CS 4720/5720 Design and Analysis of Algorithms

Homework #1

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Answers to homework problems:

1. Book Problem: 2.2.3

(a) $(n^2+1)^{10} \approx (n^2)^{10} = n^{20} \in \Theta(n^{20})$

$\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{20}} = \lim_{n\to\infty} \frac{(n^2+1)^{10}}{(n^2)^{10}} = \lim_{n\to\infty} (\frac{(n^2+1)}{n})^{10} = \lim_{n\to\infty} (1+\frac{1}{n^2})^{10} = 1 \in \Theta(n^{20})$

(b) $\sqrt{10n^2+7n+3} \approx \sqrt{10n^2} = n\sqrt{10} \in \Theta(n)$

$\lim_{n\to\infty} \frac{\sqrt{10n^2+7n+3}}{n} = \lim_{n\to\infty} \sqrt{\frac{10n^2+7n+3}{n^2}} = \lim_{n\to\infty} \sqrt{\frac{10n^2}{n^2} + \frac{7n}{n^2} + \frac{3}{n^2}} = \lim_{n\to\infty} \sqrt{10 + \frac{7}{n} + \frac{3}{n^2}} = \sqrt{10} \in \Theta(n)$

(c) $2n\log(n+2)^2 + (n+2)^2 \log \frac{n}{2} = 2n * 2\log(n+2) + (n+2)^2(\log n - \log 2) = [max(\Theta(n\log n), \Theta(n^2 \log n))] \in \Theta(n^2 \log n)$

(d) $2^{n+1} + 3^{n-1} = 2^n * 2 + 3^n * \frac{1}{3} \in [max(\Theta(2^n), \Theta(3^n))] = \Theta(3^n)$

(e) $\lfloor \log_2 n \rfloor \approx \log_2 n \in \Theta(\log n)$
$x - 1 < \lfloor x \rfloor \leq x \Rightarrow \lfloor log_2 n \rfloor \leq \log_2 n$
$\lfloor \log_2 n \rfloor > \log_2 n - 1 \geq \log_2 n - \frac{1}{2}\log_2 n = \frac{1}{2}\log_2 n \in \Theta(\log_2 n) = \Theta(\log n)$

2. Solve the recurrence relations.

(a) $T(n) = 2T(\frac{n}{2}) + n^3 \Rightarrow T(x) = 2T(\frac{x}{2}) + x^3$
Let $x = \frac{n}{2}, T(\frac{n}{2}) = 2T(\frac{n}{4}) + \frac{1}{8}n^3$
$T(n) = 2[2T(\frac{n}{4}) + \frac{1}{8}n^3] + n^3 = 4T(\frac{n}{4}) + \frac{1}{4}n^3 + n^3$
Let $x = \frac{n}{4}, T(\frac{n}{4}) = 2T(\frac{n}{8}) + \frac{1}{64}n^3$
$T(n) = 4[2T(\frac{n}{8}) + \frac{1}{64}n^3] + \frac{5}{4}n^3 = 8T(\frac{n}{8}) + \frac{1}{16}n^3 + \frac{1}{4}n^3 + n^3$
$T(n) = 2^k T(\frac{n}{2^k}) + n^3 \sum_{i=0}^{k-1} (\frac{1}{4})^i$
$\frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow \log_2 n = k \Rightarrow nT(1) + n^3 \sum_{j=0}^{\log_2 n - 1} (\frac{1}{4})^j$
$T(n) = nT(1) + n^3(\frac{(1-\frac{1}{4})^{\log_2 n}}{1-\frac{1}{4}})$
$\lim_{n\to\infty} nT(1) + n^3(\frac{(1-\frac{1}{4})^{\log_2 n}}{1-\frac{1}{4}}) = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \Rightarrow nT(1) + n^3(\frac{4}{3}) \in \Theta(n^3)$

(b) $T(n) = T(\sqrt{n})\sqrt{n} + n \Rightarrow T(x) = T(\sqrt{x})\sqrt{x} + x$
Let $x = \sqrt{n}, T(\sqrt{n}) = T(n^{\frac{1}{4}})n^{\frac{1}{4}} + n^{\frac{1}{2}}$
$T(n) = [T(n^{\frac{1}{4}})n^{\frac{1}{4}} + n^{\frac{1}{2}}]\sqrt{n} + n = T(n^{\frac{1}{4}})n^{\frac{3}{4}} + 2n$
Let $x = n^{\frac{1}{4}}, T(n^{\frac{1}{4}}) = T(n^{\frac{1}{8}})n^{\frac{1}{8}} + n^{\frac{1}{4}}$
$T(n) = [T(n^{\frac{1}{8}})n^{\frac{1}{8}} + n^{\frac{1}{4}}]n^{\frac{3}{4}} + 2n = T(n^{\frac{1}{8}})n^{\frac{7}{8}} + 3n$
$T(n) = T(n^{\frac{1}{2^k}})n^{\frac{2^k-1}{2^k}} + kn \Rightarrow \frac{2^k-1}{2^k} = 1 - 2^{-k}$
$n^{\frac{2^k-1}{2^k}} = 2 \Rightarrow (n^{\frac{2^k-1}{2^k}})^{2^k} = (2)^{2^k} \Rightarrow n = 2^{2^k} \Rightarrow \log_2 \log_2 n = k$
$T(n) = T(2)\frac{n}{2} + n\log\log n \in \Theta(n \log\log n)$

