

Probability 1

Chapter 05 : Continuous Random Variables - Part 1

Dr. Daniel Flores-Agreda,

(based on the notes of Prof. Davide La Vecchia)

Spring Semester 2021

Objectives

- Present the Exponential distribution
- Understand the consequences of the transformation of Random Variables.

1 Exponential distribution

2 Variable Transformation

- Transformation of Discrete Random Variables
- Transformation through the CDF
- Transformation of Continuous Random Variables through the PDF

1 Exponential distribution

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Exponential distribution

Definition

Let X be a continuous random variable, having the following characteristics:

- X is defined on the **positive real numbers** $(0; \infty)$ i.e. \mathbb{R}^+ ;
- the PDF and CDF are

$$f_X(x) = \lambda \exp\{-\lambda x\}, \lambda > 0; \quad F_X(x) = 1 - \exp(-\lambda x);$$

then we say that X has an **exponential distribution**.

We write $X \sim \text{Exp}(\lambda)$.

Remark (Expectation and Variance)

For $X \sim \text{Exp}(\lambda)$ we have that:

$$E[X] = \frac{1}{\lambda} \quad \text{and} \quad \text{Var}(X) = \frac{1}{\lambda^2}. \quad (1)$$

Exponential distribution

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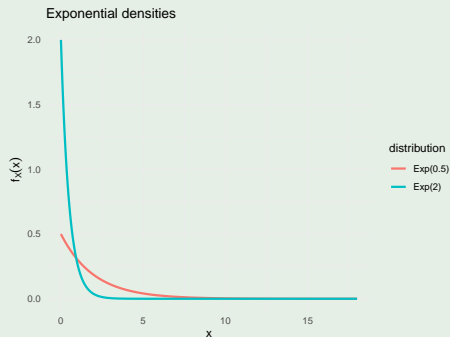
Remark (Applications and Properties)

X is typically applied to model the time until an event occurs, when events are always occurring at a rate $\lambda > 0$.

The sum of independent exponential *random variables* has a Gamma distribution (see the exercises).

Exponential distribution

Example (A graphical illustration)



Example (Illustration of use (Spoiler of the Exercises!))

The lifetime X in years of a television follows an exponential law with density:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

1. Compute the cumulative distribution function $F(x)$.
2. Compute the α -quantile $q_\alpha = F^{-1}(\alpha)$.
3. The expected life of your television is 8 years. What is the probability that the lifetime of your television is more than 8 years ? Evaluate the median.
4. Compute the variance of X for any λ .

Exponential distribution

Unlike the Normal, the CDF of an exponential has a closed-form expression

Example (CDF of Exponential)

Let $X \sim \text{Exp}(\lambda)$, with $\lambda = 0.5$. Thus

$$f_X(x) = \begin{cases} 0.5 \exp(-0.5x) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, find the CDF.

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Then, find the CDF.

For $x > 0$, we have

$$\begin{aligned} F_X(x) &= \int_0^x f_X(u) du \\ &= 0.5 \left(-2 \exp(-0.5u) \right) \Big|_{u=0}^{u=x} \\ &= 0.5(-2 \exp(-0.5x) + 2 \exp(0)) \\ &= 1 - \exp(-0.5x) \end{aligned}$$

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so, finally,

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ 1 - \exp(-0.5x) & x > 0 \end{cases}$$

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Variable Transformation

Remark (Problem)

Consider a random variable X and suppose we are interested in $Y = \psi(X)$, where ψ is a one to one function

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A **function** $\psi(x)$ is **one to one** (1-to-1) if there are no two numbers, x_1, x_2 in the domain of ψ such that $\psi(x_1) = \psi(x_2)$ but $x_1 \neq x_2$.

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Remark

A sufficient condition for $\psi(x)$ to be 1-to-1 is that it be monotonically increasing (or decreasing) in x .

