Buckley-Leverett theory.

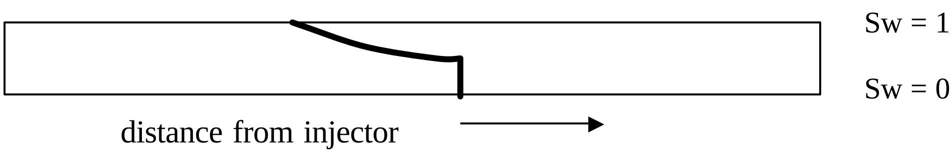
Aim: Formulate and solve equations for two-phase flow in 1D porous media. The solution gives saturation as function of position and time.

Assumptions:

- 1D horizontal reservoir with length L and constant cross section area A
- immiscible flow of two incompressible fluids water and oil
- homogeneous, incompressible reservoir, i.e. constant porosity ϕ and constant absolute permeability k
- capillary pressure is zero
- reservoir initially filled with oil
- constant injection rate of water at one end and production at the other end
- a unique physical solution exists.

Example.
Water injection in a 1D reservoir originally filled with oil.
Saturation distribution after injection of 0.5 pore volume of water

Schematic picture of water saturation profile:



Denote by Q the constant volumetric injection rate. Incompressibility implies that the total fluid rate across any cross section of the reservoir will be equal to Q. It is convenient to use Darcy velocity u define by $u = Q/A$ in the equations.

Fluid properties needed to formulate equations are viscosities, μ_w for water and μ_o for oil. In addition relative permeabilities k_{rw} and k_{ro} must be specified as functions of saturations S_w, S_o

Differential equations for mass balance.

Mass balance equations for water and oil with the above assumptions can be written

$$\frac{\partial}{\partial x}\left(\frac{kk_{rw}}{\mu_w}\frac{\partial p}{\partial x}\right) = \phi\,\frac{\partial S_w}{\partial t} \qquad (1)$$

$$\frac{\partial}{\partial x}\left(\frac{kk_{ro}}{m_o}\frac{\partial p}{\partial x}\right)= j\,\frac{\partial S_o}{\partial t} \tag{2}$$

Introducing fluid mobilities $l_w = kk_{rw}/m_w$ and $l_o = kk_{ro}/m_o$, equations (1) and (2) can be written

$$\frac{\partial}{\partial x}\left(l_w\frac{\partial p}{\partial x}\right)= j\,\frac{\partial S_w}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial x}\left(l_o\frac{\partial p}{\partial x}\right)= j\,\frac{\partial S_o}{\partial t}\ .$$

Adding the equations above and using the constraint $S_o + S_w = 1$ implies that

$$\frac{\partial}{\partial x}\left[l_T\frac{\partial p}{\partial x}\right]=0, \qquad l_T = l_w + l_o\ ,$$

and

$$u = -l_T\frac{\partial p}{\partial x} \tag{3}$$

is constant (independent of position x). Defining fluid fluxes (Darcy velocities)

$$u_w = -l_w\frac{\partial p}{\partial x}, \quad u_o = -l_o\frac{\partial p}{\partial x}\ ,$$

$u = u_w + u_o$ is the total flux (total Darcy velocity) and from equation (3) it follows that

$$\frac{\partial p}{\partial x} = -\frac{u}{l_T}\ . \tag{4}$$

Eliminating the pressure term in the mass conservation equation for water using relation (4) gives the Buckley-Leverett equation. Indeed, mass balance of water is expressed by

$$j\,\frac{\partial S_w}{\partial t} - \frac{\partial}{\partial x}\left(l_w\frac{\partial p}{\partial x}\right)=0$$

and the result after substitution becomes

$$j\,\frac{\partial S}{\partial t} + u\frac{\partial f}{\partial x} = 0 \tag{5}$$

where S is water saturation and f denotes water fractional flow function defined by

$$f = \frac{l_w}{l_T} = \frac{k_{rw}/m_w}{k_{rw}/m_w + k_{ro}/m_o}\ . \tag{6}$$

Since viscosities are constant and $S_w + S_o = 1$, f is a function of one saturation only, say water saturation S. The solution of equation (6) with appropriate boundary conditions gives S as a function of position and time.

