Classical Optimization Problems

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TU Delft

3TU/LNMB Course

De Uithof, Utrecht

September 18 - December 4, A.D. 2006

Outline

- Heron's problem
- Dido's (or isoperimetric) problem
- Snel's law of refraction
- Maxima and minima in geometry
- Maxima and minima in algebra and analysis
- Inequalities
- Problems posed and solved by Kepler
- The Brachistochrone

The presumed author of the following problem is the famous ancient mathematician Heron of Alexandria. He is most known from his invention of the formula for the area of a triangle. His book *On Mirrors*, probably written in the first century A.D., contains the following problem.

Problem 1 *A and B are two given points on the same side of a line ℓ. Find a point D on ℓ such that the sum of the distances form A to D and from D to B is a minimum.*

Dido’s (or isoperimetric) problem

The most beautiful solid is the sphere, and the most beautiful plane figure—the circle.

Pythagoras

Fleeing from persecution by her brother, the Phoenician princess Dido set off westward along the Mediterranean shore in search of a haven. A certain spot on the coast of what is now the bay of Tunis caught her fancy. Dido negotiated the sale of land with the local leader, Yarb. She asked for very little—as much as could be ‘encircled with a bull’s hide’. Dido managed to persuade Yarb, and a deal was struck. Dido then cut a bull’s hide into narrow strips, tied them together, and enclosed a large tract of land. On this land she built a fortress and, near it, the city of Carthage. There she was fated to experience unrequited love and a martyr’s death.
[Aeneid, Publius Vergilius Maro, ninth century B.C.]

Problem 2 *Among all closed plain curves of a given length, find the one that encloses the largest area.*

The refraction of light is readily apparent in nature. Ancient philosophers, like Ptolemy (second century B.C.) failed to discover the law of refraction. It was first found by the Dutch scientist Snel. Allthough not so famous as his great contemporaries Descartes, Huygens, and Fermat, Kepler regarded him as *the glory of the geometers (mathematicians) of our age*. Snel found experimentally that *the ratio of the sine of the incidence angle to the sine of the reflection angle is a constant that is independent of the incidence angle*. Fermat proved Snel's law by using the following principle:

> In an inhomogeneous medium, light travels from one point to another along the path requiring the shortest time.

Problem 3 *Given two points A and B on either side of a horizontal line ℓ separating two (homogeneous) media. Find a point D on ℓ such that the time it takes for a light ray to traverse the path ADB is a minimum.*

Huygens solved the problem by geometric arguments; Leibniz accomplished the first proof using derivatives (1684).

Problem 4 (Euclid, Elements, 4th cent. B.C.) *In a given triangle ABC inscribe a parallelogram $ADEF$ $(EF \| AB, DE \| AC)$ of maximal area.*

Problem 5 (Archimedes (287-212 B.C.)) *Among all spherical segments with the same spherical area, find the one that encloses the largest volume.*

Problem 6 (Steiner) *In the plane of a triangle, find a point such that the sum of its distances to the vertices of the triangle is minimal.*

Problem 7 (Exercise) *In the plane of a convex quadrangle, find a point such that the sum of its distances to the vertices of the quadrangle is minimal.*

Problem 8 (Largest area problem) *Given an angle and a point in its interior. To pass a line through the point that cuts off from the angle a triangle of minimal area.*

Problem 9 (Least-perimeter problem) *Given an angle and a point in its interior. To pass a line through the point that cuts off from the angle a triangle of minimal perimeter.*

Problem 10 (Tartaglia (1500–1557)) *To divide the number 8 into two parts such that the result of multiplying their product by their difference is maximal.*

Tartaglia used Cardano's formula for solving the cubic equation $x^3 + px + q = 0$ $(p > 0, q < 0)$, which is due to Scipione del Ferro (1465?-1526):

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Problem 11 *Find the maximum of the product of two numbers whose sum is given.*

Problem 12 *Find the maximal area of a right triangle whose small sides have constant sum.*

Both problems are solved by the arithmetic-geometric inequality:

$$\sqrt{ab} \leq \frac{a+b}{2}, \text{ equality holds iff } a = b.$$

Problem 13 *Given a sheet of tin $a \times b$, cut out equal squares at its corners such that the open box obtained by bending the edges has maximal volume.*

Arithmetic-geometric inequality (general):

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}$$

Arithmetic-quadratic mean inequality:

$$\frac{x_1 + \cdots + x_n}{n} \leq \left(\frac{x_1^2 + \cdots + x_n^2}{n}\right)^{\frac{1}{2}}$$

Problem 14 *Of all rectangular parallelepipeds inscribed in a sphere find the one of largest volume.*

Cauchy-Bunyakovskiĭ inequality:

$$a_1 b_1 + \cdots + a_n b_n \leq \left(a_1^2 + \cdots + a_n^2\right)^{\frac{1}{2}} \left(b_1^2 + \cdots + b_n^2\right)^{\frac{1}{2}}$$

Hőlder inequality: For nonnegative numbers a_1, $b_1, \cdots, a_n, b_n$ and for $p > 1$, $q = p/(p-1)$ one has

$$a_1 b_1 + \cdots + a_n b_n \leq \left(a_1^p + \cdots + a_n^p\right)^{\frac{1}{p}} \left(b_1^q + \cdots + b_n^q\right)^{\frac{1}{q}}$$

More inequalities can be found in the book by Hardy, Littlewood and Pólya.

