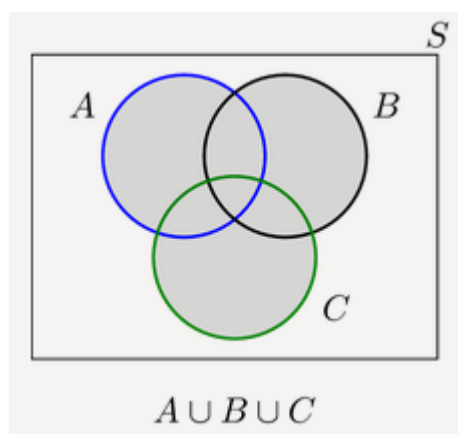


Exercise 1

- Let A , B and C be three sets as shown in the following Venn diagram.



For each of the following sets, draw a Venn diagram and shade the area representing the given set.

- $A \cup B \cup C$
 - $A \cap B \cap C$
 - $A \cup (B \cap C)$
 - $A - (B \cap C)$
 - $A \cup (B \cap C)^c$
- Using the Venn Diagrams, verify the following identities.
 - Transitive Property: If $A \subset B$ and $B \subset C$, then $A \subset C$.
 - Distributive Property: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
 - De Morgan's Laws: $(A \cap B)^c = A^c \cup B^c$, $(A \cup B)^c = A^c \cap B^c$

d) $A = (A \cap B) \cup (A - B)$

e) If A and B are finite sets, we have $|A \cup B| = |A| + |B| - |A \cap B|$

Note that the symbol $|\cdot|$ indicates the cardinality, i.e. the measure of the “number of elements of the set”.

3. Determine whether each of the following sets is countable or uncountable:

a) $A = \{(x, y) | x \in N, y \in Z\}$

b) $B = (0, 0.1]$

c) $C = \{\frac{1}{n} | n \in N\}$

d) $D = \mathbb{Q}$

Solutions:

$A = \{(x, y) | x \in N, y \in Z\}$ is countable because it is a cartesian product of two countable sets, i.e. $A = N \times Z$

$B = (0, 0.1]$ is uncountable since it is an interval of the form $(a, b]$, where $a < b$.

$C = \{\frac{1}{n} | n \in N\}$ is countable since it is a one-to-one correspondence with the set of natural numbers. In particular, you can list all the elements in the set C ,

$C = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

$D = \mathbb{Q}$ is countable since rational numbers can be expressed as the fraction of two integers. It is a one-to-one correspondence with the set of natural numbers.

Exercise 2

We consider an urn with N marbles containing r reds and $N - r$ blacks. We draw n marbles randomly without repetition.

1. How many possible outcomes do we have?
2. How many of these outcomes would include k red marbles.
3. Deduce the probability of choosing k red marbles.

Solutions:

1. C_N^n

2. $C_r^k C_{N-r}^{n-k}$

3. $P(\text{choosing } k \text{ red marbles}) = \frac{\text{number of outcomes including } k \text{ red marbles}}{\text{number of possible outcomes}} = \frac{C_r^k C_{N-r}^{n-k}}{C_N^n}$

Exercise 3

There are 3 pairs of shoes of different color in the drawer. We randomly draw 2 shoes without repetition; determine the probability associated with each of the following events:

- A : ‘they belong to the same pair’;
 B : ‘there is a right shoe and a left shoe’.

Solutions:

In the drawer there are three pairs of shoes of different color. The number of possible outcomes must first be counted when two random shoes are drawn.

We use the following notation:

- G1: Left shoe of the first pair;
- G2: Left shoe of the second pair;
- G3: Left shoe of the third pair;
- D1: Right shoe of the first pair;
- D2: Right shoe of the second pair;
- D3: Right shoe of the third pair.

All possible outcomes are:

- (G1 D1), (G1 D2), (G1 D3), (G1 G2), (G1 G3),
- (G2 D1), (G2 D2), (G2 D3), (G2 G3),
- (G3 D1), (G3 D2), (G3 D3),
- (D1 D2), (D1 D3),
- (D2 D3).

There are 15 possible outcomes. That is C_6^2 .

Note: Both events A and B do not take into account the order in which we draw the shoes. So we simplify the basic set without taking order¹. For example, drawing G1 then D1 is the same event as drawing D1 then G1, noted (G1 D1).

¹It is quite correct to count the fundamental set by taking the order into account, but it is not necessary in this case to identify events A and B .

To calculate the probability of the events A and B, as all the elements of our fundamental set are equiprobable, we use:

$$P(A) = \frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}} = \frac{\text{CF}}{\text{CP}}.$$

We count the number of favourable cases corresponding to each event:

- Event A ‘belong to the same pair’:

(G1 D1), (G2 D2), (G3 D3); **3 favourable cases**;

- Event B ‘there is a right shoe and a left shoe’:

(G1 D1), (G1 D2), (G1 D3), (G2 D1), (G2 D2), (G2 D3), (G3 D1), (G3 D2),
(G3 D3); **9 favourable cases**.

