreducible character of $W(L)$. Let $[L]$ denote the W-orbit of L in G and let $[W(L)]$ be the W-orbit of $W(L)$ in W.

3.2.1. We start by elucidating the nature of $\mathrm{Irr}(G(k))_\tau$. It consists of characters $\chi_{\theta,\rho}$ as θ runs over elements of $T(k)^\vee$ with endoscopy group satisfying $[G_\theta] = [L]$. Equivalently,

$$\mathrm{Irr}(G(k))_\tau = \{\chi_{\theta,\rho} \,|\, \theta \in T(k)^\vee,\, [W_\theta] = [W(L)]\}.$$

We refer the reader to §3.1.6 for the definition of $\chi_{\theta,\rho}$.

[4]In obtaining the above formula for $|\mathbf{X}(k)|$, we used the assumption that S is strongly regular, but not the assumption that S is generic.

3.2.2. The above discussion implies

$$S_\tau(q) \;=\; \sum_{\chi\in \mathrm{Irr}(G(k))_\tau} \chi(S) \;=\; \frac{|W(L)|}{|W|} \sum_{\substack{\theta\in \check{T}(k)\\ [W_\theta]=[W(L)]}} \chi_{\theta,\rho}(S) \;=\; \frac{|W(L)|}{|N_W(W(L))|} \sum_{\substack{\theta\in \check{T}(k)\\ W_\theta=W(L)}} \chi_{\theta,\rho}(S).$$

The second equality follows from the fact that $\chi_{\theta,\rho}$ depends only on the W-orbit of θ.

3.2.3. Recall that $[L]$ denotes the orbit of L under the action of W on subgroups of G. According to [Car72, Lemma 34], we have $|W/N_W(W(L))| = |[L]|$. Thus, we obtain

$$S_\tau(q) = \frac{|W(L)||[L]|}{|W|} \sum_{\substack{\theta\in \check{T}(k)\\ W_\theta=W(L)}} \chi_{\theta,\rho}(S).$$

Remark 18. In [Car72, Proposition 28], one finds the following formula for $|[L]|$. Let $W^\perp(L)$ be the Weyl group generated by the roots orthogonal to $\Phi(L)$. Let $A_W(L)$ denote the group of symmetries of the Dynkin diagram of L induced by elements of W. Then

$$|[L]| = \frac{|W|}{|W(L)||W^\perp(L)||A_W(L)|}.$$

3.2.4. By Theorem 14, we have

$$\chi_{\theta,\rho}(S) = \sum_{\theta'\in T(k)^\vee} \langle \chi_{\theta,\rho}, R_T^G\,\theta'\rangle\, \theta'(S).$$

The summand is zero unless θ' is in the same W-orbit as θ, in which case it equals $\dim(\rho)\,\theta'(S)$; thus,

$$\chi_{\theta,\rho}(S) = \dim(\rho) \sum_{w\in W/W_\theta} (w.\theta)(S) = \frac{\dim(\rho)}{|W_\theta|} \sum_{w\in W} \theta(w.S).$$

Note that the last sum depends only on the W-orbit of θ.

3.2.5. We now plug in the explicit formula for $\chi_{\theta,\rho}$ in the above expression and obtain:

$$S_\tau(q) = \frac{|W(L)||[L]|}{|W|} \sum_{\substack{\theta\in \check{T}(k)\\ W_\theta=W(L)}} \frac{\dim(\rho)}{|W_\theta|} \sum_{w\in W} \theta(w.S) = \frac{\dim(\rho)|[L]|}{|W|} \sum_{w\in W} \sum_{\substack{\theta\in \check{T}(k)\\ W_\theta=W(L)}} \theta(w.S)$$

3.2.6. To alleviate notation, let

$$\alpha_{L,S}(q) := \sum_{\substack{\theta\in \check{T}(k)\\ W_\theta=W(L)}} \theta(S).$$

Now, we have:

$$\boxed{S_\tau(q) = \frac{\dim(\rho)|[L]|}{|W|} \sum_{w\in W} \alpha_{L,w.S}(q).}$$

Thus, the problem of computing S_τ reduces to evaluating the character sum $\alpha_{L,S}$.

3.3. **Evaluation of $\alpha_{L,S}$.** We first consider an auxiliary sum which is easier to handle. For an endoscopy group L of G (containing T) and $S \in T(k)$, define

$$\alpha_{L,S}^{\supseteq}(q) := \sum_{\substack{\theta \in \check{T}(k) \\ W_\theta \supseteq W(L)}} \theta(S) = \sum_{\theta \in \check{T}(k)^{W(L)}} \theta(S).$$

Proposition 19. $\alpha_{L,S}^{\supseteq}(q) = \begin{cases} |\check{T}(k)^{W(L)}| & \textit{if } S \in [L(k), L(k)] \\ 0 & \textit{otherwise.} \end{cases}$

Proof. This is a straightforward application of the fact that the Pontryagin dual of the embedding of finite abelian groups $\check{T}(k)^{W(L)} \hookrightarrow \check{T}(k)$ is the canonical quotient map

$$T(k) \twoheadrightarrow \frac{T(k)}{T(k) \cap [L(k), L(k)]}.$$

See [KNP23, §5.2] for details. $\square$

3.3.1. *Description of $\check{T}^{W(L)}$.* It is proved in [KNP23] that, under the assumption that G has connected centre, we have:

Proposition 20. $\check{T}^{W(L)} \simeq Z(\check{L})$.

The above proposition implies that

$$|\check{T}(k)^{W(L)}| = |\pi_0(Z(\check{L}))(k)| \times (q-1)^{\mathrm{rank}(Z(L))}.$$

In particular, $\alpha_{L,S}^{\supseteq}(q)$ is a polynomial in q.

3.3.2. *Base change.* There are two things that can change if we replace k by a finite extension k'. First of all, $|\pi_0(Z(\check{L})(k)|$ may change, because the action of Galois group $\mathrm{Gal}(\overline{k}/k)$ is not necessarily trivial. Secondly, we may have $S \notin [L(k), L(k)]$ but $S \in [L(k'), L(k')]$. As discussed in [KNP23], both issues are resolved if k is replaced by a sufficiently large finite extension.

3.3.3. For ease of notation, let

$$|\pi_0^L| := |\pi_0(Z(\check{L}))(\overline{k})|, \qquad r(L) := \mathrm{rank}(Z(L)).$$

Then, after replacing $k = \mathbb{F}_q$ with a finite extension if necessary, we find:

Corollary 21. $\alpha_{L,S}^{\supseteq}(q) = \begin{cases} |\pi_0^L|(q-1)^{r(L)} & \textit{if } S \in [L, L] \\ 0 & \textit{otherwise.} \end{cases}$

3.3.4. *Möbius inversion.* Let μ denote the Möbius function on the poset of endoscopy subsystems of Φ, ordered by inclusion. Then by Möbius inversion, we have:

$$\alpha_{L,S} = \sum_{L'} \mu(L, L') \alpha_{L',S}^{\supseteq},$$

Here, the sum is over endoscopy groups L' of G (containing T) whose root system satisfies $\Phi(L') \supseteq \Phi(L)$ and we have abused notation and written $\mu(L, L')$ for $\mu(\Phi(L), \Phi(L'))$.

