Invariant Theory and Unipotent Groups

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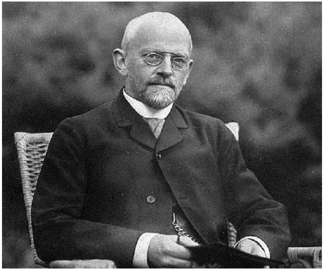
History of Hilbert's 14th Problem



In the early 20th century Hilbert posed a series of 23 open problems to the ICM.

These have been a very influential set of open problems with some still being unsolved.

Hilbert's 14th Problem

Let G be a group acting linearly on a variety X. Is the subalgebra of invariant functions,

$$k[X]^G = \{f \in k[X] \,|\, g.f = f,\ \forall g \in G\},$$

finitely generated over the ground field k?

Examples of Invariant Polynomials

Denote by $G := \mathrm{GL}_n(\mathbb{C})$ the set of $n \times n$ invertible matrices over $\mathbb{C}$, and let G act on itself by conjugation, i.e, for all $A, B \in G$, $A \cdot B = ABA^{-1}$.

Define the coordinate functions $x_{ij} \in \mathbb{C}[G]$ by $x_{ij}(A) = a_{ij}$

Then $\mathbb{C}[G]$ is the vector space with basis $\{x_{ij}\}_{1 \leq i,j \leq n}$.

Example

1) $\det \in \mathbb{C}[G]^G$ since
$(A \cdot \det)(B) = \det(A^{-1} \cdot B) = \det(A^{-1}BA) = \det(B)$ and;
2) $\mathrm{tr} \in \mathbb{C}[G]^G$ since $(A \cdot \mathrm{tr})(B) = \mathrm{tr}(A^{-1} \cdot B) = \mathrm{tr}(A^{-1}BA) = \mathrm{tr}(B)$.

In fact, the invariant functions on G are given by coefficients of the characteristic polynomial.

History of Hilbert's 14th Problem

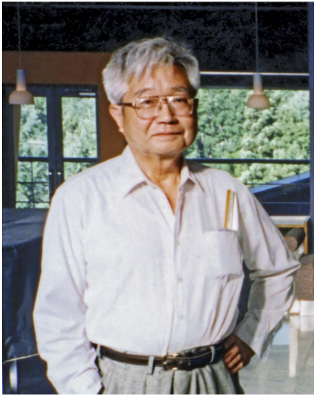
Gordan (1868), Hilbert (1890s), Hurwitz and Maurer (1897) made progress for certain classes of groups.

Hilbert showed that $\mathbb{C}[G]^G$ is always finitely generated for any reductive group G over $\mathbb{C}$.

Example

Reductive Groups	Non-reductive Groups
GL_n, SL_n, Sp_{2m}	$(\mathbb{C}, +)$, UL_n, B_n

History of Hilbert's 14th Problem



Nagata (1959) provided the first counter example by constructing an action of $\mathbb{C}^3$ on $\mathbb{C}^{32}$ whose algebra of invariants is not finitely generated.

Work by Mumford, Haboush and Nagata (1960s-1970s) settled that $k[G]^G$ is finitely generated for any reductive group G, and arbitrary field k with arbitrary characteristic.

However, results for non reductive groups are more scarce.

Subject of Thesis

Let U be a maximal unipotent subgroup of a reductive group G over an algebraically closed field k with characteristic zero acting on itself by conjugation.

Example

If $G = \mathrm{GL}_n(\mathbb{C})$, then U is the group

$$\mathrm{UL}_n(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & * & * & \dots & * \\ & 1 & * & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \right\}$$

consisting of upper triangular matrices with $1'$s on the main diagonal and $*$ denoting elements of the ground field.

Subject of Thesis

Let $k[U]$ denote the algebra of functions on U and $k[U]^U$ denote the subalgebra of functions invariant under conjugation of U.

Example

For $G = \mathrm{GL}_n(\mathbb{C})$, $\mathbb{C}[\mathrm{UL}_n] = \mathbb{C}[x_{ij}]_{1\leq i<j\leq n}$. Then, for $A \in \mathrm{UL}_n(\mathbb{C})$, the function

$$f(A) = \sum_{i=1}^{n-1} x_{i,i+1}(A) = \sum_{i=1}^{n-1} a_{i,i+1}.$$

One can show that $f \in \mathbb{C}[U]^U$.

Our goal is to give a transparent proof of the following theorem.

Theorem

$k[U]^U$ is a polynomial algebra in n variables, where n is the number of simple roots of G.

Some Preliminaries

In general, groups can be highly "nonlinear". To ammend this one can study the Lie algebra of an algebraic group.

A Lie algebra, $\mathfrak{g}$, over a field k, is a vector space together with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(x, y) \mapsto [x, y]$ (the Lie bracket) satisfying anti-commutativity and the Jacobi identity.

