Probability 1

Chapter 05 : Continuous Random Variables - Part 1

Dr. Daniel Flores-Agreda,

(based on the notes of Prof. Davide La Vecchia)

Spring Semester 2021

Objectives

Outline

- Gaussian or "Normal" Distribution
- The Chi-squared distribution
- The Student-t distribution
- The F distribution
- The lognormal distribution
- Exponential distribution

1. Variable Transformation

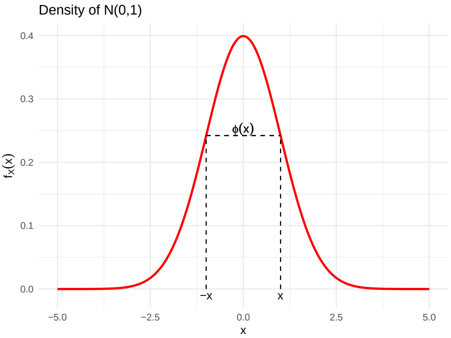
Outline

- Gaussian or "Normal" Distribution
- The Chi-squared distribution
- The Student-t distribution
- The F distribution
- The lognormal distribution
- Exponential distribution

1 Variable Transformation

Last class we said that, the Normal PDF is **Symmetric**

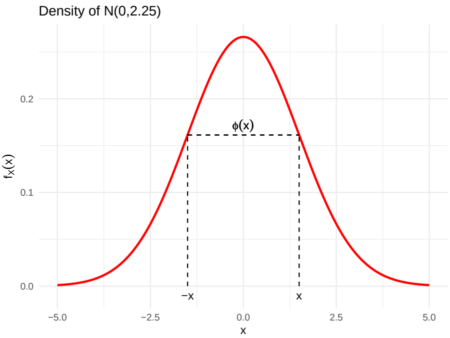
$$\phi_{(\mu,\sigma)}(-x) = \phi_{(\mu,\sigma)}(x)$$



Gaussian or "Normal" Distribution

Of course, the **Standard** Normal PDF is also **Symmetric**

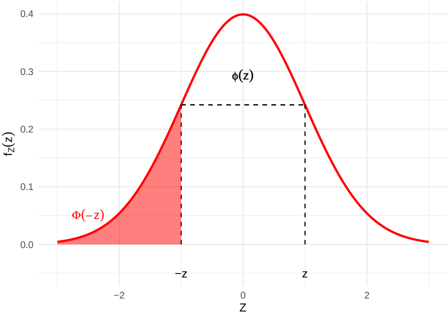
$$\phi(-x) = \phi(x)$$



Gaussian or "Normal" Distribution

Symmetry of the PDF implies that the CDF can be computed as

$$\Phi(-x) = 1 - \Phi(x)$$



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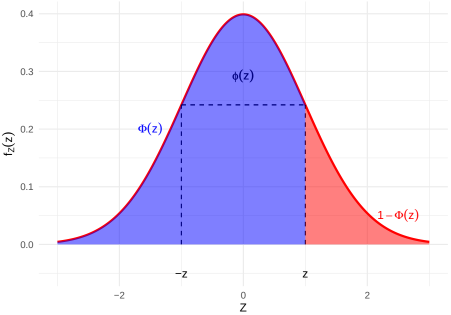

Gaussian or “Normal” Distribution

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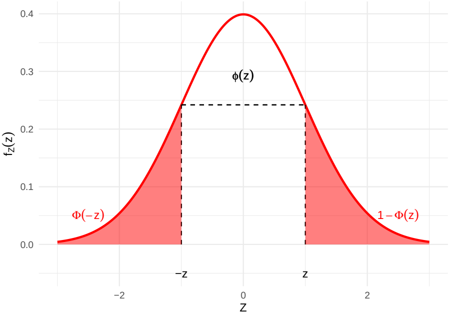

Gaussian or "Normal" Distribution

We can **shift and scale** any Normal Random Variable X and reach a **Standard Normal Random Variable** Z

$$X \sim \mathcal{N}\left(\mu, \sigma^2\right) \Longleftrightarrow Z = \frac{(X-\mu)}{\sigma} \sim \mathcal{N}(0,1)$$

- We can always transform from X to Z

$$Z = \frac{X-\mu}{\sigma} \text{ (for the random variable)} \quad \text{and} \quad z = \frac{x-\mu}{\sigma} \text{ (for its values),}$$

- and return back to X by a 're-scaling' and 're-shifting':

$$X = \sigma Z + \mu \text{ (for the random variable)} \quad \text{and} \quad x = \sigma z + \mu \text{ (for its values).}$$

Statements about a Normal Random Variable can always be translated into equivalent statements about a standard Normal Random Variable, (and vice-versa).

Gaussian or “Normal” Distribution

In particular, the **CDF of any Normal Random Variable** $X \sim \mathcal{N}(\mu, \sigma^2)$, can be **computed** with a **Standard CDF**

$$
\begin{aligned}
P(\{X \leq x\}) &= P\left(\left\{\underbrace{\frac{X-\mu}{\sigma}}_{Z} \leq \underbrace{\frac{x-\mu}{\sigma}}_{z}\right\}\right) \\
&= P(\{Z \leq z\}) \\
P(\{X \leq x\}) &= \Phi(z)
\end{aligned}
$$

Gaussian or "Normal" Distribution

Moreover, we can also compute the probabilities of any interval for $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ with the **Standard CDF**

$$
\begin{aligned}
P(\{x_1 < X \leq x_2\}) &= P\left(\left\{\frac{x_1 - \mu}{\sigma} < \frac{X - \mu}{\sigma} \leq \frac{x_2 - \mu}{\sigma}\right\}\right) \\
&= P(\{z_1 < Z \leq z_2\}) \\
&= P(\{Z \leq z_2\}) - P(\{Z \leq z_1\}) \\
P(\{x_1 < X \leq x_2\}) &= \Phi(z_2) - \Phi(z_1)
\end{aligned}
$$

where $z_1 = (x_1 - \mu)/\sigma$ and $z_2 = (x_2 - \mu)/\sigma$.