3.4. **Simplifications under the generic assumption.** So far, we have not used the generic assumption on S. We now use this assumption to simplify the expression for the polynomial $\alpha_{L,S}$ considerably. Indeed, genericness means that $S \in [L, L]$ if and only if L is isolated in G. Replacing k with a finite extension if necessary, we may assume that $S \in [L(k), L(k)]$ for every isolated endoscopy L. Note that for isolated endoscopy groups L, we have

$$Z(\check{L})^\circ = Z(L)^\circ = Z(G).$$

3.4.1. To simplify notation, let

$$\nu(L) := \sum_{L'} |\pi_0^{L'}| \mu(L, L').$$

Here, the sum runs over isolated endoscopy groups L' whose root systems contain $\Phi(L)$. Then $\nu(L)$ is an integer depending only on the root systems of L and G; in particular, it is independent of k. In fact, $\nu(L)$ depends only on W-orbit of L. In view of the above discussions, for generic S, we have

$$\alpha_{L,S}(q) = |Z(\mathbb{F}_q)| \nu(L).$$

In particular, $\alpha_{L,S}(q)$ does not depend on S. Note that in type A, the only isolated endoscopy is G itself; thus, $\nu(L) = \mu(L, G)$. Thus, we have proven the following:

Proposition 22. $S_\tau = \dim(\rho)\nu(L)|[L]||Z(\mathbb{F}_q)|$.

3.5. **Conclusion of the point count for once-punctured case.** We have seen that

$$|\mathbf{X}(k)| = \frac{|Z(k)|}{|T(k)|} \sum_{\tau=[L,\rho]\in\mathcal{T}(G)} S_\tau(q)\, m_\tau(q)^{2g-1},$$

where $S_\tau(q)$ is given by the above proposition.

3.5.1. We claim that $|\mathbf{X}(\mathbb{F}_q)|$ is a polynomial in q. To see this, note that $|\mathbf{X}(\mathbb{F}_q)|$ is clearly a rational function in q. Moreover, replacing $\mathbb{F}_q$ by $\mathbb{F}_{q^m}$ amounts to replacing q by q^m in the argument of the rational function. Thus, this rational function takes on integer values for infinitely many powers of q. It follows that it must be a polynomial.[5]

3.6. **Palindromic property.** Our goal is to explain that the polynomial $|\mathbf{X}(\mathbb{F}_q)|$ is palindromic. Recall (cf. [DM20, §7.2]) that the Alvis–Curtis duality is (up to a sign) an involution

$$\mathfrak{D} : \mathrm{Irr}(G(\mathbb{F}_q)) \to \mathrm{Irr}(G(\mathbb{F}_q)).$$

To compute the dimension of the irreducible character $\mathfrak{D}(\chi)$, we need to replace q by q^{-1} in the polynomial d_χ encoding the dimension of χ. More precisely,

$$\dim \mathfrak{D}(\chi) = q^{|\Phi^+|} d_\chi(q^{-1}).$$

[5]Another way to conclude that $|\mathbf{X}(k)|$ is a polynomial is to note that $|T(k)|\tilde{\rho}(1)$ divides $|L(k)|$, cf. [GM20, Remark 2.3.27]. Note that by definition, if $g = 0$ then $\mathbf{X}$ is empty. One can check directly that in this case the counting polynomial equal 0, cf. [Whi24, §7.8].

3.6.1. It is known that $\mathfrak{D}$ restricts to an involution on the set of irreducible constituents of $R_T^G\theta$; namely, it sends the irreducible constituent corresponding to $\rho \in \mathrm{Irr}(W_\theta)$ to the constituent corresponding to $\epsilon\rho$, where ϵ is the sign character of W_θ. In particular, we see that $\mathfrak{D}$ sends an irreducible character of type $[L,\rho]$ to one of type $[L,\epsilon\rho]$. Using these facts, it is easy to show that $|\mathbf{X}(\mathbb{F}_q)|$ is palindromic. For details, see [Whi24, §7.7].

3.7. **Multi-puncture case.** Let $\mathbf{X}$ be a character variety associated to an n-punctured surface group satisfying Assumptions 1. Then a similar analysis as in the once-punctured case gives

$$|\mathbf{X}(\mathbb{F}_q)| = \frac{|Z(\mathbb{F}_q)|}{|T(\mathbb{F}_q)|^n} \sum_{\tau\in\mathcal{T}(G)} S_\tau(q)\, m_\tau(q)^{2g-2+n},$$

where

$$S_\tau(q) := \sum_{\chi\in\mathrm{Irr}(G(\mathbb{F}_q))_\tau} \prod_{i=1}^{n} \chi(S_i) = \frac{\dim(\rho)^n\, |W|^{n-1}\, |[L]|\, \nu(L)}{|W(L)|^{n-1}} |Z(\mathbb{F}_q)|.$$

For details, see [Whi24, §6.2].

3.7.1. For the benefit of the reader, we remind what the notation means. Here, Z is the centre and T is a maximal split torus of G, $\mathcal{T}(G)$ is the set of G-types consisting of W-orbits of pairs (L,ρ), where L is an endoscopy group for G (containing T) and ρ is an irreducible character of $W(L)$. The integer $|[L]|$ is the size of the W-orbit of L, and the integer $\nu(L)$ is defined in §3.4.1 using the Mobius function on the lattice of endoscopy subsystems of Φ. Finally,

$$m_\tau(q) := q^{|\Phi^+(G)|-|\Phi^+(L)|} \frac{|L(k)|}{\dim(\tilde{\rho})} \in \mathbb{Z}[q],$$

where $\tilde{\rho}$ is the unipotent character of $L(k)$ corresponding to ρ.

4. Counting points on $\mathbf{Y}$

Give an overview of this section. Define $\mathfrak{g}$ and $\mathfrak{t}$ and the Frobenius F.

4.1. **The set-up.** Our goal in this section is to count points on the additive character variety. As in the previous section, after choosing an appropriate spreading out, we may assume that we are working over a finite field $k = \mathbb{F}_q$ whose characteristic p is *very good* for G [Let05, §2.5]. This means that p is *good* for Φ[6] and p does not divide $|\pi_1(G/Z(G))|$.

4.1.1. Fix a nontrivial additive character $\psi : k \to \mathbb{C}^\times$ and a non-degenerate G-invariant symmetric bilinear form $\mu : \mathfrak{g}(k)\times\mathfrak{g}(k) \to k$, cf. [Let05, Proposition 2.5.12]. The restriction $\mu|_{\mathfrak{t}}$ of μ to $\mathfrak{t}(k)$ is also non-degenerate. We write $x = x_s + x_n$ for the Jordan decomposition of $x \in \mathfrak{g}(k)$.

4.1.2. Since p is very good prime for G, a theorem of Springer states that there exists a G-equivariant isomorphism from the nilpotent cone of $\mathfrak{g}$ to the unipotent variety of G, cf. [Let05, §2.7.5]. We denote one such an isomorphism by ϖ.

[6] If Φ is irreducible this means that p does not divide the coefficient of the highest root. For reducible Φ, p being good means that it is good for each irreducible subsystem.

4.1.3. Let $\mathbb{C}[\mathfrak{g}(k)]^{G(k)}$ denote the space of $G(k)$-invariant functions on $\mathfrak{g}(k)$. For $x \in \mathfrak{g}(k)$, let 1_x^G denote the characteristic function of the adjoint orbit $\mathcal{O}_x = G(k).x \subseteq \mathfrak{g}(k)$. The set $\{1_x^G \,|\, x \in \mathfrak{g}(k)/G(k)\}$ is a basis of $\mathbb{C}[\mathfrak{g}(k)]^{G(k)}$. One can think of this set as the additive analogue of the set of irreducible characters of $G(k)$.

