Derivation rules:

Introduction:

$$\dfrac{A \quad B}{A \wedge B}\, I\wedge \qquad \dfrac{A}{A \vee B}\, I\vee \qquad \dfrac{B}{A \vee B}\, I\vee \qquad \dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B}\, I\supset$$

$$\dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \cup \Delta \vdash A \wedge B}\, I\wedge \qquad \dfrac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\, I\vee \qquad \dfrac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\, I\vee \qquad \dfrac{\Gamma \vdash B}{\Gamma - \{A\} \vdash B \supset A}\, I\supset$$

Elimination:

$$\dfrac{\dfrac{A \quad B}{A \wedge B}\, I\wedge \qquad \begin{matrix}[A, B]\\ \vdots \\ C\end{matrix}}{C}\, E\wedge \qquad \dfrac{\dfrac{A}{A \vee B} \qquad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\, E\vee$$

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix} \atop A \supset B \qquad A \qquad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\, E\supset$$

$$\dfrac{\Gamma \vdash A \wedge B \quad \Delta \cup \{A, B\} \vdash C}{\Gamma \cup \Delta \vdash C}\, E\wedge \qquad \dfrac{\Gamma \vdash A \vee B \quad \Delta \cup \{A\} \vdash C \quad \Theta \cup \{B\} \vdash C}{\Gamma \cup \Delta \cup \Theta \vdash C}\, \vee E$$

$$\dfrac{\Gamma \vdash A \supset B \quad \Delta \vdash A \quad \Theta \cup \{B\} \vdash C}{\Gamma \cup \Delta \cup \Theta \vdash C}\, \supset E$$

Special E rules:

$$\dfrac{A \wedge B}{A}\, E\wedge \qquad \dfrac{A \wedge B}{B}\, E\wedge \qquad \dfrac{A \supset B \quad A}{B}\, E\supset$$

$$\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\, E\wedge \qquad \dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\, E\wedge \qquad \dfrac{\Gamma \vdash A \supset B \quad \Delta \vdash A}{\Gamma \cup \Delta \vdash B}\, E\supset$$

Others:

$$\dfrac{\bot}{C}\, E\bot \qquad \dfrac{\begin{matrix}[A]\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[\neg A]\\ \vdots \\ C\end{matrix}}{C}\, Em \qquad \dfrac{\begin{matrix}[\neg A]\\ \vdots \\ \bot\end{matrix}}{A}\, Raa$$

Minimal classical logic: $I\wedge$, $I\vee$, $I\supset$, $E\wedge$, $E\vee$, $E\supset$, assumption

Intuisionistic logic: MCL + $E\bot$ (No $\neg\neg A \supset A$ or $A \vee \neg A$ or $\dfrac{\begin{matrix}\neg A\\ \vdots \\ \bot\end{matrix}}{A}$)

Classical logic: IL + Em (Raa)

Sequent calculus:

Advantages over Natural deduction:

- In ND there is no way to keep track of the process. On the other hand, SC corrects the guidance of natural deduction.
- Shows active formulas and leaves implicit the set of remaining open assumptions
- Subformula property

Key definitions:

Inversion principle: *Whatever follows from the direct grounds for deriving a proposition must follow from that proposition.*

Normal form: *A derivation in natural deduction with general elimination rules is in normal form if all major premisses of elimination rules are assumptions.*

Normalization: Procedure for converting any given derivation to normal form.

Existance of normal form: if there exists a derivation of formula A, then there exist a derivation of A in normal form.

Strong normalization: the application of conversions to a non-normal form in any order terminates.

Uniqueness of normal form: the normalization of a derivation, always ends with the same normal proof.

Principal formula, actives formulas and context:

$$\dfrac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C}\ L\wedge$$

- principal
- actives
- context

Subformula property:

Subformula property: *All formulas in a sequent calculus derivation are subformulas of the endsequent of the derivation.*

$$\begin{array}{c} \Delta \vdash \varphi' \\ \vdots \\ \hline \Gamma \vdash \varphi \end{array} \Rightarrow \varphi' \in SubFor(\varphi)$$

Closed with respect to cut: if there is a derivation of $\Gamma \vdash C$ using cut, there is also a derivation of the sequent without cut.

Proving that a system is closed with respect to cut: Show that the system is complete withouth the rule of cut, i.e All correct sequents are derivable in the system so that the addition of cut doesn't add new derivable sequents.

Cut elimination:
Eliminate cut from a derivation, using different rules.

Invertible:

classical propositional logic is **invertible**: From the derivability of a sequent of any of the forms given in the conclusions of the logical rules, the derivability of its premisses follows. Starting with the endsequent, decomposition by invertible rules gives a terminating method of proof search for classical propositional logic.

Admissibility:

Given a system of rules **G**, we say that a rule with premisses $S_1, \ldots, S_n$ and conclusion S is **admissible** in **G** if, whenever an instance of $S_1, \ldots, S_n$ is derivable in **G**, the corresponding instance of S is derivable in **G**. Structural proof theory has

Structural rules: weakeaning + contraction + cut.

In proof theory, a **structural rule** is an inference rule that does not refer to any logical connective, but instead operates on the judgment or sequents directly. Structural rules often mimic intended meta-theoretic properties of the logic. Logics that deny one or more of the structural rules are classified as substructural logics.

Exercises chapter 1:

1) Prove $A \to A$ in Hilbert axiomatic system:

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

MP
$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

$\varphi = A_1(A, A \to A) = (A \to ((A \to A) \to A))$

$\psi = A_2(A, A \to A, A) = (A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A))$

$\theta = A_1(A, A) = (A \to (A \to A))$

$$\dfrac{\theta \qquad \dfrac{\varphi \qquad \psi}{\theta \to (A \to A)}\ MP}{A \to A}\ MP$$

2) Prove in natural deduction using minimal logic:

$$\frac{A \quad B}{A \& B}\,\&I \qquad \frac{A}{A \vee B}\,\vee I_1 \qquad \frac{B}{A \vee B}\,\vee I_2 \qquad \frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B}\,\supset I \qquad \frac{A \& B \quad \begin{matrix}[A, B]\\ \vdots \\ C\end{matrix}}{C}\,\&E \qquad \frac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\,\vee E$$

$$\frac{A \supset B \quad A \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\,\supset E \qquad \frac{\bot}{C}\,\bot E \qquad \frac{A \& B}{A}\,\&E_1 \qquad \frac{A \& B}{B}\,\&E_2 \qquad \frac{A \supset B \quad A}{B}\,\supset E$$

1. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

1) $A \supset B$
 2) $B \supset C$
  3) A
  4) B $\supset E_{1,3}$
  5) C $\supset E_{2,4}$
 6) $A \supset C$ $\supset I_{3,5}$
7) $(B \supset C) \supset (A \supset C)$ $\supset I_{2,6}$
8) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$ $\supset I_{1,7}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \supset B]^1 \quad [A]^2}{B}\,\supset E \qquad [B \supset C]^3}{C}\,\supset E}{A \supset C}\,\supset I_2}{(B \supset C) \supset (A \supset C)}\,\supset I_3}{(A \supset B) \supset ((B \supset C) \supset (A \supset C))}\,\supset I_1$$

8. $(A \,\&\, (B \vee C)) \supset ((A \,\&\, B) \vee (A \,\&\, C))$

1) $A \wedge (B \vee C)$

2) A $\wedge E_1$

3) B

4) $A \wedge B$ $\wedge I_{2,3}$

5) $(A \wedge B) \vee (A \wedge C)$ $\vee I_4$

6) $B \supset (A \wedge B) \vee (A \wedge C)$ $\supset I_{3,5}$

7) C

8) $A \wedge C$ $\wedge I_{2,7}$

9) $(A \wedge C) \vee (A \wedge B)$ $\vee I_8$

10) $C \supset (A \wedge C) \vee (A \wedge B)$

11) $B \vee C$ $\wedge E_1$

12) $(A \wedge C) \vee (A \wedge B)$ $\vee E_{6,10,11}$

13) $(A \wedge (B \vee C)) \supset (A \wedge C) \vee (A \wedge B)$ $\supset I_{1,12}$