Gaussian or "Normal" Distribution

$$P(\{z_1 < Z \leq z_2\}) = P(\{Z \leq z_2\}) - P(\{Z \leq z_1\}) = \Phi(z_2) - \Phi(z_1)$$

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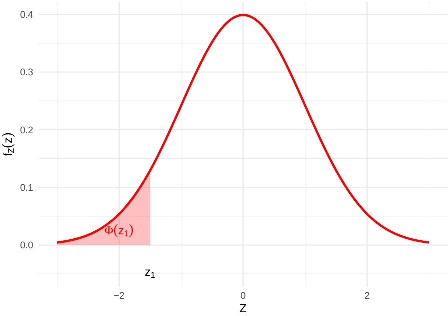
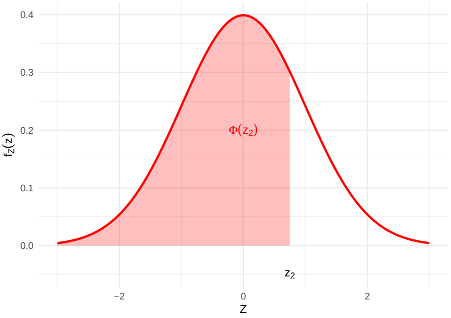
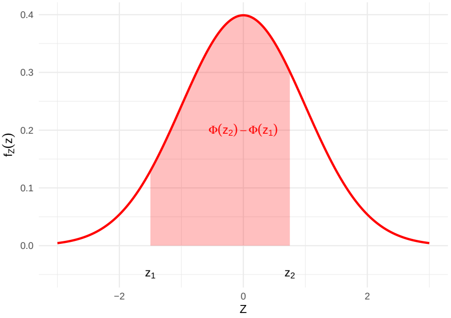


Gaussian or "Normal" Distribution

$$P(\{z_1 < Z \leq z_2\}) = P(\{Z \leq z_2\}) - P(\{Z \leq z_1\}) = \Phi(z_2) - \Phi(z_1)$$



$$P(\{z_1 < Z \le z_2\}) = P(\{Z \le z_2\}) - P(\{Z \le z_1\}) = \Phi(z_2) - \Phi(z_1)$$



The integral that defines the CDF of the standard normal:

$$P(\{Z \leq z\}) = \Phi(z) = \int_{-\infty}^{z} \phi(s)ds$$

does not have a closed-form expression.

- It can be **approximated** using a computer.
- We can rely on **Standard Normal Tables**, which show the values of $\Phi(z)$ for $z \geq 0$

Remark

We can obtain $\Phi(z)$ for $z < 0$ by symmetry of $\phi(z)$ which, again, entails:

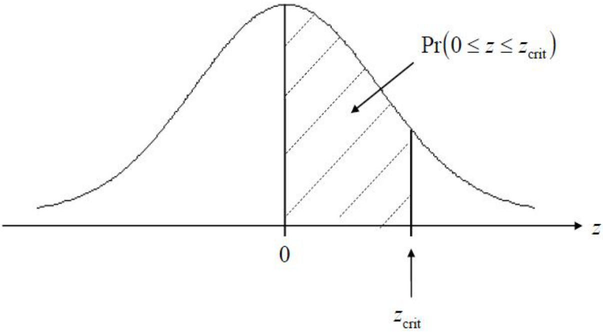
$$\Phi(-z) = 1 - \Phi(z)$$

Gaussian or "Normal" Distribution

Description of the values contained in the table:

STATISTICAL TABLES

TABLE 1: AREAS UNDER THE STANDARDIZED NORMAL DISTRIBUTION



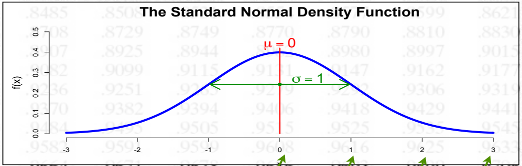
Gaussian or "Normal" Distribution

Contents of the table

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

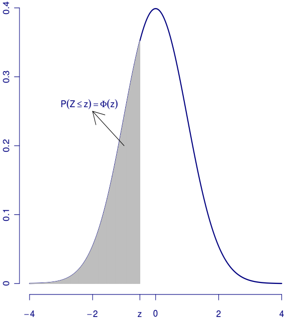
X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	[illegible]	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	[illegible]	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	[illegible]	[illegible]	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	[illegible]	[illegible]	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	[illegible]	[illegible]	[illegible]	[illegible]	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	[illegible]	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	[illegible]	[illegible]	[illegible]	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	[illegible]	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990

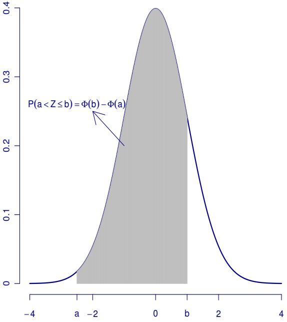
$\Phi(.46)$



Gaussian or “Normal” Distribution

One can use these tables to compute integrals/probabilities of the type:





