Probability I

Lecture 2

Davide La Vecchia

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Probability [Axiomatic definition]

Definition

We define probability a set function with values in $[0, 1]$, which satisfies the following axioms:

(i) $P(A) \geq 0$, for every event A

(ii) $P(S) = 1$

(iii) If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events (namely $A_i \cap A_j = \varnothing$, for $i \neq j$, and $i, j = 1, 2, \ldots$), and such that $A = \bigcup_{i=1}^{\infty} A_i$, then

$$P(A) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i). \tag{1}$$

Properties of $P(\cdot)$

One can make use of the three axioms to build more sophisticated statements. For instance,

Theorem

$P(\varnothing) = 0$.

Proof.

Take $A_1 = A_2 = A_3 = ... = \varnothing$. Then by (1) in Axiom (ii) we have

$$P(\varnothing) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} P(\varnothing)$$

which is true only if it is an infinite sum of zeros. Thus

$$P(\varnothing) = 0.$$

$\square$

Properties of $P(\cdot)$ (cont'd)

Theorem

If $A_1, A_2, \ldots$ are mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i). \tag{2}$$

Proof.

Let $A_{n+1} = A_{n+2} = \ldots. = \varnothing$, then $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i$, and, from (1) (see Axiom (iii)) it follows

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{n} P(A_i) + \underbrace{\sum_{i=n+1}^{\infty} P(A_i)}_{\equiv 0}.$$

$\square$

Properties of $P(\cdot)$ (cont'd)

Theorem

If A is an event, then $P(A^c) = 1 - P(A)$.

Proof.

Let $A \cup A^c = S$ and $A \cap A^c = \varnothing$, so and

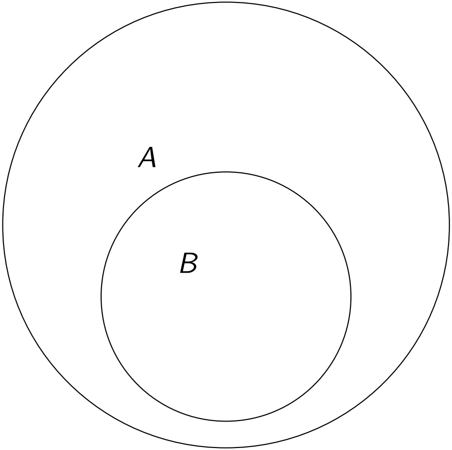
$$P(S) = P\left(A \cup A^c\right) = P(A) + P\left(A^c\right).$$

By Axiom (i) Axiom (ii) we have $P(S) = 1$, so the desired result follows from:

$$1 = P(A) + P\left(A^c\right).$$

$\square$

Properties of $P(\cdot)$ (cont'd)



$B \subset A$

Properties of $P(\cdot)$ (cont'd)

Theorem

For any two events A and B, such that $B \subset A$, we have

$$P(A) \geq P(B).$$

Proof.

Let us write

$$A = B \cup (B^c \cap A)$$

and notice that $B \cap (B^c \cap A) = \phi$, so that

$$\begin{aligned} P(A) &= P\{B \cup (B^c \cap A)\} \\ &= P(B) + P(B^c \cap A) \end{aligned}$$

which implies (since $P(B^c \cap A) \geq 0$) that

$$P(A) \geq P(B).$$

□

Properties of $P(\cdot)$ (cont'd)

Theorem (Boole's inequality)

For the events $A_1, A_2, ...A_n$,

$$P(A_1 \cup A_2 \cup .... \cup A_n) \leq \sum_{i=1}^{n} P(A_i).$$

For instance, let us consider $n = 2$. Then we have:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2)$$

since $P(A_1 \cap A_2) \geq 0$ by definition.

Remark

It is worth notice that if $A_j \cap A_i = \varnothing$, for every i and j, with $i \neq j$, then $P(A_1 \cup A_2 \cup .... \cup A_n) = \sum_{i=1}^{n} P(A_i)$, as stated in (2).

Properties of $P(\cdot)$ (cont'd)

Theorem

For any two events A and B then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof.

Consider that $A \cup B = A \cup (A^c \cap B)$, and $A \cap (A^c \cap B) = \phi$. Now remember[a] that $A^c \cap B = B - (A \cap B)$, so,

$$\begin{aligned} P(A \cup B) &= P(A) + P(A^c \cap B) \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

$\square$

[a] See Lecture 1 for the meaning of set difference.

