

# Probability 1

## Lecture 03 : Probability Axioms

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(based on the notes of Prof. Davide La Vecchia)

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- Provide the Axioms upon which the Probability Theory is built.
- Explore some rules that result from the Axioms
- Illustrate how these rules can be used to compute probabilities “in real life”
- Introduce the concepts of Independence and Conditional Probability
- Present two important Theorems and illustrate their consequences.

- 1 An Axiomatic Definition of Probability
- 2 Implications: Properties of  $P(\cdot)$
- 3 Examples and Illustrations
- 4 Conditional Probability
- 5 Independence
- 6 Theorem I: The Theorem of Total Probability
- 7 Theorem II: Bayes' Theorem

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# An Axiomatic Definition of Probability

## Definition

We define probability a set function with values in  $[0, 1]$ , which satisfies the following axioms:

- (i)  $P(A) \geq 0$ , for every event  $A$
- (ii)  $P(S) = 1$
- (iii) If  $A_1, A_2, \dots$  is a sequence of mutually exclusive events (namely  $A_i \cap A_j = \emptyset$ , for  $i \neq j$ , and  $i, j = 1, 2, \dots$ ), and such that  $A = \bigcup_{i=1}^{\infty} A_i$ , then

$$P(A) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i). \quad (1)$$

# Outline

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# Implications: Properties of $P(\cdot)$

We can use the three axioms to build more sophisticated statements, e.g.

- The Probability of the Empty Set
- The Addition Law of Probability
- The Complement Rule
- The Monotonicity Rule
- The Probability of the Union
- Boole's Inequality

# Implications: Properties of $P(\cdot)$

## Theorem (Probability of the Empty Set)

$$P(\emptyset) = 0$$

### Proof.

Take  $A_1 = A_2 = A_3 = \dots = \emptyset$ . Then by (1) in Axiom (iii) we have

$$P(\emptyset) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(\emptyset)$$

which is true only if it is an infinite sum of zeros. Thus

$$P(\emptyset) = 0.$$





# Implications: Properties of $P(\cdot)$

## Theorem (The Addition Law of Probability)

If  $A_1, A_2, \dots$  are mutually exclusive events, then

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i). \quad (2)$$

## Proof.

Let  $A_{n+1} = A_{n+2} = \dots = \emptyset$ , then  $\bigcup_{i=1}^n A_i = \bigcup_{i=1}^{\infty} A_i$ , and, from (1) (see Axiom (iii)) it follows

$$P\left(\bigcup_{i=1}^n A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^n P(A_i) + \underbrace{\sum_{i=n+1}^{\infty} P(A_i)}_{\equiv 0}.$$



# Implications: Properties of $P(\cdot)$

## Theorem (The Complement Rule)

*If  $A$  is an event, then  $P(A^c) = 1 - P(A)$ .*

## Proof.

Let  $A \cup A^c = S$  and  $A \cap A^c = \emptyset$ , so and

$$P(S) = P(A \cup A^c) = P(A) + P(A^c).$$

By Axiom (i) Axiom (ii) we have  $P(S) = 1$ , so the desired result follows from:

$$1 = P(A) + P(A^c).$$



# Implications: Properties of $P(\cdot)$

## Theorem (The Monotonicity Rule)

*For any two events  $A$  and  $B$ , such that  $B \subset A$ , we have*

$$P(A) \geq P(B).$$

## Proof.

Let us write

$$A = B \cup (B^c \cap A)$$

and notice that  $B \cap (B^c \cap A) = \phi$ , so that

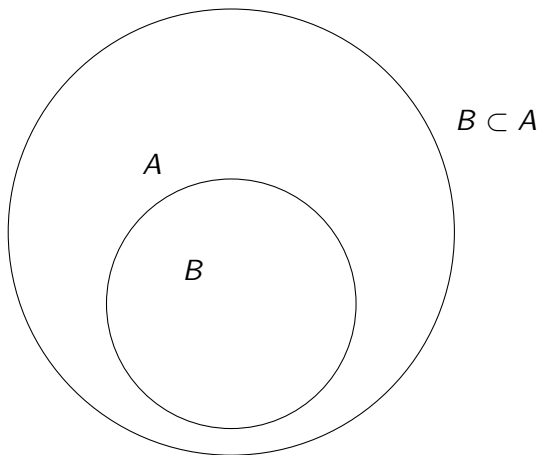
$$\begin{aligned} P(A) &= P\{B \cup (B^c \cap A)\} \\ &= P(B) + P(B^c \cap A) \end{aligned}$$

which implies (since  $P(B^c \cap A) \geq 0$ ) that

$$P(A) \geq P(B).$$



# Implications: Properties of $P(\cdot)$



# Implications: Properties of $P(\cdot)$

## Theorem (Probability of the Union of Two Events)

*For any two events  $A$  and  $B$  then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