We focus on the Lie algebra, $\mathfrak{n}$, of U

$$U \xrightarrow{\text{Linearise}} \mathfrak{n}$$

$$k[U] \xrightarrow{\text{Linearise}} k[\mathfrak{n}]$$

Linearised Problem

Example

If UL_n is the group of all $n \times n$ unipotent matrices, then the corresponding Lie algebra, $\mathfrak{n}$, is the set of all $n \times n$ upper triangular matrices.

$$\mathrm{UL}_n = \left\{ \begin{pmatrix} 1 & * & \dots & * \\ & 1 & \ddots & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix} \right\} \qquad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & * & \dots & * \\ & 0 & \ddots & \vdots \\ & & \ddots & * \\ & & & 0 \end{pmatrix} \right\}$$

The linearised problem concerns itself with the action of the group on the Lie algebra, via the adjoint action.

Lemma about Invariants of Simple Root Spaces

The adjoint representation of a finite dimensional reductive Lie algebra admits the following Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where Φ is the set of roots and $\mathfrak{g}_\alpha$ are one-dimensional root spaces.
In particular, there is a base Δ for the roots containing all simple roots.
In proving the main theorem, our approach hinges on two lemmas.

Lemma (1)

Let $x \in \mathfrak{n} := \mathrm{Lie}(U)$, where U is a maximal unipotent subgroup of a reductive group, G, over an algebraically closed field k with $\mathrm{char}(k) = 0$. Then, for all $A \in U$ and $\alpha \in \Delta$, the projection of x onto $\mathfrak{g}_\alpha$ is the same as the projection of $Ad_A(x)$ onto $\mathfrak{g}_\alpha$.

Example for GL_n

Example

For $x \in \mathfrak{n}$ and $A \in U$, consider the action of A by conjugation:

$$\begin{pmatrix} 0 & a_{12} & * & \dots & * \\ & 0 & a_{23} & * & \vdots \\ & & \ddots & \ddots & * \\ & & & 0 & a_{n-1,n} \\ & & & & 0 \end{pmatrix} \xmapsto{\mathrm{Ad}_A} \begin{pmatrix} 0 & a_{12} & {\color{orange}*} & {\color{orange}\dots} & {\color{orange}*} \\ & 0 & a_{23} & {\color{orange}*} & {\color{orange}\vdots} \\ & & \ddots & \ddots & {\color{orange}*} \\ & & & 0 & a_{n-1,n} \\ & & & & 0 \end{pmatrix}$$

The projection of x onto the simple root spaces remains unchanged.

Proof of Lemma 1

Proof.

Since U is generated by the root groups U_α, it is enough to show this on these generators.

Furthermore, the exponential map $\exp : \mathfrak{n} \to U$ is a surjection.

Hence, we can express any element $A \in U_\alpha$ as $\exp u_\alpha$, for some $u_\alpha \in \mathfrak{n}$.

Then, we compute the adjoint action on $x \in \mathfrak{n}$ by A as:

$$\begin{aligned} Ad_A(x) = Ad_{\exp u_\alpha}(x) = \exp\left(ad(u_\alpha)\right)(x) \\ = x + [u_\alpha, x] + \frac{1}{2}[u_\alpha, [u_\alpha, x]] + \cdots + \frac{1}{k!} ad(u_\alpha)^k(x). \end{aligned}$$

$\square$

Proof of Lemma 1

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Then, we compute the adjoint action on x by A as:

$$\begin{aligned} Ad_A(x) = Ad_{\exp u_\alpha}(x) &= \exp\left(ad(u_\alpha)(x)\right) \\ &= x + \underbrace{[u_\alpha, x] + \frac{1}{2}[u_\alpha, [u_\alpha, x]] + \cdots + \frac{1}{k!} ad(u_\alpha)^k(x)}_{\text{nonsimple projection}}. \end{aligned}$$

Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, this concludes the proof $\square$

Important Consequences

Corollary

The coordinate functions f_α, $\alpha \in \Delta$, are U-invariant.

We can reformulate the results above as follows:

Let $\mathfrak{n}_\Delta \subset \mathfrak{n}$ be the subset of $\mathfrak{n}$ defined as all those elements whose projection onto the nonsimple root spaces are zero, i.e.,

$$\mathfrak{n}_\Delta = \{x \in \mathfrak{n} \mid f_\alpha(x) = 0, \forall \alpha \notin \Delta\}.$$

$$\begin{pmatrix} 0 & a_{12} & 0 & \dots & 0 \\ & 0 & a_{23} & 0 & \vdots \\ & & \ddots & \ddots & 0 \\ & & & 0 & a_{n-1,n} \\ & & & & 0 \end{pmatrix}$$

Proposition

The restriction map $k[\mathfrak{n}] \to k[\mathfrak{n}_\Delta]$ defines a surjection

$$\varphi : k[\mathfrak{n}]^U \twoheadrightarrow k[\mathfrak{n}_\Delta]$$

Lemma about Adjoint Action on Regular Nilpotents

We define the set of regular nilpotent elements as

$$\mathfrak{n}^r := \{x \in \mathfrak{n} \mid f_\alpha(v) \neq 0\,,\ \forall \alpha \in \Delta\},$$

in other words, the projection onto all simple root spaces is nonzero.

$$\begin{pmatrix} 0 & a_{12} & * & \dots & * \\ & 0 & a_{23} & * & \vdots \\ & & \ddots & \ddots & * \\ & & & 0 & a_{n-1,n} \\ & & & & 0 \end{pmatrix}$$

Lemma (2)

Let $x \in \mathfrak{n}^r$. Then there exists an $A \in U$ such that the projection of $Ad_A(x)$ onto all non-simple root spaces is zero.

