University of Geneva | Bachelor in Economics and Management
Probability 1 | Bachelor in International Relations
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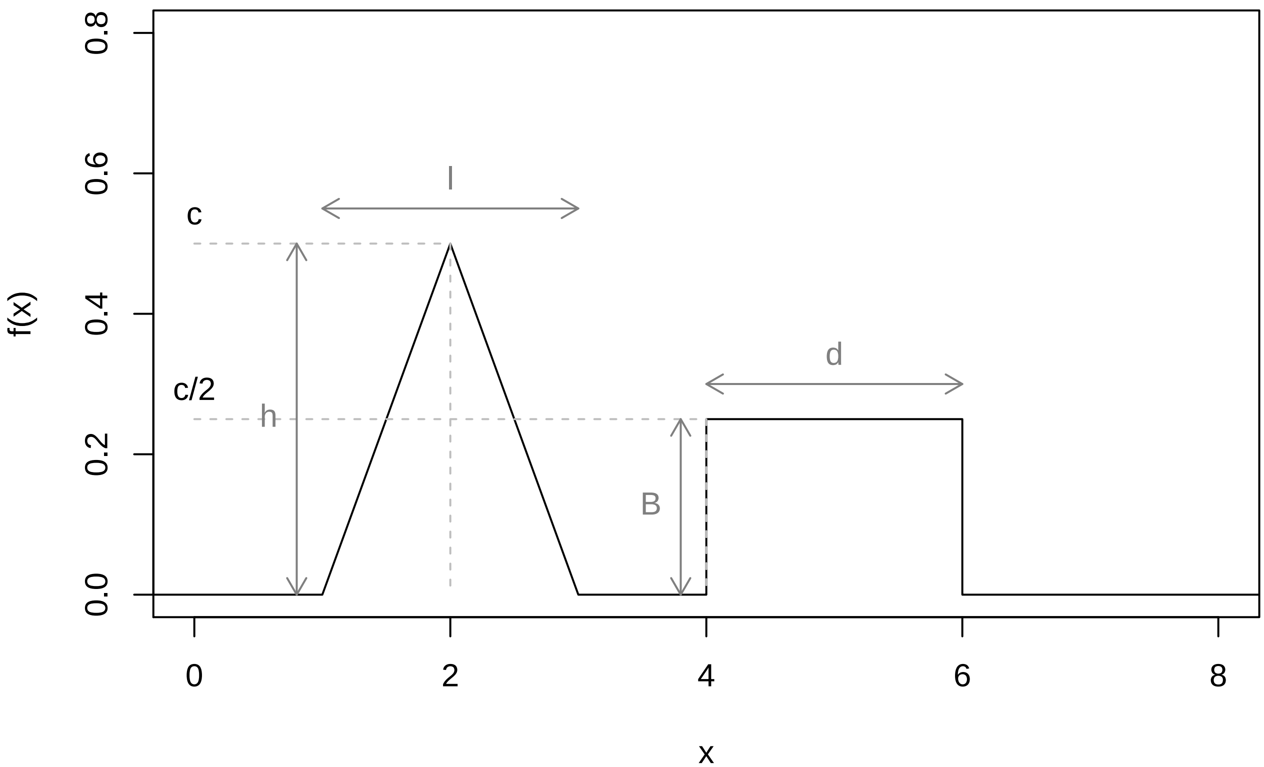
ASSIGNMENT 08

Exercise 1:



l : length of the base of the triangle

h : length of the height of the triangle

d : length of the base of the rectangle

B : length of the heightof the rectangle

1. For $f(x)$ to be a density function, it is necessary that for all x, $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$

   (a)) From the graphical representation we find that for all x we have $f(x) \geq 0$.

   (b) To verify that $\int_{-\infty}^{\infty} f(x)dx = 1$ is satisfied, it is necessary to determine if there is a value c so that the total area under the function $f(x)$ is equal to 1.

   $$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \text{triangle area} + \text{rectangle area} = \frac{lh}{2} + dB \\ &= \frac{(3-1)c}{2} + (6-4)\frac{c}{2} = 2c \end{aligned}$$

   Where we have that

   $$\int_{-\infty}^{\infty} f(x)dx = 1 \iff 2c = 1.$$

   The only solution is $c = 0.5$.

