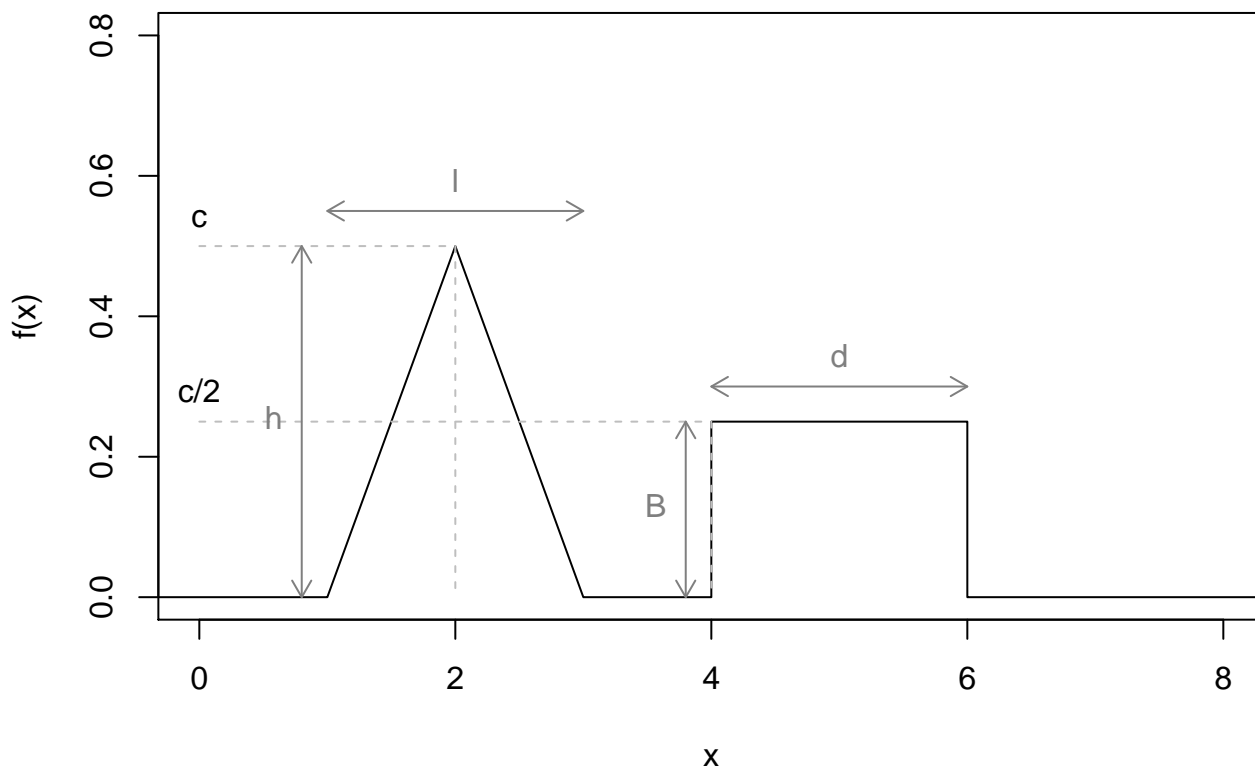


## Exercise 1:



$l$  : length of the base of the triangle

$h$  : length of the height of the triangle

$d$  : length of the base of the rectangle

$B$  : length of the height of the rectangle

1. For  $f(x)$  to be a density function, it is necessary that for all  $x$ ,  $f(x) \geq 0$  and  $\int_{-\infty}^{\infty} f(x)dx = 1$ 
  - (a) ) From the graphical representation we find that for all  $x$  we have  $f(x) \geq 0$ .
  - (b) To verify that  $\int_{-\infty}^{\infty} f(x)dx = 1$  is satisfied, it is necessary to determine if there is a value  $c$  so that the total area under the function  $f(x)$  is equal to 1.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \text{triangle area} + \text{rectangle area} = \frac{lh}{2} + dB \\ &= \frac{(3-1)c}{2} + (6-4)\frac{c}{2} = 2c\end{aligned}$$

Where we have that

$$\int_{-\infty}^{\infty} f(x)dx = 1 \iff 2c = 1.$$

The only solution is  $c = 0.5$ .

