Geodynamics - Homework 2

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All problems extracted from (Turcotte, Schubert and Turcotte, 2002).

Problem 2.2

A mountain range has an elevation of 5 km. Assuming that $\rho_m = 3300\ kg.m^{-3}$, $\rho_c = 2800\ kg.m^{-3}$, and that the reference or normal continental crust has a thickness of 35 km, determine the thickness of the continental crust beneath the mountain range. Assume that hydrostatic equilibrium is applicable.

Solution

First, we have to make a diagram of the layers:

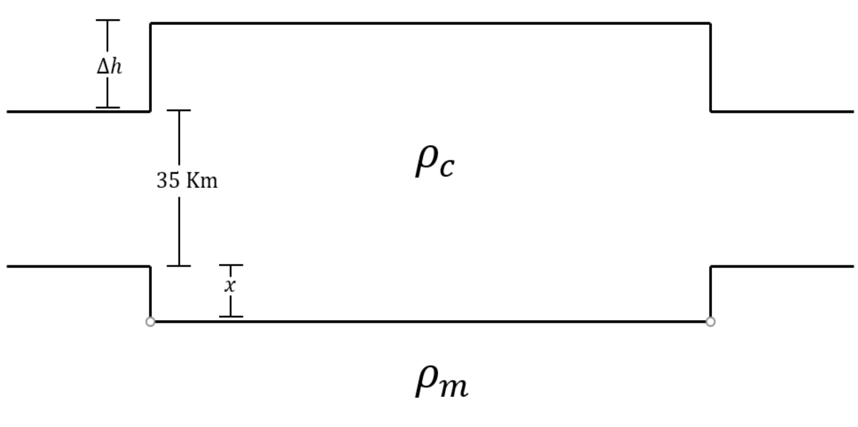

Figure 1: Scheme 2.2

Then:

$$\rho_c(\Delta h + 35\ km + x) = \rho_m(x) + \rho_c(35\ km) \tag{1}$$

$$\Delta h\rho_c + \rho_c(x) = \rho_m(x) \tag{2}$$

$$\Delta h\rho_c = x(\rho_m - \rho_c) \tag{3}$$

$$x = \frac{\Delta h\rho_c}{\rho_m - \rho_c} \tag{4}$$

$$x = \frac{5000\ m\ 2800\ kg/m^3}{500\ kg/m^3} = 28\ km \tag{5}$$

$$thickness = x + 35\ km + 5\ km = 28\ km + 40\ km = 68\ km \tag{6}$$

Problem 2.3

There is observational evidence from the continents that the sea level in the Cretaceous was 200 m higher than today. After a few thousand years, however, the seawater is in isostatic equilibrium with the ocean basins. What was the corresponding increase in the depth of the ocean basins? Take $\rho_w = 1000\ \ kg/m^3$ and the density of the displaced mantle to be $\rho_m = 3300\ \ kg/m^3$.

Solution

$\rho_w = 1000\ kg/m^3$ $\rho_m = 3300\ \ kg/m^3$

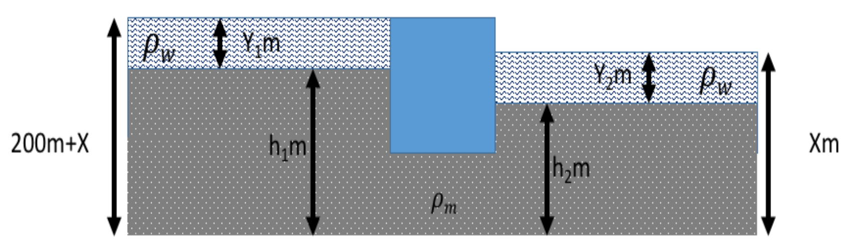

Figure 2: Representation of problem 2.3

From the Figure we obtain

$$y_1 + h_1 = 200 + y_2 + h_2 \tag{7}$$

$$y_1 - y_2 = 200 + h_2 - h_1 \tag{8}$$

Now if we calculate the the hydrostatic equilibrium in the Cretaceous and today and we equate bout we obtain

