Exercise 1

You look at the following discrete probability distributions for two random variables X and Y respectively

X values	-1	1	3	5	7
Probability	0.1	0.2	0.4	0.2	0.1

Y values	-1	1	3	5	7
Probability	0.2	0.15	0.3	0.15	0.2

Compute the mean and the variance of both variables.

Solutions:
X and Y are discrete random variables. To calculate their expectations, we use
$E(X) = \sum x \cdot P(X = x)$
$E(X) = (-1) \cdot 0.1 + 1 \cdot 0.2 + 3 \cdot 0.4 + 5 \cdot 0.2 + 7 \cdot 0.1 = 3$
$E(Y) = (-1) \cdot 0.2 + 1 \cdot 0.15 + 3 \cdot 0.3 + 5 \cdot 0.15 + 7 \cdot 0.2 = 3$
$E(X^2) = \sum x^2 \cdot P(X = x) = 1^2 \cdot 0.1 + 1^2 \cdot 0.2 + 3^2 \cdot 0.4 + 5^2 \cdot 0.2 + 7^2 \cdot 0.2 = 13.8$
$E(Y^2) = \sum y^2 \cdot P(Y = y) = 1^2 \cdot 0.2 + 1^2 \cdot 0.15 + 3^2 \cdot 0.3 + 5^2 \cdot 0.15 + 7^2 \cdot 0.2 = 16.6$
Therefore the variance is
$V(X) = E(X^2) - E(X)^2 = 13.8 - 3^2 = 4.8$
$V(Y) = E(Y^2) - E(Y)^2 = 16.6 - 3^2 = 7.6$

Exercise 2

In a TV game, a candidate faces 5 doors, one of which hides a gift. Viewers can make bets on the number of doors that the candidate will push until he finds the gift. Jules and Gaston are candidates for the game. Jules has a good memory and is not likely to push twice the same door. As far as Gaston is concerned, he has absolutely no memory.

We denote the events as follows:

- S: 'Success (there is a gift)';
- E: 'Failure (there is no gift)'.

1. Construct the probability function of the number of doors pushed by Jules. Calculate the expectation of this event.

   Let X be the random variable: Number of doors pushed by Jules. Jules remembers the doors he pushed. He will never push the same door twice:

Event	x_i	$P(X = x_i)$
{S}	1	$1/5$
{ES}	2	$4/5 \cdot 1/4 = 1/5$
{EES}	3	$4/5 \cdot 3/4 \cdot 1/3 = 1/5$
{EEES}	4	$4/5 \cdot 3/4 \cdot 2/3 \cdot 1/2 = 1/5$
{EEEES}	5	$4/5 \cdot 3/4 \cdot 2/3 \cdot 1/2 \cdot 1 = 1/5$

   So

   $$\mathrm{E}(X) = \sum_{i=1}^{5} P(X = x_i) \cdot x_i = 1/5 \cdot 1 + 1/5 \cdot 2 + 1/5 \cdot 3 + 1/5 \cdot 4 + 1/5 \cdot 5 = 15/5 = 3.$$

2. Construct the probability function of the number of doors pushed by Gaston. Calculate the expectation of this event.

   <u>Indication:</u> The following mathematical property can be used (generic geometric series):

   $$|x| < 1 \Rightarrow \sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}.$$

   Let Y be the random variable: Number of doors pushed by Gaston. Gaston does not remember the doors he pushed. It can be a bad door to infinity:

Event	y_i	$P(Y = y_i)$
{S}	1	$1/5$
{ES}	2	$4/5 \cdot 1/5 = 4/25$
{EES}	3	$4/5 \cdot 4/5 \cdot 1/5 = (4/5)^2 \cdot (1/5) = 16/125$
{EEES}	4	$4/5 \cdot 4/5 \cdot 4/5 \cdot 1/5 = (4/5)^3 \cdot (1/5) = 64/625$
...	...	...
$\{\underbrace{EE..E}_{k-1}S\}$	k	$(4/5)^{k-1} \cdot (1/5)$

   So

   $$\mathrm{E}(Y) = \sum_{i=1}^{\infty} P(Y = y_i) \cdot y_i = \sum_{i=1}^{\infty} (1/5) \cdot (4/5)^{i-1} \cdot i = \frac{1/5}{(1-4/5)^2} = \frac{(1/5)}{(1/5)^2} = 5.$$ [1]

[1] Here, we use the property $\sum_{i=1}^{\infty} x^{i-1} \cdot i = \frac{1}{(1-x)^2}$.

