Descomposición en valores singulares (SVD)

Contenido

- SVD: Matrices $m \times n$

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- SVD: Matrices $m \times n$
- Formulación SVD.

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- Formulación SVD.
- Interpretación.

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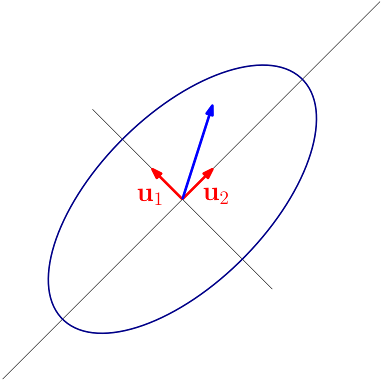
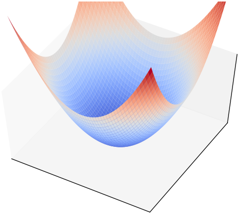
- SVD: Matrices $m \times n$
- Formulación SVD.
- Interpretación.

Matriz $\mathbf{A} \in \mathbb{R}^{n \times n}$ simétrica positiva definida





$$\mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^T$$

Matriz $\mathbf{A} \in \mathbb{R}^{m \times n}$

$$
\begin{array}{r}
\\
\text{dato } \mathbf{x}_2 \rightarrow \\
\\
\text{dato } \mathbf{x}_4 \rightarrow \\
\\
\\
\end{array}
\begin{bmatrix}
3 & 4 & 5 & 1 \\
5 & 6 & 8 & 6 \\
4 & 5 & 8 & 3 \\
3 & 4 & 5 & 8 \\
5 & 6 & 8 & 9 \\
4 & 5 & 8 & 7
\end{bmatrix}
$$

$$
\begin{bmatrix}
3 & 3 & 2 & 3 & 4 & * & 2 & 3 & 1 & 5 \\
4 & 4 & 5 & 5 & 3 & 4 & 2 & 3 & * & 3 \\
3 & * & * & 2 & 3 & * & 1 & 5 & 2 & 3 \\
5 & 4 & * & 4 & 3 & * & * & 4 & 3 & 5
\end{bmatrix}
$$

Matriz $\mathbf{A} \in \mathbb{R}^{m \times n}$

$$
\begin{array}{r l}
 & \\
\text{dato } \mathbf{x}_2 \rightarrow & \\
 & \\
\text{dato } \mathbf{x}_4 \rightarrow & \\
 & \\
 &
\end{array}
\begin{bmatrix}
3 & 4 & 5 & 1 \\
5 & 6 & 8 & 6 \\
4 & 5 & 8 & 3 \\
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5 & 6 & 8 & 9 \\
4 & 5 & 8 & 7
\end{bmatrix}
$$

$$
\begin{bmatrix}
3 & 3 & 2 & 3 & 4 & * & 2 & 3 & 1 & 5 \\
4 & 4 & 5 & 5 & 3 & 4 & 2 & 3 & * & 3 \\
3 & * & * & 2 & 3 & * & 1 & 5 & 2 & 3 \\
5 & 4 & * & 4 & 3 & * & * & 4 & 3 & 5
\end{bmatrix}
$$

- No tenemos $\mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^T$

Matriz $\mathbf{A} \in \mathbb{R}^{m \times n}$

$$
\begin{array}{r}
\\
\text{dato } \mathbf{x}_2 \rightarrow \\
\\
\text{dato } \mathbf{x}_4 \rightarrow \\
\\
\\
\end{array}
\begin{bmatrix}
3 & 4 & 5 & 1 \\
5 & 6 & 8 & 6 \\
4 & 5 & 8 & 3 \\
3 & 4 & 5 & 8 \\
5 & 6 & 8 & 9 \\
4 & 5 & 8 & 7
\end{bmatrix}
$$

$$
\begin{bmatrix}
3 & 3 & 2 & 3 & 4 & * & 2 & 3 & 1 & 5 \\
4 & 4 & 5 & 5 & 3 & 4 & 2 & 3 & * & 3 \\
3 & * & * & 2 & 3 & * & 1 & 5 & 2 & 3 \\
5 & 4 & * & 4 & 3 & * & * & 4 & 3 & 5
\end{bmatrix}
$$

- No tenemos $\mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^T$
- Algo similar?