$$\rho_w * y_1 + \rho_m * h_1 = \rho_w * y_2 + \rho_m * h_2 \tag{9}$$

$$\rho_w(y_1 - y_2) = \rho_m(h_2 - h_1) \tag{10}$$

$$\frac{\rho_w}{\rho_m} * (y_1 - y_2) = (h_2 - h_1) \tag{11}$$

using equation 8

$$\frac{\rho_w}{\rho_m} * (200 + h_2 - h_1) = (h_2 - h_1) \tag{12}$$

Now we see that the differences in depth is $h_2 - h_1$ and this is the value that we need to calculated. so let us call this differences x were $x = h_2 - h_1$

$$\frac{\rho_w}{\rho_m} * (200 + x) = x \tag{13}$$

$$\frac{\rho_w}{\rho_m} * 200 = x - \frac{\rho_w}{\rho_m} * x \tag{14}$$

$$\frac{\rho_w}{\rho_m} * 200 = x(1 - \frac{\rho_w}{\rho_m}) \tag{15}$$

Finally

$$x = \frac{\rho_w * 200}{\rho_m - \rho_w} = \frac{1000kg/m^3 * 200m}{3300kg/m^3 - 1000kg/m^3} = 86,95m \tag{16}$$

Answer is $86,95\ m + 200\ m = 286,95\ m$

Problem 2.6

Asimple model for a continental mountain belt is the crustal compression model illustrated in Figure 3. A section of the continental crust of width w_0 is compressed to a width w_{mb}. The compression factor β is defined by $\beta = \frac{w_o}{w_{mb}}$ Show that the height of the mountain belt h is given by

$$h = h_{cc}\frac{\rho_m - \rho_{cc}}{\rho_m}(\beta - 1) \tag{17}$$

Solution

In the first place, we should consider mass conservation, which translates into area conservation in figure 3:

$$w_0 h_{cc} = w_{mb}(h + h_{cc} + b) \tag{18}$$

$$\frac{w_0}{w_{mb}} h_{cc} = h + h_{cc} + b \tag{19}$$

$$h_{cc}(\beta - 1) = h + b \tag{20}$$

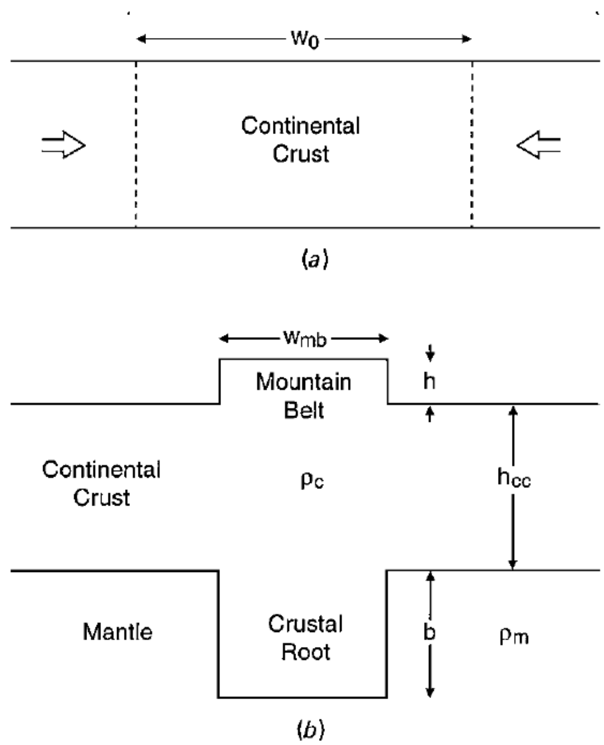

Figure 3: Crustal compression model. Taken from [?]

Then, hydrostatic equilibrium is required so we get the following

$$\rho_{cc}(h_{cc} + h + b) = \rho_{cc}h_{cc} + \rho_m b \tag{21}$$

$$\rho_{cc}(h + b) = \rho_m b \tag{22}$$

$$\rho_{cc}(h + b) + \rho_m h = \rho_m b + \rho_m h \tag{23}$$

$$\rho_{cc}(h + b) + \rho_m h = \rho_m (h + b) \tag{24}$$

$$\rho_m h = (\rho_m - \rho_{cc})(h + b) \tag{25}$$

$$h = \frac{(\rho_m - \rho_{cc})(h + b)}{\rho_m} \tag{26}$$

Finally, we simply subtitute equation 20 into 26 obtaining

$$\boxed{h = h_{cc}\frac{(\rho_m - \rho_{cc})}{\rho_m}(\beta - 1)} \tag{27}$$

Which is the same as equation 17.

qed

Problem 2.10

Consider a block of rock with a height of 1 m and horizontal dimensions of 2 m. The density of the rock is 2750 $kg.m^{-3}$. If the coefficient of friction is 0.8, what force is required to push the rock on a horizontal surface?

