

2.2.3 abuse indicate Θ class. Prove

1. $a(n^2+1)^n \approx (n^2)^n = n^{2n} \in \Theta(n^{2n})$

$$\lim_{n \rightarrow \infty} \frac{a(n^2+1)^n}{n^{2n}} = \lim_{n \rightarrow \infty} \frac{(n^2+1)^n}{(n^2)^n} = \lim_{n \rightarrow \infty} \left(\frac{n^2+1}{n^2}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^2}\right)^n = 1 \in \Theta(n^{2n})$$

b) $\sqrt{10n^2+7n+3} \approx \sqrt{10n^2} = n\sqrt{10} \in \Theta(n)$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{10n^2+7n+3}}{n} = \lim_{n \rightarrow \infty} \sqrt{\frac{10n^2+7n+3}{n^2}} = \lim_{n \rightarrow \infty} \sqrt{10 + \frac{7}{n} + \frac{3}{n^2}} = \lim_{n \rightarrow \infty} \sqrt{10 + \frac{7}{n} + \frac{3}{n^2}} = \sqrt{10} \in \Theta(n)$$

c) $2n \log(n+2)^2 + (n+2)^2 \log \frac{1}{2} = 2n \cdot 2 \log(n+2) + (n+2)^2 [\log n - \log 2]$
max of: $\Theta(n \log n) + \Theta(n^2 \log n) \in \Theta(n^2 \log n)$

d.) $2^{n+1} + 3^{n-1} = 2^n 2 + 3^n \frac{1}{3} \in \Theta(2^n) + \Theta(3^n) = \Theta(3^n)$

$$2^{n+1} = (2^n \cdot 2), \quad 3^{n-1} = (3^n \cdot 3^{-1})$$

e.) $\lfloor \log_2 n \rfloor \approx \log_2 n \in \Theta(\log n)$

$x \leq \lfloor \log_2 n \rfloor \leq \log_2 n \Rightarrow x \leq \log_2 n < 1 - x$ upper bound

lower: $\lfloor \log_2 n \rfloor > \log_2 n - 1 \geq \log_2 n - \frac{1}{2} \log_2 n = \frac{1}{2} \log_2 n$

$$\lfloor \log_2 n \rfloor \in \Theta(\log_2 n) = \Theta(\log n)$$

3. BinSearch(A[0,n-1], k, left, right)

If right < left, return -1
 $mid = floor((right+left)/2) + left$
If A[mid] == k, return mid
if k < A[mid], return BinSearch(A, k, left, mid-1)
else return BinSearch(A, k, mid+1, right)

a.) $T(n) = 2T\left(\frac{n}{2}\right) + n^3$, $T(x) = 2T\left(\frac{x}{2}\right)x^3$

$\text{Let } x = \frac{n}{2}$, $T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^3 = 2T\left(\frac{n}{4}\right) + \frac{1}{8}n^3$

$T(n) = 2\left[2T\left(\frac{n}{4}\right) + \frac{1}{8}n^3\right] + n^3 = 4T\left(\frac{n}{4}\right) + \frac{1}{4}n^3 + n^3$

$\text{Let } x = \frac{n}{4}$, $T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^3 = 2T\left(\frac{n}{8}\right) + \frac{1}{64}n^3$

$T(n) = 4\left[2T\left(\frac{n}{8}\right) + \frac{1}{64}n^3\right] + \frac{5}{4}n^3 = 8T\left(\frac{n}{8}\right) + \frac{1}{16}n^3 + \frac{1}{4}n^3 + n^3$

$\text{Let } x = \frac{n}{8}$, $T\left(\frac{n}{8}\right) = 2T\left(\frac{n}{16}\right) + \left(\frac{n}{8}\right)^3 = 2T\left(\frac{n}{16}\right) + \frac{1}{512}n^3$

$T(n) = 2^k T\left(\frac{n}{2^k}\right) + n^3 \sum_{j=0}^{k-1} \left(\frac{1}{4}\right)^j$

 $\frac{n}{2^k} = 1 \Rightarrow n = 2^k \rightarrow \log_2 n = k$

$T(n) = nT(1) + n^3 \left(\frac{(1-\frac{1}{4})^{2^k-1}}{1-\frac{1}{4}}\right) = nT(1) + n^3 \left(\frac{\frac{3}{4}^{2^k-1}}{\frac{3}{4}}\right) = nT(1) + n^3 \left(\frac{3}{4}\right)^{2^k-1} \in \Theta(n^3)$

b.) $T(n) = T(\sqrt{n})n + n$, $T(n_0) = 4$, $n \leq 2$, $T(n) \in \Theta(n \log \log n) \checkmark$

