Probability I

Lecture Intro

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Spring Semester 2020

Preliminaries

ÊTime frame:

- Lectures (no recording): Thur. from 12 to 14 (MR280)
- Exercises (no recording, text & solutions available on Chamilo): Thur. from 16 to 18 (MS130)

Ê**Exam:** Written exam **open-book, 2 hrs**

Ê**Book:** *A first course in probability*, S. Ross (any edition), Ed: Pearson New International Edition. This is the main reference book for the whole course.

Preliminaries

Let us start by a general statement:

Statement

Probability is defined as the measure of the "likelihood" of an event or random outcome.

Another way of saying that is:

Q.: "What is the chance of something happening?"

To answer this question we make use of Probability. Indeed, Probability is all about the certainty/uncertainty and the prediction of something happening. Some events are impossible, other events are certain to occur, while many are possible, but not certain to occur...

Preliminaries

'In this world there is nothing certain but death and taxes.'

Benjamin Franklin

Preliminaries

Preliminaries

Example (Stock price evolution)

Let us imagine that we are going to model the **uncertainty** characterizing the Stock price of an asset by a simple model. With probability $p = 1/2$ the stock price moves up of a factor u, and with probability $1 - p$ the price moves down of a factor d. We denote the price at time t_1 by uS_0 if the price goes up, and by dS_0 if the price goes down.

Let us set $S_0 = 1$, $u = 2$ and $d = 1/2$. Can we say something about the price at time t_2?

Example (cont'd)

The price evolution is represented by a tree:

$$
\begin{tikzcd}
 & & \$u^2 = \$4 \\
 & \$u = \$2 \arrow[ur, "p^2"] \arrow[dr, "(1-p)p"] & \\
\$1 \arrow[ur, "p"] \arrow[dr, "(1-p)"'] & & \$ud = \$du = \$1 \\
 & \$d = \$0.5 \arrow[ur, "(1-p)p"'] \arrow[dr, "(1-p)^2"] & \\
 & & \$d^2 = \$0.25
\end{tikzcd}
$$

Preliminaries

Preliminaries

Example (Quanta)

Probability models are the basis for quantum physics: they characterize the **uncertainty** of properties of single energy "quanta" emitted by a "perfect radiator" with a given temperature (T). Specifically, recall the famous Einstein's equation

$$\boxed{\mathcal{E} = mc^2}$$

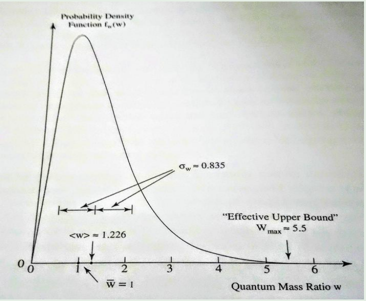
which expresses the energy $\mathcal{E}$ in terms of the mass m, a random quantity, and c, the speed of light. Moreover, consider the geometric energy mean defined by $\mathcal{E}_G = c_0 k_B T$, where $c_0 \approx 2.134$ and k_B is Boltzmann's constant. Thus, one can define the random quantity

$$W = \frac{\mathcal{E}}{\mathcal{E}_G},$$

which is called *quantum mass ratio*.

Example (cont'd)

The random behaviour of W can be described by a probability density function:



Preliminaries

In this course, we study Probability as a way to introduce Statistics.

Thus, we do not study Probability as a subject of interest in itself: we do not develop deep probability theory via measure theoretic arguments, rather we set up a few principles and methods which will be helpful for Statistics.

Preliminaries

Preliminaries

Q. Why is it important to study Probability/Statistics?

To understand why we must study Probability/Statistics, it is useful to first understand the answer to the following questions:

Q. What is Statistics? What is Probability?

A. STATISTICS deals with the collection, presentation, analysis and interpretation of data in order to make rational decisions. Since, there is uncertainty in the data, we need PROBABILITY, which deals with uncertainty.

Preliminaries

Aim & Scope

One makes use of Probability to develop methods to deal with uncertainty. Then, one applies these methods in tandem with Statistics to take decisions under uncertainty.

$\rightsquigarrow$ Uncertainty is measured in units of *probability*, which *is the currency of statistics*.

