Probability 1

Chapter 09 : Limit Theorems

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(based on the notes of Prof. Davide La Vecchia)

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Objectives

- Present two important inequalities that bound Probability statements
- Define Random Variable sequences
- Explain the limiting behaviour of these sequences and the implications
- Describe the Law of Large Numbers and the Central Limit Theory

Outline

Outline

Chebyshev's and Markov's Inequalities

Proposition (Markov's Inequality)

Let Z be random variable and $h(z)$ a non-negative valued function for all $z \in \mathbb{R}$. Then,

$$P(h(Z) \geq \zeta) \leq \frac{E[h(Z)]}{\zeta} \quad \textit{for all}\, \zeta > 0. \tag{1}$$

Proof.

$$\begin{aligned}
E[h(Z)] &= \int_{-\infty}^{\infty} h(z) f_Z(z)\, dz \\
&= \int_{\{z:h(z)\geq\zeta\}} h(z) f_Z(z)\, dz + \int_{\{z:h(z)<\zeta\}} h(z) f_Z(z)\, dz \\
&\geq \int_{\{z:h(z)\geq\zeta\}} h(z) f_Z(z)\, dz \\
E[h(Z)] &\geq \int_{\{z:h(z)\geq\zeta\}} \zeta f_Z(z)\, dz = \zeta P(h(Z) \geq \zeta),
\end{aligned}$$

giving the desired result on division by ζ. □

Example (Markov's Inequality)

Q. On the A2 highway (in the Luzern Canton), the speed limit is 80 Km/h. Most drivers are not driving so fast and the average speed on the high way is 70 Km/h. If Z denotes a randomly chosen driver's speed, what is the probability that such a person is driving faster than the speed limit?

A. Since we do not have the whole distribution of Z, but we have only limited info (i.e. we know $E[Z] = 70$ Km/h), we have to resort on Markov's inequality. So using (1) we obtain an upper bound to the probability:

$$P(Z \geq 80) \leq \frac{70}{80} = 0.875.$$

Proposition (Chebychev's Inequality)

For any random variable Z with mean μ_Z and variance $\sigma_Z^2 < \infty$

$$P\left(|Z-\mu_Z| < r\sigma_Z\right) \geq 1 - \frac{1}{r^2}$$

for all $r > 0$.

Remark

Notice that this inequality implies:

$$P\left(|Z-\mu_Z| \geq r\sigma_Z\right) \leq \frac{1}{r^2} \tag{2}$$

Put in words, the probability that a random variable lies more than r standard deviations away *from its mean value is* bounded above by $1/r^2$.

Proof.

Chebychev's inequality is, in turn, a special case of Markov's inequality.

$$h(z) = (z - \mu_Z)^2 \quad \text{and} \quad \zeta = r^2\sigma_Z^2$$

$$P(h(Z) \geq \zeta) = P[(Z - \mu_Z)^2 \geq r^2\sigma_Z^2] \leq \frac{E[h(Z)]}{\zeta} = \frac{E[(Z - \mu_Z)^2]}{r^2\sigma_Z^2} = \frac{1}{r^2}$$

□

Goal

Chebychev's inequality can be used to construct crude bounds on the probabilities associated with deviations of a random variable from its mean.

Example (Chebyshev's Inequality)

Q. On the A2 highway (in the Luzern Canton), the speed limit is 80 Km/h. Most drivers are not driving so fast and the average speed on the high way is 70 Km/h, **with variance** 9 $(Km/h)^2$. If Z denotes a randomly chosen driver's speed, what is the probability that such a person is driving faster than the speed limit?