## Proof.

Consider that  $A \cup B = A \cup (A^c \cap B)$ , and  $A \cap (A^c \cap B) = \phi$ . Now remember<sup>a</sup> that  $A^c \cap B = B - (A \cap B)$ , so,

$$\begin{aligned} P(A \cup B) &= P(A) + P(A^c \cap B) \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$



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<sup>a</sup>See Lecture 1 for the meaning of set difference.

# Implications: Properties of $P(\cdot)$

## Theorem (Boole's inequality)

For the events  $A_1, A_2, \dots, A_n$ ,

$$P(A_1 \cup A_2 \cup \dots \cup A_n) \leq \sum_{i=1}^n P(A_i).$$

For instance, let us consider  $n = 2$ . Then we have:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2)$$

since  $P(A_1 \cap A_2) \geq 0$  by definition.

## Remark

*It is worth notice that if  $A_j \cap A_i = \emptyset$ , for every  $i$  and  $j$ , with  $i \neq j$ , then  $P(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{i=1}^n P(A_i)$ , as stated in (2).*

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## Example (Flipping a coin twice)

If we flip a balanced coin twice, what is the probability of getting at least one head?

- The sample space is:  $S = \{HH, HT, TH, TT\}$
- Balanced coin: outcomes are equally likely and  $p_{HH} = p_{HT} = p_{TH} = p_{TT} = p = 1/4$
- Let  $A$  denote the event **obtaining at least one Head**, i.e.  $H = \{HH, HT, TH\}$

$$\begin{aligned} Pr(A) &= Pr(\{HH \cup HT \cup TH\}) \\ &= Pr(\{HH\}) + Pr(\{HT\}) + Pr(\{TH\}) \\ &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} \end{aligned}$$



## Example (Detecting Shoppers)

*Shopper TRK is an electronic device that counts the number of shoppers entering a shopping centre.*

*Two shoppers enter the shopping centre together, one walking in front of the other:*

- 1. There is a 0.98 probability that the first shopper is detected*
- 2. There is a 0.94 probability that the second shopper is detected.*
- 3. There is a 0.93 probability that both shoppers are detected.*

**What is the probability that the device will detect at least one of the two shoppers entering?**

## Example (Detecting Shoppers, continued)

Define the **events**:

- $D$  (shopper is detected) and
- $U$  (shopper is undetected).

Then, the Sample Space is  $S = \{DD, DU, UD, UU\}$

Define the **probabilities**:

- $Pr(DD \cup DU) = 0.98$
- $Pr(DD \cup UD) = 0.94$
- $Pr(DD) = 0.93$

## Example (Detecting Shoppers, continued)

$$\begin{aligned}Pr(DD \cup UD \cup DU) &= Pr(\{DD \cup UD\} \cup \{DD \cup DU\}) \\&= Pr(\{DD \cup UD\}) + Pr(\{DD \cup DU\}) \\&\quad - Pr(\{DD \cup UD\} \cap \{DD \cup DU\})\end{aligned}$$

The only missing probability is :  $P\{DD \cup UD\} \cap \{DD \cup DU\} = ?$

To compute it, let's use the fact that **the union is distributive with respect to the intersection** (see Ch 2):

$$(DD \cup UD) \cap (DD \cup DU) = DD \cup (UD \cap DU) = DD \cup \emptyset = DD$$

## Example (Detecting Shoppers, continued)

Graphically:

DD	DU
UD	UU

So, the desired probability is:

$$\begin{aligned}Pr(DD \cup UD \cup DU) &= Pr(\{DD \cup UD\}) + Pr(\{DD \cup DU\}) - Pr(DD) \\&= 0.98 + 0.94 - 0.93 \\&= 0.99\end{aligned}$$

## Example (De Morgan's law)

Given  $P(A \cup B) = 0.7$  and  $P(A \cup B^c) = 0.9$ , find  $P(A)$ .

