Problem 1.

a) $\lambda(\alpha_i | \omega_j) = \lambda_{ij}$ $\quad P(\omega_1) = \frac{1}{25}$ $\quad$ 1 - toxic

$P(\omega_2) = \frac{24}{25}$ $\quad$ 2 - non-toxic

$\lambda_{11} = 0 \quad \lambda_{12} = 250$ $\quad$ α - decision

ω = true state.

$\lambda_{21} = 100\,000 \quad \lambda_{22} = 0$

b) $$R(\alpha_1|x) = \lambda_{11} P(\omega_1|x) + \lambda_{12} P(\omega_2|x) = \frac{\lambda_{12} \cdot P(\omega_2) \, P(x|\omega_2)}{P(x)}$$

$$R(\alpha_2|x) = \lambda_{21} P(\omega_1|x) + \lambda_{22} P(\omega_2|x) = \frac{\lambda_{21} \, P(\omega_1) \cdot P(x|\omega_1)}{P(x)}$$

$$P(x) = P(\omega_1) \cdot P(x|\omega_1) + P(\omega_2) \cdot P(x|\omega_2)$$

c) decision boundary at $R(\alpha_1|x) = R(\alpha_2|x)$

$$\Rightarrow \lambda_{12} P(\omega_2) \cdot P(x|\omega_2) = \lambda_{21} P(\omega_1) P(x|\omega_1)$$

$(\sigma_1 = \sigma_2)$ $$\lambda_{12} P_{\omega_2} \cancel{\frac{1}{\sqrt{2\pi}\sigma_2}} e^{-\frac{(x-\mu_2)^2}{2\sigma^2}} = \lambda_{21} P_{\omega_1} \cancel{\frac{1}{\sqrt{2\pi}\sigma_1}} e^{-\frac{(x-\mu_1)^2}{2\sigma^2}}$$

$$\ln(\lambda_{12} P_{\omega_2}) - \frac{(x-\mu_2)^2}{2\sigma^2} = \ln(\lambda_{21} P_{\omega_1}) - \frac{(x-\mu_1)^2}{2\sigma^2}$$

$$(x-\mu_1)^2 - (x-\mu_2)^2 = [\ln(\lambda_{21} P_{\omega_1}) - \ln(\lambda_{12} P_{\omega_2})] 2\sigma^2$$

$$\cancel{x^2} - 2x\mu_1 + \mu_1^2 - \cancel{x^2} + 2x\mu_2 - \mu_2^2 = 2x(\mu_2 - \mu_1) - \mu_2^2 + \mu_1^2$$

$$x = \frac{[\ln(\lambda_{21} P_{\omega_1}) - \ln(\lambda_{12} P_{\omega_2})] 2\sigma^2 + \mu_2^2 - \mu_1^2}{2(\mu_2 - \mu_1)}$$

$$x = \frac{[\ln(100000 \cdot \frac{1}{25}) - \ln(250 \cdot \frac{24}{25})] 0.0001 + 0.2^2 - 0.4^2}{0.4 - 0.2}$$

$$x = 0.2986$$

Problem 1

d) $\lambda_{21} = 100\,000$

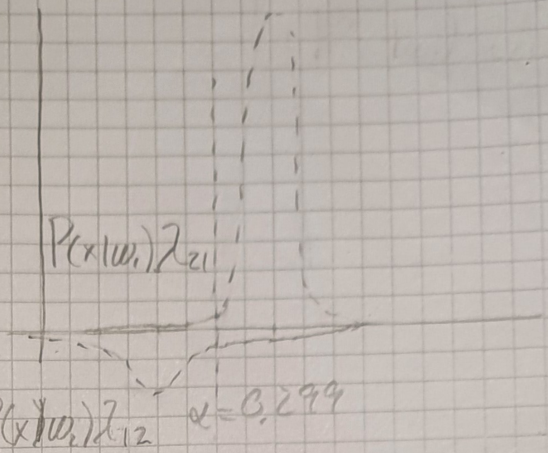
$\lambda_{12} = 250$

$P(\omega_1) = \frac{1}{25}$

$P(\omega_2) = \frac{24}{25}$

$\mu_1 = 0.4$

$\mu_2 = 0.2$



$$R = R(\alpha_1|x)P(x)\Big|_{\alpha}^{\infty} + R(\alpha_2|x)P(x)\Big|_{-\infty}^{\alpha}$$

Used norm scipy stats to calculate:

$$P(x|\omega_1)\Big|_{-\infty}^{\alpha} = 7.8 \cdot 10^{-24} \approx 0$$

$$P(x|\omega_2)\Big|_{\alpha}^{\infty} = \ldots \approx 0$$

$$R = \lambda_{12} P_{\omega_2} P(x|\omega_2)\Big|_{\alpha}^{\infty} + \lambda_{21} P_{\omega_1} P(x|\omega_1)\Big|_{-\infty}^{\alpha}$$

