Problem Set 2

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Question 1 Background:

In a parliamentary system, newly elected members of parliament choose a prime minister and a cabinet ("the government"). Typically the government stays in power for a fixed term; however, a no-confidence vote from the parliament or a decision by the prime minister can lead to early elections. Suppose we want to understand the duration of governments in a parliamentary democracy with five-year terms – that is, elections must be held at least every five years, but might be held earlier (in the case of a no-confidence vote or a prime ministerial decision to dissolve the government).

1a) If we can measure the government duration in infinitely small units of time (ie, continuously), what is the sample space for this experiment?

The sample space, if continuous, must be greater than 0 (more than no time in office) and less than or equal to five (it can technically last five years). So it would look like: $\Omega = (0, 5]$.

1b) The random variable $X(\omega)$ is the amount of time in years or fractions of years) between the last election and the calling of the next election. Suppose we know that $X(\omega)$ has the probability density function:

$$f(x) = \begin{cases} kx^3 & 0 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. What is k (hint: remember the rules of pdfs)?

In this case, k = $\frac{4}{625}$. When the highest x-value gets plugged into the CDF, the result should be 1. Our integral will thus look like: $\int kx^3 \; dx$. We can integrate that to equal $k\frac{x^4}{4}$. We know that since this has to equal 1 at the highest value (5), we can do $k\frac{5^4}{4} = 1$; $625k = 4$; $k = \frac{4}{625}$.

1c) What is the CDF of X?

We know that the CDF can be defined as $F(x) = \int_{-\infty}^{\infty} f(x) \; dx$. Thus, here we can say that for the first interval we have:

$\int_0^x \frac{4}{625}x^3 \; dx$. This equals $\frac{1}{625}x^4\Big|_0^x$, or just $\frac{1}{625}x^4$.

For the remainder, we can say the following:

$F(x) = \int_{-\infty}^{y} 0 \; dx$

Also, we already know that F(5) = 1, so F(x) = 1 for x > 5.

1d) What is the probability that the government remains in power for exactly 3 years? Why?

Well, technically, it should be 0. This is continuous, not discrete, so really the probability of the event happening at one specific point is 0. If we show it mathematically, we would take the integral from value "3" to value "3", which would end up just being 0.

1e) What is the probability that the government remains in power between 2 and 4 years?

Here, we actually need to map from x = 2 to x = 4. So this would look like:

$\int_2^4 \frac{4}{625}x^3 \, dx$. Then, we get $\frac{1}{625}(4)^4 - \frac{1}{625}(2)^4$. This equates to $\frac{256}{625} - \frac{16}{625} = \frac{240}{625}$ or about 38.4%.

1f) What is the probability that the government remains in power for < 1 or > 4 years?

Here, I will do two integrals, one from 0 to 1 and the other from 4 to 5. So:

$\int_0^1 \frac{4}{625}x^3 \, dx$ and $\int_4^5 \frac{4}{625}x^3 \, dx$.