1) $(A \wedge B) \vee (A \wedge C)$

2) $(A \wedge B)$

3) A $\wedge E_2$

4) B $\wedge E_2$

5) $B \vee C$ $\vee I_4$

6) $A \wedge (B \vee C)$ $\wedge I_{3,5}$

7) $(A \wedge B) \supset (A \wedge (B \vee C))$ $\supset I_{2,6}$

8) $A \wedge C$

9) A $\wedge E_8$

10) C $\wedge E_8$

11) $B \vee C$ $\vee I_{10}$

12) $A \wedge (B \vee C)$ $\wedge I_{9,11}$

13) $(A \wedge C) \supset (A \wedge (B \vee C))$

14) $(A \wedge (B \vee C))$ $\vee E_{1,7,13}$

15) $(A \wedge B) \vee (A \wedge C) \supset (A \wedge (B \vee C))$ $\supset I_{1,14}$

16. $((A \,\&\, B) \supset \bot) \supset (A \supset (B \supset \bot))$

1) $(A \wedge B) \supset \bot$

2) A

3) B

4) $A \wedge B$ $\wedge I_{2,3}$

5) $\bot$ $\supset E_{1,4}$

6) $B \supset \bot$ $\supset I_{3,5}$

7) $A \supset (B \supset \bot)$ $\supset I_{2,6}$

8) $((A \wedge B) \supset \bot) \supset (A \supset (B \supset \bot))$ $\supset I_{1,7}$

1) $(A \supset (B \supset \bot))$

2) $(A \wedge B)$

3) A $\wedge E_2$

4) $B \supset \bot$ $\supset E_{1,3}$

5) B $\wedge E_2$

6) $\bot$ $\supset E_{4,5}$

7) $(A \wedge B) \supset \bot$ $\supset I_{2,6}$

8) $(A \supset (B \supset \bot)) \supset ((A \wedge B) \supset \bot)$ $\supset I_{1,7}$

3) Prove using sequent calculus:

$$\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A\&B}\,R\& \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\,R\supset \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\,R\vee_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\,R\vee_2$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A\&B}\,R\& \qquad \frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\,L\vee \qquad \frac{\Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C}\,L\supset$$

} implicit contraction?

$$\frac{A, B, \Gamma \Rightarrow C}{A\&B, \Gamma \Rightarrow C}\,L\& \qquad \overline{\Gamma, A \Rightarrow A}\ \text{ax} \qquad \overline{\bot \Rightarrow C}\ L\bot \qquad \frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\,Wk$$

$$\frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\,Ctr$$

I 2. $\Rightarrow (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{(A\supset B), A \vdash A}\,Ax \quad \dfrac{\dfrac{\overline{A \vdash A}\,Ax \quad \overline{A, B \vdash B}\,Ax}{A, (A\supset B) \vdash B}\,L\supset \quad \overline{A, (A\supset B), C \vdash C}\,Ax}{(A\supset B), A, (B\supset C) \vdash C}\,L\supset}{(A\supset(B\supset C)), (A\supset B), A \vdash C}\,L\supset}{(A\supset(B\supset C)), (A\supset B) \vdash (A\supset C)}\,R\supset}{(A\supset(B\supset C)) \vdash ((A\supset B)\supset(A\supset C))}\,R\supset}{\vdash (A\supset(B\supset C))\supset((A\supset B)\supset(A\supset C))}\,R\supset$$

} implicit contraction of A

$\emptyset \Rightarrow A\vee(B\wedge C)\to(A\vee B)\wedge(A\vee C)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A}\,Ax}{A \vdash (A\vee B)}\,R\vee \quad \dfrac{\dfrac{\overline{B, C \vdash B}\,Ax}{B\wedge C \vdash B}\,L\wedge}{(B\wedge C) \vdash (A\vee B)}\,R\vee}{A\vee(B\wedge C) \vdash (A\vee B)}\,L\vee \quad \dfrac{\dfrac{\overline{A \vdash A}\,Ax}{A \vdash A\vee C}\,R\vee \quad \dfrac{\dfrac{\overline{B, C \vdash C}\,Ax}{B\wedge C \vdash C}\,L\wedge}{B\wedge C \vdash A\vee C}\,R\vee}{A\vee(B\wedge C) \vdash (A\vee C)}\,L\vee}{A\vee(B\wedge C) \vdash (A\vee B)\wedge(A\vee C)}\,R\wedge}{\vdash A\vee(B\wedge C)\supset(A\vee B)\wedge(A\vee C)}\,R\supset$$

Chapter 2:

Constructive reasoning:

- algorithm A generates the decimal expansion $0.a_1a_2a_3\ldots$
- Proof π: $\exists a_i(a_i > 0) \to \bot$
- Given two real numbers a and b, if it happens to be true that they are equal, a and b would have to be computed to infinite precision to verify $a = b$. Obviously

↳ you need to check that all decimals are the same, therefore is an infinite process, if $a, b \in$ Irrational or are both of them have infinite periodical decimals.

Definition of weight:

Definition 2.3.1: *The weight $w(A)$ of a formula A is defined inductively by*

$w(\bot) = 0,$
$w(P) = 1$ *for atoms* P,
$w(A \circ B) = w(A) + w(B) + 1$ *for conjunction, disjunction, and implication.*

It follows that $w(\sim A) = w(A) + w(\bot) + 1 = w(A) + 1$. ✓

Definition of height:

Definition 2.3.2: *A* derivation in **G3ip** *is either an axiom, an instance of* $L\bot$, *or an application of a logical rule to derivations concluding its premisses. The* height of a derivation *is the greatest number of successive applications of rules in it, where an axiom and* $L\bot$ *have height* 0.

Lemma 2.3.3:

Lemma 2.3.3: *The sequent $C, \Gamma \Rightarrow C$ is derivable for an arbitrary formula C and arbitrary context Γ.*

G3ip

G3ip

Logical axiom:

$$\overline{P, \Gamma \Rightarrow P}\ Ax$$

Logical rules:

$$\frac{A, B, \Gamma \Rightarrow C}{A \& B, \Gamma \Rightarrow C}\ L\& \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B}\ R\&$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ L\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ R\vee_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ R\vee_2$$

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C}\ L\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ R\supset$$

$$\overline{\bot, \Gamma \Rightarrow C}\ L\bot$$

$$\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ Wk \qquad \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ Ctr$$

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\ Cut$$

Proof:

Proof by induction over the weight of C:

- Base case ($w(C) \leq 1$):

Case ($C = \bot \rightarrow w(C) = 0$):

$$\overline{\bot, \Gamma \vdash \bot}\ L\bot$$

Case ($C = P \rightarrow w(C) = 1$):

$$\overline{P, \Gamma \vdash P}\ Ax$$

Case ($C = \bot \supset \bot \rightarrow w(C) = 1$):

$$\dfrac{\overline{\bot \supset \bot, \Gamma, \bot \vdash \bot}\ Ax / L\bot}{\bot \supset \bot, \Gamma \vdash \bot \supset \bot}\ R\supset$$

- Inductive case ($w(C) \leq n+1$)
- Induction hypothesis:

$C, \Gamma \vdash C$ is derivable for an arbitrary formula C and arbitrary context Γ, with $w(C) \leq n$

- We need to prove: $D, \Gamma \vdash D$ is derivable with $w(D) \geq n+1$, D an arbitrary formula and Γ an arbitrary context.

Observation 1: if $D = A \square B$ where $\square \in \{\supset, \wedge, \vee\}$, then

$w(A \square B) = w(A) + w(B) + 1 \leq n+1$

$w(A) + w(B) \leq n$

$w(A) \leq n \ \wedge \ w(B) \leq n$

Case ($D = A \wedge B$):

induction hyp with $\Gamma = B, \Gamma$ — induction hyp with $\Gamma = A, \Gamma$

$$\dfrac{\dfrac{A, B, \Gamma \vdash A \quad A, B, \Gamma \vdash B}{A, B, \Gamma \vdash A \wedge B}\ R\wedge}{A \wedge B, \Gamma \vdash A \wedge B}\ L\wedge$$

Case ($D = A \vee B$):

induction hyp with $\Gamma = \Gamma$ — induction hyp with $\Gamma = \Gamma$

$$\dfrac{\dfrac{A, \Gamma \vdash A}{A, \Gamma \vdash A \vee B}\ R\vee \quad \dfrac{B, \Gamma \vdash B}{B, \Gamma \vdash A \vee B}\ R\vee}{A \vee B, \Gamma \vdash A \vee B}\ L\vee$$

Case ($D = A \supset B$):

ind hyp with $\Gamma = A \supset B, \Gamma$ ↑ ind hyp with $\Gamma = \Gamma, A$ ↑

$$\dfrac{\dfrac{A \supset B, \Gamma, A \vdash A \quad \Gamma, A, B, \vdash B}{A \supset B, \Gamma, A \vdash B}}{A \supset B, \Gamma \vdash A \supset B} R\supset$$

$\square$

Definition:

$\vdash_n \Gamma \Rightarrow C$ will stand for: the sequent $\Gamma \Rightarrow C$ in **G3ip** is derivable with a height of derivation **at most** n.