3. Generalize the previous results to n doors.

    (a) For Jules:

Event	x_i	$P(X = x_i)$
{S}	1	$1/n$
{ES}	2	$(n-1)/n \cdot 1/(n-1) = 1/n$
{EES}	3	$(n-1)/n \cdot (n-2)/(n-1) \cdot 1/(n-2) = 1/n$
{EEES}	4	$(n-1)/n \cdot (n-2)/(n-1) \cdot (n-3)/(n-2) \cdot 1/(n-3) = 1/n$
...	...	...
$\{\underbrace{EE..E}_{n-1}S\}$	n	$1/n$

    So

$$\mathrm{E}(X) = \sum_{i=1}^{n} P(X = x_i) \cdot x_i = \frac{1}{n} \cdot (1 + 2 + 3 + ... + n) = \frac{\sum_{i=1}^{n} i}{n} = \frac{\frac{n \cdot (n+1)}{2}}{n} = \frac{n+1}{2}.$$

    (b) For Gaston:

Event	y_i	$P(Y = y_i)$
{S}	1	$1/n$
{ES}	2	$((n-1)/n) \cdot (1/n)$
{EES}	3	$((n-1)/n)^2 \cdot (1/n)$
{EEES}	4	$((n-1)/n)^3 \cdot (1/n)$
...	...	...
$\{\underbrace{EE..E}_{k-1}S\}$	k	$((n-1)/n)^{k-1} \cdot (1/n)$

    So

$$\mathrm{E}(X) = \sum_{i=1}^{\infty} P(Y = y_i) \cdot y_i = \frac{1}{n} \cdot \sum_{i=1}^{\infty} (\frac{n-1}{n})^{i-1} \cdot i = \frac{1}{n} \cdot \frac{1}{(1-\frac{n-1}{n})^2} = \frac{1}{n} \cdot \frac{1}{(1/n)^2} = n.$$

Exercise 3

Two cards are randomly chosen from a box containing the following five cards: 1, 1, 2, 2, 3. X represents the average and Y is the maximum of the two numbers drawn.

1. Calculate the probability distribution, the cumulative distribution function, the mean and the variance of X, Y and $Z = Y - X$.

   It is assumed a draw without replacement.
   The total number of possible events is: $\binom{5}{2} = 10$.

Possible combinations	(1,1)	(1,2)	(1,3)	(2,2)	(2,3)
Number of events	$\binom{2}{2}=1$	$\binom{2}{1}\binom{2}{1}=4$	$\binom{2}{1}\binom{1}{1}=2$	$\binom{2}{2}=1$	$\binom{2}{1}\binom{1}{1}=2$
x	1	1.5	2	2	2.5
y	1	2	3	2	3
z	0	0.5	1	0	0.5

The probability distribution, the distribution function, the mean and the variance can be calculated:

- For Variable X:

x	1	1.5	2	2.5
$P(X=x)$	1/10	4/10	3/10	2/10

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ 1/10 & \text{if} \quad 1 \leq x < 1.5 \\ 5/10 & \text{if} \quad 1.5 \leq x < 2 \\ 8/10 & \text{if} \quad 2 \leq x < 2.5 \\ 1 & \text{if} \quad x \geq 2.5 \end{cases}$$

$$\mathrm{E}(X) = \sum_{i=1}^{4} P(X = x_i) \cdot x_i = 1/10 \cdot 1 + 4/10 \cdot 1.5 + 3/10 \cdot 2 + 2/10 \cdot 2.5 = 1.8$$
$$\mathrm{E}(X^2) = \sum_{i=1}^{4} P(X = x_i) \cdot x_i^2 = 1/10 \cdot 1^2 + 4/10 \cdot 1.5^2 + 3/10 \cdot 2^2 + 2/10 \cdot 2.5^2 = 3.45$$
$$\mathrm{var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = 3.45 - 1.8^2 = 0.21$$

