# Quantum Computation study notes for physicists

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## 1 Introduction

These are my notes. There are many like it, but these are mine. My notes are my best friend. They are my life. I must master them as I must master my life.

Or something like that. Simply stated, these are studynotes and are not to be taken as a reference. They may contain mistakes, they will not be reviewed by others. If you do find errors, *please point them out*, my contact is at the top. The title is because I am a physicist and as such I hope someone with similar background can interpret this complete riddle I'm about to write.

### 1.1 What is a qubit?

In classic computers, information is stored as series of bits. A bit is simply a state represented by a 0 or 1 - it is binary by nature. On/off is also a binary pair of states. Sometimes a 1 represents +5 V and the 0 0 V. In spintronics, they may represent the electron's spin direction ($\uparrow$ or $\downarrow$). Independent of the physical information that the 0 or 1 classical state represents it is discrete and binary information.

Quantum computation on the other hand uses *quantum bits*, shortened to qubits. While a classical bits is either a 0 or a 1, a qubit is a superposition of both 0 and 1 states. As such, the state of the qubit is written as a superposition of both states. In these notes, I will be using mostly Dirac's notation, so get familiar with that if you're not already. For a single qubit in state $\Psi$:

$$|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle \tag{1}$$

Another useful notation is:

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \end{aligned} \tag{2}$$

Combining 1 and 2 gives us:

$$|\Psi\rangle = \frac{\alpha + \beta}{\sqrt{2}} |+\rangle + \frac{\alpha - \beta}{\sqrt{2}} |-\rangle \tag{3}$$

Both representations are the same state, simply using different base pairs - $(|0\rangle, |1\rangle)$ in equation 1 and $(|+\rangle, |-\rangle)$ in equation 3. A system of two qubits can be represented as:

$$|\Psi\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle \tag{4}$$

Comparing with a two bits system there is already a difference. Measuring a two bit system, we can only extract two values - the state of each bit. Measuring a two qubit system, we see there are 4 values that can be extracted - the square root of the probability of each state being measured.Expanding to three qubits, there are eight coefficients. Four qubits, sixteen coefficients. For $n$ qubits, $2^n$ coefficients. Considering Google's quantum circuit [1] with 53 qubits this is equivalent to 1 PB. Quantum computation advantages do not rely solely on state superposition, but we will get there soon enough.

#### 1.1.1 The Bloch sphere

Since $\alpha$ and $\beta$ are complex numbers, they can be represented in exponential forms. Which leads us to the Bloch sphere, a spherical representation of qubits. We define the positive z-axis as $|0\rangle$ and the negative as $|1\rangle$. Which leads to rewriting 1 as:

$$|\Psi\rangle = \cos\frac{\theta}{2} |0\rangle + e^{i\phi} \sin\frac{\theta}{2} |1\rangle \tag{5}$$

With the regular notation of $\theta$ being the angle between the the vector from the origin to the point P on the sphere's surface and the z-axis, and $\phi$ the angle of point's projection in the xy-plane with the x-axis. As such, operations on a single qubit can be seen as rotations in the Bloch sphere.

What do certain angles correspond to then? A pure $|0\rangle$ obviously matches a $\theta = 0$ and $|1\rangle$ corresponds to $\theta = \pi$. What if $\theta = \frac{\pi}{2}$? That leads to a superposition state with equal probability of being either $|0\rangle$ or $|1\rangle$ when measured! All possibles states are represented by points on the surface of the Bloch sphere.

### 1.2 Quantum logic gates

Similarly to classical computation, we use quantum logic gates to perform logical operations on qubits. Classically, all operations can be performed with combinations of NAND gates. Quantum speaking, that does not happen. Lets elaborate in the following sections.

#### 1.2.1 Single qubit gates

What's the simplest operation to perform on a single bit? The NOT operation: $b \to \overline{b}$. In other words, turning a 0 to a 1 and vice-versa. For easier comparison, we may write this as a vector. So if $b = 0$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{NOT}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6}$$

Writing the qubit $|\Psi\rangle$ in vector notation:

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \tag{7}$$

Comparing with the classical NOT operation, the quantum NOT is the operation:

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \xrightarrow{\text{NOT}} \begin{bmatrix}\beta\\ \alpha\end{bmatrix} \tag{8}$$

Being a matrix operation, the gate can also be represented by a matrix:

$$X = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \tag{9}$$

This is also known as the Pauli-X gate, as it represents a rotation of $\pi$ around the x-axis of the Bloch sphere. Similarly to this case, the other Pauli matrices also represent rotations of the qubit state in the Bloch sphere of $\pi$ radians around the respective axis.

$$Y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \tag{10}$$

$$Z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{11}$$

So representing $|\Psi\rangle$ as a vector, the Pauli gates act as:

$$\begin{aligned} X\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= \begin{bmatrix}\beta\\ \alpha\end{bmatrix}\\ Y\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= \begin{bmatrix}-i\beta\\ i\alpha\end{bmatrix}\\ Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= \begin{bmatrix}\alpha\\ -\beta\end{bmatrix} \end{aligned} \tag{12}$$

Following Nielsen, you can see that every unitary matrix can represent a quantum logic gate. Gates that act on a single qubit are represented by $2 \times 2$ matrices with a gate acting on n qubits being represented by a matrix of $2^n \times 2^n$. This follows from the normalization condition of the state coefficients. Logically, if all quantum gates are represented by unitary matrices, that must mean that each of them can be seen as a rotation in the Bloch sphere. To be developed in a future section.

**The Hadamard gate**

The Hadamard gate turns a "pure state" qubit into a superposition of states. What this means is that instead of the state rotating $\pi$ radians along the Bloch sphere, it rotates $\frac{\pi}{2}$. This operation is represented by the matrix:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{13}$$

So applying this to our $|\Psi\rangle$ qubit:

$$H \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \alpha + \beta \\ \alpha - \beta \end{bmatrix} = \alpha |+\rangle + \beta |-\rangle \tag{14}$$

### 1.2.2 Multiple qubit gates

# References

[1] Arya K. Babbush R. et al. Arute, F. Quantum supremacy using a programmable superconducting processor. *Nature*, 574:505–51, 2019.