How to use these properties?

Example (Real-life example I)

If we flip a balanced coin twice, what is the probability of getting at least one head?

The sample space is {*HH,HT,TH,TT*}

Since the coin is balanced, these outcomes are equally likely and we assign to each sample point probability ¼

A=event that we get at least one *H* = {*HH,HT,TH*}

$$\Pr\{A\} = \Pr\{HH \cup HT \cup TH\} = \Pr\{HH\} + \Pr\{HT\} + \Pr\{TH\}$$
$$= 1/4 + 1/4 + 1/4 = 3/4$$

How to use these properties?

Example (Real-life example II)

Shopper TRK is an electronic device designed to count the number of shoppers entering a shopping centre

When two shoppers enter the shopping centre together, one walking in front of the other the following probabilities apply

1) there is a 0.98 probability that the first shopper is detected

2) there is a 0.94 probability that the second shopper is detected

3) there is a 0.93 probability that both shoppers are detected

What is the probability that the device will detect at least one of the two shoppers entering?

How to use these properties?

Example (Real-life example II, cont'd)

Sample space $\{DD, DU, UD, UU\}$

1) $\Pr\left\{DD \cup DU\right\} = 0.98$

2) $\Pr\left\{DD \cup UD\right\} = 0.94$

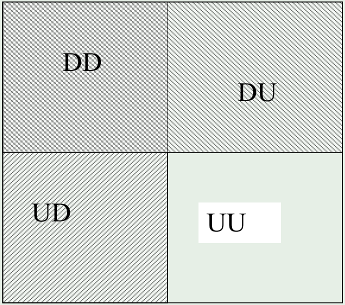
3) $\Pr\left\{DD\right\} = 0.93$

$\Pr\left\{DD \cup UD \cup DU\right\} = ?$

How to use these properties?

Example (Real-life example II, cont'd)

$$\Pr\left\{DD \cup UD \cup DU\right\} = \Pr\left\{\left(DD \cup UD\right) \cup \left(DD \cup DU\right)\right\}$$
$$= \Pr\left\{DD \cup UD\right\} + \Pr\left\{DD \cup DU\right\} - \Pr\left\{\left(DD \cup UD\right) \cap \left(DD \cup DU\right)\right\}$$



$$\boxed{\left(DD \cup UD\right) \cap \left(DD \cup DU\right) = DD}$$

Example (Real-life example II, cont'd)

So the desired probability is

$$\begin{aligned}\Pr\left\{DD \cup UD \cup DU\right\} &= \Pr\left\{DD \cup UD\right\} + \Pr\left\{DD \cup DU\right\} - \Pr\left\{DD\right\} \\ &= 0.98 + 0.94 - 0.93 = 0.99\end{aligned}$$

How to use these properties?

Example (De Morgan's law)

Given $P(A \cup B) = 0.7$ and $P(A \cup B^c) = 0.9$, find $P(A)$.

By De Morgan's law,

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.7 = 0.3$$

and similarly

$$P(A^c \cap B) = 1 - P(A \cup B^c) = 1 - 0.9 = 0.1.$$

Thus,

$$P(A^c) = P(A^c \cap B^c) + P(A^c \cap B) = 0.3 + 0.1 = 0.4,$$

so

$$P(A) = 1 - 0.4 = 0.6.$$

How to use these properties?

Example (Probability, union, and complement)

John is taking two books along on his holiday vacation. With probability 0.5, he will like the first book; with probability 0.4, he will like the second book; and with probability 0.3, he will like both books. What is the probability that he likes neither book?

Let A_i be the event that John likes book i, for $i = 1, 2$. Then the probability that he likes at least one book is (remember the short hand notation $A_1 \cap A_2 = A_1 A_2$)

$$\begin{aligned} P(\bigcup_{i=1}^{2} A_i) &= P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 A_2) \\ &= 0.5 + 0.4 - 0.3 = 0.6. \end{aligned}$$

Because the event the John likes neither books is the complement of the event that he likes at leas one of them (namely $A_1 \cup A_2$), we have

$$P(A_1^c \cap A_2^c) = P((A_1 \cup A_2)^c) = 1 - P(A_1 \cup A_2) = 0.4.$$

Example (Real-life example III)

Let A denote the event that the midtown temperature in Los Angeles (LA) is 70 F, and let B denote the event that the midtown temperature in New York (NY) is 70 F. Also, let C denote the event that the maximum of midtown temperatures in NY and LA is 70 F. If

$$P(A) = 0.3, \quad P(B) = 0.4 \quad \text{and} \quad P(C) = 0.2,$$

find the probability that the minimum of the two midtown temperatures is 70 F.