- For Variable Y:

y	1	2	3
$P(Y=y)$	1/10	5/10	4/10

$$F(y) = \begin{cases} 0 & \text{if} \quad y < 1 \\ 1/10 & \text{if} \quad 1 \leq y < 2 \\ 6/10 & \text{if} \quad 2 \leq y < 3 \\ 1 & \text{if} \quad y \geq 3 \end{cases}$$

$$\mathrm{E}(Y) = \sum_{i=1}^{3} P(Y = y_i) \cdot y_i = 1/10 \cdot 1 + 5/10 \cdot 2 + 4/10 \cdot 3 = 2.3$$
$$\mathrm{E}(Y^2) = \sum_{i=1}^{3} P(Y = y_i) \cdot y_i^2 = 1/10 \cdot 1^2 + 5/10 \cdot 2^2 + 4/10 \cdot 3^2 = 5.7$$
$$\mathrm{var}(Y) = \mathrm{E}(Y^2) - \mathrm{E}(Y)^2 = 5.7 - 2.3^3 = 0.41$$

- For Variable Z:

z	0	0.5	1
$P(Z = z)$	2/10	6/10	2/10

$$F(z) = \begin{cases} 0 & \text{if} \quad z < 0 \\ 2/10 & \text{if} \quad 0 \leq z < 0.5 \\ 8/10 & \text{if} \quad 0.5 \leq z < 1 \\ 1 & \text{if} \quad z \geq 1 \end{cases}$$

$$\mathrm{E}(Z) = \sum_{i=1}^{3} P(Z = z_i) \cdot z_i = 2/10 \cdot 0 + 6/10 \cdot 0.5 + 2/10 \cdot 1 = 0.5$$
$$\mathrm{E}(Z^2) = \sum_{i=1}^{3} P(Z = z_i) \cdot z_i^2 = 2/10 \cdot 0^2 + 6/10 \cdot 0.5^2 + 2/10 \cdot 1^2 = 0.35$$
$$\mathrm{var}(Z) = \mathrm{E}(Z^2) - \mathrm{E}(Z)^2 = 0.35 - 0.5^2 = 0.1$$

2. If W is the sum of the two numbers, what are its expectation and its variance?

   We can calculate the values of W for each possible case, and draw the table as we did for the variables X, Y and Z to obtain the probability function then the mean and the variance of W. However, it is faster to note that:

$$X = \frac{\text{Card 1} + \text{Card 2}}{2} = \frac{\text{Sum of the numbers of the two cards}}{2}.$$

   So we have:

$$W = \text{Sum of the numbers of the two cards} = 2X$$
$$\mathrm{E}(W) = \mathrm{E}(2X) = 2\mathrm{E}(X) = 2 \cdot 1.8 = 3.6$$
$$\mathrm{var}(W) = \mathrm{var}(2X) = 2^2\mathrm{var}(X) = 4 \cdot 0.21 = 0.84$$

Exercise 4

We consider a die whose faces are numbered from 1 to 6 and we define X the random variable given by the number of the upper face. It is assumed that the die is rigged so that the probability of getting a face is proportional to the number on that face.