Gaussian or "Normal" Distribution

Example (Prob of Z)

$$
\begin{aligned}
P(\{Z \leq 1\}) &\approx 0.8413 \\
P(\{Z \leq 1.96\}) &\approx 0.9750 \\
P(\{Z \geq 1.96\}) &= 1 - P(\{Z \leq 1.96\}) \approx 1 - 0.9750 = 0.0250 \\
P(\{Z \geq -1\}) &= P(\{Z \leq 1\}) \approx 0.8413 \\
P(\{Z \leq -1.5\}) &= P(\{Z \geq 1.5\}) = 1 - P(\{Z \leq 1.5\}) \approx 1 - 0.9332 = 0.0668
\end{aligned}
$$

Gaussian or "Normal" Distribution

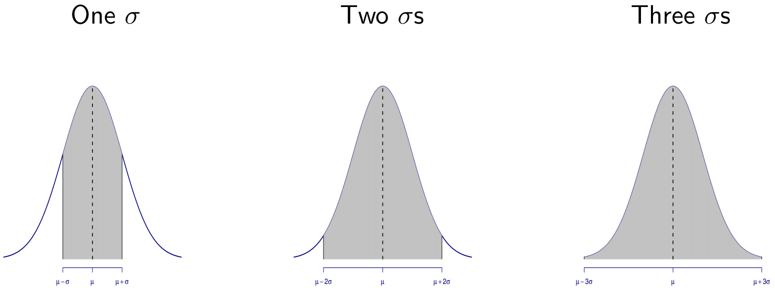
Example (continued)

$P(\{0.64 \leq Z \leq 1.96\}) =$
$P(\{Z \leq 1.96\}) - P(\{Z \leq 0.64\})$
$\approx 0.9750 - 0.7389 = 0.2361$

$P(\{-0.64 \leq Z \leq 1.96\})$
$= P(\{Z \leq 1.96\}) - P(\{Z \leq -0.64\})$
$= P(\{Z \leq 1.96\}) - (1 - P(\{Z \leq 0.64\}))$
$\approx 0.9750 - (1 - 0.7389) = 0.7139$

$P(\{-1.96 \leq Z \leq -0.64\})$
$= P(\{0.64 \leq Z \leq 1.96\})$
≈ 0.2361

Gaussian or “Normal” Distribution



The shaded areas under the pdfs are (approximately) equivalent to 0.683, 0.954 and 0.997, respectively. So we state the following ....

... rule '68 – 95 – 99.7':

If X is a Normal random variable, $X \sim \mathcal{N}(\mu, \sigma^2)$, its realization has approximately a probability of

- 68 % of being in the interval $[\mu - \sigma,\ \mu + \sigma]$;
- 95 % of being in the interval $[\mu - 2\,\sigma,\ \mu + 2\,\sigma]$;
- 99.7 % of being in the interval $[\mu - 3\,\sigma,\ \mu + 3\,\sigma]$.

Gaussian or "Normal" Distribution

- For $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$

$$E\left[X\right] = \mu \text{ and } Var\left(X\right) = \sigma^2.$$

Gaussian or "Normal" Distribution

- For $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$
$$E\left[X\right] = \mu \text{ and } Var\left(X\right) = \sigma^2.$$
- If a is a number, then
$$\begin{aligned} X + a &\sim \mathcal{N}\left(\mu + a, \sigma^2\right) \\ aX &\sim \mathcal{N}\left(a\mu, a^2\sigma^2\right). \end{aligned}$$

Gaussian or "Normal" Distribution

- For $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$
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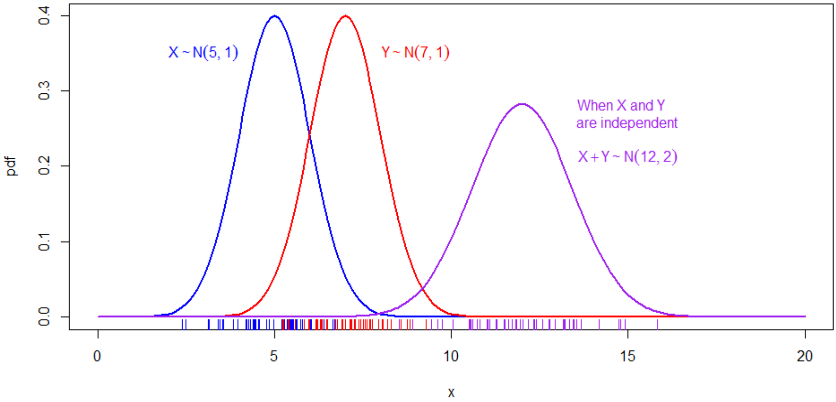
$$\begin{aligned} X + a &\sim \mathcal{N}\left(\mu + a, \sigma^2\right) \\ aX &\sim \mathcal{N}\left(a\mu, a^2\sigma^2\right). \end{aligned}$$
- If $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ and $Y \sim \mathcal{N}\left(\alpha, \delta^2\right)$, and X and Y are **independent** then
$$X + Y \sim \mathcal{N}\left(\mu + \alpha, \sigma^2 + \delta^2\right).$$

Gaussian or "Normal" Distribution



Locations of $n = 30$ sampled values of X, Y, and $X + Y$ shown as tick marks under each respective density.

Example

On the highway A2 (in the Luzern area), the speed is limited to 80 km/h. A radar measures the speeds of all the cars. Assuming that the registered speeds are distributed according to a Normal law with mean 72 km/h and standard error 8 km/h:

1. what is the proportion of the drivers who will have to pay a penalty for high speed?
2. knowing that in addition to the penalty, a speed higher than 30 km/h (over the max allowed speed) implies a withdrawal of the driving license, what is the proportion of the drivers who will lose their driving license among those who will have a to pay a fine?

Gaussian or "Normal" Distribution

Example (continued)

Let X be the random variable expressing the registered speed: $X \sim \mathcal{N}(72, 64)$.

