University of Stavanger

ELE520 Machine learning - Examples to background material

Exercise 1

[1] We will find the eigenvalue and eigenvector of the matrix

$$\mathbf{M} = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}. \tag{1}$$

We find the eigenvalues by solving the characteristic equation

$$\begin{aligned} |(\mathbf{M} - \lambda\mathbf{I})| &= 0 \\ \begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} &= 0 \\ (3-\lambda)^2(5-\lambda) - (-2)(-2)(5-\lambda) &= 0 \\ 25 - 35\lambda + 11\lambda^2 - \lambda^3 &= 0 \\ -(\lambda-1)(\lambda-5)^2 &= 0 \\ &\Downarrow \\ \lambda_1 = 1 \quad \lambda_2 &= 5 \quad \lambda_3 = 5. \end{aligned} \tag{2}$$

Next, we find the corresponding eigenvectors by substituting for the eigenvalues we found in $\mathbf{Mx} = \lambda\mathbf{x}$ and solve for $\mathbf{x}$.

[1] From H. Anton: Elementary Linear Algebra, Fifth edition, p 305

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Substituting $\lambda_1 = 1$:

$$\begin{aligned}
\mathbf{Mx} &= \lambda_1\mathbf{x} \\
(\mathbf{M} - \lambda_1\mathbf{I})\mathbf{x} &= \mathbf{0} \\
\begin{pmatrix} 3-\lambda_1 & -2 & 0 \\ -2 & 3-\lambda_1 & 0 \\ 0 & 0 & 5-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= 0 \\
\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= 0 \\
2x_1 - 2x_2 &= 0 \\
-2x_1 + 2x_2 &= 0 \\
4x_3 &= 0 \\
&\Downarrow \\
x_2 = x_1 \quad & x_1\text{arbitrarily} \quad x_3 = 0 \\
&\Downarrow \\
x_1 = t \quad & x_2 = t \quad x_3 = 0
\end{aligned} \tag{3}$$

.

Similarly we solve for $\lambda 2 = 5$:

$$\begin{aligned}
\mathbf{Mx} &= \lambda_1\mathbf{x} \\
(\mathbf{M} - \lambda_2\mathbf{I})\mathbf{x} &= \mathbf{0} \\
\begin{pmatrix} 3-\lambda_2 & -2 & 0 \\ -2 & 3-\lambda_2 & 0 \\ 0 & 0 & 5-\lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= 0 \\
\begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= 0 \\
-2x_1 - 2x_2 &= 0 \\
-2x_1 - 2x_2 &= 0 \\
0x_3 &= 0 \\
&\Downarrow \\
x_2 = -x_1 \quad & x_1\text{arbitrarily} \quad x_3\text{arbitrarily} \\
&\Downarrow \\
x_1 = -s \quad & x_2 = s \quad x_3 = t
\end{aligned} \tag{4}$$

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Here it is important to note that $\lambda_3 = \lambda_2$. This means that the solution found above is a linear combination of the two eigen vectors for $\lambda = 5$ forming a basis for this eigenvalue. The two vectors in the basis can be found from the solution given above according to

$$\mathbf{x} = \begin{pmatrix} -s \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5}$$

Thus we get two eigenvectors $\mathbf{e}_1 = \begin{pmatrix} t & t & 0 \end{pmatrix}^t$, $\mathbf{e}_2 = s\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^t$ and $\mathbf{e}_3 = s\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t$ respectively for λ_1, λ_2 and λ_3.

Values can be chosen freely for t, e.g. $\mathbf{e}_1 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^t$, $\mathbf{e}_2 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^t$ og $\mathbf{e}_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t$.

Exercise 2

[2] We want to diagonalise the matrix

$$\mathbf{M} = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}. \tag{1}$$

From example 1 we have $\lambda_1 = 1, \lambda_2 = 5, \lambda_3 = 5$ and $\mathbf{e}_1 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^t$, $\mathbf{e}_2 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^t$ and $\mathbf{e}_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t$.

