CS 4720/5720 Design and Analysis of Algorithms

Homework #1

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Answers to homework problems:

1. Book Problem: 2.2.3

   (a) $(n^2+1)^{10} \approx (n^2)^{10} = n^{20} \in \Theta(n^{20})$

   $\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{20}} = \lim_{n\to\infty} \frac{(n^2+1)^{10}}{(n^2)^{10}} = \lim_{n\to\infty} (\frac{(n^2+1)}{n})^{10} = \lim_{n\to\infty} (1+\frac{1}{n^2})^{10} = 1 \in \Theta(n^{20})$

   (b) $\sqrt{10n^2+7n+3} \approx \sqrt{10n^2} = n\sqrt{10} \in \Theta(n)$

   $\lim_{n\to\infty} \frac{\sqrt{10n^2+7n+3}}{n} = \lim_{n\to\infty} \sqrt{\frac{10n^2+7n+3}{n^2}} = \lim_{n\to\infty} \sqrt{\frac{10n^2}{n^2}+\frac{7n}{n^2}+\frac{3}{n^2}} = \lim_{n\to\infty} \sqrt{10+\frac{7}{n}+\frac{3}{n^2}} = \sqrt{10} \in \Theta(n)$

   (c) $2n\log(n+2)^2+(n+2)^2\log\frac{n}{2} = 2n*2\log(n+2)+(n+2)^2(\log n-\log 2) = [max(\Theta(n\log n), \Theta(n^2\log n))] \in \Theta(n^2\log n)$

   (d) $2^{n+1}+3^{n-1} = 2^n*2+3^n*\frac{1}{3} \in [max(\Theta(2^n), \Theta(3^n))] = \Theta(3^n)$

   (e) $\lfloor\log_2 n\rfloor \approx \log_2 n \in \Theta(\log n)$
   $x-1 < \lfloor x\rfloor \le x \Rightarrow \lfloor log_2 n\rfloor \le \log_2 n$
   $\lfloor\log_2 n\rfloor > \log_2 n - 1 \ge \log_2 n - \frac{1}{2}\log_2 n = \frac{1}{2}\log_2 n \in \Theta(\log_2 n) = \Theta(\log n)$

2. Here are the answers to a problem with sub-parts!

   (a) $T(n) = 2T(\frac{n}{2})+n^3 \Rightarrow T(x) = 2T(\frac{x}{2})+x^3$
   Let $x=\frac{n}{2}, T(\frac{n}{2}) = 2T(\frac{n}{4})+\frac{1}{8}n^3$
   $T(n) = 2[2T(\frac{n}{4})+\frac{1}{8}n^3]+n^3 = 4T(\frac{n}{4})+\frac{1}{4}n^3+n^3$
   Let $x=\frac{n}{4}, T(\frac{n}{4}) = 2T(\frac{n}{8})+\frac{1}{64}n^3$
   $T(n) = 4[2T(\frac{n}{8})+\frac{1}{64}n^3]+\frac{5}{4}n^3 = 8T(\frac{n}{8}+\frac{1}{16}n^3+\frac{1}{4}n^3+n^3$
   $T(n) = 2^kT(\frac{n}{2^k})+n^3\sum_{i=0}^{k-1}(\frac{1}{4})^i$

   (b) The 2nd part

   (c) and so on...

2.2.3 abcde indicate Θ Class. Prove

1. a.) $(n^2+1)^{10} \approx (n^2)^{10} = n^{20} \in \Theta(n^{20})$

$\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{20}} = \lim_{n\to\infty} \frac{(n^2+1)^{10}}{(n^2)^{10}} = \lim_{n\to\infty} \left(\frac{n^2+1}{n^2}\right)^{10} = \lim_{n\to\infty} \left(1+\frac{1}{n^2}\right)^{10} = 1 \in \Theta(n^{20})$

b.) $\sqrt{10n^2+7n+3} \approx \sqrt{10n^2} = n\sqrt{10} \in \Theta(n)$

$\lim_{n\to\infty} \frac{\sqrt{10n^2+7n+3}}{n} = \lim_{n\to\infty} \sqrt{\frac{10n^2+7n+3}{n^2}} = \lim_{n\to\infty} \sqrt{\frac{10n^2}{n^2}+\frac{7n}{n^2}+\frac{3}{n^2}} = \lim_{n\to\infty} \sqrt{10+\frac{7}{n}+\frac{3}{n^2}} = \sqrt{10} \in \Theta(n)$