A. Since we do not have the whole distribution of Z, but we have only limited info (i.e. we know $E[Z] = 70$ Km/h **AND** $V(Z) = 9$ $(Km/h)^2$), we have to resort on Chebyshev's inequality and give an upper bound to the probability. Thus,

$$\begin{aligned} P(Z \geq 80) &= P(Z - E[Z] \geq 80 - 70) \\ &\leq P(|Z - E[Z]| \geq 10) \leq P\left(\frac{|Z - E[Z]|}{\sqrt{V(Z)}} \geq \frac{10}{\sqrt{9}}\right) \end{aligned}$$

Using (2), with $r = \frac{10}{3}$ and $\sigma_Z = 3$, we finally get

$$P(Z \geq 80) \leq P\left(\Big|Z - E[Z]\Big| \geq \left(\frac{10}{3}\right) 3\right) \leq \frac{1}{\frac{10^2}{3^2}} \leq \frac{9}{100} \leq 0.09$$

Chebyshev's and Markov's Inequalities

Remark (A remark about Chebyshev's Inequality)

Chebyshev's inequality can be rewritten in a different way.
Indeed, for any random variable Z *with mean* μ_Z *and variance* $\sigma_Z^2 < \infty$

$$P\left(|Z - \mu_Z| \geq \varepsilon\right) \leq \frac{E[Z - \mu_Z]^2}{\varepsilon^2} = \frac{\sigma_Z^2}{\varepsilon^2}. \tag{3}$$

It's easy to check that Eq. (3) coincides with Eq. (2), setting in Eq. (3)

$$\varepsilon = r\sigma_Z.$$

Outline

Sequences of Random Variables

Definition

A sequence of random variables is an ordered list of random variables of the form

$$S_1, S_2, ..., S_n, ...$$

where, in an abstract sense, the sequence is infinitely long.

Example (Bernoulli Trials and their Sum)

Let $\tilde{Z}$ denote a dichotomous random variable with $\tilde{Z} \sim \mathcal{B}(1, p)$. A sequence of Bernoulli trials provides us with a sequence of values $\tilde{Z}_1, \tilde{Z}_2, ..., \tilde{Z}_n, ...$

$$P(''Success'') = P\left(\tilde{Z}_i = 1\right) = p \quad \text{and} \quad P(''Failure'') = P\left(\tilde{Z}_i = 0\right) = 1 - p$$

Now let

$$S_n = \sum_{s=1}^{n} \tilde{Z}_s,$$

the number of "Successes" in the first n Bernoulli trials. This yields a new sequence of random variables

$$\begin{aligned} S_1 &= \tilde{Z}_1 \\ S_2 &= \left(\tilde{Z}_1 + \tilde{Z}_2\right) \\ &\vdots \\ S_n &= \left(\tilde{Z}_1 + \tilde{Z}_2 + \cdots + \tilde{Z}_n\right) = \sum_{i=1}^{n} \tilde{Z}_i \end{aligned}$$

This new sequence is such that $S_n \sim \mathcal{B}(n, p)$ for each n.

Example (Bernoulli Trials and their Sum)

Now consider the sequence

$$P_n = \frac{S_n}{n},$$

for $n = 1, 2, \ldots$, corresponds to the **proportion of 'Successes'in the first n Bernoulli trials**.

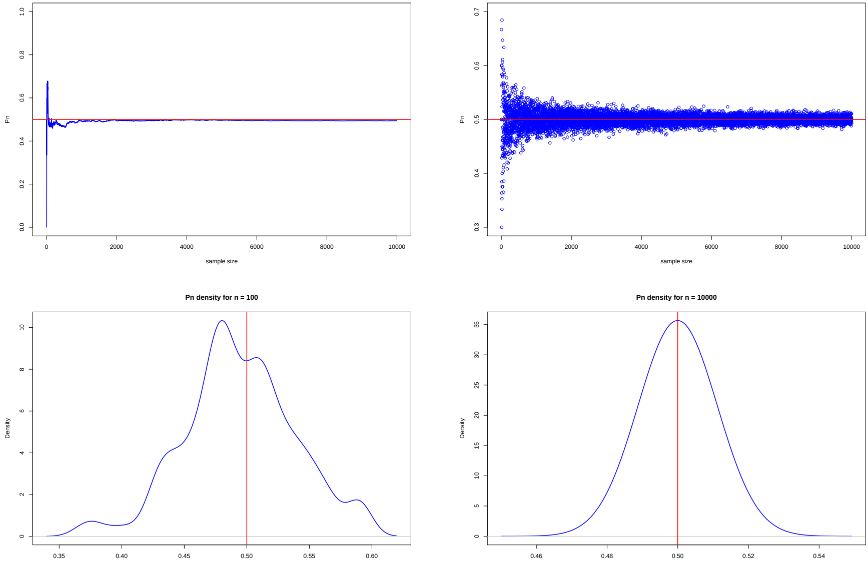
It is natural to ask how the behaviour of P_n is related to the true probability of a 'Success' (p).