$\overline{T(x)} = T(\sqrt{x})\sqrt{x} + x$

$\text{Let } x = \sqrt{n}$, $T(\sqrt{n}) = T(n^{1/4})n^{1/4} + n^{1/2}$

$T(n) = [T(n^{1/4})n^{1/4} + n^{1/2}]n^{3/4} + n = T(n^{1/4})n^{3/4} + n + n = T(n^{1/4})n^{3/4} + 2n$

$\text{Let } x = n^{1/4}$, $T(n^{1/4}) = T(n^{1/8})n^{1/8} + n^{1/4}$

$T(n) = [T(n^{1/8})n^{1/8} + n^{1/4}]n^{3/4} + 2n = T(n^{1/8})n^{7/8} + n + 2n = T(n^{1/8})n^{7/8} + 3n$

$\overline{T(n)} = T(n^{\frac{1}{2^k}})n^{\frac{2^k-1}{2^k}} + kn \xrightarrow{\frac{2^k-1}{2^k}} = 1 - 2^{-k}$

$n^{\frac{1}{2^k}} = n_0$, $n_0 \leq 2$

$\overline{T(n_0)} \Rightarrow n^{\frac{1}{2^k}} = 1 \Rightarrow n = 1$

$\overline{T(n_0=2)} \Rightarrow n^{\frac{1}{2^k}} = 2 \Rightarrow (n^{\frac{1}{2^k}})^{2^k} = (2)^{2^k} \Rightarrow n = 2^k$

for $k \in \mathbb{Z}$: $\log_2 n = 2^k \rightarrow \log_2 n = 2^k \Rightarrow \log_2 1 = 2^k \Rightarrow \log_2 n = k$

$T(n) = T(2^k) \frac{n}{2^k} + n \log \log n \in \Theta(n \log \log n)$

$$1 - c) \quad T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$$

$$\in \Theta(n \log \log n) ? \checkmark$$

$$T(x) = 2T\left(\frac{x}{2}\right) + \frac{x}{\log x}$$

$$\text{Let } x = \frac{n}{2}, \quad T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \frac{\frac{n}{2}}{\log \frac{n}{2}} = 2T\left(\frac{n}{4}\right) + \frac{n/2}{\log n - 1}$$

$$2 - \quad T(n) = 2 \left[2T\left(\frac{n}{4}\right) + \frac{n/2}{\log n - 1} \right] + \frac{n}{\log n} = 4T\left(\frac{n}{4}\right) + \frac{n}{\log n - 1} + \frac{n}{\log n}$$

$$\text{Let } x = \frac{n}{4}, \quad T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \frac{n/4}{\log \frac{n}{4}} = 2T\left(\frac{n}{8}\right) + \frac{n/4}{\log n - 2}$$

$$3 - \quad T(n) = 4 \left[2T\left(\frac{n}{8}\right) + \frac{n/4}{\log n - 2} \right] + \frac{n}{\log n} = 8T\left(\frac{n}{16}\right) + \frac{n}{\log n - 2} + \frac{n}{\log n - 1} + \frac{n}{\log n}$$

$$- \quad T(n) = 2^k T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} \frac{n}{\log n - i}$$

$$\text{Try } n = 1 \Rightarrow n = 2^k \Rightarrow \log_2 n = k \log_2 2$$

$$T(n) = 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + \frac{n}{\log_2 n - 1} \rightarrow j = \log_2 n - 1$$

$$T(n) = n^k + \sum_{j=0}^{\log_2 n} \frac{n}{2^j} \rightarrow n \log_2 n \rightarrow n \log \log n \in \Theta(n \log \log n)$$

$$d.) \quad T(n) = 3T\left(\frac{n}{2}\right) + n \log n, \quad T(x) = 3T\left(\frac{x}{2}\right) + x \log x$$

$$\text{Let } x = \frac{n}{2}, \quad T\left(\frac{n}{2}\right) = 3T\left(\frac{n}{4}\right) + \frac{n}{2} \log \frac{n}{2} = 3T\left(\frac{n}{4}\right) +$$

$$- \quad T(n) = 3 \left[3T\left(\frac{n}{4}\right) + \frac{n}{2} \log \frac{n}{2} \right] + n \log n = 9T\left(\frac{n}{4}\right) +$$

Master Theorem: $a=3, b=2, f(n) = n \log n$

$$\log_b(a - \epsilon) = \log_2 3 - \epsilon = n^{1.58}$$

$$\text{case 1 : } T(n) = \Theta(n^{1.58})$$