4.1.4. Let $R_{\mathfrak{t}}^{\mathfrak{g}} : \mathbb{C}[\mathfrak{t}(k)] \to \mathbb{C}[\mathfrak{g}(k)]^{G(k)}$ be defined by

$$R_{\mathfrak{t}}^{\mathfrak{g}}(f)(x) = \frac{1}{|G_{x_s}(k)|} \sum_{\{g \in G(k) \,|\, g.x_s \in \mathfrak{t}(k)\}} Q_T^{G_{x_s}}(\varpi(x_n)) f(g.x_s).$$

Here, G_{x_s} is the centraliser of x_s in G and Q_T^G is the Green function from unipotent elements of $G(k)$ to $\mathbb{Z}$ defined by

$$Q_T^G(u) := \left(\mathrm{Ind}_{B(k)}^{G(k)} 1\right)(u) = |\mathcal{B}_u(k)| = \frac{|G(k).u \cap B(k)||G_u(k)|}{|B(k)|}.$$

Here, $\mathcal{B}_u \subseteq G/B$ denotes the Springer fibre associated to the unipotent element $u \in G(k)$. It is known that if q is large enough, then $Q_T^G(u)$ is a polynomial in q with non-negative coefficients, cf. [Spr84, Theorem 5.2].

4.2. **Counting points: first steps.** Let $H \in \mathfrak{t}(k)$ be a regular generic element and let $\mathbf{Y}$ be the corresponding additive character variety; i.e., $\mathbf{Y} = \mathbf{A}/\!\!/G$, where

$$\mathbf{A} := \left\{ (a_1, b_1, \dots, a_g, b_g, c) \in \mathfrak{g}^{2g} \times \mathcal{O}_H \;\middle|\; \sum_{i=1}^{g} [a_i, b_i] + c = 0 \right\}$$

4.2.1. The generic property implies that the action of G on $\mathbf{A}$ has finite stabilisers. A similar argument to [?NAM, §2] then implies that $\mathbf{A}$ is smooth and

$$|\mathbf{Y}(k)| = \frac{|\mathbf{A}(k)|}{|(G/Z)(k)|}.$$

4.2.2. *Additive Frobenius Mass Formula.* Let $\mathcal{F} : \mathbb{C}[\mathfrak{g}(k)]^{G(k)} \to \mathbb{C}[\mathfrak{g}(k)]^{G(k)}$ denote the Fourier transform defined by

$$\mathcal{F}(\phi)(x) := \sum_{y \in \mathfrak{g}(k)} \psi(\mu(x, y))\phi(y).$$

The additive version of the Frobenius Mass Formula, cf. [HLRV11, Proposition 3.2.2], states

$$|\mathbf{A}(k)| = |\mathfrak{g}(k)|^{g-1} \sum_{x \in \mathfrak{g}(k)} |\mathfrak{g}_x(k)|^g \mathcal{F}(1_H^G)(x).$$

To proceed further, we need some information about $\mathcal{F}(1_H^G)(x)$.

4.2.3. Let $f_H : \mathfrak{t}(k) \to \mathbb{C}$ denote the function $\psi(\mu|_{\mathfrak{t}}(-, H))$. The following result can be found in [Let05, Theorem 7.3.3].

Theorem 23 (Kazhdan–Letellier)**.** *Suppose $H \in \mathfrak{t}(k)$ is regular and $x \in \mathfrak{g}(k)$. Then,*

$$\mathcal{F}(1_H^G)(x) = q^{|\Phi^+|} R_{\mathfrak{t}}^{\mathfrak{g}}(f_H)(x).$$

Definition 24. We call $x \in \mathfrak{g}(k)$ *split* if x_s is $G(k)$-conjugate to an element of $\mathfrak{t}(k)$.

It follows immediately from the definition that if $R_{\mathfrak{t}}^{\mathfrak{g}}(f)(x)$ is non-zero, then x is split. Thus, we obtain:

Corollary 25. *Under the assumptions of the theorem, $\mathcal{F}(1_H^G)(x) \neq 0$ only if x is split.*

These results can be considered as additive analogues of Theorem 14 and Corollary 15.

4.2.4. Let $\mathfrak{g}(k)^\circ$ denote the set of split elements of $\mathfrak{g}(k)$. The above discussions implies

$$|\mathbf{A}(k)| = |\mathfrak{g}(k)|^{g-1} \sum_{x \in \mathfrak{g}(k)^\circ} |\mathfrak{g}_x(k)|^g \mathcal{F}(1_H^G)(x).$$

4.2.5. Let $x = x_s + x_n \in \mathfrak{g}(k)^\circ$ be a split element. Then x_s is $G(k)$-conjugate to an element $t \in \mathfrak{t}(k)$, well-defined up to W-conjugation. The centraliser G_{x_s} is a Levi subgroup of G (containing T). Note that since x_n and x_s commute, we have $x_n \in \mathfrak{g}_{x_s}$; thus, we can consider the nilpotent orbit $G_{x_s}(k).x_n \subseteq \mathfrak{g}_{x_s}(k)$.

4.2.6. *Types.* Consider the set of pairs $(L, \mathcal{N})$, where L is a Levi subgroup of G (containing T) and $\mathcal{N}$ is the $L(k)$-orbit of a nilpotent element of $\mathfrak{l}(k)$. Note W acts on this set by conjugation. Let $\mathcal{T}(\mathfrak{g})$ denote the set of W-orbits. We call the elements of $\mathcal{T}(\mathfrak{g})$ *types* and denote them by $\tau = [L, \mathcal{N}]$. By the above discussion, we have a map

$$\mathfrak{g}(k)^\circ \to \mathcal{T}(\mathfrak{g}).$$

This is the additive analogue of (9).

4.2.7. Note that $\mathcal{T}(\mathfrak{g})$ is independent of the ground field. This is due to the following well-known result:

Proposition 26. *There exists a finite set, independent of the finite field k (of good characteristic), parameterising nilpotent $L(k)$-orbits in $\mathfrak{l}(k)$.*

Proof. First of all, since the characteristic of k is assumed to be good for $\mathfrak{g}$, it is also good for $\mathfrak{l}$. In this case, it is known that nilpotent orbits in $\mathfrak{l}(\overline{k})$ are in bijection with nilpotent orbits in $\mathfrak{l}(\mathbb{C})$, cf. [Pre03]. Thus, the set of nilpotent orbits of $\mathfrak{l}(\overline{k})$ is independent of k.