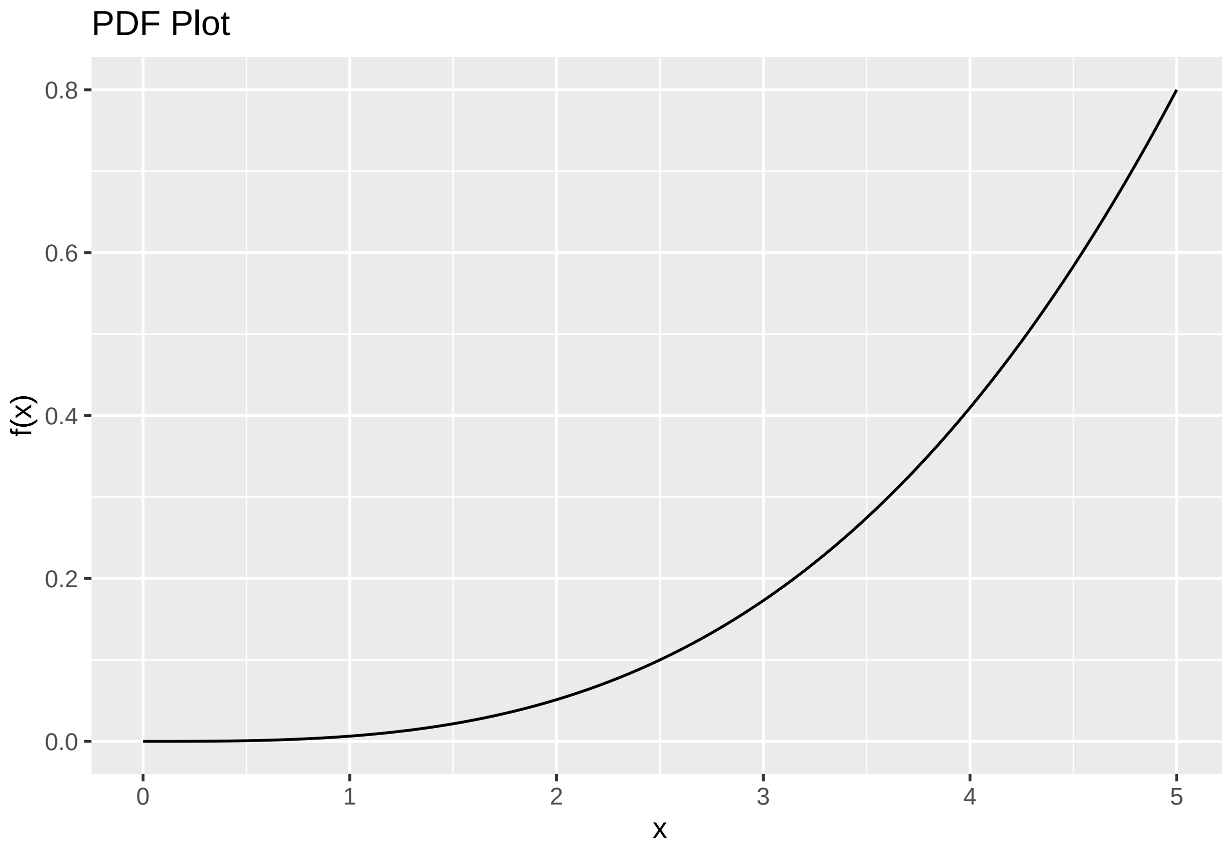
For the first integral, from this, we get $\frac{1}{625}(1)^4 - \frac{1}{625}(0)^4$. This equates to $\frac{1}{625} - 0 = \frac{1}{625}$ or about .16%.

For the second integral, from this, we get $\frac{1}{625}(5)^4 - \frac{1}{625}(4)^4$. This equates to $1 - \frac{256}{625} = \frac{369}{625}$ or about 59.04%.