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Remark

A sufficient condition for $\psi(x)$ to be 1-to-1 is that it be monotonically increasing (or decreasing) in x . Note that the inverse of a 1-to-1 function $y = \psi(x)$ is a

1-to-1 function $\psi^{-1}(y)$ such that

$$\psi^{-1}(\psi(x)) = x \text{ and } \psi(\psi^{-1}(y)) = y.$$

To transform X to Y , we need to consider all the values x that X can take
We first transform x into values $y = \psi(x)$

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Variable Transformation

Transformation of Discrete Random Variables

- To transform a discrete random variable X , into the random variable $Y = \psi(X)$, we transfer the probabilities for **each** x to the values $y = \psi(x)$:

Probability function for X

X	$P(\{X = x_i\}) = p_i$
x_1	p_1
x_2	p_2
x_3	p_3
\vdots	\vdots
x_n	p_n

\Rightarrow

Probability function for X

Y	$P(\{X = x_i\}) = p_i$
$\psi(x_1)$	p_1
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$\psi(x_n)$	p_n

Variable Transformation

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Probability function for X			Probability function for X	
X	$P(\{X = x_i\}) = p_i$	\Rightarrow	Y	$P(\{X = x_i\}) = p_i$
x_1	p_1		$\psi(x_1)$	p_1
x_2	p_2		$\psi(x_2)$	p_2
x_3	p_3		$\psi(x_3)$	p_3
\vdots	\vdots		\vdots	\vdots
x_n	p_n		$\psi(x_n)$	p_n

- Note that this is equivalent to applying the function $\psi(\cdot)$ inside the probability statements:

$$\begin{aligned}P(\{X = x_i\}) &= P(\{\psi(X) = \psi(x_i)\}) \\&= P(\{Y = y_i\}) \\&= p_i\end{aligned}$$

Variable Transformation

Transformation of Discrete Random Variables

Example (option pricing)

Let us imagine that we are tossing a balanced coin ($p = 1/2$), and when we get a “Head” (H) the stock price moves up of a factor u , but when we get a “Tail” (T) the price moves down of a factor d . We denote the price at time t_1 by $S_1(H) = uS_0$ if the toss results in head (H), and by $S_1(T) = dS_0$ if it results in tail (T). After the second toss, the price will be one of:

$$S_2(HH) = uS_1(H) = u^2S_0, \quad S_2(HT) = dS_1(H) = duS_0,$$

$$S_2(TH) = uS_1(T) = udS_0, \quad S_2(TT) = dS_1(T) = d^2S_0.$$

Indeed, after two tosses, there are four possible coin sequences,

$$\{HH, HT, TH, TT\}$$

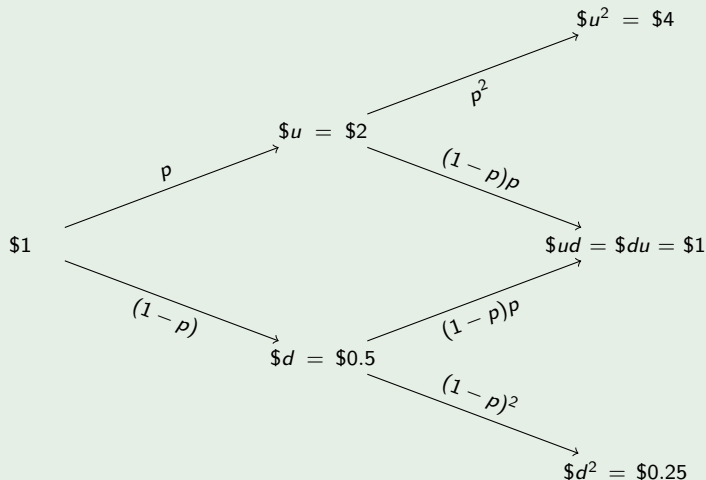
although not all of them result in different stock prices at time t_2 .

Variable Transformation

Transformation of Discrete Random Variables

Example (continued)

Let us set $S_0 = 1$, $u = 2$ and $d = 1/2$: we represent the price evolution by a tree:



Variable Transformation

Transformation of Discrete Random Variables

Example (continued)

Now consider an European option call with maturity t_2 and strike price $K = 0.5$, whose random pay-off at t_2 is $C = \max(0; S_2 - 0.5)$. Thus,

$$\begin{aligned} C(HH) &= \max(0; 4 - 0.5) = \$3.5 & C(HT) &= \max(0; 1 - 0.5) = \$0.5 \\ C(TH) &= \max(0; 1 - 0.5) = \$0.5 & C(TT) &= \max(0; 0.25 - 0.5) = \$0. \end{aligned}$$

