Sol_HW_5

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1 Mecánica Cuántica 1 -201810

1.1 Tarea # 2 - Solución

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```
In [1]: from sympy import *
        init_printing()
        from IPython.display import display
```

1.1.1 2.

Tenemos un espacio vectorial bidimensional expandido por $\{|1\rangle, |2\rangle\}$.

Además, se nos da la matriz σ_2 (segunda matriz de Pauli), para comprobar su hermicidad basta con comprobar

$$\sigma_2 = \sigma_2^\dagger.$$

```
In [2]: from sympy.physics.matrices import msigma
        sigma_2 = msigma(2)

        display(sigma_2.transpose().conjugate())
        display(sigma_2.transpose().conjugate()==sigma_2)
```

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

```
True
```

Como vemos, la relación se cumple. Además podemos notar que $\sigma_2^\dagger \sigma_2 = 1$, es decir $\sigma_2^2 = 1$ esta relación se cumple para todas las matrices $\sigma_i,\ i = 1,2,3$. Concluimos que la matriz es hermítica (y unitaria), el observable asociado a estas matrices es el spin $S_i = \frac{\hbar}{2}\sigma_i$.

A continuación calculamos los vectores y valores propios.

```
In [3]: evect = sigma_2.eigenvects()
        display(evect)
        ket_p = evect[0][-1][0]
        ket_p/=ket_p.norm()
```

```
ket_m = evect[1][-1][0]
ket_m/=ket_m.norm()
ket_1 = Matrix([[1],[0]])
ket_1/=ket_1.norm()
ket_2 = Matrix([[0],[1]])
ket_2/=ket_2.norm()
base_1 = [ket_1, ket_2]
evects_1 = [ket_p, ket_m]
```

$$\left[\left(-1, \quad 1, \quad \left[\begin{bmatrix} i \\ 1 \end{bmatrix} \right] \right), \quad \left(1, \quad 1, \quad \left[\begin{bmatrix} -i \\ 1 \end{bmatrix} \right] \right) \right]$$

La diagonalización es sencilla, se espera hecha a mano. Vemos que la matriz tiene valores propios son $\lambda_\pm = \pm 1$, con multiplicidad 1 (segunda entrada) y con los vectores propios mostrados $\{|-\rangle, \; |+\rangle\}$, respectivamente. En la base dada, podemos expresar:

$$|+\rangle = \frac{1}{\sqrt{2}}(i|1\rangle + |2\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(-i|1\rangle + |2\rangle)$$

Los proyectores a estos vectores propios son $P_\pm = |\pm\rangle\langle\pm|$. Para calcular las matrices asociadas (en la base $\{|1\rangle, |2\rangle\}$) solamente aplicamos $P_{\pm,ij} = \langle i|P_\pm|j\rangle$.

```
In [4]: def bra_ket(vec1, vec2):
            return ((vec1.transpose().conjugate())*vec2)[0]
        def build_projectors(base,pr):

            P = zeros(len(base))
            for i in range(len(base)):
                for k in range(len(base)):
                    P[i,k] = bra_ket(base[i], pr)*bra_ket(pr, base[k])
            return P
```

```
In [5]: Ps = []
        for ket in evects_1:
            P =build_projectors(base_1, ket)
            Ps.append(P)
            display('For eigket',ket)
            display ('Matrix',P,'Projector?',P**2==P,P**2,P )

        display('Closed?',reduce(lambda x, y: x+y, Ps)==eye(2))
        display('Orthogonal?',reduce(lambda x, y: x*y, Ps)==zeros(2))
```

```
'For eigket'
```

$$\begin{bmatrix} \frac{\sqrt{2}i}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

'Matrix'

$$\begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

'Projector?'

True

$$\begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

'For eigket'

$$\begin{bmatrix} -\frac{\sqrt{2}i}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

'Matrix'

$$\begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

'Projector?'

True

$$\begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

'Closed?'

True

'Orthogonal?'

True

In [6]:
```
def do_hw(matrix, base):
    evect = matrix.eigenvects()
    display('Eigenvalues, multiplicity, eigenvectors (no norm.)',evect)
    evects = []
    for i in range(len(base)):
        k =evect[i][-1][0]
        k/=k.norm()
        evects.append(k)
    Ps = []
    for ket in evects:
        P =build_projectors(base, ket)
        Ps.append(P)
        display('For eigket',ket)
        display ('Matrix',P,'Projector?',P**2==P,P**2,P )