Problems by Johannes Kepler (1571-1630)

> He is forever unaffected and true for himself. Conceit and ambition are foreign to his lofty mind. He sought neither honors nor praise. He never claims to be superior to scholars that are now virtually unknown, and all his life referred with profound respect to Maestlin, whose sole distinction is that he had the good fortune of having Kepler for a student. ... Tycho Brahe was his chief antagonist, for he rejected the Copernican theory so zealously advocated by Kepler. We know that the relations between the two great men were marred by many unpleasant incidents. And yet Kepler invariably praises Tycho, gives him due, and makes no attempt to diminish his merits .... Here, and at all times, Kepler shows himself to be a champion of the truth. Sad to say, this is all too rare ... in our time.
>
> [Predtečenskiĭ, 1921]

Problem 15 *In a given circle find a rectangle of maximal area.*

Problem 16 *In a given sphere find a cylinder of maximal volume.*

Problem 17 *In a given sphere find a rectangular parallelepiped with square base of maximal volume.*

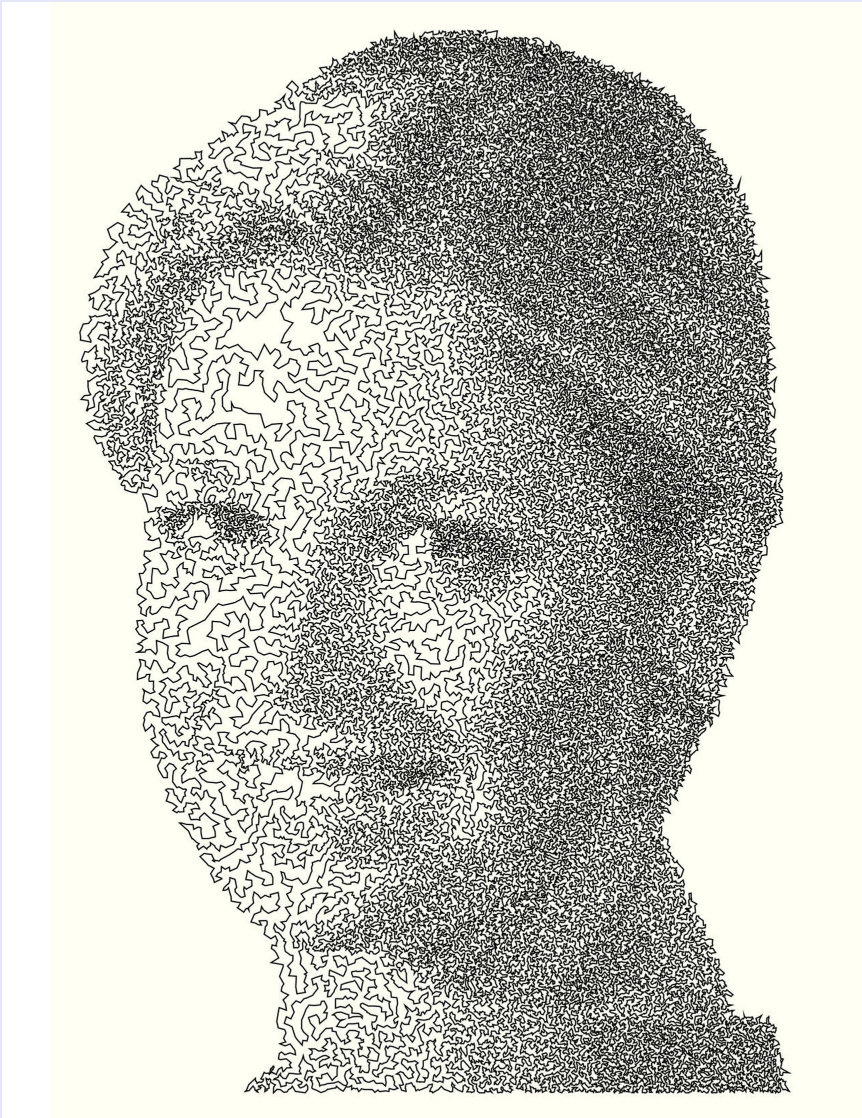
Acta Eruditorum, the first scientific journal, began publication in 1682. In the June 1696 issue, the famous Swiss scholar Johann Bernouilli published a note with the intriguing title *A new problem that mathematicians are invited to solve*:

Problem 18 *Let two points A and B be given in a vertical plane. Find the curve that a point M, moving on a path AMB must follow such that, starting from A, it reaches B in the shortest time under its own gravity.*

The curve giving the shortest time interval is called the *brachistochrone*. Many mathematicians responded to the invitation: Leibniz, Jakob Bernouilli (Johann's brother), l'Hospital, and Newton (anonymously, but Bernouilli recognized the 'lion by his claw'). They arrived at the same solution: *the brachistochrone is the cycloid*.

The brachistochrone problem was destined to play an important role in mathematical analysis. In fact, it turned out to be the first of a series of problems underlying the formulation of the *calculus of variations*.

A picture (princess) Máxima



*Thanks to Robert Bosch
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