Next, for any element $x \in \mathfrak{l}(k)$, consider the geometric orbit $L(\overline{k}).x$. Then $(L(\overline{k}).x)^F$ splits into N_x many $L(k)$-orbits, where $N_x = |H^1(F, \pi_0(L_x))|$ is the number F-conjugacy classes in $\pi_0(L_x)$, cf. [DM20, Proposition 4.2.14]. In particular, N_x is independent of the ground field. $\square$

4.2.8. Let $\tau = [L, \mathcal{N}]$ be a type. Recall that this means that $\mathcal{N}$ is the $L(k)$-orbit of a nilpotent element $n \in \mathfrak{l}(k)$. We define $d(\mathcal{N})$ to be the dimension of the corresponding L-orbit $L.n \subseteq \mathfrak{l}$. Let

$$d(\tau) := \dim(L) - d(\mathcal{N}).$$

The following is the additive analogue of Proposition 17:

Lemma 27. *Let $x \in \mathfrak{g}(k)^\circ$ be an element of type $[L, \mathcal{N}]$. Then*

$$\dim(\mathfrak{g}_x) = d(\tau).$$

Proof. Observe that $\mathfrak{g}_x = \mathfrak{g}_{x_s} \cap \mathfrak{g}_{x_n} = Z_{\mathfrak{g}_{x_s}}(x_n)$. Indeed, if an element of $\mathfrak{g}$ commutes with x, then it must commute with x_s and x_n, because these can be written as polynomials in x. Thus, we have

$$\dim(\mathfrak{g}_x) = \dim(Z_{\mathfrak{g}_{x_s}}(x_n)) = \dim(Z_{G_{x_s}}(x_n)) = \dim(G_{x_s}) - \dim(G_{x_s}.x_n) = \dim(\mathfrak{g}_{x_s}) - \dim(G_{x_s}.x_n).$$

□

4.2.9. In view of the above discussions, we can rewrite the formula for $|\mathbf{Y}(k)|$ as follows

$$|\mathbf{Y}(k)| = \frac{|Z(k)||\mathfrak{g}(k)|^{g-1}}{|G(k)|} \sum_{\tau \in \mathcal{T}(\mathfrak{g})} q^{d(\tau)g} \sum_{x \in \mathfrak{g}(k)_\tau} \mathcal{F}(1_H^G)(x).$$

For ease of notation, let

$$\boxed{H_\tau(q) := \sum_{x \in \mathfrak{g}(k)_\tau} \mathcal{F}(1_H^G)(x).}$$

Then we see that determining $|\mathbf{Y}(k)|$ is reduced to computing $H_\tau(q)$. Note that H_τ is the additive analogue of S_τ considered in §3.1.12.

4.3. **Evaluating the sum H_τ.** First, observe that $\mathcal{F}(1_H^G)$ is $G(k)$-invariant; thus, instead of summing over all elements in $\mathfrak{g}(k)_\tau$, we can sum over $G(k)$-orbits $\mathfrak{O}_x \subseteq \mathfrak{g}(k)$ of elements of $x \in \mathfrak{g}(k)_\tau$; i.e.

$$H_\tau(q) = \sum_{\mathfrak{O} \in \mathfrak{g}(k)_\tau / G(k)} |\mathfrak{O}| \mathcal{F}(1_H^G)(\mathfrak{O}).$$

We first show that the size of the orbit $|\mathfrak{O}|$ depends only on the type of $\mathfrak{O}$:

Lemma 28. *If $\mathfrak{O} \in \mathfrak{g}(k)_\tau / G(k)$, where $\tau = [L, \mathcal{N}]$, then $|\mathfrak{O}| = \dfrac{|G(k)||\mathcal{N}|}{|L(k)|}$.*

Proof. Let $x = x_s + x_n \in \mathfrak{g}(k)$. Jordan decomposition is preserved under adjoint action; thus, for $g \in G$, $g.x = g.x_s + g.x_n$ is the Jordan decomposition of $g.x$. It follows that $g.x = x$ if and only if $g.x_s = x_s$ and $g.x_n = x_n$. In other words,

$$G_x = G_{x_s} \cap G_{x_n} = Z_{G_{x_s}}(x_n).$$

Now if $x \in \mathfrak{g}(k)_\tau$, with $\tau = [L, \mathcal{N}]$, then $Z_{G_x}(x_n) = |L(k)|/|\mathcal{N}|$. Thus, if $\mathfrak{O}$ is the $G(k)$-orbit of x, we conclude

$$|\mathfrak{O}| = \frac{|G(k)|}{|G_x|} = \frac{|G(k)|}{|Z_{G_{x_s}}(x_n)|} = \frac{|G(k)||\mathcal{N}|}{|L(k)|}.$$

□

The above lemma implies

$$H_\tau(q) = \frac{|G(k)||\mathcal{N}|}{|L(k)|} \sum_{\mathfrak{O} \in \mathfrak{g}(k)_\tau / G(k)} \mathcal{F}(1_H^G)(\mathfrak{O}).$$

To proceed further, we need to understand values of the Fourier transform.

4.3.1. Recall that Theorem 23 states $\mathcal{F}(1_H^G)(x) = q^{|\Phi^+|} R_{\mathfrak{t}}^{\mathfrak{g}}(f_H)(x)$ and that $R_{\mathfrak{t}}^{\mathfrak{g}}$ is defined via a sum over the set

$$\mathcal{A}(x) := \{g \in G(k) \mid g.x_s \in \mathfrak{t}(k)\}.$$

If we assume $x_s \in \mathfrak{t}(k)$, then one readily checks

$$\mathcal{A}(x) = \bigcup_{w \in W} \dot{w} G_{x_s} = \bigsqcup_{w \in W/W_{x_s}} \dot{w} G_{x_s}.$$

Therefore, under this assumption we have:

$$R_{\mathfrak{t}}^{\mathfrak{g}}(f_H)(x) = \frac{Q_T^{G_{x_s}}(\varpi(x_n))}{|G_{x_s}(k)|} \sum_{g \in \mathcal{A}(x)} f_H(g.x_s) = \frac{Q_T^{G_{x_s}}(\varpi(x_n))}{|W_{x_s}|} \sum_{w \in W} f_H(w.x_s).$$

4.3.2. The above discussion implies that for a type $\tau = [L, \mathcal{N}]$, we have

$$H_\tau = \frac{|G(k)||\mathcal{N}|}{|L(k)|} \sum_{x \in \mathfrak{g}(k)_\tau / G(k)} q^{|\Phi^+|} R_{\mathfrak{t}}^{\mathfrak{g}}(f_H)(x) = \frac{|G(k)||\mathcal{N}|}{|L(k)|} \frac{q^{|\Phi^+|} Q_T^L(\varpi(\mathcal{N}))}{|W(L)|} \sum_{w \in W} \sum_{x \in \mathfrak{g}(k)_\tau / G(k)} f_H(w.x_s).$$

Here, x is an orbit representative satisfying $x_s \in \mathfrak{t}(k)$.

4.3.3. The element x_s is well-defined up to W/W_{x_s}-conjugacy, thus, we can now re-write the above sum as:

$$\sum_{w \in W} \sum_{x \in \mathfrak{g}(k)_\tau / G(k)} f_H(w.x_s) = \frac{|W(L)|}{|W|} \sum_{w \in W} \sum_{\substack{y \in \mathfrak{t}(k) \\ W_y \in [W(L)]}} f_H(w.y) = \frac{|W(L)|}{|N_W(W(L))|} \sum_{w \in W} \sum_{\substack{y \in \mathfrak{t}(k) \\ W_y = W(L)}} f_H(w.y).$$

4.3.4. For ease of notation, let

$$\beta_{L,H}(q) := \sum_{\substack{y \in \mathfrak{t}(k) \\ W_y = W(L)}} f_H(y).$$

In view of the fact that $|[L]| = |W/N_W(W(L))|$ (see §3.2.3), we conclude:

$$\boxed{H_\tau(q) = \frac{q^{|\Phi^+|} Q_T^L(\varpi(\mathcal{N}))|\mathcal{N}||G(k)||[L]|}{|L(k)||W|} \sum_{w \in W} \beta_{L, w.H}(q).}$$

Thus, the problem of computing H_τ reduces to evaluating $\beta_{L,H}$.

