WORKED EXAMPLES 5

CONVERGENCE IN DISTRIBUTION

EXAMPLE 1: Continuous random variable X with range $\mathbb{X}_n \equiv (0, n]$ for $n > 0$ and cdf

$$F_{X_n}(x) = 1 - \left(1 - \frac{x}{n}\right)^n, \quad 0 < x \leq n.$$

Then as $n \to \infty$, the limiting support is $\mathbb{X} \equiv (0, \infty)$, and for all $x > 0$

$$F_{X_n}(x) \to 1 - e^{-x} \qquad \therefore \qquad F_{X_n}(x) \to F_X(x) = 1 - e^{-x},$$

and hence

$$X_n \xrightarrow{d} X, \quad X \sim Exponential(1).$$

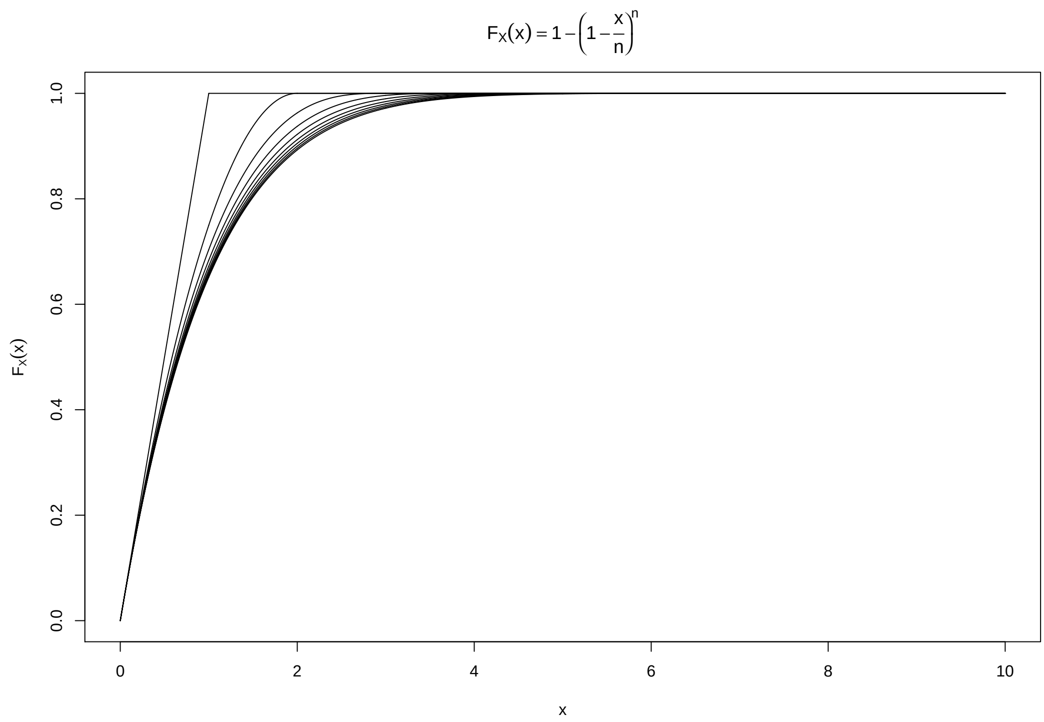

Figure 1: Example 1: for $n = 1, \ldots, 10$.

EXAMPLE 2: Continuous random variable X with range $\mathbb{X}_n \equiv \mathbb{X} = (0, \infty)$ and cdf

$$F_{X_n}(x) = \left(1 - \frac{1}{1+nx}\right)^n, \quad 0 < x < \infty.$$

Then as $n \to \infty$, for all $x > 0$

$$F_{X_n}(x) \to e^{-1/x} \qquad \therefore \qquad F_{X_n}(x) \to F_X(x) = e^{-1/x},$$

as

$$\lim_{n\to\infty} \left(1 - \frac{1}{1+nx}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{nx}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1/x}{n}\right)^n$$