Dimensionless variables.

It is convenient to introduce dimensionless position x_D and dimensionless time t_D.
Set $x_D = x / L$. Then

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x_D}\frac{\partial x_D}{\partial x} = \frac{\partial f}{\partial x_D}\frac{1}{L}$$

and equation (5) becomes

$$\frac{jL}{u}\frac{\partial S}{\partial t} + \frac{\partial f}{\partial x_D} = 0\,. \tag{7}$$

Next, set dimensionless time $t_D = ut / jL$. Then

$$\frac{\partial S}{\partial t} = \frac{\partial S}{\partial t_D}\frac{\partial t_D}{\partial t} = \frac{\partial S}{\partial t_D}\frac{u}{jL}$$

and substitution in (7)

$$\frac{\partial S}{\partial t_D} + \frac{\partial f}{\partial x_D} = 0\ .$$

Leaving out subscript *D* the equation is written

$$\frac{\partial S}{\partial t} + \frac{\partial f}{\partial x} = 0\ , \tag{8}$$

where $0 \le x \le 1$ and $t \ge 0$.

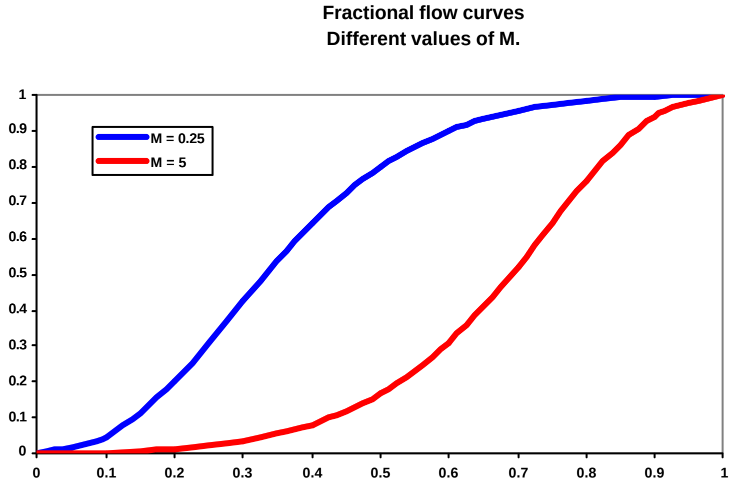
The fractional flow function *f*.

The fractional flow function defined in (6) can be written

$$f = \frac{l_w}{l_T} = \frac{k_{rw} / m_w}{k_{rw} / m_w + k_{ro} / m_o} = \frac{k_{rw}}{k_{rw} + Mk_{ro}}\ ,$$

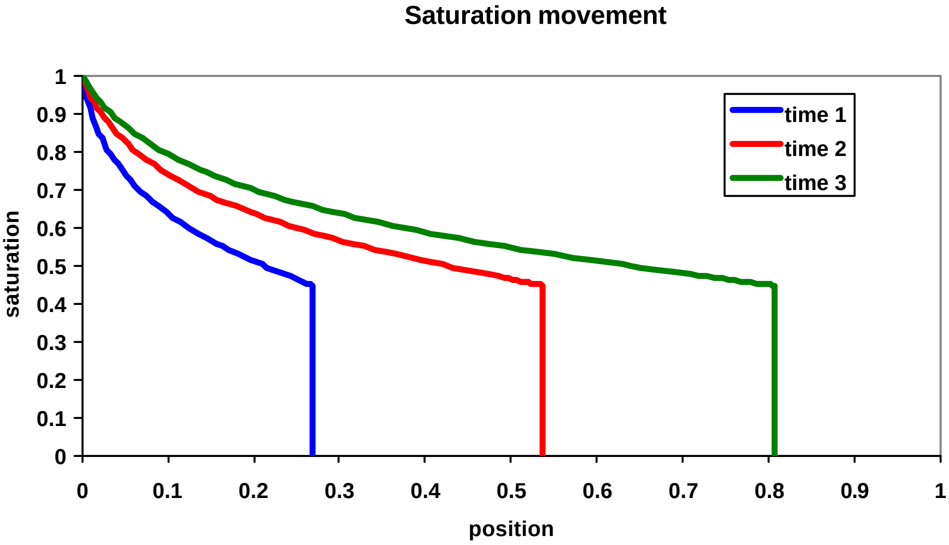
where *M* is the viscosity ratio $M = m_w / m_o$. Normalized Corey type relative permeabilities are specified using expressions $k_{rw}(S) = S^{nw},\quad k_{ro}(S) = (1-S)^{no}$. If Corey relative permeabilities are used, f is completely determined by specifying Corey exponents $n_w,\ n_o$ and viscosity ratio *M*.

