Exercise 1:

We assume that there is a probability of 0.10 to be inspected by a conductor when one takes Tram 15. Charles travels 700 times a year on this line.

1. What is the approximate probability that Charles is inspected between 60 and 100 times a year ?

2. Charles is in fact a cheater and always travels without any ticket. Given that the price of a ticket is 2 CHF, what is the minimum penalty that the TPG should fix so that Charles's probability of having a loss of profits is bigger than 75% ?

Exercise 2:

From 7 am, buses pass every 15 minutes at a given stop. So they pass at 7:00, then 7:15, etc. A user arrives between 7:00 am and 7:30 am at this stop, the exact time of his arrival being a uniform random variable over this period. Find the probability that he has to wait

1. less than 5 minutes,

2. more than 10 minutes.

Exercise 3:

The parts of a car are often copied and sold as original. We want to replace a part of a car. With probability 1/4, we buy a pirated part and probability 3/4 we buy an original part. The lifetime is a exponential random variable with expectation 2 if it is a pirated part and expectation 5 if it is an original part. Let's call T the lifetime of a part that we bought. Assume that the part has survived until time t after its installation.

1. What is the probability $\pi(t)$ that this part is pirated ? Find the limit of $\pi(t)$ when $t \to \infty$.

2. Let's formalize the problem using a random variable Y taking values 1 if the part is pirated and 0 when it is not.

   (a) What is the distribution of Y?

(b) Compute $P(T \leq t|Y = y)$, the conditional probability of T given $Y = y$.

(c) Compute $P(T \leq t \cap Y = y)$, the joint probability of T and Y.

(d) What is the marginal probability distribution of T ?

(e) Find $P(Y = y|T > t)$ and deduce $\pi(t)$.

Exercise 4:

Let S_t be the value of an asset at the end of the year t and $R_{0,n}$ be the rate of return over a horizon of n years, that is, $R_{0,n}$ is the solution of the equation

$$S_n = S_0(1 + R_{0,n})^n.$$

Let $Z_t = S_t/S_{t-1}$. Suppose Z_t follows a log-normal distribution with parameters μ and σ^2:

$$Z_t = \frac{S_t}{S_{t-1}} \sim LN(\mu, \sigma^2), \quad t = 1, 2, ..., n.$$

It is also assumed that the Z_t are independent of each other.

(a) Show that

$$R_{0,n} = \left(\prod_{t=1}^{n} \frac{S_t}{S_{t-1}}\right)^{\frac{1}{n}} - 1.$$

Indication:

$$\frac{S_n}{S_0} = \frac{S_1}{S_0} \cdots \frac{S_2}{S_1} \cdots \frac{S_t}{S_{t-1}} \cdots \frac{S_n}{S_{n-1}} = \prod_{t=1}^{n} \frac{S_t}{S_{t-1}}.$$

(b) Let

$$Y = \frac{1}{n}\sum_{t=1}^{n} \log\left(\frac{S_t}{S_{t-1}}\right).$$

Find the distribution of Y and specify its expectation and variance.

(c) Show that $R_{0,n} = \exp(Y) - 1$. Deduce the expectation and the variance of $R_{0,n}$.

(d) What happens when $n \to \infty$?

Exercise 5(Optional):

Let V a uniformly distributed random variable on $[0, 1]$, that is $V \sim U(0, 1)$.

(a) What is the cumulative distribution function of $W = \frac{-1}{\lambda}\log(V)$? Compute its density. Which distribution is it?

(b) Let $F(x)$ be some cumulative distribution function. Show that the random variable $X = F^{-1}(V)$ is distributed with cumulative distribution function $F(x)$. *Hint: $F^{-1}(x)$ is the inverse function of $F(x)$, that is it satisfies $F^{-1}(F(x)) = x$ and $F(F^{-1}(x)) = x$.*

(c) The cumulative distribution function of the Dagum distribution is:

$$F(x) = (1 + \lambda x^{-2})^{-1}$$

where $x \geq 0$ and $\lambda > 0$. From the uniformly distributed random variable V, define the transformation $g(V)$ such that $g(V)$ follows a Dagum distribution.