Solution

$$\sum F_x = F - friction = 0 \tag{28}$$

$$F = friction \tag{29}$$

$$\sum F_y = N - mg = 0 \tag{30}$$

$$N = mg \tag{31}$$

$$friction = \mu N = \mu mg = \mu \rho V g \tag{32}$$

$$F = \mu \rho V g = (0.8)(2750\ kg/m^3)(9.8\ m/s^2)(4\ m^3) = 8.6.10^4 N \tag{33}$$

Problem 2.11

Consider a rock mass resting on an inclined bedding plane as shown in Figure 2–12. By balancing the forces acting on the block parallel to the inclined plane, show that the tangential force per unit area $\sigma_{x'y'}$ on the plane supporting the block is $\rho g h \sin(\theta)$ (ρ is the density and h is the thickness of the block). Show that the sliding condition is

$$\theta = \arctan(f) \tag{34}$$

2.2 Body Forces and Surface Forces

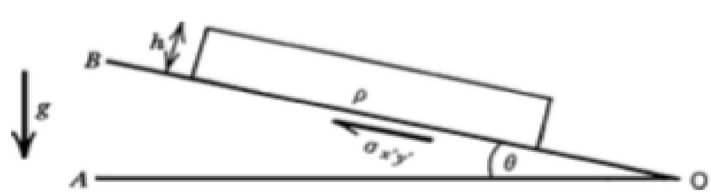

Figure 2.12 Gravitational sliding of a rock mass.

Figure 4: Inclined bedding problem 1.22

Solution

First we need to calculate the reactions that influence the system by establishing an coordinate system and proyecting the forces in it.

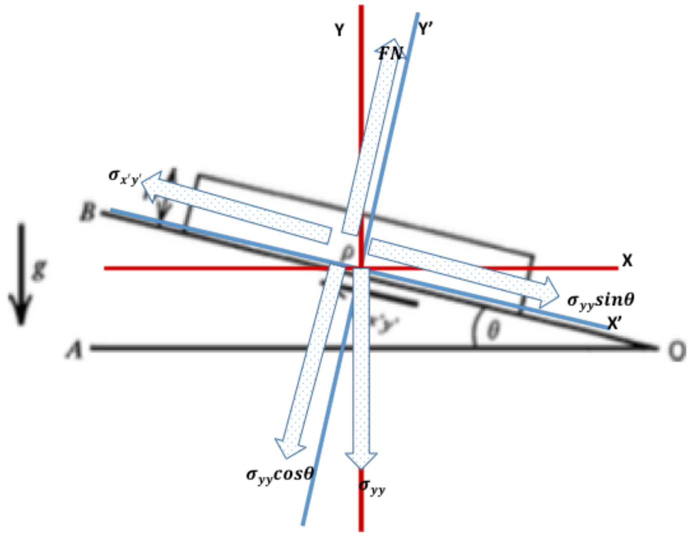

Figure 5: Free-body diagram

For the $x^{'}$ axis we have

$$\sigma_{x^{'}y^{'}} = \sigma_{yy} * \sin\theta \tag{35}$$

For the $y^{'}$ axis we have

$$FN = \sigma_{yy} * \cos\theta \tag{36}$$

In many cases it is appropriate to relate the shear stress resisting the sliding of one surface over another to the normal force. Empirically we often observe that these stresses are proportional to one another so that

$$FF = FN * f \tag{37}$$

$$\sigma_{x^{'}y^{'}} = \sigma_{yy} * \cos\theta * f \tag{38}$$

where $\sigma_y y$ is the vertical normal stress acting on the base and assuming that $\sigma_y y$ has the lithostatic value and f is the constant of proportionality, known as the coefficient of friction, we can re-write equations 35, 36 and 38 as:

$$\sigma_{x^{'}y^{'}} = \rho * g * h \sin\theta \tag{39}$$

$$FN = \rho * g * h * \cos\theta \tag{40}$$

$$\rho * g * h \sin\theta = \rho * g * h * \cos\theta * f \tag{41}$$

Doing some algebra in the last equation we obtain

$$\frac{\sin\theta}{\cos\theta} = f \tag{42}$$

$$\tan\theta = f \tag{43}$$

Finlay we demonstrate that

$$\theta = \arctan(f) \tag{44}$$

qed

Problem 2.12

The pressure p_h of fluids (water) in the pores of rocks reduces the effective normal stress pressing the surfaces together along a fault. Modify equation 45 to incorporate this effect.

