Probability 1

Lecture 01 : A reminder of Mathematics

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(based on the notes of Prof. Davide La Vecchia)

Spring Semester 2021

Outline

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Powers and Logs

Powers

- $a^m \cdot a^n = a^{m+n}$
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- $\ln(a^n) = n \cdot \ln a$
- $\ln(a \cdot b) = \ln(a) + \ln(b)$;

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Derivatives

Remember the **derivatives** of the power-to-n, exp and log functions

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Derivation Rules

- Product rule:

$$\begin{aligned} \frac{d[f(x) \cdot g(x)]}{dx} &= \frac{df(x)}{dx} g(x) + \frac{dg(x)}{dx} f(x) \\ &= f'(x)g(x) + f(x)g'(x); \end{aligned}$$

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- Chain rule:

$$\frac{df[g(x)]}{dx} = (f \circ g)'(x) = f'[g(x)] \cdot g'(x).$$

Outline

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- The integral is a **linear** operator

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- For a continuous function $f(x)$, the indefinite integral is

$$\int f(x) dx = F(x) + \text{const}$$

while the definite integral is

$$F(b) - F(a) = \int_a^b f(x) dx, \quad b \geq a.$$

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- Double sum: a sum with two indices. For instance,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j \quad = \quad x_1 y_1 + x_1 y_2 + ... + x_2 y_1 + x_2 y_2 + ...$$

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$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j &= x_1 y_1 + x_1 y_2 + ... + x_2 y_1 + x_2 y_2 + ... \\
&= \left(\sum_{i=1}^{n} x_i\right) y_1 + \left(\sum_{i=1}^{n} x_i\right) y_2 + ... + \left(\sum_{i=1}^{n} x_i\right) y_m \\
&= \sum_{i=1}^{n} x_i \sum_{j=1}^{m} y_j.
\end{aligned}$$

Outline

Combinatorial Formulas

We will rely on some **combinatorial formulas**.

- **Factorial**

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1;$$

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Useful to compute the number of **Permutations** or **Combinations** of a set.

Remark

In how many different ways can we combine n objects?

- *In the 1st place:* n *possibilities*
- *In the 2nd place:* $(n-1)$ *possibilities*
- *...*
- *Finally,* 1 *possibility*

Thus, in total: $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!$

Permutations

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Example

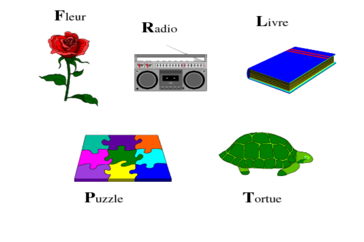
How many ways can Aline, Bridget and Carmen seat on 3 spots, from left to right?

$$(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)$$

Total # of permutations: $N = 6 = 3 \cdot 2 \cdot 1 = 3!$

Example

How many ways can we select 3 presents among 5 available presents? Assume the order does not matter!



(F, R, L)	(F, R, P)	(F, R, T)	(F, L, P)
(F, L, T)	(F, P, T)	(R, L, P)	(R, L, T)
(R, P, T)	(L, P, T)		

Total: $N = 10$

Example (Interpretation)

1. 5! gives you the total # permutations of the presents.
2. Selecting 3 presents implies **not selecting** $(5-3)$ other presents.
   - Therefore, we need to divide or **"factor out"** the $(5-3)!$ permutations of the non-selected present from the 5! permutations of all presents.

   In other words you have $5!/(5-3)!$ ways to select and order the 3 presents;
3. Finally, **we don't care about the order** of selection of the 3 presents.
   - All the permutations of this selection are deemed equivalent.
   - e.g. (F, R,L) is the same as (R, F, L)

   Since there will be 3! permutations of the selection. This number needs to be **factored out**.

Example (continued)

... so, in formula, you have

$$\frac{5!/(5-3)!}{3!}$$

ways to select the 3 presents:

$$\frac{5!/(5-3)!}{3!} = \frac{5!}{3!2!} = \binom{5}{3} = C_5^3.$$

This gives you the total # of possible ways to select the 3 presents when the order does not matter.

As an exercise, redo the exercise assuming you can select 2 presents from 3 available presents (like for instance F,L,R). Create all the combinations and then use the formulas to see how they coincide.

Combinatorics

Remark

Combinations: *How many ways can we select k objects among n?*

1. *How many ways can we combine k objects among n?*
    - *In the 1st place:* n
    - *In the 2nd place:* $(n-1)$
    - *....*
    - *In the k-th place:* $(n-k+1)$
2. *We have $k!$ ways to permute the k objects that we selected*
3. *The number of possibilities (without considering the order) is:*

$$\frac{n!/(n-k)!}{k!} = \frac{n!}{k!(n-k)!} = C_n^k$$

Remark (cont'd)

In Problem Set 2, you will have to make use of C_n^k in Ex2-Ex3-Ex5. Indeed, to compute the probability for an event E, will have to make use of the formula:

$$P(E) = \frac{\text{number of cases in } E}{\text{number of possible cases}}. \tag{1}$$

This is a first intuitive definition of probability, which we will justify in the next lecture;

For the time being, let us say that the combinatorial calculus will be needed to express both the quantities (numerator and denominator) in (1).

Outline

Limits

- The limit of a finite sum is an infinite series:

$$\lim_{n\to\infty} \sum_{i=1}^{n} x_i = \sum_{i=1}^{\infty} x_i$$

- The exp() function, characterised as a limit:

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

- The Limit of a Negative Exponential function :

$$\text{for } \alpha > 0, \quad \lim_{x\to\infty} \alpha e^{-\alpha x} = 0$$

- The Exponential function, characterised as an infinite series

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Wrap-up

Reminder of:

- The power-to-n, $\exp()$ and $\ln()$ functions.
- Elements of differential calculus (derivatives and integrals)
- Sums, Double Sums
- Limits

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- The power-to-n, $\exp()$ and $\ln()$ functions.
- Elements of differential calculus (derivatives and integrals)
- Sums, Double Sums
- Limits

Most importantly, **Combinatorics**

- A set with n elements has $n!$ **possible permutations**
- There are $n!/(n-k)!$ ways of choosing k elements from n **when the order is important**
- There are $n!/(k!(n-k)!)$ ways of choosing k elements from n **when the order is important**

In English, the symbol $\binom{n}{k} = C_k^n$ reads **"n choose k"**

Thank you for your attention!

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"See you" next week!