2. Graphically we find that the density function f (x) is a piecewise form function:

   $$f(x) = \begin{cases} f_1(x) & \text{if } 1 < x \leq 2 \\ f_2(x) & \text{if } 2 < x \leq 3 \\ f_3(x) & \text{if } 4 < x \leq 6 \\ 0 & \text{if not} \end{cases}$$

   (a) $f_1(x)$ is a line $a + bx$ found from the two points $(x_1, y_1) = (1, 0)$ and $(x_2, y_2) = (2, 0.5)$. We are looking for a and b such that $f_1(x_1) = y_1$ and $f_1(x_2) = y_2$, i.e., we look for a and b such that $a + bx_1 = y_1$ and $a + bx_2 = y_2$. We find a and b with the equations:

   $$\begin{aligned} b &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{0.5 - 0}{2 - 1} = 0.5 \\ a &= y_1 - bx_1 = 0 - 0.5 \cdot 1 = -0.5. \end{aligned}$$

   So

   $$f_1(x) = -0.5 + 0.5x.$$

   (b) Similarly, $f_2(x)$ is a line $c+dx$ that we find from the two points $(x_2, y_2) = (2, 0.5)$ and $(x_3, y_3) = (3, 0)$:

   $$\begin{aligned} d &= \frac{y_3 - y_2}{x_3 - x_2} = \frac{0 - 0.5}{3 - 2} = -0.5 \\ c &= y_2 - bx_2 = 0.5 - (-0.5) \cdot 2 = 1.5. \end{aligned}$$

   So

   $$f_2(x) = 1.5 - 0.5x.$$

(c) And, finally,

$$f_3(x) = \frac{c}{2} = \frac{0.5}{2} = 0.25.$$

3. The expectation of X is defined as $\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$.

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{1} x \cdot 0 \; dx + \int_{1}^{2} x \cdot f_1(x) \; dx + \int_{2}^{3} x \cdot f_2(x) \; dx + \int_{3}^{4} x \cdot 0 \; dx + \int_{4}^{6} x \cdot f_3(x) \; dx \\
&\quad + \int_{6}^{+\infty} x \cdot 0 \; dx \\
&= \int_{1}^{2} x \cdot (-0.5 + 0.5x) \; dx + \int_{2}^{3} x \cdot (1.5 - 0.5x) \; dx + \int_{4}^{6} x \cdot 0.25 \; dx \\
&= \int_{1}^{2} (-0.5x + 0.5x^2) \; dx + \int_{2}^{3} (1.5x - 0.5x^2) \; dx + \int_{4}^{6} 0.25x \; dx \\
&= \left[\frac{-0.5x^2}{2} + \frac{0.5x^3}{3} + c\right]_1^2 + \left[\frac{1.5x^2}{2} + \frac{-0.5x^3}{3} + c\right]_2^3 + \left[\frac{0.25x^2}{2} + c\right]_4^6 \\
&= \left(\frac{-2}{2} + \frac{4}{3} - \frac{-0.5}{2} - \frac{0.5}{3}\right) + \left(\frac{13.5}{2} + \frac{-13.5}{3} - \frac{6}{2} + \frac{4}{3}\right) + \left(\frac{9}{2} - \frac{4}{2}\right) \\
&= 3.5
\end{aligned}$$

4. Determine $P(X \geq 2)$, $P(X \leq 4.5)$ and $P(2.5 \leq X \leq 5)$.

(a) $P(X \geq 2)$: Solution from the geometrical point of view:

$$\begin{aligned}
P(X \geq 2) &= 0.5 \text{ triangle area } + \text{ rectangle area} \\
&= 0.5 \cdot 0.5 + 0.5 \\
&= 0.75
\end{aligned}$$

