Problem Set 2

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Question 1 Background:

In a parliamentary system, newly elected members of parliament choose a prime minister and a cabinet ("the government"). Typically the government stays in power for a fixed term; however, a no-confidence vote from the parliament or a decision by the prime minister can lead to early elections. Suppose we want to understand the duration of governments in a parliamentary democracy with five-year terms – that is, elections must be held at least every five years, but might be held earlier (in the case of a no-confidence vote or a prime ministerial decision to dissolve the government).

1a) If we can measure the government duration in infinitely small units of time (ie, continuously), what is the sample space for this experiment?

The sample space, if continuous, must be greater than 0 (more than no time in office) and less than or equal to five (it can technically last five years). So it would look like: $\Omega = (0, 5]$.

1b) The random variable $X(\omega)$ is the amount of time in years or fractions of years) between the last election and the calling of the next election. Suppose we know that $X(\omega)$ has the probability density function:

$$f(x) = \begin{cases} kx^3 & 0 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. What is k (hint: remember the rules of pdfs)?

In this case, k $= \frac{4}{625}$. When the highest x-value gets plugged into the CDF, the result should be 1. Our integral will thus look like: $\int kx^3 \, dx$. We can integrate that to equal $k\frac{x^4}{4}$. We know that since this has to equal 1 at the highest value (5), we can do $k\frac{5^4}{4} = 1$; $625k = 4$; $k = \frac{4}{625}$.

1c) What is the CDF of X?

We know that the CDF can be defined as $F(x) = \int_{-\infty}^{\infty} f(x) \, dx$. Thus, here we can say that for the first interval we have:

$\int_0^x \frac{4}{625}x^3 \, dx$. This equals $\frac{1}{625}x^4 \big|_0^x$, or just $\frac{1}{625}x^4$.

For the remainder, we can say the following:

$F(x) = \int_{-\infty}^{y} 0 \, dx$

Also, we already know that F(5) = 1, so F(x) = 1 for x > 5.

1d) What is the probability that the government remains in power for exactly 3 years? Why?

Well, technically, it should be 0. This is continuous, not discrete, so really the probability of the event happening at one specific point is 0. If we show it mathematically, we would take the integral from value "3" to value "3", which would end up just being 0.

1e) What is the probability that the government remains in power between 2 and 4 years?

Here, we actually need to map from x = 2 to x = 4. So this would look like:

$\int_2^4 \frac{4}{625} x^3 \, dx$. Then, we get $\frac{1}{625}(4)^4 - \frac{1}{625}(2)^4$. This equates to $\frac{256}{625} - \frac{16}{625} = \frac{240}{625}$ or about 38.4%.

1f) What is the probability that the government remains in power for < 1 or > 4 years?

Here, I will do two integrals, one from 0 to 1 and the other from 4 to 5. So:

$\int_0^1 \frac{4}{625} x^3 \, dx$ and $\int_4^5 \frac{4}{625} x^3 \, dx$.

For the first integral, from this, we get $\frac{1}{625}(1)^4 - \frac{1}{625}(0)^4$. This equates to $\frac{1}{625} - 0 = \frac{1}{625}$ or about .16%.

For the second integral, from this, we get $\frac{1}{625}(5)^4 - \frac{1}{625}(4)^4$. This equates to $1 - \frac{256}{625} = \frac{369}{625}$ or about 59.04%.

So, in total, we add .16% to 59.04, and we get 59.2%.