1. Since a driver has to pay if its speed is above 80 km/h, the proportion of drivers paying a penalty is expressed through $P(X > 80)$:

$$P(X > 80) = P\left(Z > \frac{80 - 72}{8}\right) = 1 - \Phi(1) \simeq 16\%$$

   where $Z \sim \mathcal{N}(0, 1)$.

2. We are looking for the conditional probability of a recorded speed greater than 110 given that the driver has had already to pay a fine:

$$\begin{aligned} P(X > 110 | X > 80) &= \frac{P(\{X > 110\} \bigcap \{X > 80\})}{P(X > 80)} \\ &= \frac{P(X > 110)}{P(X > 80)} = \frac{1 - \Phi((110 - 72)/8)}{1 - \Phi(1)} \approx \frac{0}{16\%} \simeq 0. \end{aligned}$$

Outline

The Chi-squared distribution

Definition

If $Z_1, Z_2, \ldots, Z_n$ are independent standard Normal random variables, then

$$X = Z_1^2 + Z_2^2 + \cdots + Z_n^2$$

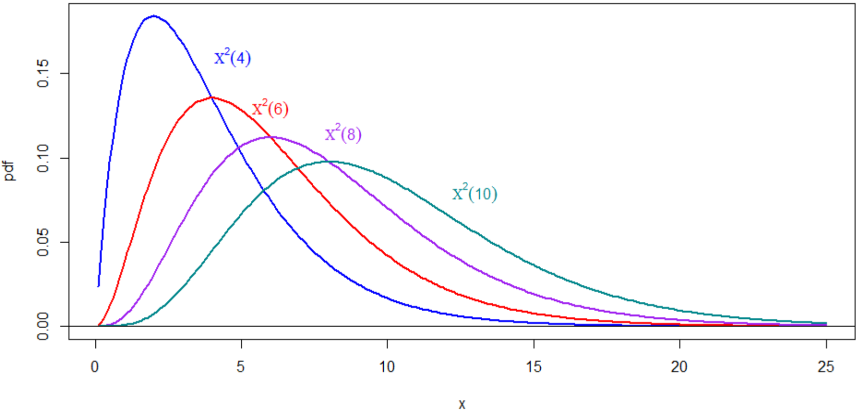
has a chi-squared distribution with n degrees of freedom. Write as $X \sim \chi^2(n)$.

$X \sim \chi^2(n)$ can take only **positive** values. Moreover, expected value and variance, for $X \sim \chi^2(n)$, are:

$$\begin{aligned} E\left[X\right] &= n \\ Var\left(X\right) &= 2n \end{aligned}$$

If $X \sim \chi^2(n)$ and $Y \sim \chi^2(m)$ are **independent** then $X + Y \sim \chi^2(n+m)$.

The Chi-squared distribution

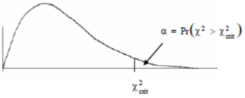


Probabilities for Chi-squared distributions may be obtained from a table

The Chi-squared distribution

TABLE 3: CHI-SQUARED DISTRIBUTION: CRITICAL VALUES

For a particular number of degrees of freedom ν, each entry represents the value of χ^2_ν corresponding to a specified upper tail area a.



	Upper Tail Areas, a										
ν	0.995	0.99	0.975	0.95	0.99	0.1	0.05	0.025	0.01	0.005	ν
1	0.000039	0.000157	0.000982	0.003932	0.000157	2.70554	3.84146	5.02390	6.63489	7.87940	1
2	0.010025	0.020100	0.050636	0.102586	0.020100	4.60518	5.99148	7.37778	9.21035	10.59653	2
3	0.071723	0.114832	0.215795	0.351846	0.114832	6.25139	7.81472	9.34840	11.34488	12.83807	3
4	0.20698	0.29711	0.48442	0.71072	0.29711	7.77943	9.48773	11.14326	13.27670	14.86017	4
5	0.41175	0.55430	0.83121	1.14548	0.55430	9.23635	11.07048	12.83249	15.08632	16.74965	5
6	0.67573	0.87208	1.23734	1.63538	0.87208	10.64464	12.59158	14.44935	16.81187	18.54751	6
7	0.98925	1.23903	1.68986	2.16735	1.23903	12.01703	14.06713	16.01277	18.47532	20.27774	7
8	1.34440	1.64651	2.17972	2.73263	1.64651	13.36156	15.50731	17.53454	20.09016	21.95486	8
9	1.73491	2.08789	2.70039	3.32512	2.08789	14.68366	16.91896	19.02278	21.66605	23.58927	9
10	2.15585	2.55820	3.24696	3.94030	2.55820	15.98717	18.30703	20.48320	23.20929	25.18805	10
11	2.60320	3.05350	3.81574	4.57481	3.05350	17.27501	19.67515	21.92002	24.72502	26.75686	11
12	3.07379	3.57055	4.40378	5.22603	3.57055	18.54934	21.02606	23.33666	26.21696	28.29966	12
13	3.56504	4.10690	5.00874	5.89186	4.10690	19.81193	22.36203	24.73558	27.68818	29.81932	13
14	4.07466	4.66042	5.62872	6.57063	4.66042	21.06414	23.68478	26.11893	29.14116	31.31943	14
15	4.60087	5.22936	6.26212	7.26093	5.22936	22.30712	24.99580	27.48836	30.57795	32.80149	15
16	5.14216	5.81220	6.90766	7.96164	5.81220	23.54182	26.29622	28.84532	31.99986	34.26705	16
17	5.69727	6.40774	7.56418	8.67175	6.40774	24.76903	27.58710	30.19098	33.40872	35.71838	17
18	6.26477	7.01490	8.23074	9.39045	7.01490	25.98942	28.86932	31.52641	34.80524	37.15639	18
19	6.84392	7.63270	8.90651	10.11701	7.63270	27.20356	30.14351	32.85234	36.19077	38.58212	19
20	7.43381	8.26037	9.59077	10.85080	8.26037	28.41197	31.41042	34.16958	37.56627	39.99686	20
21	8.03360	8.89717	10.28291	11.59132	8.89717	29.61509	32.67056	35.47886	38.93223	41.40094	21
22	8.64268	9.54249	10.98233	12.33801	9.54249	30.81329	33.92446	36.78068	40.28945	42.79566	22

The Chi-squared distribution

Example

Let X be a chi-squared random variable with 10 degrees-of-freedom. What is the value of its upper fifth percentile?