$\rightsquigarrow$ Statistics is concerned with the *study of data–driven decision making in the face of uncertainty*.

Overview

1. Introduction: some fundamental math tools
2. Basic calculus for probability
   - Random variables
   - Trees
   - Venn diagram
   - Conditional probability
   - Independence & Bayes' theorem
   - ...
3. Discrete random variables
   - Definitions
   - Expected value and variance
   - Binomial
   - Poisson
   - Negative binomial and hypergeometric

Overview

4. Continuous random variables
   - Definitions
   - Expected value and variance
   - Cumulative distribution function (cdf) and Probability density function (pdf)
   - Some important examples: Uniform, Exponential, Gamma, Normal, logNormal, Student's...
   - Relationships
5. Bivariate random variables
6. Limit theorems: Weak Law of Large Numbers (WLLN) and Central Limit Theorem (CLT)
7. Elements of simulations: numerical methods for the simulation of random variable with a given CDF...

Some mathematical formulas

To go thought this program, we need math.

For instance, we are going to make use of the following **powers and logs**:

- $a^m \cdot a^n = a^{m+n}$;
- $(a^n)^m = a^{m \cdot n}$;
- $a = \ln(\exp^a) = \ln(e^a)$;
- $\ln(a^n) = n \cdot \ln a$;
- $\ln(a \cdot b) = \ln(a) + \ln(b)$;

Some mathematical formulas (cont'd)

The **derivatives** will also play a pivotal role. For instance:

$$\frac{dx^n}{dx} = n \cdot x^{n-1}, \quad \frac{d \exp^x}{dx} = \exp^x, \quad \frac{d \ln(x)}{dx} = \frac{1}{x},$$

and we will make use of some fundamental rules, like e.g.

- Product rule:

$$\begin{aligned} \frac{d[f(x) \cdot g(x)]}{dx} &= \frac{df(x)}{dx} g(x) + \frac{dg(x)}{dx} f(x) \\ &= f'(x)g(x) + f(x)g'(x); \end{aligned}$$

- Chain rule:

$$\frac{df[g(x)]}{dx} = (f \circ g)'(x) = f'[g(x)] \cdot g'(x).$$

Some mathematical formulas (cont'd)

The **integrals** will be crucial in many tasks. For instance, recall that:

- $$\int_a^b [c \cdot f(x) + d \cdot g(x)]\, dx = c \cdot \int_a^b f(x)dx + d \cdot \int_a^b g(x)dx;$$
- If $f(x) \geq 0, \forall x \in \mathbb{R}$, then
$$\int_{\mathbb{R}} f(x)dx \geq 0.$$
- For a continuous function $f(x)$, the indefinite integral is
$$\int f(x)dx = F(x) + \text{const}$$
while the definite integral is
$$F(b) - F(a) = \int_a^b f(x)dx, \quad b \geq a.$$

Some mathematical formulas (cont'd)

Besides integrals we are also going to use **sums**:

- $$\sum_{i=1}^{n} X_i = X_1 + X_2 + ... + X_n,$$
- For every $\alpha_i \in \mathbb{R}$,
$$\sum_{i=1}^{n} \alpha_i X_i = \alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_n X_n;$$
- Double sum: a sum with two indices. For instance,
$$\begin{aligned}\sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j &= x_1 y_1 + x_1 y_2 + ... + x_2 y_1 + x_2 y_2 + ... \\ &= \left(\sum_{i=1}^{n} x_i \right) y_1 + \left(\sum_{i=1}^{n} x_i \right) y_2 + ... + \left(\sum_{i=1}^{n} x_i \right) y_m \\ &= \sum_{i=1}^{n} x_i \sum_{j=1}^{m} y_j.\end{aligned}$$

Some mathematical formulas (cont'd)

Finally, we also rely on some **combinatorial formulas**. Specifically,

- Factorial
$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1;$$
where $0! = 1$, by definition;
- Binomial coefficient, for $n \geq k$
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = C_n^k$$
which is helpful to express the Binomial Theorem
$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n;$$
or equivalently, making use of the sum notation,
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Some mathematical formulas (cont'd)

Example

Let us compute:

- for $n = 2$ we have

$$
\begin{aligned}
(x+y)^2 &= \binom{2}{0}x^2y^0 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^0y^2 \\
&= x^2 + 2xy + y^2
\end{aligned}
$$