The fractional flow function *f(S)* is an increasing function with *f(0) = 0* and *f(1) = 1*. Pictures of f with Corey exponents 2 and two different values of viscosity ratio are depicted below.



Construction of solution using characteristics.

The figure below shows the saturation distribution for three times constructed by solving the Buckley-Leverett equation (8). Main solution features are the front height, speed of front movement and the saturation distribution behind the front.



The following procedure can be used to compute the solution.

1) Determine the front saturation height. It can be done graphically as shown in the figure below.

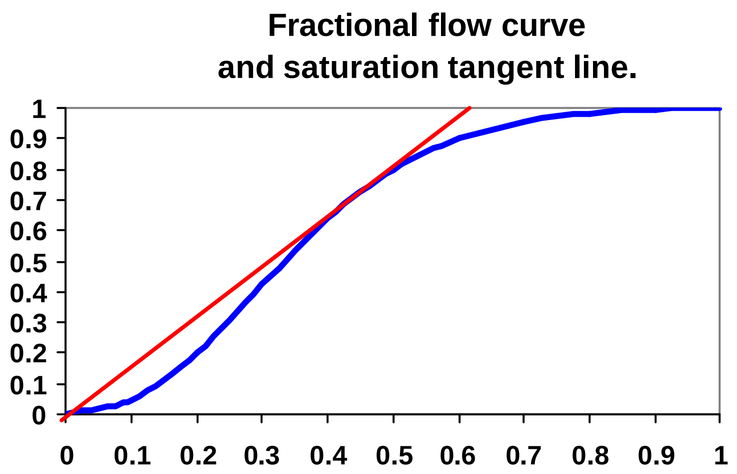
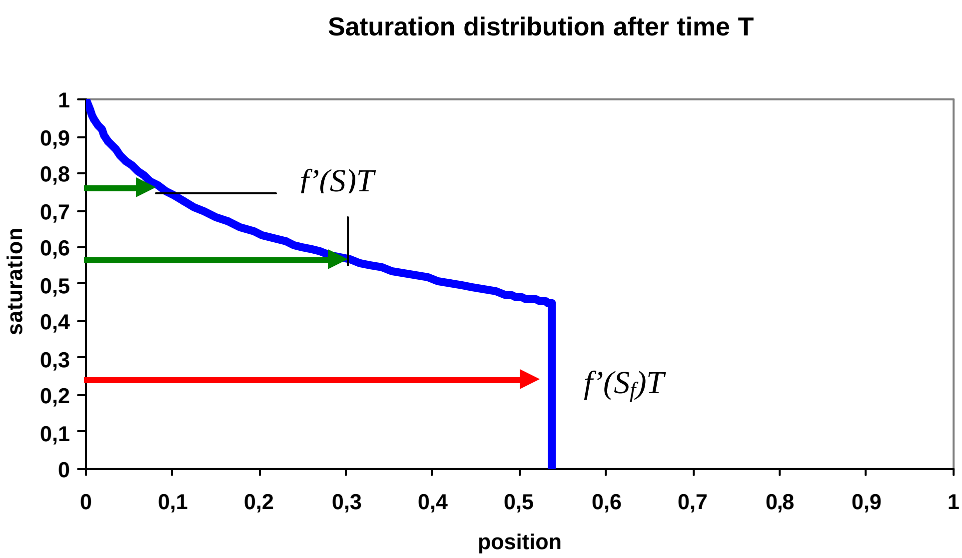


Figure 1 Graphical determination of front saturation

Draw the fractional flow curve and the tangent line passing through the origin. The front height saturation S_f corresponds to the tangent point saturation, 0.45 in the figure.