Tranformación lineal $\mathbb{R}^n \to \mathbb{R}^m$

$$\mathbf{x} \in \mathbb{R}^n \longrightarrow \boxed{\;\mathbf{A}\;} \longrightarrow \mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m$$

Tranformación lineal $\mathbb{R}^n \to \mathbb{R}^m$

$$\mathbf{x} \in \mathbb{R}^n \longrightarrow \boxed{\mathbf{A}} \longrightarrow \mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m$$

$$\color{red}{\mathbf{A}}\color{blue}{\mathbf{v}_i} = \sigma_i \color{teal}{\mathbf{u}_i}$$

Tranformación lineal $\mathbb{R}^n \to \mathbb{R}^m$

$$\mathbf{x} \in \mathbb{R}^n \longrightarrow \boxed{\mathbf{A}} \longrightarrow \mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m$$

$$\overset{m \times n}{\overset{\downarrow}{\mathbf{A}}}\mathbf{v}_i = \sigma_i \mathbf{u}_i$$

Tranformación lineal $\mathbb{R}^n \to \mathbb{R}^m$

$$\mathbf{x} \in \mathbb{R}^n \longrightarrow \boxed{\quad \mathbf{A} \quad} \longrightarrow \mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m$$

$$\overset{\overset{m \times n}{\downarrow}}{\mathbf{A}}\underset{\underset{n \times 1}{\uparrow}}{\mathbf{v}_i} = \sigma_i \mathbf{u}_i$$

Tranformación lineal $\mathbb{R}^n \to \mathbb{R}^m$

$$\mathbf{x} \in \mathbb{R}^n \longrightarrow \boxed{\quad \mathbf{A} \quad} \longrightarrow \mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m$$

$$\overset{m \times n}{\mathbf{A}}\,\underset{n \times 1}{\mathbf{v}_i} = \sigma_i \overset{m \times 1}{\mathbf{u}_i}$$

Tranformación lineal $\mathbb{R}^n \to \mathbb{R}^m$

$$\mathbf{x} \in \mathbb{R}^n \longrightarrow \boxed{\quad \mathbf{A} \quad} \longrightarrow \mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m$$

$$\overset{m \times n}{\mathbf{A}}\,\underset{n \times 1}{\mathbf{v}_i} = \underset{> 0}{\sigma_i}\,\overset{m \times 1}{\mathbf{u}_i}$$

Tranformación lineal $\mathbb{R}^n \to \mathbb{R}^m$

$$\mathbf{x} \in \mathbb{R}^n \longrightarrow \boxed{\mathbf{A}} \longrightarrow \mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m$$

$$\overset{m \times n}{\mathbf{A}}\,\underset{n \times 1}{\mathbf{v}_i} = \underset{> 0}{\sigma_i}\,\overset{m \times 1}{\mathbf{u}_i}$$

- $\mathbf{v}_i$: Base ortonormal del espacio expandido por las filas de $\mathbf{A}$.

Tranformación lineal $\mathbb{R}^n \to \mathbb{R}^m$

$$\mathbf{x} \in \mathbb{R}^n \longrightarrow \boxed{\;\mathbf{A}\;} \longrightarrow \mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^m$$

$$\overset{m \times n}{\mathbf{A}}\underset{n \times 1}{\mathbf{v}_i} = \underset{> 0}{\sigma_i}\overset{m \times 1}{\mathbf{u}_i}$$

- $\mathbf{v}_i$: Base ortonormal del espacio expandido por las filas de **A**.
- $\mathbf{u}_i$: Base ortonormal del espacio expandido por las columnas de **A**.

$$\begin{bmatrix} \mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \ldots & \mathbf{A}\mathbf{v}_r \end{bmatrix} = \begin{bmatrix} \sigma_1 \mathbf{u}_1 & \sigma_2 \mathbf{u}_2 & \ldots & \sigma_r \mathbf{u}_r \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \dots & \mathbf{A}\mathbf{v}_r & \mathbf{A}\mathbf{v}_{r+1} & \dots & \mathbf{A}\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \sigma_1\mathbf{u}_1 & \sigma_2\mathbf{u}_2 & \dots & \sigma_r\mathbf{u}_r & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \ldots & \mathbf{A}\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \sigma_1\mathbf{u}_1 & \sigma_2\mathbf{u}_2 & \ldots & \sigma_n\mathbf{u}_n \end{bmatrix}$$

