Probability 1

Lecture 02 : Elements of Set Theory

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(based on the notes of Prof. Davide La Vecchia)

Spring Semester 2021

Objective

Kickstart the mathematical characterisation of probability by:

- Introducing: **Random Experiments, Events, and Sample Spaces** and characterising them as **Sets**.
- Providing **elements of Set Theory** helpful in the study of probability.
- Giving an informal definition Probability as **Frequency**.

Outline

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Definition

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Example

Drawing a card in a BlackJack game.

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An event is an **uncertain outcome** of a random experiment.

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The card drawn is a King of Hearts

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An event can be:

- Elementary: it includes only one particular outcome of the experiment
- Composite: it includes more than one elementary outcome in the possible sets of outcomes.

Definition

The Sample Space(S) is the complete **listing of the elementary events** that can occur in a random experiment.

Example (Drawing a single card from a deck)

Suppose we choose ***one*** card at random from a deck of 52 playing cards.

S = {A♣, 2 ♣ .... K ♣, A♦, 2 ♦, ...K ♦, A♥, 2♥, ... K♥, A♠, 2♠, ... K♠}

K= event of king = { K ♣, K ♦, K♥, K♠}

H = event of heart = {A♥, 2♥, ... ,K♥}

J = event of jack or better
= {J ♣, J ♦, J ♥, J ♠, Q ♣, Q ♦, Q♥, Q ♠, K ♦, K♥, K♠, A ♣, A ♦, A♥, A♠}

Q = event of queen = {Q ♣, Q ♦, Q♥,Q ♠}

Example (Tossing two coins)

- Flipping two coins, could be seen as an **Experiment**

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- If we flip two coins and name:
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  - T for Tail,

  each of these pairs, e.g. (H, T) constitutes an **outcome**.

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- Flipping two coins, could be seen as an **Experiment**
- If we flip two coins and name:
    - H for Head and
    - T for Tail,

  each of these pairs, e.g. (H, T) constitutes an **outcome**.
- Finally, the **Sample Space** contains the following four points

$$S = \{(HH), (HT), (TH), (TT)\}.$$

Example (Measuring time on screen)

- Taking a measure of time spent on your Phone, can be considered an **experiment**.
- **Outcomes** are positive measures
- The **Sample Space** consists of all non-negative real numbers:

$$S = \{x : 0 \leq x < \infty\} \equiv \mathbb{R}^+$$

Outline

Some Definitions from Set Theory

To deal with events, we rely on the set theory

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Definition

- If every element of a set A is also an element of a set B, then A **is a subset of** B:

$$A \subset B,$$

  read as "A is contained in B"

- Two sets A and B are **equal** if

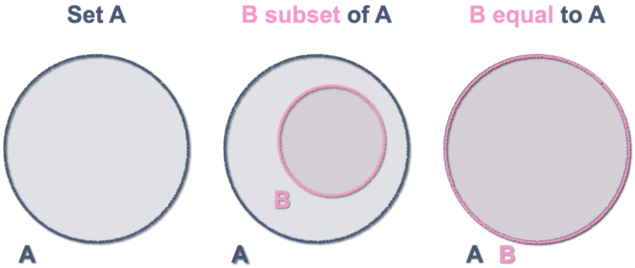
$$A \subset B \quad \text{and} \quad B \subset A;$$

- If a set A contains **no elements**, it's a *null set, or empty set*, and denoted by $\varnothing$.

Outline

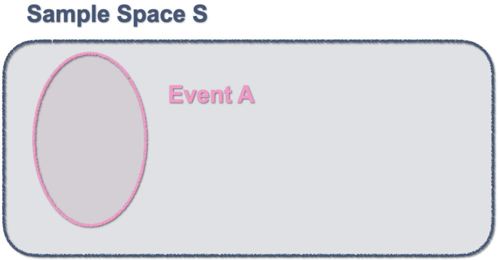
The Venn Diagram

A Venn diagram is an enclosed figure representing a set.



