PARABOLICS AND STABILITY

Let G be a connected reductive group over $\mathbb{C}$ and T a maximal torus in G. Let us call a parabolic subgroup P in G *admissible* if P contains T. Similarly we have admissible Borels.

Definition 1. An element $x \in \mathfrak{g}$ is called T-stable if it is not in any admissible parabolic $\mathfrak{p} \subset \mathfrak{g}$.

Now, let us consider the adjoint action of T on $\mathfrak{g}$.

Theorem 2. *If $x \in \mathfrak{g}$ is T-stable, then $T_x/Z(G) := \mathrm{Stab}_T(x)/Z(G)$ is finite.*

0.1. The goal of this note is to consider $B_x/Z(G) := \mathrm{Stab}_B(x)/Z(G)$ when x is T-stable. Let us consider the case $\mathfrak{g} = \mathfrak{gl}_3$. Then we know that admissible parabolic subgroups have the following form:

$$\left\{\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}\right\}, \quad \left\{\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}\right\}, \quad \left\{\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}\right\}, \quad \left\{\begin{bmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{bmatrix}\right\}, \quad \left\{\begin{bmatrix} * & 0 & * \\ * & * & * \\ * & 0 & * \end{bmatrix}\right\}.$$

Now, let us consider an element

$$x := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Obvious, this is not in any admissible parabolic subgroup, i.e., T-stable, and so $T_x/Z(G)$ if finite. Note that from the direct computation,

$$\left\{ \begin{bmatrix} a & a-b & b-a \\ 0 & b & a-b \\ 0 & 0 & a \end{bmatrix} \middle| \, a, b \neq 0 \right\} \subset B_x.$$

Remark 3. If we consider T_x, then $a = b$, so the set $\left\{ \begin{bmatrix} a & a-b & b-a \\ 0 & b & a-b \\ 0 & 0 & a \end{bmatrix} \middle| \, a, b \neq 0 \right\}$ is $Z(G)$. However, obviously, when we consider B_x, this set is strictly larger than $Z(G)$.

0.2. Let us choose $a = 2, b = 1$, and then we have the following:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \in B_x.$$

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However, for any $n \in \mathbb{N}$,

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}^n = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \in B_x$$

and $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}^n \notin Z(G)$ due to $1 \neq 2^n$ for any n. This implies that $B_x/Z(G)$ is an infinite set since $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has infinite order.

0.3. The following lists are the similar examples when we consider every Borel subgroup.

(1) When $B = \left\{ \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix} \right\}$,

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \in B_x.$$

This element has infinite order.

(2) When $B = \left\{ \begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ * & * & * \end{bmatrix} \right\}$,

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \in B_x.$$

This element has infinite order.

(3) When $B = \left\{ \begin{bmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$,

$$\begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \in B_x.$$

This element has infinite order.

(4) When $B = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{bmatrix} \right\}$,

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \in B_x.$$

This element has infinite order.

(5) When $B = \left\{ \begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & 0 & * \end{bmatrix} \right\}$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \in B_x.$$

This element has infinite order.