$$\mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \sigma_n \end{bmatrix}$$

$$
\mathbf{A}\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n\end{bmatrix}\begin{bmatrix}\sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \sigma_n\end{bmatrix}
$$

$$
\mathbf{A}\mathbf{V} = \mathbf{U}\boldsymbol{\Sigma}
$$

$$\mathbf{A}\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n\end{bmatrix}\begin{bmatrix}\sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \sigma_n\end{bmatrix}$$

$$\mathbf{A}\mathbf{V} = \mathbf{U}\boldsymbol{\Sigma}$$

$$\mathbf{A}\mathbf{V}\mathbf{V}^T = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$$

$$\mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \sigma_n \end{bmatrix}$$

$$\mathbf{A}\mathbf{V} = \mathbf{U}\boldsymbol{\Sigma}$$

$$\mathbf{A}\mathbf{V}\mathbf{V}^T = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$$

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$$

$$\mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \sigma_n \end{bmatrix}$$

$$\mathbf{A}\mathbf{V} = \mathbf{U}\boldsymbol{\Sigma}$$

$$\mathbf{A}\mathbf{V}\mathbf{V}^T = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$$

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$$

- Descomposición en valores singulares de $\mathbf{A}$.

$$\mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \sigma_n \end{bmatrix}$$

$$\mathbf{A}\mathbf{V} = \mathbf{U}\boldsymbol{\Sigma}$$

$$\mathbf{A}\mathbf{V}\mathbf{V}^T = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$$

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$$

- Descomposición en valores singulares de **A**.
- $\mathbf{U}$, $\boldsymbol{\Sigma}$, $\mathbf{V}$?

$$
\mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \sigma_n \end{bmatrix}
$$

$$
\begin{aligned}
\mathbf{AV} &= \mathbf{U\Sigma} \\
\mathbf{AVV}^T &= \mathbf{U\Sigma V}^T \\
\mathbf{A} &= \mathbf{U\Sigma V}^T
\end{aligned}
$$

- Descomposición en valores singulares de $\mathbf{A}$.
- $\mathbf{U}$, $\mathbf{\Sigma}$, $\mathbf{V}$?
  - $\mathbf{U}, \mathbf{V}$ con columnas ortonormales.

$$\mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \sigma_n \end{bmatrix}$$

$$\mathbf{AV} = \mathbf{U\Sigma}$$

$$\mathbf{AVV}^T = \mathbf{U\Sigma V}^T$$

$$\mathbf{A} = \mathbf{U\Sigma V}^T$$

- Descomposición en valores singulares de $\mathbf{A}$.
- $\mathbf{U}$, $\mathbf{\Sigma}$, $\mathbf{V}$?
  - $\mathbf{U}, \mathbf{V}$ con columnas ortonormales.
  - $\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i$, con $\sigma_i > 0$.

- Matriz $\mathbf{A}^T\mathbf{A}$:

- Matriz $\mathbf{A}^T\mathbf{A}$:
  - $n \times n$.

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  - Simétrica.

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  - Positiva (semi) definida:

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  - $n \times n$.
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  - Positiva (semi) definida:

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{x}^T\mathbf{A}^T)(\mathbf{A}\mathbf{x})$$

- Matriz $\mathbf{A}^T\mathbf{A}$:
  - $n \times n$.
  - Simétrica.
  - Positiva (semi) definida:

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{x}^T\mathbf{A}^T)(\mathbf{A}\mathbf{x}) = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) =$$

- Matriz $\mathbf{A}^T\mathbf{A}$:
  - $n \times n$.
  - Simétrica.
  - Positiva (semi) definida:

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{x}^T\mathbf{A}^T)(\mathbf{A}\mathbf{x}) = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x}\|^2$$

- Matriz $\mathbf{A}^T\mathbf{A}$:
  - $n \times n$.
  - Simétrica.
  - Positiva (semi) definida:

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{x}^T\mathbf{A}^T)(\mathbf{A}\mathbf{x}) = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x}\|^2 > 0$$

- Matriz $\mathbf{A}^T\mathbf{A}$:
  - $n \times n$.
  - Simétrica.
  - Positiva (semi) definida:

