Exercise 1:

Let X be the random variable counting the number of times Charlie is inspected during a year. The definition of X implies that $X \sim Bin(700, 0.1)$.

1. We have $\mathrm{E}(X) = np = 70$ and $\mathrm{var}(X) = np(1-p) = 63$. We can then approximate the binomial distribution by a nomal: $X \sim_{appr} N(70, 63)$ and so:

$$\begin{aligned} P(60 \leq X \leq 100) &\simeq P(X \leq 100) - P(X \leq 60) \\ &= \Phi\left(\frac{100-70}{\sqrt{63}}\right) - \Phi\left(\frac{60-70}{\sqrt{63}}\right) \\ &\approx \Phi(3.78) - \Phi(-1.26) \\ &\approx 0.9999 - (1-0.8962) = 0.9999 - 0.1038 = 0.8961. \end{aligned}$$

2. If Charles always buys his ticket, he would spend $2 \cdot 700 = 1'400$ CHF for the year. If there is a conductor, Charles has to pay the penalty a. Let F be the financial result at the end of the year. So $F = 1400 - aX$. So

$$\mathrm{E}(F) = 1400 - aE(X) = 1400 - 70a$$

and

$$\mathrm{var}(F) = a^2 \cdot \mathrm{var}(X) = 63a^2.$$

To compute the value of the minimal penalty, we have to find a such that $P(F < 0) = 0.75$.

Using the normal approximation, $F \sim N(1400 - 70a, 63a^2)$:

$$\begin{aligned} P(F<0) &\approx \Phi\left(-\frac{1400-70a}{\sqrt{63a^2}}\right) \\ &= \Phi\left(-\frac{1400-70a}{\sqrt{63}a}\right). \end{aligned}$$

We have to solve:

$$\begin{aligned} -\frac{1400-70a}{\sqrt{63}a} &\approx \Phi^{-1}(0.75) \\ &\approx 0.67. \end{aligned}$$

So finally we have:

$$a \approx \frac{1400}{70 - 0.67 \cdot \sqrt{63}} \approx 21.64.$$

Exercise 2:

1. The event 'the user waits less than 5 minutes' is the same as the event 'the user arrives between 7h10 and 7h15 or between 7h25 and 7h30', which is itself the meeting of the two disjoint events $A =$ 'the user arrives between 7h10 and 7h15' and $B =$ ''the user arrives between 7h25 and 7h30'.

   Let X be a uniform random variable on $(0, 30)$, representing the number of minutes since 7h00 until the arrival of the user.

   We then

   $$\begin{aligned} P(\text{wait less than 5 minutes}) &= P(A \cup B) = P(A) + P(B) \\ &= P(10 < X < 15) + P(25 < X < 30). \end{aligned}$$

   As X follows a uniform law on $(0, 30)$,

   $$P(10 < X < 15) + P(25 < X < 30) = \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx = \frac{1}{3}.$$

2. The event 'the user waits more than 10 minutes' is the same as the event 'the user arrives between 7h00 and 7h05 or between 7h15 and 7h20', which is itself the meeting of the two disjoint events 'the user arrives between 7h00 and 7h05' and 'the user arrives between 7h15 and 7h20'.

   We then

   $$\begin{aligned} P(\text{`wait more than 10 minutes'}) &= P(0 < X < 5) + P(15 < X < 20) \\ &= \int_{0}^{5} \frac{1}{30} dx + \int_{15}^{20} \frac{1}{30} dx = \frac{1}{3}. \end{aligned}$$

Exercise 3:

1. Let T be the lifetime of a part of the car and H the event 'the part is pirated'. We know that $P(H) = 1/4$ and then $P(\bar{H}) = 3/4$. Moreover $T|H$ follows an exponential distribution with expectation 5 and $T|\bar{H}$ follows an exponential distribution with expectation 2. The formula of the cumulative distribution function of an exponential random variable X with expectation $1/\lambda$ is $P(X \leq x) = 1 - \exp(-\lambda x)$ and then $P(X > x) = \exp(-\lambda x)$. Knowing that the part has survived until time t, the probability that it has been pirated is:

   $$\begin{aligned} \pi(t) &= P(H|T > t) = \frac{P(T > t|H)P(H)}{P(T > t|H)P(H) + P(T > t|\bar{H})P(\bar{H})} \\ &= \frac{\exp(-\frac{1}{2}t) \cdot \frac{1}{4}}{\exp(-\frac{1}{2}t) \cdot \frac{1}{4} + \exp(-\frac{1}{5}t) \cdot \frac{3}{4}} \\ &= \frac{1}{1 + 3 \cdot \exp(\frac{3}{10}t)}. \end{aligned}$$

When t goes to infinity we have:

$$\lim_{t\to\infty} \pi(t) = 0.$$

2. Let's Y be a random variable taking values 1 if the part is pirated and 0 when it is not.

   (a) Y is a bernoulli with probability 1/4. Then $P(Y = y) = \left(\frac{1}{4}\right)^y \left(\frac{3}{4}\right)^{1-y}$.