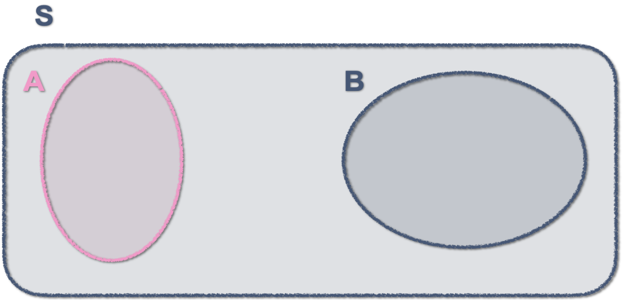
The Venn Diagram

We will use Venn Diagrams to represent **Events in the Sample Space**



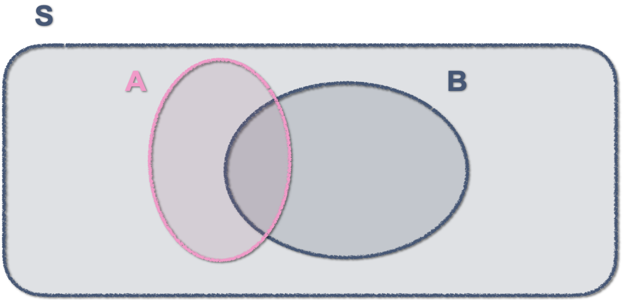
The Venn Diagram

Two events are **mutually exclusive** is they cannot occur jointly.



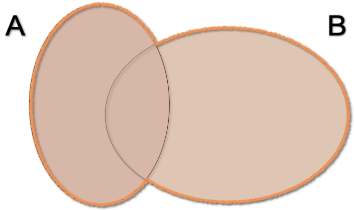
Elements of set theory (Venn diagram)

Two events are **not mutually exclusive** if they share some elements.



The Venn Diagram

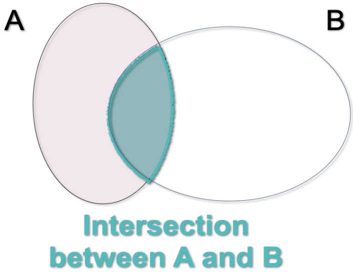
The **union of the events** A and B is the event which **occurs when either** A **or** B **occurs**: $A \cup B$



Union of A and B

The Venn Diagram

The intersection of the events A and B is the event which occurs when both A and B occur: $A \cap B$



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$$A \cup A^c = S \text{ and } A^c = S \setminus A = S - A.$$

Let A, B, and C be sets. The following laws hold:

- **Commutative laws**:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- **Associative laws**:

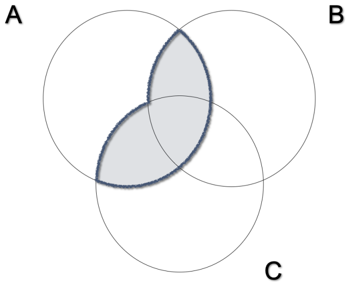
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

The Venn Diagram

Distributive laws:

The intersection is distributive with respect to the union

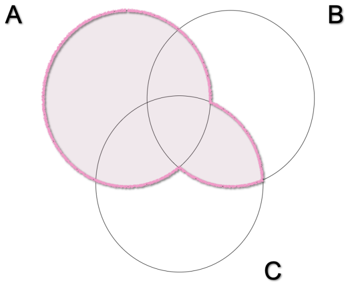
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



Distributive laws:

The union is distributive with respect to the intersection

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$



Outline

Countable and Uncountable Sets

Sets can be either **countable** or **uncountable**.

- A countable set is a set with the same number of elements as a subset of the set of natural numbers ($\mathbb{N}$).

  It can be either finite *or* countably infinite

  - Its elements can be counted one a time i.e. using $1, 2, 3, ..$
  - The counting may never finish.
- We can't do the same with uncountable sets!