By De Morgan's law,

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.7 = 0.3$$

and similarly

$$P(A^c \cap B) = 1 - P(A \cup B^c) = 1 - 0.9 = 0.1.$$

Thus,

$$P(A^c) = P(A^c \cap B^c) + P(A^c \cap B) = 0.3 + 0.1 = 0.4,$$

so

$$P(A) = 1 - 0.4 = 0.6.$$

## Example (Probability, union, and complement)

*John is taking two books along on his holiday vacation. With probability 0.5, he will like the first book; with probability 0.4, he will like the second book; and with probability 0.3, he will like both books. What is the probability that he likes neither book?*

Let  $A_i$  be the event that John likes book  $i$ , for  $i = 1, 2$ . Then the probability that he likes at least one book is:

$$\begin{aligned} P\left(\bigcup_{i=1}^2 A_i\right) &= P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \\ &= 0.5 + 0.4 - 0.3 = 0.6. \end{aligned}$$

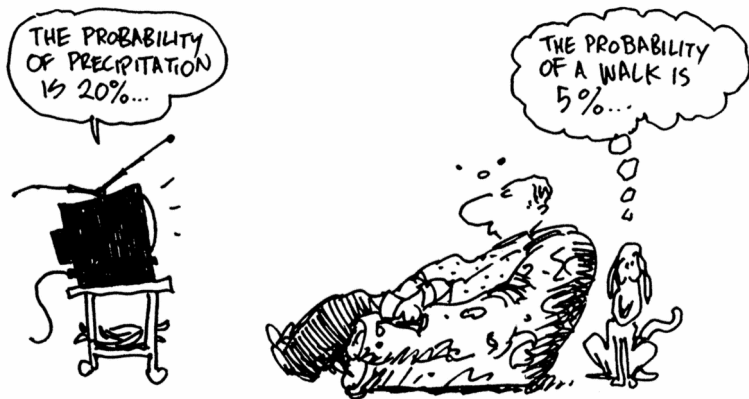
Because the event the John likes neither books is the complement of the event that he likes at least one of them (namely  $A_1 \cup A_2$ ), we have

$$P(A_1^c \cap A_2^c) = P((A_1 \cup A_2)^c) = 1 - P(A_1 \cup A_2) = 0.4.$$

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# Conditional Probability





# Conditional Probability

Sometimes, probability depends on the information available.

## Example

Suppose you have two dice and throw them; the possible outcomes are:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(6,5)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

# Conditional Probability

## Example (cont'd)

Consider the event  $A = \text{getting } 5$ , or equivalently  $A = \{5\}$ .  
What is  $P(A)$ , namely, the probability of getting 5?

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

## Example (cont'd)

The dice are fair so we can get 36 events with equal probability  $1/36$ .  
Namely:

$$Pr(i, j) = \frac{1}{36}, \quad \text{for } i, j = 1, \dots, 6$$

Thus, we can make use of the highlighted probabilities

$$\begin{aligned} P(5) &= Pr\{(1, 4) \cup (2, 3) \cup (3, 2) \cup (4, 1)\} \\ &= Pr\{(1, 4)\} + Pr\{(2, 3)\} + Pr\{(3, 2)\} + Pr\{(4, 1)\} \\ &= 1/36 + 1/36 + 1/36 + 1/36 \\ &= 4/36 \\ &= 1/9. \end{aligned}$$

# Conditional Probability

## Example (cont'd)

Now, suppose that we throw the die first and we get 2.

**What is the probability of getting 5 given that we have observed 2 in the first throw?**

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$$\Pr\{\text{getting 5 given 2 in the first throw}\} = \Pr\{\text{getting 3 in the second throw}\} = 1/6$$

# Conditional Probability

## Remark

- *Information changes the probability*
- *By knowing that we got 2 in the first throw, we have changed the sample space:*

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(6,5)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- *Probability can change drastically — e.g., suppose that in our example we have 6 in the first throw  $\Rightarrow$  the probability of observing 5 in two draws ... is zero !!!*

## Definition

Let  $A$  and  $B$  be two events. The conditional probability of event  $A$  given event  $B$ , denoted by  $P(A|B)$ , is defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad \text{if } P(B) > 0,$$

and it is left undefined if  $P(B) = 0$ .