Example for GL_n

Example

For $x \in \mathfrak{n}^r$ there is an $A \in U$, such that the action of A on x by conjugation:

$$\begin{pmatrix} 0 & a_{12} & * & \dots & * \\ & 0 & a_{23} & * & \vdots \\ & & \ddots & \ddots & * \\ & & & 0 & a_{n-1,n} \\ & & & & 0 \end{pmatrix} \xmapsto{\mathrm{Ad}_A} \begin{pmatrix} 0 & a_{12} & 0 & \dots & 0 \\ & 0 & a_{23} & 0 & \vdots \\ & & \ddots & \ddots & 0 \\ & & & 0 & a_{n-1,n} \\ & & & & 0 \end{pmatrix}$$

has a zero projection for all non-simple root spaces.

A Crucial Property of Regular Nilpotent Elements

Lemma

Let U be a unipotent algebraic group with Lie algebra $\mathfrak{n}$. The set $x + [\mathfrak{n}, \mathfrak{n}]$, where $x \in \mathfrak{n}^r$, is a single U-orbit.

Proof.

See Chriss-Ginzburg, *Representation Theory and Complex Geometry*. □

Proof of Lemma 2

Proof.

Fix an $x = \sum_{\alpha \in \Delta} a_\alpha v_\alpha + \sum_{\alpha \notin \Delta} b_\alpha v_\alpha \in \mathfrak{n}^r$

Then $x \in x' + [\mathfrak{n}, \mathfrak{n}]$, where $x' = \sum_{\alpha \in \Delta} a_\alpha v_\alpha$, since $[\mathfrak{n}, \mathfrak{n}]$ contains all those elements whose projection onto the simple root spaces is zero.

But $x' = \sum_{\alpha \in \Delta} a_\alpha v_\alpha$ also lies in space $x' + [\mathfrak{n}, \mathfrak{n}]$.

Then, since $x' + [\mathfrak{n}, \mathfrak{n}]$ is a single U-orbit, this implies the existence of $A \in U$, such that $Ad_A(x) = x'$.

□

Main Theorem

Theorem

The map

$$\varphi : k[\mathfrak{n}]^U \to k[\mathfrak{n}_\Delta]$$

is an isomorphism of k-algebras.

This, will allow us to conclude that $k[\mathfrak{n}]^U$ is a polynomial algebra in $|\Delta|$ many variables.

Proof of Theorem

Proof

Suppose $f \in \ker \varphi$. Then $f|_{\mathfrak{n}_\Delta} = 0$, i.e., $f(x) = 0$ for all $x \in \mathfrak{n}_\Delta$.

For any

$$x \in \mathfrak{n}^r = \{x \in \mathfrak{n} \mid f_\alpha(\mathfrak{n}) \neq 0, \forall \alpha \in \Delta\},$$

by Lemma (2), there exists a $A \in U$ such that $Ad_A(x) = x'$, where $x' \in \mathfrak{n}^r \cap \mathfrak{n}_\Delta$.

Since f is U-invariant $f(x) = f(x') = 0$, by assumption.

Proof of Theorem

Proof.

Hence, $f(\mathfrak{n}^r) = 0$.

But $\mathfrak{n}^r$ is a Zariski-open, dense subset of $\mathfrak{n}$ and since f is continuous, $f = 0$ on $\mathfrak{n}$, i.e, f is the constant zero function.

Thus, $\ker \varphi = \{0\}$, hence, φ is injective. $\square$

At the Group Level

Corollary

$$k[U]^U \cong k[\mathfrak{n}]^U.$$

Proof.

Since the exponential $\exp : \mathfrak{n} \to U$ is an isomorphism of affine varieties, the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{n} \arrow[r, "\mathrm{Ad}"] \arrow[d, "\exp"'] & \mathfrak{n} \arrow[d, "\exp"] \\
U \arrow[r, "\mathrm{Inn}"'] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{n} \arrow[rr, "\exp"] \arrow[drr, phantom, "\Downarrow"] & & U \\
k[U] \arrow[rr, "\exp^*"'] & & k[\mathfrak{n}]
\end{tikzcd}$$

In particular, $\mathrm{Ad}(U)$-invariant elements in $\mathfrak{n}$ get mapped to conjugation invariant elements in U under the exponential map. □

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Thank You