Specifically, the open question at this point is:

"Do these results imply that P_n collapses onto the true p as n increases, and if so, in what way?"

To gain a clue, let us see the simulated values of P_n.

Sequences of Random Variables



Sequences of Random Variables

This numerical illustration leads us to suspect that **there is a sense in which P_n converges to p**

Notice that **although the sequence is random, the 'limiting' value here is a constant (i.e. is non-random)**.

Informally, we can claim that a sequence of random variables $X_1, X_2, \dots, X_n, \dots$ is thought to **converge** if the **probability distribution of X_n gets more and more concentrated around a single point as n tends to infinity**.

Sequences of Random Variables

In general, we would like to study the behaviour of sequences of random variables n becomes larger and larger (i.e. as n tends towards infinity $n \to \infty$)

The study of such limiting behaviour is commonly called a study of 'asymptotics' , after the word asymptote used in standard calculus.

Outline

Definition (Convergence in Probability)

A sequence of random variables $X_1, X_2, ..., X_n, ...$ is said to

- **Converge in probability to a number** α if for any arbitrary constant $\varepsilon > 0$

$$\lim_{n \to \infty} P\left(|X_n - \alpha| > \varepsilon\right) = 0$$

  If this is the case, we write $X_n \xrightarrow{p} \alpha$ or $p\lim X_n = \alpha$.

- **Converge in probability to a random variable** X if for any arbitrary constant $\varepsilon > 0$

$$\lim_{n \to \infty} P\left(|X_n - X| > \varepsilon\right) = 0\,,$$

  written $X_n \xrightarrow{p} X$ or $p\lim(X_n - X) = 0$.

Convergence in Probability

Let us itemize some rules. To this end, let a be any (nonrandom) number so:

- If $X_n \xrightarrow{p} \alpha$ then
  - $aX_n \xrightarrow{p} a\alpha$ and
  - $a + X_n \xrightarrow{p} a + \alpha$,
- If $X_n \xrightarrow{p} X$ then
  - $aX_n \xrightarrow{p} aX$ and
  - $a + X_n \xrightarrow{p} a + X$
- If $X_n \xrightarrow{p} \alpha$ and $Y_n \xrightarrow{p} \gamma$ then
  - $X_n Y_n \xrightarrow{p} \alpha\gamma$ and
  - $X_n + Y_n \xrightarrow{p} \alpha + \gamma$.

- If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$ then
  - $X_n Y_n \xrightarrow{p} XY$ and
  - $X_n + Y_n \xrightarrow{p} X + Y$
- Let $g(x)$ be any (non-random) continuous function. If $X_n \xrightarrow{p} \alpha$ then

$$g(X_n) \xrightarrow{p} g(\alpha),$$

and if $X_n \xrightarrow{p} X$ then

$$g(X_n) \xrightarrow{p} g(X).$$

Suppose $X_1, X_2, ..., X_n, ...$ is a sequence of *i.i.d.* random variables with common distribution $F_X(x)$ and moments $\mu_r = E[X^r]$.

At any given point along the sequence, $X_1, X_2, ..., X_n$ constitutes a **simple random sample** of size n.