$$\Delta\sigma_{xx} = f\rho_c g L \tag{45}$$

Solution

The effective normal stress pressing the surfaces together along the fault plane is called σ_{yy} and defined by

$$\sigma_{yy} = \rho_c g h \tag{46}$$

Then, taking into account the pore pressure, this equation should me modified to

$$\sigma'_{yy} = \rho_c g h - p_h \tag{47}$$

And equation 45 should show a change in $\sigma_{yy} \to \sigma'_{yy}$

$$\Delta\sigma_{xx} = f\rho_c g L \tag{48}$$

$$\Delta\sigma_{xx} = f\rho_c g L \frac{h}{h} \tag{49}$$

$$\Delta\sigma_{xx} = f\frac{\sigma_{yy}}{h} L \tag{50}$$

$$\Delta\sigma'_{xx} = f\frac{\sigma'_{yy}}{h} L \tag{51}$$

$$\boxed{\Delta\sigma'_{xx} = f\frac{\rho_c g h - p_h}{h} L} \tag{52}$$

Problem 2.18

Consider a simple two-layer model of a planet consisting of a core of density ρ_c and radius b surrounded by a mantle of density ρ_m and thickness $a-b$. Show that the gravitational acceleration as a function of radius is given by

$$g(r) = \frac{4}{3}\pi\rho_c G r \quad 0 \le r \le b \tag{53}$$

$$g(r) = \frac{4\pi}{3}G\left[r\rho_m + \frac{b^3}{r^2}(\rho_c - \rho_m)\right] \ , \ b \le r \le a \tag{54}$$

and that the pressure as a function of radius is given by

$$\begin{aligned} p(r) =& \frac{4}{3}\pi\rho_c G b^3(\rho_c - \rho_m)(\frac{1}{r} - \frac{1}{a}) + \\ & \frac{2}{3}\pi G\rho_m^2(a^2 - r^2) \quad b \le r \le a \end{aligned} \tag{55}$$

$$\begin{aligned} p(r) =& \frac{2}{3}\pi G\rho_c^2(b^2 - r^2) + \frac{2}{3}\pi G\rho_m^2(a^2 - b^2) + \\ & \frac{4}{3}\pi G\rho_m b^3(\rho_c - \rho_m)(\frac{1}{b} - \frac{1}{a}) \quad 0 \le r \le b \end{aligned} \tag{56}$$

Apply this model to the Earth. Assume $\rho_m = 4000\ kg.m^{-3}$, $b = 3486\ km$, $a = 6371\ km$. Calculate ρ_c given that the total mass of the Earth is $5.97.10^{24}\ kg$. What are the pressures at the center of the Earth and at the core-mantle boundary? What is the acceleration of gravity at $r = b$?

Solution

To show equation 53 we may use Gauss' theorem for gravity vector field $\vec{g}$.

$$\nabla \cdot \vec{g} = 4\pi G\rho \tag{57}$$

Knowing that every vector in $\vec{g}$ is perpendicular to a gaussian sphere of radius r we can state for our earth model in the interval $r \le b$.

$$4\pi r^2 g = \frac{4}{3}\pi r^3(4\pi G\rho_c) \tag{58}$$

$$\boxed{g = \frac{4}{3}\pi G r\rho_c \ , \ r \le b} \tag{59}$$

Since the mass surrounded by the gaussian surface is $M = \frac{4}{3}\pi r^3\rho_c$.
Now, selecting a gaussian surface of radius $b \le r \le a$, so the mass surrounded by it is all of the core's mass ($M_c = \frac{4}{3}\pi b^3\rho_c$) and a portion of the mantle's mass equivalent to the mass of a radius r sphere minus a radius b sphere, both with

density ρ_m. In other words $M_m = \frac{4}{3}\pi(r^3 - b^3)\rho_m$. So the flux is