Theorem 2.3.4:

Theorem 2.3.4: Height-preserving weakening. *If* $\vdash_n \Gamma \Rightarrow C$, *then* $\vdash_n D, \Gamma \Rightarrow C$ *for arbitrary* D.

Proof:

Proof by induction over the height of a derivation (n):

Base case ($n=0$):

if $n=0$ then the derivation rules for obtaining $\Gamma \vdash C$ where A_x or $L\bot$:

$$\dfrac{}{\Gamma \vdash C} Ax \quad \dfrac{}{\Gamma \vdash C} L\bot \longrightarrow \dfrac{}{D, \Gamma \vdash C} Ax \quad \dfrac{}{D, \Gamma \vdash C} L\bot$$

both rules can still be applied, therefore $\vdash_n D, \Gamma \vdash C$ with $n=0$.

Inductive case ($n+1$):

I.H: if $\vdash_m \Gamma \vdash C$ with $m \leq n$, then $\vdash_m D, \Gamma \vdash C$ for any D.

Then, we'll do a proof by cases on the last ruled applied

Case ($L\wedge$):

let $\Gamma = A \wedge B, \Gamma'$

$$\dfrac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma' \vdash C} L\wedge$$

} Then $A, B, \Gamma \vdash C \leq n$ and by I. H, $D, A, B, \Gamma \vdash C \leq n$. applying $L\wedge$ gets us to $D, A \wedge B, \Gamma' \vdash C$ with height $\leq n+1$.

1 and 2

Generalization for the rest of the rules ($L\wedge, L\vee, L\supset, R\wedge, R\vee, R\supset$):

$\square \in \{\wedge, \vee, \supset\}$

Left rules:

$\Gamma = A \square B, \Gamma'$

$$\dfrac{S_1 \text{ or } S_1, S_2}{A \square B, \Gamma' \vdash C} L\square$$

} Then $S_1, S_2 \leq n$ and by I. H, if we add D to the antecedent $S_1 + D, S_2 + D \leq n$. Then, applying $L\square$ gives us $D, A \square B, \Gamma' \vdash C$ with height $\leq n$

Right rules:

Same argument but with $C = A \square B$ and right rules $\square$

Lemma 2.3.5:

(i) *If* $\vdash_n A\&B, \Gamma \Rightarrow C$, *then* $\vdash_n A, B, \Gamma \Rightarrow C$,

(ii) *If* $\vdash_n A \vee B, \Gamma \Rightarrow C$, *then* $\vdash_n A, \Gamma \Rightarrow C$ *and* $\vdash_n B, \Gamma \Rightarrow C$,

(iii) *If* $\vdash_n A \supset B, \Gamma \Rightarrow C$, *then* $\vdash_n B, \Gamma \Rightarrow C$.

Proof i):

Induction over n:

Base case (n=0):

$$\dfrac{}{A\wedge B, \Gamma \vdash C}\,Ax$$

} if it is concluded by axiom, then $A\wedge B$ is not the principal formula because it isn't atomic. therefore $A, B, \Gamma \vdash C$ can also be derived by axiom.

$$\dfrac{}{A\wedge B, \Gamma \vdash C}\,L\bot$$

} if it is concluded by $L\bot$ and $A\wedge B \neq \bot$, then $A, B, \Gamma \vdash C$ can also be derived by $L\bot$.

Inductive case (n+1):

IH: if $\vdash_n A\wedge B, \Gamma \vdash C$ then $\vdash_n A, B, \Gamma \vdash C$

- if $A\wedge B$ is principal formula, then

$$\dfrac{A, B, \Gamma \vdash C \quad \rightarrow n}{A\wedge B, \Gamma \vdash C \quad \rightarrow n+1}\,L\wedge$$

$\vdash_n \rightarrow$ derived in $n \geq$ steps

- if $A\wedge B$ is not principal formula then:

$$\dfrac{A\wedge B, \Gamma' \vdash C' \quad A\wedge B, \Gamma'' \vdash C'' \quad \rightarrow n}{A\wedge B, \Gamma \vdash C \quad \rightarrow n+1}\,L/R\,\square$$

$\downarrow$ I.H   same rule

$$\dfrac{A, B, \Gamma' \vdash C' \quad A, B, \Gamma'' \vdash C'' \quad \rightarrow n}{A, B, \Gamma \vdash C \quad \rightarrow n+1}\,L/R\,\square$$

$\square$

Proof ii):

Base Case: Same as i

Inductive Case:

I.H: if $\vdash_n A\vee B, \Gamma \vdash C$ then $\vdash_n A, \Gamma \vdash C$ and $\vdash_n B, \Gamma \vdash C$

- Case $A\vee B$ is the principal formula:

$$\dfrac{A, \Gamma \vdash C \quad B, \Gamma \vdash C \quad \rightarrow n}{A\vee B, \Gamma \vdash C \quad \rightarrow n+1}\,L\vee$$

- Case $A\vee B$ isn't the principal formula:

$$\dfrac{A\vee B, \Gamma' \vdash C' \quad A\vee B, \Gamma'' \vdash C'' \quad \rightarrow n}{A\vee B, \Gamma \vdash C \quad \rightarrow n+1}\,L/R\,\square$$

Same rule

$\downarrow$ I.H

$A, \Gamma' \vdash C' \quad B, \Gamma' \vdash C' \quad A, \Gamma'' \vdash C'' \quad B, \Gamma'' \vdash C'' \quad \rightarrow n$

$$\dfrac{A, \Gamma' \vdash C' \quad A, \Gamma'' \vdash C'' \quad \rightarrow n}{A, \Gamma \vdash C \quad \rightarrow n+1}\,L/R\,\square \qquad \dfrac{B, \Gamma' \vdash C' \quad B, \Gamma'' \vdash C'' \quad \rightarrow n}{B, \Gamma \vdash C \quad \rightarrow n+1}\,L/R\,\square$$

$\square$

Proof ii):
Base Case: Same as i)
Inductive Case:
I.H: if $\vdash_n A \supset B, \Gamma \vdash C$ then $\vdash_n B, \Gamma \vdash C$

- Case $A \supset B$ is the principal formula:

$$\dfrac{A \supset B, \Gamma \vdash A \quad B, \Gamma \vdash C \rightarrow n}{A \supset B, \Gamma \vdash C \rightarrow n+1} L\supset$$

- Case $A \supset B$ is not the principal formula:

$$\dfrac{A \supset B, \Gamma' \vdash C' \quad A \supset B, \Gamma'' \vdash C'' \rightarrow n}{A \supset B, \Gamma \vdash C \rightarrow n+1} L/R\,\square$$

$\downarrow$ I.H

Same rule

$$\dfrac{B, \Gamma' \vdash C' \quad B, \Gamma'' \vdash C'' \rightarrow n}{B, \Gamma \vdash C \rightarrow n+1} L/R\,\square$$

$\square$

$$\dfrac{\dfrac{}{\perp, \perp \supset \perp \Rightarrow \perp} L\perp}{\perp \supset \perp \Rightarrow \perp \supset \perp} R\supset$$

Show that $L\supset$ isn't invertible
→ with respect to its first premiss.

If $L\supset$ were invertible with respect to its first premiss, from the derivability of a sequent with an implication in the antecedent would follow the derivability of its first premiss as determined by the $L\supset$ rule. For the sequent $\perp \supset \perp \Rightarrow \perp \supset \perp$, this first premiss would be $\perp \supset \perp \Rightarrow \perp$. The sequent $\perp \Rightarrow \perp$ is an instance of $L\perp$, and $R\supset$ gives $\Rightarrow \perp \supset \perp$. An application of the cut rule now gives $\Rightarrow \perp$, which would make the system **G3ip** inconsistent. (The formula $\perp \supset \perp$ of this example is the "standard" true formula, abbreviated as $\top = \perp \supset \perp$.)

Invertible Rule → $\dfrac{\Gamma' \vdash C' \quad \Gamma'' \vdash C''}{\Gamma \vdash C} L/R\,\square$ then $\vdash_n \Gamma' \vdash C'$ and $\vdash_n \Gamma'' \vdash C''$?