1. Determine the probability function of X, then calculate its expectation and variance.

   By the proportionality assumption, we know that there exists $\alpha \in \mathbb{R}$ such that $P(X = k) = \alpha k$, $\forall k = 1, \ldots, 6$. Moreover, since the sum of the probabilities associated with the different faces must always be equal to 1, we have $\sum_{k=1}^{6} P(X = k) = 1$. By combining these two equations, we obtain $\sum_{k=1}^{6} \alpha k = \alpha \sum_{k=1}^{6} k = 21\alpha = 1 \iff \alpha = \frac{1}{21}$. We can therefore establish the probability function of X as follows:

x	1	2	3	4	5	6
$P(X=x)$	1/21	2/21	3/21	4/21	5/21	6/21

We can then calculate the expectation as follows:

$$\mathrm{E}(X)=\sum_{x=1}^{6} P(X=x)\cdot x = 1/21\cdot 1+2/21\cdot 2+3/21\cdot 3+4/21\cdot 4+5/21\cdot 5+6/21\cdot 6 = 13/3.$$

As for the variance, we have:

$$\begin{aligned}\mathrm{var}(X) &= \sum_{x=1}^{6} P(X=x)\cdot x^2 - \mathrm{E}(X)^2\\ &= 1/21\cdot 1^2+2/21\cdot 2^2+3/21\cdot 3^2+4/21\cdot 4^2+5/21\cdot 5^2+6/21\cdot 6^2-(13/3)^2\\ &= 20/9.\end{aligned}$$

2. We define $Y = 3X$. Calculate the expectation and variance of Y. Is it necessary to determine the probability function of Y for calculating its expectation and variance?

   The expectation and variance of Y can be calculated without determining its probability function. Indeed, since $Y = 3X$, we have $\mathrm{E}(Y) = \mathrm{E}(3X) = 3\mathrm{E}(X) = 3\cdot 13/3 = 13$ and $\mathrm{var}(Y) = \mathrm{var}(3X) = 3^2\mathrm{var}(X) = 9\cdot 20/9 = 20$.

3. We define $Z = \frac{1}{X}$. Calculate the expectation and variance of Z. Is it necessary to determine the probability function of Z for calculating its expectation and variance?

   Here it is necessary to determine the probability function of Z. In general $\mathrm{E}(1/X) \neq 1/\mathrm{E}(X)$. Given that $z = 1/x, \quad \forall x = 1,\ldots,6$, the probability function of Z is easily obtained:

z	1	1/2	1/3	1/4	1/5	1/6
$P(Z=z)$	1/21	2/21	3/21	4/21	5/21	6/21

Then we can calculate the expectation and the variance of Z:

$$\begin{aligned}\mathrm{E}(Z) &= \sum_{k=1}^{6} P(Z=1/k)\cdot 1/k\\ &= 1/21\cdot 1+2/21\cdot 1/2+3/21\cdot 1/3+4/21\cdot 1/4+5/21\cdot 1/5+6/21\cdot 1/6\\ &= 6/21 = 2/7\end{aligned}$$

$$\begin{aligned}\mathrm{var}(Z) &= \sum_{k=1}^{6} P(Z=1/k)\cdot (1/k)^2 - \mathrm{E}(Z)^2\\ &= 1/21\cdot 1^2+2/21\cdot(1/2)^2+3/21\cdot(1/3)^2+4/21\cdot(1/4)^2+5/21\cdot(1/5)^2\\ &\quad +6/21\cdot(1/6)^2-(2/7)^2 = 103/2940\end{aligned}$$

Exercise 5

1. The following game is considered: The player rolls a fair six sided die once. If he gets 1, 2 or 3, he wins the equivalent in francs. Otherwise, he loses 2 francs. Let X be the random variable corresponding to the player's win (a negative value indicating a loss).

   (a) Give the probability function of X and its distribution function F_X.

   Since the die is not rigged, each face has the same probability of occurrence, *i.e.* 1/6. The probability function of X is therefore the following:

x	-2	1	2	3
$P(X=x)$	1/2	1/6	1/6	1/6

   The distribution function, F_X, is written as follows:

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < -2 \\ 1/2 & \text{if} \quad -2 \leq x < 1 \\ 2/3 & \text{if} \quad 1 \leq x < 2 \\ 5/6 & \text{if} \quad 2 \leq x < 3 \\ 1 & \text{if} \quad x \geq 3 \end{cases}$$

   (b) Calculate the expectation and variance of X.