2. Graphically we find that the density function  $f(x)$  is a piecewise form function:

$$f(x) = \begin{cases} f_1(x) & \text{if } 1 < x \leq 2 \\ f_2(x) & \text{if } 2 < x \leq 3 \\ f_3(x) & \text{if } 4 < x \leq 6 \\ 0 & \text{if not} \end{cases}$$

- (a)  $f_1(x)$  is a line  $a + bx$  found from the two points  $(x_1, y_1) = (1, 0)$  and  $(x_2, y_2) = (2, 0.5)$ . We are looking for  $a$  and  $b$  such that  $f_1(x_1) = y_1$  and  $f_1(x_2) = y_2$ , i.e., we look for  $a$  and  $b$  such that  $a + bx_1 = y_1$  and  $a + bx_2 = y_2$ . We find  $a$  and  $b$  with the equations:

$$\begin{aligned}b &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{0.5 - 0}{2 - 1} = 0.5 \\ a &= y_1 - bx_1 = 0 - 0.5 \cdot 1 = -0.5.\end{aligned}$$

So

$$f_1(x) = -0.5 + 0.5x.$$

- (b) Similarly,  $f_2(x)$  is a line  $c + dx$  that we find from the two points  $(x_2, y_2) = (2, 0.5)$  and  $(x_3, y_3) = (3, 0)$ :

$$\begin{aligned}d &= \frac{y_3 - y_2}{x_3 - x_2} = \frac{0 - 0.5}{3 - 2} = -0.5 \\ c &= y_2 - dx_2 = 0.5 - (-0.5) \cdot 2 = 1.5.\end{aligned}$$

So

$$f_2(x) = 1.5 - 0.5x.$$

(c) And, finally,

$$f_3(x) = \frac{c}{2} = \frac{0.5}{2} = 0.25.$$

3. The expectation of  $X$  is defined as  $E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$ .

$$\begin{aligned} E(X) &= \int_{-\infty}^1 x \cdot 0 \, dx + \int_1^2 x \cdot f_1(x) \, dx + \int_2^3 x \cdot f_2(x) \, dx + \int_3^4 x \cdot 0 \, dx + \int_4^6 x \cdot f_3(x) \, dx \\ &\quad + \int_6^{+\infty} x \cdot 0 \, dx \\ &= \int_1^2 x \cdot (-0.5 + 0.5x) \, dx + \int_2^3 x \cdot (1.5 - 0.5x) \, dx + \int_4^6 x \cdot 0.25 \, dx \\ &= \int_1^2 (-0.5x + 0.5x^2) \, dx + \int_2^3 (1.5x - 0.5x^2) \, dx + \int_4^6 0.25x \, dx \\ &= \left[ \frac{-0.5x^2}{2} + \frac{0.5x^3}{3} + c \right]_1^2 + \left[ \frac{1.5x^2}{2} + \frac{-0.5x^3}{3} + c \right]_2^3 + \left[ \frac{0.25x^2}{2} + c \right]_4^6 \\ &= \left( \frac{-2}{2} + \frac{4}{3} - \frac{-0.5}{2} - \frac{0.5}{3} \right) + \left( \frac{13.5}{2} + \frac{-13.5}{3} - \frac{6}{2} + \frac{4}{3} \right) + \left( \frac{9}{2} - \frac{4}{2} \right) \\ &= 3.5 \end{aligned}$$

4. Determine  $P(X \geq 2)$ ,  $P(X \leq 4.5)$  and  $P(2.5 \leq X \leq 5)$ .

(a)  $P(X \geq 2)$ : Solution from the geometrical point of view:

$$\begin{aligned} P(X \geq 2) &= 0.5 \text{ triangle area} + \text{rectangle area} \\ &= 0.5 \cdot 0.5 + 0.5 \\ &= 0.75 \end{aligned}$$