Theorem: $L\supset$ is not invertible.
Proof: suppose that $L\supset$ is invertible, then:

$$\begin{matrix}\vdash_n \\ \vdash_{n+1}\end{matrix} \dfrac{A \supset B, \Gamma \vdash A \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C} L\supset \longrightarrow \vdash_n A \supset B, \Gamma \vdash A$$

Now, we have that $\vdash_n \perp \supset \perp \Rightarrow \perp \supset \perp$, then $\vdash_n \perp \supset \perp \vdash \perp$.
Using cut we have:

$$\dfrac{\dfrac{\dfrac{}{\perp \vdash \perp} L\perp}{\vdash \perp \supset \perp} R\supset \quad \dfrac{\vdots}{\perp \supset \perp \vdash \perp}}{\vdash \perp} \text{Cut}$$

would make G3ip inconsistent $\perp$. $\square$

Difference between admissible rule and invertible rule $R = \dfrac{\Gamma' \Rightarrow C' \quad \Gamma'' \Rightarrow C''}{\Gamma \Rightarrow C} R$

admissible rule R:

if $\vdash \Gamma' \Rightarrow C'$ and $\vdash \Gamma'' \Rightarrow C''$ then $\vdash \Gamma \Rightarrow C$

Invertible rule R:

if $\vdash \Gamma \Rightarrow C$ then $\vdash \Gamma' \Rightarrow C'$ and $\vdash \Gamma'' \Rightarrow C''$

Theorem 2.4.1: Height-preserving contraction. *If* $\vdash_n D, D, \Gamma \Rightarrow C$, *then* $\vdash_n D, \Gamma \Rightarrow C$.

Proof: Induction over the height n of the derivation

Base case ($n=0$):

if $n=0$ then $D, D, \Gamma \vdash C$ is an axiom or conclusion of $L\bot$, therefore C is an atom or there is a $\bot$ in the antecedent. In both cases $D, \Gamma \vdash C$ is still an axiom or it can also be a conclusion of $L\bot$.

Inductive case:

I.H: Contraction is admissible up to height n

proof by cases:

- Case D is not principal in $D, D, \Gamma \vdash C$:

$$\dfrac{D, D, \Gamma' \vdash C' \quad D, D, \Gamma'' \vdash C''}{D, D, \Gamma \vdash C} \begin{matrix} \to n \\ L \backslash R\, \square \\ \to n+1 \end{matrix}$$

Then by I.H we get

Same rule

$$\dfrac{D, \Gamma' \vdash C' \quad D, \Gamma'' \vdash C''}{D, \Gamma \vdash C} \begin{matrix} L \backslash R\, \square \\ \to n+1 \end{matrix}$$

- Case D is principal in $D, D, \Gamma \vdash C$:

- Subcase ($D = A \wedge B$):

- $$\dfrac{A, B, A \wedge B, \Gamma \vdash C}{A \wedge B, A \wedge B, \Gamma \vdash C} \begin{matrix} \to n \\ L\wedge \\ \to n+1 \end{matrix}$$

- $\vdash_n A, B, A, B, \Gamma \vdash C$ Lema 2.3.5 i)
- $\vdash_n A, B, B, \Gamma \vdash C$ I.H
- $\vdash_n A, B, \Gamma \vdash C$ I.H
- $\vdash_{n+1} A \wedge B, \Gamma \vdash C$ $L\wedge$

- Subcase ($D = A \vee B$):

$$\dfrac{\overset{1)}{A \vee B, A, \Gamma \vdash C} \quad \overset{2)}{A \vee B, B, \Gamma \vdash C}}{A \vee B, A \vee B, \Gamma \vdash C} \begin{matrix} \to n \\ L\vee \\ \to n+1 \end{matrix}$$

- $\vdash_n \overset{3)}{A, A, \Gamma \vdash C} \wedge \vdash_n B, A, \Gamma \vdash C$ Lema 2.3.5 ii) 1)
- $\vdash_n A, B, \Gamma \vdash C \wedge \vdash_n \overset{4)}{B, B, \Gamma \vdash C}$ Lema 2.3.5 ii) 2)
- $\vdash_n A, \Gamma \vdash C$ I.H 3)
- $\vdash_n B, \Gamma \vdash C$ I.H 4)
- $\vdash_{n+1} A \vee B, \Gamma \vdash C$ $L\vee$

- Subcase ($D = A \supset B$)

$$\dfrac{\overset{1)}{A \supset B, A \supset B, \Gamma \vdash A} \quad \overset{2)}{A \supset B, \Gamma, B \vdash C}}{A \supset B, A \supset B, \Gamma \vdash C} \begin{matrix} \to n \\ L\supset \\ \to n+1 \end{matrix}$$

- $\vdash_n A \supset B, \Gamma \vdash A$ I.H 1)
- $\vdash_n B, \Gamma, B \vdash C$ Lema 2.3.5 iii 2)
- $\vdash_n B, \Gamma \vdash C$ I.H
- $\vdash_{n+1} A \supset B, \Gamma \vdash C$ $L\supset$ $\square$

Definition:

Definition 2.4.2: **Cut-height.** *The* cut-height *of an instance of the rule of cut in a derivation is the sum of heights of derivation of the two premisses of cut.*

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash C}{\Gamma, \Delta \vdash C}\,Cut \longrightarrow height(\Gamma \vdash A) + height(A, \Delta \vdash C)$$

Cut-height is not monotone as we go down in a derivation; that is, a cut below another one can have a lesser cut-height: In the derivation of one of its premisses

$$\dfrac{\dfrac{\overset{\vdots}{S_1} \quad \overset{\vdots}{S_2}}{C_1}\,Cut \qquad \overset{\vdots}{S_3}}{C_2}\,Cut$$

$CH(C_1) = H(S_1) + H(S_2)$

$H(C_1) = max(H(S_1), H(S_2)) + 1$

$CH(C_2) = H(C_1) + H(S_3)$

$H(S_3) > H(S_1) + H(S_2) - max(H(S_1), H(S_2)) - 1$

$\rightarrow CH(C_1) < CH(C_2)$

$CH(C_1) < CH(C_2)$

$H(S_1) + H(S_2) < H(C_1) + H(S_3)$

$H(S_1) + H(S_2) < max(H(S_1), H(S_2)) + 1 + H(S_3)$

Theorem:

Theorem 2.4.3: *The rule of cut,*

$$\frac{\Gamma \Rightarrow D \quad D, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\,Cut$$

is admissible in **G3ip**.

Proof:

Base Cases:

Left premise instance of $L\bot$ or Ax:

- Subcase $L\bot$:

$$\frac{}{\Gamma \vdash D}\,L\bot \rightarrow \bot \in \Gamma$$

$$\rightarrow \frac{}{\Gamma, \Delta \vdash C}\,L\bot$$

- Subcase Ax:

$$\frac{}{\Gamma \vdash D}\,Ax \rightarrow D \in \Gamma$$

$$\rightarrow \frac{D, \Delta \vdash C}{\Gamma/\{D\}, D, \Delta \vdash C = \Gamma, \Delta \vdash C}\,wk$$

Right premise instance of $L\bot$ or Ax:

- Subcase Ax:

$$\frac{}{D, \Delta \vdash C}\,Ax \longrightarrow$$

- Subsubcase ($D = C$):

$$\Gamma \vdash C \rightarrow \frac{\Gamma \vdash C}{\Gamma, \Delta \vdash C}\,wk$$

- Subsubcase ($C \in \Delta$):

$$\frac{}{\Gamma, \Delta \vdash C}\,Ax$$

- Subcase $L\bot$:

$$\frac{}{D, \Delta \vdash C}\,L\bot \rightarrow$$

- Subsubcase ($\bot \in \Delta$):

$$\frac{}{\Gamma, \Delta \vdash C}\,L\bot$$

- Subsubcase ($D = \bot$):
  - Subsubsubcase ($\Gamma \vdash D = \bot$ is Ax):

$$\frac{}{\Gamma \vdash \bot}\,Ax \rightarrow \bot \in \Gamma$$

$$\rightarrow \frac{\bot, \Delta \vdash C}{\Gamma/\{\bot\}, \bot, \Delta \vdash C = \Gamma, \Delta \vdash C}\,wk$$

  - Subsubsubcase (Γ comes from $L\Box$): special case of following general cases.