By definition, the upper fifth percentile is the chi-squared value x (lower case!!!) such that the probability to the right of x is 0.05 (so the upper tail area is 5%). To find such an x we use the chi-squared table:

- setting $\mathcal{V} = 10$ in the first column on the left and getting the corresponding row
- finding the column headed by $P(X \geq x) = 0.05$.

Now, all we need to do is read the corresponding cell. What do we get? Well, the table tells us that the upper fifth percentile of a chi-squared random variable with 10 degrees of freedom is **18.30703**.

Outline

- Gaussian or "Normal" Distribution
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1 Variable Transformation

The Student-t distribution

Definition

If $Z \sim \mathcal{N}(0,1)$ and $Y \sim \chi^2(v)$ are **independent** then

$$T = \frac{Z}{\sqrt{Y/v}}$$

has a **Student-t** distribution with v degrees of freedom. Write as $T \sim t_v$.

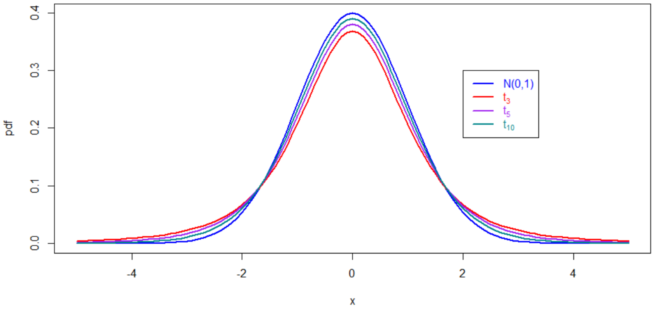
$T \sim t_v$ can take any value in $\mathbb{R}$. Expected value and variance for $T \sim t_v$ are

$$\begin{aligned} E[T] &= 0, \text{ for } v > 1 \\ Var(T) &= \frac{v}{v-2}, \text{ for } v > 2. \end{aligned}$$

The Student-t distribution

Remark

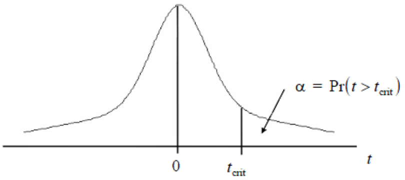
The pdf of $T \sim t_v$ is similar to a Normal (with mean zero) but with fatter tails. When v is large (typically, $v \geq 120$) t_v approaches $\mathcal{N}(0,1)$.



The Student-t distribution

TABLE 2: STUDENT *t* DISTRIBUTION: CRITICAL VALUES

For a particular number of degrees of freedom ν, each entry represents the value of t corresponding to a specified upper tail area α.



Degrees of	Upper Tail Areas, α					
Freedom ν	.25	.10	.05	.025	.01	.005
1	1.0000	3.0777	6.3137	12.7062	31.8210	63.6559
2	0.8165	1.8856	2.9200	4.3027	6.9645	9.9250
3	0.7649	1.6377	2.3534	3.1824	4.5407	5.8408
4	0.7407	1.5332	2.1318	2.7765	3.7469	4.6041
5	0.7267	1.4759	2.0150	2.5706	3.3649	4.0321
6	0.7176	1.4398	1.9432	2.4469	3.1427	3.7074
7	0.7111	1.4149	1.8946	2.3646	2.9979	3.4995
8	0.7064	1.3968	1.8595	2.3060	2.8965	3.3554
9	0.7027	1.3830	1.8331	2.2622	2.8214	3.2498
10	0.6998	1.3722	1.8125	2.2281	2.7638	3.1693
11	0.6974	1.3634	1.7959	2.2010	2.7181	3.1058
12	0.6955	1.3562	1.7823	2.1788	2.6810	3.0545
13	0.6938	1.3502	1.7709	2.1604	2.6503	3.0123
14	0.6924	1.3450	1.7613	2.1448	2.6245	2.9768
15	0.6912	1.3406	1.7531	2.1315	2.6025	2.9467
16	0.6901	1.3368	1.7459	2.1199	2.5835	2.9208
17	0.6892	1.3334	1.7396	2.1098	2.5669	2.8982
18	0.6884	1.3304	1.7341	2.1009	2.5524	2.8784
19	0.6876	1.3277	1.7291	2.0930	2.5395	2.8609
20	0.6870	1.3253	1.7247	2.0860	2.5280	2.8453
21	0.6864	1.3232	1.7207	2.0796	2.5176	2.8314

Outline

The F distribution

Definition

If $X \sim \chi^2(v_1)$ and $Y \sim \chi^2(v_2)$ are **independent**, then

$$F = \frac{\frac{X}{v_1}}{\frac{Y}{v_2}},$$

has an **F** distribution with v_1 'numerator' and v_2 'denominator' degrees of freedom. Write as $F \sim F_{v_1,v_2}$.

