

Exercise 1:

Let X be the yearly income of an employee and $E(X) = 50'000$ its expectation.

1. We use the Markov inequality to find a upper bound for the pourcentage p of the incomes bigger or equal to 80'000 \$:

$$P(X \geq 80'000) \leq \frac{50'000}{80'000} = \frac{5}{8} = 0.625.$$

Less than 62.5% of the employees of the bank earn more than 80'000 \$.

2. We know now the standard error of X and this allow us to define a more accurate upper bound using the Chebychev inequality:

$$\begin{aligned} P(X \geq 80'000) &= P(X - 50'000 \geq 30'000) \\ &\leq P(|X - 50'000| \geq 30'000) \leq \frac{10'000^2}{30'000^2} = \frac{1}{9} \approx 0.11. \end{aligned}$$

Exercise 2:

Let X_i be a Bernouilli random variable associated to the i th elector such that:

$$X_i = \begin{cases} 1 & \text{if } i \text{ vote for Barack,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = \sum_{i=1}^{1'200} X_i$ the number of voters for Barack. $Y \sim B(1200, p)$ We want to find p such that:

$$P\{Y \geq 600\} = 0.95.$$

We have $E(Y) = 1'200 \cdot p$, $Var(Y) = 1'200 \cdot p(1 - p)$ and

$$\begin{aligned} P(Y \geq 600) &= P\left(\frac{Y - 1'200 \cdot p}{\sqrt{1'200 \cdot p(1 - p)}} \geq \frac{600 - 1'200 \cdot p}{\sqrt{1'200 \cdot p(1 - p)}}\right) \\ &= 1 - P\left(\frac{Y - 1'200 \cdot p}{\sqrt{1'200 \cdot p(1 - p)}} < \frac{600 - 1'200 \cdot p}{\sqrt{1'200 \cdot p(1 - p)}}\right). \end{aligned}$$

Using the Central limit theorem:

$$P\left(\frac{Y - 1'200 \cdot p}{\sqrt{1'200p(1-p)}} < \frac{600 - 1'200 \cdot p}{\sqrt{1'200 \cdot p(1-p)}}\right) \approx \Phi\left(\frac{600 - 1'200 \cdot p}{\sqrt{1'200 \cdot p(1-p)}}\right).$$

We look then for p such that:

$$\Phi\left(\frac{600 - 1'200 \cdot p}{\sqrt{1'200 \cdot p(1-p)}}\right) = 0.05$$

so:

$$\frac{600 - 1'200 \cdot p}{\sqrt{1'200 \cdot p(1-p)}} = -1.645$$

and we find $p \approx 0.52$.

Exercise 3:

Let X be the number of registrations, it is a Poisson random variable with parameter $\lambda = 100$. We search

$$P(X \geq 120) = e^{-100} \sum_{x=120}^{\infty} \frac{100^x}{x!} = 1 - e^{-100} \sum_{x=0}^{119} \frac{100^x}{x!}.$$

The numerical calculation of such an expression is not easy.

We can use the property that a Poisson random variable with parameter 100 can be seen as the sum of 100 independent random variables of Poisson with parameter 1. We can thus use the limit central theorem. In other words, X , which follows a distribution of $Poisson(\lambda = 100)$ will also follow approximately a normal distribution of parameters.

$$\mu = E(X) = 100 \cdot 1 = 100 \quad \text{and} \quad \sigma^2 = \text{var}(X) = 100 \cdot 1 = 100.$$

So,

$$\begin{aligned} P(X \geq 120) &= P\left(\frac{X - 100}{\sqrt{100}} \geq \frac{120 - 100}{\sqrt{100}}\right) \\ &\approx 1 - \Phi(2) = 0.0228. \end{aligned}$$

Exercise 4:

1. In this table we report the joint probability distribution of the two dice:

	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

The possible values of S are 0, 1 or 2. In this table above, the left top square correspond to $S = 0$, the right bottom square to $S = 2$ and the other two correspond to $S = 1$. We have then the distribution of S :

s_i	0	1	2
$p_{s_i} = P(S = s_i)$	16/36	16/36	4/36

2. From the joint distribution of the two dice we can deduce par counting the joint probability of R and S :

$S \backslash R$	1	2	3	4	5	6
0	1/36	3/36	5/36	7/36	0	0
1	0	0	0	0	8/36	8/36
2	0	0	0	0	1/36	3/36

3. Counter-exemple: $P(R = 1, S = 1) = 0 \neq P(R = 1) \cdot P(S = 1) = (1/6) \cdot (16/36) = 16/36^2$. X et Y are then dependent.