Example (Countable and Finite sets)

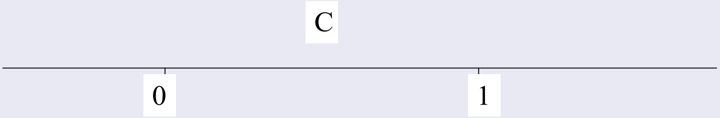
- $K \cup H$ = {A♥, 2♥, ... ,K♥,K ♣,K ♦,K♠}

- $J \cup H$ =
{J♣, J♦, J♥, J♠,Q♣,Q♦, Q♥, Q♠,K♣,K♦,K♥,K♠, A ♣,A♦,A♥,A ♠, 2♥, ...10♥}

- H^c= event of ***not*** a heart =
{A♣, 2 ♣ ,.... K ♣, A ♦, 2 ♦, ...K ♦, A♠, 2♠, ... K♠}

Exercise (Uncountable)

Sample space $= C = \{x : 0 < x < 1\}$



$A = \{x \in C : 0.3 < x < 0.8\}$
$B = \{x \in C : 0.6 < x < 1\}$

Exercise (cont'd)

and using the definition of A and B compute:

- A^c
- B^c
- $B^c \cup A$
- $B^c \cup A^c$
- $A \cup B$
- $A \cap B$
- $B \cup A^c$
- $A^c \cup A$

Exercise (Countable)

Let us consider the experiment where we flip two coins. (H stands for Head and T for Tail).

$$S = \Big\{(HH), (HT), (TH), (TT)\Big\}.$$

Then, let us consider the events:

- $A = H$ *is obtained at least once* $= \Big\{(HH), (HT), (TH)\Big\}$
- $B =$ *the second toss yields* $T = \Big\{(HT), (TT)\Big\}$

Exercise (cont'd)

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Exercise (cont'd)

and using the definitions of A and B compute:

- $A^c = \{(TT)\}$
- $B^c = \{(HH), (TH)\}$
- $B^c \cup A$
- $A \cup B$
- $A \cap B$
- $B \cup A^c$

Proposition

Let $A \subset S$ and $\varnothing$ the empty set. The following relations hold:

- $A \cap S = A$;
- $A \cup S = S$;
- $A \cap \varnothing = \varnothing$;
- $A \cup \varnothing = A$;
- $A \cap A^c = \varnothing$;
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Exercise

Verify these relations using Venn diagrams.

The above relations are helpful to define some other relations between sets/events.

Example

Let A and B be two sets in S. Then we have

$$B = (B \cap A) \cup (B \cap A^c).$$

To check it, we can proceed as follows:

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Outline

Let A and B be two sets in S. Then:

$$(A \cap B)^c = A^c \cup B^c,$$

where:

- Left hand side: $(A \cap B)^c$ represents the **set of all elements that are not both** A **and** B;
- Right hand side: $A^c \cup B^c$ represents all elements that are not A (namely they are A^c) and not B either (namely they are B^c) $\Rightarrow$ **set of all elements that are not both** A **and** B.

Put in words:

The **complement of the Intersection** is the **Union of the Complements**

Let A and B be two sets in S. Then:

$$(A \cup B)^c = A^c \cap B^c,$$

where:

- Left hand side: $(A \cup B)^c$ represents the **set of all elements that are neither** A **nor** B;
- Right hand side: $A^c \cap B^c$ represents the intersection of all elements that are not A (namely they are A^c) and not B either (namely they are B^c) $\Rightarrow$ **set of all elements that are neither** A **nor** B.

The **complement of the Union** is the **Intersection of the Complements**

De Morgan's Laws

Three sets

These laws can be extended to **Three Sets**:

Let $A_1, A_2, A_3 \subset S$, and $A_1^c = \overline{A_1})$,

(i) **First law**:

$$\overline{(A_1 \cup A_2 \cup A_3)} = \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$$

(ii) **Second Law**:

$$\overline{(A_1 \cap A_2 \cap A_3)} = \overline{A_1} \cup \overline{A_2} \cup \overline{A_3}$$

And to the union or intersection of **infinite sets** (see the Course notes)

Outline

Probability as Frequency

Primary interest will be not in events but on their *likelihood*.