4.4. **Computing $\beta_{L,H}$.** Let

$$\beta_{L,H}^{\supseteq} := \sum_{\substack{t \in \mathfrak{t} \\ W_t \supseteq W(L)}} f_H(t) = \sum_{t \in \mathfrak{t}^{W(L)}} f_H(t).$$

Let μ denote the Möbius function on the lattice of Levi subgroups of G containing T, ordered by inclusion. Möbius inversion gives

$$\beta_{L,H} = \sum_{L' \supseteq L} \mu(L, L') \beta_{L',H}^{\supseteq}$$

4.4.1. Thus, we are reduced to computing the Möbius function and the functions $\beta^{\supseteq}_{L,H}$. The latter can be computed as follows. Consider the canonical map to the canonical quotient map to the group of coinvariants

$$f_{\mathfrak{l}} : \mathfrak{t} \to \mathfrak{t}_{W(L)} = \mathfrak{t}/\mathfrak{t} \cap [\mathfrak{l}, \mathfrak{l}].$$

Then Pontryagin duality implies

$$\beta^{\supseteq}_{L,H} = \begin{cases} |\mathfrak{t}/\mathfrak{t} \cap [\mathfrak{l},\mathfrak{l}]| & \text{if } f_{\mathfrak{l}}(H) = 0 \text{; i.e. } H \in [\mathfrak{l},\mathfrak{l}]; \\ 0 & \text{otherwise.} \end{cases}$$

4.4.2. Since H is generic, we have that $H \in [\mathfrak{l}, \mathfrak{l}]$ if and only if $\mathfrak{l} = \mathfrak{g}$. Note that

$$|\mathfrak{t}/\mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}]| = |Z(\mathfrak{g})|.$$

Thus, $\beta^{\supseteq}_{L,H}$ is non-zero only if $L = G$, in which case $\mathfrak{t}/(\mathfrak{t} \cap [\mathfrak{l}, \mathfrak{l}]) = \mathrm{Lie}(Z)$. We conclude

$$\boxed{\beta_{L,H} = q^{\dim(Z)} \mu(L, G)|.}$$

4.5. **Conclusion of the point count for once-punctured case.** We have seen that

$$|\mathbf{Y}(\mathbb{F}_q)| = \frac{|Z(k)||\mathfrak{g}(k)|^{g-1}}{|G(k)|} \sum_{\tau \in \mathcal{T}(\mathfrak{g})} q^{gd(\tau)} H_\tau(q),$$

where

$$H_\tau(q) = q^{|\Phi^+| + \dim(Z)} Q^L_T(\varpi(\mathcal{N})) |\mathcal{N}| \mu(L, G) |[L]| \frac{|G(k)|}{|L(k)|}.$$

The same argument as in the previous section implies that $|\mathbf{Y}(\mathbb{F}_q)|$ is a polynomial in q.

4.6. **Multi-puncture case.** Let $\mathbf{Y}$ be a character variety associated to an n-punctured surface group satisfying Assumptions 1. Then a similar analysis as in the once-punctured case gives

$$|\mathbf{Y}(\mathbb{F}_q)| = \frac{|Z(k)||\mathfrak{g}(k)|^{g-1}}{|G(k)|} \sum_{\tau \in \mathcal{T}(\mathfrak{g})} q^{gd(\tau)} H_\tau(q)$$

where

$$H_\tau(q) := \sum_{x \in \mathfrak{g}(k)_\tau} \prod_{i=1}^n \mathcal{F}(1^G_{H_i})(x) = q^{n|\Phi^+| + \dim(Z)} \frac{|G(k)|}{|L(k)|} |\mathcal{N}| \Big(Q^L_T(\varpi(\mathcal{N})) \Big)^n \left(\frac{|W|}{|W(L)|} \right)^{n-1} |[L]| \mu(L, G)$$

For details, see [Gia24, Section number].

4.6.1. For the benefit of the reader, we recall what the notation means. Here, Z is the centre of G, $\mathcal{T}(\mathfrak{g})$ is the set of $\mathfrak{g}$-types (i.e. W-orbits of pairs $(L, \mathcal{N})$, where L is a Levi subgroup of G and $\mathcal{N}$ a nilpotent $L(k)$-orbit in $\mathfrak{l}(k)$), and $d(\tau) = \dim(L) - \dim(\mathcal{N})$. Moreover, Q^L_T is the Green function evaluated at the unipotent class $\omega(\mathcal{N}) \subseteq L(\mathbb{F}_q)$ associated to $\mathcal{N}$, $|[L]|$ is the size of the W-orbit of L, and $\mu(L, G)$ is the value of the Möbius function of the lattice of Levi subgroups of G (containing T).

5. Purity for $\mathbf{Y}$

Our approach to purity hinges on the following

Theorem 29. *Let $\mathfrak{M}$ be an orbifold, equipped with $\mathbb{G}_m$-action and a $\mathbb{G}_m$-equivariant flat map $f : \mathfrak{M} \to \mathbb{A}^1$ covering the standard action. Suppose $\lim_{c\to 0} c.m$ exists for every $m \in \mathfrak{M}$, and $\mathfrak{M}^{\mathbb{G}_m}$ is proper. Then the fibres $\mathfrak{M}_c := f^{-1}(c)$, $c \in \mathbb{A}^1$, support isomorphic pure cohomology.*

When $\mathfrak{M}$ is a smooth variety over $\mathbb{C}$, this is proved in [HLRV11, Appendix B].

5.1. **The naive family.** We first give an alternative realisation of the additive character variety $\mathbf{Y} = \mathbf{A}/\!\!/G$ associated to a generic element $H \in \mathfrak{t} \cap [\mathfrak{g}, \mathfrak{g}]$. Let

$$\mathbf{B} := \left\{ (a_1, b_1, \dots, a_g, b_g) \in \mathfrak{g}^{2g} \,\middle|\, \sum_{i=1}^{g} [a_i, b_i] = H \right\}.$$

Then T acts on $\mathbf{B}$ by the adjoint action.

Lemma 30. $\mathbf{Y} \simeq \mathbf{B}/\!\!/T$.

5.1.1. The above description of $\mathbf{Y}$ gives us an obvious family. Namely, let

$$\mathfrak{B}' := \left\{ (a_1, b_1, \dots, a_g, b_g, c) \in \mathfrak{g}^{2g} \times \mathbb{A}^1 \,\middle|\, \sum_{i=1}^{g} [a_i, b_i] = cH \right\}.$$

and let $\mathfrak{M}' := \mathfrak{B}'/\!\!/T$. The rest of this subsection concerns showing that this family many of the structures and properties required by Theorem 29, except it fails to be an orbifold, because $\mathfrak{B}'$ is singular and the action of T has infinite stabilisers. These shortcomings will be dealt with in next subsection, where we (partially) resolve the singularities of $\mathfrak{M}'$ and obtain the desired orbifold $\mathfrak{M}$.