   (b) We have $P(T \le t|Y = 1) = 1 - e^{-t/2}$ and $P(T \le t|Y = 0) = 1 - e^{-t/5}$. We can merge them in this way: $P(T \le t|Y = y) = 1 - e^{-t/(2y+5(1-y))}$.

   (c) We just have to multiply the conditional probability by $P(Y = y)$:

   $$P(T \le t \cap Y = y) = \left(1 - e^{-t/(2y+5(1-y))}\right) \left(\frac{1}{4}\right)^y \left(\frac{3}{4}\right)^{1-y}$$

   (d) We compute the cumulative distribution function of T.

   $$\begin{aligned} P(T \le t) &= \left(1 - e^{-t/5}\right) \frac{3}{4} + \left(1 - e^{-t/2}\right) \frac{1}{4} \\ &= 1 - e^{-t/5} \frac{3}{4} - e^{-t/2} \frac{1}{4} \end{aligned}$$

   (e) We can now find $P(Y = y|T \ge t)$:

   $$\begin{aligned} P(Y = y|T > t) &= \frac{P(T > t|Y = y)P(Y = y)}{P(T > t)} \\ &= e^{-t/(2y+5(1-y))} \frac{\left(\frac{1}{4}\right)^y \left(\frac{3}{4}\right)^{1-y}}{e^{-t/5}\frac{3}{4} + e^{-t/2}\frac{1}{4}} \end{aligned}$$

   We can then deduce $\pi(t)$:

   $$\begin{aligned} \pi(t) &= P(Y = 1|T > t) = e^{-t/2} \frac{\frac{1}{4}}{e^{-t/5}\frac{3}{4} + e^{-t/2}\frac{1}{4}} \\ &= \frac{1}{1 + 3 \cdot \exp(\frac{3}{10}t)} \end{aligned}$$

Exercise 4:

(a) We start by isolating $R_{0,n}$ from the equation of the statement:

$$R_{0,n} = \left(\frac{S_n}{S_0}\right)^{\frac{1}{n}} - 1.$$

We use the indication to obtain:

$$R_{0,n} = \left(\prod_{t=1}^{n} \frac{S_t}{S_{t-1}}\right)^{\frac{1}{n}} - 1.$$

(b) As $S_t/S_{t-1} \sim LN(\mu, \sigma^2)$ we have $\log(S_t/S_{t-1}) \sim N(\mu, \sigma^2)$.

It is known that a linear combination of random variables with normal distributions also follows a normal distribution. So we calculate the expectation of Y:

$$\mathrm{E}(Y) = \mathrm{E}\left[\frac{1}{n}\sum_{t=1}^{n} \log\left(\frac{S_t}{S_{t-1}}\right)\right] = \frac{1}{n}\sum_{t=1}^{n} \mathrm{E}\left[\log\left(\frac{S_t}{S_{t-1}}\right)\right] = \frac{1}{n}\sum_{t=1}^{n} \mu = \mu$$

and the variance of Y:

$$\mathrm{var}(Y) = \mathrm{var}\left[\frac{1}{n}\sum_{t=1}^{n} \log\left(\frac{S_t}{S_{t-1}}\right)\right] = \frac{1}{n^2}\sum_{t=1}^{n} \mathrm{var}\left[\log\left(\frac{S_t}{S_{t-1}}\right)\right] = \frac{1}{n^2}\sum_{t=1}^{n} \sigma^2 = \frac{\sigma^2}{n}.$$

So we get that

$$Y = \frac{1}{n}\sum_{t=1}^{n} \log\left(\frac{S_t}{S_{t-1}}\right) \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