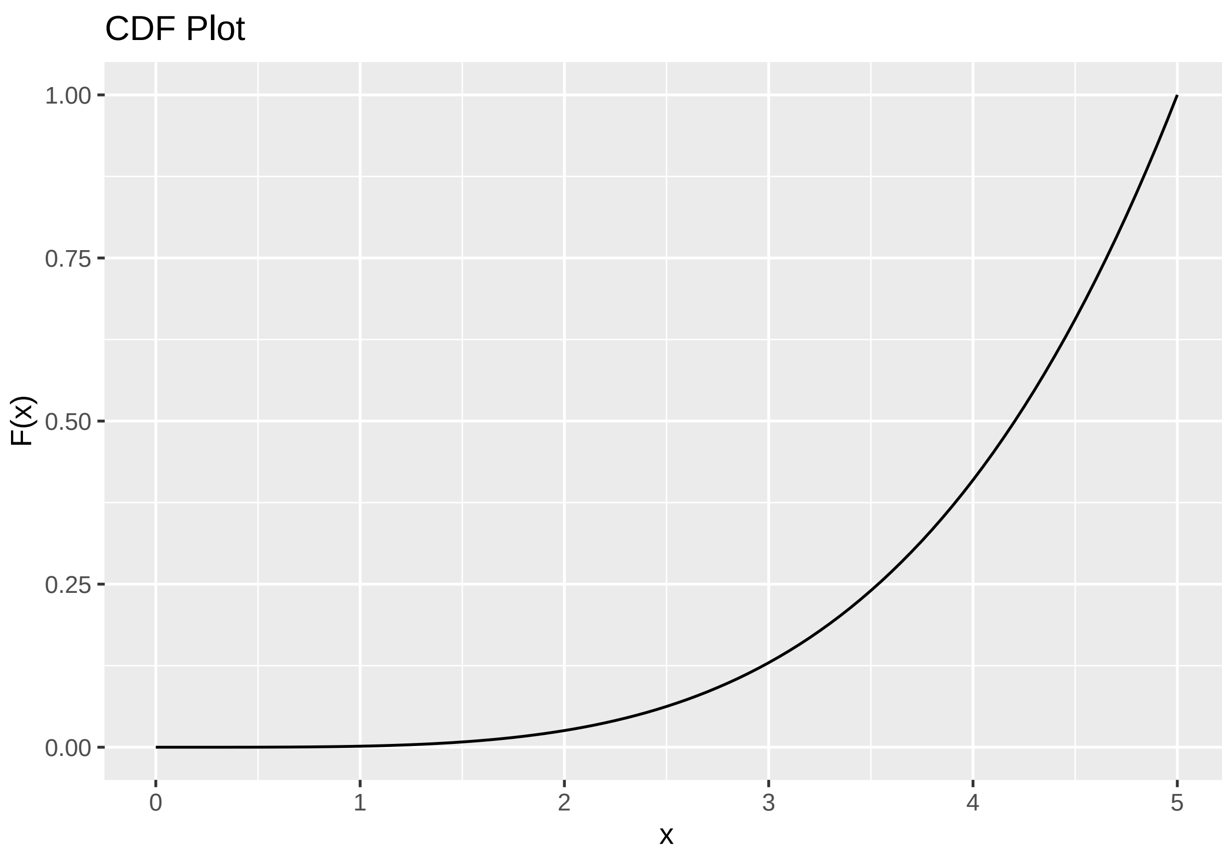
So, in total, we add .16% to 59.04, and we get 59.2%.

1g) Plot the pdf and cdf in ggplot2, with axis labels. Describe, in words, what your findings mean for the survival of parliamentary cabinets.

```
# Set limits

x <- 0:5

df <- data.frame(x)

# Create function and plot for pdf

pdfplot <- ggplot(df, aes(x)) +
  stat_function(fun = function (x) (4/625) * x ^ 3) +
  labs(title = "PDF Plot",
       x = "x",
       y = "f(x)")

pdfplot
```




```r
# Create function and plot for cdf

cdfplot <- ggplot(df, aes(x))+
stat_function(fun = function (x) (1/625) * x ^ 4) +
  labs(title = "CDF Plot",
       x = "x",
       y = "F(x)")


cdfplot
```



Question 2 Background:

In a donation experiment, two participants (P1 and P2) can give up to \$5 to the other player, in \$1 dollar increments. Let's say we expect that participants donate randomly between 0 and 5, and let's define the random variables X, Y, Z as follows:

$$\text{X=The amount that } P1 \text{ gives}$$

$$\text{Y=The number of participants that give \$5.}$$

$$Z = \begin{cases} 0 & \text{if the two give the same amount} \\ 1 & \text{if P2 gives more than P1} \\ 2 & \text{if P1 gives more than P2} \end{cases}$$

For example, if P1 gives \$5 and P2 gives \$3, then $X = 5$, $Y = 1$, *and* $Z = 2$.