5.1.2. We have the projection map

$$\gamma : \mathfrak{B}' \to \mathbb{A}^1, \qquad (a_1, b_1, \dots, a_g, b_g, c) \mapsto c$$

which is T-equivariant; therefore, it descends a map $\mathfrak{M}' \to \mathbb{A}^1$. We have an action of $\mathbb{G}_m$ on $\mathfrak{M}'$ coming from the multiplication action on $\mathfrak{g}^{2g}$. Clearly the origin $\mathbf{0} \in \mathfrak{g}^{2g} \times \mathbb{A}^1$ is in $\mathfrak{B}'$. We denote its image in $\mathfrak{M}'$ by $\mathbf{0}$ as well. This is the unique fixed point for the action of $\mathbb{G}_m$ on $\mathfrak{B}'$ and on $\mathfrak{M}'$. Moreover, we have

$$\lim_{c\to 0} c.m = \mathbf{0}, \qquad \forall m \in \mathfrak{M}'.$$

5.1.3. *Dimension of the fibres.* If $c \neq 0$, then a standard application of the regular value theorem implies that $\mathfrak{B}'_c := \gamma^{-1}(cH)$ is smooth of dimension

$$\dim(\mathfrak{B}'_d) = (2g-1)\dim(\mathfrak{g}) + \dim(Z).$$

We have a similar result for the special fibre:

Proposition 31. *We have $|\mathfrak{B}'_0(\mathbb{F}_q)|$ is a monic polynomial in q; moreover,*

$$\dim(\mathfrak{B}'_0) = \begin{cases} \dim(\mathfrak{B}'_1) & \text{if } g > 1 \\ \dim(\mathfrak{B}'_1) + \dim(T) - \dim(Z) & \text{if } g = 1. \end{cases}$$

Proof. The argument is analogous to the multiplicative version given in [BK22, Corollary 5], so we will brief. The additive Frobenius Mass Formula gives

$$|\mathfrak{B}'_0(k)| = |\mathfrak{g}(k)|^g \sum_{x \in \mathfrak{g}(k)/G(k)} |\mathfrak{g}_x(k)|^{g-1}$$

If $g = 1$, then

$$|\mathfrak{B}'_0(k)| = q^{\dim(g)} |\mathfrak{g}(k)/G(k)|$$

and $|\mathfrak{g}(k)/G(k)|$ is a monic polynomial of degree $\dim(T)$.

To see that $\mathfrak{B}'_0(k)|$ is a polynomial in q, we argue as follows. To every $x \in \mathfrak{g}$, we can associate a W-orbit of the pair $(L, \mathcal{N})$, where L is a Levi subgroup of G (not necessarily containing T) and $\mathcal{N}$ is a nilpotent $L(k)$-orbit in $\mathfrak{l}(k)$. We call these pairs a type and denote their set by $T(\mathfrak{g})$. We can then re-write the above sum as

$$|\mathfrak{B}'_0(k)| = |\mathfrak{g}(k)|^g \sum_{\tau \in T(\mathfrak{g})} |\tau(k)|^{g-1} |\mathfrak{g}(k)_\tau|.$$

Here, $\mathfrak{g}(k)_\tau$ is the set of elements of $\mathfrak{g}(k)$ of type τ and $|\tau(k)|$ is defined to be $|\mathfrak{g}_x(k)|$ for some $x \in \mathfrak{g}(k)_\tau$. Each summand is a polynomial in q and $T(\mathfrak{g})$ is independent of q; thus, $|\mathfrak{B}'_0(k)|$ is a polynomial in q. Now one can show that an element $x \in \mathfrak{g}(k)$ contributes to the leading term if and only if $x \in Z(\mathfrak{g}(k))$. Thus, the leading term has degree $(2g-1)\dim(\mathfrak{g}) + \dim(Z(\mathfrak{g}))$. $\square$

Corollary 32. *If $g > 1$, then the map $\gamma : \mathfrak{B}' \to \mathbb{A}^1$ is flat.*

5.2. **Desirable family.** We assume $g \geq 1$. For ease of notation, let $V = \mathfrak{g}^{2g} \times \mathbb{A}^1$. Note that T acts on V by the adjoint action (the action on the last component is trivial). Also, $\mathbb{G}_m$ acts on V by

$$d.(a_1, b_1, ..., a_g, b_g, c) = (da_1, db_1, ..., da_g, db_g, d^2c).$$

5.2.1. Let $\chi \in \mathrm{Hom}(T, \mathbb{G}_m)$ be a generic character. Let V^χ denote the χ-semistable part of V, cf. [Kin94]. This is a $T \times \mathbb{G}_m$-invariant open dense subvariety of V. By Proposition 12, the generic assumption implies that the χ-semi-stability and χ-stability coincide. Thus, the action of T on V^χ has finite stabilisers.

5.2.2. Now let

$$\mathfrak{B} := V^\chi \cap \mathfrak{B}', \qquad \mathfrak{M} := \mathfrak{B} /\!\!/ T.$$

Since T acts on $\mathfrak{B}$ with finite stabilisers, a similar argument as in [KNP23, §2], implies that $\mathfrak{B}$ is smooth. Thus, $\mathfrak{M}$ is an orbifold of dimension $\dim(\mathfrak{B}) - \dim(T)$.

5.2.3. The projection map $\gamma : \mathfrak{B}' \to \mathbb{A}^1$ restricts to a map $\gamma : \mathfrak{B} \to \mathbb{A}^1$. This map is $T \times \mathbb{G}_m$-equivariant; therefore, it induces a $\mathbb{G}_m$-equivariant morphism

$$f : \mathfrak{M} \to \mathbb{A}^1.$$

Let $\mathfrak{B}_c := \gamma^{-1}(c)$ and $\mathfrak{M}_c := f^{-1}(c)$.

5.2.4. If $c \neq 0$, then every point in $\mathfrak{B}_c$ is stable; therefore, χ-stable. Thus, for $c \neq 0$, the map p restricts to an isomorphism

$$\mathfrak{M}_c \simeq \mathfrak{M}'_c \simeq \mathbf{Y}.$$

On the other hand, $\mathfrak{M}_0$ is not isomorphic to $\mathbf{Y}$; nonetheless, since T acts on $\mathfrak{B}_0$ with finite stabilisers, we have

$$\dim(\mathfrak{M}_0) = \dim(\mathfrak{B}_0) - T. \tag{10}$$

Lemma 33. *Suppose $g > 1$. Then f is a flat morphism.*

Proof. By Corollary 32, $\mathfrak{B}'$ is connected. Thus, $\mathfrak{B}$ and therefore $\mathfrak{M}$ are also connected. Next, $\mathfrak{M}$ has quotient singularities; therefore, it is Cohen-Macauley. By miracle flatness (cf. [GW20, Corollary 14.128]), it is sufficient to show that f has equidimensional fibres.