Inductive cases: (neither premiss an axiom or $L\bot$):

- Subcase (cut formula not principal in either premiss (D not principal in left premiss):

if D not principal in $\Gamma \vdash D$, that means that $\Gamma \vdash D$ must have been derived by a Left rule:

Case $L\wedge$:

(cut height : ch, height : h)

Let $\Gamma = A\wedge B, \Gamma'$:

$$\dfrac{\dfrac{\overset{s_1}{A, B, \Gamma' \vdash D}}{A\wedge B, \Gamma' \vdash D}L\wedge \qquad \overset{s_2}{D, \Delta \vdash C}}{A\wedge B, \Gamma', \Delta \vdash C}cut \quad \rightarrow ch = h(s_1) + 1 + h(s_2)$$

move cut

reduce at height

$$\dfrac{\dfrac{A, B, \Gamma' \vdash D \qquad \Delta, D \vdash C}{A, B, \Gamma', \Delta \vdash C}cut}{A\wedge B, \Gamma', \Delta \vdash C}L\wedge \quad \rightarrow ch = h(s_1) + h(s_2)$$

Case $L\vee$:

Let $\Gamma = A\vee B, \Gamma'$:

$$\rightarrow Ch = \max(h(s_1), h(s_2)) + 1 + h(s_3)$$

$$\dfrac{\dfrac{\overset{s_1}{\Gamma', A \vdash D} \qquad \overset{s_2}{\Gamma', B \vdash D}}{\Gamma', A\vee B \vdash D}L\vee \qquad \overset{s_3}{D, \Delta \vdash C}}{\Gamma', A\vee B, \Delta \vdash C}cut$$

move cut

reduce cut height

$$\dfrac{\dfrac{\Gamma', A \vdash D \quad D, \Delta \vdash C}{\Gamma', \Delta, A \vdash C}cut \qquad \dfrac{\Gamma', B \vdash D \quad D, \Delta \vdash C}{\Gamma', \Delta, B \vdash C}cut}{\Gamma', A\vee B, \Delta \vdash C}$$

$\rightarrow Ch_1 = h(s_1) + h(s_3)$

$Ch_2 = h(s_2) + h(s_3)$

Case $L\supset$:

Let $\Gamma = A\supset B, \Gamma'$:

$$\rightarrow ch = \max(h(s_1), h(s_2)) + 1 + h(s_3)$$

$$\dfrac{\dfrac{\overset{s_1}{A\supset B, \Gamma' \vdash A} \qquad \overset{s_2}{B, \Gamma' \vdash D}}{A\supset B, \Gamma' \vdash D}L\supset \qquad \overset{s_3}{\Delta, D \vdash C}}{A\supset B, \Gamma', \Delta \vdash C}Cut$$

reduce ch

move cut

$$\rightarrow Ch = h(s_2) + h(s_3)$$

$$\dfrac{\dfrac{\overset{s_1}{A\supset B, \Gamma' \vdash A}}{A\supset B, \Gamma', \Delta \vdash A}wk \qquad \dfrac{\overset{s_2}{B, \Gamma' \vdash D} \qquad \overset{s_3}{D, \Delta \vdash C}}{B, \Gamma', \Delta \vdash C}cut}{A\supset B, \Gamma', \Delta \vdash C}L\supset$$

i.e not principal in right.

• Subcase (cut formula D only principal in Left premiss):

Case $L\wedge$:

Let $\Delta = A\wedge B, \Delta'$:

$$\dfrac{\overset{S_1}{\Gamma \vdash D} \qquad \dfrac{\overset{S_2}{D, A, B, \Delta' \vdash C}}{D, A\wedge B, \Delta' \vdash C}\,L\wedge}{\Gamma, A\wedge B, \Delta' \vdash C}\,cut \quad \rightarrow \quad ch = h(S_1) + h(S_2) + 1$$

reduce cut height

$$\dfrac{\dfrac{\overset{S_1}{\Gamma \vdash D} \qquad \overset{S_2}{D, A, B, \Delta' \vdash C}}{\Gamma, A, B, \Delta' \vdash C}\,cut}{\Gamma, A\wedge B, \Delta' \vdash C}\,L\wedge \quad \rightarrow \quad ch = h(S_1) + h(S_2)$$

Case $L\vee$:

Let $\Delta = A\vee B, \Delta'$:

$$\dfrac{\overset{S_1}{\Gamma \vdash D} \qquad \dfrac{\overset{S_2}{D, A, \Delta' \vdash C} \qquad \overset{S_3}{D, B, \Delta' \vdash C}}{D, A\vee B, \Delta' \vdash C}\,L\vee}{\Gamma, A\vee B, \Delta' \vdash C}\,cut \qquad ch = h(S_1) + \max(h(S_2), h(S_3)) + 1$$

reduce cut height

$$\dfrac{\dfrac{\overset{S_1}{\Gamma \vdash D} \qquad \overset{S_2}{D, \Delta', A \vdash C}}{\Gamma, \Delta', A \vdash C}\,cut \qquad \dfrac{\overset{S_1}{\Gamma \vdash D} \qquad \overset{S_3}{D, \Delta', B \vdash C}}{\Gamma, \Delta', B \vdash C}\,cut}{\Gamma, A\vee B, \Delta' \vdash C}\,L\vee \qquad \begin{array}{l} ch_1 = h(S_1) + h(S_2) \\ ch_2 = h(S_1) + h(S_3) \end{array}$$

Case $L\supset$:

Let $\Delta = A\supset B, \Delta'$:

$$\dfrac{\overset{S_1}{\Gamma \vdash D} \qquad \dfrac{\overset{S_2}{D, A\supset B, \Delta' \vdash A} \qquad \overset{S_3}{D, B, \Delta' \vdash C}}{D, A\supset B, \Delta' \vdash C}\,L\supset}{\Gamma, A\supset B, \Delta' \vdash C}\,cut \quad \rightarrow \quad ch = h(S_1) + \max(h(S_2), h(S_3)) + 1$$

reduce cut height

$$\dfrac{\dfrac{\overset{S_1}{\Gamma \vdash D} \qquad \overset{S_2}{D, A\supset B, \Delta' \vdash A}}{\Gamma, A\supset B, \Delta' \vdash A}\,cut \qquad \dfrac{\overset{S_1}{\Gamma \vdash D} \qquad \overset{S_3}{D, B, \Delta' \vdash C}}{\Gamma, B, \Delta' \vdash C}\,cut}{\Gamma, A\supset B, \Delta' \vdash C}\,L\supset \qquad \begin{array}{l} \rightarrow ch_1 = h(S_1) + h(S_2) \\ \rightarrow ch_2 = h(S_1) + h(S_3) \end{array}$$

Case $R\wedge$:

Let $C = A\wedge B$

$$\dfrac{\overset{S_1}{\Gamma\vdash D} \qquad \dfrac{\overset{S_2}{D,\Delta\vdash A} \qquad \overset{S_3}{D,\Delta\vdash B}}{D,\Delta\vdash A\wedge B}\,R\wedge}{\Gamma,\Delta\vdash A\wedge B}\,cut \qquad ch = h(S_1) + \max(h(S_2), h(S_3)) + 1$$

$\downarrow$ reduce height

$$\dfrac{\dfrac{\overset{S_1}{\Gamma\vdash D} \quad \overset{S_2}{D,\Delta\vdash A}}{\Gamma,\Delta\vdash A}\,cut \qquad \dfrac{\overset{S_1}{\Gamma\vdash D} \quad \overset{S_3}{D,\Delta\vdash B}}{\Gamma,\Delta\vdash B}\,cut}{\Gamma,\Delta\vdash A\wedge B}\,R\wedge \qquad \begin{array}{l} ch_1 = h(S_1) + h(S_2) \\ ch_2 = h(S_1) + h(S_3) \end{array}$$

Case $R\vee$:

Let $C = A\vee B$ — sin perdida de generalidad se escoge A.

$$\dfrac{\overset{S_1}{\Gamma\vdash D} \qquad \dfrac{\overset{S_2}{D,\Delta\vdash A}}{D,\Delta\vdash A\vee B}\,R\vee}{\Gamma,\Delta\vdash A\vee B}\,cut \qquad ch = h(S_1) + h(S_2) + 1$$

$\downarrow$ reduce cut height

$$\dfrac{\dfrac{\overset{S_1}{\Gamma\vdash D} \qquad \overset{S_2}{D,\Delta\vdash A}}{\Gamma,\Delta\vdash A}\,cut}{\Gamma,\Delta\vdash A\vee B}\,R\vee \qquad ch = h(S_1) + h(S_2)$$