So

$$P(A) = \frac{\text{CF}}{\text{CP}} = \frac{3}{15} = \frac{1}{5};$$

$$P(B) = \frac{\text{CF}}{\text{CP}} = \frac{9}{15} = \frac{3}{5}.$$

Exercise 4

Let A be a die whose faces display the values 2, 2, 4, 4, 9, 9. Let B be a die whose faces display the values 1, 1, 6, 6, 8, 8. We roll the two dice.

1. Write all the possible outcomes.

$$S = \{(2; 1), (2; 6), (2; 8), (4; 1), (4; 6), (4; 8), (9; 1), (9; 6), (9; 8)\}$$

There are a total of 9 equiprobable events (possible cases). ²

²In this exercise, we could also define the fundamental set in more detail if we consider that among the 6 faces of two dice. For example, the value ‘4’ actually includes two different cases: The first face of A ‘4’ could be noted 4.1 and the second face of A ‘4’ could be noted 4.2. In this case, we would have:

$$\begin{aligned} S = & \{(2.1; 1.1), (2.1; 1.2), (2.2; 1.1), (2.2; 1.2) \\ & (2.1; 6.1), (2.1; 6.2), (2.2; 6.1), (2.2; 6.2) \\ & (2.1; 8.1), (2.1; 8.2), (2.2; 8.1), (2.2; 8.2) \\ & \dots \\ & (9.1; 8.1), (9.1; 8.2), (9.2; 8.1), (9.2; 8.2)\} \end{aligned}$$

a fundamental set of 36 equiprobable elements. But as we see, each element of type (2,1) is obtained in 4 different ways. So we can rewrite the fundamental set in a compact way as we did before, because the possible cases remain equiprobable.

2. What is the probability that the result of A is greater than the result of B ?

We count the number of favorable outcomes corresponding to the event ' $A > B$ ':
(2; 1), (4; 1), (9;1), (9;6), (9;8); 5 favorable cases. So

$$P(A > B) = \frac{\text{CF}}{\text{CP}} = \frac{5}{9}.$$

3. What is the probability that the sum of the two dice equals 10?

We count the number of favorable outcomes corresponding to the event ' $A + B = 10$ ': (2;8), (4;6), (9;1). So

$$P(A + B = 10) = \frac{\text{CF}}{\text{CP}} = \frac{3}{9} = \frac{1}{3}.$$

Exercise 5

A cafeteria offers a three-course menu. We choose a main course, a starch and a dessert.
The possible choices are given below:

- 'Main course': Chicken (C) or roast beef (B);
- 'Starch': Pasta (P) or rice (R) or potatoes (T);
- 'Dessert': Ice cream(I) or jelly (J) or apple pie (A) or peach pie (P).

A person chooses a dish from each category.

1. How many possible menus are there in the basic set?

The basic set is:

$$S = \{(C; P; I), (C; P; J), (C; P; A), (C; P; P), (C; R; I), (C; R; J), (C; R; A), (C; R; P), \\ (C; T; I), (C; T; J), (C; T; A), (C; T; P), (B; P; I), (B; P; J), (B; P; A), (B; P; P), \\ (B; R; I), (B; R; J), (B; R; A), (B; R; P), (B; T; I), (B; T; J), (B; T; A), (B; T; P)\}.$$

24 possible menus. ($C_2^1 C_3^1 C_4^1 = 24$)

2. Let A be the event: 'we choose ice cream'. How many menus are there in A ?

In event A , we take dessert 'ice cream' with certainty. So the set of possible outcomes in this case is:

$$S_A = \{(C; P; I), (C; R; I), (C; T; I), (B; P; I), (B; R; I), (B; T; I)\}.$$

So, 6 menus are possible in event A . ($C_2^1 C_3^1 = 6$)

3. Let B be the event: ‘we choose the chicken’. How many menus are there in B ?
In event B , we take the chicken with certainty. The set of possible menus then becomes:

$$S_B = \{(C; P; I), (C; P; J), (C; P; A), (C; P; P), (C; R; I), (C; R; J), (C; R; A), (C; R; P), (C; T; I), (C; T; J), (C; T; A), (C; T; P)\}$$

So, 12 menus belong to Event B . ($C_3^1 C_4^1 = 12$)

4. Give all the possible menus of the event $A \cap B$.
In event $A \cap B$ we take chicken and ice cream. Hence the set of possible menus becomes:

$$S_{A \cap B} = \{(C; P; I), (C; R; I), (C; T; I)\}.$$

3 possible menus are in $A \cap B$. ($C_3^1 = 3$)

5. Let C be the event: ‘we choose rice’. How many menus are there in C ?
In event C , rice is taken with certainty. So, the set of possible menus is:

$$S_C = \{(C; R; I), (C; R; J), (C; R; A), (C; R; P), (B; R; I), (B; R; J), (B; R; A), (B; R; P)\}.$$

There are 8 possible menus. ($C_2^1 C_4^1 = 8$)

6. It is assumed that a person randomly selects his menu by associating an equal probability with all the options for each category. What is the probability that the chosen menu belongs to event A ? Answer the same questions for event B and for event C .