$F \sim F_{v_1,v_2}$ can take only **positive** values. Expected value and variance for $F \sim F_{v_1,v_2}$ (note that the order of the degrees of freedom is important!).

$$\begin{aligned} E[F] &= \frac{v_2}{v_2 - 2}, \text{ for } v_2 > 2 \\ Var(F) &= \frac{2v_2^2(v_1 + v_2 - 2)}{v_1(v_2 - 2)^2(v_2 - 4)}, \text{ for } v_2 > 4. \end{aligned}$$

The F distribution

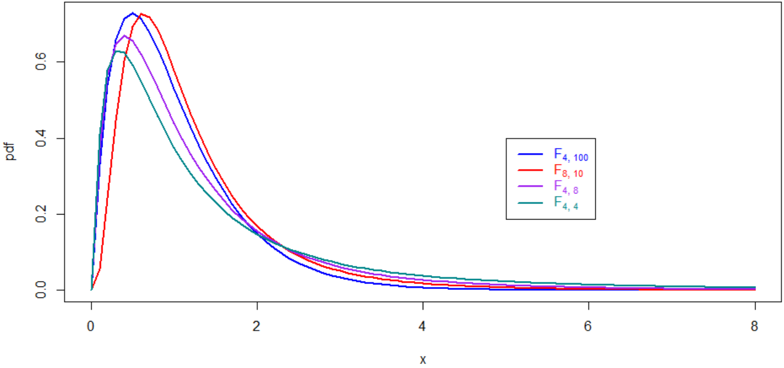
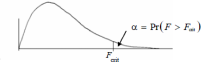


TABLE 4: F_{ν_1,ν_2} DISTRIBUTION: $\alpha = 0.05$

CRITICAL VALUES



For a particular pair of degrees of freedom, ν_1 : numerator and ν_2 : denominator, each entry represents the value of F_{ν_1,ν_2} corresponding to the upper tail area α .

ν_1

ν_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞	ν_2
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.90	245.95	248.02	249.05	250.10	251.14	252.20	253.25	254.32	1
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50	2
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53	3
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63	4
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37	5
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67	6
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23	7
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93	8
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71	9
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54	10
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40	11
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30	12
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21	13
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13	14
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07	15
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01	16
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96	17
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92	18
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88	19
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84	20
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81	21
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78	22
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76	23
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73	24
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71	25
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69	26
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67	27
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65	28

Outline

- Gaussian or "Normal" Distribution
- The Chi-squared distribution
- The Student-t distribution
- The F distribution
- The lognormal distribution
- Exponential distribution

1 Variable Transformation

The lognormal distribution

Definition

Y has a **lognormal distribution** when

$$\ln(Y) = X$$

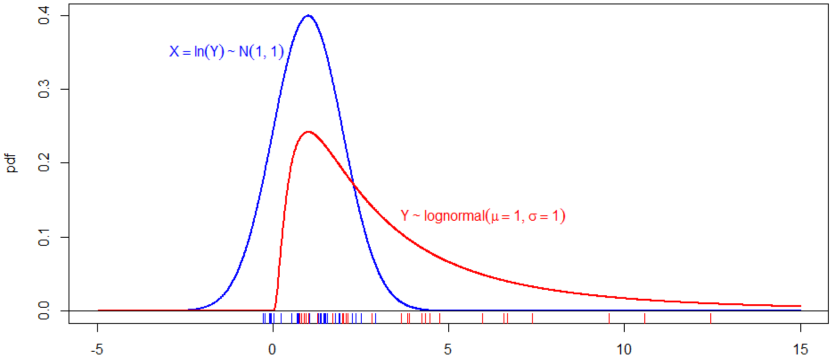
has a Normal distribution. We write $Y \sim \textit{lognormal}(\mu, \sigma^2)$.

If $Y \sim \textit{lognormal}(\mu, \sigma^2)$ then

$$\begin{aligned} E[Y] &= \exp\left(\mu + \frac{1}{2}\sigma^2\right) \\ Var(Y) &= \exp\left(2\mu + \sigma^2\right)\left(\exp\left(\sigma^2\right) - 1\right). \end{aligned}$$

The lognormal distribution

Let us just see some plots... more to come later...



Outline

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1 Variable Transformation

Exponential distribution

Definition

Let X be a continuous random variable, having the following characteristics:

- X is defined on the positive real numbers $(0;\infty)$ — namely $\mathbb{R}^+$;
- the pdf and CDF are

$$f_X(x) = \lambda \exp -\lambda x, \lambda > 0; \quad F_X(x) = 1 - \exp(-\lambda x);$$

then we say that X has an exponential distribution. We write $X \sim \text{Exp}(\lambda)$.

For $X \sim \text{Exp}(\lambda)$ we have that:

$$E[X] = \int_0^\infty x f_X(x) dx = 1/\lambda \quad \text{and} \quad Var(X) = \int_0^\infty x^2 f_X(x) dx - E^2(X) = 1/\lambda^2.$$

Remark

X is typically applied to model the waiting time until an event occurs, when events are always occurring at a random rate $\lambda > 0$. Moreover, the sum of independent exponential random variables has a Gamma distribution (see tutorial).

Exponential distribution

Example

Let $X \sim \text{Exp}(\lambda)$, with $\lambda = 0.5$. Thus

$$f_X(x) = \begin{cases} 0.5 \exp(-0.5x) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, find the CDF.