First, let's make a table showing all possible outcomes. We get a 6x6 table of (x, y, z) values:

```
P1_0 <- c(("(0, 0, 0)"), ("(0, 0, 1)"), ("(0, 0, 1)"), ("(0, 0, 1)"), ("(0, 0, 1)"), ("(0, 1, 1)"))
P1_1 <- c(("(1, 0, 2)"), ("(1, 0, 0)"), ("(1, 0, 1)"), ("(1, 0, 1)"), ("(1, 0, 1)"), ("(1, 1, 1)"))
P1_2 <- c(("(2, 0, 2)"), ("(2, 0, 2)"), ("(2, 0, 0)"), ("(2, 0, 1)"), ("(2, 0, 1)"), ("(2, 1, 1)"))
P1_3 <- c(("(3, 0, 2)"), ("(3, 0, 2)"), ("(3, 0, 2)"), ("(3, 0, 0)"), ("(3, 0, 1)"), ("(3, 1, 1)"))
P1_4 <- c(("(4, 0, 2)"), ("(4, 0, 2)"),("(4, 0, 2)"),("(4, 0, 2)"), ("(4, 0, 0)"), ("(4, 1, 1)"))
P1_5 <- c(("(5, 1, 2)"), ("(5, 1, 2)"), ("(5, 1, 2)"), ("(5, 1, 2)"), ("(5, 1, 2)"), ("(5, 2, 0)"))
```

```r
outcomes <- data.frame(P1_0, P1_1, P1_2, P1_3, P1_4, P1_5)
rownames(outcomes) <- c("P2_0", "P2_1", "P2_2", "P2_3", "P2_4", "P2_5")

outcomes
```

```
##           P1_0      P1_1      P1_2      P1_3      P1_4      P1_5
## P2_0 (0, 0, 0) (1, 0, 2) (2, 0, 2) (3, 0, 2) (4, 0, 2) (5, 1, 2)
## P2_1 (0, 0, 1) (1, 0, 0) (2, 0, 2) (3, 0, 2) (4, 0, 2) (5, 1, 2)
## P2_2 (0, 0, 1) (1, 0, 1) (2, 0, 0) (3, 0, 2) (4, 0, 2) (5, 1, 2)
## P2_3 (0, 0, 1) (1, 0, 1) (2, 0, 1) (3, 0, 0) (4, 0, 2) (5, 1, 2)
## P2_4 (0, 0, 1) (1, 0, 1) (2, 0, 1) (3, 0, 1) (4, 0, 0) (5, 1, 2)
## P2_5 (0, 1, 1) (1, 1, 1) (2, 1, 1) (3, 1, 1) (4, 1, 1) (5, 2, 0)
```

2a) Write down a table showing the joint probability mass function for X and Y.

Joint probability is calculated with the following equation: $P(AandB) = P(A|B) * P(B)$. We will define A as X, and B as Y, so it will look like $P(X|Y) * P(Y)$.

So, there are 3 possibilities for Y: 0, 1, and 2. There are 6 possibilities for X: 0, 1, 2, 3, 4, and 5. Luckily, I made this handy table above with all of the combinations, so we can really just look visually to give us a good sense as to all of these values. For P(Y = 0), the probability is 25/36; in 25 of our 36 instances, Y = 0. P(Y = 1) = 10/36 – only one person chooses "5" in ten instances. Finally, P(Y = 2) = 1/36 – when both people give 5 dollars.

P(A|B) for each value is a bit more complicated. Mainly, the big differences here come when X = 5. Below, in the code, I explain how I came up with my vectors.

```r
# First, for P(A = 0|B = 0): There are 25 instances when B = 0. Of those,
# five occur when X = 0. Thus, P(A|B) in this instance = 5/25, or 1/5.
# 1/5 * 25/36 (P(B = 0)) = 5/36. A similar operation is done for when P(A = 0|B = 1).
# There are 10 instances where B = 1. Of those, one occurs when X = 1. Thus, P(A|B)
# in this instance = 1/10. 1/10 * 10/36 (P(B = 1)) = 1/36. Finally,
# there are no instances of Y = 2 outside of where X = 5. So our vector is:

X0 <- c((5/36), (1/36), (0))

# The next four vectors (X = 1, X = 2, X = 3, X = 4) are all the same.

X1 <- c((5/36), (1/36), (0))
X2 <- c((5/36), (1/36), (0))
X3 <- c((5/36), (1/36), (0))
X4 <- c((5/36), (1/36), (0))

# When X = 5, there are 0 of 25 instances of Y = 0, 5 of 10 instances
# of Y = 1, and 1/1 instance of Y = 2. So, combining them by their respective
# P(Y) we get:

X5 <- c((0), (5/36), (1/36))

twoa <- data.frame(X0, X1, X2, X3, X4, X5)

rownames(twoa) <- c("Y0", "Y1", "Y2")

twoa
```

```
##            X0         X1         X2         X3         X4         X5
## Y0 0.13888889 0.13888889 0.13888889 0.13888889 0.13888889 0.00000000
## Y1 0.02777778 0.02777778 0.02777778 0.02777778 0.02777778 0.13888889
## Y2 0.00000000 0.00000000 0.00000000 0.00000000 0.00000000 0.02777778
```

2b, 2c) What is the marginal probability of Y?