Solution with density:

$$\begin{aligned}
\int_{2}^{3} (1.5 - 0.5x) \; dx + \int_{4}^{6} 0.25 \; dx &= \left[1.5x - \frac{0.5x^2}{2} + c\right]_2^3 + [0.25x + c]_4^6 \\
&= 0.25 + 0.5 = 0.75
\end{aligned}$$

(b) $P(X \leq 4.5)$: Solution from the geometrical point of view:

$$\begin{aligned}
P(X \leq 4.5) &= \text{ triangle area } + 0.25 \text{ rectangle area} \\
&= 0.5 + 0.25 \cdot 0.5 \\
&= 0.625
\end{aligned}$$

Solution with density:

$$\begin{aligned}
P(X \leq 4.5) &= 1 - P(X > 4.5) = 1 - \int_{4.5}^{6} 0.25 \; dx \\
&= 1 - [0.25x + c]_{4.5}^{6} = 1 - 0.375 = 0.625
\end{aligned}$$

(c) $P(2.5 \leq X < 5)$: Here it is necessary to integrate the first part $P(2.5 \leq X < 3)$of the probability because the cut at 2.5 is placed in the middle of the lowering of the triangle.

$$\begin{aligned}
P(2.5 \leq X < 5) &= \int_{2.5}^{3} (1.5 - 0.5x)\ dx + 0.5 \text{ rectangle area} \\
&= \left[1.5x - \frac{0.5x^2}{2} + c\right]_{2.5}^{3} + 0.5 \cdot 0.5 \\
&= 1.5 \cdot 3 - \frac{0.5 \cdot 3^2}{2} - 1.5 \cdot 2.5 + \frac{0.5 \cdot 2.5^2}{2} + 0.5 \cdot 0.5 \\
&= 0.3125
\end{aligned}$$

Exercise 2:

Let T be the random variable representing the waiting time of the bus, with $T \sim U(0, 30)$. Its density function is therefore

$$f_T(t) = \begin{cases} \frac{1}{30} & \text{if } 0 < t < 30 \\ 0 & \text{if not} \end{cases}$$

1. The shape of the density implies that the distribution function is a piecewise function of form

$$F_T(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ F_{T,1}(t) & \text{if } 0 < t < 30 \\ 1 & \text{if } t \geq 30 \end{cases}$$

By applying the definition $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\ du$, we get $F_{T,1}(t)$:

$$F_{T,1}(t) = \int_0^t \frac{1}{30} du = \left[\frac{u}{30} + c\right]_0^t = \frac{t}{30}.$$

2. The probability of waiting more than 10 minutes is given by $P(T > 10)$, which can be calculated using the distribution function:

$$P(T > 10) = 1 - P(T \leq 10) = 1 - F(10) = 1 - \frac{10}{30} = \frac{2}{3} = 0.\bar{6}.$$

3. We search for the conditional probability $P(T > 25 \mid T > 15)$. So

$$P(T > 25 \mid T > 15) = \frac{P(T > 25\ \cap\ T > 15)}{P(T > 15)} = \frac{P(T > 25)}{P(T > 15)} = \frac{1 - P(T \leq 25)}{1 - P(T \leq 15)} = \frac{1 - 25/30}{1 - 15/30} = \frac{1}{3} = 0.\bar{3}.$$

Exercise 3:

1. The random variable Z takes values between 0 and 1, as $P(Z \leq 1) = 1$ so we have $F(1) = 1$. Thus:

$$\begin{aligned} F(1) = k \cdot \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) &= 1 \\ k \cdot \left(\frac{10 - 15 + 6}{30} \right) &= 1 \end{aligned}$$

   and so $k = 30$.

   The density function $f(z)$ is the derivative of the distribution function, so

$$\begin{aligned} f(z) &= F'(z) \\ &= 30 \left(\frac{z^3}{3} - \frac{z^4}{2} + \frac{z^5}{5} \right)' \\ &= 30 \left(\frac{3z^2}{3} - \frac{4z^3}{2} + \frac{5z^4}{5} \right) \\ &= 30 \left(z^2 - 2z^3 + z^4 \right) \\ &= 30z^2(1-z)^2 \end{aligned}$$

   for $0 \leq z \leq 1$.