1g) Plot the pdf and cdf in ggplot2, with axis labels. Describe, in words, what your findings mean for the survival of parliamentary cabinets.
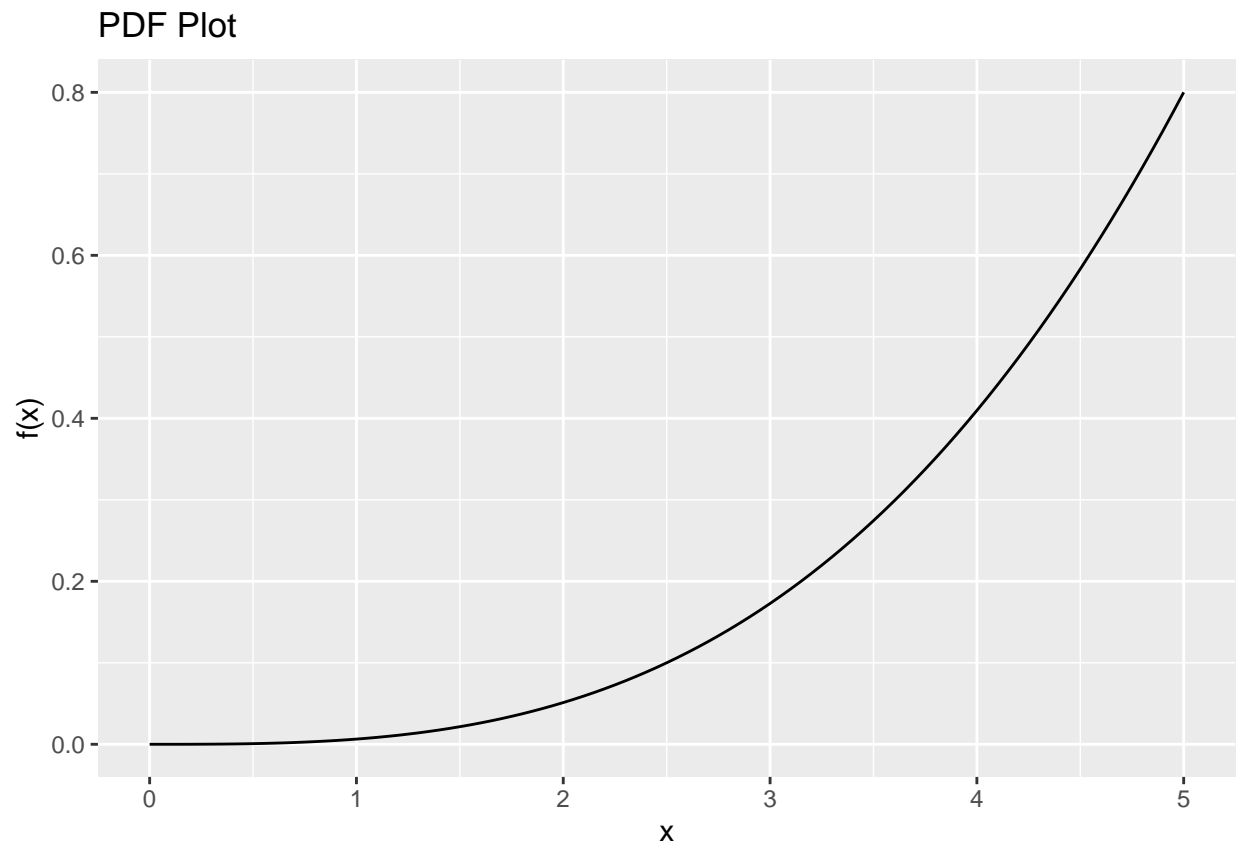
```
# Set limits

x <- 0:5

df <- data.frame(x)

# Create function and plot for pdf

pdfplot <- ggplot(df, aes(x)) +
  stat_function(fun = function (x) (4/625) * x ^ 3) +