Solution with density:

$$\begin{aligned} \int_2^3 (1.5 - 0.5x) \, dx + \int_4^6 0.25 \, dx &= \left[ 1.5x - \frac{0.5x^2}{2} + c \right]_2^3 + [0.25x + c]_4^6 \\ &= 0.25 + 0.5 = 0.75 \end{aligned}$$

(b)  $P(X \leq 4.5)$ : Solution from the geometrical point of view:

$$\begin{aligned} P(X \leq 4.5) &= \text{triangle area} + 0.25 \text{ rectangle area} \\ &= 0.5 + 0.25 \cdot 0.5 \\ &= 0.625 \end{aligned}$$

Solution with density:

$$\begin{aligned} P(X \leq 4.5) &= 1 - P(X > 4.5) = 1 - \int_{4.5}^6 0.25 \, dx \\ &= 1 - [0.25x + c]_{4.5}^6 = 1 - 0.375 = 0.625 \end{aligned}$$

- (c)  $P(2.5 \leq X < 5)$ : Here it is necessary to integrate the first part  $P(2.5 \leq X < 3)$  of the probability because the cut at 2.5 is placed in the middle of the lowering of the triangle.

$$\begin{aligned}
 P(2.5 \leq X < 5) &= \int_{2.5}^3 (1.5 - 0.5x) dx + 0.5 \text{ rectangle area} \\
 &= \left[ 1.5x - \frac{0.5x^2}{2} + c \right]_{2.5}^3 + 0.5 \cdot 0.5 \\
 &= 1.5 \cdot 3 - \frac{0.5 \cdot 3^2}{2} - 1.5 \cdot 2.5 + \frac{0.5 \cdot 2.5^2}{2} + 0.5 \cdot 0.5 \\
 &= 0.3125
 \end{aligned}$$

## Exercise 2:

Let  $T$  be the random variable representing the waiting time of the bus, with  $T \sim U(0, 30)$ . Its density function is therefore

$$f_T(t) = \begin{cases} \frac{1}{30} & \text{if } 0 < t < 30 \\ 0 & \text{if not} \end{cases}$$

1. The shape of the density implies that the distribution function is a piecewise function of form

$$F_T(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ F_{T,1}(t) & \text{if } 0 < t < 30 \\ 1 & \text{if } t \geq 30 \end{cases}$$

By applying the definition  $F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du$ , we get  $F_{T,1}(t)$  :

$$F_{T,1}(t) = \int_0^t \frac{1}{30} du = \left[ \frac{u}{30} + c \right]_0^t = \frac{t}{30}.$$

2. The probability of waiting more than 10 minutes is given by  $P(T > 10)$ , which can be calculated using the distribution function:

$$P(T > 10) = 1 - P(T \leq 10) = 1 - F(10) = 1 - \frac{10}{30} = \frac{2}{3} = 0.\bar{6}.$$

3. We search for the conditional probability  $P(T > 25 \mid T > 15)$ . So

$$P(T > 25 \mid T > 15) = \frac{P(T > 25 \cap T > 15)}{P(T > 15)} = \frac{P(T > 25)}{P(T > 15)} = \frac{1 - P(T \leq 25)}{1 - P(T \leq 15)} = \frac{1 - 25/30}{1 - 15/30} = \frac{1}{3} = 0.\bar{3}.$$

### Exercise 3:

1. The random variable  $Z$  takes values between 0 and 1, as  $P(Z \leq 1) = 1$  so we have  $F(1) = 1$ . Thus:

$$\begin{aligned} F(1) &= k \cdot \left( \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = 1 \\ k \cdot \left( \frac{10 - 15 + 6}{30} \right) &= 1 \end{aligned}$$

and so  $k = 30$ .

The density function  $f(z)$  is the derivative of the distribution function, so

$$\begin{aligned} f(z) &= F'(z) \\ &= 30 \left( \frac{z^3}{3} - \frac{z^4}{2} + \frac{z^5}{5} \right)' \\ &= 30 \left( \frac{3z^2}{3} - \frac{4z^3}{2} + \frac{5z^4}{5} \right) \\ &= 30 (z^2 - 2z^3 + z^4) \\ &= 30z^2(1 - z)^2 \end{aligned}$$

for  $0 \leq z \leq 1$ .