For $x > 0$, we have

$$\begin{aligned} F_X(x) &= \int_0^x f_X(u) du \\ &= 0.5\Big(-2\exp(-0.5u)\Big)\Big|_{u=0}^{u=x} \\ &= 0.5(-2\exp(-0.5x) + 2\exp(0)) \\ &= 1 - \exp(-0.5x) \end{aligned}$$

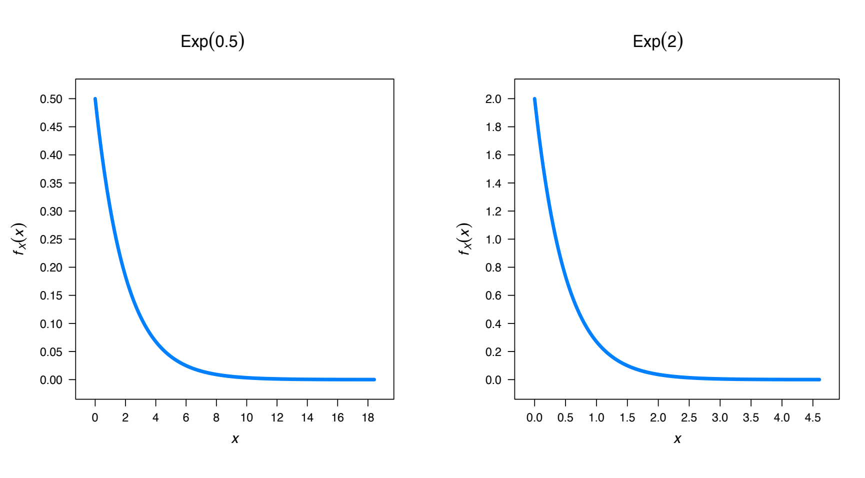
so, finally,

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ 1 - \exp(-0.5x) & x > 0 \end{cases}$$

Exponential distribution

Example (continued)

...and a graphical illustration, with varying λ



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  - A sufficient condition for $\psi(x)$ to be 1-to-1 is that it be monotonically increasing (or decreasing) in x.
  - Note that the **inverse** of a 1-to-1 function $y = \psi(x)$ is a 1-to-1 function $\psi^{-1}(y)$ such that

$$\psi^{-1}(\psi(x)) = x \text{ and } \psi\left(\psi^{-1}(y)\right) = y.$$

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$$\psi^{-1}(\psi(x)) = x \text{ and } \psi\left(\psi^{-1}(y)\right) = y.$$

- To transform X to Y, we need to consider all the values x that X can take
- We first transform x into values $y = \psi(x)$

- To transform a discrete random variable X, into the random variable $Y = \psi(X)$, we transfer the probabilities for **each** x to the values $y = \psi(x)$:

Probability function for X

X	$P(\{X = x_i\}) = p_i$
x_1	p_1
x_2	p_2
x_3	p_3
$\vdots$	$\vdots$
x_n	p_n

$\Rightarrow$

Probability function for X

Y	$P(\{X = x_i\}) = p_i$
$\psi(x_1)$	p_1
$\psi(x_2)$	p_2
$\psi(x_3)$	p_3
$\vdots$	$\vdots$
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$\vdots$	$\vdots$
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- Note that this is equivalent to applying the function $\psi(\cdot)$ inside the probability statements:

$$
\begin{aligned}
P(\{X = x_i\}) &= P(\{\psi(X) = \psi(x_i)\}) \\
&= P(\{Y = y_i\}) \\
&= p_i
\end{aligned}
$$

Example (option pricing)

Let us imagine that we are tossing a balanced coin ($p = 1/2$), and when we get a "Head" (H) the stock price moves up of a factor u, but when we get a "Tail" (T) the price moves down of a factor d. We denote the price at time t_1 by $S_1(H) = uS_0$ if the toss results in head (H), and by $S_1(T) = dS_0$ if it results in tail (T). After the second toss, the price will be one of:

$$S_2(HH) = uS_1(H) = u^2S_0, \quad S_2(HT) = dS_1(H) = duS_0,$$

$$S_2(TH) = uS_1(T) = udS_0, \quad S_2(TT) = dS_1(T) = d^2S_0.$$

Indeed, after two tosses, there are four possible coin sequences,

$$\{HH, HT, TH, TT\}$$

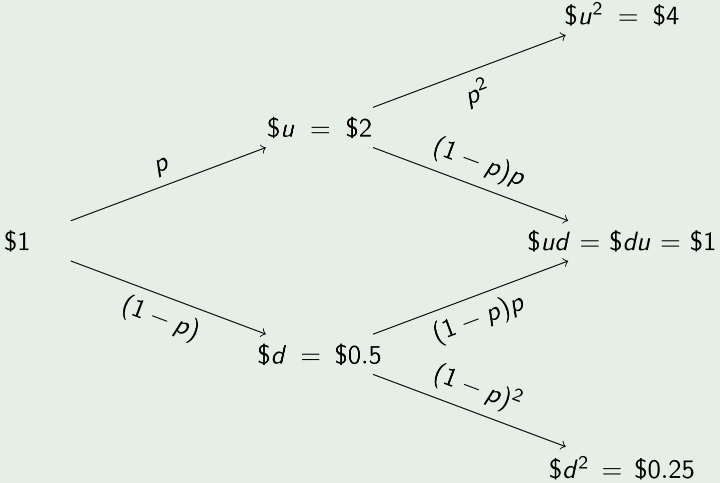
although not all of them result in different stock prices at time t_2.