For each fixed sample size n, the r**th sample moment** is:

$$M_{(r,n)} = \frac{1}{n}\left(X_1^r + X_2^r + \cdots + X_n^r\right) = \frac{1}{n}\sum_{s=1}^{n} X_s^r \,,$$

and we can verify that:

$$E[M_{(r,n)}] = \mu_r \quad \text{and} \quad Var(M_{(r,n)}) = \frac{1}{n}(\mu_{2r} - \mu_r^2)\,.$$

Convergence in Probability

Convergence of Sample Moments as a motivation...

Now consider the sequence of sample moments $M_{(r,1)}, M_{(r,2)}, \dots, M_{(r,n)}, \dots$ or, equivalently, $\{M_{(r,i)}\}_{i=1}^{n}$.
The **distribution of** $M_{(r,n)}$ (which is unknown because $F_X(x)$ has not been specified) is **concentrated around** μ_r for all n, with a **variance which tends to zero as** n **increases**.

So the distribution of $M_{(r,n)}$ becomes **more and more concentrated around** μ_r as n increases and therefore we might *anticipate* that

$$M_{(r,n)} \xrightarrow{p} \mu_r.$$

In fact, this result follows from what is known as the **Weak Law of Large Numbers** (WLLN).

Outline

The Weak Law of Large Numbers (WLLN)

Proposition

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of i.i.d. random variables with common probability distribution $F_X(x)$, and let $Y = h(X)$ be such that

$$\begin{aligned} E[Y] = E\left[h(X)\right] &= \mu_Y \\ Var(Y) = Var\left(h(X)\right) &= \sigma_Y^2 < \infty\,. \end{aligned}$$

Set

$$\overline{Y}_n = \frac{1}{n}\sum_{s=1}^{n} Y_s \quad \textit{where} \quad Y_s = h(X_s)\,, \quad s = 1, \ldots, n\,.$$

Then for any two numbers ε and δ satisfying $\varepsilon > 0$ and $0 < \delta < 1$

$$P\left(\left|\overline{Y}_n - \mu_Y\right| < \varepsilon\right) \geq 1 - \delta$$

for all $n > \sigma_Y^2/(\varepsilon^2\delta)$. Choosing both ε and δ to be arbitrarily small implies that $p\lim_{n\to\infty}(\overline{Y}_n - \mu_Y) = 0$, or equivalently $\overline{Y}_n \xrightarrow{p} \mu_Y$.

The Weak Law of Large Numbers (WLLN)

Proof.

- First note that $E[\overline{Y}_n] = \mu_Y$ and $Var(\overline{Y}_n) = \sigma_Y^2/n$.
- Now, according to **Chebyshev's inequality**

$$
\begin{aligned}
P\left(|\overline{Y}_n - \mu_Y| < \varepsilon\right) &\geq 1 - \frac{E\left[\left(\overline{Y}_n - \mu_Y\right)^2\right]}{\varepsilon^2} \\
&= 1 - \frac{\sigma_Y^2/n}{\varepsilon^2} \\
&= 1 - \frac{\sigma_Y^2}{n\varepsilon^2} \geq 1 - \delta
\end{aligned}
$$

for all $n > \sigma_Y^2/(\varepsilon^2\delta)$.

- Note that by considering the limit as $n \to \infty$ we also have

$$
\lim_{n\to\infty} P\left(\left|\overline{Y}_n - \mu_Y\right| < \varepsilon\right) \geq \lim_{n\to\infty}\left(1 - \frac{\sigma^2}{n\varepsilon^2}\right) = 1\,,
$$

again implying that $\left(\overline{Y}_n - \mu_Y\right) \xrightarrow{p} 0$.