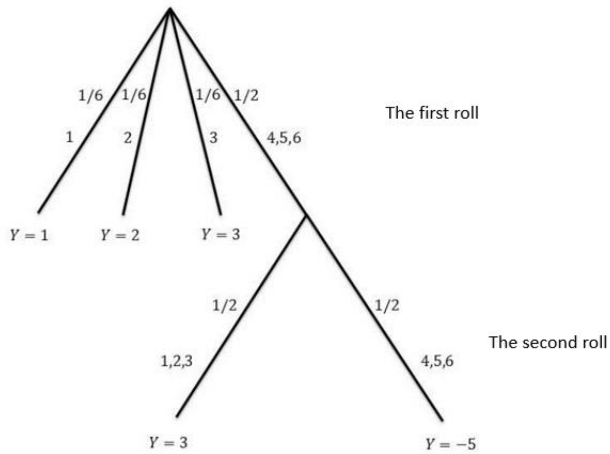
   Using the probability function of X, we can easily calculate the expectation and the variance:

$$\mathrm{E}(X) = 1/2 \cdot (-2) + 1/6 \cdot 1 + 1/6 \cdot 2 + 1/6 \cdot 3 = 0;$$
$$\mathrm{var}(X) = 1/2 \cdot (-2)^2 + 1/6 \cdot 1^2 + 1/6 \cdot 2^2 + 1/6 \cdot 3^2 - 0^2 = 13/3.$$

2. We modify the game as follows: The winnings remain the same for the results 1, 2 or 3, but if the player gets something else, he rolls the die again. If he then gets 3 or less, he wins 3 francs, otherwise he loses 5 francs. We define Y as the random variable corresponding to the gain of the player in this new game.

   (a) Give the probability function of Y and its distribution function F_Y.

   We can start by considering the outcomes where the game stops after the first roll of the die. There are three possibilities, each having a probability of 1/6, and the values of Y associated with these three outcomes are 1, 2, and 3. If the player must raise the die again (which happens with a probability of 1/2), then there are two possibilities, each having a conditional probability (of having to roll the die at the second time) of 1/2, and the values of Y are then 3 and -5 . It can be useful to represent this game by means of a probability tree:



Therefore, the probability function of Y is [2]:

y	-5	1	2	3
$P(Y = y)$	$1/4$	$1/6$	$1/6$	$5/12$

The distribution function, F_Y, is the following:

$$F_Y(y) = \begin{cases} 0 & \text{if } \; y < -5 \\ 1/4 & \text{if } \; -5 \leq y < 1 \\ 5/12 & \text{if } \; 1 \leq y < 2 \\ 7/12 & \text{if } \; 2 \leq y < 3 \\ 1 & \text{if } \; y \geq 3 \end{cases}$$

(b) Calculate the expectation and variance of Y.

Using the probability function of Y, expectancy and variance are easily calculated:

$$\mathrm{E}(Y) = 1/4 \cdot (-5) + 1/6 \cdot 1 + 1/6 \cdot 2 + 5/12 \cdot 3 = 1/2;$$
$$\mathrm{var}(Y) = 1/4 \cdot (-5)^2 + 1/6 \cdot 1^2 + 1/6 \cdot 2^2 + 5/12 \cdot 3^2 - (1/2)^2 = 127/12.$$

3. Which game is the most advantageous for the player? To justify.

It depends on the player's preferences. Indeed, we notice that $0 = \mathrm{E}(X) < \mathrm{E}(Y)$, which would seem to indicate that the second variable is preferable because, on average, the player should earn money with this but not with the first variable. However, we also have $\mathrm{var}(X) < \mathrm{var}(Y)$, sign of a greater variability of the gain with the second variable. The maximum potential loss in this second variable of the game is also much larger (in relative terms). If the player desires to avoid it at all costs, he may prefer the first game despite a lower expectation of gain.

[2] $P(Y = 3) = 1/6 + 1/2 \cdot 1/2 = 5/12$.