and for any z

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z.$$

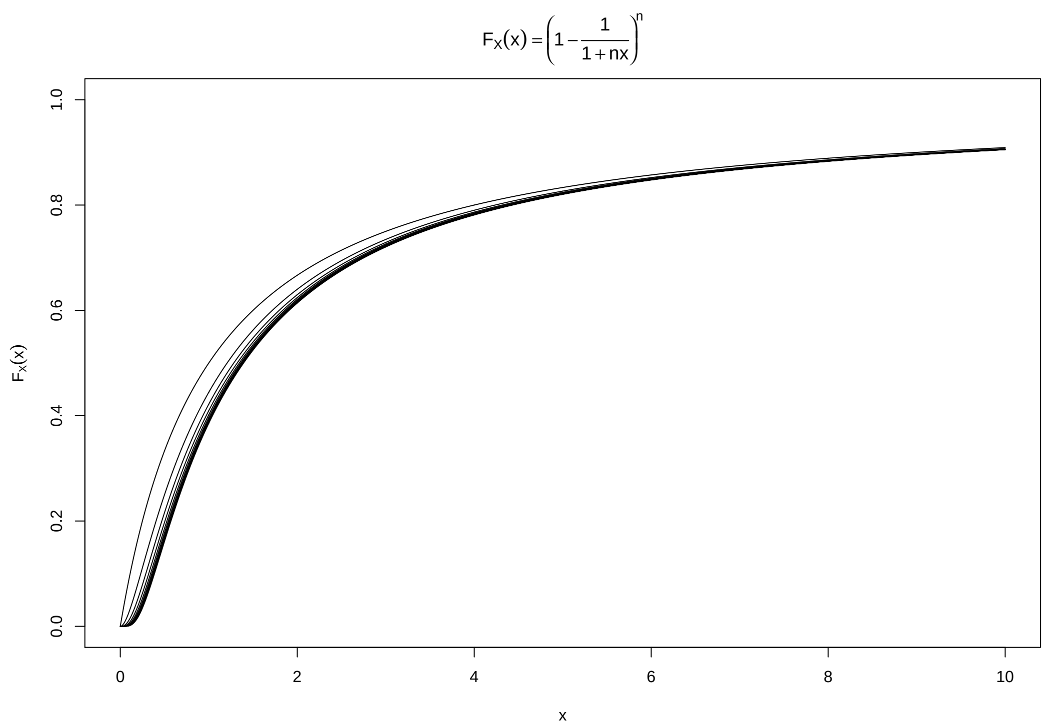

Figure 2: Example 2: for $n = 1, \ldots, 10$.

EXAMPLE 3: Continuous random variable X with range $\mathbb{X}_n \equiv \mathbb{X} = [0,1]$ and cdf