    display('Closed?',reduce(lambda x, y: x+y, Ps)==eye(len(base)))
    display('Orthogonal?',reduce(lambda x, y: x*y, Ps)==zeros(len(base)))
```

In [7]:
```
second_mat = Matrix([[2, sqrt(2)*I],[-sqrt(2)*I,3]])
do_hw(second_mat, base_1)
```

'Eigenvalues, multiplicity, eigenvectors (no norm.)'

$$\left[\left(1, \quad 1, \quad \left[\left[\begin{matrix} -\sqrt{2}i \\ 1 \end{matrix} \right] \right] \right), \quad \left(4, \quad 1, \quad \left[\left[\begin{matrix} \frac{\sqrt{2}i}{2} \\ 1 \end{matrix} \right] \right] \right) \right]$$

'For eigket'

$$\left[\begin{matrix} -\frac{\sqrt{6}i}{3} \\ \frac{\sqrt{3}}{3} \end{matrix} \right]$$

'Matrix'

$$\left[\begin{matrix} \frac{2}{3} & -\frac{\sqrt{2}i}{3} \\ \frac{\sqrt{2}i}{3} & \frac{1}{3} \end{matrix} \right]$$

'Projector?'

True

$$\begin{bmatrix} \frac{2}{3} & -\frac{\sqrt{2}i}{3} \\ \frac{\sqrt{2}i}{3} & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{3} & -\frac{\sqrt{2}i}{3} \\ \frac{\sqrt{2}i}{3} & \frac{1}{3} \end{bmatrix}$$

'For eigket'

$$\begin{bmatrix} \frac{\sqrt{3}i}{3} \\ \frac{\sqrt{6}}{3} \end{bmatrix}$$

'Matrix'

$$\begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}i}{3} \\ -\frac{\sqrt{2}i}{3} & \frac{2}{3} \end{bmatrix}$$

'Projector?'

True

$$\begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}i}{3} \\ -\frac{\sqrt{2}i}{3} & \frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}i}{3} \\ -\frac{\sqrt{2}i}{3} & \frac{2}{3} \end{bmatrix}$$

'Closed?'

True

'Orthogonal?'

True

```
In [8]: h = Symbol('hbar')
        third_mat = (h/(2*I))*Matrix([[0,sqrt(2), 0], [-sqrt(2),0,sqrt(2)], [0,-sqrt(2),0]])
        ket_1 = Matrix([[1],[0],[0]])
        ket_2 = Matrix([[0],[1],[0]])
        ket_3 = Matrix([[0],[0],[1]])
        base_2 = [ket_1, ket_2, ket_3]
        do_hw(third_mat, base_2)
```

'Eigenvalues, multiplicity, eigenvectors (no norm.)'

$$\left[\left(0, \quad 1, \quad \left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right] \right), \quad \left(-\hbar, \quad 1, \quad \left[\begin{bmatrix} -1 \\ \sqrt{2}i \\ 1 \end{bmatrix} \right] \right), \quad \left(\hbar, \quad 1, \quad \left[\begin{bmatrix} -1 \\ -\sqrt{2}i \\ 1 \end{bmatrix} \right] \right) \right]$$

'For eigket'

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

'Matrix'

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

'Projector?'

True

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

'For eigket'

$$\begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{2}i}{2} \\ \frac{1}{2} \end{bmatrix}$$

'Matrix'

$$\begin{bmatrix} \frac{1}{4} & \frac{\sqrt{2}i}{4} & -\frac{1}{4} \\ -\frac{\sqrt{2}i}{4} & \frac{1}{2} & \frac{\sqrt{2}i}{4} \\ -\frac{1}{4} & -\frac{\sqrt{2}i}{4} & \frac{1}{4} \end{bmatrix}$$

'Projector?'

True

$$\begin{bmatrix} \frac{1}{4} & \frac{\sqrt{2}i}{4} & -\frac{1}{4} \\ -\frac{\sqrt{2}i}{4} & \frac{1}{2} & \frac{\sqrt{2}i}{4} \\ -\frac{1}{4} & -\frac{\sqrt{2}i}{4} & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} & \frac{\sqrt{2}i}{4} & -\frac{1}{4} \\ -\frac{\sqrt{2}i}{4} & \frac{1}{2} & \frac{\sqrt{2}i}{4} \\ -\frac{1}{4} & -\frac{\sqrt{2}i}{4} & \frac{1}{4} \end{bmatrix}$$

'For eigket'

$$\begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{2}i}{2} \\ \frac{1}{2} \end{bmatrix}$$

'Matrix'

$$\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{2}i}{4} & -\frac{1}{4} \\ \frac{\sqrt{2}i}{4} & \frac{1}{2} & -\frac{\sqrt{2}i}{4} \\ -\frac{1}{4} & \frac{\sqrt{2}i}{4} & \frac{1}{4} \end{bmatrix}$$

'Projector?'

True

$$\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{2}i}{4} & -\frac{1}{4} \\ \frac{\sqrt{2}i}{4} & \frac{1}{2} & -\frac{\sqrt{2}i}{4} \\ -\frac{1}{4} & \frac{\sqrt{2}i}{4} & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{2}i}{4} & -\frac{1}{4} \\ \frac{\sqrt{2}i}{4} & \frac{1}{2} & -\frac{\sqrt{2}i}{4} \\ -\frac{1}{4} & \frac{\sqrt{2}i}{4} & \frac{1}{4} \end{bmatrix}$$