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{x}^T\mathbf{A}^T)(\mathbf{A}\mathbf{x}) = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x}\|^2 > 0$$

  - Podemos factorizar:

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^T$$

- Matriz $\mathbf{A}^T\mathbf{A}$:
  - $n \times n$.
  - Simétrica.
  - Positiva (semi) definida:

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{x}^T\mathbf{A}^T)(\mathbf{A}\mathbf{x}) = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x}\|^2 > 0$$

  - Podemos factorizar:

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^T$$

- En SVD:

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\Sigma\mathbf{U}^T\mathbf{U}\Sigma\mathbf{V}^T$$

- Matriz $\mathbf{A}^T\mathbf{A}$:
  - $n \times n$.
  - Simétrica.
  - Positiva (semi) definida:

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{x}^T\mathbf{A}^T)(\mathbf{A}\mathbf{x}) = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x}\|^2 > 0$$

  - Podemos factorizar:

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^T$$

- En SVD:

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\Sigma\mathbf{U}^T\mathbf{U}\Sigma\mathbf{V}^T = \mathbf{V}\Sigma^2\mathbf{V}^T$$

- Matriz $\mathbf{A}^T\mathbf{A}$:
  - $n \times n$.
  - Simétrica.
  - Positiva (semi) definida:

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{x}^T\mathbf{A}^T)(\mathbf{A}\mathbf{x}) = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x}\|^2 > 0$$

  - Podemos factorizar:

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^T$$

- En SVD:

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\Sigma\mathbf{U}^T\mathbf{U}\Sigma\mathbf{V}^T = \mathbf{V}\Sigma^2\mathbf{V}^T$$

- Entonces:
  - Columnas de $\mathbf{V}$ son vectores propios de $\mathbf{A}^T\mathbf{A}$ ortonormales.

- Matriz $\mathbf{A}^T\mathbf{A}$:
  - $n \times n$.
  - Simétrica.
  - Positiva (semi) definida:

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{x}^T\mathbf{A}^T)(\mathbf{A}\mathbf{x}) = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) = \|\mathbf{A}\mathbf{x}\|^2 > 0$$

  - Podemos factorizar:

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^T$$

- En SVD:

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\Sigma\mathbf{U}^T\mathbf{U}\Sigma\mathbf{V}^T = \mathbf{V}\Sigma^2\mathbf{V}^T$$

- Entonces:
  - Columnas de $\mathbf{V}$ son vectores propios de $\mathbf{A}^T\mathbf{A}$ ortonormales.
  - Valores propios $\lambda_i = \sigma_i^2$.

- $\mathbf{u}_i$:

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i$$

- $\mathbf{u}_i$:

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i \Rightarrow \mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$$

- $\mathbf{u}_i$:

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i \Rightarrow \mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$$

- Ortonormales?

$$\mathbf{u}_i{}^T \mathbf{u}_j$$

- $\mathbf{u}_i$:

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i \Rightarrow \mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$$

- Ortonormales?

$$\mathbf{u}_i{}^T \mathbf{u}_j = \frac{1}{\sigma_i}(\mathbf{A}\mathbf{v}_i)^T$$

- $\mathbf{u}_i$:

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i \Rightarrow \mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$$

- Ortonormales?

$$\mathbf{u}_i{}^T \mathbf{u}_j = \frac{1}{\sigma_i}(\mathbf{A}\mathbf{v}_i)^T \frac{1}{\sigma_j}\mathbf{A}\mathbf{v}_j$$

- $\mathbf{u}_i$:

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i \Rightarrow \mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$$

- Ortonormales?

$$\mathbf{u}_i{}^T \mathbf{u}_j = \frac{1}{\sigma_i}(\mathbf{A}\mathbf{v}_i)^T \frac{1}{\sigma_j}\mathbf{A}\mathbf{v}_j = \frac{1}{\sigma_i \sigma_j}\mathbf{v}_i \mathbf{A}^T \mathbf{A}\mathbf{v}_j$$

- $\mathbf{u}_i$:

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i \Rightarrow \mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$$

- Ortonormales?

$$\mathbf{u}_i{}^T \mathbf{u}_j = \frac{1}{\sigma_i}(\mathbf{A}\mathbf{v}_i)^T \frac{1}{\sigma_j}\mathbf{A}\mathbf{v}_j = \frac{1}{\sigma_i \sigma_j}\mathbf{v}_i \mathbf{A}^T \mathbf{A}\mathbf{v}_j = \frac{1}{\sigma_i \sigma_j}\sigma_j^2 \mathbf{v}_i^T \mathbf{v}_j$$