Exercise 6 (Optional)

The random variable X is Bernoulli(p) distribution if its probability mass function is given by:

$$P(X = x) = p^x(1-p)^{1-x} \text{for } x = 0, 1$$

where $0 < p < 1$. Compute the Mean and the Variance of the Bernoulli distribution.

Solutions:
$E(X) = \sum x \cdot P(X = x) = 1 \cdot p + 0 \cdot (1-p) = p$
$E(X^2) = \sum x^2 \cdot P(X = x) = 1^2 \cdot p + 0^2 \cdot (1-p) = p$
Therefore the variance is
$V(X) = E(X^2) - E(X)^2 = p - p^2 = p(1-p)$

Exercise 7 (Optional)

Let X a discrete random variable following a Poisson distribution with parameter λ. Its probability function is given by:

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{for} \quad k = 0, 1, ...$$

1. Check that $P(X = k)$ is a probability function.
2. Proove that its expectation is equal to λ.
3. Find the variance of X.
4. Compare the expectation and the variance of X with the expectation and the variance of S found in Exercise 4 above, for a very large n.

Solutions:

1. We have to check that the probability function is positive and that it sums up to 1.

$$\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$$

2.

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!} = \lambda\sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda\sum_{j=0}^{\infty} e^{-\lambda}\frac{\lambda^j}{(j)!} = \lambda \quad \text{for } j = k-1 \end{aligned}$$

3.

$$\begin{aligned}
E(X^2) &= \sum_{k=0}^{\infty} k^2 e^{-\lambda}\frac{\lambda^k}{k!} = \lambda \sum_{k=1}^{\infty} k e^{-\lambda}\frac{\lambda^k}{(k-1)!} \\
&\overset{\text{j=k-1}}{=} \sum_{j=0}^{\infty}(j+1)e^{-\lambda}\frac{\lambda^{j+1}}{(j)!} \\
&= \lambda\{\underbrace{\sum_{j=0}^{\infty} j e^{-\lambda}\frac{\lambda^j}{(j)!}}_{E(X)=\lambda} + \sum_{j=0}^{\infty} e^{-\lambda}\frac{\lambda^j}{(j)!}\} \\
&= \lambda(\lambda+1)
\end{aligned}$$

$V(X) = E(X^2) - E(X)^2 = \lambda(\lambda+1) - \lambda^2 = \lambda$

4. Let us denote the expected value of the binomial distribution, np by λ, which means $p = \frac{n}{\lambda}$. Now, if we use this to rewrite P(S=x) in terms of λ, n, and x, we obtain

$$\begin{aligned}
P(S=x) &= \binom{n}{x}(\frac{n}{\lambda})^x(1-\frac{n}{\lambda})^{n-x} = \frac{n(n-1)\cdots(n-x+1)}{(x)!}\frac{\lambda^x}{n^x}(1-\frac{\lambda}{n})^{n-x} \\
&= \frac{n}{n} \times \frac{n-1}{n} \cdots \frac{n-x+1}{n} \times \frac{\lambda^x}{(x)!}(1-\frac{\lambda}{n})^{n-x} \\
&= \frac{n}{n} \times \frac{n-1}{n} \cdots \frac{n-x+1}{n} \times \frac{\lambda^x}{(x)!}(1-\frac{\lambda}{n})^{n}(1-\frac{\lambda}{n})^{-x}
\end{aligned}$$

If we tool the limit as n approaches infinity, keeping x and λ fixed,the first x fractions in this expression would tend towards 1, as would the last factor in the expression. $\lim_{x\to\infty}(1-\frac{\lambda}{n})^n = e^{-\lambda}, so \lim_{x\to\infty} P(S=x) = \frac{e^{-\lambda}\lambda^x}{x!}$
So we can say expectations and variances of X and S are equivalent for a very large n. The Poisson distribution is actually a limiting case of a Binomial distribution when the number of trials, n, gets very large and p, the probability of success, is small.