c.) $2n\log(n+2)^2 + (n+2)^2\log\frac{n}{2} = 2n\cdot 2\log(n+2) + (n+2)^2[\log n - \cancel{\log 2}^{\,1}]$

max of: $\Theta(n\log n) + \Theta(n^2\log n) \in \Theta(n^2\log n)$

d.) $2^{n+1} + 3^{n-1} = 2^n 2 + 3^n \frac{1}{3} \in \Theta(2^n) + \Theta(3^n) = \Theta(3^n)$

$2^{n+1} = (2^n \cdot 2^1)$, $3^{n-1} = (3^n \cdot 3^{-1})$

e.) $\lfloor \log_2 n \rfloor \approx \log_2 n \in \Theta(\log n)$

$x-1 < \lfloor x \rfloor \le x \Rightarrow \lfloor \log_2 n \rfloor \le \log_2 n$ upper bound

Lower: $\lfloor \log_2 n \rfloor > \log_2 n - 1 \ge \log_2 n - \frac{1}{2}\log_2 n = \frac{1}{2}\log_2 n$

$\lfloor \log_2 n \rfloor \in \Theta(\log_2 n) = \Theta(\log n)$

3.
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BinSearch(A[0,n-1], k, left, right)
  If right < left, return -1
  Mid = floor((right-left)/2)+left
  If A[Mid] == k, return Mid
  if k < A[Mid], return BinSearch(A, k, left, Mid-1)
  else return BinSearch(A, k, Mid+1, right)
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a.) $T(n) = 2T\left(\frac{n}{2}\right) + n^3$, $T(x) = 2T\left(\frac{x}{2}\right) + n^3$ $\qquad n^3 \sum_{i=0}^{k-1} \left(\frac{1}{4}\right)^i$

Let $x = \frac{n}{2}$, $T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^3 = 2T\left(\frac{n}{4}\right) + \frac{1}{8}n^3$

2 $T(n) = 2\left[2T\left(\frac{n}{4}\right) + \frac{1}{8}n^3\right] + n^3 = \underline{4T\left(\frac{n}{4}\right) + \frac{1}{4}n^3 + n^3}$

Let $x = \frac{n}{4}$, $T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^3 = 2T\left(\frac{n}{8}\right) + \frac{1}{64}n^3$

3 $T(n) = 4\left[2T\left(\frac{n}{8}\right) + \frac{1}{64}n^3\right] + \frac{5}{4}n^3 = \underline{8T\left(\frac{n}{8}\right) + \frac{1}{16}n^3 + \frac{1}{4}n^3 + n^3}$

Let $x = \frac{n}{8}$, $T\left(\frac{n}{8}\right) = 2T\left(\frac{n}{8}\right) + \left(\frac{n}{8}\right)^3 = 2T\left(\frac{n}{8}\right) + \frac{1}{512}n^3$

$T(n) = 2^k T\left(\frac{n}{2^k}\right) + n^3 \sum_{i=0}^{k-1} \left(\frac{1}{4}\right)^i$

$\frac{n}{2^k} = 1 \Rightarrow n = 2^k \rightarrow \log_2 n = k \cancel{\log_2 2} \Rightarrow n\,T(1) + n^3 \sum_{j=0}^{\log_2 n - 1} \left(\frac{1}{4}\right)^j$

$T(n) = nT(1) + n^3 \left(\frac{1 - (1/4)^{\log_2 n}}{1 - \frac{1}{4}}\right) \lim_{n\to\infty} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \Rightarrow nT(1) + n^3\left(\frac{4}{3}\right) \in \Theta(n^3)$

1 b.) $T(n) = T(\sqrt{n})\sqrt{n} + n$, $T(n_0) = 1$ $n \le 2$ $\qquad T(n) \in \Theta(n \log\log n)$ ✓

$T(x) = T(\sqrt{x})\sqrt{x} + x$

Let $x = \sqrt{n}$, $T(\sqrt{n}) = T(n^{1/4})n^{1/4} + n^{1/2}$

2 $T(n) = \left[T(n^{1/4})n^{1/4} + n^{1/2}\right]\sqrt{n} + n = T(n^{1/4})n^{3/4} + n + n = T(n^{1/4})n^{3/4} + 2n$