- $\mathbf{u}_i$:

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i \Rightarrow \mathbf{u}_i = \frac{1}{\sigma_i}\mathbf{A}\mathbf{v}_i$$

- Ortonormales?

$$\mathbf{u}_i{}^T\mathbf{u}_j = \frac{1}{\sigma_i}(\mathbf{A}\mathbf{v}_i)^T\frac{1}{\sigma_j}\mathbf{A}\mathbf{v}_j = \frac{1}{\sigma_i\sigma_j}\mathbf{v}_i\mathbf{A}^T\mathbf{A}\mathbf{v}_j = \frac{1}{\sigma_i\sigma_j}\sigma_j^2\mathbf{v}_i^T\mathbf{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Descomposición de **A** en valores singulares (SVD)

$$\mathbf{A} = \underbrace{\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_r \end{bmatrix}}_{\text{base span de columnas}} \underbrace{\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \sigma_r \end{bmatrix}}_{\text{valores singulares}} \left.\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_r^T \end{bmatrix}\right\} \text{base span de filas}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

Descomposición de $\mathbf{A}$ en valores singulares (SVD)

$$\mathbf{A} = \underbrace{\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_r \end{bmatrix}}_{\text{base span de columnas}} \underbrace{\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \sigma_r \end{bmatrix}}_{\text{valores singulares}} \left.\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_r^T \end{bmatrix}\right\} \text{base span de filas}$$

$$\mathbf{A} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

Descomposición de $\mathbf{A}$ en valores singulares (SVD)

$$\mathbf{A} = \underbrace{\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_r \end{bmatrix}}_{\text{base span de columnas}} \underbrace{\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \sigma_r \end{bmatrix}}_{\text{valores singulares}} \left.\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_r^T \end{bmatrix}\right\} \text{base span de filas}$$

$$\mathbf{A} = \underbrace{\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T}_{r \text{ matrices de rango } 1}$$

Teorema de Eckart-Young

$$\mathbf{A} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

Teorema de Eckart-Young

$$\mathbf{A} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

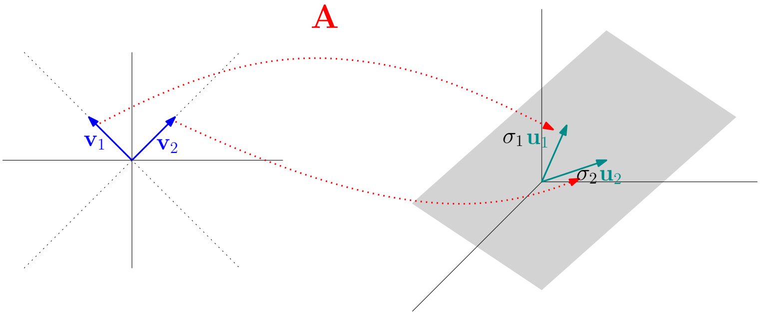
$$\mathbf{A}_k = \underbrace{\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T}_{\text{rango } k}$$

Teorema de Eckart-Young

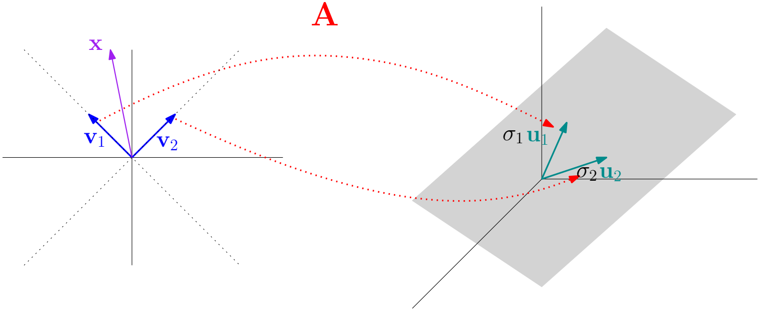
$$\mathbf{A} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

$$\mathbf{A}_k = \underbrace{\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T}_{\text{rango } k}$$