We have already seen that for $c \neq 0$, $\mathfrak{M}_c$ is isomorphic to the generic character variety $\mathbf{Y}$. On the other hand, by Proposition 31, if $g > 1$, we have $\dim(\mathfrak{B}'_0) = \dim(\mathfrak{B}'_c)$. By (10),

$$\dim(\mathfrak{M}_0) = \dim(\mathfrak{B}_0) - \dim(T) = \dim(\mathfrak{B}'_0) - \dim(T) = \dim(\mathfrak{B}'_c) - \dim(T) = \dim(\mathbf{Y}).$$

The only remaining part is to show that $\dim(\mathfrak{B}_0) = \dim(\mathfrak{B}'_0)$. This is equivalent to requiring that $\mathfrak{B}_0$ is non-empty. In other words, we want to show that $\mathfrak{B}'_0$ has a χ-stable element. This would follow if knew that $\mathfrak{B}'_0$ stable element. $\square$

5.2.5. By definition, we have a canonical *proper* map

$$p : \mathfrak{M} \to \mathfrak{M}'.$$

In view of the discussions of the previous subsection, this implies that for every $m \in \mathfrak{M}$, $\lim_{c\to 0} c.m$ exists and belongs to $p^{-1}(\mathbf{0})$. Moreover, we have $\mathfrak{M}^{\mathbb{G}_m}$ is a closed subvariety of $p^{-1}(\mathbf{0})$ and is, therefore, proper. Thus, p is a birational map and a partial resolution of singularities.

5.2.6. *Conclusion.* We have shown that the family $\mathfrak{M}$ has all the structures and properties of Theorem 29. This implies that the cohomology of each fibre carries a pure hodge structure. The generic fibre of $\mathfrak{M}$ is isomorphic to the additive character variety $\mathbf{Y}$. Thus, we have shown that $\mathbf{Y}$ has pure cohomology.

5.3. **Multipunctured case.** Let

$$\mathbf{B} := \left\{ (a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_{n-1}) \in \mathfrak{g}^{2g} \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_{n-1} \,\middle|\, \sum_{i=1}^{g} [a_i, b_i] + x_1 + \cdots + x_{n-1} + H = 0 \right\}.$$

Then T acts on $\mathbf{B}$ by the adjoint action and $\mathbf{Y} \simeq \mathbf{B}/\!\!/T$. Note that while $(\mathcal{O}_1, ..., \mathcal{O}_n)$ is assumed to be generic, $(\mathcal{O}_1, ..., \mathcal{O}_{n-1})$ need not be generic.

5.3.1. Let

$$\mathfrak{B}' := \{(a_1, b_1, ..., a_g, b_g, x_1, ..., x_{n-1}, c) \in \mathfrak{g}^{2g} \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_{n-1} \times \mathbb{A}^1 \mid \sum_{i=1}^{g} [a_i, b_i] + \sum_{j=1}^{n-1} c x_j + cH = 0\}.$$

Then $\mathfrak{B}'$ is a $T \times \mathbb{G}_m$-subvariety of V.

5.3.2. Let $\chi \in \mathrm{Hom}(T, \mathbb{G}_m)$ be a generic character and $\mathfrak{B}$ the χ-semistable part of $\mathfrak{B}'$. Since χ is generic, χ-stability coincides with χ-semistability. Thus, T acts on $\mathfrak{B}$ with finite stabilisers and a similar argument as in [KNP23, §2], implies that $\mathfrak{B}$ is smooth. Therefore,

$$\mathfrak{M} := \mathfrak{B}/\!\!/T.$$

is an orbifold.

5.3.3. We have a canonical map $f : \mathfrak{M} \to \mathbb{A}^1$ induced by the projection

$$(a_1, b_1, ..., a_g, b_g, x_1, ..., x_{n-1}, c) \mapsto c.$$

Let $\mathfrak{M}_c := f^{-1}(c)$. Then for $c \neq 0$, $\mathfrak{M}_c \simeq \mathbf{Y}$.

6. Examples

6.1. **The group $G = \mathrm{SO}_5$.**

$\tau = [L, \rho]$	$m_\tau(q)$	$S_\tau(q)$	$\lvert L(\mathbb{F}_q)\rvert$	$\tilde{\rho}(1)$	$\rho(1)$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	π_0^L	$\nu(L)$
$[\mathrm{SO}_5, \left(\begin{smallmatrix}2\\ \ \end{smallmatrix}\right)]$	$q^4\Phi_1^2\Phi_2^2\Phi_4$	$2\Phi_1^2$	$q^4\Phi_1^2\Phi_2^2\Phi_4$	1	1	8	1	2	2
$[\mathrm{SO}_5, \left(\begin{smallmatrix}0 & & 1\\ & 2 & \end{smallmatrix}\right)]$	$2q^3\Phi_1^2\Phi_2^2$	$2\Phi_1^2$	$q^4\Phi_1^2\Phi_2^2\Phi_4$	$\frac{1}{2}q\Phi_4$	1	8	1	2	2
$[\mathrm{SO}_5, \left(\begin{smallmatrix}1 & & 2\\ & 0 & \end{smallmatrix}\right)]$	$2q^3\Phi_1^2\Phi_2^2$	$2\Phi_1^2$	$q^4\Phi_1^2\Phi_2^2\Phi_4$	$\frac{1}{2}q\Phi_4$	1	8	1	2	2
$[\mathrm{SO}_5, \left(\begin{smallmatrix}0 & & 2\\ & 1 & \end{smallmatrix}\right)]$	$2q^3\Phi_1^2\Phi_4$	$2^{n+1}\Phi_1^2$	$q^4\Phi_1^2\Phi_2^2\Phi_4$	$\frac{1}{2}q\Phi_2^2$	2	8	1	2	2
$[\mathrm{SO}_5, \left(\begin{smallmatrix}0 & & 1 & & 2\\ & 1 & & 2 & \end{smallmatrix}\right)]$	$\Phi_1^2\Phi_2^2\Phi_4$	$2\Phi_1^2$	$q^4\Phi_1^2\Phi_2^2\Phi_4$	q^4	1	8	1	2	2
$[A_1 \times A_1, 2^1 \otimes 2^1]$	$q^4\Phi_1^2\Phi_2^2$	$2^n\Phi_1^2$	$q^2\Phi_1^2\Phi_2^2$	1	1	4	1	4	2
$[A_1 \times A_1, 2^1 \otimes 1^2]$	$q^3\Phi_1^2\Phi_2^2$	$2^n\Phi_1^2$	$q^2\Phi_1^2\Phi_2^2$	q	1	4	1	4	2
$[A_1 \times A_1, 1^2 \otimes 2^1]$	$q^3\Phi_1^2\Phi_2^2$	$2^n\Phi_1^2$	$q^2\Phi_1^2\Phi_2^2$	q	1	4	1	4	2
$[A_1 \times A_1, 1^2 \otimes 1^2]$	$q^2\Phi_1^2\Phi_2^2$	$2^n\Phi_1^2$	$q^2\Phi_1^2\Phi_2^2$	q^2	1	4	1	4	2
$[A_1, 2^1]$	$q^4\Phi_1^2\Phi_2$	$-2 \cdot 4^n\Phi_1^2$	$q\Phi_1^2\Phi_2$	1	1	2	2		-4
$[A_1, 1^2]$	$q^3\Phi_1^2\Phi_2$	$-2 \cdot 4^n\Phi_1^2$	$q\Phi_1^2\Phi_2$	q	1	2	2		-4
$[A_1', 2^1]$	$q^4\Phi_1^2\Phi_2$	$-4^n\Phi_1^2$	$q\Phi_1^2\Phi_2$	1	1	2	2		-2
$[A_1', 1^2]$	$q^3\Phi_1^2\Phi_2$	$-4^n\Phi_1^2$	$q\Phi_1^2\Phi_2$	q	1	2	2		-2
$[T, 1^1]$	$q^4\Phi_1^2$	$8^n\Phi_1^2$	Φ_1^2	1	1	1	1		8

Figure 2. The fourteen SO_5-types. Explain Weyl group character notation.