  labs(title = "PDF Plot",
       x = "x",
       y = "f(x)")

pdfplot
```
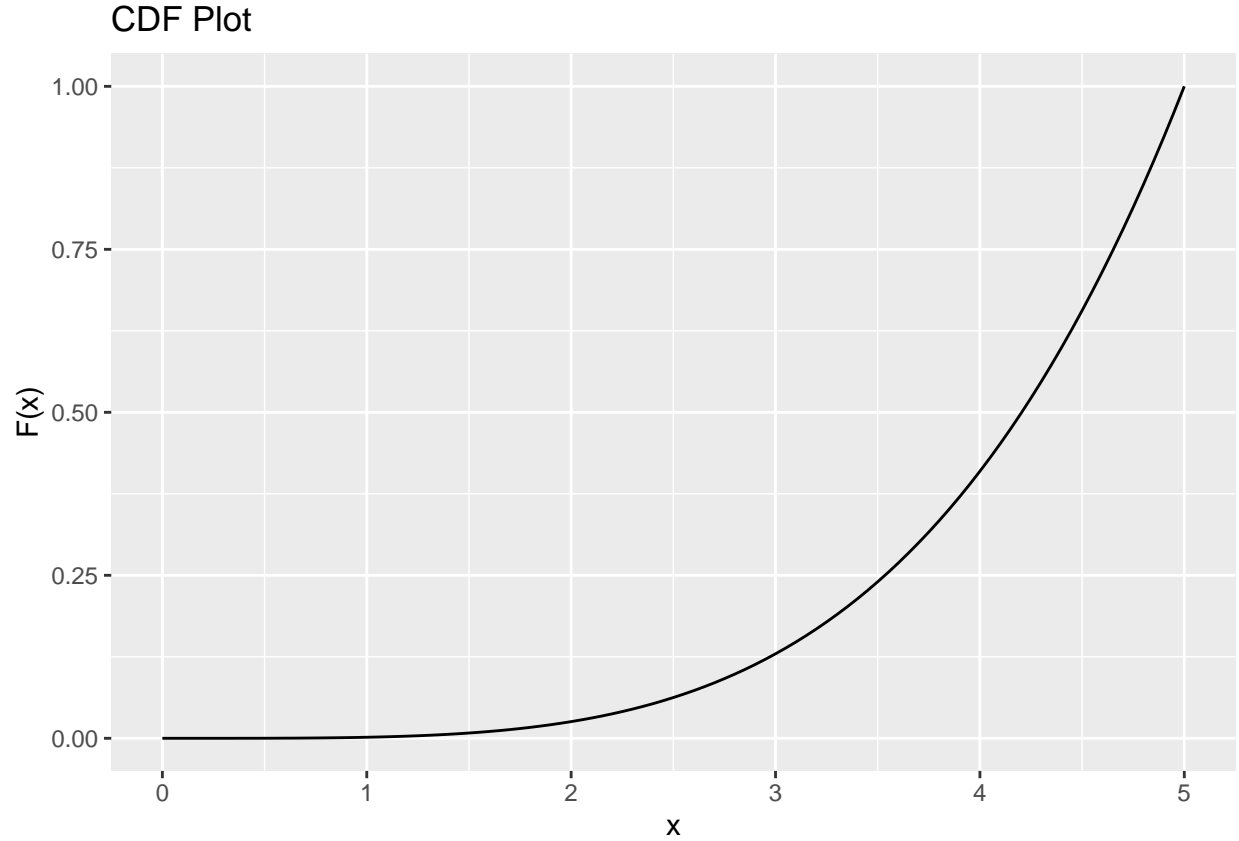
PDF Plot


```r
# Create function and plot for cdf

cdfplot <- ggplot(df, aes(x))+
stat_function(fun = function (x) (1/625) * x ^ 4) +
  labs(title = "CDF Plot",
       x = "x",
       y = "F(x)")


cdfplot
```