$\square$

Outline

Convergence in Distribution

Definition (Convergence in Distribution)

Consider, a sequence of random variables $X_1, X_2, ..., X_n, ...$

with corresponding CDFs $F_{X_1}(x), F_{X_2}(x), ..., F_{X_n}(x), ...$

We say that the sequence $X_1, X_2, ..., X_n, ...$ **converges in distribution** to the random variable X, having probability distribution $F_X(x)$, if and only if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

at all points x **where** $F_X(x)$ **is continuous**.

In this case we write $X_n \xrightarrow{D} X$

- If $p\lim_{n\to\infty}(X_n - X) = 0$ then $X_n \stackrel{D}{\to} X$.
- Let a be any real number. If $X_n \stackrel{D}{\to} X$, then $aX_n \stackrel{D}{\to} aX$
- If $Y_n \stackrel{p}{\to} \phi$ and $X_n \stackrel{D}{\to} X$, then
  - $Y_n X_n \stackrel{D}{\to} \phi X$, and
  - $Y_n + X_n \stackrel{D}{\to} \phi + X$
- If $X_n \stackrel{D}{\to} X$ and $g(x)$ is any continuous function, then $g(X_n) \stackrel{D}{\to} g(X)$

Convergence in Distribution

Example (Poisson and Normal Approximations to the Binomial Distribution)

Suppose $X_1, X_2, ..., X_n, ...$ is a sequence of independent random variables where $X_n \sim \mathcal{B}(n, p)$ with probability of "Success" p.

- We already know that, if $p = \lambda/n$, where $\lambda > 0$ is fixed, then as n goes to infinity, $F_{X_n}(x)$ converges to the probability distribution of a *Poisson* (λ) random variable. So, $X_n \xrightarrow{D} X$, where $X \sim \mathcal{P}(\lambda)$

- Now consider another case. If p is fixed, the probability distribution of

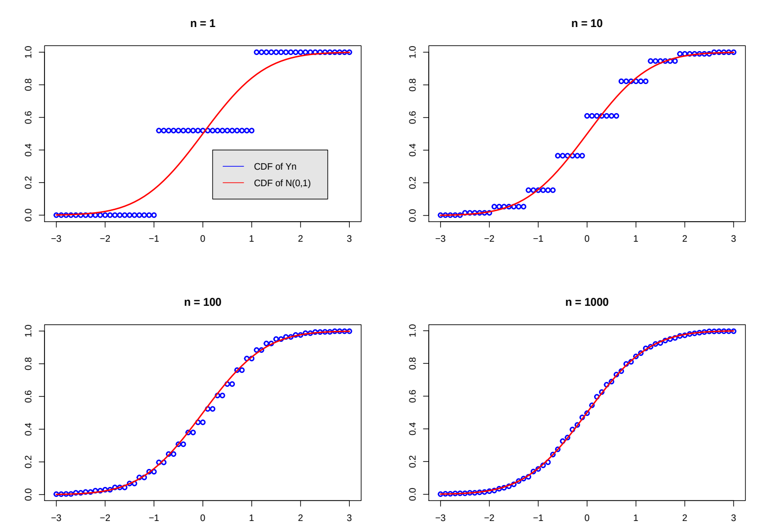
$$Y_n = \frac{X_n - np}{\sqrt{np(1-p)}}$$

converges, as n goes to infinity, to that of a standard Normal random variable [Theorem of De Moivre-Laplace]. So, $Y_n \xrightarrow{D} Y$, where $Y \sim \mathcal{N}(0, 1)$.

Convergence in Distribution

Example cont'd (visualize Y_n)

Example



Convergence in Distribution

Example (Convergence to an Exponential Random Variable)

Let us consider a sequence of continuous r.v.'s $X_1, X_2, ..., X_n, ...$, where X_n has range $(0, n]$, for $n > 0$ and CDF

$$F_{X_n}(x) = 1 - \left(1 - \frac{x}{n}\right)^n, \quad 0 < x \leq n.$$

Then, as $n \to \infty$, the limiting support is $(0, \infty)$, and $\forall x > 0$, we have

$$F_{X_n}(x) \to F_X(x) = 1 - e^{-x}$$

which is the CDF of an exponential r.v. (at all continuity points).