$$4\pi r^2 g = \left[\frac{4}{3}\pi(r^3 - b^3)\rho_m + \frac{4}{3}\pi b^3 \rho_c\right](4\pi G) \tag{60}$$

$$g = \left[\frac{1}{3}\pi(r - \frac{b^3}{r^2})\rho_m + \frac{1}{3}\pi\frac{b^3}{r^2}\rho_c\right](4\pi G) \tag{61}$$

$$\boxed{g = \frac{4\pi}{3}G\left[r\rho_m + \frac{b^3}{r^2}(\rho_c - \rho_m)\right] \ , \ b \le r \le a} \tag{62}$$

qed

Now, the pressures can be demonstrated by:
First, for $b \le r \le a$:

$$\frac{dP}{dr} = -\rho g \tag{63}$$

$$\int_0^P dP = -\int_a^{r\in[b,a]} \rho_m g dr \tag{64}$$

Note that the integration limits are the interval $[a, r]$ since pressure is determined by the mass **above** the point considered.

$$P = -\frac{4\pi G}{3}\int_a^r [r\rho_m^2 + \frac{b^3}{r'^2}\rho_m(\rho_c - \rho_m)]dr' \tag{65}$$

$$P = -\frac{4\pi G}{3}(\int_a^r (r'\rho_m^2)dr + \int_a^r \frac{b^3\rho_m}{r'^2}(\rho_c - \rho_m)dr') \tag{66}$$

$$P = -\frac{4\pi G}{3}[\frac{\rho_m^2}{2}(r^2 - a^2) - b^3\rho_m(\rho_c - \rho_m(\frac{1}{r} - \frac{1}{a})] \tag{67}$$

$$\boxed{P(r) = \frac{4}{3}\pi\rho_c G b^3(\rho_c - \rho_m)(\frac{1}{r} - \frac{1}{a}) + \frac{2}{3}\pi G\rho_m^2(a^2 - r^2)} \tag{68}$$

qed

For $0 \le r \le b$:

$$P = \int_a^{r\le b} -\rho(r')g(r')dr' \tag{69}$$

$$P = \int_a^b -\rho_m g_{r\in[b,a]}dr' + \int_b^r -\rho_c g_{r\le b}dr' \tag{70}$$

$$P = \int_a^b -\rho_m\frac{4\pi G}{3}(r'\rho_m + \frac{b^3}{r'^2}(\rho_c - \rho_m))dr' + \int_b^r -\rho_c\frac{4\pi G}{3}\rho_c r' dr' \tag{71}$$

$$P = \frac{-\rho_m^2 2\pi G}{3}(b^2 - a^2) + \frac{4\pi\rho_m^2 G b^3}{3}(\rho_c - \rho_m)(\frac{1}{b} - \frac{1}{a}) - \frac{2\pi\rho_c^2 G}{3}(r^2 - b^2) \tag{72}$$

$$\boxed{P(r) = \frac{2}{3}\pi G\rho_c^2(b^2 - r^2) + \frac{2}{3}\pi G\rho_m^2(a^2 - b^2) + \frac{4}{3}\pi G\rho_m b^3(\rho_c - \rho_m)(\frac{1}{b} - \frac{1}{a})} \tag{73}$$

qed

If $\rho_m = 4000\ kgm^{-3}$, $b = 3486\ km$, $a = 6371\ km$ and $m_{total} = 5.97.10^{24}\ kg$, ρ_c could be calculated by:

$$m_a = \rho_m(\frac{4\pi a^3}{3}) - \rho_m(\frac{4\pi b^3}{3}) \tag{74}$$

$$m_a = 4,33.10^{24}\ kg - 7,098.10^{23}\ kg = 3,62.10^{24}\ kg \tag{75}$$

Then, the mass of $r = b$ will be:

$$m_b = m_{total} - m_a = 2,35.10^{24}\ kg \tag{76}$$

Finally,

$$\rho_c = \frac{m_b}{V_b} = \frac{3.2,35.10^{24}}{4\pi b^3} = 13243,33\ kg/m^3 \tag{77}$$

The pressures at the center of the earth, if $r = 0$ will be:

$$P(0) = \frac{2}{3}\pi G\rho_c^2(b^2) + \frac{2}{3}\pi G\rho_m^2(a^2 - b^2) + \frac{4}{3}\pi G\rho_m b^3(\rho_c - \rho_m)(\frac{1}{b} - \frac{1}{a}) \tag{78}$$