Case $R\supset$:

Let $C = A\supset B$:

$$\dfrac{\overset{S_1}{\Gamma\vdash D} \qquad \dfrac{\overset{S_2}{D,\Delta,A\vdash B}}{D,\Delta\vdash A\supset B}\,R\supset}{\Gamma,\Delta\vdash A\supset B}\,cut \qquad ch = h(S_1) + h(S_2) + 1$$

$\downarrow$ reduce cut height

$$\dfrac{\dfrac{\overset{S_1}{\Gamma\vdash D} \qquad \overset{S_2}{D,\Delta,A\vdash B}}{\Gamma,\Delta,A\vdash B}\,cut}{\Gamma,\Delta\vdash A\supset B}\,R\supset \qquad ch = h(S_1) + h(S_2)$$

• Subcase (D is principal in both premisses):

Case $D = A\wedge B$:

$$\dfrac{\dfrac{\overset{S_1}{\Gamma\vdash A} \quad \overset{S_2}{\Gamma\vdash B}}{\Gamma\vdash A\wedge B}\,R\wedge \qquad \dfrac{\overset{S_3}{A,B,\Delta\vdash C}}{A\wedge B,\Delta\vdash C}\,L\wedge}{\Gamma,\Delta\vdash C}\,cut \qquad ch = \max(h(S_1), h(S_2)) + 1 + h(S_3) + 1$$

$\downarrow$

$$\dfrac{\dfrac{\overset{S_1}{\Gamma\vdash A} \qquad \dfrac{\overset{S_2}{\Gamma\vdash B} \quad \overset{S_3}{B,A,\Delta\vdash C}}{A,\Gamma,\Delta\vdash C}\,cut_1}{\Gamma,\Gamma,\Delta\vdash C}\,cut_2}{\Gamma,\Delta\vdash C}\,ctr \qquad \begin{array}{l} ch_1 = h(S_2) + h(S_3) \quad \text{reduce } ch \\ \text{reduce formula only weight } (A\wedge B\to A) \\ ch_2 = \max(h(S_2), h(S_3)) + 1 + h(S_1) \end{array}$$

Case $D = A \vee B$:

sin perdidad de generalidad se escoge A.

$$\dfrac{\dfrac{\overset{S_1}{\Gamma \vdash A}}{\Gamma \vdash A \vee B}\,R\vee \qquad \dfrac{\overset{S_2}{A, \Delta \vdash C} \quad \overset{S_3}{B, \Delta \vdash C}}{A \vee B, \Delta \vdash C}\,L\vee}{\Gamma, \Delta \vdash C}\,cut \qquad ch = h(S_1) + 1 + \max(h(S_2), h(S_3)) + 1$$

reduce cut height

$$\dfrac{\overset{S_1}{\Gamma \vdash A} \quad \overset{S_2}{A, \Delta \vdash C}}{\Gamma, \Delta \vdash C}\,cut \qquad ch = h(S_1) + h(S_2)$$

Case $D = A \supset B$:

$$\dfrac{\dfrac{\overset{S_1}{A, \Gamma \vdash B}}{\Gamma \vdash A \supset B}\,R\supset \qquad \dfrac{\overset{S_2}{A \supset B, \Delta \vdash A} \quad \overset{S_3}{\Delta, B \vdash C}}{A \supset B, \Delta \vdash C}\,L\supset}{\Gamma, \Delta \vdash C}\,cut \qquad ch = h(S_1) + 1 + \max(h(S_2), h(S_3)) + 1$$

$$\dfrac{\dfrac{\dfrac{\overset{S_1}{\Gamma, A \vdash B}}{\Gamma \vdash A \supset B}\,R\supset \quad \overset{S_2}{A \supset B, \Delta \vdash A}}{\Gamma, \Delta \vdash A}\,cut_3 \qquad \dfrac{\overset{S_1}{\Gamma, A \vdash B} \quad \overset{S_3}{\Delta, B \vdash C}}{\Gamma, \Delta, A \vdash C}\,cut_2}{\Gamma, \Delta \vdash C}\,cut_1$$

$ch_3 = h(S_1) + 1 + h(S_2)$
$ch_2 = h(S_1) + h(S_3)$

reduce cut height

$ch_1 = \max(h(S_1) + 1, h(S_2)) + 1 + \max(h(S_1), h(S_3)) + 1$

reduce formula weight $(A \supset B \to A)$

Q.E.D

Chapter 3:

G3cp

Logical axiom:

$P, \Gamma \Rightarrow \Delta, P$

Logical rules:

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A\&B, \Gamma \Rightarrow \Delta}\,L\& \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A\&B}\,R\&$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\,L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\,R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\,L\supset \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\,R\supset$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\,L\bot$$

Structural Rules

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\,LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\,RW \qquad \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\,LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\,RC$$

Theorem 3.1.1: Height-preserving inversion. *All rules of* **G3cp** *are invertible, with height-preserving inversion.*

Proof: (Left rules)

for the premisses $A, B, \Gamma \vdash \Delta$ $\;A, \Gamma \vdash \Delta$ $\;B, \Gamma \vdash \Delta$ and $B, \Gamma \vdash \Delta$, these are derivable by Lemma 2.3.5.

Premiss ($\Gamma \vdash \Delta, A$):

Proof by induction over height:

I.H: $\Gamma \vdash \Delta, A$ derivable with height $\leq n$ if $A \supset B, \Gamma \vdash \Delta$ derivable heigh $\leq n+1$

Case $A \supset B$ is principal:

$$\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}\,L\supset \quad \begin{matrix}\to n \leq \\ \to n+1 \leq\end{matrix}$$

Case $A \supset B$ is not principal:

$$\frac{A \supset B, \Gamma' \vdash \Delta' \qquad A \supset B, \Gamma'' \vdash \Delta''}{A \supset B, \Gamma \vdash \Delta}\,L/R \;\square$$

$\hookrightarrow$ I.H $\to$ $\Gamma' \vdash \Delta', A$ and $\Gamma'' \vdash \Delta'', A$ in $\leq n$

$$\frac{\Gamma' \vdash \Delta', A \qquad \Gamma'' \vdash \Delta'', A}{\Gamma \vdash \Delta, A}\,L/R \;\square \quad \begin{matrix}\to \leq n \\ \to \leq n+1\end{matrix}$$

same rule

Right rules:

$R\wedge$:

Base Cases:

$$\frac{}{\Gamma \vdash \Delta, A \wedge B}\,ax \to$$

$\left\{ \begin{array}{l} A \wedge B \text{ not atom, therefore } \exists p (p \in \Gamma \wedge p \in \Delta), \\ \text{then } \Gamma \vdash \Delta, A \text{ and } \Gamma \vdash \Delta, B \text{ are also axioms} \end{array} \right.$

$$\frac{}{\Gamma \vdash \Delta, A \wedge B}\,L\bot \to$$

$\left\{ \begin{array}{l} \bot \in \Gamma, \text{ then } \Gamma \vdash \Delta, A \text{ and } \Gamma \vdash \Delta, B \text{ can also be} \\ \text{derived by } L\bot \end{array} \right.$

Inductive Cases:

Case $A \wedge B$ principal:

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\,R\wedge \quad \begin{matrix}\to \leq n \\ \to \leq n+1\end{matrix}$$

Case $A \wedge B$ not principal:

$$\frac{\Gamma' \vdash \Delta', A \wedge B \qquad \Gamma'' \vdash \Delta'', A \wedge B}{\Gamma \vdash \Delta, A \wedge B}\,L/R \;\square \quad \begin{matrix}\to \leq n \\ \to \leq n+1\end{matrix}$$

$\hookrightarrow$ I.H: $\Gamma' \vdash \Delta', A$, $\Gamma' \vdash \Delta', B$, $\Gamma'' \vdash \Delta'', A$, $\Gamma'' \vdash \Delta'', B$ derivable $\leq n$

$$\frac{\Gamma' \vdash \Delta', A \qquad \Gamma'' \vdash \Delta'', A}{\Gamma \vdash \Delta, A}\,L/R \;\square \quad \begin{matrix}\to \leq n \\ \to \leq n+1\end{matrix}$$

$$\frac{\Gamma' \vdash \Delta', B \qquad \Gamma'' \vdash \Delta'', B}{\Gamma \vdash \Delta, B}\,L/R \;\square \quad \begin{matrix}\to \leq n \\ \to \leq n+1\end{matrix}$$