2. We calculate the expectation of  $Z$  using the density found above:

$$\begin{aligned} E(Z) &= \int_0^1 z \cdot f(z) dz \\ &= \int_0^1 z \cdot 30(z^2 - 2z^3 + z^4) dz \\ &= 30 \int_0^1 (z^3 - 2z^4 + z^5) dz \\ &= 30 \left[ \frac{z^4}{4} - \frac{2z^5}{5} + \frac{z^6}{6} + c \right]_0^1 \\ &= 30 \left( \frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) = 30 \cdot \frac{25 - 24 + 10}{60} \\ &= 0.5. \end{aligned}$$

The variance of  $Z$  is given by the formula  $\text{var}(Z) = E(Z^2) - E(Z)^2$ .

We calculate first  $E(Z^2)$ :

$$\begin{aligned} E(Z^2) &= \int_0^1 z^2 \cdot f(z) dz \\ &= \int_0^1 z^2 \cdot 30(z^2 - 2z^3 + z^4) dz \\ &= 30 \int_0^1 (z^4 - 2z^5 + z^6) dz \\ &= 30 \left[ \frac{z^5}{5} + \frac{2z^6}{6} + \frac{z^7}{7} + c \right]_0^1 \\ &= 30 \cdot \frac{21 - 35 + 15}{105} = \frac{2}{7}. \end{aligned}$$

Then

$$\begin{aligned} \text{var}(Z) &= E(Z^2) - E(Z)^2 \\ &= \frac{2}{7} - \left(\frac{1}{2}\right)^2 \\ &= \frac{8 - 7}{28} \approx 0.0357. \end{aligned}$$

3. We calculate  $P(0.75 < Z < 1.5)$ :

$$\begin{aligned} P(0.75 < Z < 1.5) &= P(Z \leq 1.5) - P(Z \leq 0.75) \\ &= 1 - F(0.75) \\ &= 1 - 30 \cdot 0.75^3 \cdot \left( \frac{1}{3} - \frac{0.75}{2} + \frac{0.75^2}{5} \right) \\ &\approx 0.104; \end{aligned}$$

and  $P(Z \geq 0.15)$ :

$$\begin{aligned} P(Z \geq 0.15) &= 1 - P(Z < 0.15) \\ &= 1 - F(0.15) \\ &= 1 - 30 \cdot 0.15^3 \cdot \left( \frac{1}{3} - \frac{0.15}{2} + \frac{0.15^2}{5} \right) \\ &\approx 0.973. \end{aligned}$$

4. The probability that  $Z$  is greater than 0.5 in the case  $Z > 0.25$  is given by

$$P(Z \geq 0.5 \mid Z > 0.25).$$

We then calculate  $P(Z \geq 0.5 \mid Z > 0.25)$ :

$$\begin{aligned}
 P(Z \geq 0.5 \mid Z > 0.25) &= \frac{P(Z \geq 0.5 \cap Z > 0.25)}{P(Z > 0.25)} \\
 &= \frac{P(Z \geq 0.5)}{P(Z > 0.25)} \\
 &= \frac{1 - P(Z < 0.5)}{1 - P(Z \leq 0.25)} \\
 &= \frac{1 - F(0.5)}{1 - F(0.25)} \\
 &= \frac{1 - 30 \cdot 0.5^3 \left( \frac{1}{3} - \frac{0.5}{2} + \frac{0.5^2}{5} \right)}{1 - 30 \cdot 0.25^3 \left( \frac{1}{3} - \frac{0.25}{2} + \frac{0.25^2}{5} \right)} \\
 &\approx 0.558.
 \end{aligned}$$

## Exercise 4:

1. The cumulative distribution of  $X$  for  $x \geq 0$  is:

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = -e^{-\lambda t} \Big|_0^x = 1 - e^{-\lambda x},$$

So:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

2. The alpha quantile satisfies :

$$\begin{aligned}
 F(q_\alpha) &= \alpha \\
 1 - e^{-\lambda q_\alpha} &= \alpha \\
 q_\alpha &= \frac{\log(1 - \alpha)}{-\lambda}
 \end{aligned}$$