(c) Using the formula of point 1 we have that

$$R_{0,n} = \left(\prod_{t=1}^{n} \frac{S_t}{S_{t-1}}\right)^{\frac{1}{n}} - 1 = \exp\left[\log\left(\prod_{t=1}^{n} \frac{S_t}{S_{t-1}}\right)^{\frac{1}{n}}\right] - 1 = \exp\left[\frac{1}{n}\sum_{t=1}^{n} \log\left(\frac{S_t}{S_{t-1}}\right)\right] - 1$$

and so

$$R_{0,n} = \exp(Y) - 1.$$

We know that $\exp(Y)$ follows a log-normal distribution with parameters μ and σ^2/n. Applying the expectation formula of the log-normal distribution, we find the expectation of $R_{0,n}$:

$$\mathrm{E}(R_{0,n}) = \mathrm{E}[\exp(Y)] - 1 = \exp\left(\mu + \frac{\sigma^2}{2n}\right) - 1$$

and the variance $R_{0,n}$:

$$\mathrm{var}(R_{0,n}) = \mathrm{var}[\exp(Y)] = \exp\left(2\mu + \frac{\sigma^2}{n}\right)\left[\exp\left(\frac{\sigma^2}{n}\right) - 1\right].$$

(d) When $n \to \infty$, $\mathrm{E}(R_{0,n}) = \exp(\mu) - 1$ and $\mathrm{var}(R_{0,n}) = \exp(2\mu) \cdot (1 - 1) = 0$.

Exercise 5 (Optional):

$V \sim \mathcal{U}(0,1)$. The density function of V: $f_V(v) = 1, when\ v \in (0,1)$. CDF of V is:

$$F_V(v) = \begin{cases} 0 & \text{if } v \leq 0, \\ v & \text{if } 0 < v < 1 \\ 1 & \text{if } v \geq 1. \end{cases}$$

(a) Let's denote $F_W(w)$ its density.

$$\begin{aligned} 1.1, \quad \text{if} \quad \lambda > 0, F_W(w) &= P(W \leq w) = P(\frac{-1}{\lambda}\log(V) \leq w) = P(V > \exp(-\lambda w)) \\ &= 1 - F_V(\exp(-\lambda w)) = 1 - \exp(-\lambda w), \end{aligned}$$

The random variable follows an exponential distribution with density:

$$f_W(w) = \lambda \exp(-\lambda w), \ \text{for } w > 0.$$

$$\begin{aligned} 1.2, \quad \text{if} \quad \lambda < 0, F_W(w) &= P(W \leq w) = P(\frac{-1}{\lambda}\log(V) \leq w) = P(V < \exp(-\lambda w)) \\ &= F_V(\exp(-\lambda w)) = \exp(-\lambda w), \end{aligned}$$

$$f_W(w) = -\lambda \exp(-\lambda w), \ \text{for } w < 0$$

(b) V is uniformly distributed. Let's compute the cumulative distribution of X:

$$P(X \leq x) = P(F^{-1}(V) \leq x) = P(V \leq F(x)) = F_V(F(x)) = F(x).$$

(c) Solution 1: From point 2. we know that $g(x) = F^{-1}(x)$. So we just have to invert the cumulative distribution F:

$$\begin{aligned} F(x) &= v \\ \frac{1}{1+\frac{\lambda}{x^2}} &= v \\ 1 &= v\left(1 + \frac{\lambda}{x^2}\right) \\ \frac{1}{v} - 1 &= \frac{\lambda}{x^2} \\ x &= \sqrt{\frac{\lambda}{\frac{1}{v} - 1}} \end{aligned}$$

The transformed random variable $g(V) = \sqrt{\frac{\lambda}{\frac{1}{V}-1}}$ has then a Dagum distribution.

Solution 2: Let $Y = g(V)$, show that $Y \sim$Dagum distri.
i) suppose $g(V)$ is a 1-to-1 and increasing function.

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(g(V) \leq y) = P(V \leq g^{-1}(y)) \\
&= F(g^{-1}(y)) = g^{-1}(y) = v \\
&= (1 + \lambda y^{-2})^{-1} \\
y &= \sqrt{\frac{\lambda}{\frac{1}{v} - 1}} \; (0 < v < 1) \; \Rightarrow \; Y = g(V) = \sqrt{\frac{\lambda}{\frac{1}{V} - 1}}
\end{aligned}$$

ii) suppose $g(V)$ is a 1-to-1 and decreasing function.

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(g(V) \leq y) = P(V \geq g^{-1}(y)) \\
&= 1 - F(g^{-1}(y)) = 1 - g^{-1}(y) = 1 - v \\
&= (1 + \lambda y^{-2})^{-1} \\
y &= \sqrt{\frac{\lambda}{\frac{1}{1-v} - 1}} \; (0 < v < 1) \; \Rightarrow \; Y = g(V) = \sqrt{\frac{\lambda}{\frac{1}{1-V} - 1}}
\end{aligned}$$