```
'Closed?'
```

```
True
```

```
'Orthogonal?'
```

```
True
```

1.1.2 3.

Para los kets dados, debemos comprobar

$$\langle \psi_i | \psi_i \rangle = 1$$

Comencemos con $|\psi_0\rangle$:

$$\langle \psi_0 | \psi_0 \rangle = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1,$$

entonces, podemos concluir que este ket está normalizado. Note que los únicos términos sobrevivientes del bracket son los $\langle u_i | u_i \rangle$, ya que los demás son cero por ortogonalidad. Ahora,

$$\langle \psi_1 | \psi_1 \rangle = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \neq 1,$$

por lo tanto, el segundo ket no está normalizado. Al normalizarlo, tenemos que

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} |u_1\rangle + \frac{i}{\sqrt{2}} |u_3\rangle.$$

Luego, debemos definir los operadores

$$\rho_i = |\psi_i\rangle\langle\psi_i|$$

```
In [9]: psi_0 = Matrix([[1/sqrt(2)],[I/2],[nsimplify(1./2)]])
        rho_0 = psi_0*psi_0.transpose().conjugate()
        display('rho_0 = ',rho_0)
        display('adj(rho_0) = ',rho_0.transpose().conjugate())
        display('Hermitian?',rho_0.transpose().conjugate()==rho_0)
```

'rho_0 = '

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{2}i}{4} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}i}{4} & \frac{1}{4} & \frac{i}{4} \\ \frac{\sqrt{2}}{4} & -\frac{i}{4} & \frac{1}{4} \end{bmatrix}$$

'adj(rho_0) = '

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{2}i}{4} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}i}{4} & \frac{1}{4} & \frac{i}{4} \\ \frac{\sqrt{2}}{4} & -\frac{i}{4} & \frac{1}{4} \end{bmatrix}$$

'Hermitian?'

True

```
In [10]: psi_1 = Matrix([[1/sqrt(2)],[0],[1/sqrt(2)]])
         rho_1 = psi_1*psi_1.transpose().conjugate()
         display('rho_1 = ',rho_1)
         display('adj(rho_1) = ',rho_1.transpose().conjugate())
         display('Hermitian?',rho_1.transpose().conjugate()==rho_1)
```

'rho_1 = '

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

'adj(rho_1) = '

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

'Hermitian?'

True

1.1.3 5.

Finalmente, tenemos que dados P_i ortogonales, es decir $P_i = P_i^\dagger$ (autoadjuntas), una proyección P_1P_2 es ortogonal, si

$$P_1P_2 = (P_1P_2)^\dagger = P_2^\dagger P_1^\dagger = P_2P_1,$$

es decir, si

$$[P_1, P_2] = 0.$$

Sea $\mathcal{V}$ el subespacio al cual proyecta P_1P_2, es decir, $P_1P_2|\alpha\rangle = |\alpha\rangle$ implica que $|\alpha\rangle \in \mathcal{V}$. Ahora, si aplicamos $P_1|\alpha\rangle = P_1^2P_2|\alpha\rangle = P_1P_2|\alpha\rangle = |\alpha\rangle$ nos damos cuenta que $|\alpha\rangle \in \mathcal{S}_1$. Si repetimos el proceso con P_2, tenemos que $P_2|\alpha\rangle = P_2P_1P_2|\alpha\rangle = P_2^2P_1|\alpha\rangle = P_2P_1|\alpha\rangle = P_1P_2|\alpha\rangle = |\alpha\rangle$ es evidente que $|\alpha\rangle \in \mathcal{S}_2$. De todo lo anterior, podemos concluir que si $|\alpha\rangle$ está tanto en $\mathcal{S}_1$ como en $\mathcal{S}_2$, el subespacio al que proyecta el producto debe ser $\mathcal{V} = \mathcal{S}_1 \cap \mathcal{S}_2$.