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1. To be **positive** or more generally non-negative (it can be zero);
2. The $\text{pr}(S) = 1$ and $\text{pr}(\varnothing) = 0$;
3. The probability of two (or more) **mutually exclusive events** is the **sum** of the probabilities of each event.

Probability as Frequency

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or equivalently

$$P(\{i\}) = \frac{1}{N}, \text{ for } i = 1, 2, ..., N$$

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Example

Think of when you throw a die!

Probability as Frequency

If we define a **composite event** A, there exist N_A realizations having the same likelihood (i.e. probability) in the event A, so:

$$\boxed{P(A) = \frac{N_A}{N} = \frac{\#\text{ of favorable outcomes}}{\text{total }\#\text{ of outcomes}} = \frac{\#\text{ of outcomes in }A}{\#\text{ of outcomes in }S}}$$

(here $\#$ means "number").

Probability as Frequency

Example

We roll a fair die and we define the event

$$A = \text{the outcome is an even number} = \{2, 4, 6\}.$$

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Sample Space:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

$$P(A) = \frac{N_A}{N} = \frac{\text{3 favorable outcomes}}{\text{6 total outcomes}} = \frac{1}{2}.$$

Probability as Frequency

Building on this intuition we state a first **informal definition of probability**, namely in terms of *relative frequency*.

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Definition (Informal)

An experiment with sample space is S, is repeatedly performed under exactly the same conditions.
For each event $A \subset S$, let $n(A)$ the **number of times** A **occurs** in the **first** n **repetitions** of the experiment.

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An experiment with sample space is S, is repeatedly performed under exactly the same conditions.
For each event $A \subset S$, let $n(A)$ the **number of times** A **occurs** in the **first** n **repetitions** of the experiment.
Then, the **probability of the event** A is:

$$P(A) = \lim_{n \to \infty} \frac{n(A)}{n},$$

Example (Tossing a well-balanced coin)

- 2 mutually exclusive equiprobrable outcomes: H and T.
- Let $A = \{H\}$, $P(A) = 1/2$.
- We can toss the coin a large number of times (each under identical conditions) and count the times we have H.

Let n **total # of tosses** and $n(A)$ the **# of times in which we observe** A. Thus

$$P(A) \sim \frac{n(A)}{n}, \quad \text{for large } n.$$

Example (cont'd)

***Example* 4.** Table 1 shows the results of a series of 10,000 coin tosses,[3] grouped into 100 different series of $n = 100$ tosses each. In every case, the table shows the number of tosses $n(A)$ leading to the occurrence of a head. It is clear that the relative frequency of occurrence of "heads" in each set of 100 tosses differs only slightly from the probability $\mathbf{P}(A) = \frac{1}{2}$ found in Example 1. Note that the relative frequency of occurrence of "heads" is even closer to $\frac{1}{2}$ if we group the tosses in series of 1000 tosses each.

Table 1. Number of heads in a series of coin tosses

Number of heads in 100 series of 100 trials each										*Number of heads in 10 series of 1000 trials each*[4]
54	46	53	55	46	54	41	48	51	53	501
48	46	40	53	49	49	48	54	53	45	485
43	52	58	51	51	50	52	50	53	49	509
58	60	54	55	50	48	47	57	52	55	536
48	51	51	49	44	52	50	46	53	41	485
49	50	45	52	52	48	47	47	47	51	488
45	47	41	51	49	59	50	55	53	50	500
53	52	46	52	44	51	48	51	46	54	497
45	47	46	52	47	48	59	57	45	48	494
47	41	51	48	59	51	52	55	39	41	484

Clearly,

$$0 \leq n(A) \leq n, \quad \text{so} \quad 0 \leq P(A) \leq 1.$$

Thus, we say that:

the probability is a **set function**[1] that **associates a number between 0 and 1** to each set/event.

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Next week:
We'll further formalise this notion, by adding **conditions** to the three desired properties...

[1] A function that operates on a Set

Wrap-up

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“See you” next week!