Converting all units to the international system of units we get a value in Pa, then:

$$P(0) = 418,39\ GPa \tag{79}$$

The pressures at the core - mantle boundary, if $r = b$ will be:

$$P(b) = \frac{2}{3}\pi G\rho_c^2(b^2 - b^2) + \frac{2}{3}\pi G\rho_m^2(a^2 - b^2) + \frac{4}{3}\pi G\rho_m b^3(\rho_c - \rho_m)(\frac{1}{b} - \frac{1}{a}) \tag{80}$$

$$P(b) = \frac{2}{3}\pi G\rho_m^2(a^2 - b^2) + \frac{4}{3}\pi G\rho_m b^3(\rho_c - \rho_m)(\frac{1}{b} - \frac{1}{a}) \tag{81}$$

Converting all units to the international system of units we get a value in Pa, then:

$$P(b) = 120,47\ GPa \tag{82}$$

The acceleration of gravity when $r = b$ will be:

$$g(b) = \frac{4}{3}\pi\rho_c G b = 12.9\ m/s^2 \tag{83}$$

Problem 2.20

The measured horizontal principal stresses at a depth of 200 m are given in Table 2–1 as a function of distance from the San Andreas fault. What are the values of maximum shear stress at each distance?

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Table 2.1 *Stress Measurements at 200 m Depth vs. Distance from the San Andreas Fault*

Distance from Fault (km)	Maximum Principal Stress (MPa)	Minimum Principal Stress (MPa)
2	9	8
4	14	8
22	18	8
34	22	11

Solution

In order to calculated the maximum shear stress also know as τ_{max} we plot Mohr's Circle and we see that the value of τ_{max} is the same as the magnitude of the radios. So whit the principal stresses we are able to calculate the radios as:

$$Radios = \tau_{max} = \frac{\sigma_1 - \sigma_2}{2} \tag{84}$$

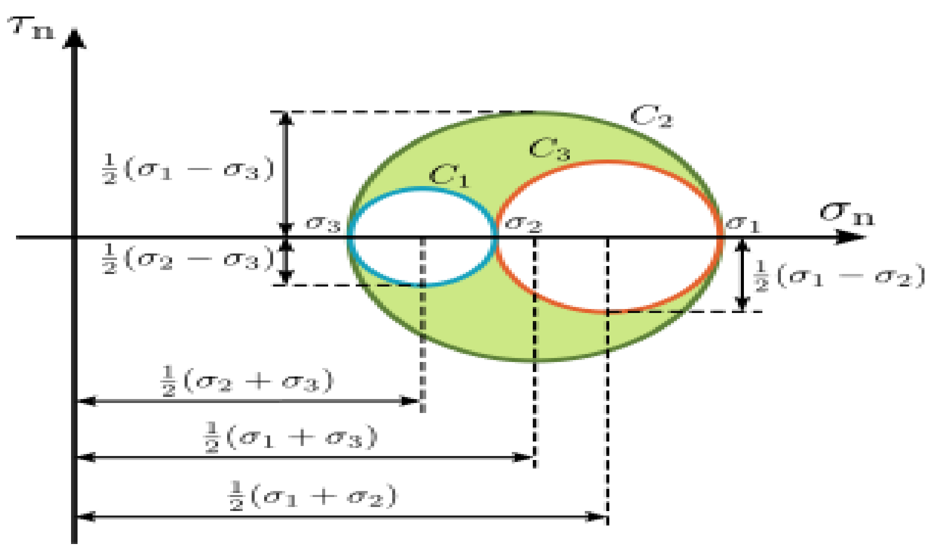

Figure 6: Mohr's circle

Distances from fault=2km

$$\tau_{max} = \frac{9MPa - 8MPa}{2} = 0,5MPa \tag{85}$$

Distances from fault=4km

$$\tau_{max} = \frac{14MPa - 8MPa}{2} = 3MPa \tag{86}$$

Distances from fault=22km

$$\tau_{max} = \frac{18MPa - 8MPa}{2} = 5MPa \tag{87}$$

Distances from fault=34km

$$\tau_{max} = \frac{22MPa - 11MPa}{2} = 5,5MPa \tag{88}$$

References

Turcotte, D., Schubert, G. and Turcotte, D. (2002). Geodynamics. Cambridge: Cambridge University Press.