2) The speed of the saturation front is equal to the steepness of the tangent line used to determine front saturation, i.e. $f'(S_f)$. Hence, after time T the front has moved a distance $f'(S_f)T$.

3) The position of saturations $S > S_f$ after time T is given by $f'(S)T$.



Mathematical derivation of the solution procedure.

The Buckley-Leverett equation (8) can be written

$$\frac{\partial S}{\partial t} + f'(S)\frac{\partial S}{\partial x} = 0 \ , \qquad (9)$$

using the chain rule for derivation

$$\frac{\partial f(S)}{\partial x} = f'(S)\frac{\partial S}{\partial x} \quad .$$

Suppose appropriate boundary conditions are specified and that a unique solution *S(x,t)* of (9) exists. Let *g* = [*x(t),t*] be a smooth curve in the x-t plane and consider the single argument function *h(t) = S(x(t),t)*. Then

$$\frac{dh}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}\frac{dx}{dt} \quad ,$$

and if the curve g satisfies the condition

$$\frac{dx}{dt} = f'(S) \qquad (10)$$

then

$$\frac{dh}{dt} = \frac{\partial S}{\partial t} + f'(S)\frac{\partial S}{\partial x} \ .$$

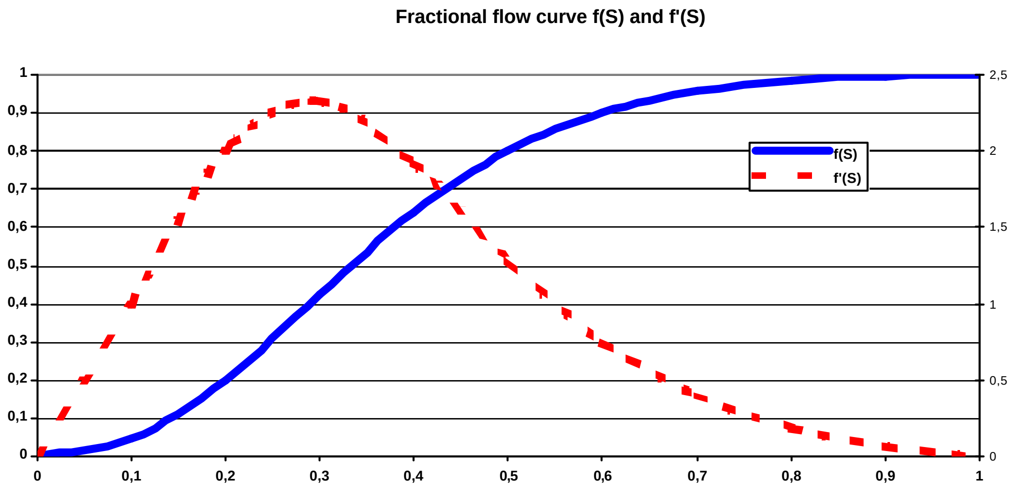
Since *S* is assumed to be a solution of equation (9), *h* is constant and hence *S* is constant, say S_0, on the curve *g*. According to (10) the curve on which *S* is constant equal to S_0, is given by

$$x = X_0 + f'(S_0)t \ . \qquad (11)$$

Start with an initial solution *S(x,0)* equal to zero everywhere except at an infinitesimal interval starting at *x = 0* where the saturation decreases from 1 to 0. With this initial state equation (11) becomes

$$x = f'(S_0)t \ , \qquad (12)$$

and an initial saturation S_0 is moved to position $f'(S_0)T$ at time *T*.



Moving saturations S a distance $f'(S)T$, T elapsed time, gives an unphysical solution since there are positions where two saturation values are computed.