# Conditional Probability

## Example (A check)

Let us define the set  $B$  as:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(6,5)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

So we have

$$\begin{aligned}P(B) &= Pr\{(2,1) \cup (2,2) \cup (2,3) \cup (2,4) \cup (2,5) \cup (2,6)\} \\&= Pr(2,1) + Pr(2,2) + Pr(2,3) + Pr(2,4) + Pr(2,5) + Pr(2,6) \\&= 6/36 = 1/6\end{aligned}$$

# Conditional Probability

## Example (A check)

and consider  $A \cap B$

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

So we have  $P(A \cap B) = Pr(2, 3) = 1/36$  so,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{1/6} = \frac{1}{6}.$$



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## Definition

Two events  $A$  and  $B$  are independent if the occurrence of one event has no effect on the probability of occurrence of the other event. Thus,

$$P(A|B) = P(A)$$

or equivalently

$$P(B|A) = P(B)$$

Clearly, if  $P(A|B) \neq P(A)$ , then  $A$  and  $B$  are *dependent*.

# Independence

## A second characterisation

Two events  $A$  and  $B$  are independent if

$$P(A|B) = P(A),$$

now by definition of conditional probability we know that

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

so we have

$$P(A) = \frac{P(A \cap B)}{P(B)},$$

and rearranging the terms, we find that two events are independent iif

$$P(A \cap B) = P(A)P(B).$$

## Example

A coin is tossed three times and the eight possible outcomes  
HHH, HHT, HTH, THH, HTT, THT, TTH, TTT  
are assumed to be equally likely with probability  $1/8$

Define

$A$  = an H occurs on each of the first two tosses

$B$  = T occurs in the third toss

$D$  = two Ts occur in the three tosses

**Q1:** Are  $A$  and  $B$  independent?

**Q2:** Are  $B$  and  $D$  independent?

## Example (cont'd)

We have

$$A = \{HHH, HHT\}$$

$$\Pr\{A\} = 2/8 = 1/4$$

$$B = \{HHT, HTT, THT, TTT\}$$

$$\Pr\{B\} = 4/8 = 1/2$$

$$D = \{HTT, THT, TTH\}$$

$$\Pr\{D\} = 3/8$$

$$A \cap B = \{HHT\}$$

$$\Pr\{A \cap B\} = 1/8$$

$$B \cap D = \{HTT, THT\}$$

$$\Pr\{B \cap D\} = 2/8 = 1/4$$

$$\Pr\{A\}\Pr\{B\} = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} \quad \Pr\{A \cap B\} = \frac{1}{8} \Rightarrow A, B \text{ independent}$$

$$\Pr\{B\}\Pr\{D\} = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16} \quad \Pr\{B \cap D\} = \frac{1}{4} \Rightarrow B \text{ and } D \text{ dependent}$$

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# Theorem I: The Theorem of Total Probability

## Theorem (Total Probabilities)

Let  $B_1, B_2, \dots, B_k, \dots, B_n$  be mutually disjoint events, satisfying  $S = \cup_{i=1}^n B_i$ , and  $P(B_i) > 0$ , for every  $i = 1, 2, \dots, n$  then for every  $A$  we have that:

$$P(A) = \sum_{i=1}^n P(A|B_i)P(B_i). \quad (3)$$

## Proof.

Write  $A = A \cap S = A \cap (\cup_{i=1}^n B_i) = \cup_{i=1}^n (A \cap B_i)$ . Since the  $\{B_i \cap A\}$  are mutually disjoint, we have

$$P(A) = P(\cup_{i=1}^n (A \cap B_i)) = \sum_{i=1}^n P(A \cap B_i) = \sum_{i=1}^n P(A|B_i)P(B_i).$$



# Theorem I: The Theorem of Total Probability

**Remark.** The theorem remains valid even if  $n = \infty$  in Eq. (3).

## Corollary

*Let  $B$  satisfy  $0 < P(B) < 1$ ; then for every event  $A$ :*

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

## Proof.

Exercise [Hint:  $S = B \cup B^c$ ].





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## Theorem II: Bayes' Theorem

Theorem I can be applied to derive the well-celebrated Bayes' Theorem.

### Theorem (Bayes' Theorem)

*Let  $B_1, B_2, \dots, B_k, \dots, B_n$  be mutually disjoint events, satisfying*

$$S = \cup_{i=1}^n B_i,$$

*and  $P(B_i) > 0$ , for every  $i = 1, 2, \dots, n$ . Then for every event  $A$  for which  $P(A) > 0$ , we have that*

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^n P(A|B_i)P(B_i)}. \quad (4)$$

## Theorem II: Bayes' Theorem

Proof.