• Same logic for the remaining 2 right rules. □

Q: if in G3cp all rules are invertible and the decomposition into top sequents unique, then if the sequent is provable, it is not possible to get to a dead end? (like in intuitionistic where $L\supset$ is not invertible)

Definition 3.1.3: *A* regular sequent *is a sequent of the form* $P_1, \ldots, P_m \Rightarrow Q_1, \ldots, Q_n, \perp, \ldots, \perp$ *where* $P_i \neq Q_j$, *the antecedent is empty if* $m = 0$, *and the succedent is* $\perp$ *if* $n = 0$. *The* trace formula *of a regular sequent is*

1. $P_1 \& \ldots \& P_m \supset Q_1 \vee \ldots \vee Q_n$ *for* $m, n > 0$,
2. $Q_1 \vee \ldots \vee Q_n$ *for* $m = 0, n > 0$,
3. $\sim(P_1 \& \ldots \& P_m)$ *for* $m > 0, n = 0$,
4. $\perp$ *for* $m, n = 0$,

where possible repetitions of the P_i *or* Q_j *in the regular sequent are deleted.*

the order in the disjunctions and conjunctions. By the invertibility of the rules of **G3cp**, a regular sequent with trace formula C is derivable if and only if the sequent $\Rightarrow C$ is derivable. It follows that a formula is equivalent to the conjunction

Proof:

$\Rightarrow$) if the trace formula C is derivable, then its regular sequent $P_1 \ldots P_m \vdash Q_1 \ldots Q_m$ is derivable. then applying $L\wedge$ on all the antecedent we get $P_1 \wedge \ldots \wedge P_m \vdash Q_1 \ldots Q_n$. Same goes for the succedent with $R\vee$ which give us $P_1 \wedge \ldots \wedge P_m \vdash Q_1 \vee \ldots \vee Q_n$. Finally apply $R\supset$ to get $\vdash P_1 \wedge \ldots \wedge P_m \supset Q_1 \ldots Q_m = \vdash C$.

$\Leftarrow$) if $\vdash C$ is derivable then apply $R\supset$, $L\wedge$ and $R\vee$ to get to $P_1 \ldots P_m \vdash Q_1 \ldots Q_m$. because $\vdash C$ is derivable and $P_1 \ldots P_m \vdash Q_1 \ldots Q_m$ is in middle of the derivation of $\vdash C$, the it is also derivable. Finally if $P_1 \ldots P_m \vdash Q_1 \ldots Q_m$ is derivable, the C is derivable. □

Theorem 3.2.1: Height-preserving weakening. *If* $\vdash_n \Gamma \Rightarrow \Delta$, *then* $\vdash_n A, \Gamma \Rightarrow \Delta$. *If* $\vdash_n \Gamma \Rightarrow \Delta$, *then* $\vdash_n \Gamma \Rightarrow \Delta, A$.

Proof:

Theorem 3.2.2: Height-preserving contraction. *If* $\vdash_n C, C, \Gamma \Rightarrow \Delta$, *then* $\vdash_n C, \Gamma \Rightarrow \Delta$. *If* $\vdash_n \Gamma \Rightarrow \Delta, C, C$, *then* $\vdash_n \Gamma \Rightarrow \Delta, C$.

Proof:

proof by induction over the height n:

Base case ($n=0$):

- $C, C, \Gamma \vdash \Delta$ is Ax or $L\perp$ then $C, \Gamma \vdash \Delta$ is also Ax or $L\perp$
- $\Gamma \vdash \Delta, C, C$ is Ax or $L\perp$ then $\Gamma \vdash \Delta, C$ is also Ax or $L\perp$

Inductive case:

I. H: if $\vdash_n C, C, \Gamma \vdash \Delta$ / $\vdash_n \Gamma \vdash \Delta, C, C$ then $\vdash_n C, \Gamma \vdash \Delta$ / $\vdash_n \Gamma \vdash C, \Delta$

• if C is not principal then:

same rule

$$\dfrac{C, C, \Gamma' \vdash \Delta' \qquad C, C, \Gamma'' \vdash \Delta''}{C, C, \Gamma \vdash \Delta} L/R\Box \quad \begin{matrix}\to n-1 \\ \to n\end{matrix} \quad \xrightarrow{I.H} \quad \dfrac{C, \Gamma' \vdash \Delta' \qquad C, \Gamma'' \vdash \Delta''}{C, \Gamma \vdash \Delta} L/R\Box \quad \begin{matrix}\to n-1 \\ \to n\end{matrix}$$

Same process for the left contraction

. if C is principal:

Case $L\wedge$:

Let $C = A \wedge B$:

$$\frac{A, B, A\wedge B, \Gamma \vdash \Delta \quad \to n}{A\wedge B, A\wedge B, \Gamma \vdash \Delta \quad \to n+1} L\wedge$$

- $\vdash_n A, B, A, B, \Gamma \vdash \Delta$ (Lemma 2.3.5)
- $\vdash_n A, B, B, \Gamma \vdash \Delta$ (I.H)
- $\vdash_n A, B, \Gamma \vdash \Delta$ (I.H)
- $\vdash_{n+1} A\wedge B, \Gamma \vdash \Delta$ ($L\wedge$)

Case $L\vee$:

Let $C = A \vee B$:

$$\frac{A, A\vee B, \Gamma \vdash \Delta \qquad B, A\vee B, \Gamma \vdash \Delta \quad \to n}{A\vee B, A\vee B, \Gamma \vdash \Delta \quad \to n+1} L\vee$$

- $\vdash_n A, A, \Gamma \vdash \Delta$ (Lemma 2.3.5) *
- $\vdash_n B, B, \Gamma \vdash \Delta$ (Lemma 2.3.5) **
- $\vdash_n A, \Gamma \vdash \Delta$ (I.H over *)
- $\vdash_n B, \Gamma \vdash \Delta$ (I.H over **)
- $\vdash_{n+1} A\vee B, \Gamma \vdash \Delta$ ($L\vee$)

Case $R\wedge$:

Let $C = A\wedge B$:

$$\vdash \frac{\Gamma \vdash \Delta, A\wedge B, A \qquad \Gamma \vdash \Delta, A\wedge B, B}{\Gamma \vdash \Delta, A\wedge B, A\wedge B} R\wedge$$
$\vdash_{n+1}$

By height-preserving invertibility, we obtain $\vdash_n \Gamma \Rightarrow \Delta, A, A$ and $\vdash_n \Gamma \Rightarrow \Delta, B, B$, and the inductive hypothesis gives $\vdash_n \Gamma \Rightarrow \Delta, A$ and $\vdash_n \Gamma \Rightarrow \Delta, B$. The

why cant I use I.H directly?

→ it is OK

$$\frac{\Gamma \vdash \Delta, A, A \qquad \Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A\wedge B, A} R\wedge \qquad \frac{\Gamma \vdash \Delta, A, B \qquad \Gamma \vdash \Delta, B, B}{\Gamma \vdash \Delta, A\wedge B, B} R\wedge$$

Derivable in $m < n$ steps

↳ then by invertibility $\Gamma \vdash \Delta, A, A$ and $\Gamma \vdash \Delta, B, B$ derivable in n steps

↳ then by I.H $\vdash_n \Gamma \vdash \Delta, A$ and $\vdash_n \Gamma \vdash \Delta, B$ and apply $R\wedge$ to get $\vdash_n \Gamma \vdash \Delta, A\wedge B$.