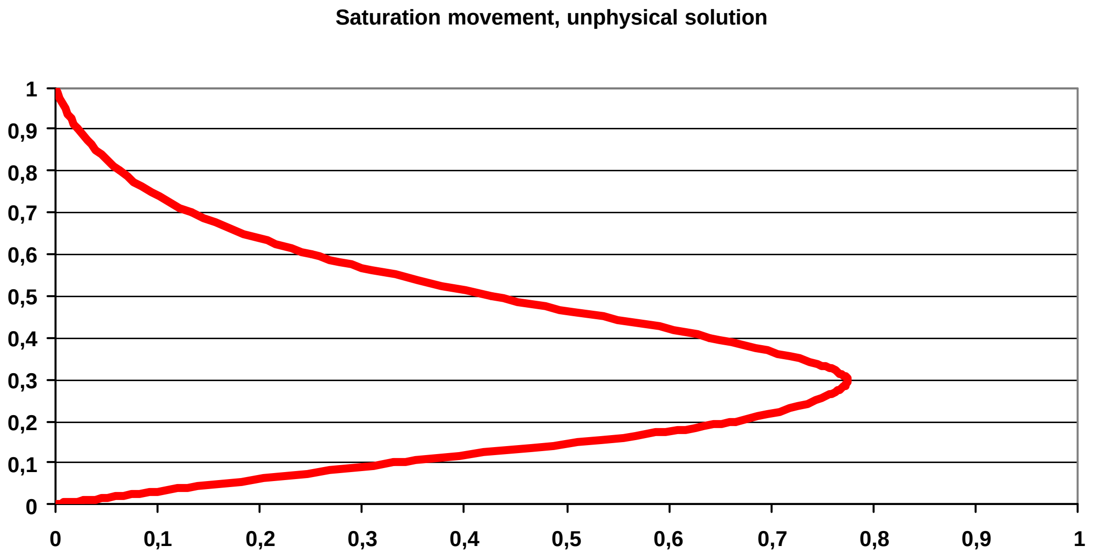


Figure 2 Unphysical solution

To obtain a physical solution, a shock front is introduced, and the solution becomes

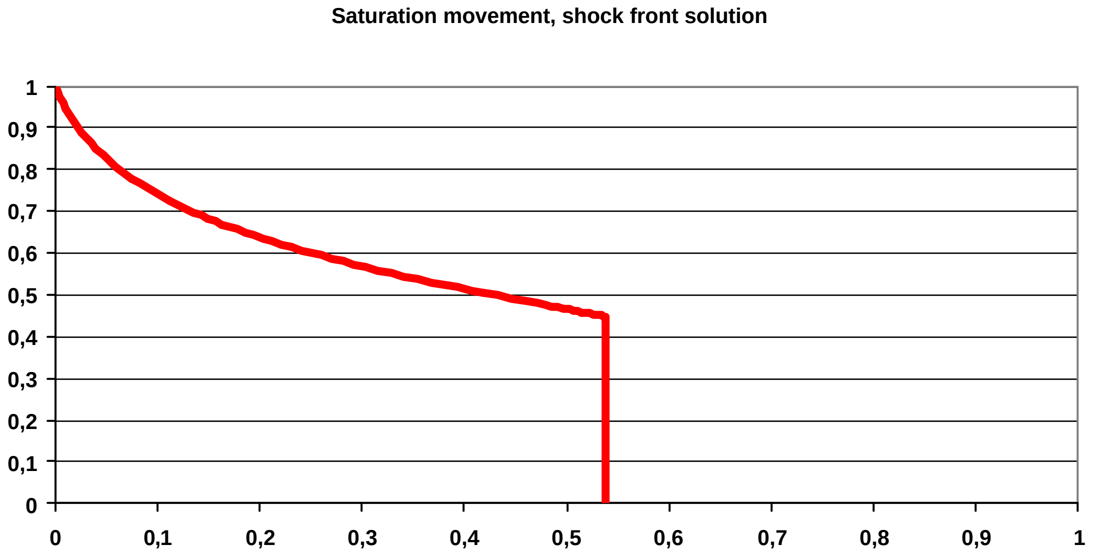


Figure 3 Physical solution with shock front

The position of the shock front can be determined using a material balance argument. Indeed, integrating the functions depicted in Figures 2 and 3 over the saturation interval [0,1] will give the same result. For the case of Figure 2 the result is