So, we conclude that X_n convergences in distribution to an exponential r.v., that is

$$X_n \xrightarrow{D} X, \quad X \sim \exp(1).$$

Outline

The Central Limit Theorem (CLT)

The following theorem is often said to be one of the most important results. Its significance lies in the fact that it allows accurate probability calculations to be made without knowledge of the underlying distributions!

Theorem

Let $X_1, X_2, ..., X_n, ...$ be a sequence of i.i.d. random variables and let $Y = h(X)$ be such that

$$\begin{aligned} E[Y] = E\left[h(X)\right] &= \mu_Y \\ Var(Y) = Var\left(h(X)\right) &= \sigma_Y^2 < \infty\,. \end{aligned}$$

Set

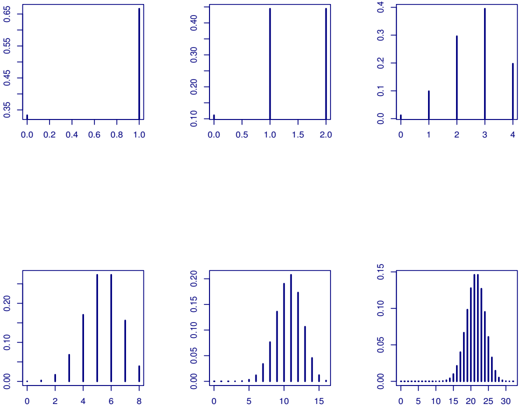
$$\overline{Y}_n = \frac{1}{n}\sum_{s=1}^{n} Y_s \quad where \quad Y_s = h(X_s)\,, \quad s = 1, \ldots, n\,.$$

Then (under quite general regularity conditions)

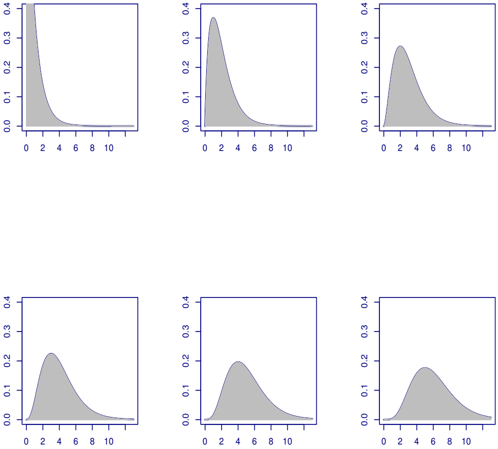
$$\frac{\sqrt{n}\left(\overline{Y}_n - \mu_Y\right)}{\sigma_Y} \xrightarrow{D} N\left(0,1\right).$$

The Central Limit Theorem (CLT)

CLT: sum of Bernoulli Random Variables



Distributions de la somme de 1, 2, 4, 8, 16 *et* 32 *variables aléatoires indépendantes de Bernoulli avec* $p = 2/3$, *i.e.* $B(1, 2/3)$.



Distributions de la somme de 1, 2, 3, 4, 5 *et* 6 *variables aléatoires indépendantes exponentielles avec* $\lambda = 1$, *i.e.* $\mathrm{Exp}(1)$.

The Central Limit Theorem (CLT)

Remark

Several generalizations of this statement are available.

For instance, one can state a CLT for data which are independent but NOT identically distributed.

Another possibility is to define a CLT for data which are NOT independent, namely for dependent data (e.g. in the case of Time Series)

- Chebyshev's and Markov's inequalities are important tools to give upper bounds for CDF's. They also play a role in some theoretical results

- Linear Combinations of elements in an iid Random Sequence converge in probability to the Expectation

- Linear Combinations of elements in an iid Random Sequence can converge in distribution to the a Normal

Thank You for your Attention!

"See you" Next Week