3. First we should find  $\lambda$  such that the  $E(X) = 8$ .

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x} dx$$

Integrating by part:

$$\begin{aligned}
 E(X) &= -xe^{-\lambda x} \Big|_0^\infty - \int_0^\infty -e^{-\lambda x} dx \\
 &= -xe^{-\lambda x} \Big|_0^\infty - \frac{1}{\lambda} e^{-\lambda x} \Big|_0^\infty \\
 &= \frac{1}{\lambda}
 \end{aligned}$$

Then  $E(X) = \frac{1}{\lambda} = 8$  implies that  $\lambda = \frac{1}{8}$ . We can compute then the probability that lifetime of the television is more than 8 years:

$$P(X > 8) = 1 - P(X \leq 8) = 1 - (1 - e^{-\frac{1}{8}8}) = e^{-1} \simeq 0.37.$$

The median is  $q_{\frac{1}{2}}$ :

$$q_{\frac{1}{2}} = \frac{\log(1 - \frac{1}{2})}{-\lambda} = 8 \log(2) \simeq 5.54$$

4. Let's first compute  $E(X^2)$ :

$$\begin{aligned} E(X^2) &= \int_0^\infty \frac{x^2}{8} e^{-\frac{x}{8}} dx = -x^2 e^{-\frac{x}{8}} \Big|_0^\infty - \int_0^\infty -2x e^{-\frac{x}{8}} dx \\ &= -x^2 e^{-\frac{x}{8}} \Big|_0^\infty + 2 \cdot 8 \int_0^\infty \frac{x}{8} e^{-\frac{x}{8}} dx = 0 + 2 \cdot 8 E(X) \\ &= 2 \cdot 8^2 = 128 \end{aligned}$$

Then we can compute the variance:

$$Var(X) = E(X^2) - E(X)^2 = 128 - 8^2 = 64 = E(X)^2$$

## Exercise 5:

1. To determine  $C$  and  $\lambda$  we will use:

$$\begin{aligned} \int_0^\infty C v \exp(-\lambda v) dv &= 1 \\ \text{and} \quad E(V) &= \int_0^\infty C v^2 \exp(-\lambda v) dv = 30. \end{aligned}$$

(a) Let's integrate by part:

$$\begin{aligned} \int_0^\infty C v \exp(-\lambda v) dv &= \frac{-1}{\lambda} C v \exp(-\lambda v) \Big|_0^\infty + C \int_0^\infty \frac{1}{\lambda} \exp(-\lambda v) dv \\ &= 0 - \frac{C}{\lambda^2} \exp(-\lambda v) \Big|_0^\infty = \frac{C}{\lambda^2} = 1 \Leftrightarrow C = \lambda^2 \end{aligned}$$

(b)

$$\begin{aligned} C \int_0^\infty v^2 \exp(-\lambda v) dv &= \frac{-1}{\lambda} C v^2 \exp(-\lambda v) \Big|_0^\infty + C \int_0^\infty \frac{1}{\lambda} 2v \exp(-\lambda v) dv \\ &= \frac{2}{\lambda} \int_0^\infty C v \exp(-\lambda v) dv = \frac{2}{\lambda} = 30 \\ \Leftrightarrow \lambda &= \frac{1}{15} \Rightarrow C = \frac{1}{225} \end{aligned}$$



2. Let  $T = \frac{15}{V}$ . It's cumulative distribution function is:

$$F_T(t) = P(T < t) = P\left(\frac{15}{V} < t\right) = P\left(V > \frac{15}{t}\right) = 1 - F_V\left(\frac{15}{t}\right).$$

We deduce its density:

$$f_T(t) = \frac{15}{t^2} f_V\left(\frac{15}{t}\right) = \frac{1}{t^3} \exp\left(-\frac{1}{t}\right).$$

Finally we can compute the expectation of  $T$ :

$$\begin{aligned} E(T) &= \int_0^\infty t \frac{1}{t^3} \exp\left(-\frac{1}{t}\right) dt \\ &= \exp\left(-\frac{1}{t}\right) \Big|_{t=0}^{t=\infty} = 1 \end{aligned}$$