DISPLACEMENT OF TWO FLUIDS DESCRIBED THROUGH A NONLINEAR CONSERVATION LAW

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Abstract. The purpose of this project is to explore a conservation law which can capture main mechanisms involved when one fluid displaces another fluid in a porous media setting, where the two fluids have different properties (viscosity, weight, etc). More precisely, you are asked to deal with the following aspects:

(i) Starting with two mass balance and momentum balance equations, derive the nonlinear conservation law $s_t + f(s)_x = 0$ where the shape of $f(s)$ is determined and the role of different parameters are involved. Herein, s represents the volume fraction of one fluid whereas $1 - s$ represents the other fluid.

(ii) Use/develop an algorithm for computing the analytical solution as well as the numerical approximation through a discrete scheme.

(iii) Explore how different parameters that determine the magnitude of various forces that are involved will give rise to different shapes of $f(s)$ and corresponding different fluid displacement patterns.

1. Introduction

1.1. **Background information.** The mathematical equations needed to describe how a fluid (w) and a fluid (o) move in a porous media with porosity ϕ are given by

$$\begin{aligned}
&\phi(s_w)_t + (U_w)_x = 0, \qquad x \in [0, L] \\
&\phi(s_o)_t + (U_o)_x = 0 \\
&U_w = -\frac{Kk_{rw}(s_w)}{\mu_w} P_x = -K\lambda_w(s_w)P_x \qquad \text{(Darcy's equation)} \\
&U_o = -\frac{Kk_{ro}(s_w)}{\mu_o} P_x = -K\lambda_o(s_w)P_x
\end{aligned} \tag{1}$$

where s_w is volume fraction of fluid (w) and s_o is volume fraction of fluid (o) such that

$$s_w + s_o = 1. \tag{2}$$

Moreover, P is fluid pressure which is common for both fluids, K is absolute permeability, k_{rw}, k_{ro} are relative permeability given in terms of s_w-dependent functions, and μ_w, μ_o are fluid viscosity.

It can be shown from (1) and (2) that we may obtain a conservation law for the behavior of the volume fraction s_w of the form (we have set $s = s_w$ and $s_o = 1 - s$)

$$s_t + f(s)_x = 0, \qquad x \in [0, 1] \tag{3}$$

where x and t are -dimensionless variables.

1.2. **A discrete scheme for computing a numerical approximation of general conservation law.** Consider a discretization of the spatial domain $[0, 1]$ into M cells. This gives rise to grid points $\{x_j\}_{j=1}^{M}$ where x_j is located in the center of the cell $I_j = [x_{j-1/2}, x_{j+1/2}]$. The length of each cell is $\Delta x = x_{j+1/2} - x_{j-1/2}$. All cells are of equal length.

We also consider a discretization of the time interval $[0, T]$ into N steps (of the same length) represented by times $\{t^n\}_{n=1}^{N}$ where the length of each time step is $\Delta t = t^{n+1} - t^n$.

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A discrete version of the problem (3) with an explicit discretization in time then takes the following form:

$$\frac{s_j^{n+1} - s_j^n}{\Delta t} = -\frac{1}{\Delta x}\Big(F_{j+1/2}^n - F_{j-1/2}^n\Big), \qquad j = 1, \dots, M, \tag{4}$$

where $F_{j+1/2}^n \approx f(s(x_{j+1/2}, t))$ for $t \in [t^n, t^{n+1}]$.

Central based flux. We consider the a numerical flux $F_{j+1/2}$ of the following form

$$F_{j+1/2}^n = \frac{1}{2}[f(s_j^n) + f(s_{j+1}^n)] - \frac{a}{2}[s_{j+1}^n - s_j^n], \qquad j = 1, \dots, M-1. \tag{5}$$

where the parameter $a > 0$ is chosen such that

$$\max_s |f'(s)| \le a. \tag{6}$$

In other words, a is an upper bound for the speed $f'(s)$. Note that the flux $F_{j+1/2}^n$ given by (5) is composed of a *central based* discretization term $\frac{1}{2}[f(s_j^n) + f(s_{j+1}^n)]$ followed by a correction term $\frac{a}{2}[s_{j+1}^n - s_j^n]$. We have seen before that the use of the flux

$$F_{j+1/2}^n = \frac{1}{2}[f(s_j^n) + f(s_{j+1}^n)],$$

gives and unconditionally *unstable* scheme. However, the correction term will make the scheme (4) based on (5) stable under the stability condition (CFL-condition)

$$a\frac{\Delta t}{\Delta x} \le 1. \tag{7}$$

The numerical flux (5) might be understood as a discretization of the *vanishing viscosity* model (see the note "1-Intro-ConsLaw" for a brief discussion)

$$s_t + f(s)_x = \varepsilon s_{xx}, \tag{8}$$

or

$$s_t + [f(s) - \varepsilon s_x]_x = 0, \tag{9}$$

where $\varepsilon \sim a\frac{\Delta x}{2}$. In other words, the original model has been replaced by a model where we have added a "viscous" term (damping term) of the form εs_{xx} where ε will vanish as Δx tends to zero (finer and finer grid is employed).

2. **Tasks**

Write a report where you describe how you have solved the following 3 tasks:

Task 1. By using information taken from the document "Theory-Displacement", explain how to obtain the conservation law of the form (3).
(i) Specify assumptions used to derive (3) from (1) and (2). Describe how to obtain explicit expression for the total velocity $U_T = U_w + U_o$, which is used to obtain an expression for P_x. Describe how you obtain a dimensionless form as given by (3) described on the interval $[0, 1]$.

(ii)Identify the flux function $f(s)$ and the role played by the parameter $M = \frac{\mu_w}{\mu_o}$ which dictates the shape of $f(s)$.

(iii) Make a plot of $f(s)$ for different $M \in [0.5, 5]$ when $k_{rw}(s) = s^{nw}$ and $k_{ro}(s) = (1-s)^{no}$ where $nw = 3$ and $no = 2$.
Consider the initial data

$$s(x, t=0) = s_0(x) = \begin{cases} 1.0 & \text{for } x \in [0, 0.1) \\ 0 & \text{for } x \in [0.1, 1] \end{cases} \tag{10}$$

Task 2. This is a continuation of Task 1.
(i) Construct an exact solution at time $T = 0.5$. You may rewrite the matlab script "Example-nonconvex-Analytical-1" where you now only have one decreasing jump involved in the initial data.

(ii) Compare exact solution with a numerical solution. You may use the script "Example-nonconvex-Numerical-1" for that purpose (after a suitable modification).
How many grid cells do you need to obtain a reasonable good approximation to the extact solution?
Vary the physical parameter M and explain what happens with the solution. How do you interpret the simulation results?

Task 3. Next, we focus on the extended case where gravity is included in Darcy's equations. We refer to Exercise 1 in "Intro-ConsLaw-NonConvex-2021" (page 7) for a description.
- Carry out the tasks (i)–(v) listed on page (8) to show how to obtain the conservation law (*) with a specific choice of the flux function $g(s_w)$.
- Compute the exact solution of the conservation law (*) with the two initial data (A) and (B), respectively.