How to use these properties?

Example (cont'd)

Let D denote the event that the minimum temperature is 70 F. Then

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) = 0.7 - P(AB) \\ P(C \cup D) &= P(C) + P(D) - P(CD) = 0.2 - P(D) - P(DC). \end{aligned}$$

Since

$$A \cup B = C \cup D \quad \text{and} \quad AB = CD,$$

subtracting one of the preceding equations from the other we get

$$\begin{aligned} P(A \cup B) - P(C \cup D) &= 0.7 - P(AB) - [0.2 - P(D) - P(DC)] \\ &= 0.5 - P(D) = 0, \end{aligned}$$

thus $P(D) = 0.5$.

Conditional probability

As a measure of uncertainty, the probability depends on the information available.

Example

Suppose you have two dice and throw them; the possible outcomes are:

(1,1)	**(1,2)**	**(1,3)**	**(1,4)**	**(1,5)**	**(1,6)**
(2,1)	**(2,2)**	**(2,3)**	**(2,4)**	**(2,5)**	**(2,6)**
(3,1)	**(3,2)**	**(3,3)**	**(3,4)**	**(3,5)**	**(3,6)**
(4,1)	**(4,2)**	**(4,3)**	**(4,4)**	**(4,5)**	**(4,6)**
(5,1)	**(5,2)**	**(5,3)**	**(5,4)**	**(5,5)**	**(6,5)**
(6,1)	**(6,2)**	**(6,3)**	**(6,4)**	**(6,5)**	**(6,6)**

Conditional probability

Example (cont'd)

Let us define A = getting 5, or equivalently $A = \{5\}$. What is $P(A)$, namely, the probability of getting 5?

(1,1)	**(1,2)**	**(1,3)**	**(1,4)**	**(1,5)**	**(1,6)**
(2,1)	**(2,2)**	**(2,3)**	**(2,4)**	**(2,5)**	**(2,6)**
(3,1)	**(3,2)**	**(3,3)**	**(3,4)**	**(3,5)**	**(3,6)**
(4,1)	**(4,2)**	**(4,3)**	**(4,4)**	**(4,5)**	**(4,6)**
(5,1)	**(5,2)**	**(5,3)**	**(5,4)**	**(5,5)**	**(6,5)**
(6,1)	**(6,2)**	**(6,3)**	**(6,4)**	**(6,5)**	**(6,6)**

Conditional probability

Example (cont'd)

The dice are fair so we can get 36 events with equal probability $1/36$. Namely:

$$Pr(i,j) = \frac{1}{36}, \quad \text{for} \quad i,j = 1,..,6$$

Thus, we can make use of the blue guys as

$$\begin{aligned}
P(5) &= Pr\{(1,4) \cup (2,3) \cup (3,2) \cup (4,1)\} \\
&= Pr\{(1,4)\} + Pr\{(2,3)\} + Pr\{(3,2)\} + Pr\{(4,1)\} \\
&= 1/36 + 1/36 + 1/36 + 1/36 \\
&= 4/36 \\
&= 1/9.
\end{aligned}$$

Conditional probability

Example (cont'd)

Now, suppose that we throw the die first and e we get 2.

Q. What is the probability of getting 5 given that we have observed 2 in the first throw?

(1,1)	**(1,2)**	**(1,3)**	**(1,4)**	**(1,5)**	**(1,6)**
(2,1)	**(2,2)**	**(2,3)**	**(2,4)**	**(2,5)**	**(2,6)**
(3,1)	**(3,2)**	**(3,3)**	**(3,4)**	**(3,5)**	**(3,6)**
(4,1)	**(4,2)**	**(4,3)**	**(4,4)**	**(4,5)**	**(4,6)**
(5,1)	**(5,2)**	**(5,3)**	**(5,4)**	**(5,5)**	**(6,5)**
(6,1)	**(6,2)**	**(6,3)**	**(6,4)**	**(6,5)**	**(6,6)**

A.