Case $R\vee$:

Let $C = A\vee B$

$$\frac{\Gamma \vdash \Delta, A\vee B, A, B \quad \to n}{\Gamma \vdash \Delta, A\vee B, A\vee B \quad \to n+1} R\vee$$

- $\vdash_{n-1} \Gamma \vdash \Delta, A, B, A, B$ ($R\vee$)
- $\vdash_n \Gamma \vdash \Delta, A, B, A, B$ (hight-preserving invertibility)
- $\vdash_n \Gamma \vdash \Delta, A, B, B$ (I.H)
- $\vdash_n \Gamma \vdash \Delta, A, B$ (I.H)
- $\vdash_{n+1} \Gamma \vdash \Delta, A\wedge B$ ($R\vee$)

X } $\vdash_n \Gamma \vdash \Delta, A, B, A, B$ (hight-preserving invertibility)

Case $L\supset$:

Let $C = A\supset B$:

$$\dfrac{\Gamma, A\supset B \vdash \Delta, A \quad \Gamma, A\supset B, B \vdash \Delta \to h}{\Gamma, A\supset B, A\supset B \vdash \Delta \to h+1} L\supset$$

$\vdash_{n-1}$ $\Gamma \vdash \Delta, A, A$ and $\Gamma, B, B \vdash \Delta$ ($L\supset$)
$\vdash_n$ $\Gamma \vdash \Delta, A, A$ and $\Gamma, B, B\vdash\Delta$ (hight preserving invertibility)
$\vdash_n$ $\Gamma\vdash\Delta, A$ and $\Gamma, B\vdash\Delta$ (I. H)
$\vdash_{n+1}$ $\Gamma, A\supset B\vdash\Delta$ ($R\supset$)

Case $R\supset$:

Let $C = A\supset B$:

$$\dfrac{\Gamma, A\vdash\Delta, A\supset B, B \to h}{\Gamma\vdash\Delta, A\supset B, A\supset B \to n+1} R\supset$$

$\vdash_{n-1}$ $\Gamma, A, A\vdash\Delta, B, B$ ($R\supset$)
$\vdash_n$ $\Gamma, A\ A\vdash\Delta, B, B$ (hight preserving invertibility)
$\vdash_n$ $\Gamma, A\vdash\Delta, B, B$(I. H)
$\vdash_n$ $\Gamma, A\vdash\Delta, B$ (I. H)
$\vdash_{n+1}$ $\Gamma\vdash\Delta, A\supset B$ ($R\supset$)

$\square$

Theorem 3.2.3: *The rule of cut,*

$$\dfrac{\Gamma\Rightarrow\Delta, D \quad D, \Gamma'\Rightarrow\Delta'}{\Gamma, \Gamma'\Rightarrow\Delta, \Delta'} Cut$$

is admissible in **G3cp**.

Right P	Left P
1) Ax 1, 2, 3, 4	1) Ax 1, 2, 3, 4
2) $L\bot$ 1, 2, 3, 4	2) $L\bot$ 1, 2, 3, 4
3) D prin 1, 2	3) D prin 1, 2
4) D not prin 1, 2	4) D not prin 1, 2

Proof:

induction over height and weight:

Base cases:

Case Left is Ax or $L\bot$:

- if $\bot\in\Gamma$ then $\Gamma, \Gamma'\vdash\Delta, \Delta'$ can be derived by $L\bot$
- if $D\in\Gamma$ then using the right premiss and weakening (RW and LW) with $\Gamma/\{D\}$ and Δ we derive $D, \Gamma/\{D\}, \Gamma'\vdash\Delta', \Delta = \Gamma, \Gamma'\vdash\Delta, \Delta'$.
- if Δ shares an atom with Γ, then $\Gamma, \Gamma'\vdash\Delta, \Delta'$ is also derivable by Ax.

Case right is Ax or $L\bot$:

- if $\bot\in\Gamma'$ then $\Gamma, \Gamma'\vdash\Delta, \Delta'$ can also be derived by $L\bot$
- if Γ' and Δ' share an atom, then $\Gamma, \Gamma'\vdash\Delta, \Delta'$ can be derived by axiom.
- if $D\in\Delta'$ then by Rw and Lw with Γ' and $\Delta'/\{D\}$ we obtain $\Gamma, \Gamma'\vdash D, D'$
- if $D = \bot$ then:

Subcase Left is Ax or $L\bot$:

- Done previously

Subcase Left ($\Gamma\vdash\Delta, \bot$) is derived by other rules:

- Special cases of following proofs (Case D not principal in Left premiss).

Inductive cases:

Case D is not principal in Left: $\Gamma \vdash D, \Delta \qquad \Gamma', D \vdash \Delta'$

Case $R\wedge$:

Let $\Delta = A \wedge B, \Delta''$

$$\dfrac{\dfrac{\overset{S_1}{\Gamma \vdash D, A, \Delta''} \quad \overset{S_2}{\Gamma \vdash D, B, \Delta''}}{\Gamma \vdash D, A\wedge B, \Delta''}\,R\wedge \qquad \overset{S_3}{\Gamma', D \vdash \Delta'}}{\Gamma, \Gamma' \vdash A\wedge B, \Delta'', \Delta'}\,\text{cut}$$

$ch = h(S_3) + \max(h(S_1), h(S_2)) + 1$

$\downarrow$ reduced ch

$$\dfrac{\dfrac{\overset{S_1}{\Gamma \vdash D, A, \Delta''} \quad \overset{S_3}{D, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash A, \Delta'', \Delta}\,\text{cut} \qquad \dfrac{\overset{S_2}{\Gamma \vdash D, B, \Delta''} \quad \overset{S_3}{D, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash B, \Delta'', \Delta'}\,\text{cut}}{\Gamma, \Gamma' \vdash A\wedge B, \Delta'', \Delta'}\,R\wedge$$

$ch_1 = h(S_1) + h(S_3)$

$ch_2 = h(S_2) + h(S_3)$

Case $L\wedge$:

$\Gamma \vdash D \qquad D, \Delta \vdash C$

$\Gamma, \Delta \vdash C$

Let $\Gamma = A\wedge B, \Gamma''$:

$$\dfrac{\dfrac{\overset{S_1}{\Gamma'', A, B \vdash D, \Delta}}{\Gamma'', A\wedge B \vdash D, \Delta}\,L\wedge \qquad \overset{S_2}{D, \Gamma' \vdash \Delta'}}{\Gamma'', A\wedge B, \Gamma' \vdash \Delta, \Delta'}\,\text{cut}$$

$ch = h(S_1) + 1 + h(S_2)$

$\downarrow$ reduced ch

$$\dfrac{\dfrac{\overset{S_1}{\Gamma'', A, B \vdash D, \Delta} \qquad \overset{S_2}{D, \Gamma' \vdash \Delta'}}{\Gamma'', A, B, \Gamma' \vdash \Delta, \Delta'}\,\text{cut}}{\Gamma'', A\wedge B, \Gamma' \vdash \Delta, \Delta'}\,L\wedge$$

$ch = h(S_1) + h(S_2)$

Case $R\vee$:

Let $\Delta = A\vee B, \Delta''$:

$$\dfrac{\dfrac{\overset{S_1}{\Gamma \vdash D, A, B, \Delta''}}{\Gamma \vdash D, A\vee B, \Delta''}\,R\vee \qquad \overset{S_2}{D, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash A\vee B, \Delta'', \Delta'}\,\text{cut}$$

$ch = h(S_1) + 1 + h(S_2)$

$\downarrow$ reduced ch

$$\dfrac{\dfrac{\overset{S_1}{\Gamma \vdash D, A, B, \Delta''} \qquad \overset{S_2}{D, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash A, B, \Delta'', \Delta'}\,\text{cut}}{\Gamma, \Gamma' \vdash A\vee B, \Delta'', \Delta'}\,R\vee$$

$ch = h(S_1) + h(S_2)$

Case $L\vee$:

Let $\Gamma = \Gamma'', A\vee B$

$$\dfrac{\dfrac{\overset{S_1}{\Gamma'', A \vdash D, \Delta} \quad \overset{S_2}{\Gamma'', B \vdash D, \Delta}}{\Gamma'', A\vee B \vdash D, \Delta}\,L\vee \qquad \overset{S_3}{D, \Gamma' \vdash \Delta'}}{\Gamma'', A\vee B, \Gamma' \vdash \Delta, \Delta'}\,\text{cut}$$

$ch = \max(h(S_1), h(S_2)) + 1 + h(S_3)$

$\downarrow$ reduced ch

$ch_1 = h(S_1) + h(S_3)$

$ch_2 = h(S_2) + h(S_3)$

$$\dfrac{\dfrac{\overset{S_1}{\Gamma'', A \vdash D, \Delta} \quad \overset{S_3}{D, \Gamma' \vdash \Delta'}}{\Gamma'', A, \Gamma' \vdash \Delta, \Delta'}\,\text{cut} \qquad \dfrac{\overset{S_2}{\Gamma'', B \vdash D, \Delta} \quad \overset{S_3}{D, \Gamma' \vdash \Delta'}}{\Gamma'', B, \Gamma' \vdash \Delta, \Delta'}\,\text{cut}}{\Gamma'', A\vee B, \Gamma' \vdash \Delta, \Delta'}\,L\vee$$