$$R = 0$$

Problem 2

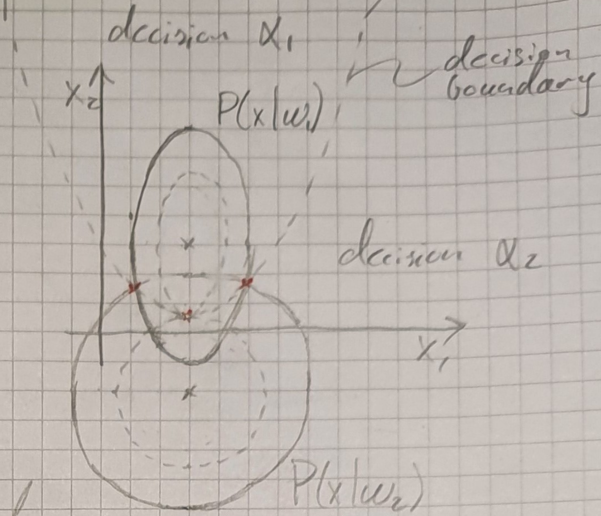
a) $P(\omega_1) = P(\omega_2) = \frac{1}{2}$

$\mu_1 = (3, 3)^T$

$\mu_2 = (3, -2)^T$

$\Sigma_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$

$\Sigma_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$



b) Since priors are equal

$$\text{Decide} \begin{cases} \omega_1 & \text{if } P(x|\omega_1) > P(x|\omega_2) \\ \omega_2 & \text{else} \end{cases}$$

Since $\Sigma_1 \neq \Sigma_2$ we have non-linear boundary of form: Using online matrix calc to find:

$$g_i(x) = \underline{x}^T \Theta_i \underline{x} + \theta_i^T \underline{x} + \theta_{i0}$$

$$\Theta_i = -\frac{1}{2} \Sigma_i^{-1} \quad i=1 \Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix} \quad i=2 \Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\theta_i = \Sigma_i^{-1} \mu_i \quad i=1 \Rightarrow \begin{bmatrix} 6 \\ 3/2 \end{bmatrix} \quad i=2 \Rightarrow \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$$

$$\theta_{i0} = -\frac{1}{2} \mu_i^T \Sigma_i^{-1} \mu_i - \frac{1}{2} \ln|\Sigma_i| + \ln P(\omega_i)$$

$$i=1 \Rightarrow -\frac{45}{4} - 0 + \ln\frac{1}{2} = -11.94$$

$$i=2 \Rightarrow -\frac{13}{4} - \frac{1}{2}\ln 4 + \ln\frac{1}{2} = -4.64$$

$$g_1(x) = -x_1^2 - \frac{1}{4}x_2^2 + 6x_1 + \frac{3}{2}x_2 - 11.94$$

$$g_2(x) = -\frac{1}{4}x_1^2 - \frac{1}{4}x_2^2 + \frac{3}{2}x_1 - x_2 - 4.64$$

$$-\frac{3}{4}x_1^2 + 0x_2^2 + 4.5x_1 + \frac{5}{2}x_2 - 7.3$$

$$x_2 = \frac{2}{5}\left(\frac{3}{4}x_1^2 - \frac{9}{2}x_1 + 7.3\right) = \frac{6}{20}x_1^2 - \frac{18}{10}x_1 + 2.92$$

Problem 3

$$\cancel{}(\mu_i - \mu_j)^T\left(x - \frac{1}{2}(\mu_i + \mu_j)\right) + \frac{\sigma^2}{\|\mu_i - \mu_j\|^2} \ln \frac{P_{\omega_i}}{P_{\omega_j}} (\mu_i - \mu_j)$$

$$g_i(x) = g_j(x)$$

$$-\frac{\|x - \mu_i\|^2}{2\sigma^2} + \ln P(\omega_i) + \frac{\|x - \mu_j\|^2}{2\sigma^2} - \ln P(\omega_j) = 0$$

$$\|x - \mu_j\|^2 - \|x - \mu_i\|^2 + 2\sigma^2 \ln \frac{P_{\omega_i}}{P_{\omega_j}} = 0$$

$$(x - \mu_j)^T(x - \mu_j) - (x - \mu_i)^T(x - \mu_i)$$

$$\cancel{x^T x} - 2\mu_j^T x + \mu_j^T \mu_j - \cancel{x^T x} + 2\mu_i^T x - \mu_i^T \mu_i$$

$$2x(\mu_i - \mu_j)^T - (\mu_i - \mu_j)^T(\mu_i + \mu_j)$$

$$(\mu_i - \mu_j)^T\left(2x - (\mu_i + \mu_j)\right) + \frac{2\sigma^2 (\mu_i - \mu_j)}{(\mu_i - \mu_j)^T(\mu_i - \mu_j)} \ln \frac{P(\omega_i)}{P(\omega_j)}$$

$$(\mu_i - \mu_j)^T\left(x - \frac{1}{2}(\mu_i + \mu_j)\right) + \frac{\sigma^2}{\|\mu_i - \mu_j\|^2} \ln \frac{P(\omega_i)}{P(\omega_j)} (\mu_i - \mu_j)$$

$$x_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{\sigma^2}{\|\mu_i - \mu_j\|^2} \ln \frac{P(\omega_i)}{P(\omega_j)} (\mu_i - \mu_j)$$

$$\Rightarrow (\mu_i - \mu_j)^T(x - x_0) = 0$$

$$\theta = \mu_i - \mu_j$$

$$\Rightarrow \theta^T(x - x_0) = 0$$