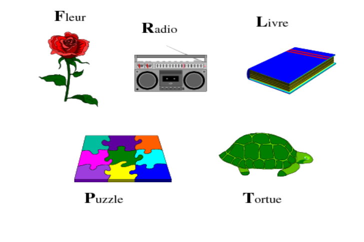
- for $n = 3$ we have

$$
\begin{aligned}
(x+y)^3 &= \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \cdots + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3 \\
&= y^3 + 3xy^2 + 3x^2y + x^3
\end{aligned}
$$

Some mathematical formulas (cont'd)

Example

How many ways can we select 3 presents among the 5 available presents (see figure below)? Assume the order does not matter!



(F, R, L)	(F, R, P)	(F, R, T)	(F, L, P)
(F, L, T)	(F, P, T)	(R, L, P)	(R, L, T)
(R, P, T)	(L, P, T)		

Total: $N = 10$

Some mathematical formulas (cont'd)

Example (cont'd by Explanation)

1. 5! gives you the total # of possible choices when you can select 5 presents (so-called "permutations", see next slides);
2. If you're going to select 3 presents from the list, then you have $(5-3)$ presents that you're not going to select. Therefore, you need to divide out the $(5-3)!$ different ways you can order the presents you are not going to select from the 5! possible choices of all presents. In other words you have $5!/(5-3)!$ ways to select and order the 3 presents;
3. Finally, remember you don't care about the order in which you select the 3 presents. So, in how many ways can you select 3 presents from the *n* available ones? The problem is the same as in [1] above except that now you don't care about the order of the 3 presents, and therefore you also need to divide out the 3! different ways you can order the presents.

Some mathematical formulas (cont'd)

Example (cont'd by Explanation)

... so, in formula, you have

$$\frac{5!/(5-3)!}{3!}$$

ways to select the 3 presents:

$$\frac{5!/(5-3)!}{3!} = \frac{5!}{3!2!} = \binom{5}{3} = C_5^3.$$

This gives you the total # of possible ways to select the 3 presents when the order does not matter.

As a recommendation, redo the exercise assuming you can select 2 presents from 3 available presents (like for instance F,L,R)...

Some mathematical formulas (cont'd)

Remark

Permutations: *How many different ways can we combine n objects?*

- *In the 1st place: n possibilities*
- *In the 2nd place:* $(n-1)$ *possibilities*
- *...*
- *Finally,* 1 *possibility*

Thus, in total we have $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!$

Example

How many ways can Aline, Brigitte and Carmen seat on 3 spots, from left to right? Possible outcomes:

$$(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)$$

Total $\#$ of permutations: $N = 6 = 3 \cdot 2 \cdot 1 = 3!$

Some mathematical formulas (cont'd)

Remark (cont'd)

Combinations: *How many ways can we select k objects among n? To answer this question, we proceed as follows:*

1. *How many ways can we combine k objects among n?*
   - *In the 1st place:* n
   - *In the 2nd place:* $(n-1)$
   - *....*
   - *In the k-th place:* $(n-k+1)$
2. *We have $k!$ ways to permute the k objects that we selected*
3. *The number of possibilities (without considering the order) is:*

$$\frac{n!/(n-k)!}{k!} = \frac{n!}{k!(n-k)!} = C_n^k$$

Some mathematical formulas (cont'd)

Remark (cont'd)

For the Problem Set 2, you will have to make use of C_n^k in Ex2-Ex3-Ex5. Indeed, to compute the probability for an event E, will have to make use of the formula

$$P(E) = \frac{\text{number of cases in } E}{\text{number of possible cases}}. \tag{1}$$

This is a first intuitive definition of probability, which we will justify in the next lecture; see Lecture 1, slide 28. For the time being, let us say that the combinatorial calculus will be needed to express both the quantities (numerator and denominator) in (1).

Some mathematical formulas (cont'd)

Finally, the following **limits** will be crucial in many tasks:

- $$\lim_{n \to \infty} \sum_{i=1}^{n} x_i = \sum_{i=1}^{\infty} x_i$$

- $$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

- for $\alpha > 0$
$$\lim_{x \to \infty} \alpha e^{-\alpha x} = 0$$

- $$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$