Let $x = n^{1/4}$, $T(n^{1/4}) = T(n^{1/8})n^{1/8} + n^{1/4}$

3 $T(n) = \left[T(n^{1/8})n^{1/8} + n^{1/4}\right]n^{3/4} + 2n = T(n^{1/8})n^{7/8} + n + 2n = T(n^{1/8})n^{7/8} + 3n$

$T(n) = T\left(n^{\frac{1}{2^k}}\right) n^{\frac{(2^k - 1)}{2^k}} + kn \longrightarrow \frac{2^k - 1}{2^k} = 1 - 2^{-k}$

$n^{1/2^k} = n_0$, $n_0 \le 2$

~~Try $n_0 = 1 \Rightarrow n^{(1/2^k)} = 1 \Rightarrow n = 1$~~

Try $n_0 = 2 \Rightarrow n^{(1/2^k)} = 2 \Rightarrow \left(n^{1/2^k}\right)^{2^k} = (2)^{2^k} \Rightarrow n = 2^{2^k}$

for k: $\log_2 n = 2^k \cancel{\log_2 2} \rightarrow \log_2 n = 2^k \Rightarrow \underline{\log_2 \log_2 n = k \cancel{\log_2 2}}$

$T(n) = T(2)\frac{n}{2} + n \log\log n \in \Theta(n \log\log n)$

1 — c.) $T(n) = 2T(\frac{n}{2}) + \frac{n}{\log n}$ $\quad \in \Theta(n \log\log n)$? ✓

$T(x) = 2T(\frac{x}{2}) + \frac{x}{\log x}$

Let $x = \frac{n}{2}$, $T(\frac{n}{2}) = 2T(\frac{n}{4}) + \frac{n}{2} / \log\frac{n}{2} = 2T(\frac{n}{4}) + \frac{n/2}{\log n - 1}$

2 — $T(n) = 2\left[2T(\frac{n}{4}) + \frac{n/2}{\log n - 1}\right] + \frac{n}{\log n} = 4T(\frac{n}{4}) + \frac{n}{\log n - 1} + \frac{n}{\log n}$

Let $x = \frac{n}{4}$, $T(\frac{n}{4}) = 2T(\frac{n}{8}) + \frac{n/4}{\log\frac{n}{4}} = 2T(\frac{n}{8}) + \frac{n/4}{\log n - 2}$

3 — $T(n) = 4\left[2T(\frac{n}{8}) + \frac{n/4}{\log n - 2}\right] + \frac{n}{\log n - 1} + \frac{n}{\log n} = 8T(\frac{n}{8}) + \frac{n}{\log n - 2} + \frac{n}{\log n - 1} + \frac{n}{\log n}$

— $T(n) = 2^k T(\frac{n}{2^k}) + \sum_{i=0}^{k-1} \frac{n}{\log n - i}$

Try $n_0 = 1$

$\frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow \log_2 n = k \cancel{\log_2 2}^{1}$

$T(n) = 2^{\log_2 n} T(\frac{n}{n}) + \sum_{i=0}^{\log_2 n - 1} \frac{n}{\log_2 n - i} \rightarrow j = \log_2 n - 1$

$T(n) = n\& + \sum_{j=0}^{\log_2 n} \frac{n}{j} \rightarrow n H_{\log_2 n} \rightarrow n \log\log n \in \Theta(n \log\log n)$

d.) $T(n) = 3T(\frac{n}{2}) + n \log n$, $T(x) = 3T(\frac{x}{2}) + x \log x$

Let $x = \frac{n}{2}$, $T(\frac{n}{2}) = 3T(\frac{n}{4}) + \frac{n}{2}\log\frac{n}{2} = 3T(\frac{n}{4}) +$

— $T(n) = 3\left[3T(\frac{n}{4}) + \frac{n}{2}\log\frac{n}{2}\right] + n\log n = 9T(\frac{n}{4}) +$

Master Theorem: $a = 3$, $b = 2$, $f(n) = n \log n$

$n^{\log_b (a - \varepsilon)} = n^{\log_2 3 - \varepsilon} = n^{1.58}$

case 1: $T(n) = \Theta(n^{1.58})$