4. By definition:

$$P(R = r_i | S = 1) = \frac{P(R = r_i, S = 1)}{P(S = 1)}, \quad \text{with } r_1 = 1, \dots, r_6 = 6.$$

As $P(S = 1) = 16/36$, we obtain:

$r_i S = 1$	5	6
$P(R = r_i S = 1)$	1/2	1/2

Exercise 5:

1. The probability distribution of X is

$$P(X = x) = \frac{1}{4}$$

for all $x = 1, 2, 3, 4$. And the distribution of $Y | X = x$ is:

$$P(Y = y | X = x) = \frac{1}{10 - (x - 1)} \quad (1)$$

for all $y = x, x + 1, \dots, 10$.

2. The joint distribution of X and Y can be obtained by the following connection:

$$P(X = x, Y = y) = P(Y = y | X = x)P(X = x) = \frac{1}{10 - (x - 1)} \times \frac{1}{4}$$

for $y \geq x$ and $x = 1, 2, 3, 4$.

In the table it is:

	$Y = 1$	$Y = 2$	$Y = 3$	$Y = 4$	$Y = 5$	$Y = 6$	$Y = 7$	$Y = 8$	$Y = 9$	$Y = 10$
$X = 1$	1/40	1/40	1/40	1/40	1/40	1/40	1/40	1/40	1/40	1/40
$X = 2$	0	1/36	1/36	1/36	1/36	1/36	1/36	1/36	1/36	1/36
$X = 3$	0	0	1/32	1/32	1/32	1/32	1/32	1/32	1/32	1/32
$X = 4$	0	0	0	1/28	1/28	1/28	1/28	1/28	1/28	1/28

Exercise 6 (Optional):

- Using the hint, we know $P(Z \geq \zeta) = P(\exp^{\{Z\}} \geq \exp^{\{\zeta\}})$

Let $\exp^{\{Z\}}$ be a random variable X . So we can write down its distribution:

x_i	$\exp^0 = 1$	\exp^1
$p_{x_i} = P(X = x_i)$	$1 - p$	p

$$P(Z \geq \zeta) = P(\exp^{\{Z\}} \geq \exp^{\{\zeta\}}) = P(X \geq \exp^{\{\zeta\}})$$

By making use of Markov's inequality,

$$P(X \geq \exp^{\{\zeta\}}) \leq \frac{E(X)}{\exp^{\{\zeta\}}} = \frac{p \cdot 1 + (1 - p) \cdot \exp^{\{1\}}}{\exp^{\{\zeta\}}} = \frac{1 - p(1 - \exp^{\{1\}})}{\exp^{\{\zeta\}}}$$

- X is the sum of a sequence of independent and identically distributed Bernoulli random variables. So $X \sim \text{Binomial}(n, p)$.

2.1 $X \sim B(n, p)$. X is a discrete random variable. $E(X) = np$

2.2 We know $E(X) = np$, so we replace np in Equation (1) by $E(X)$.

$$\begin{aligned}
 RHD &= E[X(X - 1)] + np(1 - np) \\
 &= E[X(X - 1)] + E(X)[1 - E(X)] \\
 &= E(X^2) - E(X) + E(X) - E(X)^2 \\
 &= E(X^2) - E(X)^2 \\
 &= \text{Var}(X)
 \end{aligned}$$

2.3 We know $E[X^2] \leq n^2\beta^2$, and we subtract $E(X)^2$ for two sides, finding that

$$E[X^2] - E(X)^2 \leq n^2\beta^2 - E(X)^2$$

The left hand side is $\text{Var}(X)$, and the right hand side is $n^2\beta^2 - n^2p^2$.
So $\text{Var}(X) \leq n^2(\beta^2 - p^2)$

2.4 By making use of Chebyshev's inequality,

$$P(|X - np| \geq nE[Y]) \leq \frac{\text{Var}(X - np)}{(nE(Y))^2}$$

$$\text{Var}(X - np) = \text{Var}(X) = E[X(X - 1)] + np(1 - np) \leq n^2(\beta^2 - p^2)$$

As Y is a Poisson random variable, $E(Y) = \lambda$.

So

$$\begin{aligned} P(|X - np| \geq nE[Y]) &\leq \frac{E[X(X - 1)] + np(1 - np)}{(nE[Y])^2} \\ &\leq \frac{n^2(\beta^2 - p^2)}{(n\lambda)^2} = \left(\frac{\beta - p}{\lambda}\right) \left(\frac{\beta + p}{\lambda}\right) \end{aligned}$$