$\Pr\{\text{getting 5 given 2 in the first throw}\} = \Pr\{\text{getting 3 in the second throw}\} = 1/6$

Conditional probability

Remark

- *Information changes the probability*
- *By knowing that we got 2 in the first throw, we have changed the sample space:*

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(6,5)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- *Probability can change drastically — e.g., suppose that in our example we have 6 in the first throw* $\Rightarrow$ *the probability of observing 5 in two draws ... is zero !!!*

Conditional probability

Definition

Let A and B be two events. The conditional probability of event A given event B, denoted by $P\left(A|B\right)$, is defined by:

$$P\left(A|B\right) = \frac{P(A \cap B)}{P(B)}, \quad \text{if} \quad P(B) > 0,$$

and it is left undefined if $P(B) = 0$.

Conditional probability

Example (A check)

Let us define the set B as:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(6,5)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

So we have

$$
\begin{aligned}
P(B) &= Pr\left\{(2,1)\cup(2,2)\cup(2,3)\cup(2,4)\cup(2,5)\cup(2,6)\right\} \\
&= Pr(2,1)+Pr(2,2)+Pr(2,3)+Pr(2,4)+Pr(2,5)+Pr(2,6) \\
&= 6/36 = 1/6
\end{aligned}
$$

Example (A check)

and consider $A \cap B$

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(6,5)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

So we have $P(A \cap B) = Pr(2,3) = 1/36$ so,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{1/6} = \frac{1}{6}.$$

Independence

Definition

Two events A and B are independent if the occurrence of one event has no effect on the probability of occurrence of the other event. Thus,

$$P(A|B) = P(A)$$

or equivalently

$$P(B|A) = P(B)$$

Clearly, if $P(A|B) \neq P(A)$, then A and B are *dependent*.

Independence – another characterization

Two events A and B are independent if

$$P(A|B) = P(A),$$

now by definition of conditional probability we know that

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

so we have

$$P(A) = \frac{P(A \cap B)}{P(B)},$$

and rearranging the terms, we find that two events are independent iif

$$P(A \cap B) = P(A)P(B).$$

Independence

Example

A coin is toss three times and the eight possible outcomes

HHH, HHT, HTH, THH, HTT, THT, TTH, TTT

are assumed to be equally likely with probability 1/8

Define

A= an H occurs on each of the first two tosses

B= T occurs in the third toss

D= two Ts occur in the three tosses

Q1: Are A and B independent?

Q2: Are B and D independent?

Example (cont'd)

We have

$$A=\{HHH,\ HHT\} \qquad \Pr\{A\}=2/8=1/4$$

$$B=\{HHT,\ HTT,\ THT,\ TTT\} \qquad \Pr\{B\}=4/8=1/2$$

$$D=\{HTT,\ THT,\ TTH\} \qquad \Pr\{D\}=3/8$$

$$A\cap B=\{HHT\} \qquad \Pr\{A\cap B\}=1/8$$

$$B\cap D=\{HTT, THT\} \qquad \Pr\{B\cap D\}=2/8=1/4$$

$$\Pr\{A\}\Pr\{B\}=\frac{1}{4}\times\frac{1}{2}=\frac{1}{8} \qquad \Pr\{A\cap B\}=\frac{1}{8} \Rightarrow A,B \text{ independent}$$

$$\Pr\{B\}\Pr\{D\}=\frac{1}{2}\times\frac{3}{8}=\frac{3}{16} \qquad Pr\{B\cap D\}=\frac{1}{4} \Rightarrow B \text{ and } D \text{ dependent}$$

Theorem (Th. of total probabilities)

Let $B_1, B_2, ..., B_k, ..., B_n$ *be mutually disjoint events, satisfying* $S = \cup_{i=1}^n B_i$*, and* $P(B_i) > 0$*, for every* $i = 1, 2, ..., n$ *then for every* A *we have that:*

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i). \tag{3}$$

Proof.

Write $A = A \cap S = A \cap (\cup_{i=1}^n B_i) = \cup_{i=1}^n (A \cap B_i)$. Since the $\{B_i \cap A\}$ are mutually disjoint, we have

$$P(A) = P\left(\cup_{i=1}^n (A \cap B_i)\right) = \sum_{i=1}^{n} P\left(A \cap B_i\right) = \sum_{i=1}^{n} P(A|B_i)P(B_i).$$

$\square$

Theorem I (cont'd)

Remark. The theorem remains valid even if $n = \infty$ in Eq. (3). (Double check, and re-do the proof using $n = \infty$.)