The marginal probability $P_Y(y)$ can be described as $\sum_x Pr(X = x, Y = y)$

The marginal probability $P_X(x)$ can be described as $\sum_y Pr(Y = y, X = x)$.

So we simply need to add across or down for each value of X or Y. First, I add a column to express the marginal probability of Y:

```
twoa <- twoa %>%
  rowwise %>%
  mutate("Xmarginal" = sum(across(X0:X5)))
```

Next, I create a new vector showing the totals for X and use rbind() to bind it together with the broader dataset.

```
# Used an extra rep of 1/6 as a placeholder -- will remove at end.

df <- rep(1/6, 7)
dg <- c("X0", "X1", "X2", "X3", "X4", "X5", "Xmarginal")

data <- data.frame(df, dg) %>%
  pivot_wider(names_from = dg, values_from = df)

twoupdate <- rbind(twoa, data)

twoupdate <- data.frame(twoupdate)

rownames(twoupdate) <- c("Y0", "Y1", "Y2", "Ymarginal")

# As mentioned, I set the bottom right corner to NA.

twoupdate[4,7] = NA
```

2d) Write down a table showing the joint probability mass function for X and Z.

Here we will perform a similar procedure as in 2a. Joint probability is calculated with the following equation: $P(AandB) = P(A|B) * P(B)$. We will define A as X, and B as Z, so it will look like $P(X|Z) * P(Z)$.

There are 3 possibilities for Z: 0, 1, and 2. There are 6 possibilities for X: 0, 1, 2, 3, 4, and 5. Going off of the table again, we can just look visually to give us a good sense as to all of these values. For P(Z = 0), the probability is 6/36 or 1/6; in 6 of our 36 instances, Z = 0 (P2 = P1). P(Z = 1) = 15/36 – P2 > P1 in 15 instances. Finally, P(Z = 2) = 15/36 – when P1 > P2.

Calculating P(A|B) (otherwise known as P(X|Z)) will be a bit more difficult; more difficult than 2a. Below, in the code, I explain how I came up with my first vector; the remaining ones follow the same logic but are not explained in detail.

```r
# First, for P(A = 0|B = 0): There are 6 instances when B = 0. Of those,
# one occurs when X = 0. Thus, P(A|B) in this instance = 1/6. 1/6 * 1/6 (P(B = 0))
# = 1/36. A similar operation is done for when P(A = 0|B = 1).
# There are 15 instances where B = 1. Of those, five occur when X = 1. Thus, P(A|B)
# in this instance = 5/15, or 1/3. 1/3 * 15/36 (P(B = 1)) = 5/36. Finally,
# there are no instances of Y = 2 where X = 0. So our vector is:

X_0 <- c(1/36, 5/36, 0)

# Apart from Z = 0, Z values differ a fair amount. The next vector will look
# like:

X_1 <- c(1/36, 4/36, 1/36)

# And so on:

X_2 <- c(1/36, 3/36, 2/36)

X_3 <- c(1/36, 2/36, 3/36)

X_4 <- c(1/36, 1/36, 4/36)

X_5 <- c(1/36, 0, 5/36)

twod <- data.frame(X_0, X_1, X_2, X_3, X_4, X_5)

rownames(twod) <- c("Z0", "Z1", "Z2")

twod
```

```
##           X_0        X_1        X_2        X_3        X_4        X_5
## Z0 0.02777778 0.02777778 0.02777778 0.02777778 0.02777778 0.02777778
## Z1 0.13888889 0.11111111 0.08333333 0.05555556 0.02777778 0.00000000
## Z2 0.00000000 0.02777778 0.05555556 0.08333333 0.11111111 0.13888889
```