Then we get the diagonal matrix as

$$\begin{aligned} \mathbf{E} &= \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \mathbf{\Lambda} &= \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{2}$$

The eigenvectors are arranged according to th ecorresonding eigenvalues sorted in descending order.

Exercise 3

A random variable x is characterised by the probability density function $p(x)$ as illustrated in figure 1.

a) Find c.

[2]From H. Anton: Elementary Linear Algebra, Fifth edition, s 312

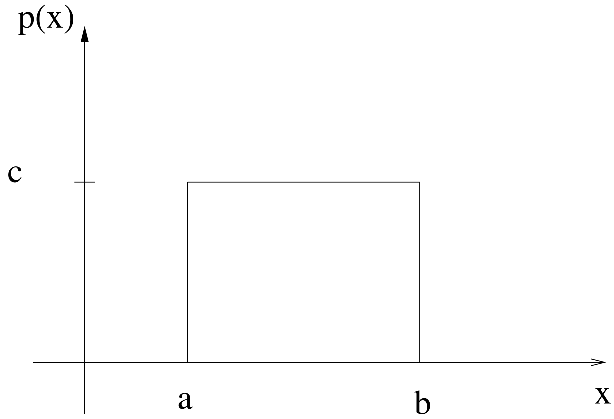

Figure 1: Uniform propability density function.

b) Find the expected value of x according to its *definition*.

c) Find the variancer of x according to its definition.

d) Repeat a) and b) for the following probability density function (figure 2):

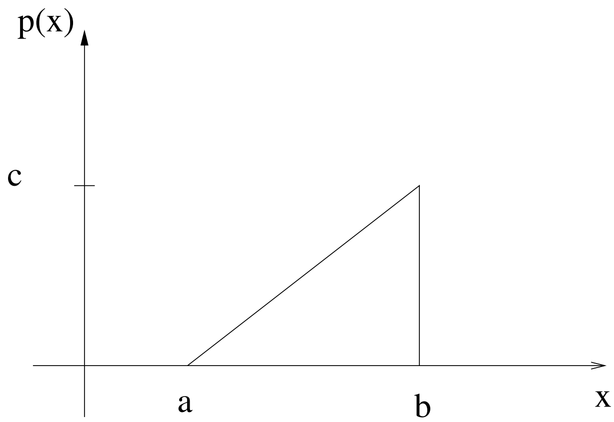

Figure 2: Probability density function for subproblem d).

a) c is a constant and the relationship that the area under the PDF equals 1 gives us:

$$\int_{-\infty}^{\infty} p(x)dx = \int_{a}^{b} cdx = c(b-a) = 1,$$

meaning that $c = \frac{1}{b-a}$.

b)

$$\mu = \mathbb{E}[x] = \int_{-\infty}^{\infty} xp(x)dx = \int_{a}^{b} x\frac{1}{b-a}dx = \frac{a+b}{2}.$$

c)

$$\begin{aligned}
\sigma^2 &= \mathbb{E}[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 p(x) dx \\
&= \int_{-\infty}^{\infty} x^2 p(x) dx - \mu^2 \\
&= \frac{1}{3(b-a)}[x^3]_a^b - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}.
\end{aligned}$$

d) Here we get

$$p(x) = \begin{cases} \frac{c}{b-a}x - \frac{ca}{b-a} & \text{hvis } a < x < b \\ 0 & \text{ellers,} \end{cases}$$

and furthermore:

$$\int_a^b \left(\frac{c}{b-a}x - \frac{ca}{b-a} \right) dx = c\frac{b-a}{2} = 1,$$

meaning that $c = \frac{2}{b-a}$.

The expected value of x is:

$$\mu = \mathbb{E}[x] = \int_{-\infty}^{\infty} xp(x)dx = \int_a^b x \left(\frac{2x}{(b-a)^2} - \frac{2a}{(b-a)^2} \right) dx = \frac{1}{3}\frac{2b^2 - a^2 - ab}{b-a}.$$