Variable Transformation

Transformation of discrete random variables

Example (continued)

Let us set $S_0 = 1$, $u = 2$ and $d = 1/2$: we represent the price evolution by a tree:



Example (continued)

Now consider an European option call with maturity t_2 and strike price $K = 0.5$, whose random pay-off at t_2 is $C = \max(0; S_2 - 0.5)$. Thus,

$$C(HH) = \max(0; 4 - 0.5) = \$3.5 \quad C(HT) = \max(0; 1 - 0.5) = \$0.5$$
$$C(TH) = \max(0; 1 - 0.5) = \$0.5 \quad C(TT) = \max(0; 0.25 - 0.5) = \$0.$$

Thus at maturity t_2 we have

Probability function for S_2

S_2	$P(\{X = x_i\}) = p_i$
$\$u^2$	p^2
$\$ud$	$2p(1-p)$
$\$d^2$	$(1-p)^2$

$\Rightarrow$

Probability function for C

C	$P(\{C = c_i\}) = p_i$
$\$3.5$	p^2
$\$0.5$	$2p(1-p)$
$\$0$	$(1-p)^2$

Since $ud = du$ the corresponding values of S_2 and C can be aggregated, without loss of info.

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Variable Transformation

Transformation of variables using the CDF

- We can use the same logic for CDF probabilities, whether the random variables are **discrete or continuous**
- Let $Y = \psi(X)$ with $\psi(x)$ 1-to-1 and monotone increasing. Then

$$\begin{aligned} F_Y(y) &= P(\{Y \leq y\}) \\ &= P(\{\psi(X) \leq y\}) = P\left(\{X \leq \psi^{-1}(y)\}\right) \\ &= F_X\left(\psi^{-1}(y)\right) \end{aligned}$$

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Example

Let $Y = \psi(X) = \exp X$ where $X \sim F_X$ on all values $x \in \mathbb{R}$

$$\begin{aligned} F_Y(y) &= P(\{Y \leq y\}) \\ &= P(\{\exp X \leq y\}) = P(\{X \leq \ln(y)\}) \\ &= F_X(\ln(y)) \text{ only for } y > 0. \end{aligned}$$

- Monotone decreasing functions work in a similar way, but require changing of the inequality sign

Example

Example: let $Y = \psi(X) = -\exp X$ where $X \sim F_X$ on all values $x \in \mathbb{R}$

$$
\begin{aligned}
F_Y(y) &= P(\{Y \leq y\}) = P\left(\{-\exp^X \leq y\}\right) \\
&= P(\{\exp X \geq -y\}) = P(\{X \geq \ln(-y)\}) \\
&= 1 - F_X(\ln(-y)) \text{ only for } y < 0.
\end{aligned}
$$

- Monotone decreasing functions work in a similar way, but require changing of the inequality sign
- Let $Y = \psi(X)$ with $\psi(x)$ 1-to-1 and **monotone decreasing**. Then

$$\begin{aligned} F_Y(y) &= P(\{Y \leq y\}) \\ &= P(\{\psi(X) \leq y\}) = P\left(\{X \geq \psi^{-1}(y)\}\right) \\ &= 1 - F_X\left(\psi^{-1}(y)\right) \end{aligned}$$

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$$F_Y(y) = F_X\left(\psi^{-1}(y)\right)$$
- Notice this implies that the pdf of $Y = \psi(X)$ must satisfy

$$\begin{aligned}
f_Y(y) &= \frac{dF_Y(y)}{dy} = \frac{dF_X\left(\psi^{-1}(y)\right)}{dy} \\
&= \frac{dF_X(x)}{dx} \times \frac{d\psi^{-1}(y)}{dy} \qquad \text{(chain rule)} \\
&= f_X(x) \times \frac{d\psi^{-1}(y)}{dy} \qquad \text{(derivative of CDF (of } X\text{) is pdf)} \\
&= f_X\left(\psi^{-1}(y)\right) \times \frac{d\psi^{-1}(y)}{dy} \qquad \text{(substitute } x = \psi^{-1}(y)\text{)}
\end{aligned}$$

- What happens when $\psi(x)$ 1-to-1 and monotone **decreasing**? We have

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- This expression (called Jacobian-formula) is valid for $\psi(x)$ 1-to-1 and monotone (whether increasing or decreasing)

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- Using $\psi(x) = \exp x$ we know we'll have possible values for Y only on $0 < y < \infty$

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- $\Rightarrow$ the *pdf* of Y is

$$f_Y(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left(\ln(y) - \mu\right)^2\right\}$$

for any $0 < y < \infty$

Example (continued)

- Both the Normal and the lognormal are characterized by only two parameters (μ and σ). The *median* of the lognormal distribution is $\exp\mu$, since

$$P(\{X \leq \mu\}) = 0.5,$$

and hence

$$\begin{aligned} 0.5 &= P(\{X \leq \mu\}) \\ &= P(\{\exp X \leq \exp\mu\}) \\ &= P(\{Y \leq \exp\mu\}). \end{aligned}$$

More generally, for $\alpha \in [0,1]$, the α-th quantile of a r.v. X is the value x_α such that $P(\{X \leq x_\alpha\}) \geq \alpha$. If X si a continuous r.v. we can set $P(\{X \leq x_\alpha\}) = \alpha$ (as we did, e.g., for the lognormal).

When X and Y are two random variables, we should pay attention to their transformations. For instance, let us consider

$$X \sim \mathcal{N}(\mu, \sigma^2) \quad \text{and} \quad Y \sim Exp(\lambda).$$

Then, let's transform X and Y

- in a linear way: $Z = X + Y$. We know that

$$E[Z] = E[X + Y] = E[X] + E[Y]$$

- in a nonlinear way $W = X/Y$. One can show that

$$E[W] = E\left[\frac{X}{Y}\right] \neq \frac{E[X]}{E[Y]}.$$

Despite exotic names, the common distributions relate to each other in intuitive and interesting ways. Several follow naturally from the Bernoulli distribution, for example.

- ▷ *'Common probability distributions: the data scientist's crib sheet'* (goo.gl/NJRlXn):