$$F_{X_n}(x) = x - \frac{\sin(2n\pi x)}{2n\pi}, \quad 0 \le x \le 1.$$

Then as $n \to \infty$, and for all $0 \le x \le 1$

$$F_{X_n}(x) \to x \qquad \therefore \qquad F_{X_n}(x) \to F_X(x) = x$$

and hence

$$X_n \xrightarrow{d} X, \quad \text{where } X \sim Uniform(0,1).$$

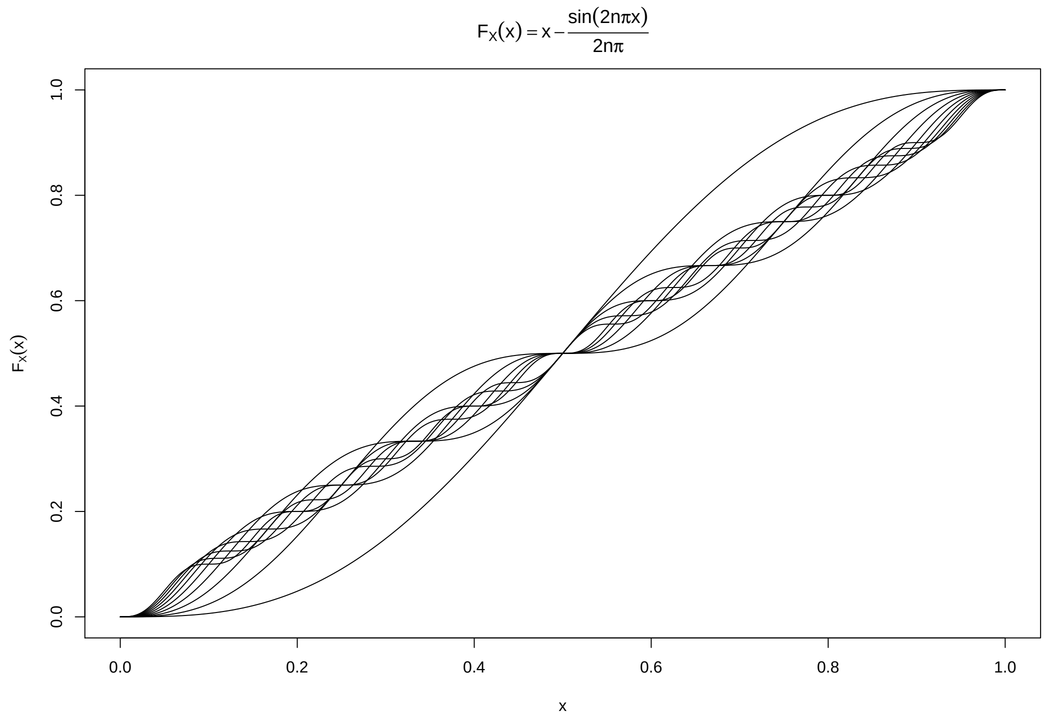

Figure 3: Example 3: for $n = 1, \ldots, 10$.

NOTE: for the pdf

$$f_{X_n}(x) = 1 - \cos(2n\pi x), \quad 0 \le x \le 1,$$

there **is no limit as** $n \to \infty$.

EXAMPLE 4: Continuous random variable X with range $\mathbb{X}_n \equiv \mathbb{X} = [0,1]$ and cdf

$$F_{X_n}(x) = 1 - (1-x)^n\,, \quad 0 \le x \le 1.$$

Then as $n \to \infty$, and for $x \in \mathbb{R}$

$$F_{X_n}(x) \to \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}.$$

This limiting form is **not** continuous at $x = 0$ and the **ordinary definition of convergence in distribution cannot be immediately applied to deduce convergence in distribution or otherwise**. However, it is clear that for $\epsilon > 0$,

$$P\left[|X| < \epsilon\right] = 1 - (1-\epsilon)^n \to 1 \text{ as } n \to \infty,$$

so it is correct to say

$$X_n \xrightarrow{d} X, \quad \text{where } P\left[X = 0\right] = 1,$$

so the limiting distribution is **degenerate at** $x = 0$.