$\tau = [L, \mathcal{N}]$	$q^{gd(\tau)}$	$H_\tau(q)$	$\lvert L(\mathbb{F}_q)\rvert$	$\lvert\mathcal{N}\rvert$	$Q_L^T(\varpi(\mathcal{N}))$	$\lvert W(L)\rvert$	$\lvert[L]\rvert$	$\mu(L, G)$
$[\mathrm{SO}_5, 1^4]$	a	b	c	d	e	f	g	h
$[\mathrm{SO}_5, 2^1 1^2]$	a	b	c	d	e	f	g	h
$[\mathrm{SO}_5, 2^2]$	a	b	c	d	e	f	g	h
$[\mathrm{SO}_5, 2^2_\star]$	a	b	c	d	e	f	g	h
$[\mathrm{SO}_5, 4^1]$	a	b	c	d	e	f	g	h
$[A_1, 1^2]$	a	b	c	d	e	f	g	h
$[A_1, 2^1]$	a	b	c	d	e	f	g	h
$[A_1', 1^2]$	a	b	c	d	e	f	g	h
$[A_1', 2^1]$	a	b	c	d	e	f	g	h
$[T, 1^1]$	a	b	c	d	e	f	g	h

Figure 3. The ten $\mathfrak{so}_5$-types. Explain rational orbit notation.

6.2. **The group $G = G_2$.**

$\tau = [L, \rho]$	$m_\tau(q)$	$S_\tau(q)$	$\|L(\mathbb{F}_q)\|$	$\rho(1)$	$\tilde{\rho}(1)$	$\|W(L)\|$	$\|[L]\|$	π_0^L	$\nu(L)$
$[G_2, \phi_{1,0}]$	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	Φ_1^2	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	1	1	12	1	1	1
$[G_2, \phi'_{1,3}]$	$3q^5\Phi_1^2\Phi_2^2$	Φ_1^2	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\frac{1}{3}q\Phi_3\Phi_6$	1	12	1	1	1
$[G_2, \phi''_{1,3}]$	$3q^5\Phi_1^2\Phi_2^2$	Φ_1^2	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\frac{1}{3}q\Phi_3\Phi_6$	1	12	1	1	1
$[G_2, \phi_{2,1}]$	$6q^5\Phi_1^2\Phi_6$	$2^n\Phi_1^2$	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\frac{1}{6}q\Phi_2^2\Phi_3$	2	12	1	1	1
$[G_2, \phi_{2,2}]$	$2q^5\Phi_1^2\Phi_3$	$2^n\Phi_1^2$	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\frac{1}{2}q\Phi_2^2\Phi_6$	2	12	1	1	1
$[G_2, \phi_{1,6}]$	$\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	Φ_1^2	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	q^6	1	12	1	1	1
$[A_2, 3^1]$	$q^6\Phi_1^2\Phi_2\Phi_3$	$2^n\Phi_1^2$	$q^3\Phi_1^2\Phi_2\Phi_3$	1	1	6	1	3	2
$[A_2, 2^1 1^1]$	$q^5\Phi_1^2\Phi_3$	$4^n\Phi_1^2$	$q^3\Phi_1^2\Phi_2\Phi_3$	$q\Phi_2$	2	6	1	3	2
$[A_2, 1^3]$	$q^3\Phi_1^2\Phi_2\Phi_3$	$2^n\Phi_1^2$	$q^3\Phi_1^2\Phi_2\Phi_3$	q^3	1	6	1	3	2
$[A_1 \times A'_1, 2^1 \otimes 2^1]$	$q^6\Phi_1^2\Phi_2^2$	$3^n\Phi_1^2$	$q^2\Phi_1^2\Phi_2^2$	1	1	4	3	2	1
$[A_1 \times A'_1, 2^1 \otimes 1^2]$	$q^5\Phi_1^2\Phi_2^2$	$3^n\Phi_1^2$	$q^2\Phi_1^2\Phi_2^2$	q	1	4	3	2	1
$[A_1 \times A'_1, 1^2 \otimes 2^1]$	$q^5\Phi_1^2\Phi_2^2$	$3^n\Phi_1^2$	$q^2\Phi_1^2\Phi_2^2$	q	1	4	3	2	1
$[A_1 \times A'_1, 1^2 \otimes 1^2]$	$q^4\Phi_1^2\Phi_2^2$	$3^n\Phi_1^2$	$q^2\Phi_1^2\Phi_2^2$	q^2	1	4	3	2	1
$[A_1, 2^1]$	$q^6\Phi_1^2\Phi_2$	$-2\cdot 6^n\Phi_1^2$	$q\Phi_1^2\Phi_2$	1	1	2	3		-4
$[A_1, 1^2]$	$q^5\Phi_1^2\Phi_2$	$-2\cdot 6^n\Phi_1^2$	$q\Phi_1^2\Phi_2$	q	1	2	3		-4
$[A'_1, 2^1]$	$q^6\Phi_1^2\Phi_2$	$-6^n\Phi_1^2$	$q\Phi_1^2\Phi_2$	1	1	2	3		-2
$[A'_1, 1^2]$	$q^5\Phi_1^2\Phi_2$	$-6^n\Phi_1^2$	$q\Phi_1^2\Phi_2$	q	1	2	3		-2
$[T, 1^1]$	$q^6\Phi_1^2$	$12^n\Phi_1^2$	Φ_1^2	1	1	1	1		12

FIGURE 4. The eighteen G_2-types. Explain Weyl group character notation.

$\tau = [L, \mathcal{N}]$	$q^{gd(\tau)}$	$H_\tau(q)$	$\|L(\mathbb{F}_q)\|$	$\|\mathcal{N}\|$	$Q_L^T(\varpi(\mathcal{N}))$	$\|W(L)\|$	$\|[L]\|$	$\mu(L,G)$
$[G_2, 0]$	a	b	c	d	e	f	g	h
$[G_2, A_1]$	a	b	c	d	e	f	g	h
$[G_2, \tilde{A}_1]$	a	b	c	d	e	f	g	h
$[G_2, G_2(a_1)_{1^3}]$	a	b	c	d	e	f	g	h
$[G_2, G_2(a_1)_{2^1 1^1}]$	a	b	c	d	e	f	g	h
$[G_2, G_2(a_1)_{3^1}]$	a	b	c	d	e	f	g	h
$[G_2, G_2]$	a	b	c	d	e	f	g	h
$[A_1, 1^2]$	a	b	c	d	e	f	g	h
$[A_1, 2^1]$	a	b	c	d	e	f	g	h
$[A'_1, 1^2]$	a	b	c	d	e	f	g	h
$[A'_1, 2^1]$	a	b	c	d	e	f	g	h
$[T, 1^1]$	a	b	c	d	e	f	g	h

FIGURE 5. The twelve $\mathfrak{g}_2$-types. We are using the notation of [Car93, p. 427] for the unipotent (geometric) conjugacy classes of G_2. The component group of $G_2(a_1)$ is S_3, so rational classes are parameterised by partitions of 3.

6.3. **Euler characteristic.**

6.3.1. *The case $g = 1$ and $\dim(Z) = 0$.* In this case, the Euler characteristic is given by

6.3.2. *The case $g = 0$ and $n \geq 3$.*

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