UNIVERSITY OF GENEVA
Probability 1
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ASSIGNMENT 01

BACHELOR IN ECONOMICS AND MANAGEMENT
BACHELOR IN INTERNATIONAL RELATIONS
Spring 2021
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Before Start

- problem sets are not graded
- the problem set will be posted on-line one week before its correction in class
- if you have any problem you can write me an e-mail: chaonan.jiang@unige.ch

1 Integrals

Compute the following integrals:

$$\int 5x^3 - 10x^{-6} + 4dx$$

$$\int \pi^8 + \pi^{-8}dx$$

$$\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}dx$$

$$\int \frac{15}{x}dx$$

$$\int exp(x)dx$$

$$\int_1^2 y^2 + y^{-2}dy$$

$$\int_{-3}^1 6y^2 - 5y + 2dy$$

$$\int_0^u \lambda exp(\lambda y)dy$$

$$\int_{-\infty}^u \frac{1}{2b}exp\left(-\frac{|y-\mu|}{b}\right)dy, \quad \text{for} \quad \mu \in \mathbb{R} \quad \text{and} \quad b \in \mathbb{R}^+.$$

Solution

$$
\begin{aligned}
&F(x) = \frac{5}{4}x^4 + \frac{-10}{-5}x^{-5} + 4x + c\\
&F(x) = \pi^8 \int dx + \pi^{-8} \int dx = x(\pi^8 + \pi^{-8}) + c\\
&F(x) = \int 3x^{\frac{3}{4}} + 7x^{-5} + \frac{1}{6}x^{-\frac{1}{2}}dx = 3\frac{1}{\frac{7}{4}}x^{\frac{7}{4}} - \frac{7}{4}x^{-4} + \frac{1}{6}\frac{1}{\frac{1}{2}}x^{\frac{1}{2}} + c\\
&F(x) = 15log|x| + c\\
&F(y) = \left[\frac{1}{3}y^3 - \frac{1}{y}\right]_1^2 = \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 = \frac{17}{6}\\
&F(y) = \left[2y^3 - \frac{5}{2}y^2 + 2y\right]_{-3}^1 = 2 - \frac{5}{2} + 2 - \big(- 54 - \frac{45}{2} - 6\big) = 84\\
&F(x) = exp(x)\\
&F(y) = \left[\frac{\lambda}{\lambda}exp(\lambda y)\right]_0^u = exp(\lambda u) - 1\\
&F(y) = \begin{cases} \frac{1}{2}exp[(u-\mu)/b] & \text{if } u < \mu \\ 1 - \frac{1}{2}exp[-(u-\mu)/b] & \text{if } u \geq \mu \end{cases}
\end{aligned}
$$

2 Combinatronics

1. we draw (with replacement) k elements from n objects.
   Drawing the 1_{st} element has n possible outcomes. Since we draw with replacement, the 1_{st} element we choose could be drawn again. Then drawing the 2_{nd} element also has n possibilities. So we can say, each element has the same n possibilities. Then we have $\underbrace{n \times n \cdots n \times n}_{\text{k times}} = n^k$ total possibilities.

2. we draw (without replacement) n elements from n objects.
   Drawing the 1_{st} element has n possible outcomes. The 2_{nd} has the remaining $n-1$ possibilities. For the n_{th} element, we have only one left. Then we have $\underbrace{n \times (n-1) \cdots 2 \times 1}_{\text{n times}} = n!$ total possibilities.

3. we draw (without replacement) k elements from n objects.
   Drawing the 1_{st} element has n possible outcomes. The 2_{nd} has the remaining $n-1$ possibilities. For the k_{th} element, we have $n-(k-1) = n-k+1$ elements left. Then we have $\underbrace{n \times (n-1) \cdots (n-k+2) \times (n-k+1)}_{\text{k times}} = \dfrac{n!}{(n-k)!}$ total possibilities.

4. Same as a partial permutation but the order of the items doesn't count.
   For example, we draw two from elements A, B, C. We have {A, B}, {B, C}, {A, C} total outcomes. {A, B}={B, A}, {B, C}= {C, B}, {A, C}= {C, A}. We can find

$P_2^3 = P_2^2 \times \binom{3}{2}$. In general, drawing k from n without order and repetition, we have $\binom{n}{k} = \frac{P_k^n}{P_k^k}$ possibilities.

To proof Theorem:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \quad \text{for } 1 \leq r \leq n \tag{1}$$

In the left hand side, we draw r from n, so we have $\binom{n}{r}$ outcomes.
In the right hand side, we first take one element aside:

- Suppose that this element is inside the r elements we choose, we have $r-1$ which need to be drawn from the remaining $n-1$ elements, that is $\binom{n-1}{r-1}$.
- Suppose that we draw r elements without this element, we draw r from the remaining $n-1$ elements, that is $\binom{n-1}{r}$.

So we have $\binom{n-1}{r-1} + \binom{n-1}{r}$ possible outcomes. Left = Right

1) Assume there are 6 women and 4 men taking an exam. Then they are ranked according to their grades.

- 10 people are ranked. That's the case of permutation without repetition. So there are 10! possible outcomes.
- There are 6! possibilities for women and 4! possibilities for men. So there are $6! \cdot 4!$ total possibilities.

2) How many car number plates with 7 items does it exist:

- if the first two are letters and the five last are digits?
  permutation with repetition: we have 26 letters and 10 digits. For the first two letters, we have 26^2 outcomes. For the last five digits, there are 10^5 possibilities. In total, $26^2 \cdot 10^5$ possible outcomes.
- same question but we assume that there is no repetition of the letters and the digits.
  permutation without repetition: we have 26 letters and 10 digits. For the first two letters, we have $26 \cdot 25$ outcomes. For the last five digits, there are $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ possibilities. In total, $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ possible outcomes.

3) Suppose that a fair coin is tossed five times. The outcome of the experiment is registered with H if the result of the tossing is head and with T if the outcome is tail (the outcomes would look like $HHTHT, THTTH$ and so on).

- How many possible outcomes do we have?
  permutation with repetition: 2^5 possible outcomes.
- How many of these five dimensional outcomes would include two heads?
  We draw 2 heads from 5 without order: $\binom{5}{2} = \frac{5!}{2! \cdot (5-2)!} = \frac{5 \cdot 4}{2} = 10$

4) There are $\binom{20}{3} = 1140$ possible committees.

3 Taylor Expansions

1) Using binomial theorem, compute the following expansions:

$$
\begin{aligned}
&(x+y)^2 \\
&(x-y)^2 \\
&x^2-y^2 \\
&(x+y)^3 \\
&x^3+y^3
\end{aligned}
$$

Solution

- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x-y)^2 = x^2 - 2xy + y^2$
- $x^2 - y^2 = (x+y)(x-y)$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
  $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$
- $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
  $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

2) Find the Maclaurin expansions of the following functions $f(x) = \frac{1}{1-x}$ and $g(x) = exp(x)$.

Solution

$$
\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + x^5 + ... \\
exp(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + ...
\end{aligned}
$$

4 Sums

Compute the sum of the first 100 integers, namely find the value of

$$\sum_{i=1}^{100} i = 1 + 2 + 3 + ... + 100.$$

Solution

$$\sum_{i=1}^{100} i = \frac{100(100+1)}{2} = 5050,$$

or more generally, for $n > 1$ and $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$