Thus at maturity t_2 we have

Probability function for S_2		Probability function for C	
S_2	$P(\{X = x_i\}) = p_i$	C	$P(\{C = c_i\}) = p_i$
$\$u^2$	p^2	$\$3.5$	p^2
$\$ud$	$2p(1-p)$	$\$0.5$	$2p(1-p)$
$\$d^2$	$(1-p)^2$	$\$0$	$(1-p)^2$

Since $ud = du$ the corresponding values of S_2 and C can be aggregated, without loss of info.

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Transformation through the CDF

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Transformation through the CDF

- We can use the same logic for CDF probabilities, whether the random variables are **discrete or continuous**
- Let $Y = \psi(X)$ with $\psi(x)$ **1-to-1 and monotone increasing**. Then

$$\begin{aligned}F_Y(y) &= P(\{Y \leq y\}) \\&= P(\{\psi(X) \leq y\}) = P(\{X \leq \psi^{-1}(y)\}) \\&= F_X(\psi^{-1}(y))\end{aligned}$$

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Example

Let $Y = \psi(X) = \exp X$ where $X \sim F_X$ on all values $x \in \mathbb{R}$

$$\begin{aligned}F_Y(y) &= P(\{Y \leq y\}) \\&= P(\{\exp X \leq y\}) = P(\{X \leq \ln(y)\}) \\&= F_X(\ln(y)) \text{ only for } y > 0.\end{aligned}$$

Variable Transformation

Transformation through the CDF

- **Monotone decreasing functions** work in a similar way, but require **changing the sense of the inequality**.
- Let $Y = \psi(X)$ with $\psi(x)$ 1-to-1 and **monotone decreasing**. Then

$$\begin{aligned}F_Y(y) &= P(\{Y \leq y\}) \\&= P(\{\psi(X) \leq y\}) = P(\{X \geq \psi^{-1}(y)\}) \\&= 1 - F_X(\psi^{-1}(y))\end{aligned}$$

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Example

Example: let $Y = \psi(X) = -\exp X$ where $X \sim F_X$ on all values $x \in \mathbb{R}$

$$\begin{aligned}F_Y(y) &= P(\{Y \leq y\}) = P(\{-\exp^X \leq y\}) \\&= P(\{\exp X \geq -y\}) = P(\{X \geq \ln(-y)\}) \\&= 1 - F_X(\ln(-y)) \text{ only for } y < 0.\end{aligned}$$

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$$F_Y(y) = F_X(\psi^{-1}(y))$$

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Transformation of Continuous Random Variables through the PDF

- For continuous random variables, if $\psi(x)$ 1-to-1 and monotone **increasing**, we have

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- Notice this implies that the pdf of $Y = \psi(X)$ must satisfy

$$\begin{aligned} f_Y(y) &= \frac{dF_Y(y)}{dy} = \frac{dF_X(\psi^{-1}(y))}{dy} \\ &= \frac{dF_X(x)}{dx} \times \frac{d\psi^{-1}(y)}{dy} && \text{(chain rule)} \\ &= f_X(x) \times \frac{d\psi^{-1}(y)}{dy} && \text{(derivative of CDF (of } X) \text{ is pdf)} \\ &= f_X(\psi^{-1}(y)) \times \frac{d\psi^{-1}(y)}{dy} && \text{(substitute } x = \psi^{-1}(y) \text{)} \end{aligned}$$

Variable Transformation

Transformation of Continuous Random Variables through the PDF

- What happens when $\psi(x)$ 1-to-1 and monotone **decreasing**? We have

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- What happens when $\psi(x)$ 1-to-1 and monotone **decreasing**? We have

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- but $\frac{d\psi^{-1}(y)}{dy} < 0$ since here $\psi(\cdot)$ is monotone decreasing, hence we can write

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- This expression (called Jacobian-formula) is valid for $\psi(x)$ 1-to-1 and monotone (whether increasing or decreasing)