Let us write

$$\begin{aligned}P(B_k|A) &= \frac{P(A \cap B_k)}{P(A)} \\&= \frac{P(A \cap B_k)}{\sum_{i=1}^n P(A|B_i)P(B_i)} \\&= \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^n P(A|B_i)P(B_i)}\end{aligned}$$

That concludes the proof. □

# Theorem II: Bayes' Theorem

## Example

Let us consider a special case, where we have only two events  $A$  and  $B$ .

From the definition of conditional probability

$$\Pr\{A|B\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}} \qquad \Pr\{B|A\} = \frac{\Pr\{A \cap B\}}{\Pr\{A\}}$$

This can be written as

$$\Pr\{A \cap B\} = \Pr\{A|B\} \Pr\{B\} \qquad \Pr\{A \cap B\} = \Pr\{B|A\} \Pr\{A\}$$

That is

$$\Pr\{A|B\} \Pr\{B\} = \Pr\{B|A\} \Pr\{A\}$$

Rearranging this we have

$$\Pr\{A|B\} = \frac{\Pr\{A\}}{\Pr\{B\}} \times \Pr\{B|A\} \quad \Leftarrow \text{Bayes' Theorem}$$

... so thanks to Bayes' Theorem we can reverse the role of  $A|B$  and  $B|A$ .

# Theorem II: Bayes' Theorem

## Application

### Example (Guessing in a multiple choice exam)

**Example 1.13** In answering a question on a multiple choice test a student either knows the answer or he guesses. Let  $p$  be the probability that she knows the answer and  $1 - p$  the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability  $1/m$ , where  $m$  is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?

**Solution:** Let  $C$  and  $K$  denote respectively the event that the student answers the question correctly and the event that she actually knows the answer. Now

$$\begin{aligned}P(K|C) &= \frac{P(KC)}{P(C)} = \frac{P(C|K)P(K)}{P(C|K)P(K) + P(C|K^c)P(K^c)} \\&= \frac{p}{p + (1/m)(1 - p)} \\&= \frac{mp}{1 + (m - 1)p}\end{aligned}$$

Thus, for example, if  $m = 5$ ,  $p = \frac{1}{2}$ , then the probability that a student knew the answer to a question she correctly answered is  $\frac{5}{6}$ . ♦

# Theorem II: Bayes' Theorem

## Application

### Example

Members of a consulting firm in Geneva rent cars from two rental agencies:

- 60% from AVIS
- 40% from Mobility

Now consider that

- 9% of the cars from AVIS need a tune-up
- 20% of the cars from Mobility need a tune-up

If a car delivered to the consulting firm needs a tune-up, what is the probability that the care came from AVIS?

### Aim

$A := \{\text{car rented from AVIS}\}$  and  $B := \{\text{car needs a tune-up}\}$ . We know  $P(B|A)$  and we look for  $P(A|B) \Rightarrow$  Bayes' theorem!!

# Theorem II: Bayes' Theorem

## Application

### Example (cont'd)

$$\Pr\{A\}=0.6$$

$$\Pr\{B|A\}=0.09$$

$$\Pr\{B|A^C\}=0.2$$

$$\Pr\{B\}=\Pr\{(B\cap A)\cup(B\cap A^C)\}$$

$$=\Pr\{B\cap A\}+\Pr\{B\cap A^C\}$$

$$=\Pr\{B|A\}\Pr\{A\}+\Pr\{B|A^C\}\Pr\{A^C\}$$

$$=0.09\times 0.6+0.2\times 0.4$$

$$=0.134$$

$$\Pr\{A|B\}=\frac{\Pr\{A\}}{\Pr\{B\}}\times\Pr\{B|A\}=\frac{0.6}{0.134}0.09=0.402985$$

## Take-home message

- The Probability Axioms are important, since they imply **computation rules**.
- These rules are key to compute probabilities “in real life”.
- Probability of an event can be **conditional** on the realisation of another event.
- When an event has the **same probability** regardless of another it is called independent
- Theorem I (basic)

$$P(A) = P(A \cap B) + P(A \cap B^c) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

- Theorem II (basic) If we have  $P(A|B)$ , we can find  $P(B|A)$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$



Thank you for your attention!  
“See you” next week!