Probability I

Lecture 8 (additional material)

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Sequences of real numbers

Definition

A sequence is an ordered list of real numbers of the form

$$a_1, a_2, ..., a_n, ...$$

where each natural number $n \in \mathbb{N}$ corresponds exactly to a real number $a_n \in \mathbb{R}$. A sequence is denoted by $\{a_n\}_{n\in\mathbb{N}}$, where n is called the index of the sequence and a_n is its n-th term.

Remark: the sequence can contain infinite terms...

Example

The list of numbers

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$$

is a sequence, where each natural number corresponds the real number $a_n = \frac{1}{n}$.

Limit of a sequences of real numbers

Definition

A real number $a \in \mathbb{R}$ (a is a finite number) is called the limit of a sequence $\{a_n\}_{n\in\mathbb{N}}$ if, for any $\epsilon > 0$, a natural number $n(\epsilon) \in \mathbb{N}$ exists such that

$$|a_n - a| < \epsilon \quad \text{for all} \quad n \geq n(\epsilon). \tag{1}$$

If for a given sequence $\{a_n\}_{n\in\mathbb{N}}$ the real number a satisfies (1), then we write

$$a = \lim_{n\to\infty} a_n.$$

Example

The sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_n = \frac{1}{n}$, converges to zero.

Remark: loosely speaking, Eq. (1) states that, for $n \geq n(\epsilon)$, a_n is **always** close to a (or equivalently, the difference in absolute value between a_n and a is **never** large).

Series

Definition

Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence. The sum of the first n terms of $\{a_k\}_{k\in\mathbb{N}}$:

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

is called the n-th partial sum of $\{a_k\}_{k\in\mathbb{N}}$. The sequence $\{s_n\}_{n\in\mathbb{N}}$ of partial sums is called a series.

Remark: $s_n = s_{n-1} + a_n$.

Example

Let us consider again the sequence $\{a_k\}_{k\in\mathbb{N}}$ with $a_k = \frac{1}{k}$. Its partial sums are

$$s_n = \sum_{k=1}^{n} a_k.$$

For instance, when $n = 1, 2, 3$ we have:

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\ s_3 &= 1 + \frac{1}{2} + \frac{1}{3} = s_2 + \frac{1}{3} = \frac{11}{6}. \end{aligned}$$