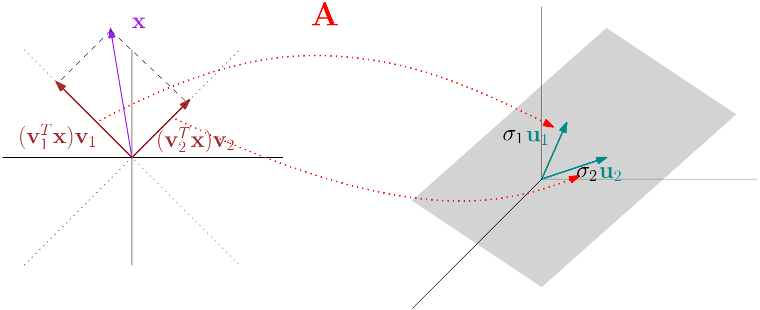
- $\mathbf{A}_k$ es la mejor aproximación de rango k de $\mathbf{A}$



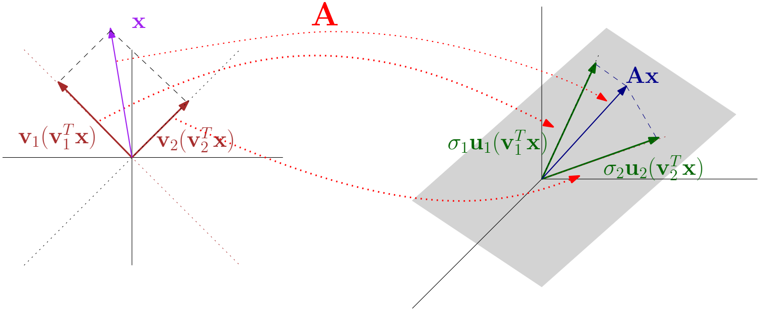
A
v1
v2
σ1 u1
σ2 u2



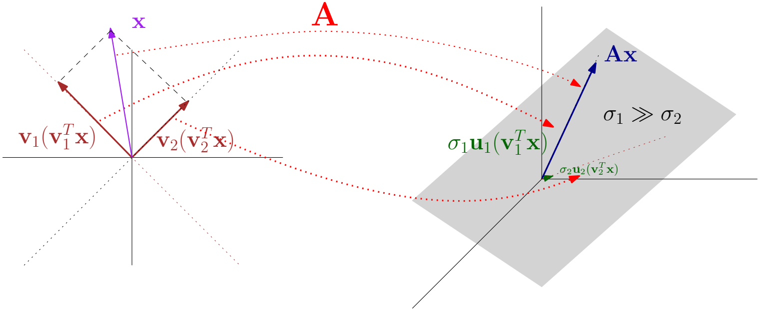
A
x
v1
v2
σ1 u1
σ2 u2



$\mathbf{x}$
$\mathbf{A}$
$(\mathbf{v}_1^T\mathbf{x})\mathbf{v}_1$
$(\mathbf{v}_2^T\mathbf{x})\mathbf{v}_2$
$\sigma_1\,\mathbf{u}_1$
$\sigma_2\,\mathbf{u}_2$



$\mathbf{x}$
$\mathbf{A}$
$\mathbf{v}_1(\mathbf{v}_1^T\mathbf{x})$
$\mathbf{v}_2(\mathbf{v}_2^T\mathbf{x})$
$\mathbf{Ax}$
$\sigma_1\mathbf{u}_1(\mathbf{v}_1^T\mathbf{x})$
$\sigma_2\mathbf{u}_2(\mathbf{v}_2^T\mathbf{x})$



$\mathbf{A}$
$\mathbf{x}$
$\mathbf{Ax}$
$\sigma_1 \gg \sigma_2$
$\mathbf{v}_1(\mathbf{v}_1^T\mathbf{x})$
$\mathbf{v}_2(\mathbf{v}_2^T\mathbf{x})$
$\sigma_1\mathbf{u}_1(\mathbf{v}_1^T\mathbf{x})$
$\sigma_2\mathbf{u}_2(\mathbf{v}_2^T\mathbf{x})$

Resumen

- SVD: Relación entre bases de espacio de filas y espacio de columnas.

Resumen

- SVD: Relación entre bases de espacio de filas y espacio de columnas.
- Magnitud valores singulares $\rightarrow$ Aproximar matriz original por una matriz de rango menor.