(c) $T(n) = 2T(\frac{n}{2}) + \frac{n}{\log n} \Rightarrow T(x) = 2T(\frac{x}{2}) + \frac{x}{\log x}$
Let $x = \frac{n}{2}, T(\frac{n}{2}) = 2T(\frac{n}{4}) + \frac{\frac{n}{2}}{\log \frac{n}{2}} = 2T(\frac{n}{4}) + \frac{\frac{n}{2}}{\log n - 1}$

$T(n) = 2[2T(\frac{n}{4}) + \frac{\frac{n}{2}}{\log \frac{n}{2}}] + \frac{n}{\log n} = 4T(\frac{n}{4}) + \frac{n}{\log n-1} + \frac{n}{\log n}$
Let $x = \frac{n}{4}, T(\frac{n}{8}) = 2T(\frac{n}{8}) + \frac{\frac{n}{4}}{\log \frac{n}{4}} = 2T(\frac{n}{8}) + \frac{\frac{n}{4}}{\log n-2}$
$T(n) = 4[2T(\frac{n}{8}) + \frac{\frac{n}{4}}{\log \frac{n}{4}}] + \frac{n}{\log n-1} + \frac{n}{\log n} = 8T(\frac{n}{4}) + \frac{n}{\log n-2} + \frac{n}{\log n-1} + \frac{n}{\log n}$
$T(n) = 2^k T(\frac{n}{2^k}) + \sum_{i=0}^{k-1} \frac{n}{\log n-i}$
$\frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow \log_2 n = k$
$T(n) = 2^{\log_2 n} T(\frac{n}{n}) + \sum_{i=0}^{\log_2 n-1} \frac{n}{\log_2 n-i} \Rightarrow j = \log_2 n - i$
$T(n) = nd + \sum_{j=0}^{\log_2 n-1} \frac{n}{j} \Rightarrow nH \log_2 n \Rightarrow n \log\log n \in \Theta(n \log\log n)$

(d) Using Master Thm: Case 1: $a = 3, b = 2, f(n) = nlogn$
$n^{\log_2(a-\epsilon)} = n^{\log_2 3-\epsilon} = n^{1.58} \Rightarrow T(n) = \Theta(n^{1.58})$

3. BinarySearch(A[0, n-1], key, lower, upper)
if upper >= lower and A size > 0
mid = lower + (upper - lower) /2
if key == A[mid], return mid
else if key < A[mid] return BinarySearch(A, key, lower, mid-1)
else return BinarySearch(A, key, mid+1, upper)
else return -1

$T(n) = T(\frac{n}{2}) + 1, T(x) = T(\frac{x}{2}) + 1$
Let $x = \frac{n}{2}, T(\frac{n}{2}) = T(\frac{n}{4}) + 1$
$T(n) = T(\frac{n}{4}) + 1 + 1$
Let $x = \frac{n}{4}, T(\frac{n}{4}) = T(\frac{n}{8}) + 1$
$T(n) = T(\frac{n}{8}) + 1 + 1 + 1$
$T(n) = T(\frac{n}{2^k}) + k$
$\frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow \log_2 n = k$
$T(n) = T(\frac{n}{n}) + \log_2 n = T(1) + \log_2 n = 1 + \log_2 n \in \Theta(\log n)$

TernarySearch(A[0, n-1], key, lower, upper)
if upper >= lower and A size > 0
Lmid = lower + (upper - lower) /3
Umid = Lmid + (upper - lower) /3
if key == A[Lmid], return Lmid
else if key == A[Rmid], return Umid
else if key < A[Lmid] return TernarySearch(A, key, lower, Lmid-1)
else if key > A[Umid] return TernarySearch(A, key, Umid+1, upper)
else return TernarySearch(A, key, Lmid+1, Umid-1)
else return -1

$T(n) = T(\frac{n}{3}) + 1, T(x) = T(\frac{x}{3}) + 1$
Let $x = \frac{n}{3}, T(\frac{n}{3}) = T(\frac{n}{9}) + 1$
$T(n) = T(\frac{n}{9}) + 1 + 1$
Let $x = \frac{n}{9}, T(\frac{n}{9}) = T(\frac{n}{27}) + 1$
$T(n) = T(\frac{n}{27}) + 1 + 1 + 1$
$T(n) = T(\frac{n}{3^k}) + k$
$\frac{n}{3^k} = 1 \Rightarrow n = 3^k \Rightarrow \log_3 n = k$
$T(n) = T(\frac{n}{n}) + \log_3 n = T(1) + \log_3 n = 1 + \log_3 n \in \Theta(\log n)$

The graphs below both follow somewhat similar data trends. The experimental results might be slightly different because of the random numbers that had been generated for the problem.