Corollary

Let B *satisfy* $0 < P(B) < 1$*; then for every event* A*:*

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

Proof.

Exercise [Hint: $S = B \cup B^c$]. □

Theorem II

Theorem I can be applied to derive the well-celebrated Bayes' Theorem.

Theorem (Bayes' Theorem)

Let $B_1, B_2, ..., B_k, ..., B_n$ be mutually disjoint events, satisfying

$$S = \cup_{i=1}^{n} B_i,$$

and $P(B_i) > 0$, for every $i = 1, 2, ..., n$. Then for every event A for which $P(A) > 0$, we have that

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}. \tag{4}$$

Theorem II (cont'd)

Proof.

Let us write

$$\begin{aligned} P(B_k|A) &= \frac{P(A \cap B_k)}{P(A)} \\ &= \frac{P(A \cap B_k)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)} \\ &= \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)} \end{aligned}$$

That concludes the proof. $\square$

Example

Let us consider a special case, where we have only two events A and B.

From the definition of conditional probability

$$\Pr\{A|B\}=\frac{\Pr\{A\cap B\}}{\Pr\{B\}} \qquad \Pr\{B|A\}=\frac{\Pr\{A\cap B\}}{\Pr\{A\}}$$

This can be written as

$$\Pr\{A\cap B\}=\Pr\{A|B\}\Pr\{B\} \qquad \Pr\{A\cap B\}=\Pr\{B|A\}\Pr\{A\}$$

That is

$$\Pr\{A|B\}\Pr\{B\}=\Pr\{B|A\}\Pr\{A\}$$

Rearranging this we have

$$\Pr\{A|B\}=\frac{\Pr\{A\}}{\Pr\{B\}}\times\Pr\{B|A\} \quad \Leftarrow \text{ Bayes' Theorem}$$

... so thanks to Bayes' Theorem we can reverse the role of $A|B$ and $B|A$.

Example (Guessing in a multiple choice exam)

Example 1.13 In answering a question on a multiple choice test a student either knows the answer or he guesses. Let p be the probability that she knows the answer and $1 - p$ the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?

Solution: Let C and K denote respectively the event that the student answers the question correctly and the event that she actually knows the answer. Now

$$P(K|C) = \frac{P(KC)}{P(C)} = \frac{P(C|K)P(K)}{P(C|K)P(K) + P(C|K^c)P(K^c)}$$

$$= \frac{p}{p + (1/m)(1 - p)}$$

$$= \frac{mp}{1 + (m - 1)p}$$

Thus, for example, if $m = 5$, $p = \frac{1}{2}$, then the probability that a student knew the answer to a question she correctly answered is $\frac{5}{6}$. ◆

Another real-life example

Example

Members of a consulting firm in Geneva rent cars from two rental agencies:

- 60% from AVIS
- 40% from Mobility

Now consider that

- 9% of the cars from AVIS need a tune-up
- 20% of the cars from Mobility need a tune-up

If a car delivered to the consulting firm needs a tune-up, what is the probability that the care came from AVIS?

Aim

Let us set: $A := \{\textit{car rented from AVIS}\}$ *and* $B := \{\textit{car needs a tune-up}\}$. *We know* $P(B|A)$ *and we look for* $P(A|B) \Rightarrow$ *Bayes' theorem!!*

A real-life example (cont'd)

Example (cont'd)

$$\Pr\{A\}=0.6$$

$$\Pr\{B|A\}=0.09$$

$$\Pr\left\{B|A^C\right\}=0.2$$

$$\begin{aligned}\Pr\{B\}&=\Pr\left\{(B\cap A)\cup\left(B\cap A^C\right)\right\}\\&=\Pr\{B\cap A\}+\Pr\left\{B\cap A^C\right\}\\&=\Pr\{B|A\}\Pr\{A\}+\Pr\left\{B|A^C\right\}\Pr\left\{A^C\right\}\\&=0.09\times0.6+0.2\times0.4\\&=0.134\end{aligned}$$

$$\Pr\{A|B\}=\frac{\Pr\{A\}}{\Pr\{B\}}\times\Pr\{B|A\}=\frac{0.6}{0.134}0.09=0.402985$$