CDF Plot

Question 2 Background:

In a donation experiment, two participants (P1 and P2) can give up to $5 to the other player, in $1 dollar increments. Let's say we expect that participants donate randomly between 0 and 5, and let's define the random variables X, Y, Z as follows:

$$X = \text{The amount that } P1 \text{ gives}$$

$$Y = \text{The number of participants that give \$5.}$$

$$Z = \begin{cases} 0 & \text{if the two give the same amount} \\ 1 & \text{if P2 gives more than P1} \\ 2 & \text{if P1 gives more than P2} \end{cases}$$

For example, if P1 gives $5 and P2 gives $3, then $X = 5$, $Y = 1$, and $Z = 2$.

First, let's make a table showing all possible outcomes. We get a 6x6 table of (x, y, z) values:

```
P1_0 <- c(("(0, 0, 0)"), ("(0, 0, 1)"), ("(0, 0, 1)"), ("(0, 0, 1)"), ("(0, 0, 1)"), ("(0, 1, 1)"))
P1_1 <- c(("(1, 0, 2)"), ("(1, 0, 0)"), ("(1, 0, 1)"), ("(1, 0, 1)"), ("(1, 0, 1)"), ("(1, 1, 1)"))
P1_2 <- c(("(2, 0, 2)"), ("(2, 0, 2)"), ("(2, 0, 0)"), ("(2, 0, 1)"), ("(2, 0, 1)"), ("(2, 1, 1)"))
P1_3 <- c(("(3, 0, 2)"), ("(3, 0, 2)"), ("(3, 0, 2)"), ("(3, 0, 0)"), ("(3, 0, 1)"), ("(3, 1, 1)"))
P1_4 <- c(("(4, 0, 2)"), ("(4, 0, 2)"),("(4, 0, 2)"),("(4, 0, 2)"), ("(4, 0, 0)"), ("(4, 1, 1)"))
P1_5 <- c(("(5, 1, 2)"), ("(5, 1, 2)"), ("(5, 1, 2)"), ("(5, 1, 2)"), ("(5, 1, 2)"), ("(5, 2, 0)"))
```

```
outcomes <- data.frame(P1_0, P1_1, P1_2, P1_3, P1_4, P1_5)
rownames(outcomes) <- c("P2_0", "P2_1", "P2_2", "P2_3", "P2_4", "P2_5")

outcomes
```

```
##           P1_0      P1_1      P1_2      P1_3      P1_4      P1_5
## P2_0 (0, 0, 0) (1, 0, 2) (2, 0, 2) (3, 0, 2) (4, 0, 2) (5, 1, 2)
## P2_1 (0, 0, 1) (1, 0, 0) (2, 0, 2) (3, 0, 2) (4, 0, 2) (5, 1, 2)
## P2_2 (0, 0, 1) (1, 0, 1) (2, 0, 0) (3, 0, 2) (4, 0, 2) (5, 1, 2)
## P2_3 (0, 0, 1) (1, 0, 1) (2, 0, 1) (3, 0, 0) (4, 0, 2) (5, 1, 2)
## P2_4 (0, 0, 1) (1, 0, 1) (2, 0, 1) (3, 0, 1) (4, 0, 0) (5, 1, 2)
## P2_5 (0, 1, 1) (1, 1, 1) (2, 1, 1) (3, 1, 1) (4, 1, 1) (5, 2, 0)
```

2a) Write down a table showing the joint probability mass function for X and Y.

There are 3 possibilities for Y: 0, 1, and 2. There are 6 possibilities for X: 0, 1, 2, 3, 4, 5, and 6. We will assume that each player is equally likely to give all increments of 1 dollar, so the probability of each value of X is 1/6. For Y, the probability of Y = 2 is 1/36 – when both people give 5 dollars. The probability of Y = 1 is 10/36 – when only one person gives 5 dollars. And the probability of Y = 0 is 25/36 – when neither person gives 5 dollars. Multiplying values together, we end up with:

```
# Set up 6 identical vectors with three values: 1/6 (probability of
# all six possible values of X) multiplied by 25/36 (the probability
# of Y = 0), 10/36 (the probability of Y = 1), and 1/36 (the probability
# of Y = 2). Then, I bound them into a dataframe.

X0 <- c((25/216), (10/216), (1/216))
X1 <- c((25/216), (10/216), (1/216))
X2 <- c((25/216), (10/216), (1/216))
X3 <- c((25/216), (10/216), (1/216))
X4 <- c((25/216), (10/216), (1/216))
X5 <- c((25/216), (10/216), (1/216))

twoa <- data.frame(X0, X1, X2, X3, X4, X5)

rownames(twoa) <- c("Y0", "Y1", "Y2")

twoa
```

```
##            X0         X1         X2         X3         X4         X5
## Y0 0.11574074 0.11574074 0.11574074 0.11574074 0.11574074 0.11574074
## Y1 0.04629630 0.04629630 0.04629630 0.04629630 0.04629630 0.04629630
## Y2 0.00462963 0.00462963 0.00462963 0.00462963 0.00462963 0.00462963
```