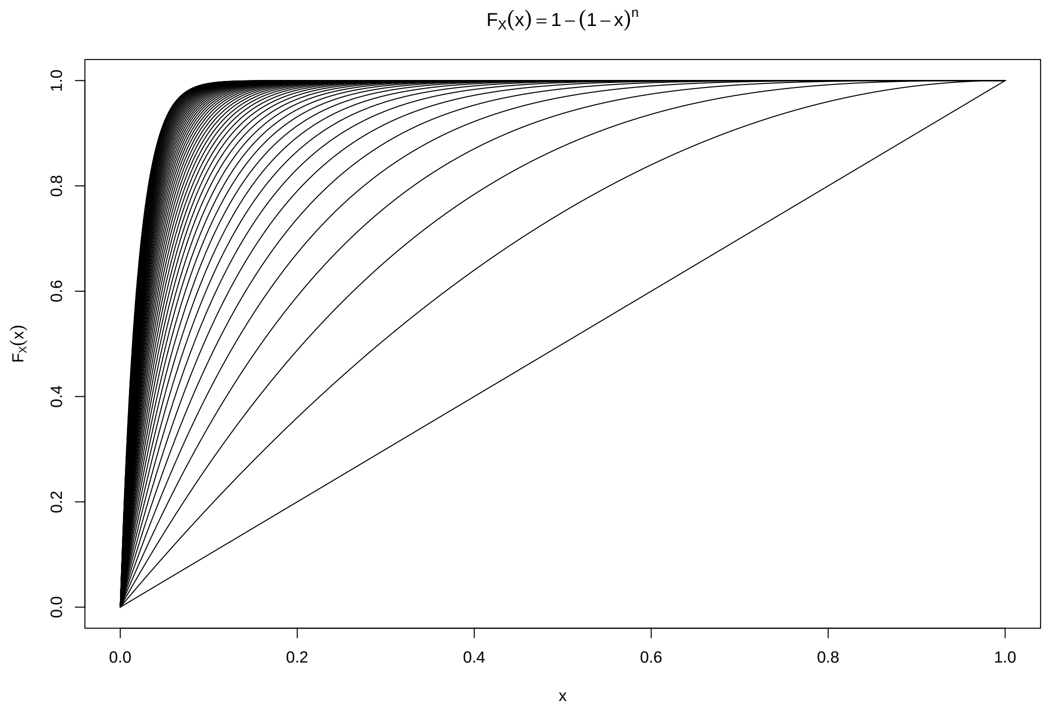

Figure 4: Example 4: for $n = 1, \ldots, 50$.

EXAMPLE 5: Continuous random variable X with range $\mathbb{X}_n \equiv \mathbb{X} = (0, \infty)$ and cdf

$$F_{X_n}(x) = \left(\frac{x}{1+x}\right)^n, \quad x > 0.$$

Then as $n \to \infty$, and for any $x > 0$

$$F_{X_n}(x) \to 0.$$

Thus there is **no limiting distribution**.

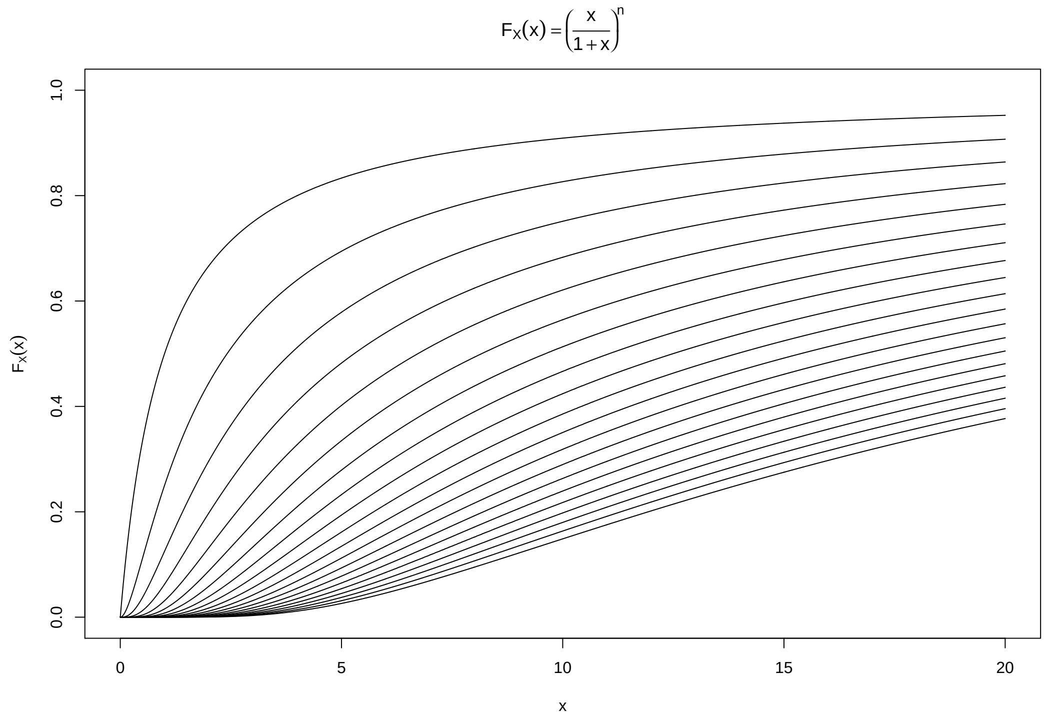

Figure 5: Example 5: for $n = 1, \ldots, 20$.