$$\int_0^1 f'(z)T dz = [f(1) - f(0)]T = T .$$

If S_f denotes the front saturation the integral of the function in Figure 3 becomes

$$\int_{S_f}^1 f'(z)T dz + f'(S_f)TS_f = [1 - f(S_f)]T + f'(S_f)TS_f ,$$

and the relation

$$f'(S_f)S_f = f(S_f) \qquad (13)$$

determines S_f. The steepness of the line passing through the points (0,0) and $(S_f, f(S_f))$ in Fugure 1 is equal to $f(S_f)/S_f$ and this line will be identical to the tangent line at the point $(S_f, f(S_f))$ because of relation (13).

Existence of a unique solution

The procedure outlined above relies on the existence of a unique solution to the Buckley-Leverett equation with appropriate boundary conditions. In fact, additional conditions must be introduced in order to obtain physical and unique solutions. Rigorous derivations of such conditions are generally difficult and tedious and are beyond the scope of this text.

Computation of recovery before water break-through.

Denote dimensionless time corresponding to real time t by T.
The recovery factor is defined as produced (displaced) oil divided by original oil in place. Before break-through the amount of produced oil is equal to the amount of injected water. The amount of injected water is uAt, where A denotes cross section area, and original oil in place is equal to jAL. Hence, recovery factor before break-through is

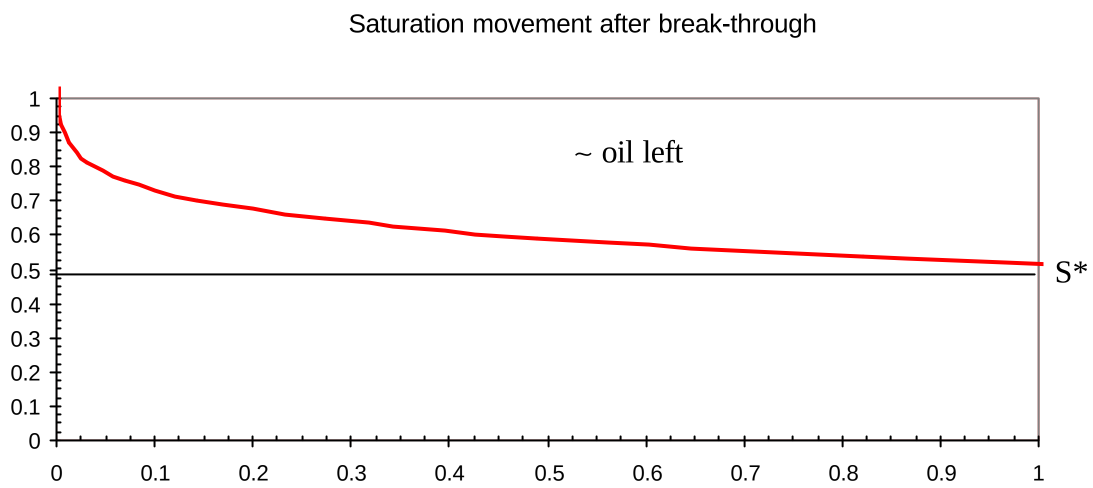
$$R = uAt / jAL = T \ ,$$

i.e. *recovery factor before break-through is equal to dimensionless time.*

Computation of recovery after water break-through.

After break-through the amount of oil left is computed and then recovery is obtained as

(volume of original oil in place – volume of oil left)/volume of original oil in place



The first step will be to compute the saturation S* at distance 1 using the equation f '(S*)T = 1. The area above the saturation curve is computed:

$$1 - S^* - T\int_{S^*}^{1} f'(S)\,dS = 1 - S^* - T(1 - f(S^*)) \ .$$

The amount of oil left is equal to

$$\left[1 - S^* - T(1 - f(S^*))\right] jLA$$

and recovery is obtained using

R = oil in place/original oil in place
R = (original oil in place – oil left)/original oil in place
R = S + T(1 - f(S*)).*