As each option in each dish is equiprobable, each menu has the same probability of being taken. We define X as the chosen menu. So,

$$P(X \in A) = \frac{6}{24} = \frac{1}{4}.$$

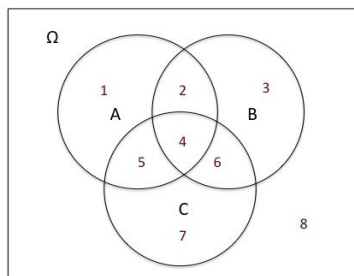
Similarly for events B and C we have:

$$P(X \in B) = \frac{12}{24} = \frac{1}{2};$$

$$P(X \in C) = \frac{8}{24} = \frac{1}{3}.$$

Exercise 6 (Optional)

In this problem, you are given descriptions in words of certain events (e.g., at least one of the events A,B,C occurs). For each one of these descriptions, identify the correct symbolic description in terms of A, B, C from Events E1-E7 below. Also identify the correct description in terms of regions (i.e., subsets of the sample space Ω) as depicted in the Venn diagram below. (For example, Region 1 is the part of A outside of B and C.)



Symbolic descriptions:

Event E1: $A \cap B \cap C$ covers Region 4

Event E2: $(A \cap B \cap C)^c$ covers Region 1,2,3,5,6,7,8

Event E3: $A \cap B \cap C^c$ covers Region 2

Event E4: $B \cup (B^c \cap C^c)$ covers Region 1,2,3,4,6,8

Event E5: $A^c \cap B^c \cap C^c$ covers Region 8

Event E6: $(A \cap B) \cup (A \cap C) \cup (B \cap C)$ covers Region 2,4,5,6

Event E7: $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$ covers Region 1,3,7

Which Event or Regions satisfy the following conditions.

1. At least two of the events A, B, C occur.
Region 2,4,5,6 (E6) satisfy this condition.
2. At most two of the events A, B, C occur.
Region 1,2,3,5,6,7,8 (E2) satisfy this condition.
3. None of the events A, B, C occurs.
Region 8 (E5) satisfy this condition.
4. All three events A, B, C occur.
Region 4 (E1) satisfy this condition.
5. Exactly one of the events A, B, C occurs.
Region 1,3,7 (E7) satisfy this condition.
6. Events A and B occur, but C does not occur.
Region 2 (E3) satisfy this condition.
7. Either event B occurs or, if not, then C also does not occur.
Region 1,2,3,4,6,8 (E4) satisfy this condition.

Exercise 7 (Optional)

A, B and C take turns flipping a coin, that is: A flips first, then B, then C, then A and so on. The first one get a head wins.

1. Define the sample space S of all the possible outcomes.

H: Head, T: Tail

$$S = \{H, TH, TTH, TTTH, TTTTH, \dots, \underbrace{TT \dots TT}_{n \text{ times}} H, \dots, \underbrace{TT \dots TT}_{\text{infinite T}}\}$$

2. Define the following events in terms of S :

- (a) The event A : A wins.

The time that A flips are $1_{st}, 4_{th}, 7_{th}, \dots, (3k+1)_{th}, \dots$.

$$Event A = \{H, TTH, TTTTH, \underbrace{TT \dots TT}_{3k} H, \dots\}$$

- (b) The event B : B wins.

The time that B flips are $2_{nd}, 5_{th}, 8_{th}, \dots, (3k+2)_{th}, \dots$.

$$Event B = \{TH, TTTTH, TTTTTH, \underbrace{TT \dots TT}_{3k+1} H, \dots\}$$

- (c) $(A \cup B)^c$

$$(A \cup B)^c = \{C \text{ wins}\} + \{\text{no one wins}\}$$

The time that C flips are $3_{rd}, 6_{th}, 9_{th}, \dots, (3k+3)_{th}, \dots$.

$$\{C \text{ wins}\} = \{TTH, TTTTH, TTTTTH, \underbrace{TT \dots TT}_{3k+2} H, \dots\}$$

$$\{\text{no one wins}\} = \{\underbrace{TT \dots TT}_{\text{infinite T}}\}$$

$$\text{So } (A \cup B)^c = \{TTH, TTTTH, TTTTTH, \underbrace{TT \dots TT}_{3k+2} H, \dots, \underbrace{TT \dots TT}_{\text{infinite T}}\}$$