2. We calculate the expectation of Z using the density found above:

$$\begin{aligned} \mathrm{E}(Z) &= \int_0^1 z \cdot f(z)\, dz \\ &= \int_0^1 z \cdot 30(z^2 - 2z^3 + z^4) dz \\ &= 30 \int_0^1 (z^3 - 2z^4 + z^5) dz \\ &= 30 \left[\frac{z^4}{4} - \frac{2z^5}{5} + \frac{z^6}{6} + c \right]_0^1 \\ &= 30 \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) = 30 \cdot \frac{25 - 24 + 10}{60} \\ &= 0.5. \end{aligned}$$

   The variance of Z is given by the formula $\mathrm{var}(Z) = \mathrm{E}(Z^2) - \mathrm{E}(Z)^2$.

We calculate first $\mathrm{E}(Z^2)$:

$$\begin{aligned}
\mathrm{E}(Z^2) &= \int_0^1 z^2 \cdot f(z)\, dz \\
&= \int_0^1 z^2 \cdot 30(z^2 - 2z^3 + z^4)\, dz \\
&= 30 \int_0^1 (z^4 - 2z^5 + z^6)\, dz \\
&= 30 \left[\frac{z^5}{5} + \frac{2z^6}{6} + \frac{z^7}{7} + c \right]_0^1 \\
&= 30 \cdot \frac{21 - 35 + 15}{105} = \frac{2}{7}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{var}(Z) &= \mathrm{E}(Z^2) - \mathrm{E}(Z)^2 \\
&= \frac{2}{7} - \left(\frac{1}{2}\right)^2 \\
&= \frac{8-7}{28} \approx 0.0357.
\end{aligned}$$

3. We calculate $P(0.75 < Z < 1.5)$:

$$\begin{aligned}
P(0.75 < Z < 1.5) &= P(Z \leq 1.5) - P(Z \leq 0.75) \\
&= 1 - F(0.75) \\
&= 1 - 30 \cdot 0.75^3 \cdot \left(\frac{1}{3} - \frac{0.75}{2} + \frac{0.75^2}{5} \right) \\
&\approx 0.104;
\end{aligned}$$

and $P(Z \geq 0.15)$:

$$\begin{aligned}
P(Z \geq 0.15) &= 1 - P(Z < 0.15) \\
&= 1 - F(0.15) \\
&= 1 - 30 \cdot 0.15^3 \cdot \left(\frac{1}{3} - \frac{0.15}{2} + \frac{0.15^2}{5} \right) \\
&\approx 0.973.
\end{aligned}$$

4. The probability that Z is greater than 0.5 in the case $Z > 0.25$ is given by

$$P(Z \geq 0.5 \mid Z > 0.25).$$

We then calculate $P(Z \geq 0.5 \mid Z > 0.25)$:

$$\begin{aligned}
P(Z \geq 0.5 \mid Z > 0.25) &= \frac{P(Z \geq 0.5 \ \cap \ Z > 0.25)}{P(Z > 0.25)} \\
&= \frac{P(Z \geq 0.5)}{P(Z > 0.25)} \\
&= \frac{1 - P(Z < 0.5)}{1 - P(Z \leq 0.25)} \\
&= \frac{1 - F(0.5)}{1 - F(0.25)} \\
&= \frac{1 - 30 \cdot 0.5^3 \left(\frac{1}{3} - \frac{0.5}{2} + \frac{0.5^2}{5}\right)}{1 - 30 \cdot 0.25^3 \left(\frac{1}{3} - \frac{0.25}{2} + \frac{0.25^2}{5}\right)} \\
&\approx 0.558.
\end{aligned}$$

Exercise 4:

1. The cumulative distribution of X for $x \geq 0$ is:

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = -e^{-\lambda t}\Big|_0^x = 1 - e^{-\lambda x},$$

So:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

2. The alpha quantile satisfies :

$$\begin{aligned}
F(q_\alpha) &= \alpha \\
1 - e^{-\lambda q_\alpha} &= \alpha \\
q_\alpha &= \frac{\log(1-\alpha)}{-\lambda}
\end{aligned}$$