Now let $V_n = X_n/n$. Then $\mathbb{V}_n \equiv \mathbb{V} = (0, \infty)$ and the cdf of V_n is

$$F_{V_n}(v) = P\left[V_n \leq v\right] = P\left[X_n/n \leq v\right] = P\left[X_n \leq nv\right] = F_{X_n}(nv) = \left(\frac{nv}{1+nv}\right)^n, \quad v > 0,$$

and as $n \to \infty$, for all $v > 0$

$$F_{V_n}(v) \to e^{-1/v} \qquad \therefore \qquad F_{V_n}(v) \to F_V(v) = e^{-1/v},$$

and the limiting distribution of V_n **does** exist.

EXAMPLE 6: Continuous random variable X with range $\mathbb{X}_n \equiv \mathbb{X} = (-\infty, \infty)$ and cdf

$$F_{X_n}(x) = \frac{\exp(nx)}{1+\exp(nx)}, \quad x \in \mathbb{R}.$$

Then as $n \to \infty$

$$F_{X_n}(x) \to \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & x = 0, \\ 1, & x > 0, \end{cases} \qquad x \in \mathbb{R}.$$

This limit is **not** a cdf, as it is not right continuous at $x = 0$. However, as $x = 0$ is not a point of continuity, convergence in distribution, or otherwise, is not immediately obvious from the definition. However, it is clear that for $\epsilon > 0$,

$$P\left[|X| < \epsilon\right] = \frac{\exp(n\epsilon)}{1+\exp(n\epsilon)} - \frac{\exp(-n\epsilon)}{1+\exp(-n\epsilon)} \to 1 \text{ as } n \to \infty,$$

so it is correct to say

$$X_n \xrightarrow{d} X, \quad \text{where } \mathrm{P}\left[X = 0\right] = 1,$$

and the limiting distribution is **degenerate at** $x = 0$.

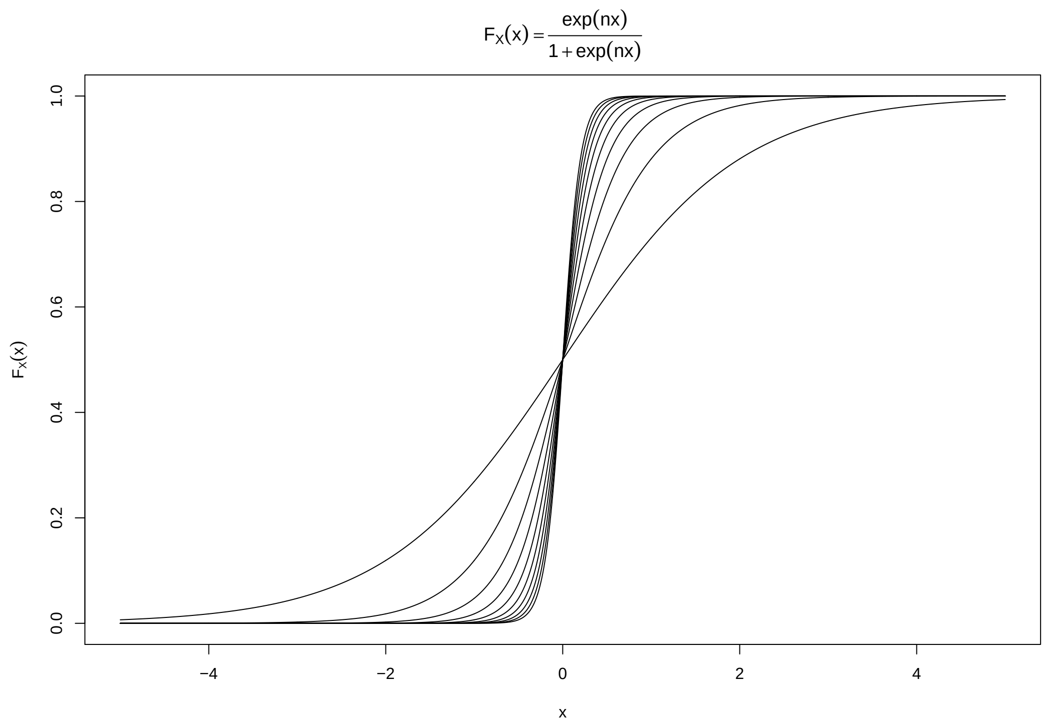

Figure 6: Example 6: for $n = 1, \ldots, 10$.