Variable Transformation

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Variable Transformation

Transformation of Continuous Random Variables through the PDF

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- So what is the pdf for the lognormal distribution?
- Recall that Y has a **lognormal distribution** when $\ln(Y) = X$ has a Normal distribution
- \Rightarrow if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = \exp X \sim \text{lognormal}(\mu, \sigma^2)$

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- The *pdf* of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x - \mu)^2\right\}$$

for any $-\infty < x < \infty$

Variable Transformation

Transformation of Continuous Random Variables through the PDF

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- Using $\psi(x) = \exp x$ we know we'll have possible values for Y only on $0 < y < \infty$

Variable Transformation

Transformation of Continuous Random Variables through the PDF

Example (continued)

- We know that

$$f_Y(y) = f_X(\psi^{-1}(y)) \times \left| \frac{d\psi^{-1}(y)}{dy} \right|$$

Variable Transformation

Transformation of Continuous Random Variables through the PDF

Example (continued)

- We know that

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- And since $\psi^{-1}(y) = \ln(y)$ then

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- \Rightarrow the *pdf* of Y is

$$f_Y(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{1}{2\sigma^2} (\ln(y) - \mu)^2 \right\}$$

for any $0 < y < \infty$

Variable Transformation

Transformation of Continuous Random Variables through the PDF

Example (continued)

- Both the Normal and the lognormal are characterized by only two parameters (μ and σ). The *median* of the lognormal distribution is $\exp \mu$, since

$$P(\{X \leq \mu\}) = 0.5,$$

and hence

$$\begin{aligned} 0.5 &= P(\{X \leq \mu\}) \\ &= P(\{\exp X \leq \exp \mu\}) \\ &= P(\{Y \leq \exp \mu\}). \end{aligned}$$

Variable Transformation

A Word of Warning

When X and Y are two random variables, **we should pay attention to their transformations.**

For instance, let us consider

$$X \sim \mathcal{N}(\mu, \sigma^2) \quad \text{and} \quad Y \sim \text{Exp}(\lambda).$$

Then, let's transform X and Y

- in a linear way: $Z = X + Y$. We know that

$$E[Z] = E[X + Y] = E[X] + E[Y]$$

- in a nonlinear way $W = X/Y$. One can show that

$$E[W] = E\left[\frac{X}{Y}\right] \neq \frac{E[X]}{E[Y]}.$$

Variable Transformation

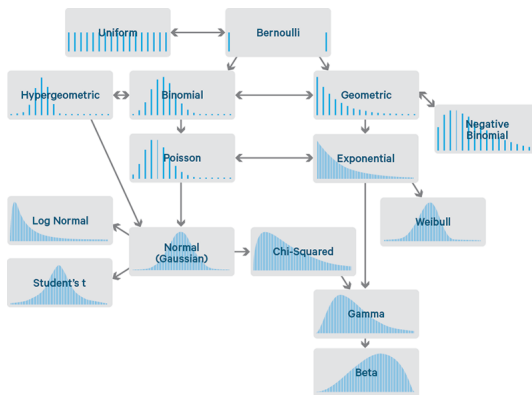
The big picture

Despite exotic names, the common distributions relate to each other in intuitive and interesting ways. Several follow naturally from the Bernoulli distribution, for example.

Variable Transformation

The big picture

- ▷ 'Common probability distributions: the data scientist's crib sheet'
(goo.gl/NJRIXn):



- The Exponential distribution helps to model *duration* data.
- To compute the probabilities of transformed Discrete Random Variables we proceed on the Probability Function:

$$P(X = x) = P(\psi(X) = \psi(x)) = P(Y = \psi(x))$$

- This principle applies to the inequalities in the CDF (give or take the sense of the monotonicity)

$$F_X(x) = P(X \leq x) = P(\psi(X) \leq \psi(x)) = P(Y \leq \psi(x)) = F_Y(\psi(x))$$

$$F_X(x) = P(X \leq x) = P(\psi(X) \geq \psi(x)) = P(Y \geq \psi(x)) = 1 - F_Y(\psi(x))$$

- The density of a transformed variable can be found with the formula:

$$f_Y(y) = f_X(\psi^{-1}(y)) \times \left| \frac{d\psi^{-1}(y)}{dy} \right|$$

Thank You for your Attention!

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“See you” Next Week