3. First we should find λ such that the $E(X) = 8$.

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x} dx$$

Integrating by part:

$$\begin{aligned}
E(X) &= -xe^{-\lambda x}\Big|_0^\infty - \int_0^\infty -e^{-\lambda x} dx \\
&= -xe^{-\lambda x}\Big|_0^\infty - \frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty \\
&= \frac{1}{\lambda}
\end{aligned}$$

Then $E(X) = \frac{1}{\lambda} = 8$ implies that $\lambda = \frac{1}{8}$. We can compute then the probability that lifetime of the television is more than 8 years:

$$P(X > 8) = 1 - P(X \leq 8) = 1 - (1 - e^{-\frac{1}{8}8}) = e^{-1} \simeq 0.37.$$

The median is $q_{\frac{1}{2}}$:

$$q_{\frac{1}{2}} = \frac{\log(1-\frac{1}{2})}{-\lambda} = 8\log(2) \simeq 5.54$$

4. Let's first compute $E(X^2)$:

$$\begin{aligned} E(X^2) &= \int_0^\infty \frac{x^2}{8} e^{-\frac{x}{8}} dx = -x^2 e^{\frac{-x}{8}}\Big|_0^\infty - \int_0^\infty -2xe^{\frac{-x}{8}} dx \\ &= -x^2 e^{\frac{-x}{8}}\Big|_0^\infty + 2 \cdot 8 \int_0^\infty \frac{x}{8} e^{\frac{-x}{8}} dx = 0 + 2 \cdot 8 E(X) \\ &= 2 \cdot 8^2 = 128 \end{aligned}$$

Then we can compute the variance:

$$Var(X) = E(X^2) - E(X)^2 = 128 - 8^2 = 64 = E(X)^2$$

Exercise 5:

1. To determine C and λ we will use:

$$\begin{aligned} \int_0^\infty Cv\exp(-\lambda v)dv &= 1 \\ \text{and} \qquad E(V) &= \int_0^\infty Cv^2 \exp(-\lambda v) dv = 30. \end{aligned}$$

(a) Let's integrate by part:

$$\begin{aligned} \int_0^\infty Cv\exp(-\lambda v)dv &= \frac{-1}{\lambda} Cv\exp(-\lambda v)\Big|_0^\infty + C\int_0^\infty \frac{1}{\lambda}\exp(-\lambda v)dv \\ &= 0 - \frac{C}{\lambda^2}\exp(-\lambda v)\Big|_0^\infty = \frac{C}{\lambda^2} = 1 \Leftrightarrow C = \lambda^2 \end{aligned}$$

(b)

$$\begin{aligned} C\int_0^\infty v^2\exp(-\lambda v)dv &= \frac{-1}{\lambda}Cv^2\exp(-\lambda v)\Big|_0^\infty + C\int_0^\infty \frac{1}{\lambda}2v\exp(-\lambda v)dv \\ &= \frac{2}{\lambda}\int_0^\infty Cv\exp(-\lambda v)dv = \frac{2}{\lambda} = 30 \\ \Leftrightarrow \lambda &= \frac{1}{15} \Rightarrow C = \frac{1}{225} \end{aligned}$$

2. Let $T = \frac{15}{V}$. It's cumulative distribution function is:

$$F_T(t) = P(T < t) = P\left(\frac{15}{V} < t\right) = P\left(V > \frac{15}{t}\right) = 1 - F_V\left(\frac{15}{t}\right).$$

We deduce its density:

$$f_T(t) = \frac{15}{t^2} f_V\left(\frac{15}{t}\right) = \frac{1}{t^3}\exp\left(-\frac{1}{t}\right).$$

Finally we can compute the expectation of T:

$$\begin{aligned} E(T) &= \int_0^\infty t\frac{1}{t^3}\exp\left(-\frac{1}{t}\right) dt \\ &= \exp\left(-\frac{1}{t}\right)\Bigg|_{t=0}^{t=\infty} = 1 \end{aligned}$$